PERIODIC SOLUTIONS OF INTEGRO-DIFFERENTIAL EQUATIONS WHICH ARISE IN POPULATION DYNAMICS

Thesis by

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In Partial Fulfillment of the Requirements

for the Degree of Doctor of Philosophy

California Institute of Technology

Pasadena, California

(1979)

(Submitted June 12, 1978)

ACKNOWLEDGMENTS

I wish to thank Professor Donald S. Cohen for suggesting the topic of this thesis and for his invaluable help and lively encouragement in the course of its completion. I would also like to thank Professors Herbert Keller and Paco Lagerstrom for their attention and help in this and other matters. I am also indebted to my fellow graduate students for the helpful suggestions they have provided. I thank Pam Hagan for the excellent job of typing and Deborah Massey for offering to do it.

During my stay at Caltech financial support was provided by Fellowships, Graduate Teaching Assistantships, and a Graduate Research Assistantship.

This thesis is dedicated to Clyde Dubbs and Alan Sharples for their friendship and support during the years.

ABSTRACT

The problem of the existence and stability of periodic solutions of infinite-lag integro-differential equations is considered. Specifically, the integrals involved are of the convolution type with the dependent variable being integrated over the range $(-\infty, t]$, as occur in models of population growth. It is shown that Hopf bifurcation of periodic solutions from a steady state can occur, when a pair of eigenvalues crosses the imaginary axis. Also considered is the existence of traveling wave solutions of a model population equation allowing spatial diffusion in addition to the usual temporal variation. Lastly, the stability of the periodic solutions resulting from Hopf bifurcation is determined with aid of a Floquet theory.

The first chapter is devoted to linear integro-differential equations with constant coefficients utilizing the method of semi-groups of operators. The second chapter analyzes the Hopf bifurcation providing an existence theorem. Also, the two-timing perturbation procedure is applied to construct the periodic solutions. The third chapter uses two-timing to obtain traveling wave solutions of the diffusive model, as well as providing an existence theorem. The fourth chapter develops a Floquet theory for linear integro-differential equations with periodic coefficients again using the semi-group approach. The fifth chapter gives sufficient conditions for the stability or instability of a periodic solution in terms of the linearization of the equations. These results are then applied to the Hopf bifurcation problem and to a certain population equation modeling periodically fluctuating environments to deduce the stability of the corresponding periodic solutions.

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INTRODUCTION

We study periodic solutions of nonlinear integro-differential equations which arise in population dynamics. More precisely, we examine the bifurcation of such solutions for equations of the form

$$\frac{dN_i}{dt} = f_i\left(N_j(t), \sum_m \int_{-\infty}^{t} K_{jm}(t-s)N(s)\,ds\right),$$

where $i, j, m = 1, \ldots, n$, and the f_i are arbitrary functions of the dependent variables N_i and the convolution integrals.

Equations of this type were first introduced by Volterra [17] as models of population growth and incorporate the essential feature of time delays or lags in the effects that the various controlling parameters have on the population. Examples of this are age variations within the population, self pollution, and in general any biological process whose influence may not be immediately felt. For example, if the young of a population do not receive proper nourishment the population growth is not affected until they mature when the birth rate of that species may decline. Usually $f_i = N_i g_i$ so that for one species N say,

$$\frac{dN}{dt} = bN\left(1 - \frac{1}{c}\int_{-\infty}^{t} K(t-s)N(s)\,ds\right)$$

where the integral takes into account the effects of a limited food supply causing the growth rate to decrease as the population increases, but with a time lag due perhaps to hoarding of the food (b is the birth). It is in this context of biological population dynamics that the integro-differential equations we consider have received the most attention (see, for example, Cushing [3]).

It is known that the equations can have periodic solutions even in the case of one dependent variable. Suppose, for example, that the maximum influence on growth rate at time t is due to a population density at a previous time $t - T$. If T increases past a critical value, T_0, a stable equilibrium state for the population can become unstable. Thus, the possibility of the bifurcation of a periodic solution from the steady state presents itself. We examine this question for the general case given above with $\underset{\sim}{f}$ depending on a single parameter. We obtain sufficient conditions for the bifurcation and construct the periodic solutions. The method we use for the construction is a singular perturbation procedure. Even though many periodic solutions of infinite-lag integro-differential equations have been found it has been impossible up to now to determine their stability (except in isolated special cases). We provide a method for deciding this. For the stability analysis we utilize the theory of semi-groups of operators.

In Chapter I we study in detail linear integro-differential equations with constant coefficients. This is done by the method of semi-groups following the analysis of Hale [4] for the analogous case of delay-differential equations with finite lags (as opposed to our case of infinite lags). We take this approach to introduce the essential ideas which are amenable to future generalization in Chapter IV to case of linear equations with periodic coefficients. In the present case we obtain a spectral analysis of the equations. We then introduce the adjoint equation and use this to study the forced linear equation obtaining a variation of constants formula in terms of the semi-group.

In Chapter II we extend the Hopf bifurcation of periodic solutions to nonlinear integro-differential equations and prove the corresponding existence theorem. We then construct the solutions by means of a two-timing perturbation scheme obtaining behavior analogous to that of ordinary differential equations. This then leads us to conjecture about the stability of the periodic solution, which we consider fully in Chapter V.

In Chapter III we derive and examine a model population equation allowing spatial diffusion in addition to the usual temporal variation. This model is a generalization of constant time lags to explain traveling waves observed in certain predator-prey situations. By the use of the two-timing perturbation scheme we also obtain the bifurcation of traveling waves. The corresponding existence theorem is proved.

In Chapter IV we begin to examine the question of stability of periodic solutions by studying linear integro-differential equations with periodic coefficients. Again, by following the methods of Hale [4], we obtain a spectral analysis of these equations using the semi-group approach. The results are strikingly similar to the Floquet theory of ordinary differential equations. We prove a Fredholm Alternative theorem for the existence of periodic solutions of periodically forced linear systems, again using a variation of constants formula.

In Chapter V we solve the problem of stability by giving sufficient conditions for the stability or instability of a periodic solution of a system of nonlinear integro-differential equations. In the case of autonomous equations, conditions for orbital stability are given generalizing the classical Poincaré theorem. The methods are completely analogous to those of ordinary differential equations, transforming the equation into a nonlinear Volterra integral equation by use of the variation of constants formula. Then techniques such as those in Coddington and Levinson [1] are applied to this form of the equations. It is here that we see the advantage of the semi-group approach. Finally we apply the results to deduce the stability of the periodic solutions constructed in Chapter II; and to a certain nonautonomous population equation. The method is general enough to apply to any case of bifurcating periodic solutions from a steady state.

CHAPTER I

CHAPTER I

LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

For later use it will be convenient to first establish certain results for the simplest class of integro-differential equations, namely linear equations with constant coefficients. For finite-lag integro-differential equations a rather complete theory has been given by J. Hale [4]. For our case, namely infinite-lag equations, a direct extension of Hale's semi-group methods is possible, and we get a spectral characterization of the solutions, much as with ordinary differential equations. Thus, solutions are divided into various classes of exponential growth or decay and this is what we need to analyze the problems that confront us in later chapters. There is a more fundamental reason this semi-group approach is used, however, rather than a more direct Laplace transform approach, (cf. Miller [12]). In Chapter IV we need results for linear equations with periodic coefficients, and there the methods used here are necessary. The point of view we adopt is a new and different way of looking at solutions of integro-differential equations and is credited by Hale to Krasovskii. Instead of considering the solution $\underset{\sim}{x}(t)$ as a collection of points, one for each value of t , we think of it as a collection of <u>functions</u> x_t , for each value of t , where each x_t contains all of the values $\underset{\sim}{x}(s)$ for $s \leq t$. Thus, knowing just x_t for some t , one could find $\underset{\sim}{x}(t)$ for a bit larger value of t directly from the equation since the integral that occurs could be evaluated. We shall see that this way of looking at the problem is very fruitful for yielding the spectral decomposition.

We now consider

$$\frac{d\underset{\sim}{x}}{dt} = A\underset{\sim}{x}(t) + \int_0^\infty K(s)\,\underset{\sim}{x}(t-s)\,ds \tag{1.1}$$

where A is a constant $n \times n$ matrix, $\underset{\sim}{x}(t)$ a column vector in $\mathbb{R}^n$, and $K(s)$, the kernel, is any $n \times n$ matrix function defined for $s \geq 0$ which satisfies certain conditions we specify. Throughout this and the following chapters we shall assume that $K(s)$ is continuous and exponentially decreasing as $s \to \infty$: $K(s) = O(e^{-\gamma s})$ for some $\gamma > 0$. This is the case in the majority of applications. Since the kernel always occurs in convolution with the unknown, we adopt the following notation:

$$(K * x)(t) = \int_0^\infty K(s)\,\underset{\sim}{x}(t-s)\,ds.$$

Also for brevity define the linear operator L :

$$L\underset{\sim}{x} \equiv A\underset{\sim}{x} + K * \underset{\sim}{x}.$$

The inhomogeneous version of (1.1) is

$$\dot{\underset{\sim}{x}} = \frac{d\underset{\sim}{x}}{dt} = L\underset{\sim}{x} + \underset{\sim}{f}(t) \tag{1.2}$$

where $\underset{\sim}{f}(t)$ is a continuous column n-vector. Equation (1.1) in addition to the specification of arbitrary initial data for $\underset{\sim}{x}(t)$ on $(-\infty, t_0]$ constitutes a well-posed initial value problem. Thus, if $\underset{\sim}{x}(t) = \underset{\sim}{\phi}(t)$ for $t \leq t_0$ then (1.1) has a unique solution for all $t > t_0$. Also, if $\underset{\sim}{\phi}$ generates $\underset{\sim}{x}(t)$ for t up to τ, then the solution that is in turn generated by $\underset{\sim}{x}$ for $t \geq \tau$, say $\underset{\sim}{y}(t)$, coincides with the solution generated originally by $\underset{\sim}{\phi}$. Thus we may view the solution of (1.1) as a succession of functions from some space of initial values. We view $\underset{\sim}{x}(t)$ as a family of functions $\underset{\sim}{x}_t$ each of which is an element of a space C of functions defined on $(-\infty, 0]$ such that

$$\underset{\sim}{x}_t(\theta) = \underset{\sim}{x}(t+\theta), \qquad \theta \in (-\infty, 0].$$

The process of obtaining $\underset{\sim}{x}_t$ from $\underset{\sim}{\phi}$ is that of a linear transformation T on C into C. Thus,

$$\underset{\sim}{x}_t = T(t, 0)\,\underset{\sim}{\phi}, \qquad t \geq 0,$$

where $\underset{\sim}{x} = \underset{\sim}{\phi}$ for $t \leq 0$. If the initial data is meant for all t up to s instead of zero then

$$\underset{\sim}{x}_t = T(t,s)\,\underset{\sim}{\phi}, \qquad t \geq s.$$

It is easy to see that for $\tau \leq s \leq t$, $\underset{\sim}{x}_t = T(t,s)\,\underset{\sim}{x}_s = T(t,s)\,T(s,\tau)\,\underset{\sim}{x}_\tau$, so that

$$T(t,\tau) = T(t,s)\,T(s,\tau), \qquad \tau \leq s \leq t.$$

Also if $\underset{\sim}{x}(t)$ solves (1.1) then $\underset{\sim}{x}(t+c)$ is also a solution. Thus $T(t,s) = T(t+c, s+c)$ and in particular $T(t,s) = T(t-s, 0)$. We adopt the notation $T(t) \equiv T(t,0)$ for $t \geq 0$, so

$$\underset{\sim}{x}_t = T(t)\,\underset{\sim}{\phi}\ ,$$

with

$$T(t)\,T(s) = T(t+s), \qquad t,s \geq 0,$$
$$T(0) = I$$

where I is the identity on C. We shall take

$$C = \{\underset{\sim}{\phi} \text{ continuous on } (-\infty,0] \mid \sup_{\theta\leq 0} |e^{\gamma^+\theta}\underset{\sim}{\phi}(\theta)| < \infty,$$
$$\text{and } \lim_{\theta\to-\infty} e^{\gamma^+\theta}\phi(\theta) \text{ exists}\},$$

$$\|\phi\| = \sup_{\theta\leq 0} |e^{\gamma^+\theta}\phi(\theta)|,$$

where γ^+ is any number arbitrarily close to γ but $<\gamma$, the decay constant for the kernel. Then C forms a Banach space. Also we use the L_2-norm on $\mathbb{R}^n \subset \mathbb{C}^n$: $|p| = (\sum_1^n |p_i|^2)^{1/2}$ and $p\cdot\psi = \sum_1^n \bar{p}_i\psi_i$. The growth restrictions imposed on $\underset{\sim}{\phi} \in C$ are those that guarantee the convergence of $K * x$. Also the solution $\underset{\sim}{x}_t$ generated by $\underset{\sim}{\phi}$ is itself in C since $\underset{\sim}{x}_t(\theta) = \underset{\sim}{\phi}(t+\theta)$ for sufficiently large $|\theta|$. This is why we are viewing the solution as a collection of functions $\underset{\sim}{x}_t$, namely it enables us to introduce the linear operators $T(t)$ acting on a Banach space. We can then deduce their spectral properties which will characterize the solution of (1.1). We turn to this next.

We see that $T(t)$ is a bounded linear operator on C. This can be shown with the aid of the resolvent of (1.1). The resolvent is defined as an $n\times n$ matrix function $R(t)$ solving the equation

$$\frac{d}{dt} R(t) = R(t)A + \int_0^t R(t-u)K(u)\,du,$$
$$R(0) = I.$$

It is easily verified that this is equivalent to the Volterra integral equation

$$R(t) = I + \int_0^t R(t-s)\left[A + \int_0^s K(u)\,du\right] ds.$$

Standard existence theorems (cf. Miller [11]) assert that a continuously differentiable $R(t)$ exists for $t \geq 0$, and also it is easily shown that if $\underset{\sim}{x}$ solves

$$\frac{d\underset{\sim}{x}}{dt} = A\underset{\sim}{x} + \int_0^t K(t-s)\underset{\sim}{x}(s)\,ds + \underset{\sim}{g}(t)$$

then

$$\underset{\sim}{x}(t) = R(t)\underset{\sim}{x}(0) + \int_0^t R(t-s)\underset{\sim}{g}(s)\,ds.$$

In our case if $\underset{\sim}{x}$ solves (1.1) then by writing $\underset{\sim}{g}(t) = \int_{-\infty}^0 K(t-u)\underset{\sim}{x}(u)\,du$ we have

$$\underset{\sim}{x}(t) = R(t)\underset{\sim}{\phi}(0) + \int_0^t R(t-s)\int_{-\infty}^0 K(s-u)\phi(u)\,du\,ds$$

where $\underset{\sim}{x}_t = \underset{\sim}{\phi}$ at $t=0$. This gives an explicit formula for $\underset{\sim}{x}(t)$ in terms of the initial data $\underset{\sim}{\phi}$ and we have

$$[T(t)\phi](\theta) = \begin{cases} R(t+\theta)\,\phi(0) + \int_0^{t+\theta} R(t+\theta-s)\int_{-\infty}^{0} K(s-u)\,\phi(u)\,du\,ds, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad -t \le \theta \le 0, \\ \phi(t+\theta), \qquad\qquad\qquad\qquad\qquad \theta \le -t. \end{cases}$$

Thus,

$$\begin{aligned} \|T(t)\phi\| &= \sup_{\theta\le 0} |e^{\gamma^+\theta} x_t(\theta)| \\ &\le \max\Big\{ \sup_{-t\le\theta\le 0} \Big[|R(t+\theta)\,e^{\gamma^+\theta}| + \\ &\qquad \int_0^{t+\theta} |e^{\gamma^+(\theta-s)} R(t+\theta-s)| \int_{-\infty}^{0} |K(s-u)\,e^{-\gamma^+(s-u)}|\,du\,ds \Big] \|\phi\|, \\ &\qquad e^{-\gamma^+ t}\|\phi\| \Big\} \le A_1 \|\phi\| \end{aligned}$$

for some constant A_1 so $T(t)$ is a bounded linear operator. Similarly we can show that $T(t)$ is strongly continuous, i.e. for each $\phi \in C$

$$\lim_{t\to s} \|T(t)\phi - T(s)\phi\| = 0.$$

Thus $\{T(t) : t \ge 0\}$ forms a semi-group of linear operators on C. Following Hale [4] we now appeal to the large literature on semi-groups of operators to derive certain specific results.

Let $\mathcal{A}$ denote the infinitesimal operator given by

$$\mathcal{A}\phi = \lim_{t\to 0^+} \frac{T(t)\phi - \phi}{t}$$

for each $\phi \in D(\mathcal{A})$,

$$D(\mathcal{A}) = \{ \underset{\sim}{\phi} \in C : \dot{\underset{\sim}{\phi}} \in C , \dot{\underset{\sim}{\phi}}(0) = A\underset{\sim}{\phi}(0) + \int_0^\infty K(u) \underset{\sim}{\phi}(-u)\, du \}.$$

$\mathcal{A}$ is a closed linear operator and

$$(\mathcal{A}\phi)(\theta) = \begin{cases} \dot{\phi}(\theta), & \theta < 0, \\ A\phi(0) + \int_0^\infty K(u)\, \phi(-u)\, du, & \theta = 0. \end{cases}$$

It is easy to see that $(\mathcal{A}\phi)(\theta) = \dot{\phi}(\theta)$ for $\theta < 0$ and for $\theta = 0$

$$(\mathcal{A}\phi)(0) = \lim_{t\to 0^+} \frac{x_t(0) - \phi(0)}{t} = \lim \frac{x(t) - x(0)}{t} = \dot{x}(0) = (Lx)(0).$$

Also $\mathcal{A}\phi \in C$ implies $\mathcal{A}\phi$ is a continuous function so $\dot{\phi}(0) = (L\phi)(0)$. Appealing to Theorem 10.3.3. of Hille and Phillips [5] we have that $D(\mathcal{A})$ is dense in C and for $\phi \in D(\mathcal{A})$, $T(t)\phi \in D(\mathcal{A})$ and

$$\frac{d}{dt} T(t)\phi = T(t)\mathcal{A}\phi = \mathcal{A}\, T(t)\phi \tag{1.3}$$

for $t \geq 0$. Thus, the semi-group operator satisfies a differential equation in C , and we may analyze its spectrum by examining that of $\mathcal{A}$. Letting $\rho(\mathcal{A})$, $\Sigma(\mathcal{A})$ denote the resolvent set and spectrum, respectively, of $\mathcal{A}$, we have that $\lambda \in \rho(\mathcal{A})$ iff for every ψ in C there is $\phi \neq 0$, $\phi \in D(\mathcal{A})$, such that $(\mathcal{A} - \lambda I)\phi = \psi$. Hence,

$$\dot{\phi}(\theta) - \lambda\, \phi(\theta) = \psi(\theta), \qquad \theta \leq 0,$$

so that

$$\phi(\theta) = e^{\lambda\theta}\,\underset{\sim}{b} + \int_0^{\theta} e^{\lambda(\theta-\xi)}\,\psi(\xi)\,d\xi, \tag{1.4}$$

with $\underset{\sim}{b} \in \mathbb{R}^n$. Then $\phi \in D(\mathcal{A})$ iff $\dot{\phi} \in C$ and $\dot{\phi}(0) = (L\phi)(0)$, i.e. $Re\,\lambda > -\gamma^+$ and

$$\dot{\phi}(0) = \lambda\underset{\sim}{b} + \psi(0) = L\phi(0) = A\underset{\sim}{b} + \int_0^{\infty} K(s)\left\{ e^{-\lambda u}\underset{\sim}{b} + \int_0^{-u} e^{\lambda(-u-\xi)}\psi(\xi)\,d\xi \right\} du.$$

This implies that

$$[\lambda I - A - \hat{K}(\lambda)]\underset{\sim}{b} = -\psi(0) + \int_0^{\infty} K(u) \int_0^{-u} e^{-\lambda(u+\xi)}\psi(\xi)\,d\xi\,du, \tag{1.5}$$

where we denote the Laplace transform

$$\hat{K}(\lambda) = \int_0^{\infty} e^{-\lambda s}\,K(s)\,ds.$$

Now, the right side of (1.5) spans $\mathbb{R}^n$ as ψ spans C. If we assume that $\lambda I - A - \hat{K}(\lambda)$ is a nonsingular matrix then $\underset{\sim}{b}$ exists, and $\phi \in D(\mathcal{A})$ exists for all $\psi \in C$. Thus the range $\mathcal{R}(\mathcal{A} - \lambda I) = C$ and the inverse $(\mathcal{A} - \lambda I)^{-1}$ exists and is bounded. Letting

$$\Delta(\lambda) = \lambda I - A - \hat{K}(\lambda)$$

we have that $\det \Delta(\lambda) \neq 0$ implies $\lambda \in \rho(\mathcal{A})$. If $\det \Delta(\lambda) = 0$ then $\phi(\theta) = \underset{\sim}{b} e^{\lambda\theta}$ where $\Delta(\lambda)\underset{\sim}{b} = 0$, $\underset{\sim}{b} \neq 0$, satisfies $(\mathcal{A} - \lambda I)\phi = 0$. Also if we assume $\mathrm{Re}\,\lambda > -\gamma^+$ then $\phi \in C$ and so $\lambda \in \Sigma(\mathcal{A})$. Now $\phi = (\mathcal{A} - \lambda I)^{-1}\psi$ is given by (1.4) with

$$\underset{\sim}{b} = \frac{1}{\det \Delta(\lambda)}\Big[\text{right side of (1.5)}\Big].$$

Since $\hat{K}(\lambda)$ is an analytic function of λ for $\mathrm{Re}\,\lambda > -\gamma$ then $\det \Delta(\lambda)$ is analytic for such λ and its zeros are isolated and of finite multiplicity and finite in number in any region $\mathcal{R}_c$ of the form $\mathrm{Re}\,\lambda \geq c > -\gamma^+$. Thus $(\mathcal{A} - \lambda I)^{-1}$ is an analytic function in $\mathcal{R}_c$ except for isolated poles and by Theorem 5.8 - A of Taylor [16], for a pole at λ, $\mathcal{A} - \lambda I$ has finite ascent, i.e. for some integer $\nu_\lambda \geq 1$

$$\mathcal{N}(\mathcal{A} - \lambda I)^{\nu_\lambda + 1} = \mathcal{N}(\mathcal{A} - \lambda I)^{\nu_\lambda}$$

($\mathcal{N}$ denotes null space) and

$$\dim \mathcal{N}(\mathcal{A} - \lambda I)^{\nu_\lambda} < \infty.$$

For brevity we shall say that the generalized null space $\mathcal{N}(\mathcal{A} - \lambda I)^{\nu_\lambda}$ of $\mathcal{A} - \lambda I$ is finite dimensional. Furthermore, the theorem states that

$$C = \mathcal{N}(\mathcal{A} - \lambda I)^{\nu_\lambda} \oplus \mathcal{R}(\mathcal{A} - \lambda I)^{\nu_\lambda}.$$

Let $\phi_{1\lambda}, \ldots, \phi_{k\lambda}$ ($k = \dim \mathcal{N}(\mathcal{A} - \lambda I)^{\nu_\lambda} \equiv d_\lambda$) be a basis for the generalized null space $\mathcal{M}_\lambda$ of $\mathcal{A} - \lambda I$. Let

$$\Phi_\lambda = (\phi_{1\lambda}, \dots, \phi_{k\lambda})$$

a row vector in $C^k = C \times \cdots \times C$. Then since $\operatorname{span}\{\phi_{i\lambda}\}$ is invariant under $\mathcal{A}$, we have for some constant $k \times k$ matrix B_λ that

$$\mathcal{A}\,\Phi_\lambda = \Phi_\lambda B_\lambda .$$

By definition of $\mathcal{A}$ we have

$$\frac{d}{d\theta}\,\Phi_\lambda(\theta) = \Phi_\lambda(\theta)\,B_\lambda , \qquad \theta \le 0,$$

so that

$$\Phi_\lambda(\theta) = \Phi_\lambda(0)\,e^{B_\lambda \theta} .$$

Also, by (1.3),

$$\frac{d}{dt}\,T(t)\,\Phi_\lambda = T(t)\,\mathcal{A}\,\Phi_\lambda = T(t)\,\Phi_\lambda B_\lambda$$

so that

$$T(t)\,\Phi_\lambda = \Phi_\lambda\, e^{B_\lambda t},$$

$$[T(t)\,\Phi_\lambda](\theta) = \Phi_\lambda(\theta)\,e^{B_\lambda t} = \Phi_\lambda(0)\,e^{B_\lambda(t+\theta)}$$

and Φ_λ generates the solution of (1.1):

$$(\underset{\sim}{x}_1(t), \ldots, \underset{\sim}{x}_k(t)) \equiv \Phi_\lambda(0)\, e^{B_\lambda t}, \qquad -\infty < t < \infty.$$

Clearly the matrix B_λ has only the single eigenvalue λ so that the $\underset{\sim}{x}_i(t)$ above are finite sums of the form $t^r e^{\lambda t}$. Now if the same process is applied to $\mathcal{R}(\mathcal{A}-\lambda I)^{\nu_\lambda}$ instead of C, and on which $\mathcal{A}$ has spectrum $\Sigma(\mathcal{A}) - \{\lambda\}$ we arrive at a decomposition of C into a direct sum

$$C = \mathcal{M}_{\lambda_1} \oplus \cdots \oplus \mathcal{M}_{\lambda_q} \oplus R$$

where $Re\,\lambda_i \geq c > -\gamma^+$ and R is a closed subspace. Those ϕ in $\mathcal{M}_{\lambda_i}$ generate solutions of the form $t^r e^{\lambda_i t}$. Later we shall show that $\phi \in R$ generates solutions of the form $O(e^{ct})$ as $t \to \infty$. Any solution of (1.1) is a linear combination of those described above. This completes the spectral analysis.

In summary we have

Theorem 1. The space of initial data can be decomposed

$$C = \mathcal{M}_{\lambda_1} \oplus \cdots \oplus \mathcal{M}_{\lambda_q} \oplus R$$

where the eigenvalues λ_i satisfy

$$\det[\lambda_i I - A - \hat{K}(\lambda_i)] = 0,$$

$$Re\,\lambda_i \geq c > -\gamma^+, \qquad i = 1, \ldots, q,$$

and $\phi \in \mathcal{M}_{\lambda_i}$ generates a solution $\underset{\sim}{x}(t) =$ finite sum of terms $t^r e^{\lambda_i t}$. Also $\phi \in \mathcal{R}$ generates a solution which is $O(e^{ct})$ as $t \to \infty$.

Next we turn to an examination of the inhomogeneous equation (1.2). We shall find an explicit form for its solution in terms of the semi-group, namely a variation of constants formula. Then we find necessary and sufficient conditions for the existence of a periodic solution of (1.2) when the forcing term $\underset{\sim}{f}(t)$ is periodic, and this is the Fredholm alternative. To carry this out we introduce the adjoint problem to (1.1). Along with this a bilinear form connecting the two problems is introduced and this will enable us to find an explicit form for the projection operators onto the eigenspace $\mathcal{M}_\lambda$.

The solution of (1.2) for initial data ϕ specified up to time σ is

$$\underset{\sim}{x}_t = T(t-\sigma)\underset{\sim}{\phi} + \int_\sigma^t [T(t-s)X_0]\cdot \underset{\sim}{f}(s)\,ds \qquad (1.6)$$

where X_0 is an $n\times n$ matrix whose columns T acts on, the result being dotted into $\underset{\sim}{f}$, and where

$$X_0(\theta) = \begin{cases} I, & \theta = 0, \\ 0, & \theta < 0. \end{cases}$$

That this solves (1.2) can be verified by substitution into the equation and using the properties of T.

To examine the question of existence of solutions of (1.2) under periodic boundary conditions we shall need to introduce the adjoint equation to (1.1). Following Hale [4], Chapter 21 we define the adjoint integro-differential equation

$$\dot{\underset{\sim}{y}}(s) = -\underset{\sim}{y}(s)A - \int_{-\infty}^{0} \underset{\sim}{y}(s-\xi)\, K(-\xi)\, d\xi, \tag{1.7}$$

where $\underset{\sim}{y}(s)$ is a row vector in $\mathbb{R}^n$. The initial value space is the Banach space

$$\tilde{C} = \{\psi(\xi) \text{ continuous on } [0,\infty) \mid \sup_{\xi \geq 0} |e^{-\gamma^+\xi}\psi(\xi)| < \infty\},$$

$$\|\psi\| = \sup_{\xi \geq 0} |e^{-\gamma^+\xi}\psi(\xi)|$$

Let $\underset{\sim}{y}_s \in \tilde{C}$ for all $s \leq t$ where

$$\underset{\sim}{y}_s(\xi) \equiv \underset{\sim}{y}(s+\xi), \qquad \xi \geq 0.$$

If $\underset{\sim}{y}_\tau = \underset{\sim}{\psi}$ is specified initial data then a unique solution to (1.7) is generated (backwards) for all $s \leq \tau$, and is given by the family of elements of $\tilde{C}$, $\underset{\sim}{y}_s$, as defined above.

Next we define the (usually degenerate) bilinear form (,) on $\tilde{C} \times C$:

$$(\psi, \phi) \equiv \psi(0)\cdot\phi(0) - \int_{-\infty}^{0} \left[\int_{-\theta}^{0} \psi(\xi)\, K(-\theta)\, \phi(\xi+\theta)\, d\xi\right] d\theta \tag{1.8}$$

for $\psi \in \tilde{C}$, $\phi \in C$. The importance of this bilinear form is that it provides the connection between (1.1) and its adjoint (1.7). In fact we prove that if $\underset{\sim}{x}_t$ solves (1.2) and $\underset{\sim}{y}_t$ solves (1.7), for $\sigma \leq t \leq \tau$, then

$$(\underset{\sim}{y}_t, \underset{\sim}{x}_t) = (\underset{\sim}{y}_\sigma, \underset{\sim}{x}_\sigma) + \int_\sigma^t \underset{\sim}{y}(s)\cdot \underset{\sim}{f}(s)\, ds. \tag{1.9}$$

In particular if $\underset{\sim}{x}(t)$ satisfies (1.1) then $(\underset{\sim}{y}_t, \underset{\sim}{x}_t)$ is constant on $[\sigma, T]$. This will be useful later for finding the projection operators onto $\mathcal{M}_\lambda$. To prove (1.9) we have

$$(y_t, x_t) = y_t(0)\cdot x_t(0) + \int_{-\infty}^{0}\left[\int_0^\theta y(t+\xi)\, K(-\theta)\, x(t+\xi+\theta)\, d\xi\right] d\theta$$

$$= y(t)\cdot x(t) + \int_0^\infty y(t+\xi) \int_{-\infty}^{-\xi} K(-\theta)\, x(t+\xi+\theta)\, d\theta\, d\xi$$

$$= y(t)\cdot x(t) + \int_t^\infty y(\xi') \int_{-\infty}^{-\xi'+t} K(-\theta)\, x(\xi'+\theta)\, d\theta\, d\xi'.$$

Thus, on using (1.2), we have

$$\frac{d}{dt}(y_t, x_t) = \dot{y}\cdot x + y\cdot\dot{x} - y(t)\int_{-\infty}^{0} K(-\theta)\, x(t+\theta)\, d\theta$$

$$+ \int_t^\infty y(\xi')\, K(\xi'-t)\, x(t)\, d\xi' =$$

$$= (\dot{y} + yA)\, x(t) + y(t)\cdot f(t) + \left(\int_0^\infty y(t+\xi) K(\xi) d\xi\right) x(t)$$

By (1.7) this equals $y(t)\cdot f(t)$. Integrating the resulting equation yields (1.9).

Returning to the solutions of (1.7), let $\tilde{T}$ be the semi-group of operators corresponding to (1.7) so $y_t = \tilde{T}(t)\psi,\ t \leq 0$. Let $\tilde{\mathcal{A}}$ be the corresponding infinitesimal operator

$$\tilde{\mathcal{A}}\psi = \lim_{t\to 0^-} \frac{\tilde{T}(t)\psi - \psi}{t},$$

$$D(\tilde{\mathcal{A}}) = \{\psi \in \tilde{C} \mid \dot{\psi} \in \tilde{C},\ \dot{\psi}(0) = -\psi(0)A - \int_{-\infty}^{0} \psi(-\xi)K(\xi)d\xi\},$$

$$(\tilde{\mathcal{A}}\psi)(\xi) = \begin{cases} -\dot{\psi}(\xi), & \xi < 0, \\ \psi(0)A + \int_{-\infty}^{0} \psi(-\theta)K(\theta)d\theta, & \xi = 0. \end{cases}$$

The analysis that was performed on the semi-group T can now be applied to $\tilde{T}$ to deduce that the spectrum of $\tilde{\mathcal{A}}$ in $Re\lambda \geq c > -\gamma^+$ consists of a finite number of point eigenvalues, zeros of

$$\det[\lambda I - A - \hat{K}(\lambda)]$$

with a corresponding eigenfunction $\psi(\xi) = e^{-\lambda\xi}\underset{\sim}{b}^T, (\xi \geq 0)$, $\underset{\sim}{b}^T$ a left null-vector of $\lambda I - A - \hat{K}(\lambda)$. Note that $Re\lambda > -\gamma^+$ implies that $\psi \in \tilde{C}$. Thus the spectra of $\mathcal{A}$ and $\tilde{\mathcal{A}}$ are identical in the half-plane $Re\lambda > -\gamma$. Several facts will now be stated which show more clearly the relationships between (1.1) and (1.7). Their derivations are given afterwards.

Lemma 1. For $\phi \in D(\mathcal{A})$, $\alpha \in D(\tilde{\mathcal{A}})$

$$(\alpha, \mathcal{A}\phi) = (\tilde{\mathcal{A}}\alpha, \phi).$$

Lemma 2. Given $\psi \in C$, k an integer ≥ 1,

$$(\mathcal{A} - \lambda I)^k \phi = \psi$$

has a solution $\phi \in C$ iff $(\alpha, \psi) = 0$ for all $\alpha \in \mathcal{N}(\tilde{\mathcal{A}} - \lambda I)^k$.

Lemma 3. $\mathcal{N}(\mathcal{A} - \lambda I)^k$ coincides with all $\phi \in C$ where

$$\phi(\theta) = \sum_{j=0}^{k-1} \underset{\sim}{\gamma}_{j+1} \frac{\theta^j}{j!} e^{\lambda\theta}, \qquad \theta \le 0$$

where $\underset{\sim}{\gamma}_i$ are column n-vectors and $\underset{\sim}{\gamma} = \mathrm{col}\,(\gamma_1, \dots, \gamma_k) \in \mathbb{R}^{nk}$ satisfies $A_k \underset{\sim}{\gamma} = 0$ where

$$A_k \equiv \begin{bmatrix} \Delta & \Delta^{(1)} & \frac{1}{2!}\Delta^{(2)} & \cdots & \frac{1}{(k-1)!}\Delta^{(k-1)} \\ & \Delta & \Delta^{(1)} & \cdots & \frac{1}{(k-2)!}\Delta^{(k-2)} \\ & & \ddots & \ddots & \vdots \\ & 0 & & \Delta & \Delta^{(1)} \\ & & & & \Delta \end{bmatrix}$$

$$\Delta^{(j)} \equiv \frac{d^j \Delta(\lambda)}{d\lambda^j}, \qquad j = 0, 1, 2, \dots,$$

$$\Delta(\lambda) \equiv \lambda I - A - \hat{K}(\lambda).$$

Furthermore, $\phi \in \mathcal{N}(\mathcal{A} - \lambda I)^k$ generates the solution $(x_0 = \phi)$ to (1.1):

$$\underset{\sim}{x}(t) = \sum_{j=0}^{k-1} \underset{\sim}{\gamma}_{j+1} \frac{t^j}{j!} e^{\lambda t}, \qquad -\infty < t < \infty. \tag{1.10}$$

Lemma 4 $\mathcal{N}(\tilde{\mathcal{A}} - \lambda I)^k$ coincides with $\psi \in \tilde{C}$ where

$$\psi(\xi) = \sum_{j=1}^{k} \underset{\sim}{\beta}_j \frac{(-\xi)^{k-j}}{(k-j)!} e^{-\lambda\xi}, \qquad \xi \ge 0$$

where $\underset{\sim}{\beta} = \text{row}(\beta_1, \ldots, \beta_k)$ satisfies $\underset{\sim}{\beta} A_k = 0$. $\psi \in \mathcal{N}(\tilde{\mathcal{A}} - \lambda I)^k$ generates the solution $(y_0 = \psi)$ of (1.7):

$$\underset{\sim}{y}(s) = \sum_{j=1}^{k} \underset{\sim}{\beta_j} \frac{(-s)^{k-j}}{(k-j)!} e^{-\lambda s}, \qquad -\infty < s < \infty.$$

Thus the dimensions of the generalized null spaces (and the ascents) of $\mathcal{A} - \lambda I$, $\tilde{\mathcal{A}} - \lambda I$ are equal, being $\dim \mathcal{N}(A_k)$, $k = d_\lambda$.

Lemma 5. Let $\lambda \in \Sigma(\mathcal{A})$, $\mathrm{Re}\,\lambda > -\gamma^+$

$$\Phi_\lambda = \text{row}(\phi_1, \ldots, \phi_p),$$

$$\Psi_\lambda = \text{col}(\psi_1, \ldots, \psi_p), \qquad p = d_\lambda = \dim \mathcal{M}_\lambda = \dim \tilde{\mathcal{M}}_\lambda$$

be bases for $\mathcal{M}_\lambda$, $\tilde{\mathcal{M}}_\lambda$ respectively. Then the matrix $(\Psi_\lambda, \Phi_\lambda)$ is nonsingular and may be taken to be the $d_\lambda \times d_\lambda$ identity matrix I. Define the projection operators P_λ, Q_λ:

$$P_\lambda \phi \equiv \Phi_\lambda (\Psi_\lambda, \phi),$$

$$Q_\lambda \equiv I - P_\lambda.$$

Assume $C = \mathcal{N}(\mathcal{A} - \lambda I)^p \oplus \mathcal{R}(\mathcal{A} - \lambda I)^p$ as in the theorem. Then P_λ is the projection onto $\mathcal{M}_\lambda$ along $\mathcal{R}(\mathcal{A} - \lambda I)^p$ and Q_λ is the complementary projection. In fact

$$\mathcal{R}(Q_\lambda) = \{\phi \in C : (\Psi_\lambda, \phi) = 0\},$$

$$\mathcal{R}(P_\lambda) = \{\phi \in C : \phi = \Phi_\lambda \underset{\sim}{b}, \ \underset{\sim}{b} \in \mathbb{R}^{d_\lambda}\}.$$

Lemma 6. $P\Sigma\,(T(t))$ = point spectrum of $T(t)$ in $|\mu| \geq e^{ct}$ $(c > -r^*)$ $= \exp\, t\, \Sigma(\mathcal{A})$. More specifically $\mu \in P\Sigma\,(T(t))$ implies there is $\lambda \in \Sigma(\mathcal{A})$ such that $\mu = \exp \lambda t$ and if $\{\lambda_n\}$ consists of all distinct points in $\Sigma(\mathcal{A})$ such that $e^{\lambda_n t} = \mu$ then $\mathcal{N}(T(t) - \mu I)^k$ equals the closed span of the subspaces $\mathcal{N}(\mathcal{A} - \lambda_n I)^k$.

