INVARIANTS, LIE-BÄCKLUND OPERATORS AND BÄCKLUND TRANSFORMATIONS

Thesis by

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ABSTRACT

This thesis is mainly concerned with the application of groups of transformations to differential equations and in particular with the connection between the group structure of a given equation and the existence of exact solutions and conservation laws. In this respect the Lie-Bäcklund groups of tangent transformations, particular cases of which are the Lie tangent and the Lie point groups, are extensively used.

In Chapter I we first review the classical results of Lie, Bäcklund and Bianchi as well as the more recent ones due mainly to Ovsjannikov. We then concentrate on the Lie-Bäcklund groups (or more precisely on the corresponding Lie-Bäcklund operators), as introduced by Ibragimov and Anderson, and prove some lemmas about them which are useful for the following chapters. Finally we introduce the concept of a conditionally admissible operator (as opposed to an admissible one) and show how this can be used to generate exact solutions.

In Chapter II we establish the group nature of all separable solutions and conserved quantities in classical mechanics by analyzing the group structure of the Hamilton-Jacobi equation. It is shown that consideration of only Lie point groups is insufficient. For this purpose a special type of Lie-Bäcklund groups, those equivalent to Lie tangent groups, is used. It is also shown how these generalized groups induce Lie point groups on Hamilton's equations. The generalization of the above results to any first order equation, where the dependent variable does not appear

explicitly, is obvious. In the second part of this chapter we investigate admissible operators (or equivalently constants of motion) of the Hamilton-Jacobi equation with polynomial dependence on the momenta. The form of the most general constant of motion linear, quadratic and cubic in the momenta is explicitly found. Emphasis is given to the quadratic case, where the particular case of a fixed (say zero) energy state is also considered; it is shown that in the latter case additional symmetries may appear. Finally, some potentials of physical interest admitting higher symmetries are considered. These include potentials due to two centers and limiting cases thereof. The most general two-center potential admitting a quadratic constant of motion is obtained, as well as the corresponding invariant. Also some new cubic invariants are found.

In Chapter III we first establish the group nature of all separable solutions of any linear, homogeneous equation. We then concentrate on the Schrödinger equation and look for an algorithm which generates a quantum invariant from a classical one. The problem of an isomorphism between functions in classical observables and quantum observables is studied concretely and constructively. For functions at most quadratic in the momenta an isomorphism is possible which agrees with Weyl's transform and which takes invariants into invariants. It is not possible to extend the isomorphism indefinitely. The requirement that an invariant goes into an invariant may necessitate variants of Weyl's transform. This is illustrated for the case of cubic invariants. Finally, the case of a specific value of energy is considered; in this case Weyl's transform does not yield an isomorphism even for the quadratic case.

However, for this case a correspondence mapping a classical invariant to a quantum one is explicitly found.

Chapters IV and V are concerned with the general group structure of evolution equations. In Chapter IV we establish a one to one correspondence between admissible Lie-Bäcklund operators of evolution equations (derivable from a variational principle) and conservation laws of these equations. This correspondence takes the form of a simple algorithm.

In Chapter V we first establish the group nature of all Bäcklund transformations (BT) by proving that any solution generated by a BT is invariant under the action of some conditionally admissible operator. We then use an algorithm based on invariance criteria to rederive many known BT and to derive some new ones. Finally, we propose a generalization of BT which, among other advantages, clarifies the connection between the wave-train solution and a BT in the sense that, a BT may be thought of as a variation of parameters of some special case of the wave-train solution (usually the solitary wave one). Some open problems are indicated.

Most of the material of Chapters II and III is contained in [I], [II], [III] and [IV] and the first part of Chapter V in [V].

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CHAPTER I

1.1 INTRODUCTION

This thesis is mainly concerned with the application of groups of transformations to differential equations and in particular with the connection between the group properties of a given equation and the existence of exact solutions and conservation laws. In this respect it is essential to define what group of transformations we are concerned with. S. Lie developed and applied in his study of differential equations what we shall call Lie point and Lie tangent groups of transformations. These groups although very useful in practice are quite restricted and are not adequate for the complete analysis of many physical phenomena. The classical literature evidenced two directions of efforts to generalize these transformations: first, a search for groups of higher-order (but finite) tangent transformations which was essentially abortive; second, a development of a special type of surface transformation (first discovered by Lie and then formally generalized by Bäcklund) which led to what was later called a Bäcklund transformation. Recently the search for groups of higher-order tangent transformations has been realized with the notion of Lie-Bäcklund (LB) groups which are infinite-order tangent transformations. These groups are characterized infinitesimally by the Lie-Bäcklund operators which are extensively used in this work.

In this chapter, after presenting a rather detailed account of the development of the theory of surface transformations and its application to differential equations, we concentrate on LB operators. We prove different theorems about them (most of which are new) which clarify their nature and also show how they can be used effectively for deriving exact solutions. This naturally leads to consideration of admissible and conditionally admissible LB operators.

1.2 HISTORICAL INTRODUCTION

In this section we present the fundamental ideas, notions and results of the classical papers of Lie and Bäcklund as well as the more recent development of Lie's theory due mainly to Ovsjannikov.

1.2.1 Groups of Lie Point Transformations

We shall briefly review those definitions and results from the theory of continuous groups necessary to the understanding of the succeeding sections. This material can be found in [1], [2], [3] and some additional examples in [4] and [5].

We shall consider a one parameter continuous group of transformations of R^n,

$$\bar{x}_i = f_i(x;\alpha), \qquad 1 \leqslant i \leqslant n, \tag{1.1}$$

where $x = (x_1, \ldots, x_n)$, $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$, α is a real parameter and R^n is an n-dimensional real space. For brevity we shall denote the above as

$$\bar{x} = f(x;\alpha), \quad \text{or} \quad \bar{x} = T_\alpha x. \tag{1.2}$$

We assume that each such transformation is <u>invertible</u> and in addition; i) There exists an α_0 such that

$$x = f(x;\alpha_0) \quad \text{for all} \quad x.$$

ii) For any two values α_1 and α_2 of the parameter α, there exists a unique α_3 such that

$$f(x;\alpha_3) = f(f(x,\alpha_1),\alpha_2) \quad \text{for all} \quad x.$$

iii) For each α_1 there is a <u>unique</u> α_2 such that

$$x = f^{-1}(\bar{x},\alpha_1) = f(f(x,\alpha_1),\alpha_2) \quad \text{for all} \quad x.$$

A. Infinitesimal formulation.

S. Lie was the first to systematically develop an infinitesimal characterization of the above group of transformation. He considered the tangent vector field ξ of the above group, defined by

$$\xi_i(x) = \frac{\partial f_i}{\partial \alpha}(x, \alpha_0), \qquad 1 \leq i \leq n,$$

and associated with ξ an infinitesimal generator, or Lie operator X, defined by

$$X = \sum_j \xi_j \frac{\partial}{\partial x_j}. \tag{1.3}$$

Geometrically X is the operator of differentiation in the direction of the curve $\alpha \to T_\alpha x$. Lie proved that X uniquely specifies the group defined by (1.1) through the solution of equations

$$\frac{df_i}{d\alpha} = \xi_i(f), \qquad f_i(x, \alpha_0) = x_i. \tag{1.4}$$

In practice, most of the time we only consider X and not the group defined by (1.1), which sometimes is called the global group; (however, this terminology is unfortunate since all considerations in Lie's theory are local).

B. Invariants.

A function $I(x)$ is an invariant under the action of the group $\bar{x} = T_\alpha x$ if

$$I(T_\alpha x) = I(x) \quad \text{for all } x \text{ and } \alpha.$$

It is easily proved that $I(x)$ is an invariant iff

$$XI = \sum_j \xi_j \frac{\partial I}{\partial x_j} = 0. \tag{1.5}$$

C. Invariant Manifolds.

Associated with any set of functions $\psi_\nu(x)$, $\nu = 1, \ldots, \rho$, is a manifold (surface) M consisting of all x satisfying

$$\psi_\nu(x) = 0, \qquad \nu = 1, \ldots, \rho. \tag{1.6}$$

We say that M is an invariant manifold for the group $\bar{x} = T_\alpha x$, if $T_\alpha x$ lies in M for every x initially in M (and any α sufficiently close to α_0). It is easily proved that M is an invariant manifold iff

$$X\psi_\nu = 0 \ \underline{\text{on} \ \ M}, \qquad \nu = 1, \ldots, \rho. \tag{1.7}$$

The above equation is denoted as

$$X\psi_\nu \Big|_{\psi_\nu = 0} = 0 \ , \qquad \nu = 1, \ldots, \rho. \tag{1.8}$$

D. Application to differential equations.

Suppose we are given the system of differential equations

$$F_\nu(x, u, \underset{1}{u}, \ldots, \underset{s}{u}) = 0, \qquad \nu = 1, \ldots, N, \tag{1.9}$$

where $x \in R^M$, $u \in R^N$, $N+M = n$ and $\underset{k}{u}$ denotes the set of all kth order derivatives of u with respect to x. We may think of (1.9) as defining a manifold in $(x, u, \underset{1}{u}, \ldots, \underset{s}{u})$-space. In order to examine the group properties of this manifold we must extend the group $\bar{x} = T_\alpha x$, so that it can act on the above space. To fix the ideas let us consider the case of one independent and one dependent variable. Consider the group of transformation G defined by

$$\bar{x} = f(x, u; \alpha), \tag{1.10a}$$

G:

$$\bar{u} = g(x, u; \alpha). \tag{1.10b}$$

We must extend G to include as many derivatives as specified by the given equation under consideration. The easiest way to achieve this is by the chain rule. The first extension of G, denoted by $\underset{1}{G}$ is given by equations (1.10a), (1.10b) and

$$\bar{u}_{\bar{x}} = \phi(x, u, u_x; \alpha), \tag{1.10c}$$

where

$$\phi = \frac{df}{dx}\Big/\frac{dg}{dx}. \tag{1.11}$$

Let us now obtain an infinitesimal characterization of $\underset{1}{G}$: Suppose

$$\left.\frac{\partial f}{\partial \alpha}\right|_{\alpha=0} = \xi(x, u), \qquad \left.\frac{\partial g}{\partial \alpha}\right|_{\alpha=0} = \eta(x, u),$$

$$\left.\frac{\partial \phi}{\partial \alpha}\right|_{\alpha=0} = \zeta_1(x, u, u_x), \text{ (where we have assumed } \alpha_0 = 0)$$

Then using (1.11),

$$u_x + \alpha\zeta_1 + O(\alpha^2) = \frac{u_x + \alpha D_x \eta}{1 + \alpha D_x \xi} = u_x + \alpha[D_x\eta - u_x D_x \xi] + O(\alpha^2).$$

Therefore,

$$\zeta_1 = D_x \eta - u_x D_x \xi, \tag{1.12}$$

where D_x is the total derivate $\partial/\partial x + u_x \partial/\partial u + \ldots$. The above analysis can be easily extended to any order. Let us summarize: Suppose we are given an equation of sth order,

$$F(x, u, u_x, \ldots, u_{\underbrace{x\ldots x}_{s}}) = 0. \tag{1.13}$$

In order to examine its group properties with respect to a group G, specified by the Lie operator

$$X = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u},$$

we extend this operator to

$$\underset{s}{X} = X + \zeta_1\frac{\partial}{\partial u_x} +,\dots,\zeta_s\frac{\partial}{\partial u_{\underbrace{x\dots x}_{s}}}, \tag{1.14}$$

where ζ_1 is defined by (1.12), $\zeta_2 = D_x\zeta_1 - u_{xx}D_x\xi$, etc.

There is also another way of defining the extended group $\underset{1}{G}$ (which can be trivially generalized to cover any extended group $\underset{s}{G}$): We must choose ϕ in such a way that the group of transformations defined by (1.10a,b,c) (and obviously extended to differentials $d\bar{x}$, $d\bar{u}$, $d\bar{u}_{\bar{x}}$), leaves the first order tangency condition

$$du - u_x dx = 0$$

invariant. We will return to this point of view later.

E. Admissible operators.

Having obtained the extended Lie operator (1.14) we can now examine the action of the group $\underset{s}{G}$ on the manifold defined by equation (1.13). In accordance with our definition of an invariant manifold, the equation $F = 0$ is invariant under G, or more precisely, the manifold defined by $F = 0$ in the $(x, u, u_x, \dots, u_{\underbrace{x\dots x}_{s}})$-space is an invariant manifold for G, iff

$$\underset{s}{X}F\Big|_{F=0} = 0 \ . \tag{1.15}$$

In this case we say that X (or $\underset{s}{X}$) is an admissible (Lie point) operator for equation $F = 0$. Writing out equation (1.15) we obtain a set of <u>linear overdetermined</u> equations for ξ and η

the solution of which specifies X. If this system has r solutions, with corresponding Lie operators $X_1, \ldots, X_r$, we say that equation $F = 0$ is invariant under an r-parameter group of transformations. It can be proved that these operators form a Lie Algebra with commutator

$$[X_i, X_j] = X_i X_j - X_j X_i .$$

F. Invariant solutions.

The solution manifolds of equations (1.9) are submanifolds of the equation manifold. An admissible operator takes a solution manifold into a (possibly different) solution manifold. A solution manifold is called invariant with respect to some operator X if under its actions it is taken into itself. The corresponding solutions are called invariant, or similarity solutions. The interesting question arising is the following: Given a set of equations

$$F_\nu(x, u, \underset{1}{u}, \ldots, \underset{s}{u}) = 0, \qquad \nu = 1, \ldots, N, \tag{1.9}$$

where $x \in R^{n-N}$, $u \in R^N$ and an r parameter group of transformations G_r which leave (1.9) invariant, (or equivalently r Lie point operators $X_1, \ldots, X_r$) when can we find invariant solutions and what is their form? A partial answer to this question was given by Ovsjannikov who found a necessary condition for the existence of invariant solutions as well as their general form: Given X_i, $1 \leq i \leq r$ and solving

$$X_i J(x, u) = 0, \qquad 1 \leq i \leq r,$$

we find $n-\mu$ invariants $J_\tau(x, u)$, $1 \leq \tau \leq n-\mu$, where μ is the rank of the matrix of the infinitesimal generators $\xi(x, u)$

associated with X_i. Let ρ be the rank of the matrix $(\partial J_\tau/\partial u)$. Then a necessary condition for the existence of invariant solutions is $\rho = N$, and the invariant solutions are given by

$$\phi_\tau(J_1, \ldots, J_{n-\mu}) = 0, \qquad \tau = 1, \ldots, N.$$

Clearly the invariant manifold just defined has dimension $n-N-\mu$ in contrast with any non-invariant manifold which has dimension $n-N$. After having developed the general mechanism for obtaining invariant solutions one has to decide which of them are essentially distinct, i.e. which is the basic set of invariant solutions; basic in the sense that all other invariant solutions can be obtained from this set with the aid of the group G_r. This problem has been solved also by Ovsjannikov with the use of the adjoint group, see [1].

We conclude this subsection with pointing out that there are two types of groups which can be found by inspection; translations $\bar{x} = x+\alpha$, $\bar{u} = u+\beta$ and stretchings $\bar{x} = \gamma x, \bar{u} = \delta u$. The invariant solutions corresponding to the stretching group can also be obtained using dimensional analysis. However, this is the only type of invariant solutions obtained through dimensional analysis. Let us give a trivial example, where we can find groups by inspection: Consider the heat equation

$$u_t - u_{xx} = 0. \tag{1.16}$$

Letting $x \to x+\alpha_1$, $t \to t+\alpha_2$, $u \to u+\alpha_3$, $u \to (1+\alpha_4)u$, $x \to (1+\alpha_5)x$, $t \to (1+\alpha_5)^2 t$, equation (1.16) remains invariant. The operators corresponding to the parameters α_i, $1 \leq i \leq 5$ are

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = \frac{\partial}{\partial u}, \quad X_4 = u\frac{\partial}{\partial u},$$

$$X_5 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}.$$

Let us find the invariant solutions under the operator $X = X_5+\lambda X_4$. The invariants are $J_1 = u/t^{\lambda/2}$, $J_2 = x^2/t$. The necessary condition of Ovsjannikov's theorem is satisfied and therefore, the similarity solution is specified by

$$\phi(\frac{u}{t^{\frac{\lambda}{2}}}, \frac{x^2}{t}) = 0,$$

or

$$u = t^{\frac{\lambda}{2}}F(\frac{x^2}{t}).$$

Substituting the above in (1.16) we obtain an ordinary equation for F. The case $\lambda = -1$ corresponds to the source solution.

1.2.2 Groups of Lie Tangent Transformations

The transformations defined by equations (1.10a, b, c) although very useful in practice are quite restricted. In the last one hundred years many generalizations of the above transformations have been proposed. A crucial step in this process was made by S. Lie who formulated two very important questions in his 1874 paper [6]. These questions are presented in the next two subsections. For their understanding the introduction of what we call Lie tangent transformations is essential; The historical perspective and the basic material of this subsection as well as that of §§1-4 may be found in [7] and [8].

A first generalization of the transformations defined by (1.10a, b, c) was given by Lie himself: Consider the group G of point transformations

$$G: \quad \begin{aligned} \bar{x} &= f_1(x,u,u_x;\alpha), \\ \bar{u} &= f_2(x,u,u_x;\alpha), \\ \bar{u}_{\bar{x}} &= f_3(x,u,u_x;\alpha), \end{aligned} \qquad (1.17)$$

in the space of independent variables (x,u,u_x); this group can be trivially extended to differentials $(d\bar{x}, d\bar{u}, d\bar{u}_{\bar{x}})$. Let us call the extended group $\tilde{G}$, which now acts in the space of independent variables (x,u,u_x,dx,du,du_x). Lie called the group G a group of contact transformations, if the equation

$$du - u_x dx = 0, \qquad (1.18)$$

is invariant with respect to the extended group $\tilde{G}$. Hereafter, we refer to this particular group as the group of <u>Lie tangent transformations</u>. A priori it is not obvious that such a group exists (other than the particular case of a Lie point group): Let us work infinitesimally; ξ and η will depend now on u_x as well, therefore ζ_1 will in general depend also on u_{xx}. For the existence of a Lie tangent group it is necessary to eliminate the dependence of ζ_1 on u_{xx}. Using equation (1.12) to express ζ_1 in terms of η and ξ and then equating the coefficient of u_{xx} to zero we obtain

$$\eta_{u_x} - u_x \xi_{u_x} = 0. \qquad (1.19)$$

The general solution of (1.19) is given by

$$\xi = -\frac{\partial W}{\partial u_x}, \qquad \eta = W - u_x \frac{\partial W}{\partial u_x}, \qquad (1.20)$$

where W is some function of x,u,u_x. Therefore, equations (1.17) form a group of Lie tangent transformations if there exists a function $W(x,u,u_x)$ such that equations (1.20) hold. The above analysis can be generalized when x is a vector, but it <u>cannot</u> be generalized when u is a vector.

1.2.3 Lie's First Question

Let us now consider the group G^n of the point transformations

$$G^n: \quad \begin{aligned} \bar{x} &= g_1(x, u, \underset{1}{u}, \dots, \underset{n}{u}; \alpha), \\ \bar{u} &= g_2(x, u, \underset{1}{u}, \dots, \underset{n}{u}; \alpha), \\ &\;\vdots \\ \underset{n}{\bar{u}} &= g_{n+2}(x, u, \underset{1}{u}, \dots, \underset{n}{u}; \alpha), \end{aligned} \tag{1.21}$$

in the space of the variables $(x, u, \underset{1}{u}, \dots, \underset{n}{u})$. In this case we say that G^n is a group of contact transformations of nth-order if

$$\begin{aligned} du - \underset{1}{u}dx &= 0, \\ d\underset{1}{u} - \underset{2}{u}dx &= 0 \\ &\;\vdots \\ d\underset{n-1}{u} - \underset{n}{u}dx &= 0, \end{aligned}$$

are invariant with respect to the group $\tilde{G}^n$ obtained by the extension of the group G to the differentials $d\bar{x}, d\bar{u}, \dots, d\underset{n}{\bar{u}}$. After Lie's success in obtaining groups of tangent (or contact of first order) transformations, which are not just extensions of Lie point transformations (i.e. ξ and/or η depend necessarily upon u_x), the next question is to ask if there exist groups of contact transformations of nth-order, $n > 1$, which are not just extensions of Lie point or Lie tangent groups. This is essentially the first question of S. Lie. He predicted a negative answer to this question. The results in Bäcklund's first papers [9], [10] can be interpreted as proving Lie's conjecture; the proof is geometrical. An analytical proof is given in [8]. Bäcklund's results can be summarized in the following statement: There are no nontrivial higher-order generalizations of Lie tangent transformations if one understands a transformation as an invertible one to one map in a finite-dimensional space.

1.2.4 Lie's Second Question

Now let us go back to point transformations defined by equation (1.17). If these transformations are viewed as surface-transformations then necessarily they are single-valued surface-transformations acting invariantly in a finite-dimensional space. Lie, in his second important question asked, if there exist any useful many-valued surface-transformations. Before explaining what we mean by useful let us define such a transformation. To fix the ideas we take the case of two independent variables and one dependent variable; let

$$p = z_x, \quad q = z_y, \quad r = z_{xx}, \quad s = z_{xy}, \quad t = z_{yy}. \qquad (1.22)$$

The capital letters will denote the corresponding transformed quantities. A many-valued surface-transformation is one which takes the surface element $(x, y, z(x, y), z_x, z_y)$ to one-fold infinity elements (X, Y, Z, P, Q), which actually are surface elements; i.e.

$$\frac{\partial P}{\partial Y} - \frac{\partial Q}{\partial X} = 0, \quad \text{on} \quad z = z(x, y), \qquad (1.23)$$

where $z = z(x, y)$ defines a surface in the (x, y, z)-space. This transformation (although not single-valued) can still be useful in the transformation theory of differential equations if it satisfies two requirements: i) if it transforms a given differential equation to one of the same or lower order; ii) if it becomes a surface transformation on any surface which belongs in the family of solution surfaces of the given differential equation.

Lie's analytical treatment of Bianchi's geometrical construction of a transformation of surfaces of constant curvature, was the first example of a useful many-valued surface-transformation. He considered the transformation

$$\begin{aligned} &(x-X)^2 + (y-Y)^2 + (z-Z)^2 = a^2, \\ &p(x-X) + q(y-Y) - (z-Z) = 0, \\ &P(x-X) + Q(y-Y) - (z-Z) = 0, \\ &pP + qQ + 1 = 0. \end{aligned} \tag{1.24}$$

The above equations are the equivalent analytical form of Bianchi's geometrical construction. First we observe, that given any surface element (x, y, z, p, q) equations (1.24) give a one-fold infinity of potential surface elements (X, Y, Z, P, Q). Lie proved that if $z = z(x, y)$ is a surface of constant curvature $-1/a^2$, i.e. if it solves

$$s^2 - rt = (1 + p^2 + q^2)/a^2, \tag{1.25}$$

then the element (X, Y, Z, P, Q) is a surface element (i.e. equation (1.23) is satisfied) and further the surface $Z = Z(X, Y)$ is also a surface of constant curvature $-1/a^2$. Equations (1.24) define what today we call a Bäcklund transformation. Therefore, the first Bäcklund transformation was due to Lie!

A generalization of the above transformation was introduced by Bäcklund [11] who considered four general relations between two sets of surface elements:

$$F_i(x,y,z,p,q,X,Y,Z,P,Q) = 0, \qquad i = 1,2,3,4. \quad (1.26)$$

A literal repetition of Lie's considerations and techniques for treating (1.24), applied to (1.26) leads to what is called in the literature a Bäcklund transformation . Further, without loss of generality we can take equations (1.26) to be

$$\begin{aligned} &X-x = 0, \\ &Y-y = 0, \\ &F_1(x,y,z,p,q,Z,P,Q) = 0, \\ &F_2(x,y,z,p,q,Z,P,Q) = 0. \end{aligned} \qquad (1.27)$$

As Goursat [12] has remarked one can generalize this form in many ways, including increasing the dimension of the underlying space, the order of the surface elements and the number of relations in (1.27), etc.

1.3 GROUPS OF LIE-BÄCKLUND (LB) TANGENT TRANSFORMATIONS

As it was pointed out in §1.2.3 Bäcklund proved that there are no nontrivial finite higher-order generalizations of Lie-tangent transformations. Expressing this fact in terms of an infinitesimal operator we may say that, if the operator X_s, given by (1.14) characterizes a group of contact transformations of sth-order then, either

$$\xi = \xi(x, u), \qquad \eta = \eta(x, u) \tag{1.28}$$

and X_s is a Lie-point operator, or

$$\xi = \xi(x, u, u_x), \qquad \eta = \eta(x, u, u_x), \tag{1.29}$$

where ξ and η satisfy equations (1.20), and X_s is a Lie tangent operator. The above groups of transformations express symmetries of a geometrical origin. However, in trying to explain some physical phenomena (for example, the conservation of the quantum mechanical analogue of the Runge-Lenz vector for the hydrogen atom) the "dynamical symmetries" were introduced. These were symmetries of a non-geometrical origin. This led to a consideration of infinitesimal operators of the form (1.14), where ξ and η depended on higher derivatives (see for example [13]). It was clearly shown that there exist infinitesimal transformations depending on higher derivatives, which leave equations of physical importance (like the Schrödinger's equation) invariant. Although the group nature of those transformations was not very clear, the necessity for a generalization of Lie theory was evident. However, Bäcklund's result indicates that such a generalization for finite-order contact transformations is impossible. Recently, Ibragimov and Anderson established rigorously the only possible generalization, of the original Lie formulation, based on the notion of infinite-order contact transformations. They called these generalized transformation groups Lie-Bäcklund (LB) tangent transformations. Because our work is heavily based on the existence of these groups we briefly summarize some results of [8], [14], [15]:

A. Definition.

Let $x = (x_1, \dots, x_n) \in R^N$, $u = (u^1, \dots, u^M) \in R^M$ and for every $k = 1,2,3,\dots,$ $\underset{k}{u}$ be the set of partial derivatives $u^{\alpha}_{i_1 \dots i_k}$, $(\alpha = 1, \dots, M; i_1, \dots, i_k = 1, \dots, N)$, symmetric in their lower indices. Let us consider a one-parameter group G of point transformations

$$
\begin{aligned}
G: \quad x'_i &= f_i(x, u, \underset{1}{u}, \underset{2}{u}, \dots; a), \\
u'^{\alpha} &= \phi^{\alpha}(x, u, \underset{1}{u}, \underset{2}{u}, \dots; a), \qquad (1.30) \\
u'^{\alpha}_i &= \psi^{\alpha}_i(x, u, \underset{1}{u}, \underset{2}{u}, \dots; a), \\
&\cdots\cdots\cdots
\end{aligned}
$$

in the infinite dimensional space $(x, u, \underset{1}{u}, \underset{2}{u}, \dots)$. In the above equations, the number of arguments of each of the functions f_i, $\phi, \dots,$ is a priori arbitrary and may be finite or infinite. Together with the group G, we consider its extension $\widetilde{G}$ to the differentials dx_i, du^{α}, $du^{\alpha}_i \dots$.

A group G is called a group of Lie-Bäcklund tangent transformations if the infinite system of equations:

$$du^\alpha - u_j^\alpha dx_j = 0,$$

$$du_i^\alpha - u_{ij}^\alpha dx_j = 0,$$

$$du_{i_1 i_2}^\alpha - u_{i_1 i_2 j}^\alpha dx_j = 0,$$

$$\cdots\cdots\cdots$$

is invariant with respect to the group $\widetilde{G}$. (The summation convention will be implied for repeated indices.)

B. Infinitesimal characterizations.

In order to give an infinitesimal characterization of the group G we define the operator

$$Z = \xi_i \frac{\partial}{\partial x_i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \ldots + \zeta_{i_1 \ldots i_k}^\alpha \frac{\partial}{\partial u_{i_1 \ldots i_k}^\alpha} + \ldots \tag{1.31}$$

where

$$\xi_i = \left.\frac{\partial f_i}{\partial \alpha}\right|_{a=0}, \qquad \eta^\alpha = \left.\frac{\partial \phi^\alpha}{\partial a}\right|_{a=0},$$

$$\zeta_{i_1 \ldots i_k}^\alpha = \left.\frac{\partial \psi_{i_1 \ldots i_k}^\alpha}{\partial a}\right|_{a=0}, \qquad k = 1, 2, 3, \ldots . \tag{1.32}$$

The operator (1.2) fully characterizes the group G if we ensure the existence and uniqueness of the solution of the Lie equation:

$$\frac{dF}{da} = \Theta(F), \qquad F\Big|_{a=0} = z,$$

where

$$z = (x_i, u^\alpha, u_i^\alpha \ldots),$$

$$F = (f_i, \phi^\alpha, \psi_i^\alpha, \ldots),$$

$$\Theta = (\xi_i, \eta^\alpha, \zeta_i^\alpha, \ldots).$$

The group F of transformations (1.30)is a group of Lie-Bäcklund tangent transformations if and only if coordinates of the infinitesimal operator (1.31) satisfy the equations:

$$\zeta_i^\alpha = D_i(\eta^\alpha) - u_j^\alpha D_i(\xi_j),$$
$$\zeta_{i_1 i_2}^\alpha = D_{i_2}(\zeta_{i_1}^\alpha) - u_{i_1 j}^\alpha D_{i_2}(\xi_j), \tag{1.33}$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

where

$$D_i = \frac{\partial}{\partial x_i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ii_1}^\alpha \frac{\partial}{\partial u_{i_1}^\alpha} + u_{ii_1 i_2}^\alpha \frac{\partial}{\partial u_{i_1 i_2}^\alpha} + \ldots . \tag{1.34}$$

C. Application to differential equations.

Consider a given system of differential equations

$$\underline{\Omega}(x, u, \underset{1}{u}, \ldots, \underset{n}{u}) = 0. \tag{1.35}$$

The equation $\underline{\Omega} = 0$ together with all its differential consequences:

$$\underline{\Omega} = 0, \quad D_i \underline{\Omega} = 0, \quad D_{i_1}(D_{i_2} \underline{\Omega}) = 0, \ldots, \tag{1.36}$$

defines a manifold Ω in the $(x, u, \underset{1}{u}, \ldots,)$-space. The system of equations (1.35) is called invariant with respect to a Lie-Bäcklund group G if the manifold Ω is invariant under the transformations (1.30). In this case G is called admissible for the equation (1.35). The criterion for invariance of the system (1.35) is

$$Z\Omega\big|_{(1.36)} = 0, \qquad (1.37)$$

where the subscript means that (1.36) are assumed.

1.4 BÄCKLUND TRANSFORMATIONS

BT were introduced in §1.2.4. We will consider in Chapter V BT admitted by (mainly) evolution equations. In this section we introduce general BT, a clear understanding of which will be essential for understanding Chapter V.

We will consider BT which map an n^{th} order surface element $(x_1, x_2, v, \underset{1}{v}, \ldots, \underset{n}{v})$ into a family of n^{th}-order surface elements $(x_1, x_2, u, \underset{1}{u}, \ldots, \underset{n}{u})$. Without loss of generality, we may take such a BT in the form

$$
\begin{aligned}
u_i &= \psi_i(x, u, v, \underset{1}{v}, \ldots, \underset{n}{v}),\\
u_{ij} &= \psi_{ij}(x, u, v, \underset{1}{v}, \ldots, \underset{n}{v}),\\
&\ \ \vdots\\
u_{i_1 \ldots i_n} &= \psi_{i_1 \ldots i_n}(x, u, v, \underset{1}{v}, \ldots, \underset{n}{v}),
\end{aligned}
\tag{1.38}
$$

where $i, j, i_1, \ldots, i_n = 1, 2$ and $\underset{1}{u}, \ldots, \underset{n}{u}$ denote the set of $1^{st}, \ldots,$ n^{th} order derivatives, respectively and $x = (x_1, x_2)$. Observe that less restrictive BT are obtained if we require only

$$u_1 = \psi_1(x, u, v, \underset{1}{v}, \ldots, \underset{n}{v}), \tag{1.39a}$$

$$u_2 = \psi_2(x, u, v, \underset{1}{v}, \ldots, \underset{n}{v}). \tag{1.39b}$$

The terminology "less restrictive" is employed here in the sense that in general the set of u and v satisfying (1.39) is larger than the one satisfying (1.38). Further, observe that if (1.39) is admitted by the differential equation

$$\omega(x, u, \underset{1}{u}, \ldots, \underset{m}{u}) = 0, \tag{1.40}$$

whenever v satisfies the differential equation

$$\Omega(x, v, \underset{1}{v}, \dots, \underset{m}{v}) = 0, \tag{1.41}$$

then in general the least restrictive of the transformations of type (1.39) are those for which the system (1.39b), (1.40), (1.41) implies (1.39a) through the process of differentiation and elimination. We shall make this assumption here; furthermore we assume that (1.39b) is independent of x and that (1.40) and (1.41) are evolution equations. These restrictions are imposed in order to make the reasoning more transparent. Thus we shall study the following situation: The BT determined by

$$u_2 - \psi_2(u, v, \underset{1}{v}, \dots, \underset{n}{v}) = 0, \tag{1.42}$$

is admitted by the evolution equation

$$u_1 + F(u, u_2, u_{22}, \dots, u_{\underbrace{2\dots2}_{m}}) = 0, \tag{1.43}$$

whenever v satisfies the (possibly different) evolution equation

$$v_1 + G(v, v_2, v_{22}, \dots, v_{\underbrace{2\dots2}_{m}}) = 0 \tag{1.44}$$

The assumption that (1.43) and (1.44) are of the same order is made only for convenience of writing.

1.5 MATHEMATICAL PRELIMINARIES

A LB operator as defined by (1.31) acts on an infinite-dimensional space. However, the following points should be stressed:

i) A LB operator is uniquely specified if ξ and η are given (using equations (1.32)). Therefore the defining part of the LB operator (1.31) is

$$\xi_i \frac{\partial}{\partial x_i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}.$$

Hereafter we only give the defining part of a LB; however we should remember that a LB contains implicitly its infinitely many extensions. The operator (1.31) will be denoted as

$$Z = \xi_i \frac{\partial}{\partial x_i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \ldots, \tag{1.45}$$

where the $(+\ldots)$ sometimes will be dropped for convenience of writing.

ii) In a given problem only a finite number of extensions of the defining part is needed. We shall call the minimal extension necessary the relevant part. What is relevant will vary from problem to problem.

We now prove some results which are of general mathematical nature.

1.5.1 Computation of Commutators

It is convenient in practice to consider LB operators with $\xi = 0$. This can be done without loss of generality (see §1.5.2). Next we shall give a lemma regarding the computations of the commutator of two such operators. We restrict ourselves to one independent and one dependent variable only. This is done only for convenience of writing; the idea of the proof is equally valid for several independent variables.

Lemma 1.1. Let $Y_k = A_k\frac{\partial}{\partial u}$, $k = 1, 2$, $A_k = A_k(x, u, u_x \ldots)$. If we regard the Y_k as infinitely extended operators and compute their commutator Y_3 in the ordinary Lie sense then the defining part of the resulting LB operator is,

$$Y_3 = (Y_1A_2 - Y_2A_1)\frac{\partial}{\partial u}.$$

Proof. If the extension is written explicitly we have

$$Y_k = A_k\frac{\partial}{\partial u} + (DA_k)\frac{\partial}{\partial u_x} + (D^2A_k)\frac{\partial}{\partial u_{xx}} + \ldots$$

where the operator D is

$$D = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + \ldots .$$

Then

$$[Y_1, Y_2] = Y_1Y_2 - Y_2Y_1$$

$$= (Y_1A_2 - Y_2A_1)\frac{\partial}{\partial u} + (Y_1DA_2 - Y_2DA_1)\frac{\partial}{\partial u_x} + (Y_1D^2A_2 - Y_2D^2A_1Y_2)\frac{\partial}{\partial u_{xx}} + \ldots$$

$$+ (A_2Y_1 - A_1Y_2)\frac{\partial}{\partial u} + (DA_2Y_1 - DA_1Y_2)\frac{\partial}{\partial u_x} + (D^2A_2Y_1 - D^2A_1Y_2)\frac{\partial}{\partial u_{xx}} + \ldots$$

$$= Z + \widetilde{Z}$$

It is easily verified that $\widetilde{Z}$ is zero; also D and Y_k commute, see [14]. Hence

$$[Y_1, Y_2] = (Y_1A_2 - Y_2A_1)\frac{\partial}{\partial u} + D(Y_1A_2 - Y_2A_1)\frac{\partial}{\partial u_x} + \ldots$$

The first term above is the defining part of an operator and the other term is part of the extension. The lemma is thus proved.

The formula given above has an especially simple form if the A_k have a special form occurring in many applications. This is shown in the following corollary of the first lemma:

Lemma 1.2. Let Y_k have the special form

$$Y_k = A_k(x_j, S_{x_j})\frac{\partial}{\partial S}, \qquad k = 1,2.$$

Then $[Y_1, Y_2] = [A_1, A_2]_P\frac{\partial}{\partial S}$, where the coefficient of $\frac{\partial}{\partial S}$ is computed as a Poisson bracket,

$$[A_1, A_2]_P = \sum_j \left(\frac{\partial A_1}{\partial x_j}\frac{\partial A_2}{\partial S_{x_j}} - \frac{\partial A_2}{\partial x_j}\frac{\partial A_1}{\partial S_{x_j}}\right).$$

Proof. From Lemma 1.1 (generalized to several independent variables),

$$[Y_1, Y_2] = (Y_1A_2 - Y_2A_1)\frac{\partial}{\partial S} = [A_1\frac{\partial A_2}{\partial S} + (\frac{\partial A_1}{\partial x_j} +$$

$$S_{x_k x_j}\frac{\partial A_1}{\partial S_{x_k}})\frac{\partial A_2}{\partial S_{x_j}} - A_2\frac{\partial A_1}{\partial S} - (\frac{\partial A_2}{\partial x_j} +$$

$$S_{x_k x_j}\frac{\partial A_2}{\partial S_{x_k}})\frac{\partial A_1}{\partial S_{x_j}}]\frac{\partial}{\partial S} = [A_1, A_2]_P\frac{\partial}{\partial S}.$$

1.5.2 A Commutation Relation as a Condition for Admissibility

Consider an arbitrary differential equation,

$$B(x, u, u_x, u_{xx}, \ldots) = 0. \tag{1.46}$$

(For simplicity we assume only one independent variable; the generalization is obvious). With (1.46) we associate the LB operator $Y = B\frac{\partial}{\partial u}$. The following lemma will be important later.

Lemma 1.3. The equation (1.46) admits the LB operator

$$X = A(x, u, u_x, u_{xx}, \ldots)\frac{\partial}{\partial u} \quad \text{iff} \tag{1.47}$$

$$[X, Y]_{B=0} = 0.$$

The subscript $B=0$ means that (1.46) and its differential consequences $D_x B=0$, $D_{xx}B=0$, etc., are assumed.

