EXACT SOLUTIONS AND TRANSFORMATION PROPERTIES OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS FROM GENERAL RELATIVITY

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ABSTRACT

Various families of exact solutions to the Einstein and Einstein-Maxwell field equations of General Relativity are treated for situations of sufficient symmetry that only two independent variables arise. The mathematical problem then reduces to consideration of sets of two coupled non-linear differential equations.

The physical situations in which such equations arise include: a) the external gravitational field of an axi-symmetric, uncharged steadily rotating body, b) cylindrical gravitational waves with two degrees of freedom, c) colliding plane gravitational waves, d) the external gravitational and electromagnetic fields of a static, charged axisymmetric body, and e) colliding plane electromagnetic and gravitational waves. Through the introduction of suitable potentials and coordinate transformations, a formalism is presented which treats all these problems simultaneously. These transformations and potentials may be used to generate new solutions to the Einstein-Maxwell equations from solutions to the vacuum Einstein equations, and vice-versa.

The calculus of differential forms is used as a tool for generation of similarity solutions and generalized similarity solutions. It is further used to find the invariance group of the equations; this in turn leads to various finite transformations that give new, physically distinct solutions from old. Some of the above results are then generalized to the case of three independent variables.

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INTRODUCTION

This thesis is devoted to a study of the Einstein and Einstein-Maxwell equations in two and three independent variables. Relationships between the vacuum Einstein equations and the coupled Einstein-Maxwell equations are found, and these relationships may be exploited to obtain new exact solutions to physically distinct problems from known solutions. In addition, the calculus of differential forms is shown to be a powerful tool for generation of exact solutions to these problems.

The first chapter presents a very brief discussion of the Einstein and Einstein-Maxwell equations. The concept of a Killing vector field is introduced, and the importance of exact solutions is discussed.

The second chapter is concerned with space-times containing two commuting Killing vectors. There are five distinct physical problems (a) the external gravitational field of an axisymmetric, uncharged, steadily rotating body (b) cylindrical gravitational waves, (c) colliding plane gravitational waves, (d) cylindrically symmetric static Einstein-Maxwell fields, and (e) colliding plane gravitational and plane electromagnetic waves. The field equations for all the above problems are discussed, and it is then shown how all of these field equations may be brought into an identical form. This result shows that there is a duality between the particular component of the gravitational field due to stationary rotational motion and the electromagnetic field. Vacuum solutions with two Killing vectors are also seen to be Einstein-Maxwell solutions.

The third chapter begins by demonstrating that the well known Curzon solution is a generalized similarity solution. The Kerr and Tomimatsu-Sato vacuum solutions are discussed as solutions to the Einstein-Maxwell equations. The calculus of differential forms is then used to find the isogroup of the equations. This group is then used to generate some similarity solutions. The finite transformation generated by the isogroup is found, and the well known Ehlers transformation is seen to be a special case of this transformation. Finally, some new generalized similarity solutions are presented, and soliton-like solutions of Harrison are discussed.

In Chapter 4, we extend some of the above results to the more general case of problems with one Killing vector. The Einstein-Maxwell equations with one Killing vector are presented. The results of Chapter 2 are then extended to this case by studying various special cases of the equations. We then show that the finite invariance transformation of Chapter 3 is also an invariance transformation for these problems as well. These results are even more striking than those of Chapter 2, since we are dealing with a more general space-time. Again there is a dualism between rotational motion and electromagnetism. This is very important for the three variable case, since three variable solutions are rare and we now have three solutions corresponding to any one known solution.

Finally, an appendix is added to discuss the differential form techniques used in Chapter 3 to find similarity and generalized similarity solutions.

CHAPTER 1

GENERAL RELATIVITY AND EXACT SOLUTIONS

In this chapter we present a brief discussion of the Einstein field equations of General Relativity. Although a detailed account of Riemannian geometry and the Theory of General Relativity is beyond the scope of the present treatise, we present here a brief description which is intended to familiarize the reader with some basic solution techniques of the field equations.

1.1 The Einstein Field Equations

In the General Theory of Relativity, we seek a metric

$$ds^2 = g_{\mu\nu}\,dx^\mu dx^\nu \qquad \mu,\nu = 1,\ldots 4 \tag{1.1.1}$$

with signature $(+,-,-,-)$ or $(-,+,+,+)$ which describes the local geometry at each point of a Riemannian manifold with coordinates x^μ. The $g_{\mu\nu}$ are the components of a symmetric covariant tensor, so there are in fact only ten independent $g_{\mu\nu}$, which take the place of the classical Newtonian gravitational potential. The $g_{\mu\nu}$ are to be found as solutions to the Einstein field equations

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \tag{1.1.2}$$

where $G_{\mu\nu}$ is the Einstein tensor,

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R. \tag{1.1.3}$$

$R_{\mu\nu}$ is the Ricci (contracted) curvature tensor, and R is the scalar curvature. $T_{\mu\nu}$ is the stress-energy tensor due to any matter or electromagnetic fields present in the region of space-time where we wish to solve (1.1.2). In the absence of such sources, $T_{\mu\nu}$ is identically zero, and (1.1.2) reduces to

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 . \tag{1.1.4}$$

Contraction of (1.1.4) yields $R = 0$, so that equivalent to (1.1.4) is

$$R_{\mu\nu} = 0 . \tag{1.1.5}$$

(1.1.5) is a set of ten coupled nonlinear second order partial differential equations. The ten equations are not independent, however, due to the Bianchi identities. There are in fact six independent equations contained in (1.1.5). The procedure for finding an exact solution of (1.1.5) usually consists of partially determining the form of the metric (1.1.1) by geometric and physical considerations, and then substituting this form of the $g_{\mu\nu}$ into (1.1.5).

The first known exact solution was found in this fashion by Schwarzschild, and is a good example of this procedure. We seek the external gravitational field of a static, spherically symmetric mass distribution. We would expect the external field of such a distribution to also be static and spherically symmetric, and hence we may write

$$ds^2 = -W(r)dt^2 + U(r)dr^2 + V(r)(r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2) \tag{1.1.6}$$

with time coordinate t and spherical spatial coordinates r, θ, and φ . U , V and W are functions of r only.

Letting

$$r_1^2 = r^2 V(r)$$

(1.1.6) becomes

$$ds^2 = -W_1(r_1)dt^2 + U_1(r_1)dr_1^2 + r_1^2 d\theta^2 + r_1^2 \sin^2\theta d\phi^2 \tag{1.1.7}$$

where U_1 and W_1 are arbitrary functions of r_1. We may view this as merely a rescaling of the coordinate r, and thus drop the suffix, writing (1.1.7) in the form

$$ds^2 = -e^{\nu}dt^2 + e^{\lambda}dr^2 + r^2 \sin^2\theta d\phi^2 + r^2 d\theta^2 \tag{1.1.8}$$

Upon substituting (1.1.8) into (1.1.5), we arrive at the following set of ordinary differential equations:

$$R_{11} = \frac{\nu''}{2} - \frac{1}{4}\lambda'\nu' + \frac{1}{4}\nu'^2 - \frac{\lambda'}{r} = 0 \tag{1.1.9a}$$

$$R_{22} = e^{-\lambda}\left(1 + \frac{r}{2}(\nu' - \lambda')\right) - 1 = 0 \tag{1.1.9b}$$

$$R_{33} = \sin^2\theta\left(e^{-\lambda}\left(1 + \frac{r}{2}(\nu' - \lambda')\right) - 1\right) = 0 \tag{1.1.9c}$$

$$R_{44} = e^{\nu-\lambda}\left(-\frac{1}{2}\nu'' + \frac{1}{4}\lambda'\nu' - \frac{1}{4}\nu'^2 - \frac{\nu'}{r}\right) = 0 \tag{1.1.9d}$$

where a prime denotes differentiation with respect to r. The remaining components of $R_{\mu\nu}$ are identically zero. Since (1.1.9c) is a repetition of (1.1.9b), we must in fact solve only (1.1.9a,b) and (d). From (a) and (d) we have $\lambda' = -\nu'$.

As r tends to infinity we will have λ and ν tend to zero, so that the metric reduces to a Minkowski one there.

We consequently obtain

$$\lambda = -\nu .$$

(1.1.9b) then becomes

$$e^{\nu}(1 + r\nu') = 1 .$$

Setting

$$e^{\nu} = \gamma$$

this becomes

$$\gamma + r\gamma' = 1 .$$

Integration yields

$$\gamma = 1 - \frac{2M}{r}$$

where $2M$ is a constant of integration. Hence the metric (1.1.8) is

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \qquad (1.1.10)$$

This is the celebrated Schwarzschild solution. From it, Einstein was able to make his calculations of the perihelion shift of Mercury and the bending of light rays by the sun. It is also used in work on black holes. The constant M is interpreted as the total mass of the distribution.

For completeness, we present a brief discussion of the Einstein-Maxwell equations. These field equations hold when the energy-momentum content of space-time is due solely to electromagnetic fields, and this is sometimes referred to as the electrovac case of the Einstein equations. Maxwell's equations are then coupled to the Einstein equations as follows. In (1.1.2), the stress-energy tensor is given by

$$T_{\mu\nu} = (F_{\mu\sigma} F^{\sigma}{}_{\nu} - \frac{1}{4} F_{\tau\sigma} F^{\tau\sigma} g_{\mu\nu}) \tag{1.1.11}$$

where $F_{\mu\nu}$ is the electromagnetic field tensor. Maxwell's equations in curved space-time are identical to those in flat space-time, but partial derivatives are replaced by covariant derivatives with respect to the metric tensor $g_{\mu\nu}$:

$$F_{\alpha\beta;\gamma} + F_{\beta\gamma;\alpha} + F_{\gamma\alpha;\beta} = 0 \tag{1.1.12a}$$

$$F^{\alpha\beta}{}_{;\beta} = 4\pi J^{\alpha} \tag{1.1.12b}$$

where a semicolon denotes covariant differentiation and J^{α} is the current density. (1.1.2) and (1.1.12) are then to be solved together, consistently, in the same fashion as (1.1.5). In the absence of sources, $J^{\alpha} = 0$.

1.2 Killing Vector Symmetries

A geometrical concept of symmetry often used in formulating solutions to the Einstein field equations is that of the Killing vector field. We introduce this concept in the following manner: let the metric components $g_{\mu\nu}$ relative to some particular coordinates x^{μ} be independent of one of the coordinates, x^1 , so that

$$\frac{\partial g_{\mu\nu}}{\partial x^1} = 0.$$

Geometrically, this says that any curve

$$x^{\alpha} = C^{\alpha}(\lambda)$$

(where λ parametrizes the curve) can be translated in the

x^1 direction by the coordinate shift $\Delta x^1 = \varepsilon$ to form a congruent curve given by

$$x^\alpha = c^\alpha(\lambda) \qquad \text{for} \quad \alpha \neq 1$$

and

$$x^1 = c^1(\lambda) + \varepsilon .$$

Let the original curve run from λ_1 to λ_2 and have length L given by

$$L = \int_{\lambda_1}^{\lambda_2} \left[g_{\mu\nu}(x(\lambda)) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right]^{1/2} d\lambda .$$

The displaced curve then has length given by

$$L(\varepsilon) = \int_{\lambda_1}^{\lambda_2} \left[\left\{ g_{\mu\nu}(x(\lambda)) + \varepsilon \frac{\partial g_{\mu\nu}}{\partial x^1} \right\} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right]^{1/2} d\lambda .$$

Since the coefficient of ε in the integrand is zero, the lengths of the two congruent curves are the same. In general relativity, the basic physics is determined by the measurement of length (more properly, interval) along curves. An invariance of the $g_{\mu\nu}$ such as described here thus reflects a symmetry of the physics. In fact, in the particular coordinates x^μ we have described a vector field $\vec{\xi} = \frac{\partial}{\partial x^1}$. If such a "Killing" field exists, it provides an infinitesimal generator of a one parameter group of length preserving translations. We now show that such a vector field satisfies, in a general coordinate system, a set of partial differential equations called Killing's equations:

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0 . \qquad (1.2.1)$$

Since this condition is expressed in covariant form, we need only establish it in the preferred coordinate system used above in order that it hold in all coordinate systems. In the preferred coordinate system, the vector field $\vec{\xi}$ has components

$$\xi^\mu = 0, \ \mu \neq 1$$
$$\xi^1 = 1.$$

Thus

$$\mathop{\pounds}_{\vec{\xi}} g_{\mu\nu} \equiv g_{\mu\nu,\sigma}\xi^\sigma + g_{\mu\sigma}\xi^\sigma{}_{,\nu} + g_{\sigma\nu}\xi^\sigma{}_{,\mu} = g_{\mu\nu,1} = 0$$

where $\mathop{\pounds}_{\vec{\xi}}$ denotes the Lie derivative with respect to the vector field $\vec{\xi}$. (1.2.1) is merely a covariant way of writing the Lie derivative of $g_{\mu\nu}$.

Conversely, if a metric has coefficients which are independent of a coordinate X^K , then the geometry described by that metric possesses a Killing vector field $\frac{\partial}{\partial X^K}$. For example, the Schwarzschild solution, (1.1.10), is independent of t and φ and possesses by inspection two Killing vector fields:

$$\frac{\partial}{\partial t}$$
$$\frac{\partial}{\partial \varphi}.$$

The metric (1.1.10) in fact contains two additional Killing vector fields. To see this, we first transform (1.1.10) to isotropic coordinates defined by

$$r = \left(1 + \frac{M}{2\bar{r}}\right)^2 \bar{r}.$$

(1.1.10) then becomes

$$ds^2 = -\left(1+\frac{M}{2\bar{r}}\right)^2\left(d\bar{r}^2+\bar{r}^2d\theta^2+\bar{r}^2\sin^2\theta\, d\varphi^2\right)+\frac{\left(1-\frac{M}{2\bar{r}}\right)^2}{\left(1+\frac{M}{2\bar{r}}\right)^2}\,dt^2 \ . \quad (1.2.2)$$

If we now make the change to isotropic rectangular coordinates given by

$$x = \bar{r}\sin\theta\cos\varphi$$
$$y = \bar{r}\sin\theta\sin\varphi$$
$$z = \bar{r}\cos\theta$$

then (1.2.2) becomes

$$ds^2 = -\left(1+\frac{2M}{\bar{r}}\right)^4\left(dx^2+dy^2+dz^2\right)+\frac{\left(1-\frac{M}{2\bar{r}}\right)^2}{\left(1+\frac{M}{2\bar{r}}\right)^2}\,dt^2 \qquad (1.2.3)$$

with

$$\bar{r} = (x^2+y^2+z^2)^{1/2}.$$

In these coordinates, one may verify that in addition to the timelike Killing vector field $\frac{\partial}{\partial t}$, there are three spacelike ones given by

$$\vec{\xi}_x = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$$

$$\vec{\xi}_y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$$

$$\vec{\xi}_z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \ .$$

The $\vec{\xi}_z$ Killing vector field is equivalent to the $\frac{\partial}{\partial \varphi}$ one, so there are in fact four independent Killing vector fields for the Schwarzschild geometry.

Most of the recent work on exact solutions falls into the category of solutions with Killing vectors, as we shall see in the next chapter. The geometrical property of the Killing vector is used to assert the existence of a special coordinate system adapted to the particular physical situation, and so to reduce the number of independent variables that actually appear in the field equations. Most known solutions possess at least two Killing vectors, since then one has only two independent variables in the field equations.

1.3 The Importance of Exact Solutions

When one begins the task of solving a set of nonlinear partial differential equations such as Einstein's equations, there are basically three approaches available: exact solutions, approximation schemes, and numerical computation. Let us consider their advantages and disadvantages with regard to General Relativity.

Approximate schemes have of course been much used in Relativity: there are the weak-field and slow-motion approximations, perturbation expansions about known exact solutions, and so on. A serious criticism is that many of these schemes have not been rigorously shown to be valid. Many questions as to the uniform validity of perturbation expansions, error estimates, etc., are still unanswered. (Ehlers, et al, 1976).

