THE STOCHASTIC EXIT PROBLEM
FOR DYNAMICAL SYSTEMS

Thesis by

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In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

California Institute of Technology
Pasadena, California

(1978)

(Submitted June 22, 1977)

ACKNOWLEDGMENTS

I am deeply indebted to Professor Joel N. Franklin for his assistance and patience with this work. Professor Franklin has proven to be a source of faith and kind words, most necessary commodities at times, so my debt to him is one which can never really be repaid.

I with to thank the faculty, fellow graduate students, and the secretaries of the Applied Mathematics Department. In particular, Dr. Benjamin S. White is deserving of special thanks. The problems and methods which he suggested are displayed in this work; the amount of time we spent in fruitful discussion can only be acknowledged. Mrs. Vivian Davies has done a masterful job of typing a most difficult manuscript.

Financial support for my studies at Caltech was provided in part by Graduate Teaching Assistantships and a Graduate Research Assistantship.

Finally, I dedicate this thesis to my wife, Susan, who has borne the brunt of my frustrations quietly and cheerfully.

ABSTRACT

The problem of "exit against a flow" for dynamical systems subject to small Gaussian white noise excitation is studied. Here the word "flow" refers to the behavior in phase space of the unperturbed system's state variables. "Exit against a flow" occurs if a perturbation causes the phase point to leave a phase space region within which it would normally be confined. In particular, there are two components of the problem of exit against a flow:

i) the mean exit time

ii) the phase-space distribution of exit locations.

When the noise perturbing the dynamical systems is small, the solution of each component of the problem of exit against a flow is, in general, the solution of a singularly perturbed, degenerate elliptic-parabolic boundary value problem.

Singular perturbation techniques are used to express the asymptotic solution in terms of an unknown parameter. The unknown parameter is determined using the solution of the adjoint boundary value problem.

The problem of exit against a flow for several dynamical systems of physical interest is considered, and the mean exit times and distributions of exit positions are calculated. The systems are then simulated numerically, using Monte Carlo techniques, in order to determine the validity of the asymptotic solutions.

INTRODUCTION

It is known that dynamical systems, even asymptotically stable systems, will exit from any bounded domain in phase space if they are perturbed with white noise for a suitably long period of time. It is the purpose of this thesis to study this problem of exit for asymptotically stable dynamical systems which are forced with small Gaussian white noise in order to determine the mean exit time and the distribution of exit positions. To this end, the first chapter consists of the mathematical formulation of the appropriate boundary value problems.

In the second chapter, we demonstrate that regular perturbation techniques are inapplicable to the boundary value problems. We use singular perturbation techniques to generate uniformly valid, asymptotic solutions to the boundary value problems in terms of an unknown parameter which we are unable to determine using singular perturbation principles. Instead, we apply methods suggested by Matkowsky and Schuss to determine the unknown parameter. We modify the technique of Matkowsky and Schuss in order to predict the mean exit time and the distribution of exit positions from an asymptotically stable limit cycle as well as asymptotically stable equilibrium points.

A comparison of our results with the results of other authors is made in the third chapter. The theoretical results of Ventsel' and Freidlin are studied as are the results of Matkowsky and Schuss.

The asymptotic results of the mean exit time problem, calculated using the results of Miller and Ludwig, are also compared with our results from the second chapter.

The fourth chapter is devoted to a study of the distribution of exit positions for various dynamical systems. We demonstrate that the asymptotic results of the second chapter agree with the asymptotic approximation of the exact solution in the case of the Ornstein-Uhlenbeck process. We study the asymptotic distribution of exit positions for two problems of physical interest: a damped linear harmonic oscillator and a damped pendulum. We conclude the chapter with a study of the asymptotic distribution of exit positions for a dynamical system with a limit cycle.

We study the mean exit time for these same four dynamical systems in the fifth chapter. We show that the asymptotic results of the second chapter agree asymptotically with the exact solution of the mean exit time problem for the Ornstein-Uhlenbeck process. We calculate the mean exit times for a damped linear harmonic oscillator and a damped pendulum, and compare the results. Then the mean exit time for a process with a limit cycle is determined.

In order to answer how small a small parameter must be for various dynamical systems, we present the results of Monte Carlo simulations in the sixth chapter. We test the hypothesis that a dynamical system will exit at the most probable point on the boundary as the noise parameter becomes small. We use the simulations to study the mean exit time and the distribution of exit positions for the damped, linear harmonic oscillator, the damped pendulum, and a system with a limit cycle. We conclude the chapter with a discussion of the possible sources of discrepancy between the simulated results and the predicted results.

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CHAPTER I

The aim of this chapter is to review known results for infinitesimal operators of Markov processes. We begin by examining a process whose behavior is governed by a stochastic differential equation and then we derive the infinitesimal generator for the process. We shall conclude the chapter by giving interpretations to two different boundary value problems associated with the infinitesimal generator.

1.1 Infinitesimal Generators for Markov Processes

We wish to study the effect of perturbing dynamical systems with Gaussian white noise. Let Ω be a bounded domain in $\mathbb{R}^n$ whose boundary $\partial\Omega$ is smooth. Let $\underline{x}(t) \equiv \mathrm{col}\left(x_1(t), x_2(t), \cdots, x_n(t)\right)$ represent the behavior in time of some dynamical system or process. Let $\underline{b}\left(\underline{x}(t)\right) \equiv \mathrm{col}\left(b^1(\underline{x}(t)),\ b^2(\underline{x}(t)),\ \cdots,\ b^n(\underline{x}(t))\right)$ be a bounded, smooth vector field in $\overline{\Omega}$. We now consider a dynamical system, or process, governed by the differential equation

$$\frac{d}{dt}\,\underline{x}(t) = \underline{b}(\underline{x}(t)) \tag{1.1.1}$$

It is often more convenient to consider the differential form of (1.1.1):

$$d\,\underline{x}(t) = \underline{b}\,(\underline{x}(t))dt \tag{1.1.2}$$

If the deterministic system (1.1.2) is perturbed by Gaussian white noise, the resulting motion $\underline{x}_{\varepsilon}(t)$ satisfies

$$d\underline{x}_\varepsilon(t) = \underline{b}(\underline{x}_\varepsilon(t))dt + \varepsilon\sigma(\underline{x}_\varepsilon(t))d\underline{w}(t) \tag{1.1.3}$$

where $\sigma(\underline{x}_\varepsilon(t))$ is the diffusion matrix, $\underline{w}(t)$ is an n-dimensional Wiener process (brownian motion), and ε is a small, real parameter.

We observe that (1.1.3) is the form for which Ito's Lemma for a stochastic calculus is most useful. We also note from (1.1.3) that if we know $\underline{x}_\varepsilon(t)$, then we do not need $\underline{x}_\varepsilon(s)$, $s < t$, in order to calculate $\underline{x}_\varepsilon(t+\tau)$, $\tau > 0$. Thus $\underline{x}_\varepsilon(t)$ is a Markov process.

We consider Markov processes whose transition probabilities

$$p(t,\underline{\xi};s,\underline{x})d\underline{\xi} \equiv \Pr\left\{\underline{x}_\varepsilon(t)\in(\underline{\xi},\underline{\xi}+d\underline{\xi})\,\middle|\,\underline{x}_\varepsilon(s) = \underline{x}\right\}$$

satisfy the following conditions: for $\delta > 0$,

$$\left.\begin{aligned}
&\lim_{\Delta t\downarrow 0}\ \underset{\|\underline{\xi}-\underline{x}\|\le\delta}{\int\cdots\int} p(s+\Delta t,\underline{\xi};s,\underline{x})(\xi_i - x_i)d\underline{\xi} = b_i(\underline{x})\Delta t + o(\Delta t) \qquad 1\le i\le n\\
&\lim_{\Delta t\downarrow 0}\ \underset{\|\underline{\xi}-\underline{x}\|\le\delta}{\int\cdots\int} P(s+\Delta t,\underline{\xi};s,\underline{x})(\xi_i - x_i)(\xi_j - x_j)d\underline{\xi} = \varepsilon^2\sum_{k=1}^{n}\sigma_{ik}(\underline{x})\sigma_{kj}(\underline{x})\Delta t + o(\Delta t)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1\le i,j\le n\\
&\lim_{\Delta t\downarrow 0}\ \underset{\|\underline{\xi}-\underline{x}\|>\delta}{\int\cdots\int} p(s+\Delta t,\underline{\xi};s,\underline{x})d\underline{\xi} = o(\Delta t)
\end{aligned}\right\} \tag{1.1.4}$$

Notice that the vector $\underline{b}(\underline{x}_\varepsilon(t))$ characterizes the average trend of evolution of the random process $\underline{x}_\varepsilon(t)$ in a small increment of time from s to $s+\Delta t$, subject to $\underline{x}_\varepsilon(s) = x$, and is called the drift coefficient.

We now invoke Ito's Lemma for the n-dimensional Markov process $\underline{x}_\varepsilon(t)$:

Let $$f : \mathbb{R}^n \rightarrow \mathbb{R}^1$$

Let $$z(t) = f(\underline{x}_\varepsilon(t)).$$

Expand z(t) in a Taylor series, retaining the first two terms:

$$z(t+dt) = z(t) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_\varepsilon^i} dx_\varepsilon^i(t) + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_\varepsilon^i \partial x_\varepsilon^j} dx_\varepsilon^i(t) dx_\varepsilon^j(t) \qquad (1.1.5)$$

Thus we see that

$$dz(t) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_\varepsilon^i} \left[b_i(\underline{x}_\varepsilon(t))dt + \varepsilon \sum_{j=1}^{n} \sigma_{ij}(\underline{x}_\varepsilon(t))dw_j(t) \right]$$

$$+ \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_\varepsilon^i \partial x_\varepsilon^j} \left[b_i(\underline{x}_\varepsilon(t))dt + \varepsilon \sum_{k=1}^{n} \sigma_{ik}(\underline{x}_\varepsilon(t))dw_k(t) \right] \cdot$$

$$\cdot \left[b_j(\underline{x}_\varepsilon(t))dt + \varepsilon \sum_{\ell=1}^{n} \sigma_{j\ell}(\underline{x}_\varepsilon(t))dw_\ell(t) \right]$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x_\varepsilon^i} \left[b_i(\underline{x}_\varepsilon(t))dt + \varepsilon \sum_{j=1}^{n} \sigma_{ij}(\underline{x}_\varepsilon(t))dw_j(t) \right]$$

$$+ \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_\varepsilon^i \partial x_\varepsilon^j} \Big[b_i(\underline{x}_\varepsilon(t))b_j(\underline{x}_\varepsilon(t)\ (dt)^2$$

$$+ \varepsilon\, b_i(\underline{x}_\varepsilon(t)) \sum_{\ell=1}^{n} \sigma_{j\ell}(\underline{x}_\varepsilon(t))dw_\ell(t)\, dt$$

$$+ \varepsilon\, b_j(\underline{x}_\varepsilon(t) \sum_{k=1}^{n} \sigma_{ik}(\underline{x}_\varepsilon(t))dw_k(t)\, dt$$

$$+ \varepsilon^2 \sum_{k,\ell=1}^{n} \sigma_{ik}(\underline{x}_\varepsilon(t))\, \sigma_{j\ell}(\underline{x}_\varepsilon(t))dw_k(t)\, dw_\ell(t) \Big] \qquad (1.1.6)$$

When we apply Ito's multiplication table for the infinitesimals of the n-dimensional Markov process $\underline{x}_\varepsilon(t)$:

	dt	$dw_1(t)$	$dw_2(t)$	$\cdots$	$dw_n(t)$
dt	0	0	0	$\cdots$	0
$dw_1(t)$	0	dt	0		0
$dw_2(t)$	0	0	dt		0
$\vdots$	$\vdots$			$\ddots$	0
$dw_n(t)$	0	0	0		dt

Figure 1.1.1

we find that

$$dz(t) = \left\{ \frac{\varepsilon^2}{2} \sum_{i,j=1}^{n} a_{ij}\left(\underline{x}_\varepsilon(t)\right) \frac{\partial^2 f}{\partial x_\varepsilon^i \partial x_\varepsilon^j} + \sum_{i=1}^{n} b_i(\underline{x}_\varepsilon(t)) \frac{\partial f}{\partial x_\varepsilon^i} \right\} dt$$

$$+ \varepsilon \sum_{i,j=1}^{n} \sigma_{ij}(\underline{x}_\varepsilon(t)) \frac{\partial f}{\partial x_\varepsilon^i} dw_j(t) \qquad (1.1.7)$$

where $\left(a^{ij}(\underline{x}_\varepsilon(t))\right) = a(\underline{x}_\varepsilon(t)) = \sigma(\underline{x}_\varepsilon(t))\, \sigma^T(\underline{x}_\varepsilon(t))$. A proof of Ito's Lemma can be found in McKean [13].

We can write this more compactly as

$$dz(t) = (Af)(\underline{x}_\varepsilon(t))dt + \varepsilon \sum_{i,j=1}^{n} \sigma_{ij}(\underline{x}_\varepsilon(t)) \frac{\partial f}{\partial x_\varepsilon^i} dw_j(t) \qquad (1.1.8)$$

We call the operator A the infinitesimal generator for the Markov process $\underline{x}_\varepsilon(t)$.

1.2 An Interpretation of Problems Involving Infinitesimal Generators for Markov Processes

If we know $\underline{x}_\varepsilon(t)$ for some time t_0, we define a bounded Markov time τ for the process $\underline{x}_\varepsilon(t)$ to be a bounded time for an event which is independent of $\underline{x}_\varepsilon(s)$ for $s < t_0$. In particular, let τ be the time at which the process $\underline{x}_\varepsilon(t)$ first reaches the boundary of Ω, $\partial\Omega$, provided that $\underline{x}_\varepsilon(0) \in \overline{\Omega}$. We define $\tau \equiv 0$ if $\underline{x}_\varepsilon(0) \in \partial\Omega$. We integrate (1.1.8) between 0 and τ to find

$$\int_0^\tau dz(t) = z(\tau) - z(0)$$

$$= f(\underline{x}_\varepsilon(\tau)) - f(\underline{x}_\varepsilon(0))$$

$$= \int_0^\tau (Af)(\underline{x}_\varepsilon(t))dt + \varepsilon \int_0^\tau \sum_{i,j=1}^{n} \sigma_{ij}(\underline{x}_\varepsilon(t)) \frac{\partial f}{\partial x_\varepsilon^i} dw_j(t) \qquad (1.2.1)$$

Let $E_{\underline{x}}[\cdot] \equiv E[\cdot \mid \underline{x}_\varepsilon(0) = x]$ be the conditional expectation given that the process $\underline{x}_\varepsilon(t)$ begins at the position $\underline{x} \in \overline{\Omega}$. If we apply the operator $E_{\underline{x}}[\cdot]$ to both sides of (1.2.1) we find

$$E_{\underline{x}}\left[f(\underline{x}_\varepsilon(\tau)) - f(\underline{x}_\varepsilon(0))\right] =$$

$$E_{\underline{x}}\left[\int_0^\tau (Af)(\underline{x}_\varepsilon(t))dt + \varepsilon \int_0^\tau \sum_{i,j=1}^{n} \sigma_{ij}(\underline{x}_\varepsilon(t) \frac{\partial f}{\partial x_\varepsilon^i} dw_j(t)\right] \qquad (1.2.2)$$

Since $\underline{w}(t)$ is a Wiener process, the second term on the right hand side of (1.2.2) vanishes and we find

$$E_{\underline{x}}\left[f(\underline{x}_{\varepsilon}(\tau)) - \int_0^{\tau} (Af)(\underline{x}_{\varepsilon}(t))dt\right] = f(\underline{x}) \qquad (1.2.3)$$

This is Dynkin's formula for the n-dimensional Markov process, $\underline{x}_{\varepsilon}(t)$.

We now note that

$$u(\underline{x}) \equiv E_{\underline{x}}\left[f(\underline{x}_{\varepsilon}(\tau))\right]$$

solves the boundary value problem

$$\begin{aligned} Au(\underline{x}) &= 0 \qquad & \underline{x} \in \Omega \\ u(\underline{x}) &= f(\underline{x}) \qquad & \underline{x} \in \partial\Omega \end{aligned} \qquad (1.2.4)$$

To see this, we observe that due to the definition of the Markov time τ, $\underline{x}_{\varepsilon}(t) \in \Omega$ for $t < \tau$. Suppose we can find a function $u(\underline{x})$ which satisfies (1.2.4). Then Dynkin's formula (1.2.3) becomes

$$u(\underline{x}) = E_{\underline{x}}\left[u(\underline{x}_{\varepsilon}(\tau))\right] \qquad (1.2.5)$$

Since the boundary condition in (1.2.4) states $u(\underline{x}_{\varepsilon}(\tau)) = f(\underline{x}_{\varepsilon}(\tau))$, the result follows.

Thus $u(\underline{x})$ represents the conditional expectation of an arbitrary function f of the exit position of the Markov process $\underline{x}_{\varepsilon}(t)$ from Ω. Since $u(\underline{x})$ can be expressed as the integral around

$\partial\Omega$ of a certain kernel multiplied by the function $f(\underline{x}_\varepsilon(\tau))$, we find that the kernel represents the probability distribution of the exit position on $\partial\Omega$ of $\underline{x}_\varepsilon(t)$. For purposes of calculations, we generally assume that the function f is a smooth (C^∞) function of the boundary values $\underline{x}_\varepsilon(\tau)$.

A second problem associated with the process $\underline{x}_\varepsilon(t)$, the mean exit time problem, can be formulated in the following manner: Suppose we can find a solution $v(\underline{x})$ of the boundary value problem

$$\begin{aligned} Av(\underline{x}) &= -1 \qquad \underline{x} \in \Omega \\ v(\underline{x}) &= 0 \qquad \underline{x} \in \partial\Omega \end{aligned} \tag{1.2.6}$$

When we apply Dynkin's formula (1.2.3) we see that

$$\begin{aligned} v(\underline{x}) &= E_{\underline{x}}\left[v(\underline{x}_\varepsilon(\tau)) - \int_0^\tau Av(\underline{x}_\varepsilon(t))dt\right] \\ &= E_{\underline{x}}\left[\int_0^\tau dt\right] \\ &= E_{\underline{x}}\left[\tau\right]. \end{aligned} \tag{1.2.7}$$

We see that the solution $v(\underline{x})$ of 1.2.6) represents the conditional expectation of the exit time τ of the Markov process $\underline{x}_\varepsilon(t)$ from Ω.

CHAPTER II

The purpose of this chapter is to study the asymptotic behavior of the mean exit time and the probability distribution of exit positions of the Markov process $\underline{x}_\varepsilon(t)$ from Ω. We are primarily interested in studying the solution of these problems for small ε in the case of diffusion against a flow. Singular perturbation techniques are used to demonstrate the existence of boundary layers and a method for determining unknown constants which appear in the solution is developed.

2.1 Introduction

We now consider the problem of exit. Due to the presence of noise, the Markov process $\underline{x}_\varepsilon(t)$ does not follow a trajectory which is known a priori because there is diffusion present. When the parameter ε is small, note that the infinitesimal generator A becomes a singularly perturbed differential operator. Then Ventsel' and Freidlin [15] tell us that there are three distinctly different types of diffusion problems to consider:

a) diffusion along a flow

b) diffusion across a flow

c) diffusion against a flow

For diffusions of type (a), trajectories given by the deterministic equation (1.1.1) exit from Ω (see Fig. 2.1.1a)

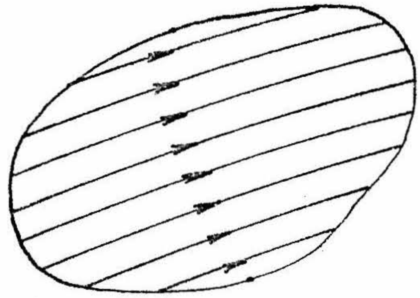


Figure 2.1.1a

Results for the singularly perturbed boundary value problems of this type were first given by N. Levinson [9] in 1950.

For diffusions of type (b), trajectories given by the deterministic equation (1.1.1) do not exit from Ω. A particular example is the case where the trajectories are concentric circles (see Fig. 2.1.1b).

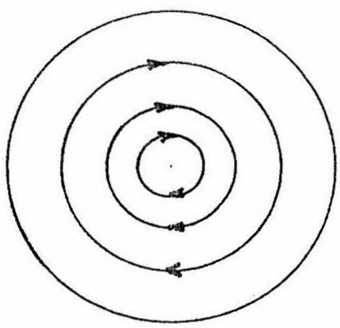


Figure 2.1.1b

In general, the critical point for (1.1.1), a center, will lie in Ω. Khasminskii [6] was the first to use singular perturbation techniques to compute the probability distribution of the exit points for this type of diffusion.

Diffusions of type (c) also have trajectories which do not exit from Ω (see Figure 2.1.1c).

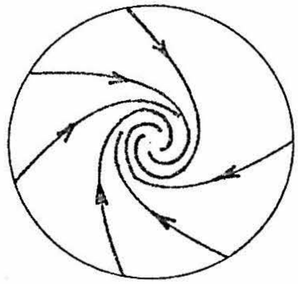


Figure 2.1.1c

The difference between type(b) and type(c) is that the trajectories of the latter type pass from the boundary to some limit set (limit cycle, focal point, star point, etc.) which is asymptotically stable in the absence of noise. The problem for small ε has been studied by Ventsel' and Freidlin [15], [16], Ludwig [10], Matkowsky and Schuss [11], [12] and others. We are concerned with the study of diffusion problems of this type.

In the second, third, and fourth sections, the singularly perturbed nature of the solution will be examined for general problems of diffusions of type (c), and a method will be described to determine the solution of the boundary value problem.

2.2 Outer Solution of the Singularly Perturbed Boundary Value Problem

Consider the process $\underline{x}_\varepsilon(t)$ which is governed by (1.1.3). Let $\underline{\nu}(\underline{x}) \equiv \mathrm{col}\left(\nu_1(\underline{x}),\ \nu_2(\underline{x}), \cdots, \ \nu_n(\underline{x})\right)$ denote the outer normal vector to $\partial\Omega$. We require that all trajectories of the deterministic system (1.1.1) converge to a limit set as time t increases and that the drift vector $\underline{b}(\underline{x})$ satisfy

$$\underline{b}(\underline{x}) \cdot \underline{\nu}(\underline{x}) = \sum b^i(\underline{x})\, \nu_i(\underline{x}) \leq 0 \qquad \underline{x} \in \partial\Omega \qquad (2.2.1)$$

This is the requirement that ensures that the diffusion problem (1.1.3) is of type (c).

We wish to apply perturbation techniques to the study of the boundary value problem

$$\begin{aligned} Au(\underline{x}) &= g(\underline{x}) \qquad && \underline{x} \in \Omega \\ u(\underline{x}) &= f(\underline{x}) \qquad && \underline{x} \in \partial\Omega \end{aligned} \qquad (2.2.2)$$

in the case where the parameter ε tends to zero. The solution of this boundary value problem can represent either the mean exit time of the Markov process $\underline{x}_\varepsilon(t)$ from Ω or it can represent the probability distribution of exit positions; in the former case, we set $g(\underline{x}) = -1$ and $f(\underline{x}) = 0$, and in the latter case, we set $g(\underline{x}) = 0$ and $f(\underline{x})$ to be an arbitrary smooth function. We require that the solution of the mean exit time problem be nonnegative in $\overline{\Omega}$ and the solution of the problem of the probability distribution of exit positions be the integral of $f(\underline{x})$ multiplied by a nonnegative kernel which can be suitably normalized.

In accordance with regular perturbation theory, we begin by assuming that we can represent the solution $u(\underline{x};\varepsilon)$ and $f(\underline{x};\varepsilon)$ as power series in ε:

$$u(\underline{x};\varepsilon) \sim u^{(0)}(\underline{x}) + \varepsilon^2 u^{(1)}(\underline{x}) + \varepsilon^4 u^{(2)}(\underline{x}) + \cdots$$

$$f(\underline{x};\varepsilon) \sim f^{(0)}(\underline{x}) + \varepsilon^2 f^{(1)}(\underline{x}) + \varepsilon^4 f^{(2)}(\underline{x}) + \cdots \qquad (2.2.3)$$

Substitute this ansatz into (2.2.2), equate the coefficients of the various powers of ε^2. We find

$$\left.\begin{aligned} \sum_{i=1}^{n} b^i(\underline{x}) \frac{\partial}{\partial x_i} u^{(0)}(\underline{x}) &= g(\underline{x}) && \underline{x} \in \Omega \\ u^{(0)}(\underline{x}) &= f^{(0)}(\underline{x}) && \underline{x} \in \partial\Omega \end{aligned}\right\} \qquad (2.2.4a)$$

$$\left.\begin{aligned} \sum_{i=1}^{n} b^i(\underline{x}) \frac{\partial}{\partial x_i} u^{(k)}(\underline{x}) &= -\frac{1}{2} \sum_{i,j=1}^{n} a^{ij}(\underline{x}) \frac{\partial^2}{\partial x_i \partial x_j} u^{(k-1)}(\underline{x}) && \underline{x} \in \Omega \\ u^{(k)}(\underline{x}) &= f^{(k)}(\underline{x}) && \underline{x} \in \partial\Omega \\ k &= 1, 2, 3, \cdots \end{aligned}\right\} \qquad (2.2.4b)$$

We see that the components of the drift vector $\underline{b}(\underline{x})$, the sub-characteristics of the problem, determine the leading order asymptotic behavior of the solution $u(\underline{x};\varepsilon)$. Since (2.2.4a) is a linear, first order, partial differential equation, we solve it using the method of characteristics. We introduce characteristic curves given by

$$\frac{d}{dt}\underline{x}(t) = \underline{b}(\underline{x}(t)) \,. \qquad (2.2.5)$$

As an initial condition, we set $\underline{x}(0) \epsilon \partial\Omega$. Along these characteristic curves, we note that

$$\frac{d}{dt}\, u^{(0)}(\underline{x}(t)) = g(\underline{x}(t)) \tag{2.2.6}$$

At this point, we consider the problems of the mean exit time and the probability distribution of exit positions separately.