Proof. By definition (1.46) admits X iff

$$XB = F(B)$$

where $F(B)$ is a function of B, $D_x B$, etc., which vanishes when (1.46) holds. From Lemma 1.1

$$[X, Y] = C\frac{\partial}{\partial u}, \quad C = XB - YA = XB - (B\frac{\partial A}{\partial u} + (D_x B)\frac{\partial A}{\partial u_x} + \ldots).$$

From (1.46) it follows that the expression in parenthesis equals zero and hence that $[X, Y]_{B=0} = (XB)_{B=0}\frac{\partial}{\partial u}$. Thus $[X, Y]_{B=0} = 0$ iff X is admissible.

Comment on preceding lemma. From (1.47) we see that a sufficient condition for admissibility of X is $[X, Y] = 0$. However, a necessary condition is only that this relation be valid on the manifold $B=0$ in $(x, u, u_x, \ldots)$-space. Consider now the Hamilton-Jacobi equation $H=E$ for a general value of E. By giving the constant E all values consistent with the problem, we get a continuum of equations. If we require X to be admissible for all such equations we must require that the commutator of X and Y be identically zero. In other words a constant of motion is a dynamical variable which is constant along the path of a particle, no matter what energy surface the path lies on; the union of all energy surfaces is the entire phase space. On the other hand if we only require an operator to be admissible for one fixed value of E (which we can take to be zero after shifting the value of the potential by a constant) then admissibility of X requires only that X commutes with Y on the special manifold $B \equiv H - E = 0$. This is the reason why additional symmetries may appear if $E = 0$, see Chapter II.

1.5.3 First Correspondence Rule

This rule has been used by many authors, but its content will be stated more explicitly below:

Theorem 1.1. Consider the correspondence rule

$$\widetilde{X} \equiv \left(\sum_j \xi_j \frac{\partial}{\partial x_j}\right) + \eta \frac{\partial}{\partial v} \leftrightarrow X \equiv \left(-\sum_j \xi_j v_{x_j} + \eta\right) \frac{\partial}{\partial v} \tag{1.48}$$

where ξ_j and η may depend on x, v, v_{x_j}, etc. Then: a) (1.48) is an isomorphism, b) an equation admits X iff it admits $\widetilde{X}$.

Proof. Part a) is proved by direct computation of the commutator of an arbitrary pair of operators of the type X and the commutator of the pair of corresponding operators. Part b) is a consequence of the isomorphism and Lemma 1.3.

1.5.4 Second Correspondence Rule

In (1.47) only first-order differential operators occur although their coefficients may depend on higher-order derivatives. In quantum mechanics and more generally in the study of linear differential equations, see for instance [16], one often uses higher-order operators of the form

$$A = a(\underline{x}) + \sum_j a_j(\underline{x}) \frac{\partial}{\partial x_j} + \sum_{j,k} a_{jk}(\underline{x}) \frac{\partial^2}{\partial x_j \partial x_k} + \dots . \tag{1.49}$$

(As usual $a(\underline{x})$, regarded as an operator, takes the function $f(\underline{x})$

into $a(\underline{x})f(\underline{x})$). Consider the special case for which only the second term of (1.49) occurs. The operator is then an infinitesimal Lie point operator in $\underline{x}$-space. One might let the a_j depend on v also and add a term $\eta(\underline{x}, v)\frac{\partial}{\partial v}$; one then obtains generators of Lie point transformations in $(\underline{x}, v)$-space. The space of such operators has two important properties. 1) It is closed under commutation and is thus a Lie algebra. 2) The question of its admissibility is well-defined for any linear or nonlinear equation. This case may be contrasted with the case of second-order differential operators of the form (1.49): 1) The space of such operators is not closed under commutation since the commutator of two such operators is in general a third-order differential operator. 2) What is more important, the question whether such operators are admitted by a differential equation makes sense only for linear equations. To remedy these shortcomings we shall recast operators of the form (1.49) in Lie-Bäcklund form. Such a correspondence rule is actually given by Anderson and Ibragimov in [7]. Our version will be given in Theorem 1.2 below. It differs from that of [7] by a minus sign. This sign is irrelevant for the purposes of [7] but needed here since we want the correspondence to be an isomorphism.

Theorem 1.2. Let A be defined by (1.49) and $\overline{A}$ be defined by

$$\overline{A} = -(Av)\frac{\partial}{\partial v}.$$

Then, a) the mapping $A \to \overline{A}$ is an isomorphism and b) a linear equation admits A iff it admits $\overline{A}$.

Proof. The proof is parallel to that of Theorem 1.1 and

and need not be given here. We observe that $\overline{A}$ may be an admissible operator for a nonlinear equation. In this case the question whether A is admissible does not make sense (unless admissibility of A is defined as the admissibility of the corresponding $\overline{A}$).

1.6 LB OPERATORS AND VARIATIONAL EQUATIONS

As it was pointed out earlier, in practice, it is very convenient to cast every LB operator in the form

$$Z = \eta^\alpha \frac{\partial}{\partial u^\alpha} + \dots \tag{1.51}$$

One of the reasons for doing it is that the above operator can be extended very easily; (using equations (1.33) with $\xi_i = 0$). Using the first correspondence rule (1.48) (obviously extended to many dependent variables) every LB operator, and in particular every Lie point or Lie tangent operator, can be cast in the form (1.51). Therefore, hereafter when proving general theorems about LB operators we shall always use the form (1.51). One example of how convenient it is to work with this form is provided by the following lemma:

Lemma 1.4. Let

$$B = 0 \tag{1.46}$$

be some nonlinear equation for u. This equation admits the LB operator (1.47), iff A solves the variational equation associated with equation (1.46).

Proof. The proof is a direct consequence of the definition of an admissible operator of the form (1.46). Let us give some examples: (prime denotes differentiation with respect to x)

i)

$$u'' + uu' = 0. \tag{1.52}$$

The linearized equation is

$$v'' + uv' + vu' = 0. \tag{1.53}$$

It is well known that, because (1.52) does not depend on x

explicitly, $v_1 = u'$ is a solution of (1.53). From the group theoretical point of view this solution is a consequence of the invariance of the equation (1.52) under translation in x: Letting $x \to x + \alpha$ equation (1.52) remains invariant, therefore the Lie operator

$$\tilde{X}_1 = \frac{\partial}{\partial x},$$

is an admissible operator. Using (1.48), $\tilde{X}_1$ is equivalent to

$$X_1 = u'\frac{\partial}{\partial u}$$

Therefore, $v = u'$ is a solution of equation (1.53). How can we find a second solution of equation (1.53)? One may think of trying variation of parameters but the answer would be in terms of an integral. However, letting $x \to \alpha x$, $u \to \alpha^{-1}u$, equation (1.52) remains invariant. Therefore the operator

$$\tilde{X}_2 = x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u},$$

is an admissible operator, which is equivalent to

$$X_2 = (xu'+u)\frac{\partial}{\partial u}.$$

Therefore, using lemma 1.4 , $v = u+xu'$ is another solution of equation (1.53).

Note. Another way of finding solutions of the variational equation, when u is known and depending on some parameters, is to differentiate with respect to these parameters. Here $u = \alpha \tanh \alpha/2(x+\beta)$, and therefore $\partial u/\partial\alpha$, $\partial u/\partial\beta$ are solutions of (1.53). It is easily seen that $\partial u/\partial\beta = u'$ and $\partial u/\partial\alpha = 1/\beta(u+xu')$. This result is not surprising since α and β are the group parameters corresponding to stretching and translation respectively.

ii)

$$u'' + \frac{1}{x}u' + u'^2 = 0. \tag{1.54}$$

Letting $x \to \alpha x$ equation (1.54) remains invariant. Therefore

$$\tilde{X} = x\frac{\partial}{\partial x}, \qquad X = xu'\frac{\partial}{\partial u},$$

and xu' solves the variational equation. Also $u \to u + \alpha$ indicates that constant is another solution.

1.7 ADMISSIBLE AND CONDITIONALLY ADMISSIBLE LB OPERATORS. INVARIANT SOLUTIONS.

One of the main goals of this research is to provide a systematic group theoretical characterization of many exact solutions of physical interest. The main tool for implementing this will be a LB operator. Most of the time we will be concerned with one dependent and several independent variables. Then the most general LB operator is of the form

$$Z = A(x, u, \underset{1}{u}, \ldots, \underset{k}{u}) \frac{\partial}{\partial u} + (D_j A) \frac{\partial}{\partial u_{x_j}} + \ldots, \tag{1.55}$$

where $x \in R^n$ and D_j denotes total differentiation with respect to x_j, (summation of j is assumed).

The question we address in this section is central to our work: <u>Given an equation</u>

$$\omega(x, u, \underset{1}{u}, \ldots) = 0, \quad x \in R^n \tag{1.56}$$

<u>what requirements must be satisfied by a group of transformations, (or more precisely by a LB operator of the form (1.55)), in order for this group to be used for obtaining exact solutions of equation (1.56)?</u> Two different types of requirements lead to consideration of i) <u>admissible LB operators</u> and ii) <u>conditionally admissible LB operators</u> (CAO).

1.7.1 Admissible LB Operators

An obvious requirement is to require the operator Z to leave equation (1.56) invariant. Then the operator Z is called admissible and can be found by solving

$$Z\omega\Big|_{\omega=0} = 0, \tag{1.57}$$

where equation (1.57) means that the operator Z is applied to equation (1.56) and then evaluated on the manifold defined by equation (1.56) and all its differential consequences.

In order to see how an admissible LB operator may be used for obtaining exact solutions let us first recall the case of a Lie point admissible operator: If $A = \hat{A}(x,u,\underset{1}{u})$ in (1.55) and furthermore if $\hat{A}$ depends on $\underset{1}{u}$ linearly, then the operator (1.55) is equivalent to a Lie point operator

$$\tilde{Z} = \xi_j(x,u)\frac{\partial}{\partial x_j} + \eta(x,u)\frac{\partial}{\partial u} + \dots, \tag{1.58}$$

where

$$\hat{A} = \eta - \xi_j u_{x_j}. \tag{1.59}$$

In §1.2.1 it was shown how admissible Lie point operators may be used for obtaining invariant (or similarity) solutions. The analysis presented there was based on the assumption that the invariants of the given operators may be found. However, in order to characterize the similarity solutions this assumption is not necessary, see [5]: Assume that equation (1.56) admits the operator (1.58). Then the class of solutions of equation (1.56), which remain invariant under the action of this operator is characterized by the simultaneous validity of equation (1.56) and of

$$\hat{A} = 0. \tag{1.60}$$

Writing the Lie point operator (1.58) in the standard form (1.55) we see that $\hat{A}$ is the coefficient of $\frac{\partial}{\partial u}$. This is another advantage of using the latter form. As an illustration let us recall the example given in §1.2.1: Equation (1.59) now becomes

$$xu_x + 2tu_x - \lambda u = 0. \tag{1.61}$$

Integrating (1.61) we obtain $u = t^{\lambda/2}F(x^2/t)$, where F is determined by requiring it to solve equation (1.16). This procedure is typical when dealing with Lie point operators: Equation (1.59) is always a quasilinear first order equation for u and therefore in general it is easier to solve than equation (1.56); after solving equation (1.59) and obtaining the general form of u finally we use equation (1.56).

The notion of invariant solutions and in particular the above characterization may be directly carried over when dealing with LB operators. This is given in the form of a theorem:

Theorem 1.3. Assume that equation (1.59) admits the LB operator given by (1.55), i.e.,

$$Z\omega \Big|_{\omega=0} = 0. \tag{1.57}$$

Then the class of solutions invariant under the action of Z may be characterized by the simultaneous validity of equation (1.58) and of

$$A = 0, \tag{1.62}$$

where A is the coefficient of $\partial/\partial u$ in (1.55).

Equation (1.62) in contrast with equation (1.60)) is not a first order equation for u and it might be harder to solve than equation (1.56) itself. However, in general it is easier solving the system of equations (1.56), (1.62) than just solving equation (1.56); this will be illustrated in Chapters II and III where the separable solutions of a given equation will be considered.

The above theorem indicates how different types of exact solutions may be characterized group theoretically: Regard them as solutions of a system of two equations and then prove that one of the two equations corresponds to equation (1.62). This will be elaborated in Chapters II, III and V.

1.7.2 Conditionally Admissible LB Operators

An admissible operator (or actually the corresponding group) takes i) a solution manifold into a (possibly different) solution manifold, and ii) an invariant solution manifold (assuming that such a manifold exists) into itself. In obtaining invariant solutions we only use the second (ii) property of an admissible operator. This motivates us to look for a LB operator which does not in general take solution manifolds of a given equation into solution manifolds of the same equation; however, we require a certain class of solution manifolds, also called invariant manifolds, to be taken into themselves under the action of this LB operator. Such a LB operator is then called CAO and can be found by solving

$$Z\omega\Big|_{\omega=0} = F(x, u, \underset{1}{u}, \ldots; D_j, D_{ij}, \ldots)A \qquad (1.63)$$

where F is a linear operator in D_j, $D_{ij}, \ldots$ with coefficients depending on $x, u, \underset{1}{u}, \ldots$. Equation (1.63) is denoted as

$$Z\omega\Big|_{\omega=0, A=0} = 0. \qquad (1.64)$$

The solutions corresponding to the above invariant manifold are also called invariant solutions (or generalized invariant solutions) and are characterized by the simultaneous validity of equations (1.58) and (1.62) where A is now the coefficient of $\partial/\partial u$ of the CAO satisfying equation (1.64). Actually particular solutions of the above type have been sought in [17] and [18]; however it was thought/that in [18] these solutions are not physically interesting. In Chapter V we present a detailed algorithm for handling such solutions and further we establish their importance (both mathematically and physically) by proving that the solutions obtained through Bäcklund transformations are of the above type.

CHAPTER II

2.1 INTRODUCTION

The use of group theoretical methods for the understanding and solving of problems arising in classical mechanics has been well established. In spite of this, it seems that many fundamental questions still exist. For example, it is known that in Hamilton's canonical equations the constants of motion linear in the momenta are related to Lie point groups of the Hamilton-Jacobi equation. However, in general, there is no way of using Lie point theory on the Hamilton-Jacobi equation to explain the existence of conserved quantities which are nonlinear in the momenta. Another open question is the group theoretical characterization of all separable solutions of the Hamilton-Jacobi equation. It seems that by using Lie point theory we can characterize only some of the separable solutions.

Similar problems appear in quantum mechanics. E. Noether has established a connection between conservation laws and invariant properties of a given system of equations, under the assumption that the system possesses a Langrangian [19]. However, Noether's theorem cannot guarantee that every conservation law comes from Lie point groups. Actually, in quantum mechanics, some conserved quantities were discovered which were not the consequence of Lie groups (for example, the quantum mechanical analogue of the Runge-Lenz vector for the hydrogen atom). In order to explain these conservation laws the "dynamical symmetries" were introduced, see for example [20]. These were symmetries of a non-geometrical origin. The "dynamical symmetries" in quantum mechanics were, at an early stage, related to the problem of separation of variables of the Schrödinger equation. It was noted that potentials admitting "higher symmetries" also allow separation in more coordinate systems (for example, the equation for the hydrogen atom also separates in parabolic coordinates). The connection between group theory and separation of variables was discussed for many interesting equations of mathematical physics. However, using Lie point theory it seems that the characterization of separable solutions is incomplete [16].

In spite of the great applicability of group theoretical methods in quantum mechanics, and the fact that classical mechanics is the geometrical limit of quantum mechanics, the group theoretical consideration of the above problems in classical mechanics has not been extensive [21]. Actually, it was thought that the meaning of dynamical invariance groups in classical mechanics is less straightforward [20]. We think the investigation of the above questions in classical mechanics will clarify the role played by higher (Lie-Bäcklund) symmetries and will provide a better understanding of the corresponding problems in quantum mechanics.

2.1.1 Outline of this Chapter

In §2.2 we present a general analysis of the group structure of the Hamilton-Jacobi equation. More specifically: In §2.2.1 we give an isomorphism between invariants of Hamilton's equation and admissible LB operators of the Hamilton-Jacobi equation. More generally, this isomorphism relates dynamical variables (that is, functions in phase space) and LB operators. This isomorphism is not surprising since Hamilton's equations and the Hamilton-Jacobi equation are mathematically equivalent formulations of the same physical theory. However, this result is interesting in the following sense: If we want to explain the existence of conserved quantities in classical mechanics analyzing the group structure of Hamilton's equations we see that not every symmetry of Hamilton's equations produces a conserved quantity (see theorem 2.2); however, theorem 2.1 indicates that every symmetry of the Hamilton-Jacobi equation generates a conserved quantity. In §2.2.2 we show how Lie-Bäcklund groups of the Hamilton-Jacobi equation induce Lie point groups of Hamilton's equations. These groups were emphasized in [22], but a logical explanation for their existence was not given. In §2.2.3 we establish a group-theoretic characterization of all the separable solutions of the Hamilton-Jacobi equation. This result reveals the group basis of total separation as well as partial separation. We emphasize the latter case which does not seem to have been sufficiently considered in the literature (the connection between separation of variables and degeneracy is thoroughly discussed in [23]).

The above results illustrate the importance of Lie-Bäcklund groups in classical mechanics. Their existence leads to:

i) conserved quantities of Hamilton's equations.

ii) separable solutions of Hamilton-Jacobi equation.

iii) Lie point groups of Hamilton's equations.

We want to stress at this point that the LB operators associated with the group structure of the Hamilton-Jacobi equation are a special class of LB operators, which are equivalent to Lie tangent operators (see §2.1.2).

In §2.3 we consider the equation

$$\frac{1}{2} S_{x_i} S_{x_i} + \hat{V}(\underline{x}) = 0, \quad i = 1,2,3 \qquad (2.13)$$

(a special case of which is the time-independent Hamilton-Jacobi equation, $\hat{V} = V - E$) and investigate constants of motion (or admissible operators) with polynomial dependence on the momenta. Concrete results are obtained for the cases of constants of motion linear, quadratic and cubic in the momenta. Emphasis is given to the linear and quadratic cases, for which the particular case of zero energy state is also investigated; it is shown that in the latter case additional symmetries may appear.

In §2.4 we present some applications which include:

i) Investigation of potentials depending only on the distance to one or two fixed centers. It is found that the most general case of a two center-potential having a nontrivial quadratic invariant is that of two Newtonian centers of arbitrary strength at arbitrary locations with two superimposed harmonic centers of equal strength at the same locations (equivalently, we may have one harmonic center of arbitrary strength located at the midpoint of the two Newtonian centers), we call this potential a "mixed" one. The corresponding quadratic invariant is also given.

ii) Investigation of central fields for the case of zero energy state.

iii) Characterization of all separable solutions of the one-body Keplerian problem.

In §2.5 we extend and discuss further some of the above results: From the "mixed" potential we obtain by limit processes all known cases of potentials with quadratic invariants. The same limit processes also give us invariants of the limiting potentials. We discuss not only the Lie algebra of these invariants† but also show how linearly independent elements may be functionally dependent; this leads naturally to a discussion of the concept of degeneracy. We have thus shown, for a class of important nontrivial nonlinear invariants that they are special or limiting cases of one single case. A different and in a sense complementary point of view is also interesting: Let the potentials V_1 and V_2 each have quadratic invariants. Under which conditions is the same true for the combined potential $V_1 + V_2$? The answer is found to be that the strictly quadratic terms of each invariant must be the same. A simple rule is given for the construction of the new invariant. Finally the new invariants cubic in the momenta are found.

2.1.2 An Important Equivalence

In this chapter we shall be primarily concerned with LB operators of the form

$$Y = A(x_i, S, S_{x_i})\frac{\partial}{\partial S}, \qquad i = 1, \ldots, n.$$

We remind the reader (see §1.2.2) that an operator

$$X = \xi_i\frac{\partial}{\partial x_i} + \eta\frac{\partial}{\partial S},$$

where ξ_i and η depend on x_i, S, S_{x_i} is a Lie tangent operator provided that there exists a function $W(x_i, S, S_{x_i})$ such that

† For the case of a constant force field we get a basic set of linearly independent invariants which is not closed under commutation. This leads to the unresolved problem of determining the structure of its symmetry group.

$$\xi_i = -\frac{\partial W}{\partial S_{x_i}}, \qquad \eta = W - S_{x_i}\frac{\partial W}{\partial S_{x_i}}.$$

In our case $\xi_i = 0$ and $\eta = A$. Therefore the operator Y is not a Lie tangent operator. However, in [7] it is shown that, every operator Y is equivalent to a Lie tangent operator (two LB operators are equivalent if they differ by an operator of the type $a_i D_i = a_i \partial/\partial x_i + a_i u_i \partial/\partial u + \dots$). So, although we still call the operator Y a LB operator we stress that it belongs in a special class of LB operators, which are equivalent to Lie tangent operators.

Taking into consideration the above discussion as well as the correspondence rule (1.48) (see §1.5.3) we may now state:

i) If A (the coefficient of $\partial/\partial u$) is linear in S_{x_i}, then the LB operator Y is equivalent to a Lie point operator.

ii) If A is nonlinear in S_{x_i} then Y is equivalent to a Lie tangent operator.

From theorem 2.1 it will then follow that constants of motion linear in the momenta correspond to Lie point groups while nontrivial constants of motion nonlinear in the momenta correspond to Lie tangent groups.

Nontrivial constants of motion.

What we mean by "nontrivial" is best understood with the aid of an example: Consider the case of a central potential in two dimensions; the angular momentum m_3 is conserved. m_3 is linear in the momenta and therefore its conservation is a consequence of a Lie point group (rotation about the z axis). Of course m_3^2 is also a constant of motion, which we call trivial; it is quadratic in the momenta but it still reflects a Lie point symmetry in the sense that the LB operator corresponding to m_3^2 belongs in the enveloping algebra of the Lie algebra of Lie point operators. Nontrivial constants of motion nonlinear in the momenta are those which cannot be expressed in terms of linear constants of motion (for example, the Runge-Lenz vector is a nontrivial quadratic constant of motion).

We conclude the introduction by reminding the reader that if a canonical transformation is made in which the new coordinates depend only on the old coordinates then the new momenta are linear in the old momenta. If a new coordinate (say angles in spherical symmetry) does not appear in the new Hamiltonian, then the corresponding momentum (say angular momentum) is constant; this constant reflects a geometrical symmetry. Obviously, a nonlinear constant represents a more sophisticated symmetry (dynamical or more general Lie-Bäcklund).

2.2 A GROUP ANALYSIS OF THE HAMILTON-JACOBI EQUATION

The Hamilton-Jacobi equation is regarded as the basic equation of classical mechanics. An analysis of its group structure provides the explanation for the existence of all conserved quantities and separable solutions in classical mechanics. Furthermore it clarifies the origin of the Lie point symmetries of Hamilton's equations. Also, an understanding of the group structure of the Hamilton-Jacobi equation is essential for the LB group analysis of the Schrödinger equation, to follow in the next chapter.

2.2.1 Lie-Bäcklund Groups of the Hamilton-Jacobi Equation and Constants of Motion of Hamilton's Equations

Let

$$\Omega \equiv S_t + H(t, x_i, S_{x_i}) = 0, \qquad i = 1, \ldots, n \tag{2.1}$$

be the Hamilton-Jacobi equation describing the motion of a given dynamical system [23]. The corresponding Hamilton's equations are:

$$\begin{aligned} \dot{x}_i &= H_{p_i}, \\ \dot{p}_i &= -H_{x_i}. \end{aligned} \tag{2.2}$$

Theorem 2.1.

The Hamilton-Jacobi equation (2.1) admits the Lie-Bäcklund operator $Y = A(t, x_i, S_{x_i})\partial/\partial S$, $i = 1, \ldots, n$, if and only if $A(t, x_i, p_i)$ is a constant of motion of Hamilton's equations. Furthermore the correspondence

$$A(t, x_i, p_i) \leftrightarrow A(t, x_i, S_{x_i})\frac{\partial}{\partial S}, \tag{2.3}$$

is an isomorphism for any A which is a function of the variables indicated.

Proof

The proof that (2.3) is an isomorphism is a consequence of the fact that a commutator of two constants of motion is their Poisson bracket and of lemma 1.2. The first part of the theorem then follows using lemma 1.3 (trivially extended to the case of many independent variables).

An illustration of this Theorem is the following: Assume (2.1) is invariant under rotation about the x_3 - axis. Then it admits the operator $Y = x_1\partial/\partial x_2 - x_2\partial/\partial x_1$, which is equivalent to $Y = (x_1 S_{x_2} - x_2 S_{x_1})\partial/\partial S$. Therefore $m_3 = x_1 p_2 - x_2 p_1$, the x_3 - component of the angular momentum is conserved.

2.2.2 Groups of Hamilton's Equations Induced by Groups of the Hamilton-Jacobi Equation.

Let $S(t, x_i)$ be a solution of (2.1). Let (2.1) be invariant under the group G_S.

$$G_S: \begin{aligned} &S \rightarrow S + \epsilon A, \\ &t \rightarrow t + \epsilon T, \\ &x_i \rightarrow x_i + \epsilon X_i, \end{aligned}$$

where A, T, X_i are functions of t, x_i, S_{x_i}, $i = 1, \ldots, n$. Define

$$F: \quad \begin{aligned} &p_i = \frac{\partial S}{\partial_{x_i}}, \\ & \qquad\qquad i = 1, \ldots, n, \\ &q_i = x_i, \end{aligned}$$

then p_i, q_i satisfy the Hamilton's equations (2.2). The transformation F maps G_S onto some group G_H of equations

(2.2) defined by:

$$G_H: \quad \begin{aligned} p_i &\to p_i + \epsilon P_i \\ t &\to t + \epsilon T \\ q_i &\to q_i + \epsilon Q_i \end{aligned}$$

where

$$X_i = Q_i$$

$$P_i = \frac{dA}{dx_i} - S_{x_j}\frac{dX_j}{dx_i} - S_t\frac{dT}{dx_i}\bigg|_\Omega \tag{2.4}$$

From the above it is clear that if a group of (2.1) is given, the induced group on (2.2) can be found using (2.4). Suppose Y is a Lie-Bäcklund operator of (2.1). To find the induced operator of (2.2), extend Y and keep the coefficients of $\partial/\partial x_i$, $\partial/\partial t$ and $\partial/\partial S_{x_i}$.

<u>Theorem 2.2.</u> If $Y = A(t, x_i, S_{x_i})\partial/\partial S$, $i = 1, \ldots, n$, is an admissible Lie-Bäcklund operator of the Hamilton-Jacobi equation, then $Z = \partial/\partial t + A_{x_i}\partial/\partial p_i - A_{p_i}\partial/\partial_{x_i}$ is an admissible Lie point operator of the corresponding Hamilton's equations.

<u>Proof</u>. Extend Y:

$$\overline{Y} = A\frac{\partial}{\partial S} + (A_t + S_{tx_i}A_{S_{x_i}})\frac{\partial}{\partial S_t} + (A_{x_i} + S_{x_ix_j}A_{S_{x_j}})\frac{\partial}{\partial S_{x_i}}.$$

Adding and subtracting $A_{S_{x_i}}\partial/\partial x_i$ we get:

$$\overline{Y} = A\frac{\partial}{\partial S} + A_t\frac{\partial}{\partial S_t} + A_{x_i}\frac{\partial}{\partial S_{x_i}} - A_{S_{x_i}}\frac{\partial}{\partial x_i} + \Phi$$

where

$$\Phi = S_{tx_i}A_{S_{x_i}}\frac{\partial}{\partial S_t} + S_{x_ix_j}A_{S_{x_j}}\frac{\partial}{\partial S_{x_j}} + A_{S_{x_i}}\frac{\partial}{\partial x_i}.$$

Apply Φ to (2.1)

$$\Phi(2.1) = A_{S_{x_i}} D_i(2.1).$$

Therefore

$$\overline{Y}|_{\Omega} = A\frac{\partial}{\partial S} + \frac{\partial}{\partial t} + A_{x_i}\frac{\partial}{\partial S_{x_i}} - A_{S_{x_i}}\frac{\partial}{\partial x_i},$$

and the induced group on (2.2) is Z. Q.E.D.

2.2.3 Separation of the Hamilton-Jacobi Equation

Lemma 2.1. Let

$$\psi(x_i, S_{x_i}, \phi(x, S_x)) = 0, \qquad i = 1, \ldots, n-1 \tag{2.5}$$

be a first order partial differential equation in which x and S_x enter only in some combination $\phi(x, S_x)$ not involving the other coordinates. Assume that S does not appear explicitly in (2.5). Then (2.5) admits $Y = \phi(x, S_x)\partial/\partial S$.

Proof. Extending Y and applying it to (2.5) we get:

$$\overline{Y}\psi = S_{xx_i}\phi_{S_x}\frac{\partial\psi}{\partial S_{x_i}} + (\phi_x + S_{xx}\phi_{S_x})\phi_{S_x}\frac{\partial\psi}{\partial\phi} = \phi_{S_x}D_x(\psi)$$

therefore

$$\overline{Y}\psi\Big|_{\psi} = 0. \quad \text{Q.E.D.}$$

The equation (2.5) is also invariant under translation in S, i.e., it admits the operator $Y_0 = \lambda\partial/\partial S$, where λ any constant.

Theorem 2.3. Suppose an additive separable solution exists for equation (2.5) of the form

$$S = \hat{S}(x_i) + \overline{S}(x). \tag{2.6}$$

Then S is an invariant solution of (2.5) under the action of the admissible LB operator

$$\hat{Y} = Y - Y_0 = [\phi(x, S_x) - \lambda]\partial/\partial S. \qquad (2.7)$$

Proof. The solution of equation (2.5) invariant under the action of the operator $\hat{Y}$ given by (2.7), is specified by the simultaneous validity of equation (2.5) and of

$$\phi(x, S_x) - \lambda = 0, \qquad (2.8)$$

(see theorem 1.4). However, by definition a separable solution also satisfies equation (2.8) for some constant λ. Q.E.D.

From the above it is clear that every additively separable solution of (2.5) is invariant under a LB operator. If the separable coordinates are known, this operator is found by inspection.

In the special case for which (2.5) is the Hamilton-Jacobi equation theorem 2.1 establishes a way of evaluating λ, the constant of separation: $\lambda = \phi(x, p)$.

The above results complete our analysis of the group properties of the Hamilton-Jacobi equation and Hamilton's equations. However, Hamilton's equations are the characteristic equations of the Hamilton-Jacobi equation. This provides the motivation for the following generalization.

2.2.4 Lie-Bäcklund Groups of Some First-Order Partial Differential Equations

Let

$$\Omega(x_i, S, S_{x_i}) = 0, \qquad i = 1, \ldots, n. \qquad (2.9)$$

Recall [24] that the characteristic equations of (2.9) are given by:

$$\frac{dx_i}{d\lambda} = \frac{\partial\Omega}{\partial p_i},$$

$$\frac{dp_i}{d\lambda} = - p_i\frac{\partial\Omega}{\partial S} - \frac{\partial\Omega}{\partial x_i}, \tag{2.10}$$

$$\frac{dS}{d\lambda} = p_j\frac{\partial\Omega}{\partial p_j},$$

where

$$p_i \equiv \frac{\partial S}{\partial x_i}.$$

Lemma 2.2. The operation $Y = A(x_i, S, S_{x_i})\partial/\partial S$ is an admissible operator of (2.9) if and only if

$$[\frac{dA}{d\lambda} + A\Omega_S]_{\Omega} = 0 \tag{2.11}$$

Proof. Extending Y, applying it to (2.9), and using

$$D_i\Omega|_{\Omega=0} = 0$$

we obtain,

$$\overline{Y}\Omega|_{\Omega=0} = (A - S_{x_i}A_{S_{x_i}})\frac{\partial\Omega}{\partial S} - A_{S_{x_i}}\frac{\partial\Omega}{\partial x_i} + (A_{x_i} + A_S S_{x_i})\frac{\partial\Omega}{\partial S_{x_i}}\bigg|_{\Omega=0}.$$

Now using (2.10) we find,

$$\overline{Y}\Omega|_{\Omega=0} = [\frac{dA}{d\lambda} + A\Omega_S]_{\Omega=0} \qquad \text{Q.E.D.}$$

From (2.11) it is clear that if (2.9) does not involve S explicitly, then $\Omega = 0$ admits the operator Y if and only if A is a conserved quantity of the characteristic equations of $\Omega = 0$.

2.3 CONSTANTS OF MOTION WITH POLYNOMIAL DEPENDENCE ON THE MOMENTA

In order to gain some insight of the problem we start with the quadratic case.

2.3.1. Constants of Motion Quadratic in the Momenta

A. Hamilton-Jacobi equation for the zero energy state.

In this subsection we restrict ourselves to operators of the from

$$Y \equiv (a_i S_{x_i} + b_{ij} S_{x_i} S_{x_j} + 2c)\frac{\partial}{\partial S} \tag{2.12}$$

where

$$a_i = a_i(x_k),^{\dagger} \quad b_{ij} = b_{ji} = b_{ij}(x_k), \quad c = c(x_k), \quad i,k = 1,2,3.$$

Let

$$H(x_i, S_{x_i}) \equiv \frac{1}{2} S_{x_i} S_{x_i} + \hat{V}(x_k) = 0 \;^{\dagger\dagger} \tag{2.13}$$

In theorem 2.1 we proved that the operator $Y \equiv A(x_i, S_{x_i})\partial/\partial S$ is an admissible operator of (2.13) if and only if

$$[H_{x_i}\frac{\partial A}{\partial S_{x_i}} - H_{S_{x_i}}\frac{\partial A}{\partial x_i}]_{H=0} = 0, \quad i = 1,2,3.$$

Substituting for H from (2.13), for A from (2.12), equating to zero the coefficients of $S_{x_i} S_{x_j} S_{x_k}$, $i,j,k = 0,1,2,3,$ we obtain the following set of equations:

---

† It is convenient to use various notations. The position vector $\underline{x}$ will be denoted by (x, y, z) or $x_i = (x_1, x_2, x_3)$.

†† Equation (2.13) can be thought of as the eikonal equation of some generalised Helmholtz equation, (where the frequency depends on $\underline{x}$), or as the Hamilton-Jacobi equation for the zero energy state.

$$a_{1\,x_1} = a_{2\,x_2} \qquad (\alpha.1)$$

$$a_{1\,x_2} = -\,a_{2\,x_1} \qquad (\alpha.2)$$

$$a_i\hat{V}_{x_i} + 2a_{3\,x_3}\hat{V} = 0, \quad i = 1,2,3 \qquad (\hat{\alpha})$$

$$(b_{11} - b_{33})_{x_1} = 2b_{13\,x_3} \qquad (\beta.1)$$

$$(b_{11} - b_{33})_{x_3} = -\,2b_{13\,x_1} \qquad (\beta.2)$$

$$b_{12\,x_3} + b_{23\,x_1} + b_{31\,x_2} = 0 \qquad (\beta.7)$$

$$c_{x_1} = b_{1i}\hat{V}_{x_i} + \hat{V}b_{11\,x_1}, \quad i = 1,2,3 \qquad (\gamma.1)$$

$(\alpha.k)$, $(\beta.k)$, $3 \leqslant k \leqslant 6$, $(\gamma.2)$ and $(\gamma.3)$ by cyclic permutation $1 \rightarrow 2 \rightarrow 3$.

The compatibility equations of the set (γ) are the following:

$$(b_{11}-b_{22})\hat{V}_{x_1x_2}+b_{13}\hat{V}_{x_2x_3}-b_{23}\hat{V}_{x_3x_1}+b_{12}(\hat{V}_{x_2x_2}-\hat{V}_{x_1x_1})+$$
$$2b_{12\,x_2x_2}\hat{V}+3b_{12\,x_2}\hat{V}_{x_2}-3b_{12\,x_1}\hat{V}_{x_1}+(b_{13\,x_2}-b_{23\,x_1})\hat{V}_{x_3} = 0, \qquad (\delta.1)$$

$(\delta.2)$ and $(\delta.3)$ by cyclic permutation.

It should be noted that the equations determining a_i and b_{ij} are uncoupled. Therefore, operators linear and quadratic in the derivatives respectively, can be found independently.

Let

$$Y_1 = a_i S_{x_i}\frac{\partial}{\partial S} \qquad (2.14)$$

and

$$Y_2 = (b_{ij}S_{x_i}S_{x_j} + 2c(\underline{x}))\frac{\partial}{\partial S}, \quad i,j = 1,2,3 \qquad (2.15)$$

To find an admissible operator of the form (2.14), equations (α), which are independent of $\hat{V}$, are completely solved. Then, substituting a_i in equation $(\hat{\alpha})$ we determine which groups, if any, are admitted by a given $\hat{V}$. Similarly to find an admissible operator of the form (2.15), equations (β) are completely solved. Then, substitute b_{ij} in equations (δ) we determine which groups are admitted by a given $\hat{V}$. Finally integration of equations (γ) determines $c(\underline{x})$.

Solving equations (α) and (β) we obtain (see Appendix I):

$$\begin{aligned} a_1 &= A_3(x^2-y^2-z^2) + 2(A_1xy + A_2xz) + B_1z - B_2y + Cx + D_1 \\ a_2 &= A_1(y^2-x^2-z^2) + 2(A_2yz + A_3xy) + B_2x - B_3z + Cy + D_2 \\ a_3 &= A_2(z^2-x^2-y^2) + 2(A_3xz + A_1yz) + B_3y - B_1x + Cz + D_3 \end{aligned} \tag{2.16}$$

$$\begin{aligned} b_{12} = {} & \alpha_1+\alpha_2y+\alpha_3x+\alpha_4z-\alpha_5z^2+\beta_5xz+\gamma_5yz+\alpha_6xy+\alpha_7(x^2-y^2) - \\ & \beta_7xz+\gamma_7yz-\alpha_8z^3-2\beta_8z^3+\alpha_8(3y^2z-3x^2z)+(\alpha_9+2\gamma_{10})y(y^2-3x^2) + \\ & (\alpha_{10}+2\beta_9)x(x^2-3y^2)+3\alpha_9z^2y+3\alpha_{10}xz^2-6(\beta_{10}+\gamma_9)xyz + \\ & 2\alpha_{11}zy(3x^2-z^2-y^2)+\beta_{11}(z^4-y^4-x^4+6x^2y^2)+2\gamma_{11}xz(3y^2-x^2-z^2) + \\ & 3\alpha_{12}xyz^2+(\gamma_{12}-\beta_{12})xy(y^2-x^2). \end{aligned} \tag{2.17a}$$

b_{23}, b_{31} by cyclic permutation where $\alpha_i \rightarrow \beta_i \rightarrow \gamma_i$ and $\gamma_4 = -(\alpha_4 + \beta_4)$, $\gamma_8 = -(\alpha_8 + \beta_8)$, $\gamma_{12} = -(\alpha_{12} + \beta_{12})$.

$$b_{11} = 0^{\dagger}$$

---

† Without loss of generality we can assume $b_{11} = 0$, as we can always eliminate S_x^2 from the operator (2.12) using equation (2.13).

$$b_{22} = -2[\alpha_0+\alpha_2 x-\gamma_2 z-\alpha_3 y+\beta_3 z-\beta_5 zy+\gamma_5 zx+\frac{\alpha_6}{2}(x^2-y^2) +$$

$$\beta_6 \frac{z^2}{2} -\gamma_6 \frac{z^2}{2} -2\alpha_7 xy+\beta_7 zy+\gamma_7 zx+6\alpha_8 xyz+\alpha_9 x(3z^2+3y^2-x^2) +$$

$$2\beta_9 y(y^2-3x^2)+\gamma_9 z(-3x^2+3y^2-z^2)+\alpha_{10} y(-3x^2-3z^2+y^2) +$$

$$\beta_{10} z(-3x^2+3y^2+z^2)+2\gamma_{10} x(3y^2-x^2)+2\alpha_{11} xz(x^2-z^2-3y^2) +$$

$$4\beta_{11} xy(x^2-y^2)+2\gamma_{11} zy(3x^2-y^2+z^2)+\frac{3}{2}\alpha_{12} z^2(x^2-y^2) +$$

$$\frac{1}{4}(\beta_{12}-\gamma_{12})(x^4+y^4+z^4-6x^2y^2)]. \qquad (2.18a)$$

$$b_{33} = -2[\beta_0+\beta_2 y-\gamma_2 z-\alpha_3 y+\gamma_3 x+\alpha_5 xy-\beta_5 yz-\frac{\alpha_6}{2}y^2+\beta_6 \frac{y^2}{2}+\frac{\gamma_6}{2}(x^2-z^2) -$$

$$\alpha_7 xy-\beta_7 yz+2\gamma_7 zx-6\gamma_8 xyz+2\alpha_9 x(3z^2-x^2)+\beta_9 y(-3x^2+3z^2+y^2) +$$

$$\gamma_9 z(-3x^2-3y^2+z^2)+\alpha_{10} y(-3x^2+3z^2-y^2)+2\beta_{10} z(z^2-3x^2) +$$

$$\gamma_{10} x(3y^2+3z^2-x^2)+4\alpha_{11} xz(x^2-z^2)+2\beta_{11} xy(x^2-3z^2-y^2) +$$

$$2\gamma_{11} yz(3x^2+y^2-z^2)+\frac{3}{2}\gamma_{12} y^2(x^2-z^2) +$$

$$\frac{1}{4}(\beta_{12}-\alpha_{12})(x^4+y^4+z^4-6x^2z^2)]. \qquad (2.18b)$$

where α_i, β_i, γ_i; $1 \leqslant i \leqslant 12$ and A_i, B_i, C, D_i, $i = 1,2,3$ are constant parameters.