Many realistic problems in gravitation so far lie outside the domain of approximation schemes. An excellent example of this is the production of large amplitude gravitational waves which accompanies the formation of a neutron star. The disturbances in problems of this type are too large to be covered by any perturbation scheme, and we must thus choose between exact solutions and numerical analysis. The application of numerical methods to Einstein's equations, however, is only just recently beginning to yield quantitative results (Smarr, 1977). Exact solutions offer an alternative and complementary approach.

The discovery of exact solutions in the past has been rather erratic, depending more on guesswork and intuition than on any systematic methods.

In the following chapters, we present a more systematic treatment of some problems in Relativity. Much of this work may be viewed as an extension of the work of Kinnersley, (Kinnersley, 1975) who published a comprehensive survey of axially symmetric exact solutions in Relativity, of Ernst (Ernst, 1968), of Harrison (Harrison, 1968), and of Harrison and Estabrook (Harrison and Estabrook, 1971).

CHAPTER 2

THE NONLINEAR EQUATIONS FOR VARIOUS SOURCE-FREE EINSTEIN AND EINSTEIN-MAXWELL PROBLEMS WITH TWO INDEPENDENT VARIABLES

In this chapter, we present a discussion of the source-free Einstein and Einstein-Maxwell equations in two independent variables. These problems all possess two commuting Killing vectors. There are five physically relevant problems of this type, treated in sections 2.1 to 2.5. In section 2.6 we present a formalism to treat these problems simultaneously.

2.1 External Gravitational Field of an Axisymmetric, Uncharged, Steadily Rotating Body

The metric in this case may be put into a canonical form first introduced by Lewis (Lewis, 1932) and now known as the Weyl metric. We assume the existence of two Killing vector fields, which we write as $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \varphi}$. The metric must therefore be independent of t , φ . For axial symmetry, the metric must also be invariant under the transformation

$$t \rightarrow -t$$
$$\varphi \rightarrow -\varphi$$

(See Synge, 1960). Thus the metric cannot contain the terms $dx_1 d\varphi$, $dx_2 d\varphi$, $dx_1 dt$, $dx_2 dt$, while the term $d\varphi\, dt$ may appear. We thus have

$$ds^2 = g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + 2g_{12}dx^1dx^2 + g_{33}d\varphi^2 + 2g_{03}d\varphi dt + g_{00}dt^2 \quad (2.1.1)$$

where the g_{ik} are independent of t , φ and x^1 , x^2 are asymptotically space-like coordinates. The two-dimensional metric

$$ds_2^2 = g_{11}(dx^1)^2 + 2g_{12}dx^1dx^2 + g_{22}(dx^2)^2$$

must be conformally flat, since the Weyl conform tensor vanishes identically in two dimensions. There thus exists a transformation to new coordinates

$$x^{1'} = x^{1'}(x^1, x^2) \qquad x^{2'} = x^{2'}(x^1, x^2)$$

in terms of which the two-dimensional metric takes the form

$$ds_2^2 = e^{\mu(x^{1'}, x^{2'})}\left[(dx^{1'})^2 + (dx^{2'})^2\right].$$

The metric (2.1.1) then becomes

$$ds^2 = e^{\mu}\left[(dx^{1'})^2 + (dx^{2'})^2\right] + g_{33}d\theta^2 + 2g_{03}d\theta dt + g_{00}dt^2 \qquad (2.1.2)$$

By appropriate choice of the coordinates $x^{1'}$, $x^{2'}$ and the form of the functions μ, g_{33}, g_{03}, and g_{00}, (2.1.2) finally may be put in the form (See Reina and Treves, 1976)

$$ds^2 = f(dt + \omega d\theta)^2 - f^{-1}\left[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\theta^2\right] \qquad (2.1.3)$$

where f, ω, and γ are functions of ρ and z only. If we regard ρ, z, θ as cylindrical coordinates in a flat space, a gradient operation ∇ is defined which is convenient to use in what follows.

We now distinguish two cases of (2.13): $\omega \equiv 0$ or $\omega \not\equiv 0$. In the ρ, θ, z, t coordinate system, the Killing vector field $\vec{V} = \partial/\partial t$ has contravariant coordinate components given by

$$V^{\mu} = (0, 0, 0, 1).$$

Using the metric (2.1.3), we see that the covariant components of $\vec{V}$ are given by

$$\vec{V}_{\mu} = (0, f\omega, 0, f).$$

For a static space-time we require the existence of instantaneous space-like 3-surfaces that are orthogonal to the time-like Killing congruence, and thus orthogonal to the vector field

$$V_\mu = (0, \omega, 0, 1).$$

This requires that V_μ be proportional to the gradient of a scalar:

$$V_\mu = h\,\Psi_{,\mu}$$

where h is an arbitrary function of the coordinates. Differentiating, we find

$$V_{\mu,\nu} - V_{\nu,\mu} = h_{,\nu} V_\mu - h_{,\mu} V_\nu .$$

However, if we write out the components of the above equation, recalling that ω is a function of ρ and z only, we find no nontrivial solution, unless $\omega \equiv 0$. Therefore $\omega \equiv 0$ is the necessary and sufficient condition for a static metric. If $\omega \not\equiv 0$, the metric is called stationary. The metric and the resulting physics are then invariant along the lines of the time-like Killing congruence, but invariant orthogonal 3-surfaces do not exist.

Similarly, the vector field $\vec{V} = \frac{\partial}{\partial \phi}$ has contravariant coordinate components given by

$$V^\mu = (0, 1, 0, 0)$$

and covariant coordinate components given by

$$V_\mu = (0, f\omega^2 - f^{-1}\rho^2, 0, f\omega) .$$

In a manner identical to that above, we find that $\omega \equiv 0$ is the necessary and sufficient condition for the existence of 3-surfaces that are orthogonal to the space-like Killing congruence.

Furthermore, we see that these two Killing vector fields commute. The commutator of two vector fields $\vec{U}$ and $\vec{V}$ is defined as

$$[\vec{U},\vec{V}] = (U^\alpha V^\beta{}_{,\alpha} - V^\alpha U^\beta{}_{,\alpha})\frac{\partial}{\partial x^\beta}.$$

In the ρ, θ, z, t coordinate system, we see that the commutator of the space-like and time-like Killing vector fields defined above vanishes for any value of ω. These Killing vector fields thus define a family of invariant 2-surfaces and are called 2-forming.

The Einstein vacuum field equations $R_{\mu\nu}=0$ for the metric (2.1.3) reduce to

$$\nabla\cdot[f^{-1}\nabla f + \rho^{-2}f^2\omega\nabla\omega] = 0 \qquad (2.1.4a)$$

$$\nabla\cdot[\rho^{-2}f^2\nabla\omega] = 0 \qquad (2.1.4b)$$

and

$$\gamma_{,\rho} = \tfrac{1}{4}\rho f^{-2}(f_{,\rho}^2 - f_{,z}^2) - \tfrac{1}{4}\rho^{-1}f^2(\omega_{,\rho}^2 - \omega_{,z}^2) \qquad (2.1.5)$$

$$\gamma_{,z} = \tfrac{1}{2}\rho f^{-2} f_{,\rho} f_{,z} - \tfrac{1}{2}\rho^{-1} f^2 \omega_{,\rho}\,\omega_{,z} \quad .$$

When (2.1.4) are satisfied, (2.1.5) are integrable and determine γ up to a constant. If f becomes negative, we must use this constant to maintain the correct signature of the metric, by replacing

$$\gamma \rightarrow \gamma + \frac{i\pi}{2}.$$

In this case, $\mathcal{Q}$ becomes a time-like coordinate. Otherwise, γ may be ignored in the process of finding solutions.

We thus concentrate on solving (2.1.4). (2.1.4b) is the integrability condition for the existence of a related function Ω, defined by

$$\nabla\Omega = \rho^{-1} f^2 \hat{e}_{\mathcal{Q}} \times \nabla\omega \tag{2.1.6}$$

where $\hat{e}_{\mathcal{Q}}$ is a unit vector in the $\mathcal{Q}$ direction. Ω is usually referred to in the literature as the "twist" potential, since its essential effect is to interchange the components of $\nabla\omega$. Written explicitly, (2.1.6) becomes

$$\begin{aligned} \Omega_{,\rho} &= \rho^{-1} f^2 \omega_{,z} \\ \Omega_{,z} &= -\rho^{-1} f^2 \omega_{,\rho} \end{aligned} \tag{2.1.7}$$

where the so-called "twisting" is displayed explicitly. We may eliminate ω for Ω to obtain an alternative pair of field equations equivalent to (2.1.4):

$$\begin{aligned} &\nabla\cdot[f^{-2}(f\nabla f + \Omega\nabla\Omega)] = 0 \\ &\nabla\cdot[f^{-2}\nabla\Omega] = 0 . \end{aligned} \tag{2.1.8}$$

Letting $f = e^{U}$, we may write the above equations explicitly as

$$U_{,\rho\rho} + \frac{U_{,\rho}}{\rho} + U_{,zz} = -e^{-2U}(\Omega_{,\rho}^{2} + \Omega_{,z}^{2}) \tag{2.1.9a}$$

$$\Omega_{,\rho\rho} + \frac{\Omega_{,\rho}}{\rho} + \Omega_{,zz} = 2(\Omega_{,\rho}U_{,\rho} + \Omega_{,z}U_{,z}) \tag{2.1.9b}$$

These are the coupled nonlinear equations upon which we will focus our attention. A most comprehensive survey of this problem is given in the review article by Reina and Treves

(Reina and Treves, 1976). Cohen (Cohen, 1976) considers the problems that arise in classifying stationary axisymmetric gravitational fields. The familiar Weyl solutions are included as the case $\Omega = 0$, in which case U satisfies the ordinary cylindrical Laplace's equation.

A complex formulation of (2.1.8) was first introduced by Ernst (Ernst, 1968). We introduce a complex potential, $\mathcal{E}$, defined by

$$\mathcal{E} = f + i\Omega \tag{2.1.10}$$

in terms of which (2.1.8) become

$$(\mathrm{Re}\,\mathcal{E})\nabla^2\mathcal{E} = \nabla\mathcal{E}\cdot\nabla\mathcal{E}. \tag{2.1.11}$$

It is also sometimes convenient to introduce a different complex potential E defined by

$$\mathcal{E} = (E-1)/(E+1). \tag{2.1.12}$$

(2.1.11) then becomes

$$(EE^*-1)\nabla^2 E = 2E^*\nabla E\cdot\nabla E. \tag{2.1.13}$$

(2.1.13) is called the Ernst equation.

The metric functions f, ω, and γ and the potential Ω are given in terms of E by

$$f = \mathrm{Re}\,\frac{E-1}{E+1} \qquad \Omega = \mathrm{Im}\,\frac{E-1}{E+1} \tag{2.1.14}$$

$$\nabla\omega = \frac{2\rho}{(EE^*-1)^2}\,\mathrm{Im}\left[(E^*+1)^2(\hat{Q}\times\nabla E)\right] \tag{2.1.15}$$

$$\gamma_{,\rho} = \frac{\rho}{(EE^*-1)^2}\left(E_{,\rho}E^*_{,\rho} - E_{,z}E^*_{,z}\right) \tag{2.1.16}$$

$$\gamma_{,z} = \frac{2\rho}{(EE^*-1)^2}\,\mathrm{Re}\left(E_{,\rho}E^*_{,z}\right) .$$

The most important solutions to (2.1. 8) to be recently discovered using (2.1.13) are the Tomimatsu-Sato solutions (Tomimatsu and Sato, 1973). These are asymptotically flat solutions, most of which have naked singularities outside the event horizon. The Kerr solution is the simplest twisting solution of the Tomimatsu-Sato form, and if $\omega \to 0$ in it one obtains the Schwarzschild solution already discussed.

2.2 Cylindrical Gravitational Waves

If we let

$$t = i\tilde{z}$$
$$z = i\tilde{t}$$
$$\omega = i\tilde{\omega}$$

the metric (2.1.3) becomes

$$ds^2 = f^{-1}e^{2\gamma}(d\tilde{t}^2 - d\rho^2) - f(d\tilde{z} + \tilde{\omega}\, d\theta)^2 - f^{-1}\rho^2 d\theta^2 \qquad (2.2.1)$$

The Killing vector fields are now

$$\frac{\partial}{\partial \tilde{z}}$$

and

$$\frac{\partial}{\partial \theta}.$$

The surfaces $\tilde{t} =$ constant, $\rho =$ constant are intrinsically flat, but we identify points with θ differing by 2π.

If we let

$$Q = -U + \log \rho$$

then the field equations (2.1.4) for Q, $\tilde{\omega}$ become

$$Q_{,\rho\rho} + \frac{Q_{,\rho}}{\rho} - Q_{,tt} = e^{-2Q}(\tilde{\omega}_{,t}^2 - \tilde{\omega}_{,\rho}^2) \qquad (2.2.2a)$$

$$\tilde{\omega}_{,\rho\rho} + \frac{\tilde{\omega}_{,\rho}}{\rho} - \tilde{\omega}_{,tt} = 2(\tilde{\omega}_{,\rho}Q_{,\rho} - \tilde{\omega}_{,t}Q_{,t}). \qquad (2.2.2b)$$

The metric (2.2.1) describes Jordan-Ehlers waves (Jordan et al, 1960) which are cylindrical gravitational waves propagating in vacuum with two degrees of freedom, corresponding to two available wave polarizations. The familiar Einstein-Rosen waves are included in the case $\tilde{\omega} = 0$, in which case Q is a solution of the ordinary cylindrical wave equation.

2.3 Colliding Plane Gravitational Waves

If we let

$$t = i\hat{\rho}$$
$$\rho = i\hat{t}$$
$$Q = i\hat{Q}$$

the metric (2.1.1) becomes

$$ds^2 = f^{-1}e^{2\gamma}(d\hat{t}^2 - dz^2) - f(d\hat{\rho} + \omega d\hat{Q})^2 - f^{-1}\hat{t}^2 dQ^2 \quad . \qquad (2.3.1)$$

The Killing vector fields are now

$\frac{\partial}{\partial \hat{\rho}}$

and

$\frac{\partial}{\partial \hat{Q}}$.

The surfaces $\hat{t}$ = constant and $\hat{z}$ = constant are flat, taken to be Euclidean planes. The solutions are now independent of $\hat{\rho}$, $\hat{Q}$ and the waves propagate along the z- axis. The field equations (2.1.9) for U, Ω become

$$U_{,\hat{t}\hat{t}} + \frac{U_{,\hat{t}}}{\hat{t}} - U_{,zz} = -e^{-2U}(\Omega_{,z}^2 - \Omega_{,\hat{t}}^2) \qquad (2.3.2a)$$

$$\Omega_{,\hat{t}\hat{t}} + \frac{\Omega_{,\hat{t}}}{\hat{t}} - \Omega_{,zz} = 2(U_{,\hat{t}}\Omega_{,\hat{t}} - U_{,z}\Omega_{,z}) \qquad (2.3.2b)$$

This problem has recently received attention by Szekeres

(Szekeres, 1972) who interprets (2.3.1) as the metric describing the interaction region of two colliding plane waves.

The preceding three problems are those discussed by Kinnersley (Kinnersley, 1975). To these problems we now add two more problems concerning the source-free Einstein-Maxwell equations.