Problem 1: Probability distribution of exit positions

In this problem, $g(\underline{x}(t)) \equiv 0$. Thus $u^{(0)}(\underline{x}(t))$ remains constant along the subcharacteristics. The value of $u^{(0)}(\underline{x}(t))$ is of course determined by where the subcharacteristic crosses the boundary. However, as the parameter t increases, all subcharacteristics converge to a limit set. At the limit set, the values of $u^{(0)}(\underline{x}\ (t))$ from all subcharacteristics must be identical. We are forced to conclude that we cannot satisfy the boundary conditions and the consistency conditions at the limit set simultaneously. We assume then that the solution $u^{(0)}(\underline{x}(t))$ is composed of two parts: an "outer solution" which is valid in most of Ω and an "inner solution" which is valid near the boundary $\partial\Omega$. Thus we have a singularly perturbed boundary value problem. For the "outer solution," we take $u_{outer}(\underline{x}(t))$ to be an unknown constant. We shall determine the "inner solution," or boundary layer correction in the next section.

Problem 2: Mean exit time

For this problem, we set $g(\underline{x}(t)) = -1$ and $f(\underline{x}) = 0$. Then

$u^{(0)}(\underline{x}(t))$ decreases monotonically along the subcharacteristics. But $u^{(0)}(\underline{x}(0))$ vanishes, so we see that the solution we have generated violates the physical requirements of the solution. If we assume

$$u(\underline{x};\varepsilon) \sim \varepsilon^{-2p}u^{(-p)}(\underline{x}) + \varepsilon^{-2p+2}u^{(-p+1)}(\underline{x}) + \varepsilon^{-2p+4}u^{(-p+2)}(\underline{x}) + \cdots \quad (2.2.7)$$

for some positive integer p, substitute this ansatz into (2.2.2), and equate the coefficients of the various powers of ε^2, we find

$$\left.\begin{aligned} &\sum_{i=1}^{n} b_i(\underline{x}) \frac{\partial}{\partial x_i} u^{(-p)}(\underline{x}) = 0 && \underline{x}\in\Omega \\ &u^{(-p)}(\underline{x}) = 0 && \underline{x}\in\partial\Omega \end{aligned}\right\} \quad (2.2.8a)$$

$$\left.\begin{aligned} &\sum_{i=1}^{n} b_i(\underline{x}) \frac{\partial}{\partial x_i} u^{(k)}(\underline{x}) = -\delta_{k,0} - \frac{1}{2}\sum_{i,j=1}^{n} a^{ij}(\underline{x}) \frac{\partial^2}{\partial x_i \partial x_j} u^{(k-1)}(\underline{x}) && \underline{x}\in\Omega \\ &u^{(k)}(\underline{x}) = 0 && \underline{x}\in\partial\Omega \\ &k = -p+1,\ -p+2,\ -p+3,\ \cdots \end{aligned}\right\} \quad (2.2.8b)$$

where $\delta_{k,0}$ is the Kronecker delta. When we solve (2.2.8a), we find that $u^{(-p)}(\underline{x})$ is constant along the subcharacteristics. Since $u^{(-p)}(\underline{x})$ vanishes on $\partial\Omega$, we conclude that $u^{(-p)}(\underline{x})$ vanishes identically. Since p is an arbitrary positive integer, we are forced to conclude that there is no uniformly valid solution of (2.2.2) in the form of (2.2.7) for any positive integer p. This conclusion, combined with the observation that the solution must grow unbounded for all points in Ω in the absence of noise since deterministic system will always remain in Ω, suggests that the solution $u(\underline{x};\varepsilon)$ might be transcendentally large compared with any finite power of ε as ε tends to zero in some portion of Ω. This immediately suggests that there should be a boundary layer somewhere in Ω, because the solution would be transcendentally large in some portion of Ω and would vanish on the boundary $\partial\Omega$. Again, the problem is a singularly perturbed boundary value problem.

In order to test this hypothesis, we rescale $u(\underline{x};\varepsilon)$ as

$$u(\underline{x};\varepsilon) = C(\varepsilon)\, v(\underline{x};\varepsilon) \qquad (2.2.9)$$

where $C(\varepsilon)$ is transcendentally large compared with any finite power of ε as ε tends to zero and $v(\underline{x};\varepsilon)$ remains bounded as ε tends to zero. In addition, we assume that $v(\underline{x};\varepsilon)$ can be expanded as a power series in ε

$$v(\underline{x};\varepsilon) \sim v^{(0)}(\underline{x}) + \varepsilon^2 v^{(1)}(\underline{x}) + \varepsilon^4 v^{(2)}(\underline{x}) + \cdots \qquad (2.2.10)$$

Substitute the ansatz for $u(\underline{x};\varepsilon)$ into (2.2.2) and divide by the scaling factor $C(\varepsilon)$ to find

$$Av(\underline{x};\varepsilon) = \text{T.S.T.} \qquad \underline{x}\in\Omega$$

$$v(\underline{x};\varepsilon) = 0 \qquad \underline{x}\in\partial\,\Omega \qquad (2.2.11)$$

Substitute the ansatz (2.2.10) into (2.2.11) and equate the coefficients of the various powers of ε^2. We find

$$\left.\begin{aligned} \sum_{i=1}^{n} b^i(\underline{x}) \frac{\partial}{\partial x_i} v^{(0)}(\underline{x}) &= 0 \qquad \underline{x} \in \Omega \\ v^{(0)}(\underline{x}) &= 0 \qquad \underline{x} \in \partial\Omega \end{aligned}\right\} \tag{2.2.12a}$$

$$\left.\begin{aligned} \sum_{i=1}^{n} b^i(\underline{x}) \frac{\partial}{\partial x_i} v^{(k)}(\underline{x}) &= -\frac{1}{2} \sum_{i,j=1}^{n} a^{ij}(\underline{x}) \frac{\partial^2}{\partial x_i \partial x_j} v^{(k-1)}(\underline{x}) \qquad \underline{x} \in \Omega \\ v^{(k)}(\underline{x}) &= 0 \qquad \underline{x} \in \partial\Omega \\ k &= 1, 2, 3 \cdots, \end{aligned}\right\} \tag{2.2.12b}$$

We solve (2.2.12a) by the method of characteristics to find that $v^{(0)}$ is constant. If we apply the boundary conditions, we would be forced to conclude that $v^{(0)}(\underline{x})$ vanishes identically in Ω, and hence by induction, all $v^{(k)}(\underline{x})$ would vanish identically in Ω. This is clearly unacceptable, and since we already had suspicions of the existence of a boundary layer somewhere in Ω, we conclude that $v^{(0)}(\underline{x})$ represents an "outer solution" which is valid away from the boundary $\partial\Omega$.

Thus we are forced to conclude that the "outer solution" of the (2.2.2) is given by

$$u^{(0)}_{outer}(\underline{x};\varepsilon) \sim \text{constant.} \tag{2.2.13}$$

Perturbation theory has forced us to the conclusion that the solution of (2.2.2) behaves differently in different portions of Ω. For that part of Ω which is away from the boundary, we conclude that the solution is the "outer solution" which is constant. For that of Ω which is near the boundary, the solution will be the "inner solution." In the next section, we examine the "inner solution" of (2.2.2) and determine a uniformly valid asymptotic representation for the solution of (2.2.2).

2.3 Inner Solution of the Singularly Perturbed Boundary Value Problem

We saw in the previous section that the behavior of the solution of (2.2.2) was different near the boundary layer than it was elsewhere in Ω. We shall now construct a boundary layer expansion using singular perturbation theory.

In accordance with standard singular perturbation practice, we wish to couple the higher order operator in (2.2.2), $\varepsilon^2/2 \sum a^{ij}(\underline{x}) \partial/\partial x_i \; \partial/\partial x_j$, with the drift term, $\sum b^i(\underline{x}) \partial/\partial x_i$. We would like to do this by introducing a local coordinate system near the boundary, stretching one of the coordinates appropriately, and applying matching conditions on the boundary $\partial\Omega$ and as the stretched variable grows unbounded. Unfortunately, there is a subtle difficulty in this procedure; we do not know how to extend the boundary values into the interior of Ω. Thus, we do not know how to fully define the boundary layer correction.

Instead, we assume that the boundary layer correction has the form

$$u_{inner}(\underline{x}) \sim C_0 + z(\underline{x}) \exp\left\{-\frac{\zeta(\underline{x})}{\varepsilon^2}\right\} \qquad (2.3.1)$$

where

$$\zeta(\underline{x}) = 0 \qquad \underline{x} \in \partial\Omega$$

$$z(\underline{x}) = f(\underline{x}) - C_0 \qquad \underline{x} \in \partial\Omega \qquad (2.3.2)$$

Substitute (2.3.1) into (2.2.2), and equate the coefficients of the various powers of ε^2. The leading order equation, $O(\varepsilon^{-2})$, is the eiconal equation for $\zeta(\underline{x})$. In particular, $\zeta(\underline{x})$ satisfies

$$\frac{1}{2}\sum_{i,j=1}^{n} a^{ij}(\underline{x}) \frac{\partial\zeta(\underline{x})}{\partial x_i} \frac{\partial\zeta(\underline{x})}{\partial x_j} - \sum_{i=1}^{n} b^i(\underline{x}) \frac{\partial\zeta(\underline{x})}{\partial x_i} = 0 . \qquad \underline{x} \in \Omega \qquad (2.3.3)$$

Since the boundary $\partial\Omega$ is the level surface $\zeta(\underline{x}) \equiv 0$, we note

$$\nabla\zeta(\underline{x}) = |\nabla\zeta(\underline{x})|\, \underline{n}(\underline{x}) \qquad \underline{x} \in \partial\Omega \qquad (2.3.4)$$

where $\underline{n}(\underline{x})$ = unit inner normal vector to the boundary. We can now determine $|\nabla\zeta(\underline{x})|$. We observe that if we substitute (2.3.4) into (2.3.3), we obtain

$$\frac{1}{2}\sum_{i,j=1}^{n} a^{ij}(\underline{x}) n_i n_j |\nabla\zeta(\underline{x})|^2 - \sum_{i=1}^{n} b^i(\underline{x}) n_i |\nabla\zeta(\underline{x})| = 0 \qquad \underline{x} \in \partial\Omega \qquad (2.3.5)$$

Then we ignore the solution $|\nabla\zeta(\underline{x})| \equiv 0$ and find

$$|\nabla \zeta(\underline{x})| = \frac{2\underline{b}(\underline{x}) \cdot n(\underline{x})}{n^T(\underline{x})\, a(\underline{x})\, n(\underline{x})} \geq 0 \qquad \underline{x} \in \partial\Omega \qquad (2.3.6)$$

where it is understood that the magnitude of the gradient may be infinite and that L'Hopital's rule may be required to determine the magnitude. Now that we have determined the magnitude of $\nabla \zeta(\underline{x})$ on the boundary, we solve (2.3.3) using the method of rays. Let $\underline{p} \equiv \nabla \zeta(\underline{x})$. Then (2.3.3) corresponds to the Hamiltonian

$$H(\underline{x}, \underline{p}) = \frac{1}{2} \sum_{i,j=1}^{n} a^{ij}(\underline{x})\, p_i p_j - \sum_{i=1}^{n} b^i(\underline{x}) p_i = 0 \qquad (2.3.7)$$

The corresponding system of ordinary differential equations for the rays is

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i} = \sum_{j=1}^{n} a^{ij}(\underline{x}) p_j - b^i(\underline{x}) \qquad i = 1, \cdots, n \qquad (2.3.8)$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i} = -\frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial}{\partial x_i}\left(a^{jk}(\underline{x})\right) p_j p_k - \sum_{j=1}^{n} \frac{\partial}{\partial x_i}\left(b^j(\underline{x})\right) p_j$$

$$i = 1, \cdots, n \qquad (2.3.9)$$

Along such a system of trajectories, we set

$$\frac{d\zeta}{dt} = -H(\underline{x}, \underline{p}) + \sum_{i=1}^{n} p_i \frac{dx_i}{dt} = \frac{1}{2} \sum_{i,j=1}^{n} a^{ij}(\underline{x}) p_i p_j \geq 0. \qquad (2.3.10)$$

We solve the equations (2.3.8)-(2.3.10) subject to the initial conditions that

$$\underline{x}(0) \in \partial\Omega$$

$$p_i(0) \equiv \frac{\partial}{\partial x_i} \zeta\left(\underline{x}(0)\right) = |\nabla \zeta(\underline{x})| n_i\left(\underline{x}(0)\right) \tag{2.3.11}$$

$$\zeta(0) \equiv \zeta\left(\underline{x}(0)\right) = 0$$

The next leading term in the perturbation hierarchy of equations, $O(\varepsilon^0)$, is the transport equation for $z(\underline{x})$. In particular, $z(\underline{x})$ satisfies

$$-\frac{1}{2}\sum_{i,j=1}^{n} a^{ij}(\underline{x})\left[\frac{\partial^2\zeta(\underline{x})}{\partial x_i \partial x_j} z + 2\frac{\partial\zeta(\underline{x})}{\partial x_j}\frac{\partial z(\underline{x})}{\partial x_i}\right] + \sum_{i=1}^{n} b^i(\underline{x})\frac{\partial z(\underline{x})}{\partial x_i} = g(\underline{x})$$

$$\underline{x} \in \overline{\Omega} \tag{2.3.12}$$

We can again relate the partial derivatives to derivatives along the rays to find that $z(t) \equiv z(\underline{x}(t))$ satisfies

$$\frac{dz}{dt} + \frac{1}{2}\sum_{i,j=1}^{n} a^{ij}(\underline{x}(t))\frac{\partial^2\zeta(\underline{x}(t))}{\partial x_i \partial x_j} z(t) = -g(\underline{x}(t)) \quad \underline{x}(t) \in \overline{\Omega} \tag{2.3.13}$$

subject to the initial condition that

$$z(0) \equiv z(\underline{x}(0)) = f(\underline{x}(0)) - C_0 \tag{2.3.14}$$

When we solve for $\zeta(\underline{x})$, we have found the leading order term of the "inner solution" of (2.2.2). When we match the "inner solution" and the "outer solution" of (2.2.2), we find that the leading term in the uniformly valid asymptotic expansion for $u(x;\varepsilon)$

in (2.2.2) is given by

$$u_{u.v.}(\underline{x};\varepsilon) \sim C_0 + z(\underline{x}) \exp\{-\frac{\zeta(\underline{x})}{\varepsilon^2}\} . \qquad (2.3.15)$$

It should be noted that the only requirement on the matrix $\left(a^{ij}(\underline{x})\right)$ is that it be symmetric and positive semi-definite. If the matrix is singular, there may be a set of nonattainable, or inaccessible points on the boundary. Exit from Ω is impossible with probability one on this nonattainable set of points unless the process $\underline{x}_\varepsilon(t)$ is initially at some point in the set. An example of a dynamical system with this type of behavior will be given in Chapter IV. In the next section, we present a method for determining the unknown constant C_0 in (2.3.15) by using the solution of the homogeneous adjoint problem of (2.2.2).

2.4 Determination of the Unknown Parameter

We see from (2.3.15) that we have determined the leading order asymptotic solution to (2.2.2) in terms of an unknown parameter C_0. That this is so is not particularly surprising since the unknown parameter is a global constant for the problem and the underlying tenent of singular perturbation theory is to solve a series of local problems and then match the solutions in such a manner as to generate a uniformly valid asymptotic representation for the solution. Thus in problems where the so-called "outer solution" is not required to meet prescribed boundary conditions, we can expect that the solutions will be expressed in terms of unknown parameters. We now present a method for determining the unknown parameter(s).

We begin by multiplying both sides of (2.2.2) by a function $v(\underline{x};\varepsilon)$ which will be determined later and integrating over Ω. We find

$$\int_{\Omega}\cdots\int\Big\{\frac{\varepsilon^2}{2}\sum_{i,j=1}^{n} a^{ij}(\underline{x})\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_{i=1}^{n} b^i(\underline{x})\frac{\partial u}{\partial x_i}\Big\}v(\underline{x};\varepsilon)d\underline{x}=\int_{\Omega}\cdots\int g(\underline{x})v(\underline{x};\varepsilon)d\underline{x} \tag{2.4.1}$$

Integrate the left hand side of (2.4.1) by parts to find

$$\begin{aligned}\int_{\partial\Omega}\Big\{\frac{\varepsilon^2}{2}\sum_{i,j=1}^{n} a^{ij}(\underline{x})\frac{\partial u}{\partial x_j}\nu_i v(\underline{x};\varepsilon)-\frac{\varepsilon^2}{2}\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\Big(a^{ij}(\underline{x})v(\underline{x};\varepsilon)\Big)\nu_i u(\underline{x};\varepsilon)\\+\sum_{i=1}^{n} b^i(\underline{x})u(\underline{x};\varepsilon)v(\underline{x};\varepsilon)\nu_i\Big\}dS\\+\int_{\Omega}\cdots\int\Big\{\frac{\varepsilon^2}{2}\sum_{i,j=1}^{n}\frac{\partial^2}{\partial x_i\partial x_j}\Big(a^{ij}(\underline{x})v(\underline{x};\varepsilon)\Big)-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\Big(b^i(\underline{x})v(\underline{x};\varepsilon)\Big)\Big\}u(\underline{x};\varepsilon)d\underline{x}\\=\int_{\Omega}\cdots\int g(\underline{x})v(\underline{x};\varepsilon)d\underline{x}\,.\end{aligned} \tag{2.4.2}$$

We see that if $v(\underline{x};\varepsilon)$ is a solution of the boundary value problem

$$\frac{\varepsilon^2}{2}\sum_{i,j=1}^{n}\frac{\partial^2}{\partial x_i\partial x_j}\Big(a^{ij}(\underline{x})v(\underline{x};\varepsilon)\Big)-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\Big(b^i(\underline{x})v(\underline{x};\varepsilon)\Big)=0.\quad \underline{x}\,\varepsilon\,\overline{\Omega}$$

$$\frac{\varepsilon^2}{2}\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\Big(a^{ij}(\underline{x})v(\underline{x};\varepsilon)\Big)\nu_i-\sum_{i=1}^{n} b^i(\underline{x})v(\underline{x};\varepsilon)\nu_i=0\qquad \underline{x}\in\partial\Omega \tag{2.4.3}$$

then (2.4.2) reduces to

$$\int_{\partial\Omega} \frac{\varepsilon^2}{2} \sum_{i,j=1}^{n} a^{ij}(\underline{x})\nu_i \frac{\partial u}{\partial x_j} v(\underline{x};\varepsilon)dS = \int \cdots \int_{\Omega} g(\underline{x})v(\underline{x};\varepsilon)d\underline{x} \qquad (2.4.4)$$

We seek a solution of (2.4.3) of the form

$$v(x;\varepsilon) \sim \exp\left\{-\frac{\varphi(\underline{x})}{\varepsilon^2}\right\}\left[w^{(0)}(\underline{x}) + \varepsilon^2 w^{(1)}(\underline{x}) + \varepsilon^4 w^{(2)}(\underline{x}) + \cdots\right] \qquad (2.4.5)$$

where $v(\underline{x};\varepsilon) = 1$ at the limit set. Substitute this ansatz into (2.4.3) and equate the various powers of ε^2. We find that $\varphi(\underline{x})$ satisfies the eiconal equation

$$\frac{1}{2}\sum_{i,j=1}^{n} a^{ij}(\underline{x}) \frac{\partial \varphi(\underline{x})}{\partial x_i}\frac{\partial \varphi(\underline{x})}{\partial x_j} + \sum_{i=1}^{n} b^i(\underline{x}) \frac{\partial \varphi(\underline{x})}{\partial x_i} = 0 \qquad \underline{x} \in \Omega$$

$$\frac{1}{2}\sum_{i,j=1}^{n} a^{ij}(\underline{x})\nu_i(\underline{x}) \frac{\partial \varphi(\underline{x})}{\partial x_j} + \sum_{i=1}^{n} b^i(\underline{x})\nu_i(\underline{x}) = 0 \qquad \underline{x} \in \partial\Omega \qquad (2.4.6)$$

Again, we associate (2.4.6) with the Hamiltonian

$$H(\underline{x}, \underline{p}) = \frac{1}{2}\sum_{i,j=1}^{n} a^{ij}(\underline{x})p_i p_j + \sum_{i=1}^{n} b^i(\underline{x})p_i \qquad \underline{x} \in \overline{\Omega} \qquad (2.4.7)$$

The corresponding system of ordinary differential equations for the rays is

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i} = \sum_{j=1}^{n} a^{ij}(\underline{x})p_j + b^i(\underline{x}) \qquad i = 1, \cdots, n \qquad (2.4.8)$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i} = -\frac{1}{2}\sum_{j,k=1}^{n} \frac{\partial}{\partial x_i}\left(a^{jk}(\underline{x})\right)p_j p_k - \sum_{j=1}^{n} \frac{\partial}{\partial x_i}\left(b^j(\underline{x})\right)p_j$$
$$i = 1, \cdots, n. \qquad (2.4.9)$$

Along such a system of trajectories, we set

$$\frac{d\varphi}{dt} = -H(\underline{x}, \underline{p}) + \sum_{i=1}^{n} p_i \frac{dx_i}{dt} = \frac{1}{2}\sum_{i,j=1}^{n} a^{ij}(\underline{x})p_i p_j \geq 0 \ . \qquad (2.4.10)$$

We now examine the behavior of $\varphi(\underline{x})$ near a single, stable limit point at the deterministic system (1.1.1). At the limit point, we take $\varphi(\underline{x}) \equiv 0$ and require that $\varphi(\underline{x})$ achieve a minimum value there.

We solve (2.4.6) using a method employed by Ludwig [10]. Cover Ω with a family of rays which depend upon the parameters $t, \theta_1, \cdots, \theta_{n-1}$. Then

$$\underline{x} = \underline{x}\left(t, \theta_1, \cdots, \theta_{n-1}\right)$$
$$\underline{p} = \underline{p}\left(t, \theta_1, \cdots, \theta_{n-1}\right) \qquad (2.4.11)$$

Define the Jacobian of the transformation between $\underline{x}$ and $(t, \underline{\theta})$ by

$$J = \begin{vmatrix} \frac{dx_1}{dt} & \frac{\partial x_1}{\partial \theta_1} & \cdots & \frac{\partial x_1}{\partial \theta_{n-1}} \\ \vdots & \vdots & & \vdots \\ \frac{dx_n}{dt} & \frac{\partial x_n}{\partial \theta_1} & \cdots & \frac{\partial x_n}{\partial \theta_{n-1}} \end{vmatrix} \qquad (2.4.12)$$

If $J \neq 0$, then locally, the trajectories give a simple covering of the $\underline{x}$ - space. Calculate the matrix of second derivatives of φ at the equilibrium point $\underline{y}$:

$$\left(\frac{\partial^2 \varphi(\underline{y})}{\partial x_i \partial x_j} \right) \equiv S^{-1} \tag{2.4.13}$$

Then the covariance matrix S satisfies

$$a^{ij}(\underline{x}) + \sum_{k=1}^{n} S_{ik} \frac{\partial b^k(\underline{x})}{\partial x_j} + \sum_{k=1}^{n} \frac{\partial b^i(\underline{x})}{\partial x_k} S_{kj} = 0 \tag{2.4.14}$$

Since $\varphi(\underline{x})$ and its first derivatives vanish at the limit point, we approximate $\varphi(\underline{x})$ and $\underline{p}(\underline{x})$ in the neighborhood of the limit point as

$$\varphi(\underline{x}) = \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 \varphi(\underline{y})}{\partial x_i \partial x_j} (x_i - y_i)(x_j - y_j) + o\left(|\underline{x}-\underline{y}|^2 \right) \tag{2.4.15}$$

$$p_i(\underline{x}) = \sum_{j=1}^{n} \frac{\partial^2 \varphi(\underline{y})}{\partial x_i \partial x_j} (x_j - y_j) + o\left(|\underline{x}-\underline{y}| \right) \tag{2.4.16}$$

The rays cannot be chosen to emanate from the limit point since it is a singular point of (2.4.8) and (2.4.9). Instead, choose $\underline{x}$ to initially be on an ellipsoid

$$\underline{x}(t_0, \delta, \underline{\theta}) = \delta S^{1/2} U(\underline{\theta}) \tag{2.4.17}$$

where δ is a small parameter, $S^{1/2}$ denotes the square root of the matrix S and $U(\underline{\theta})$ is a unit vector which depends on $\underline{\theta}$. Then

(2.4.15) implies that

$$\varphi\left(\underline{x}(t_0, \delta, \underline{\theta})\right) = \frac{1}{2}\delta^2 + o(\delta^2) \tag{2.4.18}$$

Initial data for $\underline{p}$ are provided by neglecting the remainder in (2.4.16). Thus we can integrate along the rays and construct a solution in the neighborhood of the ellipsoid. The function $w^{(0)}(\underline{x})$ satisfies the transport equation

$$\sum_{i,j=1}^{n} a^{ij}(\underline{x}) \frac{\partial \varphi(\underline{x})}{\partial x_j} \frac{\partial w^{(0)}(\underline{x})}{\partial x_i} + \sum_i b^i(\underline{x}) \frac{\partial w^{(0)}(\underline{x})}{\partial x_i}$$
$$+ \left[\left(\sum_{i,j=1}^{n} \frac{\partial a^{ij}(\underline{x})}{\partial x_i} \frac{\partial \varphi(\underline{x})}{\partial x_j} + \frac{1}{2} a^{ij}(\underline{x}) \frac{\partial^2 \varphi(\underline{x})}{\partial x_i \partial x_j}\right) + \sum_{i=1}^{n} \frac{\partial b^i(\underline{x})}{\partial x_i}\right] w^{(0)}(\underline{x}) = 0 \tag{2.4.19}$$

We would like to write (2.4.1) as an ordinary differential equation along the rays given by (2.4.8), and (2.4.9). Since the rays cannot emanate from the limit point, expand $w^{(0)}(\underline{x})$ in a Taylor series about the limit point:

$$w^{(0)}(\underline{x}) = w^{(0)}(\underline{y}) + \sum_{i=1}^{n} \frac{\partial w^0(\underline{y})}{\partial x_i}(x_i - y_i) + o\left(|\underline{x}-\underline{y}|\right)$$

$$= 1 + O(\delta). \tag{2.4.20}$$

We find the initial condition for $w^{(0)}\left(\underline{x}(t_0, \delta, \underline{\theta})\right)$ on the initial ellipsoid by ignoring the remainder in (2.4.20). Then we treat (2.4.19) as an initial value problem, starting from the initial ellipsoid, along each ray. Observe that the limiting values of $\varphi(\underline{x})$

and $w^{(0)}(\underline{x})$ give the correct limiting value in (2.4.5) as $\delta \to 0$. If the deterministic system (1.1.1) possesses an asymptotically stable limit cycle, we must slightly modify the previous results. In order to determine $\varphi(\underline{x})$ in the neighborhood of the limit cycle, set $\varphi(\underline{x}) \equiv 0$ on the limit cycle and study the eiconal equation. Since the limit cycle is a level curve for $\varphi(\underline{x})$, note

$$\nabla\varphi(\underline{x}_L) = |\nabla\varphi(\underline{x}_L)|\underline{\nu}(\underline{x}_L) \qquad (2.4.21)$$

where $\underline{\nu}(\underline{x}_L)$ is the unit outer normal to the limit cycle at the point $\underline{x}_L$. Substitute this into the eiconal equation (2.4.6) to find

$$\frac{1}{2}\sum_{i,j=1}^{n} a^{ij}(\underline{x}_L)\nu_i(\underline{x}_L)\nu_j(\underline{x}_L)|\nabla\varphi(\underline{x}_L)|^2 + \sum_{i=1}^{n} b^i(\underline{x}_L)\nu_i(\underline{x}_L)|\nabla\varphi(\underline{x}_L)| = 0 \quad (2.4.22)$$

Since $\underline{b}(\underline{x}_L)\cdot\underline{\nu}(\underline{x}_L) = 0$ on the limit cycle, we conclude that $|\nabla\varphi(\underline{x}_L)| = \nabla\varphi(\underline{x}_L) \equiv 0$ there.