Using equations (2.16), (2.17) and (2.18) we can find all distinct operators of the form (2.14) and (2.15). However, in order to reveal their structure, it is better to express them in an alternative form, although in the new form some of the quadratic operators are equivalent.†

---

† Two admissible operators of equation (2.13) are equivalent if one can be obtained from the other with the aid of equation (2.13).

Let

$$P_i \equiv S_{x_i},\ \Delta \equiv x_i P_i,\ M_i \equiv \epsilon_{ijk} x_j P_k,\ K_i = 2x_i x_j P_j - r^2 P_i, \qquad (2.19)$$

where

$$r^2 = x_i x_i;\ i,j,k = 1,2,3.$$

Then

$$Y_1 = (\kappa_i P_i + \lambda_i M_i + \mu_i K_i + \nu\Delta)\frac{\partial}{\partial S}, \qquad (2.20)$$

$$Y_2 = (\alpha_{ij} P_i P_j + \beta_{ij} P_i M_j + \gamma_{ij} M_i M_j + \delta\Delta^2 + \epsilon_i P_i \Delta + \zeta_i M_i \Delta + \eta_i K_i \Delta + \theta_{ij} P_i K_j + \xi_{ij} M_i K_j + \sigma_{ij} K_i K_j + 2c(\underline{x}))\frac{\partial}{\partial S} \qquad (2.21)$$

where all lower-case Greek letters denote constant parameters symmetric in their indices.

The operators Y_1 represent the Lie algebra of the conformal group, P_i and M_i generate the Euclidean group of motions, K_i are the generators of the special conformal transformations and Δ generates the dilatations.

It is clear that when $c(\underline{x}) = 0$ the operator Y_2 belongs in the enveloping algebra of the conformal group. However, when $c(\underline{x}) \neq 0$ Y_2 is Lie-Bäcklund (independently of $\hat{V}$).

B. The Hamilton-Jacobi equation (for arbitrary value of E).

Let

$$\hat{S}_t + \frac{1}{2}\hat{S}_{x_i}\hat{S}_{x_i} + V(\underline{x}) = 0, \quad i = 1,2,3 \qquad (2.22)$$

be the time-dependent Hamilton-Jacobi equation, and

$$\frac{1}{2}S_{x_i}S_{x_i} + V(\underline{x}) - E = 0, \quad i = 1,2,3 \qquad (2.23)$$

the corresponding time-independent equation obtained by additive separation of variables, $\hat{S}(\underline{x},t) = S(\underline{x}) - Et$. It is obvious that the operator

$$\hat{Z} = A(x_i, \hat{S}_{x_i})\frac{\partial}{\partial S}, \qquad i = 1,2,3 \tag{2.24}$$

is an admissible operator of the equation (2.22), iff the operator

$$Z = A(x_i, S_{x_i})\frac{\partial}{\partial S}, \qquad i = 1,2,3$$

is an admissible operator of equation (2.23).

Therefore, looking for admissible operators of the form (2.24), it is sufficient to consider equation (2.23). Equation (2.23) is a special case of (2.13), where $\hat{V} = V - E$.† Equations (2.17) indicate that a necessary condition for an operator of the form (2.12) to be admitted by equation (2.23) is to be admitted by a constant potential. Hence

$$b_{12_{yy}} = b_{23_{zz}} = b_{31_{xx}} = 0$$

i.e., the relevant group parameters are $\alpha_i, \beta_i, \gamma_i, \; 0 \leq i \leq 6$.

Using the notation introduced in (2.19) we can express the relevant admissible operators as:

$$Z_1 = (\kappa_i P_i + \lambda_i M_i)\frac{\partial}{\partial S}, \qquad i = 1,2,3 \tag{2.25}$$

$$Z_2 = (\alpha_{ij}P_iP_j + \beta_{ij}P_iM_j + \gamma_{ij}M_iM_j + 2c(\underline{x}))\frac{\partial}{\partial S}, \quad i,j = 1,2,3. \tag{2.26}$$

---

† For a fixed function V (which is the potential energy of the dynamical system), the energy E may vary continuously over all positive numbers (e.g. the harmonic oscillator) or all real numbers (e.g. the Keplerian problem).

Equation (2.23) is the eikonal equation of the time-independent normalised Schrödinger equation

$$\frac{1}{2} u_{x_i x_i} + (V(\underline{x}) - E)\underline{u} = 0, \qquad i = 1,2,3$$

Therefore, being an approximate equation, its group-theoretical analysis is simpler. However, it is of interest that its group theoretical consideration was preceded by that of (2.23). The quantum mechanical analogues of Z_1, Z_2 have been found in [20]. By taking the classical mechanics limit of these operators we obtain Z_1, Z_2. Since the Hamilton-Jacobi equation is a limiting case of the Schrödinger equation we expect the symmetry group of the latter to be the same or a proper subgroup of the former. Here we see that the groups actually are the same (if the nature of the potential is not considered), at least up to second order operators.

2.3.2 Constants of Motion Cubic in the Momenta

We now concentrate on the more interesting case of an arbitrary range of values of E (the case $E = 0$ is not very interesting physically). Also we only present the results for the two dimensional case; the extention to three dimensions is obvious. For completeness we also include the relevant results of the previous lemma.

Lemma 2.3 The most general constants of motion of the time-independent Hamilton-Jacobi equation (2.3), linear, quadratic and cubic in the momenta are,

$$I_1 = (\text{linear combinations with constant coefficients of } p_1, p_2, m_3)$$
$$\equiv \sum_j a_j(\underline{x}) p_j \ , \qquad (2.27a)$$

$$I_2 = (\text{quadratic combinations of } p_1, p_2, m_3) + c(\underline{x})$$
$$\equiv \sum_{j,k} b_{jk}(\underline{x}) p_j p_k + c(\underline{x}) \ , \qquad (2.27b)$$

$$I_3 = (\text{cubic combinations of } p_1, p_2, m_3) + \sum_j a_j(\underline{x}) p_j \qquad (2.27c)$$
$$\equiv d_{111}(\underline{x}) p_1^3 + d_{222}(\underline{x}) p_2^3 + d_{112}(\underline{x}) p_1^2 p_2 + d_{221}(\underline{x}) p_2^2 p_1 + \sum_j a_j(\underline{x}) p_j \ ,$$

where

$$m_3 = xp_2 - yp_1 \ .$$

The a_j in (2.27a) are coupled with the potential V through the equation

$$a_1 V_x + a_2 V_y = 0 \ ;$$

the b_{jk} in (2.27b) through the equation

$$(b_{11}-b_{22})V_{xy} + b_{12}(V_{yy}-V_{xx}) + 3b_{12_y} V_y - 3b_{12_x} V_x = 0 \ .$$

The $c(\underline{x})$ satisfies (2.28a)

$$c_x = b_{11} V_x + b_{12} V_y \qquad (2.28b)$$

(from the above by cyclic permutation) . (2.28c)

The $d_{jk\ell}$ and a_j in (2.27c) are coupled with the potential V through the following equations

$$\hat{a}_{1_x} - 3V_x d_{111} - V_y d_{112} = 0 \ , \qquad (2.29a)$$

(from the above by cyclic permutation), (2.29b)

$$\hat{a}_{2_x} + \hat{a}_{1_y} - 2d_{112} V_x - 2d_{221} V_y = 0 \ , \qquad (2.29c)$$

$$\hat{a}_1 V_x + \hat{a}_2 V_y = 0 \ . \qquad (2.29d)$$

Proof. The derivation of the above results, although very cumbersome, is similar to that of §2.3.1 and is therefore omitted.

Note that equations (2.27) define the a_j, b_{jk}, $d_{jk\ell}$ to within constant parameters. For example, $a_1 = \alpha - \beta y$, α and β arbitrary constants, etc. Also (2.28a) is linear in V and the $c(\underline{x})$ is uncoupled from the b_{ij} and the V. This may be contrasted with the case of a constant of motion cubic in the momenta where the $\hat{a}_i$ are coupled with the $d_{jk\ell}$ and the V. Let us now come back to equations (2.29). First note that they can be solved for $\hat{a}_1$ and $\hat{a}_2$;

$$\hat{a}_1 = \frac{3Vy[d_{111}\phi^3 - d_{112}\phi^2 + d_{221}\phi - d_{222}]}{\phi\phi_x - \phi_y}, \tag{2.29e}$$

$$\hat{a}_2 = \phi a_1, \quad \phi \equiv -\frac{Vx}{Vy}. \tag{2.29f}$$

Then substituting (2.9e,f) in (2.29a,b) we obtain two equations relating $d_{jk\ell}$ with V. However, these equations are nonlinear in V and therefore are not very suitable for discovering which potentials admit a given invariant or which invariants are admitted by a given potential. (For a discussion of the corresponding problems for the quadratic case see §2.4). In order to obtain a linear equation relating $d_{jk\ell}$ with V we disregard for the moment equation (2.29d) and eliminate $\hat{a}_1$ and $\hat{a}_2$ from equations (2.27a,b,c). This yields the equation below, (which is necessary but not sufficient)

$$\begin{aligned} &d_{221}V_{xxx} + d_{112}V_{yyy} + (3d_{222} - 2d_{112})V_{xxy} + (3d_{111} - 2d_{221})V_{yyx} \\ &2(d_{221_x} - d_{112_y})(V_{xx} - V_{yy}) + (6d_{111_y} + 6d_{222_x} - 2d_{221_y} - 2d_{112_x})V_{xy} \\ &+ (3d_{222_{xx}} + d_{112_{yy}} - 2d_{221_{xy}})V_y + (3d_{111_{yy}} + d_{221_{xx}} - 2d_{112_{xy}})V_x = 0. \end{aligned} \tag{2.29h}$$

2.4 APPLICATIONS TO CLASSICAL MECHANICS

2.4.1 The Hamilton-Jacobi Equation (for arbitrary value of E).

In §2.3 we showed how one can find admissible operators (and hence invariants) of equations (2.13) and (2.23) when the functions $\hat{V}$ and V, respectively, are given.

In the present subsection we shall first consider the inverse problem: Given an operator, find all V which admit it. Some classes of solutions to this problem are given in §2.4.1 A, as well as an example illustrating the incompleteness of these results. In §2.4.1 B we shall consider potentials of a prescribed general form which are of physical interest and shall find all potentials of this form which give nontrivial invariants as well as the corresponding invariants. It is believed that some of these invariants are new. Their extension to quantum mechanics is given in Chapter III and some important limiting cases in §2.5.1. In §2.4.1 C we shall reveal the group-theoretical nature of all the separable solutions of the one-body Keplerian problem.

A. The inverse problem.

α. Operators linear in the momenta.

Instead of integrating $(\hat{\alpha})$ directly we follow [20] and perform a Euclidean coordinate transformation:

$$x_i' = \alpha_{ik}x_k+\beta_i, \quad \text{where} \quad \alpha_{ik}\alpha_{ij} = \delta_{kj}. \quad i,j,k = 1,2,3 \qquad (2.30)$$

such that the operator Z_1 of (6.14) takes a simple (normalized) form Z_1', where

$$Z_1' = (aM_1'+bP_1')\frac{\partial}{\partial S}, \quad a^2 = a_ia_i, \quad a_ib_i = ab; \quad i = 1,2,3.$$

Therefore, in the normalized coordinates, the general solution of $(\hat{\alpha})$ is:

$$V = F(y'^2+z'^2, \alpha x'-b\theta'); \qquad \theta' = \tan^{-1}\frac{z'}{y'}$$

where for physically meaningful potentials the first derivatives of F must be periodic in θ'.

β. Operators quadratic in the momenta (without linear terms)

Solving equations (δ) completely is quite complicated. While one can easily find some solutions of these equations, it is very difficult to determine when one has found all solutions. The corresponding problem in quantum mechanics has been considered in [20] and [15]. In looking for potentials admitting second order operators Winternitz and others considered a set of equations equivalent to (δ). Initially, they performed a coordinate transformation of the form (2.30), rather than integrating the relevant equations directly. Their procedure, when applied here, corresponds to using (2.30) to simplify the operator Z_2 of (2.26), and then integrating equations (δ). The results of [20] and [25] are directly applicable here:

1) In two dimensions all the potentials admitting symmetries of the form (2.25) are those which allow separation of equation (2.23) in one of the four coordinate systems: cartesian, polar, parabolic and elliptic.

2) In three dimensions, all the potentials admitting two commuting operators of the form (2.25), are those which allow complete separation of the time-independent Hamilton-Jacobi equation.

The above relationship between symmetries and complete separation of variables is very interesting. Actually the corresponding problem in quantum mechanics was the motivation for the group characterisation of complete separation of variables. However, in our view, the problem of partial separation has been unduly neglected. Theorem (2.3) expresses the group nature of the partial separation and provides a way of finding a more general class of potentials admitting symmetries: Any potential allowing partial separation of the Hamilton-Jacobi equation must be a solution of (δ).

To obtain simple solutions of (δ) directly, we let all the independent parameters in (2.17) and (2.18) be zero except one, which is put equal to unity. In this way we derive potentials covered by 1) and 2) above. However, other solutions may be obtained using the parameters α_4, β_4, γ_4. If we put $\beta_4 = 1$ and all other group parameters equal to zero, equations (δ) yield:

$$V = \kappa(\varphi^2 + \frac{z^2}{\rho^2}) + \lambda\varphi + \mu\frac{z}{\rho^2} + f(\rho)$$

where $\rho^2 = x^2+y^2$, $\varphi = \tan^{-1}\frac{y}{x}$, κ, λ, μ arbitrary constants, $f(\rho)$ arbitrary function of ρ. However, for physically meaningful potential, the first derivatives of V must be single-valued, therefore $\kappa = 0$.

The admissible operator corresponding to this potential is:

$$Z = (S_\varphi S_z + 2\kappa\varphi z + \lambda z + \mu\varphi)\frac{\partial}{\partial S}.$$

The potentials and invariants obtained by replacing β_4 by α_4 or γ_4 can be obtained by cyclic permutation.

B. Potentials due to one or two fixed centers.

We now investigate a class of potentials of physical interest, namely potentials due to one or two fixed centers. Their interrelation will be discussed in §2.5.1 with the aid of various limit processes, which also will yield new potentials not considered here. It will then become clear that the two fixed centers have a logical priority to the one fixed center. However, we shall start with the latter case, i.e. of a central field in three dimensions. It is well known that for a fixed center there exists only one type of geometrical symmetry, the spherical symmetry, which leads to the conservation of the angular momentum vector. We shall check using our methods that the only central potentials which give nontrivial invariants (quadratic in the momenta) are of form $V \sim r^2$ or $V \sim r^{-1}$. The first case is that of an isotropic harmonic oscillator and the second case occurs in Newtonian gravitational theory. In the above cases we shall refer to the origin as a harmonic or a Newtonian center, respectively.

A natural generalization of the fixed center potential is, especially in view of an old discovery of Euler (see below), the potential due to two fixed centers. Since the spherical symmetry is now reduced to cylindrical we consider this problem in the (x, y) plane. Then the invariants can be extended to three dimensions using the cylindrical symmetry. It turns out that the only two centers which admit a geometrical symmetry are the two harmonic centers. Also, the only two centers which admit nongeometrical symmetries are two Newtonian or two harmonic centers. In both cases the distance between and the strength of each center are arbitrary. It is interesting that a curious hybrid also exists: two harmonic centers at the same location as two Newtonian centers, provided the former have the same strength.

α. Central Fields

To find admissible operators linear in the momenta, let

$A_i = C = 0$, $i = 1,2,3$ in (2.16), $\hat{V} = V(r)$ in $\hat{\alpha}$, and substitute (2.16) in $\hat{\alpha}$. Then integrating $\hat{\alpha}$, the only invariants found are M_i, $i = 1,2,3$ as expected.

To find admissible operator quadratic in the momenta, let

$$V(r) = F(r^2), \qquad F' = \frac{dF}{d(r^2)}, \qquad F'' = \frac{dF'}{d(r^2)}.$$

Substituting (2.17) and (2.18) in (6.1) we obtain:

$$\begin{aligned}&[(y\alpha_2 - x\alpha_3)(x^2+y^2) + (y\gamma_3 - x\beta_2)z^2 + 2xy\alpha_0 + z(y\gamma_1 - x\gamma_1) + \\ &(y^2-x^2)\alpha_1 + xyz(\beta_3-\gamma_2) + z(y^2-x^2)(\alpha_4+\gamma_4)]F'' + \\ &[(y\alpha_2 - x\alpha_3) + \frac{2}{3}(\gamma_4-\beta_4)]F' = 0,\end{aligned} \tag{2.31a}$$

(7.1.2), (7.1.3) by cyclic permutation.

i) Note that the parameters α_i, β_i, γ_i, $i = 5,6$ do not appear in (2.13). Therefore the operators corresponding to those parameters are admissible for any $V(r)$. Abbreviating Z_2 of (2.26) to

$$X_{ij} = A_{ij}\frac{\partial}{\partial S}, \tag{2.32}$$

we find that now $A_{ij} = M_i M_j$, $i,j = 1,2,3$.

ii) Let $\alpha_i = \beta_i = \gamma_i = 0$, $i = 2,3,4$

Then $F'' = 0$ and α_i, β_i, γ_i, $i = 0,1$ are arbitrary. Hence $V = \alpha r^2$

$$A_{ij} = P_i P_j + 2\alpha x_i x_j, \qquad i,j = 1,2,3 \tag{2.33}$$

iii) Let $\alpha_i = \beta_i = \gamma_i = 0$, $i = 0,1,4$.

Also $\alpha_3 = \beta_2$, $\beta_3 = \gamma_2$, $\gamma_3 = \alpha_2$

$\alpha_2 = \gamma_3$, $\beta_2 = \alpha_3$, $\gamma_2 = \beta_3$

Therefore $F'' r^2 + \frac{3}{2}F' = 0$ and α_2, β_2, γ_2 are arbitrary

Then $V = \alpha r^{-1}$

$$A_i = \epsilon_{ijk} P_j M_k + \alpha x_i r^{-1}, \qquad i,j,k = 1,2,3 \tag{2.34}$$

Comments

1. Clearly the operator (2.32) where $A_{ij} = M_i M_j$, $i,j = 1,2,3$ although nonlinear in the momenta, is a trivial consequence of the spherical symmetry (therefore it is of type (ii) of (1.4)). This symmetry also allows separation of variables of the Hamilton-Jacobi equation in spherical coordinates.

2. The use of the tensor (2.33) and the Runge-Lenz vector (2.34) have been well established in the literature. We just stress that their existence is the consequence of non-geometrical symmetries. These strictly Lie-Bäcklund symmetries also lead to separation of variables in cartesian and parabolic coordinates respectively. The separation of the equation for the harmonic oscillator is trivial. The separation of the equation for the one-body Keplerian problem will be discussed later. The Lie-Bäcklund algebras of the above groups are isomorphic to algebras of finite dimensions [2.6].

β. Two fixed centers.

Let

$$V = V_1(\rho) + V_2(\rho_0)$$

where

$$\rho^2 = x^2 + y^2, \quad \rho_0^2 = (x - x_0)^2 + y^2.$$

An analysis similar to that of 4) yields (see Appendix III):

i) $$V = \alpha\rho^2 + \beta\rho_0^2$$

Then

$$A_0 = (x - \frac{x_0\beta}{\alpha+\beta})P_2 - yP_1 \tag{2.35}$$

$$A_{ij} = P_i P_j + 2[\alpha x_i x_j + \beta(x_i - x_{0\,i})(x_j - x_{0\,j})], \quad i,j = 1,2 \tag{2.36a}$$

$$B = M_3\hat{M} - \frac{2y^2 x_0^2 \beta^2}{\alpha+\beta} \tag{2.36b}$$

where

$$x_{0_i} = \left[\begin{array}{ll} x_0, & i = 1 \\ 0, & i = 2 \end{array}\right., \qquad \hat{M} = \left(x - \frac{2x_0\beta}{\alpha+\beta}\right)P_2 - yP_1$$

ii) $V = \alpha_0 \rho^2 + \alpha_0 \rho_0^2 + \alpha\rho^{-1} + \beta\rho_0^{-1}$

Then

$$A_1 = M_3 M_0 - \alpha_0 y^2 x_0^2 - \frac{\alpha x_0 x}{\rho} + \frac{\beta x_0 (x - x_0)}{\rho_0} \tag{2.37}$$

where

$$M_0 = (x - x_0)P_2 - yP_1 .$$

Comments

1. It is interesting that, for every symmetry of the one harmonic center corresponds one for two harmonic centers. This is a consequence of the fact that the two harmonic centers are equivalent to one, located at $(\frac{x_0\beta}{\alpha+\beta}, 0)$ and having strength $(\alpha+\beta)$.

2. The invariant (2.36b) is a trivial consequence of the invariants A_0 and A_{22} since $B = A_0^2 - \frac{x_0^2}{\alpha+\beta} A_{22}$. However, the form (2.36b) is also useful as it illustrates the connection of (2.36b) and (2.37). As α and $\beta \to \alpha_0$ $B \to M_3 M_0 - \alpha_0 y^2 x_0^2$.

3. The invariant for the two Newtonian centers was found by Euler in 1760 and is discussed in [27], using elliptic coordinates which are the appropriate separating coordinates for this problem. A more transparent form of this invariant, namely the one obtained from (2.37) putting $\alpha_0 = 0$, was used in [28].

4. The invariant corresponding to the "mixed" case is given by (2.37). The spherical symmetry is destroyed, but this time (as in the case of two Newtonian centers) it is replaced by

a Lie-Bäcklund one.

C. One-Body Keplerian Problem

As an illustration of Theorem 2.3 we investigate the group-theoretical nature of all separable solutions of the one-body Keplerian problem:

$$S_t + \frac{1}{2} S_{x_i} S_{x_i} + \frac{\mu}{r} = 0, \qquad i = i, 2, 3. \tag{2.38}$$

The following admissible operators will be used:

$$Y_i = A_i \frac{\partial}{\partial S}, \quad 0 \leq i \leq 6, \quad \text{where } A_0 = 1, \; A_1 = S_t,$$

$$A_2 = M_3, \; A_3 = M_1, \; A_4 = M_2, \; A_5 = A_2^2 + A_3^2 + A_4^2,$$

$$A_6 = P_1 M_2 - P_2 M_1 + \frac{\mu z}{r}.$$

Spherical

Writing (2.38) and the operators

$$Y_1 - \lambda_1 Y_0, \quad y_2 - \lambda_2 Y_0, \quad Y_5 - \lambda_3 Y_0,$$

in spherical coordinates we obtain the separated equations:

$$S_t + \frac{1}{2} S_r^2 + \frac{1}{2r^2}\left(S_\theta^2 + \frac{S_\varphi^2}{\sin^2\theta}\right) + \frac{\mu}{r} = 0$$

$$S_t = \lambda_1$$

$$S_\varphi = \lambda_2$$

$$S_\theta^2 + \frac{1}{\sin^2\theta} S_\varphi^2 = \lambda_3 .$$

Using Theorem (2.1): $\lambda_1 = -E$, $\lambda_2 = p^\varphi$, $\lambda_3 = M^2$, where p^φ is the φ component of the angular momentum and M the total angular momentum.

Parabolic

Writing (2.38) and the operators

$$Y_1 - \lambda_1 Y_0, \quad Y_2 - \lambda_2 Y_0, \quad Y_6 - \lambda_3 Y_0,$$

in parabolic coordinates we obtain:

$$S_t + 2\xi S_\xi^2 + \mu + \xi S_t + \frac{S_\varphi^2}{2\xi} + 2\eta S_\eta^2 + \eta S_t + \frac{S_\varphi^2}{2\eta} = 0$$

$$S_t = \lambda_1$$

$$S_\varphi = \lambda_2$$

$$-2\xi S_\xi^2 - \mu + E\xi - \frac{1}{2\xi} S_\varphi^2 = \lambda_3 .$$

Using Theorem 2.3: $\lambda_3 = L_3$, the z component of the Runge-Lenz vector.

We see that for the complete characterization of the separable solutions in parabolic coordinates, the operator Y_6 which is strictly a Lie-Bäcklund operator is necessary.

2.4.2 Hamilton-Jacobi equation for the zero energy state.

It was shown in §2.3 that, when $E = 0$, the Hamilton-Jacobi equation may possess additional symmetries, not present for $E \neq 0$. As in §2.4.1A we first give some classes of solutions of the inverse problem. We then concentrate on potentials due to one fixed center. Although the zero energy state is a very special case the results (which are new) may be of some interest.†

A. The inverse problem.

α. Operators linear in the momenta.

Abbreviating the operator (2.20) to $Y_1 = A\frac{\partial}{\partial S}$ and solving $(\hat{\alpha})$ we obtain:

1) $\kappa_i = \lambda_i = \mu_i = 0, \quad \nu \neq 0; \quad i = 1,2,3$

$$\hat{V} = \frac{1}{x^2} F\left(\frac{y}{x}, \frac{z}{x}\right); \quad A = \Delta \tag{2.39}$$

† As an analogy we point out that certain dynamical systems admit periodic solutions only under very special conditions. These special solutions may, however be of interest.

2) $\kappa_i = \lambda_i = \nu = 0, \quad \mu_i \neq 0; \quad i = 1,2,3$

$$\hat{V} = \frac{1}{z^2} F\left(\frac{y}{z}, \frac{x^2-y^2-z^2}{z}\right); \qquad A = K_1 \tag{2.40}$$

Additional potentials can be obtained by cyclic permutation, as well as combining different group parameters.

β. Operators quadratic in the momenta (without linear terms).

The possible systems of coordinates allowing additive separation of (2.13) have been found [29]. Again, using Theorem 2.3, any function $\hat{V}(\underline{x})$ allowing partial or total separation of (2.13) is a solution of (δ). We emphasize the case of partial separation, which in three dimensions can be quite useful in deriving invariants.

B. Central Potentials.

Assume $\hat{V} = \hat{V}(r)$ in (2.13).

α. Operators linear in the momenta.

Integrating $(\hat{\alpha})$ We obtain:

Potential	Invariant	
$\hat{V}(r)$	$M_i, \quad i = 1,2,3$	(2.41a)
αr^{-2}	Δ	(2.41b)
αr^{-4}	$K_i, \quad i = 1,2,3$	(2.41c)

β. Operators quadratic in the momenta.

Integrating (γ) and (δ), in addition to the cases considered in §2.4.1 we find

Potential	Invariant	
αr^{-2}	$\Delta M_i, \quad i = 1,2,3$	(2.42a)
$\alpha r^{-3}+\beta r^{-4}$	$\Delta K_i + \alpha x_i r^{-1} + 2\beta x_i r^{-2}, \quad i = 1,2,3$	(2.42b)

$$\alpha r^{-6}+\beta r^{-4} \qquad\qquad K_iK_j+2\alpha x_ix_jr^{-4} \quad i,j = 1,2,3 \qquad (2.43c)$$

The invariant (2.42a) is of course a trivial consequence of (2.41).

2.5 FURTHER APPLICATIONS

2.5.1 Potentials Generated by Fixed Centers and Limiting Cases. Associated Invariants.

In the present subsection we consider an important class of potentials. A fixed point P_1 is a center if there is a potential V_1 whose value at an arbitrary point Q depends only on the distance $|QP_1|$. If V_2 is a potential depending only on the distance to the point P_2, one may form a two-center potential by superposition, $V_1 + V_2$. By varying the strength of the potentials and the distance between P_1 and P_2 one can obtain various limiting cases which need not be center-potentials.

In discussing invariants in this section we shall deal only with constants of motion of Hamilton's equations, that is with functions $f(x_j, p_j)$, $j = 1, 2, 3$, constant along any orbits. The commutator of two functions f and g is the Poisson bracket which we define as

$$[f, g] = \sum_{j=1}^{3} \left(\frac{\partial f}{\partial x_j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial x_j}\right). \qquad (2.44)$$

In Chapter III we shall discuss the relations between such invariants and the corresponding operators admitted by the Schrödinger equation. For the purposes of Chapter III it is instructive to deal with potentials having nontrivial invariants nonlinear in the momenta. The potentials discussed below are interesting in their own right, and they will also be used to illustrate the use of limit processes for potentials and their associated invariants.

Below we shall mainly discuss the case $x_3 = 0$; the extension to three dimensions is easily made. We consider a two-center potential; without loss of generality we may place the centers at $(0,0)$ and $(x_*, 0)$. It was found in §2.4.1 that the most general two-center potential with a nontrivial quadratic invariant has the form†

$$V = \frac{\alpha}{2}(\rho^2 + \rho_*^2) + \beta\rho^{-1} + \gamma\rho_*^{-1}, \qquad (2.45a)$$

where

$$\rho^2 = x^2 + y^2, \qquad \rho_*^2 = (x - x_*)^2 + y^2, \qquad (2.45b)$$

$\alpha, \beta, \gamma, x_*$ arbitrary constants.

The nonlinear invariant is

$$C = m_3 m_* - \frac{\alpha}{2}x_*^2 y^2 - \beta x_* x \rho^{-1} + \gamma x_*(x - x_*)\rho_*^{-1}, \qquad (2.46)$$

where

$$m_* = (x - x_*)p_2 - yp_1.$$

We shall now study special and limiting cases of the potential (2.45) and the Lie algebra structure of the associated invariants as well as nonlinear relations between linearly independent invariants. We first discuss Newtonian centers and then harmonic centers; in each case various limits will be discussed.

A. Newtonian Centers.

Putting $\alpha = 0$ in (2.45) we obtain the potential due to two fixed Newtonian centers. The invariant C reduces to the invariant found by Euler in 1760. Table 2.1 below shows various limiting cases:

† The case of two harmonic centers of different strengths is included in (2.45) in a sense to be discussed later.

Limit	Potential	Constant of Motion
$\alpha \to 0$ in (2.45) & (2.46)	$V_1 = \beta\rho^{-1} + \gamma\rho_*^{-1}$	$C_1 = m_3 m_* - \beta x_* x\rho^{-1} + \gamma x_*(x - x_*)\rho_*^{-1}$
$\left.\begin{array}{l}\gamma = \varepsilon x_*^{-1},\\ \beta = \delta - \varepsilon x_*^{-1}\end{array}\right\} x_* \to 0$	$V_1 \to V_2 = \delta\rho^{-1} - \varepsilon x\rho^{-3}$	$C_2 = m_3^2 - 2\varepsilon x\rho^{-1}$
Not a limit	$V_3 = f(\rho) - \varepsilon x\rho^{-3}$	$C_3 = C_2$
$\delta \to 0$	$V_2 \to V_4 = -\varepsilon x\rho^{-3}$	$C_4 = C_2$
$\gamma = \zeta x_*^2$; $x_* \to \infty$	$V_1 \to V_5 = \beta\rho^{-1} + \zeta x$	$-x_*^{-1}C_1 - \zeta x_*^2 \to C_5 = m_3 p_2 + \beta x\rho^{-1} - \frac{1}{2}\zeta y^2$
$x_* \to 0$	V_1 (→ V_6)	$C_1 \to C_6^{(1)} = m_3^2$
$\zeta \to 0$	$V_5 \to V_6 = \tilde{\beta}\rho^{-1}$	$C_5 \to C_6^{(2)} = m_3 p_2 + \beta x\rho^{-1}$
$\gamma \to 0$	V_1 (→ V_6)	$C_1 \to C_6^{(3)} = C_6^{(1)} - x_* C_6^{(2)}$
$\beta \to 0$	$V_5 \to V_7 = \zeta x$	$C_5 \to C_7 = m_3 p_2 - \frac{1}{2}\zeta y^2$ (incomplete)

where

$$m_* = (x - x_*)p_2 - yp_1.$$

Table 2.1. Potential and Constants of Motion (Invariants) for Two Newtonian Centers, and Limiting Cases.†

† See also [30, IX] and, for some specific computations, [23 §48].

V_2 and C_2 are obtained from V_1 and C_1 by a dipole limit, more precisely by a limit which yields one center and one dipole directed along the x-axis, at the origin (δ and ϵ are considered fixed as the limit is taken). If one uses (6.1) of §2.3.1 to find the general potential at the origin which has the invariant C_2 one finds V_3 which is due to an arbitrary center and a Newtonian dipole, both at the origin. (See also discussion of invariants of superimposed potentials at the end of this section). Putting the strength of the center equal to zero yields V_4 which still has the invariant C_2. V_5 is obtained from V_1 by the constant-force limit: The second center recedes to infinity and its strength increases in such a way that a constant force field remains. (For a Newtonian center the strength has to increase as x_*^2 because of the inverse-square force law.) The resulting potential may be called the Stark potential. To obtain C_5 from C_1 some care must be exercised; one reason is indicated in the table, it is also necessary to expand ρ_* to second order in x_*^{-1}:

$$-\zeta x_*^2 (x-x_*)\rho_*^{-1} = \zeta x_*^2 (1-xx_*)^{-1} (1+xx_*^{-1} - \tfrac{1}{2} y^2 x_*^{-2} + x^2 x_*^{-2} + \dots) =$$

$$\zeta x_*^2 - \frac{1}{2}\zeta y^2 + O(x_*^{-1})$$

In C_5 one may regard p_2 as the angular momentum about the center at infinity. More precisely $-x_*^{-1} m_* \to p_2$ as $x_* \to \infty$. The case of the potential V_6, that is of one Newtonian center, is instructive. The separating coordinates corresponding to C_1 are elliptic-hyperbolic with foci at $(0,0)$ and $(x_*, 0)$. If one obtains one center by merging $(x_* \to 0)$ these coordinates become polar and the corresponding invariant is the trivial one, $C_6^{(1)} = m_3^2$.

However, if one lets x_* tend to infinity the coordinates become parabolic. The corresponding invariant is C_5 for the Stark potential and $C_6^{(2)}$ in the limiting case $\zeta = 0$. The connection between separation of variables and admissible operators of the Hamilton-Jacobi equation is explained in §2.2.3 and the connection between admissible operators and constants of motion in §2.2.1. Note that $C_6^{(2)}$ is equal to A_1, the x-component of the Runge-Lenz vector.† Thus the fact that the Hamilton-Jacobi equation with a Newtonian potential separates in two distinct coordinate systems (polar and parabolic) and has two distinct invariants, angular momentum and Runge-Lenz, is a direct consequence of Euler's discovery that C_1 is an invariant associated with V_1 for arbitrary values of x_*, and γ. Note further that the third way of obtaining $V_6(\gamma \to 0)$ gives an invariant $C_6^{(3)}$ which is a linear combination of $C_6^{(1)}$ and $C_6^{(2)}$. The corresponding separating coordinates are elliptic. This may be formulated as a general principle. Assume that the Hamilton-Jacobi equation separates in two distinct coordinate systems. To these correspond two independent quadratic invariants. From these invariants one finds an infinity of invariants by linear combinations. By this detour one discovers that two separating coordinate systems generate a one-parameter family of separating systems. This family, however, does not give any new independent invariants.

For one Newtonian center the problem has spherical symmetry and all components of the angular momentum vector are

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† Since V_6 could also be obtained as a limit of two centers on the y-axis it follows that A_2 is also invariant.

geometric invariants. Similarly all components of the Runge-Lenz vector are dynamic invariants. These may be obtained from the Runge-Lenz component shown in Table 2.1 by forming the commutator with the components of the angular momentum. As is well-known, see e.g. [31], when the energy E is negative (elliptic orbits) the six invariants form the Lie algebra $O(4;R)$. When $E > 0$ (hyperbolic orbits) the algebra is $O(3,1;R)$. The case $E = 0$ (parabolic orbits) is a limiting case of both $E < 0$ and $E > 0$. Thus we expect the Lie algebra for $E = 0$ to contain the algebra of Euclidean motions in three dimensions, since this algebra is obtained by the limit process called contraction†† from both of the two algebras mentioned; as is easily seen it is actually the full symmetry algebra. Obviously, as long as quadratic invariants are considered the Hamiltonian function H itself may be counted. (The special role of the Hamiltonian is discussed below). However, by definition H commutes with every constant of motion. Thus by adding H to the invariant one obtains the direct product of a one-dimensional Lie algebra and whatever algebra one has without H. For instance, adding H to the algebra of the harmonic oscillator, which is $su(3)$ (see below), gives $u(3)$.

Finally we remark that the invariant C_7 obtained in Table 2.1 does not give the complete set of invariants for V_7. We shall, however, see below that a complete set of invariants is obtained by way of the harmonic center.

B. Harmonic Centers.

Considering now harmonic centers one finds that due to the

†† See, for instance [32].

quadratic nature of the potential there are actually only two cases: One harmonic center, or a constant force field (which corresponds to a harmonic center located at infinity). Since $(x-x_*)^2 = x^2 - 2xx_* + \text{constant}$, one harmonic center at a point is the same as one harmonic center at a displaced point and a constant-force potential. Displacing two centers to the same point, chosen such that the constant forces cancel, one finds that $\beta\rho^2 + \gamma\rho_*^2$ gives the same force field as $(\beta + \gamma)\rho_e^2$ where $\rho_e^2 = (x - x_e)^2 + y^2$, $x_e = \gamma x_*/(\beta+\gamma)$. If $\beta + \gamma = 0$, the effective location of the center is at infinity and using a proper limit one gets a constant force field. Similarly, $\beta\rho^2 + \zeta x$, $\beta \neq 0$ gives one harmonic center at $(x_e, 0)$, $x_e = -\zeta/2\beta$. Finally, the dipole limit and the constant-force limits yield the same results.