2.4 Cylindrically Symmetric Static Einstein-Maxwell Fields

We consider first a metric of the form

$$ds^2 = f\,dt^2 - f^{-1}\left[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2\right] \qquad (2.4.1)$$

which, if f and γ are functions of ρ, z only, is cylindrically symmetric and static. (2.4.1) is identical to (2.1.3) with the cross term $\omega = 0$. A vacuum solution would describe the external gravitational field of an axisymmetric static body (Weyl solution). If we now allow the body to have charge, its exterior fields must satisfy the Einstein-Maxwell equations (1.1.2), (1.1.11) and (1.1.12), which reduce to:

$$U_{,\rho\rho} + \frac{U_{,\rho}}{\rho} + U_{,zz} = -e^{-2U}(\Omega_{,\rho}{}^2 + \Omega_{,z}{}^2) \qquad (2.4.2a)$$

$$\Omega_{,\rho\rho} + \frac{\Omega_{,\rho}}{\rho} + \Omega_{,zz} = 2(\Omega_{,\rho}U_{,\rho} + \Omega_{,z}U_{,z}) \qquad (2.4.2b)$$

plus equations for $\gamma_{,\rho}$ and $\gamma_{,z}$ which are integrable when (2.4.2) are satisfied and determine γ up to a constant. We assume that the electromagnetic field also depends only on ρ, z and hence, due to the form of the metric (2.4.1), we may express it in terms of a single potential Ω. The electromagnetic field tensor in this case is given in terms of Ω by

$$F_{ij} = \Omega_{,K} \exp(f_i + f_j - f_K - f_3) \quad (i,j,K = 1,2,3 \text{ in cyclic order})$$

$$F_{i3} = -\Omega_{,i} \quad (i = 1,2,3)$$

where

$$x_0 = z$$
$$x_1 = \rho$$
$$x_2 = Q$$
$$x_3 = t$$

and

$$f_1 = \gamma - U$$
$$f_2 = \ln \rho - U$$
$$f_3 = U.$$

All other components of $F_{\mu\nu}$ vanish.

2.5 Colliding Plane Gravitational and Plane Electromagnetic Waves

If we let

$$z = i\tilde{t}$$
$$t = i\tilde{z}$$
$$\gamma \to \gamma + i\pi$$

the metric (2.4.1) becomes

$$ds^2 = f^{-1} e^{2\gamma}(d\rho^2 - d\tilde{t}^2) - f^{-1}\rho^2 dQ^2 - f\, d\tilde{z}^2 \qquad (2.5.1)$$

with Killing vector fields

$$\frac{\partial}{\partial z}$$

and

$$\frac{\partial}{\partial Q}.$$

Surfaces ρ = constant, $\tilde{t}$ = constant are flat, identified as

Euclidean planes. The Einstein-Maxwell equations (2.4.2) become

$$U_{,\rho\rho} + \frac{U_{,\rho}}{\rho} - U_{,\tilde{t}\tilde{t}} = e^{-2U}\left(\Omega_{,\tilde{t}}^{2} - \Omega_{,\rho}^{2}\right) \tag{2.5.2a}$$

$$\Omega_{,\rho\rho} + \frac{\Omega_{,\rho}}{\rho} - \Omega_{,\tilde{t}\tilde{t}} = 2\left(\Omega_{,\rho}U_{,\rho} - \Omega_{,\tilde{t}}U_{,\tilde{t}}\right) \tag{2.5.2b}$$

where Ω is again a potential for the electromagnetic field. (2.4.2) and (2.5.2) were derived by Harrison (Harrison, 1965) as the field equations for cylindrical Einstein-Maxwell fields.

Recently, the problem of colliding plane electromagnetic waves and plane gravitational waves has been treated by Bell and Szekeres (Bell and Szekeres, 1974). To treat this colliding wave problem, they used the Newman-Penrose spinor version of the Einstein-Maxwell equations. We will not discuss the Newman-Penrose formulation, since such a discussion is beyond the scope of the present treatise (see Newman and Penrose, 1962). The problem is set up in much the same way as the colliding plane gravitational wave problem treated by Szekeres.

Space-time is divided up into four regions, one of which is flat, two of which correspond, respectively, to incoming gravitational (with $\omega = 0$) and electromagnetic plane waves, while the fourth region is the interaction region, the region of interest. The O'Brien-Synge jump conditions (O'Brien and Synge, 1952) are then used together with the Einstein-Maxwell equations in the interaction region. If a potential Ω is introduced for the electromagnetic field tensor, the field equations of Bell and Szekeres may be shown to reduce to (2.5.2) after a suitable transformation of the independent coordinates, while the metric reduces to (2.5.1). We omit the details here, since they are essentially an exercise in the Newman-Penrose formalism and provide no great insight into the problems we will treat in this discourse.

2.6 Simultaneous Treatment

We now note the following remarkable fact: equations (2.1.9) for $(\mathcal{U}, \Omega)$ are identical to equations (2.4.2). Furthermore, equations (2.2.2), involving $(Q, \tilde{\omega})$, are identical in form to equations (2.3.2) and (2.5.2), which involve $(\mathcal{U}, \Omega)$. Also, by letting $z = i\tilde{f}$ (2.1.9) are transformed into (2.5.2). We may thus draw the following diagram:

Axisymmetric stationary vacuum (2.1.9)	Complex $\longleftrightarrow$ coordinate transformation	Colliding plane gravitational waves in vacuum (2.3.2)
$\updownarrow$ "twist" potential		$\updownarrow$ "twist" potential
Cylindrically symmetric static Einstein-Maxwell (2.4.2)	Complex $\longleftrightarrow$ coordinate transformation	Cylindrically symmetric Jordan-Ehlers Waves in vacuum (2.2.2) Colliding plane gravitational and electromagnetic waves (2.5.2)

Thus, through the use of the potential (2.1.6) and the simple complex coordinate transformations given above, we may consider problems 2.1-2.5 simultaneously, at least when searching for exact solutions. The analogy becomes even more evident when we consider the boundary conditions associated with problems 2.1 and 2.5. In problem 2.1, the metric must be flat at infinity, or

$$U \to 0$$
$$\Omega \to 0$$

as $\rho \to \infty$. In problem 3.4, the metric must be flat and the electromagnetic field must vanish at spatial infinity, which translates into the same boundary conditions on U, Ω as those above.

We note that the electromagnetic potential Ω seems to take the place of the "twist" potential (2.1.6). We will refer to this fact again when we discuss three variable solutions with one Killing vector in Chapter 4.

Some of the above ideas have recently been independently realized and published by Catenacci and Alonso (Catenacci and Alonso, 1976). They noted that the invariance groups of problems 3.1 to 3.5 were identical.

CHAPTER 3

VARIOUS FAMILIES OF EXACT SOLUTIONS

In this chapter we present a discussion of various exact solutions and transformation theorems for the problems discussed in Chapter 2. The calculus of differential forms is used extensively. A discussion of the techniques involving differential forms used in this chapter is presented in the appendix, and the reader is referred there for further details.

3.1 The Curzon Solution as a Generalized Similarity Solution

In this section a generalized isovector of (2.1.9) is presented for the case $\omega = 0$. The corresponding generalized similarity solution is found and shown to be the well known Curzon solution. The corresponding wave solution is also discussed.

When $\omega = 0$, the field equations (2.1.9) reduce to the cylindrical Laplacian

$$U_{,\rho\rho} + \frac{U_{,\rho}}{\rho} + U_{,zz} = 0. \tag{3.1.1}$$

A suitable ideal of differential forms corresponding to (3.1.1) is

$$\begin{aligned} \alpha &= dU - A\,dz - B\,d\rho \\ d\alpha &= -dA\wedge dz - dB\wedge d\rho \\ \gamma &= dB\wedge dz + dA\wedge d\rho + \frac{B}{\rho}\, d\rho\wedge dz. \end{aligned} \tag{3.1.2}$$

The ideal (3.1.2) is closed under exterior differentiation. There are five variables, one one-form, and two two-forms, so the Cartan criteria are satisfied.

We now present a generalized isovector of the ideal (3.1.2). Consider the vector $\vec{V}$ with components

$$V^z = -\rho$$
$$V^\rho = z$$
$$V^u = 0 \qquad (3.1.3)$$
$$V^A = -B$$
$$V^B = A.$$

The necessary augmented forms are

$$\vec{V} \rfloor \alpha = \rho A - Bz$$
$$\vec{V} \rfloor d\alpha = Bdz - \rho dA - Ad\rho + zdB \qquad (3.1.4)$$
$$\vec{V} \rfloor \gamma = Adz + \rho dB - Bd\rho - zdA + \frac{B}{\rho}(zdz + \rho d\rho).$$

The Lie derivatives with respect to $\vec{V}$ of the forms in the augmented ideal (3.1.2) and (3.1.4) are found to be:

$$\underset{\vec{V}}{\mathcal{L}} \alpha = \underset{\vec{V}}{\mathcal{L}} d\alpha = \underset{\vec{V}}{\mathcal{L}} (\vec{V} \rfloor \alpha) = \underset{\vec{V}}{\mathcal{L}} (\vec{V} \rfloor d\alpha) = 0$$

$$\underset{\vec{V}}{\mathcal{L}} \gamma = \frac{(\vec{V} \rfloor \alpha)\, d\rho \wedge dz}{\rho^2}$$

$$\underset{\vec{V}}{\mathcal{L}} (\vec{V} \rfloor \gamma) = -\frac{(\vec{V} \rfloor \alpha)}{\rho^2} z dz.$$

We see that the Lie derivative of the augmented ideal is contained in the augmented ideal and hence the vector (3.1.3) is a generalized isovector. Annuling the first, scalar, form of (3.1.4) implies the functional form

$$U = U(\rho^2 + z^2).$$

We could of course also choose U to be any function of a function of $\rho^2 + z^2$. To simplify the following calculations, we choose

$$U = U(\sqrt{\rho^2 + z^2}) = U(r).$$

Substitution of the above form for U into (3.1.1) yields the ordinary differential equation

$$\frac{d^2U}{d\eta^2}+\frac{1}{\eta}\frac{dU}{d\eta}=0. \tag{3.1.5}$$

The general solution of (3.1.5) is

$$U=-\frac{M}{\eta}+C \tag{3.1.6}$$

where M and C are constants of integration. We set $C=0$ so that $U\to 0$ as $\rho\to\infty$, making the solution asymptotically flat. This yields

$$f=e^{2U}=e^{-2M/\eta}.$$

From (2.1.5) we have

$$\gamma_{,\rho}=\frac{\rho}{4}\left(U_{,\rho}^2-U_{,z}^2\right)=\frac{M^2\rho^3}{4}(\rho^2-z^2)(\rho^2+z^2)^{-3}$$

$$\gamma_{,z}=\frac{1}{2}U_{,\rho}U_{,z}=\frac{1}{2}M^2\rho z(\rho^2+z^2)^{-3}.$$

This yields

$$\gamma=-\frac{1}{2}M^2(\rho^2+z^2)^{-2}.$$

The metric is then given by

$$ds^2=\exp\left[-2M(\rho^2+z^2)^{-1/2}\right]dt^2-\exp\left[2M(\rho^2+z^2)^{-1/2}-\frac{M^2}{2}\rho^2(\rho^2+z^2)^{-2}\right]\cdot$$
$$\cdot(d\rho^2+dz^2)-\exp\left[2M(\rho^2+z^2)^{-1/2}\right]\rho^2d\varphi^2. \tag{3.1.7}$$

This metric was first studied by Curzon (Curzon, 1924). This solution has long been considered a mathematical curiosity with a strange physical interpretation (see Synge, 1960). Recently, however, Voorhees (Voorhees, 1970) has suggested that the Curzon metric corresponds to the external field of a disk of radius M. Since $\sqrt{\rho^2+z^2}\sim r$, where r is the spherical Schwarzschild coordinate, we may obtain the physical meaning of the constant M by expanding $g_{00} = f$ for $r \to \infty$. This gives $g_{00} \to 1 - 2\frac{M}{r}$ as $r \to \infty$ and we see that M is the mass of the source.

Similarly, if we set $\tilde{\omega} = 0$ in (2.2.2), those equations reduce to the cylindrical wave equation

$$Q_{,\rho\rho} + \frac{Q_{,\rho}}{\rho} - Q_{,tt} = 0.$$

Recalling

$$Q = -U + \log \rho,$$

we find that U also satisfies the cylindrical wave equation:

$$U_{,\rho\rho} + \frac{U_{,\rho}}{\rho} - U_{,tt} = 0.$$

If we want a source term at the origin $\rho = 0$ at time $t = 0$, the appropriate problem to solve is

$$U_{,\rho\rho} + \frac{U_{,\rho}}{\rho} - U_{,tt} = \frac{-\delta(\rho)\delta(t)}{2\pi\rho}$$

with initial conditions $Q(\rho, 0^-) = Q_{,t}(\rho, 0^-) = 0$.

Working as before, we find that

$$U = \frac{-1}{2\pi\sqrt{t^2 - \rho^2}}$$

with $U = 0$ for $\rho > t$.

This solution corresponds to a gravitational wave pulse emitted at $\rho = 0$, $t = 0$. It vanishes outside the light cone $\rho = \pm t$.

3.2 The Kerr and Tomimatsu-Sato Solutions and Transformations

According to section 2.6, the functions (f, Ω) describing stationary solutions to (2.1.9) are also solutions to (2.4.2) describing static cylindrically symmetric Einstein-Maxwell fields. The Ernst potential Ω of the stationary fields becomes the electromagnetic potential for the Einstein-Maxwell fields. In this section we discuss some physically relevant, known solutions of (2.1.9). By displaying f and Ω explicitly, we also arrive at new solutions to the static Einstein-Maxwell equations (2.4.2).

The solutions are most easily discussed in the complex Ernst formalism of section 2.1. It is convenient to work in prolate spheroidal coordinates defined by

$$\rho = k(x^2-1)^{1/2}(1-y^2)^{1/2}$$
$$z = kxy.$$

x and y are given explicitly in terms of ρ, z by

$$x = \frac{1}{2k}\left[\sqrt{(z+k)^2+\rho^2} + \sqrt{(z-k)^2+\rho^2}\right]$$
$$y = \frac{1}{2k}\left[\sqrt{(z+k)^2+\rho^2} - \sqrt{(z-k)^2+\rho^2}\right]$$

where k is an arbitrary constant.

We now assume that the Ernst potential E is of the form

$$E = \frac{\alpha}{\beta} \tag{3.2.1}$$

where α and β are complex polynomials of x and y. Substitution of (3.2.1) into (2.1.13) yields, in the x, y coordinates:

$$(x^2-1)(\alpha\alpha^* - \beta\beta^*)(\alpha_{,xx}\beta - \alpha\beta_{,xx}) + \{2x(\alpha\alpha^* - \beta\beta^*) - 2(x^2-1)\cdot$$
$$\cdot(\alpha^*\alpha_{,x} - \beta^*\beta_{,x})\}(\alpha_{,x}\beta - \alpha\beta_{,x}) - [\text{the same expression replacing } x \text{ by } y] = 0. \tag{3.2.2}$$

The first solution to (3.2.2) was found by Ernst (Ernst, 1968) and is given by

$$\alpha = px - iqy \qquad (3.2.3)$$
$$\beta = 1$$

where p and q are real parameters related by

$$p^2 + q^2 = 1. \qquad (3.2.4)$$

The case $p=1, q=0$ yields the Schwarzschild solution. In fact, (3.2.3) is isometric to the Kerr solution (Reina and Treves, 1976). From (2.1.14) and (2.1.16) we obtain f, Ω, ω and γ as

$$f = \frac{p^2x^2 + q^2y^2 - 1}{(px+1)^2 + q^2y^2} \qquad (3.2.5)$$
$$\Omega = -\frac{2qy}{(px+1)^2 + q^2y^2}$$
$$\omega = -\frac{2q(1-y^2)(px+1)}{p^2x^2 + q^2y^2 - 1}$$
$$e^{2\gamma} = \frac{p^2x^2 + q^2y^2 - 1}{p^2(x^2-y^2)}.$$

The metric is given by

$$ds^2 = K^2\Big[\frac{(px+1)^2 + q^2y^2}{p^2}\Big(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}\Big) + \frac{(px+1)^2 + q^2y^2}{p^2x^2 + q^2y^2 - 1} \cdot$$
$$\cdot (x^2-1)(1-y^2)\,d\varphi^2 - \frac{(p^2x^2+q^2y^2-1)}{[(px+1)^2 + q^2y^2]}\Big(dt + \frac{2q(1-y^2)(px+1)\,d\varphi}{(p^2x^2+q^2y^2-1)}\Big)^2\Big]$$

where K is the arbitrary constant which appears in the definition of prolate spheroidal coordinates. The mapping

$$px+1 = r/M$$
$$qy = a/M$$

with

$$p = K/M$$
$$q = a/M$$
$$K = \sqrt{M^2-a^2}$$

maps the above metric into the Boyer-Lindquist form of the Kerr metric:

$$ds^2 = (r^2+a^2\cos^2\theta)\left(d\theta^2 + \frac{dr^2}{r^2+a^2-2Mr}\right) + (r^2+a^2)\sin^2\theta\, d\varphi^2$$
$$-dt^2 + \frac{2Mr}{r^2+a^2\cos^2\theta}(dt + a\sin^2\theta\, d\varphi)^2 \,.$$

If $q=0$ then $a=0$ and the resulting metric reduces to the Schwarzschild metric.