Near the limit cycle, introduce a local, orthonormal coordinate system $\left(\tau(\underline{x}_L), \underline{\nu}(\underline{x}_L)\right)$ where $\underline{\tau}(\underline{x}_L)$ is the unit tangent vector to the limit cycle at the point $\underline{x}_L$ and $\underline{\nu}(\underline{x}_L)$ is the unit normal vector to the limit cycle at the same point. Since $\varphi(\underline{x}) \equiv 0$ on the limit cycle, all derivatives of $\varphi(\underline{x})$, evaluated on the limit cycle, vanish in the tangential direction. Then for points $\underline{x}$ near the limit cycle, expand $\varphi_i(\underline{x})$, $a^{ij}(\underline{x})$, and $b^i(\underline{x})$ in a Taylor series about points on the limit cycle. If

$$\underline{x} = \underline{x}_L + \delta\underline{x} \qquad (2.4.23)$$

where $\delta \underline{x}$ is normal to the limit cycle at the point $\underline{x}_L$, then

$$\left.\begin{aligned} p_i(\underline{x}) &= \sum_{k=1}^{n} \frac{\partial^2 \varphi(\underline{x}_L)}{\partial x_i \partial x_k} \delta x^k + \frac{1}{2} \sum_{k,\ell=1}^{n} \frac{\partial^3 \varphi(\underline{x}_L)}{\partial x_i \partial x_k \partial x_\ell} \delta x^k \delta x^\ell + o\left(|\delta \underline{x}|^3\right) \\ a^{ij}(\underline{x}) &= a^{ij}(\underline{x}_L) + \sum_{k=1}^{n} \frac{\partial a^{ij}(\underline{x}_L)}{\partial x_k} \delta x^k + o(\delta \underline{x}) \\ b^i(\underline{x}) &= b^i(\underline{x}_L) + \sum_{k=1}^{n} \frac{\partial b^i(\underline{x}_L)}{\partial x_k} \delta x^k + o(\delta \underline{x}) \end{aligned}\right\} \quad (2.4.24)$$

Substitute these three quantities into the eiconal equation and equate the coefficients of the various powers of the incremental vector $\delta \underline{x}$. We find that the coefficient of $(\delta \underline{x})$ is given by

$$\sum_{i=1}^{n} b^i(\underline{x}_L) \frac{\partial^2 \varphi(\underline{x}_L)}{\partial x_i \partial x_k} \delta x^k = 0 \qquad k = 1, \cdots, n \quad (2.4.25)$$

This is just

$$\text{constant} \cdot \underline{\nu}^T(\underline{x}_L) \cdot \nabla\left(p_k(\underline{x}_L)\right) \cdot \underline{b}(\underline{x}_L)$$

$$= \text{constant} \cdot \underline{\nu}^T(\underline{x}_L) \cdot \frac{d}{dt} \underline{p}(\underline{x}_L)$$

Thus, (2.4.24) is consistent with (2.4.21). We find that the coefficient of $(\delta \underline{x})^2$ is

$$\left[\frac{1}{2}\sum_{i,j=1}^{n} a^{ij}(\underline{x}_L)\,\frac{\partial^2\varphi(\underline{x}_L)}{\partial x_i\partial x_k}\,\frac{\partial^2\varphi(\underline{x}_L)}{\partial x_j\partial k_\ell} + \sum_{i=1}^{n}\frac{\partial b^i(\underline{x}_L)}{\partial x_k}\,\frac{\partial^2\varphi(\underline{x}_L)}{\partial x_i\partial x_\ell} + \right.$$

$$\left. + \frac{1}{2}\, b^i(\underline{x}_L)\,\frac{\partial^3\varphi(\underline{x}_L)}{\partial x_i\partial x_L\partial x_\ell}\right]\delta x^k\delta x^\ell = 0 \qquad (2.4.26)$$

If

$$P = \left(\frac{\partial^2\varphi(\underline{x}_L)}{\partial x_k\partial x_\ell}\right)$$

$$A = \left(a^{k\ell}(\underline{x}_L)\right)$$

$$B = \left(\frac{\partial b^k(\underline{x}_L)}{\partial x_\ell}\right)$$

then we note

$$\underline{\nu}^T(\underline{x}_L)\left[PAP + PB + B^TP + \frac{dP}{dt}\right]\underline{\nu}(\underline{x}_L) = 0 \qquad (2.4.27)$$

on the limit cycle. When we rotate coordinates to the tangential-normal coordinate frame, we note $\underline{\tau}^T P\tau = \varphi_{\tau\tau} = 0$. We assume we can write P as

$$P(\underline{x}_L(t)) = \beta(t)\,\underline{\tau}(\underline{x}_L(t))\,\underline{\nu}^T(\underline{x}_L(t)) + \beta(t)\,\underline{\nu}(\underline{x}_L(t))\,\underline{\tau}^T(\underline{x}_L(t))$$

$$+\ \gamma(t)\,\underline{\nu}(\underline{x}_L(t))\,\nu^T(\underline{x}_L(t)) \qquad (2.4.28)$$

Then $\beta(t)$ satisfies

$$\beta(t)\left[\underline{\nu}^T \frac{d\underline{\tau}}{dt} + \frac{d\underline{\tau}^T}{dt}\underline{\nu} + \underline{\tau}^T B\underline{\nu} + \underline{\nu}^T B^T \underline{\tau}\right] = 0 \qquad (2.4.29)$$

Hence $\beta(t) \equiv 0$ on the limit cycle. Also, $\gamma(t)$ satisfies

$$\frac{d\gamma(t)}{dt} + \gamma^2(t)\,\underline{\nu}^T(\underline{x}_L(t))A\nu(\underline{x}_L(t)) + \gamma(t)\left[\underline{\nu}^T(\underline{x}_L(t))(B+B^T)\underline{\nu}(\underline{x}_L(t))\right] = 0$$

(2.4.30)

on the limit cycle, since the Frenet formulas from differential geometry tell us that $d\underline{\nu}/dt \propto \underline{\tau}$. We do not want $\gamma(t)$ to vanish on the limit cycle, so we note that (2.4.28) is a Riccati equation and make the substitution

$$\mu(t) = \gamma^{-1}(t) \qquad (2.4.31)$$

Then (2.4.29) becomes

$$\frac{d}{dt}\mu(t) - 2\mu(t)\left[\underline{\nu}^T(\underline{x}_L(t))B\underline{\nu}(\underline{x}_L(t))\right] = \underline{\nu}^T(\underline{x}_L(t))A\underline{\nu}(\underline{x}_L(t)) \qquad (2.4.32)$$

We find that $\mu(t)$ is given by

$$\mu(t) = \int_0^t ds\, \underline{\nu}^T(\underline{x}_L(s))A\underline{\nu}(\underline{x}_L(s)) \exp\left\{2\int_s^t ds_1\, \underline{\nu}^T(\underline{x}_L(s_1))B\,\underline{\nu}(\underline{x}_L(s_1))\right\}$$

$$+ \mu(0)\exp\left\{2\int_0^t ds_1\, \underline{\nu}^T(\underline{x}_L(s_1))B\underline{\nu}(\underline{x}_L(s_1))\right\} \qquad (2.4.33)$$

Since the motion on the limit cycle is periodic with period t^*, we choose $\mu(0)$ to make $\mu(t)$ t^*-periodic. Hence

$$\mu(t) = \left[1 - \exp\{2\int_0^{t^2} ds_1 \, \underline{\nu}^T(\underline{x}_L(s_1))B\underline{\nu}(\underline{x}_L(s_1))\}\right]^{-1} \cdot$$

$$\cdot \left[\int_0^t ds \, \underline{\nu}^T(\underline{x}_L(s))A\underline{\nu}(\underline{x}_L(s)) \exp\{2\int_s^t ds_1 \underline{\nu}^T(\underline{x}_L(s_1))B\underline{\nu}(\underline{x}_L(s_1))\}\right.$$

$$+ \int_t^{t^*} ds \, \underline{\nu}^T(\underline{x}_L(s))A\underline{\nu}(\underline{x}_L(s)) \exp\{2\int_s^t ds_1 \underline{\nu}^T(\underline{x}_L(s_1))B\underline{\nu}(\underline{x}_L(s_1))$$

$$\left. + 2\int_0^{t^*} ds_1 \underline{\nu}^T(\underline{x}_L(s_1))B\underline{\nu}(\underline{x}_L(s_1))\}\right] \qquad (2.4.34)$$

We can simplify (2.4.32) by first noting

$$\underline{\nu}^T(\underline{x}_L(s))B\underline{\nu}(\underline{x}_L(s)) = \text{tr } B - \underline{\tau}^T(\underline{x}_L(s))B\underline{\tau}(\underline{x}_L(s))$$

$$= \text{tr } B - \frac{\underline{b}^T(\underline{x}_L(s))B\underline{b}(\underline{x}_L(s))}{|\underline{b}(\underline{s}_L(s))|^2}$$

$$= \text{tr } B - \frac{1}{2}\frac{d}{dx}\left[\ell n \, |\underline{b}(\underline{x}_L(s))|^2\right] \qquad (2.4.35)$$

Define

$$\exp\{\int_0^{t^*} \text{tr } B(\underline{x}_L(s))ds\} = \lambda \qquad (2.4.36)$$

Then (2.4.32) becomes

$$\mu(t) = \left[1 - \lambda^2\right]^{-1} \cdot \left[\frac{1}{|b(\underline{x}_L(t))|^2} \int_0^t ds \, |\underline{b}(\underline{x}_L(s))|^2 \, \underline{\nu}^T(\underline{x}_L(s))A\underline{\nu}(\underline{x}_L(s))\right.$$

$$\exp\{2\int_s^t ds_1 \text{tr } B\} \qquad (2.4.37)$$

$$+ \frac{\lambda^2}{|\underline{b}(\underline{x}_L(t))|^2} \int_t^{t^*} ds\, |\underline{b}(\underline{x}_L(s))|^2\, \underline{\nu}^T(\underline{x}_L(s)) A \underline{\nu}(\underline{x}_L(s)) \exp\left\{2\int_s^t ds_1 \mathrm{tr} B\right\}\Bigg]$$

Thus

$$P = \left(\frac{\partial^2 \varphi(\underline{x}_L(t))}{\partial x_k \partial x_\ell}\right)$$

$$= \mu^{-1}(t)\, \underline{\nu}(\underline{x}_L(t))\, \underline{\nu}^T(\underline{x}_L(t))$$

The rays cannot be chosen to emanate from the limit cycle since it is a singular solution of (2.4.8) and (2.4.9). Instead, choose $\underline{x}$ to initially be on a δ-tube

$$\underline{x}(t_0, \delta) = \underline{x}_L + \delta\, \underline{\nu}(\underline{x}_L) \tag{2.4.38}$$

where δ is a small parameter. Then in the neighborhood of the limit cycle

$$\varphi\left(\underline{x}(t_0, \delta)\right) = \frac{1}{2}\gamma(t_0)\delta\underline{\nu}^T\left(x_L(t_0)\right)\delta\underline{\nu}\left(\underline{x}_L(t_0)\right) + o\left(|\delta\underline{\nu}|^2\right) \tag{2.4.39}$$

$$p\left(\underline{x}(t_0, \delta)\right) = \gamma(t_0)\delta\underline{\nu}\left(\underline{x}_L(t_0)\right) + o\left(|\delta\underline{\nu}|\right) \tag{2.4.40}$$

Initial data for $\underline{p}$ are provided by neglecting the remainder in (2.4.40). Thus we can integrate along the rays and construct a solution of the eiconal equation in the neighborhood of the δ-tube about the limit cycle.

The solution $w^{(0)}(\underline{x})$ satisfies the transport equation (2.4.19).

We can write the transport equation as an ordinary differential equation along the rays. In particular the equation for $w^{(0)}(\underline{x})$ along the limit cycle becomes

$$\frac{d}{dt} w^{(0)}(\underline{x}_L(t)) + w^{(0)}(\underline{x}_L(t))\left[tr(\tfrac{1}{2}AP + B)\right] = 0 \quad (2.4.41)$$

We can immediately integrate (2.4.40) to find

$$w^{(0)}(\underline{x}_L(t)) = w^{(0)}(\underline{x}_L(0)) \exp\left\{-\int_0^t tr(\tfrac{1}{2}AP + B)ds\right\} \quad (2.4.42)$$

If we perform a coordinate rotation, the trace remains invariant. Thus, if $Q = [\underline{\tau}, \underline{\nu}]$, then

$$tr(\tfrac{1}{2}AP+B) = tr(\tfrac{1}{2}Q^T APQ + Q^T BQ)$$

$$= \frac{1}{2}\tau^T(\underline{x}_L)A\mu(t)\underline{\nu}(\underline{x}_L)\underline{\nu}^T(\underline{x}_L)\underline{\tau}(\underline{x}_L) + \frac{1}{2}\underline{\nu}^T(\underline{x}_L)A\mu(t)\underline{\nu}(\underline{x}_L)\underline{\nu}^T(\underline{x}_L)\underline{\nu}(\underline{x}_L)$$

$$+ \underline{\tau}^T(\underline{x}_L)B\underline{\tau}(\underline{x}_L) + \underline{\nu}^T(\underline{x}_L)B\underline{\nu}(\underline{x}_L)$$

$$= -\frac{1}{2\gamma(t)}\frac{d}{dt}\gamma(t) + \underline{\tau}^T(\underline{x}_L)B\tau(\underline{x}_L) \quad (2.4.43)$$

This last result is an immediate consequence of (2.4.29). Thus

$$tr(\tfrac{1}{2}AP+B) = \frac{1}{2}\frac{d}{dt}\ell n\left(\frac{|\underline{b}(\underline{x}_L(t))|^2}{\gamma(t)}\right) \quad (2.4.44)$$

Then

$$w^{(0)}(\underline{x}_L(t)) = w^{(0)}(\underline{x}_L(0)) \exp\left\{ -\int_0^t \frac{1}{2} \frac{d}{ds} \ln\left(\frac{|b(\underline{x}_L(s))|^2}{\gamma(s)} \right) \right\}$$

$$= w^{(0)}(\underline{x}_L(0)) \sqrt{\frac{|\underline{b}(\underline{x}_L(0))|^2}{\gamma(0)}} \sqrt{\frac{\gamma(t)}{|\underline{b}(\underline{x}_L(t))|^2}} \qquad (2.4.45)$$

Observe that due to the t*-periodicity of $\underline{b}(\underline{x}_L(t))$ and $\gamma(t)$ $w^{(0)}(\underline{x}_L(t))$ is also t*-periodic. Again, we determine $w^{(0)}(\underline{x})$ on a δ-tube about the limit cycle and then integrate along the rays given by (2.4.8) and (2.4.9). Also note that $w^0(\underline{x}_L)$ is proportional to the reciprocal of the speed $|\underline{b}(\underline{x}_L)|$. In order to determine the unknown parameter C_0, let

$$\varphi_{\min} = \min_{\underline{x} \in \partial\Omega} \varphi(\underline{x}) \qquad (2.4.46)$$

At the point or points on $\partial\Omega$ where $\varphi(\underline{x}) = \varphi_{\min}$, the level surface $\varphi(\underline{x}) = \varphi_{\min}$ is tangent to $\partial\Omega$. Thus, the boundary condition in (2.4.3) is satisfied there to leading order in ε since $\nabla\varphi(\underline{x}) \equiv |\nabla\varphi(\underline{x})|\underline{\nu}$. At all other points on $\partial\Omega$, $v(x;\varepsilon)$ is transcendentally small compared with $\exp\{-\varphi_{\min}/\varepsilon^2\}$ as $\varepsilon \downarrow 0$. Thus, $v(\underline{x};\varepsilon)$ given by the leading term in (2.4.5) represents an asymptotic solution to (2.4.3).

Now that we have determined an asymptotic representation for $v(\underline{x};\varepsilon)$, we substitute the result into (2.4.4) to find

$$\int_{\partial\Omega}\left\{\left[\frac{1}{2}\sum_{i,j=1}^{n} a^{ij}(\underline{x})\left[C_0 - f(\underline{x})\right]\frac{\partial\zeta(\underline{x})}{\partial x_j}\,\nu_i + O(\varepsilon^2)\right]w^{(0)}(\underline{x})\exp\left\{-\frac{\varphi(\underline{x})}{\varepsilon^2}\right\}\right.$$

$$\left. + O\left(\varepsilon^2\exp\left\{-\frac{\varphi_{\min}}{\varepsilon^2}\right\}\right)\right\} ds$$

$$= \int\cdots\int g(\underline{x})\, w^{(0)}(\underline{x})\exp\left\{-\frac{\varphi(\underline{x})}{\varepsilon^2}\right\} d\underline{x} \qquad (2.4.47)$$

provided that the leading term on the left hand side of (2.4.47) is $O\left(\exp\{-\varphi_{\min}/\varepsilon^2\}\right)$. The integral on the left hand side of (2.4.47) is to be evaluated using Laplace's Method. We find

$$C_0 \sim \frac{N}{D} \qquad (2.4.48)$$

where

$$N = \int_{\Omega}\cdots\int g(\underline{x})\, w^{(0)}(\underline{x})\exp\left\{-\frac{\varphi(\underline{x})}{\varepsilon^2}\right\} d\underline{x}$$

$$+ \int_{\partial\Omega}\frac{1}{2}\sum_{i,j=1}^{n} a^{ij}(\underline{x})\,\nu_i\,\frac{\partial\zeta(\underline{x})}{\partial x_j}\, f(\underline{x})w^{(0)}(\underline{x})\exp\left\{-\frac{\varphi(\underline{x})}{\varepsilon^2}\right\} dS$$

$$+ O\left(\varepsilon^2\exp\left\{-\frac{\varphi_{\min}}{\varepsilon^2}\right\}\right) \qquad (2.4.49)$$

and where

$$D = \int_{\partial\Omega}\frac{1}{2}\sum_{i,j=1}^{n} a^{ij}(\underline{x})\,\nu_i\,\frac{\partial\zeta(\underline{x})}{\partial x_j}\, w^{(0)}(\underline{x})\exp\left\{-\frac{\varphi(\underline{x})}{\varepsilon^2}\right\} dS. \qquad (2.4.50)$$

But in both the numerator and denominator, we note that we can apply (2.3.3) and (2.3.4) so that the expressions reduce to

$$N = \int_{\Omega}\cdots\int g(\underline{x})w^{(0)}(\underline{x})\exp\left\{-\frac{\varphi(\underline{x})}{\varepsilon^2}\right\}d\underline{x}$$

$$+\int_{\partial\Omega}\sum_{i=1}^{n} b^i(\underline{x})\nu_i f(\underline{x})w^{(0)}(\underline{x})\exp\left\{-\frac{\varphi(\underline{x})}{\varepsilon^2}\right\}dS + O\left(\varepsilon^2\exp\left\{-\frac{\varphi_{min}}{\varepsilon^2}\right\}\right) \quad (2.4.51)$$

and

$$D = \int_{\partial\Omega}\sum_{i=1}^{n} b^i(\underline{x})\,\nu_i w^{(0)}(\underline{x})\exp\left\{-\frac{\varphi(\underline{x})}{\varepsilon^2}\right\}dS \qquad (2.4.52)$$

If the leading term on the left hand side of (2.4.47) is not $O\left(\exp\{-\varphi_{min}/\varepsilon^2\}\right)$, then the terms which we have ignored may be significant. In that case, the dominant contribution to the boundary integrals will still occur in the neighborhood of the point or points where $\varphi(\underline{x}) = \varphi_{min}$, but Watson's Lemma must be invoked in order to evaluate the integrals asymptotically.

Now that we have determined the unknown parameter C_0 for the general problem (2.2.2), we can restrict ourselves to the problems of the mean exit time and the probability distribution of exit positions.

Problem 1: Probability distribution of exit positions.

We set $\quad g(\underline{x}) \equiv 0$

$f(x)$ is an arbitrary smooth function

We find

$$C_0 \sim \frac{\int_{\partial\Omega} f(\underline{x})\,\underline{b}(\underline{x}) \cdot \underline{\nu}(\underline{x})\, w^{(0)}(\underline{x}) \exp\{-\frac{\varphi(\underline{x})}{\varepsilon^2}\}\, dS}{\int_{\partial\Omega} \underline{b}(\underline{x}) \cdot \underline{\nu}(\underline{x})\, w^{(0)}(\underline{x}) \exp\{-\frac{\varphi(\underline{x})}{\varepsilon^2}\}\, dS} \qquad (2.4.53)$$

We first note that we meet the consistency condition that $C_0 \sim 1$ if $f(\underline{x}) \equiv 1$. Thus the process will exit from Ω with probability one in the presence of noise. Notice also that if there is a unique point on $\partial\Omega$ where $\varphi(\underline{x}) = \varphi_{min}$, then the probability distribution tends to a δ-function at that point. If there is not a unique point on the boundary where $\varphi(\underline{x}) = \varphi_{min}$, then the effects of the transport term $w^{(0)}(\underline{x})$ become important. Further comparison with the results of other authors for this problem will be made in the next chapter.

Problem 2: Mean exit time problem

We set

$$g(\underline{x}) \equiv -1$$

$$f(x) \equiv 0$$

We find

$$C_0 \sim \frac{-\int \cdots \int_{\Omega} w^{(0)}(\underline{x}) \exp\{-\frac{\varphi(\underline{x})}{\varepsilon^2}\}\, d\underline{x}}{\int_{\partial\Omega} \underline{b}(\underline{x}) \cdot \underline{\nu}(\underline{x})\, w^{(0)}(\underline{x}) \exp\{-\frac{\varphi(\underline{x})}{\varepsilon^2}\}\, dS} \qquad (2.4.54)$$

We observe that the initial guess regarding the magnitude of the solution C_0 was correct since

$$C_0 = O\left(\exp\left\{\frac{\varphi_{\min}}{\varepsilon^2}\right\}\right). \tag{2.4.55}$$

If the deterministic system (1.1.1) possesses a limit cycle, then we determine even more about the form of C_0. We evaluate the numerator by noting that $\varphi(\underline{x})$ is minimized on the limit cycle so that we can determine the numerator by considering $\varphi(\underline{x})$ in the neighborhood of the limit cycle, rotating coordinates into the tangential-normal coordinate system along the limit cycle, applying Laplace's method to evaluate the integral in the direction normal to the limit cycle, and integrating the results around the limit cycle. Observe in (2.4.44) that

$$w^{(0)}(x_L(t)) = k \cdot \sqrt{\frac{\gamma(t)}{|b(\underline{x}_L(t))|^2}} \quad \text{where } k = \text{constant}. \tag{2.4.56}$$

Thus

$$C_0 \sim \frac{k\int_0^{t^*} \sqrt{\frac{2\pi\varepsilon^2}{\gamma(t)}} \sqrt{\frac{\gamma(t)}{|\underline{b}(\underline{x}_L)(t))|^2}}\, dt}{\int_{\partial\Omega} \underline{b}(\underline{x}) \cdot \underline{\nu}(\underline{x})\, w^{(0)}(\underline{x}) \exp\left\{-\frac{\varphi(\underline{x})}{\varepsilon^2}\right\} dS}$$

$$= \frac{\varepsilon\sqrt{2\pi} \int_0^{t^*} \frac{dt}{|\underline{b}(\underline{x}_L(t))|}}{\int_{\partial\Omega} \underline{b}(\underline{x}) \cdot \underline{\nu}(\underline{x}) \frac{w^0(\underline{x})}{k} \exp\left\{-\frac{\varphi(\underline{x})}{\varepsilon^2}\right\} dS} \tag{2.4.57}$$

We will compare the results for the mean exit problem given by other authors with the results of this chapter in the next section.