We shall define the potential of a harmonic center (at the origin) by

$$V = \frac{\delta}{2}\sum_j x_j^2 . \tag{2.47a}$$

In one dimension δ is then the ordinary spring constant if it is positive. We may, however, allow δ to be any real number. As stated above the number of cases is small. However, the number of linearly independent invariants is larger than in the Newtonian case. The spherical symmetry gives the obvious geometrical invariants m_j. In addition there are dynamical invariants which form a symmetric tensor.

$$A_{jk} = p_j p_k + \delta x_j x_k . \tag{2.47b}$$

The Hamiltonian equals half the trace of this tensor, If we exclude the Hamiltonian we find that the A_{jk} gives five linearly independent invariants. The m_j gives three additional ones and the linear space spanned by these eight invariants is closed under commutation. In fact restricting ourselves to the (x_1, x_2)-plane a simple computation yields, for $\delta > 0$, that $A_{12}/2\sqrt{\delta}$, $(A_{22}-A_{11})/4\sqrt{\delta}$, $m_3/2$ commute like m_1, m_2, m_3, that is the algebra is su(2). Since the same is true for the (x_2, x_3)-plane and the (x_3, x_1)-plane the algebra, including the Hamiltonian, is u(3) and, excluding the Hamiltonian, su(3). This is a well-known result, see for instance [31].

We now consider the limiting case of a harmonic center at infinity, that is a constant force field. A harmonic center at $(x_*, 0)$ has potential $\delta/2[(x-x_*)^2 + y^2]$. If one subtracts the constant $\delta x_*^2/2$ and puts $\delta = -\zeta x_*^{-1}$ one obtains the potential ζx as $x_* \to \infty$, ζ fixed. Putting $m_* = (x - x_*)p_2 - yp_1$ one finds

$$- x_*^{-1} m_* \to p_2 . \tag{2.48a}$$

Furthermore, in the limit described above

$$A_{11} \to B_1 = p_1^2 + 2\zeta x \tag{2.48b}$$

$$A_{12} \to B_2 = p_1 p_2 + \zeta y \tag{2.48c}$$

The invariant A_{22} yields p_2^2 which is a trivial result since p_2 is invariant. Above the A_{jk} are of course adjusted to the new location of the center, that is $A_{11} = (x-x_*)^2 + p_1^2$, etc. Comparing with Table 2.1 one sees that the invariants B_1 and B_2 are not obtained as limits if one lets a Newtonian center tend to infinity. On the other hand another quadratic invariant, called there C_7, was obtained. The same invariant may also be obtained in the harmonic case. Define A by

$$A = m_*^2 - x_*^2 A_{22}.$$

Then

$$-\frac{A}{2x_*} \rightarrow B_3 = m_3 p_2 - \frac{\zeta}{2} y^2 \qquad (2.48d)$$

Note that A is obviously dependent on m_* and A_{22}. Below, see (2.52b) we shall show a nonlinear relation between p_2 and the B_j.

The commutation relations between the invariants are

$$[p_2, B_1] = 0, \quad [p_2, B_2] = -\zeta, \quad [p_2, B_3] = B_2,$$
$$[B_1, B_2] = 2\zeta p_2, \quad [B_1, B_3] = -2p_2B_2, \qquad (2.49)$$
$$[B_2, B_3] = p_2(B_1 - p_2^2).$$

In previous cases discussed here the linear closure of the basic invariants was also closed under commutation. This is not true in the present case (although the right-hand sides of the above equations belong to the enveloping algebra of the basic invariants). Clearly, the commutator of two functions linear and homogeneous in the momenta is also linear and homogeneous. However, one expects commutators of quadratic functions to be cubic and by repeated application of the commutator to obtain functions of arbitrarily high degree in the momenta. This argument is analogous to the reasoning showing that the proper generalization of Lie point operators are LB operators which in principle form an infinite-dimensional space. The linear and quadratic invariants of a Newtonian center and those of a harmonic center have a very special form and their linear closure happens to be closed under commutation; this cannot be expected in general. The symmetry properties of a harmonic potential are said to be described by su(3).

How do the relations (2.49) describe the symmetry of a constant-force field? We remark only that it is surprising that this question arises for such an elementary problem. The discussion of the symmetry algebra will not be pursued further in this work.

C. The Hamiltonian as a Distinguished Invariant.

To find the symmetry algebra of the Newtonian center one replaces the Hamiltonian $H(p_j, x_j) \equiv 1/2 \sum_j p_j^2 + V$ by its constant value E. This leads to different algebras for the different cases $E < 0$, $E = 0$, $E > 0$. However, in (3.5) we did not give constant values to the invariants appearing in the right-hand sides of the equations. The reason the Hamiltonian is distinguished is that in testing an invariant we restrict ourselves to the manifold defined by the equation, in the present case $H(S_{x_j}, x_j) - E = 0$. Thus we may replace H by E.

D. Functional versus Linear Dependence.

Consider for example the harmonic oscillator in three dimensions. Its symmetry algebra is su(3); it has eight linearly independent invariants. However, clearly there can be only five functionally independent invariants since the phase-space is six-dimensional. Thus a linear Lie-algebra description is insufficient. For the potentials studied in this section the nonlinear functional relations are actually simple polynomials in the enveloping algebra. We shall give these relations below. First, however, let us consider the simplest possible case, namely the motion of a free particle. There are six linear invariants namely p_j and m_j, $j = 1, 2, 3$. Since the angular momentum vector is by its definition orthogonal to the momentum vector we have, for any potential,

$$\sum_j p_j m_j = 0. \tag{2.50}$$

For the free particle this equation is a functional relation between invariants. The Hamiltonian is always an invariant. In the present case $H = \Sigma p_j^2$. It is functionally dependent on the linear invariants as are in fact all other invariants. Thus there are exactly five functionally independent invariants. Next we consider the potentials in Table 2.1. We shall restrict ourselves to motions in the (x, y)-plane. Since phase space then is four-dimensional we can have at most three functionally independent invariants. For V, given by (2.45) and for V_1 to V_5 (see Table 2.1) we have two linearly independent invariants, namely C or C_1 to C_5 and H. There is no functional relation between the two invariants. However for one Newtonian center (V_6) there are four invariants, namely the Hamiltonian, m_3, and two components of the Runge-Lenz vector A_1, A_2 $\left(A_1 = m_3 p_2 + \beta x \rho^{-1}\right)$. The functional relation is

$$2m_3^2 H = A_1^2 + A_2^2 - \beta^2 . \tag{2.51}$$

For the constant-force field (V_7) we have five linearly independent invariants: H, p_2, B_1, B_2, B_3. The functional relations are

$$H = \frac{p_2^2}{2} + \frac{1}{2} B_1, \tag{2.52a}$$

and

$$p_2^2 B_1 = B_2^2 + 2\zeta B_3. \tag{2.52b}$$

Finally, for the harmonic center the linearly independent invariants are m_3, A_{11}, A_{12}, A_{22}, the last three defined by (2.47). The Hamiltonian is linearly dependent on these

$$2H = A_{11} + A_{22} \tag{2.53a}$$

and we have the additional relation

$$A_{11}A_{22} = A_{12}^2 + \delta m_3^2 \tag{2.53b}$$

2.5.2 Complete Set of Invariants. Degeneracy.

In Hamiltonian mechanics a problem in 2n-dimensional phase space is called completely integrable if we have n constants of motion in evolution. The time-dependent Hamilton-Jacobi equation can then be completely separated (see for instance [23, §47 and §48]). The Hamiltonian itself (assumed not to involve time explicitly) accounts for separating out the time-dependence. Thus we find a complete integral of the Hamilton-Jacobi equation.

Existence of invariants is also tied to degeneracy. For classical mechanics this concept is discussed in [23, §52]. In quantum mechanics degeneracy means degeneracy of an eigenvalue of the energy, namely the existence of several linearly independent eigenfunctions belonging to the same eigenvalue. The number of such functions may depend on the eigenvalue. We therefore propose to define the degree of degeneracy as the number of functionally independent invariants minus one. To illustrate the idea consider motions in the (x, y)-plane. If the potential is due to any center the problem is completely integrable since we have two independent invariants, the Hamiltonian and the angular momentum. Thus there is degeneracy of degree one. If the center is Newtonian or harmonic the degree of degeneracy is maximal (=3) due to the added invariants. For bounded motion we can use an alternative formulation. We consider the dimension of the topological closure† of the

† This is a more precise notion than that of "space-filling" used in [30].

orbit in phase-space. Harmonic and Newtonian centers have maximal degeneracy. The dimension of the orbit (as defined above) is one, and the motion is periodic. Other centers have only two invariants, the motion is not periodic (we exclude isolated periodic motions) and the dimension of the orbit is two. Finally, if we have no invariants except the Hamiltonian the dimension is three. Ergodic motion, for which the orbit comes arbitrarily close to any point on a hypersurface in phase space given by H = constant, thus has no degeneracy. Degeneracy and "ergodicity" are complementary concepts. In §2.2.3 we considered the relation between invariants and separability of the time-independent Hamilton-Jacobi equation. Any additive separation of the Hamilton-Jacobi equation is equivalent to a multiplicative separation of the Schrödinger equation (cf. also §3.2). To obtain a variety of examples we consider three-dimensional motions. If the degree of degeneracy is one, that is there is only one invariant (in addition to the Hamiltonian), the (time-independent) Hamilton-Jacobi equation separates once so that S may be expressed as the sum of one function of one variable and one function of two variables. Examples are potentials cylindrical around the x-axis with m_1 as the only invariant The potential given by (2.45a), and V_1 and V_5 of Table 2.1 have degeneracy two (since m_1 is also invariant). The Hamilton-Jacobi equation then separates completely and the problem is completely integrable. (See, for instance, the end §48 of [23]). A central potential, excepting the Newtonian and Hamiltonian centers, has degeneracy three. The most famous example is the Newtonian potential with an Einsteinian correction. The motion of the perihelion causes the orbit to fill a two-dimensional region in configuration space and in phase space. Finally, Newtonian and harmonic centers have degeneracy four. Complete separation of variables is possible in two independent systems of coordinates, and bounded orbits are always one-dimensional and periodic.

2.5.3 Invariants of Superimposed Potentials.

In §2.5.1 the approach of limit processes was used to obtain new potentials admitting higher symmetries from known ones. In this subsection a different and in a sense complementary approach is used. This is expressed in the form of the following lemma:

Lemma 2.5. Let the potentials V_1 and V_2 each have quadratic invariants. Then if the strictly quadratic terms of each invariant are the same, the combined potential V_1+V_2 has also a quadratic invariant given by the sum of the two invariants minus the common part.

Proof. This is a consequence of the general form of an invariant quadratic in the momenta and the linearity of equations (δ) in §2.4.1. Let us give some illustrations of the above lemma.

Example 1.

The potential $V = \hat{\alpha}\rho^2 + \hat{\beta}\rho_*^2$, where ρ_* is given by (2.45) has the invariant (see 2.36b)

$$A = m_3\left[\left(x - \frac{2x_*\hat{\beta}}{\hat{\alpha}+\hat{\beta}}\right)p_2 - yp_1\right] - \frac{2x_*^2\hat{\beta}^2y^2}{\hat{\alpha}+\hat{\beta}}$$

The potential $V_1 = \alpha\rho^{-1} + \beta\rho_*^{-1}$ has the invariant (see (2.37) with $\alpha_0 = 0$)

$$C_1 = m_3[(x-x_*)p_2 - yp_1] - \beta x_* x\rho^{-1} + \gamma x_*(x-x_*)\rho_*^{-1}.$$

Then according to the lemma 2.5 we can superimpose V and V_1

without breaking the symmetry if $x_* = 2x_*\hat{\beta}/(\hat{\alpha} + \hat{\beta})$, that is $\hat{\alpha} = \hat{\beta}$. Therefore, if we have two Newtonian centers and add two harmonic centers at the same locations the latter must be of equal strength if we want to have a dynamic invariant. The invariant of the superimposed potential is given by equation (2.46), where $\alpha/2 = \hat{\alpha}$.

Example 2.

Consider the potential V_3 in Table 2.1. The strictly quadratic part of the corresponding invariant C_2 is m_3^2. Now consider any central potential $g(\rho)$; this potential has m_3^2 as an invariant. Therefore to the potential V_3 we can add any potential depending only on ρ without breaking the symmetry.

2.5.4 Some New Cubic Invariants.

In Appendix II we investigate the existence of invariants cubic in the momenta, admitted by potentials of the general form $V = V(x^2 + \nu y^2)$, ν = constant. The following potentials and corresponding invariants are found:

Potential	Invariant
$\frac{1}{2}x^2 + \frac{9}{2}y^2$,	$p_1^2 m + 3x^2 y p_1 - \frac{1}{3}x^3 p_2$,
$\frac{1}{2}x^2 + \frac{1}{18}y^2$,	$p_2^2 m + \frac{1}{27}y^3 p_1 - \frac{1}{3}xy^2 p_1$,
$(x^2 - y^2)^{-\frac{2}{3}}$,	$(p_1^2 - p_2^2)m - 4(x^2 - y^2)^{-\frac{2}{3}}(yp_1 + xp_2)$.

3.1 INTRODUCTION

In Chapter II we considered two ways of describing the motion of a particle: 1) Hamilton's equations and 2) The Hamilton-Jacobi equation. The group properties of Hamilton's equations can be analyzed with the aid of Lie point groups, whereas the group analysis of the Hamilton-Jacobi equation necessitates the introduction of a special type of LB groups, which are equivalent to Lie tangent groups. In this chapter we describe the motion of a particle using the Schrödinger equation. In this case an invariant is expressed by an admissible LB operator; the equivalence of this description to the customary one follows from the correspondence rule discussed in §1.5.4. The main goal of this chapter is to relate the group structure of the Hamilton-Jacobi equation to that of the Schrödinger equation.

In discussing the group structure connected with the existence of invariants we shall use the word "group" somewhat loosely: It may refer to a global group, but usually refers to the corresponding infinitesimal operator, that is a Lie algebra. This last term is taken in the technical sense of a linear space (which may be infinite-dimensional) for which a bilinear skew product (commutation) obeying Jacobi's identity is defined. Thus there are Lie algebras of LB operators although such operators are not of a type envisaged by Lie. In all Lie algebras considered there is also an associative (often noncommutative) product defined. Thus we may speak of the (associative) enveloping algebra of a Lie algebra. Invariants linear† in the momenta are usually associated

†This term is well-defined provided we use as canonical coordinates either Cartesian coordinates and their conjugate momenta or coordinates obtained therefrom by an extended point transformations.

with geometric symmetries and the nontrivial nonlinear ones with dynamic symmetries. For lack of a better alternative we still use this unfortunate terminology. Actually a linear invariant depends on a geometrical symmetry which can be expressed in configuration space; the other invariants may often be related to geometrical symmetries in other spaces whose physical meaning may not be immediately obvious (see for instance [30]).

As is well known, symmetries, or invariants, are related to the possibility of separating variables. In §3.2 we show the correspondence between a (partial) multiplicative separation of an arbitrary linear homogeneous equation and an admissible LB operator, and point out how this result may be used. In §3.3 we give an algorithm for constructing a LB operator, admitted by the Schrödinger equation, from a LB operator, at most cubic in the momenta, admitted by the Hamilton-Jacobi equation. This is of practical interest: In [20] the general form of quadratic invariants for the stationary Schrödinger equation with a general value of the energy E was determined. The corresponding problem for the Hamilton-Jacobi equation is much simpler to solve, as was shown in §2.3.1. Also, the more difficult (although presumably less important) problem for the case $E = 0$ was solved in §2.3.1. The correspondence rule of §3.3 immediately transfers these results to the Schrödinger equation. In §3.4 we discuss relations between quantum mechanics, as expressed by the Schrödinger equation, and classical mechanics, as expressed by Hamilton's equation or, equivalently, by the Hamilton-Jacobi equation. First we look for a mapping between classical observables and quantum-mechanical observables; the former are expressed as functions in phase space and the latter as LB operators. The Weyl transform gives a mapping between the two sets of objects. It can be used to describe general canonical transformations in quantum mechanics and Weyl shows that the equation of motion, that is the Schrödinger equation or equivalent formulations, may be expressed in a form which does not depend on any specific canonical coordinates. Thus Weyl has shown that an essential fact of classical mechanics is also true for quantum mechanics. Weyl is not interested in finding an isomorphic transform. In fact it can be shown that this is impossible. However, Weyl's transform justifies the basic assumption of quantum mechanics that commutation relations between any set of canonical variables are isomorphically mirrored by relations between the corresponding operators. Our approach is to show concretely how, starting with a correspondence rule for very simple functions in phase space, the requirement of isomorphism extends the correspondence to more complicated functions. We also use a different method of constructing correspondence rules, namely requiring that classical invariants map into quantum-mechanical invariants. The question arises whether these two methods give the same results and whether they give the Weyl transform. For invariants at most quadratic in the momenta we find that our construction agrees with Weyl's transform, and that the correspondence rule is an isomorphism. For cubic invariants this is no longer true; Weyl's rule does not necessarily take an invariant into an invariant. However, one can sometimes achieve this by modifying Weyl's rule. The case of quantities which are invariant for one value of energy only, which we may normalize to be zero, may not be physically very important but leads to mathematically interesting problems as discussed in §3.4.

We are left with various mathematical questions. Are there variations of Weyl's rule which still solve the problems Weyl posed? In particular, in special cases the requirement of admissibility leads to variations of Weyl's rule. Can this be formulated in a general way? In view of the impossibility of a complete isomorphism, how do we describe the difference in group structure of classical and quantum mechanics? Our method of constructing correspondence rules by a combination of the requirement of isomorphism and the requirement of invariants being mapped on invariants seems fruitful but needs further discussion.

3.2 SEPARATION OF VARIABLES IN ANY LINEAR HOMOGENEOUS EQUATION

The problem of determining dynamical symmetries of the Schrödinger equation was at an early stage related to the problem of separation of variables. For more recent studies of this problem see for instance [33] and [25]. This gave rise to a systematic study of the connection between group theory and separation of variables for various important equations. An account of this research is given by one of the main investigators in this field in [16]. Let us recall very briefly the relevant ideas. Suppose we are given a linear, homogeneous second-order partial differential equation

$$\Omega = 0. \tag{3.1}$$

Together with (3.1) we consider the equation

$$Au = \lambda u \tag{3.2}$$

where A is a second order linear operator. The separable solutions are exactly those solutions of (3.1) which are simultaneously eigenfunctions of (3.2). Here λ is the separation constant. Obviously we do not know a priori the operator A but with every separable coordinate system of equation (3.1) we can easily associate an operator A. There are two cases to be distinguished:

i) The operator A belongs in the enveloping algebra of some Lie algebra G of Lie point operators of equation (3.1).

ii) The operator A does not belong in the enveloping algebra of G.

The separation associated with the first case can be completely explained using Lie point operators. However, for the group-theoretical characterization of the separation associated with the second case the Lie point theory is insufficient. In this section we use LB theory to characterize completely all separable solutions of any linear homogeneous equation:

Lemma 1.1. Let u depend on $\underline{x} = (x_1, \ldots, x_n)$ and y, and let

$$\Omega \equiv L(\underline{x}, u, \underset{1}{u}, \ldots, \underset{m}{u}) + f(\underline{x})[g(y)u + \sum_{j=1}^{m} g_j(y)\, \partial_y^j u] = 0 \qquad (3.31)$$

be any linear, homogeneous equation of order m in $(n+1)$ dimensions, separable in the y coordinate. Here L is linear in u and its derivatives, and

$$\partial_y^j u \equiv \frac{\partial^j u}{\partial y^j},$$

$$\underset{k}{u} \equiv \{u_{i_1 \ldots i_k}\},\ 1 \leq i_1, \ldots i_k \leq n,\ 1 \leq k \leq m .$$

Then the LB operator T,

$$T \equiv (Au)\frac{\partial}{\partial u}, \quad \text{where} \quad A = g(y) + \sum_1^m g_j(y)\,\partial_y^j, \qquad (3.4)$$

is an admissible operator of (3.3).

Proof.

$$T\Omega = L_u(Au) + \sum_1^m L_{\underset{k}{u}}(\underset{k}{Au}) + f(\underline{x})[g(y)Au + \sum_1^m g_j(y)\,\partial_y^j(Au)] .$$

Therefore,

$$T\Omega = g(y)(\Omega - L) + \sum_1^m g_j(y)\,\partial_y^j \Omega + g(y)[uL_u + \sum_1^m \underset{k}{u} L_{\underset{k}{u}}] .$$

However,

$$uL_u + \sum_1^m \underset{k}{u} L_{\underset{k}{u}} = L, \quad \text{hence} \quad T\Omega = g(y)\Omega + \sum_1^m g_j(y)\,\partial_y^j \Omega .$$

Thus $T\Omega\Big|_{\Omega=0} = 0$, and (3.3) admits T.

Equation (3.3) also admits the stretching operator

$$T_0 = \lambda u \frac{\partial}{\partial u}, \quad \lambda = \text{constant}, \tag{3.5}$$

since it is linear and homogeneous. Therefore equation (3.3) admits the LB operator

$$\hat{T} \equiv T - T_0 = (Au - \lambda u)\frac{\partial}{\partial u}, \tag{3.6}$$

where A is defined by (3.4).

Theorem 3.1. If a multiplicatively separable solution for (3.3) of the form $u = \hat{u}(\underline{x})\bar{u}(y)$ exists, then u is an invariant solution of (3.3) under the action of the LB operator $\hat{T}$ defined by (3.6).

Proof. The solution of equation (3.3) invariant under the action of the operator $\hat{T}$ given by (3.6), is specified by the simultaneous validity of equation (3.3) and of

$$Au - \lambda u = 0, \tag{3.7}$$

(see Theorem 1.4). However, by definition of a separable solution/it also satisfies equation (3.7) for some constant λ. Q.E.D.

From the above it is clear that every separable solution of (3.3) is invariant under a LB operator. If the separable coordinates are known, this operator is found by inspection. This can be quite useful in obtaining admissible operators and hence conservation laws provided we know the separable coordinates. Conversely, knowing an admissible operator the corresponding separable coordinates can be found.

Let us now give some illustrations of the above theorem:

Example 1. Consider the Helmholtz equation

$$u_{xx} + u_{yy} - Ku = 0. \tag{3.8}$$

It obviously admits the operators

$$\hat{X}_1 = \frac{\partial}{\partial x}, \quad \hat{X}_2 = u\frac{\partial}{\partial u}. \tag{3.9}$$

Therefore, it also admits the LB operator

$$X_1 = (u_{xx} - \lambda u)\frac{\partial}{\partial u} \equiv A_1\frac{\partial}{\partial u}, \tag{3.10}$$

which belongs in the enveloping algebra of the Lie point operators (3.9) (i.e. it is of type i) defined above). The solution of equation (3.8) invariant under the action of X_1 (which is obtained by solving (3.8) together with $A_1 = 0$), is the separable solution in cartesian coordinates of equation (3.8).

Example 2. Consider the Schrödinger equation for the hydrogen atom,

$$\frac{1}{2}u_{xx} + \frac{1}{2}u_{yy} + (\frac{1}{\rho} - E)u = 0. \tag{3.11}$$

The LB operator

$$X_2 = (xu_{yy} - yu_{xx} - \frac{1}{2}u_x + \frac{xu}{\rho})\frac{\partial}{\partial u} \equiv A_2\frac{\partial}{\partial u}, \tag{3.12}$$

is an admissible operator of equation (3.11); it does not belong in the enveloping algebra of any Lie algebra of equation (3.11) (i.e. it is of type ii)). The solution of equation (3.11) invariant under the action of X_2 is the separable solution in parabolic coordinates of equation (3.11). This is easily seen by writing equation $A_2 = 0$ in parabolic coordinates: Let

$$\begin{aligned} x &= \frac{1}{2}(\xi^2 - \eta^2), \\ y &= \xi\eta. \end{aligned} \tag{3.13}$$

Then $A_2 = 0$ becomes

$$u_{\xi\xi} - u_{\eta\eta} - (2K)(\xi^2 - \eta^2)u = 0. \tag{3.14}$$

Solving equation (3.14) together with equation (3.11) (written in parabolic coordinates), we obtain the parabolic solution.

In §2.4 we saw that the Hamilton-Jacobi equation with potential $1/\rho$ has two distinguished properties: i) it possesses an additional (to angular momentum) conserved quantity (the Runge-Lenz vector) and ii) it also separates in parabolic coordinates. Both of these properties are a consequence of the existence of a dynamical symmetry expressed by some LB operator. This LB operator is mirrored in quantum mechanics to the operator X_2 (see §3.3) which leads to the conservation of the quantum mechanical analogue of the Runge-Lenz vector and to the separation of the Schrödinger equation in parabolic coordinates.

Example 3. We now consider Tricomi's equation

$$xu_{yy} - u_{xx} = 0. \tag{3.15}$$

Looking for second order LB operators we obtain (see Appendix IV):

$$X_i = A_i \frac{\partial}{\partial u}, \quad i = 0,1,\ldots,7,$$

where

$$A_0 = 1,\ A_1 = u,\ A_2 = u_y,\ A_3 = 4xu_x + 6yu_y,$$
$$A_4 = yu + 4xyu_x + (3y^2 + \tfrac{4}{3}x^3)u_y,\ A_5 = 4xu_{xy} +$$
$$6yu_{yy},\ A_6 = u_{yy},\ A_7 = 4xyu_{xy} + (3y^2 + \tfrac{4}{3}x^3)u_{yy} -$$
$$\tfrac{2}{3}xu_x \tag{3.16}$$

Note that the operators X_i, $i = 5,6,7$, belong in the enveloping algebra of the Lie point operators X_j, $j = 0,1,2,3,4$. The solution of equation (3.15) invariant under the LB operator

$$X = \tfrac{2}{3}X_3 + X_6 + X_7 \equiv A\frac{\partial}{\partial u}, \tag{3.17}$$

is the separable solution in elliptic coordinates of equation (3.15). This solution was recently obtained by Cole [40]: Let $\tau = 2/3(-x)^{3/2}$ and equation (3.15) becomes

$$u_{yy} + u_{\tau\tau} + \frac{1}{3\tau} u_{\tau} = 0. \tag{3.18}$$

Now let

$$\begin{aligned} \tau &= \sinh\xi \, \sin\eta, \\ y &= \cosh\xi \, \cos\eta, \end{aligned} \tag{3.19}$$

and write equations (3.18) and $A = 0$ (where A is defined by (3.17)) in ξ and η variables. This yields the sought separable solution.

3.3 RELATIONS BETWEEN ADMISSIBLE LB OPERATORS OF THE SCHRÖDINGER EQUATION AND THOSE OF THE HAMILTON-JACOBI EQUATION

3.3.1 Operators Quadratic in the Momenta

We shall consider a time-independent Schrödinger equation for a single particle of unit mass,

$$U \equiv \frac{1}{2} \sum_{j=1}^{3} u_{x_j x_j} - V(\underline{x})u = 0 \quad , \tag{3.20}$$

and its associated eikonal equation (Hamilton-Jacobi equation)

$$H \equiv \frac{1}{2} \sum_{j=1}^{3} S^2_{x_j} + V(\underline{x}) = 0 \, . \tag{3.21}$$

Theorem 3.2. Equation (3.21) admits the operator

$$Y = \sum_{j,k} \left[a_j(\underline{x})\, S_{x_j} + b_{jk}(\underline{x})\, S_{x_j} S_{x_k} + c(\underline{x}) \right] \frac{\partial}{\partial S} \, ,$$

where $b_{jk} = b_{kj}$, iff (2.1) admits the operator

$$X = \sum_{j,k} \left[(a_j + \bar{a}_j)\, u_{x_j} + b_{jk}\, u_{x_j x_k} + (\bar{c} - c)\, u \right] \frac{\partial}{\partial u} \, .$$

Here

$$\bar{a}_j = \sum_{k \neq j} \frac{\partial b_{jk}}{\partial x_k} \quad , \quad 2\bar{c} = \frac{\partial a_1}{\partial x_1} + F \, ,$$

$$\frac{\partial F}{\partial x_1} = \frac{\partial^3 b_{23}}{\partial x_1 \partial x_2 \partial x_3} \quad ,$$

and $\quad \dfrac{\partial F}{\partial x_2}$ and $\dfrac{\partial F}{\partial x_3}$

are obtained by cyclic permutation. Note that F itself is determined only within a constant. This constant is irrelevant since a linear homogeneous equation for u admits the operator $u\partial/\partial u$.

Proof. In §2.3.1 the conditions on a_j, b_{jk} and c for Y to be an admissible operator of (3.21) were completely determined. The results needed for the proof of the present theorem are reviewed below: An operator of the general form $A(x_\ell, S_{x_\ell})\,\partial/\partial S$ is an admissible operator of (3.21) iff

$$\sum_j \left[\frac{\partial H}{\partial x_j} \frac{\partial A}{\partial S_{x_j}} - \frac{\partial H}{\partial S_{x_j}} \frac{\partial A}{\partial x_j} \right] \Bigg|_{H=0} = 0$$

Applying this result to the operator Y we find,

$$(b_{11} - b_{33})_{x_1} = 2b_{13\,x_3} \qquad (3.22a)$$

$$(b_{11} - b_{33})_{x_3} = -2b_{13\,x_1} \qquad (3.22b)$$

(by cyclic permutations) (3.22c, d, e, f)

$$b_{12\,x_3} + b_{23\,x_1} + b_{31\,x_2} = 0 \qquad (3.22g)$$

$$a_{1\,x_1} = a_{2\,x_3}\,, \quad a_{1\,x_2} = -a_{2\,x_1}\,, \qquad (3.23a, b)$$

(by cyclic permutations) (3.23c, d, e, f)

$$c_{x_k} = 2\sum_j b_{kj}V_{x_j} + 2Vb_{kk_{x_k}}, \quad k = 1, 2, 3, \tag{3.24a, b, c}$$

$$\sum_j a_j V_{x_j} + 2\alpha_{3_{x_3}} V = 0. \tag{3.25}$$

First we find some corollaries of the above equations. The compatibility conditions for (3.22) are

$$b_{12_{x_1x_1}} + b_{12_{x_2x_2}} = 0 \text{ , etc. by cyclic permutation,} \tag{3.26a, b, c}$$

$$b_{12_{x_2x_2}} - b_{13_{x_2x_3}} + b_{23_{x_1x_3}} = 0 \text{ , etc. by cyclic permutation.} \tag{3.27a, b, c}$$

Using (3.22f), (3.26) and (3.27) we find,

$$b_{12_{x_1x_1x_3x_3}} = b_{12_{x_2x_2x_3x_3}} = 0, \text{ etc. by cyclic permutation.} \tag{3.28}$$

The compatibility equations of (3.23) imply

$$\sum_j (\nabla^2 a_j)_{x_j} = 0 \tag{3.29}$$

where ∇^2 is the Laplacian operator in three dimensions.

Next we derive some formulas for an admissible operator of the Schrödinger equation. An operator of the form

$$X = \Big(\sum_{j,k} A_j u_{x_j} + B_{jk} u_{x_j x_k} + Cu\Big)\frac{\partial}{\partial u} ,$$

is an admissible operator of (3.20) iff

$$XU\Big|_{U=0} = 0.$$

This leads to the following set of equations:

$$(B_{11} - B_{33})_{x_1} = 2B_{13\,x_3} \quad , \tag{3.30a}$$

$$(B_{11} - B_{33})_{x_3} = -2B_{13\,x_1} \quad , \tag{3.30b}$$

(by cyclic permutation) , (3.30c, d, e, f)

$$B_{12\,x_3} + B_{23\,x_1} + B_{31\,x_2} = 0 \quad , \tag{3.30g}$$

$$\left.\begin{aligned}
&2A_{1\,x_1} - 2A_{3\,x_3} + \nabla^2 (B_{11} - B_{33}) = 0 \quad ,\\
&2A_{2\,x_2} - 2A_{3\,x_3} + \nabla^2 (B_{22} - B_{33}) = 0 \quad ,\\
&A_{2\,x_1} + A_{1\,x_2} + \nabla^2 B_{12} = 0 \quad ,\\
&A_{3\,x_1} + A_{1\,x_3} + \nabla^2 B_{13} = 0 \quad ,\\
&A_{3\,x_3} + A_{2\,x_3} + \nabla^2 B_{23} = 0 \quad ,
\end{aligned}\right\} \tag{3.31}$$

$$C_{x_k} = -2\sum_j B_{kj}V_{x_j} - 2VB_{kk_{x_k}} + \tfrac{1}{2}\nabla^2 A_k, \qquad k = 1, 2, 3 \tag{3.32a, b, c}$$

$$2\sum_{j,k}\left(A_j V_{x_j} + 2A_{3\,x_3}V + V_{x_j x_k} B_{jk} + 2V_{x_j} B_{jj_{x_j}}\right) + 2V\nabla^2 B_{33} + \nabla^2 C = 0. \tag{3.33}$$

We may now establish relations between the operators Y and X. The set (3.30) is identical with the set (3.22) if one replaces B_{jk} by b_{jk}, and the homogeneous part of the set (3.31) is equivalent to the set (3.23) if one replaces A_j by a_j. Therefore, we propose $B_{jk} = b_{jk}$ and $A_j = a_j + \overline{a}_j$. Using (3.25) to simplify (3.31) we find a particular solution

$$\overline{a}_j = \sum_k b_{jk_{x_k}} , \qquad j = 1, 2, 3 . \tag{3.34}$$

Now (3.32) becomes

$$C_{x_j} = -c_{x_j} + \tfrac{1}{2}\left(b_{23_{x_1 x_2 x_3}} + a_{1_{x_1 x_1}}\right), \quad j = 1, 2, 3. \tag{3.35, a, b, c}$$

Before proceeding further we must prove that the set (3.35) is compatible, i.e., that $C_{x_j x_k} = C_{x_k x_j}$. The compatibility of the terms c_{x_i}, $a_{j_{x_i x_i}}$ is obvious. Furthermore,

$$b_{23_{x_1 x_2 x_3 x_2}} - b_{13_{x_1 x_2 x_3 x_1}} = b_{12_{x_3 x_3 x_2 x_2}} = 0$$

where we have used (3.26), (3.27), and (3.28). The set is thus compatible and

$$C = -c + \tfrac{1}{2}\left(a_{1_{x_1}} + F\right) \tag{3.36}$$

The equations for the first derivatives of F follow from (3.35). Finally, using (3.34) and (3.36) in (3.33), and noting that from (3.22f) and (3.29)

$$\sum_j \left(\nabla^2 A_j\right)_{x_j} = \left(B_{12_{x_3}} + B_{23_{x_1}} + B_{31_{x_2}}\right)_{x_1 x_2 x_3} + \sum_j \left(\nabla^2 a_j\right)_{x_j} = 0 ,$$

we see that (3.33) reduces to (3.25). This concludes the proof of the theorem.

We shall now derive a corollary from Theorem 3.2 which gives correspondence between the operators Y and X in a more transparent form. In §2.3.1 it was shown that the most general quadratic operator admitted by (3.21) is the sum of a linear operator Y_1, given by equation (2.20), and a quadratic operator without linear terms Y_2, given by equation (2.21). Using these results and the notation introduced by equations (2.19) we may now state the corollary.

Corollary of Theorem 3.2. The correspondence rule of Theorem 3.2 may now be expressed as the following substitutions

$$Y = \left[A + c(\underline{x})\right] \frac{\partial}{\partial S} \rightarrow X = \left[\overline{A} - c(\underline{x})\right] u \frac{\partial}{\partial u} \tag{3.37a}$$

where $\overline{A}$ is obtained from A as follows

$$P_j \rightarrow \overline{P_j} = \frac{\partial}{\partial x_j}, \; M_j \rightarrow \overline{M}_j = \sum_{k,\ell} \varepsilon_{jk\ell}\, x_k \overline{P}_\ell$$
$$\Delta \rightarrow \overline{\Delta} = \frac{1}{2} + \sum_j x_j \overline{P}_j \,, \; K_j \rightarrow \overline{K}_j = x_j + 2\sum_j x_j x_k \overline{P}_k - r^2 \overline{P}_j \tag{3.37b}$$

If A and B are two of the quantities P_j, M_j, etc., then $AB \rightarrow 1/2\{\overline{A}, \overline{B}\} \equiv 1/2\left(\overline{A}\,\overline{B} + \overline{B}\,\overline{A}\right)$ with the following two exceptions

$$\Delta^2 \rightarrow \overline{\Delta}^2 - \frac{1}{4}\,, \; P_j K_\ell \rightarrow \frac{1}{2}\{\overline{P}_j\,, \overline{K}_\ell\} - \frac{1}{2}\delta_{j\ell}. \tag{3.37c}$$

We start with $P_j \rightarrow \overline{P}_j$. However, since the differential operator $\overline{P}_j$ no longer commutes with functions of the x_j the correct ordering has to be found. The two basic rules are to let the functions of x_j precede the differential operator (in the linear case) and to use the anticommutator for purely quadratic terms. The same function $c(\underline{x})$ is used in Y and X. The additional terms 1/2 and x_j appearing in $\overline{\Delta}$ and K_j

contribute to $\overline{c}(\underline{x})$, as shown in the examples below. The basic reason for these terms is that the stretching operator (corresponding to Δ) and the special conformal operators K_j leave the Laplacean of u invariant only if the u is suitably modified. Note that the formula for the quantum mechanical analogue of the Runge-Lenz vector (see, for instance, [31]) becomes a special case of our use of anticommutators.

Rather than give a general proof of the corollary, we shall give two special examples. First consider the special case

$$Y = \left[\Delta + c(\underline{x})\right] \frac{\partial}{\partial x}$$

Then $a_j = x_j$, $b_{jk} = 0$, $\overline{a}_j = 0$, $2\overline{c} = a_{1_{x_1}} = 1$. Thus according to Theorem 3.2

$$X = \left(\sum_j a_j u_{x_j} + \overline{c}\,u - c(\underline{x})u\right)\frac{\partial}{\partial u}$$

$$= \left(\sum_j x_j \frac{\partial}{\partial x_j} + \frac{1}{2} - c(\underline{x})\right) u \frac{\partial}{\partial u} = \left(\overline{\Delta} - c(\underline{x})\right) u \frac{\partial}{\partial u}$$

Secondly, let

$$Y = \left(M_1 K_2 + c(\underline{x})\right) \frac{\partial}{\partial S}$$

Using the explicit form of Y we find from (3.34)

$$\overline{a}_1 = -x_1 x_3\,,\ \overline{a}_2 = -4x_2 x_3\,,\ \overline{a}_3 = \frac{1}{2}\left(5x_2^2 - 3x_3^2 - x_1^2\right),$$

and from (3.35) and (3.36)

$$F_{x_1} = F_{x_2} = 0\,,\ F_{x_3} = -1,\ 2\overline{c} = -x_3\,.$$

We can now write X from Theorem 3.2. On the other hand, a straightforward calculation of $1/2\{\overline{M}_1, \overline{K}_2\}$, which we advise the reader to carry out, shows that the substitutions described in (3.37) give the same result.