The Kerr solution was the first stationary (but non-static) exact vacuum solution found. It is thought to describe the gravitational field external to a spinning object (Kerr, 1963). It is asymptotically flat, so that (3.2.4) also describes the external gravitational and electromagnetic fields of a static distribution of charge and mass.

Much more general solutions to (3.2.2) were presented by Tomimatsu and Sato (Tomimatsu and Sato, 1973). They found solutions assuming α and β are polynomials of degree δ^2 and δ^2-1, respectively, where δ is an integer. Explicit solutions were given for $\delta = 1, 2, 3, 4$. For $\delta=1$, the solution is (3.2.3). For $\delta=2$, they obtained

$$E = \frac{p^2x^4 + q^2y^4 - 1 - 2ipqxy(x^2-y^2)}{2px(x^2-1) - 2iqy(1-y^2)} \tag{3.2.6}$$

where p and q are related by (3.2.4).

(3.2.6) yields the following for f and Ω:

$$f = A/B \tag{3.2.7}$$

$$\Omega = C/B$$

where

$$A = p^4(x^2-1)^4 + q^4(1-y^2)^4 - 2p^2q^2(x^2-1)(1-y^2)[2(x^2-1)^2 + 2(1-y^2)^2 + 3(x^2-1)(1-y^2)]$$

$$B = [p^2(x^4-1) - q^2(1-y^4) + 2px(x^2-1)]^2 + 4q^2y^2[px(x^2-1) + (px+1)(1-y^2)]^2$$

$$C = (p^2x^4 + q^2y^4 - 1)(1-y^2) - 2p^2x^2(x^2-1)(x^2-y^2) .$$

(3.2.7) is again an asymptotically flat solution corresponding either to an uncharged stationary distribution or a charged static one. For $\delta = 3, 4$ the expressions are even more complicated and we refrain from discussing them here.

Ernst (Ernst, 1968) noted that a phase transformation of the solutions of (2.1.13)

$$E = e^{i\alpha} E_0$$

yields new solutions which are not asymptotically flat. The previous solutions may thus be considered as a special member (with $\alpha = 0$) of a more general family. For example, the NUT solution (Newman, Unti, and Tamburino, 1963) given by

$$E = e^{i\alpha} x$$

is seen to be a generalized Schwarzschild solution. Similarly, Demianski and Newman (Demianski and Newman, 1966) obtained a generalized Kerr solution

$E = e^{i\alpha}(px + iqy).$

We refrain from discussing such solutions further since their physical interpretation is questionable.

3.3 The Isogroup of the Two-Variable Equations and Some Similarity Solutions

In this section, we discuss the isogroups of the field equations of sections 2.1 to 2.5 and use them to derive some similarity solutions.

We begin by repeating the two sets of field equations under consideration. For the physical cases of sections 2.2, 2.3, and 2.5 the field equations are

$$U_{,\rho\rho} + \frac{U_{,\rho}}{\rho} - U_{,tt} = e^{-2U}(\Omega_{,t}^{2} - \Omega_{,\rho}^{2}) \tag{3.3.1a}$$

$$\Omega_{,\rho\rho} + \frac{\Omega_{,\rho}}{\rho} - \Omega_{,tt} = 2(\Omega_{,\rho}U_{,\rho} - \Omega_{,t}U_{,t}). \tag{3.3.1b}$$

If we let $z = it$, then (3.3.1) correspond to the cases of sections 2.1 and 2.4:

$$U_{,\rho\rho} + \frac{U_{,\rho}}{\rho} + U_{,zz} = -e^{-2U}(\Omega_{,\rho}^{2} + \Omega_{,z}^{2}) \tag{3.3.2a}$$

$$\Omega_{,\rho\rho} + \frac{\Omega_{,\rho}}{\rho} + \Omega_{,zz} = 2(\Omega_{,\rho}U_{,\rho} + \Omega_{,z}U_{,z}). \tag{3.3.2b}$$

Harrison and Estabrook (Harrison and Estabrook, 1971) have presented a discussion of (3.3.1) and have calculated the isogroup. An appropriate ideal of differential forms for (3.3.1) is

$$\begin{aligned}
\alpha &= dU - A\,dt - B\,d\rho & d\alpha &= -dA\wedge dt - dB\wedge d\rho \\
\beta &= d\Omega - F\,dt - G\,d\rho & d\beta &= -dF\wedge dt - dG\wedge d\rho \\
\gamma &= dB\wedge dt + dA\wedge d\rho - \left[e^{-2U}(F^{2} - G^{2}) - \frac{B}{\rho}\right]d\rho\wedge dt & &
\end{aligned} \tag{3.3.3}$$

$\delta = dG\wedge dt + dF\wedge d\rho - [2(BG-AF) - \frac{G}{\rho}]d\rho\wedge dt$.

This set of forms is closed under exterior differentiation. There are eight variables, two one-forms, and four two-forms. Thus the ideal (3.3.3) meets the Cartan criteria and is a well-set ideal. The isogroup of (3.3.3) is given in Table I (page 37), where a description of each type of transformation is provided when feasible. If we let $z = it$, we obtain the isogroup of (3.3.2). Although one might have guessed isovectors 1 and 3 from inspection of (3.3.1), the other isovectors are more complicated and cannot be found by inspection.

We now seek similarity solutions using the isogroup. Isovector 2 leads to the functional dependence

$$U = U(\rho/t) = U(\eta)$$
$$\Omega = \Omega(\rho/t) = \Omega(\eta) . \qquad (3.3.4)$$

Substitution of (3.3.4) into (3.3.1) results in the ordinary differential equations

$$U''(\eta^2-1) + U'(2\eta - \frac{1}{\eta}) = e^{-2U}\ \Omega'^2(1-\eta^2) \qquad (3.3.5a)$$
$$\Omega''(\eta^2-1) + \Omega'(2\eta - \frac{1}{\eta}) = 2U'\Omega'(\eta^2-1) \qquad (3.3.5b)$$

where a prime denotes differentiation with respect to η .

We proceed to find solutions of (3.3.5) as follows. (3.3.5b) may be rewritten as

$$\frac{\Omega''}{\Omega'} + \frac{2\eta - \frac{1}{\eta}}{\eta^2 - 1} = 2U'.$$

Integration of the above yields

$$\Omega' = \frac{C_1 e^{2U}}{\eta\sqrt{\eta^2-1}} \qquad (3.3.6)$$

Table I Isogroup of (3.3.1)

No.	Coordinate Components								Type
	t	ρ	u	Ω	A	B	F	G	
1	1	0	0	0	0	0	0	0	time translation
2	t	ρ	0	0	$-A$	$-B$	$-F$	$-G$	ρ, t scale change
3	0	0	0	1	0	0	0	0	potential transformation
4	0	0	1	Ω	0	0	F	G	u, Ω scale change
5	0	0	Ω	$\frac{1}{2}(\Omega^2 - e^{2u})$	F	G	$\Omega F - Ae^{2u}$	$\Omega G - Be^{2u}$	

where C_1 is a constant of integration. Substitution of (3.3.6) into (3.3.5a) yields

$$U''(\eta^2-1)+U'\left(2\eta-\frac{1}{\eta}\right)=-\frac{e^{2U}C_1^2}{\eta^2}$$

which may be rewritten as

$$U''(\eta^2-\eta^4)+U'(\eta-2\eta^3)=e^{2U}C_1^2\,. \qquad (3.3.7)$$

The equation obtained by multiplying (3.3.7) by U' may be written as

$$\frac{d}{d\eta}\left(U'^2(\eta^2-\eta^4)\right)=\frac{d}{d\eta}C_1^2e^{2U}$$

which integrates to

$$U'^2(\eta^2-\eta^4)=C_1^2e^{2U}$$

where we choose the constant of integration to be zero so that we obtain a solution explicitly up to a quadrature. Integration of the above yields

$$e^{-U}=C_1\operatorname{sech}^{-1}\eta+C_2 \qquad (3.3.8a)$$

where C_2 is another constant of integration. (3.3.6) then gives Ω as a quadrature:

$$\Omega=C_1\int_{C_3}^{\eta}\frac{e^{2U(\sigma)}}{\sigma\sqrt{\sigma^2-1}}\,d\sigma \qquad (3.3.8b)$$

where C_3 is a third integration constant. (3.3.8) comprise an exact solution of (3.3.1).

We could of course similarly use isovector 2 in (3.3.2) to obtain

$$U=U(\rho|z)=U(\eta) \qquad (3.3.9)$$
$$\Omega=\Omega(\rho|z)=\Omega(\eta)\,.$$

Substitution of (3.3.9) into (3.3.2) yields the ordinary differential equations

$$U''(\eta^2+1)+U'\left(2\eta+\tfrac{1}{\eta}\right)=-e^{-2U}\Omega'^2(\eta^2+1)$$
$$\Omega''(\eta^2+1)+\Omega'\left(2\eta+\tfrac{1}{\eta}\right)=2U'\Omega'(\eta^2+1)\,. \tag{3.3.10}$$

(3.3.10) may be solved in a manner analogous to (3.3.5). The solution is

$$e^{2U}=\frac{c_2}{c_1^2}\left\{1+4\coth^2\left(\sqrt{c_2}\,\ln\left[\frac{(\eta^2+1)^{1/2}-1}{\eta}+c_3\right]\right)\right\}$$
$$\Omega=\int_{c_4}^{\eta}\frac{e^{2U}}{\eta\sqrt{\eta^2+1}}\,. \tag{3.3.11}$$

Unfortunately, we cannot interpret (3.3.11) as the external gravitational field of an isolated axisymmetric body since e^{2U} does not have the proper asymptotic behavior.

We may use other isovectors to reduce the number of independent variables. For example, using a combination of isovector 2 + isovector 3, we find

$$U=U(\rho/t)=U(\eta)$$
$$\Omega=\ln t+\Psi(\rho/t)=\ln t+\Psi(\eta)$$

where U and Ψ satisfy the ordinary differential equations

$$U''(\eta^2-1)+U'\left(2\eta-\tfrac{1}{\eta}\right)=-e^{2U}(1+\Psi'^2\eta^2-2\Psi'\eta-\Psi'^2)$$
$$\Psi''(\eta^2-1)+\Psi'\left(2\eta-\tfrac{1}{\eta}\right)=1+2\Psi'(\eta^2-U')-2U'\eta\,.$$

Use of isovector 2 + isovector 4 yields

$$U=\ln t+\phi(\rho/t)=\ln t+\phi(\eta)$$
$$\Omega=t\Psi(\rho/t)=t\,\Psi(\eta)$$

where ϕ and Ψ satisfy

$$\phi''(\eta^2-1)+\phi'(2\eta-\tfrac{1}{\eta})-1=e^{-2\phi}[\psi^2-2\eta\psi\psi'+(\eta^2-1)\psi'^2]$$
$$\psi''(\eta^2-1)-\psi'(\eta-\tfrac{1}{\eta})-\psi=\phi'(\eta\psi-2\psi'-\eta^2\psi').$$

Using α(isovector 2) + β(isovector 3) where α and β are constants yields

$$U=\phi(\rho/t)=\phi(\eta)$$
$$\Omega=\sigma\ln t+\psi(\rho/t)=\sigma\ln t+\psi(\eta)$$

where ϕ and ψ satisfy

$$\phi''(1-\eta^2)-\phi'(2\eta-\tfrac{1}{\eta})=e^{-2\phi}(\sigma^2+\eta^2\psi'^2-2\sigma\eta\psi'-\psi'^2)$$
$$\psi''(\eta^2-1)-\sigma+\psi'(2\eta-\tfrac{1}{\eta})=2\phi'(\eta^2\psi'-\psi'-\sigma\eta)$$

and $\sigma=\beta/\alpha$.

We remark that although we have not been able to find exact solutions of the above equations, these coupled ordinary differential equations would be easier to solve numerically than the original partial differential equations (3.3.1).

3.4 The Ehlers Transformation and Invariance Transformations

In this section we discuss how the isogroup may be used to generate finite transformations which give new, distinct solutions from old ones. We see that the isogroup of Table I (properly, the generators of infinitesimal invariances) separates into two subgroups, which respectively transform only the original independent (ρ, t) or dependent (U, Ω) variables. The second group, vectors 3-5, is integrable in closed form, giving a three parameter set of finite invariance transformations.

Integration of

$$\frac{du}{d\tau} = V^u \qquad (3.4.1)$$

$$\frac{d\Omega}{d\tau} = V^\Omega$$

yields finite transformations which may be put in the form (Harrison and Estabrook, 1971)

$$\Omega = \Upsilon + \frac{\mathcal{L}(\Omega_0 - \sigma)}{(\Omega_0-\sigma)^2 + f_0^2} \qquad (3.4.2)$$

$$f = \frac{\mathcal{L} f_0}{(\Omega_0-\sigma)^2 + f_0^2}$$

where $f = e^u$, $f_0 = e^{u_0}$, and Υ, $\mathcal{L}$, and σ are constants. To establish (3.4.2), we integrate (3.4.1) first for isovector 5:

$$\frac{du}{d\tau} = 2\Omega \qquad (3.4.3a)$$

$$\frac{d\Omega}{d\tau} = \Omega^2 - e^{2u} . \qquad (3.4.3b)$$

Differentiation of (3.4.3a) and substitution into (3.4.3b) results in a single equation for u :

$$\frac{1}{2}\frac{d^2u}{d\tau^2} = \frac{1}{4}\left(\frac{du}{d\tau}\right)^2 - e^{2u} .$$

If we make the substitution

$$\omega = e^{-u/2} \qquad (3.4.4)$$

we obtain an equation for ω ,

$$\frac{d^2\omega}{d\tau^2} = \omega^{-3}$$

which integrates to

$$\left(\frac{d\omega}{d\tau}\right)^2 = b^2 - \frac{1}{\omega^2}$$

where b^2 is a constant of integration. The above is separable:

$$\pm d\tau = \frac{\omega\, d\omega}{\sqrt{b^2\omega^2 - 1}} \,.$$

The substitution

$$x^2 = b^2\omega^2 - 1 \tag{3.4.5}$$

brings the expression for $d\tau$ into the form

$$\pm d\tau = \frac{dx}{b^2}$$

which integrates to

$$x = \pm b^2\tau \pm v$$

where v is a constant of integration. Using (3.4.4) and (3.4.5) we find

$$b^2\omega^2 = 1 + (b^2\tau + v)^2 = b^2 e^{-u}$$

or

$$e^u = \frac{b^2}{1 + (b^2\tau + v)^2} \,. \tag{3.4.6a}$$

(3.4.3a) then gives Ω as

$$\Omega = -\frac{b^2(b^2\tau + v)}{1 + (b^2\tau + v)^2} \tag{3.4.6b}$$

Setting $\tau = 0$ in (3.4.6) we find

$$e^u\big|_{\tau=0} = e^{u_0} = \frac{b^2}{1+v^2} \tag{3.4.7}$$

$$\Omega\big|_{\tau=0} = \Omega_0 = \frac{-b^2 v}{1+v^2} \,.$$

Letting

$$l = b^2 \tau$$

and solving for v and b from (3.4.7), (3.4.6) may be written as

$$\begin{aligned} e^{u} &= \frac{e^{u_0}\left(1+\Omega_0^2 e^{-2u_0}\right)}{\left(l-\Omega_0 e^{-u_0}\right)^2+1} \\ \Omega &= \frac{e^{u_0}\left(1+\Omega_0^2 e^{-2u_0}\right)\left(-l+\Omega_0 e^{-u_0}\right)}{\left(l-\Omega_0 e^{-u_0}\right)^2+1} \,. \end{aligned} \tag{3.4.8}$$

Setting

$$f = e^{u}$$

(3.4.8) becomes

$$\begin{aligned} f &= \frac{f_0\left(f_0^2+\Omega_0^2\right)}{\left(\Omega_0 - l f_0^2\right)+f_0^2} \\ \Omega &= \frac{\left(\Omega_0 - l f_0\right)\left(f_0^2+\Omega_0^2\right)}{\left(\Omega_0 - l f_0\right)^2+f_0^2} \,. \end{aligned} \tag{3.4.9}$$

If we let

$$l f_0 = \sigma$$

and make a scale change on (f, Ω) to get rid of a factor

$$\frac{1}{l}\left(f_0^2+\Omega_0^2\right)$$

(3.4.9) takes the form

$$\begin{aligned} f &= -\frac{\sigma f_0}{\left(\Omega_0-\sigma\right)^2+f_0^2} \\ \Omega &= -\frac{\sigma\left(\Omega_0-\sigma\right)}{\left(\Omega_0-\sigma\right)^2+f_0^2} \,. \end{aligned} \tag{3.4.10}$$

We next integrate (3.4.2) for isovector 4:

$$\frac{du}{d\tau} = 1 \tag{3.4.11a}$$

$$\frac{d\Omega}{d\tau} = \Omega \ . \qquad (3.4.11b)$$

Integration of (3.4.11) yields

$$U = a + \tau \qquad (3.4.12)$$
$$\Omega = ce^{\tau}$$

where a and c are constants. Setting $\tau = 0$, we get

$$U_0 = a$$
$$\Omega_0 = c$$

and (3.4.12) may be written (using $f = e^{U}$) as

$$f = f_0 e^{\tau}$$
$$\Omega = \Omega_0 e^{\tau}.$$

Letting

$$\beta = e^{\tau}$$

this becomes

$$f = \beta f_0 \qquad (3.4.13)$$
$$\Omega = \beta \Omega_0 \ .$$

We may now combine transformation (3.4.13) with (3.4.10), using (f, Ω) in (3.4.10) as the (f_0, Ω_0) in (3.4.13).