Lemma 7. Letting

$$\Lambda(c) = \{ \lambda \in \Sigma(\mathcal{A}) \mid Re\, \lambda \geq c > -r^* \}$$

and denoting

$$P_\Lambda = \sum_{\lambda \in \Lambda(c)} P_\lambda \quad ,$$

$$Q_\Lambda = I - P_\Lambda \quad ,$$

so that P_Λ projects on $\sum_{\lambda \in \Lambda} m_\lambda$ and Q_λ onto the residual space R as occur in Theorem 1, then there are constants $K, \epsilon > 0$ such that for all $\phi \in C$

$$\| T(t)\, Q_\Lambda \phi \| \leq K\, e^{(c-\epsilon)t} \, \| Q_\Lambda \phi \| \, , \quad t \geq 0. \tag{1.11}$$

Lemma 8.

$$\tilde{\mathcal{A}}\, \Psi_\lambda = B_\lambda\, \Psi_\lambda \quad ,$$

$$\tilde{T}(t)\, \Psi_\lambda = e^{-B_\lambda t}\, \Psi_\lambda \quad ,$$

$$\Psi_\lambda(\theta) = e^{-B_\lambda \theta}\, \Psi_\lambda(0)$$

where B_λ has been defined before: $\mathcal{A}\, \Phi_\lambda = \Phi_\lambda B_\lambda$ (here we assume

$(\Psi_\lambda, \Phi_\lambda) = I$).

The proofs of these statements are as follows.

Proof of Lemma 1: For $\phi \in D(\mathcal{A})$, $\alpha \in D(\tilde{\mathcal{A}})$

$$
\begin{aligned}
(\alpha, \mathcal{A}\phi) &= \alpha(0) \cdot \left[A\phi(0) + \int_{-\infty}^{0} K(-\theta)\phi(\theta)\,d\theta \right] \\
&\qquad - \int_{-\infty}^{0} \int_{-\theta}^{0} \alpha(\xi) K(-\theta) \dot{\phi}(\xi+\theta)\,d\xi\,d\theta \\
&= \alpha(0) \cdot \left[A\phi(0) + \int_{-\infty}^{0} K(-\theta)\phi(\theta)\,d\theta \right] - \int_{-\infty}^{0}\int_{0}^{\theta} \alpha(\xi'-\theta) K(-\theta) \dot{\phi}(\xi')\,d\xi'\,d\theta \\
&= \alpha(0) \cdot \left[A\phi(0) + \int_{-\infty}^{0} K(-\theta)\phi(\theta)\,d\theta \right] - \int_{-\infty}^{0} \alpha(\xi'-\theta) K(-\theta) \phi(\xi') \Big|_{0}^{\theta} d\theta \\
&\qquad + \int_{-\infty}^{0} \int_{0}^{\theta} \dot{\alpha}(\xi'-\theta) K(-\theta) \phi(\xi')\,d\xi'\,d\theta \\
&= \alpha(0) \cdot \left[A\phi(0) + \int_{-\infty}^{0} K(-\theta)\phi(\theta)\,d\theta \right] \\
&\qquad - \int_{-\infty}^{0} \left[\alpha(0) K(-\theta)\phi(\theta) - \alpha(-\theta) K(-\theta)\phi(0) \right] d\theta + \int_{-\infty}^{0}\int_{-\theta}^{0} \dot{\alpha}(\xi) K(-\theta) \phi(\xi+\theta)\,d\xi\,d\theta \\
&= \left[\alpha(0) A + \int_{-\infty}^{0} \alpha(-\theta) K(-\theta)\,d\theta \right] \phi(0) \\
&\qquad + \int_{-\infty}^{0}\int_{-\theta}^{0} \dot{\alpha}(\xi) K(-\theta) \phi(\xi+\theta)\,d\xi\,d\theta \\
&= (\tilde{\mathcal{A}}\alpha, \phi).
\end{aligned}
$$

To prove Lemmas 2, 3, 4, $(\mathcal{A} - \lambda I)^k \phi = \psi$ iff $\phi \in D(\mathcal{A})$ and

$$
\left(\frac{d}{d\theta} - \lambda \right)^k \phi(\theta) = \psi(\theta), \qquad \theta \le 0,
$$

iff $\phi \in D(\mathcal{A})$ and

$$\phi(\theta) = \sum_{j=0}^{k-1} \gamma_{j+1} \frac{\theta^j}{j!} e^{\lambda\theta} + \int_0^\theta e^{\lambda(\theta-\xi)} \frac{(\theta-\xi)^{k-1}}{(k-1)!} \psi(\xi)\, d\xi,$$

where the γ_{j+1} are arbitrary column n-vectors to be determined so that $\phi \in D(A)$. It is easily shown that

$$D^{(m)}\phi(\theta) \equiv \left(\frac{d}{d\theta} - \lambda\right)^m \phi(\theta) =$$

$$\sum_{j=0}^{k-1} \gamma_{m+j+1} \frac{\theta^j}{j!} e^{\lambda\theta} + \int_0^\theta e^{\lambda(\theta-\xi)} \frac{(\theta-\xi)^{k-m-1}}{(k-m-1)!} \psi(\xi)\, d\xi .$$

Now $\phi \in D(A-\lambda I)^k$ iff $D^{(m)}\phi \in D(A)$ for $m = 0, 1, 2, \dots k-1$ iff $\frac{d}{d\theta}(D^{(m)}\phi)(0) = L[D^{(m)}\phi]$, $0 \le m \le k-1$. Now $D^{(m)}\phi \in D(A)$ for $m = 0, 1, \dots, k-2$ iff (since $\frac{d}{d\theta}(D^{(m)}\phi)(0) = \lambda\gamma_{m+1} + \gamma_{m+2}$)

$$\lambda\gamma_{m+1} + \gamma_{m+2} = A\gamma_{m+1} + \sum_{j=0}^{k-m-1} \frac{1}{j!}\left(\int_0^\infty K(u)(-u)^j e^{-\lambda u} du\right)\gamma_{m+j+1}$$

$$+ \int_0^\infty K(u) \int_0^{-u} e^{-\lambda(u+\xi)} \frac{(-u-\xi)^{k-m-1}}{(k-m-1)!} \psi(\xi)\, d\xi\, du$$

iff

$$\Delta\gamma_{m+1} + \sum_{j=1}^{k-m-1} \frac{\Delta^{(j)}}{j!} \gamma_{m+j+1} =$$

$$\int_0^\infty K(u) \int_0^{-u} e^{-\lambda(u+\xi)} \frac{(-u-\xi)^{k-m-1}}{(k-m-1)!} \psi(\xi)\, d\xi\, du$$

iff

$$\Delta\gamma_{m+1} + \Delta^{(1)}\gamma_{m+2} + \cdots + \frac{\Delta^{(k-m-1)}}{(k-m-1)!}\gamma_k =$$

$$= \int_0^\infty K(u) \int_0^{-u} \left[e^{-\lambda s} \frac{(-s)^{k-m-1}}{(k-m-1)!} \right]_{s=u+\xi} \psi(\xi)\, d\xi\, du$$

$$= \int_0^\infty K(u) \int_0^{-u} \alpha_{m+1}(u+\xi)\ \psi(\xi)\, d\xi\, du$$

$$= -(\alpha_{m+1}, \psi)$$

where we define

$$\alpha_j(s) = \frac{(-s)^{k-j}}{(k-j)!} e^{-\lambda s} \qquad j = 1, \dots, k.$$

Similarly $D^{(k-1)} \phi \in \mathcal{D}(\mathcal{A})$ iff (since $\frac{d}{d\theta}[D^{(k-1)}\phi](0) = \lambda r_k + \psi(0)$)

$$\Delta r_k = -\psi(0) + \int_0^\infty K(u) \int_0^{-u} \alpha_k(u+\xi)\ \psi(\xi)\, d\xi\, du$$

$$= -(\alpha_k, \psi).$$

Thus $\phi \in \mathcal{D}(\mathcal{A})$ iff

$$A_k \underset{\sim}{r} = -(\Psi_0, \psi),$$

$$\Psi_0 \equiv \text{diag}\,(\alpha_1 I, \dots, \alpha_k I).$$

This has a solution iff $b^T(\Psi_0, \psi) = 0$ for all b^T left null vectors of A_k. Calculations analogous to those above show that $\alpha \in \mathcal{N}(\mathcal{A}-\lambda I)^k$ iff $\alpha = b^T \Psi_0$ where $b^T A_k = 0$. Thus $(\alpha, \psi) = 0$ is necessary and sufficient. To show that the solution $\underset{\sim}{x}(t)$ generated by ϕ $(x_0 = \phi)$

has the form indicated, note that $x(t) = x_t(0)$ and $x_t(0)$ is obtained from $\phi(\theta)$ by replacing θ by t. The formula can also be verified to satisfy (1.1) by direct and tedious calculation. This proves Lemmas 2 and 3; Lemma 4 being proved similarly.

To show Lemma 5, if $(\Psi_\lambda, \Phi_\lambda)a = 0$ then $(\Psi_\lambda, \Phi_\lambda a) = 0$ so by Lemma 1 $\Phi_\lambda a \in R(A - \lambda I)^p$. But $\Phi_\lambda a \in \mathcal{N}(A - \lambda I)^p$ and these spaces are complementary so $\Phi_\lambda a = 0$, i.e. $a = 0$. Thus, by multiplying Ψ_λ, Φ_λ by nonsingular matrices, we may take $(\Psi_\lambda, \Phi_\lambda) = I_{d_\lambda \times d_\lambda}$. Then $P_\lambda \phi = \Phi_\lambda(\Psi_\lambda, \phi)$ clearly satisfies $P_\lambda^2 = P_\lambda$ and $R(P_\lambda) = \{\Phi_\lambda b\}$ and $P_\lambda \Phi_\lambda = \Phi_\lambda$. Also $R(Q_\lambda) = \mathcal{N}(P_\lambda) = \{\phi : (\Psi_\lambda, \phi) = 0\}$ by Lemma 1.

To show Lemma 6 we refer to an easy extension of Theorem 16.7.2 of Hille and Phillips [5]. To show Lemma 7 we let $X \equiv R(Q_\lambda) = R(A - \lambda I)^p$ which is itself a Banach space. Then we show that $\| T(t)\phi \| \le K e^{(c-\epsilon)t} \|\phi\|$ for $\phi \in X$. First note that $T(t)$ maps X into itself since:

(i) $D(A) \cap X$ is dense in X since by Theorem 5.8-A of Taylor [16] there is a projection $Q' : C$ onto X such that $Q' D(A) \subset D(A)$ and $D(A)$ is dense in C.

(ii) $T(t)$ maps $D(A) \cap X$ into X since $\phi \in D(A) \cap X \Rightarrow$ $\phi = (A - \lambda I)^p \psi$ so $T(t)\phi = (A - \lambda I)^p T(t)\psi$ by commutativity of $T(t)$ and A on $D(A)$ so $T(t)\phi \in X$ by definition.

(iii) $T(t) : X \to X$ by the continuity of $T(t)$.

Let $T_1(t)$ be the restriction of $T(t)$ to X. Then the infinitesimal operator corresponding to T_1 is $A_1 = A/_X$. Then since $\rho(A_1) \supset \{\mathrm{Re}\,\lambda > c_1\}$ where c_1 is some $-\gamma^+ < c_1 < c$ we have by the

Hille-Yosida Theorem (cf. Martin [10], Chapter 7) that $\mu[\mathcal{A}_1] \le c_1$ so $\nu[T_1] = \mu[\mathcal{A}_1]$ and $\| T_1(t) \| \le e^{c_1 t}$ for all $t \geq 0$.

<u>To prove Lemma 8</u> we have $\tilde{\mathcal{A}} \Psi_\lambda = \tilde{B}_\lambda \Psi_\lambda$ for some $\tilde{B}_\lambda$ and $\mathcal{A} \Phi_\lambda = \Phi_\lambda B_\lambda$. Then $(\Psi_\lambda, \mathcal{A}\Phi_\lambda) = (\Psi_\lambda, \Phi_\lambda B_\lambda) = (\Psi_\lambda, \Phi_\lambda) B_\lambda = B_\lambda =$ $(\tilde{\mathcal{A}} \Psi_\lambda, \Phi_\lambda) = \tilde{B}_\lambda (\Psi_\lambda, \Phi_\lambda) = \tilde{B}_\lambda$; and noting

$$\frac{d}{dt} \tilde{T}(t) \psi = -\tilde{T}(t) \tilde{\mathcal{A}} \psi = -\tilde{\mathcal{A}} \tilde{T}(t) \psi ;$$

for $\psi \in \mathcal{D}(\tilde{\mathcal{A}})$ we have $\tilde{T}(t) \Psi_\lambda = e^{-B_\lambda t} \Psi_\lambda$. This completes the proofs of Lemmas 1 through 8.

Now we have a complete picture of what the solution set of (1.1) is like. Since the initial value space C can be decomposed into a certain finite number of subspaces that in turn generate solutions $\underset{\sim}{x}(t)$ of the form $t^r e^{\lambda_i t}$ or $O(e^{-\gamma^* t})$ then any solution $x(t)$ is a (finite) sum of such terms. This achieves the spectral decomposition. The exact form of the sums containing $t^r e^{\lambda_i t}$ is given in (1.10) and note it depends only on $\Delta(\lambda)$.

Finally we turn to the problem of determining periodic solutions of (1.2) when the forcing term $\underset{\sim}{f}(t)$ is periodic. For this we shall need to examine the variation of constants formula (1.6) more closely. If $\Lambda(c)$ is as in Lemma 7, C is decomposed as in Theorem 1 and P_Λ is the corresponding projection. Let us denote

$$x_t^P = P_\Lambda x_t = \Phi(\Psi, x_t),$$

$$x_t^Q = Q_\Lambda x_t .$$

The solution of (1.7) generated by Ψ is $y_t = e^{-Bt}\Psi$, $y(s) = e^{-Bs}\Psi(0)$ so by (1.9)

$$(e^{-Bt}\Psi, x_t) = \int_\sigma^t e^{-Bs}\Psi(0) f(s)\,ds + (e^{-B\sigma}\Psi, x_\sigma).$$

Thus if $x_\sigma = \phi$ we have

$$x_t^P = \int_\sigma^t \Phi\, e^{B(t-s)}\, \Psi(0)\, f(s)\, ds + \Phi\, e^{B(t-\sigma)}(\Psi, \phi)$$

$$= \int_\sigma^t [T(t-s)\,\Phi]\,\Psi(0)\, f(s)\, ds + T(t-\sigma)\,\Phi\cdot(\Psi, \phi).$$

Thus

$$x_t^P = T(t-\sigma)\,\phi^P + \int_\sigma^t [T(t-s)\,X_0^P]\cdot f(s)\, ds \tag{1.12}$$

where we define $X_0^P = \Phi\cdot\Psi(0)$, an $n\times n$ matrix of scalars (Φ is $n\times d_\lambda$, Ψ is $d_\lambda \times n$) and $T(t-s)$ acts on its columns. Also we note that if we define $q(t) \equiv (\Psi, x_t)$ then

$$q(t) = e^{B(t-\sigma)} q(\sigma) + \int_\sigma^t e^{B(t-s)}\,\Psi(0)\, f(s)\, ds, \tag{1.13}$$

$$\dot q(t) = B\, q(t) + \Psi(0)\, f(t). \tag{1.14}$$

Similarly,

$$x_t^Q = T(t-\sigma)\,\phi^Q + \int_\sigma^t [T(t-s)\,X_0^Q]\cdot f(s)\, ds \tag{1.15}$$

with $X_0^Q \equiv X_0 - X_0^P$. (1.12) and (1.15) provide the variation of constants formulas for the projections of the solution x_t .

We are now in a position to prove the Fredholm alternative for periodic forcing. Let

$$\Lambda_1 = \{\lambda : \operatorname{Re}\lambda > 0 , \lambda \in \Sigma(A)\} ,$$

$$\Lambda_0 = \{\lambda : \operatorname{Re}\lambda = 0 , \lambda \in \Sigma(A)\} .$$

Then we have the decomposition

$$x_t = x_t^{P_0} + x_t^{P_1} + x_t^{Q}$$

with $x_t^{P_i}$, x_t^Q satisfying (1.12), (1.15) respectively. We may indeed write, by letting $\sigma \to \infty$ or $\sigma \to -\infty$

$$x_t^Q = \int_{-\infty}^{t} T(t-s) X_0^Q f(s)\, ds ,$$

$$x_t^{P_1} = \int_{\infty}^{t} T(t-s) X_0^{P_1} f(s)\, ds. \tag{1.16}$$

That the integrals converge may be seen by use of (1.11). Notice that if $\underset{\sim}{f}$ is periodic of period ω then x_t^Q and $x_t^{P_1}$ are ω-periodic ($x_{t+\omega}^Q = x_t^Q$ by a simple change of variables in (1.16)).

We have the Alternative Theorem:

<u>Theorem 2</u>. If $\underset{\sim}{f}(t+\omega) = \underset{\sim}{f}(t)$ then a necessary and sufficient condition for the existence of an ω-periodic solution for all t of (1.2) is that

$$\int_0^\omega \underset{\sim}{y}(t) \cdot \underset{\sim}{f}(t)\, dt = 0 \tag{1.17}$$

for all ω - periodic solutions of the adjoint equation (1.7).

Proof: The necessity of (1.17) follows from (1.9) since $(y_{t+\omega}, x_{t+\omega}) = (y_t, x_t)$ for ω -periodic y_t, x_t as follows from (1.8).

To prove the converse it suffices by the note following (1.16) to find an ω -periodic $x_t^{P_0}$. Putting $q(t) = (\Psi_{P_0}, x_t)$ we have equation (1.14) (with $\Psi = \Psi_{P_0}$). Also, the eigenvalues of B are all pure imaginary. Now a periodic solution of (1.14) exists if

$$\int_0^\omega u(t)\ \Psi(0)\ f(t)\, dt = 0$$

for all ω -periodic solutions of $\dot{u} = -uB$ by the Fredholm Alternative for ordinary differential equations. This implies

$$\int_0^{\omega_0} \underset{\sim}{u}_0\, e^{-Bt}\ \Psi(0)\ f(t)\, dt = 0$$

for all $\underset{\sim}{u}_0$ such that $\underset{\sim}{u}_0\, e^{-Bt}$ is ω -periodic. But $\underset{\sim}{u}_0 e^{-Bt}\ \Psi(0)$ for such $\underset{\sim}{u}_0$ form the set of all ω-periodic solutions of the adjoint (1.7). Thus by (1.17) an ω -periodic $q(t)$ exists so $x_t^{P_0} = \Phi_{P_0} q(t)$ is ω -periodic, and finally x_t is ω -periodic. ∎

We note that all ω -periodic solutions of (1.2) are obtained by collecting all $\underset{\sim}{u}_0$ as in the proof.

This completes the theory of linear integro-differential equations with constant coefficients that we will consider. The results will be used in the next chapter.

CHAPTER II

HOPF BIFURCATION

In this chapter we examine periodic solutions of nonlinear integro-differential equations. More specifically, we look for periodic solutions bifurcating from a steady state solution of the equations. This is an extension to integro-differential equations of the Hopf bifurcation developed originally for ordinary differential equations. We shall examine the question by use of a singular perturbation method known as two-timing or the method of multiple scales. First, we give conditions under which bifurcation can occur and then apply the perturbation method to formally construct the periodic solution. The method yields not only the periodic solution but a neighboring family of slowly-varying solutions which approach either the steady state or the periodic solution asymptotically. Thus, the stability of the periodic solution is also resolved. Finally, we give a rigorous proof that the periodic solution so constructed is indeed a solution of the equations.

Hopf bifurcation for finite-delay differential equations has been extensively researched with approaches via the center manifold theorem, the method of averaging, the implicit function theorem, and the method of Poincaré-Linstedt, cf. Marsden and Mc Cracken [9] and Kazarinoff, et al. [8]. We extend the latter two methods to our case of infinite-lag integro-differential equations.

We now examine the method in detail for an equation general enough to bring out the main features. Consider

$$\frac{d\underset{\sim}{N}}{dt} = \underset{\sim}{f}\left(\underset{\sim}{N}, \int_0^\infty \mathcal{K}(\lambda,s)\,\underset{\sim}{N}(t-s)\,ds, \lambda\right), \tag{2.1}$$

where $\mathcal{K}$ is the kernel depending on a parameter λ and $\mathcal{K}(\lambda, s) = O(e^{-\gamma s})$ as $s \to \infty$ for some $\gamma > 0$. We assume that there exists a constant solution (steady state) of (2.1) and that, by shifting $\underset{\sim}{N}$ if necessary, this steady state is $\underset{\sim}{N}(t) \equiv 0$. Further, assume that $\underset{\sim}{f}$ is expanded near $\underset{\sim}{N} = 0$ in a Taylor series with <u>no</u> quadratic terms. Thus

$$\frac{d\underset{\sim}{N}}{dt} = \mathcal{L}(\underset{\sim}{N}, K(\lambda) * \underset{\sim}{N}, \lambda) + Q_3(\underset{\sim}{N}, \mathcal{K}(\lambda) * \underset{\sim}{N}, \lambda) + Q_4 \tag{2.2}$$

where $\mathcal{L}$ is linear in $\underset{\sim}{N}$, and

$$K(\lambda) * \underset{\sim}{N} \equiv \int_0^\infty K(\lambda, s) \underset{\sim}{N}(t-s)\, ds.$$

Thus,

$$\mathcal{L}(\underset{\sim}{N}, K(\lambda) * \underset{\sim}{N}, \lambda) \equiv L(\lambda) \underset{\sim}{N} + \int_0^\infty K(\lambda, s) \underset{\sim}{N}(t-s)\, ds$$

(L is an $n \times n$ matrix), Q_3 is cubic in $\underset{\sim}{N}$ and $\mathcal{K}(\lambda) * \underset{\sim}{N}$, and Q_4 is quartic and higher.

We next examine the linearized problem about the steady state $N \equiv 0$:

$$\frac{d\underset{\sim}{p}}{dt} = L(\lambda) \underset{\sim}{p} + \int_0^\infty K(\lambda, s) \underset{\sim}{p}(t-s)\, ds. \tag{2.3}$$

From the theory of Chapter I we know that the eigenvalues σ of this equation are the zeros of the determinant of

$$\Delta(\sigma, \lambda) = \sigma I - L(\lambda) - \int_0^\infty K(\lambda, s)\, e^{-\sigma s} ds, \qquad (2.4)$$

$$Re\ \sigma \geq -\gamma^+ > -\gamma.$$

Assume that for some value of λ, say λ_1, there are eigenvalues with positive real part and that for some other value, say λ_2, all the eigenvalues have negative real parts. For $\lambda = \lambda_1$ the steady state is unstable since there are solutions $\underset{\sim}{p} = \underset{\sim}{p}_0 e^{\sigma t}$ with $Re\ \sigma > 0$. For $\lambda = \lambda_2$ the steady state is stable (Theorem 1 of Chapter I) since all $Re\ \sigma < 0$. Thus for some λ between λ_1 and λ_2 there is a critical value λ_0 where some eigenvalue σ_0 crosses the imaginary axis. Since the roots of (2.4) occur in complex conjugate pairs and since we assume only one pair crosses the axis, say at $\pm i\mu$, $(\mu \neq 0)$, then (2.3) will have solutions $e^{\pm i\mu t}$, i.e. $\cos \mu t$ and $\sin \mu t$. Thus, the linearized problem has periodic solutions for $\lambda = \lambda_0$. We will show that for λ near λ_0 and to one side of λ_0 there is a periodic solution of (2.2) of amplitude $O(\lambda - \lambda_0)^{1/2}$ when certain conditions are satisfied, namely that all $Re\ \sigma < 0$ except for the pair near $\pm i\mu$ and that if we parametrize this pair $\sigma(\lambda)$ then we require that

$$Re\ \frac{d}{d\lambda}\sigma(\lambda_0) \neq 0,$$

i.e. the pair crosses the imaginary axis not tangentially.

Since the linearization (2.3) turns out to yield all the sufficient conditions for bifurcation, we examine it in more detail. At $\lambda = \lambda_0$ we want $\sigma = \pm i\mu$ so that

$$\det \Delta(\lambda_0, \pm i\mu) = 0. \tag{2.5}$$

Also, we want no other pair to cross the axis so assume that (2.5) has only the single pure imaginary solution $\pm i\mu$. Also, we assume that the pair $\sigma(\lambda)$ that crosses at $\pm i\mu$ is simple, i.e. the dimension of the generalized null space corresponding to $i\mu$ (or to $-i\mu$) is equal to one (cf. Chapter I).

For later use we shall need an expression for $\sigma'(\lambda_0)$, i.e. the speed with which the pair crosses the imaginary axis. Near $\lambda = \lambda_0$

$$\sigma(\lambda) = i\mu + \sigma'(\lambda_0)(\lambda - \lambda_0) + O(\lambda - \lambda_0)^2.$$

Since $\sigma(\lambda)$ is simple there is a single right null vector $\underset{\sim}{x}(\lambda)$ and left null vector $\underset{\sim}{y}^T(\lambda)$ of $\Delta(\lambda, \sigma(\lambda))$. Thus

$$\Delta(\lambda, \sigma(\lambda))\, x(\lambda) = \left[\sigma(\lambda) I - L(\lambda) - \int_0^\infty k(\lambda, s)\, e^{-\sigma(\lambda) s} ds \right] x(\lambda) = 0.$$

Differentiating with respect to λ and denoting $\underset{\sim}{x}(\lambda_0) = \underset{\sim}{x}_0$, we obtain

$$\frac{\partial \Delta}{\partial \lambda}(\lambda_0, i\mu)\, \underset{\sim}{x}_0 + \frac{\partial \Delta}{\partial \sigma}(\lambda_0, i\mu)\, \underset{\sim}{x}_0\, \sigma'(\lambda_0) + \Delta(\lambda_0, i\mu)\, \underset{\sim}{x}'(\lambda_0) = 0.$$

Multiply on the left by $\underset{\sim}{y}_0^T$ to get

$$\sigma'(\lambda_0) = - \frac{\underset{\sim}{y}_0^T \left(\frac{\partial \Delta}{\partial \lambda}\right)_0 \underset{\sim}{x}_0}{\underset{\sim}{y}_0^T\, \Delta^{(1)}\, \underset{\sim}{x}_0} \tag{2.6}$$

where $\Delta^{(1)} \equiv \frac{\partial \Delta}{\partial \sigma}(\lambda_0, i\mu)$. We show later that $\underset{\sim}{y}_0^T \cdot \Delta^{(1)} \cdot \underset{\sim}{x}_0 \neq 0$. Also

$$\left(\frac{\partial \Delta}{\partial \lambda}\right)_0 = -L'(\lambda_0) - \int_0^\infty \frac{\partial K}{\partial \lambda}(\lambda_0, s)\, e^{-i\mu s} ds,$$

$$\Delta^{(1)} = I + \int_0^\infty s K(\lambda_0, s)\, e^{-i\mu s}\, ds. \tag{2.7}$$

(2.6) gives the desired formula; i.e., the speed at which the eigenvalue crosses the imaginary axis.

Next we examine (2.3) for $\lambda = \lambda_0$. Put $L \equiv L(\lambda_0)$, $K(s) \equiv K(\lambda_0, s)$,

$$\frac{d\underset{\sim}{p}}{dt} = L\underset{\sim}{p} + K * \underset{\sim}{p} \equiv \mathcal{L}\underset{\sim}{p}. \tag{2.8}$$

Of course this has the eigenvalues $\pm i\mu$ with corresponding eigenfunctions $e^{\pm i\mu t}$. Since we assume that these are simple, then the matrix A_k in Lemma 3, Chapter I has null space of dimension 1 for each $k = 1, 2, 3, \ldots$ (with $\Delta = \Delta(\lambda_0, i\mu)$). We show that this is equivalent to

$$\underset{\sim}{y}_0^T \, \Delta^{(1)} \underset{\sim}{x}_0 \neq 0. \tag{2.9}$$

Δ has the single null vectors $\underset{\sim}{x}_0$, $\underset{\sim}{y}_0^T$, $\Delta \underset{\sim}{x}_0 = \underset{\sim}{y}_0^T \Delta = 0$. The corresponding solutions of (2.8) are the real and imaginary parts of $\underset{\sim}{x}_0 e^{i\mu t}$

$$Re\,(\underset{\sim}{x}_0\, e^{i\mu t}),$$

$$Im\,(\underset{\sim}{x}_0\, e^{i\mu t}).$$

Now if $\begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}$ is a null vector for A_2, then $\Delta \gamma_2 = 0$ and $\Delta \gamma_1 + \Delta^{(1)} \gamma_2 = 0$.

This has solution $r_2 = x_0$ and $r_1 = 0$. It is easy to see that this is the only solution iff $y_0^T \Delta^{(1)} x_0 \neq 0$. Thus $\dim \mathcal{N}(A_k) = 1$ for $k \geq 1$ iff $y_0^T \Delta^{(1)} x_0 \neq 0$ as was to be proved.

Next we consider the following inhomogeneous equations (for n = integer)

$$\frac{d\underset{\sim}{p}}{dt} - L\underset{\sim}{p} - K * \underset{\sim}{p} = \underset{\sim}{D} \cos n\mu t + \underset{\sim}{E} \sin n\mu t. \tag{2.10}$$

The right side can be written

$$Re\,(\underset{\sim}{r}\, e^{in\mu t})$$

with $\underset{\sim}{r} = \underset{\sim}{D} - i\underset{\sim}{E}$. First consider the case $n \neq 1$. We try a particular solution $\underset{\sim}{p} = Re\,(\underset{\sim}{\rho}\, e^{in\mu t})$ with $\underset{\sim}{\rho}$ undetermined. From (2.10) we get

$$Re\,(\,[\,in\mu I - L - \int_0^\infty K(s)\, e^{-in\mu s} ds\,]\,\underset{\sim}{\rho}\, e^{in\mu t}) =$$

$$Re\,(\underset{\sim}{r}\, e^{in\mu t}).$$

So we choose $\Delta(\lambda_0, ni\mu)\underset{\sim}{\rho} = \underset{\sim}{r}$. But $\Delta(\lambda_0, ni\mu)$ is nonsingular by hypothesis so $\underset{\sim}{\rho} = \Delta^{-1}(\lambda_0, ni\mu)\,\underset{\sim}{r}$. The general solution is

$$\underset{\sim}{p} = Re\,[\,\underset{\sim}{p_0}\, e^{i\mu t} + \Delta^{-1}(ni\mu)\,\underset{\sim}{r}\, e^{in\mu t}\,]$$

for an arbitrary complex constant $\underset{\sim}{p_0}$. Now consider the case $n = 1$. Try the particular solution $\underset{\sim}{p} = [\,\underset{\sim}{\rho_1}\, t e^{i\mu t} + \underset{\sim}{\rho_2}\, e^{i\mu t}\,]$. The left member of (2.9) is

$$Re\,[\,t\,\Delta\underset{\sim}{\rho}_1\,e^{i\mu t} + (I + \int_0^\infty s\,K(s)\,e^{-i\mu s}\,ds)\,\underset{\sim}{\rho}_1\,e^{i\mu t}$$
$$+\ \Delta\underset{\sim}{\rho}_2\,e^{i\mu t}\,]\,.$$

(2.7), (2.10) imply

$$\Delta\underset{\sim}{\rho}_1 = 0\ ,$$
$$\Delta\underset{\sim}{\rho}_2 + \Delta^{(1)}\underset{\sim}{\rho}_1 = \underset{\sim}{r}\,.$$

Thus $\underset{\sim}{\rho}_1 = c\underset{\sim}{x}_0$ for some $c \in \mathbb{C}$ and $\Delta\underset{\sim}{\rho}_2 = \underset{\sim}{r} - c\,\Delta^{(1)}\underset{\sim}{x}_0$. This has a solution iff $\underset{\sim}{y}_0^T(\underset{\sim}{r} - c\cdot\Delta^{(1)}\underset{\sim}{x}_0)$, so that $c = \underset{\sim}{y}_0^T\underset{\sim}{r}/\underset{\sim}{y}_0^T\Delta^{(1)}\underset{\sim}{x}_0$. Then $\underset{\sim}{\rho}_2 = \underset{\sim}{\rho}_2^0 + c_1\underset{\sim}{x}_0$, $c_1 \in \mathbb{C}$. The general solution of (2.10) in this case is

$$\underset{\sim}{p} = Re\,[\,c\,\underset{\sim}{x}_0\,t\,e^{i\mu t} + \underset{\sim}{\rho}_2\,e^{i\mu t}\,]$$

with $c = \dfrac{y_0^T r}{y_0^T\Delta^{(1)}x_0}$ and $\underset{\sim}{\rho}_2$ determined up to an arbitrary complex multiple of $\underset{\sim}{x}_0$. This has an unbounded term $t\,e^{i\mu t}$ and a necessary and sufficient condition for the suppression of this term is that $c = 0$, i.e. $\underset{\sim}{y}_0^T\underset{\sim}{r} = 0$, i.e.

$$\underset{\sim}{D} = \underset{\sim}{E} = 0 \tag{2.11}$$

This condition also follows from the Fredholm alternative (Theorem 2) of Chapter I where it is required that

$$\int_0^{\frac{2\pi}{\mu}} (e^{-i\mu t} \underset{\sim}{y}_0{}^T)\ Re\ (\underset{\sim}{r}\ e^{i\mu t})\ dt = 0$$

since the periodic solutions of the adjoint to (2.8)

$$-\frac{d\underset{\sim}{y}}{dt} = \underset{\sim}{y} L + \int_0^\infty \underset{\sim}{y}(s+\xi)\ K(\xi)\, d\xi$$

are the real and imaginary parts of $e^{-i\mu t}\ \underset{\sim}{y}_0{}^T$ where

$$\underset{\sim}{y}_0{}^T \left[-i\mu I + L + \int_0^\infty e^{-i\mu\xi} K(\xi)\, d\xi \right] = 0,$$

i.e. $y_0{}^T \Delta(i\mu) = 0$. The condition reduces to $\underset{\sim}{y}_0{}^T \underset{\sim}{r} = 0$. The Fredholm alternative condition (2.11) will be used repeatedly in what follows.

We are now in a position to introduce the perturbation scheme. A two-time scale solution will be constructed that tends asymptotically to the required periodic solution. We introduce a small parameter ϵ in terms of which all quantities will be expressed. The two time scales are denoted t^*, τ the fast and slow time respectively. Expanding in powers of ϵ :

$$\underset{\sim}{N} = \underset{\sim}{N}(t^*, \tau) = \epsilon \underset{\sim}{N}_1 + \epsilon^2 \underset{\sim}{N}_2 + \epsilon^3 \underset{\sim}{N}_3 + \cdots ,$$

$$t^* = \rho(\epsilon)\, t ,$$

$$\tau = \epsilon^2 t ,$$

$$\rho(\epsilon) = 1 + \epsilon\rho_1 + \epsilon^2 \rho_2 + \cdots ,$$

$$\lambda(\epsilon) = \lambda_0 + \epsilon\lambda_1 + \epsilon^2\lambda_2 + \cdots .$$

Before preceeding it will be convenient to first calculate $K*N$. Writing t^* and τ as functions of t we have

$$\underset{\sim}{N}(t) \equiv \underset{\sim}{N}(t^*(t), \tau(t)).$$

Then $\underset{\sim}{N}(t-s) = \underset{\sim}{N}(t^*(t-s), \tau(t-s)) = \underset{\sim}{N}(t^* - \rho s, \tau - \epsilon^2 s)$ so

$$K*\underset{\sim}{N} = \int_0^\infty K(s)\,\underset{\sim}{N}(t-s)\,ds = \int_0^\infty K(s)\,\underset{\sim}{N}(t^*-\rho s, \tau-\epsilon^2 s)\,ds$$

$$= \int_0^\infty K(s)\Big\{ N(t^*-s,\tau) - (\epsilon\rho_1 + \epsilon^2\rho_2 + \dots)\, s\, \frac{\partial N}{\partial t^*}(t^*-s,\tau)$$

$$- \epsilon^2 s\, \frac{\partial N}{\partial \tau}(t^*-s,\tau) + \frac{1}{2}(\epsilon\rho_1 + \dots)^2 s^2 \frac{\partial^2 N}{\partial t^{*2}}(t^*-s,\tau)$$

$$+ O(\epsilon^3 |N|)\Big\}\, ds.$$

Notice that all the convolution integrals ignore the slow time τ so that any functions of τ may be treated as constants. We adopt the notation for any function $\underset{\sim}{N}$ of t^* and τ

$$K*N \equiv \int_0^\infty K(s)\, N(t^*-s,\tau)\, ds,$$

$$sK*N \equiv \int_0^\infty s\,K(s)\, N(t^*-s,\tau)\, ds,$$

etc. Now expanding $\underset{\sim}{N}$ we have

$$K*\underset{\sim}{N} = \epsilon\,[K*N_1] +$$

$$+ \quad \epsilon^2 \left[K * N_2 - \rho_1 \left(sK * \frac{\partial N_1}{\partial t^*} \right) \right]$$

$$+ \epsilon^3 \left[K * N_3 - \rho_1 \left(sK * \frac{\partial N_2}{\partial t^*} \right) - \rho_2 \left(sK * \frac{\partial N_1}{\partial t^*} \right) \right.$$

$$\left. + \frac{1}{2} \rho_1^2 \left(s^2 K * \frac{\partial^2 N_1}{\partial t^{*2}} \right) - \left(sK * \frac{\partial N_1}{\partial T} \right) \right]$$

$$+ O(\epsilon^4).$$

Now substitute all expansions into (2.2) and equate the coefficients of powers of ϵ. There will arise a hierarchy of problems to be solved where the functions N_i will be successively found. We constrain the N_i to be bounded in t as $t \to \infty$ (there are no secular terms) and to be asymptotically periodic or zero. This is sufficient to determine the various parameters λ_i, ρ_i as well as the N_i. We expect that as $t \to \infty$ (or possibly $-\infty$) $\underset{\sim}{N}$ becomes a $\frac{2\pi}{\mu}$ - periodic function of t^* only. This means $\underset{\sim}{N}(t)$ has period $2\pi/\mu\rho(\epsilon)$ so $\rho(\epsilon)$ is proportional to the frequency.

The coefficient of ϵ yields

$$\frac{\partial \underset{\sim}{N}_1}{\partial t^*} - \mathcal{L} \underset{\sim}{N}_1 = 0,$$

which has solution

$$\underset{\sim}{N}_1 = R(T) \ \mathrm{Re} \left\{ \underset{\sim}{x}_0 \exp i(\mu t^* + \theta(T)) \right\}$$

where we shall always ignore the solutions of (2.8) that decay exponentially. R_1, θ are to be found later. We denote

$$\phi \equiv \mu t^* + \theta(\tau).$$

The coefficient of ϵ^2 yields

$$\frac{\partial \underset{\sim}{N}_2}{\partial t^*} - \mathcal{L} \underset{\sim}{N}_2 = -\rho_1 \frac{\partial \underset{\sim}{N}_1}{\partial t^*} + \lambda_1 L'(\lambda_0) \underset{\sim}{N}_1 + \lambda_1 \left(\frac{\partial K}{\partial \lambda}(\lambda_0) * \underset{\sim}{N}_1\right)$$
$$- \rho_1 \left(s K * \frac{\partial \underset{\sim}{N}_1}{\partial t^*}\right)$$

$$= -\rho_1 (I + sK *) \frac{\partial \underset{\sim}{N}_1}{\partial t^*} + \lambda_1 \left(L'(\lambda_0) + \frac{\partial K}{\partial \lambda}(\lambda_0) *\right) \underset{\sim}{N}_1$$

$$= R \cdot \mathrm{Re}\left\{ -\mu \rho_1 [\Delta^{(1)} \underset{\sim}{x}_0] \, i e^{i\phi} - \lambda_1 \left[\left(\frac{\partial \Delta}{\partial \lambda}\right)_0 \underset{\sim}{x}_0\right] e^{i\phi}\right\}$$

by use of (2.7). Since $\underset{\sim}{N}_2$ is bounded there are no secular terms so (2.11) implies

$$\underset{\sim}{y}_0^T \left[-i\mu\rho_1 \Delta^{(1)} \underset{\sim}{x}_0 - \lambda_1 \left(\frac{\partial \Delta}{\partial \lambda}\right)_0 \underset{\sim}{x}_0 \right] = 0.$$

By (2.6) we have $-i\mu\rho_1 + \lambda_1 \sigma'(\lambda_0) = 0$. Thus

$$\begin{cases} \lambda_1 \, \mathrm{Re}\, \sigma'(\lambda_0) = 0, \\ -\mu\rho_1 + \lambda_1 \, \mathrm{Im}\, \sigma'(\lambda_0) = 0; \end{cases}$$

and so

$$\lambda_1 = \rho_1 = 0$$

since $Re\ \sigma'(\lambda_0) \neq 0$ by hypothesis. Then $\frac{\partial N_2}{\partial t^*} - \mathcal{I} N_2 = 0$ so we take $\underset{\sim}{N}_2 = 0$.