CHAPTER III

In this chapter we compare the results obtained in Chapter II with the results of other authors. Since the results of the previous chapter pertain to both the mean exit time problem and the problem of the probability distribution of exit positions, we compare results for both types of problems. The first section is devoted to a comparison of results for the problem of the probability distribution of exit positions. The results of the mean exit problem are compared in the second section.

3.1 A Comparison of Results for the Problem of the Probability Distribution of Exit Positions

Results for this problem have been published in the literature for only about a decade. Early results can be attributed to Ventsel' and Freidlin [15] who studied the case where the matrix $\left(a^{ij}(\underline{x})\right)$ is nonsingular. They proved, using probabilistic arguments, that the problem of determining the exit position can be reduced to determining the point on the boundary where a certain function V(y) attains its minimum value. In particular, the origin is a unique asymptotically stable point and

$$V(\underline{y}) = \inf_{\underline{\varphi} \in H(\underline{0}, \underline{y})} I_{T_1, T_2}(\underline{\varphi}) \tag{3.1.1}$$

where

$H(\underline{x}, \underline{y})$ is the set of all absolutely continuous functions $\varphi(t)$ such that $\underline{\varphi}(T_1) = \underline{0}$ and $\underline{\varphi}(T_2) = \underline{y}$ and T_1 and T_2 are arbitrary (3.1.2)

$$I_{T_1, T_2}(\underline{\varphi}) = \int_{T_1}^{T_2} \sum_{i,j=1}^{n} a_{ij}\left(\underline{\varphi}(y)\right)\left(\dot{\varphi}^i - b^i(\underline{\varphi}(t))\right)\left(\dot{\varphi}^j - b^j(\underline{\varphi}(t))\right) dt \qquad (3.1.3)$$

$$\left(a_{ij}(\cdot)\right) = \left(a^{ij}(\cdot)\right)^{-1} \qquad (3.1.4)$$

Thus Ventsel' and Freidlin assume that there is a unique point on $\partial\Omega$ at which $V(\underline{y})$ attains its minimum value, so the probability distribution of exit positions must be a δ-function centered at that unique point.

In order to compare the results of Chapter II with the results of Ventsel' and Freidlin, observe from (3.1.3) that $I_{T_1, T_1}(\underline{\varphi}) = 0$. If we regard the integrand in (3.1.3) as a Lagrangian

$$L = L\left(\underline{\varphi}(t) \; , \; \frac{d\underline{\varphi}}{dt}\right) \qquad (3.1.5)$$

then it can be shown that the Lagrangian corresponds to a Hamiltonian

$$H = H(\underline{y}, \underline{p}) = \underline{p} \cdot \frac{d\underline{y}}{dt} - L(y, \frac{d\underline{y}}{dt}) \qquad (3.1.6)$$

where

$$p_i = \sum_{j=1}^{n} a_{ij}(\underline{y})\left(\frac{dy_i}{dt} - b^j(\underline{y})\right) \tag{3.1.7}$$

Thus, if $V(\underline{y})$ is defined as in (3.1.1), then it satisfies the Hamilton-Jacobi equation

$$\frac{1}{2}\sum_{i,j=1}^{n} a^{ij}(\underline{y})\frac{\partial V}{\partial y_i}\frac{\partial V}{\partial y_j} + \sum_{i=1}^{n} b^i(\underline{y})\frac{\partial V}{\partial y_i} = 0 \tag{3.1.8}$$

But this is exactly the eiconal equation (2.4.6) for a function which we called $\varphi(\underline{x})$. Note that $V(\underline{0}) = 0$ as does $\varphi(\underline{x})$ at the limit point. Thus, the function which we called $\varphi(\underline{x})$ is a solution which Ventsel' and Freidlin would denote by $V(y)$. In particular, if there is a unique point on $\partial\Omega$ which minimizes $\varphi(\underline{x})$ (or equivalently $V(y)$), then we see from the discussion following (2.4.53) that we have obtained the correct probability density for the exit position. Ventsel' and Freidlin did not demonstrate a method to construct the solution, and the mathematical tools which they used are incapable of determining the distribution of exit positions in the case where the point which minimizes $\varphi(\underline{x})$ (or $V(\underline{y})$) is not unique.

Matkowsky and Schuss [11], [12] have been able to extend the results of Ventsel' and Freidlin. They also restrict themselves to the case where the matrix $\left(a^{ij}(\underline{x})\right)$ is nonsingular and to where the equilibrium points are distinct. They demonstrate the existence of the boundary layer for the solution of (2.2.2). They also obtain exactly the same equation for the unknown parameter C_0 as (2.4.53).

Matkowsky and Schuss also examine the case where the vector $\underline{b}(x)$ is the gradient of some scalar field $\psi(\underline{x})$. In that case, they were explicitly able to determine the function $\varphi(\underline{x})$. The only differences in the two results are in the boundary layer correction and in the calculation of the asymptotic representation of the solution of the homogeneous adjoint problem, $v(\underline{x};\varepsilon)$.

The first difference lies in the calculation of the boundary layer correction and is rather subtle. The boundary value for the problem is $u(\underline{x}) = f(\underline{x})$ which is an arbitrary unknown function. It is not a priori clear how to extend this unknown function into the interior of Ω in order to obtain a uniformly valid expression for the solution $u(\underline{x};\varepsilon)$. There would also appear to be a question about how the distance between an arbitrary point $\underline{x}$ and the boundary is to be defined so that an "inner solution" of (2.2.2) can be constructed. We have chosen to circumvent these questions by assuming a typical form for the boundary layer correction and then determining the various unknown functions from the boundary layer equation and the boundary values.

The second difference lies in the construction of the solution $v(\underline{x};\varepsilon)$. We both assume the same form for $V(\underline{x};\varepsilon)$, but we construct the solution in different manners. Matkowsky and Schuss choose to solve the eiconal equation and the transport equation by starting at the boundary and integrating their ray equations into the interior of Ω. They then try to meet a final condition on $\varphi(\underline{x})$ and $w^{(0)}(\underline{x})$ at the limit point. This is unsatisfying since they are unable to prescribe $\nabla\varphi(\underline{x})$ on the boundary as an initial condition for the integration of the ray equations. Instead, we integrate the ray equations (2.4.8) and (2.4.9) from the limit point outward to the boundary, using the technique of Ludwig [10]. Again, this is a subtle point. Also, Matkowsky and Schuss do not prescribe boundary values for the adjoint problem, as in (2.4.3).

Thus we have been able to duplicate previous results in the case where the matrix $\left(a^{ij}(\underline{x})\right)$, is nonsingular in Ω. The results of Chapter II indicate that this restriction is, in fact, unnecessary in the general case. In the next section, we compare results for the mean exit time problem.

3.2 A Comparison of Results for the Mean Exit Time Problem

Various authors have published results in this area for about fifteen years. Miller [14] developed a technique to study the persistence of dynamical systems in a genetics problem with one dimension. The problem was such that the infinitesimal generator degenerated, i.e., both $b(x)$ and $a(x)$ vanished on the boundary. Miller started with the Fokker-Planck equation, and assumed that he could find an eigenfunction expansion where the minimum eigenvalue would be a reasonable approximation to the reciprical of the mean exit time. Specifically, if A^*v is the adjoint operator to Au, Miller wanted to determine the minimum eigenvalue, λ_{min}, such that

$$A^*v + \lambda_{min} v = 0 . \qquad (3.2.1)$$

He integrated (3.2.1) over Ω to find

$$\lambda_{min} \sim \frac{-\int_\Omega A^* v \, dx}{\int_\Omega v \, dx} \qquad (3.2.2)$$

Hence,

$$E[\tau] = \frac{\int_\Omega v \, dx}{-\left[\frac{\varepsilon^2}{2} \frac{\partial}{\partial x}\left(a(x) v\right)\right]_{\partial\Omega}} \qquad (3.2.3)$$

Miller required that the solution be integrable in Ω, that $b(x)$ have a simple zero at an interior point $\hat{x}$ and that $v(x;\varepsilon)$ could be written asymptotically as

$$v(x;\varepsilon) \sim \frac{\theta(x)}{a(x)} \exp\left\{-\frac{\varphi(x)}{\varepsilon^2}\right\} \qquad (3.2.4)$$

where $\theta(x)$ is a smooth function with $\theta(x) \equiv 1$ in the neighborhood of $\hat{x}$, and $\theta(x) \equiv 0$ in the neighborhood of the endpoints. He determined the eiconal equation for $\varphi(x)$, set $\varphi(\hat{x}) = 0$, and evaluated the integral using Laplace's method. He found

$$E[\tau] \sim \frac{\frac{2}{a(\hat{x})} \left| \frac{2\pi\varepsilon^2}{\varphi''(\hat{x})} \right|^{1/2}}{\left[\varphi'(x) \exp\left\{-\frac{\varphi(x)}{\varepsilon^2}\right\}\right]_{\partial\Omega}} \qquad (3.2.5)$$

Ludwig [10] extended the results of Miller to higher dimensions. He assumed that

$$v(\underline{x}) \sim z(\underline{x}) \exp\{-\frac{\varphi(\underline{x})}{\varepsilon^2}\} \tag{3.2.6}$$

Thus, to leading order

$$\lambda \sim \frac{\int_{\partial\Omega} \frac{1}{2} \sum_{i,j=1}^{n} a^{ij}(\underline{x}) \frac{\partial \varphi(\underline{x})}{\partial x_j} \nu_i \, z(\underline{x}) \exp\{-\frac{\varphi(\underline{x})}{\varepsilon^2}\} dS}{\int \cdots \int_{\Omega} z(\underline{x}) \exp\{-\frac{\varphi(\underline{x})}{\varepsilon^2}\} d\underline{x}} \tag{3.2.7}$$

This expression is equivalent to (2.4.27), provided that $C_0 \sim \lambda^{-1}$ since

$$\frac{1}{2} \sum_{i,j=1}^{n} a^{ij}(\underline{x}) \frac{\partial \varphi(\underline{x})}{\partial x_j} \nu_i = \sum_{i=1}^{n} b^i(\underline{x}) \nu_i \; ; \; \underline{x} \in \partial\Omega \tag{3.2.8}$$

when $\varphi(\underline{x}) = \varphi_{min}$. Thus the results of Chapter II are consistent with Ludwig's results.

Ventsel' and Freidlin [16] have also examined the asymptotic behavior of the mean exit time for the case where the parameter ε is small and where the matrix $\left(a^{ij}(\underline{x})\right)$ is nonsingular. Using probabilistic methods, they proved that

$$\lim_{\varepsilon \downarrow 0} 2\varepsilon^2 \, \ell n \, E_{\underline{x}_0} [\tau] = \min_{\underline{y} \in \partial\Omega} V(\underline{x}_0, \underline{y}) \tag{3.2.9}$$

where $\underline{x}_0$ is an asymptotically stable limit point in Ω, and $V(\underline{x}_0, \underline{y})$ is defined as in Section 3.1. Observe that these results are precisely the same as we observed in (2.4.55). We have been able to determine the asymptotic constant which Ventsel' and

Freidlin were unable to determine.

Finally, Matkowsky and Schuss [12] have studied the mean exit time problem. The method which they employed is the same as Ludwig's method. An analysis of the validity of each method will not be presented here, but we will note the restrictions and summarize the results. The two authors restrict themselves to the case where the matrix $\left(a^{ij}(\underline{x})\right)$ is nonsingular and the drift vector $\underline{b}(\underline{x})$ is essentially the gradient of a scalar field. They demonstrate the existence of a boundary layer for the solution of the mean exit time problem and calculate the boundary layer correction. They determine the solution of the homogeneous adjoint problem and find an equation from which they determine the unknown parameter C_0. The equation for the parameter C_0 is exactly the same as (2.4.54).

Thus we have been able to reproduce previous results when we restrict ourselves to the conditions which the original authors used. We have been able to extend the results to the general problem of exit against a flow.

CHAPTER IV

We devote this chapter to a study of asymptotic representations for probability distributions of exit positions for various dynamical systems. We are interested in the asymptotic results when the magnitude of the noise perturbing the systems is small. In particular, we are concerned with a study of the probability distribution of exit positions for the Ornstein-Uhlenbeck process as well as a damped linear harmonic oscillator and a damped pendulum subject to Gaussian white noise excitation. We also study an example of exiting from the domain of attraction of a stable limit cycle.

The chapter is divided into five sections. In the first section, we present the asymptotic evaluation of the exact solution of the distribution of exit positions for the Ornstein-Uhlenbeck process. The results of the second section predict the asymptotic distribution of exit positions, using the results of Chapter II. The third section is concerned with a study of the asymptotic distribution of exit positions for the damped linear harmonic oscillator. In the fourth section, we study the asymptotic distribution of exit positions for the damped pendulum. Finally, we study a process diffusing from an asymptotically stable limit cycle.

4.1 Asymptotic Evaluation of the Probability Distribution of Exit Positions for the Ornstein-Uhlenbeck Process

We begin with a study of the probability distribution of exit positions for the Ornstein-Uhlenbeck process. This process models a damped linear harmonic oscillator with a negligibly small spring constant. The equation of motion becomes

$$\ddot{x}(t) + \beta \dot{x}(t) = \varepsilon \frac{d}{dt} w(t) \qquad (4.1.1)$$

If we write $y(t) \equiv \dot{x}(t)$ and take $\beta = 1$, we can rewrite (4.1.1) as

$$dy(t) = -y(t)dt + \varepsilon dw(t) \; . \qquad (4.1.2)$$

Take $\Omega = \{y: -a < y < b\}$ where $a > 0$, $b > 0$. The boundary value problem which one must solve in order to study the distribution of exit positions for the Ornstein-Uhlenbeck process is given by

$$\left.\begin{aligned} \frac{\varepsilon^2}{2} \frac{d^2 u}{dy^2} - y \frac{du}{dy} &= 0 \qquad && y \in \Omega \\ u(y;\varepsilon) &= f(y) \qquad && y \in \partial\Omega \end{aligned}\right\} \qquad (4.1.3)$$

We can formally integrate the equation twice to find

$$u(y;\varepsilon) = f(-a) + \alpha \int_{-a}^{y} \exp\{\frac{z^2}{\varepsilon^2}\} \, dz \; . \qquad (4.1.4$$

When we apply the boundary condition at $y = b$, we find

$$u(y;\varepsilon) = \left[\int_{-a}^{b} \exp\{\frac{z^2}{\varepsilon^2}\} dz\right]^{-1} \cdot \left[f(b)\int_{-a}^{y} \exp\{\frac{z^2}{\varepsilon^2}\} dz + f(-a)\int_{y}^{b} \exp\{\frac{z^2}{\varepsilon^2}\} dz\right] \qquad (4.1.5)$$

In the notation of Chapter I, we note

$$P_y\{x_\varepsilon(\tau)=b\} = \frac{\int_{-a}^{y} \exp\{\frac{z^2}{\varepsilon^2}\}\,dz}{\int_{-a}^{b} \exp\{\frac{z^2}{\varepsilon^2}\}\,dz} \qquad (4.1.6)$$

$$P_y\{x_\varepsilon(\tau)=-a\} = \frac{\int_{y}^{b} \exp\{\frac{z^2}{\varepsilon^2}\}\,dz}{\int_{-a}^{b} \exp\{\frac{z^2}{\varepsilon^2}\}\,dz} \qquad (4.1.7)$$

Observe that the integrals involved are such that the maximum contribution occurs at the end points. When we evaluate the integrals using Laplace's method, we find we can distinguish three separate cases:

Case 1: $a < b$.

Provided that y is away from either end point, we see that $P_y\{x_\varepsilon(\tau)=b\} \sim 0$ and $P_y\{x_\varepsilon(\tau)=-a\} \sim 1$. If y is near b, then $P_y\{x_\varepsilon(\tau)=b\} \sim 1$ and $P_y\{x_\varepsilon(\tau)=-a\} \sim 0$. Conversely, if y is near -a, then $P_y\{x_\varepsilon(\tau)=b\} \sim 0$ and $P_y\{x_\varepsilon(\tau)=-a\} \sim 1$. Thus we expect that the Ornstein-Uhlenbeck process will be far more likely to exit at the point $y=-a$, provided that the process does not start too close to the point $y=b$ initially.

Case 2: $a = b$.

If y is away from either endpoint, we see that $P_y\{x_\varepsilon(\tau)=b\} \sim$

$P_y\{x_\varepsilon(\tau) = -a\} \sim 1/2$. If y is sufficiently near either endpoint, then the probability of exit at that point will tend to one. Physically we would expect that the process would be about equally likely to exit from either side due to symmetry, provided that it was not too close to either endpoint initially.

Case 3: $a > b$.

We find that the results in this case are exactly the opposite of the results in case 1. Again, we would expect that the Ornstein-Uhlenbeck process would be most likely to exit at the point closest to the origin, provided that the process was not too close to the other boundary initially.

These results agree with physical intuition and are presented only because we can solve the boundary value problem exactly. In the next section, we use the theory developed in Chapter II to predict the distribution of exit positions for the Ornstein-Uhlenbeck process. We then compare the results of this section and the next one.

4.2 Predicted Probability Distribution of Exit Positions for the Ornstein-Uhlenbeck Process

We know from the previous section that the boundary value problem which we must solve in order to determine the probability distribution of exit positions for the Ornstein-Uhlenbeck process is

$$\left.\begin{aligned} &\frac{\varepsilon^2}{2}\frac{d^2u}{dy^2} - y\frac{du}{dy} = 0 \qquad\qquad y \in (-a, b) \\ &u(-a) = f(-a) \;;\; u(b) = f(b) \end{aligned}\right\} \tag{4.2.1}$$

When we apply the theory of Chapter II, we assume

$$u(y;\varepsilon) \sim C_0 + z_1(y)\exp\{-\frac{\zeta_1(y)}{\varepsilon^2}\} + z_2(y)\exp\{-\frac{\zeta_2(y)}{\varepsilon^2}\} \tag{4.2.2}$$

where $\zeta_1(y)$ is small near $y = -a$ and $\zeta_2(y)$ is small near $y = b$. Because the boundary $\partial\Omega$ consists only of separate points, rather than being smooth, we observe in (4.2.2) that we do not have smooth functions $z(y)$ and $\zeta(y)$. We find that we have two separate boundary layer corrections which become transcendentally small compared with any finite power of ε^2 at the opposite endpoint. The homogeneous adjoint problem is given by

$$\left.\begin{aligned} &\frac{\varepsilon^2}{2}\frac{d^2v}{dy^2} + \frac{d}{dy}(yv) = 0 \qquad\qquad y \in (-a, b) \\ &\frac{\varepsilon^2}{2}\frac{dv}{dy} + yv = 0 \end{aligned}\right\} \tag{4.2.3}$$

By inspection, the properly normalized solution of (4.2.3) is

$$v(y;\varepsilon) \sim \exp\{-\frac{y^2}{\varepsilon^2}\} \tag{4.2.4}$$

The parameter C_0 is given by

$$C_0 \sim \frac{bf(b)\exp\{-\frac{b^2}{\varepsilon^2}\} + a\,f(-a)\exp\{-\frac{a^2}{\varepsilon^2}\}}{b\exp\{-\frac{b^2}{\varepsilon^2}\} + a\exp\{-\frac{a^2}{\varepsilon^2}\}} \quad . \tag{4.2.5}$$

In order to specify the solution $u(y;\varepsilon)$, we must now solve for $z_1(y)$, $\zeta_1(y)$, $z_2(y)$, and $\zeta_2(y)$. For y near the point $y = -a$, we note

$$\frac{1}{2}\left(\frac{d\zeta_1(-a)}{dy}\right)^2 - a\,\frac{d\,\zeta_1(-a)}{dy} = 0 \tag{4.2.6}$$

Take

$$\frac{d\zeta_1(-a)}{dy} = 2a$$

So

$$\zeta_1(y) = 2a(a + y) \tag{4.2.7}$$

From (2.3.13) we see that $z_1(y)$ satisfies

$$\frac{dz_1}{dt} = 0 \tag{4.2.8}$$

along the rays, with $z_1\big(y(0)\big) = f(-a) - C_0$. Thus

$$z_1\big(y(t)\big) \equiv z_1(y) \equiv f(-a) - C_0 \tag{4.2.9}$$

Similarly

$$\zeta_2(y) = 2b(b-y) \tag{4.2.10}$$

$$z_2(y) \equiv f(b) - C_0 \tag{4.2.11}$$

Hence

$$u(y;\varepsilon) \sim \frac{bf(b)\exp\{-\frac{b^2}{\varepsilon^2}\} + a\, f(-a)\exp\{-\frac{a^2}{\varepsilon^2}\}}{b\exp\{-\frac{b^2}{\varepsilon^2}\} + a\exp\{-\frac{a^2}{\varepsilon^2}\}}$$

$$\frac{a\left[f(b)-f(-a)\right]\exp\{-\frac{a^2+2b(b-y)}{\varepsilon^2}\} + b\left[f(-a)-f(b)\right]\exp\{-\frac{b^2+2a(a+y)}{\varepsilon^2}\}}{b\exp\{-\frac{b^2}{\varepsilon^2}\} + a\exp\{-\frac{a^2}{\varepsilon^2}\}} \qquad (4.2.12)$$

Thus, using the notation of Chapter I, we see that

$$P_y\{x_\varepsilon(\tau) = b\} = \frac{b\exp\{-\frac{b^2}{\varepsilon^2}\} + a\exp\{-\frac{a^2+2b(b-y)}{\varepsilon^2}\} - b\exp\{-\frac{b^2+2a(a+y)}{\varepsilon^2}\}}{b\exp\{-\frac{b^2}{\varepsilon^2}\} + a\exp\{-\frac{a^2}{\varepsilon^2}\}} \qquad (4.2.13)$$

$$P_y\{x_\varepsilon(\tau) = -a\} = \frac{a\exp\{-\frac{a^2}{\varepsilon^2}\} - a\exp\{-\frac{a^2+2b(b-y)}{\varepsilon^2}\} + b\exp\{-\frac{b^2+2a(a+y)}{\varepsilon^2}\}}{b\exp\{-\frac{b^2}{\varepsilon^2}\} + a\exp\{-\frac{a^2}{\varepsilon^2}\}} \qquad (4.2.14)$$

Again, we find that we can distinguish three separate cases:

Case 1: $a < b$

We see that if y is away from either endpoint, then $P_y\{x_\varepsilon(\tau) = -a\} \sim 1$ and $P_y\{x_\varepsilon(\tau) = b\} \sim 0$. If y is near b, then $P_y\{x_\varepsilon(\tau) = b\} \sim 1$. Conversely, if y is near -a, then $P_y\{x_\varepsilon(\tau) = -a\} \sim 1$.

Case 2 : a = b

We see that if y is away from either endpoint, then $P_y\{x_\varepsilon(\tau) = -a\} \sim P_y\{x_\varepsilon(\tau) = b\} \sim 1/2$. As in the first case, the probabilities tend to the limiting 0-1 probabilities as the initial point y moves to either endpoint.

Case 3: a > b

The results for this case are exactly the oppositive of case 1.

We see that we have obtained exactly the same results using the theory developed in Chapter II as we found in the first section of this chapter. In the next section, we examine the problem of the probability distribution of exit positions for the damped linear harmonic oscillator.

4.3 Probability Distribution of Exit Positions for the Damped Linear Harmonic Oscillator

We now turn to a study of the distribution of exit positions of a damped linear harmonic oscillator subject to Gaussian white noise excitation. We can write the equation as

$$\ddot{x}(t) + 2\beta\,\dot{x}(t) + x(t) = \varepsilon\, dw(t) \tag{4.3.1}$$

In differential matrix form, we can write this as

$$d\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ -2\beta_y(t)-x(t) \end{pmatrix} dt + \varepsilon \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} dw_1(t) \\ dw_2(t) \end{pmatrix} \tag{4.3.2}$$

The boundary value problem which we must solve in order to determine the distribution of exit positions for this dynamical system is given by

$$\left.\begin{aligned} \frac{\varepsilon^2}{2}\frac{\partial^2 u}{\partial y^2} - (2\beta y + x)\frac{\partial u}{\partial y} + y\frac{\partial u}{\partial x} &= 0 \qquad (x,y)\in\Omega \\ u(x,y) &= f(x,y) \qquad (x,y)\in\partial\Omega \end{aligned}\right\} \tag{4.3.3}$$

We take $\Omega = \{(x,y) : x^2+y^2 < r^2\}$ so that the boundary represents a surface of constant energy in the phase plane. When we apply the theory of Chapter II, we assume that we can represent $u(x,y;\varepsilon)$ as

$$u(x,y;\varepsilon) \sim C_0 + z(x,y)\exp\left\{-\frac{\zeta(x,y)}{\varepsilon^2}\right\} \tag{4.3.4}$$

In order to determine the unknown parameter C_0, we must solve the general boundary value problem (2.4.3) which for this process is

$$\left.\begin{aligned} \frac{\varepsilon^2}{2}\frac{\partial^2 v}{\partial y^2} + \frac{\partial}{\partial y}\Big[(2\beta y + x)v\Big] - \frac{\partial}{\partial x}\Big[yv\Big] &= 0 \qquad (x,y)\in\overline{\Omega} \\ \left(\frac{\varepsilon^2}{2}\frac{\partial v}{\partial y} + (2\beta y + x)v\right)\nu_y - yv\nu_x &= 0 \qquad (x,y)\in\partial\Omega \end{aligned}\right\} \tag{4.3.5}$$

We assume that $v(x,y;\varepsilon)$ has the asymptotic form (2.4.5) so that the eiconal equation and boundary condition for $\varphi(x,y)$ become

$$\left.\begin{aligned} &\frac{1}{2}\left(\frac{\partial \varphi}{\partial y}\right)^2 - (2\beta y + x)\frac{\partial \varphi}{\partial y} + y\frac{\partial \varphi}{\partial x} = 0 \qquad (x, y) \in \overline{\Omega} \\ &\left(\frac{1}{2}\frac{\partial \varphi}{\partial y} - 2\beta y - x\right)\nu_y + y\nu_x = 0 \qquad (x, y) \in \partial\Omega \end{aligned}\right\} \tag{4.3.6}$$

In order to calculate $\varphi(x, y)$, we must calculate the matrix of second derivatives at the origin. The covariance matrix S satisfies the linear matrix equation (2.4.14) which, for this problem, we write as a system of simultaneous equations:

$$s_{12} + s_{21} = 0 \tag{4.3.7a}$$

$$-s_{11} - 2\beta s_{12} + s_{22} = 0 \tag{4.3.7b}$$

$$s_{22} - s_{11} - 2\beta s_{21} = 0 \tag{4.3.7c}$$

$$1 - s_{21} - 2\beta s_{22} - s_{12} - 2\beta s_{22} = 0 \tag{4.3.7d}$$

Thus, one finds that $S = \frac{1}{4\beta} I$. On the initial ellipsoid about the origin we assume that

$$\left.\begin{aligned} \varphi(x, y) &= 2\beta(x^2 + y^2) + o(x^2 + y^2) \\ \varphi_x(x, y) &= 4\beta x + o\left(\sqrt{x^2 + y^2}\right) \\ \varphi_y(x, y) &= 4\beta y + o\left(\sqrt{x^2 + y^2}\right). \end{aligned}\right\} \tag{4.3.8}$$

Starting from the initial ellipsoid, we integrate $\varphi\big(x(t), y(t)\big)$ along the rays given by

$$\left.\begin{aligned} \dot{x}(t) &= y(t) \\ \dot{y}(t) &= -\big(2\beta y(t) + x(t)\big) + q(t) \\ \dot{p}(t) &= q(t) \\ \dot{q}(t) &= 2\beta\, q(t) - p(t) \end{aligned}\right\} \tag{4.3.9}$$

to find that

$$\varphi(x, y) = 2\beta\,(x^2 + y^2) \tag{4.3.10}$$

is the solution of the eiconal equation. At the boundary, the outer normal unit vector is given by

$$\underline{\nu}(x, y) = \frac{1}{r}\begin{pmatrix} x \\ y \end{pmatrix} \tag{4.3.11}$$

Then $\varphi(x, y)$, as given by (4.3.10) also satisfies the boundary condition of (4.3.6). Furthermore, $\varphi(x, y)$ is properly normalized at the origin.