General value of E. As shown in §2.3.1, if $V(\underline{x})$ is replaced by $\hat{V}(\underline{x}) - E$ where $\hat{V}$ is fixed and the energy E may take any value consistent with the problem the general form of Y reduces to

$$Y = \sum_{j,k} \left(\alpha_{jk} P_j P_k + \beta_{jk} P_j M_j + \gamma_{jk} M_{jk} + \kappa_j P_j + \lambda_j M_j + c(\underline{x}) \right) \frac{\partial}{\partial S} . \tag{3.38}$$

The corresponding operator for X is then

$$X = \sum_{j,k} \left(\alpha_{jk} \overline{P}_j \overline{P}_k + \frac{1}{2} \beta_{jk} \{\overline{P}_j, \overline{M}_k\} + \frac{1}{2} \gamma_{jk} \{\overline{M}_j, \overline{M}_k\} + \kappa_j \overline{P}_j + \lambda_j \overline{M}_j - c(\underline{x}) \right) u \frac{\partial}{\partial u} . \tag{3.39}$$

3.3.2 Operators Cubic in the Momenta

In this subsection we only consider the more interesting case of arbitrary values of E, we also restrict ourselves to two dimensions. The most general invariant of the Hamilton-Jacobi equation, cubic in the momenta, for the above case, was given in §2.3.2 by equation (2.27c). To this invariant corresponds a quantum mechanical one, given by equation (3.40) below iff equation (3.42) holds.

Theorem 3.3. To the invariant I_3 defined by (2.27c) of §2.3.2 corresponds the quantum-mechanical invariant

$$X = \left(d_{111} u_{xxx} + d_{222} u_{yyy} + d_{112} u_{xxy} + d_{221} u_{yyx} + \sum_{j,k} (B_{jk} u_{x_j x_k} + A_j u_{x_j}) + Cu\right) \frac{\partial}{\partial u} , \tag{3.40}$$

where

$$B_{11} = \tfrac{1}{2} d_{112_y} + b_{11}, \quad B_{22} = \tfrac{1}{2} d_{221_x} + b_{22}, \quad B_{12} = \tfrac{1}{2} d_{112_x} + b_{12}, \tag{3.41}$$

$$A_j = -\hat{a}_j + a_j, \qquad C = \tfrac{1}{2} A_{1_x} + \tfrac{1}{2} A_{2_y},$$

iff

$$d_{111} V_{xxx} + d_{222} V_{yyy} + d_{112} V_{xxy} + d_{221} V_{yyx} + (4a_1 - 2d_{112\,xy}) V_x + (4a_2 - 2d_{221\,xy}) V_y + 4 \sum_{j,k}' b_{jk} V_{x_j x_k} = 0 . \tag{3.42}$$

In (3.41), the functions $a_j(x)$ and $b_{jk}(x)$ are defined by (2.27a,b). For example, $a_1 = \alpha - \beta y$, α and β arbitrary constants, etc.

<u>Proof.</u> The proof although cumbersome is similar to that of Theorem 3.2 and is omitted. Let us give an illustration.

<u>Example.</u> We now consider a non-trivial invariant, cubic in the momenta. Consider the special case of a non-isotropic oscillator with potential

$$V = x^2 + \frac{1}{9} y^2 .$$

Then

$$A_3 = m_3 p_2^2 + \frac{y^3}{27} p_1 - \frac{xy^2}{3} p_2$$

is a classical invariant. (See Appendix II). Therefore

$$d_{222} = x, \quad d_{221} = -y, \quad \text{all other } d_{jk\ell} = 0,$$

$$\hat{a}_1 = \frac{y^3}{27}, \quad \hat{a}_2 = -\frac{xy^2}{3} .$$

In order to satisfy equation (3.42) take

$$\alpha_1 = \frac{1}{2} d_{112_{xy}} = 0, \quad \alpha_2 = \frac{1}{2} d_{221_{xy}} = 0,$$

$$b_{jk} = 0 .$$

Equations (3.41) indicate that

$$B_{11} = 0, \quad B_{22} = 0, \quad B_{12} = 0,$$

$$C = -\frac{1}{2}\hat{a}_{1_x} - \frac{1}{2}\hat{a}_{2_y} = \frac{1}{3}xy.$$

Therefore to the classical invariant A_3 corresponds the quantum invariant

$$\overline{A}_3 = xu_{yyy} - yu_{xyy} + \frac{1}{3}xyu - \frac{y^3}{27}u_x + \frac{xy^2}{3}u_y.$$

The question of general correspondence rules between classical dynamical variables and quantum-mechanical observables will be discussed further in §3.4.

3.4 AN ISOMORPHIC CORRESPONDENCE, WEYL'S TRANSFORM AND THEIR LIMITATIONS

3.4.1 The Problem of an Isomorphic Correspondence

We shall now discuss correspondences between classical quantities and quantum-mechanical quantities. The former will be represented by dynamical variables, that is functions of the x_j and p_j (but not of time). The latter are the quantum mechanical observables which normally are expressed as Hermitian differential operators. Since infinitesimal unitary operators are skew-Hermitian and since the commutator of two Hermitian operators is skew the number $i = \sqrt{-1}$ occurs frequently in quantum mechanical formulas. However, as will be seen, this can be avoided, see also [34, CH. 16]. Furthermore, we shall express all operators in Lie-Bäcklund form, the correspondence rule 1.49, given in §1.5.4 establishes the connection with other forms used in the literature.

In the Heisenberg approach to quantum mechanics it is assumed that to a set of classical canonical variables (q_j, p_j) correspond quantum mechanical operators (Q_j, P_j) such that the classical commutation rules

$$[q_j, p_k] = \delta_{jk}, \quad [q_j, q_k] = [p_j, p_k] = 0, \tag{3.43a}$$

are mirrored by the rules

$$[Q_j, P_k] = c\delta_{jk}I, \quad [Q_j, Q_k] = [P_j, P_k] = 0. \tag{3.43b}$$

Here c is a constant and I is the identity operator. For a discussion of this approach, see for instance [35, Ch. IV].

This basic correspondence principle just described was assumed on physical grounds. It involves a <u>limited</u> isomorphism.

It assumes that there is a mapping from functions in phase space to a space of operators which preserves certain commutation relations. Note that, from a mathematical point of view, the correspondence rule is incomplete: Assume that we have made the correspondence for rectangular coordinates $q_j = x_j$ and their conjugate momenta p_k and then make a canonical transformation to variables q'_j, p'_k. A constructive formula giving the corresponding Q'_j P'_j as functions of Q_j and P_j is needed. Also it must be shown that assuming (3.43b) for both systems is consistent. Weyl [36, II.10 and IV 15] made a profound investigation of some relevant mathematical problems: 1) Assume that the operators corresponding to x_j and p_j have been found. Weyl gave a rule, called the Weyl transform, for constructing an operator corresponding to any (reasonable!) function of x_j and p_j. 2) He showed that applying this rule to any classical canonical transformation one gets a quantum-mechanical canonical transformation which has the desired property that the equation of motion (in this case the Schrödinger equation) can be formulated in a coordinate-free way. The relations (3.43b) follow for any set of canonically conjugate variables.

Weyl only shows a limited isomorphism, and he never claims that his transform is a full isomorphism†. In fact, Van Hove [37] shows that a full isomorphism is not possible. Here we shall show this same result in an elementary way. Our approach will be to assume a correspondence rule for very simple dynamical variables and then try to extend them to more complicated ones so that isomorphism is retained as long as possible. The rules so obtained will be identical with Weyl's. The Weyl transforms of certain simple variables are given explicitly in [38]. The discussion by Hermann [34, Ch. 16] is a useful reference; we agree with him that there are many mathematical problems in elementary quantum

† Dirac [35, p.87] considers the assumption of a full isomorphism but quickly retreats to physically safe ground, namely to (3.43).

mechanics to which mathematicians have paid insufficient attention.

Step A. (Linear Quantities). As stated above we shall proceed from simple cases to successively more complicated ones. One measure of simplicity is the degree in the momentum variable. We note that the difficulties arise because x_j and p_j do not commute. For this reason a term such as p_1p_2 may for practical purposes be counted as linear. Thus many of the essential points can be shown by considering only one pair of conjugate variables (x, p). Using this simplification, we may write the correspondence rules for linear operators as

$$f(x) \leftrightarrow f(x)u\frac{\partial}{\partial u}, \qquad p \leftrightarrow u_x\frac{\partial}{\partial u}, \qquad g(x)p \leftrightarrow (g(x)u_x + \frac{1}{2}g'(x)u)\frac{\partial}{\partial u}. \qquad (3.44a, b, c)$$

The second rule is, of course, only a very special case of the third. However, it is written out explicitly because the third rule may be derived if we assume the first two rules and isomorphism. The correspondence (3.44) has all the desired properties: The linear quantities (classical and quantum mechanical) are closed under commutation and are hence Lie algebras. The correspondence (3.44) is a (complete) isomorphism. If one quantity happens to be an invariant then so is the corresponding quantity. Weyl's transformation rule also gives (3.44).

Step B. We now add p^2 to the linear quantities. Classically, we find

$$[g(x)p, p^2] = 2g'(x)p^2 \equiv h(x)p^2. \qquad (3.45)$$

Thus, we no longer have a closed Lie algebra. We require a limited isomorphism in the sense that one application of the commutator to any linear combinations of the basic set $f(x)$, $g(x)p$,

p^2 should give an isomorphism (repeated application of the commutator would give cubic quantities and are not considered at this step.) The following correspondence rule leads to an isomorphism

$$h(x)p^2 \leftrightarrow (hu_{xx} + h'u_x + \frac{1}{4}h''u)\frac{\partial}{\partial u}. \quad (3.46)$$

This includes as a special case $p^2 \leftrightarrow u_{xx}\partial/\partial u$. Conversely, assuming the special rule and requiring isomorphism leads to the general rule (3.46)

Invariants will be considered after the next step.

Step C. The function $h(x)p^2$ was not among our basic set of quantities $f(x)$, $g(x)p, p^2$ but was obtained from them by commutation. We now add $h(x)p^2$ to our collection, or rather generalize p^2 to $h(x)p^2$. Any commutator of a linear combination of the first three quantities will be a linear combination of the augmented set and the correspondence rules yield an isomorphism. If we now let $h(x)p^2$ be a factor in a commutator with the first three, in particular if we form $[p^2, h(x)p^2]$ and require isomorphism we are led to the rule (3.47) below. However, if we also consider $[g(x)p, h(x)p^2]$ we see that (3.47) gives an isomorphism iff $g''(x) = 0$. Thus adding the rule

$$k(x)p^3 \leftrightarrow (ku_{xxx} + \frac{3}{2}k'u_{xx} + \frac{3}{4}k''u_x + \frac{1}{8}k'''u)\frac{\partial}{\partial u} \quad (3.47)$$

to the previous rules we have isomorphisms for commutators of linear combinations of the set

$$f(x),\ p,\ xp,\ h(x)p^2. \quad (3.48)$$

The rule (3.47) is still a special case of Weyl's transform. Note that in generalizing p^2 to $h(x)p^2$ we have to specialize the function $g(x)p$ considered in Step B to the two cases p or xp.

Even with the specialization mentioned the rules obtained are useful for discussing the invariants. The most general quadratic invariant in two dimensions is, for a general value of E, a linear combination of p_1^2, p_2^2, m_3^2, m_3p_1, m_3p_2, $c(\underline{x})$ (see §2.3.1). Let us consider for instance m_3^2 in detail. It is $x^2p_2^2 + y^2p_1^2 - 2xyp_1p_2$. As remarked earlier only canonically conjugate variables can cause trouble. Thus the first two terms are essentially of the form p^2 and the second term of the form xp. The set (3.48) thus gives rise to all possible quadratic invariants, including the Hamiltonian. Thus, at this state our correspondence rules still take invariants into invariants as seen by using Lemma 1.3 of §1.5.2. Note that for this to be true the specialization $g(x)p$ to p or xp turned out to be harmless whereas it is essential to keep the function $f(x)$ in full generality.

<u>Step D.</u> Adding the function $k(x)p^3$ to the basic set (3.48) we obtain the set

$$f(x),\ p,\ xp,\ h(x)p^2,\ k(x)p^3. \tag{3.49}$$

Taking the commutator of linear combinations of the old set with $k(x)p^3$ we obtain a correspondence rule for $l(x)p^4$ which agrees with Weyl's rule (an explicit formula, in different notation from ours, is given in [38]). However, in order to obtain an isomorphism it is necessary to specialize the basic set by the requirement $f''' = 0$. This restriction is necessary even if we specialize $h(x)p^2$ to p^2. It is clear from the results of §2.3.2 that the general cubic invariant is not a linear combination of the set (3.49) with $f(x)$ restricted to be at most quadratic. Thus we cannot use Lemma 1.3 of §1.5.2 to derive admissibility. (According to

[39], if one considers the basic set (x, x^2, xp, p^2) and their Weyl transforms, and then forms the commutators with the Weyl transform of an arbitrary function $F(x,p)$, an isomorphism is obtained.)

3.4.2 Weyl's Transform Versus the Rules Derived in §3.3

A. Arbitrary value of E.

α. Operators linear and quadratic in the momenta.

The most general classical invariant is given by (3.38) and the corresponding quantum invariant by (3.39). The latter agrees with both the construction given above, based on the requirement of an isomorphism (see equations (3.44), (3.45), (3.46)), as well as with the results obtained using the Weyl's transform. Therefore, for invariants at most quadratic, the "quantizations" obtained through Weyl's transform, the requirement of an isomorphism and our rule are identical.

β. Operators cubic in the momenta.

It was shown in §3.4.1 that we cannot extend the isomorphism indefinitely (see step D). Trying to achieve an isomorphism for as long as possible we obtained some correspondence rules which were identical with Weyl's rules. However, because these correspondence rules do not form an isomorphism they will not in general take a classical invariant into a quantum mechanical one. On the other hand our rules derived in §3.3 will achieve this if equation (3.42) is satisfied. It turns out that Weyl's transform is a special case of our rule given by (3.41), where

$$b_{jk} = 0, \quad 2a_1 = d_{112xy}, \quad 2a_2 = d_{221xy}. \tag{3.50}$$

In this case (3.42) reduces to

$$d_{111}V_{xxx} + d_{222}V_{yyy} + d_{112}V_{xxy} + d_{221}V_{yyx} = 0. \tag{3.51}$$

Let us summarize: For potentials V, where V satisfies equation (3.51) our rule and Weyl's rule are identical. However for potentials V which do not satisfy equation (3.51), but satisfy equation (3.42), the two rules are different and only our rule generates a quantum invariant.

In example 1 below, which is very simple, equation (3.51) is satisfied identically. This is no longer the case for example 2.

Example 1. a) Let $V = F(y)$. Then p_1 and hence p_1^3 are invariants. Put $d_{jk\ell}$ equal to zero except that $d_{111} = 1$. Then (3.51) is trivially satisfied. b) Let $V = G(x^2+y^2)$, then m_3 and hence m_3^3 are invariants. Let $d_{111} = -y^3$, $d_{222} = x^3$, $d_{112} = 3xy^2$, $d_{221} = -3x^2y$ and (3.51) is satisfied. Thus, if p_1, or p_2, or m_3 are invariants then Weyl's transform gives the quantum-mechanical analogues of p_1^3, or p_2^3, or m_3^3.

Example 2. Let $v = (x^2+y^2)^{-1/2}$. Then the x-component of the Runge-Lenz vector $m_3p_2 + x/\rho$ is an invariant. Multiplying this invariant by m_3 we obtain another (trivial) invariant. Letting $d_{111} = 0$, $d_{222} = x^2$, $d_{112} = y^2$, $d_{221} = -2xy$, the left hand side of (3.51) gives $3y(x^2+y^2)^{-3/2}$. Therefore (3.51) is not satisfied and Weyl's rule does not give a quantum invariant. Now we return to equation (3.42) and to more general rules (3.41). In this case it turns out that (3.42) can be satisfied if we take $a_1 = 0$, $b_{ij} = 0$, $a_2 = -\frac{1}{4}$ (the Weyl rules give $b_{ij} = 0$, $a_1 = 0$, $a_2 = -1$).

Example 3. For the example presented in §3.3.2 equation (3.51) is satisfied identically and therefore in this case (3.41) reduces to Weyl's rule.

B. The case $E = 0$.

The general form of a quadratic classical invariant for this case is given by the operator Y_2 (2.21). Operators of this type (or rather the corresponding function in phase space) are no longer, in general, in the linear closure of the set (3.48). Therefore, if we find the corresponding operator by Weyl's rules we do not expect it to be admissible; since Weyl's rules do not form an isomorphism and lemma 1.3 can not be applied. Actually one can verify directly that the correspondence rule derived in §3.3.1 (based on admissibility) is different from Weyl's rule.

The following point is worth noting: Consider the special case of (2.21), $Z = (\Delta^2 + c(\underline{x}))\frac{\partial}{\partial S}$. This operator is in the linear closure of (3.48), but its Weyl transform is still not an admissible operator. The reason is that in proving that an operator is admissible for $E = 0$, (but not for general values of E), we assume $H = 0$ as well as $D_{x_j} H = 0$, $D_{x_j} D_{x_k} H = 0$ and then the isomorphism is destroyed.

C. Generalizations.

The correspondence rules derived from isomorphism can be very useful for relating invariants of two equations, provided the equations belong in the closure of the rules. As an example consider the following corollary of step C of §3.4.1: Assume that

$$A = g(x)S_x^2 + f(x) - E = 0,$$

admits the LB operator

$$B = [G(x)S_x^2 + \alpha x S_x + F(x)]\frac{\partial}{\partial S}.$$

Then, if $\overline{A}$ and $\overline{B}$ are constructed from A and B with the aid of formulae (3.44), (3.45), (3.46), the equation $\overline{A} = 0$ admits the LB operator $\overline{B}\frac{\partial}{\partial u}$.

However, deriving correspondence rules on the assumption that these rules form an isomorphism is not always possible. Then a synthesis of the algebraic approach used above and of the direct approach used in §3.2 is very fruitful in dealing with the problem of relating the group structure of two different equations.

CHAPTER IV

4.1 INTRODUCTION

In this and the next chapter we shall be concerned with the general group properties of evolution equations. The aim of this chapter is to establish a one-to-one correspondence between admissible operators of evolution equations (derivable from a variational principle) and conservation laws of these equations. This correspondence takes the form of a simple algorithm given in §4.2.

The existence of a connection between the conservation laws, for differential equations obtained from a variational principle, and the invariance of the corresponding variational integral was established in the works of Jacobi, Klein and Noether. Jacobi [41] considered the equations of classical mechanics, Klein [42], [43] the equations of general relativity, and Noether [19] an arbitrary system of differential equations. Noether's result, now known as Noether's theorem†, says that if the values of a variational integral, for arbitrary admissible functions, are invariant with respect to an r-parameter continuous group of transformations of the dependent and independent variables, then the Euler equations, for the extremals of the functional under consideration, have r linearly independent conservation laws. All these conservation laws can be obtained by a certain standard formula. Noether's theorem has two limitations: i) it provides a sufficient condition for the existence of conservation laws and ii) the order of the derivatives on which the conservation law depends does not exceed the largest order of the derivatives which appear in the corresponding Lagrange function (i.e. LB groups of transformations were not

†Actually this is the first of the two important theorems of [19].

considered). The inverse of Noether's theorem (i.e. given a conservation law find the generator of the corresponding group) was considered by Bessel-Hagen [44] in 1921. Since then many generalizations of Noether's theorem and its inverse have been investigated, see for example [45] , [46] , [47] , [48] . In particular, Ibragimov [49] observed that in considering the invariance properties of the variational integral, it is sufficient to consider only its extremal values (i.e. the values obtained when the functions occurring in the integrand satisfy Euler's equations) instead of all its admissible values; this weak invariance condition turns out to be a necessary and sufficient condition for the validity of the conservation laws considered in Noether's work. Also, Ibragimov [15] recently proposed another generalization of Noether's theorem based on the notion of a weak Lagrangian and the concept of LB groups of tangent transformations [15] .

However, in spite of all the above generalizations, any approach based on Noether's theorem has the following disadvantages: i) Noether's theorem (and its generalizations) does not relate admissible operators of Euler's equations to conservation laws; it relates admissible operators of the variational integral to conservation laws (of Euler's equations). The latter admissible operators, are, of course, also admitted by Euler's equations; but not every admissible operator of Euler's equations is an admissible operator of the variational integral. Therefore, given an admissible operator of Euler's equation we must first check if it is also an admissible operator of the variational integral; only then can we construct a conservation law (using a standard algorithm).

ii) The inverse of Noether's theorem provides a way of finding an admissible operator of Euler's equations given a conservation law, which however may be only of formal value (different quotients are involved which may not be well defined).†

For the above reasons we shall examine the connection between admissible operators of evolution equations and conservation laws using a rather direct approach which is not based on Noether's theorem.

Before presenting this result we first review in §4.1.1 some methods of obtaining LB operators and then review in §4.1.2 different methods of obtaining conservation laws.

4.1.1 Methods of Obtaining LB Operators

1) The first natural way is to use the definition of a LB admissible operator (see §1.7 equation (1.57)). This classical approach, although in principle straightforward, is in practice very cumbersome. It has been used by Kumei [68] for obtaining LB operators of the Korteweg-de Vries equation and of the cubic Schrödinger equation.

2) Kumei [50] used a series expansion (given by Scott et al. [51]) based on a Bäcklund transformation (BT), to obtain LB operators for the Sine-Gordon equation. This approach can also be used for other equations possessing BT [52].

3) Olver [53] uses what he calls a recursion operator $\triangle$ to obtain new LB operators from known ones. The idea is

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†Note added in proof: However, Olver [72] has recently obtained results similar to the ones presented here, by analyzing further Noether's theorem. I thank Professor G.B. Whitham for communicating to me this interesting preprint.

quite simple: Let $\Omega = 0$ be the equation under consideration. Define the operator $A(\Omega)$ by $X\Omega \equiv A(\Omega)\eta$, where $X = \eta\partial/\partial u$ is some LB operator. A recursion operator Δ is one which satisfies $[A(\Omega),\Delta] = 0$. Assume that the operator X is an admissible LB operator; then, by definition

$$X\Omega\Big|_{\Omega=0} = 0 \qquad \text{or} \qquad A(\Omega)\eta\Big|_{\Omega=0} = 0$$

or

$$\Delta A(\Omega)\eta\Big|_{\Omega=0} = 0 \qquad \text{or} \qquad A(\Omega)\Delta\eta\Big|_{\Omega=0} = 0.$$

Therefore, if the operator $X = \eta\partial/\partial u$ is an admissible LB operator and Δ is a recursion operator, then $Y = \Delta\eta\partial/\partial u$ is also an admissible operator. The above approach is used in [53] for obtaining LB operators for the Korteweg-de Vries (KdV) equation, for the modified KdV equation, for the Burger's equation and for the Sine-Gordon equation. This approach has the disadvantage that a recursion operator must be found, whose general form is not a priori known.

4.1.2 Methods of Obtaining Conservation Laws

For the sake of completeness we first present some early important results in the study of evolution equations in general and of the KdV equation in particular: Let us consider conservation laws of the KdV equation,

$$u_t + uu_x + u_{xxx} = 0, \tag{4.1}$$

in the form

$$\rho_t + B_x = 0 \tag{4.2}$$

where ρ, the conserved density, and $-B$, the flux of ρ, are functionals of u. The following four functionals are among the polynomial conserved densities of the KdV equation:

$$\rho_1 = u, \qquad \rho_2 = \frac{1}{2}u^2, \qquad \rho_3 = \frac{1}{3}u^3 - u_x^2,$$

$$\rho_4 = \frac{1}{4}u^4 - 3uu_x^2 + \frac{9}{5}u_{xx}^2 . \tag{4.3}$$

ρ_1 originates from writing KdV itself in a conservation form and ρ_2 follows after multiplying the KdV equation by u. These are obvious and correspond to conservation of mass and momentum. The ρ_3 was found by Whitham [54] in his development of a variational approach to the study of nonlinear dispersive phenomena. ρ_4 and ρ_5 (which is not given here) were found by Kruskal and Zabusky [55] in their development of a nonlinear extension of the WKB method. Five more explicit conservation laws were given in [56] and it was conjectured that there were infinitely many of them. Similar conserved densities were also found for the modified KdV equation. This apparent distinguished feature of both the KdV equation and the modified KdV equation led Miura to conjecture that the solutions of these two equations are related. Miura observed that u occurs in powers of 1, 2, 3, 4 in the conserved densities given by (4.3), whereas v (the dependent variable for the modified KdV) occurs in powers of 1, 2, 4, 6 in the corresponding conserved densities, see [57]. This led him [58] to the discovery of the nonlinear transformation

$$u = v^2 \pm (-6)^{1/2} v_x, \tag{4.4}$$

which transforms solutions of the KdV equation (4.1), to solutions of the modified KdV equation

$$v_t+v^2v_x+v_{xxx} = 0. \tag{4.5}$$

Transformation (4.4) defines "half" a Bäcklund transformation (see Chapter V). Regarded as an equation for v, it is of the Ricatti type and therefore it can be linearized to become a Schrödinger equation. This provides the starting point for the inverse scattering method. Furthermore a modification of equation (4.4), yields a powerful way of constructing conservation laws (see A1 below).

After the above historical remarks we now review different ways for obtaining conservation laws.

A. Methods based on a BT.

1. Consider the following generalization of Miura's transformation, which was proposed by Gardner,

$$u = \omega + \epsilon\omega_x + \epsilon^2\omega^2 \tag{4.6}$$

which relates solutions of

$$u_t - 6uu_x + u_{xxx} = 0, \tag{4.7}$$

to solution of

$$\omega_t + (-3\omega^2-2\epsilon^2\omega^3+\omega_{xx})_x = 0. \tag{4.8}$$

Solving (4.6) for ω in a form of a formal power series in ϵ, with coefficients which are functions of u and x-derivatives of u, we obtain

$$\omega = u-\epsilon u_x-\epsilon^2(u^2-u_{xx}) + \dots . \tag{4.9}$$

Substituting the above in equation (4.8), the coefficient of each power of ϵ generates a conservation law. It can be shown [56] that the coefficients of the even powers of ϵ give nontrivial conservation laws.

The methods 2 and 3 below are simple variants of the above.

2. Scott et al.[51] derived a power series, similar to one given by (4.9), from the BT of the Sine-Gordon equation. It is clear that substitution of that series in any conservation law yields infinitely many conservation laws; Scott et al. derived such a series of conservation laws by using the law of conservation of energy associated with the Sine-Gordon equation.

3. Wadati et al.[59] used an approach similar to the above to construct conservation laws for the KdV, modified KdV and the Sine-Gordon equations. However, instead of substituting the power series obtained through "half" a BT into some conservation law, they substituted it into the other "half" BT which they wrote in a form of a conservation law.

4. Steudel [60] was the first, to our knowledge, to consider the conservation laws of evolution equations from a group theoretical point of view. Scott et al.[51] posed the problem of using Noethers theorem to explain the existence of infinitely many conservation laws. Steudel, starting from the BT of the Sine-Gordon equation, constructed infinitesimal invariant transformations (generators of LB groups), which he proved were Noether transformations (i.e. they left the variational integral invariant). Then, using Noether's theorem obtained a series of conservation laws. He also used the above approach for the KdV equation [61].

B. <u>Other Methods.</u>

1. Another way of finding conservation laws is to use the fact that the KdV equation can be viewed as a completely integrable

Hamiltonian system [62]. For details see [63], [64] and [65].

2. Pseudo potentials, as developed by Wahlquist and Estabrook provide another way of obtaining conservation laws [66], [67]. It is interesting that these conservation laws proceed in an "opposite direction" to the ones found in [56], in the sense that they depend on integrals of u instead on derivatives of u.

3. Given a conservation law and a LB operator X it is possible to derive a new (sometimes nontrivial) conservation law. If ρ is a conserved density and X an admissible LB operator, then $X\rho$ is also a conserved density see [14] and [50]. This approach yields nontrivial conservation laws for the Sine-Gordon equation [50], but trivial ones for the KdV equation [68]. Recently Kumei [69] established a connection between LB operators and conservation laws of nonlinear field equations in Hamilton's canonical form.

4. As was noted earlier Steudel was the first to consider the group-theoretical nature of the conservation laws of the Sine-Gordon equation and of the KdV equation. His approach was based on Noether's theorem. Kumei considered the same problem but he used rather direct approaches: For the Sine-Gordon equation he used the approach presented in B3; for the KdV he considered the equation

$$u_t + a(u)u_x + u_{xxx} = 0 \,, \tag{4.10}$$

(where $a(u)$ is a function of u only) and the constant of motion

$$I(u) = \int \rho(u)dx \,, \tag{4.11}$$

where $\rho(u) = \rho(x,t,u,u_x,\ldots)$. If Γ is the gradient of the functional $I(u)$, then the LB operator

$$X = (D\Gamma)\frac{\partial}{\partial u}, \qquad D \equiv D_x \tag{4.12}$$

is an admissible operator for equation (4.10). The proof of the last statement (for this particular equation (4.10)) is very simple and it is not clear how it could be generalized to cover other equations. However, it is well known that equation (4.10) can be derived from a variational principle; this motivated us to seek for a generalization of the above result, which will be presented in §4.2.

Equation (4.12) was used in [68] to explain why the approach used in [55] yields equations describing soliton interactions. Let us summarize: Kruskal and Zabusky observed that by considering the extremal value of ρ_3 (see equations (4.3)) subject to the constraint ρ_2 = constant an equation is obtained which yields the soliton solution; let $\rho_{23} \equiv \rho_3 - c\rho_2$, then

$$\Gamma_{23} = u^2 + 2u_{xx} - cu. \tag{4.13}$$

Now according to equation (4.12) the operator $X_{23} = (D\Gamma_{23})\partial/\partial u$ is an admissible operator for the KdV equation (4.1). Actually

$$X_{23} = (u_t + \frac{c}{2}u_x)\frac{\partial}{\partial u} \sim \frac{\partial}{\partial t} + \frac{c}{2}\frac{\partial}{\partial x}, \tag{4.14}$$

(where we used (4.1) to replace $u_{xxx} + uu_x$ by $-u_t$), is obviously an admissible operator, as the KdV is invariant under translations in x and t. Similarly, let $\rho_{234} \equiv \rho_4 + \alpha\rho_3 + \beta\rho_2$; it was shown numerically by Kruskal and Zabusky that the equation

$$\Gamma_{234} = 0, \tag{4.15}$$

contains a two soliton solution. Lax [70] proved this analytically. In [68] it is shown that the operator $X_{234} = (D\Gamma_{234})\partial/\partial u$ is an admissible operator of the KdV equation. It was then claimed that the two soliton solution of the KdV equation is the invariant solution of the operator X_{234}. However, the two soliton solution is just a member of a subclass of the class of invariant solutions of (4.1) under the action of X_{234}. We shall discuss the above problems further in §5.6.

4.2 ADMISSIBLE LB OPERATORS AND CONSERVATION LAWS OF EVOLUTION EQUATIONS

In this section the following notation for a LB operator will be used,

$$X_B \equiv B\frac{\partial}{\partial u}. \tag{4.16}$$

The lemma given below will be essential for the proof of the main theorems given later:

Lemma 4.1. Assume that $A = A(x, u, u_x, \ldots)$ can be expressed in terms of a variational derivative, i.e. there exists an $L = L(x, u, u_x, \ldots)$ such that

$$A = \frac{\delta L}{\delta u}, \tag{4.17}$$

where

$$\frac{\delta}{\delta u} \equiv \frac{\partial}{\partial u} - D\frac{\partial}{\partial u_x} + \ldots . \tag{4.18}$$

Then

$$\int BX_\phi A dx = \int \phi X_B A dx, \tag{4.19}$$

where $B = B(x, t, u, u_x, u_{xx}, \ldots)$, $\phi = \phi(x, t, u, u_x, u_{xx}, \ldots)$, the integrals defined in (4.19) are extended over an arbitrary region of the space of the independent variable x, and the function ϕ must vanish on the boundary of this region.

Note. It will be clear from the proof that the result of the lemma is stronger than what follows from the above proposition.

Proof. There are two possible ways to proceed. The first way is to assume that $A = \delta L/\delta u$ and then prove equation (4.19); the second way is not to assume equation (4.17) a priori and to discover the necessary restrictions on A such that equation (4.19) is satisfied. We shall follow the second way and we shall assume for the sake of concreteness that A depends on no higher derivative than the fourth x-derivative; the extension to higher derivatives is straightforward. Let

$$A_i \equiv \frac{\partial A}{\partial u_{\underbrace{x\ldots x}_{i}}}, \qquad i = 0,\ldots, \tag{4.20}$$

(i.e., $A_0 = \partial A/\partial u$, $A_1 = \partial A/\partial u_x, \ldots$). Then

$$J \equiv \int BX_\phi A dx = \int B \sum_{j=0}^{4} (D^j\phi)A_j dx = \int \phi \sum_{j=0}^{4} (-D)^j [BA_j] dx,$$

where the first equality follows by the definition of a LB operator and the second follows by integrating by parts. Using Leibniz's rule in the third integral and rearranging we obtain:

$$J = \int \phi \Big[\sum_{j=0}^{4} (D^j B)A_j - [2(DB)+BD][\sum_{j=0}^{3} (-D)^j A_{j+1}] - [2(D^3B)+3(D^2B)D+(DB)D^2][A_3 - 2DA_4]\Big] dx$$

Therefore, equation (4.19) holds for any ϕ and B iff

$$\sum_{j=0}^{3} (-D)^j A_{j+1} = 0, \tag{4.21a}$$

and

$$A_3 - 2DA_4 = 0. \tag{4.21b}$$

Equations (4.21) express the necessary restrictions on a function $A = A(x, u, u_x, \ldots, u_{xxxx})$, such that equation (4.19) holds. It can be easily checked, using the identity

$$\Big[\frac{\partial}{\partial u_{\underbrace{x\ldots x}_{i}}}, D\Big] = \frac{\partial}{\partial u_{\underbrace{x\ldots x}_{i-1}}}, \tag{4.22}$$

given in Gelfand and Dikiǐ [71] that if there exists an L such that

$A = \delta L/\delta u$, equations (4.21) follow. Q.E.D.

From the above proof it is clear that equation (4.17) is a sufficient condition. It is natural to ask the following question: Is equation (4.17) also a necessary condition? That is, does there exist an A which satisfies equation (4.19) and which cannot be expressed as a variational derivative? We expect a negative answer to this question and we verify this for the case that

$$A = A(x, u, u_x, u_{xx}). \tag{4.23}$$

Then A satisfies equation (4.19) iff (see equations (4.21))

$$A_1 = DA_2. \tag{4.24}$$

The left hand side of the above equation does not depend on u_{xxx}, whereas the right hand side contains the term $u_{xxx}A_{22}$. Therefore $A_{22} = 0$ and

$$A = a_{11}(x, u, u_x)u_{xx} + b(x, u, u_x), \tag{4.25}$$

where a and b are arbitrary functions of x, u, u_x and a_{11} is defined by (4.20). Substituting the above in (4.24) we obtain

$$b_1 = a_{x11} + a_{011}u_x,$$

therefore,

$$b = a_{x1} + u_x a_{01} - a_0 + c_u(x, u).$$

Therefore, the most general function A of the form (4.23) for which equation (4.19) is valid is given by (using now explicit notation),

$$A = \frac{\partial^2 a}{\partial u_x^2}(x, u, u_x)u_{xx} + \frac{\partial^2 a}{\partial x \partial u_x} + u_x \frac{\partial^2 a}{\partial u \partial u_x} - \frac{\partial a}{\partial u} +$$

$$\frac{\partial c}{\partial u}(x, u), \tag{4.26}$$

where $a = a(x, u, u_x)$ and $c = c(x, u)$ are arbitrary functions of the arguments indicated. Furthermore it is easily verified that $A = \delta L/\delta u$, where

$$L = - a(x, u, u_x) + c(x, u). \tag{4.27}$$

From the theorems to be proved below it will be clear that the algorithm relating LB operators to conservation laws does not depend explicitly on the Lagrangian L; only its existence is required, not its particular form. Given an A therefore, we just have to see if equation (4.21) (suitably extended if A depends on higher derivatives than the fourth x-derivative) are satisfied for the theorems to hold. This is much easier to check than finding an L associated with the given A (see Example 2). This justifies in our opinion the note made after the statement of the lemma 4.1.

It is well known that an evolution equation in the form

$$u_t + N(u) = 0, \tag{4.28}$$

where $N(u)$ denotes a function of x, u, and x-derivatives of u, can never be derived from a variational integral (because of the u_t term). However, there exist two tricks for writing equation (4.28) as the Euler equation of some variational problem: The first trick is to differentiate equation (4.28) with respect to x and the second is to replace in equation (4.28) u by v_x. These two tricks lead to two different algorithms for relating admissible LB operators to conservation laws:

Theorem 4.1. Assume that the x-derivative of the evolution equation

$$u_t + K(u) = 0, \tag{4.29}$$

(where $K(u) = K(x, u, u_x, \ldots)$) is the Euler's equation of some variational problem, i.e. there exists a Lagrangian L such that

$$DK = \frac{\delta L}{\delta u}. \qquad (4.30)$$

Further assume that

$$I(u) = \int \rho(u) dx, \qquad (4.31)$$

(where $\rho(u) = \rho(x, t, u, u_x, \ldots)$) is a constant of motion of equation (4.29). Then Γ is the gradient of a constant of motion I iff the LB operator $X_{(D^{-1}\Gamma)}$ is an admissible operator for equation (4.29). That is

$$\Gamma \equiv \frac{\delta \rho}{\delta u} \leftrightarrow X = (D^{-1}\Gamma)\frac{\partial}{\partial u}, \qquad (4.32)$$

where the above correspondence means that ρ is a conserved density iff X is an admissible LB operator.

Proof. Let us consider the infinitesimal transformation $u' = u + \epsilon\phi + O(\epsilon^2)$. This transformation is an invariant one for equation (4.29) iff

$$X_\phi[u_t + K]\Big|_{(4.29)} = 0 \,. \qquad (4.33)$$

Considering the effect of this transformation on a constant of motion $I(u)$ it is easily seen (see [64], [70]) that

$$D_t \int \Gamma \phi dx = 0, \qquad (4.34)$$

where Γ is defined in (4.32). Using the fact that X_ϕ and D commute, equation (4.33) yields

$$X_\phi[u_{tx} + DK]\Big|_{(4.29)} = 0 \,. \qquad (4.35)$$

Multiplying the above by $(D^{-1}\Gamma)$, integrating (and dropping the subscript (4.29)) we obtain

$$\int (D^{-1}\Gamma)[\phi_{tx} + X_\phi A] dx = 0, \qquad (4.36)$$

where $A \equiv DK$. Now integrating the first term in the above by parts and using lemma 4.1 for the second term, equation (4.36) yields,

$$\int [-\Gamma\phi_t + \phi X_{(D^{-1}\Gamma)} A]\,dx = 0.$$

Using equation (4.34) we obtain

$$\int \phi[D_t\Gamma + X_{(D^{-1}\Gamma)} A]\,dx = 0.$$

Finally, noting that $D_t\Gamma = D_t D[D^{-1}\Gamma]$, the above equation reduces to

$$\int \phi X_{(D^{-1}\Gamma)}[u_{tx} + DK] = 0.$$

Therefore, the LB operator $X_{(D^{-1}\Gamma)}$ is an admissible operator for the x-derivative of equation (4.29); hence, it is also an admissible operator of equation (4.29) (using again the fact that X and D commute). Q.E.D.