The coefficient of ϵ^3 yields,

$$\frac{\partial N_3}{\partial t^*} - \mathcal{I} N_3 = -\rho_2 \frac{\partial N_1}{\partial t^*} - \frac{\partial N_1}{\partial T} + \lambda_2 L'(\lambda_0) N_1$$

$$+ \lambda_2 \frac{\partial K}{\partial \lambda}(\lambda_0) * N_1 - \rho_2 \left(sK * \frac{\partial N_1}{\partial t^*}\right) - \left(sK * \frac{\partial N_1}{\partial T}\right)$$

$$+ Q_3 \left(N = R\ Re(\underset{\sim}{x}_0\, e^{i\phi}),\ \lambda = \lambda_0 \right).$$

Since Q_3 is cubic it yields only the first and third harmonics. Let us suppose

$$Q_3 = R^3 \cdot (\underset{\sim}{\alpha} \sin\phi + \underset{\sim}{\beta} \cos\phi) + \text{ third harmonics}$$

$$= R^3\ Re\left[(\underset{\sim}{\beta} - i\underset{\sim}{\alpha})\, e^{i\phi}\right] + \text{ third harmonics}.$$

The right member above is

$$-\rho_2 \left(I + sK * \right) \frac{\partial N_1}{\partial t^*} + \lambda_2 \left(L'(\lambda_0) + \frac{\partial K}{\partial \lambda}(\lambda_0) * \right) N_1$$

$$-\left(I + sK * \right) \frac{\partial N_1}{\partial T} + Q_3$$

Denoting $\dot{R} \equiv \frac{dR}{dT}$ etc., we obtain, using (2.7) that this right hand side is

$$\mathrm{Re}\,\Big\{ e^{i\phi}\big[-\mu\rho_2 i R\,[\Delta^{(1)}\underset{\sim}{x}_0] \quad -\lambda_2 R\,[(\tfrac{\partial\Delta}{\partial\lambda})_0 \underset{\sim}{x}_0]$$

$$-(\dot{R}+iR\dot{\theta})\,[\Delta^{(1)}\underset{\sim}{x}_0] \; + R^3(\underset{\sim}{\beta}-i\underset{\sim}{\alpha})\big]\Big\}$$

plus third harmonics. The condition that the secular terms be absent implies

$$\underset{\sim}{y}_0^T\cdot\Big[-\mu\rho_2\, iR\;\Delta^{(1)}\cdot\underset{\sim}{x}_0 \;-(\dot{R}+iR\dot{\theta})\,\Delta^{(1)}\underset{\sim}{x}_0 \;-\lambda_2 R\,(\tfrac{\partial\Delta}{\partial\lambda})_0\,\underset{\sim}{x}_0$$

$$+R^3(\underset{\sim}{\beta}-i\underset{\sim}{\alpha})\Big] \;=\; 0.$$

Using (2.6)

$$-i\mu\rho_2 R + \lambda_2\,\sigma'(\lambda_0)\,R \;-\;(\dot{R}+iR\dot{\theta})$$

$$+\;R^3\,\frac{\underset{\sim}{y}_0^T(\underset{\sim}{\beta}-i\underset{\sim}{\alpha})}{\underset{\sim}{y}_0^T\,\Delta^{(1)}\,\underset{\sim}{x}_0} \;=\; 0.$$

Equating the real and imaginary parts to zero gives the <u>modulation equations</u>

$$\begin{cases} \dot{R} = \lambda_2\,\mathrm{Re}\,\sigma'(\lambda_0)\,R + (\mathrm{Re}\,\delta)\,R^3, \\ \dot{\theta} = [-\mu\rho_2 + \lambda_2\,\mathrm{Im}\,\sigma'(\lambda_0)] + (\mathrm{Im}\,\delta)\,R^2, \end{cases} \tag{2.12}$$

where $\delta \equiv \underset{\sim}{y}_0^T\cdot(\underset{\sim}{\beta}-i\underset{\sim}{\alpha})/\underset{\sim}{y}_0^T\Delta^{(1)}\cdot\underset{\sim}{x}_0$. These have the solution

$$R(\tau) = \left\{ \frac{-\dfrac{\lambda_2 \, Re\, \sigma'(\lambda_0)}{Re\, \delta}}{1 \pm \exp(-2\lambda_2 \, Re\, \sigma'(\lambda_0)\, \tau)} \right\}^{1/2}$$

Depending on the sign of $\lambda_2 \, Re\, \sigma'(\lambda_0)$, as $\tau \to \infty$ or $-\infty$, R tends to a nonzero constant R_0. Since ϵ can be scaled arbitrarily and multiplies R we may assume $R_0 = 1$. This implies

$$\lambda_2 = - \frac{\beta_2}{2 \, Re\, \sigma'(\lambda_0)} ,$$

$$\beta_2 \equiv 2 \, Re\, \delta .$$

Hence,

$$R(\tau) = \left(\frac{1}{1 \pm e^{\beta_2 \tau}} \right)^{1/2} .$$

Also, as $\tau \to \pm\infty$ we expect the phase Θ to become constant (the asymptotic phase), so

$$-\mu \rho_2 + \lambda_2 \, Im\, \sigma'(\lambda_0) + Im\, \delta = 0$$

determines ρ_2. $\underset{\sim}{N}_3$ can now be found and the procedure repeated.

Thus we have the solution

$$\underset{\sim}{N} = \epsilon R(\tau) \cdot Re\left\{ \underset{\sim}{x}_0 \exp i(\mu t^* + \Theta(\tau)) \right\} + O(\epsilon^2)$$

which represents oscillations of period $2\pi/\mu\rho(\epsilon)$ with slowly varying amplitude and phase. We have $\rho(\epsilon) = 1 + \epsilon^2\rho_2 + O(\epsilon^3)$ and $\underset{\sim}{N}$ solves (2.2) for $\lambda = \lambda_0 + \epsilon^2\lambda_2 + O(\epsilon^3)$. Also $\Theta(\tau) \to$ constant, $R(\tau) \to 1$ for either $\tau \to \infty$ or $-\infty$ depending on $\operatorname{sgn} \beta_2$. The periodic solution corresponds to $R \equiv 1$ and $\Theta \equiv$ constant. Also if $\beta_2 < 0$ then $R \to 1$ as $t \to \infty$, and if $\beta_2 > 0$ then $R \to 0$ or ∞ as $t \to \infty$ so one could conjecture that the periodic solution is stable if $\beta_2 < 0$ and unstable if $\beta_2 > 0$. This question of stability will be resolved in a later chapter. Finally we note that $\lambda_2 \operatorname{Re} \sigma'(\lambda_0)$ and β_2 are of opposite signs. Since $\operatorname{Re} \sigma(\lambda) = \epsilon^2 \lambda_2 \operatorname{Re} \sigma'(\lambda_0) + O(\epsilon^3)$ the steady state is stable iff $\operatorname{Re} \lambda_2 \sigma'(\lambda_0) < 0$ iff the periodic solution is unstable. Thus we always have an exchange of stability between the periodic solution and the steady state.

The form of the modulation equations given in (2.12) is completely general and in the Appendix we treat the general case. We summarize the results.

Consider the system

$$\frac{d\underset{\sim}{N}}{dt} = \underset{\sim}{f}(\underset{\sim}{N}, \mathcal{K}(\lambda) * \underset{\sim}{N}, \lambda) \tag{2.13}$$

where $\underset{\sim}{N}$ is an n-vector, $\mathcal{K}(\lambda, s)$ is an $n \times n$ matrix kernel which is $O(e^{-\gamma s})$ as $s \to \infty$ and λ a real parameter. Assume that the steady state $\underset{\sim}{N} \equiv 0$ is a solution and the linearization of (2.13) about is given by

$$\frac{d\underset{\sim}{p}}{dt} = \mathcal{L}\underset{\sim}{p} = L(\lambda)\underset{\sim}{p} + \int_0^\infty K(\lambda, s)\, \underset{\sim}{p}(t-s)\, ds. \tag{2.14}$$

Assume further that at λ_0 a pair of eigenvalues $\sigma(\lambda)$ of (2.14) crosses the imaginary axis at $\pm i\mu, \mu \neq 0$; that no other pair does so at $\lambda = \lambda_0$; that $\sigma(\lambda)$ is a <u>simple</u> pair (the generalized null spaces are each of dimension one), and that the real parts of all the eigenvalues of (2.14), except the pair that crosses, are negative. Suppose the secular matrix

$$\Delta = i\mu I - L(\lambda_0) - \int_0^\infty K(\lambda_0, s)\, e^{-i\mu s}\, ds$$

has right null vector $\underset{\sim}{x}_0$, left null vector $\underset{\sim}{y}_0^T$.

Then assuming the transversality condition

$$\operatorname{Re} \sigma'(\lambda_0) \neq 0$$

(2.13) has a formal solution for all ϵ near zero

$$\underset{\sim}{N} = \epsilon R(T) \operatorname{Re}\left\{ e^{i\mu t^* + i\Theta(T)} \underset{\sim}{x}_0 \right\} + O(\epsilon^2)$$

where

$$t^* = \rho(\epsilon)\, t = (1 + \epsilon^2 \rho_2 + O(\epsilon^3))\, t,$$

$$T = \epsilon^2 t,$$

for

$$\lambda = \lambda_0 + \epsilon^2 \lambda_2 + O(\epsilon^3)$$

with

$$R(\tau) = \left(\frac{1}{1 \pm e^{\beta_2 \tau}}\right)^{\frac{1}{2}},$$

$$\lambda_2 = -\frac{\beta_2}{2\,Re\,\sigma'(\lambda_0)},$$

$$\rho_2 = \frac{1}{\mu}\lambda_2\, Im\,\sigma'(\lambda_0) + \frac{1}{2\mu}\beta_2',$$

for some constants β_2, β_2'. The periodic solution is given by $R \equiv 1$, $\theta \equiv$ constant.

Now that we have a formal periodic solution in the form of a perturbation expansion we examine the question of its validity as an actual solution of the equations. To this end we state and prove the following theorem. Assume $\underset{\sim}{f}$ is twice continuously differentiable.

<u>Theorem</u> Consider the equation (2.13) with steady state $\underset{\sim}{N} \equiv 0$ and the linearization (2.14). Assume the hypotheses stated after (2.13) up to and including the transversality condition $Re\,\sigma'(\lambda_0) \neq 0$. Then (2.13) has a nontrivial periodic solution $\underset{\sim}{N}(t, \epsilon)$ for all ϵ near zero of the form $\underset{\sim}{N}(t, \epsilon) = \epsilon\, Re(\underset{\sim}{x}_0 e^{i\mu t^*}) + w(t^*, \epsilon)$ corresponding to $\lambda(\epsilon)$, where $t^* = \rho(\epsilon) t$ and $\lambda(0) = \lambda_0$, $\rho(0) = 1$. Furthermore, $w = O(\epsilon^2)$ and satisfies

$$\int_0^{\frac{2\pi}{\mu}} y_0^T \cdot w(t^*, \epsilon)\, e^{i\mu t^*}\, dt^* = 0,$$

i.e. is orthogonal to the periodic solutions of the adjoint of (2.8).

<u>Proof</u>: Let us assume without loss of generality that $\lambda_0 = 0$, so λ is near zero. Write (2.13) as

$$\frac{d\underset{\sim}{N}}{dt} = \mathcal{L}(\underset{\sim}{N}, \mathcal{K}(\lambda) * \underset{\sim}{N}, \lambda) + g(\underset{\sim}{N}, \mathcal{K}(\lambda) * \underset{\sim}{N}, \lambda)$$

with $\underset{\sim}{g}$ consisting of only nonlinear terms in $\underset{\sim}{N}, \mathcal{K}(\lambda) * \underset{\sim}{N}$. Put

$$L = L(\lambda_0),$$

$$K(s) = K(\lambda_0, s),$$

where $\mathcal{L}\underset{\sim}{N} = L(\lambda)\underset{\sim}{N} + \int_0^\infty K(\lambda, s)\underset{\sim}{N}(t-s)ds$. We shall use the notation concerning the linearization as adopted following (2.14). Put

$$\underset{\sim}{x}_1(t) = Re(\underset{\sim}{x}_0 e^{i\mu t}), \qquad \underset{\sim}{x}_2(t) = Im(\underset{\sim}{x}_0 e^{i\mu t}),$$

$$\underset{\sim}{y}_1^T(t) = Re(\underset{\sim}{y}_0^T e^{-i\mu t}), \qquad \underset{\sim}{y}_2(t) = Im(\underset{\sim}{y}_0^T e^{-i\mu t}).$$

The x_i are the periodic solutions of (2.8) and the y_i^T are the periodic solutions of its adjoint. We rescale t by defining $t^* = \frac{t}{1+\tau(\epsilon)}$ so $\rho(\epsilon) = \frac{1}{1+\tau(\epsilon)}$ and $\tau(\epsilon) = O(\epsilon)$. Define

$$\tilde{N}(t^*) = N(t) = N[(1+\tau)t^*].$$

We seek a $\frac{2\pi}{\mu}$ - periodic solution $\tilde{N}$ which will be the fixed point of an operator equation in the Banach space

$$\mathcal{B} = \Big\{ \underset{\sim}{u}(t) \;\Big|\; \underset{\sim}{u} \text{ is continuously differentiable and } \frac{2\pi}{\mu} \text{ - periodic} \Big\}$$

$$\|u\| = \sup_{0 \le t \le \frac{2\pi}{\mu}} |u(t)| + \sup_{0 \le t \le \frac{2\pi}{\mu}} |\dot{u}(t)|$$

Also define the closed subspace

$$\mathcal{B}_0 = \Big\{ \underset{\sim}{u} \in \mathcal{B} \;\Big|\; \int_0^{2\pi/\mu} \underset{\sim}{y}_i^T(t) \cdot \underset{\sim}{u}(t)\, dt = 0 \quad , \; i = 1,2 \Big\} .$$

We have

$$\int_0^\infty K(s)\, \underset{\sim}{p}(t-s)\, ds = \int_0^\infty K(s)\, \underset{\sim}{p}\left[(1+\tau)t^* - s\right] ds$$

$$= \int_0^\infty K(s)\, \tilde{\underset{\sim}{p}}\left(t^* - \frac{s}{1+\tau}\right) ds ,$$

$$K * \underset{\sim}{p} = \int_0^\infty K(s)\, \tilde{p}(t^* - s)\, ds + d(\tilde{p}, \tau, K)$$

where

$$d(\tilde{p}, \tau, K) \equiv \int_0^\infty K(s)\left[\tilde{p}\left(t^* - \frac{s}{1+\tau}\right) - \tilde{p}(t^* - s)\right] ds$$

and is of order $\tau(\epsilon) \frac{d\tilde{p}}{dt^*}$. This is a bounded linear transformation on $\mathcal{B}$. Put $L(\lambda) = L + \lambda L_1(\lambda)$, $K(\lambda, s) = K(s) + \lambda K_1(\lambda, s)$. Then $\tilde{N}$ satisfies

$$\frac{d\tilde{N}}{dt^*} = (1+T)\Big[L\tilde{N} + K*\tilde{N} + \mathscr{L}(\tilde{N}, T, K) + \lambda L_1(\lambda)\tilde{N} + \lambda K_1(\lambda)*\tilde{N} + \mathscr{L}(\tilde{N}, T, \lambda K_1(\lambda)) + g\big(\tilde{N}, \mathscr{K}(\lambda)*\tilde{N} + \mathscr{L}(\tilde{N}, T, \mathscr{K}(\lambda)), \lambda\big)\Big].$$

Put $\tilde{\underset{\sim}{N}} = \epsilon\, \underset{\sim}{p}(t^*) + \epsilon\, \underset{\sim}{z}(t^*, \epsilon)$ where

$$p(t^*) = \gamma_1\, \underset{\sim}{x}_1(t^*) + \gamma_2\, \underset{\sim}{x}_2(t^*) = \mathrm{Re}\,(\gamma\, \underset{\sim}{x}_0\, e^{i\mu t^*})$$

(with $\gamma = \gamma_1 - i\gamma_2 \neq 0$) which is some periodic solution of (2.8). We wish $z = O(\epsilon)$ and $z \in \mathscr{B}_0$. Then we wish to solve the following for z :

$$\frac{dz}{dt} - \mathcal{L} z = \mathscr{L}(p, T, K) + \lambda L_1(\lambda)\, p + \lambda K_1(\lambda) * p + T\mathcal{L}p + G(z, \lambda, T, \epsilon) \tag{2.15}$$

where

$$G = (1+T)\Big[\mathscr{L}(z, T, K) + \lambda L_1 z + \lambda K_1 * z + \mathscr{L}(p, T, \lambda K_1) + \mathscr{L}(z, T, \lambda K_1) + \epsilon h\Big] + T\Big[\mathscr{L}(p, T, K) + \lambda L_1 p + \lambda K_1 * p + Lz + K*z\Big]$$

where we have dropped the star on t^* and have put

$$\epsilon h = \frac{1}{\epsilon} g(\epsilon p + \epsilon z,\ \epsilon \mathcal{K} * p + \epsilon \mathcal{K} * z +$$

$$\epsilon \mathcal{J}(p, T, \mathcal{K}) + \epsilon \mathcal{J}(z, T, \mathcal{K})\ ; \lambda)\ ;$$

(ϵ has been scaled out of g which is nonlinear). The right member of (2.15) is of the form $F(z, \lambda, T, \epsilon)$ and for (2.15) to have a periodic solution it is necessary and sufficient that

$$y_i \cdot F = \frac{1}{2\pi/\mu} \int_0^{2\pi/\mu} y_i(t) \cdot F(z(t), \lambda, T, \epsilon)\, dt = 0$$

for $i = 1, 2$. These two conditions will determine λ and T as functions of z and ϵ both sufficiently small with $\lambda = T = 0$ when $\epsilon = z = 0$. Put

$$h_i(z, \lambda, T, \epsilon) \equiv y_i \cdot F .$$

To solve $h_i = 0$ for λ, T it is sufficient that the Jacobian $\partial(h_1, h_2)/\partial(\lambda, T)$ evaluated at $\lambda = T = \epsilon = z = 0$ not have vanishing determinant. This follows from the implicit function theorem for Banach spaces. We evaluate this Jacobian. The terms in G all involve either ϵ or products of the four variables, so gives zero contribution to the Jacobian. In fact, Jacobian =

$$\begin{bmatrix} y_1 \cdot [L_1(0) p + K_1(0) * p] & y_2 \cdot [L_1(0) p + K_1(0) * p] \\ y_1 \cdot [\mathcal{L} p + \mathcal{J}_T^0] & y_2 \cdot [\mathcal{L} p + \mathcal{J}_T^0] \end{bmatrix} \tag{2.16}$$

where

$$\mathscr{A}_T^0 \equiv \frac{d}{d\tau}\mathscr{A}\Big|_0 = \int_0^\infty sK(s)\,\frac{dp}{dt}(t-s)\,ds\,.$$

Also

$$\mathscr{L}p + \mathscr{A}_T^0 = \dot{p} + sK*\dot{p} = Re\,\{\, i\mu\gamma\,\Delta^{(1)}\underset{\sim}{x}_0\, e^{i\mu t}\}$$

by (2.7), and

$$L_1(0)p + K_1(0)*p = Re\,\Big\{\gamma e^{i\mu t}\Big(\frac{d}{d\lambda}L(\lambda) + \int_0^\infty \frac{\partial}{\partial\lambda}K(\lambda,s)\,e^{-i\mu s}ds\Big)\Big|_{\lambda=0}\underset{\sim}{x}_0\Big\}$$

$$= Re\,\Big\{\,\gamma\, e^{i\mu t}\Big(-\frac{\partial\Delta}{\partial\lambda}\Big)_0\cdot\underset{\sim}{x}_0\Big\}$$

by (2.7). Thus

$$y_1\cdot(\mathscr{L}p + \mathscr{A}_T^0) = \frac{1}{4}\cdot\frac{\mu}{2\pi}\int_0^{2\pi/\mu}(\underset{\sim}{y}_0^T e^{-i\mu t} + c.c.)(i\mu\gamma\,\Delta^{(1)}\underset{\sim}{x}_0 e^{i\mu t} + c.c.)\,dt$$

$$= \frac{1}{4}\,[\,i\mu\gamma\,(\underset{\sim}{y}_0^T\Delta^{(1)}\underset{\sim}{x}_0) + c.c.]$$

$$= \frac{\mu}{2}\;Re\,[\,i\gamma\,(\underset{\sim}{y}_0^T\cdot\Delta^{(1)}\cdot\underset{\sim}{x}_0)]\,.$$

Similarly,

$$y_2\cdot(\mathscr{L}p + \mathscr{A}_T^0) = \frac{\mu}{2}\;Im\,[\,i\gamma\,(\underset{\sim}{y}_0^T\Delta^{(1)}\underset{\sim}{x}_0)]\,,$$

$$y_1\cdot(L_1p + K_1*p) = \frac{1}{2}\;Re\,\Big[-\gamma\,\Big(\underset{\sim}{y}_0^T\Big(\frac{\partial\Delta}{\partial\lambda}\Big)_0\underset{\sim}{x}_0\Big)\Big]\,,$$

$$y_2\cdot(L_1p + K_1*p) = \frac{1}{2}\;Im\,\Big[-\gamma\,\Big(\underset{\sim}{y}_0^T\Big(\frac{\partial\Delta}{\partial\lambda}\Big)_0\underset{\sim}{x}_0\Big)\Big]\,.$$

Putting $\nu = i\gamma\,(\underset{\sim}{y}_0^T \Delta^{(1)} \underset{-}{x}_0)$ and noting (2.6) we have

$$\det(\text{Jac.}) = \frac{\mu}{4}\left\{ Re\left(\frac{\sigma'(\lambda_0)\nu}{i}\right) Im\,\nu - Im\left(\frac{\sigma'(\lambda_0)\nu}{i}\right) Re\,\nu \right\}$$

$$= -\frac{\mu}{4}\; Im\left(\frac{\sigma'(\lambda_0)\,|\nu|^2}{i}\right)$$

$$= \frac{\mu}{4}\; |\nu|^2\; Re\,\sigma'(\lambda_0)$$

$$= \frac{\mu}{4}\; |\gamma|^2\; |y_0^T\cdot\Delta^{(1)}\cdot x_0|^2\; Re\,\sigma'(\lambda_0)\,.$$

This is nonzero by the hypotheses and $\gamma \neq 0$. Thus, by the implicit function theorem there are unique functions $\lambda(z,\epsilon)$, $\tau(z,\epsilon)$ defined for all $|\epsilon| \leq \epsilon^*$ and $\underset{\sim}{z} \in S_{\epsilon^*} \equiv \{ z \in \mathcal{B}_0 \,|\, \|z\| \leq \epsilon^* \}$ such that $\lambda = \tau = 0$ when $\epsilon = z = 0$.

First note that $\tau(z,0) \equiv 0$, $\lambda(z,0) \equiv 0$ since $h_i = 0$ is satisfied for $\tau = \lambda = \epsilon = 0$ and all $\underset{\sim}{z} \in S_{\epsilon^*}$. Thus by uniqueness of λ, τ we must have $\tau(z,0) = \lambda(z,0) = 0$.

Consider now the equation

$$\dot{v} - \mathcal{L} v = q(t)$$

for any $q \in \mathcal{B}_0$. Appealing to the Fredholm alternative theorem of Chapter I, a periodic solution v can be found and made unique by requiring $v \in \mathcal{B}_0$. Thus $v = \hat{A} q$ for some linear operator $\hat{A}$ on $\mathcal{B}_0$. That $\hat{A}$ is continuous can be seen as follows. In the proof of Theorem 2 of Chapter I $V_t^{P_1}$ and V_t^{Q} and their derivatives depend continuously on $q(t)$ as in (1.16). Next, $V_t^{P_0}$ is defined in terms of a function $y(t)$ satisfying (1.14). There we have

$$y(t) = e^{Bt}y(0) + \int_0^t e^{B(t-s)}\Psi(0)\,q(s)\,ds,$$

and $y(0)$ satisfies

$$(e^{B\omega}-I)\,y(0) = -e^{B\omega}\int_0^\omega e^{-Bs}\Psi(0)\,q(s)\,ds,$$

and is determined continuously in terms of q up to a null vector of $e^{B\omega}-I$ (corresponding to $v(0)$ unique up to a multiple of $\underset{\sim}{x}_0$) and which is made unique by requiring $v \in \mathcal{B}_0$. Thus $y, \dot{y}$ depend continuously on q, so $v_t^{P_0}$ does likewise. Thus the map $\mathcal{F}(z,\epsilon) = \hat{A} \circ F(z, \lambda(z,\epsilon), \tau(z,\epsilon), \epsilon)$ is defined on $S_{\epsilon^*} \times [-\epsilon^*, \epsilon^*]$ and is continuous. Now as $\epsilon \to 0$, $\lambda, \tau \to 0$ so $F \to 0$ and $\mathcal{F} \to 0$. Thus for some $\epsilon_0 > 0$ we have $|\epsilon| \le \epsilon_0$ implying $\|\mathcal{F}(z,\epsilon)\| \le \epsilon^*$ for all $z \in S_{\epsilon^*}$ so $\mathcal{F}$ maps S_{ϵ^*} into itself. Also $\frac{\partial \mathcal{F}}{\partial z}\big|_{\epsilon=0} = 0$ since $\frac{\partial \tau}{\partial z}\big|_{\epsilon=0} = \frac{\partial \lambda}{\partial z}\big|_{\epsilon=0} = 0$. Then by choosing ϵ_0 smaller we have $\|\mathcal{F}(z_1,\epsilon) - \mathcal{F}(z_2,\epsilon)\| \le k_0 \|z_1 - z_2\|$ for all $z_i \in S_{\epsilon^*}$ and $|\epsilon| \le \epsilon_0$; and $0 < k_0 < 1$. Thus, for each ϵ, $\mathcal{F}$ is a contraction map on S_{ϵ^*} and has a unique fixed point $z(\epsilon)$ where $z = \mathcal{F}(z,\epsilon)$. Also $z \in \mathcal{B}_0$. Then $\dot{z} - \mathcal{J}z = F(z,\lambda,\tau,\epsilon)$ so z solves (2.15) for $|\epsilon| \le \epsilon_0$. Put $\tau(\epsilon) = \tau(z(\epsilon),\epsilon)$, $\lambda(\epsilon) = \lambda(z(\epsilon),\epsilon)$. Thus τ, λ, z and therefore τ, λ, N are determined with the properties stated in the theorem. Note $N \not\equiv 0$ since otherwise $p = -z$ but $p \notin \mathcal{B}_0$. ∎

As a final point we consider the case that $\underset{\sim}{f}$ in (2.13) depends analytically on all its arguments and $\mathcal{K}(s)$ is analytic. We briefly show that the periodic solution $\underset{\sim}{N}(t,\epsilon)$ given in the theorem is analytic in t and ϵ. Furthermore $\lambda(\epsilon)$ and $\rho(\epsilon)$ are analytic in ϵ.

Since L and K are analytic then the resolvent $R(t)$ is analytic in $t \geq 0$. Denote by $\mathcal{B}^{\omega}$ the set of $u(t) \in \mathcal{B}$ analytic in t, with $\mathcal{B}$ as in the proof of the theorem. Also τ and λ are determined as functions of z, ϵ. If $z(t,\eta) \in \mathcal{B}^{\omega}$ depends analytically on η (as well as t) then $F(z(t,\eta), \lambda, \tau, \epsilon)$ is analytic in $\lambda, \tau, \epsilon, \eta$ so the h_i do likewise. Thus $\tau = \tau(z(\eta), \epsilon)$, $\lambda = \lambda(z(\eta), \epsilon)$ are analytic in η, ϵ by the implicit function theorem. We deduce $\mathcal{F}(z,\epsilon) = \hat{A} \circ F(z, \lambda(z,\epsilon), \tau(z,\epsilon), \epsilon)$ depends analytically on η, ϵ. The image of $\mathcal{F}$ is in $\mathcal{B}^{\omega}$ since $\mathcal{B}$ is invariant under $\hat{A}$ (use the formula for the solution of (1.2) in terms of the resolvent R, which is analytic.) Thus the various iterations used in the contraction mapping principle yield an analytic $z(t,\epsilon) \in \mathcal{B}^{\omega}$ if we put $\eta = \epsilon$. Then λ, τ are likewise analytic.

CHAPTER III

A TIME-LAG DIFFUSIONAL MODEL

In the previous chapter we examined some integro-differential equations describing, for example, the time variation of a biological population. These equations involved one independent variable, t . In this chapter we examine the effects of an added space variable, $\underset{\sim}{x}$, on such equations, these involving derivatives with respect to $\underset{\sim}{x}$ and t and convolution integrals with respect to t only.

We shall study an equation modeling a predator-prey relationship. An example of such a situation is that of herbivorous copepods living off of phytoplankton in the sea. It has been observed by Steele [14] that traveling waves can occur in these populations. It is suggested by Murray [13] that this is due to the combined effects of diffusion in space of the population with the inherent time delay of the life processes involved. An example of the latter is provided by noting that in any population there is a variation in individual maturity. This has an important effect since the amount of food available to the young directly affects the number of individuals reaching maturity (and this in turn controls the birth rate.) Diffusion arises from the tendency of a species to migrate toward regions of lower population density. This will be true if food is continuously and homogeneously supplied in time and space. Then in regions of high density population the food will become scarce and individuals will tend to migrate to (or will have a higher expectation of surviving in) the regions of lower population density. Thus one may conclude that the flux of individuals across any surface is proportional to the population gradient at any position and time. With these assumptions in mind we shall construct a model of the predator-prey situation, describing, for example, the copepod-plankton relationship.

The governing equation is derived in a manner similar to that of the heat equation. Given a fixed volume $\mathcal{V}$ in space, the time rate of change of the population size within $\mathcal{V}$ is equal to the flux across the boundary of $\mathcal{V}$ plus the change in population size due to the birth and death of individuals within the population. In the absence of predators the prey population obeys such a rule. Denote the prey population density by a function $V(\underset{\sim}{x}, t)$, $\underset{\sim}{x}$ being position in space. In this model we shall assume that the growth and decay processes are described by some function of V not only at the present instant but in the past also. This is represented by the lags or delays described above and are due to age variations within the population, seasonal changes, and so on. We denote this function by

$$h\left(\int_0^\infty K(s)\, V(\underset{\sim}{x}, t-s)\, ds\right).$$

Notice that the delay is independent of $\underset{\sim}{x}$. This follows from the assumption that the migration is on a much longer time scale than the lag. The kernel $K(s)$, a weighting function for past effects, is arbitrary except that we assume it decays at least exponentially as $s \to \infty$. We have also assumed that the flux (migration velocity) is proportional to the gradient $\nabla V(\underset{\sim}{x}, t)$ (choose the constant of proportionality equal to unity.) Then we have

$$\frac{d}{dt}\int_{\mathcal{V}} V(\underset{\sim}{x},t)\,d\underset{\sim}{x} = \int_{\partial\mathcal{V}} \underset{\sim}{\nabla} V\cdot d\underset{\sim}{a} + \int_{\mathcal{V}} h\,d\underset{\sim}{x}\,.$$

By the divergence theorem we obtain

$$\frac{\partial V}{\partial t} = h\left(\int_0^\infty K(s)\,V(\underset{\sim}{x},t-s)\,ds\right) + \nabla^2 V. \tag{3.1}$$

As noted above, h incorporates the birth and death processes of the population and depends on the situation being modeled.

Now if we assume that the predator consumption depends only on their capacity b for prey, the full predator-prey relationship is given by

$$\frac{\partial V}{\partial t} = h\left(\int_0^\infty K(s)\,V(\underset{\sim}{x},t-s)\,ds\right) - bW + \nabla^2 V \tag{3.2}$$

where $W(\underset{\sim}{x},t)$ is the density of predators. Clearly, since the predators depend exclusively on the prey for food, W depends on V. Furthermore, if there is not enough food for the predator at some time, then in the future that population will decline (the usual delays are at work.) Thus W depends on V at an earlier time and we assume

$$bW(\underset{\sim}{x},t) = m\left(\int_0^\infty K_1(s)\,V(\underset{\sim}{x},t-s)\,ds\right).$$

For simplicity we take the two kernels equal

$$K(s) = K_1(s).$$

We consider only the one-space-dimensional problem

$$\frac{\partial V}{\partial t} = h(K*V) - m(K*V) + \frac{\partial^2 V}{\partial x^2} \tag{3.3}$$

where we define

$$(K*V)(\underset{\sim}{x},t) \equiv \int_0^\infty K(s)\, V(\underset{\sim}{x}, t-s)\, ds.$$

Since h takes into account overcrowding, in which case the population tends to decline due to self-poisoning and depletion of food, h must decrease with V for V large enough. Thus there is a $V = V_0$ where $h(K*V_0) - m(K*V_0)$. We assume V_0 to be a constant. From (3.3) we see that V_0 is an equilibrium point of the population. We wish to examine the solutions near this steady state. Assuming with Stirzaker [15] that h and m are odd functions about V_0 we have, putting

$$u = V - V_0,$$

$$\frac{\partial u}{\partial t} = \alpha\, K*u + \epsilon\,(K*u)^3 + \frac{\partial^2 u}{\partial x^2} \tag{3.4}$$

for constants α and ϵ. We examine (3.4) for small ϵ so that the nonlinear effects are small (but crucial).

In the case that $K(s) = \delta(s-T)$, the Dirac delta function, (3.4) becomes a delay-differential equation. Murray [13] has produced traveling wave solutions of this equation thus lending weight to the hypothesis that diffusion and time lags are at work. We shall perform a similar analysis of (3.4) for a general kernel (with the weak conditions on K mentioned above), showing that traveling wave solutions of the equation exist. This will confirm Murray's hypothesis in the case of continuously distributed time delays.

We look for traveling waves of the form

$$u = u\left(t - \frac{x}{c}\right)$$

where c is the (constant) phase speed of the wave. Putting

$$z = t - \frac{x}{c}$$

we get an ordinary integro-differential equation in z :

$$\frac{1}{c^2}\frac{d^2u}{dz^2} - \frac{du}{dz} + \alpha K*u + \epsilon (K*u)^3 = 0 \tag{3.5}$$

where

$$K*u \equiv \int_0^\infty K(s)\, u(z-s)\, ds .$$

We examine (3.5) for the bifurcation of a periodic solution from the steady state $u \equiv 0$ for ϵ near zero. Since we wish to keep α and K fixed this will force us to consider c as a function of ϵ. This, as well as the oscillation frequency, will be determined in the course of the calculations.

Now that we have an equation in one independent variable the technique of the previous chapter can be employed. We begin by examining the linearization of (3.5) about $u \equiv 0$ for periodic solutions, i.e. for eigenvalues which are pure imaginary (and nonzero). The linearization is

$$\frac{1}{c_0^2} u'' - u' + \alpha K * u = 0 \tag{3.6}$$

and has eigenvalues $\pm i\mu$, $\mu \neq 0$, when the secular equation is satisfied

$$-\frac{\mu^2}{c^2} - i\mu + \alpha \int_0^\infty K(s)\, e^{-i\mu s}\, ds = 0, \tag{3.7}$$

i.e.

$$\begin{cases} \alpha A(\mu) - \dfrac{\mu^2}{c_0^2} = 0, \\ \alpha B(\mu) + \mu = 0, \end{cases} \tag{3.8}$$

where $\quad A(\mu) + i B(\mu) \equiv \int_0^\infty K(s)\, e^{-i\mu s}\, ds.$

Equations (3.8) give two conditions on the wave speed c_0 and frequency μ for the periodic solution of (3.6). Assume that some such pair exists. We now look for a periodic solution of (3.5) with $c(\epsilon)$ near c_0 and $u(z,\epsilon)$ near $\cos \mu z$ for ϵ near 0.
Using (3.8), note that (3.6) can be written

$$\frac{1}{c_0^2} \mathcal{L} u = 0$$

where

$$\mathcal{L} u \equiv u'' + \frac{\mu B}{A} u' + \frac{\mu^2}{A} (K * u). \tag{3.9}$$

Thus (3.5) can be put in the form

$$\mathcal{L}u = \nu(\epsilon)\, u'' - \epsilon c_0^2 (K * u)^3 \tag{3.10}$$

where

$$\nu(\epsilon) \equiv 1 - \left(\frac{c_0}{c(\epsilon)}\right)^2$$

$$= \frac{2c_1}{c_0}\,\epsilon + O(\epsilon^2).$$

Before applying the perturbation procedure of Chapter II to (3.10) we examine the linearized equation

$$u'' + \frac{\mu B}{A} u' + \frac{\mu^2}{A}(K * u) = \nu u'' \tag{3.11}$$

with

$$\nu = \epsilon \nu_1 + O(\epsilon^2).$$

This has eigenvalue $\sigma = \sigma(\epsilon) = i\mu + O(\epsilon)$ with eigenfunction $u = e^{\sigma z}$. From (3.11) we obtain the secular equation

$$\Delta(\nu, \sigma) = \sigma^2 + \frac{\mu B}{A}\sigma + \frac{\mu^2}{A}\int_0^\infty K(s) e^{-\sigma s} ds - \nu\sigma^2 = 0. \tag{3.12}$$

Thus

$$\left(\frac{d\sigma}{d\nu}\right)_{\nu=0} = -\left(\frac{\frac{\partial\Delta}{\partial\nu}}{\frac{\partial\Delta}{\partial\sigma}}\right)_{\nu=0} = -\frac{\mu^2}{\Delta^{(1)}} \tag{3.13}$$

where

$$\Delta^{(1)} \equiv \left(\frac{\partial\Delta}{\partial\sigma}\right)_{\nu=0} = 2i\mu + \frac{\mu B}{A} - \frac{\mu^2}{A}\int_0^\infty s\, K(s)\, e^{-i\mu s} ds. \tag{3.14}$$

This expression will be needed later. The inhomogeneous problem

$$\mathcal{L}u = a\cos n\mu z + b\sin n\mu z \tag{3.15}$$

is solved as follows. The right side is $Re\,[\gamma \exp in\mu z]$ with $\gamma = a - ib$. A particular solution is

$$u = Re\left[\frac{\gamma}{\Delta(n\mu,0)}\, e^{in\mu z}\right], \qquad n \neq 1, \tag{3.16}$$

$$u = Re\left[\frac{\gamma}{\Delta^{(1)}}\, z\, e^{i\mu z}\right], \qquad n = 1. \tag{3.17}$$

For (3.16), (3.17) to be valid we require

$$\Delta(n\mu, 0),\ \Delta^{(1)} \neq 0 \quad , \qquad \forall n \neq \pm 1.$$

Equivalently, for given c_0 and μ satisfying (3.8) there is no solution $c_0, n\mu$ of (3.8) for $n \neq \pm 1$, and the eigenvalues $\pm i\mu$ are each simple. Furthermore, we assume the secular equation (3.12) has all its other roots with negative real parts. Finally we note that (3.15) for $n = 1$ has a bounded solution iff $\gamma = 0$, i.e. $a = b = 0$. This is the Fredholm alternative revisited.