Letting

$$-\beta\sigma = \mathcal{L}$$

the combined two-parameter transformation is

$$f = \frac{\mathcal{L} f_0}{(\Omega_0 - \sigma)^2 + f_0^2} \qquad (3.4.14)$$

$$\Omega = \frac{\mathcal{L}(\Omega_0 - \sigma)}{(\Omega_0 - \sigma)^2 + f_0^2} \ .$$

Finally, (3.4.2) for isovector 3 yields

$$\frac{dU}{d\tau} = 0 \qquad (3.4.15a)$$

$$\frac{d\Omega}{d\tau} = 1. \tag{3.4.15b}$$

Integration of (3.4.15) yields the transformation

$$U = U_0 \tag{3.4.16}$$
$$\Omega = \Omega_0 + \tau .$$

We may now combine transformation (3.4.16) with (3.4.14) as before to establish (3.4.2). (3.4.2) generates new physically distinct solutions (f, Ω) from known solutions (f_0, Ω_0).

If we let

$$\gamma = K^{-2}$$
$$\sigma = \tau = -K^{-1}$$

in (3.4.2) and replace Ω_0 by ($-\Omega_0$) (this is permissible, since replacing Ω by ($-\Omega$) leaves (3.3.1) unaltered), we obtain

$$\Omega = \frac{\Omega_0 - K(f_0^2 + \Omega_0^2)}{(1 - K\Omega_0)^2 + K^2 f_0^2} \tag{3.4.17}$$
$$f = \frac{f_0}{(1 - K\Omega_0)^2 + K^2 f_0^2} .$$

(3.4.17) is the well known Ehlers transformation (Ehlers, 1957). We see that the Ehlers transformation is a special case of the finite transformation generated by the isogroup. For a discussion of how (3.4.17) is used to generate solutions, we refer the reader to Kinnersley (1975) where it is discussed how the Ehlers transformation leads from the Weyl solutions to the vacuum Papapetrou solutions, as well as from the Schwarzschild solution to the NUT solution.

3.5 Some Generalized Similarity Solutions

In this section, we present a generalized isovector of (3.3.1) and (3.3.2) and use it to find generalized similarity solutions.

Consider the vector $\vec{V}$ with components

$$
\begin{aligned}
V^t &= \rho \\
V^\rho &= t \\
V^\psi &= V^\Omega = 0 \\
V^A &= -B \\
V^B &= -A \\
V^F &= -G \\
V^G &= -F .
\end{aligned}
\tag{3.5.1}
$$

We first verify that this vector is a generalized isovector of the ideal (3.3.3). The Lie derivatives of the ideal and augmented ideal with respect to (3.5.1) are found to be as follows:

$$\underset{\vec{V}}{\mathcal{L}}\alpha = \underset{\vec{V}}{\mathcal{L}}\beta = \underset{\vec{V}}{\mathcal{L}}d\beta = \underset{\vec{V}}{\mathcal{L}}d\alpha = \underset{\vec{V}}{\mathcal{L}}\sigma = \underset{\vec{V}}{\mathcal{L}}d\sigma = \underset{\vec{V}}{\mathcal{L}}\tau = \underset{\vec{V}}{\mathcal{L}}d\tau = 0$$

$$\underset{\vec{V}}{\mathcal{L}}\gamma = \frac{\sigma\, d\rho \wedge dt}{\rho^2}$$

$$\underset{\vec{V}}{\mathcal{L}}\delta = \frac{\tau \wedge d\rho \wedge dt}{\rho^2}$$

$$\underset{\vec{V}}{\mathcal{L}}\mu = -\frac{t\,\sigma\, dt}{\rho^2}$$

$$\underset{\vec{V}}{\mathcal{L}}\nu = \tau \wedge \left(\frac{t\,dt}{\rho^2} - \frac{d\rho}{\rho}\right)$$

where

$$\sigma = \vec{V} \rfloor \alpha$$
$$\tau = \vec{V} \rfloor \beta$$
$$\mu = \vec{V} \rfloor \gamma$$
$$\nu = \vec{V} \rfloor \delta .$$

Since the Lie derivative of the augmented ideal is contained in the augmented ideal, (3.5.1) is a generalized isovector. It is in fact the extension of the generalized isovector (3.1.3) to the ideal (3.3.3). We now proceed as before to find the functional forms of U and Ω:

$$U = U(\rho^2 - t^2) = U(\eta) \qquad (3.5.2)$$
$$\Omega = \Omega(\rho^2 - t^2) = \Omega(\eta)$$

Substitution of (3.5.2) into (3.3.1) yields the following ordinary differential equations for U and Ω:

$$2\eta U'' + 3U' = -2\eta e^{-2U} \Omega'^2 \qquad (3.5.3a)$$
$$2\eta \Omega'' + 3\Omega' = 4\eta \Omega' U' . \qquad (3.5.3b)$$

Solutions of (3.5.3) may be found as follows. (3.5.3b) may be rewritten as

$$2\frac{\Omega''}{\Omega'} + \frac{3}{\eta} = 4U'$$

which integrates to

$$\Omega' = C_1 e^{2U} \eta^{-3/2} \qquad (3.5.4)$$

where C_1 is a constant of integration. Substitution of (3.5.4) into (3.5.3a) yields

$$2\eta^3 U'' + 3\eta^2 U' = -2C_1^2 e^{-2U} .$$

If we multiply the above by U', we may rewrite the resulting equation as

$$\frac{d}{d\eta}\eta^3 U'^2 = -\frac{d}{d\eta} C_1^2 e^{2U}$$

which integrates to

$$\eta^3 U'^2 = -C_1^2 e^{2U} + C_2$$

where C_2 is another constant of integration. This may be rewritten as

$$\frac{dU}{(C_2 - C_1^2 e^{2U})^{1/2}} = \eta^{-3/2} d\eta .$$

We see that C_2 must be chosen so that

$$C_2 > C_1^2 e^{2U} .$$

Integration of the above then yields

$$e^{2U} = \frac{C_2}{C_1^2}\left\{1 - 4\coth^2\left[\sqrt{C_2}\,(2\eta^{-1/2} + C_3)\right]\right\} \qquad (3.5.5a)$$

where C_3 is another constant of integration. (3.5.4) then gives Ω as a quadrature:

$$\Omega = C_1 \int_{C_4}^{\eta} e^{2U(\sigma)} \sigma^{-3/2} d\sigma \qquad (3.5.5b)$$

where C_4 is a fourth integration constant. (3.5.5) represents a new generalized similarity solution to (3.3.1).

If we let $z = it$, the vector (3.5.1) is transformed into a generalized isovector of (3.3.2). The functional form is now

$$U = U(\rho^2 + z^2) = U(\eta)$$
$$\Omega = \Omega(\rho^2 + z^2) = \Omega(\eta)$$

with U and Ω satisfying (3.5.3). The solution is again

(3.5.5). This solution has the same symmetry as the Curzon solution (3.1.6). The solution in this case is asymptotically flat, since, as $\eta \to \infty$ we see from (3.5.5a) that

$$e^{2U} \to \frac{C_2}{C_1^2}\left[1 - 4\coth^2\sqrt{C_2 C_3}\right].$$

The integral for Ω, (3.5.5b), also converges as $\eta \to \infty$, since for large σ the integrand behaves like $\sigma^{-3/2}$. The constants C_1, C_2, C_3 may be chosen so that $e^{2U} \to 1$ as $\eta \to \infty$. We may set C_4 in (3.5.5b) to ∞ to give

$$\Omega = -C_1 \int_\eta^\infty e^{2U(\sigma)} \sigma^{-3/2} d\sigma \qquad (3.5.6)$$

so that $\Omega \to 0$ as $\eta \to \infty$.

This asymptotically flat solution represents either the external gravitational field of a stationary rotating body or the external gravitational and electromagnetic fields of a static body as in sections 2.1 and 2.4, respectively.

3.6 Soliton-Like Colliding Wave Solutions

In this section we present some previously known solutions due to Harrison (Harrison, 1965) and show that these solutions behave like solitons. Solitons are a special class of solitary wave, which is essentially a localized traveling wave. Solitons are solitary waves that emerge from collision with each other having the same shapes and velocities with which they entered. For a more detailed account of solitons, see (Whitham 1974), or Scott, Chu, and McLaughlin (1973).

Harrison considered solutions of (3.3.1) in the context of colliding plane gravitational and electromagnetic waves (Section 2.5). He considered, in particular, functionally dependent solutions

$$\Omega = \Omega(H)$$
$$V = V(H) \tag{3.6.1}$$

where $V = e^{u}$ and H is a new function of ρ and t. Substitution of (3.6.1) into (3.3.1) yields

$$V'\left(H_{,\rho\rho} + \frac{H_{,\rho}}{\rho} - H_{,tt}\right) = \left(H_{,\rho}^{2} - H_{,t}^{2}\right)\left(\left(V'^{2} - \Omega'^{2}\right)/V - V''\right) \tag{3.6.2a}$$

$$\Omega'\left(H_{,\rho\rho} + \frac{H_{,\rho}}{\rho} - H_{,tt}\right) = \left(H_{,\rho}^{2} - H_{,t}^{2}\right)\left(2\Omega' V'/V - \Omega''\right) \tag{3.6.2b}$$

where the prime denotes differentiation with respect to H. If we choose H so that

$$H_{,\rho\rho} + \frac{H_{,\rho}}{\rho} - H_{,tt} = 0$$
$$H_{,\rho}^{2} - H_{,t}^{2} \neq 0 \tag{3.6.3}$$

(3.6.2) becomes

$$V'' = V^{-1}\left(V'^{2} - \Omega'^{2}\right)$$
$$\Omega'' = 2V^{-1}V'\Omega' .$$

The solution of these equations, slightly simplified, is

$$V = \lambda \operatorname{sech} H$$
$$\Omega = \lambda \tanh H \tag{3.6.4}$$

where λ is a constant. Here we have a linear wave equation (3.6.3) for H, so that solutions are readily obtained and superimposed. The resulting nonlinear superposition rules for the metric functions V and Ω (given by 3.6.4) are, however, remarkably similar to rules for superposition of solitons arising in other equations.

Set

$$H = H_1(\rho,t) + H_2(\rho,t) \tag{3.6.5}$$

where $H_1 \neq H_2$ and H_1 and H_2 are each solutions of (3.6.3). Substitution into (3.6.4) yields

$$V = \lambda \operatorname{sech}(H_1+H_2) = \lambda(\cosh H_1 \cosh H_2 + \sinh H_1 \sinh H_2)^{-1} \tag{3.6.6a}$$

$$\Omega = \lambda \tanh(H_1+H_2) = \lambda \frac{(\tanh H_1 + \tanh H_2)}{1+\tanh H_1 \tanh H_2} \tag{3.6.6b}$$

Now, in regions where $H_1 \approx 0$ and $H_2 \neq 0$, we have, from (3.6.6),

$$V \sim \lambda \operatorname{sech} H_2$$

$$\Omega \sim \lambda \tanh H_2$$

This is the solution for $H = H_2$. Similarly, if $H_2 \approx 0$, $H_1 \neq 0$, we find

$$V \sim \lambda \operatorname{sech} H_1$$

$$\Omega \sim \lambda \tanh H_1$$

which is the solution for $H = H_1$. In this sense, the above solutions represent a two-soliton solution of (3.3.1), where the solutions corresponding to H_1 and H_2 are the original solitons, localized solutions of the cylindrical wave equation. This property of superimposed solutions (3.6.4) may be shown to hold for any number of solutions to (3.6.3):

$$H = \sum_{k=1}^{n} \alpha_n H_n(\rho,t)$$

and these solutions correspond to n-soliton solutions.

CHAPTER 4

EXTENSIONS TO THREE-VARIABLE SOLUTIONS

4.1 Einstein-Maxwell Equations with One Killing Vector

In this chapter we extend some of the previous results to solutions of the Einstein-Maxwell equations in three independent variables (one Killing vector). This section is devoted to a discussion of the appropriate Einstein-Maxwell equations and extension of the results of Chapter 2 to these equations.

A detailed account of the derivation of the Einstein-Maxwell equations in the presence of one Killing vector is given in Harrison (1968). We outline the important parts of the derivation as follows.

The metric is assumed to have the form

$$-ds^2 = \varepsilon e^{2u}\left(dx^K + a f_\alpha dx^\alpha\right)^2 + a^2 e^{-2u}\gamma_{\alpha\beta}dx^\alpha dx^\beta \qquad (4.1.1)$$

where a is an arbritrary constant, $\varepsilon = \pm 1 =$ sign (g_{KK}). K is some particular value of 0,1,2,3. Greek letters take all values of 0,1,2,3 except K, and all metric coefficients are independent of x^K. Therefore, a Killing vector field is generated by translation along x^K. Latin letters (except K) take on all values of 0,1,2,3. If $\varepsilon = -1$, the Killing vector is timelike, if $\varepsilon = +1$, the Killing vector is space-like. The metric is not specialized in any other way; (4.1.1) is a general four-dimensional metric with one Killing vector.

For notational convenience, we define the differential parameters

$$\Delta_1(F) = \gamma^{\alpha\beta} F_{,\alpha} F_{,\beta}$$
$$\Delta_1(F,G) = \gamma^{\alpha\beta} F_{,\alpha} G_{,\beta} \tag{4.1.2}$$
$$\Delta_2(F) = \gamma^{\alpha\beta} F_{;\alpha\beta}$$

where $\gamma^{\alpha\beta}$ is the inverse of the three-dimensional metric $\gamma_{\alpha\beta}$. $\gamma_{\alpha\beta}$ is a sort of "background" 3-metric in the 3-space that is the quotient of the 4-space by the Killing vector. A semicolon denotes covariant differentiation with respect to $\gamma_{\alpha\beta}$.

We assume that all metric coefficients and the electromagnetic field tensor are independent of X^K. We now wish to solve (1.1.2), (1.1.11) and (1.1.12) with this assumption.