We must now find the transport term $w^{(0)}(x, y)$. The transport equation is

$$4\beta\, y\, \frac{\partial w^{(0)}}{\partial y} + y\, \frac{\partial w^{(0)}}{\partial x} - (2\beta y + x)\, \frac{\partial w^{(0)}}{\partial y} - \left[\frac{1}{2}\cdot 4\beta - 2\beta\right] w^{(0)} = 0 \tag{4.3.12}$$

We can rewrite this as an ordinary differential equation along the rays:

$$\frac{d}{dt}\, w^{(0)}\Big(x(t), y(t)\Big) = 0 \tag{4.3.13}$$

Since $w^{(0)}(x, y)$ is assumed to be unity on the initial ellipsoid, we find that $w^{(0)}(x, y) \equiv 1$ in Ω .

Then the unknown parameter C_0 is determined by

$$C_0 \sim \frac{\int_{\partial\Omega} f(x, y)\, \underline{b}(x, y) \cdot \underline{\nu}(x, y) \exp\left\{-\frac{\varphi(x, y)}{\varepsilon^2}\right\} dS}{\int_{\partial\Omega} \underline{b}(x, y) \cdot \underline{\nu}(x, y) \exp\left\{-\frac{\varphi(x, y)}{\varepsilon^2}\right\} dS}$$

$$= \frac{-\int_0^{2\pi} f(r, \theta)\, 2\beta r \sin^2\theta \exp\left\{-\frac{2\beta r^2}{\varepsilon^2}\right\} d\theta}{-\int_0^{2\pi} 2\beta r \sin^2\theta \exp\left\{-\frac{2\beta r^2}{\varepsilon^2}\right\} d\theta}$$

$$= \frac{1}{\pi} \int_0^{2\pi} f(r, \theta) \sin^2\theta \, d\theta \tag{4.3.14}$$

Now that we have determined the parameter C_0, we can calculate the boundary layer correction. We find from (2.3.6) that

$$\left| \nabla \zeta(x, y) \right| = 4\beta r \tag{4.3.15}$$

Thus we see

$$\nabla \zeta(x, y) = -4\beta \begin{pmatrix} x \\ y \end{pmatrix} \tag{4.3.16}$$

So

$$\zeta(x, y) = 2\beta(r^2 - x^2 - y^2)$$

Along the rays defined by

$$\left.\begin{aligned} \dot{x}(t) &= -y(t) \\ \dot{y}(t) &= 2\beta\, y(t) + x(t) + q(t) \\ \dot{p}(t) &= q(t) \\ \dot{q}(t) &= 2\beta\, q(t) - p(t) \end{aligned}\right\} \qquad (4.3.17)$$

we find that the transport term $z(x, y)$ satisfies the initial value problem

$$\left.\begin{aligned} \frac{d}{dt} z\big(x(t), y(t)\big) - 2\beta\, z\big(x(t), y(t)\big) &= 0 \\ z\big(x(0), y(0)\big) = f\big(x(0), y(0)\big) &- C_0 \end{aligned}\right\} \qquad (4.3.18)$$

So we see that $z\big(x(t), y(t)\big) = \Big[f\big(x(0), y(0)\big) - C_0\Big]\, e^{2\beta t}$.

The form for $z\big(x(t), y(t)\big)$ tells us how to extend the boundary conditions into the interior of Ω. As t becomes large, $z\big(x(t), y(t)\big)$ will tend toward $z(0, 0)$ due to the requirement that the rays converge to the limit set for large time. However, due to the form of $\zeta(x, y)$ we observe that the region where the boundary layer correction term is significant is only $O(\varepsilon)$ wide. In order to predict the distribution of exit positions from a point (x, y) in the boundary layer, we would trace backward along the ray passing through (x, y) to find from where on the boundary the ray emanated since the trajectory of the process is most likely to be near the given ray for short distances. If the initial point (x, y) does not lie in the boundary layer, then the distribution of exit positions is $\frac{1}{\pi}\sin^2\theta$ where θ is the conventional polar angle. Note that the points $\theta = 0$ and $\theta = \pi$ are asymptotically inaccessible points on the boundary, provided that the initial state of the oscillator is not at either point, since $\sin^2\theta$ vanishes there. This is to be expected since the direction of the noise is tangent to the boundary at $\theta = 0$ and $\theta = \pi$. Thus we have been able to mirror a physical phenomenon in the mathematics.

4.4 Probability Distribution of Exit Positions for the Damped Pendulum

We now consider the problem of determining the distribution of exit positions for a damped pendulum subject to Gaussian white noise excitation. We write the equation of motion as

$$\ddot{\theta}(t) + 2\beta\dot{\theta}(t) + \sin\theta(t) = \varepsilon dw(t) . \qquad (4.4.1)$$

In differential matrix form, we can write this as

$$d\begin{pmatrix}\theta(t)\\ \omega(t)\end{pmatrix} = \begin{pmatrix}\omega(t)\\ -2\ \omega(t)-\sin\theta(t)\end{pmatrix} dt + \varepsilon\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}dw_1(t)\\ dw_2(t)\end{pmatrix} \qquad (4.4.2)$$

The boundary value problem which we must solve in order to determine the distribution of exit positions for this particular dynamical system is

$$\left.\begin{aligned} &\frac{\varepsilon^2}{2}\frac{\partial^2 u}{\partial\omega^2} - (2\beta\omega+\sin\theta)\frac{\partial u}{\partial\omega} + \omega\frac{\partial u}{\partial\theta} = 0 && (\theta,\omega)\in\Omega \\ &u(\theta,\omega) = f(\theta,\omega) && (\theta,\omega)\in\partial\Omega \end{aligned}\right\} \quad (4.4.3)$$

We take $\Omega = \{(\theta,\omega) : \omega^2 - 2\cos\theta < 2\}$. When we apply the theory of Chapter II, we assume that we can represent $u(\theta,\omega;\varepsilon)$ as

$$u(\theta,\omega;\varepsilon) \sim C_0 + z(\theta,\omega)\exp\left\{-\frac{\zeta(\theta,\omega)}{\varepsilon^2}\right\} \quad (4.4.4)$$

In order to determine the unknown parameter C_0, we must solve the general boundary value problem (2.4.3). For this particular dynamical system, we must solve

$$\left.\begin{aligned} &\frac{\varepsilon^2}{2}\frac{\partial^2 v}{\partial\omega^2} + \frac{\partial}{\partial\omega}\left[(2\beta\omega+\sin\theta)v\right] - \frac{\partial}{\partial\theta}\left[\omega v\right] = 0 && (\theta,\omega)\in\overline{\Omega} \\ &\left[\frac{\varepsilon^2}{2}\frac{\partial v}{\partial\omega} + (2\beta\omega+\sin\theta)v\right]\nu_\omega - \omega v\nu_\theta = 0 && (\theta,\omega)\in\partial\Omega \end{aligned}\right\} \quad (4.4.5)$$

We assume the general asymptotic form (2.4.5) for $v(\theta,\omega;\varepsilon)$ so that the eiconal equation and boundary condition for $\varphi(\theta,\omega)$ become

$$\left.\begin{aligned} &\frac{1}{2}\left(\frac{\partial\varphi}{\partial\omega}\right)^2 - (2\beta\omega+\sin\theta)\frac{\partial\varphi}{\partial\omega} + \omega\frac{\partial\varphi}{\partial\theta} = 0 && (\theta,\omega)\in\overline{\Omega} \\ &\left(\frac{1}{2}\frac{\partial\varphi}{\partial\omega} + 2\beta\omega + \sin\theta\right)\nu_\omega - \omega\nu_\theta = 0 && (\theta,\omega)\in\partial\Omega \end{aligned}\right\} \quad (4.4.6)$$

We observe that the origin is an asymptotically stable limit point in Ω and that the points $(\pm\pi, 0)$ on the boundary are asymptotically unstable. In order to determine $\varphi(\theta,\omega)$, we must calculate the matrix of second derivatives at the origin. The covariance matrix S satisfies the linear matrix equation (2.4.14) which we write as a system of simultaneous equations

$$s_{21} + s_{12} = 0 \tag{4.4.7a}$$

$$s_{22} - s_{11} - 2\beta s_{12} = 0 \tag{4.4.7b}$$

$$-s_{11} - 2\beta s_{21} + s_{22} = 0 \tag{4.4.7c}$$

$$1 - s_{12} - 2\beta s_{22} - s_{21} - 2\beta s_{22} = 0 \tag{4.4.7d}$$

Thus we see that $S = \frac{1}{4\beta} I$. Then on the initial ellipsoid about the origin we assume

$$\left.\begin{aligned} \varphi(\theta,\omega) &= 2\beta(\theta^2+\omega^2) + o(\theta^2+\omega^2) \\ \varphi\ (\theta,\omega) &= 4\beta\,\theta + o(\sqrt{\theta^2+\omega^2}\,) \\ \varphi_\omega(\theta,\omega) &= 4\beta\,\omega + o(\sqrt{\theta^2+\omega^2}\,) \end{aligned}\right\} \tag{4.4.8}$$

Starting from the initial ellipsoid, we integrate $\varphi\big(\theta(t),\omega(t)\big)$ along the rays given by

$$\left.\begin{aligned} \dot{\theta}(t) &= \omega(t) \\ \dot{\omega}(t) &= -\Big(2\beta\omega(t) + \sin\theta(t)\Big) + q(t) \\ \dot{p}(t) &= \cos\theta(t)\, q(t) \\ \dot{q}(t) &= -p(t) + 2\beta q(t) \end{aligned}\right\} \qquad (4.4.9)$$

to find that

$$\varphi(\theta, \omega) = 2\beta(\omega^2 + 2 - 2\cos\theta) \qquad (4.4.10)$$

is a solution of the eiconal equation. At the boundary, the outer normal unit vector is given by

$$\underline{\nu}(\theta, \omega) = \frac{1}{\sqrt{\omega^2 + \sin^2\theta}} \begin{pmatrix} \sin\theta \\ \omega \end{pmatrix} \qquad (4.4.11)$$

Then $\varphi(\theta, \omega)$, as given by (4.4.10) also satisfies the boundary condition of (4.3.6). Furthermore, $\varphi(\theta, \omega)$ is properly normalized at the origin.

We must now find the transport term $w^{(0)}(\theta, \omega)$. The transport equation for $w^{(0)}(\theta, \omega)$ is given by

$$4\beta\omega \frac{\partial w^{(0)}}{\partial \omega} + \omega \frac{\partial w^{(0)}}{\partial \theta} - (2\beta\omega + \sin\theta) \frac{\partial w^{(0)}}{\partial \omega} + \left[\frac{1}{2} \cdot 4\beta - 2\beta\right] w^{(0)} = 0 \quad (4.4.12)$$

Again, we rewrite this as an ordinary differential equation along the rays:

$$\frac{d}{dt} w^{(0)}\Big(\theta(t), \omega(t)\Big) = 0 \qquad (4.4.13)$$

Since $w^{(0)}(\theta, \omega)$ is assumed to be unity on the initial ellipsoid, we find that $w^{(0)}(\theta, \omega) \equiv 1$ in Ω.

Then the unknown parameter C_0 is determined by

$$C_0 \sim \frac{\int_{\partial\Omega} f(\theta, \omega)\, \underline{b}(\theta, \omega) \cdot \underline{\nu}(\theta, \omega) \exp\left\{ - \frac{\varphi(\theta, \omega)}{\varepsilon^2} \right\} dS}{\int_{\partial\Omega} \underline{b}(\theta, \omega) \cdot \underline{\nu}(\theta, \omega) \exp\left\{ - \frac{\varphi(\theta, \omega)}{\varepsilon^2} \right\} dS}$$

$$= \frac{-\int_{\partial\Omega} f(\theta, \omega) \dfrac{2\beta\omega^2}{\sqrt{\omega^2 + \sin^2\theta}}\, dS \exp\left\{ - \frac{8\beta}{\varepsilon^2} \right\}}{-\int_{\partial\Omega} \dfrac{2\beta\,\omega^2}{\sqrt{\omega^2 + \sin^2\theta}}\, dS \exp\left\{ - \frac{8\beta}{\varepsilon^2} \right\}}$$

$$= \frac{N}{D} \qquad (4.4.14)$$

where

$$N = 2\beta \int_{\pi}^{-\pi} f\left(\theta, 2\cos\frac{\theta}{2} \right) \frac{2(1+\cos\theta)}{\sqrt{2+2\cos\theta + \sin^2\theta}} \cdot \frac{\sqrt{2+2\cos\theta+\sin^2\theta}}{\sqrt{2+2\cos\theta}}\, d\theta$$

$$+ 2\beta \int_{-\pi}^{\pi} f\left(\theta, -2\cos\frac{\theta}{2}\right) \frac{2(1+\cos\theta)}{\sqrt{2+2\cos\theta+\sin^2\theta}} \cdot \frac{\sqrt{2+2\cos\theta+\sin^2\theta}}{\sqrt{2+2\cos\theta}}\, d\theta$$

$$= +2\beta \int_{\pi}^{-\pi} f(\theta, 2\cos\frac{\theta}{2}) \cdot 2\cos\frac{\theta}{2} d\theta$$

$$+ 2\beta \int_{-\pi}^{\pi} f(\theta, -2\cos\frac{\theta}{2}) \cdot 2\cos\frac{\theta}{2} d\theta \qquad (4.4.15)$$

$$D = 2\beta \int_{\pi}^{-\pi} \frac{2(1+\cos\theta)}{\sqrt{2+2\cos\theta + \sin^2\theta}} \cdot \frac{\sqrt{2+2\cos\theta+\sin^2\theta}}{\sqrt{2+2\cos\theta}}\, d\theta$$

$$+ 2\beta \int_{-\pi}^{\pi} \frac{2(1+\cos\theta)}{\sqrt{2+2\cos\theta+\sin^2\theta}} \cdot \frac{\sqrt{2+2\cos\theta+\sin^2\theta}}{\sqrt{2+2\cos\theta}}\, d\theta$$

$$+ 4\beta \int_{-\pi}^{\pi} 2\cos\frac{\theta}{2}\, d\theta$$

$$= 32\beta \tag{4.4.16}$$

Thus we find that

$$C_0 \sim \frac{1}{8}\int_{\pi}^{-\pi} f\left(\theta, 2\cos\frac{\theta}{2}\right)\cos\frac{\theta}{2}d\theta + \frac{1}{8}\int_{-\pi}^{\pi} f\left(\theta, -2\cos\frac{\theta}{2}\right)\cos\frac{\theta}{2}d\theta \tag{4.4.17}$$

Now that we have determined the parameter C_0, we can calculate the boundary layer correction. We find from (2.3.6) that

$$|\nabla\zeta(\theta,\omega)| = 4\beta\sqrt{\omega^2 + \sin^2\theta} \tag{4.4.18}$$

Thus we see

$$\nabla\zeta(\theta,\omega) = -4\beta\begin{pmatrix}\sin\theta\\ \omega\end{pmatrix} \tag{4.4.19}$$

So

$$\zeta(\theta,\omega) = 2\beta(2-\omega^2 + 2\cos\theta)$$

Then along the rays defined by

$$\left.\begin{aligned} \dot{\theta}(t) &= -\omega(t) \\ \dot{\omega}(t) &= 2\beta\omega(t) + \sin\theta(t) + q(t) \\ \dot{p}(t) &= \cos\theta(t)\, q(t) \\ \dot{q}(t) &= 2\beta q(t) - p(t) \end{aligned}\right\} \qquad (4.4.20)$$

we find that the transport term $z(\theta, \omega)$ satisfies the initial value problem

$$\left.\begin{aligned} \frac{d}{dt} z(\theta(t), \omega(t)) - 2\beta z(\theta(t), \omega(t)) &= 0 \\ z(\theta(0), \omega(0)) &= f(\theta(0), \omega(0)) - C_0 \end{aligned}\right\} \qquad (4.4.21)$$

So we see that $z(\theta(t), \omega(t)) = \left[f(\theta(0), \omega(0)) - C_0\right] e^{2\beta t}$.

The form for $z(\theta(t), \omega(t))$ tells us how to extend the boundary conditions into the interior of Ω. As in the previous example, we again note that as t becomes large, $z(\theta(t), \omega(t))$ will tend toward $z(0, 0)$ due to the requirement that the rays converge to the limit set for large time. Again, due to the form of $\zeta(\theta,\omega)$, we observe that the region where the boundary layer correction term is significant is only $O(\varepsilon)$ wide. In order to predict the distribution of exit positions from a point (θ, ω) in the boundary layer, we would trace backward along the ray passing through (θ, ω) to find from where on the boundary the ray emanated. If the initial point (θ, ω) is not in the boundary layer, then the distribution of exit positions is $\frac{1}{8} \cos \frac{\theta}{2}$

in both the upper half-plane and the lower half-plane where θ now measures the angular deflection of the pendulum from the vertical. Note that the probability of escape through the asymptotically unstable critical points at $(\pm \pi, 0)$ vanishes, provided that the initial state of the pendulum is not at either point, since $\cos \theta/2$ vanishes there. Again, this is to be expected because the direction of the tangent to the boundary at those two points.

It is interesting to note that $\varphi(\theta, \omega)$ in (4.4.10) and $\varphi(x,y)$ in (4.3.10) represent multiples of the total energy for the undamped pendulum and the undamped linear harmonic oscillator, respectively. This is entirely due to the fact that the damping is linear in both problems. If the damping had been nonlinear in either case, then the function φ would not have represented an energy. It is also interesting to note that the nonlinear restoring force for the pendulum has tended to spread the distribution of exit positions from that of the linear harmonic oscillator. Again, the solution for the unknown parameter C_0 predicts physical phenomena.

4.5 Asymptotic Evaluation of the Probability Distribution of Exit Positions for a Dynamical System with a Limit Cycle

In this section, we apply the theory developed in Chapter II to predict the probability distribution of exit positions for a process whose deterministic trajectories wind onto an asymptotically stable limit cycle. Consider the stochastically perturbed dynamical system

$$dx(t) = \left[x(t) + y(t) - x(t)\left(x^2(t) + y^2(t)\right)\right]dt + \varepsilon\, dw_1(t)$$

$$dy(t) = \left[y(t) - x(t) - y(t)\left(x^2(t) + y^2(t)\right)\right]dt + \varepsilon\, dw_2(t) \qquad (4.5.1)$$

The boundary value problem which we must solve in order to study the distribution of exit positions for this system is

$$\frac{\varepsilon^2}{2}\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] + \left[x+y-x(x^2+y^2)\right]\frac{\partial u}{\partial x} + \left[y-x-y(x^2+y^2)\right]\frac{\partial u}{\partial y} = 0 \quad (x,y)\in\Omega$$

$$u(x,y) = f(x,y) \qquad (x,y)\in\partial\Omega$$

$$(4.5.2)$$

Take $\Omega = \left\{(x,y) : x^2+y^2 < 4\right\}$.

If we multiply the deterministic portion of the first equation in (4.5.1) by $x(t)$, multiply the deterministic portion of the second equation in (4.5.1) by $y(t)$, add the two equations, and consider polar coordinates, we find

$$\frac{1}{2}\frac{d}{dt} r^2(t) = r^2(t)\left(1-r^2(t)\right) \qquad (4.5.3)$$

Thus the origin is an asymptotically unstable critical point and the limit cycle $r^2(t) = 1$ is asymptotically stable.

In order to apply the theory developed in Chapter II. we assume we can represent $u(x,y;\varepsilon)$ as

$$u(x, y; \varepsilon) \sim C_0 + z(x, y) \exp\left\{- \frac{\zeta(x, y)}{\varepsilon^2}\right\} \tag{4.5.4}$$

We must then solve the homogeneous boundary value problem

$$\left.\begin{aligned} &\frac{\varepsilon^2}{2}\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right] - \frac{\partial}{\partial x}\left[\left(x+y-x(x^2+y^2)\right)v\right] - \frac{\partial}{\partial y}\left[\left(y-x-y(x^2+y^2)\right)v\right] = 0 \\ &\hspace{28em} (x,y) \in \overline{\Omega} \\ &\left[\frac{\varepsilon^2}{2}\frac{\partial v}{\partial x} - \left(x+y-x(x^2+y^2)\right)v\right]\nu_x + \left[\frac{\varepsilon^2}{2}\frac{\partial v}{\partial y} - \left(y-x-y(x^2+y^2)\right)v\right]\nu_y = 0. \\ &\hspace{28em} (x,y) \in \partial\Omega \end{aligned}\right\} \tag{4.5.5}$$

We assume that the solution $v(x, y; \varepsilon)$ has the asymptotic form (2.4.5) so that the eiconal equation and the boundary condition for $\varphi(x, y)$ become

$$\left.\begin{aligned} &\frac{1}{2}\left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2\right] + \left[x+y-x(x^2+y^2)\right]\frac{\partial \varphi}{\partial x} + \left[y-x-y(x^2+y^2)\right]\frac{\partial \varphi}{\partial y} = 0 \quad (x,y) \in \overline{\Omega} \\ &\left[\frac{1}{2}\frac{\partial \varphi}{\partial x} + x+y-x(x^2+y^2)\right]\nu_x + \left[\frac{1}{2}\frac{\partial \varphi}{\partial y} + y-z-y(x^2+y^2)\right]\nu_y = 0 \qquad (x,y) \in \partial\Omega. \end{aligned}\right\} \tag{4.5.6}$$

In order to determine $\varphi(x, y)$ we must determine the unit normal to the limit cycle, $\underline{\nu}(\underline{x}_L)$, as well as the function $\gamma(t)$ which is the curvature of $\varphi(x, y)$ in the direction normal to the limit cycle, evaluated on the limit cycle. Now

$$\underline{\nu}(\underline{x}_L) = \begin{pmatrix} x \\ y \end{pmatrix} = \text{radial vector} \tag{4.5.7}$$

On the limit cycle, the function $\gamma(t)$ satisfies the differential equation

$$\frac{d\gamma(t)}{dt} + \gamma^2(t)(x^2+y^2) - 4\gamma(t)(x^3+2x^2+y^4) = 0 . \tag{4.5.8}$$

Since (4.5.8) is a Riccati equation, make the substitution $\mu(t) = \gamma^{-1}(t)$ and note that $(x^2+y^2) = 1$ on the limit cycle. Then

$$\frac{d\mu(t)}{dt} + 4\mu(t) = 1 \tag{4.5.9}$$

The only periodic solution is $\mu(t) \equiv \frac{1}{4}$, so $\gamma(t) = 4$ everywhere on the limit cycle. On the initial δ-tube about the limit cycle, we assume that

$$\left.\begin{aligned} \varphi(x,y) &= 2(x^2+y^2-1) + o\left(|1-x^2-y^2|\right) \\ \varphi_x(x,y) &= 4x + o\left(\sqrt{|1-x^2-y^2|}\right) \\ \varphi_y(x,y) &= 4y + o\left(\sqrt{|1-x^2-y^2|}\right) \end{aligned}\right\} \tag{4.5.10}$$

Starting from the initial ellipsoid, we integrate $\varphi\left(x(t), y(t)\right)$ along the rays given by

$$\left.\begin{aligned} \dot{x}(t) &= \left(x(t) + y(t) - x(t)(x^2(t)+y^2(t))\right) + p(t) \\ \dot{y}(t) &= \left(y(t) - x(t) - y(t)(x^2(t)+y^2(t))\right) + q(t) \\ \dot{p}(t) &= \left[1-3x^2(t)-y^2(t)\right]p(t) - \left[1+2x(t)y(t)\right]q(t) \\ \dot{q}(t) &= \left[1-2x(t)y(t)\right]p(t) + \left[1-x^2(t)-3y^2(t)\right]q(t). \end{aligned}\right\} \tag{4.5.11}$$

to find that

$$\varphi(x, y) = \frac{1}{2}\left(x^2 + y^2 - 1\right)^2 \qquad (4.5.12)$$

is a solution of the eiconal equation. At the boundary, the outer normal unit vector is given by

$$\underline{\nu}(x, y) = \frac{1}{2}\begin{pmatrix} x \\ y \end{pmatrix} \qquad (4.5.13)$$

Then $\varphi(x, y)$, as given by (4.3.10) also satisfies the boundary condition of (4.5.6) and is properly normalized at the limit cycle.