We can now proceed to the perturbation analysis of (3.10). We shall seek a periodic solution of (3.10) for all ϵ near 0 which reduces to $R\cos\mu z$ for $\epsilon = 0$ (this is a periodic solution of (3.11) at $\nu = 0$). Since it is expected that the nonlinearity will change the frequency and wave speeds slightly and modulate the wavetrain $R\cos\mu z$

on a time-scale which is longer than that of the oscillations, we write the solution as a function of two time scales. Thus, much the same scheme of Chapter II applies. We note briefly that the Hopf bifurcation of Chapter II applied to this problem yields the one-parameter family $u = R\cos\mu z$ for all R and solving (3.5) for $\epsilon \equiv 0$. This is a degenerate case for the Hopf bifurcation and does not yield solutions of the perturbed equation. Instead we construct a different one-parameter family of solutions.

Assume an expansion of the form

$$u = R(\zeta) \cos(\mu z^* + \theta(\zeta)) + \epsilon u_1(z^*, \zeta) + \cdots ,$$

$$\nu(\epsilon) = \nu_1 \epsilon + O(\epsilon^2),$$

where

$$z^* = \rho(\epsilon) z ,$$

$$\zeta = \epsilon z ,$$

are the two time scales, and

$$\rho(\epsilon) = 1 + \epsilon\rho_1 + \epsilon^2\rho_2 + O(\epsilon^3)$$

is the adjusted frequency ratio. Put

$$u_0 \equiv R(\zeta) \cos(\mu z^* + \theta(\zeta)) .$$

As before

$$K * u = K * u_0 + \epsilon\left[K * u_1 - \rho_1\left(sK * \frac{\partial u_0}{\partial z^*}\right) - \left(sK * \frac{\partial u_0}{\partial \zeta}\right)\right] + O(\epsilon^2)$$

where

$$K * f \equiv \int_0^\infty k(s)\, f(z^* - s, \zeta)\, ds \tag{3.18}$$

for any function f of z^* and ζ . Also

$$\frac{d}{dz} = (1 + \epsilon\rho_1 + O(\epsilon^2))\frac{\partial}{\partial z^*} + \epsilon\frac{\partial}{\partial \zeta} .$$

Substituting these into (3.10) and equating the coefficients of the powers of ϵ , gives for $O(\epsilon^0)$:

$$\mathcal{L}u_0 \equiv \frac{\partial^2 u_0}{\partial z^{*2}} + \frac{\mu B}{A}\frac{\partial u_0}{\partial z^*} + \frac{\mu^2}{A} K * u_0 = 0$$

with $K*$ as in (3.17). Thus $u_0 = R\cos(\mu z^* + \theta) = Re[R\, e^{i(\mu z^* + \theta)}]$ as we already knew. The $O(\epsilon')$ coefficient yields

$$\mathcal{L}u_1 = -2\rho_1\frac{\partial^2 u_0}{\partial z^{*2}} - 2\frac{\partial^2 u_0}{\partial z^*\partial\zeta} - \frac{\mu B}{A}\rho_1\frac{\partial u_0}{\partial z^*} - \frac{\mu B}{A}\frac{\partial u_0}{\partial \zeta} + \frac{\mu^2}{A}\rho_1\left(sK * \frac{\partial u_0}{\partial z^*}\right) + \frac{\mu^2}{A}\left(sK * \frac{\partial u_0}{\partial \zeta}\right) + \nu_1\frac{\partial^2 u_0}{\partial z^{*2}} - c_0^2\,(K * u_0)^3$$

$$= \operatorname{Re}\left\{ \left(-\rho_1 \frac{\partial}{\partial z^*} - \frac{\partial}{\partial \zeta}\right)\left(2\frac{\partial u_0}{\partial z^*} + \frac{\mu B}{A} u_0 - \frac{\mu^2}{A}(s K * u_0)\right)\right.$$

$$\left. + \nu_1 \; (i\mu)^2 \; R \; e^{i\mu z^* + i\theta} \right\}$$

$$- c_0^2 \; R^3 \left[A \cos(\mu z^* + \theta) + B \sin(\mu z^* + \theta)\right]^3$$

(A and B are defined after (3.8))

$$= \operatorname{Re}\left\{ \Delta^{(1)} \left(-\rho_1 \frac{\partial}{\partial z^*} - \frac{\partial}{\partial \zeta}\right) \left(R e^{i\mu z^* + i\theta}\right)\right.$$

$$\left. - \nu_1 \mu^2 \; R \; e^{i\mu z^* + i\theta} \right\}$$

$$- \frac{3}{4} c_0^2 R^3 \left(A^3 \cos(\mu z^* + \theta) + A^2 B \sin(\mu z^* + \theta) + \right.$$

$$\left. + A B^2 \cos(\mu z^* + \theta) + B^3 \sin(\mu z^* + \theta)\right) + 3^{rd}$$

$$= \operatorname{Re}\left\{ \Delta^{(1)} \left[-i\mu\rho_1 - (\dot{R} + i R\dot{\theta})\right] e^{i\mu z^* + i\theta}\right.$$

$$\left. - \nu_1 \mu^2 \; R \; e^{i\mu z^* + i\theta} \right\}$$

$$- \frac{3}{4} c_0^2 R^3 (A^2 + B^2) \left(A \cos(\mu z^* + \theta) + B \sin(\mu z^* + \theta)\right) + 3^{rd}$$

$$= \; - \operatorname{Re}\left\{ e^{i\mu z^* + i\theta} \left[(i\mu\rho_1 \Delta^{(1)} + \nu_1 \mu^2) R \right.\right.$$

$$\left.\left. + \Delta^{(1)} (\dot{R} + i R \dot{\theta}) + \frac{3}{4} c_0^2 R^3 (A^2 + B^2)(A - iB)\right]\right\}$$

$$+ 3^{rd}$$

where the 3^{rd} denotes third harmonics. Since we force u_1 to be bounded, noting (3.17) we require

$$(i\mu\rho_1 \Delta^{(1)} + \nu_1\mu^2)R + \Delta^{(1)}(\dot{R} + iR\dot{\theta})$$
$$+ \frac{3}{4} c_0^2 (A^2+B^2)(A-iB) \; R^3 = 0.$$

Dividing by $\Delta^{(1)}$ and taking the real and imaginary parts yields the modulation equations

$$\begin{aligned} \dot{R} &= -\nu_1 \mu^2 \left(\mathrm{Re}\,\frac{1}{\Delta^{(1)}}\right) R - \frac{3}{4} c_0^2 (A^2+B^2)\left(\mathrm{Re}\,\frac{A-iB}{\Delta^{(1)}}\right) R^3, \\ \dot{\theta} &= -\mu\rho_1 - \nu_1\mu^2\left(\mathrm{Im}\,\frac{1}{\Delta^{(1)}}\right) - \frac{3}{4} c_0^2 (A^2+B^2)\left(\mathrm{Im}\,\frac{A-iB}{\Delta^{(1)}}\right) R^2. \end{aligned} \tag{3.19}$$

These equations have already been analyzed in Chapter II with the results

$$\begin{aligned} \nu_1 &= -\frac{3}{4} \frac{c_0^2}{\mu^2} (A^2+B^2) \frac{\mathrm{Re}\,\frac{A-iB}{\Delta^{(1)}}}{\mathrm{Re}\,\frac{1}{\Delta^{(1)}}} R_0^2, \\ \rho_1 &= -\nu_1\mu \,\mathrm{Im}\,\frac{1}{\Delta^{(1)}} - \frac{3}{4}\frac{c_0^2}{\mu^2}(A^2+B^2)\,\mathrm{Im}\,\frac{A-iB}{\Delta^{(1)}} \, R_0^2, \end{aligned} \tag{3.20}$$

where $R \to R_0$ as $\zeta \to \infty$ or $-\infty$, the sign equalling $\mathrm{sgn}\left[\mathrm{Re}\,\frac{A-iB}{\Delta^{(1)}}\right]$. In fact

$$R = \frac{R_0}{(1 \pm \exp \beta\zeta)^{1/2}}$$

with

$$\beta = -\frac{3}{2} c_0^2 (A^2 + B^2) \; Re \frac{A - iB}{\Delta^{(1)}} .$$

Since $\nu_1 = \frac{2c_1}{c_0}$, we have

$$c_1 = -\frac{3}{8} \frac{c_0^2}{\mu^2} (A^2 + B^2) \frac{Re \frac{A-iB}{\Delta^{(1)}}}{Re \frac{1}{\Delta^{(1)}}} R_0^2 . \tag{3.21}$$

<u>Note</u> that c_1 and ρ_1 are proportional to R_0^2. Finally, note that (3.21) requires $Re \frac{1}{\Delta^{(1)}} \neq 0$, i.e.

$$Re \; \Delta^{(1)} \neq 0 .$$

From (3.13) we get the equivalent <u>transversality</u> condition $Re \frac{d\sigma}{d\nu}(0) \neq 0$. In either case, from (3.14), this gives

$$\int_0^\infty K(s) \sin \mu s \, ds - \mu \int_0^\infty s K(s) \cos \mu s \, ds \neq 0 .$$

Summarizing, under the hypotheses

(i) $\Delta(n\mu, 0) \neq 0$, n = integer $\neq \pm 1$,

(ii) $\Delta^{(1)} \neq 0$,

(iii) $Re \; \Delta^{(1)} \neq 0$, $Re \frac{d\sigma}{d\nu}(0) \neq 0$,

(iv) $\beta \neq 0$

a periodic traveling wave solution of (3.4) bifurcates from the zero solution with period $\frac{2\pi}{\mu \rho(\epsilon)}$ and wave speed $c(\epsilon)$ where

$$c(\epsilon) = c_0 + c_1 \epsilon + O(\epsilon^2),$$

$$\rho(\epsilon) = 1 + \rho_1 \epsilon + O(\epsilon^2);$$

ρ_1, c_1 given in (3.20) and (3.21). The form of the solution is

$$u = R(\zeta) \cos(\mu z^* + \theta(\zeta)) + O(\epsilon) \qquad (3.22)$$

where $\zeta = \epsilon z, \quad z^* = \rho(\epsilon) z,$

and

$$z = t - \frac{x}{c(\epsilon)}$$

with

$$R(\zeta) = \frac{R_0}{(1 \pm e^{\beta\zeta})^{1/2}},$$

and u tends asymptotically to the periodic solution.

Note that the leading term in (3.22) can be written

$$R \cos(\omega t - kx)$$

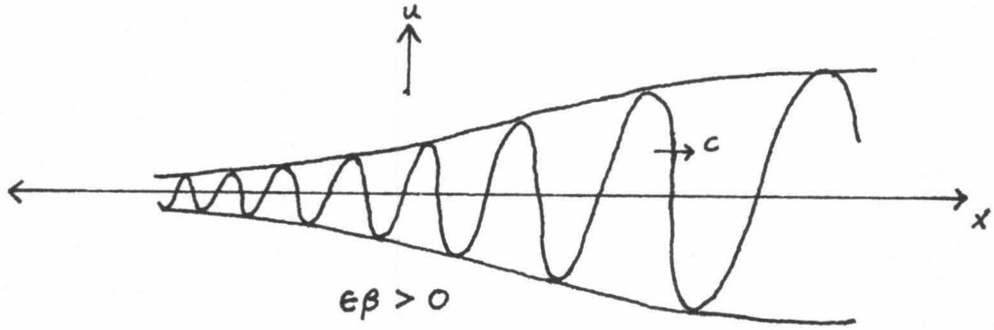
with

$$\omega = \omega(\epsilon) = \mu \rho(\epsilon) = \mu + O(\epsilon R_0^2),$$

$$k = k(\epsilon) = \frac{\mu \rho(\epsilon)}{c(\epsilon)} = \frac{\mu}{c_0} + O(\epsilon R_0^2).$$

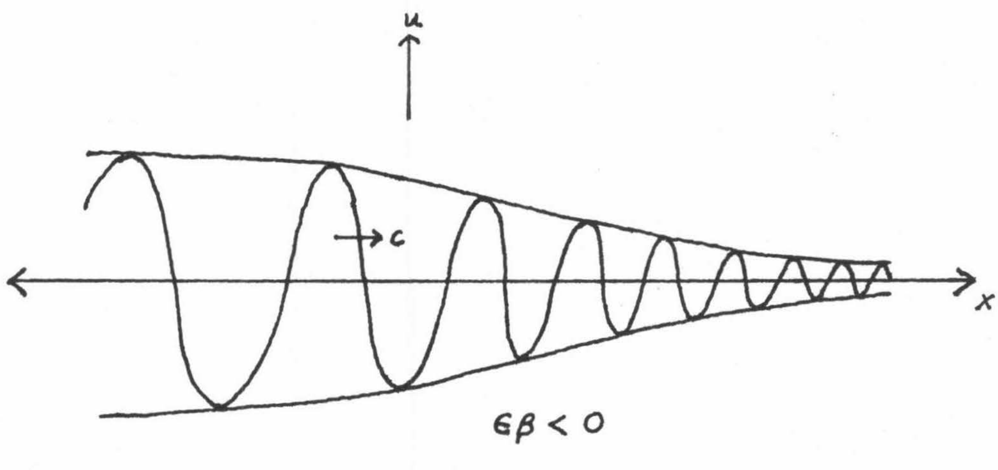
Thus we have a wave propagating with frequency ω and wave number k that depend on the amplitude of the wave. This corresponds to the general result of nonlinear dispersive waves that the dispersion relation depends on the amplitude, cf. Whitham [18].

Assume $c_0 > 0$



Profile moves to right at speed $x/t = c$.

Oscillations die to zero as $t \to \infty$ for fixed x .



Oscillations become periodic as $t \to \infty$ for fixed x .

Now that we have a formal periodic solution of (3.5) in the form of a perturbation expansion we examine the question of its validity as an actual solution of the equation. Since (3.5) contains second order derivatives we reduce it to a system of two first-order equations by the usual device of defining $p_1 = u$, $p_2 = \frac{du}{dz}$. Thus we have

$$\frac{dp_1}{dz} = p_2 ,$$

$$\frac{dp_2}{dz} = c^2 p_2 - \alpha c^2 K * p_1 - \epsilon c^2 (K * p_1)^3 .$$

Putting $\eta = c^2 - c_0^2$, we obtain

$$\frac{d}{dz}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = C \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \mathcal{K} * \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

$$+ \eta \left(C_1 \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \mathcal{K}_1 * \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \right)$$

$$+ \epsilon \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

for some constant $2 * 2$ matrices C, C_1; 2×2 matrix kernels $\mathcal{K}$, $\mathcal{K}_1$; and functions f_i of $p_1, p_2, K*p_1, K*p_2$ and η which contain only quadratic and higher order terms in these variables. In general let $\underset{\sim}{p}$ denote a column n-vector; C, K, C_1, K_1 $n \times n$ matrices; and $\underset{\sim}{f}$ an n-vector function of $\underset{\sim}{p}$, $K * \underset{\sim}{p}$, η and ϵ; and $\underset{\sim}{f}$ vanishes when $\underset{\sim}{p} = 0$ and contains only quadratic and higher order terms. Then we prove the following.

<u>Theorem</u>. Consider the system

$$\frac{d\underset{\sim}{p}}{dt} = C\underset{\sim}{p} + K*\underset{\sim}{p} + \eta\left[C_1\underset{\sim}{p} + K_1*\underset{\sim}{p}\right] \tag{3.23}$$
$$+ \epsilon\, \underset{\sim}{f}(\underset{\sim}{p}, K*\underset{\sim}{p}, \eta, \epsilon)$$

with $\underset{\sim}{f}$ nonlinear and η a real parameter. Assume that the eigenvalues of the linear equation

$$\frac{d\underset{\sim}{p}}{dt} = C\underset{\sim}{p} + K*\underset{\sim}{p} + \eta(C_1\underset{\sim}{p} + K_1*\underset{\sim}{p})$$

for all η sufficiently small are all in the left half-plane except for a pair that crosses the imaginary axis at $\pm i\mu$, $\mu \neq 0$, for $\eta = 0$. Assume this pair is simple and that the transversality condition holds, i.e.

$$Re\ \frac{d\sigma}{d\eta}(0) \neq 0$$

for the eigenvalue pair $\sigma(\eta)$ described.

Then (3.23) has a nontrivial one-parameter family of T-periodic solutions for all ϵ near 0 of the form

$$\underset{\sim}{p} = \underset{\sim}{y}\left(\frac{2\pi t}{\mu T}\right) + \underset{\sim}{w}\left(\frac{2\pi t}{\mu T}, \epsilon\right)$$

corresponding to the parameter value $\eta(\epsilon)$, where $\underset{\sim}{y}$ is a $\frac{2\pi}{\mu}$-periodic solution of

$$\frac{d\underset{\sim}{y}}{dt} = C\underset{\sim}{y} + K * \underset{\sim}{y} \tag{3.24}$$

and

$$T = T(\epsilon) = \frac{2\pi}{\mu}(1+\tau(\epsilon)).$$

Here w, η, τ are $O(\epsilon)$ as $\epsilon \to 0$. Furthermore, w can be chosen orthogonal to the periodic solutions of the adjoint of (3.24).

Proof: The proof is the same as that of the Hopf Bifurcation Theorem of Chapter II except for some initial differences. We note that the role played by ϵ is quite different in the two cases. We use the same notation as regards the linear equation (3.24). First rescale t by defining $t^* = \frac{t}{1+\tau(\epsilon)}$ and $\tilde{p}(t^*) \equiv p(t)$. Then putting

$$\underset{\sim}{\tilde{p}} = \underset{\sim}{y}(t^*) + \underset{\sim}{w}(t^*, \epsilon)$$

where

$$\underset{\sim}{y}(t^*) = Re(\gamma \underset{\sim}{x_0} e^{i\mu t^*}),$$

we obtain an equation for $\underset{\sim}{w}$ analogous to (2.15) (drop the asterisk on t.)

$$\frac{d\underset{\sim}{w}}{dt} - \mathcal{L}\underset{\sim}{w} = \underset{\sim}{\mathcal{Q}}(y, \tau, K) + \eta\cdot(C_1\underset{\sim}{y} + K_1 * \underset{\sim}{y})$$

$$+ \tau\mathcal{L}\underset{\sim}{y} + \underset{\sim}{G}(w, \eta, \tau, \epsilon) \tag{3.25}$$

where

$$G = (1+\tau)[\mathcal{Q}(w, \tau, K) + \eta C_1 w + \eta K_1 * w +$$

$$+ \eta \mathcal{J}(y, \tau, K_1) + \eta \mathcal{J}(w, \tau, K_1) + \epsilon f]$$

$$+ \tau [\mathcal{J}(y, \tau, K) + \eta C_1 y + \eta K_1 * y$$

$$+ Cw + K * w],$$

and where

$$\mathcal{J}(w, \tau, K) \equiv \int_0^\infty K(s) [\underset{\sim}{w} (t - \tfrac{s}{1+\tau}) - w(t-s)] ds$$

$$= O(\tau \tfrac{dw}{dt}) ,$$

and

$$\mathcal{L} w \equiv Cw + K * w$$

as in Chapter II. Now notice that the right member of (3.25) is exactly of the form of the right member in (2.15). Thus, since we are trying to solve only (3.25), the rest of the proof is exactly the same as that following (2.15). Putting the right side in (3.25) equal to $F(w, \eta, \tau, \epsilon)$, we try to solve the orthogonality conditions

$$h_i \equiv \int_0^{2\pi/\mu} y_i(t) \cdot F(w(t), \eta, \tau, \epsilon) \, dt = 0, \qquad i = 1, 2,$$

for η, τ in terms of w, ϵ. The relevant Jacobian is

$$\det \left. \frac{\partial(h_1, h_2)}{\partial(\eta, \tau)} \right|_{\eta = \tau = w = \epsilon = 0} =$$

$$\det \begin{bmatrix} y_1 \cdot [C_1 y + K_1 * y] & y_2 \cdot [C_1 y + K_1 * y] \\ y_1 \cdot [\mathcal{L} y + \ell_T^0] & y_2 \cdot [\mathcal{L} y + \ell_T^0] \end{bmatrix}$$

which is exactly as in (2.16). Since η here plays the role of λ, the Jacobian equals

$$\frac{\mu}{4} \, |\gamma|^2 \, | \underset{\sim}{y}_0^T \Delta^{(1)} \underset{\sim}{x}_0 |^2 \left(\mathrm{Re} \frac{d\sigma}{d\nu}(0) \right)$$

which is nonzero by hypothesis. The rest of the proof proceeds as follows (2.16). ∎

We make some final remarks on stability. It is easy to deduce as in Chapter II the stability properties of (3.22) as a solution of (3.5). However this is stability with respect to perturbations which also solve (3.5). This class of perturbations is a very small subset of all possible perturbations which solve the full equation (3.4). Thus a valid stability analysis would require an examination of the linearization of (3.4) about (3.22) and this is <u>not</u> the corresponding linearization of (3.5). At the moment this is an open question.

CHAPTER IV

LINEAR EQUATIONS WITH PERIODIC COEFFICIENTS

1. Introduction

The previous two chapters have been devoted to the problem of finding periodic solutions of nonlinear integro-differential equations. It would be desirable to determine the stability of these solutions, and this question will be considered in the next chapter. There it will be required to solve linear integro-differential equations with periodic coefficients. In this chapter we will analyze this class of equations, and to accomplish this the semi-group idea of Chapter I is applied. This will result in a spectral (Floquet) theory for the linear equations with periodic coefficients much like the one in Chapter I for constant coefficients. However, here a completely different approach must be taken due to the inability to define an infinitesimal operator for the semi-group. There is a striking analogy of the theory with the corresponding Floquet theory for ordinary differential equations, cf. Coddington and Levinson [1], Chapter 3. The analysis we perform follows closely that of Hale [4] which is for finite-lag equations. The important results we are able to carry over to the infinite-lag case. We show that the general solution is a sum of eigenfunctions each of which correspond to a characteristic (Floquet) multiplier the totality forming the spectrum of the equation under consideration. The method used by Hale to analyze this spectrum cannot be used in the infinite-lag case, and we take a different approach. We then deduce an explicit form for the eigenfunctions thus characterizing the behavior of the general solution. Finally we obtain a Fredholm Alternative theorem giving necessary and sufficient conditions for the existence of a periodic solution of the equation with periodic forcing.

We now develop the Floquet theory for linear equations with periodic coefficients. The methods we use are those of Chapter I with modifications and generalizations to the present problem. Again we follow Hale [4], Chapters 35-37.

2. The Semi-group

Consider

$$\frac{d\underset{\sim}{x}}{dt} = A(t)\,\underset{\sim}{x}(t) + \int_0^\infty K(t,s)\,\underset{\sim}{x}(t-s)\,ds \tag{4.1}$$

where $\underset{\sim}{x}(t)$ is a column vector in $\mathbb{R}^n$, $A(t)$ is an $n\times n$ matrix function, periodic of period $\omega > 0$, $K(t,s)$ is the kernel, an $n\times n$ matrix function defined for $s \geq 0$, periodic of period ω in its first argument. Thus $A(t+\omega) = A(t)$, $K(t+\omega,s) = K(t,s)$ for all t. We shall always assume that $K(t,s) = O(e^{-\gamma s})$ as $s \to \infty$ uniformly in t for some $\gamma > 0$. Let us denote

$$\mathcal{L}(t,\underset{\sim}{x}) \equiv A(t)\,\underset{\sim}{x} + \int_0^\infty K(t,s)\,\underset{\sim}{x}(t-s)\,ds.$$

As in Chapter I we view the solution $\underset{\sim}{x}(t)$ of (4.1) as a collection of functions $\underset{\sim}{x}_t$ each of which is defined on the interval $(-\infty, 0]$. Specifically, we define

$$\underset{\sim}{x}_t(\theta) = \underset{\sim}{x}(t+\theta), \qquad \theta \in (-\infty, 0],\ t \in \mathbb{R}.$$

A well-posed initial value problem results if $\underset{\sim}{x}$ is given for all $t \leq 0$. Then a solution of (4.1) exists for all $t \geq 0$ coinciding with the initial data for $t \leq 0$. We view the situation as follows: the initial data are given by ϕ where $\underset{\sim}{\phi}(\theta) = \underset{\sim}{x}(\theta)$, $\theta \in (-\infty, 0]$. This generates a solution $\underset{\sim}{x}(t)$ for $t \geq 0$. This is equivalently written as x_t and the initial conditions demand that $x_t = \phi$ for $t = 0$. Thus, x_t is a curve with initial point ϕ in some space C of functions defined on $(-\infty, 0]$. The process of obtaining x_t from ϕ is viewed as a linear transformation T from C into itself. We write

$$x_t = T(t, 0)\, \phi, \qquad t \geq 0,$$

where $x_0 = \phi$. If we shift the initial data from $(-\infty, 0]$ to $(-\infty, s]$ then a solution x_t is generated for $t \geq s$, and we write

$$x_t = T(t, s)\, \phi, \qquad t \geq s,$$

where $x_s = \phi$. Let us adopt the notation $x_t(s, \phi) = T(t, s)\phi$. Some obvious properties of T are

$$T(t, \tau) = T(t, s)\, T(s, \tau), \qquad \tau \leq s \leq t,$$

$$T(s, s) = I,$$

where I is the identity transformation. Thus, $T(t, s)$ forms a semi-group of operators on C.

We shall take C to be given as follows:

$$C = \left\{ \phi \text{ continuous on } (-\infty, 0] \;\middle|\; \sup_{\theta \le 0} |e^{\gamma^+\theta}\phi(\theta)| < \infty, \right.$$

$$\left. \lim_{\theta \to -\infty} e^{\gamma^+\theta}\phi(\theta) \text{ exists} \right\}, \quad \|\phi\| = \sup_{\theta \le 0} |e^{\gamma^+\theta}\phi(\theta)|,$$

with $\gamma^+ < \gamma$ and γ^+ arbitrarily near γ.

That each $T(t,s)$ is bounded can be shown with the aid of the resolvent of (4.1). The resolvent $R(t,s)$ is an $n \times n$ matrix function solving the equation

$$\frac{\partial R}{\partial s}(t,s) = -R(t,s)A(s) - \int_s^t R(t,u)K(u,u-s)\,du, \qquad s \le t,$$

$$R(t,t) = I.$$

This equivalent to the Volterra integral equation

$$R(t,s) = I + \int_s^t R(t,u)\left[A(u) - L(u,u-s)\right] du, \qquad s \le t, \tag{4.2}$$

$$R(t,s) \equiv 0, \qquad t < s,$$

where $L(\alpha,\beta) \equiv -\int_0^\beta K(\alpha,\gamma)\,d\gamma,\ \beta \ge 0,\ \alpha \in \mathbb{R}$. Existence theorems such as in Miller [11] assert that a continuously differentiable R exists and that if $\underset{\sim}{y}$ solves

$$\frac{d\underset{\sim}{y}}{dt} = A(t)\underset{\sim}{y} + \int_0^{t-\sigma} K(t,s)\,\underset{\sim}{y}(t-s)\,ds + \underset{\sim}{g}(t)$$

then

$$\underset{\sim}{y}(t) = R(t,\sigma)\,\underset{\sim}{y}(\sigma) + \int_\sigma^t R(t,s)\,\underset{\sim}{g}(s)\,ds, \qquad t \ge \sigma.$$

In our case if $\underset{\sim}{x}$ solves (4.1), then by putting $\underset{\sim}{g}(t) = \int_{t-\sigma}^{\infty} K(t,u)\, \underset{\sim}{x}(t-u)\, du$ we obtain an explicit formula for $\underset{\sim}{x}(t)$:

$$\underset{\sim}{x}(t) = R(t,\sigma)\, \underset{\sim}{x}(\sigma) + \int_{\sigma}^{t} R(t,s) \Big[\int_{s-\sigma}^{\infty} K(s,u)\, \underset{\sim}{x}(s-u)\, du \Big] ds. \quad (4.3)$$

If the initial data is ϕ , specified up to time σ (so $x(\sigma+\theta) = \phi(\theta)$, $\theta \leq 0$) , then we can write this as

$$\underset{\sim}{x}(t) = T(t,\sigma)\phi = R(t,\sigma)\, \underset{\sim}{\phi}(0) + \int_{-\infty}^{0} \Big[\int_{\sigma}^{t} R(t,s)\, K(s, s-\sigma-v)\, ds \Big] \phi(v)\, dv,$$

for $t \geq \sigma$. Thus,

$$[T(t,\sigma)\phi](\theta) = \begin{cases} R(t+\theta,\sigma)\,\phi(0) + \\ \quad \int_{-\infty}^{0} \Big[\int_{\sigma}^{t+\theta} R(t+\theta,s)\, K(s,s-\sigma-v)\, ds \Big] \phi(v)\, dv, & \sigma - t \leq \theta \leq 0, \\ \phi(\theta+\sigma-t), & \theta \leq \sigma - t. \end{cases}$$

This is similar to the corresponding formula of Chapter I and a repetition of the arguments there proves that $\| T(t,s) \|$ is finite for all $t \geq s$, and that the semi-group is strongly continuous:

$$\lim_{t \to \tau} \| T(t,s)\phi - T(\tau,s)\phi \| = 0$$

for each $\phi \in C$, $s \in \mathbb{R}$.

In addition to the transitivity property

$$T(t,s)\ T(s,u) = T(t,u), \qquad u \le s \le t,$$

we have

$$T(t+\omega, s) = T(t,s)\ T(s+\omega, s) \qquad (4.4)$$

where ω is the period of the coefficients in (4.1). This proved by first showing that

$$T(t, s-\omega) = T(t+\omega, s), \qquad t \geq s. \qquad (4.5)$$

This is demonstrated by the following argument. Let x_t solve (4.1) where the initial point is s : $x_s = \phi$. Define $\underset{\sim}{y}(t) = \underset{\sim}{x}(t+\omega)$ for all t . It is easy to show that $\underset{\sim}{y}$ also solves (4.1) by the periodicity of the coefficients. Also $y_{s-\omega} = x_s = \phi$ so $y_t = T(t, s-\omega)\ \phi$. But clearly $y_t = x_{t+\omega} = T(t+\omega, s)\ \phi$ and (4.5) follows. To show (4.4), $T(t,s)\ T(s+\omega, s) = T(t,s)\ T(s, s-\omega) = T(t, s-\omega) = T(t+\omega, s)$ using (4.5) and transitivity.

3. The Spectrum

With these preliminaries we proceed to the analysis of the structure of the semi-group. This we do in a manner similar to that of the Floquet theory for ordinary differential equations, cf. Coddington and Levinson [1], Chapter 3, Section 5. There the circulant matrix $\Phi(\omega)$ (where $\Phi(t)$ is the fundamental matrix solution) is spectrally decomposed into Jordan canonical form and $\Phi(t)$ is subsequently determined. It is shown that all solutions are linear combinations of terms of the form $P(t)\, t^r e^{\lambda t}$ where r is a nonnegative integer, $e^{\lambda\omega}$ is an eigenvalue of $\Phi(\omega)$, and $P(t)$ is some ω-periodic function. In our case we examine

$$U = T(\omega, 0)$$

for its spectrum and then determine $T(t,0)$ for all $t \geq 0$. Since $T(t,s)$ is similarly found by considering $U(s) = T(s+\omega, s)$ we limit ourselves to the case $s = 0$.

The operator U is a bounded linear transformation on the space C. Thus its spectrum is confined to the disc $\{z \mid |z| \leq \Omega_0\}$ in the complex plane $\mathbb{C}$ where Ω_0 is the spectral radius of U. We shall show that for all γ' such that $0 < \gamma' < \gamma^+$ the spectrum of U in $|z| \geq e^{-\gamma'\omega}$ consists of finitely many point eigenvalues. This is done by showing that U is nearly equal to an operator of finite rank (whose spectrum is known to consist of finitely many point eigenvalues).

Letting $x_t = T(t,0)\phi$ we have

$$[U\phi](\theta) = \begin{cases} x(\omega+\theta), & \theta \in [-\omega, 0], \\ \phi(\omega+\theta), & \theta \in (-\infty, -\omega]. \end{cases}$$

Let $f(\theta)$ be a continuous function such that $0 \leq f \leq 1$, with $f = 1$ on $[-a, 0]$, and $f = 0$ on $(-\infty, -\omega]$ where $-\omega \leq -a$ and a arbitrarily near ω. Define the operator U' on C (cf. (4.3), with $t = \theta + \omega$, $\sigma = 0$):

$$[U'\phi](\theta) = \begin{cases} f(\theta)R(\omega+\theta,0)\phi(0) + \int_0^{\theta+\omega} R(\theta+\omega,s)\left[\int_s^\infty K(s,u)\phi(s-u)\,du\right]ds, \\ \qquad\qquad\qquad \theta\in[-\omega,0], \\ 0, \qquad \theta\in(-\infty,-\omega]. \end{cases}$$

Clearly $U'\phi$ is in C. Also U' is a bounded linear transformation. In effect U' replaces the initial data part of x_ω by zero allowing it to become a compact operator. Next, we have

$$\begin{aligned} \|U\phi - U'\phi\| &= \sup_{\theta\le 0} |e^{\gamma^+\theta}(U\phi - U'\phi)| \\ &= \max\Big\{\sup_{\theta\le-\omega} |e^{\gamma^+\theta}\phi(\omega+\theta)|, \\ &\qquad \sup_{-\omega\le\theta\le-a} |e^{\gamma^+\theta}[f(\theta)R(\omega+\theta,0) - R(\omega+\theta,0)]\phi(0)|\Big\} \\ &\le \max\Big\{\sup_{\theta\le-\omega} |e^{\gamma^+(\theta+\omega)}\phi(\theta+\omega)|\, e^{-\gamma^+\omega}, \\ &\qquad \sup_{-\omega\le\theta\le-a} |R(\omega+\theta,0)|\, e^{-\gamma^+a}\, |\phi(0)|\Big\} \\ &\le \max\Big\{\|\phi\|\, e^{-\gamma^+\omega},\ e^{-\gamma^+a}\|\phi\| \sup_{0\le\theta'\le\omega-a} |R(\theta',0)|\Big\} \\ &\le [e^{-\gamma^+\omega} + \eta(a)]\, \|\phi\|, \end{aligned}$$

where $\eta(a)\to 0$ as $a\to\omega$ independently of ϕ. Then $\|U-U'\| \le e^{-\gamma^+\omega} + \eta(a)$. Finally, we approximate U' arbitrarily closely by an operator W having finite dimensional range. Note that $U'\phi$ has the form

$$[U'\phi](\theta) = \begin{cases} g_1(\theta)\phi(0) + \int_0^{\theta+\omega} g_2(\theta,s)\left[\int_s^\infty K(s,u)\phi(s-u)\,du\right]ds, \\ \qquad\qquad\qquad -\omega\le\theta\le 0, \\ 0, \qquad \theta\le-\omega, \end{cases}$$

where g_1, g_2 are continuous functions that vanish for $\theta\le-\omega$. Choose any $\epsilon>0$. Divide $[-\omega,0]$ into a finite number of subintervals

$[\theta_1 = -\omega, \theta_2]$, $[\theta_2, \theta_3]$, $\ldots$, $[\theta_{m-1}, \theta_m = 0]$ such that on each the variation of $g_2(\theta, s)$ in θ is less than $\epsilon / (2 \max_{0 \le v \le \omega} \int_0^\infty |K(v,u)| e^{\gamma^* u} du)$ for all $s \in [0, \omega]$ and the variation of g_1 is $< \epsilon/2$. Let h_i be defined on $[-\omega, 0]$ by $(i = 1, \cdots, m)$

$$h_i(\theta) = \begin{cases} 1, & \theta = \theta_i, \\ 0, & \theta = \theta_j, \quad j \neq i, \end{cases}$$

with linear interpolation between mesh points. Let

$$k_{1,2}(\theta) = \sum_{i=1}^{m} h_i(\theta) \, g_{1,2}(\theta_i).$$

These functions are approximations of g_1, g_2 respectively, and clearly $\max_{\theta \in [-\omega, 0]} | k_1(\theta) - g_1(\theta)| < \epsilon/2$, etc. . Now define the linear operator W,

$$[W\phi](\theta) = \begin{cases} \sum_{i=1}^{m} h_i(\theta) \, [U'\phi](\theta_i), & \theta \in [-\omega, 0], \\ 0, & \theta \in (-\infty, -\omega). \end{cases} \tag{4.6}$$

First note that $\| U' - W \| < \epsilon$ since if $\|\phi\| = 1$,

$$\| U'\phi - W\phi \| \le \sup_{-\omega \le \theta \le 0} [\, | k_1(\theta) - g_1(\theta)| +$$

$$+ \left| k_2(\theta) - g_2(\theta) \right| \omega \max_s \int_0^\infty |K(s,u)| e^{\gamma^+ u} du \Big],$$

$$< \epsilon .$$

Secondly, note that W has finite dimensional range being a subspace of $\text{span}\{h_1, \ldots, h_m\}$. Finally, we have

$$\| U - W \| \leq e^{-\gamma^+ \omega} + \eta(a) + \epsilon$$

Choose ϵ so small and positive, and a so near ω that $e^{-\gamma^+\omega} + \eta + \epsilon < e^{-\gamma'\omega}$ with γ' a prespecified number in $[0, \gamma^+)$ and arbitrarily near γ^+. We now show that the spectrum of U has the properties mentioned above. Put $\Omega = e^{-\gamma'\omega}$ and $H = U - W$. Then since $\|H\| < \Omega$, for $|\lambda| > \Omega$ we have that $H_\lambda \equiv H - \lambda I$ is nonsingular, i.e. one-one, onto, and has bounded inverse. Thus H_λ^{-1} is analytic in the region $|\lambda| > \Omega$ and can in fact be expanded in a power series in λ^{-1}, cf. Taylor [16]. Also

$$U_\lambda = U - \lambda I = W + H_\lambda = H_\lambda (H_\lambda^{-1} W + I).$$

By (4.6) W can be written

$$W = \sum_{i=1}^{m} h_i \alpha_i$$

where $h_i \in C$ and the α_i are bounded linear functionals on C

given by $\alpha_i(\phi) = [U'\phi](\theta_i)$. Thus

$$Q(\lambda) \equiv H_\lambda^{-1} W = \sum_{i=1}^{m} h_i(\lambda)\, \alpha_i$$

where $h_i(\lambda) = H_\lambda^{-1} h_i$ are elements of C depending analytically on λ in $|\lambda| > \Omega$. We show that $U_\lambda^{-1} = (Q(\lambda) + I)^{-1} H_\lambda^{-1}$ has at most finite-order poles in $|\lambda| > \Omega$. These obviously can occur only at the poles of $(Q(\lambda) + I)^{-1}$. Also note that the spectrum of U is confined to $|\lambda| \leq \Omega^*$ where Ω^* is larger than the spectral radius of U and such that $\Omega < \Omega^*$. Thus we need only examine the spectrum of $Q(\lambda) + I$ in the annulus $\Omega \leq |\lambda| \leq \Omega^*$.