We define an antisymmetric tensor in the quotient 3-space:

$$h_{\alpha\beta} = f_{\alpha,\beta} - f_{\beta,\alpha} \tag{4.1.3a}$$

and note that the integrability condition on $h_{\alpha\beta}$ is

$$h_{\alpha\gamma,\delta} + h_{\delta\alpha,\gamma} + h_{\gamma\delta,\alpha} = 0. \tag{4.1.3b}$$

The $h_{\alpha\beta}$ has a dual axial vector:

$$h_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma}\, \gamma^{\gamma\delta} z_\delta (-\gamma)^{1/2} . \tag{4.1.4}$$

In terms of z_α (4.1.3b) becomes

$$\gamma^{\alpha\beta} z_{\alpha;\beta} = 0. \tag{4.1.5}$$

The Maxwell equations (1.1.12) can be split into equations involving the index K and equations not involving K. Recalling that all quantities are assumed independent of X_K, the Maxwell equations become

$$[(-g)^{1/2} F^{K\alpha}]_{,\alpha} = 0$$
$$[(-g)^{1/2} F^{\beta\alpha}]_{,\alpha} = 0$$
$$F_{\alpha K,\beta} + F_{K\beta,\alpha} = 0$$
$$\varepsilon^{\alpha\beta\gamma} F_{\alpha\beta,\gamma} = 0$$

where $\varepsilon^{\alpha\beta\gamma}$ is the alternating 3-index symbol. The second and third of these equations may be satified by choosing potentials A and B :

$$F^{\alpha\beta} = (-g)^{-1/2} \varepsilon^{\alpha\beta\gamma} A_{,\gamma}$$
$$F_{K\alpha} = B_{,\alpha} .$$

Using these potentials, the remaining Maxwell equations become, in terms of z_α,

$$\Delta_2(A) - 2\Delta_1(U,A) + e^{2U} \gamma^{\alpha\beta} B_{,\alpha} z_\beta = 0 \qquad (4.1.6a)$$

$$\Delta_2(B) - 2\Delta_1(U,B) - e^{2U} \gamma^{\alpha\beta} A_{,\alpha} z_\beta = 0 \qquad (4.1.6b)$$

The Einstein field equations (1.1.2), (1.1.11) become

$$\Delta_2(U) + \tfrac{1}{2} e^{4U} \gamma_{\alpha\beta} z_\alpha z_\beta = -\varepsilon e^{-2U}[\Delta_1(A) + \Delta_1(B)] \qquad (4.1.7a)$$

$$\varepsilon^{\gamma\alpha\beta}(z_{\alpha,\beta} + 4 z_\alpha U_{,\beta} + 4\varepsilon e^{-4U} B_{,\alpha} A_{,\beta}) = 0 \qquad (4.1.7b)$$

$$P_{\alpha\beta} - 2U_{,\alpha}U_{,\beta} - \tfrac{1}{2} e^{4U} z_\alpha z_\beta = 2\varepsilon e^{-2U}(A_{,\alpha}A_{,\beta} + B_{,\alpha}B_{,\beta}) \qquad (4.1.7c)$$

where $P_{\alpha\beta}$ is the Ricci tensor for the background metric $\gamma_{\alpha\beta}$. (4.1.7b) may be satisfied identically by choosing z_α as

$$z_\alpha = e^{-4U}[\phi_{,\alpha} + 2\varepsilon(BA_{,\alpha} - AB_{,\alpha})] \qquad (4.1.8)$$

where ϕ is a new scalar "twist" potential. ϕ is very similar to the "twist" potential introduced in section 2.1. In fact, if $A = B = 0$ corresponding to pure vacuum solutions, we see that

$$h_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma}\,\gamma^{\gamma\delta}(-\gamma)^{1/2}e^{-4u}\,\phi_{,\delta}$$

and ϕ is seen to be the extension to three variables of the "twist" potential Ω of section 2.1. We recall that it was Ω that allowed us to transform the vacuum Einstein equations and the Einstein-Maxwell equations of Chapter 2 into identical form. We might now look for similar phenomena to occur in this case of three independent variables.

(4.1.6) and (4.1.7) for A, B, u and ϕ become:

$$\Delta_2(\phi) - 4\Delta_1(u,\phi) + 2\varepsilon B[\Delta_2(A) - 4\Delta_1(u,A)] - \\ - 2\varepsilon A[\Delta_2(B) - 4\Delta_1(u,B)] = 0$$

$$\Delta_2(A) - 2\Delta_1(u,A) + e^{-2u}[\Delta_1(\phi,B) + 2\varepsilon B\Delta_1(A,B) - \\ - 2\varepsilon A\Delta_1(B)] = 0$$

$$\Delta_2(B) - 2\Delta_1(u,B) - e^{-2u}[\Delta_1(\phi,A) + 2\varepsilon B\Delta_1(A) - \\ - 2\varepsilon A\Delta_1(A,B)] = 0$$

$$\Delta_2(u) + \varepsilon e^{-2u}[\Delta_1(A) + \Delta_1(B)] + \tfrac{1}{2}e^{-4u}[\Delta_1(\phi) + \\ + 4\varepsilon B\Delta_1(\phi,A) - 4\varepsilon A\Delta_1(\phi,B) + 4B^2\Delta_1(A) - \\ - 8AB\Delta_1(A,B) + 4A^2\Delta_1(B)] = 0$$

We now consider some special cases:

Case (i) $A = B = 0$

This is the vacuum Einstein case. The field equations reduce to

$$\Delta_2(\Psi) = -e^{-2\Psi}\Delta_1(\phi) \tag{4.1.9a}$$

$$\Delta_2(\phi) = 2\Delta_1(\Psi, \phi) \tag{4.1.9b}$$

$$P_{\alpha\beta} = \frac{\Psi_{,\alpha}\Psi_{,\beta}}{2} + \frac{1}{2}e^{-2\Psi}\phi_{,\alpha}\phi_{,\beta} \tag{4.1.9c}$$

where $\Psi = 2U$

Case (ii) $\phi = B = 0$

This is a case of the Einstein-Maxwell equations with a particular type of electromagnetic field. The field equations reduce to

$$\Delta_2(U) = -\varepsilon e^{-2U}\Delta_1(A) \tag{4.1.10a}$$

$$\Delta_2(A) = 2\Delta_1(U, A) \tag{4.1.10b}$$

$$P_{\alpha\beta} = 2U_{,\alpha}U_{,\beta} + 2\varepsilon e^{-2U}A_{,\alpha}A_{,\beta} \tag{4.1.10c}$$

Case (iii) $\phi = A = 0$

This is a case of the Einstein-Maxwell equations with a different electromagnetic field than case (ii). The field equations reduce to

$$\Delta_2(U) = -\varepsilon e^{-2U}\Delta_1(B) \tag{4.1.11a}$$

$$\Delta_2(B) = 2\Delta_1(U, B) \tag{4.1.11b}$$

$$P_{\alpha\beta} = 2U_{,\alpha}U_{,\beta} + 2\varepsilon e^{-2U}B_{,\alpha}B_{,\beta} \tag{4.1.11c}$$

The (c) equations for the $P_{\alpha\beta}$ are equations for the background metric $\gamma_{\alpha\beta}$. The integrability conditions

on the Ricci tensor $P_{\alpha\beta}$ are the Bianchi identities
$(P^{\alpha}{}_{\beta} - \frac{1}{2}\delta^{\alpha}{}_{\beta}P)_{;\alpha} = 0.$

When the (a) and (b) equations are satisfied, these integrability conditions are also satisfied. We may therefore ignore the (c) equations for the moment, and we see that the (a) and (b) equations are all identical if $\varepsilon = +1$ (space-like Killing vector). This is the extension to three variables of the results of Chapter 2, since (i) is the vacuum Einstein case and (ii) and (iii) are Einstein-Maxwell cases. We see that again the electromagnetic potential A or B takes the place of the "twist" potential ϕ. Furthermore, in cases (ii) and (iii) $\phi = 0$ which means that $f_{\alpha} = 0$ and there are no cross terms in the metric of the form $dx^{K} dx^{\alpha}$. This is also analogous to Chapter 2, where we noted that the stationary vacuum metrics (with a cross term) have the same field equations as the static (no cross term) Einstein-Maxwell problems.

The problems treated in Chapter 2 all had at least one space-like Killing vector. Assuming a second Killing vector results in specializing the functions U, ϕ, A, B and the background metric $\gamma_{\alpha\beta}$ still further. All the results of Chapter 2 may be obtained from the $\varepsilon = +1$ case here by further specialization. This means that the extension to $\varepsilon = +1$ in three variables is the natural extension of the results of Chapter 2. We may make these results complete by noting that if $\varepsilon = -1$ (time-like Killing vector) we may still make the (a) and (b) equations of cases (ii) and (iii) identical to those of case (i) by the complex coordinate transformation

$$\tilde{x}_K = i x_0$$
$$\tilde{x}_0 = i x_K$$

This has the effect of changing the sign of ε from -1 to $+1$. Alternatively, we could set

$$\phi = i\tilde{\phi}$$

if $\varepsilon = -1$ to make all the (a) equations identical.

4.2 Invariance Transformations

If we concentrate on the (a) and (b) equations of the preceding section, they may all be written in the form

$$\Delta_2(V) = V^{-1}(\Delta_1(V) - \Delta_1(\Omega)) \tag{4.2.1a}$$
$$\Delta_2(C) = 2V^{-1}\Delta_1(V,\Omega) \tag{4.2.1b}$$

where $V = e^{\psi}$ or e^{u} and Ω is any one of ϕ, A, B.

We now consider extending the invariance transformation (3.4.2) to (4.2.1). Consider barred quantities $\bar{V}$ and $\bar{C}$ which are functions of the unbarred quantities:

$$\bar{V} = \bar{V}(V,C)$$
$$\bar{C} = \bar{C}(V,C)$$

The differential parameters of the barred quantities are linear functions of the differential parameters of the unbarred ones. Using (4.1.2) and the chain rule, one finds:

$$\Delta_1(\bar{V}) = \bar{V}_{,V}{}^2\Delta_1(V) + 2\bar{V}_{,V}\bar{V}_{,C}\Delta_1(V,C) + \bar{V}_{,C}{}^2\Delta_1(C)$$
$$\Delta_1(\bar{V},\bar{C}) = \bar{V}_{,V}\bar{C}_{,V}\Delta_1(V) + \bar{V}_{,C}\bar{C}_{,C}\Delta_1(C) + (\bar{V}_{,V}\bar{C}_{,C} + \bar{V}_{,C}\bar{C}_{,V})\Delta_1(V,C)$$
$$\Delta_2(\bar{V}) = \bar{V}_{,V}\Delta_2(V) + \bar{V}_{,C}\Delta_2(C) + \bar{V}_{,VV}\Delta_1(V) + \bar{V}_{,CC}\Delta_1(C) + 2\bar{V}_{,VC}\Delta_1(V,C)$$

where a comma denotes partial differentiation. We may now look for finite invariance transformations as follows. We write (4.2.1) using $\bar{V}$ and $\bar{C}$ and use the assumed functional dependence to express these equations in terms of the differential parameters of the unbarred functions. This gives

$$\bar{V}_{,V}\Delta_2(V)+\bar{V}_{,C}\Delta_2(C)+\bar{V}_{,VV}\Delta_1(V)+\bar{V}_{,CC}\Delta_1(C)+2\bar{V}_{,VC}\Delta_1(V,C)$$
$$=\bar{V}^{-1}[\bar{V}_{,V}^2\Delta_1(V)+2\bar{V}_{,V}\bar{V}_{,C}\Delta_1(V,C)+\bar{V}_{,C}^2\Delta_1(C)-\bar{C}_{,V}^2\Delta_1(V)-2\bar{C}_{,V}\bar{C}_{,C}\Delta_1(V,C)$$
$$-\bar{C}_{,C}^2\Delta_1(C)]$$

$$\bar{C}_{,V}\Delta_2(V)+\bar{C}_{,C}\Delta_2(C)+\bar{C}_{,VV}\Delta_1(V)+\bar{C}_{,CC}\Delta_1(C)+2\bar{C}_{,VC}\Delta_1(V,C)$$
$$=2V^{-1}[\bar{V}_{,V}\bar{C}_{,V}\Delta_1(V)+\bar{V}_{,C}\bar{C}_{,C}\Delta_1(C)+(\bar{V}_{,V}\bar{C}_{,C}+\bar{V}_{,C}\bar{C}_{,V})\Delta_1(V,C)] .$$

We now substitute for the Δ_2's of the unbarred functions from (4.2.1) and equate the coefficients of each of the Δ_1's to zero. This means that we are looking for new solutions ($\bar{V}$, $\bar{C}$) which are functionally dependent on known solutions (V , C). This last step results in six second order partial differential equations:

$$\begin{aligned}
&\bar{V}_{,VV}+\tfrac{1}{V}\bar{V}_{,V}=\bar{V}^{-1}(\bar{V}_{,V}^2-\bar{C}_{,V}^2)\\
&\bar{V}_{,CC}-\tfrac{1}{V}\bar{V}_{,V}=\bar{V}^{-1}(\bar{V}_{,C}^2-\bar{C}_{,C}^2)\\
&2V^{-1}\bar{V}_{,C}+2\bar{V}_{,VC}=2\bar{V}^{-1}(\bar{V}_{,V}\bar{V}_{,C}-\bar{C}_{,V}\bar{C}_{,C})\\
&\bar{C}_{,VV}+\frac{\bar{C}_{,V}}{V}=2\bar{V}^{-1}\bar{V}_{,V}\bar{C}_{,V}\\
&\bar{C}_{,CC}-\frac{\bar{C}_{,V}}{V}=2\bar{V}^{-1}\bar{V}_{,C}\bar{C}_{,C}\\
&\bar{C}_{,VC}+2\bar{C}_{,C}V^{-1}=2\bar{V}^{-1}(\bar{V}_{,V}\bar{C}_{,C}+\bar{V}_{,C}\bar{C}_{,V}) .
\end{aligned}\tag{4.2.2}$$

We now must solve the above equations for $\bar{V}$, $\bar{C}$ as functions of V , C . By substitution one may verify that the transformation (3.4.2), written here as

$$\bar{V}=\frac{\mathcal{L}V}{(C-\sigma)^2+V^2}\qquad \bar{C}=\tau+\frac{\mathcal{L}(C-\sigma)}{(C-\sigma)^2+V^2}\tag{4.2.3}$$

where $\mathcal{L}$, Υ, and σ are constants, satisfies (4.2.2). On one hand this should not be too surprising, since the equations of Chapter 2 may be written, using the differential parameters, as

$$\Delta_2(U) = -e^{-2U}\Delta_1(\Omega) \qquad (4.2.4)$$
$$\Delta_2(\Omega) = 2\Delta_1(U,\Omega)$$

if we consider the differential parameters as being taken with respect to a metric given by $ds^2 = d\rho^2 + \rho^2 d\theta^2 + dz^2$ for sections 2.1 and 2.4 and by $ds^2 = d\rho^2 + \rho^2 d\theta^2 - dt^2$ for sections 2.2, 2.3, and 2.5. If we now let

$$V = e^U$$

(4.2.4) are seen to be identical to (4.2.1).

On the other hand, this result is very surprising, since it implies that the particular part of the isogroup that transforms only dependent variables is the same for (4.2.1) and (3.3.1) or (3.3.2). (Recall that (4.2.1) are written with respect to the curved background metric $\gamma_{\alpha\beta}$). Although these two sets of equations are formally identical, when written out explicitly they are very different.

(4.2.3) may be used to give new, physically distinct solutions to (4.2.1) from old ones. Since (4.2.1) correspond to three different physical situations, we see that any solution of the vacuum case (i) gives two more electromagnetic solutions. (4.2.3) may then be used to generate new solutions for all three cases. Since three variable solutions are rare, this result is important as we now get more information out of any one such solution.