We must now find the transport term $w^{(0)}(x, y)$. Along the limit cycle, we find

$$\left|\underline{b}\left(\underline{x}_L(t)\right)\right|^2 = \left[x(t)\left(1-x^2(t)-y^2(t)\right) + y(t)\right]^2 + \left[y(t)\left(1-x^2(t)-y^2(t)\right) - x(t)\right]^2$$

$$= x^2(t) + y^2(t) = 1 \, . \qquad (4.5.14)$$

Thus $w^{(0)}\left(\underline{x}_L(t)\right) = \text{constant}$. We normalize $w^{(0)}\left(\underline{x}_L(t)\right)$ by setting the constant to unity. On the initial δ-tube about the limit cycle, we set $w^{(0)}(x, y) = 1$ and integrate the transport equation along the rays to determine $\varphi(x, y)$ elsewhere in Ω. For this problem, we find that the transport equation is

$$\left(2x^3 + 2xy^2 - 2x + x+y - x^3 - xy^2\right) \frac{\partial w^{(0)}}{\partial x}$$

$$+ \left(2y^3 + 2x^2 - 2y + y - x - y^3 - x^2 y\right) \frac{\partial w^{(0)}}{\partial y}$$

$$+ \left[3x^2 + y^2 - 1 + 1 - 3x^2 - y^2 + 3y^2 + x^2 - 1 + 1 - 3y^2 - x^2\right] w^{(0)} = 0 \quad (4.5.15)$$

We can rewrite this more simply as an ordinary differential equation along the rays:

$$\frac{d}{dt} w^{(0)}\left(x(t), y(t)\right) = 0 \qquad (4.5.16)$$

Thus $w^{(0)}(x, y) \equiv 1$ in Ω, and we observe that $v(x, y; \varepsilon)$, as given by (2.4.5) is actually an exact solution of (4.5.5). Then the unknown parameter C_0 is determined by

$$C_0 \sim \frac{\int_{\partial\Omega} f(x, y)\, \underline{b}(x, y) \cdot \underline{\nu}(x, y) \exp\left\{- \frac{\varphi(x, y)}{\varepsilon^2}\right\} dS}{\int_{\partial\Omega} \underline{b}(x, y) \cdot \underline{\nu}(x, y) \exp\left\{- \frac{\varphi(x, y)}{\varepsilon^2}\right\} ds}$$

$$= \frac{-6 \int_0^{2\pi} f(2, \theta) d\theta \exp\left\{- \frac{9}{2\varepsilon^2}\right\}}{-6 \int_0^{2\pi} d\theta \exp\left\{- \frac{9}{2\varepsilon^2}\right\}} = \frac{1}{2\pi} \int_0^{2\pi} f(2, \theta)\, d\theta \ . \qquad (4.5.17)$$

For this particular process, we see that the probability distribution of exit positions is uniformly distributed on $[0, 2\pi]$.

Now that we have determined the parameter C_0, we can calculate the boundary layer correction term. We find from (2.3.6) that

$$\left| \nabla \zeta(x, y) \right| = 12 \qquad (x, y) \in \partial\Omega \tag{4.5.19}$$

So we see

$$\nabla \zeta(x, y) = -6\begin{pmatrix} x \\ y \end{pmatrix} \qquad (x, y) \in \partial\Omega \tag{4.5.20}$$

Thus

$$\zeta(x, y) = 3\left(4-x^2-y^2\right)$$

Then the transport term z(x, y) satisfies the initial value problem

$$\begin{aligned} &\frac{d}{dt} z\Big(x(t), y(t)\Big) - 24z\Big(x(t), y(t)\Big) = 0 \\ &z\Big(x(0), y(0)\Big) = f\Big(x(0), y(0)\Big) - C_0 \end{aligned} \tag{4.5.21}$$

along the rays defined by

$$\begin{aligned} \dot{x}(t) &= -x(t)-y(t) + x(t)\left(x^2(t)+y^2(t)\right) + p(t) \\ \dot{y}(t) &= -y(t) + x(t) + y(t)\left(x^2(t) + y^2(t)\right) + q(t) \\ \dot{p}(t) &= \left[3x^2(t) + y^2(t) - 1\right]p(t) + \left[1+2x(t)y(t)\right]q(t) \\ \dot{q}(t) &= \left[2x(t)y(t)-1\right]p(t) + \left[3y^2(t) + x^2(t) - 1\right]q(t) \end{aligned}$$

Hence $\quad z\Big(x(t), y(t)\Big) = \Big[f\Big(x(0), y(0)\Big) - C_0\Big] e^{24t}$

Again, as is typical in problems of this type, we now know how to extend the boundary values into the interior of Ω. Furthermore, the boundary layer correction term will be significant only in a region of width $O(\varepsilon)$. Outside that region, the solution will be asymptotic to the parameter C_0.

In this chapter, we have presented results for the distributions of exit positions of dynamical systems diffusing against flows. For the examples of the damped linear harmonic oscillator and the damped pendulum, the theory of Chapter II predicted behavior which is consistent with physical intuition. The results for the distribution of exit positions for the Ornstein-Uhlenbeck process demonstrate that the theory predicts results which are consistent with the asymptotic representation of the exact solution when the exact solution is known. The last example is not of interest by itself since it is not really a physical problem, but again, the theory predicted results which agree with intuition due to the radial nature of the problem.

CHAPTER V

This chapter is devoted to the study of mean exit time problems for various dynamical systems. We are interested in determining the asymptotic behavior of the mean exit times when the magnitude of the noise perturbing the systems is small. In particular, we shall be concerned with the study of mean exit times for the Ornstein-Uhlenbeck process as well as a damped linear harmonic oscillator subject to Gaussian white noise excitation. We are also concerned with the mean exit time for a damped pendulum and a problem with a limit cycle.

The chapter is divided into five sections. In the first section, the problem of the mean exit time for the Ornstein-Uhlenbeck process is solved exactly, and then evaluated asymptotically. The second section contains asymptotic results for the mean exit time of the Ornstein-Uhlenbeck process, as predicted by the results of Chapter II. Results are presented in the third section for the mean exit time of the damped linear harmonic oscillator. In the fourth section, we predict the mean exit time for a damped pendulum. Finally, we study the mean exit time for a process with a limit cycle.

5.1 Asymptotic Evaluation of the Mean Exit Time for the Ornstein-Uhlenbeck Process

We begin this chapter with a study of the mean exit time of the Ornstein-Uhlenbeck process. Using (4.1.2), we find that we must solve the boundary value problem

$$\left.\begin{aligned} &\frac{\varepsilon^2}{2}\frac{d^2u}{dy^2} - y\frac{du}{dy} = -1 \qquad\qquad y \in (-a, b) \\ &u(-a) = u(b) = 0 \end{aligned}\right\} \tag{5.1.1}$$

Let $z = y/\varepsilon$ be a new stretched independent variable and let $v(z) = u(y)$ be the new dependent variable. The rescaled boundary value problem then becomes

$$\left.\begin{aligned} &\frac{1}{2}\frac{d^2v}{dz^2} - z\frac{dv}{dz} = -1 \\ &v\left(-\frac{a}{\varepsilon}\right) = v\left(\frac{b}{\varepsilon}\right) = 0 \end{aligned}\right\} \tag{5.1.2}$$

We can make (5.1.2) self-adjoint by introducing the introducing factor $\exp\{-z^2\}$. We find

$$\left.\begin{aligned} &\frac{d}{dz}\left(\exp\{-z^2\}\frac{dv}{dz}\right) = -2\exp\{-z^2\} \\ &v\left(-\frac{a}{\varepsilon}\right) = v\left(\frac{b}{\varepsilon}\right) = 0 \end{aligned}\right\} \tag{5.1.3}$$

We formally integrate (5.1.3) twice to find [8]

$$\begin{aligned} v(z) &= -2\int_0^z dx\,\exp\{s^2\}\int_0^t dt\,\exp\{-t^2\} + \alpha'\int_0^z ds\,\exp\{s^2\} + \beta \\ &= -\sqrt{\pi}\int_0^z ds\,\exp\{s^2\}\,\mathrm{erf}(s) + \alpha'\int_0^z ds\,\exp\{s^2\} + \beta \end{aligned} \tag{5.1.4}$$

where the constants α' and β are determined from the boundary conditions. Let $\alpha' = \alpha\sqrt{\pi}$ and note that the problem is symmetric under reflections. Thus we find

$$v(z) = \sqrt{\pi}\left(\alpha \operatorname{sgn}(z)-1\right)\int_0^{|z|} ds \exp\{s^2\} + \sqrt{\pi}\int_0^{|z|} ds \exp\{s^2\}\operatorname{erfc}(s) + \beta \quad (5.1.5)$$

This result is valid for all z in $-\frac{a}{\varepsilon} < z < \frac{b}{\varepsilon}$. From Abramowitz and Stegun [1], we find

$$\frac{1}{s+\sqrt{s^2+2}} < \frac{\sqrt{\pi}}{2}\exp\{s^2\}\operatorname{erfc}(s) \le \frac{1}{s+\sqrt{s^2+4/\pi}} \quad \text{for } s \ge 0 .$$

Hence

$$2\int_0^{|z|} \frac{ds}{s+\sqrt{s^2+2}} < \sqrt{\pi}\int_0^{|z|} ds \exp\{s^2\}\operatorname{erfc}(s) \le 2\int_0^{|z|} \frac{ds}{s+\sqrt{s^2+4/\pi}} \quad (5.1.6)$$

We find

$$|z|\sqrt{z+2} + 2\ln\left(\frac{|z|+\sqrt{z^2+2}}{2}\right) - z^2 < \sqrt{\pi}\int_0^{|z|} ds \exp\{s^2\}\operatorname{erfc}(s)$$

$$\le \frac{\pi}{2}\left[\,|z|\sqrt{z^2+4/\pi} + \frac{4}{\pi}\ln\left[\frac{\pi\left(|z|+\sqrt{z^2+4/\pi}\right)}{4}\right]-z^2\right. \quad (5.1.7)$$

We now consider $\int_0^{|z|} ds \exp\{s^2\}$. The maximum contribution to the integral comes from near the upper limit, so we evaluate the integral using Laplace's method. Provided that $|z|$ is away from the origin, we find

$$\int_0^{|z|} ds \exp\{s^2\} = 2\sinh(z^2)\Big(\frac{1}{2|z|} + \frac{1}{4|z|^3} + \frac{3}{8|z|^5} + \cdots$$

$$-\exp\{-z^2\}\Big[\tfrac{1}{2}\big(|z| + |z|^{-1}\big) + \tfrac{1}{4}\big(|z|^3 + 2|z| + 3|z|^{-1} + 3|z|^{-3}\big) + \cdots\Big] \qquad (5.1.8)$$

We also note that the integral vanishes, by definition, at the origin, so we are really most concerned with z in the neighborhood of the end points. We find that we can place the following bounds on the solution v(z) for z bounded away from the origin.

$$\sqrt{\pi}\big(\alpha\,\mathrm{sgn}(z)-1\big)\Big[2\sinh z^2\big(\tfrac{1}{2}|z|^{-1} + O(|z|^{-3})\big) - O\big(\exp\{-z^2\}\big)\Big] + |z|\sqrt{z^2+2}$$

$$+ 2\ln\Big(\frac{|z| + \sqrt{z^2+2}}{2} - z^2 + \beta < v(z)$$

$$\leq \sqrt{\pi}\,\big(\alpha\,\mathrm{sgn}(z)-1\big)\Big[2\sinh z^2\big(\tfrac{1}{2}|z|^{-1} + O(|z|^{-3})\big) - O\big(\exp\{-z^2\}\big)$$

$$+ \frac{\pi|z|}{2}\sqrt{z^2+4/\pi} + 2\ln\Big(\frac{\pi|z| + \sqrt{z^2+4/\pi}}{4}\Big) - \frac{\pi z^2}{2} + \beta \qquad (5.1.9)$$

We substitute $z = y/\varepsilon$ and find that

$$u(y;\varepsilon) \sim \sqrt{\pi}\big(\alpha\,\mathrm{sgn}(y)-1\big)\,\frac{\varepsilon}{|y|}\,\sinh\Big(\frac{y^2}{\varepsilon^2}\Big) + \beta.$$

When we apply the boundary conditions, we see

$$\alpha \cong \frac{\frac{1}{b}\sinh\Big(\frac{b^2}{\varepsilon^2}\Big) - \frac{1}{a}\sinh\Big(\frac{a^2}{\varepsilon^2}\Big)}{\frac{1}{b}\sinh\Big(\frac{b^2}{\varepsilon^2}\Big) + \frac{1}{a}\sinh\Big(\frac{a^2}{\varepsilon^2}\Big)}$$

$$\beta \cong \frac{2\sqrt{\pi}\,\varepsilon}{ab} \quad \frac{1}{\dfrac{1}{b\sinh\left(\dfrac{a^2}{\varepsilon^2}\right)} + \dfrac{1}{a\sinh\left(\dfrac{a^2}{\varepsilon^2}\right)}}$$

Thus,

$$u(y;\varepsilon) \sim \sqrt{\pi}\left[\frac{1}{b}\sinh\left(\frac{b^2}{\varepsilon^2}\right) + \frac{1}{a}\sinh\left(\frac{a^2}{\varepsilon^2}\right)\right]^{-1} \cdot \left[\frac{2\varepsilon}{ab}\sinh\left(\frac{a^2}{2}\right)\sinh\left(\frac{b^2}{2}\right)\right.$$

$$+\left[\operatorname{sgn}(y)\left(\frac{1}{b}\sinh\left(\frac{b^2}{2}\right) - \frac{1}{a}\sinh\left(\frac{a^2}{2}\right)\right) - \frac{1}{b}\sinh\left(\frac{b^2}{2}\right) - \frac{1}{a}\sinh\frac{a^2}{2}\right] \cdot$$

$$\left.\frac{\varepsilon}{|y|}\sinh\left(\frac{y^2}{\varepsilon^2}\right)\right] \tag{5.1.11}$$

Hence, if $b > a$ and $y > 0$,

$$u(y;\varepsilon) \sim \frac{2\varepsilon}{a}\sqrt{\pi}\sinh\left(\frac{a^2}{\varepsilon^2}\right)\left[\frac{1}{b}\sinh\left(\frac{b^2}{\varepsilon^2}\right) - \frac{1}{y}\sinh\left(\frac{y^2}{\varepsilon^2}\right)\left[\frac{1}{b}\sinh\left(\frac{b^2}{\varepsilon^2}\right) + \right.\right.$$

$$\left.\frac{1}{a}\sinh\left(\frac{a^2}{\varepsilon^2}\right)\right]^{-1} \tag{5.1.12}$$

Similarly, if $b < a$ and $y < 0$,

$$u(y;\varepsilon) \sim \frac{2\varepsilon}{b}\sqrt{\pi}\sinh\left(\frac{b^2}{\varepsilon^2}\right)\left[\frac{1}{a}\sinh\left(\frac{a^2}{\varepsilon^2}\right) + \frac{1}{y}\sinh\left(\frac{y^2}{\varepsilon^2}\right)\right]\left[\frac{1}{b}\sinh\left(\frac{b^2}{\varepsilon^2}\right) + \right.$$

$$\left.\frac{1}{a}\sinh\left(\frac{a^2}{\varepsilon^2}\right)\right]^{-1} \tag{5.1.13}$$

We note from (5.1.12) and (5.1.13) that $u(y;\varepsilon)$ is asymptotic to a very large constant within most of Ω and that the only regions of significant change are near the boundaries. Furthermore, the magnitude of the solution agrees with the results of Ventsel' and Freidlin [16]. In the next section, we apply the theory of Chapter II to the mean exit time problem for the Ornstein-Uhlenbeck process and compare the results with the results in this section.

5.2 Predicted Mean Exit Time for the Ornstein-Uhlenbeck Process

We know from the previous section that we must solve the following boundary value problem

$$\left.\begin{aligned} &\frac{\varepsilon^2}{2}\frac{d^2u}{dy^2} - y\frac{du}{dy} = -1 \qquad y\varepsilon(-a,b) \\ &u(-a) = u(b) = 0 \end{aligned}\right\} \tag{5.2.1}$$

When we apply the theory of Chapter II, we assume

$$u(y;\varepsilon) \sim C_0 + z(y)\exp\left\{-\frac{\zeta(y)}{\varepsilon^2}\right\}. \tag{5.2.2}$$

Using (4.2.3) and (4.2.4), we see that the properly normalized solution of the homogeneous adjoint problem is given by

$$v(y;\varepsilon) \sim \exp\left\{-\frac{y^2}{\varepsilon}\right\}. \tag{5.2.3}$$

We substitute this expression into (2.4.27) in order to determine the unknown parameter C_0. Thus

$$C_0 \sim \frac{\int_{-a}^{b}\exp\left\{-\frac{y^2}{\varepsilon^2}\right\}dy}{b\exp\left\{-\frac{b^2}{\varepsilon^2}\right\} + a\exp\left\{-\frac{a^2}{\varepsilon^2}\right\}} \tag{5.2.4}$$

$$= \frac{2\varepsilon\sqrt{\pi}}{(a\wedge b)} \exp\left\{ \frac{(a\wedge b)^2}{\varepsilon^2} \right\} \left(1 + O(\varepsilon^2)\right) \qquad \text{for } a \neq b$$

where $(a \wedge b) \equiv \min(a, b)$. We employ the calculations (4.2.7), (4.2.9), (4.2.10), and (4.2.11) to find

$$u(y;\varepsilon) \sim \frac{2\varepsilon\sqrt{\pi}}{(a\wedge b)} \exp\left\{\frac{(a\wedge b)^2}{\varepsilon^2}\right\}\left(1-\exp\left\{\frac{-2a(a+y)}{\varepsilon^2}\right\} - \exp\left\{\frac{-(2b(b-y)}{\varepsilon^2}\right\}\right) \quad (5.2.5)$$

If $a = b$, we find

$$C_0 \sim \frac{\varepsilon\sqrt{\pi}}{b} \exp\left\{\frac{b^2}{\varepsilon^2}\right\}\left(1 + O(\varepsilon^2)\right) \qquad (5.2.6)$$

and

$$u(y;\varepsilon) \sim \frac{\varepsilon\sqrt{\pi}}{b} \exp\left\{\frac{b^2}{\varepsilon^2}\right\}\left(1 - \exp\left\{\frac{2b(b-|y|)}{\varepsilon^2}\right\}\right) \qquad (5.2.7)$$

Thus we see that the results of the theory in Chapter II agree with the asymptotic evaluation of the exact solution as well as the results of Ventsel' and Freidlin.

5.3. Mean Exit Time for the Damped Linear Harmonic Oscillator

The mean exit time for a damped linear harmonic oscillator satisfies the boundary value problem

$$\left.\begin{aligned} \frac{\varepsilon^2}{2} \frac{\partial^2 u}{\partial y^2} - (2\beta y + x)\frac{\partial u}{\partial y} + y\frac{\partial u}{\partial x} &= -1 \qquad (x,y) \in \Omega \\ u(x,y) &= 0 \qquad (x,y) \in \partial\Omega \end{aligned}\right\} \qquad (5.3.1)$$

Take $\Omega = \{(x, y) : x^2+y^2< r^2\}$. When we apply the theory of Chapter II, we assume

$$u(x, y; \varepsilon) \sim C_0 + z(x, y) \exp\{- \frac{\zeta(x, y)}{\varepsilon^2}\} \tag{5.3.2}$$

We assume that the solution of the homogeneous adjoint boundary value problem (2.4.3) is of the form

$$v(x, y; \varepsilon) \sim w^{(0)}(x, y) \exp\{- \frac{\varphi(x, y)}{\varepsilon^2}\} \tag{5.3.3}$$

From (4.3.10) and (4.3.13), we see that the properly normalized solution for $v(x, y; \varepsilon)$ is

$$v(x, y; \varepsilon) \sim \exp\{- \frac{2\beta(x^2+y^2)}{\varepsilon^2}\} . \tag{5.3.4}$$

At the boundary, the outer normal unit vector is

$$\underline{\nu}(x, y) = \frac{1}{r}\begin{pmatrix} x \\ y \end{pmatrix} . \tag{5.3.5}$$

Then the unknown parameter C_0 is determined by

$$C_0 \sim \frac{-\iint_{\Omega} \exp\{-\frac{2\beta(x^2+y^2)}{\varepsilon^2}\} dx\,dy}{\int_{\partial\Omega} \underline{b}(x,y)\cdot \nu(x,y) \exp\{-\frac{2\beta(x^2+y^2)}{\varepsilon^2}\} dS}$$

$$= \frac{2\pi \int_0^{v} \rho \exp\{-\frac{2\beta\rho^2}{\varepsilon^2}\} d\rho}{\int_0^{2\pi} 2\beta r \sin^2\theta \, d\theta \exp\{-\frac{2\beta r^2}{\varepsilon^2}\}} = \frac{\varepsilon^2}{2\beta^2 r} \exp\{\frac{\beta r^2}{\varepsilon^2}\} \sinh\left(\frac{\beta r^2}{\varepsilon^2}\right) \qquad (5.3.6)$$

Then using (4.3.18) and (2.3.13) we can determine the boundary layer correction $z(x,y)\exp\{-\zeta(x,y)/\varepsilon^2\}$. Again, we note that the boundary layer width is nominally $O(\varepsilon)$, but the region where the solution is $O(1)$ is transcendentally thin.

5.4 Mean Exit Time for the Damped Pendulum

The mean exit time for the damped pendulum satisfies the boundary value problem

$$\left.\begin{aligned} &\frac{\varepsilon^2}{2}\frac{\partial^2 u}{\partial w^2} - (2\beta\omega + \sin\theta)\frac{\partial u}{\partial \omega} + \omega \frac{\partial u}{\partial \theta} = -1 && (\theta,\omega)\in\Omega \\ &u(\theta,\omega) = 0 && (\theta,\omega)\in\partial\Omega \end{aligned}\right\} \qquad (5.4.1)$$

Take $\Omega = \{(\theta,\omega) : \omega^2 - 2\cos\theta = 2\}$. When we apply the theory of Chapter II, we assume

$$u(\theta,\omega;\varepsilon) \sim C_0 + z(\theta,\omega)\exp\{-\frac{\zeta(\theta,\omega)}{\varepsilon^2}\} \qquad (5.4.2)$$

We assume that the solution of the homogeneous adjoint boundary value problem (2.4.3) is of the form

$$v(\theta, \omega; \varepsilon) \sim w^{(0)}(\theta, \omega) \exp\left\{\frac{\varphi(\theta, \omega)}{\varepsilon^2}\right\} \tag{5.4.3}$$

From (4.4.10) and (4.4.13) we see that the properly normalized solution for $v(\theta, \omega; \varepsilon)$ is

$$v(\theta, \omega; \varepsilon) \sim \exp\left\{- \frac{2\beta(\omega^2 + 2 - 2\cos\theta)}{\varepsilon^2}\right\} \tag{5.4.4}$$

At the boundary, the outer normal unit vector is

$$\underline{\nu}(\theta, \omega) = \frac{1}{\sqrt{\omega^2 + \sin^2\theta}} \begin{pmatrix} \sin\theta \\ \omega \end{pmatrix} \tag{5.4.5}$$

Then the unknown parameter C_0 is determined by

$$C_0 \sim \frac{-\int_{\theta=-\pi}^{\theta=\pi} d\theta \int_{\omega=-\sqrt{2+2\cos\theta}}^{\omega=\sqrt{2+2\cos\theta}} d\omega \exp\left\{- \frac{2\beta(\omega^2+2-2\cos\theta)}{\varepsilon^2}\right\}}{\int_{\partial\Omega} \underline{b}(\theta, \omega) \cdot \underline{\nu}(\theta, \omega) \exp\left\{- \frac{2\beta(\omega^2+2-2\cos\theta)}{\varepsilon^2}\right\} dS}$$

$$\cong \frac{\varepsilon\sqrt{\frac{\pi}{2\beta}} \int_{-\pi}^{\pi} d\theta \exp\left\{- \frac{4\beta(1-\cos\theta)}{\varepsilon^2}\right\} \operatorname{erf}\left(\frac{2}{\varepsilon}\sqrt{\beta(1+\cos\theta)}\right)}{\exp\left\{- \frac{8\beta}{\varepsilon^2}\right\} \int_{\partial\Omega} \underline{b}(\theta, \omega) \cdot \underline{\nu}(\theta, \omega) dS}$$

$$\cong \frac{\varepsilon\sqrt{\frac{\pi}{2\beta}} \sqrt{\frac{2\pi\varepsilon^2}{4\beta}} \operatorname{erf}\left(\frac{2\sqrt{2\beta}}{\varepsilon^2}\right)}{32\beta \exp\left\{- \frac{8\beta}{\varepsilon^2}\right\}} \tag{5.4.6}$$

So

$$C_0 \sim \frac{\varepsilon^2 \pi}{64\beta^2} \exp\left\{\frac{8\beta}{\varepsilon^2}\right\} . \qquad (5.4.7)$$

If we desire, we can now determine the boundary layer correction. The region where the solution is O(1) is transcendentally thin compared with any power of ε^2, so the solution is asymptotically transcendentally large almost everywhere in Ω.