Theorem 4.2. Assume that if u is replaced by v_x in the evolution equation

$$u_t + M(u) = 0, \tag{4.37}$$

then equation (4.37) is the Euler's equation of some variational problem, i.e. there exists a Lagrangian $\hat{L}$ such that

$$M(v_x) = \frac{\delta \hat{L}}{\delta v}. \tag{4.38}$$

Then

$$\Gamma \equiv \frac{\delta\rho}{\delta u} \leftrightarrow X = (D\Gamma)\frac{\partial}{\partial u}, \tag{4.39}$$

where the above correspondence means that ρ is a conserved density of (4.37), iff X is an admissible operator.

Proof. The proof is analogous to the previous one: To equation (4.35) corresponds the following equation,

$$X_{(D^{-1}\phi)}[v_{tx} + M(v_x)]\Big|_{(4.37)} = 0.$$

Then proceeding as before (where now the above equation is multiplied by Γ and then integrated) we obtain

$$\int (D^{-1}\phi) X_{\Gamma}[\, v_{tx} + M(v_x)]\, dx = 0.$$

Therefore, the LB operator X_{Γ} is an admissible operator of equation $v_{tx} + M(v_x) = 0$; hence, $X_{D\Gamma}$ is an admissible operator of equation (4.37) (it is clear that if $w(u) = 0$ admits X_{η}, then $w(v_x) = 0$ admits $X_{D^{-1}\eta}$). Q.E.D.

<u>EXAMPLE 1.</u> Consider the evolution equation

$$u_t + a''(u) u_x + u_{\underbrace{x\ldots x}_{2r+1}} = 0, \tag{4.40}$$

where $a''(u) = \frac{d^2 a}{du^2}$ and $a(u)$ some function of u. This equation can be obtained from a variational formulation after replacing u by v_x. Therefore, formula (4.39) can be used. Writing the above in a conservation form we obtain

$$\rho_1 = u. \tag{4.41a}$$

Equation (4.40) is invariant under x-translation, i.e. X_{u_x} is an admissible operator. Therefore $\Gamma = u$ and

$$\rho_2 \equiv \frac{1}{2} u^2. \tag{4.41b}$$

Equation (4.40) is also invariant under t-translation, i.e. X_{u_t} is an admissible operator. Therefore $\eta = u_t$ or $\eta = u_{\underbrace{x\ldots x}_{2r+1}} + a''(u) u_x$ and $\Gamma = u_{\underbrace{x\ldots x}_{2r}} + a'(u)$. Therefore,

$$\rho_3 = \frac{1}{2}(-1)^r (u_{\underbrace{x\ldots x}_{r}})^2 + a(u). \tag{4.41c}$$

Therefore, every evolution equation of the form (4.40) has at least three conservation laws, whose conserved densities are given by equations (4.41).

Example 2. In chapter V we shall consider the equation

$$u_t + u_5 + \alpha u u_3 + (60-\alpha) u_1 u_2 + 6\alpha u^2 u_1 = 0, \tag{4.42}$$

where

$$u_n \equiv u_{\underbrace{x \ldots x}_{n}} .$$

For $\alpha = 20$ the above equation becomes the first member of the Lax's heirarchy associated with the KdV equation. Let $u = v_x$ and then

$$A = v_6 + \alpha v_1 v_4 + (60-\alpha) v_2 v_3 + 6\alpha v_1^2 v_2. \tag{4.43}$$

When $A = A(x, t, v, v_1, \ldots v_6)$, to equations (4.20) there correspond the following equations;

$$\begin{aligned} M_1 &\equiv \sum_{j=0}^{5} (-D)^j A_{j+1} = 0 , \\ M_2 &\equiv A_3 - 2DA_4 + 5D^3 A_6 = 0 , \\ M_3 &\equiv A_5 - 3DA_6 = 0 . \end{aligned} \tag{4.44}$$

Substituting A as defined by (4.43) in equations (4.44) we obtain

$$M_1 = M_3 = 0 , \quad M_2 = (60-3\alpha) v_2 .$$

Therefore, theorem 4.2. applies to equation (4.42) iff $\alpha = 20$.

CHAPTER V

5.1 INTRODUCTION

The last decade has clearly shown that Bäcklund transformations (BT) play many useful roles in the analysis of nonlinear phenomena. They provide a powerful way of analyzing soliton interaction, they can be used for constructing conservation laws and they provide a constructive method for finding eigenvalue problems associated with the inverse scattering method, see Appendix VII. For this reason, it is certainly worth investigating them independently and trying to find their essential mathematical structure. Also, the deeper understanding of their basic nature will, hopefully, provide a better algorithm for finding them.

Generally speaking, the use of BT is a mathematical technique for producing physically meaningful exact solutions. The main question we raise in this chapter is: Is it possible for these solutions to be characterized group theoretically? An affirmative answer to the above question would have two implications. Firstly, a theoretical one as it would unify our view of exact solutions. Two seemingly different classes of exact solutions, one obtained by the customary Lie-Ovsjannikov analysis of a given equation (similarity solutions) and one obtained through BT will turn out to have similar theoretical origin. Secondly, a practical one as the techniques used in deriving invariant solutions could be used to obtain BT.

In §5.2,3,4 the above question is considered. This leads to a new method for obtaining BT, which is investigated in the remaining subsections. Some new BT are obtained.

5.2 CONDITIONALLY ADMISSIBLE OPERATORS

The ideas will be introduced with the aid of an example. Consider the Korteweg-de Vries potential equation in the form

$$K(u_1, u_2, u_{222}) \equiv u_1 + u_{222} + \gamma u_2^2 = 0 . \tag{5.1}$$

"Half" of the Bäcklund transformation admitted by (5.1) is, see [73]

$$T(u, v, u_2, v_2) \equiv u_2 + v_2 + \frac{\gamma}{6}(u-v)^2 = 0 , \tag{5.2}$$

where v satisfies

$$v_1 + v_{222} + \gamma v_2^2 = 0 . \tag{5.3}$$

In analyzing group theoretically the solutions obtained from (5.1), (5.2), (5.3), we shall use two, in a sense complementary approaches. The first approach is to regard equation (5.2) as imposing some group theoretical constraint on the solutions of (5.1) and (5.3). The second approach is to regard (5.1) and (5.3) as equations imposing some constraint on solutions of (5.2). The notion of a conditionally admissible operator is necessary to describe the first point of view.

First approach.

For the sake of clarity we start with the special solution of equation (5.1) which can be obtained by the Bäcklund transformation (5.2) with $v = 0$. (This happens to be the soliton solution). Then the above system reduces to (5.1) and to

$$T_0 = 0 \tag{5.4}$$

where

$$T_0 \equiv u_2 + \frac{\gamma}{6} u^2 . \tag{5.5}$$

In order to reveal the group nature of the constraint imposed by equation (5.4) on the class of solutions of (5.1) we determine the action of the LB operator

$$X_0 \equiv T_0 \frac{\partial}{\partial u} + \sum_{j=1,2} (D_j T_0) \frac{\partial}{\partial u_j} + (D_2^3 T_0) \frac{\partial}{\partial u_{222}} + \cdots \tag{5.6}$$

on equation (5.1):

$$X_0 K = D_1 T_0 + 2\gamma u_2 D_2 T_0 + D_2^3 T_0 . \tag{5.7}$$

The next step is to restrict the action of X_0 to the solution of (5.1) and its differential consequences. This can be accomplished by substituting for the various x_1-derivatives. The only term affected by this is $D_1 T_0$ (since the other terms do not involve a x_1-derivative which can be replaced by x_2-derivatives using (5.1) and its differential consequences). Since

$$D_1 T_0 = u_{12} + \frac{\gamma}{3} u u_1 , \tag{5.8}$$

we conclude from (5.1)

$$D_1 T_0 \Big|_{K=0} = -u_{2222} - 2\gamma u_2 u_{22} - \frac{\gamma}{3} u(u_{22} + \gamma u_2^2) . \tag{5.9}$$

Examining the right hand side of (5.9) we see that it can be written in terms of T_0 as

$$D_1 T_0 \Big|_{K=0} = -D_2^3 T_0 - \gamma u_2 D_2 T_0 . \tag{5.10}$$

Therefore, equation (5.7) finally yields

$$X_0 K \Big|_{K=0} = \gamma u_2 D_2 T_0 . \tag{5.11}$$

What is the meaning of equation (5.11)? Clearly, the operator

X_0 is not an admissible operator of equation (5.1) because $X_0 K\big|_{K=0} \neq 0$. However, the operator X_0 is an admissible operator on a subclass of solutions of (5.1), namely the ones for which (5.4) is also valid. It is in this sense that the operator X_0 is a conditionally admissible operator. The operator X_o defines locally a group of transformations which in general maps solutions of (5.1) into solutions of some other equation, say, $K' = 0$. However, there exists a special class of solutions of (5.1), namely the one characterized by the nontrivial simultaneous validity of (5.1) and (5.4), for which this group maps solutions of (5.1) into themselves (so that $K' = K$). This class of solutions is precisely the invariant class of solutions of (5.1) under the action of operator X_0. With this special example in mind we give a general definition of a conditionally admissible operator.

Definition

The LB operator

$$C = B(x, u, \underset{1}{u}, \ldots \underset{n}{u}) \frac{\partial}{\partial u} + \sum_j (D_j B) \frac{\partial}{\partial u_j} + \ldots \qquad (5.12)$$

is a conditionally admissible operator for the equation

$$W(x, u, \underset{1}{u}, \ldots, \underset{m}{u}) = 0$$

iff

$$CW\big|_{W=0} \equiv 0, \quad \text{nontrivially when } B = 0. \qquad (5.13)$$

Equation (5.13) is written for simplicity as

$$CW\big|_{W=0,\, B=0} = 0 , \qquad (5.14)$$

However, we stress that the order in which $W=0$ and $B=0$ are assumed is of vital importance (otherwise the whole concept becomes trivial).

The extension of the above to systems of equations is exactly parallel to the extension of the notion of an admissible operator to systems of equations. As an example we consider the systems of equations (5.1) and (5.3). Consider the action of the LB operator

$$X \equiv T\frac{\partial}{\partial u}+\ldots = [u_1 + v_1 + \frac{\gamma}{6}(u-v)^2]\frac{\partial}{\partial u}+\ldots \tag{5.15}$$

on equations (5.1) and (5.3). The action of X on (5.3) is trivial. Applying X to (5.1) we obtain

$$XK = \overline{D}_1 T + \overline{D}_2^3 T + 2\gamma u_2 \overline{D}_2 T\ , \tag{5.16}$$

where (we use a bar to remind the reader that the total derivatives now are applied to functions which also involve v)

$$\overline{D}_i = \frac{\partial}{\partial x_i} + u_i\frac{\partial}{\partial u} + v_i\frac{\partial}{\partial v}+\ldots, \qquad i=1, 2. \tag{5.17}$$

The next step is the crucial one. Before assuming $T=0$ we must utilize (5.1) and (5.3) to replace u_1, v_1, u_{12}, v_{12} by x_2-derivatives. At this stage it is not a priori obvious that what we get after doing this will depend on $T, \overline{D}_2T, \ldots$ in such a way that it will vanish when $T, \overline{D}_2T, \ldots$ vanish. However, in the present case

$$XK\Big|_{(5.1),\,(5.3)} = \gamma(u_2-v_2)\overline{D}_2 T\ ,$$

and hence

$$XK\Big|_{(5.1),\,(5.3)} = 0 \qquad \text{when} \qquad T=0\ .$$

In this particular example, therefore, we see that the solutions obtained from the system of equations (5.1), (5.2), (5.3) are the invariant solutions of the system of equations (5.1), (5.3) under the action of the CAO X.

The above shows that if we want a group theoretical characterization of the Bäcklund solutions starting with a given manifold specified by equations (5.1) and (5.3) we need to extend our customary class of invariant solutions to include the ones which are invariant only under the action of conditionally admissible operators. This already suggests an alternative way of deriving BT. However, before elaborating on this, we give another group theoretical characterization of the above solutions.

<u>Second approach.</u>

We now regard (5.2) as our basic equation and (5.1), (5.3) as equations imposing some constraint of group theoretical nature on the manifold defined by (5.2). We therefore examine the action of the LB operator

$$\hat{Y} = (u_1 + u_{222} + \gamma u_2^2)\frac{\partial}{\partial u} + (v_1 + v_{222} + \gamma v_2^2)\frac{\partial}{\partial v} \quad (5.18)$$

on equation (5.2). Clearly equation (5.2) is invariant under translation in x_1, i.e. it admits the LB operator $Y_0 = u_1\frac{\partial}{\partial u} + v_1\frac{\partial}{\partial v}$. Therefore, we only have to examine the action of the LB operator

$$Y = (u_{222} + \gamma u_2^2)\frac{\partial}{\partial u} + (v_{222} + \gamma v_2^2)\frac{\partial}{\partial v}$$

on equation (5.2). A straightforward computation yields

$$YT\Big|_{(5.2)} = 0$$

i.e. Y and therefore $\hat{Y}$ is an admissible LB operator of equation (5.2).

Therefore, the Bäcklund solutions may alternatively be thought of as the invariant solutions of equation (5.2) under the action of the admissible LB operator (5.18).

5.3 CONDITIONALLY ADMISSIBLE VERSUS ADMISSIBLE LB OPERATORS

Before concentrating on the algorithmic implications of the above results we elaborate on the equivalence between the two approaches considered in section 5.2. This equivalence is presented in the form of the following lemma:

Lemma 5.1 The LB operator

$$X = T(u, v, u_2, v_2) \frac{\partial}{\partial u} + \overline{D}_1 T \frac{\partial}{\partial u_1} + \overline{D}_2 T \frac{\partial}{\partial u_2} + \dots ,$$

is a CAO for the evolution equation

$$u_1 + F(u, u_2, u_{22}, \dots u_{\underbrace{2\dots 2}_{m}}) = 0 , \tag{5.19}$$

for all v satisfying

$$v_1 + G(v, v_2, v_{22}, \dots v_{\underbrace{2\dots 2}_{m}}) = 0 , \tag{5.20}$$

iff the LB operator

$$Y = F \frac{\partial}{\partial u} + G \frac{\partial}{\partial v} \tag{5.21}$$

is an admissible operator of equation

$$T(u, v, u_2, v_2) = 0 . \tag{5.22}$$

Proof. The proof is constructive. Assume that X is a CAO for (5.19) whenever (5.20) is satisfied. Then

$$[\overline{D}_1 T + F_u T + F_{u_2} \overline{D}_2 T + \dots F_{u_{\underbrace{2\dots 2}_{m}}} \overline{D}_2^m T]_{(5.19), (5.20)} = 0 ,$$

when $T = 0$.

The only term effected by assuming (5.19), (5.20) is $\overline{D}_1 T$. This is important for the computations, as all the other terms of the above equation can be ignored immediately. Assuming (5.19), (5.20) to evaluate $\overline{D}_1 T$ and then assuming $T = 0$, the above reduces to

$$\left[T_u F + T_v G + T_{u_2} \overline{D}_2 F + T_{v_2} \overline{D}_2 G \right]\Big|_{T=0} = 0. \tag{5.23}$$

However, equation (5.23) is precisely the condition that the equation $T = 0$ admits the LB operator defined by (5.22). Q.E.D.

This proof is easily extended to the case

$T = T(u, v, u_2, v_2, \ldots u_{\underbrace{2\ldots2}_{n}}, v_{\underbrace{2\ldots2}_{n}})$, see [V] .

From equation (5.23) we see that the problem of finding a conditionally admissible operator is in a sense complementary to that of finding an admissible operator. The question of finding T can be stated as follows: Find a function $T(u, v, u_2, v_2)$ such that the equation $T = 0$ admits the LB operator Y.

In section 5.2 we considered the solutions of the Korteweg-de Vries equation obtained through the BT (5.2) and proved that they are group-theoretically characterizable in two ways. In order to characterize them the first way (which is also the most natural) we introduced the notion of a CAO. Further, we proved the equivalence of the two ways for any evolution equation.

The natural question arising is the following: Are the Bäcklund solutions of any evolution equation characterizable in the above two ways? The answer is affirmative and the proof is given in [V].

5.4 A FIRST WAY OF DERIVING BT.

In this section we utilize the group nature of BT to derive them. However, although we established their basic structure, we still denote them in the traditional way. This complicates the algorithm and hides the connection of the BT to the existence of a "nice" (in a sense to be defined in section 5.5.1) soliton solution. This connection will become clear in the next section.

5.4.1 Burgers Equation and Generalizations.

We consider a generalization of Burger's equation given by

$$u_t + u_{xx} + F(u, u_x) = 0 , \tag{5.24}$$

and we are looking for a BT which linearizes it, in particular one which maps solutions of (5.24) into solutions of

$$v_t + v_{xx} = 0 . \tag{5.25}$$

At this point it is necessary to assume the form of the BT (this is a definite weakness of any method concerned with BT; we shall elaborate more on this point in the next section). So, let "half" of the BT be

$$v_x - f(u, v) = 0 . \tag{5.26}$$

The problem of finding f reduces to the following: Find the function $f(u, v)$ such that equation (5.26) admits the LB operator

$$Y = v_{xx} \frac{\partial}{\partial v} + [u_{xx} + F(u, u_x)] \frac{\partial}{\partial u} + v_{xxx} \frac{\partial}{\partial v_x} + \ldots . \tag{5.27}$$

Applying Y to the left hand side of (5.26) we obtain,

$$Y(5.26) = v_{xxx} - f_1(u_{xx} + F) - f_2 v_{xx} , \tag{5.28}$$

where

$$f_1 = \frac{\partial f}{\partial u}\ ,\quad f_2 = \frac{\partial f}{\partial v}\ .$$

For Y to be admissible we require

$$Y(5.26)\Big|_{(5.26)} = 0\ . \tag{5.29}$$

The rest is the usual routine applied when looking for admissible operators: We first find the relevant differential consequences of (5.26),

$$v_x = f$$

$$v_{xx} = f_1 u_x + ff_2$$

$$v_{xxx} = f_{11}u_x^2 + 2f_{12}fu_x + f_1 u_{xx} + f_{22}f^2 + f_2 f_1 u_x + ff_2^2\ .$$

Then we substitute the above expressions in (5.29) to obtain, (after some cancellation)

$$f_{11}u_x^2 + 2ff_{12}u_x + f^2 f_{22} - f_1 F = 0\ , \tag{5.30}$$

or

$$F(u, u_x) = \frac{f_{11}}{f_1}u_x^2 + \frac{2ff_{12}}{f_1}u_x + \frac{f^2 f_{22}}{f_1}\ . \tag{5.31}$$

($f_1 \neq 0$, otherwise the BT is trivial)

Equation (5.31) implies that the following expressions depend on u only,

$$\frac{f_{11}}{f_1} = A(u),\quad \frac{ff_{12}}{f_1} = A_2(u),\quad \frac{f^2 f_{22}}{f_1} = A_3(u)\ .$$

The last equations determine f and then equation (5.31) determines F:

$$F = A(u)u_x^2 + 2\alpha B(u)u_x \ , \tag{5.32}$$

$$v_x = f = (\alpha v + \beta)B(u) \ , \tag{5.33}$$

where

$$B(u) = \int_0^u e^{\int_0^\xi A(\tau)d\tau} d\xi + \gamma \ . \tag{5.34}$$

Therefore, we conclude that the most general equation of the form (5.24) which admits a linearization of the form (5.26) is

$$u_t + u_{xx} + A(u)u_x^2 + 2\alpha B(u)u_x = 0 \ , \tag{5.35}$$

where $A(u)$ is arbitrary and $B(u)$ depends on $A(u)$ as given by (5.34)†.

Particular cases.

Equation (5.35) takes a simple form when the integration of $A(u)$ defined by (5.34) is simple. For example, let $A(u) = \dfrac{C''(u)}{C'(u)}$, then $B(u) = C(u) + \gamma$, and equations (5.30), (5.31) yield,

$$u_t + u_{xx} + \frac{C''(u)}{C'(u)} u_x^2 + 2\alpha C(u)u_x = 0 \ , \tag{5.36}$$

$$C(u) = \frac{v_x}{\alpha v + \beta} - \gamma \tag{5.37}$$

where $C(u)$ is an arbitrary function of u and α, β are constant parameters.

---

† Note that by putting $\alpha = 0$ in (5.35) we see that equation $u_t + u_{xx} + A(u)u_x^2 = 0$ linearizes for any $A(u)$. This is well known, see §5.5.2.

A well known particular case of (5.36) is the Burger's equation. If we let $C(u) = u$ in the above equations they reduce to

$$u_t + u_{xx} + 2\alpha u u_x = 0 ,$$

$$u = \frac{v_x}{\alpha v+\beta} ,$$

i.e. to Burger's equation and to the Cole-Hopf transformation.

If $C(u) = e^{\lambda u}$, equations (5.36), (5.37) give

$$u_t + u_{xx} + \lambda u_x^2 + \mu e^{\lambda u} u_x = 0 ,$$

$$u = \frac{1}{\lambda} \ell n \left(\frac{v_x}{\frac{1}{2}\mu v+\beta}\right) .$$

A generalization of the BT given by (5.26) is $u_x - f(u, v, v_x) = 0$. This leads to a new class of equations, of the general form (5.24), which may be linearized. Assuming other forms of BT new classes of equations may be obtained. However, the above approach has the disadvantage that every time a new form of BT is assumed the whole algorithm must be repeated. An alternative approach without this disadvantage is considered in §5.5.2.

5.4.2 KdV Equation and Generalizations.

We consider a generalization of KdV's potential equation given by

$$u_t + u_{xxx} + \gamma u_x^{\alpha} = 0 , \tag{5.38}$$

and we are looking for a BT which maps solutions of (5.38) to solutions of

$$v_t + v_{xxx} + \gamma v_x^{\alpha} = 0 . \tag{5.39}$$

We are taking "half" of the BT to be of the form

$$v_x - \beta u_x - \Phi(u+\lambda v) = 0 . \tag{5.40}$$

(If we start with $u_x - f(u, v, v_x)$, a lengthy analysis shows that (5.40) is a proper form). The determination of β and Φ follows from the requirement that equation (5.40) admits the LB operator

$$Y = (v_{xxx} + \gamma v_x^{\alpha})\frac{\partial}{\partial v} + (u_{xxx} + \gamma u_x^{\alpha})\frac{\partial}{\partial u} \tag{5.41}$$

i.e.,

$$[v_{xxxx} + \alpha\gamma v_x^{\alpha-1} v_{xx} - \beta u_{xxxx} - \beta\alpha\gamma u_x^{\alpha-1} u_{xx} - \Phi'(u_{xxx} + \gamma u_x^{\alpha} + \lambda v_{xxx} + \lambda\gamma v_x^{\alpha})]\Big|_{(5.40)} = 0$$

Using equation (5.40) and its differential consequences to replace $v_x, v_{xx}, v_{xxx}, v_{xxxx}$ by u derivatives, the above equation finally yields

$$\begin{aligned} &u_{xx}[-3(1+\lambda\beta)^2 u_x \Phi'' - 3\lambda(1+\lambda\beta)\Phi\Phi'' + \alpha\beta\gamma u_x^{\alpha-1} - \alpha\beta\gamma(\beta u_x + \Phi)^{\alpha-1}] - \\ &u_x^3(1+\lambda\beta)^3\Phi''' - 3\lambda u_x^2(1+\lambda\beta)^2(\Phi\Phi'' + \Phi'\Phi'') - 3\lambda^2 u_x(1+\lambda\beta)(2\Phi\Phi'\Phi'' + \Phi^2\Phi''') \\ &-\lambda^3(\Phi^3\Phi''' + 3\Phi^2\Phi'\Phi'') - \alpha\gamma(\beta u_x + \Phi)^{\alpha-1}[(1+\lambda\beta)\Phi' u_x + \lambda\Phi\Phi'] + \Phi'\gamma[u_x^{\alpha} + \lambda(\beta u_x + \Phi)^{\alpha}] = 0. \end{aligned} \tag{5.42}$$

Now, we equate the coefficients of $u_{xx}, u_x^3, u_x^2, u_x$ to zero. Looking at the coefficient of u_{xx} we deduce that in order for it to vanish, u_x must appear in it linearly or quadratically, i.e. $\alpha = 2$ or $\alpha = 3$.

<u>i) $\alpha = 2$ (KdV potential equation)</u>

Now equating to zero the coefficient of u_{xx} we obtain;

$$[3(1+\lambda\beta)^2\Phi'' + 2\beta\gamma(\beta-1)]u_x + 3\lambda(1+\lambda\beta)\Phi'' + 2\beta\gamma = 0 .$$

Therefore,

$$3(1+\lambda\beta)^2\Phi'' + 2\beta\gamma(\beta-1) = 0 , \tag{5.43.a}$$

and

$$3\lambda(1+\lambda\beta)\Phi'' + 2\beta\gamma = 0 . \qquad (5.43.b)$$

From (5.43) we deduce that

$$\frac{\beta-1}{1+\lambda\beta} = \frac{1}{\lambda}, \quad \text{or} \quad \lambda = -1 .$$

The coefficient of u_x^3 vanishes identically. The coefficient of u_x^2 vanishes iff

$$3(1-\beta)^2(\Phi\Phi''' + \Phi'\Phi'') + \gamma(1-\beta)^2\Phi' = 0 .$$

From the above equation we choose the solution satisfying

$$3\Phi'' + \gamma = 0 . \qquad (5.44)$$

Comparing (5.44) to (5.43) we find $\beta = -1$. Further, if equation (5.44) holds the coefficients of u_x and of the term independent of u_x vanish identically. Therefore, a BT for the equation

$$u_t + u_{xxx} + \gamma u_x^2 = 0 , \qquad (5.45)$$

is given by

$$u_x + v_x + \frac{\gamma}{6}(u-v)^2 + A(u-v) + B = 0 , \qquad (5.46)$$

where A and B are arbitrary constants.

ii) $\alpha = 3$ (Modified KdV potential equation)

Equating to zero the coefficient of u_{xx} we obtain

$$\beta\gamma(\beta^2-1)u_x^2 + [(1+\lambda\beta)^2\Phi'' + 2\beta^2\gamma]u_x + [\lambda(1+\beta)\Phi'' + \beta\gamma\Phi]\Phi = 0$$

Therefore,

$$\beta^2 = 1, \quad \lambda = \beta, \quad 2\Phi'' + \gamma\phi = 0 .$$

The coefficients of u_x^3, u_x^2, u_x, u_x^0 also vanish when the above equations hold. Therefore, a BT for the equation

$$u_t + u_{xxx} + \gamma u_x^3 = 0 , \qquad (5.47)$$

is given by

$$u_x - \beta v_x + A \cos\left(\frac{\gamma}{2}\right)^{1/2}(u+\beta v) + B \sin\left(\frac{\gamma}{2}\right)^{1/2}(u+\beta v) = 0\,, \qquad (5.48)$$

where $\beta^2 = 1$ and A, B arbitrary constants.

The BT given by equations (5.46), (5.48) have been previously obtained by classical methods, see [73] and [74]. The above results are further discussed and generalized in §5.5.1.

5.4.3 Sine-Gordon Equation and Generalizations.

The discussion so far has been limited to evolution equations for which general theorems have been proved. However, the idea of a CAO can be used for the group theoretical characterization of Bäcklund solutions for any equation. As an example we consider the equation

$$u_{xy} - F(u) = 0\,, \qquad (5.49)$$

and we look for a BT which maps solutions of (5.49) to solutions of

$$v_{xy} - G(v) = 0\,. \qquad (5.50)$$

Let us assume that "half" of the BT is

$$u_x - v_x - \Phi(u+v) = 0\,. \qquad (5.51)$$

(If we start with $u_x - f(u, v, v_x) = 0$, we will discover that (5.51) is a proper form, see Appendix V). To determine Φ, for a given F, we require that the operator

$$X = [u_x - v_x - \Phi(u+v)]\frac{\partial}{\partial u} + \ldots \equiv B\frac{\partial}{\partial u} + \ldots \qquad (5.52)$$

is a CAO for equation (5.49). Applying the operator X to (5.49) we obtain

$$X(u_{xy} - F(u)) = u_{xxy} - v_{xxy} - (u_x + v_x)(u_y + v_y)\Phi'' - (u_{xy} + v_{xy})\Phi' - (u_x - v_x - \Phi)F'(u).$$

Assuming (5.49), (5.50) the above becomes

$$X(u_{xy}-F)\Big|_{(5.49),(5.50)} = F'(u)u_x - G'(v)v_x - (u_x+v_x)(u_y+v_y)\Phi'' - (F(u)+G(v))\Phi' - (u_x-v_x-\Phi)F'(u). \tag{5.53}$$

However,

$$D_yB = u_{xy} - v_{xy} - (u_y+v_y)\Phi' ,$$

or

$$u_y+v_y = \frac{F(u)-G(v)}{\Phi'} - \frac{D_yB}{\Phi'} . \tag{5.54}$$

Also

$$u_x = B + v_x + \Phi . \tag{5.55}$$

Susbtituting (5.54), (5.55) in (5.53) and then assuming $B = D_yB = 0$ we finally obtain

$$X(u_{xy}-F(u))\Big|_{(5.49),(5.50),(5.51)} =$$

$$\left\{F'(u)-G'(v)-2\frac{\Phi''}{\Phi'}[F(u)-G(v)]\right\}v_x + F'(u)\Phi-\Phi'[F(u)+G(v)] - \frac{\Phi\Phi''}{\Phi'}[F(u)-G(v)].$$

Therefore,

$$\frac{F'(u)-G'(v)}{F(u)-G(v)} = \frac{2\Phi''(u+v)}{\Phi'(u+v)} , \tag{5.56}$$

and

$$F'(u) - \frac{\Phi'}{\Phi}[F(u)+G(v)] - \frac{\Phi''}{\Phi'}[F(u)-G(v)] = 0 . \tag{5.57}$$

Replacing in (5.57)

$$\frac{\Phi''}{\Phi'}[F(u)-G(v)] \qquad \text{by} \qquad \tfrac{1}{2}[F'(u)-G'(v)]$$

we obtain

$$\frac{F'(u)+G'(v)}{F(u)+G(v)} = 2\,\frac{\Phi'(u+v)}{\Phi(u+v)} . \tag{5.58}$$

Equations (5.56) and (5.58) determine which equations admit a BT of the form (5.51). We shall now discuss some special cases.

i) Linearizations

Let us first look for equations (5.49), which can be linearized under the BT (5.51). Putting $G(v) = 0$ in equations (5.56), (5.58) we obtain

$$\frac{F'(u)}{F(u)} = 2\,\frac{\Phi'(u+v)}{\Phi(u+v)} = 2\,\frac{\Phi''(u+v)}{\Phi'(u+v)} . \tag{5.59}$$

The above equations yield

$$F = \beta e^{\alpha u} , \quad \Phi = \gamma\, e^{\frac{\alpha}{2}(u+v)} ,$$

where α, β, γ are constants.

Therefore, the only equation of the form (5.49) which linearizes under the BT (5.51) is

$$u_{xy} = \beta e^{\alpha u} ,$$

and the linearizing BT is given by

$$u_x - v_x - \gamma\, e^{\frac{\alpha}{2}(u+v)} = 0 ,$$

where v satisfies $v_{xy} = 0$.

ii) Restricted BT

By restricted BT we mean transformations which map solutions of some equation among themselves. Putting $G(v) = F(v)$ in equations (5.56), (5.58), the equation (5.58) yields $F = \beta \sin \alpha u$. Then

$$\frac{\Phi'(u+v)}{\Phi(u+v)} = \frac{\alpha}{2} \tan \tfrac{1}{2}\alpha(u+v).$$

Therefore,

$$\Phi = \gamma \sin \tfrac{1}{2}\alpha(u+v).$$

Substituting the above forms of F and Φ in (5.56) it is seen that (5.56) is satisfied identically. Therefore, the only equation of the form (5.49) admitting a restricted BT of the form (5.51) is the Sine-Gordon equation

$$u_{xy} = \beta \sin \alpha u ,$$

and the BT is determined by

$$u_x - v_x - \gamma \sin \tfrac{1}{2}\alpha(u+v) = 0 .$$

In the Appendix V it is shown that the most general BT of the form $u_x - f(u, v, v_x)$ is given by $u_x - \alpha v_x - \Phi(u+\alpha v+g(u-\alpha v))$. Starting with this form (instead of (5.51)) it can be shown that the only equations (5.49) which admit a restricted BT are those for which $F''(u) \pm \lambda F(u) = 0$. Similar non-existence proofs are given in the literature, see for example [75]. However, we think that their use is very limited since $u_x - f(u, v, v_x) = 0$ defines a BT of a very restricted form. A more general BT is, for example, $u_x - f(u, v, v_x, v_{xx}, \ldots) = 0$, for which a non-existence proof would be very difficult. In §5.5 a generalization of (5.51) is proposed. This also suggests an alternative way of obtaining non-existence proofs.

5.5 A SECOND WAY OF DERIVING BT GENERALIZATIONS.

In §5.2 we established the basic nature of solutions obtained through BT and indicated two equivalent ways for characterizing them. In §5.4 we used an algorithm based on invariance criteria to rederive some of the well known BT and to derive some new ones (see §5.4.1). The basic idea was that if an equation

$$u_t + \tilde{F}(u, u_x, \ldots) = 0 \quad , \tag{5.60}$$

admits a BT of the form

$$u_x - T(u, v, v_x, \ldots) = 0 \quad , \tag{5.61}$$

which maps solutions of (5.60) to solutions of

$$v_t + G(v, v_x, \ldots) = 0 \quad , \tag{5.62}$$

then the operator

$$\hat{X} = [u_x - T(u, v, v_x, \ldots)] \frac{\partial}{\partial u} \tag{5.63}$$

is a CAO for the above evolution equations. However, the form of $\hat{X}$ is quite restricted. A more general operator is

$$X = [u_x - g(x, t, u)] \frac{\partial}{\partial u} \quad . \tag{5.64}$$

Clearly this class of operators <u>includes</u> the class of the form $\hat{X}$. In this section we investigate CAO of the form given by (5.64). This new approach has the following advantages: i) when we are looking for BT of the form (5.61) we have to <u>assume</u> the arguments of T. As it was pointed out earlier, this is a definite weakness of any method used for obtaining BT. In our approach this is not necessary, since the class of operators of the form (5.63) is a subclass of the class of operators defined by (5.64). ii) It clarifies the connection between the wave-train solution and the existence of a BT and in that respect provides a very easy test for expecting BT based on the general structure of the wave-train solution (see §5.5.1). iii) It obtains exact solutions in some cases where BT do not exist. We will elaborate more on the above points in section 5.6.

In §5.3 the equivalence between CAO of the form $\hat{X}$ and admissible operators was proved. Here we give a similar result for the case that the CAO is of the form (5.64):

Lemma 5.2 The LB operator

$$X = [u_x - g(x,t,u)]\frac{\partial}{\partial u} \tag{5.64}$$

is a CAO for the evolution equation

$$u_t + \tilde{F}(u, u_x, \ldots u_{\underbrace{x\ldots x}_{m}}) = 0\ , \tag{5.65}$$

iff the LB operator

$$Y = (u_t + \tilde{F})\frac{\partial}{\partial u} \tag{5.66}$$

is an admissible operator for equation

$$u_x - g = 0\ . \tag{5.67}$$

Proof. X is a CAO for (5.65) iff

$$X(u_t + \tilde{F})\Big|_{(5.65),(5.67)} = 0\ . \tag{5.68}$$

Applying X to (5.65) we obtain

$$X(u_t + \tilde{F}) = u_{xt} - D_t g + \tilde{F}_u D_x(u_x - g) + \ldots \tilde{F}_{u_{\underbrace{x\ldots x}_{m}}} D_x^m (u_x - g)\ . \tag{5.69}$$

Assuming (5.65), i.e. replacing t-derivatives, (5.69) yields

$$X(u_t+\tilde{F})\Big|_{(5.65)} = -D_x\tilde{F}-g_t + g_u\tilde{F}+\tilde{F}_u D_x(u_x-g)+\dots \tilde{F}_{u_{\underbrace{x\dots x}_{m}}} D^m_x(u_x-g) .$$

Finally assuming (5.67), equation (5.68) gives

$$[-D_x\tilde{F}-g_t + g_u\tilde{F}]_{(5.67)} = 0 . \tag{5.70}$$

Y is an admissible operator of (5.67) iff

$$Y(u_x-g)\Big|_{(5.67)} = 0 ,$$

or

$$[u_{tx}+ D_x\tilde{F} -(u_t+\tilde{F})g_u]\Big|_{(5.67)} = 0 . \tag{5.71}$$

Differentiating equation (5.67) with respect to x we obtain

$$u_{tx} - u_t g_u = g_t. \tag{5.72}$$

Using (5.72), equation (5.71) reduces to (5.70). Q.E.D.

5.5.1 KdV and Generalizations.

In this section we investigate CAO of the form

$$X = [u_x-g(x,t,u)]\frac{\partial}{\partial u} , \tag{5.64}$$

for evolution equations of the form

$$u_t + u_{xxx} + F(u,u_x) = 0 . \tag{5.73}$$

Requiring X to be a CAO for equation (5.73), (or equivalently $Y = (u_t + u_{xxx} + F)\frac{\partial}{\partial u}$ to be an admissible operator for equation (5.67)), equation (5.70) yields (with $\tilde{F} = u_{xxx} + F$)

$$[u_{xxxx} + F_1 u_x + F_2 u_{xx} + g_2-g_3(u_{xxx} + F)]\Big|_{u_x=g} = 0 , \tag{5.74}$$

where

$$F_1 = \frac{\partial F}{\partial u}, \quad F_2 = \frac{\partial F}{\partial u_x}, \quad g_1 = \frac{\partial g}{\partial x}, \quad g_2 = \frac{\partial g}{\partial t}, \quad g_3 = \frac{\partial g}{\partial u} \tag{5.75}$$

Now we have to replace the x-derivatives of u using the following equations

$$\begin{aligned}
u_x &= g, \\
u_{xx} &= g_1 + gg_3, \\
u_{xxx} &= g_{11} + 2gg_{13} + g_1 g_3 + gg_3^2 + g^2 g_{33}, \qquad (5.76)\\
u_{xxxx} &= g_{111} + 3gg_{113} + 3g_1 g_{13} + 3g^2 g_{133} + 5gg_3 g_{13} + 3gg_1 g_{33} + \\
&\qquad g_3 g_{11} + g_3^2 g_1 + gg_3^3 + 4g^2 g_3 g_{33} + g^3 g_{333}.
\end{aligned}$$

This yields (after some cancellation)

$$\begin{aligned}
&g_{111} + g_2 + F_2 g_1 + 3gg_{113} + 3g_1 g_{13} + 3g^2 g_{133} + 3gg_3 g_{13} + \\
&3gg_1 g_{33} + g^2 \left(\frac{F}{g}\right)_3 + (g^3 g_{33})_3 = 0, \qquad (5.77)
\end{aligned}$$

where now

$$F = F(u, g), \quad F_1 = \frac{\partial F}{\partial u}, \quad F_2 = \frac{\partial F}{\partial g}.$$

Noticing that $(g^3 g_{33})_3 = g^2 (gg_3)_{33}$ and rearranging, equation (5.77) can be written as

$$g_{111} + g_2 + F_2 g_1 + \frac{3}{2} g(g^2)_{331} + 3[(gg_{13})_1 - g_3(gg_{13})] + g^2 [\tfrac{1}{2}(g^2)_{33} + \frac{F}{g}]_3 = 0. \tag{5.78}$$

Let us summarize: We are investigating the existence of CAO given by (5.64) for equation (5.73). It turns out that g satisfies a nonlinear equation in three independent variables x, t, u. Solving equation (5.78) seems much more difficult than solving equation (5.73). However, we are only looking for particular solutions of (5.78). So we have inflated our problem (in the sense that (5.78) is more complicated than (5.73)) but have deflated our goal (in the sense that we are looking for particular solutions only).