Putting $T = Q(\lambda) + I$; for any $\phi \in C$

$$\psi = T\phi = \sum_{i=1}^{m} h_i(\lambda)\, \alpha_i(\phi) + \phi .$$

If $T\phi = 0$ then

$$\sum_{j=1}^{m} c_{ij}(\lambda)\, \alpha_j(\phi) + \alpha_i(\phi) = 0 , \qquad i = 1, \ldots, m, \tag{4.7}$$

where

$$c_{ij}(\lambda) = \alpha_i(h_j(\lambda))$$

is analytic in the annulus. Equation (4.7) implies

$$\det [c_{ij}(\lambda) + \delta_{ij}] = 0 ; \tag{4.8}$$

and the set Σ of λ such that this occurs in the annulus is finite (each point in Σ is isolated). Clearly Σ is the spectrum of $Q(\lambda)+I$ in the annulus. For $\lambda \notin \Sigma$ we can invert T as follows. Put $a_{ij}(\lambda) = c_{ij}(\lambda) + \delta_{ij}$. Then $\alpha_i(\psi) = \sum_j a_{ij}(\lambda)\alpha_j(\phi)$, and since the matrix $[a_{ij}(\lambda)]$ is invertible $\alpha_i(\phi) = \sum_j a_{ij}^{-1}\alpha_j(\psi)$. Thus $\phi = \psi - \sum h_i(\lambda)\alpha_i(\phi) = \psi - \sum_{ij} h_i(\lambda)\, a_{ij}^{-1}\, \alpha_j(\psi) \equiv T^{-1}\psi$. By Cramer's rule a_{ij}^{-1} is analytic except at $\lambda \in \Sigma$ where (4.8) holds. Thus we conclude that $Q(\lambda)+I$ is nonsingular in the annulus except at $\lambda \in \Sigma$ where its inverse has a pole. Thus, the spectrum of $U - \lambda I$ has exactly the same properties in the annulus. We now appeal to Theorem 5.8-A of Taylor [16] and summarize the results as follows:

> Inside the annulus $e^{-\gamma'\omega} \le |\lambda| \le \Omega^*$ the spectrum Σ of U consists of finitely many point eigenvalues with finite dimensional generalized null space, i.e. the ascent ν_λ of $U_\lambda \equiv U - \lambda I$ is finite:
>
> $$n(U_\lambda^{\nu_\lambda}) = n(U_\lambda^{\nu_\lambda+1}),$$
> $$\dim n(U_\lambda^{\nu_\lambda}) < \infty .$$
>
> Furthermore, C can be decomposed into a direct sum of the null and range spaces of $U_\lambda^{\nu_\lambda}$:
>
> $$C = n(U_\lambda^{\nu_\lambda}) \oplus R(U_\lambda^{\nu_\lambda})$$
>
> The range $R(U_\lambda^{\nu_\lambda})$ is closed.

This characterizes the part of the spectrum of U we are concerned with. These results are similar to those of Chapter I, Lemma 6. Next we turn to an analysis of the operator $T(t,s)$ using this information.

Let μ be an eigenvalue of U, and we shall always assume $|\mu| > e^{-\gamma'\omega}$. We call μ a <u>characteristic</u> (Floquet) <u>multiplier</u>. Define the corresponding characteristic (Floquet) exponent, λ, by $\mu = e^{\lambda\omega}$. Then $Re\,\lambda > -\gamma'$. Denote

$$E_\mu = \mathcal{N}\left((U-\mu I)^{\nu_\mu} \right),$$

$$R_\mu = \mathcal{R}\left((U-\mu I)^{\nu_\mu} \right).$$

We have the following

(i) E_μ is finite dimensional

(ii) $C = E_\mu \oplus R_\mu$,

(iii) $U(E_\mu) \subset E_\mu$, $U(R_\mu) \subset R_\mu$,

(iv) the spectrum of U restricted to E_μ, $\Sigma(U|_{E_\mu})$ equals $\{\mu\}$. Also $\Sigma(U|_{R_\mu}) = \Sigma - \{\mu\}$.

Let $\{\phi_1, \cdots, \phi_{d_\mu}\}$ be a basis for E_μ ($d_\mu \equiv \dim E_\mu$). By (iii) there is a $d_\mu \times d_\mu$ matrix $M_\mu = (m_{ij})$ of real numbers such that $U\phi_i = \sum_j \phi_j m_{ji}$,

$$U\Phi = \Phi M_\mu,$$

where $\Phi = \text{row}(\phi_1, \cdots, \phi_{d_\mu})$, a row-vector of elements of C. Property (iv) above implies that M_μ has the sole eigenvalue $\mu\,(\neq 0)$, so there exists a $d_\mu \times d_\mu$ matrix B such that $B = \frac{1}{\omega}\log M$. Define the row vector $P(t)$ (elements in C),

$$P(t) \equiv [T(t,0)\,\Phi]\, e^{-Bt}, \qquad t \geq 0.$$

Then by (4.4), $P(t+\omega) = T(t+\omega,0)\,\Phi\, e^{-B\omega} e^{-Bt} = T(t,0)\, T(\omega,0)\,\Phi\, e^{-B\omega} e^{-Bt}$ $= T(t,0)\, U\Phi\, e^{-B\omega} e^{-Bt} = T(t,0)\,\Phi\, e^{-Bt} = P(t)$. Extend the definition of

P in an obvious manner to all $t \leq 0$ so P is ω-periodic. Thus $P(t)\, e^{Bt}$ is defined for all $-\infty < t < \infty$ and $T(t,0)\,\Phi = P(t)\, e^{Bt}$. Thus $x_t(0,\phi) = P(t)\, e^{Bt}\, \underset{\sim}{b}$, where $\phi = \Phi \underset{\sim}{b}$, for some scalar column d_μ-vector $\underset{\sim}{b}$. We seek the function $x(t)$ represented by x_t. For all $\theta \leq 0$, $x_t(\theta) = x(t+\theta) = x_{t+\theta}(0) = P(t+\theta)(0)\, e^{B(t+\theta)}\underset{\sim}{b}$. Define the row d_μ-vector function $P_0(s) = P(s)(0)$ (all $s \in \mathbb{R}$). Note $P_0(s+\omega) = P_0(s)$. Then $\underset{\sim}{x}(t) = P_0(t)\, e^{Bt}\underset{\sim}{b}$, and we conclude

> E_μ generates solutions of (4.1) of the form
>
> $$P_0(t)\, e^{Bt}\, \underset{\sim}{b}$$
>
> with P_0 an ω-periodic row-vector function, B a matrix with sole eigenvalue $\frac{1}{\omega} \log \mu$, and $\underset{\sim}{b}$ a column scalar vector (all of dimension d_μ).

This form of the solution x is in complete analogy with that of the Floquet theory for ordinary differential equations, cf. Coddington and Levinson [1], Chapter 3, Section 5 (where B would be a Jordan block of the logarithm of the circulant matrix.)

We have deduced the set of characteristic multipliers corresponding to solutions generated at starting time $t = 0$. We need to show that the characteristic multipliers are independent of the starting time. For any such time s (≥ 0) let $U(s) \equiv T(s+\omega, s)$. All the above deduced for U can be similarly deduced for $U(s)$ thus obtaining generalized null space $E_\mu(s)$ with corresponding basis $\Phi(s)$, and range $R_\mu(s)$ for any $\mu \in \Sigma(U(s))$. As with the case $s = 0$ we can define $T(t,s)\, \Phi(s)$ for all $t \in \mathbb{R}$. Then for any τ

$$\begin{aligned} U(\tau)\, T(\tau,s)\, \Phi(s) &= T(\tau+\omega,\tau)\, T(\tau,s)\, \Phi(s) \\ &= T(\tau+\omega,s)\, \Phi(s) = T(\tau,s)\, T(s+\omega,s)\, \Phi(s) \\ &= T(\tau,s)\, U(s)\, \Phi(s) = T(\tau,s)\, \Phi(s)\, M_\mu(s). \end{aligned}$$

Thus

$$(U(\tau)-\mu I)\; T(\tau,s)\,\Phi(s) = T(\tau,s)\,\Phi(s)\,(M_\mu(s)-\mu I)$$

and it is easy to show that

$$(U(\tau)-\mu I)^k\; T(\tau,s)\,\Phi(s) = T(\tau,s)\,\Phi(s)\;(M_\mu-\mu I)^k.$$

Letting $k =$ ascent of $U(s)$, we have

$$(U(\tau)-\mu I)^k\; T(\tau,s)\,\Phi(s) = 0.$$

If we could show that $T(\tau,s)\,\Phi(s)\,\underset{\sim}{b} = 0$ implies $\underset{\sim}{b} = 0$, then we could deduce that μ is also an eigenvalue of $U(\tau)$ and that $\dim E_\mu(\tau) \geq \dim E_\mu(s)$. Reversing the roles of s and τ would prove that the dimensions are equal and that $T(\tau,s)\,\Phi(s)$ is a basis for $E_\mu(\tau)$, which is the desired result. To prove the above fact, choose m an integer large enough so that $s+m\omega > \tau$. Then $0 = T(s+m\omega,\tau)\cdot T(\tau,s)\,\Phi(s)\,\underset{\sim}{b} = T(s+m\omega,s)\,\Phi(s)\,\underset{\sim}{b} = [U(s)]^m\,\Phi(s)\,\underset{\sim}{b} = \Phi(s)\,M^m\,\underset{\sim}{b}$ implies $M^m\,\underset{\sim}{b} = 0$, i.e. $\underset{\sim}{b} = 0$ since the eigenvalue of M^m is $\mu^m \neq 0$. Lastly we note an explicit expression for the basis of $E_\mu(t)$,

$\Phi(t) \equiv T(t,0)\,\Phi(0) \equiv T(t,0)\,\Phi = P(t)\,e^{Bt}$.

Having achieved the decomposition of C into a direct sum of E_μ and R_μ (which are closed and invariant under U) we can similarly decompose R_μ into the generalized null and range spaces of $U-\mu' I$ restricted to R_μ for some other eigenvalue μ' of U . Repeating this procedure for any finite subset $\{\mu_1, \dots, \mu_n\}$ of $\Sigma(U)$ we have

$$C = E_{\mu_1} \oplus E_{\mu_2} \oplus \cdots \oplus E_{\mu_n} \oplus R$$

where is R some closed, invariant residual subspace. Any ϕ in $E_{\mu_1} \oplus \cdots \oplus E_{\mu_n}$ generates a solution x_t of the form $\sum_{i=1}^{n} P_{\mu_i}(t) \cdot$ $\cdot\, e^{B_{\mu_i} t} \underset{\sim}{b}_i$. Note that x_t has exponential growth $e^{\lambda t}$ as $t \to \infty$ where $\lambda = \max\{Re\lambda_i\}$; $\lambda_i = \frac{1}{\omega} \log \mu_i$ are the characterisitc multipliers. Suppose that $\{\mu_1, \dots, \mu_n\}$ consists of all eigenvalues in $\Sigma(U)$ such that $|\mu_i| \geq e^{\alpha\omega}$ where $\alpha > -\gamma'$. We wish to show that if ϕ is in the residual space R then the solution it generates, $x_t = T(t,0)\phi$, has exponential growth at mast $e^{\alpha t}$ as $t \to \infty$. First define the projections Q_i onto E_{μ_i} along R_{μ_i} . Each Q_i is bounded since E_{μ_i}, R_{μ_i} are closed. We later give an explicit form to these projections using an adjoint characterization of the problem. These projections are useful for separating out the various eigenfunctions that make up a solution x_t . Thus, given an arbitrary $\phi \in C$, the $Q_i\phi$ generate these eigenfunctions, denoted $x_t(0, Q_i\phi)$. We now prove

Given $\alpha > -\gamma'$ there are positive constants β, M depending on α only such that

$$\| x_t(s,\phi) - \sum_{|\mu_i| \geq \exp \alpha\omega} x_t(s, Q_i\phi) \| \leq M e^{(\alpha-\beta)(t-s)} \|\phi\| \quad , \quad \forall t \geq s.$$

In fact $0 < \beta < \alpha - \alpha'$ where $e^{\alpha'\omega}$ equals the magnitude of the first excluded μ in the sum.

We follow Hale [4], Theorem 35.1. Since $C = E_{\mu_1}(s) \oplus \cdots \oplus E_{\mu_n}(s) \oplus R(s)$, where $|\mu_1| \geq |\mu_2| \geq \cdots |\mu_n| \geq \exp \alpha\omega > |\mu_{n+1}|$, the spectrum of $U(s)$ restricted to $R(s)$ is $\Sigma(U) - \{\mu_1 \cdots \mu_n\}$. If $\pi\phi = \phi - \sum_1^n Q_i\phi$ then $\pi\phi \in R(s)$. Let $U_1(s) = U(s)$ restricted to $R(s)$. Then the spectral radius of U_1 equals $|\mu_{n+1}| = e^{\alpha'\omega}$. Proceeding as in Lemma 22.2 of Hale, for any $0 < \beta' < \alpha - \alpha'$, $e^{-\beta'\omega} = \lim_{n\to\infty} e^{-(\alpha'+\beta')\omega} \| U_1^n(s) \|^{1/n}$. Thus $\lim_{n\to\infty} e^{-(\alpha'+\beta')\omega} \| U_1^n(s) \| = \lim_n e^{-n\beta'\omega} = 0$. By strong continuity of $T(t,s)$ there exists N such that $\| T(t,s)|_R \| \leq N$ for $s \leq t \leq s+\omega$, $s \in \mathbb{R}$ (by (4.5) we may assume $0 \leq s \leq \omega$). Define $K(\beta') = N e^{|\alpha'+\beta'|\omega} \max_{n\geq 0} e^{-(\alpha'+\beta')n\omega} \| U_1^n(s) \|$. For $n\omega \leq t-s \leq (n+1)\omega$, $n = 0,1,2,\ldots$, $s \in \mathbb{R}$, $\phi \in R(s)$ we have $\| T(t,s)\pi\phi \| \leq \| T(t,s+n\omega)\, T(s+n\omega,s)\pi\phi \| \leq N \| U_1^n(s) \| \, \|\pi\phi\| = \left(N e^{|\alpha'+\beta'|(t-s)} \| U_1^n(s) \|\right) \cdot \left(e^{(\alpha'+\beta')(t-s)} \|\pi\phi\| \right) \leq K e^{(\alpha'+\beta')(t-s)} \|\pi\phi\| \leq K \|\pi(s)\| e^{(\alpha'+\beta')(t-s)} \|\phi\|$. Now let $\beta = \alpha - \alpha' - \beta'$. Since $U(s)$ is periodic in s, so is $\pi(s)$. Let $M = K \max_{0\leq s\leq\omega} \|\pi(s)\|$. Since $x_t(s,\phi) - \sum_i x_t(s, Q_i\phi) = T(t,s)\pi\phi$ we have completed the proof.

We summarize our results.

Given any γ' in $(0, \gamma^+)$, there are a finite number of characteristic multipliers μ such that $e^{-\gamma'\omega} \leq |\mu|$. Corresponding to each such μ and $s \in \mathbb{R}$ is a finite-dimensional subspace $E_\mu(s)$ of C such that $\phi \in E_\mu(s)$

generates a solution $x_t = T(t,s)\phi$ of (4.1) of the form

$$x_t(s,\phi) = P_\mu(t)\, e^{B_\mu t}\, \underset{\sim}{b}, \quad \forall t \geq s.$$

Here $P_\mu(t)$ is a row d_μ -vector (elements in $\mathbb{C}$) and is ω -periodic; B_μ is a $d_\mu \times d_\mu$ constant matrix with sole eigenvlaue $\frac{1}{\omega} \log \mu$; and $\underset{\sim}{b}$ is an arbitrary column (constant) d_μ -vector such that when it varies over $\mathbb{R}^{d_\mu}$, $\phi = P_\mu(s)\underset{\sim}{b}$ varies over $E_\mu(s)$. Furthermore, given $\alpha > -\gamma'$, an arbitrary $\phi \in C$, then the solution $x_t(s,\phi)$ generated is decomposed

$$x_t = \sum_{|\mu_i| \geq \exp \alpha\omega} x_t(s, Q_i\phi) + y_t$$

where $Q_i\phi \in E_{\mu_i(s)}$ and

$$\|y_t\| \leq M\, e^{(\alpha-\beta)(t-s)} \|\phi\|$$

for some $\beta, M > 0$.

The above completes our characterization of the solution set of (4.1) and provides a classification which will be needed later when we consider the question of the stability of periodic solutions of nonlinear equations. Notice that the circle $|\mu| = 1$ is included in the region of interest $|\mu| \geq e^{-\gamma^+ \omega}$. This circle is the dividing line between the stability and instability properties of the solutions of (4.1). If some Floquet multiplier μ has magnitude greater than one, then the corresponding Floquet exponent λ has real part greater than zero and there exists a solution of (4.1) which is unbounded as $t \to \infty$. If all the multipliers are inside the circle then all solutions of (4.1) tend to zero as $t \to \infty$.

The problem of the calculation of the Floquet multipliers is in general quite difficult. However, in the case of bifurcation, the multipliers vary continuously with some small parameter ϵ such that when $\epsilon = 0$ all the multipliers are known. Thus, by the use of a perturbation expansion the multipliers can be calculated, and the stability of the bifurcated solution is established. We shall consider this in the next chapter.

4. The Inhomogeneous Equation

Our next objective is to solve the inhomogeneous problem corresponding to (4.1). Consider the equation

$$\frac{d\underset{\sim}{x}}{dt} = A(t)\,\underset{\sim}{x}(t) + \int_0^\infty K(t,s)\,\underset{\sim}{x}(t-s)\,ds + \underset{\sim}{f}(t) \tag{4.9}$$

where $\underset{\sim}{f}$ is some continuous vector-function defined for $t \geq \sigma$. If $\underset{\sim}{x}$ is prespecified for $t \in (-\infty, \sigma]$ then (4.9) has a unique solution given by

$$\begin{aligned}\underset{\sim}{x}(t) = {} & R(t,\sigma)\,\underset{\sim}{x}(\sigma) + \int_\sigma^t R(t,s)\left[\int_{s-\sigma}^\infty K(s,u)\,\underset{\sim}{x}(s-u)\,du\right] ds \\ & + \int_\sigma^t R(t,s)\,\underset{\sim}{f}(s)\,ds,\end{aligned} \tag{4.10}$$

where R is the resolvent. Now let

$$X_0(\theta) = \begin{cases} I, & \theta = 0, \\ 0, & \theta < 0 \end{cases}$$

where X_0 is an $n \times n$ constant matrix defined on $(-\infty, 0]$. Then from (4.10) we see that

$$[T(t,\sigma)X_0](\theta) = \begin{cases} R(t+\theta,\sigma), & \theta \in [\sigma-t, 0], \\ I, & \theta = \sigma - t, \\ 0, & \theta < \sigma - t. \end{cases}$$

From this we can deduce the solution of (4.9) (letting the initial data be $x_\sigma = \phi$)

$$x_t(\sigma,\phi) = T(t,\sigma)\phi + \int_\sigma^t [T(t,s)X_0]\cdot f(s)\,ds, \qquad (4.11)$$

the <u>variation of constants formula.</u>

Up to now we have been concerned with the properties of the T operator. We have cast the problem of the structure of the solution set of (4.1) into the language of semi-groups with the result that every solution is a linear combination of certain eigenfunctions. We shall need a convenient way of projecting out these various components of the solution and an adjoint characterization of the problem (4.1) will provide this. After obtaining an appropriate adjoint equation, we proceed as in the case of ordinary differential equations and introduce a bilinear form connecting the original and adjoint formulations. This is done with the object of providing explicit formulas for the projections.

We turn to the adjoint problem. Letting $\underset{\sim}{y}(s)$ be a row n-vector, the <u>adjoint equation</u> is

$$\frac{d\underset{\sim}{y}}{ds} = -\underset{\sim}{y}(s)\,A(s) - \int_{-\infty}^{0} \underset{\sim}{y}(s-\xi)\,K(s-\xi,-\xi)\,d\xi \qquad (4.12)$$

for all $s \le \tau$, with initial data $y_\tau = \psi$ given on $[\tau, \infty)$. Note $\underset{\sim}{y}$ is generated backwards. The set of initial data forms a Banach space

$$\tilde{C} = \{\underset{\sim}{\psi} \text{ continuous on } [0,\infty) \mid \sup_{\xi \ge 0} |e^{-r^+\xi}\psi(\xi)| < \infty\},$$

$$\|\psi\| = \sup_{\xi \ge 0} |e^{-r^+\xi}\psi(\xi)|.$$

As usual $\underset{\sim}{y}_t \in \tilde{C}$ means $\underset{\sim}{y}_t(\xi) = \underset{\sim}{y}(t+\xi)$ for all $\xi \ge 0$.

Define a bilinear form on $\tilde{C} \times C$ into $\mathbb{R}$ depending on t parametrically, such that for $\phi \in C$, $\psi \in \tilde{C}$

$$(\psi, \phi)_t \equiv \underset{\sim}{\psi}(0)\cdot\underset{\sim}{\phi}(0) - \int_{-\infty}^{0}\left[\int_{-\theta}^{0} \psi(\xi)\, K(t+\xi,-\theta)\, \phi(\xi+\theta)\, d\xi\right] d\theta. \quad (4.13)$$

This form is usually degenerate but it is useful for connecting problems (4.1) and (4.12). As a first result in this direction we show that if x_t solves the inhomogeneous problem (4.9) and y_t solves (4.12) then

$$(y_t, x_t)_t = (y_\sigma, x_\sigma)_\sigma + \int_\sigma^t \underset{\sim}{y}(s)\cdot\underset{\sim}{f}(s)\, ds \qquad (4.14)$$

for all $t \ge \sigma$. This will be used many times. Thus,

$$\begin{aligned}(y_t, x_t)_t &= y_t(0)\cdot x_t(0) - \int_{-\infty}^{0}\left[\int_{-\theta}^{0} y(t+\xi)\, K(t+\xi,-\theta)\, x(t+\xi+\theta)\, d\xi\right] d\theta \\ &= y(t)\cdot x(t) + \int_0^\infty y(t+\xi)\left[\int_{-\infty}^{-\xi} K(t+\xi,-\theta)\, x(t+\xi+\theta)\, d\theta\right] d\xi \\ &= y(t)\cdot x(t) + \int_t^\infty y(\xi)\left[\int_{-\infty}^{-\xi-t} K(\xi,-\theta)\, x(\xi+\theta)\, d\theta\right] d\xi.\end{aligned}$$

Then

$$\frac{d}{dt}(y_t, x_t)_t = \dot{y}(t)\cdot x(t) + y(t)\cdot\dot{x}(t) - y(t)\int_{-\infty}^{0} K(t,-\theta)\, x(t+\theta)\,d\theta$$

$$+ \int_t^{\infty} y(\xi)\; K(\xi, \xi - t)\; x(t)\, d\xi$$

$$= [\dot{y}(t) + y(t)A]\,x(t) + \int_0^{\infty} y(\xi)\, K(\xi,\xi-t)\,d\xi \cdot x(t)$$

$$+ y(t)\cdot f(t) = y(t)\cdot f(t).$$

Integrating yields (4.14).

5. The Spectrum of the Adjoint

We now proceed to the spectral properties of the adjoint. Since (4.12) is an integro-differential equation with periodic coefficients, we see that whatever qualitative results were deduced for (4.1) will be true of (4.12). Corresponding to (4.12) is a semi-group $\tilde{T}(s,\tau)$ for $s \leq \tau$ and $\tilde{U} = \tilde{T}(0,\omega)$ (more generally $\tilde{U}(s) = \tilde{T}(s, s+\omega)$). The spectrum $\tilde{\Sigma}$ of $\tilde{U}(s)$ allows a decomposition

$$\tilde{C} = \sum_{\mu \in \tilde{\Sigma}} \tilde{E}_\mu(s) \oplus \tilde{R}$$

with $\tilde{E}_\mu(s) = \mathcal{N}(\tilde{U}(s) - \mu I)^{\tilde{\nu}_\mu}$ having basis $\Psi(s)$, a column vector (elements in $\tilde{C}$.) We shall prove that $\tilde{\Sigma} = \Sigma$, $\tilde{\nu}_\mu = \nu_\mu$ (ascents are equal), $\tilde{d}_\mu = d_\mu$ (dimensions of $\tilde{E}_\mu$, E_μ are equal.) Thus the characteristic multipliers and the corresponding eigenspaces are in a one-one correspondence.

As a preliminary to this proof we obtain an equivalent integral equation formulation of (4.12). Suppose $\sigma < t$, $z_t = \psi$ is given, $z_\sigma = \tilde{T}(\sigma, t)\, z_t$. Then integrating (4.12) from σ to t yields

$$z(t) - z(\sigma) = -\int_{\sigma}^{t} z(s)\,A(s) - \int_{\sigma}^{t}\int_{s}^{\infty} z(\xi)\,K(\xi, \xi-s)\,d\xi\,ds.$$

Interchanging the order of integration in the double integral gives

$$z(\sigma) = -\int_{\sigma}^{t} z(s)\left[L(s, s-\sigma) - A(s)\right]ds + z(t) +$$

$$\int_{t}^{\infty} z(\xi)\left[L(\xi, \xi-t) - L(\xi, \xi-\sigma)\right]d\xi$$

where

$$L(\alpha,\beta) \equiv -\int_{0}^{\beta} K(\alpha,\gamma)\,d\gamma, \qquad \beta \geq 0. \tag{4.15}$$

Since $z_t = \psi$, this gives

$$z(\sigma) = -\int_{\sigma}^{t} z(s)\left[L(s, s-\sigma) - A(s)\right]ds$$

$$+ \psi(0) + \int_{t}^{\infty} \psi(\xi-t)\left[L(\xi, \xi-t) - L(\xi, \xi-\sigma)\right]d\xi,$$

i.e.

$$z(\sigma) = -\int_{\sigma}^{t} z(s)\left[L(s, s-\sigma) - A(s)\right]ds$$

$$+ \psi(0) + \int_{0}^{\infty} \psi(\theta)\left[L(\theta+t, \theta) - L(\theta+t, \theta-(\sigma-t))\right]d\theta. \tag{4.16}$$

Thus we have an equivalent Volterra integral equation to be satisfied for the solution $z(\sigma)$ of (4.12).

Next we show that the adjoint equation to (4.12) is just the original equation (4.1). Thus the adjoint of the adjoint is the original. Write (4.12) in a form similar to (4.1) by putting $t = -s$ and $y_1(t) = y^T(-t) = y^T(s)$ (superscript T denotes transpose.) Then

$$\dot{y}_1(t) = A^T(-t)\, y_1(t) + \int_{-\infty}^{0} K^T(t-\xi, -\xi)\, y_1(t+\xi)\, d\xi.$$

Putting $K_1(t,-\xi) = K^T(-t-\xi, -\xi)$, and $A_1(t) = A^T(-t)$, we obtain

$$\dot{y}_1(t) = A_1(t)\, y_1(t) + \int_0^{\infty} K_1(t,\xi)\, y_1(t-\xi)\, d\xi \;;$$

exactly the form in (4.1). The adjoint of this is

$$\dot{x}_1(s) = -x_1(s)\, A_1(s) - \int_{-\infty}^{0} x_1(s-\xi)\, K_1(s-\xi, -\xi)\, d\xi$$

$$= -x_1(s)\, A^T(-s) - \int_{-\infty}^{0} x_1(s-\xi)\, K^T(-s, -\xi)\, d\xi.$$

Going back to the original variables $t = -s$ and $x(t) = x_1^T(-t) = x_1^T(s)$ we get

$$\dot{x}(t) = A(t)\, x(t) + \int_{-\infty}^{0} K(t, -\xi)\, x(t+\xi)\, d\xi,$$

which is exactly the equation (4.1). This result shows the complete duality that exists between (4.1) and (4.12) and is again reminiscent of that for the case of ordinary differential equations.

We know that if $x_t = T(t,\sigma)\phi$ and $y_t = \tilde{T}(t,\tau)\psi$ solve (4.1) and (4.12) respectively, with $\sigma \le t \le \tau$, then by (4.14)

$$(y_t, x_t)_t = \text{constant.}$$

Thus $(\tilde{T}(t,\tau)\psi,\ T(t,\sigma)\phi)_t = (\tilde{T}(\tau,\tau)\psi,\ T(\tau,\sigma)\phi)_\tau = (\psi,\ T(\tau,\sigma)\phi)_\tau$. Similarly this equals $(\tilde{T}(\sigma,\tau)\psi,\ \phi)_\sigma$, so for $\sigma \leq \tau$

$$(\psi,\ T(\tau,\sigma)\,\phi)_\tau = (\tilde{T}(\sigma,\tau)\psi,\ \phi)_\sigma\ . \qquad (4.17)$$

In particular, putting $\tau = \omega$ and $\sigma = 0$ we have

$$(\psi,\ U\phi)_\omega = (\tilde{U}\psi,\ \phi)_0\ .$$

However

$$(\psi, U\phi)_\omega = \psi(0)\ [U\phi](0) - \int_{-\infty}^{0}\int_{-\theta}^{0}\psi(\xi)\ K(\omega+\xi,-\theta)\ [U\phi](\xi+\theta)\ d\xi\, d\theta$$

$$= (\psi, U\phi)_0$$

by the periodicity of K. Finally, we have

$$(\psi, U\phi)_0 = (\tilde{U}\psi,\ \phi)_0\ . \qquad (4.18)$$

At this point we can see that if the bilinear form were an inner product then by standard techniques of Hilbert spaces we could deduce the required properties of the spectrum of $\tilde{U}$. Since this is in general not the case we must proceed differently. To do this we go to the functional analytic adjoint operator U^* of U, and connect it with $\tilde{U}$.

We introduce the functional analytic adjoint space C^* to C

consisting of the bounded linear functionals acting on C. This is not to be confused with the formal adjoint given by (4.12) and denoted with a tilde. First we show that C is isometric to the space

$$\mathfrak{X} = \{f \text{ continuous on } [0,1]\},$$

$$\|f\| = \sup_{0 \le u \le 1} |f(u)|.$$

This is accomplished via the map $\pi : C \to \mathfrak{X}$ given by

$$[\pi\phi](u) = u^{\gamma^+}\phi(\log u), \qquad u \in [0,1].$$

In fact $\|\pi\phi\| = \sup_{0 \le u \le 1} |u^{\gamma^+}\phi(\log u)| = \sup_{\theta \le 0} |e^{\gamma^+\theta}\phi(\theta)| = \|\phi\|$ (putting $\theta = \log u$). The map π is onto, since, given $f \in \mathfrak{X}$, let $\phi(\theta) = e^{-\gamma^+\theta} f(e^{\theta})$. Then clearly $\phi \in C$.

The adjoint of $\mathfrak{X}$, namely the set of all bounded linear functionals acting on $\mathfrak{X}$, is given by (cf. Taylor [16])

$$\mathfrak{X}^* = \{\eta \text{ row vector defined on } [0,1] \,/\, \eta \text{ is of bounded variation on } [0,1]\}$$

where $\hat{\eta} \in \mathfrak{X}^*$ acts on $\mathfrak{X}$ according to $\hat{\eta}(f) = \int_0^1 d\eta(u)\, f(u)$. Thus $\hat{\eta}(f) = \int_0^1 d\eta(u)\, u^{\gamma^+}\phi(\log u)$. $\mathfrak{X}^*$ has norm $\|\hat{\eta}\| = \operatorname{Var}\eta = \int_0^1 |d\eta|$. Let ρ be defined on $(-\infty, 0]$ by $\rho(\log u) = \int_1^u d\eta(v)\, v^{\gamma^+}$ so if $\theta = \log u$, $d\rho(\theta) = d\eta(u)\, u^{\gamma^+}$. Also $\hat{\eta}(f) = \int_{-\infty}^0 d\rho(\theta)\,\phi(\theta)$, and $\|\hat{\eta}\| = \int_0^1 |d\eta| = \int_0^1 u^{-\gamma^+} |d\rho(\log u)| = \int_{-\infty}^0 e^{-\gamma^+\theta} |d\rho(\theta)|$. We can now define the set of bounded linear functionals acting on C to be (note ρ is a row vector)

$$C^* = \Big\{ \rho \text{ a function defined on } (-\infty, 0] \Big| \int_{-\infty}^{0} e^{-r^+\theta} |d\rho(\theta)| < \infty \Big\}$$

which is a Banach space with norm

$$\|\hat{\rho}\| = \int_{-\infty}^{0} e^{-r^+\theta} |d\rho(\theta)| .$$

This is valid because C^* is the isometric image of X^* via the identification $\hat{\eta} \leftrightarrow \hat{\rho}$. The linear functionals in C^* act on C via

$$\hat{\rho}(\phi) \equiv \langle \rho, \phi \rangle = \int_{-\infty}^{0} d\rho(\theta)\, \phi(\theta),$$

thus defining the bilinear form $\langle , \rangle$ on $C^* \times C$. We note ρ is determined up to a constant and we identify in C^* such ρ as differ by a constant.

Now that we have found C^*, standard results of functional analysis may be employed to deduce that the functional analytic adjoint U^* of U has the same spectrum as U, for each $\mu \in \Sigma$ the ascent ν_μ of $U^* - \mu I$ equals that of $U - \mu I$, and that $\dim (U^* - \mu I)^{\nu_\mu} = \dim (U - \mu I)^{\nu_\mu}$. Before relating U^* and $\tilde{U}$ we provide a connection between the bilinear forms $(,)_t$ and $\langle , \rangle$. Define the map $S(t)$ for each $t \in \mathbb{R}$, $S(t) : \tilde{C} \to C^*$ by

$$[S(t)\eta](\xi) = \begin{cases} \int_0^\infty \eta(\theta)\,[L(\theta+t,\theta) - L(\theta+t,\,\theta-\xi)]\,d\theta \\ \qquad\qquad\qquad + \eta(0), & \xi < 0, \\ 0, & \xi = 0. \end{cases} \tag{4.19}$$

By (4.15) $L(\theta+t,\,\theta-\xi) = O(e^{\gamma\xi})$ as $\xi \to -\infty$ so $S(t)\eta$ is indeed in C^*. Let $\eta \in \tilde{C}$ and $\pi = S(t)\eta$. Then for $\phi \in C$

$$\begin{aligned} \langle \pi, \phi \rangle &= \int_{-\infty}^0 d\pi(\xi)\,\phi(\xi) \\ &= -\eta(0)\,\phi(0) - \int_{-\infty}^0 \int_0^\infty \eta(\theta)\,K(\theta+t,\,\theta-\xi)\,\phi(\xi)\,d\theta\,d\xi \\ &= -\eta(0)\,\phi(0) - \int_0^\infty \eta(\theta) \int_{-\infty}^0 K(\theta+t,\,\theta-\xi)\,\phi(\xi)\,d\xi\,d\theta \\ &= -\eta(0)\,\phi(0) - \int_0^\infty \eta(\theta) \int_{-\infty}^{-\theta} K(\theta+t,\,-\xi')\,\phi(\theta+\xi')\,d\xi'\,d\theta \\ &= -\eta(0)\,\phi(0) - \int_{-\infty}^0 \int_0^{-\xi'} \eta(\theta)\,K(\theta+t,\,-\xi')\,\phi(\theta+\xi')\,d\theta\,d\xi' \\ &= -(\eta, \phi)_t\,. \end{aligned}$$

Thus for each $\phi \in C$, $\eta \in \tilde{C}$

$$(\eta, \phi)_t = \langle -S(t)\eta, \phi \rangle. \tag{4.20}$$

By (4.17), for $\sigma \le t$,

$$(\tilde{T}(\sigma,t)\eta, \phi)_\sigma = (\eta, T(t,\sigma)\phi)_t\,,$$

so

$$\langle S(\sigma)\,\tilde{T}(\sigma,t)\eta,\ \phi\rangle = \langle S(t)\eta,\ T(t,\sigma)\phi\rangle.$$

Letting $T^*(t,\sigma)$ denote the functional-analytic adjoint of $T(t,\sigma)$ this equals $\langle T^*(t,\sigma)\,S(t)\eta,\ \phi\rangle$. Since this holds for all $\phi \in C$, $\eta \in \tilde{C}$ we have

$$S(\sigma)\,\tilde{T}(\sigma,t) = T^*(t,\sigma)\,S(t).$$

Putting $\sigma = 0$, $t = \omega$ and noting from (4.19) that $S(\omega) = S(0)$, $S(0)\,\tilde{T}(0,\omega) = T^*(\omega,0)\,S(0)$, we have

$$S(0)\,\tilde{U} = U^*\,S(0). \tag{4.21}$$

This provides the connection between $\tilde{U}$ and U^*. Unfortunately, $S(0)$ is not invertible but (4.21) is still useful. First we note that for any positive integer k

$$(U^* - \mu I)^k\,S(0) = S(0)\,(\tilde{U} - \mu I)^k. \tag{4.22}$$

Let μ be in the spectrum of $\tilde{U}$, $\tilde{\nu}_\mu$ the ascent of $\tilde{U} - \mu I$, and $\psi_1, \dots, \psi_{\tilde{d}_\mu}$ a basis for $\mathcal{N}(\tilde{U} - \mu I)^{\tilde{\nu}_\mu}$ (dimension $\tilde{d}_\mu$). Proceeding as we did for equation (4.1), put $\Psi = \mathrm{col}(\psi_1 \dots \psi_{\tilde{d}_\mu})$. Then there are $\tilde{d}_\mu \times \tilde{d}_\mu$ matrices $\tilde{M}$ and $\tilde{B}$, $\tilde{U}\Psi = \tilde{M}\Psi$, $\tilde{B} = \frac{1}{\omega}\log\tilde{M}$; $\tilde{B}$ has the sole eigenvalue $\frac{1}{\omega}\log\mu$. Define the ω-periodic column-vector function

$\tilde{P}(t) = e^{\tilde{B}t}\,\tilde{T}(t,0)\Psi$ for $t<0$, and extend it to $t \geq 0$. Then the solution of (4.12) generated by Ψ is

$$y_t = \tilde{T}(t,0)\,\Psi = e^{-\tilde{B}t}\,\tilde{P}(t),$$

and in fact solves (4.12) for all $-\infty < t < \infty$. The function corresponding to y_t is $Y(t)$ defined for all t and

$$Y(t) = e^{-\tilde{B}t}\,\tilde{P}(t)(0).$$

We can show as before that if for some constant $\tilde{d}_\mu$ -row vector $\underset{\sim}{b}$ and $t \leq 0$, $\underset{\sim}{b}\,y_t = \underset{\sim}{b}\,\tilde{T}(t,0)\Psi = 0$, then $\underset{\sim}{b} = 0$.

We now prove that the set of $S(0)\,\Psi_i$, $1 \leq i \leq \tilde{d}_\mu$, is linearly independent, i.e. if for some constant row $\tilde{d}_\mu$-vector $\underset{\sim}{b}$, $\underset{\sim}{b}\,S(0)\Psi = 0$ then $\underset{\sim}{b} = 0$. Put $\Gamma = S(0)\,\Psi$. We show that for $\sigma \leq 0$

$$Y(\sigma) = -\int_\sigma^0 d\Gamma(\theta)\, R(\theta,\sigma) \tag{4.23}$$

where R is the resolvent matrix corresponding to (4.1). Infact if we let $z(\sigma) = -\int_\sigma^0 d\Gamma(\theta)\,R(\theta,\sigma)$ for $\sigma<0$, then by (4.2),

$$\begin{aligned} z(\sigma) &= -\int_\sigma^0 d\Gamma(\theta)\left[I + \int_\sigma^\theta R(\theta,u)\left[A(u) - L(u,u-\sigma)\right] du\right] \\ &= \Gamma(\sigma) - \Gamma(0) - \int_\sigma^0 d\Gamma(\theta)\int_\sigma^0 R(\theta,u)\left[A(u) - L(u,u-\sigma)\right] du, \end{aligned}$$

since $R(\theta,u) = 0$ for $u > \theta$,

$$= \Gamma(\sigma) - \Gamma(0) - \int_\sigma^0 \left(\int_\sigma^0 d\Gamma(\theta) R(\theta,u) \right) [A(u) - L(u,u-\sigma)]\, du$$

$$= \Gamma(\sigma) - \Gamma(0) + \int_\sigma^0 z(u) [A(u) - L(u,u-\sigma)]\, du .$$

By definition of $S(0)$, $\Gamma(0)=0$. So z satisfies the integral equation

$$z(\sigma) + \int_\sigma^0 z(u) [A(u) - L(u,u-\sigma)]\, du = \Gamma(\sigma) =$$

$$[S(0)\Psi](\sigma) = \Psi(0) + \int_0^\infty \Psi(\theta) [L(\theta,\theta) - L(\theta,\theta-\sigma)]\, d\theta.$$

Thus z satisfies (4.16) with $t=0$, and initial data equal to Ψ on $[0,\infty)$. This is the integral form of the adjoint equation (4.12). We know one solution to be $Y(\sigma)$, so by the uniqueness of the solutions of (4.16), $z(\sigma) = Y(\sigma)$ for all $\sigma \le 0$. This is what we wished to show. Now $\underset{\sim}{b}\,\Gamma = 0$ so by (4.23) $0 = \underset{\sim}{b}\, Y(\sigma) = \underset{\sim}{b}\, \tilde{T}(\sigma,0)\Psi$, i.e. $\underset{\sim}{b} = 0$. Thus we have proved that the $S(0)\psi_i$ are linearly independent.