It is interesting to note the difference in the values of C_0 in (5.3.6) and (5.4.7) if we require that both the damped linear harmonic oscillator and the damped pendulum exit from regions bounded by curves which represent the same energy levels. The total energy for the pendulum on the curve

$$C = \left\{(\theta, \omega) : \omega^2 - 2\cos\theta = 2\right\} \qquad (5.4.8)$$

is 2 units. For the oscillator, the curve which corresponds to a curve with a total energy of 2 units is a circle with radius 2. We can then compare the exit times by noting that for small ε

$$\frac{E_{\underline{0}}[\tau_{pen}]}{E_{\underline{0}}[\tau_{osc}]} \sim \frac{\pi}{8} \qquad (5.4.9)$$

Thus the nonlinear restoring force for the pendulum has substantially shortened the mean exit time for the pendulum from that of the linear harmonic oscillator.

5.5 Predicted Mean Exit Time for a Dynamical System with a Limit Cycle

We again consider the physical example of Section 4.5:

$$\left.\begin{aligned} dx(t) &= \left[x(t) + y(t) - x(t)\left(x^2(t)+y^2(t)\right)\right]dt + \varepsilon\, dw(t) \\ dy(t) &= \left[y(t) - x(t) - y(t)\left(x^2(t)+y^2(t)\right)\right]dt + \varepsilon\, dw(t) \end{aligned}\right\} \quad (5.5.1)$$

The mean exit time for this process satisfies the boundary value problem

$$\left.\begin{aligned} &\frac{\varepsilon^2}{2}\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] + \left[x+y-x(x^2+y^2)\right]\frac{\partial u}{\partial x} + \left[y-x-y(x^2+y^2)\right]\frac{\partial u}{\partial y} = -1 \quad (x,y)\in\Omega \\ &u(x,y) = 0 \qquad\qquad (x,y)\in\partial\Omega\,. \end{aligned}\right\} \quad (5.5.2)$$

Again, we take $\Omega = \{(x,y) : x^2+y^2 < 4\}$. We assume that $u(x,y)$ has the asymptotic representation

$$u(x,y;\varepsilon) \sim C_0 + z(x,y)\exp\left\{-\frac{\zeta(x,y)}{\varepsilon^2}\right\} \quad (5.5.3)$$

Let $v(x,y;\varepsilon)$ satisfy the homogeneous adjoint boundary value problem

$$\left.\begin{aligned} &\frac{\varepsilon^2}{2}\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right] - \frac{\partial}{\partial x}\left(x+y-x(x^2+y^2)\right)v\Big] - \frac{\partial}{\partial y}\left[\left(y-x-y(x^2+y^2)\right)v\right] = 0 \quad (x,y)\in\Omega \\ &\left[\frac{\varepsilon^2}{2}\frac{\partial v}{\partial x} - \left(x+y-x(x^2+y^2)\right)v\right]\nu_x + \left[\frac{\varepsilon^2}{2}\frac{\partial v}{\partial y} - \left(y-x-y(x^2+y^2)\right)v\right]\nu_y = 0. \quad (x,y)\in\partial\Omega \end{aligned}\right\}$$

(5.5.4)

We assume that we can write $v(x, y; \varepsilon)$ as

$$v(x, y; \varepsilon) \sim w^{(0)}(x, y) \exp\left\{- \frac{\varphi(x, y)}{\varepsilon^2}\right\} . \tag{5.5.5}$$

Then $\varphi(x, y)$ satisfies the eiconal equation

$$\frac{1}{2}\left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2\right] + \left[x+y-x(x^2+y^2)\right]\frac{\partial \varphi}{\partial x} + \left[y-x-y(x^2+y^2)\right]\frac{\partial \varphi}{\partial y} = 0 \qquad (x, y) \in \Omega$$

$$\left[\frac{1}{2}\frac{\partial \varphi}{\partial x} + \left(x+y-x(x^2+y^2)\right)\right]\nu_x + \left[\frac{1}{2}\frac{\partial \varphi}{\partial y} + \left(y-x-y(x^2+y^2)\right)\right] = 0 \qquad (x, y) \in \partial\Omega \tag{5.5.6}$$

The solution to this problem, as given by (4.5.12), is

$$\varphi(x, y) = \frac{1}{2}\left(1-x^2-y^2\right)^2 \tag{5.5.7}$$

The function $w^{(0)}(x, y)$ is again identically unity. Then the unknown parameter C_0 is given by

$$C_0 \sim \frac{-\int_\Omega \exp\left\{-\frac{\varphi(x, y)}{\varepsilon^2}\right\}dxdy}{\int_{\partial\Omega} \underline{b}(x, y)\cdot\underline{\nu}(x, y)\exp\left\{-\frac{\varphi(x,y)}{\varepsilon^2}\right\}dS} = \frac{\int_0^{2\pi} d\theta \int_0^2 r\,dr \exp\left\{-\frac{(1-r^2)^2}{2\varepsilon^2}\right\}}{24\pi \exp\left\{-\frac{9}{2\varepsilon^2}\right\}}$$

$$= \frac{1}{24\pi}\exp\left\{\frac{9}{2\varepsilon^2}\right\}\int_0^{2\pi} d\theta \sqrt{\frac{4\varepsilon^2\pi}{8}}\left[\frac{1}{2}\operatorname{erf}\left(\frac{3}{\varepsilon\sqrt{2}}\right) + \frac{1}{2}\operatorname{erf}\left(\frac{1}{\varepsilon\sqrt{2}}\right)\right] \tag{5.5.8}$$

So

$$C_0 \sim \frac{\varepsilon}{12}\sqrt{\frac{\pi}{2}}\exp\left\{\frac{9}{2\varepsilon^2}\right\} \tag{5.5.9}$$

We could also determine the boundary layer correction, if we desired, by writing the transport equation as an initial value problem along the rays and then determining the ray parameter t as t(x, y).

In this chapter, we have worked several examples of mean exit time problems for dynamical systems subject to small Gaussian white noise excitation. The results are all similar in that the solution is transcendentally large compared with any finite power of ε almost everywhere. This is not surprising since the deterministic systems are asymptotically stable. In the case where the exact solution can be determined and then evaluated asymptotically, the theory of Chapter II predicts the same asymptotic behavior. In other cases, the theory predicts results which are consistent with other authors.

CHAPTER VI

In this chapter, we present the results of numerical simulation of asymptotically stable dynamical systems subject to small Gaussian white noise excitation. We use several different values of the noise parameter, ε, for each process we simulate in order to check the validity of the theory presented in Chapter II. All calculations presented in this chapter were performed on a Digital Equipment Corporation PDP-10 computer.

The chapter is divided into four sections. In the first section, the method used to simulate the various dynamical systems is discussed and error estimates for the accuracy of the calculations are presented. The second section is devoted to a study of the sample paths of the various dynamical systems which exit against a flow. In the third section, we use numerical techniques to study the distribution of exit positions for the damped linear harmonic oscillator, the damped pendulum, and a problem with a limit cycle. Finally, we present numerical results pertaining to the study of mean exit times for these same three dynamical systems.

6.1 A Description of the Numerical Methods Used in the Simulations

In order to numerically simulate the solution of the differential matrix equation (1.1.3), we approximate the random vector $\underline{x}_\varepsilon(t)$ by the vector $\underline{x}_\varepsilon^{(n)}$ defined by the difference equation

$$\underline{x}_{\varepsilon}^{(k+1)} = \underline{x}_{\varepsilon}^{(k)} + \Delta t\, \underline{b}(\underline{x}_{\varepsilon}^{(k)}) + \varepsilon \underline{g}^{(k)}$$

$$\underline{x}_{\varepsilon}(0) = \underline{x}_0 \qquad (6.1.1)$$

where $t = k\Delta t$ and $\underline{g}^{(k)}$ is a pseudo-random vector simulating a random Gaussian vector $\underline{g}$ where

$$\begin{aligned} E\, g_i &= 0 & i &= 1, \cdots, n \\ E\, g_i g_j &= \sigma_{ij}\left(\underline{x}_{\varepsilon}(t)\right)\Delta t & i,j &= 1, \cdots, n. \end{aligned} \qquad (6.1.2)$$

We do not use a more sophisticated numerical technique than (6.1.1), such as Adams or Runge-Kutta, because these methods require bounded derivatives of $\underline{b}\left(\underline{x}_{\varepsilon}(t)\right)$ and $\sigma\left(x_{\varepsilon}(t)\right) d\underline{w}(t)$, and any sample of a white noise process, with positive variance is required to be everywhere unbounded, discontinuous, and nondifferentiable.

It is numerically convenient to generate a sequence of Gaussian pseudo-random numbers with zero mean and unit variance and then multiply the numbers by $\sqrt{\Delta t}$. This produces a sequence of pseudo-random numbers with the proper mean and variance.

It is well-known that the Euler Method for solving (1.1.3), as given by (6.1.1), is numerically accurate to $O(\Delta t)$ [3]. Thus, it is advantageous to pick the time-step, Δt, as small as possible. However, this increases the amount of time necessary to compute the time of exit because more iterations are required. For reference purposes, the value of Δt will always be given, and when possible, several sets of calculations with various values of Δt will be shown.

6.2 An Examination of Exit Trajectories

In this section, we use (6.1.1) to simulate the Ornstein-Uhlenbeck process, the damped linear harmonic oscillator, the damped pendulum, and a process with a limit cycle. For each dynamical system, we examine the exit trajectories in order to determine whether the hypotheses of Chapter II are consistent with experiment.

We first consider a two-dimensional version of the Ornstein-Uhlenbeck process in a region whose boundary is not symmetric about the origin. In particular, (6.1.1) becomes

$$
\begin{aligned}
x^{(k+1)} &= x^{(k)} - x^{(k)}\Delta t + \varepsilon\sqrt{\Delta t}\, g_1 \\
y^{(k+1)} &= y^{(k)} - y^{(k)}\Delta t + \varepsilon\sqrt{\Delta t}\, g_2
\end{aligned}
\qquad (6.2.1)
$$

where g_1, g_2 are Gaussian pseudo-random numbers with zero mean and unit variance. We take $\Delta t = 0.01$, and then vary the parameter ε. The boundary of the region Ω is assumed to be a circle with radius 0.75, centered at (-0.25, 0). Thus, the boundary has a unique nearest point to the origin at (0.5, 0). We pick the initial point to be the origin, and then integrate the equation of motion using (6.2.1) The results for five escape trajectories are given in

Fig. 6.2.1 and Fig. 6.2.2; in the former figure, we set $\varepsilon = 0.5$, and in the latter figure, we set $\varepsilon = 0.3$. In Fig. 6.2.1, we note that the trajectories cover most of Ω and that there is certainly no reason to conclude that the process is most likely to exit at the point on the boundary nearest the origin. Fig. 6.2.2 is much closer to what we would expect, based on the results of Chapter II. The trajectories tend to cluster around the origin, with periodic excursions away from the origin. Observe that the diffusion against a flow is not a slow process, but rather, consists of excursions in the phase plane which cover a finite amount of distance in a relatively short amount of time. If the process does not reach the boundary, then it is attracted back to a neighborhood from where it begins another excursion at a later time. The five exit points all lie in the neighborhood of the point on the boundary which is nearest the origin. We would expect that as we take values of ε sufficiently small, we would concentrate the exit points in an even smaller neighborhood of $(0.5, 0)$.

In Fig. 6.2.3, we study the exit trajectories for a damped linear harmonic oscillator exiting from a unit circle in the phase plane, centered at the origin. Then (6.1.1) becomes

$$\begin{aligned} x^{(k+1)} &= x^{(k)} + y^{(k)}\Delta t \\ y^{(k+1)} &= y^{(k)} - (2\beta y^{(k)} + x^{(k)})\Delta t + \varepsilon\sqrt{\Delta t}\, g_1 \end{aligned} \tag{6.2.2}$$

The initial point is taken to be the origin, we set $\varepsilon = 0.3535$, $\beta = 0.25$, and let $\Delta t = 0.01$. We then use (6.2.2) to determine the exit trajectories. Fig. 6.2.3 is a plot of five escape trajectories for the oscillator.

As we expect, the escape trajectories are generally concentrated near the origin, but make periodic excursions outward toward the boundary along the rays. If the excursion does not reach the boundary, then it is attracted back toward the origin from where it tries repeatedly to escape. The region where the escape trajectories are most heavily concentrated is a circle with a radius of about 0.3. This is approximately the standard deviation of the noise, and is consistent with what one would expect based upon physical intuition.

In Fig. 6.2.4 we plot five exit trajectories for a damped pendulum exiting from the boundary $\omega^2 = 2\cos^2 \frac{\theta}{2}$. Then (6.1.1) becomes

$$\begin{aligned} \theta^{(k+1)} &= \theta^{(k)} + \omega^{(k)} \Delta t \\ \omega^{(k+1)} &= \omega^{(k)} - \left(2\beta\omega^{(k)} + \sin\theta^{(k)}\right) \Delta t + \varepsilon\sqrt{\Delta t}\, g_1 \end{aligned} \qquad (6.2.3)$$

The initial point is taken to be the origin, we set $\varepsilon = 0.5$, $\beta = 0.125$, and take $\Delta t = 0.01$. We then use (6.2.3) to determine the exit trajectories.

The escape trajectories are generally to be found in the neighborhood of the origin, with the heaviest concentration occurring in an ellipse with a semi-major axis of about 1.0 in the θ-direction and a semi-minor axis of about 0.7 in the ω-direction. The trajectories for the damped pendulum appear to spread more rapidly than the trajectories for the damped oscillator because the damping parameter for the former problem is smaller and the noise parameter is larger. Again, we observe that the escape trajectories generally spiral outward along the rays, and that if the trajectories do not successfully exit, then they are attracted back to the neighborhood of the origin.

Finally, we present a plot of five escape trajectories for a process with a limit cycle. In this example, (6.1.1) becomes

$$x^{(k+1)} = x^{(k)} + \left[x^{(k)} + y^{(k)} - x^{(k)}\left((x^{(k)})^2 + (y^{(k)})^2\right)\right]\Delta t + \varepsilon\sqrt{\Delta t}\, g_1$$

$$y^{(k+1)} = y^{(k)} + \left[y^{(k)} - x^{(k)} - y^{(k)}\left((x^{(k)})^2 + (y^{(k)})^2\right)\right]\Delta t + \varepsilon\sqrt{\Delta t}\, g_2 \qquad (6.2.4)$$

We assume the initial point is on the limit cycle at (1, 0), and take $\varepsilon = 0.707$ and $\Delta t = 0.01$. As we can see from Fig. 6.2.5, the escape trajectories are most heavily concentrated in the neighborhood of the limit cycle. The concentration is not as heavy about the limit set as in previous examples because the magnitude of the noise parameter is greater. Hence, individual steps in the random walk approximation to the diffusion against the flow will tend to be longer. The excursions from the limit cycle occur both toward and away from the boundary, as we would expect.

The results of this section are in qualitative agreement with the results of Chapter II. The function $v(\underline{x};\varepsilon)$ is proportional to the stationary probability distribution for the process being studied. In each of the figures, the dynamical system was most likely to be near the limit set within a region whose width was approximately ε. It is apparent that the predicted results are more in agreement with numerical simulation as the noise parameter becomes smaller, but numerical simulations with smaller values of the noise could not be conveniently performed to more fully check this hypothesis.

6.3 Numerical Simulation to Determine the Distribution of Exit Positions

In this section, we present results of the numerical simulations of a lightly damped linear harmonic oscillator, a damped pendulum, and a dynamical system with a limit cycle, in order to study the distribution of exit positions for these systems. In all examples in this section, we let the particular process being studied exit 250 times in order to determine the distribution of exit positions.

In Figures 6.3.1 and 6.3.2, we plotted histograms of the exit positions of a damped linear harmonic oscillator, which exited from a unit circle, centered at the origin, in the phase plane. The left end of the axis corresponds to the angle 0 in polar coordinates and the right end corresponds to the angle 2π. In Fig. 6.3.1, we set $\varepsilon = 0.707$, $\beta = 0.25$, and $\Delta t = 0.01$, and used (6.2.1) to calculate the exit trajectories. In Fig. 6.3.2, we set $\varepsilon = 0.3535$, $\beta = 0.25$, and $\Delta t = 0.01$, and then repreated the calculations. In both cases, all trajectories started at the origin. The predicted distribution, shown as a broken line, is $(250/\pi) \sin^2\theta$, which is symmetric about $\theta = \pi$. Note that Fig. 6.3.1 is not particularly symmetric and that the simulated exits were generally distributed about $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$. When the magnitude of the noise parameter was decreased, then the distribution of exit positions, as shown in Fig. 6.3.2, became much more symmetric. The distribution broadened itself out and the predicted curve fit it much better. There are still more exits near $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$ than Chapter II predicts, but this is to be expected since these are the most probable positions of exit for the system, and the tails of the distribution near $\theta = 0, \pi, 2\pi$ are the least probable positions of exit.

In Figs. 6.3.3 and 6.3.4, we plotted histograms of the exit positions of a damped pendulum, which exited from the region bounded by $\omega^2 = 2\cos^2\frac{\theta}{2}$ in the phase plane. The left half of the axis represents the boundary in the lower half of the phase plane, and the right half of the axis represents the boundary in the upper half of the phase plane. In Fig. 6.3.3, we set $\varepsilon = 0.707$, $\beta = 0.125$, and $\Delta t = 0.01$ and then used (6.2.3) to calculate the exit trajectories. In Fig. 6.3.4, we assumed $\varepsilon = 0.5$, $\beta = 0.125$ and $\Delta t = 0.01$, and then repeated the calculations. As before, all trajectories began at the origin. In both figures, there are generally more exits near the peaks of the predicted curves, shown as broken lines, than the results of Chapter II would predict, and there are not as many exits near the tails of the distribution. Again, as the noise parameter was decreased, the number of exits near the tails of the distribution increased. Again, it appears that the predicted distribution of exit positions is approximating the simulated distribution better as the noise parameter ε becomes smaller.

Finally, we consider Figs. 6.3.5 and 6.3.6, which are histogram plots of the exit position of a dynamical system with a limit cycle. The left end of the axis represents the polar angle O and the right end represents the polar angle 2π. In Fig. 6.3.5, we set $\varepsilon = 0.8$ and $\Delta t = 0.01$, and used (6.2.4) to calculate the exit trajectories. In Fig. 6.3.6, we set $\varepsilon = 0.707$ and $\Delta t = 0.01$, and repeated the calculations. In both cases, all trajectories began at the point (1, 0) on the limit cycle. In both figures, we observe that the distribution is subject to a great deal of irregularity. Qualitatively, the distribution in Fig. 6.3.6 might appear smoother because the excursions from an approximate mean level are not as large in magnitude. However, due to the relatively great size of the small parameter ε, it is difficult to determine whether the predicted distribution of exit positions is really meaningful.

In order to more fully demonstrate that the theory of Chapter II predicts correct results, we would need to decrease the magnitude of the noise parameter ε and increase the number of exits used to calculate the final distributions. This would be most costly in terms of computer time. It is apparent, however, that the goal of reasonable computability of exit trajectories conflicts with the desire to take the noise parameter ε as small as possible. Furthermore, in order to justify the asymptotic results obtained in Chapter II, we should pick ε significantly smaller than the values which we assumed when performing the calculations presented in this section. Based upon the results for the damped oscillator and the damped pendulum, we are confident that the results of Chapter II are correct for sufficiently small ε.

6.4 Numerical Simulation to Determine Mean Exit Times

In this section we present the results of numerical simulations of a damped linear harmonic oscillator and a damped pendulum which were made in order to study the mean exit times for these dynamical systems. In Figs. 6.4.1 and 6.4.2, mean exit times and the standard deviations of the mean exit times are given for a damped oscillator and a damped pendulum, respectively. The standard deviation of the mean was computed using the formula

$$s = \sqrt{\frac{1}{n(n-1)} \sum_{i=1}^{n} (\tau_i - \bar{\tau})^2} \qquad (6.4.1)$$

The sample mean is based upon 100 exit times. The parameter values for each process are given in the appropriate figure.

If one examines the figures, one notes that the predicted value is generally lower than the sample mean. The predicted values range from about 1.3 times too great when $\varepsilon = 0.707$ to about 0.7 times the predicted value when $\varepsilon = 0.3535$. We expect the discrepancy between prediction and simulation to have two sources; the value of the terms in (2.4.48) is approximated incorrectly, and the numerical approximation used in the simulation provides an answer which is too low. We shall now comment on both sources of error. The greatest amount of error is probably caused by making an asymptotic approximation with a parameter which is too large. We ignored terms of order $\varepsilon^2 \exp\{-\frac{\varphi_{min}}{\varepsilon^2}\}$, but in the numerical simulations, ε is not particularly small, especially when compared with the damping parameter β. It is encouraging that the ratio of the predicted mean exit time to the simulated mean exit time does not continue to decrease as ε becomes smaller. The second possible source of error is much more subtle. We are trying to approximate a system of differential equations with a finite difference numerical scheme. The analytic solution should always be greater than the numerical solution because of the time lag in the Euler scheme (6.1.1). This difference will become smaller as the parameter Δt decreases because the time lag for the effects of the stochastic perturbation to be felt is smaller.

The results of the numerical simulations given in this chapter indicate that we are faced with a dilemma. We would like to check the accuracy of the results of Chapter II using numerical simulations, but the computations become intractable. Nonetheless, predictions based upon the results of Chapter II do appear to match simulated values as the noise parameter ε becomes smaller. It is on this basis as well as the agreement of asymptotic results when exact solutions are known which gives us confidence that the theory proposed in Chapter II is valid.