A. Wave-train solution.

We will discuss different ways for obtaining particular solutions of equation (5.78). However, we start with an obvious one, which also turns out to be very important: Assume $g = \Phi(u)$. This corresponds to CAO of the form

$$X_0 = [u_x - \Phi(u)] \frac{\partial}{\partial u} \quad . \tag{5.79}$$

The invariant solutions corresponding to X_0 are the wave-train solutions, as seen by the following lemma:

Lemma 5.3. Assume that equation (5.73) possesses a wave-train solution. Then this solution is the invariant solution of equation (5.73) under the action of the CAO X_0 given by (5.79).

Proof. The wave-train solution of (5.73) is given by $u = u(x-Ut)$, and for such a solution $u_t = -Uu_x$. Then (5.73) becomes

$$u_{xxx} + F(u, u_x) - U u_x = 0 \ , \tag{5.80}$$

Let $u_x = \Phi(u)$ in (5.80) to obtain

$$\Phi(\Phi\Phi')' + F - U\Phi = 0 \ . \tag{5.81}$$

If the LB operator X_0 is a CAO for equation (5.73), then equation (5.78) yields (with $g = \Phi(u)$)

$$[\tfrac{1}{2}(\Phi^2)'' + \frac{F}{\Phi}]' = 0 \ . \tag{5.82}$$

Comparing (5.82) with (5.81) concludes the proof.

B. Bäcklund transformations.

We now come back to equation (5.78) and look for other particular solutions. We just proved that $g = \Phi(u)$ corresponds to the wave-train solution. What choice of g corresponds to interaction of solitons? The answer to this question is not a priori obvious. One of the well known tricks for finding particular solutions of differential equations is to use variation of parameters on the homogeneous solution. This trick is also used here: We look for solutions of (5.78) which satisfy

$$[\tfrac{1}{2}(g^2)_{33} + \frac{F}{g}]_3 = 0 \ . \qquad (5.83)$$

This is equivalent to look for $g = \Phi(u)$, i.e. for the wave-train solution, and then to let the parameters depend on x and t. Let us look at some particular cases.

1. $F = \gamma u_x^{\alpha}$

If we look for functions g of the form $g = \Phi(u)$, equations (5.77) or (5.78) indicate that we must solve

$$\tfrac{1}{2}(\Phi^2)'' + \frac{F(u, \Phi)}{\Phi} - U = 0 \ , \qquad (5.84)$$

or equivalently,

$$(\Phi^3\Phi'')' + \Phi^2(\frac{F}{\Phi})' = 0 \ . \qquad (5.85)$$

In this case $F = \gamma\,\Phi^{\alpha}$, therefore $\Phi^2(\frac{F}{\Phi})' = \gamma\frac{(\alpha-1)}{(\alpha+1)}(\Phi^{\alpha+1})'$, and (5.85) gives

$$\Phi'' + \gamma\frac{(\alpha-1)}{(\alpha+1)}\Phi^{\alpha-2} = c_0\Phi^{-3} \ , \qquad (5.86)$$

where c_0 is a constant of integration. The above equation can be integrated again to give

$$\tfrac{1}{2}\Phi'^2 = c_1\Phi^{-2} + c_2 - \frac{\gamma}{\alpha+1}\Phi^{\alpha-1} \ , \qquad (5.87)$$

where c_1, c_2 are constants.

i) $\alpha = 2$ (KdV potential equation)

Assuming $c_0 = 0$ in equation (5.86) we obtain

$$\Phi = -\frac{\gamma}{6}u^2 + Au + B , \tag{5.88}$$

where A, B are constants. If we look for a solution of equation (5.78) in the form

$$g = -\frac{\gamma}{6}u^2 + A(x,t)u + B(x,t) , \tag{5.89}$$

the bracket $g^2[\cdots]$ is identically zero. Substituting (5.89) in (5.78) we see that the left hand side of (5.78) becomes,

$$(3A_{xx}+3AA_x+\gamma B_x)u^2 + [A_t + A_{xxx} + 3A_x^2 + A(3A_{xx}+3AA_x+\gamma B_x)]u +$$
$$B_t + B_{xxx} + 3A_xB_x + B(3A_{xx} + 3AA_x + \gamma B_x) .$$

Therefore, g defined by (5.89) is a solution of (5.78) iff

$$3A_{xx} + 3AA_x + \gamma B_x = 0 , \tag{5.90.a}$$

$$A_t + A_{xxx} + 3A_x^2 = 0 , \tag{5.90.b}$$

$$B_t + B_{xxx} + 3A_xB_x = 0 . \tag{5.90.c}$$

Equation (5.90.a) implies that

$$B = -\frac{3}{\gamma}(A_x + \frac{A^2}{2}) + c(t) .$$

Substituting the above in (5.90.c) and assuming A satisfies (5.90.b) we get c(t) = constant. Therefore

$$g = -\frac{\gamma}{6}u^2 + Au - \frac{3}{\gamma}(A_x + \frac{A^2}{2}) + k , \tag{5.91}$$

where A satisfies (5.90.b). To express the BT in the customary form let

$$A = \frac{\gamma}{3}v + \lambda ,$$

then equation (5.91) gives

$$g = -\frac{\gamma}{6}(u-v)^2 - v_x + \lambda(u-v) + \mu , \qquad (5.92)$$

where v satisfies

$$v_t + v_{xxx} + \gamma v_x^2 = 0 ,$$

and λ, $\mu = k - \frac{3\lambda^2}{2\gamma}$, are constants. Equation (5.92) defines a BT identical to the one given by (5.46).

Before proceeding further let us extract the essentials of the above example. After establishing the basic nature of Bäcklund solutions, as invariant solutions under some CAO, we were naturally led to consider CAO of the form $X = [u_x - g(x,t,u)] \frac{\partial}{\partial u}$, where $g(x,t,u)$ satisfies (5.78). We then observed that a particular solution of equation (5.78), namely $g = \Phi(u; A, B, C)$ where A, B, C constants, corresponds to the wave-train solution. Further we noticed that if $C = 0$ (which corresponds to the solitary-wave solution) the form of g is very simple; we then looked for a solution of equation (5.78) in the form $g = \Phi(u; A(x,t), B(x,t))$. With this choice of g some terms of equation (5.78) are identically zero; requiring the remaining terms of (5.78) to vanish too, we obtained some equations for A and B. In the example just considered $B = -\frac{3}{\gamma}(A_x + \frac{A^2}{2}) + k$, and A any solution of (5.90.b).

The obvious question arising is the following: Given a particular case of a wave-train solution in the form $g = \Phi(u; A, B)$ and then letting A and B depend on x and t, when do we expect the system of equations for A and B to have a non-trivial solution? From the analysis carried above it is clear that a necessary condition is Φ to be a "nice" function of u, for example polynomial of integer powers of u or some trigonometric function of u. Still, this does not guarantee that non-trivial A and B exist, as in general A and B satisfy a system of <u>overdetermined equations</u> (in the above example A and B satisfy a system of three equations). The following cases are possible: i) The only solution for A and B is A, B constants. This corresponds to the case that the only exact solution obtained through this approach is the wave-train solution. ii) A, B are any solutions of some differential equations (like in the example just considered). This corresponds to the case where a BT exists for the given equation. iii) A, B are some <u>given</u> functions of x, t. This corresponds to the case where a BT does not exist, however an exact solution can still be found, which may be different than the wave-train solution. In the remaining examples the above points will be clarified further.

ii) <u>$\alpha = 3$ (Modified KdV potential equation)</u>

Assuming $c_0 = 0$ in equation (5.86) we obtain

$$\Phi = A\cos\left(\frac{\gamma}{2}\right)^{1/2} u + B\sin\left(\frac{\gamma}{2}\right)^{1/2} u\,. \tag{5.93}$$

If we now substitute

$$g = A(x,t)\cos\left(\frac{\gamma}{2}\right)^{1/2} u + B(x,t)\sin\left(\frac{\gamma}{2}\right)^{1/2} u\,,$$

in equation (5.78) we get an overdetermined system of equations for A and B the solution of which leads to a BT identical to the one given by (5.49).

2. $F = \gamma u^{\alpha} u_x$

In this case equation (5.84) can be integrated twice to give

$$\tfrac{1}{2}\Phi^2 = -\frac{\gamma u^{\alpha+2}}{(\alpha+1)(\alpha+2)} + \tfrac{1}{2}Uu^2 + c_1 u + c_2 \ . \tag{5.94}$$

i) $\alpha = 1$

Then the above equation yields

$$\tfrac{1}{2}\Phi^2 = -\frac{\gamma}{6}u^3 + \tfrac{1}{2}Uu^2 + c_1 u + c_2 \ . \tag{5.95}$$

Because of the u^3 term there is no way to make the right hand side of equation (5.95) a perfect square, so we do not expect a BT for this case.

ii) $\alpha = 2$ (Modified KdV equation)

Equation (5.94) yields

$$\tfrac{1}{2}\Phi^2 = -\frac{\gamma}{12}u^4 + \tfrac{1}{2}Uu^2 + c_1 u + c_2 \ . \tag{5.96}$$

Taking $c_1 = 0$ we can write the right hand side of (5.96) as a perfect square and

$$\Phi = \left(-\frac{\gamma}{6}\right)^{1/2} u^2 + A \ . \tag{5.97}$$

Now letting A to depend on x, t and substituting $g = \left(-\frac{\gamma}{6}\right)^{1/2} u^2 + A(x,t)$ in (5.78) we obtain

$$A_t + A_{xxx} + 6\left(-\frac{\gamma}{6}\right)^{1/2} AA_x = 0 \ . \tag{5.98}$$

Therefore,

$$u_x = \left(-\frac{\gamma}{6}\right)^{1/2} u^2 + A(x,t) \ , \tag{5.99}$$

is a BT mapping solutions of equation

$$u_t + u_{xxx} + \gamma u^2 u_x = 0 \ , \tag{5.100}$$

to solutions of equation (5.98). Equation (5.99) defines the well known Miura transformation [58].

5.5.2 Hierarchies of KdV equations. A new BT.

It is well known that associated with the KdV equation there exists a hierarchy of equations, first found by Lax [64], [70], each member of which has the same interesting properties possessed by the KdV equation, namely

i) each equation can be solved by the inverse-scattering method and therefore has N-soliton solutions;

ii) each equation possesses the same BT possessed by the KdV equation (from which the N-soliton ladder can be generated);

(iii) each equation possesses an infinity of independent polynomial conservation laws.

One way of looking at these equations is to regard them as LB symmetries of the KdV equation. Let us be more specific; it was noted in §4.1.1. that one method of obtaining LB operators is to use a recursion operator Δ (If $X = \eta\ \partial/\partial u$ is an admissible LB operator of a given equation and Δ is a recursion operator for the same equation then the LB operator $(\Delta\eta)\partial/\partial u$ is also admissible). It was shown in [53] that the operator

$$\Delta = D^2 + 8u + 4u_1 D^{-1}, \tag{5.101}$$

where

$$D \equiv D_x , \quad u_i \equiv u_{\underbrace{x\ldots x}_{i}} , \tag{5.102}$$

is a recursion operator for the KdV equation

$$u_t + u_3 + 12uu_1 = 0 . \tag{5.103a}$$

The LB operator $u_1\ \partial/\partial u$ is a trivial admissible operator of (5.103a)(translation in x). Then the operators $(\Delta^j u_1)\,\partial/\partial u$, $j=1,2,\ldots$

are also admissible; finally the operators $[u_t + (\Delta^j u_1)]\,\partial/\partial u$ are also admissible because equation (5.103a) admits the operator $u_t\,\partial/\partial u$ (translation in t). The equations

$$u_t + (\Delta^j u_1) = 0\ , \quad j=1,2,3,\ldots$$

give Lax's hierarchy (this method of construction differs from Lax's original method). The value $j=1$ defines the KdV itself and $j=2$ gives

$$u_t + u_5 + 40u_1u_2 + 20uu_3 + 120u^2u_1 = 0, \tag{5.103b}$$

where u_1, u_2, u_3, u_5 are defined by (5.102). Similar hierarchies may be obtained for the modified KdV, the Sine-Gordon and the Burgers' equations. In this way we obtain evolution equations, of order higher than three, which admit a BT; however, to obtain such a BT we need only consider the first member of the corresponding hierarchy which is at most of third order. In this sense we may claim that the only BT obtained so far in the literature are essentially admitted by at most third-order equations. In this subsection we shall derive a BT for the equation

$$u_t + u_5 + 30uu_3 + 30u_1u_2 + 180u^2u_1 = 0. \tag{5.104}$$

However, we consider the above equation not only because it is of fifth order, without being a LB symmetry of some other lower order evolution equation, but also for the following reason:
P.J. Caudrey et al. [80] considered equation (5.104) and noted that although it possesses multiple soliton solutions and some higher conservation laws, it does not appear to fit the present inverse-scattering formulism and does not seem to possess a BT; in this sense it seems to be the <u>only</u> known evolution equation not fitting

the usual scheme (BT-inverse scattering-multiple solitons- infinite conservation laws). However, if we regard the multisoliton solutions as invariant solutions, we do expect that the existence of multisolitons implies the existence of a BT. This BT will be given below; however let us first motivate equation (5.104) mathematically. Caudrey et al. noted that each of the two equations (5.103b) and (5.104) is a special case of the more general equation

$$u_t + u_5 + \alpha u u_3 + (60-\alpha)u_1 u_2 + 6\alpha u^2 u_1 = 0, \qquad (4.41)$$

where $\alpha = 20$ gives (5.103b) and $\alpha = 30$ gives (5.104). Equation (4.41) has a single soliton solution for all values of α. We suggest here the following method for obtaining (4.41): The recursion operator for the KdV equation Δ commutes by construction with $A \equiv D_t + D^3 + 12uD + 12u_1$. In order to obtain equations admitting at least single solitons we look for an operator $\overline{\Delta}$ which commutes only with the x-dependent part of A. Then it is easily found that $\overline{\Delta} = D^2 + (\alpha-12)u + (24-\alpha)u_1 D^{-1}$, where α is an arbitrary constant parameter. Any member of the class of equations

$$u_t + \overline{\Delta}^j u_1 = 0, \quad j=1,2,\ldots$$

has (at least) single soliton solutions; j=1 gives the KdV itself and j=2 gives equation (4.41). After this mathematical digression let us derive the new BT.

A New BT.

For simplicity we consider the potential version of equation (4.41)

$$w_t + w_5 + \alpha w_1 w_3 + (30-\alpha)w_2^2 + 2\alpha w_1^3 = 0, \qquad (5.105)$$

where $w \equiv u_x$. Motivated by the discussion in §5.5.1 we look for a BT of the form

$$w_1 = -w^2 + A(x,t)w + B(x,t) , \tag{5.106}$$

(compare with (5.89), where $\gamma = 6$); recall that equation (5.106) (solved together with (5.105) when A, B are constants generates the single soliton solution. The functions $A(x,t)$ and $B(x,t)$ are found by the requirement that the operator $(w_1 + w^2 - Aw - B)\partial/\partial u$ is a CAO for equation (5.105). This yields

$$B_t + A_t w + w_6 + (60-\alpha)w_2w_3 + \alpha w_1w_4 + 6\alpha w_1^2 w_2 + 2ww_5$$
$$2\alpha ww_1w_3 + 4\alpha ww_1^3 - Aw_5 - \alpha Aw_1w_3 - 2\alpha Aw_1^3 - (30-\alpha)Aw_2^2 +$$
$$2(30-\alpha)ww_2^2 = 0, \quad \text{when (5.105) holds.}$$

Using (5.105) and (5.106) in the above, we obtain

$$(\ )w^7 + (\ )w^6 + (\ )w^5 + \ldots (\ )w + (\ ) = 0 ,$$

where the parentheses enclose some combinations of derivatives of A and B. Actually the coefficients of w^7, w^6, w^5 are identically equal to zero; equating to zero the coefficients of w^4 we obtain

$$(\alpha-30)(AA_1 + 2B_1 + A_2) = 0 ,$$

i.e. either $\alpha = 30$ and A, B arbitrary, or α = arbitrary and $B = -A_1/2 - A^2/4 + k$, k = constant. The coefficient of w^3 equals to zero iff the coefficients of w^4 equals to zero; equating to zero the coefficient of w^2 we obtain that either

$$\alpha = 20 \quad \text{and} \quad B = -\frac{A_1}{2} - \frac{A^2}{4} + k ,$$

or

$$\alpha = 30 \quad \text{and} \quad B = -\frac{A_1}{4} - \frac{A^2}{4} .$$

i) $\underline{\alpha = 20}$

To express the BT in the customary form let $A = 2v$; then $B = -v_1 - v^2 + k$ and (5.106) gives

$$w_1 = -(w-v)^2 - v_1 + k \ .$$

The equation which is satisfied by v is determined by the coefficient of w^j, $j=1,0$. These coefficients equal to zero iff

$$v_t + v_5 + 20v_1v_3 + 10v_2^2 + 40v_1^3 = 0;$$

i.e. as it was expected v and u satisfy the same equation (the potential version of the second member of Lax's hierarchy).

ii) $\underline{\alpha = 30}$

A similar analysis shows that

$$w_1 + \frac{v_1}{2} + (w-v)^2 = 0 \ , \tag{5.107}$$

where

$$v_t + v_5 + 30v_1v_3 + 60v_1^3 + \frac{45}{2}v_2^2 = 0. \tag{5.108}$$

Therefore, the equation (5.104) admits the BT given by equation (5.107) whenever v satisfies equation (5.108). The interesting feature of this BT is that equation (5.108) is different from equation (5.104). This exemplifies another advantage of our method: when the classical approach is used one must assume i) the general form of (5.107); ii) the equation satisfied by v (in practice it is assumed that either v satisfies the same equation as u, or its linearized version). None of the above assumptions is made when our method is used.

Introducing a parameter in the BT.

The BT defined by equation (5.107) is not in a proper form for discussing solitons because it does not depend on a free parameter; furthermore equation (5.108) does not possess a soliton solution. In the known BT the free parameter reflects the existence of a Lie point symmetry of the equation under consideration (coordinate stretchings for the Sine-Gordon equation, Galilean invariance for the KdV, etc., see also §5.5.3). However, equation (5.104) admits no nontrivial Lie point groups, therefore we cannot introduce a parameter in equation (5.107) using an invariant transformation of equation (5.104). Instead, we introduce a parameter in (5.107) using a coordinate transformation which affects only equation (5.108), requiring only that the transformed equation obtained from (5.108) admits a soliton. This can be achieved by considering

$$V = v + \frac{2}{3}\lambda x - \frac{160}{9}\lambda^3 t ,$$

where the coefficient of t is chosen so that the constant term in the transformed equation is identically equal to zero (otherwise the transformed equation will not admit $V = 0$ as a solution). Then equations (5.107), (5.108) become

$$W_1 + \frac{V_1}{2} + (W-V)^2 - \lambda = 0 , \qquad (5.109)$$

$$V_t + V_5 + 30V_1V_3 + 60V_1^3 + \frac{45}{2}V_2^2 - 20\lambda\left[V_3 + 6V_1^2 - 4\lambda V_1\right] = 0, \qquad (5.110)$$

where

$$W = w + \frac{2}{3}\lambda x - \frac{160}{9}\lambda^3 t .$$

(i) $\underline{V = 0}$

Then

$$(W_S)_1 = -W_S^2 + \lambda, \quad W_S = W_S(\zeta), \quad \zeta = x - 16\lambda^2 t .$$

Therefore

$$w_S = W_S - \frac{2}{3}\lambda x + \frac{160}{9}\lambda^3 t$$

is the potential version of the soliton solution.

(ii) W = 0.

Then $\frac{1}{2}(V_S)_1 + V_S^2 - \lambda = 0$, $V_S = V_S(\zeta)$, $\zeta = x-16\lambda^2 t$ and V_S is the potential version of the soliton solution of (5.110).

Inserting the above solution in (5.109) we obtain the two-soliton solution. However, in order to have a nice algebraic formulation of the multisoliton solution we need to find a superposition principle (analogous to the Bianchi diagram) for the Riccati equation when V_S is not the solution obtained with $W = 0$; this problem is under consideration.

5.5.3 Burgers' Equation and Generalizations.

In this section we investigate CAO of the form

$$X = [u_x - g(x, t, u)] \frac{\partial}{\partial u}, \tag{5.64}$$

for evolution equations of the form

$$u_t + u_{xx} + F(u, u_x) = 0. \tag{5.24}$$

Requiring X to be a CAO for equation (5.24), equation (5.70) yields (with $\tilde{F} = u_{xx} + F$)

$$[g_2 + u_{xxx} + F_1 u_x + F_2 u_{xx} - g_3(u_{xx}+F)]\Big|_{u_x=g} = 0, \tag{5.111}$$

where $F_1 = \frac{\partial F}{\partial u}$, $F_2 = \frac{\partial F}{\partial u_x}$ and g_1, g_2, g_3 defined by (5.75). Replacing the x-derivatives of u, using (5.76a, b, c) we obtain

$$g_2 + g_{11} + 2gg_{13} + F_2 g_1 + g^2[g_3 + \frac{F(u, g)}{g}]_3 = 0. \tag{5.112}$$

In order to find invariant solutions of equation (5.24) we must find particular solutions of equation (5.112). Motivated by the discussion in §5.5.1 we seek solutions such that $[g_3 + \frac{F}{g}]_3 = 0$. Therefore, we week solutions which satisfy simultaneously

$$g_3 + \frac{F}{g} = A(x,t) \; , \tag{5.113}$$

$$g_2 + g_{11} + 2gg_{13} + g_1 F_2 = 0 \; . \tag{5.114}$$

Let us look at some particular cases.

1. $\underline{F = a'(u)u_x}$

Then equation (5.113) yields

$$g = -a(u) + A(x,t)u + B(x,t) \; . \tag{5.115}$$

Substituting the above in (5.114) we obtain

$$(B_t+B_{xx}+2A_xB) + (A_t+A_{xx}+2AA_x)u + (-2a+ua')A_x + B_xa'(u) = 0. \tag{5.116}$$

The above equation has non-trivial solutions for A and B iff $a'(u) = \text{constant}$ or $2a-ua' = \text{constant}$. Therefore, it is sufficient to consider $a(u) = 0$ or $a(u) = \alpha u^2$ (as we can get rid of the linear terms in the equation using a linear transformation).

i) $\underline{a(u) = 0}$

Then equation (5.116) yields

$$A_t + A_{xx} + 2AA_x = 0 \; , \tag{5.117}$$

$$B_t + B_{xx} + 2BA_x = 0 \; . \tag{5.118}$$

Clearly $B = kA$, where $k = \text{constant}$. Therefore, the equation

$$u_t + u_{xx} = 0 \; , \tag{5.119}$$

admits the BT

$$u_x = A(u+k) , \tag{5.120}$$

where A satisfies (5.117). The above corresponds to the Cole-Hopf [78], [79] linearization of Burgers equation.† From our point of view it is interesting that this BT can also be thought as a generalization of $u_x = \Phi(u; A, B)$, when A and B are allowed to depend on x and t.

ii) $a(u) = \alpha u^2$

Then equation (5.116) yields (with $B = 0$)

$$A_t + A_{xx} + 2AA_x = 0 , \tag{5.117}$$

and equation (5.115) indicates that the BT is given by

$$u_x = -\alpha u^2 + Au .$$

However, letting $A = \alpha v$, v satisfies the same equation as u. Therefore, the BT

$$u_x = -\alpha u^2 + \alpha uv , \tag{5.121}$$

maps solutions of equation

$$u_t + u_{xx} + 2\alpha uu_x = 0 , \tag{5.122}$$

among themselves. This seems to be a new result. Let us summarize: It is well known that the Burgers' equation can be linearized. The linearizing transformation, as was pointed out by many investigators   is a BT. Many investigators tried unsuccessfully to obtain linearizations for other interesting evolution equations (KDV, etc.,). However, it was discovered that some of these higher order evolution equations possess BT which map solutions among themselves. Here, it has been shown that the Burgers equation also possesses such a BT given by equation (5.121).

2. $g = a'(u)u_x^2$

Now equation (5.113) yields

$$g = Ae^{-a}\int e^{a(u)}d_u + Be^{-a}. \tag{5.123}$$

Substituting the above in (5.114) we obtain

$$(A_t + A_{xx} + 2AA_x)e^{-a}\int e^{a(u)}du + (B_t + B_{xx} + 2BA_x)e^{-a} = 0\ . \tag{5.124}$$

The above equation has non-trivial solutions for A and B for any A and B for <u>any</u> a(u).

α) $A = 0$

The BT

$$u_x = Be^{-a(u)}, \tag{5.125}$$

maps solutions of equation

$$u_t + u_{xx} + a'(u)u_x^2 = 0\ , \tag{5.126}$$

to solutions of equation

$$B_t + B_{xx} = 0\ . \tag{5.127}$$

β) $B = 0$

The BT

$$u_x = Ae^{-a(u)}\int e^{a(u)}du\ , \tag{5.128}$$

maps solutions of equation (5.126), to solutions of Burgers' equation (5.117). Therefore, equation (5.126) can be linearized and also can be transformed to Burgers' equation for <u>any</u> a(u). This result can be rederived very easily (and in a sense explained) as follows: Let φ satisfy the equation

$$\varphi_t + \varphi_{xx} = 0 , \qquad (5.129)$$

and let $\varphi = A(u)$. Then equation (5.129) becomes equation (5.126) with $a'(u) = \frac{A''(u)}{A'(u)}$, which shows that equation (5.126) can be linearized for any $a(u)$. By the way, the last approach provides the easiest way for obtaining the Cole-Hopf transformation: Let $\frac{A''(u)}{A'(u)} = \beta$, therefore $A = \frac{1}{\beta} e^{\beta u}$ and $\varphi = \frac{1}{\beta} e^{\beta u}$. Differentiating the last equation with respect to x we obtain $\varphi_x = \beta u_x \varphi$. Equation (5.126), with $a'(u) = \beta$ becomes $u_t + u_{xx} + \beta u_x^2 = 0$.

Differentiating this with respect to x and letting $v = u_x$ we obtain $v_t + v_{xx} + 2\beta v v_x = 0$. The BT now becomes $\varphi_x = \beta v \varphi$.

There are different ways by which the above results can be generalized. First we can look for solutions of equation (5.112) in the form $g(x, t, u) = g(v, u)$; in this way the results of §5.4.1 may be rederived. Alternatively, instead of generalizing the solution $g = \phi(u; A, B)$ by $g = \Phi(u; A(x, t), B(x, t))$, we can generalize it by $g = \Phi(u; A(v), B(v))$. For example, the BT given by (5.121) generalizes to the BT

$$u_x = -\alpha u^2 + \alpha u A(v) ,$$

which maps solutions of the Burger's equation (5.122) to solutions of equation

$$v_t + v_{xx} + \frac{A''(v)}{A'(v)} v_x^2 + 2\alpha A(v) v_x = 0 . \qquad (5.130)$$

From the above it is clear that the new approach is quite powerful: By generalizing the solution $g = \Phi(u; A, B)$ (letting A and B to depend on x and t) some basic BT can be derived; by generalizing it further $A = A(v)$, $B = B(v)$ more general BT may obtained.

A. Introducing a parameter in a BT.

Here we shall use the BT given by (5.121) to illustrate i) how a parameter can be introduced in a BT and ii) the use of Bianchi diagrams.

A trivial change of variables transforms equation (5.121) and (5.122) to

$$u_t + uu_x - \nu u_{xx} = 0 , \qquad (5.131)$$

and

$$2\nu u_x = u^2 - uv . \qquad (5.132)$$

Equation (5.131) admits the following obvious Lie point groups

$$t' = t + \alpha , \; x' = x + \beta , \; u' = u , \qquad (5.133)$$

$$t' = \gamma^2 t, \; x' = \gamma x, \; u' = \gamma^{-1} u , \qquad (5.134)$$

$$t' = t, \; x' = x - Ut \; u' = u - U; \; U = \text{const.} \qquad (5.135)$$

The transformations defined by (5.133) and (5.134) leave the equation (5.132) invariant. However, using the transformations (5.135) (and dropping the primes) equation (5.132) becomes

$$2\nu u_x = u^2 - uv + U(u-v). \qquad (5.136)$$

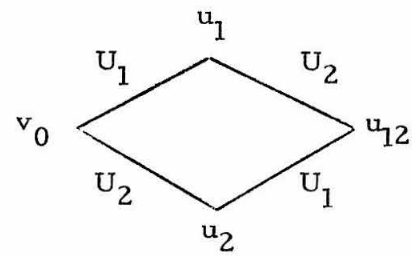
Note further that equation (5.131) admits the following Lie point group of transformations

$$t' = \frac{t}{1-\tau t} , \quad x' = \frac{x}{1-\tau t} , \quad u' = u + \tau(x - ut). \qquad (5.137)$$

Using the above group, we obtain the following BT which depends explicitly on x and t;

$$2\nu u_x = u^2 - uv + \frac{U(u-v)}{1+\tau t} + \frac{\tau}{1+\tau t}\left[x(u-v) - 2\nu \right] . \qquad (5.138)$$

B. Bianchi diagrams.



Let us call u_1 the solution obtained from (5.136) when $v = v_0$ and $U = U_1$, and u_2 the solution obtained from (5.136) when $v = v_0$ and $U = U_2$. The Bianchi diagram indicates that the solution u_{12} can be obtained either using $v = u_1$ and $U = U_2$, or $v = u_2$ and $U = U_1$. Therefore the following equations hold;

$$2\nu u_{1_x} = u^2 - uv_0 + U_1(u_1 - v_0), \tag{5.139a}$$

$$2\nu u_{2_x} = u^2 - uv_0 + U_2(u_2 - v_0), \tag{5.139b}$$

$$2\nu u_{12_x} = u_{12}^2 - uu_1 + U_2(u_{12} - u_1), \tag{5.139c}$$

$$2\nu u_{12_x} = u_{12}^2 - uu_2 + U_1(u_{12} - u_2) . \tag{5.139d}$$

Equations (5.139c,d) imply that

$$u_{12} = \frac{U_2 u_1 - U_1 u_2}{(u_2 - u_1) + (U_2 - U_1)} , \tag{5.140}$$

where u_1, u_2 are defined by (5.139a,b). Let $v_0 = c_2$, $U_1 = -c_1$, $U_2 = -c_3$, where c_1, c_2, c_3 are constants; then (5.139a,b) describe two single shocks, whereas (5.140) describes the interaction of two single shocks. The formulae obtained this way coincide with the corresponding ones given in [24] (see also D below). The validity of the Bianchi diagram can be proved by checking that u_{12} as defined by (5.140) solves equations (5.139c,d).

C. The Burgers hierarchy.

Let us take $\nu=1$ for convenience of writing. It is shown in [53] that the operator

$$\Delta = -D + \tfrac{1}{2}u + \tfrac{1}{2}u_1 D^{-1} , \tag{5.141}$$

is a recursion operator for the equation

$$u_t + uu_1 - u_2 = 0. \tag{5.142}$$

(where $u_i \equiv u_{\underbrace{x\ldots x}_{i}}$). Therefore, the class of equations

$$u_t + \Delta^j u_1 = 0, \quad j = 1, 2, \ldots$$

defines the hierarchy of Burgers equation. The equation (5.142) admits the BT defined by equation (5.136) (where $\nu=1$) and the linearization $u = -2\varphi_1/\varphi$. These BT are admitted by every member of the hierarchy; in particular the hierarchy can be linearized,

$$u_t + \Delta^j u_1 = 0 \quad \leftarrow u = \frac{-2\varphi_1}{\varphi} \rightarrow \quad \varphi_t - \varphi_{j+1} = 0, \ j=1,2,\ldots$$

D. The invariance of shock solutions.

Using the recursion operator Δ defined by (5.141) we have the following collection of admissible LB operators

$$X_j = (\Delta^j u_1)\frac{\partial}{\partial u} \quad , \ j=0,1,2,\ldots$$

It turns out, that the solution describing the interaction of n shocks is invariant under the action of the LB operator

$$X_n \equiv \sum_{j=0}^{n} (\alpha_j \Delta^j u_1)\frac{\partial}{\partial u} \quad ,$$

where α_j are constant parameters.

Let us illustrate the above result for the case of the interaction of two shocks; we must prove that the solution obtained by solving simultaneously equation (5.142) together with

$$\alpha_0 u_1 - \alpha_1 u_t + \alpha_2(-u_3 + \frac{3}{2}uu_2 + \frac{3}{2}u_1^2 - \frac{3}{4}u^2 u_1) = 0, \qquad (5.143)$$

describes the interaction of two shocks. Using the linearization $u = -2\varphi_1/\varphi$ equations (5.142) and (5.143) become

$$\varphi_t - \varphi_2 = 0 ,$$

$$\alpha_0\varphi_1 - \alpha_1\varphi_t - \alpha_2\varphi_3 = 0 .$$

Solving the above we obtain

$$\varphi = f_1 + f_2 + f_3, \quad f_i = \exp(-\frac{c_i x}{2} + \frac{c_i^2 t}{4} - b_i),$$

where c_i, b_i are constants. Then the solution

$$u = \frac{c_1 f_1 + c_2 f_2 + c_3 f_3}{f_1 + f_2 + f_3} ,$$

which is identical with u_{12} defined by (5.140) is the sought solution (see [24] page 111).

5.6 CONCLUSIONS

Let us try to summarize the main results obtained in this chapter. First of all we proved that all solutions of evolution equations obtained through BT can be characterized as invariant solutions under the action of some CAO. We also gave an alternative way of characterizing them, as invariant solutions of the equation defining the BT under the action of some admissible LB operator, which is completely determined by the given evolution equation. The above characterizations, (which were proved to be equivalent), led to an algorithm for obtaining BT based on invariance criteria. In this way many known BT were rederived and the problem of linearizing the equation $u_t + u_{xx} + F(u, u_x) = 0$, using the BT $v_x - f(u, v) = 0$, was completely solved; it was also shown how to extend the above ideas to non evolution equations, for example the Sine-Gordon equation.

After establishing the basic nature of Bäcklund solutions we generalized the BT in the following way: Instead of looking for CAO of the form $\hat{X} = [u_x - f(u, v, v_x, \ldots)] \frac{\partial}{\partial u}$, we look for CAO of the form $X = [u_x - g(x, t, u)] \frac{\partial}{\partial u}$. Clearly every operator $\hat{X}$ is a <u>special case</u> of some operator X. Requiring X to be a CAO for the given equation, we obtain a nonlinear equation for g in three independent variables x, t, u. The main problem is how to find particular solutions of this equation. One way, not very efficient in practice, is to let $g = f(u, v, v_x, \ldots)$. The obvious question is the following: What is the advantage of starting with $g(x, t, u)$ and then letting $g = f(u, v, v_x, \ldots)$, instead of the classical approach starting directly with $f(u, v, v_x, \ldots)$? In practice, when looking for BT we have to assume the arguments of f, and if we want to change the arguments of f the whole process must be repeated. In our approach this is not necessary as we only have to check if a particular form of g is actually a solution of a given equation.

However, we have developed a more efficient way of finding particular solutions of the equation for g: It turns out that there is a particular choice of g which is physically very interesting, $g = \Phi(u)$. This corresponds to the wave-train solution of the given equation (i.e. the solution of the equation invariant under the action of $X_0 = [u_x - \Phi(u)] \frac{\partial}{\partial u}$, is the wave-train solution, assuming that such a solution exists). In general this solution will depend on some constant parameters say A_i, $i = 1, 2, \ldots$. So let us denote it by writing $g = \Phi(u; A_i)$. One very efficient way of finding other particular solutions for g is to let $g = \Phi(u; A_i(x,t))$, where it might be necessary to set some of the $A_i \equiv 0$ in order to simplify the form of g; i.e., we do not generalize the most general wave-train solution but just a particular one, usually the solitary-wave solution. This is interesting both mathematically and physically as the solutions representing soliton interaction can be thought as variation of parameters of some wave-train solution (usually the solitary wave solution). In this way new BT can be obtained. (Also in this way some exact solutions have been found for cases that BT do not exist, for example for the equation $u_{xy} = u^3$; however, these solutions do not seem physically interesting and therefore are not presented here).

Using the above approach a BT was obtained for a *fifth* order evolution equation first introduced by Caudrey et al. [80] and known to have some interesting properties (multisolitons and some conservation laws). The discovery of this BT for the only known evolution equation not fitting the usual scheme (BT, inverse scattering, multi-solitons, conservation laws) strengthens our belief in the existence of a one to one correspondence between multisolitons (or multishocks) and BT (this will be exemplified below). This BT has the interesting feature that it maps solutions of the above equation to solutions of another fifth order equation. This poses the problem of finding a new superposition principle for the corresponding Riccati equation and also formulating a new inverse-scattering scheme; this problem is under consideration.

Using the above approach also, a new BT was obtained which maps solutions of Burgers' equation among themselves. This BT was used to illustrate; i) how a parameter can be introduced in a BT using an admissible Lie point group of the equation under consideration; ii) the use of Bianchi diagrams for constructing a multi-shock (or multisoliton) solution; iii) that the hierarchy obtained from a given equation possesses the BT possessed by the first member of the hierarchy (the members of the hierarchy are viewed here as LB symmetries of the given equation); iv) that the multishocks (or multisolitons) solutions are invariant under the action of some LB operators uniquely defined by the above hierarchy; v) that it is possible to have a BT without having infinite conservation laws or solitons.

The results of chapter IV and V (and in particular the consideration of the above comments) justify in our opinion the following statements:

i) An evolution equation possessing an infinite number of symmetries or a BT or multisolitons (multishocks) automatically possesses the other two.

ii) An evolution equation possesses N-solitons for any N iff it possesses an infinite number of conservation laws.

iii) An evolution equation possessing infinite symmetries and a Lagrangian possesses infinite conservation laws.

Further discussion and some open problems.