We are now in a position to connect the spectra of U and $\tilde{U}$. Let μ be an eigenvalue of $\tilde{U}$, k an integer $1 \le k \le \tilde{\nu}_\mu$; and $\psi_1 \ldots \psi_m$ a basis for the null space $\mathcal{N}(\tilde{U} - \mu I)^k$. Then since Ψ can be chosen to include $\psi_1 \ldots \psi_m$ we deduce that the set $\rho_i = S(0)\psi_i$, $1 \le i \le m$, is linearly independent. Also it is clear from (4.22) that

$$(U^* - \mu I)^k \rho_i = 0, \qquad i = 1, \ldots, m.$$

Thus μ is in the spectrum of U^* and U, and

$$\dim (U^* - \mu I)^k \geq \dim (\tilde{U} - \mu I)^k.$$

Now $\dim (U^* - \mu I)^k = \dim (U - \mu I)^k$ for the following reasons. Since U_μ is of the form $U - \mu I = B(Q + I)$ with B nonsingular, Q of finite rank, then U_μ^k has the same form and we may without loss of generality assume $k = 1$. Then the null space of U_μ corresponds to the null space of some finite dimensional matrix D. It is easy to see that the null space of U_μ^* corresponds to that of the transposed matrix D^T. These have the same dimension and we are done. Thus

$$\Sigma \supset \tilde{\Sigma}$$

$$\dim (U - \mu I)^k \geq \dim (\tilde{U} - \mu I)^k, \qquad \forall k. \tag{4.24}$$

By duality we can reverse the roles of U and $\tilde{U}$ in (4.24) to deduce the reverse inclusion and inequality. We summarize the results :

> The spectra of U and $\tilde{U}$ for $|\mu| \geq e^{-r'\omega}$ coincide. $U - \mu I$ and $\tilde{U} - \mu I$ have equal (finite) ascents ν_μ, and $\dim (U - \mu I)^{\nu_\mu} = \dim (\tilde{U} - \mu I)^{\nu_\mu} = d_\mu$; i.e. the generalized null spaces of the two operators are in one-to-one correspondence.

This completes our demonstration of the duality between equation (4.1) and its formal adjoint (4.12).

We can now turn to the projection operators, but first we need some preliminaries. As above let Φ, Ψ be bases for the generalized null spaces of $U - \mu I$ and $\tilde{U} - \mu I$ respectively. We may choose these such that

$$(\Psi, \Phi)_0 = I \tag{4.25}$$

(the $d_\mu \times d_\mu$ identity) since $(\Psi, \Phi)_0$ is nonsingular. To show its nonsingularity, if $(\Psi, \Phi)_0 \underset{\sim}{b} = 0$ then $(\Psi, \Phi \underset{\sim}{b}) = 0$, i.e. $\langle \Gamma, \Phi \underset{\sim}{b} \rangle = 0$ with Γ defined above. Since Γ is a basis for $\mathcal{N}(U^* - \mu I)^{\nu_\mu}$ we have that $\Phi \underset{\sim}{b}$ annihilates this space so $\Phi \underset{\sim}{b} \in \mathcal{R}(U - \mu I)^{\nu_\mu}$ (cf. Taylor [16], Theorem 5.8-A). But then since $\Phi \underset{\sim}{b} \in \mathcal{N}(U - \mu I)^{\nu_\mu}$ we have $\Phi \underset{\sim}{b} = 0$, i.e. $\underset{\sim}{b} = 0$.

Now since $U\Phi = \Phi M$, $\tilde{U}\Psi = \tilde{M}\Psi$, we have $(\Psi, U\Phi)_0 = (\Psi, \Phi M)_0 = (\Psi, \Phi)_0 M = M$. By (4.18) this equals $(\tilde{U}\Psi, \Phi)_0 = \tilde{M}(\Psi, \Phi)_0 = \tilde{M}$, so $M = \tilde{M}$, and $B = \tilde{B}$. Thus

$$T(t, 0)\,\Phi = P(t)\, e^{Bt}, \qquad t > 0,$$

$$\tilde{T}(t, 0)\,\Psi = e^{-Bt}\, \tilde{P}(t), \qquad t < 0,$$

are the eigenfunction solutions of (4.1) and (4.12) respectively corresponding to μ. Let us denote $\Phi(t) = T(t, 0)\Phi$, $\Psi(t) = \tilde{T}(t, 0)\Psi$ and note $T(t, s)\,\Phi(s) = \Phi(t)$. Also note that (4.25) implies

$$(\Psi(t), \Phi(t))_t = I. \tag{4.26}$$

We can now specify the projection onto the generalized null space of $U - \mu I$. Let μ be a characteristic multiplier with $\Psi(t)$, $\Phi(t)$ as above. Define the projection $Q_\mu(t)$ onto $\mathcal{N}(U_\mu(t))^{\nu_\mu}$ along

$R(U(t)-\mu I)^{\nu_\mu}$ via

$$Q_\mu(t)\,\phi \equiv \Phi(t)\,(\Psi(t),\,\phi)_t \tag{4.27}$$

for each t . That this is the desired projection is clear except that we show $\phi \in R(U(t)-\mu I)^{\nu_\mu}$ implies $Q_\mu(t)\,\phi = 0$. In fact $\phi =$ $[U(t)-\mu I]^{\nu_\mu}\phi_1$ implies $Q_\mu(t)\,\phi = \Phi(t)\;(\Psi(t),\,[U(t)-\mu I]^{\nu_\mu}\phi_1)_t =$ $\Phi(t)\;([\tilde{U}(t)-\mu I]^{\nu_\mu}\Psi(t),\;\phi_1)_t = 0.$

6. <u>Decomposition of the Solutions of the Inhomogeneous Equation</u>

Let us now return to the inhomogeneous equation (4.9):

$$\dot{x} - \mathcal{L}(t,x) = f(t). \tag{4.9'}$$

Let a solution of this be x_t and let w_t be the projection of x_t onto $E_\mu(t)$:

$$w_t = \Phi(t)\;(\Psi(t),\,x_t)_t\,. \tag{4.28}$$

An expression for x_t is provided by the variation of constants formula (4.11). The projection of X_0 onto $E_\mu(t)$ is

$$X_0^Q(t) = \Phi(t)\;(\Psi(t),\,X_0)_t = \Phi(t)\;\Psi(t)(0). \tag{4.29}$$

We show that w_t satisfies

$$\dot{w} - \mathcal{L}(t, w) = X_0^Q(t) f(t) \tag{4.30}$$

and that, for $\sigma \leq t$,

$$\begin{aligned} w_t &= \int_\sigma^t [T(t,s) X_0^Q(s)] f(s)\, ds + T(t,\sigma) w_\sigma \\ &= \Phi(t) \int_\sigma^t \Psi(s)(0) f(s)\, ds + T(t,\sigma) w_\sigma . \end{aligned} \tag{4.31}$$

It is easy to show that (4.30) follows from (4.31). To show the latter we use (4.14) to get

$$\begin{aligned} w_t &= \Phi(t) (\Psi(\sigma), x_\sigma)_\sigma + \Phi(t) \int_\sigma^t \Psi(s)(0) f(s)\, ds \\ &= T(t,\sigma) \Phi(\sigma) (\Psi(\sigma), x_\sigma)_\sigma + \int_\sigma^t T(t,s) X_0^Q(s) f(s) ds. \end{aligned}$$

which is (4.31). Also, since $\Phi(t) = P(t) e^{Bt}$, $\Psi(t) = e^{-Bt} \tilde{P}(t)$ letting

$$w_t = P(t) q(t) \tag{4.32}$$

where $q(t) = e^{Bt}(\Psi(t), x_t)_t$ is a column d_μ-vector, we have that q satisfies the ordinary differential equation

$$\dot{q}(t) = B q(t) + \tilde{P}(t)(0) f(t). \tag{4.33}$$

This follows by use of (4.14), $\dot{q}(t) = B q(t) + e^{Bt} \frac{d}{dt} (\Psi(t), x_t)_t =$

$$Bq + e^{Bt} \frac{d}{dt} \int_{\sigma}^{t} \Psi(s)(0) f(s)\, ds = Bq + e^{Bt} \Psi(t)(0) f(t).$$

With this result on the projection of x_t, $Q_{\mu}(t) x_t$, we can prove the Fredholm Alternative Theorem giving necessary and sufficient conditions for the existence of an ω-periodic solution of (4.9') in the case that $f(t)$ is ω-periodic. Now the homogeneous equation (4.1) has spectrum that can be separated into three parts:

$$\Lambda_0 = \{ \mu \mid |\mu| = 1 \},$$

$$\Lambda_+ = \{ \mu \mid |\mu| > 1 \},$$

and all $|\mu| < 1$.

Denote by $E^{\Lambda_0}(t)$ the span of the generalized null spaces corresponding to $\mu \in \Lambda_0$, and similarly for $E^{\Lambda_+}(t)$. Let $\Phi^0(t)$, $\Phi^+(t)$ be the corresponding bases with projections $Q_0(t)$, $Q_+(t)$. For the formal adjoint let these be $\tilde{E}^{\Lambda_0}(t)$, $\tilde{E}^{\Lambda_+}(t)$, $\Psi^0(t)$, $\Psi^+(t)$. Letting x_t be any solution of (4.9') we have the projections

$$x_t^0 = Q_0(t) x_t = \Phi^0(t) \left(\Psi^0(t), x_t \right)_t ,$$

$$x_t^+ = Q_+(t) x_t = \Phi^+(t) \left(\Psi^+(t), x_t \right)_t .$$

Also

$$\frac{d}{dt} x_t^0 - \mathcal{L}(t, x_t^0) = X_0^{\Lambda_0}(t) f(t),$$

$$\frac{d}{dt} x_t^+ - \mathcal{L}(t, x_t^+) = X_0^{\Lambda_+}(t) f(t),$$

$$x_t^0 = T(t,\sigma) x_\sigma^0 + \int_\sigma^t [T(t,s) X_0^{\Lambda_0}(s)] f(s)\, ds,$$

$$x_t^+ = T(t,\sigma) x_\sigma^+ + \int_\sigma^t [T(t,s) X_0^{\Lambda_+}(s)] f(s)\, ds, \tag{4.34}$$

where

$$X_0^{\Lambda_0}(t) = Q_0(t) X_0 ,$$

$$X_0^{\Lambda_+}(t) = Q_+(t) X_0 ,$$

as we know from (4.30) and (4.31) ($Q_0 = \sum_{\mu \in \Lambda_0} Q_\mu$ etc.). We now prove the <u>Fredholm Alternative Theorem.</u>

<u>Theorem</u>: If $\underset{\sim}{f}(t+\omega) = \underset{\sim}{f}(t)$ in (4.9') then a necessary and sufficient condition for the existence of an ω-periodic solution for all t is that

$$\int_0^\omega \underset{\sim}{y}(t) \cdot \underset{\sim}{f}(t)\, dt = 0 \tag{4.35}$$

for all ω-periodic solutions $\underset{\sim}{y}$ of the adjoint equation (4.12).

<u>Proof.</u> The necessity of (4.35) follows easily using (4.14). Suppose x_t is an ω-periodic solution of (4.9'). Then for any $\underset{\sim}{y}(t)$ an ω-periodic adjoint solution we have $\int_0^\omega y(t) f(t)\, dt = (y_\omega, x_\omega)_\omega - (y_0, x_0)_0$ which is zero by the periodicity of y, x and K.

For the converse suppose (4.35) holds for all ω-periodic $\underset{\sim}{y}(t)$

solving (4.12). Letting $x_t^- = x_t - x_t^0 - x_t^+$ we have besides (4.34)

$$x_t^- = T(t,\sigma)\, x_\sigma^- + \int_\sigma^t \left[T(t,s)\, X_0^{\Lambda -}(s)\right] f(s)\, ds \qquad (4.36)$$

where $X_0^{\Lambda -}(s) \equiv X_0 - X_0^{\Lambda 0}(s) - X_0^{\Lambda +}(s)$. Since $T(t,s)\, X_0^{\Lambda +}(s) = \Phi^+(t)\ \Psi^+(s)(0)$ we have $\| T(t,s)\, X_0^{\Lambda +}(s) \| = O(e^{\alpha(t-s)})$ as $|t-s| \to \infty$ where $\alpha = \max\{ Re\,(\frac{1}{\omega} \log \mu)\}$. If Λ_+ is empty we put $x_t^+ = 0$. Otherwise $\alpha > 0$. Next we know that $\| T(t,s)\, X_0^{\Lambda -}(s) \| = O(e^{\beta(t-s)})$ as $t - s \to \infty$ where β is between 0 and $\max\{ Re\,(\frac{1}{\omega} \log \mu) \mid \mu \notin \Lambda_+ \cup \Lambda_0 \}$; so $\beta < 0$. Thus in (4.34) we may let $\sigma \to \infty$ so

$$x_t^+ = \int_\infty^t \left[T(t,s)\, X_0^{\Lambda +}(s) \right] f(s)\, ds , \qquad (4.37)$$

the integral converging. Similarly we let $\sigma \to -\infty$ in (4.36)

$$x_t^- = \int_{-\infty}^t \left[T(t,s)\, X_0^{\Lambda -}(s)\right]\ f(s)\, ds. \qquad (4.38)$$

The functions $x_t^{\pm}$ as given in (4.37) and (4.38) are ω-periodic since

$$x_t^{\pm} = \int_{\pm\infty}^0 \left[T(t,\ s+t)\ X_0^{\Lambda \pm}(s+t)\right] f(t+s)\, ds$$

and $T(t+\omega, s+t+\omega) = T(t, s+t)$ by (4.5), and f and $X_0(t+s)$ are ω-periodic ($X_0^{\Lambda +}(t) = \Phi^+(t)\ \Psi^+(t)(0) = P^+(t)\ \tilde{P}^+(t)(0)$ is ω-periodic, similarly for $X_0^{\Lambda 0}(t)$ and $X_0^{\Lambda -}(t) = X_0 - X_0^{\Lambda 0}(t) - X_0^{\Lambda +}(t)$).

Thus we are reduced to finding some x_σ in $E^{\Lambda 0}(\sigma)$ such that

$$z_t = T(t,\sigma)x_\sigma + \int_\sigma^t [T(t,s)X_0{}^{\Lambda_0}(s)] f(s)\,ds$$

is ω-periodic since then $x_t = x_t^+ + x_t^0 + x_t^-$ would solve (4.9) and be periodic. Defining $q(t) = e^{Bt}(\Psi^0(t), z_t)_t$ as in (4.33) where $\Phi^0(t) = P^0(t)e^{Bt}$, $\Psi^0(t) = e^{-Bt}\tilde{P}^0(t)$, we have the ordinary differential equation to be solved,

$$\dot{q}(t) = Bq(t) + \tilde{P}^0(t)(0)f(t) \tag{4.39}$$

Clearly if an ω-periodic solution $q(t)$ of (4.39) exists then $z_t = P(t)q(t)$ would suffice. Now such a $q(t)$ exists iff

$$\int_0^\omega r^T(t)\,\tilde{P}^0(t)(0)f(t)\,dt = 0 \tag{4.40}$$

for all $r^T(t)$ ω-periodic solutions of the adjoint of (4.39). This follows by the Fredholm Alternative for ordinary differential equations. The adjoint of (4.39) is $\dot{r}^T = -r^T B$ and has solutions $r^T(t) = r_0{}^T e^{-Bt}$ so (4.40) becomes

$$\int_0^\omega r_0{}^T e^{-Bt}\,\tilde{P}^0(t)(0)f(t)\,dt = 0,$$

i.e.

$$\int_0^\omega r_0{}^T\,\Psi^0(t)(0)\,f(t)\,dt = 0$$

where $r_0{}^T$ is chosen such that $r_0{}^T e^{-Bt}$ is ω-periodic. But this set of $r_0{}^T\Psi^0(t)$ is precisely the set of ω-periodic solutions of (4.12). Thus (4.40) holds and $q(t)$ exists. ∎

7. A Structural Stability Theorem

As an application of the Fredholm Alternative Theorem we prove a theorem about periodic solutions of nonlinear integro-differential equations.

Consider

$$\frac{d\underset{\sim}{x}}{dt} = \underset{\sim}{f}(\underset{\sim}{x}, \mathcal{K}(\epsilon) * \underset{\sim}{x}, \epsilon), \tag{4.41}$$

where $\mathcal{K}(\epsilon) * \underset{\sim}{x} \equiv \int_0^\infty \mathcal{K}(\epsilon, s)\, \underset{\sim}{x}(t-s)\, ds$, ϵ is some parameter, and $\underset{\sim}{f}$ is a twice continuously differentiable function.

Suppose that at $\epsilon = 0$ this equation has an ω_0-periodic solution $w(t)$. Then the linearization of (4.41) about $x = w$ and at $\epsilon = 0$ is of the form

$$\begin{aligned} \frac{d\underset{\sim}{v}}{dt} &= A(t)\,\underset{\sim}{v}(t) + \int_{-\infty}^{0} K(t,-\theta)\, \underset{\sim}{v}(t+\theta)\, d\theta \\ &\equiv \mathcal{L}(t, \underset{\sim}{v}), \end{aligned} \tag{4.42}$$

where $A(t)$ and $K(t,-\theta)$ in its first argument are ω_0-periodic. As usual we assume $\mathcal{K}(\epsilon, s) = O(e^{-\gamma s})$ as $s \to \infty$ for some $\gamma > 0$ so likewise $K(t,s) = O(e^{-\gamma s})$. Now we note that $v_p \equiv \frac{dw}{dt}$ is a solution of (4.42) and is ω_0-periodic. Thus $\mu = 1$ is a characteristic multiplier for (4.42). We shall say that $\mu = 1$ is <u>simple</u> if the ascent ν_μ of $U - \mu I$ is equal to one and $\dim n(U - \mu I) = 1$, i.e. the generalized null space has dimension one.

<u>Theorem:</u> Consider the equation (4.41) for ϵ near zero. At $\epsilon = 0$ let (4.41) have an ω_0-periodic solution $w(t)$ and suppose the linearization of (4.41) about $x = w(t)$ at $\epsilon = 0$ has $\mu = 1$ as a simple

characteristic multiplier. Then for all ϵ sufficiently near zero there exists an $\omega(\epsilon)$-periodic solution $w(t,\epsilon)$ of (4.41) where $\omega(\epsilon) \to \omega_0$, $w(t,\epsilon) \to w(t)$ as $\epsilon \to 0$.

Proof: The proof is similar to that of the Hopf bifurcation in Chapter II. We first scale t since the period $\omega(\epsilon)$ is expected to vary: $t^* = t/(1+\tau(\epsilon))$ where $\tau(\epsilon) = O(\epsilon)$ and $\omega(\epsilon) = \omega_0(1+\tau(\epsilon))$. Define $\underset{\sim}{y}(t^*) = \underset{\sim}{x}(t) = \underset{\sim}{x}[(1+\tau)t]$. We seek an ω_0-periodic solution $\underset{\sim}{y}$ which is the fixed point of an operator equation in the Banach space ($\|u\| = \sup|u| + \sup|\dot{u}|$)

$$\mathcal{B} = \Big\{ \underset{\sim}{u}(t) \;\Big|\; \underset{\sim}{u} \text{ is continuously differentiable and } \omega_0\text{-periodic} \Big\}.$$

We are assuming that the linearization (4.42) has $\mu = 1$ as simple. Then the adjoint of (4.42) will have only one ω_0-periodic solution since $\mu = 1$ is likewise simple. Denote this adjoint solution by $p(t)$. Define the closed subspace

$$\mathcal{B}_0 = \Big\{ u \in \mathcal{B} \;\Big|\; \int_0^{\omega_0} \underset{\sim}{p}(t) \cdot \underset{\sim}{u}(t)\,dt = 0 \Big\}.$$

We have $\int_0^\infty \mathcal{K}(\epsilon,s)\, x(t-s)\,ds = \int_0^\infty \mathcal{K}(\epsilon,s)\, x[(1+\tau)t^* - s]\,ds = \int_0^\infty \mathcal{K}(\epsilon,s)\, y(t^* - \frac{s}{1+\tau})\,ds$. So $\mathcal{K}(\epsilon) * x = \int_0^\infty \mathcal{K}(\epsilon,s)\, y(t^*-s)\,ds + \mathcal{J}(y,\tau,\mathcal{K}(\epsilon))$ where $\mathcal{J}(y,\tau,\mathcal{K}(\epsilon)) \equiv \int_0^\infty \mathcal{K}(\epsilon,s)[\,y(t^* - \frac{s}{1+\tau}) - y(t^*-s)\,ds]$ and is of order $\tau(\epsilon)\frac{dy}{dt^*}$ for small τ. This is clearly a bounded linear transformation on $\mathcal{B}$.

The linearization of f about $w(t)$ is

$$f_1^\circ[v(t)] \quad + \quad f_2^\circ\left[\int_0^\infty \mathcal{K}(0,s)\, v(t-s)\, ds\right]$$

where f_1° and f_2° are the partial derivatives of f with respect to the first and second arguments, the first evaluated at $w(t)$, and the second at $\int_0^\infty \mathcal{K}(0,s)\, w(t-s)\, ds$, and $\epsilon = 0$. Denoting $A(t) = f_1^\circ$, $(t) = f_2^\circ$ we see that A, B are $n \times n$ matrix functions of t that are ω_0-periodic. Note that the kernel $K(t,s)$ in (4.42) equals $B(t)\, \mathcal{K}(0,s)$.

Put $y(t^*, \epsilon) = w(t^*) + z(t^*, \epsilon)$; then z satisfies (dropping the star on t)

$$\frac{dz}{dt} = (1+T)\, f(w+z,\ \mathcal{K}(\epsilon) * w + \mathcal{K}(\epsilon) * z + \mathcal{J}(w, T, \mathcal{K}(\epsilon))$$
$$+ \mathcal{J}(z, T, \mathcal{K}(\epsilon)), \epsilon) - f(w, \mathcal{K}(0) * w, 0).$$

The right side is a function of z, T, ϵ and when we Taylor expand it in powers of these variables we get

$$\frac{dz}{dt} = T\, f(w, \mathcal{K}(0) * w, 0) + f_1^\circ z +$$
$$f_2^\circ\left[\mathcal{J}(w, T, \mathcal{K}(0)) + \mathcal{K}(0) * z + \epsilon \frac{\partial \mathcal{K}}{\partial \epsilon}(0) * w\right]$$
$$+ \left(\frac{\partial f}{\partial \epsilon}\right)_0 \epsilon + g(z, T, \epsilon; t)$$

where g involves only quadratic and higher powers in z, T, ϵ ($\frac{\partial g}{\partial z} = \frac{\partial g}{\partial T} = \frac{\partial g}{\partial \epsilon} = 0$ at $z = T = \epsilon = 0$). We rewrite this, using $\mathcal{J}(w, T, \mathcal{K}(0)) =$ $-T\int_0^\infty \mathcal{K}(0,s)\, s\, \dot{w}(t-s)\, ds + O(T^2)$,

$$\frac{dz}{dt} = A(t)\,z + \int_0^\infty K(t,\theta)\; z(t-\theta)\,d\theta$$

$$+ \; T\left[\, \dot{w}(t) - B(t)\int_0^\infty \chi(0,s)\; s\; \dot{w}(t-s)\,ds \right]$$

$$+ \; \epsilon\left[\left(\frac{\partial f}{\partial \epsilon}\right)_0 + B(t)\int_0^\infty \frac{\partial \chi}{\partial \epsilon}(0,s)\; w(t-s)\,ds \right]$$

$$+ \; g'(z, T, \epsilon\,;\, t)$$

where g' is of the form of g . Thus,

$$\frac{dz}{dt} - \mathcal{I}(t,z) = T\left[\, \dot{w}(t) - \int_0^\infty K(t,s)\; s\; \dot{w}(t-s)\,ds \right]$$

$$+ \; \epsilon\left[\left(\frac{\partial f}{\partial \epsilon}\right)_0 + B(t)\int_0^\infty \frac{\partial K}{\partial \epsilon}(0,s)\; w(t-s)\,ds \right] \tag{4.43}$$

$$+ \; g'(z, T, \epsilon\,;\, t)$$

$$\equiv \; F(z, T, \epsilon\,;\, t)$$

By the Fredholm Alternative Theorem (4.43) has a periodic solution iff

$$h(z, T, \epsilon) \equiv \int_0^{\omega_0} \underset{\sim}{p}(t) \cdot \underset{\sim}{F}(z(t), T, \epsilon\,;\, t)\,dt = 0 .$$

This condition determines T as a function of z and ϵ (for these small) with $T = 0$ when $z = \epsilon = 0$. To solve $h = 0$ it is sufficient that $\frac{\partial h}{\partial T} \neq 0$ at $T = z = \epsilon = 0$. We have

$$\left(\frac{\partial h}{\partial T}\right)_0 = \int_0^{\omega_0} p(t) \cdot \left[\, \dot{w}(t) - \int_0^\infty K(t,s)\; s\; \dot{w}(t-s)\,ds \right] dt .$$

That this is nonzero is seen from the following argument. If it were zero then the equation

$$\dot{v} - \mathcal{L}(t,v) = \dot{w}(t) - \int_0^\infty k(t,s)\, S\,\dot{w}(t-s)\, ds$$

would have a periodic solution $v_0(t)$ by the Fredholm Alternative again. But clearly $t\,\dot{w}(t)$ is a particular solution so the homogeneous equation (4.42) would have $k(t) \equiv -v_0(t) + t\,\dot{w}(t)$ as a solution. Since v_0 and $\dot{w}$ are periodic, $k \not\equiv 0$; so k is an eigenfunction corresponding to multiplier $\mu = 1$, since $(\dot{w}_t\,,\, k_t) = \Phi(t) = P(t)e^{Bt}$ with $P(t) = (\dot{w}_t\,, -v_{0t})$, $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. This implies that the generalized null space of $U - I$ has dimension greater than one, contradicting the simplicity of $\mu = 1$.

Thus, by the implicit function theorem, we have $T = T(z,\epsilon)$ defined for $|\epsilon| \le \epsilon^*$, $z \in S_{\epsilon^*} = \{z \in \mathcal{B} \mid \|z\| \le \epsilon^*\}$ for some $\epsilon^* > 0$, and such that $T(0,0) = 0$. Next we note that $\left(\frac{\partial T}{\partial z}\right)_{z=\epsilon=0} = 0$ since $\left(\frac{\partial T}{\partial z}\right)_0 = -(h_z/h_T)_{z=T=\epsilon=0}$ and $(h_z)_0 = 0$ since F contains no linear terms in z . Consider now the equation

$$\dot{v} - \mathcal{L}(t,v) = m(t)$$

for any $m \in \mathcal{B}_0$. The Fredholm Alternative allows us to conclude that a periodic solution v exists, and can be made unique by some arbitrary requirement. Thus $v = \hat{A}\,m$ for some linear operator $\hat{A} : \mathcal{B}_0 \to \mathcal{B}$. That $\hat{A}$ is continuous can be seen by examining (4.37) and (4.38). Analogously we define $v_t^{\pm}$ and see that these and their derivatives depend continuously on the inhomogeneity m . Also $v_t - v_t^{+} - v_t^{-} =$

$= P^{o}(t)\, n(t)$ where $\dot{n}(t) - B\,n(t) = \tilde{P}^{o}(t)(o)\, m(t)$. We can choose $n(0) = 0$, say, for uniqueness. Clearly n and $\dot{n}$ depend continuously on m. Thus v_t does likewise and $\hat{A}$ is bounded.

Thus the map $\mathcal{F}(z,\epsilon) = \hat{A} \circ F(z, T(z,\epsilon), \epsilon\,; t) : \mathcal{B} \to \mathcal{B}$ is defined on $S_{\epsilon^*} \times [-\epsilon^*, \epsilon^*]$ and is continuous. Also $\left(\frac{\partial \mathcal{F}}{\partial z}\right)_{z=\epsilon=0} = 0$ since $\left(\frac{\partial T}{\partial z}\right)_{z=\epsilon=0} = 0$. Thus by choosing ϵ_0 small enough $\| \mathcal{F}(z_1, \epsilon) - \mathcal{F}(z_2, \epsilon) \| \leq k_0 \| z_1 - z_2 \|$ for $z_i \in S_{\epsilon_0}$ and $|\epsilon| \leq \epsilon_0$ and $0 < k_0 < 1$. Thus for each fixed ϵ, $\mathcal{F}$ is a contraction mapping on S_{ϵ_0} and has a unique fixed point $z(\epsilon)$ where $z = \mathcal{F}(z,\epsilon)$. Then $\dot{z} - \mathcal{J}(t,z) = F(z, T, \epsilon\,; t)$ so z is the required periodic solution of (4.43). Putting $T(\epsilon) = T(z(\epsilon), \epsilon)$ finishes the proof. ∎

We note the close similarity of this and Theorem 2.1, Chapter 14 of Coddington and Levinson [1]. These assert that if a periodic solution of a system of nonlinear equations is nondegenerate in some sense then if the equations are perturbed slightly in any desired way they will still have a periodic solution lying in a neighborhood of the original one. This is a sort of structural stability theorem.

CHAPTER V

STABILITY OF PERIODIC SOLUTIONS

In Chapters II and III we constructed periodic solutions of nonlinear integro-differential equations. In this chapter we examine the stability of these solutions. We shall prove some theorems that will assert the stability or instability of the periodic solution given certain information concerning the linearization of the equations about the solution. This linear equation has periodic coefficients so we can utilize the results of Chapter IV. Next we apply the theorems to deduce the stability of the periodic solution which was constructed in Chapter II by perturbation methods (the Hopf bifurcation). We also analyze the stability of a bifurcating periodic solution of a certain nonautonomous equation arising from population dynamics. These examples should suffice to demonstrate the applicability of the method in the case of bifurcating solutions.

Suppose we have a system of nonlinear equations

$$\frac{d\underset{\sim}{z}}{dt} = \underset{\sim}{F}(\underset{\sim}{z}, \mathcal{K} * \underset{\sim}{z}, t), \tag{5.1}$$

where $\underset{\sim}{z} \in \mathbb{R}^n$, $\mathcal{K} = \mathcal{K}(s)$ is an exponentially decaying matrix kernel, and $\underset{\sim}{F}$ is twice continuously differentiable. As usual

$$(\mathcal{K} * \underset{\sim}{z})(t) \equiv \int_0^\infty \mathcal{K}(s)\, \underset{\sim}{z}(t-s)\, ds.$$

Suppose further that equation (5.1) has a known ω-periodic solution $\underset{\sim}{p}(t)$, so that $\underset{\sim}{F}$ is ω-periodic in its last argument.

We wish to examine the stability of $\underset{\sim}{p}(t)$. First we define this concept. Suppose $\mathcal{K}(s) = O(e^{-\gamma s})$ as $s \to \infty$, $\gamma > 0$. Then, as in Chapter IV we have γ', γ^+ such that $\gamma' < \gamma^+ < \gamma$ and arbitrarily near γ.

<u>Definition</u>: The solution $\underset{\sim}{p}(t)$ of (5.1) is <u>stable at σ</u> iff given $\epsilon > 0$ there exists a $\delta > 0$ such that if initial data $\underset{\sim}{u}(t)$ is given for $(-\infty, \sigma]$ satisfying $\sup_{t \le \sigma} |(\underset{\sim}{p}(t) - \underset{\sim}{u}(t))\, e^{-\gamma^+(t-\sigma)}| < \delta$ then the solution it generates, $\underset{\sim}{z}(t)$, is such that $|\underset{\sim}{z}(t) - \underset{\sim}{p}(t)| < \epsilon$ for all $t \ge \sigma$. If in addition $|\underset{\sim}{z}(t) - \underset{\sim}{p}(t)| \to 0$ as $t \to \infty$ then $\underset{\sim}{p}$ is <u>asymptotically stable</u>. $\underset{\sim}{p}$ is <u>unstable at σ</u> if it is not stable at σ.

Thus small initial perturbations on the interval $(-\infty, \sigma]$ generate small perturbations to the solution, where "small" is in the sense of some norm given above. The definition can be restated as follows. Recall from Chapter IV that $\underset{\sim}{p}(t)$ is represented by $\underset{\sim}{p}_t \in C$ and $C = \{\phi(\theta)$ defined on $\theta \le 0 \mid \sup_{\theta \le 0} |e^{\gamma^+\theta}\phi(\theta)| < \infty\}$. Then p is <u>stable at σ</u> iff given $\epsilon > 0$ there exists $\delta > 0$ such that if $\phi \in C$ and $\|\phi - p_\sigma\| < \delta$ then the solution z_t that ϕ generates at σ $(z_\sigma = \phi)$ satisfies $|z_t(0) - p_t(0)| < \epsilon$ for all $t \ge \sigma$. The solution p is <u>asymptotically stable</u> if $|z(t) - p(t)| \to 0$ as $t \to \infty$. Next let us change the dependent variable from z to x, putting

$$\underset{\sim}{z}(t) = \underset{\sim}{p}(t) + \underset{\sim}{x}(t)$$

so $\underset{\sim}{x}$ satisfies the equation

$$\frac{d\underset{\sim}{x}}{dt} = A(t)\,\underset{\sim}{x}(t) + \int_0^\infty K(t,s)\,\underset{\sim}{x}(t-s)\,ds + \underset{\sim}{f}(\underset{\sim}{x}, \mathcal{K} * \underset{\sim}{x}, t) \quad (5.2)$$

where $K(t,s) = B(t)\mathcal{X}(s)$ and A, B are ω-periodic. Also, $\underset{\sim}{f}$ is nonlinear in $\underset{\sim}{x}$ and ω-periodic in its last argument: $f(0,0,t) = f_x(0,0,t) = f_{\mathcal{X}*x}(0,0,t) = 0$. We shall in fact suppose, given $\epsilon > 0$, there exists $\eta > 0$ such that $|f(x_1, \mathcal{X}*x_1, t) - f(x_2, \mathcal{X}*x_2, t)| < \epsilon \|x_1 - x_2\|$ for $\|x_i\| \leq \eta$, $i = 1, 2$, and all $t \in \mathbb{R}$.

The linearization of (5.1) about $\underset{\sim}{z} = \underset{\sim}{p}(t)$ is given by the first two terms of the right side of (5.2); and has periodic coefficients:

$$\frac{d\underset{\sim}{v}}{dt} = A(t)\underset{\sim}{v} + \int_0^\infty K(t,s)\,\underset{\sim}{v}(t-s)\,ds. \tag{5.3}$$

Suppose (5.3) has some characteristic exponent with positive real part. Then we show that for all σ, the solution $\underset{\sim}{x} \equiv 0$ of (5.2) is unstable at σ. Equivalently, $\underset{\sim}{p}(t)$ is unstable as a solution of (5.1).

<u>Theorem 1</u>. Suppose equation (5.1) has an ω-periodic solution $\underset{\sim}{p}(t)$ and that its linearization about $\underset{\sim}{p}$ is given by (5.3). Suppose that (5.3) has a characteristic exponent with positive real part. Then for every σ, the periodic solution $\underset{\sim}{p}(t)$ is unstable at σ.

<u>Proof</u>: To prove this we proceed as follows. Fix σ throughout and for brevity denote $f(t, x_t) = f(x, \mathcal{X}*x, t)$. We divide the characteristic multipliers of (5.3) into two parts

$$\Lambda^+ = \{\mu \mid |\mu| > 1\},$$

$$\Lambda^- = \{\mu \mid |\mu| \leq 1\}.$$

Denote $\Lambda^+ = \{\mu_1 \cdots \mu_m\}$, $\lambda_i = \frac{1}{\omega}\log\mu_i$ and $\lambda = \min_i \operatorname{Re}\lambda_i$. Let $\Phi_i(t)$, $\Psi_i(t)$ be bases for the generalized null spaces $E_{\mu_i}(t)$, $\tilde{E}_{\mu_i}(t)$

of $U - \mu_i I$, $\tilde{U} - \mu_i I$ respectively; and $\Phi_i(t) = P_i(t)e^{B_i t}$, $\Psi_i(t) = e^{-B_i t}\tilde{P}_i(t)$ where B_i has the only eigenvalue λ_i. Combine the bases into $\Phi(t) = (\Phi_1, \dots, \Phi_m) = P(t)e^{Bt}$, $\Psi(t) = e^{-Bt}\tilde{P}(t)$, $B = \mathrm{diag}(B_1, \dots, B_m)$, $P(t) = \mathrm{row}(P_1 \cdots P_m)$, $\tilde{P}(t) = \mathrm{col}(\tilde{P}_1 \cdots \tilde{P}_m)$. These form bases for the span of the $E_{\mu_i}(t)$, $\tilde{E}_{\mu_i}(t)$ respectively.

Let x_t be an arbitrary solution of (5.2) and define its projections

$$x_t^+ = \Phi(t)\ (\Psi(t), x_t)_t \ ,$$

$$x_t^- = x_t - x_t^+ \ ,$$

cf. (4.28)-(4.31). We assume B is in Jordan canonical form with superdiagonal ν and $\nu > 0$ can be chosen arbitrarily small, say $\nu < \frac{\lambda}{10}$.

To prove the theorem we assume on the contrary that $\underset{\sim}{x} \equiv 0$ is stable and derive a contradiction. For each $\epsilon > 0$ (to be fixed later) there is $\eta > 0$ such that $|f(t,\phi_1) - f(t,\phi_2)| \leq \epsilon \|\phi_1 - \phi_2\|$ for all t and $\|\phi_i\| \leq \eta$. Then by the assumption of stability there is $\delta > 0$ such that $\|x_\sigma\| \leq \delta$ implies $\|x_t\| \leq \eta$ for all $t \geq \sigma$. With this hypothesis we choose $x_\sigma = x_\sigma^+ + x_\sigma^-$ with $x_\sigma^- = 0$, $x_\sigma^+ \neq 0$ so that x has initial value in the space of eigenvalues with positive real part. We are assuming the ensuing solution x_t is bounded in norm by η.