APPENDIX

DIFFERENTIAL FORMS AND PARTIAL DIFFERENTIAL EQUATIONS

In this appendix we discuss how the calculus of differential forms may be used to find special solution sets of partial differential equations. The calculus of differential forms is essentially the calculus of surfaces or submanifolds of various dimensions; it systematizes the use of Stokes' Theorems and continuous transformation groups. If we recall that a partial differential equation may be interpreted as an equation defining a family of surfaces, then it is not surprising that differential forms are found very useful when dealing with differential equations and their invariances. This idea is not at all new, having its beginnings in the work of the French mathematician Élie Cartan (Cartan, 1946). Our present purpose is only to show how the differential form calculus can be used as a tool, and the reader is referred to the works of Flanders (Flanders, 1963) and Slebodzinski (Slebodzinski, 1970) for detailed accounts of the differential form calculus itself. A somewhat briefer account that is more in the spirit of the present discussion may be found in Estabrook (Estabrook, 1976a).

A.1 Basic Identities

In this section we summarize without proof the notation and basic identities of the calculus of differential forms. We work in an n-dimensional differentiable manifold spanned by a set of scalar fields (coordinates) X^i,

$i = 1, \ldots, n$, each with a continuous range of values. The basic geometric entities to be manipulated are vectors and 1-forms. These exist at each coordinate point in auxilliary (tangent) and dual-linear vector spaces.

The coordinate differentials dx^i furnish a basis for the 1-forms. A general 1-form is then given by

$$\omega = \sum_{i=1}^{n} A_i dx^i$$

where the A_i are scalar functions of the coordinates. The total differential dx^1 describes, at each point x^i, the family of (n-1)-surfaces x^1 = constant, with similar interpretations for dx^2, dx^3, etc. The general 1-form, being an arbitrary linear superposition of basis 1-forms, may be thought of as a local, oriented, spaced set of surfaces at each point.

To describe families of (n-2)-surfaces, we introduce the operation of exterior multiplication, denoted by $\wedge$. The exterior product of two basis 1-forms, dx^i and dx^j, is written $dx^i \wedge dx^j$. This product describes the family of (n-2)-surfaces x^1 = constant, x^2 = constant. The $\wedge$ operation is completely antisymmetric:

$$dx^i \wedge dx^j = - dx^j \wedge dx^i . \qquad \text{(A.1.1)}$$

The exterior product is also associative:

$$dx^i \wedge (dx^j + dx^k) = dx^i \wedge dx^j + dx^i \wedge dx^k .$$

The basis 2-forms are all the 2-forms $dx^i \wedge dx^k$ where dx^i and dx^k are basis 1-forms. A general 2-form is then an arbitrary linear superposition on basis 2-forms:

$$\sigma = \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij}(x^k)\, dx^i \wedge dx^j$$

where the f_{ij} are again scalar functions of the coordinates.

In an obvious extension of the above, we may introduce basis p-forms (p=3,..., n) by utilizing the exterior product. The basis p-forms can be defined as the exterior products of the basis (p-1)-forms with the basis 1-forms. The general p-forms are then similarly defined as arbitrary linear superpositions of basis p-forms.

We next define the operation of exterior differentiation, denoted by d, which takes p-forms into (p+1)-forms. For scalar functions (0-forms) ϕ, we have

$$d\phi = \sum_{i=1}^{n} \phi_{,i}\, dx^i. \qquad \text{(A.1.2)}$$

This definition may now be used to define the exterior derivative of a 1-form ω:

$$d\omega = \sum_{j=1}^{n} \sum_{i=1}^{n} A_{i,j}\, dx^i \wedge dx^j.$$

This is clearly a generalization of the curl operation in three dimensions, and shows its non-metric character. Again, in an obvious way, exterior differentiation may be defined for p-forms.

A common 1-form is the gradient $d\phi$ of a scalar function ϕ given by (A.1.2). Taking the exterior derivative of (A.1.2) and using (A.1.1) we find

$$d^2\phi = \sum_{j=1}^{n} \sum_{i=1}^{n} \phi_{,ij}\, dx^i \wedge dx^j = 0 \quad \text{, identically.}$$

This illustrates an important theorem that is true for any form ω :

$$d^2\omega = 0 . \tag{A.1.3}$$

If ω and σ are forms of order p and q, respectively, we have from the above

$$\omega \wedge \sigma = (-1)^{pq} \sigma \wedge \omega \tag{A.1.4}$$
$$d(\omega\wedge\sigma) = d\omega\wedge\sigma + (-1)^p \omega\wedge d\sigma .$$

If c is a constant we clearly have

$$dc = 0. \tag{A.1.5}$$

A differential form σ is said to be exact if $d\sigma = 0$. From (A.1.3) we see that any exact form σ may locally be written as $\sigma = d\omega$ (we are not concerned with the global topological considerations that may vitiate this in the large).

We next introduce the (contravariant) vector fields $\vec{V}$ as linear superpositions of basis vectors at each point. The basis vectors are dual to the basis 1-forms. A basis vector $\vec{\lambda}_i$ can be represented as a (in some sense, infinitesimal) displacement along the lines of intersection of the n-1 coordinate (n-1)-surfaces, along which all but one coordinate (x^i) is held constant. The general vector is then $\vec{V} = \sum_{i=1}^{n} V^i \vec{\lambda}_i$. We think of it as a <u>finite</u> entity in the "tangent" vector space at each point.

The duality of the basis vectors and the basis 1-forms is expressed by

$$\vec{\lambda}_i \rfloor dx^j = \delta_i^j$$

where $\rfloor$ denotes the operation of inner product, or contraction. The contraction of a general vector $\vec{V}$ with the basis 1-forms displays the (scalar) components of the vector:

$$\vec{V} \rfloor dx^i = V^i.$$

The contraction of $\vec{V}$ onto a basis 2-form is given by

$$\vec{V} \rfloor dx^i \wedge dx^j = V^i dx^j - V^j dx^i$$

with obvious extensions to p-forms.

This process of contraction is linear, and from the above we have

$$(\vec{V}\rfloor + \vec{W}\rfloor)\sigma = \vec{V}\rfloor\sigma + \vec{W}\rfloor\sigma \qquad \text{(A.1.6a)}$$

$$(f\vec{V})\rfloor\sigma = f(\vec{V}\rfloor\sigma) \qquad \text{(A.1.6b)}$$

$$\vec{V}\rfloor(\omega\wedge\sigma) = (\vec{V}\rfloor\omega)\wedge\sigma + (-1)^p\omega\wedge(\vec{V}\rfloor\sigma) \qquad \text{(A.1.6c)}$$

where $\vec{V}$ and $\vec{W}$ are vectors, ω and σ are forms of rank p and q respectively and f is any scalar function. Contraction clearly takes p-forms into (p-1)-forms.

We next introduce the Lie derivative with respect to a vector field $\vec{V}$, denoted by $\mathcal{L}_{\vec{V}}$. We may think of the Lie derivative as a directional derivative taken in the direction $\vec{V}$. For any scalar Q we have

$$\mathcal{L}_{\vec{V}} Q = \vec{V}\rfloor dQ$$

while for any basis 1-form we have

$$\mathcal{L}_{\vec{V}} dx^i = V^i.$$

By requiring that the Lie derivative be a derivation, and satisfy the Leibniz rule, for higher rank forms constructed by the exterior product rule one finds the Lie derivative of any form ω to be expressed as

$$\underset{\vec{V}}{\mathcal{L}}\,\omega = \vec{V} \rfloor d\omega + d(\vec{V} \rfloor \omega) \tag{A.1.7}$$

and we see that Lie differentiation takes p-forms into p-forms. For any exact form $d\omega$, we have, from (A.1.3) and (A.1.7),

$$\underset{\vec{V}}{\mathcal{L}}\,d\omega = d(\vec{V} \rfloor d\omega) = d\left(\underset{\vec{V}}{\mathcal{L}}\,\omega\right) \tag{A.1.8}$$

and we see that the operations of d and $\underset{\vec{V}}{\mathcal{L}}$ commute.

For any two forms ω and σ we have

$$\underset{\vec{V}}{\mathcal{L}}(\omega \wedge \sigma) = \left(\underset{\vec{V}}{\mathcal{L}}\,\omega\right)\wedge\sigma + \omega \wedge \left(\underset{\vec{V}}{\mathcal{L}}\,\sigma\right). \tag{A.1.9}$$

For any two vector fields $\vec{V}$ and $\vec{W}$ we have

$$\underset{\vec{V}}{\mathcal{L}}(\vec{W} \rfloor \omega) = [\vec{V},\vec{W}] \rfloor \omega + \vec{W} \rfloor \left(\underset{\vec{V}}{\mathcal{L}}\,\omega\right) \tag{A.1.10}$$

where $[\vec{V}, \vec{W}]$ is the commutator, or Lie bracket, of the two vector fields:

$$[\vec{V},\vec{W}]^i = \vec{V} \rfloor dW^i - \vec{W} \rfloor dV^i. \tag{A.1.11}$$

We finally introduce the process of restricting differential forms to submanifolds of the original manifold. We recall that in dealing with differential forms we make no distinction between dependent and independent variables.

In dealing with differential equations, however, this distinction is important. If we impose this difference between independent and dependent variables on forms, then we are restricting the forms to certain submanifolds of the original manifold, which are coordinatized by the independent variables. If we denote independent variables by x^A and dependent variables by z^i, then in these restricted submanifolds the exterior derivatives of these restricted quantities x^A and z^i (denoted by $\tilde{x}^A$ and $\tilde{z}^i$) are given by

$$d\tilde{x}^A = dx^A$$

$$d\tilde{z}^i = \sum_A \frac{\partial z^i}{\partial x^A} dx^A .$$

A.2 Cartan Theory

We now consider the problem of representing a given partial differential equation by an appropriate set of differential forms. This set of differential forms should have the following property: the exterior derivative of any form in the set can be expressed in terms of the original forms in the set. The set is then said to be closed. This is an important property, since it implies that no further integrability conditions can be derived from the set of forms. This set of forms is then the basis of a differential ideal of the Grassman algebra of forms on the manifold.

A submanifold of the differentiable manifold that annuls - gives zero values to - all forms in the set (and hence in the ideal) when they are restricted to the submanifold is called, by Cartan, an integral manifold. We wish to obtain conditions that tell us when the integral manifolds of a set of forms correspond to solutions of the corresponding set of differential equations.

These ideas are expounded in Estabrook (1976**a**). **We** illustrate **them** here with an example. We use Burger's equation, one of the more important nonlinear wave equations (Whitham, 1974). Burger's equation is

$$\phi_{,t} + \phi\phi_{,x} - \phi_{,xx} = 0. \qquad \text{(A.2.1)}$$

In practice, the given equation is first rewritten as a first order system. A corresponding set of forms is then easily found. We first write (A.2.1) as a first order system:

$$\phi_{,x} - u = 0 \qquad \text{(A.2.2a)}$$

$$\phi_{,t} - \omega = 0 \qquad \text{(A.2.2b)}$$

$$\omega + \phi u - u_{,x} = 0\,. \qquad \text{(A.2.2c)}$$

We see that there are two independent variables (x, t) and three dependent variables (ϕ, u, ω). Clearly, if we want solutions of (A.2.2) to correspond to the integral manifolds of a set of forms, we should at least require that when we restrict the forms to independent and dependent variables the integral manifolds reduce to (A.2.2). This may in fact be used to write a set of forms for (A.2.2). We see that (A.2.2a,b) may be satisfied when we restrict and annul the 1-form

$$\alpha = d\phi - u\,dx - \omega\,dt$$

since restricting α yields

$$\tilde{\alpha} = (\phi_{,x} - u)\,dx + (\phi_{,t} - \omega)\,dt\,.$$

Setting the coefficients of independent basis 1-forms to zero then gives (A.2.2.a, b). Similarly the form

$$\beta = \omega\, dx \wedge dt + \phi u\, dx \wedge dt - du \wedge dt$$

will yield (A.2.2c) upon restricting and annulling.

The next procedure is to check that the set of forms is closed, i.e. that $d\alpha$, $d\beta$ are contained in the ideal generated by α and β . We find

$$d\alpha = -du \wedge dx - d\omega \wedge dt$$
$$d\beta = d\omega \wedge dx \wedge dt + \phi\, du \wedge dx \wedge dt + u\, d\phi \wedge dx \wedge dt$$
$$= dx \wedge d\alpha - \phi\, d\alpha\, dt + u\alpha \wedge dx \wedge dt.$$

We see that $d\alpha$ is not in the ideal, but also if it were, $d\beta$ would be. Thus, just the 2-form $d\alpha$ must be added to our original set of forms α and β . Since $d^2\alpha = 0$, the resulting set of forms is closed. If we restrict and then annul $d\alpha$, we find

$$u_{,t} = w_{,x}$$

which is the integrability condition on (A.2.2a,b).

For convenience, we write the generators of the closed ideal α , β , $d\alpha$ for Burger's equation (A.2.1) together here:

$$\alpha = d\phi - u\, dx - \omega\, dt$$
$$d\alpha = -du \wedge dx - d\omega \wedge dt \qquad \text{(A.2.3)}$$
$$\beta = \omega\, dx \wedge dt + \phi u\, dx \wedge dt - du \wedge dt.$$

The integral manifolds of an ideal, like the solutions of a partial differential equation, may be classified as either regular or singular. The general manifolds are those which may be obtained by a sequence of Cauchy-Kowaleski integrations, starting with one-dimensional integral manifolds and giving a chain of integral manifolds of every dimensionality up to a maximum, g. Cartan considers the criteria for an ideal of forms to be "well set" in the sense that the maximum dimensional regular integral manifolds of the forms represent solutions of the corresponding set of first order partial differential equations. **These are the so called Cartan Criteria. Essentially we want to be sure that the surface elements** locally defined by stepwise integrations along vectors that annul the ideal of forms mesh together correctly to define solution surfaces of the partial differential equation.

Although this may be done for ideals involving forms of any order we restrict ourselves here to differential equations with two independent variables and ideals generated only by 1-forms and 2-forms. Then the Cartan criteria reduce to the following: the number of dependent variables in the set of first order partial differential equations must equal the number of independent generating forms. In our example of Burger's equation, there are three dependent variables in (A.2.2) and three independent forms in (A.2.3). The ideal (A.2.3) therefore constitutes a well set ideal for Burger's equation. If an ideal meets the Cartan criteria, it is said to be in involution with respect to the independent variables.

A.3 The Isogroup

We now consider how special geometric properties of the ideal may lead to the construction of special solutions of partial differential equations. One such special property is the existence of vector fields $\vec{V}$ that take the families of surfaces corresponding to the ideal into themselves under the "active" coordinate transformation generated by $\vec{V}$. If we denote the collection of forms in the ideal by $\{I\}$, then we are looking for vector fields $\vec{V}$ such that

$$\underset{\vec{V}}{\mathcal{L}}\{I\} \in \{I\}. \qquad \text{(A.3.1)}$$

Any such $\vec{V}$ is called an isovector of $\{I\}$ and the collection of all such $\vec{V}$ is called the isogroup, or invariance group, of $\{I\}$. By (A.3.1) we see that all transformations $\vec{V}$ will preserve the form of the original system of partial differential equations.