In §5.5 it was shown how to relate a CAO to an admissible LB operator (see lemma 5.2). However, this was achieved only by interchanging the roles of the given evolution equation and the equation imposing some group theoretical constraint on this evolution equation: Consider the equation

$$u_t + \tilde{F}(u, u_x, \ldots, u_{\underbrace{x\ldots x}_{m}}) = 0 , \tag{5.65}$$

together with the equation

$$u_x - g = 0 . \tag{5.67}$$

The LB operator

$$X = (u_x - g)\frac{\partial}{\partial u} \tag{5.64}$$

is a CAO for the equation (5.65), iff the LB operator

$$Y = (u_t + \tilde{F})\frac{\partial}{\partial u}$$

is an admissible operator for the equation (5.67).

We would like now to relate a CAO of a given evolution equation to an admissible LB operator of the <u>same</u> equation. That is, now we raise the following question:

<u>Given the evolution equation (5.65) and the CAO (5.64), is it always possible to find an admissible operator Z for equation (5.65)?</u>

We conjecture an affirmative answer to this question. Let us give a very simple example supporting our conjecture. Consider the KdV potential equation

$$w_t + w_{xxx} - \tfrac{1}{2} w_x^2 = 0 \ . \tag{5.144}$$

It was shown in §5.5 (see equation (5.88)) that the soliton solution of this equation is the invariant solution under the action of the <u>CAO</u>

$$X_0 = [w_x - (\tfrac{1}{2} w^2 + Aw + B)] \frac{\partial}{\partial w} \ . \tag{5.145}$$

However, any wave-train solution of (5.144) is an invariant solution under the action of the <u>admissible</u> LB operator

$$Z_0 = (w_t - Uw_x) \frac{\partial}{\partial w} \ . \tag{5.146}$$

Therefore, to the CAO X_0 must correspond the admissible operator Z_0. This is actually easily verified: From equation (5.144) it follows that

$$w_t = -w_{xxx} + \tfrac{1}{2} w_x^2 \ . \tag{5.147}$$

Using the equation

$$w_x = \frac{1}{12} w^2 + Au + B \ ,$$

(and its differential consequences) to simplify the right hand side of equation (5.147) we obtain

$$w_t = (A^2 + \frac{2}{3} B) w_x \ .$$

Therefore, to the CAO (5.145) there corresponds the admissible operator (5.146), where $U = (A^2 + \frac{2}{3}B)^{1/2}$. Notice however, that obtaining an admissible operator from a CAO requires a process of differentiation; therefore the class of the invariant solutions corresponding to a CAO, will be just a subclass of the class of the invariant solutions corresponding to the admissible operator. In the above example, the soliton solution is just a special case of the wave-train solution.

Now let us consider the case of the two-soliton solution of the KdV equation

$$u_t + uu_x + u_{xxx} = 0 . \tag{5.148}$$

The two-soliton solution of the above equation is the invariant solution under the action of the CAO

$$X = (-u-w^2 + 2v(\xi)w-\hat{U})\frac{\partial}{\partial u} , \tag{5.149}$$

where w is a solution of (5.144), $v(\xi)$ is the one-soliton solution with speed $4U$, and $\hat{U}$ is a constant. To the above CAO corresponds the admissible operator

$$Z = [\tfrac{3}{5}u_{xxxxx} + uu_{xxx} + 2u_x u_{xx} + \tfrac{1}{2}u^2 u_x + U_1(u_{xxx}+uu_x) + U_2 u_x]\frac{\partial}{\partial u} \equiv \Omega\frac{\partial}{\partial u} , \tag{5.150}$$

where U_1 and U_2 are functions of U and $\hat{U}$. The above operator was obtained in [68], see also the discussion in §4.1.2B. The invariant solution corresponding to the CAO (5.149) is the two-soliton solution; what is the invariant solution corresponding to the admissible operator (5.150)? We conjecture that this solution is the one describing the interaction of two wave-train solutions. The conjecture is based on the analogy with the one-soliton case: The CAO X_0 characterizes the soliton solution and the corresponding admissible operator Z_0 characterizes the wave-train solution.

Similarly the CAO X characterizes the two-soliton solution; therefore, we expect the corresponding admissible operator Z to characterize the two-wave-train solution. Our conjecture is consistent with the fact that the equation $\Omega = 0$ (where Ω is defined in (5.150)) contains the two-soliton solution; (this was proved by Lax [70]). Our speculation would be proved by proving that the solution of equation $\Omega = 0$ describes the interaction of two wave-train solutions of the KdV equation.

Our conjecture is further supported by considering the case of a BT: A BT defines infinitely many CAO; however, it also defines infinitely many admissible operators (see for example [50]). Therefore, in this case to each CAO there corresponds an admissible operator. Furthermore each admissible operator generates a conservation law (see §4.2); this underlines the connection between BT and conservation laws.

We remark that proving our conjecture would also clarify the connection between the existence of a two-soliton solution and the existence of an additional conservation law for the equation possessing the two-soliton solution (when a BT does not exist): The two-soliton is the invariant solution under some CAO; to this CAO corresponds some admissible operator which generates a conservation law (using the algorithm developed in chapter IV).

In concluding we point out that proving the above conjecture would unify the two approaches proposed here for obtaining invariant solutions. Let us call (for the sake of this argument) a solution invariant under the action of an admissible LB operator an invariant solution and the solution invariant under the action of a CAO a restricted invariant solution. Proving our conjecture would mean that every restricted invariant solution may be regarded as an invariant solution. However, the concept of a CAO would still be useful because it is in general easier to find a restricted invariant solution than the corresponding invariant solution.

APPENDICES

In Appendix I, the sets of equations (α) and (β) (see §2.3) are completely solved.

In Appendix II, some invariants cubic in the momenta are derived for potentials of the general form

$$V = V(x^2 + \nu y^2), \quad \nu = \text{constant}.$$

In Appendix III, the calculations of §2.4B, regarding the most general two-center potentials admitting quadratic invariants, are elaborated.

In Appendix IV, all the LB operators, linear in u and in all first and second order derivatives of u, which leave Tricomi's equation

$$xu_{yy} - u_{xx} = 0,$$

invariant are found. This example also serves as an illustration of how to obtain admissible LB operators for linear equations.

In Appendix V it is shown that the most general BT of the form

$$u_x - f(u, v, v_x) = 0,$$

mapping solutions of the equation

$$u_{xy} - F(u) = 0,$$

among themselves is given by

$$u_x - \alpha v_x - \Phi(u + \alpha v - g(u - \alpha v)) = 0.$$

In Appendix VI it is shown that the only type of equations, of the general form

$$u_{xy} = F(u), \qquad F''(u) \neq 0,$$

which admit higher order symmetries (i.e. LB as opposed to Lie point) are those for which

$$F''(u) - \lambda F(u) = 0.$$

This is consistent with the fact that the above type of equations are the only ones admitting a BT. This appendix also provides an illustration of how to derive admissible LB operators for nonlinear equations.

It has been observed that all the well known BT can be transformed to the Riccati equation. Further, it is well known that the Riccati equation possesses a nonlinear superposition principle. The nonlinear superposition laws have been emphasized by Ames [81]. In Appendix VII we review some of the above results and show that all BT considered in Chapter V are special cases of the generalized Riccati equation.

APPENDIX I

I. The general solution of the set (α)

The compatibility equations for a_1 are:

$$a_{1\,x_1x_1} + a_{1\,x_2x_2} = 0 \qquad (1)$$

$$a_{1\,x_1x_1} + a_{1\,x_3x_3} = 0 \qquad (2)$$

$$(a_{1\,x_2x_2} + a_{1\,x_3x_3})_{x_2} = 0 \qquad (3)$$

Therefore

$$a_{1\,x_1x_1x_2} = a_{1\,x_2x_2x_2} = a_{1\,x_3x_3x_2} = 0\,, \qquad (4.1)$$

(4.2), (4.3) by cyclic permutation. Using (4) in the set (α) we obtain:

$$a_{1\,x_1x_1x_3} = a_{1\,x_2x_2x_3} = a_{1\,x_3x_3x_3} = a_{1\,x_1x_2x_3} = 0 \qquad (5.1)$$

Therefore

$$a_1 = \hat{a}_1(x) + \alpha_1 y^2 x + \beta_1 z^2 x + \gamma_1 xy + \delta_1 xz + \epsilon_1 yz + \zeta_1 y^2 + \eta_1 z^2 + \theta_1 y + \kappa_1 z,$$

a_2, a_3 by cyclic permutation. Substituting the above in (α) we obtain (2.16).

II. The general solution of the set (β)

The compatibility equations for b_{ij} are:

$$\nabla^2_{x_i, x_j}(b_{ij}) = 0, \quad i \neq j; \quad i,j = 1,2,3 \qquad (6)$$

where

$$\nabla^2_{x_i, x_j} \equiv \left(\frac{\partial}{\partial x_i}\right)^2 + \left(\frac{\partial}{\partial x_j}\right)^2.$$

Also

$$b_{12\,x_2x_2} - b_{13\,x_2x_3} + b_{23\,x_1x_3} = 0, \qquad (7.1)$$

(7.2), (7.3) are obtained by cyclic permutation. Eliminating b_{12} from (7.2), (7.3) and replacing $-b_{23\,x_2x_2}$ by $b_{23\,x_3x_3}$ we obtain

$$\nabla^2_{x_1, x_2}(b_{13\,x_1} - b_{23\,x_2}) = 0. \qquad (8.1)$$

However, $(b_{13_{x_2}} + b_{23_{x_1}}) = - b_{12_{x_3}}$ (from (β.7)) and therefore

$$\nabla^2_{x_1, x_2}(b_{13_{x_2}} + b_{23_{x_1}}) = 0. \qquad (8.2)$$

(8.1) together with (8.2) yield :

$$\nabla^4_{x_1, x_2} b_{13} = \nabla^4_{x_1, x_2} b_{23} = 0.$$

Cyclically we obtain the following set for b_{12}:

$$\nabla^4_{x_1, x_3} b_{12} = 0 \qquad (9.1)$$

$$\nabla^4_{x_2, x_3} b_{12} = 0 \qquad (9.2)$$

and

$$\nabla^4_{x_1, x_2} b_{12} = 0. \qquad (9.3)$$

Adding (9.1) and (9.2), and using (9.3):

$$\left[\left(\frac{\partial}{\partial x_3}\right)^4 + \left(\frac{\partial}{\partial x_1}\right)^4 \right] b_{12} = 0. \qquad (9.4)$$

Finally using (9.4) in (9.1) and (9.2):

$$b_{12_{x_3 x_3 x_1 x_1}} = b_{12_{x_3 x_3 x_2 x_2}} = 0. \qquad (10)$$

Integrating equation (10) we get:

$$b_{12} = A(z)xy + A_1(z)x + A_2(z)y + A_3(z) + A_4(x, y)z + A_5(x, y), \qquad (11.1)$$

b_{23}, b_{31} are obtained by cyclic permutation where $x \to y \to z$, $A \to B \to C$, $A_i \to B_i \to C_i$, $1 \leq i \leq 5$.

To determine the form of A, A_i, B, B_i, C, C_i we must substitute (13) in (8) and (9). Substituting (13) in (8):

$$\nabla^2_{x, y} A_4 = \nabla^2_{x, y} A_5 = 0, \qquad (12.1)$$

(12.2), (12.3) are obtained cyclically.

Substituting (11) in (7.1):

$$A_{4_{yy}}(x, y)z + A_{5_{yy}}(x, y) + B'(x)y + B_2'(x) + B_{4_z}(y, z) - C'(y)x -$$

$$C_1'(y) - C_{4_z}(z, x) = 0 \qquad (13)$$

Differentiating the above with respect to z and using

(12.3):

$$A_{4_{yy}}(x,y) + B_{4_{zz}}(y,z) + C_{4_{xx}}(z,x) = 0.$$

Therefore

$$A_{4_{yy}} = a_{41}(x) + a''_{42}(y)$$

$$B_{4_{zz}} = b_{41}(y) + b''_{42}(z)$$

$$C_{4_{xx}} = c_{41}(z) + c''_{42}(x) \qquad (14)$$

where

$$C''_{42}(x) = -a_{41}(x), \quad c_{41}(z) = -b''_{42}(z), \quad b_{41}(y) = -\alpha''_{42}(y). \qquad (15)$$

Therefore

$$A_4 = a_{41}(x)\frac{y^2}{2} + a_{42}(y) + a_{43}(x)y + a_{44}(x), \qquad (16.1)$$

similarly for B_4, C_4.

Using (16) in (12) we obtain:

$$a_{41}(x) = \alpha_{41}\frac{x^2}{2} + \alpha_{42}x + \alpha_{43}$$

$$a_{42}(y) = -\alpha_{41}\frac{y^4}{24} - \alpha_{44}\frac{y^3}{6} + \alpha_{45}\frac{y^2}{2} + \alpha_{46}y + \alpha_{47}$$

$$a_{43}(x) = \alpha_{44}\frac{x^2}{2} + \alpha_{48}x + \alpha_{49}$$

$$a_{44}(x) = -\alpha_{41}\frac{x^4}{24} - \alpha_{42}\frac{x^3}{6} - (\alpha_{45} + \alpha_{43})\frac{x^2}{2} + \alpha_{410}x + \alpha_{411}.$$

(17.1)

(15) yields:

$$\alpha_{41} = \gamma_{41}, \quad \alpha_{42} = \gamma_{44}, \quad \gamma_{43} = -\gamma_{45}. \qquad (18.1)$$

Using the above expressions (and the ones obtained from the above cyclically) in (13) we get:

$$A_{5_{yy}} = -B'(x)y + C'(y)x - B'_2(x) + C'_1(y) + \gamma_{42}\frac{x^2}{2} + \gamma_{48}x$$

$$-\beta_{44}\frac{y^2}{2} - \beta_{48}y - \beta_{49} - \beta_{46} + \gamma_{410}. \qquad (19.1)$$

Using (19.1) and $A_{5_{xx}} + A_{5_{yy}} = 0$:

$$B(x) = \beta_1 \frac{x^4}{24} + \beta_2 \frac{x^3}{6} + \beta_3 \frac{x^2}{2} + \beta_4 x + \beta_5$$

$$B_2(x) = \gamma_2 \frac{x^4}{24} + \beta_6 \frac{x^3}{6} + \beta_7 \frac{x^2}{2} + \beta_8 x + \beta_9$$

$$C_1(y) = \beta_2 \frac{y^4}{24} + \beta_6 \frac{y^3}{6} + c_{10} \frac{y^2}{2} + c_{11} y + c_{12} \tag{20}$$

Also A_5 is now completely determined. A, C, A_2, C_2, A_1, B_1 are obtained by cyclic permutation where $\beta_1 = \alpha_1 = \gamma_1$. Substituting (16), (17), (20) in (11) and then (11) in (β.7) we obtain (2.17), (2.18).

APPENDIX II

In this appendix we shall investigate cubic invariants admitted by potentials of the general form

$$V = V(x^2 + \nu y^2). \tag{1}$$

The most general invariant cubic in the momenta is given by (see lemma 2.1)

$$\begin{aligned} I_3 = {} & \alpha p_1^3 + \beta p_2^3 + \gamma p_1^2 p_2 + \delta p_2^2 p_1 + \varepsilon m^3 + \zeta p_1 m^2 + \theta p_2 m^2 \\ & + k p_1^2 m + \lambda p_2^2 m + \mu p_1 p_2 m + \alpha_1(\underline{x}) p_1 + \alpha_2(\underline{x}) p_2 . \end{aligned} \tag{2}$$

Equations (2) and (2.27c) define the $d_{jk\ell}$ within the constant parameters $\alpha, \beta \ldots \mu$. Substituting these $d_{jk\ell}$ in (2.29h) and using equation (1) to evalute the relevant derivatives of V, we obtain an equation coupling V and the parameters $\alpha, \beta \ldots \mu$. Solving this equation we find that

i) For $\nu = 1$, no new nontrivial cubic invariants are found.

ii) For $\nu \neq 1$, the parameters k, λ generate nontrivial cubic invariants: The parameters k, λ correspond to

$$d_{111} = -ky, \quad d_{222} = \lambda x, \quad d_{112} = kx, \quad d_{221} = -\lambda y. \tag{3}$$

Using (2) and (3) in (2.27h) we find

$$\left[(3\nu-13)\nu k - (3-13\nu)\lambda \right] V'' + 2\left[(-\lambda - 2k\nu + 3\nu\lambda)x^2 + (k\nu - 3k + 2\lambda)\nu^2 y^2 \right] V''' = 0, \tag{4}$$

where

$$V' = \frac{dV}{dz}, \quad z = x^2 + \nu y^2 .$$

1. $V''=0$ $(V = \frac{1}{2}x^2 + \frac{1}{2}\nu y^2)$.

Then k, λ are arbitrary. Using (3) and (1) in (2.29e, f) we find

$$a_1 = 3y(kx^2 + \nu^2\lambda y^2), \quad a_2 = -\frac{3x}{\nu}(kx^2 + \nu^2\lambda y^2) . \qquad (5)$$

However, because equation (2.29h) is only a necessary condition we have to check if the $d_{jk\ell}$ as defined by (3) and the a_i as defined by (5) satisfy equations (2.9) with $V = \frac{1}{2}x^2 + \frac{1}{2}\nu y^2$. A straightforward substitution shows that this is the case iff

$$k(\nu-9) = 0, \quad \lambda(9\nu-1) = 0 .$$

Therefore, either $\nu=9$, $\lambda=0$ and k=aribtrary or $\nu = \frac{1}{9}$, $k=0$, λ=arbitrary. Finally using these values in (5) we obtain:

Potential	Invariant
$\frac{1}{2}x^2 + \frac{9}{2}y^2$,	$p_1^2 m + 3x^2 y p_1 - \frac{1}{3}x^3 p_2$,
$\frac{1}{2}x^2 + \frac{1}{18}y^2$,	$p_2^2 m + \frac{1}{27}y^3 p_1 - \frac{1}{3}xy^2 p_1$.

2. $V'' \neq 0$

Then equation (4) makes sense iff

$$(k\nu-3k+2\lambda)\nu = (-\lambda-2k\nu + 3\nu\lambda), \quad \text{or} \quad \nu k = \lambda.$$

Then it becomes $3zV''' + 8V' = 0$, or

$$V = Az^{-\frac{2}{3}} + Bz. \qquad (6)$$

Letting

$$d_{111} = -y, \; d_{222} = \nu x, \; d_{112} = x, \; d_{221} = -\nu y \qquad (7)$$

in (2.29e, f) we find

$$a_1 = 6y(x^2 + \nu^3 y^2)V', \quad a_2 = \frac{-6x}{\nu}(x^2 + \nu^3 y^2)V'. \qquad (8)$$

Finally using (6), (7), (8) in (2.9) we find $B=0$, $\nu = -1$.

APPENDIX III

In this appendix it is proved that the most general two-center potentials admitting constants of motion quadratic in the momenta are:

i) $$V_1 = \alpha\rho^2 + \beta\rho_0^2 , \quad (1)$$

with corresponding invariants given by equations (2.35), (2.36), and

ii) $$V_2 = \alpha_0\rho^2 + \alpha_0\rho_0^2 + \alpha\rho^{-1} + \beta\rho^{-1} , \quad (2)$$

with corresponding invariant given by equation (2.37).

In §2.5. it was shown that the most general constant of motion quadratic in the momenta is given by

$$I_2 = \hat{\alpha}p_1^2 + \hat{\beta}p_2^2 + \gamma p_1p_2 + \delta p_1m + \varepsilon p_2m + \zeta m^2 + c(x, y), \quad (3)$$

where all the greek letters denote constant parameters. Letting

$$I_2 = b_{11}p_1^2 + b_{22}p_2^2 + 2b_{12}p_1p_2 + c(x, y), \quad (4)$$

we obtain

$$\begin{aligned} b_{11} &= \hat{\alpha} - \delta y + \zeta y^2 , \\ b_{22} &= \hat{\beta} + \varepsilon x + \zeta x^2 , \\ 2b_{12} &= \gamma + \delta x - \varepsilon y - 2\zeta xy . \end{aligned} \quad (5)$$

Let

$$V = F(R) + G(R_0) , \quad (6)$$

where

$$R \equiv \rho^2 = x^2 + y^2, \quad R_0 = \rho_0^2 = (x-x_0)^2 + y^2 . \quad (7)$$

The b_{ij}, defined by equation (5), are coupled with the potential V through the equation (δ.1) (when restricted to two dimensions).

Letting

$$F' = \frac{dF}{dR}, \quad G' = \frac{dG}{dR_0},$$

equation (δ.1) yields:

$$\{2(\hat{\alpha}-\hat{\beta})xy + \gamma(y^2-x^2)-\delta xR-\varepsilon yR\}F'' + \frac{3}{2}\{-\varepsilon y-\delta x\}F' +$$

$$\{2(\hat{\alpha}-\hat{\beta})(x-x_0)y + \gamma[y^2-(x-x_0)^2] + \delta[-x(R+x_0^2) + 2x_0R] \tag{8}$$

$$-\varepsilon y(R-x_0^2)-2\zeta x_0y(R-2x_0x)\}G'' + \frac{3}{2}\{-\varepsilon(1+\frac{2x_0\zeta}{\varepsilon})y-\delta x-x_0\delta\}G' = 0$$

i) $F' = \alpha, \; G' = \beta$.

Then the potential V takes the form

$$V = \alpha R + \beta R_0 . \tag{9}$$

From equation (8) it follows that $\hat{\alpha}, \hat{\beta}, \gamma$ are arbitrary, and $\delta = 0$; also

$$\alpha + \beta + \frac{2x_0\zeta\beta}{\varepsilon} = 0 \tag{10}$$

Therefore,

$$\zeta = \text{arbitrary and } \varepsilon = -\frac{2x_0\zeta}{1+\frac{\alpha}{\beta}} . \tag{11}$$

Therefore the potential V given by equation (9) admits the following invariants:

$$\begin{aligned} \alpha &: p_1^2 + c_1 \\ \beta &: p_2^2 + c_2 \\ \gamma &: p_1p_2 + c_3 \\ \varepsilon,\zeta &: m^2 - \frac{2x_0}{1+\frac{\alpha}{\beta}}p_2m + c_4 . \end{aligned} \tag{12}$$

Integrating equations (γ) we determine the functions c_i, $1 \leq i \leq 4$. Then the formulae (2.35), (2.36) follow.

ii) $G'' \neq 0$.

From equation (8) it follows that $\hat{\alpha} = \hat{\beta} = \gamma = \delta = 0$; also $\varepsilon \neq 0$, iff

$$RF'' + \frac{3}{2}F' = -[R - x_0^2 + \frac{2\zeta x_0}{\varepsilon}(R - 2x_0 x)]G'' - \frac{3}{2}(1 + \frac{2x_0\zeta}{\delta})G' . \quad (13)$$

The above equation has a nontrivial solution iff the coefficient of G'' is a function of R_0 only. Let

$$\frac{\zeta x_0}{\varepsilon} = -1 \quad (14)$$

and this coefficient becomes R_0. Then equation (13) reduces to

$$RF'' + \frac{3}{2}F' = R_0 G'' + \frac{3}{2}G' ,$$

which can be solved to yield the potential (2). The invariant corresponding to ζ = arbitrary and $\varepsilon = -\zeta x_0$ (see equation (14)) is given by

$$I = m^2 - x_0 p_2 m + c .$$

Integrating equations (γ) we determine $c(x, y)$ and then the invariant given by (2.37) follows.

APPENDIX IV.

In this appendix we derive LB operators of the form

$$X = \eta\frac{\partial}{\partial u} , \tag{1}$$

admitted by Tricomi's equation

$$xu_{yy}-u_{xx} = 0 , \tag{3.15}$$

where

$$\eta = au + bu_x + cu_y + du_{xy} + fu_{yy} + g \tag{2}$$

and a, b, d, f, g are functions of x and y.

Note that the operator (1) with η defined by (2), is the most general LB operator linear in u and in all first and second order derivatives of u; the term u_{xx} is missing, but this is without loss of the generality as u_{xx} can be expressed in terms of u_{yy}, using equation (3.15).

Extending the operator X and applying it to equation (3.15) we obtain

$$X(3.15) = x(D_y)^2\eta-(D_x)^2\eta .$$

Therefore, X is admissible iff

$$x(D_y)^2\eta - (D_x)^2\eta \,\Big|_{(3.15)} = 0 \tag{3}$$

Writing the above equation in full, using equation (3.15) and its differential consequences to eliminate u_{xx} and higher x-derivatives, and then equating to zero the coefficients of u and its derivatives we obtain:

$$u: \; xa_{yy} - a_{xx} = 0 \tag{4}$$

$$u_x: \; xb_{yy} - b_{xx} - 2a_x = 0 \tag{5}$$

$$u_y: \; xc_{yy} - c_{xx} + 2xa_y = 0 \tag{6}$$

$$u_{yy}: \; xf_{yy} - f_{xx} + 2xc_y - 2xb_x - b = 0 \tag{7}$$

$$u_{xy}: \; xd_{yy} - d_{xx} + 2xb_y - 2c_x = 0 \tag{8}$$

$$u_{yyx}: \; 2xd_y - 2f_x = 0 \tag{9}$$

$$u_{yyy}: \; 2xf_y - 2xd_x - d = 0 \tag{10}$$

$$(u)^0: \; xg_{yy} - g_{xx} = 0 \tag{11}$$

To solve equations (5) and (6) we introduce the auxiliary functions ψ and Ω defined by:

$$\psi_x \equiv xb_y \; , \tag{12.a}$$

$$-2a + \psi_y \equiv b_x \; , \tag{12.b}$$

$$\Omega y \equiv c_x \; , \tag{13.a}$$

$$-2xa + \Omega_x \equiv xc_y . \tag{13.b}$$

Note that the compatibility equation of equations (12) ($\psi_{xy} = \psi_{yx}$) implies equation (5) and the compatibility equation of equations (13) implies (6). Now equation (8) yields

$$\tfrac{1}{2}\left(\frac{d}{x}\right)_x + 2\psi_x - 2c_x = 0 \; ,$$

or

$$d = xA(y) + 4xc - 4x\psi \; . \tag{14}$$

Equation (9) implies

$$f_x = x\, d_y \; . \tag{15}$$

Then, using equation (14) and (15) equation (7) can be integrated to yield

$$b = 4xa - \tfrac{1}{2}xA'(y). \tag{16}$$

Now solving equation (5) for a, we obtain

$$a = -\frac{1}{60}x^3A'''(y) + B(y) . \tag{17}$$

Using the above in (4) we find

$$A'''' = 0 \quad ; \quad B'' = -\frac{1}{10}A''' \tag{18}$$

To completely determine d we need ψ, which is found by integrating equations (12);

$$\psi = -\frac{1}{906}x^6A'''' + \frac{4}{3}x^3B' - \frac{1}{6}x^3A'' + 6\int Bdy - \tfrac{1}{2}A + K, \tag{19}$$

where K is a constant. Then d and f follow from equation (14) and (15):

$$d = -\frac{4}{30}x^4A'' + x[4D(y) - 24\int Bdy + 3A - 4K] , \tag{20}$$

$$f = -\frac{1}{45}x^6A''' + \frac{x^3}{3}[4D' - 24B + 3A'] + \varphi(y). \tag{21}$$

Now using equation (8) we can determine c:

$$c = \frac{4}{3}x^3a_y - \frac{1}{5}x^3A''(y) + D(y). \tag{22}$$

Using the above expressions in (6) and (7) we deduce that,

$$A'' = 0, \; D'' = 6B', \; \frac{2}{3}\varphi' = 4D - 24\int Bdy + 3A - 4K. \tag{23}$$

Equations (23) and (18) yield,

$$B = \alpha y + \beta , \quad A = \gamma y + \delta; \quad D = 3\alpha y^2 + 6\beta y + \zeta, \tag{24}$$

where all the greek letters denote constant parameters. Therefore,

$$a = B, \quad b = 4xB - \tfrac{1}{2}xA', \quad c = \frac{4}{3}x^3B' + D,$$
$$d = \frac{2}{3}x\,\varphi', \quad f = \frac{1}{3}x^3 - \frac{2}{3}\varphi' + \varphi . \tag{25}$$

Finally, equations (23), (24) and (25) completely determine a, b, c, d, f; and g is any solution of equation (3.15). Hence the LB operators given by equations (3.16) follow.

APPENDIX V

In this appendix it is shown that the most general BT of the form

$$u_x - f(u, v, v_x) = 0 ,$$

mapping solutions of the equation

$$u_{xy} - F(u) = 0 , \qquad (5.49)$$

among themselves, is given by

$$u_x - \alpha v_x - \phi(u + \alpha v - g(u - \alpha v)) = 0 , \qquad (1)$$

where α = constant, and ϕ and g depend on $F(u)$. A special case of the above is

$$u_x - v_x - \phi(u+v) = 0 , \qquad (5.51)$$

which was used in §5.4.3.

Assuming that

$$X = [u_x - f(u, v, v_x)] \frac{\partial}{\partial u} \equiv T \frac{\partial}{\partial u}$$

is a CAO for equation (5.49) we obtain

$$[u_{xxy} - D_{xy} f - F'(u)(u_x - f)]_{(5.49)} = 0 , \quad \text{when} \quad T = 0.$$

Therefore,

$$F'(u)u_x - u_x(f_{11}u_y + f_{12}v_y + f_{13}F(v)) - v_x(f_{21}u_y + f_{22}v_y + f_{23}F(v))$$
$$-v_{xx}(f_{31}u_y + f_{32}v_y + f_{33}F(v)) - f_1F(u) - f_2F(v) - f_3F'(v)v_x - F'(u)(u_x - f) =$$
$$0 , \quad \text{when} \quad T = 0 , \qquad (2)$$

where $f_1 \equiv \frac{\partial f}{\partial u}$, $f_2 \equiv \frac{\partial f}{\partial v}$, $f_3 \equiv \frac{\partial f}{\partial v_x}$. Taking the y-derivative of T,

$$u_y = \frac{1}{f_1}(F(u) - f_2 v_y - f_3 - D_y T) . \qquad (3)$$

To solve (2) we replace u_x by f, u_y be the right hand side of (3) (with $T = 0$), and then equate to zero the coefficients of v_{xx}, $v_x v_y$, v_x and v_y. From the coefficients of v_{xx} we deduce that

$$f_{13} = f_{23} = f_{33} = 0 ,$$

or

$$f = \alpha v_x + \phi(u, v) . \tag{4}$$

From the coefficient of $v_x v_y$ we deduce that

$$\frac{\alpha\phi_{21}+\phi_{22}}{\phi_2} = \frac{\alpha\phi_{11}+\phi_{12}}{\phi_1} , \tag{5}$$

where $\phi_1 \equiv \frac{\partial\phi}{\partial u}$, $\phi_2 = \frac{\partial\phi}{\partial v}$. Let

$$\bar{u} \equiv u + \alpha v, \quad \bar{v} \equiv u-\alpha v , \tag{6}$$

and equation (5) becomes

$$\frac{1}{\phi_2}(\phi_2)_{\bar{u}} = \frac{1}{\phi_1}(\phi_1)_{\bar{u}} ,$$

or

$$\phi_2-\hat{g}(\bar{v})\phi_1 = 0 . \tag{7}$$

Writing equation (7) in terms of the $\bar{u}, \bar{v}$ variables and integrating we find

$$\phi = \phi(u+\alpha v, \; g(u-\alpha v)) .$$

To completely determine ϕ in terms of $F(u)$ we must equate to zero the coefficients of v_x and v_y. This is done in §5.4.3 for the special case that $\phi = \phi(u+\alpha v)$.

APPENDIX VI

In this appendix it is shown that the only type of equations, of the general form

$$u_{xy} = F(u), \quad F''(u) \neq 0 \tag{5.49}$$

which admit higher order symmetries are those for which

$$F''(u) - \lambda F(u) = 0 . \tag{1}$$

To prove the above statement we should consider the operator

$$X = \eta \frac{\partial}{\partial u} , \tag{2}$$

where $\eta = \eta(u, u_x, u_y, u_{xx}, u_{xxx}, u_{yy}, u_{yyy}, \ldots)$. However, we only present the proof for the case that

$$\eta = \eta(u, u_x, u_{xx}, u_{xxx}); \tag{3}$$

the generalization is straightforward. The operator X, where η is defined by (3) is an admissible LB operator of equation (5.49) iff

$$D_{xy}\eta - F'(u)\eta \Big|_{(5.49)} = 0 \tag{4}$$

Writing the above equation in full, using equation (5.49) and its differential consequences to replace u_{xy}, u_{xyx}, u_{xyxxx} by lower x-derivatives, and then equating to zero the coefficients of u_{xxxx}, $u_y u_x$, ..., we obtain the following: From the coefficient of u_{xxxx} we deduce that

$$\eta = f(u, u_x, u_{xx}) + u_{xxx}. \tag{5}$$

Then, from the coefficients of $u_y u_x$, $u_y u_{xx}$, $u_y u_{xxx}$, $u_x u_{xxx}$, u_{xxx} we deduce that

$$f = \alpha u + \beta u_{xx} + a(u_x) , \qquad (6)$$

where α and β are constants and $a(u_x)$ is an arbitary function of u_x. Then, equation (4) reduces to

$$[F(u)a''(u_x) + 3u_x F''(u)]u_{xx} + \alpha g + u_x F'(u)a' + \beta u_x^3 F''(u) + + u_x^2 F'''(u) - \alpha u F'(u) - F'(u)a = 0 \qquad (7)$$

Equating the coefficient of u_{xx} to zero, in equation (7), we obtain

$$\frac{F''(u)}{F(u)} = -\frac{1}{3}\frac{a''(u_x)}{u_x} = \lambda , \qquad (8)$$

where λ is a constant. Therefore

$$F''(u) - \lambda F(u) = 0 , \qquad (1)$$

and

$$a(u_x) = -\frac{\lambda}{2}u_x^3 + \gamma u_x + \delta , \qquad (9)$$

where γ and δ are arbitrary constants. Substituting $a(u_x)$, as defined by equation (9), in equation (7) and using (1) (where we assume $\lambda \neq 0$) we deduce that (7) is identically satisfied iff

$$\alpha=0, \ \beta=0, \ \delta=0, \ \gamma = \text{arbitrary constant} \qquad (10)$$

Equations (5), (6), (9) and (10) imply that, the equation (5.49) admits the LB operator (2), where η is given by (3) iff equation (1) holds; then the operator X is given by

$$X = (-\frac{\lambda}{2}u_x^3 + u_{xxx} + \gamma u_x)\frac{\partial}{\partial u} . \qquad (11)$$

Note that the pressence of the term γu_x, in the above expression is well justified as the equation (5.49) is invariant under x-translation, i.e. if admits the operator

$$\frac{\partial}{\partial x} \sim u_x \frac{\partial}{\partial u} .$$

APPENDIX VII

In this appendix we review some results about the nonlinear supperposition of solutions of nonlinear first order equations. These results are important in connection with the BT considered in chapter V, because all these BT can be thought of as a special case of a generalized Riccati equation (see below for the definition of this equation).

Vessiot [82] in 1893! posed the following interesting question: Which equations of the form

$$u_x = F(u, x), \tag{1}$$

have the property that, <u>any</u> solution u of equation (1) can be expressed in terms of a fixed function of n other solutions of the same equation (1)? That is, for which equations (1) there exist a function f such that

$$u = f(u_1, \ldots, u_n) , \tag{2}$$

where u, u_i ; $1 \leq i \leq n$, are solutions of (1)? Vessiot with the above question posed the problem of determining all equations (1) which possess the <u>nonlinear superposition</u> law expressed by equation (2). He solved this problem, using group-theoretical arguments:

An equation (1) possesses a nonlinear superposition law expressed by (2) iff it can be expressed in one of the following three forms:

1) $$u_x = \hat{U}(u)\hat{X}(x) , \tag{3}$$

where $\hat{U}$ and $\hat{X}$ are arbitrary functions of u and x respectively.

2) $$u_x = a(x)y_1(u) + b(x)y_2(u) \ , \tag{4}$$

where a and b are arbitrary functions of x and y_1, y_2 are two linearly independent solutions of

$$\frac{d^2y}{du^2} + \lambda(u)\frac{dy}{dx} + \lambda'(u)y = 0 \ , \tag{5}$$

where $\lambda(u)$ is an arbitrary function of u.

3) $$u_x = A(x)V_1(u) + B(x)V_2(u) + C(x)V_3(u) \ , \tag{6}$$

where A, B and C are arbitrary functions of x and V_1, V_2, V_3 are three linearly independent solutions of

$$\frac{d^3V}{du^3} + \mu(u)\frac{dV}{du} + \tfrac{1}{2}\mu'(u)V = 0 \ , \tag{7}$$

where $\mu(u)$ is an arbitrary function of u.

It is interesting that letting $u = \varphi(U)$ equations (3), (4) and (6) can be transformed to equations (8), (9), and (10) respectively, where

$$U_x + P(x)U = 0 \ , \tag{8}$$

$$U_x + P(x)U = Q(x), \tag{9}$$

$$U_x = p(x)U^2 + q(x)U + v(x) \tag{10}$$

Clearly equation (10), the Riccati equation, contains equations (8) and (9). Similarly equation (6) contains equations (3) and (4); equation (6) is called the generalized Riccati equation. Therefore, the Vessiot result can be restated in the following form:

An equation (1) possesses a nonlinear superposition law expressed by (2) iff under the transformation $u = \varphi(U)$, it can be transformed to the Riccati equation.

Now we show that the BT considered in chapter V are of the above type (if t is regarded as a parameter):

i) <u>The Sine-Gordon equation</u>

The BT is given by

$$u_x = (v_x) + (2\alpha \sin \frac{v}{2}) \cos \frac{u}{2} + (2\alpha \cos \frac{v}{2}) \sin \frac{u}{2} .$$

Regarding the function v as some given function of x and t, the above equation is of the type (6) iff there exists an $\mu(u)$ such that equation (7) has as solutions

$$V_1 = 1, \quad V_2 = \cos \frac{u}{2} , \quad V_3 = \sin \frac{u}{2} . \tag{11}$$

Clearly

$$\mu(u) = \frac{1}{4} , \tag{12}$$

is the proper $\mu(u)$.

ii) <u>The KDV potential equation</u>

The BT is given by (see equation (5.91))

$$u_x = - \frac{\gamma}{6} u^2 + Au - \frac{3}{\gamma} (Ax + \frac{A^2}{2}) + k .$$

The above equation is already of the Riccati type;

$$V_1 = 1, \quad V_2 = u, \quad V_3 = u^2 , \tag{13}$$

and

$$\mu = 0 \tag{14}$$

Similarly, it is easily verified that the BT of the modified KDV potential equation, as well as the Cole-Hopf linearization of Burgers equation, are of the above type. It is also interesting that the BT derived in §5.5.2 (see equation (5.121)) is also of the Ricatti type. Finally, the equation (5.107) is already of the Ricatti type.

It is well known that the Riccati equation (and therefore the generalized Riccati equation) can be linearized. This establishes the connection with the inverse scattering method. Also, for the Riccati equation, the superposition law (2) takes the form

$$\frac{U_4-U_3}{U_4-U_1} = k \frac{U_2-U_3}{U_2-U_1} , \tag{15}$$

where U_i, $1 \leq i \leq 4$ are solution of (10) and k is a constant. The above superposition law is obviously reflected to a similar one for the generalized Riccati equation. This provides the explanation for the existence of the Bianchi diagrams [83] .

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