As in Chapter IV let

$$X_0^+(t) = \Phi(t)\,(\Psi(t), X_0)_t = \Phi(t)\,\Psi(t)(0),$$

$$X_0^-(t) = X_0 - X_0^+(t),$$

so $x_t^{\pm}$ satisfy the integral equations

$$x_t^{\pm} = T(t,\sigma)\,x_\sigma^{\pm} + \int_\sigma^t T(t,s)\,X_0^{\pm}(s)\,f(s,x_s)\,ds \qquad (5.4)$$

and $x_\sigma^- = 0$. By a result of Chapter IV

$$\| T(t,s)\,X_0^-(s) \| \le K_1\, e^{\nu(t-s)}, \qquad t \ge s,$$

with ν defined as above and K_1 depending on ν only. Thus from (5.4)

$$\| x_t^- \| \le \epsilon\eta K_1 (e^{\nu(t-\sigma)} - 1) = O(e^{\nu(t-\sigma)})$$

as $t \to \infty$. Also if we write $x_\sigma^+ = \Phi(\sigma)\,c$ for some constant c then

$$x_t^+ = \Phi(t)\left[\, c + \int_\sigma^t \Psi(s)(0)\, f(s,x_s)\,ds \right].$$

Since $\Psi(s)(0) = O(e^{-Bs})$ as $s \to \infty$, then the integral $\int_t^\infty \Psi(s)(0)\,f(s,x_s)\,ds$ converges and

$$x_t^+ = \Phi(t)\left[\, c + \int_\sigma^\infty \Psi(s)(0)\,f(s,x_s)\,ds \right]$$
$$\qquad - \Phi(t)\int_t^\infty \Psi(s)(0)\,f(s,x_s)\,ds.$$

Under the assumption $\|x_s\| \le \eta$ the last term in x_t^+ is bounded since

$$\| \Phi(t) \int_t^\infty \Psi(s)(0)\ f(s,x_s)\, ds \| \le \epsilon \eta M^2 \int_t^\infty e^{(\lambda-\nu)(t-s)} ds$$

$$= \frac{\epsilon \eta M^2}{\lambda_1},$$

where $0 < \lambda - \nu = \lambda_1 \le \|B\|$ and $M = \max_{t \in \mathbb{R}} (\|P(t)\|, \|\tilde{P}(t)\|)$. The first term in x_t^+ is $\Phi(t)\, \underline{b}$ with $b = c + \int_\sigma^\infty \Psi(s)(0)\ f(s,x_s)\, ds$, and this has growth at least $e^{\lambda_1 t}$ as $t \to \infty$. Thus, since x_t is assumed bounded, we must have $b = 0$. Then x_t solves the integral equation

$$x_t = \int_\sigma^t T(t,s)\, X_0^-(s)\ f(s,x_s)\, ds$$

$$- \Phi(t) \int_t^\infty \Psi(s)(0)\ f(s,x_s)\, ds.$$

Then

$$\|x_t\| \le K_1 \epsilon \int_\sigma^t e^{\nu(t-s)} \|x_s\|\, ds$$

$$+ M^2 \epsilon \int_t^\infty e^{\lambda_1 (t-s)} \|x_s\|\, ds. \tag{5.5}$$

Choose $\rho > 0$ such that $0 < \nu < \rho < \lambda_1$; put $a_1 = \rho - \nu$, $a_2 = \lambda_1 - \rho$. Also put

$$r(t) = e^{-\rho t} \|x_t\|.$$

Then $R = \sup_{t \ge \sigma} r(t)$ exists. Furthermore, from (5.5),

$$r(t) \leq K_1 \epsilon \int_\sigma^t e^{-a_1(t-s)} r(s)\,ds + M^2 \epsilon \int_t^\infty e^{a_2(t-s)} r(s)\,ds$$

$$\leq \epsilon \left(\frac{K_1}{a_1} + \frac{M^2}{a_2} \right) R$$

$$\leq \frac{1}{2} R$$

if we choose $\epsilon < \dfrac{1}{2\left(\frac{K_1}{a_1} + \frac{M^2}{a_2}\right)}$. This holds for all $t \geq \sigma$ so

$$R = \sup r \leq \frac{1}{2} R.$$

This implies $R = 0$ so $x_t = 0$ for all $t \geq \sigma$. This is impossible since $x_\sigma \neq 0$. This gives the desired contradiction and proves the theorem. ∎

We now turn to the question of finding sufficient conditions for the stability of the periodic solution. It seems reasonable that if (5.3) has all characteristic exponents with real part negative then $\underset{\sim}{p}(t)$ is stable. In fact we prove asymptotic stability. First we note <u>Gronwall's inequality</u> (cf. Problem 1, Chapter 1, Coddington and Levinson [1]): let $\chi(t) > 0$ on $[\sigma, \infty)$ and

$$\phi(t) \leq \psi(t) + \int_\sigma^t \chi(s)\,\phi(s)\,ds.$$

Then for $t \geq \sigma$, $\phi(t) \leq \psi(t) + \int_\sigma^t \chi(s)\,\psi(s)\,e^{\int_s^t \chi(u)du}\,ds$.

<u>Theorem 2</u>: Suppose the linearization (5.3) of (5.1) about $\underset{\sim}{p}(t)$ has all its characteristic exponents with negative real parts. Then the periodic solution $\underset{\sim}{p}(t)$ is asymptotically stable.

<u>Proof</u>: Suppose the characteristic exponents λ satisfy

$$\operatorname{Re}\lambda \le -\alpha < 0.$$

Then for some constant M

$$\| T(t,s)\phi \| \le M e^{-\alpha(t-s)} \|\phi\| , \qquad t \geq s,$$

for all $\phi \in C$. The solution x_t of (5.2) satisfies

$$x_t = T(t,\sigma)x_\sigma + \int_\sigma^t T(t,s) X_0 f(s, x_s)\,ds.$$

Choose $0<\epsilon<\alpha$ and $\delta>0$ such that $\|\phi\| \le \delta$ implies $\|f(s,\phi)\| \le \frac{\epsilon}{M}\|\phi\|$, $\forall s$. Then, so long as $\|x_s\| \le \delta$,

$$e^{\alpha t}\|x_t\| \le M e^{\alpha\sigma}\|x_\sigma\| + \epsilon\int_\sigma^t e^{\alpha s}\|x_s\|\,ds.$$

Using Gronwall's inequality we find

$$e^{\alpha t}\|x_t\| \le M e^{\alpha\sigma}\|x_\sigma\| + \epsilon M e^{\alpha\sigma}\|x_\sigma\| \int_\sigma^t e^{\epsilon(t-s)}\,ds$$

$$= M e^{\alpha\sigma}\|x_\sigma\| e^{\epsilon(t-\sigma)}$$

so

$$\|x_t\| \le M e^{-(\alpha-\epsilon)(t-\sigma)}\|x_\sigma\|.$$

Thus, choosing $\|x_\sigma\| \le \frac{\delta}{M}$ implies $\|x_t\| \le \delta$ for all $t \geq \sigma$.

Furthermore $\|x_t\| \to 0$ as $t \to \infty$, so $x(t) \to 0$. ∎

For many cases these two theorems suffice to determine the stability or instability of a given periodic solution. However, in the case of autonomous equations, the hypotheses of Theorem 2 are never satisfied, and we must examine the situation more closely.

Consider the autonomous system of equations

$$\frac{dz}{dt} = F(z, \mathcal{K} * z) \tag{5.6}$$

with $\mathcal{K}, F, z$ as in equation (5.1). Note F is independent of t. If $p(t)$ is an ω-periodic solution of (5.6) then the linearization of (5.6) about p is still given by (5.3) with A, K ω-periodic. Now notice that $\dot{p} = \frac{dp}{dt}$ solves (5.3). Since $\dot{p}$ is ω-periodic we conclude that (5.3) has $\mu = 1$ as a characteristic multiplier. Thus we see that Theorem 2 cannot apply. In many cases $\mu = 1$ is <u>simple,</u> i.e. the generalized null space of $U - I$ has dimension one (being spanned by $\dot{p}_t$). If also all other $|\mu| < 1$ then we can prove that a different type of stability holds. If z_t starts close to p_t at some t then z becomes asymptotically periodic as $t \to \infty$ and even though $|p(t) - z(t)|$ may not tend to zero, the orbits described by z and p tend to coincide as $t \to \infty$. More precisely,

<u>Definition</u>. The periodic solution $p(t)$ is <u>asymptotically orbitally stable</u> iff there exists $\epsilon > 0$ such that if $\| z_{t_1} - p_{t_0} \| < \epsilon$ for some t_0, t_1, then the solution z_t of (5.6) for $t \geq t_0, t_1$ satisfies

$$\lim_{t \to \infty} |z(t) - p(t+c)| = 0$$

for some constant c, called the <u>asymptotic</u> <u>phase</u>.

<u>Theorem 3</u>: Suppose the linearization (5.3) of (5.6) about p has all characteristic multipliers $|\mu| < 1$ except $\mu = 1$, which is assumed simple (equivalently, all $\operatorname{Re} \lambda < 0$ except for a simple $\lambda = 0$, λ being a characteristic exponent). Then p is asymptotically orbitally stable.

<u>Proof</u>: We shall prove this by showing that there is a manifold orthogonal to the orbit $p(t)$ at one of its points such that if a trajectory passes through this manifold then it develops the asymptotic properties of orbital stability. We note that this is exactly the method of proof of the corresponding classical orbital stability theorem given in Theorem 2.2, Chapter 14, Coddington and Levinson [1].

Note the translation invariance of solutions of (5.6), namely that if $z(t)$ solves (5.6) then $z(t+k)$ also solves (5.6) for any constant k.

Consider the equation (5.2) obtained from (5.6), by putting $x_t = z_t - p_t$. We have the Lipschitz condition for f: for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|x\|, \|\tilde{x}\| \leq \delta$ then $|f(t, x) - f(t, \tilde{x})| < \epsilon \|x - \tilde{x}\|$ for all . (Put $f(t, x_t) = f(x, \varkappa + x, t)$). The linearization (5.3) has multiplier $\mu = 1$ and null space $E_1(0) = \mathcal{N}(U - I)$. $\Phi(t) = P(t) = \dot{p}_t$ spans $E_1(0)$. Put $C = E_1(0) \oplus R_1$. The adjoint has corresponding $\Psi(t)$, $\tilde{E}_1(0)$ with $\Psi(t)$ ω-periodic.

We now find the manifold mentioned above. Consider the integral equation

$$\theta_t(a) = T(t,0)\,a + \int_0^t T(t,s)\, X^{(1)}(s)\, f(s,\theta_s)\,ds$$
$$- \int_t^\infty \Phi(t)\,\Psi(s)(0)\, f(s,\theta_s)\,ds \qquad (5.7)$$

with $a \in R_1$.

Here $X^{(1)}(s) \equiv X_0 - \Phi(s)\,(\Psi(s), X_0)_s = X_0 - \Phi(s)\,\Psi(s)(0)$, cf. Also for some $K_2 > 0$

$$\| T(t,s)\, X^{(1)}(s) \| \leq K_2\, e^{-\sigma(t-s)},$$

$$\| \Phi(t)\, \Psi(s)(0) \| \leq K_2 , \qquad t \geq s,$$

where $\sigma = \min \{ |\lambda| / \lambda$ is a characteristic exponent such that $Re\,\lambda < 0 \}$. Also

$$\| T(t,0)\, a \| \leq K_1\, \|a\|\, e^{-\sigma t}, \qquad t \geq 0.$$

Now choose $\epsilon < \frac{\sigma}{2K_2}$ and let δ be the corresponding quantity in the Lipschitz condition for f. We show that if $\|a\| \leq \frac{\delta}{2K_1}$ and $a \in R_1$ then (5.7) has a solution $\theta_t(a)$ for $t \geq 0$ satisfying

$$\| \theta_t(a) \| \leq 2K_1\, \|a\|\, e^{-\frac{1}{2}\sigma t}, \qquad t \geq 0.$$

It is easy to show that θ_t satisfies (5.2) (cf. proof of Theorem 1). We construct θ_t by successive iterations $\theta_t^{(k)}$, $k = 0, 1, 2, \ldots$. Let $\theta_t^{(0)} = 0$, and

$$\theta_t^{(k+1)}(a) = T(t,0)\,a + \int_0^t T(t,s)\,X^{(1)}(s)\,f(s,\theta_s^{(k)})\,ds$$

$$- \int_t^\infty \Phi(t)\,\Psi(s)(0)\,f(s,\theta_s^{(k)})\,ds.$$

Then it is easy to show by induction that for all $t \geq 0$

$$\| \theta_t^{(k+1)} - \theta_t^{(k)} \| \leq \frac{k_1 \|a\|}{2^k} e^{-\frac{1}{2}\sigma t}.$$

Thus $\lim_{k\to\infty} \theta_t^{(k)} = \theta_t$ exists for each $t \geq 0$ and $\theta_t \in C$. Clearly θ_t satisfies (5.7) and

$$\| \theta_t(a) \| \leq \sum_{k=0}^{\infty} \| \theta_t^{(k+1)} - \theta_t^{(k)} \| \leq 2k_1 \|a\| e^{-\frac{1}{2}\sigma t}$$

so $\theta_t \to 0$ as $t \to \infty$. Now at $t = 0$

$$\theta_0(a) = a - \Phi(0) \int_0^\infty \Psi(s)(0)\,f(s,\theta_s(a))\,ds$$

and $a \in R_1$, $\Phi(0) \in E_1(0)$ so $\theta_0(a)$ has been decomposed according to $C = E_1(0) \oplus R_1$. Let π be the projection operator onto $E_1(0)$ along R_1, $\pi' = I - \pi$ projecting onto R_1 along $E_1(0)$. Note π, π' are bounded since $E_1(0)$, R_1 are closed. On a neighborhood of zero in R_1 define the real-valued function N

$$N(a) = -\int_0^\infty \Psi(s)(0)\,f(s,\theta_s(a))\,ds.$$

It can be shown that $\theta_t(a)$ is continuous in a so N is a nonlinear continuous function and is $o(a)$ as $a \to 0$. Also

$$\Pi \, \theta_0(a) = N(a) \, \Phi(0),$$

$$\Pi' \, \theta_0(a) = a.$$

If we can find $\phi \in C$ such that $\phi = a \oplus N(a)\,\Phi(0)$, $a \in R_1$, and $\|\phi\|$ small, then the corresponding $\theta_t(a)$ satisfies $\phi = \theta_0(a)$ and $\theta_t \to 0$ as $t \to \infty$. The set of such ϕ defines a manifold $\mathcal{M}$ given by the relation

$$\Pi \phi = N(\pi'\phi) \, \Phi(0).$$

Thus if ϕ is on $\mathcal{M}$ it generates a solution $\theta_t(\pi'\phi)$ that tends to zero as $t \to \infty$.

Having defined the desired orthogonal manifold we show that, given a solution $z(t)$ of (5.6), if $\| z_{t_1} - p_{t_0} \|$ is small for some t_0, t_1 then the trajectory z_t passes through the manifold $p_0 + \mathcal{M}$. Let z_t satisfy (5.6) and

$$\| z_{t_1} - p_{t_0} \| < \epsilon$$

with ϵ to be defined later. Put

$$\psi(t) \equiv z(t + t_1 - t_0) \quad , \qquad t \geq t_0 .$$

Clearly ψ solves (5.6) so

$$y_t \equiv \psi_t - p_t$$

solves (5.2) for $t \geq t_0$. Note $\|y_{t_0}\| < \epsilon$. Later we shall choose ϵ so small that $\|y_t\|$ remains small for $|t - t_0| < 2\omega$, no matter what y we have. Then it is clear that by shifting t if necessary we may assume that t_0 is a multiple of ω . Also put

$$u_{\tilde{t}} \equiv \psi_{\tilde{t}+t_0} - p_{t_0} \;;$$

we show that for some $\tilde{t}$, $u_{\tilde{t}}$ is on the manifold M .

The fact that N is nonlinear of higher order implies that for $\epsilon' > 0$ there is $\eta_{\epsilon'} > 0$ such that $\|a\|, \|b\| \leq \eta_{\epsilon'}$ implies $N(a)$, $N(b)$ are defined and $|N(a) - N(b)| < \epsilon' \|a - b\|$. Choose

$$\epsilon' = \frac{\|\pi\|}{\|\pi'\| \, \|\Phi(0)\|} \quad , \quad \eta = \eta_{\epsilon'} .$$

Put $q_{\tilde{t}} \equiv p_{\tilde{t}+t_0} - p_{t_0}$ for $|\tilde{t}| \leq 2\omega$. Choose $\tau > 0$ so small that $\tau < 2\omega$ and $\|q_{\tilde{t}}\| \leq \frac{\eta}{2\|\pi'\|}$ for $|\tilde{t}| \leq \tau$. For such $\tilde{t}$ we have that $N(\pi' q_{\tilde{t}})$ is defined. Put $\pi q_{\tilde{t}} = \kappa(\tilde{t})\,\Phi(0)$ and $\hat{\gamma}(\tilde{t}) = \kappa(\tilde{t}) - N(\pi' q_{\tilde{t}})$. Then since $q_{\tilde{t}} = \tilde{t}\,\dot{p}_{t_0} + O(\tilde{t}^2) = \tilde{t}\,\Phi(0) + O(\tilde{t}^2)$ we have $\hat{\gamma}(\tilde{t}) = \tilde{t} + O(\tilde{t}^2)$, and we suppose this last term in $\hat{\gamma}$ is less than $M\tilde{t}^2$ in absolute value (for $|\tilde{t}| < \tau$) . Now there exists $\rho > 0$ and $0 < \tau' < \tau$ such that

$$\tilde{t} - M\tilde{t}^2 - \rho > 0, \qquad \tilde{t} \in [\tau', \tau],$$
$$\tilde{t} + M\tilde{t}^2 + \rho < 0, \qquad \tilde{t} \in [-\tau, -\tau']. \tag{5.8}$$

Since $\|y_{t_0}\| < \epsilon$, choose $\epsilon > 0$ so small that

$$\|y_t\| \leq \min\left\{ \frac{\rho \|\Phi(0)\|}{2\|\pi\|}, \frac{\eta}{2\|\pi'\|} \right\}$$

for $|t - t_0| \leq 2\omega$.

With these preliminaries we find $\tilde{t}$ such that $u_{\tilde{t}} \in M$. Since $u_{\tilde{t}} = y_{\tilde{t}+t_0} + q_{\tilde{t}}$ we have

$$\|u_{\tilde{t}}\| \leq \frac{\eta}{\|\pi'\|}, \qquad |\tilde{t}| < \tau;$$

$N(\pi' u_{\tilde{t}})$ is defined. Put $\pi u_{\tilde{t}} = \alpha(\tilde{t})\,\Phi(0)$, and $\beta(\tilde{t}) = \alpha(\tilde{t}) - N(\pi' u_{\tilde{t}})$. We seek $\tilde{t}$ such that $\beta(\tilde{t}) = 0$ (since β and $\hat{\gamma}$ are close, $\hat{\gamma}(0) = 0$, and $\hat{\gamma}$ is linear near zero, we expect to be able to do this). Note $\pi y_{\tilde{t}+t_0} = \pi(u_{\tilde{t}} - q_{\tilde{t}}) = [\alpha(\tilde{t}) - \kappa(\tilde{t})]\Phi(0)$. Now

$$|\beta(\tilde{t}) - \hat{\gamma}(\tilde{t})| \leq |\alpha(\tilde{t}) - \kappa(\tilde{t})| + |N(\pi' u_{\tilde{t}}) - N(\pi' q_{\tilde{t}})|$$
$$\leq \|\pi y_{\tilde{t}+t_0}\| / \|\Phi(0)\| + \epsilon' \|\pi'\| \|y_{\tilde{t}+t_0}\|$$
$$\leq \frac{2\|\pi\|}{\|\Phi(0)\|} \|y_{\tilde{t}+t_0}\| < \rho.$$

Thus $\beta(\tilde{t}) = \tilde{t} + \nu$ where $|\nu| < M\tilde{t}^2 + \rho$. Thus by (5.8) $\beta(\tilde{t})$ changes sign as $\tilde{t}$ crosses zero. It has a zero at $\bar{t}$. Thus

$\Pi u_{\bar{t}} = N(\pi' u_{\bar{t}})\, \Phi(0)$ so $u_{\bar{t}} \in \mathcal{M}$. By construction of $\mathcal{M}$, $\theta_t\,(\pi' u_{\bar{t}})$ is defined for $t \geq 0$ and clearly $\theta_t = \psi_{t+\bar{t}+t_0} - p_{t+t_0}$ for $t \geq 0$. Since $z_{t+\bar{t}+t_1} = \psi_{t+\bar{t}+t_0}$ and $\theta_t \to 0$ as $t \to \infty$ then $\| z_{t+\bar{t}+t_1} - p_{t+t_0} \| \to 0$, i.e. as $t \to \infty$, $\| z_t - p_{t+c} \| \to 0$ for $c = t_0 - \bar{t} + t_1$ as we wished to prove. ∎

We now apply the results of these three theorems to some problems. First we consider a problem from Cushing [3]. He considers a certain nonautonomous generalization of the delay-logistic equation for one-species population growth. Specifically, a periodically fluctuating environment is assumed:

$$\frac{dN}{dt} = N \cdot \Big(\lambda + b(t) - a(t) N - \int_0^\infty k(t,\theta)\, N(t-\theta)\, d\theta + r(t, N, k*N) \Big), \tag{5.9}$$

where $N, a, b, \ldots$ are real numbers, N representing the population size as a function of time t . We assume a, b are ω-periodic as well as k, r in their first arguments (k is exponentially decaying as $\theta \to \infty$) . Also r is assumed to be at least quadratic in $N, k*N$. Thus the birth rate $\lambda + b(t)$, capacity coefficient $a(t)$, delay k , and higher order corrections r are periodically fluctuating with the same period. This could be due, for example, to seasonal changes, harvesting, biological rhythms, etc. It is assumed that the average value of b is zero:

$$\int_0^\omega b(t)\, dt = 0, \tag{5.10}$$

so λ represents the average birth rate. Cushing then shows that a positive periodic solution of (5.9) bifurcates from the trivial solution $N \equiv 0$ at $\lambda = 0$, assuming $\alpha, K > 0$. Then he shows that if K, r are sufficiently small, the periodic solution $\bar{N}$ is stable. We shall analyze the stability of $\bar{N}$ for any K, r, showing that if these are large enough $\bar{N}$ can become unstable. This demonstrates the general rule that sufficiently strong delays can have a destabilizing effect.

A regular perturbation procedure can be used to obtain $\bar{N}$ as follows. Put

$$N = \epsilon N_1 + \epsilon^2 N_2 + \cdots ,$$

$$\lambda = \epsilon \lambda_1 + \epsilon^2 \lambda_2 + \cdots ,$$

with the N_i ω-periodic. Substituting into (5.9) and equating like powers of ϵ yields for ϵ^1:

$$\frac{dN_1}{dt} = b(t) N_1 ,$$

i.e.

$$N_1(t) = \alpha \exp \int_0^t b(s)\,ds, \qquad \epsilon, \alpha > 0 .$$

This is ω-periodic by (5.10). The ϵ^2 terms yield:

$$\frac{dN_2}{dt} = \lambda_1 N_1 + b(t) N_2 - a(t) N_1^2 - N_1(t) \int_0^\infty K(t,\theta) N_1(t-\theta)\,d\theta ,$$

i.e.

$$\frac{dN_2}{dt} - b(t)N_2 = \alpha\, e^{\int_0^t b\,ds}\left[\lambda_1 - \alpha\, a(t)\, e^{\int_0^t b\,ds} - \alpha\int_0^\infty K(t,\theta)\, e^{\int_0^{t-\theta} b(s)ds}\, d\theta\right].$$

We require N_2 to be ω-periodic so an application of the Fredholm Alternative is needed. The adjoint ω-periodic null solution is $e^{-\int_0^t b(s)ds}$, so we have

$$\int_0^\omega\left[\lambda_1 - \alpha\, a(t)\, e^{\int_0^t b\,ds} - \alpha\int_0^\infty K(t,\theta)\; e^{-\int_{t-\theta}^0 b(s)ds}\, d\theta\right] dt = 0.$$

This gives

$$\lambda_1 = \alpha\cdot\frac{1}{\omega}\int_0^\omega a(t)\, e^{\int_0^t b\,ds}\, dt + \alpha\cdot\frac{1}{\omega}\int_0^\omega\int_0^\infty K(t,\theta)\; e^{-\int_{t-\theta}^0 b(s)ds}\, d\theta\, dt.$$

Then

$$N_2(t) = \alpha\, e^{\int_0^t b\,ds}\int_0^t\left[\lambda_1 - \alpha\, a(u)\, e^{\int_0^u b\,ds} - \alpha\int_0^\infty K(u,\theta)\; e^{-\int_{u-\theta}^0 b\,ds}\, d\theta\right] du.$$

Clearly this can be continued to find ω-periodic $\bar{N}(t)$. We stop here and turn to stability considerations.

The linearization of (5.9) about $\bar{N}$ is

$$\frac{dv}{dt} = v\left[\lambda + b(t) - a(t)\bar{N} - K(t) * \bar{N} + r(t, \bar{N}, K*\bar{N})\right]$$
$$+ \bar{N}\left[-a(t)v - K(t)*v + r_{\bar{N}}(t)[v]\right], \tag{5.11}$$

where $r_{\bar{N}}$ = linearization of r about $\bar{N}$. At $\epsilon = 0$ this is

$$\frac{dv}{dt} = b(t)\, v. \tag{5.12}$$

It is easy to see from the proof given by Cushing that if r is analytic in $N, K*N$ then $\bar{N}, \lambda$ are analytic functions of ϵ. Then the semi-group $T(t, s; \epsilon)$ associated with (5.11) is analytic in ϵ so $U(\epsilon) = T(\omega, 0; \epsilon)$ is analytic. We appeal to Theorems 1.7, 1.8, Chapter VII of Kato [7] to deduce that the spectrum of $U(\epsilon)$ varies analytically with ϵ, reducing to that of $U(0)$ as $\epsilon \to 0$. The latter consists of $\mu = 1$ with corresponding (periodic) eigenfunction $e^{\int_0^t b\, ds}$ solving (5.12). Since the characterisitc multipliers $\mu(\epsilon)$ for (5.11) vary near those of (5.12), we have one branch $\mu_s(\epsilon)$ perturbed off the value $\mu = 1$ and all others are $|\mu| < 1$. Note that μ_s is simple. If $|\mu_s(\epsilon)| > 1$ then by Theorem 1, $\bar{N}$ is unstable. If $|\mu_s| < 1$ then by Theorem 2, $\bar{N}$ is stable. We thus need only calculate $\mu_s(\epsilon)$. Put

$$\beta(\epsilon) = \frac{1}{\omega} \log \mu_s(\epsilon) = \beta_1 \epsilon + \beta_2 \epsilon^2 + \cdots ,$$

the characteristic exponent. The corresponding eigenfunction has the form

$$v(t, \epsilon) = P(t, \epsilon)\, e^{\beta(\epsilon) t},$$

with P ω-periodic, cf. Chapter IV. Let

$$P = P_0(t) + \epsilon P_1(t) + \epsilon^2 P_2(t) + \dots$$

(each P_i is ω-periodic). Then

$$V = V_0 + \epsilon V_1 + \epsilon^2 V_2 + \cdots$$

where

$$\begin{aligned} V_0 &= P_0 , \\ V_1 &= P_1 + P_0 \beta_1 t, \\ &\vdots \end{aligned}$$

Substituting these expansions and $\bar{N} = \epsilon N_1 + \cdots$, $\lambda = \epsilon \lambda_1 + \cdots$ into (5.11) yields a hierarchy to be solved.

The ϵ^1 terms yield

$$\frac{dP_0}{dt} = b(t) P_0 ,$$

i.e.

$$P_0(t) = e^{\int_0^t b(s)\,ds} .$$

The ϵ^2 terms give:

$$\begin{aligned} \frac{dV_1}{dt} - b(t) V_1 &= \left(\frac{dP_1}{dt} - b(t) P_1\right) + e^{\int_0^t b\,ds} \beta_1 \\ &= \left\{ \lambda_1 + \alpha\left[-2\,a(t) P_0 - 2\,K * P_0\right]\right\} e^{\int_0^t b\,ds} . \end{aligned}$$

We require P_1 to be ω-periodic, so the Fredholm Alternative implies

$$\int_0^\omega \{ -\beta_1 + \lambda_1 + \alpha[-2a(t)P_0 - 2k * P_0] \} \, dt = 0 ,$$

i.e.

$$\beta_1 = \alpha \cdot \frac{1}{\omega} \int_0^\omega [-a(t) \, e^{\int_0^t b \, ds} - K * e^{\int_0^t b \, ds}] \, dt = -\lambda_1 .$$

Then periodic P_1 exists, etc. We stop here since, for $\epsilon, \alpha > 0$, $\operatorname{sgn} \beta(\epsilon) = \operatorname{sgn} \beta_1$ (ϵ small). Applying Theorems 1 and 2:

$\bar{N}$ is stable if $\beta_1 < 0$, unstable if $\beta_1 > 0$.

Note, if K is small then since $a > 0$ we have $\beta_1 < 0$ so $\bar{N}$ is stable, in agreement with Cushing [3]. However, since $K > 0$ also, $\bar{N}$ is stable regardless of the size of K .

We next turn to the Hopf bifurcation of Chapter II. There exists a one-parameter family of periodic solutions $z = p(t,\epsilon)$ of

$$\frac{dz}{dt} = F(z, \mathcal{K} * z, \lambda) \tag{5.13}$$

of period $\omega(\epsilon)$ for $\lambda = \lambda(\epsilon)$, cf. Theorem of Chapter II. We shall assume F is analytic in all its arguments. Then, as the note following the theorem of Chatper II shows, $p(t,\epsilon)$ is analytic in t and ϵ , and λ, ω are analytic in ϵ . Assume $\lambda(0) = 0$ and that we have the nontrivial case $\lambda(\epsilon) \not\equiv 0$. Suppose the linearization of (5.13) about $p(t,\epsilon)$ is

$$\frac{dv}{dt} = A(t;\epsilon)\,v + \int_0^\infty k(t,s;\epsilon)\,v(t-s)\,ds, \tag{5.14}$$

where A, k (in its first argument) are $\omega(\epsilon)$-periodic, and are analytic in s, t, ϵ. At $\epsilon = 0$ this becomes the linearization of (5.13) about $z = 0$:

$$\frac{dv}{dt} = Av + \int_0^\infty k(s)\,v(t-s)\,ds, \tag{5.15}$$

so

$$A = A(t;0), \quad k(s) = k(t,s;0).$$

We shall deduce the stability of each of the periodic solutions $p(t,\epsilon)$ by calculating the characteristic exponents associated with the periodic - coefficients equation (5.14). This will then be related back to the two time-scale perturbation scheme we used in Chatper II to construct $p(t,\epsilon)$. There we also found a neighboring family of non-periodic solutions that enabled us to state a conjecture about the stability of p. This depended on a certain coefficient in the modulation equations for the amplitude and phase of p. What we shall show is that this reasoning based on the perturbation scheme is completely valid, namely that whatever stability is predicted by the modulation equations is in agreement with that deduced by finding the characteristic exponents of (5.14) and applying Theorems 1-3.

The characteristic multipliers of (5.14) are the point eigenvalues of the operators $U(\epsilon) = T(\omega(\epsilon), 0; \epsilon)$ where $T(t,s;\epsilon)$ is the semi-group associated with (5.14). Since the resolvent $R(t,s;\epsilon)$ for (5.14) is analytic in all arguments then T is likewise. Thus U is analytic in ϵ. Also, $U(0) = T(\omega_0, 0; 0) = T_0(\omega_0)$ where $\omega_0 = \omega(0)$ and T_0 is the semi-group for (5.15). Now the infinitesimal operator $\mathcal{A}$ for T_0 has eigenvalues $\lambda = \pm i\nu$ $\left(\omega_0 = \frac{2\pi}{\nu}\right)$ and all others $Re\,\lambda < 0$. Since we can view (5.15) as a periodic-coefficients equation (period ω_0) then $U(0) = T_0(\omega_0)$ has only point eigenvalues μ for $|\mu| > e^{-\gamma'\omega_0}$. But then these are of the form $e^{\lambda\omega_0}$ for each eigenvalue λ of $\mathcal{A}$, cf. Lemma 6, Chapter I. Now by Theorems 1.7, 1.8, Chapter VII of Kato [7], the point spectrum of $U(\epsilon)$ varies analytically near that of $U(0)$ for ϵ near zero. Each eigenvalue $\mu(\epsilon)$ of $U(\epsilon)$ is the branch of an analytic function in a neighborhood of $\epsilon = 0$. Thus we may expand $\mu(\epsilon)$ in a power series in $\epsilon^{\frac{1}{p}}$ for some integer $p > 0$. If $\mu(\epsilon)$ is the only branch then $p = 1$. The set of eigenvalues of $U(0)$ form two sets: $|\mu_0| < 1$ and $\mu_0 = e^{\pm i\nu\omega_0} = 1$ (double multiplicity). For ϵ sufficiently small the set of eigenvalues $\mu(\epsilon)$ of $U(\epsilon)$ that perturb off the set $|\mu_0| < 1$ will continue to have modulus less than unity (cf. the theorems cited in Kato [7]). These are irrelevant for stability purposes. What does determine stability are the eigenvalues $\mu(\epsilon)$ that branch off $\mu_0 = 1$, which has multiplicity two (equal to the dimension of the generalized null space of $U(0) - I$). As ϵ varies μ_0 will split into two branches $\mu^{(1)}(\epsilon)$, $\mu^{(2)}(\epsilon)$ each simple. Since (5.13) is autonomous, (5.14) will always have a characteristic multiplier $\mu = 1$; thus $\mu^{(1)}(\epsilon) \equiv 1$ for all ϵ. The other eigenvalue $\mu^{(2)}(\epsilon)$ will vary near 1, and is in fact an analytic function of ϵ (not a branch of one) since $\mu^{(1)}(\epsilon)$ is not the branch of a double-valued analytic function. Thus we may expand

$\mu^{(2)}(\epsilon) = 1 + O(\epsilon)$ in a power series in ϵ near 0. The eigenfunction corresponding to $\mu^{(2)}(\epsilon)$ is $P(t,\epsilon)\, e^{\beta(\epsilon)t}$ where $\beta(\epsilon) = \frac{1}{\omega} \log \mu^{(2)}(\epsilon)$ is the characteristic exponent associated with $\mu^{(2)}$, and P is $\omega(\epsilon)$-periodic in t. Clearly $\beta(\epsilon) = \beta_1 \epsilon + \beta_2 \epsilon^2 + \cdots$ and we expand $P(t,\epsilon) = P_0(t) + \epsilon P_1(t) + \epsilon^2 P_2(t) + \cdots$. Also we can expand A, K in (5.14) in powers of ϵ. These perturbation series allow us to find $P_0, P_1, \ldots, \beta_1, \beta_2, \ldots$ successively by putting $v(t,\epsilon) = P(t,\epsilon)\, e^{\beta(\epsilon)t}$ in (5.14) and solving a perturbation hierarchy. The details are carried out in the Appendix (after obtaining a canonical form for (5.13) to simplify the calculations). We state the results:

$$\beta(\epsilon) \text{ is real},$$

$$\beta_1 = 0,$$

$$\beta_2 = 2 \operatorname{Re} \delta$$

where δ is as in the modulation equations (2.12).

If $\beta_2 > 0$ then the characteristic exponent $\beta(\epsilon)$ is positive. Applying Theorem 1 we deduce that $p(t,\epsilon)$ is <u>unstable</u>. If $\beta_2 < 0$ Theorem 3 implies <u>orbital</u> <u>stability</u>. But in Chapter II we conjectured <u>stability</u> if $\operatorname{Re}\delta < 0$, and instability if $\operatorname{Re}\delta > 0$. Thus we have complete agreement and <u>the modulation equations are completely accurate in predicting the stability of the periodic solution</u>. We thus have a convenient method for finding the stability of the bifurcated periodic solution as opposed to the usually tedious procedure of calculating the characteristic exponents directly.

APPENDIX

Here we give a demonstration of two-timing as it applies to a general system of integro-differential equations in the case of Hopf bifurcation (cf. Chapter II). In order to motivate the methods we first consider the corresponding problem for ordinary differential equations. We have the equation

$$\frac{d\underset{\sim}{N}}{dt} = A(\lambda)\underset{\sim}{N} + \underset{\sim}{f}(\underset{\sim}{N}, \lambda), \tag{A1}$$

where $A(\lambda)$ is an $n \times n$ matrix, and $\underset{\sim}{f}(\underset{\sim}{N}, \lambda)$ contains only quadratic and higher order terms in $\underset{\sim}{N}$. Assuming the same hypotheses as those we presented for Hopf bifurcation in Chapter II, $A(\lambda_0)$ has eigenvalues $\pm i\mu$ with all other eigenvalues having negative real parts. See also Hopf [6]. We may assume (cf. Problem 40, Chapter 3, Coddington and Levinson [1])

$$A(\lambda) = \left[\begin{array}{cc|c} \alpha(\lambda) & \beta(\lambda) & 0 \\ -\beta(\lambda) & \alpha(\lambda) & \\ \hline 0 & & B(\lambda) \end{array}\right],$$

$$\alpha(0) = 0,$$

$$\beta(0) = \mu,$$

where the eigenvalues of B have negative real parts. We see that $N_3(t), N_4(t), \ldots, N_n(t) \to 0$ as $t \to \infty$, for small initial values, so

$\underset{\sim}{N}$ tends asymptotically to a function $\mathrm{col}\,(N_1(t), N_2(t), 0, \ldots, 0)$ where

$$\frac{dN_1}{dt} = \alpha(\lambda) N_1 + \beta(\lambda) N_2 + g_1(N_1, N_2, \lambda),$$

$$\frac{dN_2}{dt} = -\beta(\lambda) N_1 + \alpha(\lambda) N_2 + g_2(N_1, N_2, \lambda).$$

The g_i are obtained from the f_i by putting $N_3 = \cdots = N_n = 0$. We now seek a periodic solution of this system, which would then produce a periodic solution of (A1) (with $N_3 = \cdots = N_n = 0$). This problem is conveniently analyzed by using a complex form of the equations. Put $z(t) = N_1(t) + i N_2(t)$. Then z satisfies

$$\frac{dz}{dt} = \sigma(\lambda) z + g(z, \bar{z}, \lambda), \tag{A2}$$

where $\sigma(\lambda) = \alpha(\lambda) + i\beta(\lambda)$, and $g = g_1 + i g_2$ ($\bar{z}$ denotes the complex conjugate of z). The linearization has eigenvalues $\sigma(\lambda)$, $\bar{\sigma}(\lambda)$ crossing the imaginary axis at $\lambda = \lambda_0$ where $\sigma = i\mu$. Equation (A2) can now be put into a very nice form known as Poincaré normal form. By a nonlinear transformation of variables $\xi = \xi(z, \bar{z})$ with $\xi = z + O(|z|^2)$, we can write (A2) in the form (for any $m \geq 1$)

$$\frac{d\xi}{dt} = \sigma(\lambda)\xi + c_1(\lambda)\xi|\xi|^2 + c_2(\lambda)\xi|\xi|^4 + \cdots + c_m(\lambda)\xi|\xi|^{2m} + O(|\xi|^{2m+2}). \tag{A3}$$

The details can be found in Marsden and McCracken [9], Section 6A. Now we apply two-timing to this form of the equations. Thus we have

$$\xi = \epsilon\,\xi_1 + \epsilon^2\,\xi_2 + \epsilon^3\,\xi_3 + \cdots\,,$$

$$\xi_i = \xi_i\,(t^*, \tau)\,,$$

$$t^* = \rho(\epsilon)\,t\,,$$

$$\tau = \epsilon^2\,t\,,$$

$$\rho(\epsilon) = 1 + \epsilon\rho_1 + \epsilon^2\rho_2 + \cdots,$$

$$\lambda(\epsilon) = \lambda_0 + \epsilon\lambda_1 + \epsilon^2\lambda_2 + \cdots\,.$$

Substituting into (A3), the $O(\epsilon)$ terms give

$$\frac{\partial \xi_1}{\partial t^*} = i\mu\,\xi_1\,,$$

i.e. $\quad \xi_1 = R(\tau)\ \exp i\,[\,\mu t^* + \theta(\tau)] = R\,e^{i\phi}$

with $\quad \phi = \mu t^* + \theta\,.$

The $O(\epsilon^2)$ terms yield

$$\frac{\partial \xi_2}{\partial t^*} - i\mu\,\xi_2 = -\rho_1\,\frac{\partial \xi_1}{\partial t^*} + \sigma'(\lambda_0)\,\lambda_1\,\xi_1$$

$$= R\,e^{i\phi}\,[\,-i\mu\rho_1 + \sigma'(\lambda_0)\,\lambda_1\,]\,.$$

To suppress secular terms we require $-i\mu\rho_1 + \sigma'(\lambda_0)\lambda_1 = 0$, i.e.

$$\begin{cases} \alpha'(\lambda_0)\,\lambda_1 = 0\,, \\ -\mu\rho_1 + \beta'(\lambda_0)\,\lambda_1 = 0\,. \end{cases}$$

By the transversality condition $\mathrm{Re}\,\sigma'(\lambda_0) = \alpha'(\lambda_0) \neq 0$ so $\lambda_1 = \rho_1 = 0$. Then we may take $\xi_2 = 0$ since by the Hopf bifurcation theorem proved in Chapter II we take the $O(\epsilon^2)$ part of $\xi(t)$ to be orthogonal to the two periodic solutions of the linearized problem at λ_0, which here are $e^{i\mu t^*}$ and $i e^{i\mu t^*}$.