The collection of $\vec{V}$'s satisfying (A.3.1) can easily be shown to form a Lie algebra, since their commutators also satisfy (A.3.1). If there are N distinct isovectors $\vec{V}$ labeled by a subscript $, A,B = 1 \ldots N$, then the structure constants of the group are given by

$$[\vec{V}_A, \vec{V}_B] = \sum_{C=1}^{N} C^C_{AB} \vec{V}_C .$$

We now calculate the isogroup of Burger's equation. The ideal is given by (A.2.3). The first equation of (A.3.1) to consider is

$$\underset{\vec{V}}{\mathcal{L}}\alpha = \lambda\alpha$$

where λ is an arbitrary scalar function. No other term is possible on the right hand side since α is the only 1-form. This may be treated by a technique applicable whenever there is a single 1-form. Write

$$F = \vec{V} \lrcorner \alpha. \qquad \text{(A.3.3)}$$

Since

$$\underset{\vec{V}}{\mathcal{L}}\alpha = \vec{V} \lrcorner d\alpha + d(\vec{V} \lrcorner \alpha)$$

we have

$$\vec{V} \lrcorner d\alpha = \lambda\alpha - dF.$$

Expanding on the basis 1-forms dQ , dx , dt , du , dw we have

$$-V^u dx + V^x du - V^w dt + V^t dw = \lambda(dQ - u\,dx - w\,dt)$$
$$- F_{,t}dt - F_{,x}dx - F_{,u}du - F_{,w}dw - F_{,Q}\,dQ\,. \qquad \text{(A.3.4)}$$

We equate coefficients of each basis 1-form to zero in (A.3.4):

$$-V^u = -\lambda u - F_{,x}$$
$$-V^w = -\lambda w - F_{,t}$$
$$V^x = -F_{,u}$$
$$V^t = -F_{,w}$$
$$\lambda = F_{,Q}\,.$$

We next solve for the $\vec{V}^i$, obtaining

$$V^u = uF_{,\phi} + F_{,x}$$
$$V^w = wF_{,\phi} + F_{,t}$$
$$V^x = -F_{,u} \qquad \text{(A.3.5)}$$

$$V^t = -F_{,\omega}$$
$$V^Q = F - uF_{,u} - \omega F_{,\omega}.$$

If we take the exterior derivative of (A.3.2) we get

$$\underset{\vec{V}}{\mathcal{L}}\, d\alpha = (d\lambda)\wedge\alpha + \lambda\, d\alpha$$

and $\underset{\vec{V}}{\mathcal{L}}\, d\alpha$ is seen to be already in the ideal. We need only consider $\underset{\vec{V}}{\mathcal{L}}\, \beta$ to complete the calculation. We put

$$\underset{\vec{V}}{\mathcal{L}}\, \beta = \xi\beta + \eta\wedge\alpha - \mathcal{L}\, d\alpha \tag{A.3.6}$$

where ξ, $\mathcal{L}$, and η are arbitrary 0, 0, and 1-forms, respectively. Expansion of (A.3.6) yields

$$V^\omega dx\wedge dt + uV^\phi dx\wedge dt + \phi V^u dx\wedge dt + \omega\, dV^x\wedge dt - \omega\, dV^t\wedge dx$$
$$+ \phi u\, dV^x\wedge dt - \phi u\, dV^t\wedge dx - dV^u\wedge dt + dV^t\wedge du =$$
$$\xi(\omega\, dx\wedge dt + Qu\, dx\wedge dt - du\wedge dt) + \mathcal{L}(du\wedge dx + d\omega\wedge dt)$$
$$+ (A\,dt + B\,dx + C\,du + D\,d\omega)\wedge(d\phi - u\,dx - \omega\,dt).$$

ξ, $\mathcal{L}$, A, B, C, and D are arbitrary and are to be eliminated. The expression dV^i, $i = u, x, t, \omega, Q$ is just an abbreviation for $V^i_{,x^j} dx^j$. The equations obtained by equating coefficients of all basis 2-forms to zero are:

$$V^\omega + uV^\phi + \phi V^u + \omega V^x_{,x} + \omega V^t_{,t} + \phi u V^x_{,x} + \phi u V^t_{,t} - V^u_{,x}$$
$$= \xi(\omega + \phi u) + Au - B\omega$$
$$\omega V^x_{,\omega} + \phi u V^x_{,\omega} - V^u_{,\omega} = \mathcal{L} - D\omega$$
$$-\omega V^t_{,x} - \phi u V^t_{,\omega} = -Du$$
$$\phi u V^x_{,Q} - V^u_{,Q} = -A \tag{A.3.7}$$
$$-\omega V^t_{,Q} - \phi u V^t_{,Q} = -B$$
$$\omega V^x_{,u} + \phi u V^x_{,u} - V^u_{,u} - V^t_{,t} = -\xi - C\omega$$
$$D = 0$$

$$-\omega V^t{}_{,u} + \phi u V^t{}_{,u} - V^t{}_{,x} = \mathcal{L} - Cu$$
$$V^t{}_{,Q} = -C$$
$$V^t{}_{,u} = -Du.$$

Solution of (A.3.5) and (A.3.7) now yields

$$\begin{aligned}
V^x &= K_1 + K_3 x + K_4 t + K_5 xt \\
V^t &= K_2 + 2K_3 t + K_5 t^2 \\
V^\phi &= -K_3\phi + K_4 + K_5(x - t\phi) \\
V^u &= -2K_3\phi - 2K_5 ut \\
V^\omega &= -3K_3\omega - K_4 u - K_5(\phi + ux + 3\omega t).
\end{aligned} \tag{A.3.8}$$

The K_i , i=1 , ...,5 are constants. If we in turn set all but one of these to zero, the resulting five isovectors can be used to generate all possible isovectors by linear superpositions with arbitrary constant coefficients. These five independent isovectors are given in Table II (page 75). Rows 1-5 are characterized by K_{1-5}. A description of each type of transformation is provided where feasible. Each of these vectors describes an independent generator of the invariance group. Vectors 1 and 2 are obvious from inspection of (A.2.1), but the others might not have been anticipated.

Of course, the above could all be done in indicial notation without ever introducing forms and vectors; in fact, invariance groups have been traditionally calculated in this way (See Bluman and Cole, 1974). However, it seems that the use of forms facilitates calculation. In addition, the geometric insight gained is invaluable in

Table II Isogroup of (A.2.1)

No.	Coordinate Components					Type
	x	t	ϕ	u	w	
1	1	0	0	0	0	length translation
2	0	1	0	0	0	time translation
3	x	$2t$	$-\phi$	$-2u$	$-3w$	x, t, ϕ scale change
4	t	0	1	0	$-u$	Galilean transformation
5	xt	t^2	$x-t\phi$	$-2ut$	$-\phi-ux-3tw$	

discovering special classes of solutions.

A.4 Similarity Solutions

Harrison and Estabrook (Harrison and Estabrook, 1971) showed how similarity solutions can be found from ideals augmented with additional forms found by contraction with isovectors. Again representing the ideal by $\{I\}$, consider the collection of forms

$$\{\sigma\} = \vec{V} \lrcorner \{I\} \tag{A.4.1}$$

where $\vec{V}$ is now a particular isovector. Taking the Lie derivative of (A.4.1) we find, using (A.1.10),

$$\underset{\vec{V}}{\mathcal{L}}\{\sigma\} = \vec{V}\{\underset{\vec{V}}{\mathcal{L}} I\} = \vec{V} \lrcorner \{I\} = \{\sigma\}$$

where we have used (A.3.1). We see that the ideal of forms generated by the generators of $\{I\}$ and the forms $\{\sigma\}$ is invariant under the particular isovector $\vec{V}$. We call $\{I\}$ and $\{\sigma\}$ collectively the augmented ideal. Since this augmented ideal is invariant only under a particular $\vec{V}$, we may annul it to find a class of special solutions of the original equations. The augmented ideal $\{I, \sigma\}$ is closed. Since $\{I\}$ was originally closed, we must only check the forms $\{\sigma\}$. Using (A.1.7) we find

$$d\{\sigma\} = d(\vec{V} \lrcorner \{I\}) = \underset{\vec{V}}{\mathcal{L}}\{I\} - \vec{V} \lrcorner \{dI\} \subset \{I, \sigma\}$$

since $\vec{V}$ is an isovector.

We again illustrate with an example. We use the particular isovector obtained by adding a(isovector 2) + b(isovector 4) from Table II, where a and b are arbitrary constants. In components we then have

$$
\begin{aligned}
V^x &= bt \\
V^t &= a \\
V^\phi &= b \\
V^u &= 0 \\
V^\omega &= -bu .
\end{aligned}
\tag{A.4.2}
$$

The additional forms (A.4.1) are found by contracting (A.4.2) into (A.2.3):

$$\vec{V} \lrcorner \alpha = b - but - aw , \quad \vec{V} \lrcorner d\alpha = bt\,du + bu\,dt + a\,dw \tag{A.4.3}$$

$$\vec{V} \lrcorner \beta = w(bt\,dt - a\,dx) + \phi u(bt\,dt - a\,dx) + a\,du .$$

We now search for integral manifolds by simultaneously restricting and annulling (A.4.3) and (A.2.3). We already know that by restricting to independent variables (A.2.3) yields (A.2.2) and the integrability condition on u and ω. We may thus substitute (A.2.2) into (A.4.3). We first consider annulling $\vec{V} \lrcorner \alpha$. This yields

$$b - bt\,\phi_{,x} - a\,\phi_{,t} = 0 .$$

This equation may be solved to yield

$$\phi = \sigma t + f(x - \sigma t^2/2) \tag{A.4.4}$$

where f is an arbitrary function of its argument and is to be determined, and $\sigma = b/a$. We find u and ω to be

$$
\begin{aligned}
u &= \phi_{,x} = f' \\
\omega &= \phi_{,t} = \sigma(1 - tf')
\end{aligned}
$$

where a prime denotes differentiation with respect to $\eta = x - \sigma t^2/2$. We may write

$$\vec{V} \lrcorner d\alpha = bt\,du + bu\,dt + a\,dw = d(btu) + a\,dw.$$

Then, since

$$w = \sigma(1-ut)$$

from annulling (A.4.3) we see that $\vec{V} \lrcorner d\alpha$ is annulled. The remaining forms to be annulled are β and $\vec{V} \lrcorner \beta$. If we now substitute for ϕ, u, w, and

$$du = f''(dx - \sigma t\,dt)$$

into $\vec{V} \lrcorner \beta$ we obtain

$$\vec{V} \lrcorner \beta = a\,dx\left[f'' - f'(\sigma t + f) - \sigma(1 - f't)\right] + bt\,dt\left[\sigma(1 - tf') + (\sigma t + f)f' - f''\right].$$

Setting the coefficients of dx and dt to zero, we obtain a second order ordinary differential equation for f:

$$f'' - ff' - \sigma = 0. \qquad \text{(A.4.5)}$$

Annulling β also yields (A.4.5). Thus any f satisfying (A.4.5) will yield a solution to Burger's equation. (A.4.5) may be integrated once to yield

$$f' - \frac{f^2}{2} - \sigma\eta = c \qquad \text{(A.4.6)}$$

where c is a constant of integration. Solutions to (A.4.6) may be found, the solution depending on the constant c. In this way we have found a special class of solutions (or integral manifolds) of Burger's equation. It should be evident that upon obtaining the functional form of Q, (A.4.4), we could have substituted directly into the original partial differential equation (A.2.1) to obtain (A.4.5). Upon solving (A.4.5) we would have annulled the augmented ideal. This procedure is much quicker than the above, and in sections 3.1, 3.3, and 3.5 we omit this lengthy procedure and merely substitute the functional form back into the partial differential equation.

As a generalization of the above, Estabrook and Harrison considered "generalized" isovectors that preserve only the augmented ideal. Denoting the augmented ideal by $\{I'\}$, we would then have

$$\underset{\vec{V}}{\mathcal{L}}\{I'\} \subset \{I'\}. \tag{A.4.7}$$

This idea is motivated by noticing that since we are searching for an augmented ideal invariant only under a specific vector the augmented forms (A.4.1) could have been included as part of the original ideal $\{I\}$, even though their exact expression was not known since $\vec{V}$ had not yet been found. This is the essential content of (A.4.7). As opposed to (A.3.1) which yields linear equations for $\vec{V}$, (A.4.7) yields nonlinear equations. Once such a generalized isovector is found, it may be utilized in the same way as isovectors to obtain special solution sets. Although we do not present any generalized similarity solutions for Burger's equation, we do present some in sections 3.1 and 3.5 for the Einstein field equations.

Most problems of physical interest consist of a set of equations and boundary conditions. The isovectors and generalized isovectors may still be used to generate special solution sets. One would look for those combinations of isovectors and generalized isovectors that leave the boundary conditions invariant and proceed as before. Examples may be found in Bluman and Cole (1974).

A.5 Conservation Laws

Differential forms are also useful for finding conservation laws for partial differential equations. If we can find an exact 1-form, $d\psi$, in the ideal, then we have found a (differential) conservation law for the set of partial differential equations. From Stokes' Theorem we obtain

$$\int_V d\psi = \int_{\partial V} \psi$$

where V is any volume in the manifold bounded by the closed boundary manifold ∂V. If V lies in an integral manifold, then the restriction of $d\psi$ to this integral manifold, denoted by $d\tilde{\psi}$, is zero, and the above becomes

$$\int_{\partial V} \tilde{\psi} = 0$$

which is a non-trivial integral conservation law if ψ itself is not in the ideal (since then $\tilde{\psi} = 0$).

We again illustrate with Burger's equation. We seek all 1-forms ψ of the form

$$\psi = F(\phi, u, \omega)dx + G(\phi, u, \omega)dt \tag{A.5.1}$$

that satisfy

$$d\psi \subset I. \tag{A.5.2}$$

Using (A.2.3), (A.5.2) becomes

$$dx \wedge dt(\omega G_{,u} + \phi u G_{,u} - \omega F_{,\phi} + u G_{,\phi}) + d\omega \wedge dt(G_{,\omega} - F_{,u}) + d\omega \wedge dx(F_{,\omega}) = 0 .$$

Setting the coefficients of basis 2-forms to zero gives three equations for F and G :

$$\begin{aligned} &F_{,\omega} = 0 \\ &F_{,u} = G_{,\omega} \\ &\omega G_{,u} + \phi u G_{,u} - \omega F_{,\phi} + u G_{,\phi} = 0 . \end{aligned} \tag{A.5.3}$$

The general solution of (A.5.3) is

$$\begin{aligned} &F = u C(\phi) + \alpha\phi + \beta \\ &G = \omega C(\phi) + \alpha(u - \phi^2/2) + \delta \end{aligned} \tag{A.5.4}$$

where α, β, δ are arbitrary constants. The first terms correspond to a 1-form $C(\phi)\alpha$ which is already in the ideal and so a trivial generalization of the conservation law we seek. Hence $C(\phi)$ can be **set equal to zero.**

From (A.5.1) and (A.5.2) we see that we can subtract any closed form $d\sigma$ from ψ, since $d^2\sigma = 0$. σ is a new variable or coordinate. Its introduction allows the 1-form

$$\psi = -d\sigma + F dx + G dt$$

with F and G given by (A.6.4) to be added to the ideal; since the augmented ideal is still closed, the Cartan criteria remain satisfied. Annulling ψ then gives

$$\sigma_{,x} = F$$
$$\sigma_{,t} = G$$

for F and G belonging to any integral manifold. In terms of independent and dependent variables, this becomes

$$\sigma_{,x} = \alpha\phi + \beta$$
$$\sigma_{,t} = \alpha\left(\phi_{,x} - \frac{\phi^2}{2}\right) + \delta .$$

If we choose $\beta = \delta = 0$, we obtain

$$\sigma_{,x} = \alpha\phi$$
$$\sigma_{,t} = \alpha\left(\phi_{,x} - \phi^2/2\right) \qquad \text{(A.5.5)}$$

which is the Cole-Hopf transformation (Whitham, 1974). In fact, if we compute the isogroup of the augmented ideal, we find that there are now seven isovectors. One of the new isovectors has components that depend on the partial derivatives of a function $\gamma(x,t)$ which satisfies the heat equation

$$\gamma_{,xx} = \gamma_{,t} . \qquad \text{(A.5.6)}$$

There are thus an infinite number of such new isovectors corresponding to the infinitude of solutions to (A.5.6). This is an indication that the transformation (A.5.5) has linearized the equation (A.2.1) (Estabrook, 1976b). In fact, if we set $\alpha = -\frac{1}{2}$ in (A.5.5) and eliminate ϕ for σ in (A.2.1), we obtain

$$\sigma_{,xt} - 2\sigma_{,x}\sigma_{,xx} - \sigma_{,xxx} = 0$$

which integrates to

$$\sigma_{,t} - \sigma_{,x}^2 - \sigma_{,xx} = 0 \qquad \text{(A.5.7)}$$

where we have chosen the constant of integration to be zero. If we now let

$$\sigma = \log v$$

(A.5.7) becomes

$$v_{,xx} = v_{,t}$$

and the linearity is displayed explicitly.

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