The $O(\epsilon^3)$ terms yield

$$\frac{\partial \xi_3}{\partial t^*} - i\mu\,\xi_3 = -\rho_2 \frac{\partial \xi_1}{\partial t^*} - \frac{\partial \xi_1}{\partial T} + \lambda_2\,\sigma'(\lambda_0)\,\xi_1 + c_1(\lambda_0)\,\xi_1|\xi_1|^2$$

$$= e^{i\phi}\left[-i\mu\rho_2 R - \dot{R} - iR\dot{\theta} + \lambda_2\,\sigma'(\lambda_0) R + c_1(\lambda_0) R^3\right]$$

Suppressing secular terms requires that the brackets vanish. Equating real and imaginary parts to zero gives the modulation equations

$$\begin{aligned} \dot{R} &= \lambda_2\,\alpha'(\lambda_0) R + [\mathrm{Re}\,c_1(\lambda_0)]\,R^3, \\ \dot{\theta} &= [-\mu\rho_2 + \lambda_2\,\beta'(\lambda_0)] + [\mathrm{Im}\,c_1(\lambda_0)]\,R^2. \end{aligned} \tag{A4}$$

These equations have been analyzed in Chapter II where it is shown that λ_2 and ρ_2 are determined provided $\alpha'(\lambda_0) \neq 0$ (transversality condition) and $\mathrm{Re}\,c_1(\lambda_0) \neq 0$.

Before proceeding to integro-differential equations we indicate what procedure may be used in the exceptional case that $\mathrm{Re}\,c_1(\lambda_0) = 0$. Here R does <u>not</u> tend to a nonzero constant as $T \to \infty$ or $-\infty$. However, we know that it must since a periodic solution exists by the Hopf bifurcation theorem. What happens is that $\lambda_2 = 0$ and R does not depend on the time scale τ . Thus we introduce a third time scale $\tau_2 = \epsilon^4 t$ $(\tau_1 = \epsilon^2 t)$. Then R depends on τ_2 only, while θ depends on both τ_1 and τ_2 . In fact (A4) gives

$$\frac{\partial \theta}{\partial \tau_1}(\tau_1, \tau_2) = -\mu \rho_2 + [\mathrm{Im}\, c_1(\lambda_0)]\,[R(\tau_2)]^2 ,$$

$$\frac{\partial R}{\partial \tau_1} = 0 .$$

so $\theta = (\mathrm{Im}\, c_1(\lambda_0) R^2 - \mu \rho_2)\tau_1 + \theta^{(2)}(\tau_2)$. Also $\frac{\partial \xi_3}{\partial t^*} - i\mu \xi_3 = 0$ and we may take $\xi_3 = 0$. We assume that the normal form in (A3) has been accomplished for $m \geq 2$ so that $O(\epsilon^4)$ terms in the equation yield

$$\frac{\partial \xi_4}{\partial t^*} - i\mu \xi_4 = -\rho_3 \frac{\partial \xi_1}{\partial t^*} + \sigma'(\lambda_0)\lambda_3 \xi_1 .$$

Suppressing secular terms as before yields $\rho_3 = \lambda_3 = 0$. Also we may take $\xi_4 = 0$. The $O(\epsilon^5)$ terms yield

$$\frac{\partial \xi_5}{\partial t^*} - i\mu \xi_5 = -\rho_4 \frac{\partial \xi_1}{\partial t^*} + \sigma'(\lambda_0)\lambda_4 \xi_1$$
$$- \frac{\partial \xi_1}{\partial \tau_2} + c_2(\lambda_0)\xi_1 |\xi_1|^4 .$$

Suppression of secular terms gives the modulation equations

$$\frac{\partial R}{\partial \tau_2} = \alpha'(\lambda_0)\lambda_4 R + [\mathrm{Re}\, c_2(\lambda_0)]\, R^5 ,$$

$$\frac{\partial \theta}{\partial \tau_2} = [-\mu \rho_4 + \beta'(\lambda_0)\lambda_4] + [\mathrm{Im}\, c_2(\lambda_0)]\, R^4 .$$

If $\mathrm{Re}\, c_2(\lambda_0) \neq 0$, then these and (A4) determine $\lambda_4, \rho_4, \rho_2$ since

$\Theta, \theta^{(2)} \to 0$ as $\tau_1 \to \infty, \tau_2 \to \infty$. Also, $R \longrightarrow$ nonzero constant as $\tau_2 \to \infty$ or $-\infty$ depending on the sign of $Re\, c_2(\lambda_0)$. In case $Re\, c_2(\lambda_0) = 0$ we may repeat the procedure. Thus in the case of Hopf bifurcation for ordinary differential equations the solution near the steady state is of the form

$$\epsilon R \cos(\mu t^* + \Theta) \quad + O(\epsilon^2),$$

where Θ depends on $\tau_1, \tau_2, \dots, \tau_k$, and R depends on τ_k only. ($\tau_\ell = \epsilon^{2\ell} t$). The equation governing R is

$$\frac{dR}{d\tau_k} = \gamma_1 R + \gamma_2 R^{2k+1}$$

for some nonzero constants γ_i.

What happens in case all $Re\, c_j(\lambda_0) = 0$, $j = 1, 2, 3, \dots$? This is the degenerate case where all $\lambda_j = 0$. Thus $\lambda(\epsilon) \equiv \lambda_0$ for all near zero. This means that the family of periodic solutions occurs only for $\lambda = \lambda_0$, so that (A1) for λ_0

$$\frac{d\underset{\sim}{N}}{dt} = A(\lambda_0)\underset{\sim}{N} + \underset{\sim}{f}(\underset{\sim}{N}, \lambda_0)$$

has a one-parameter family of periodic solutions near $\underset{\sim}{N} = 0$ analogous to a center in the phase plane. This is the family that bifurcates according to the Hopf theorem. There are no periodic solutions of small amplitude and period near $\frac{2\pi}{\mu}$ for $\lambda \neq \lambda_0$.

We now return to the integro-differential equations. We wish to perform an analysis of the general case analogous to the one done above for ordinary differential equations. Due to the added complexity of integrals a normal form such as that in (A3) cannot be obtained. Instead we introduce a form for the equations that incorporates the features of (A3) that allow for a simple application of two-timing. The main aspect is that in (A3) the quadratic terms have been eliminated thus allowing ξ_2 to be taken to be zero. Thus the $O(\epsilon^3)$ terms are uncomplicated by ξ_2. We do the next best thing by modifying the quadratic terms so that substitution of the $O(\epsilon)$ part of the solution into the quadratic terms produces zero. Then the $O(\epsilon^2)$ term of the solution may be taken to be zero, so the equations corresponding to $O(\epsilon^3)$ are relatively simple and the modulation equations can be found explicitly. The form we choose for the equations will be used later when the question of stability of the periodic solution is considered. We shall use the notation of Chapter II throughout - see the analysis following (2.3).

Consider

$$\frac{d\underset{\sim}{p}}{dt} = \mathcal{L}(\underset{\sim}{p}, \lambda) + Q_2(\underset{\sim}{p}, \lambda) + Q_3(\underset{\sim}{p}, \lambda) + \cdots \tag{A5}$$

where $\mathcal{L}(\underset{\sim}{p}, \lambda) \equiv L(\lambda)\underset{\sim}{p} + \int_0^\infty K(\lambda, s)\underset{\sim}{p}(t-s)\,ds$ and the Q_i are homogeneous in $\underset{\sim}{p}$ and $K(\lambda) * \underset{\sim}{p}$ of degree i. Put $K(\lambda_0, s) \equiv K(s)$. The linearized problem at $\lambda = \lambda_0$

$$\frac{d\underset{\sim}{p}}{dt} = \mathcal{L}(\underset{\sim}{p}, \lambda_0)$$

has solutions $R\,\underset{\sim}{\Xi}_1(\mu t + \theta)$, $R\,\underset{\sim}{\Xi}_2(\mu t + \theta)$, where

$$\underset{\sim}{X}_1(\mu t) \equiv Re(\underset{\sim}{x}_0 e^{i\mu t}),$$

$$\underset{\sim}{X}_2(\mu t) \equiv Im(\underset{\sim}{x}_0 e^{i\mu t}),$$

and R, θ are arbitrary constants.

It is convenient to introduce an equation auxiliary to (A5)

$$\frac{d\underset{\sim}{q}}{dt} = \mathcal{L}(\underset{\sim}{q}, \lambda) + Q_2(\underset{\sim}{q}, \lambda) + Q_3(\underset{\sim}{q}, \lambda) + \cdots \qquad \text{(A5')}$$

being a copy of (A5) with $\underset{\sim}{q}$ replacing $\underset{\sim}{p}$. The solution of (A5), (A5') is

$$\underset{\sim}{p} = \epsilon R \underset{\sim}{X}_1(\mu t + \theta) + O(\epsilon^2),$$

$$\underset{\sim}{q} = \epsilon R \underset{\sim}{X}_2(\mu t + \theta) + O(\epsilon^2).$$

Define the function

$$Z(\underset{\sim}{p}, \underset{\sim}{q}) \equiv \underset{\sim}{y}_0^T \cdot (\underset{\sim}{p} + i\underset{\sim}{q}).$$

Note that for $p = R\underset{\sim}{X}_1$, $q = R\underset{\sim}{X}_2$ we have

$$Z = Re^{i(\mu t + \theta)} = Re^{i\phi}$$

choosing $\underset{\sim}{y}_0^T \cdot \underset{\sim}{x}_0 = 1$ as we shall always suppose and for convenience

$$\phi = \mu t + \theta.$$

z is a <u>linear</u> combination of components of $\underset{\sim}{p}$ and $\underset{\sim}{q}$. Now we introduce a nonlinear transformation of variables from (p, q) to (x, y) to modify Q_2 as we desire.

$$\underset{\sim}{x} = \underset{\sim}{p} + \sum_{\substack{r+s=2 \\ r,s\geq 0}} \underset{\sim}{b}_{rs} [z(p,q)]^r [\overline{z(p,q)}]^s,$$

(A6)

$$\underset{\sim}{y} = \underset{\sim}{q} + \sum_{\substack{r+s=2 \\ r,s\geq 0}} \underset{\sim}{b}'_{rs} [z(p,q)]^r [\overline{z(p,q)}]^s;$$

and to make $\underset{\sim}{x}, \underset{\sim}{y}$ real we require $\bar{b}_{rs} = b_{sr}$, $\bar{b}'_{rs} = b'_{sr}$. The b's are to be determined. The inverse transformation is

$$\underset{\sim}{p} = \underset{\sim}{x} - \sum b_{rs} [z(x,y)]^r [\overline{z(x,y)}]^s + \cdots,$$

$$\underset{\sim}{q} = \underset{\sim}{y} - \sum b'_{rs} [z(x,y)]^r [\overline{z(x,y)}]^s + \cdots,$$

where the dots $\cdots$ denote cubic and higher terms in $\underset{\sim}{x}$ and $\underset{\sim}{y}$. Now $\underset{\sim}{x}$ and $\underset{\sim}{y}$ satisfy the coupled equations

$$\dot{\underset{\sim}{x}} = \mathcal{L}(\underset{\sim}{x}, \lambda) + \widetilde{Q}_2(\underset{\sim}{x}, \underset{\sim}{y}, \lambda) + \widetilde{Q}_3(\underset{\sim}{x}, \underset{\sim}{y}, \lambda) + \cdots,$$

(A7)

$$\dot{\underset{\sim}{y}} = \mathcal{L}(\underset{\sim}{y}, \lambda) + \widetilde{Q}'_2(\underset{\sim}{x}, \underset{\sim}{y}, \lambda) + \widetilde{Q}'_3(\underset{\sim}{x}, \underset{\sim}{y}, \lambda) + \cdots,$$

with the $\widetilde{Q}_i$ homogeneous of degree i and

$$\begin{aligned}\tilde{Q}_2 = & -\sum_{r+s=2} \left[L(\lambda) + K(\lambda) * \right] Z^r \bar{Z}^s \underset{\sim}{b}_{rs} \\ & + \sum_{r+s=2} r Z^{r-1} \bar{Z}^s \left\{ \underset{\sim}{y}_0^T \cdot \left[L(\lambda) + K(\lambda) * \right] (\underset{\sim}{x} + i\underset{\sim}{y}) \right\} \underset{\sim}{b}_{rs} \\ & + \sum_{r+s=2} s Z^r \bar{Z}^{s-1} \left\{ \bar{\underset{\sim}{y}}_0^T \cdot \left[L(\lambda) + K(\lambda) * \right] (\underset{\sim}{x} - i\underset{\sim}{y}) \right\} \underset{\sim}{b}_{rs} \\ & + Q_2(\underset{\sim}{x}, \lambda) ;\end{aligned} \tag{A8}$$

where $Z = Z(x,y)$ and for $\tilde{Q}_2'$, $b_{rs}' \leftrightarrow b_{rs}$, $Q_2' \leftrightarrow Q_2$. We want $\tilde{Q}_2$, $\tilde{Q}_2'$ to vanish whenever $R\underset{\sim}{X}_1(\phi)$, $R\underset{\sim}{X}_2(\phi)$ are substituted for $\underset{\sim}{x}, \underset{\sim}{y}$ respectively. Evaluating the terms in (A8) for $\lambda = \lambda_0$ and these values of $\underset{\sim}{x}, \underset{\sim}{y}$ gives

$$\begin{aligned}\left[L(\lambda_0) + K(\lambda_0) * \right] Z^r \bar{Z}^s &= R^2 (L + K*) e^{ri\phi} e^{-si\phi} \\ &= R^2 (L + K*) e^{(r-s)i\phi} \\ &= R^2 e^{(r-s)i\phi} \left[(r-s) i\mu I - \Delta(\lambda_0, (r-s)i\mu) \right],\end{aligned}$$

$$\begin{aligned}r Z^{r-1} \bar{Z}^s \underset{\sim}{y}_0^T \cdot [L + K*](\underset{\sim}{x} + i\underset{\sim}{y}) &= R^2 r \, e^{(r-s)i\phi} \, y_0^T \cdot \left[L + \int_0^\infty K(u) e^{-i\mu u} du \right] \underset{\sim}{x}_0 \\ &= R^2 e^{(r-s)i\phi} r \, \underset{\sim}{y}_0^T \left[I - \Delta(\lambda_0, i\mu) \right] \underset{\sim}{x}_0 \\ &= i\mu R^2 e^{(r-s)i\phi} r ,\end{aligned}$$

$$s Z^r \bar{Z}^{s-1} \bar{\underset{\sim}{y}}_0^T \cdot [L + K*](\underset{\sim}{x} - i\underset{\sim}{y}) = -i\mu R^2 e^{(r-s)i\phi} s ;$$

gives

$$\tilde{Q}_2 = R^2 \sum_{r+s=2} \Big\{ \big[\Delta(\lambda_0, (r-s)i\mu) - (r-s)i\mu I \big] \underset{\sim}{b}_{rs} \, e^{(r-s)i\phi}$$

$$+ i\mu r \, \underset{\sim}{b}_{rs} \, e^{(r-s)i\phi} - i\mu s \, \underset{\sim}{b}_{rs} \, e^{(r-s)i\phi} \Big\} + Q_2$$

$$= R^2 \sum_{r+s=2} e^{(r-s)i\phi} \big[\Delta(\lambda_0, (r-s)i\mu) \big] \underset{\sim}{b}_{rs} + Q_2 .$$

Since Q_2 is quadratic it involves only second and zeroth harmonics so can be written $R^2 \sum_{r+s=2} \underset{\sim}{c}_{rs} \, e^{ri\phi} e^{-si\phi}$. Thus (A8) set to zero gives

$$R^2 \sum_{r+s=2} e^{(r-s)i\phi} \big[\Delta(\lambda_0, (r-s)i\mu) \underset{\sim}{b}_{rs} + \underset{\sim}{c}_{rs} \big] = 0$$

i.e.

$$\underset{\sim}{b}_{rs} = - \Delta^{-1}[(r-s)i\mu] \, \underset{\sim}{c}_{rs} ,$$

where Δ^{-1} exists since $(r-s)i\mu = \pm 2i\mu$ or 0 . Similarly b'_{rs} can be found. This completes the transformation.

Thus we have the equations

$$\dot{x} = \mathcal{L}(x, K(\lambda) * x, \lambda) + Q_2(x, y, \lambda) + Q_3 + \cdots ,$$

$$\dot{y} = \mathcal{L}(y, K(\lambda) * y, \lambda) + Q_2'(x, y, \lambda) + Q_3' + \cdots , \tag{A9}$$

where the Q_i, Q_i' are homogeneous of degree i , and Q_2, Q_2' equal zero when

$$\underset{\sim}{x} = R\, \underset{\sim}{X}_1(\mu t + \theta),$$
$$\underset{\sim}{y} = R\, \underset{\sim}{X}_2(\mu t + \theta), \qquad \text{(A10)}$$

for any constants R, θ . This form of the equations is suitable for the application of two-timing.

Put

$$\underset{\sim}{x} = \epsilon \underset{\sim}{x}_1 + \epsilon^2 \underset{\sim}{x}_2 + \epsilon^3 \underset{\sim}{x}_3 + \cdots,$$

$$\underset{\sim}{y} = \epsilon \underset{\sim}{y}_1 + \epsilon^2 \underset{\sim}{y}_2 + \epsilon^3 \underset{\sim}{y}_3 + \cdots,$$

where the x_i, y_i are functions of t^*, T with

$$t^* = \rho(\epsilon)\, t,$$

$$T = \epsilon^2 t,$$

$$\rho(\epsilon) = 1 + \epsilon^2 \rho_2 + \cdots,$$

$$\lambda(\epsilon) = \lambda_0 + \epsilon^2 \lambda_2 + \cdots .$$

Assume Q_3 ,evaluated at $x = R\,X_1(\phi), y = R\,X_2$, yields

$$R^3 [\underset{\sim}{\alpha} \sin\phi + \underset{\sim}{\beta} \cos\phi] \quad + \quad \text{third harmonics.}$$

We use the notation $\phi = \mu t^* + \theta$, $R = R(T)$, $\theta = \theta(T)$. As shown in Chapter II, we have

$$K(\lambda) * \underset{\sim}{x} = \epsilon \underset{\sim}{z}_1 + \epsilon^2 \underset{\sim}{z}_2 + \cdots ,$$

$$K(\lambda) * \underset{\sim}{y} = \epsilon \underset{\sim}{u}_1 + \epsilon^2 \underset{\sim}{u}_2 + \cdots ,$$

where

$$\underset{\sim}{z}_1 = K(\lambda) * \underset{\sim}{x}_1 ,$$

$$\underset{\sim}{z}_2 = K(\lambda) * \underset{\sim}{x}_2 ,$$

$$\underset{\sim}{z}_3 = K(\lambda) * \underset{\sim}{x}_3 - \rho_2 \left(s\, K(\lambda) * \frac{\partial x_1}{\partial t^*} \right) - \left(s\, K(\lambda) * \frac{\partial x_1}{\partial \tau} \right)$$

and $K(\lambda) * x_1 \equiv \int_0^\infty K(\lambda, s)\, x_1(t^* - s, \tau)\, ds$, etc., so the convolution ignores the τ variables. Substituting into (A9) and equating like powers of ϵ gives the following.

The coefficient of ϵ gives

$$\frac{\partial x_1}{\partial t^*} = \mathcal{L}(x_1, z_1, \lambda_0),$$

$$\frac{\partial y_1}{\partial t^*} = \mathcal{L}(y_1, u_1, \lambda_0),$$

which has the solution

$$x_1 = R(\tau)\, X_1(\mu t^* + \theta(\tau)) = R X_1(\phi),$$

$$y_1 = R X_2(\phi).$$

We always ignore the other solutions which decay exponentially.

The coefficient of ϵ^2 yields

$$\frac{\partial x_2}{\partial t^*} - \mathcal{L}(x_2, z_2, \lambda_0) = Q_2 \Big|_{\substack{x_1 = R\mathrm{X}_1 \\ y_1 = R\mathrm{X}_2}} .$$

Now since convolution ignores the T variables all the operations involved in evaluating the right-hand member are formally identical to those involved in evaluating Q_2 in (A9), (A10) since R, θ are treated essentially as constants. Thus the right side above is zero. We may then take $x_2 = 0$ since the Hopf bifurcation theorem in Chapter II allows this. Similarly $y_2 = 0$.

The coefficient of ϵ^3 gives

$$\frac{\partial x_3}{\partial t^*} - \mathcal{L}(x_3, K * x_3, \lambda_0) = -\rho_2 \frac{\partial x_1}{\partial t^*} - \frac{\partial x_1}{\partial T} - \rho_2 \left(s K * \frac{\partial x_1}{\partial t^*}\right)$$

$$-\left(s K * \frac{\partial x_1}{\partial T}\right) + \lambda_2 \left\{ \frac{dL}{d\lambda}(\lambda_0)\, x_1 + \frac{\partial K}{\partial \lambda}(\lambda_0) * x_1 \right\}$$

$$+ R^3 \left[\underset{\sim}{\alpha} \sin\phi + \underset{\sim}{\beta} \cos\phi \right] + \cdots\cdots$$

where the dots denote third harmonics. Notice that no terms from Q_2 arise $x_2 = y_2 = 0$. This equation has been solved in Chapter II and thus we get exactly the same modulation equations (2.12). Thus we have proved the conclusions we asserted for (2.13).

Next we calculate the Floquet exponents associated with the periodic solution. As noted in Chapter V the only important exponents are those which vary near zero. One such is $\beta^{(1)}(\epsilon) \equiv 0$ for all ϵ with corresponding eigenfunction $\frac{d\underset{\sim}{x}}{dt}(t, \epsilon)$ where $\underset{\sim}{x}$ is the periodic solution. The other is $\beta^{(2)}$ where

$$\beta^{(2)} = \beta_1 \epsilon + \beta_2 \epsilon^2 + O(\epsilon^3),$$

with corresponding eigenfunction

$$v = P(t^*, \epsilon)\, e^{\beta^{(2)}(\epsilon) t^*} \tag{A11}$$

where P is $\frac{2\pi}{\mu}$ -periodic (and real). Also, since Floquet exponents occur in conjugate pairs it must be that $\beta^{(2)}$ is real.

Consider equations (A9) and put

$$\underset{\sim}{x}(t) = \underset{\sim}{z}(t^*),$$

$$\underset{\sim}{y}(t) = \underset{\sim}{u}(t^*),$$

$$t^* = \rho(\epsilon)\, t.$$

Since

$$K * x = \int_0^\infty K(s)\, z(t^*-s)\, ds - \epsilon^2 \rho_2 \int_0^\infty s K(s)\, \dot{z}(t^*-s) ds + \ldots$$

$$= K * z - \epsilon^2 \rho_2 (sK * \dot{z}) + O(\epsilon^3),$$

the equations satisfied by z, u are

$$\rho(\epsilon) \frac{dz}{dt^*} = L(\lambda) z + K(\lambda) * z - \epsilon^2 \rho_2 (sK * \dot{z})$$
$$+ Q_2(z, u, \lambda) + Q_3(z, u, \lambda) + O(\epsilon^4), \tag{A12}$$

and the same for u except that $Q_2 \leftrightarrow Q_2'$, etc. These equations have the periodic solution

$$z = \tilde{z}(t^*, \epsilon) = \epsilon X_1(\mu t^*) + O(\epsilon^2),$$

$$u = \tilde{u}(t^*, \epsilon) = \epsilon X_2(\mu t^*) + O(\epsilon^2).$$

Put

$$\phi = \mu t^*$$

and linearize (A12) about these solutions:

$$z = \tilde{z} + v,$$

$$u = \tilde{u} + w,$$

with v, w small perturbations. The linearizations of the Q_i are

$$Q_{iz}(\tilde{z}, \tilde{u}, \lambda)[v] + Q_{iu}(\tilde{z}, \tilde{u}, \lambda)[w],$$

where Q_{iz}, Q_{iu} are Frechét derivatives-linear transformations acting on v, w respectively. Thus the linearization of (A12) is

$$\rho(\epsilon)\mathring{v} = L(\lambda)v + K(\lambda) * v - \epsilon^2\rho_2(sK * \mathring{v})$$
$$+ \mathring{Q}_{2z}[v] + \mathring{Q}_{2u}[w] + \mathring{Q}_{3z}[v] + \mathring{Q}_{3u}[w] + O(\epsilon^4), \qquad \text{(A13)}$$

$$\rho(\epsilon)\,\dot{w} = L(\lambda)\,w + K(\lambda) * w - \epsilon^2 \rho_2\,(sK * \dot{w})$$

$$+\; Q_{2z}^{\prime 0}\,[v] + Q_{2u}^{\prime 0}\,[w] + O(\epsilon^3).$$

We know one solution of these equations to be

$$v = \frac{1}{\epsilon}\,\dot{\tilde{z}}(t^*, \epsilon),$$

$$w = \frac{1}{\epsilon}\,\dot{\tilde{u}}(t^*, \epsilon),$$

i.e.

$$v = \mu\,\dot{X}_1(\phi) + O(\epsilon) = \mu\,Re\,(ie^{i\phi}\underset{\sim}{x}_0) + O(\epsilon),$$

$$w = \mu\,\dot{X}_2(\phi) + O(\epsilon) = \mu\,Im\,(ie^{i\phi}\underset{\sim}{x}_0) + O(\epsilon).$$

This corresponds to the Floquet exponent $\beta^{(1)} \equiv 0$. We now consider the other exponent $\beta^{(2)}$. Let v be as in (A11) and

$$w = P'(t^*, \epsilon)\, e^{\beta^{(2)}(\epsilon)\, t^*}$$

with P' $\frac{2\pi}{\mu}$-periodic. Expanding P and P'

$$P = P_0 + \epsilon P_1 + \epsilon^2 P_2 + \cdots,$$

$$P' = P_0' + \epsilon P_1' + \epsilon^2 P_2' + \cdots,$$

and

$$V = V_0 + \epsilon V_1 + \epsilon^2 V_2 + \cdots,$$

$$W = W_0 + \epsilon W_1 + \epsilon^2 W_2 + \cdots,$$

gives

$$V_0 = P_0,$$

$$V_1 = P_1 + P_0 \beta_1 t^*, \tag{A14}$$

$$V_2 = P_2 + P_1 \beta_1 t^* + P_0 \beta_2 t^* + \tfrac{1}{2} P_0 \beta_1^2 t^{*2}.$$

Similarly for W. Now substitute (A14) and $\lambda = \lambda_0 + \epsilon^2 \lambda_2 + \cdots$ into (A13). The coefficient of ϵ^0 is

$$\dot{V}_0 = \mathcal{L}(V_0, \lambda_0),$$

$$\dot{W}_0 = \mathcal{L}(W_0, \lambda_0).$$

Thus

$$\underset{\sim}{V}_0 = \mathrm{Re}\,(\alpha_1 e^{i\phi} \underset{\sim}{x}_0)$$

$$\underset{\sim}{W}_0 = \mathrm{Im}\,(\alpha_2 e^{i\phi} \underset{\sim}{x}_0)$$

with $\alpha_1, \alpha_2 \in \mathbb{C}$ to be found later.

We digress to evaluate

$$\overset{\circ}{Q}_{2z}\,[v_0 + \epsilon v_1] \;+\; \overset{\circ}{Q}_{2u}\,[w_0 + \epsilon w_1]\,. \tag{A15}$$

Let us assume for the moment that v_0, v_1, w_0, w_1 are linear combinations of $\cos\mu t^*$, $\sin\mu t^*$. For certain of these linear combinations we show that (A15) vanishes. Put

$$\underset{\sim}{f}_1 = R(h,\epsilon)\,\underset{\sim}{X}_1\,(\phi + \theta(h,\epsilon)),$$

$$\underset{\sim}{f}_2 = R(h,\epsilon)\,\underset{\sim}{X}_2\,(\phi + \theta(h,\epsilon)),$$

where

$$R(h,\epsilon) \equiv 1 + hF(\epsilon) = 1 + h(F_0 + \epsilon F_1),$$

$$\theta(h,\epsilon) \equiv h\,G(\epsilon) = h(G_0 + \epsilon G_1)$$

are real constants and h, ϵ are small. We know $Q_2(\underset{\sim}{f}_1, \underset{\sim}{f}_2, \lambda_0) = 0$ for all h, ϵ by construction of Q_2. Also

$$\begin{aligned}\underset{\sim}{f}_1 &= (1 + hF)\,\underset{\sim}{X}_1\,(\phi + hG)\\ &= \underset{\sim}{X}_1(\phi) + h[F\,\underset{\sim}{X}_1(\phi) + G\,\dot{\underset{\sim}{X}}_1(\phi)] + O(h^2),\end{aligned}$$

$$\underset{\sim}{f}_2 = \underset{\sim}{X}_2(\phi) + h[F\,\underset{\sim}{X}_2(\phi) + G\,\dot{\underset{\sim}{X}}_2(\phi)] + O(h^2).$$

Putting

$$\underset{\sim}{V} = F\underset{\sim}{X}_1 + G\,\dot{\underset{\sim}{X}}_1 ,$$

$$W = F\underset{\sim}{X}_2 + G\,\dot{\underset{\sim}{X}}_2 ,$$

we have the linearizations

$$\overset{\circ}{Q}_{2z}[V] + \overset{\circ}{Q}_{2u}[W] =$$

$$= \lim_{h\to 0} \frac{1}{h}\left\{ Q_2(X_1 + hV, X_2 + hW, \lambda_0) - Q_2(X_1, X_2, \lambda_0)\right\}$$

$$= \lim_{h\to 0} \frac{1}{h}\, Q_2(X_1 + hV + O(h^2),\ X_2 + hW + O(h^2), \lambda_0)$$

$$= \lim_{h\to 0} \frac{1}{h}\, Q_2(\underset{\sim}{f}_1, \underset{\sim}{f}_2, \lambda_0) = 0 .$$

Thus we choose $v_0 + \epsilon v_1 = V$, $w_0 + \epsilon w_1 = W$, i.e.

$$v_0 = F_0 X_1 + G_0 \dot{X}_1 ,$$

$$w_0 = F_0 X_2 + G_0 \dot{X}_2 ,$$

$$v_1 = F_1 X_1 + G_1 \dot{X}_1 ,$$

$$w_1 = F_1 X_2 + G_1 \dot{X}_2 .$$

It is clear that in addition to $\overset{\circ}{Q}_{2z}[v_i] + \dot{Q}_{2u}[w_i] = 0$ we have $Q_{2z}'^{\,\circ}[v_i] + Q_{2u}'^{\,\circ}[w_i] = 0$ for $i = 0, 1$.

Returning to the perturbation scheme the ϵ^1 term yields.

$$\dot{V}_1 = \mathcal{L}(v_1, \lambda_0) + Q_{2z}^{\circ}[v_0] + Q_{2u}^{\circ}[w_0]$$

$$= \mathcal{L}(v_1, \lambda_0) ;$$

also

$$\dot{w}_1 = \mathcal{L}(w_1, \lambda_0).$$

Choosing v_1, w_1 as above we see from (A14) that

$$\beta_1 = 0 .$$

The ϵ^2 term yields

$$\rho_2 \dot{v}_0 + \dot{v}_2 = \mathcal{L}(v_2, \lambda_0) - \rho_2 (sK * \dot{v}_0)$$

$$+ \lambda_2 \left[\frac{dL}{d\lambda}(\lambda_0) v_0 + \frac{\partial K}{\partial \lambda}(\lambda_0) * v_0 \right] + Q_{3z}^{\circ}[v_0] + Q_{3u}^{\circ}[w_0]$$

since $\tilde{z}_2 = \tilde{u}_2 = 0$ in the expansion $\tilde{z} = \epsilon \tilde{z}_1 + \epsilon^2 \tilde{z}_2 + \cdots$, $\tilde{u} = \epsilon \tilde{u}_1 + \epsilon^2 \tilde{u}_2 + \cdots$; and by our choice of V_1, W_1 . Thus

$$\dot{v}_2 - \mathcal{L}(v_2, \lambda_0) = -\rho_2 (I + sK*) \dot{v}_0$$

$$+ \lambda_2 \frac{\partial}{\partial \lambda} (L(\lambda) + K(\lambda)*)_0 \, v_0 + Q_{3z}^{\circ}[v_0] + Q_{3u}^{\circ}[w_0] .$$

Now $v_0 = F_0 X_1 + G_0 \dot{X}_1 = Re\{(F_0 + iG_0) e^{i\phi} \underset{\sim}{x}_0\}$, $\dot{v}_0 = \mu \cdot Re\{ i(F_0 + iG_0) e^{i\phi} \underset{\sim}{x}_0 \}$, so

$$\dot{v}_2 - \mathcal{L}(v_2, \lambda_0) = Re\left\{ e^{i\phi}\left[-i\mu \rho_2 (F_0 + iG_0)(\Delta^{(1)} \underset{\sim}{x}_0) \right.\right.$$

$$\left.\left. + \lambda_2 (F_0 + iG_0)\left(\frac{\partial \Delta}{\partial \lambda}(\lambda_0, i\mu)\, \underset{\sim}{x}_0\right)\right]\right\}$$

$$+ \overset{\circ}{Q}_{3z}[v_0] + \overset{\circ}{Q}_{3u}[w_0].$$

Now we have assumed that when $x = X_1(\phi)$, $y = X_2(\phi)$ that Q_3 equals $R^3(\underset{\sim}{\alpha}\sin\phi + \underset{\sim}{\beta}\cos\phi)$ + third harmonics. Thus

$$\overset{\circ}{Q}_{3z}[v_0] + \overset{\circ}{Q}_{3u}[w_0] =$$

$$\lim_{h \to 0} \frac{1}{h}\left\{ R(h,0)^3 (\underset{\sim}{\alpha}\sin(\phi + \theta(h,0)) + \underset{\sim}{\beta}\cos(\phi + \theta(h,0)) \right.$$

$$\left. - \underset{\sim}{\alpha}\sin\phi - \underset{\sim}{\beta}\cos\phi \right\} + 3^{rd}$$

$$= 3R(0,0)\frac{\partial R}{\partial h}(0,0)\,(\underset{\sim}{\alpha}\sin\phi + \underset{\sim}{\beta}\cos\phi)$$

$$+ (\underset{\sim}{\alpha}\cos\phi - \underset{\sim}{\beta}\sin\phi)\,\frac{\partial \theta}{\partial h}(0,0) + 3^{rd}$$

$$= \cos\phi\,[3F_0\underset{\sim}{\beta} + G_0\underset{\sim}{\alpha}]$$

$$+ \sin\phi\,[3F_0\underset{\sim}{\alpha} - G_0\underset{\sim}{\beta}] + 3^{rd}$$

$$= Re\left[(3F_0\underset{\sim}{\beta} + G_0\underset{\sim}{\alpha} - i\cdot 3F_0\underset{\sim}{\alpha} + iG_0\underset{\sim}{\beta})\,e^{i\phi}\right] + 3^{rd}$$

$$= Re\left[(3F_0 + iG_0)(\underset{\sim}{\beta} - i\underset{\sim}{\alpha})\,e^{i\phi}\right] + 3^{rd}.$$

We get

$$\dot{v}_2 - \mathcal{L}(v_2, \lambda_0) = Re\left[e^{i\phi}\left\{ (F_0 + iG_0)\left[-i\mu\rho_2\, \Delta^{(1)} \underset{\sim}{x}_0 \right.\right.\right.$$

$$\left.\left.\left. + \lambda_2 \left(\frac{\partial \Delta}{\partial \lambda}\right)_0 \cdot \underset{\sim}{x}_0\right] + (3F_0 + iG_0)(\underset{\sim}{\beta} - i\underset{\sim}{\alpha})\right\}\right] + 3^{rd}.$$

Then v_2 = periodic terms + $\mathrm{Re}\,(c t^* e^{i\mu t^*} \underset{\sim}{x}_0)$ where (cf. (2.10))

$$c = \frac{1}{y_0^T \Delta^{(1)} x_0} \quad \underset{\sim}{y}_0^T \cdot \Big[(F_0 + i G_0)(-i\mu\rho_2 \Delta^{(1)} \underset{\sim}{x}_0 + \lambda_2 \left(\frac{\partial \Delta}{\partial \lambda}\right)_0 \cdot \underset{\sim}{x}_0$$

$$+ (3F_0 + iG_0)(\underset{\sim}{\beta} - i\underset{\sim}{\alpha}) \Big]$$

$$= (F_0 + iG_0)(-i\mu\rho_2 + \lambda_2 \, \sigma'(\lambda_0)) + \delta \, (3F_0 + iG_0),$$

where $\delta = \dfrac{y_0^T \cdot (\beta - i\alpha)}{y_0^T \cdot \Delta^{(1)} x_0}$. By Hopf bifurcation

$$\begin{cases} \lambda_2 \, \mathrm{Re}\, \sigma'(\lambda_0) + \mathrm{Re}\,\delta = 0, \\ -\mu\rho_2 + \lambda_2 \, \mathrm{Im}\, \sigma'(\lambda_0) + \mathrm{Im}\,\delta = 0; \end{cases}$$

so

$$- i\mu\rho_2 + \lambda_2 \, \sigma'(\lambda_0) + \delta = 0.$$

Thus c simplifies to

$$c = 2 F_0 \, \delta. \tag{A16}$$

Now from (A14) we want

$$t^* \, \mathrm{Re}\,(c \underset{\sim}{x}_0 e^{i\phi}) = \beta_2 t^* v_0(t^*)$$

$$= \beta_2 t^* \, \mathrm{Re}\,\{ (F_0 + iG_0) \underset{\sim}{x}_0 e^{i\phi} \},$$

so

$$c = \beta_2 (F_0 + i G_0) .$$

Thus (A16) implies

$$2 F_0 \delta = \beta_2 (F_0 + i G_0).$$

Taking real and imaginary parts yields

$$\begin{cases} 2 F_0 \, Re\, \delta = \beta_2 F_0 , \\ 2 F_0 \, Im\, \delta = \beta_2 G_0 . \end{cases}$$

One solution is $\beta_2 = 0$, $F_0 = 0$, $G_0 = 1$. This corresponds to $\beta^{(1)}(\epsilon) \equiv 0$ and $V_0 = \dot{X}_1(\phi)$ as noted previously. The other solution is $F_0 \neq 0$, $G_0/F_0 = Im\,\delta / Re\,\delta$ and

$$\beta_2 = 2 \, Re\, \delta \tag{A17}$$

and represents the desired exponent. This completes the calculations and we have the second Floquet exponent

$$\beta^{(2)} = 2 \epsilon^2 \, Re\, \delta + O(\epsilon^3) .$$

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