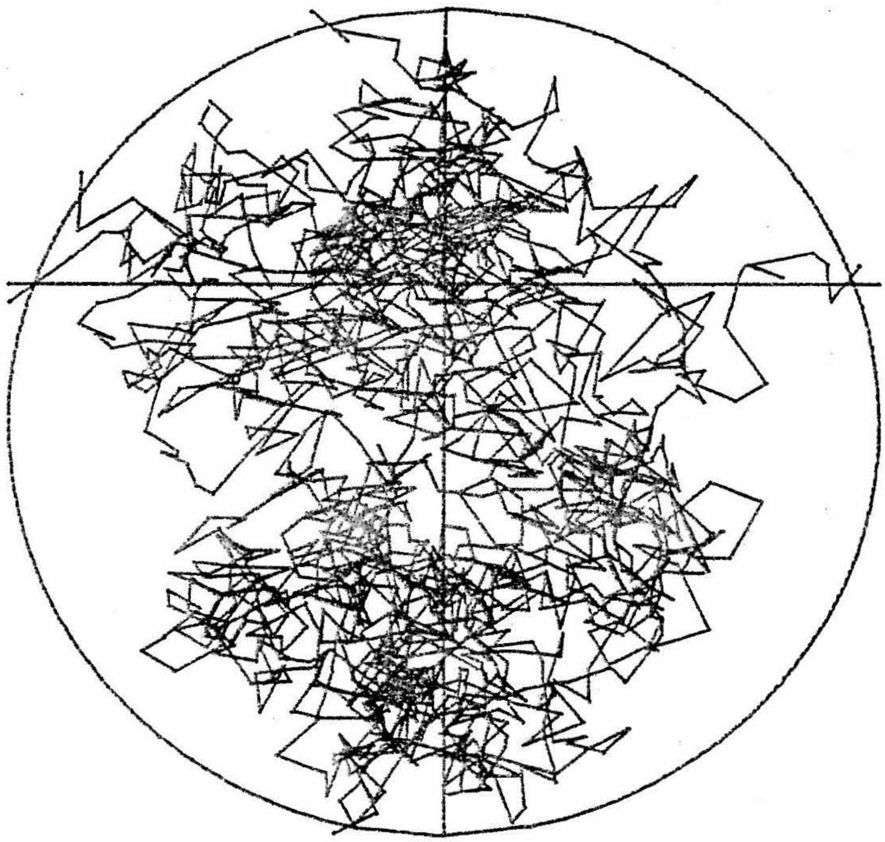


Figure 6.2.1

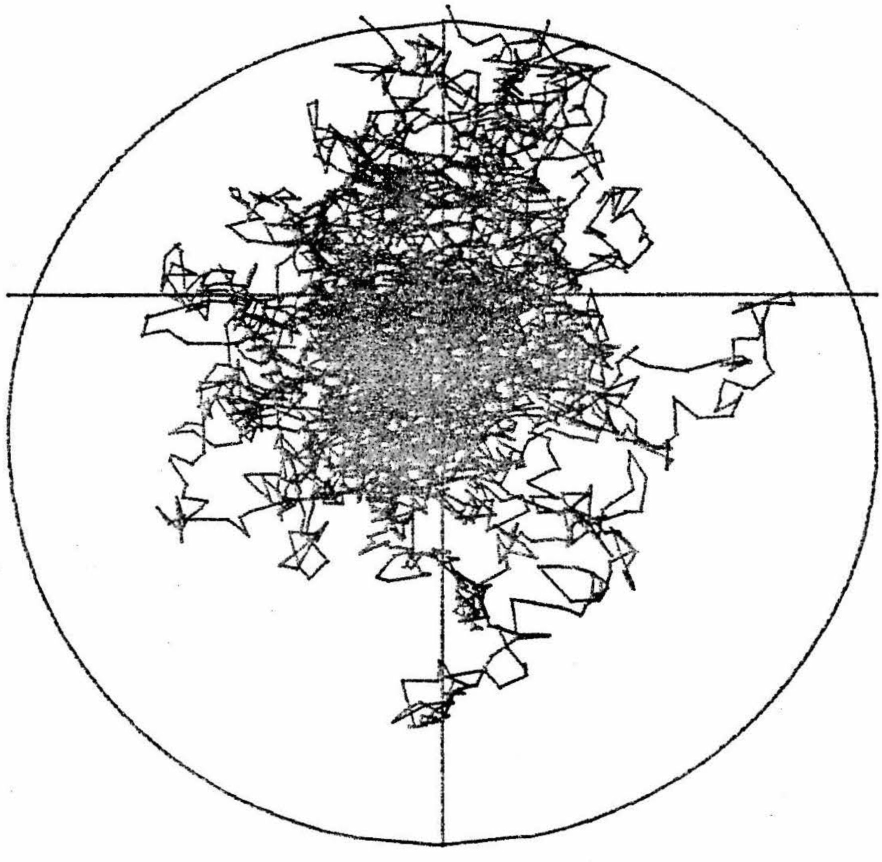


Figure 6.2.2

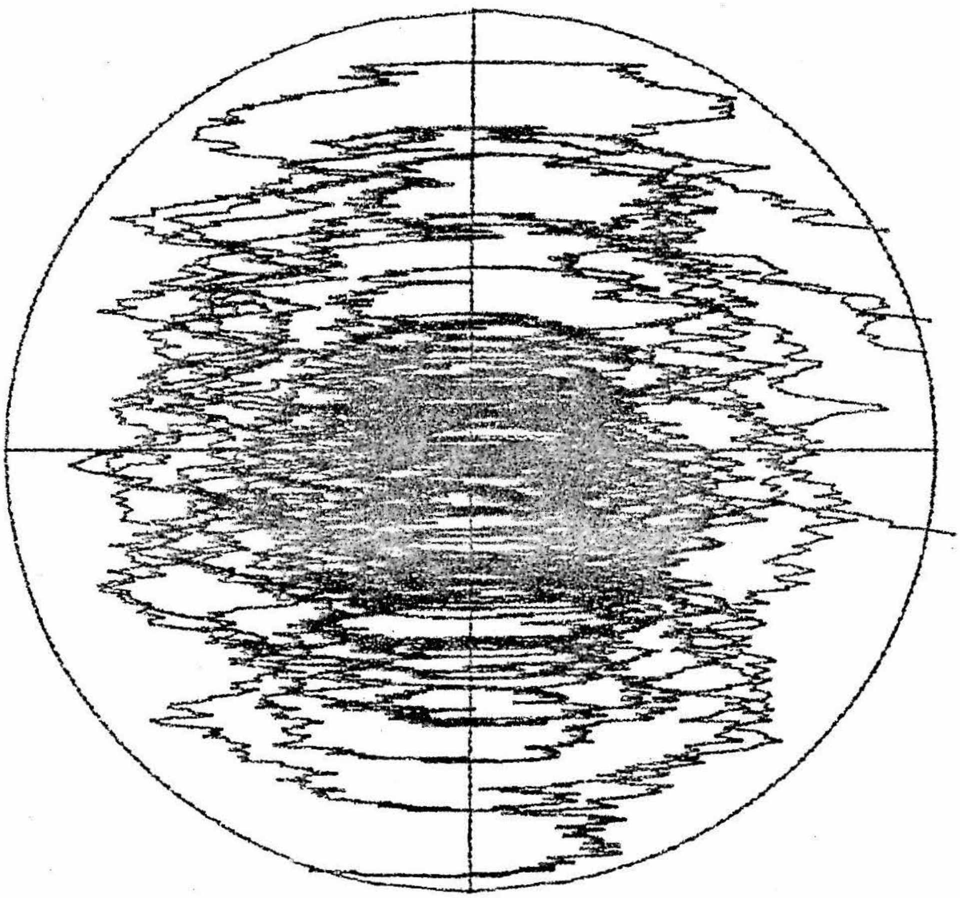


Figure 6.2.3

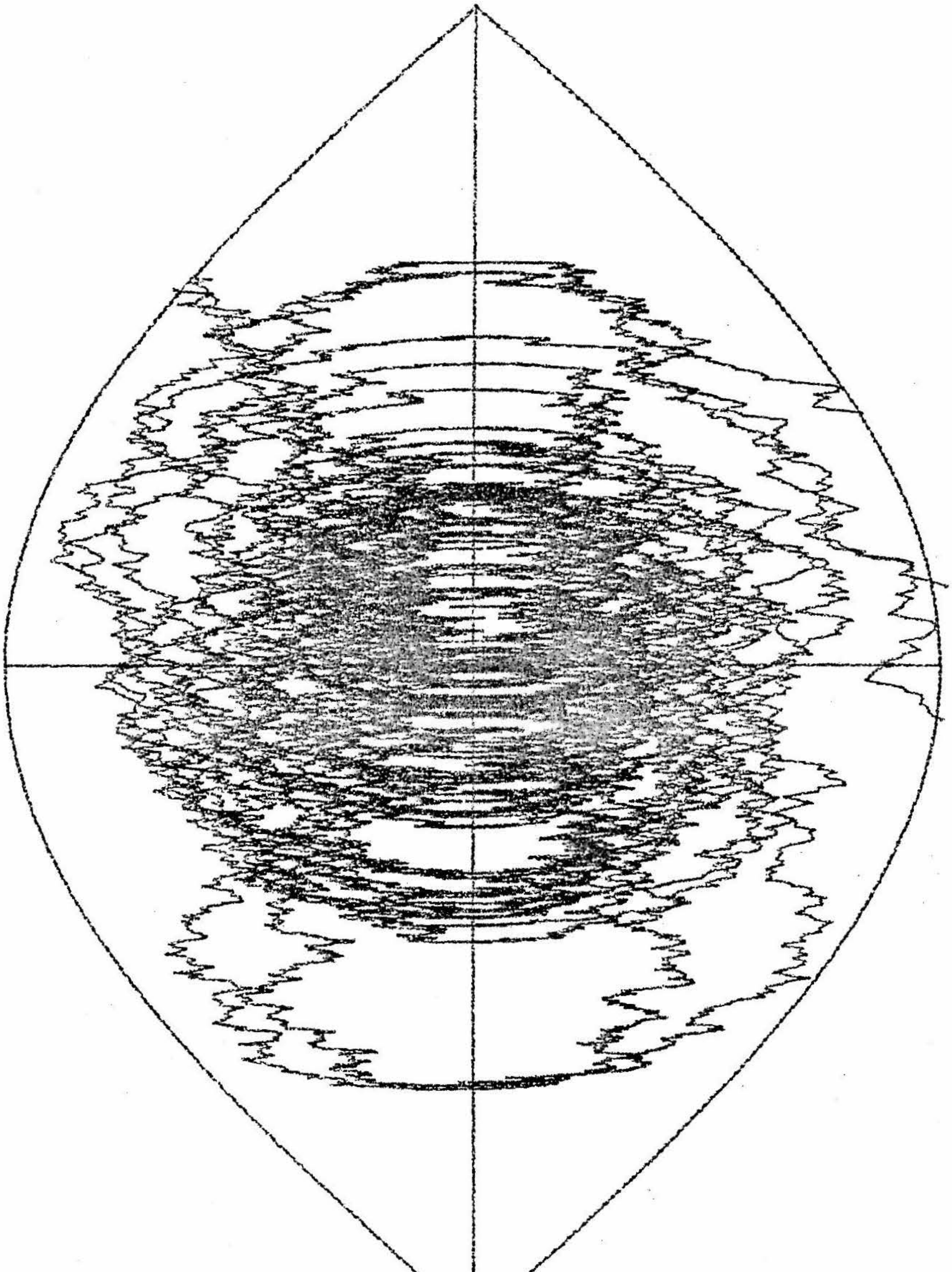


Figure 6.2.4

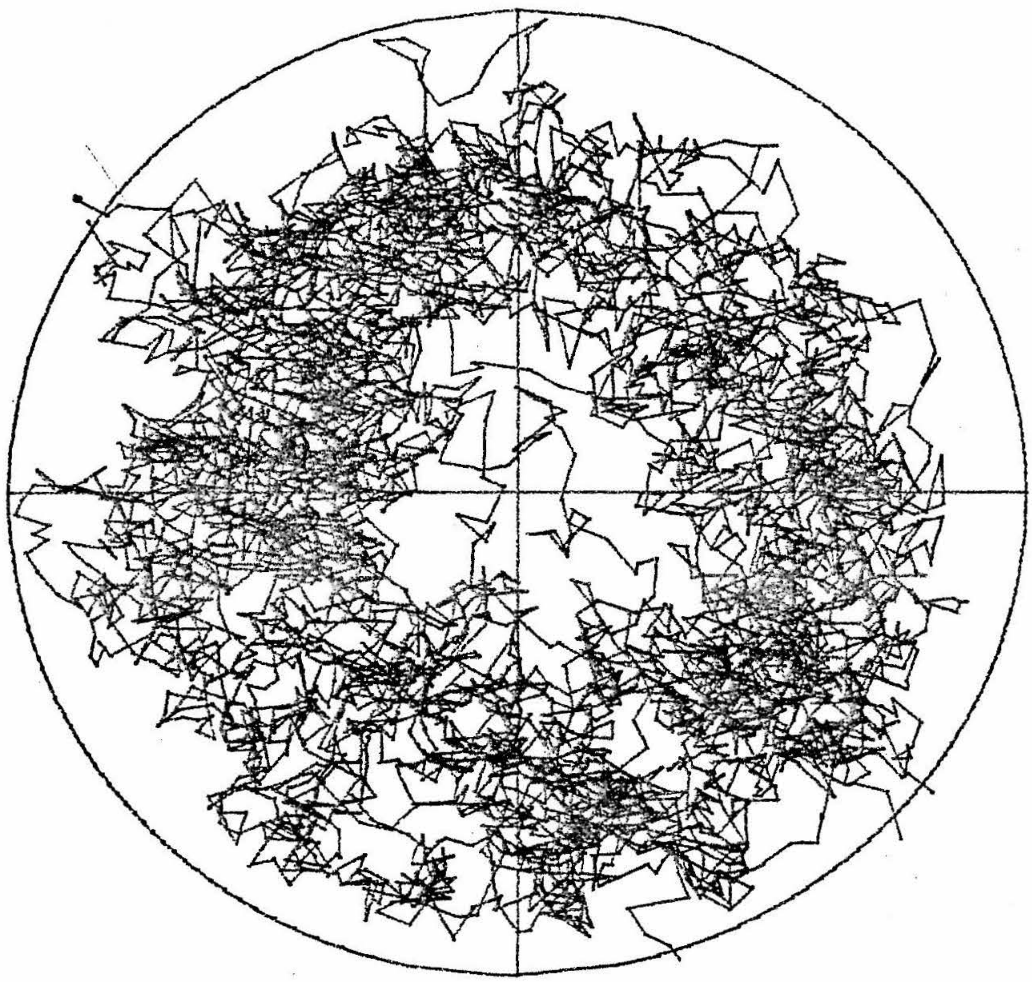


Figure 6.2.5

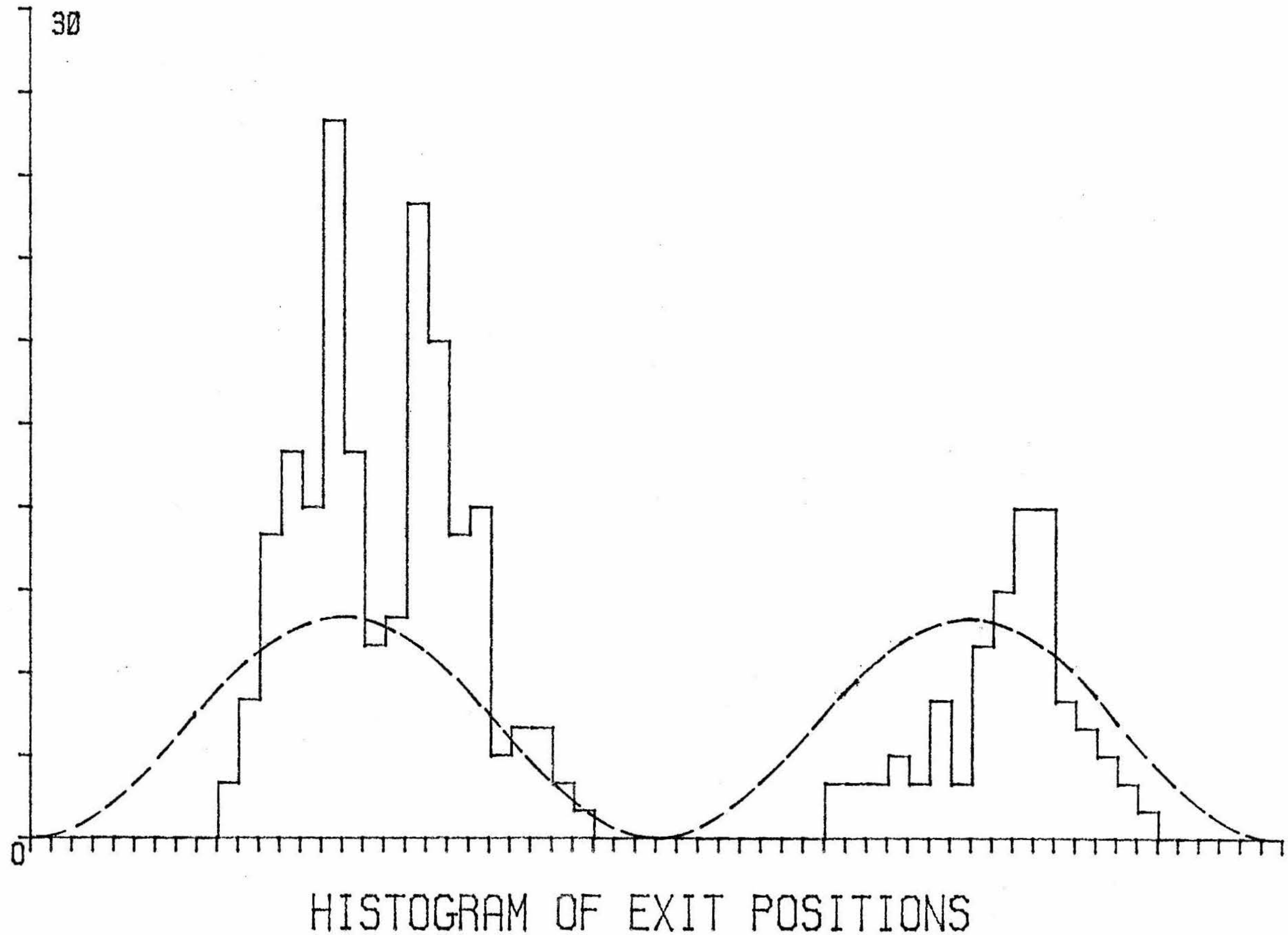
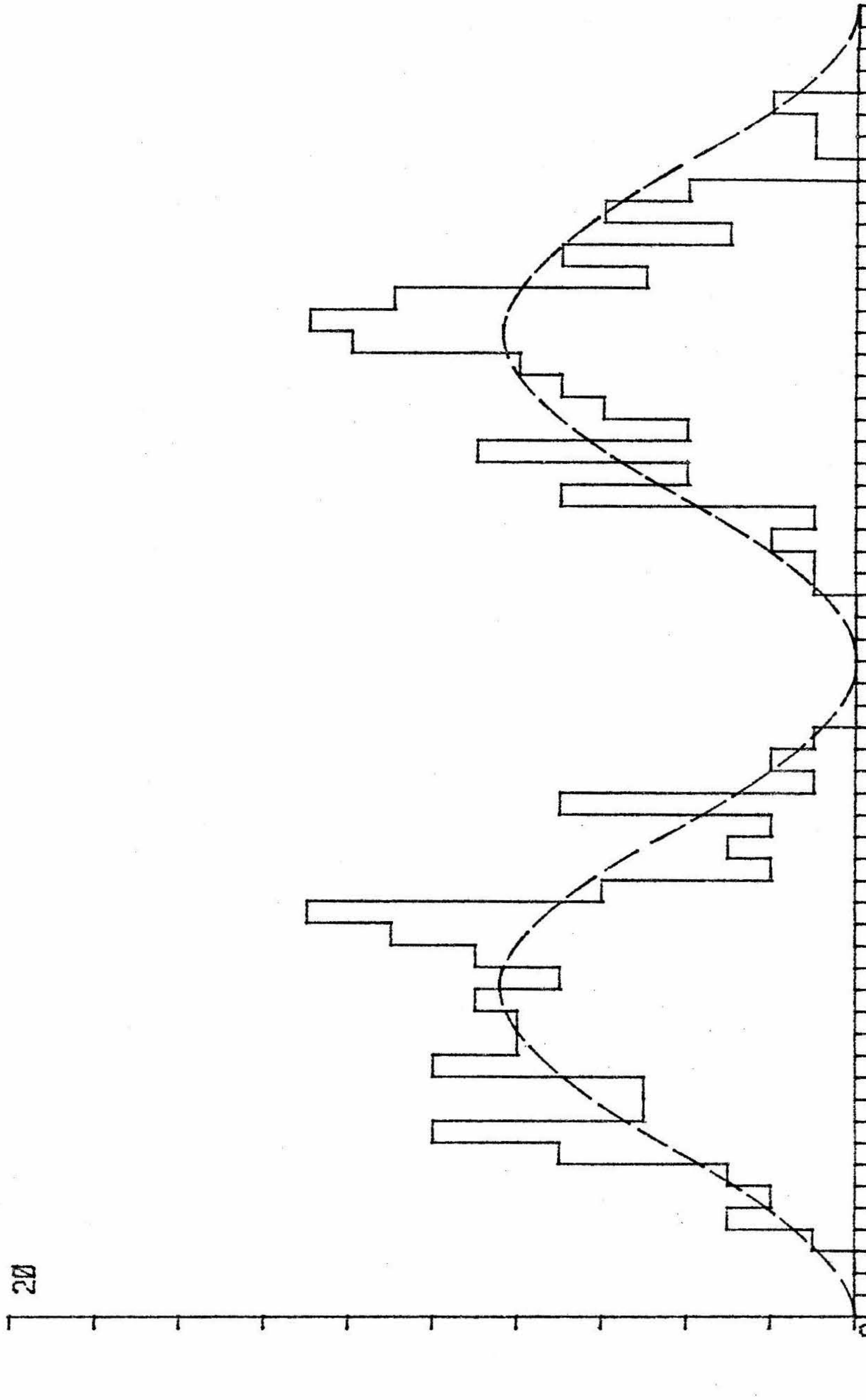


Figure 6.3.1



HISTOGRAM OF EXIT POSITIONS

Figure 6.3.2

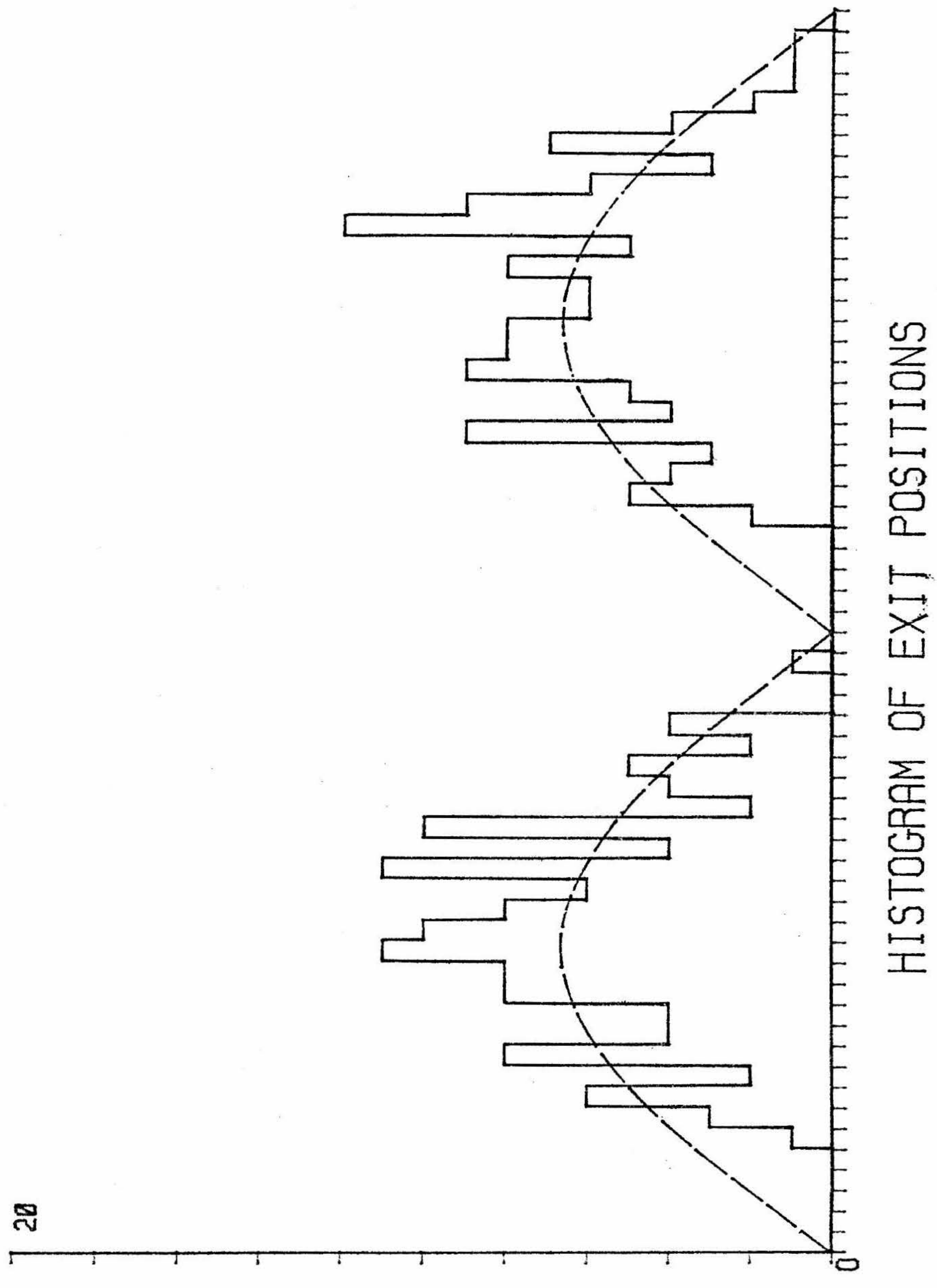
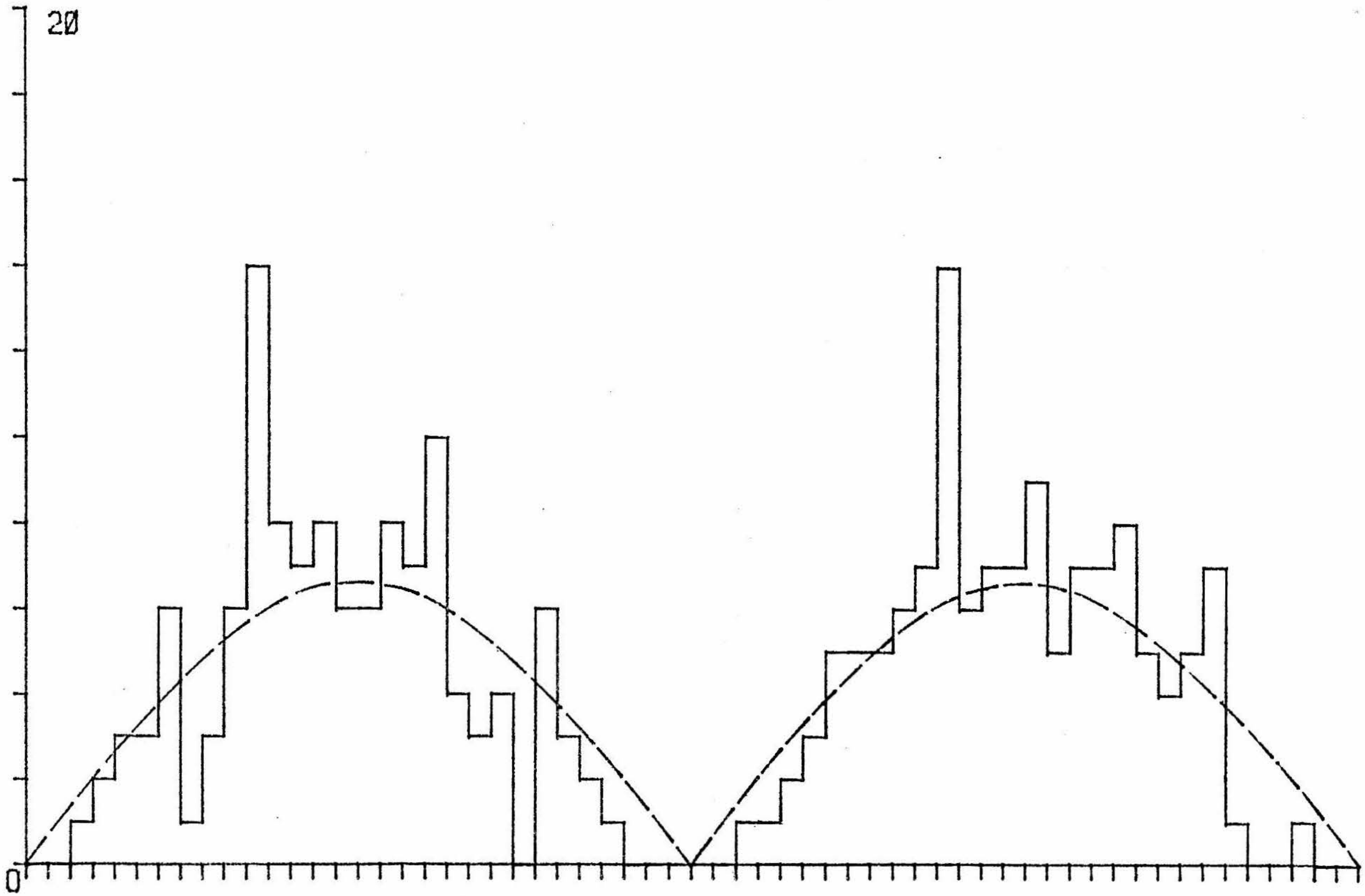
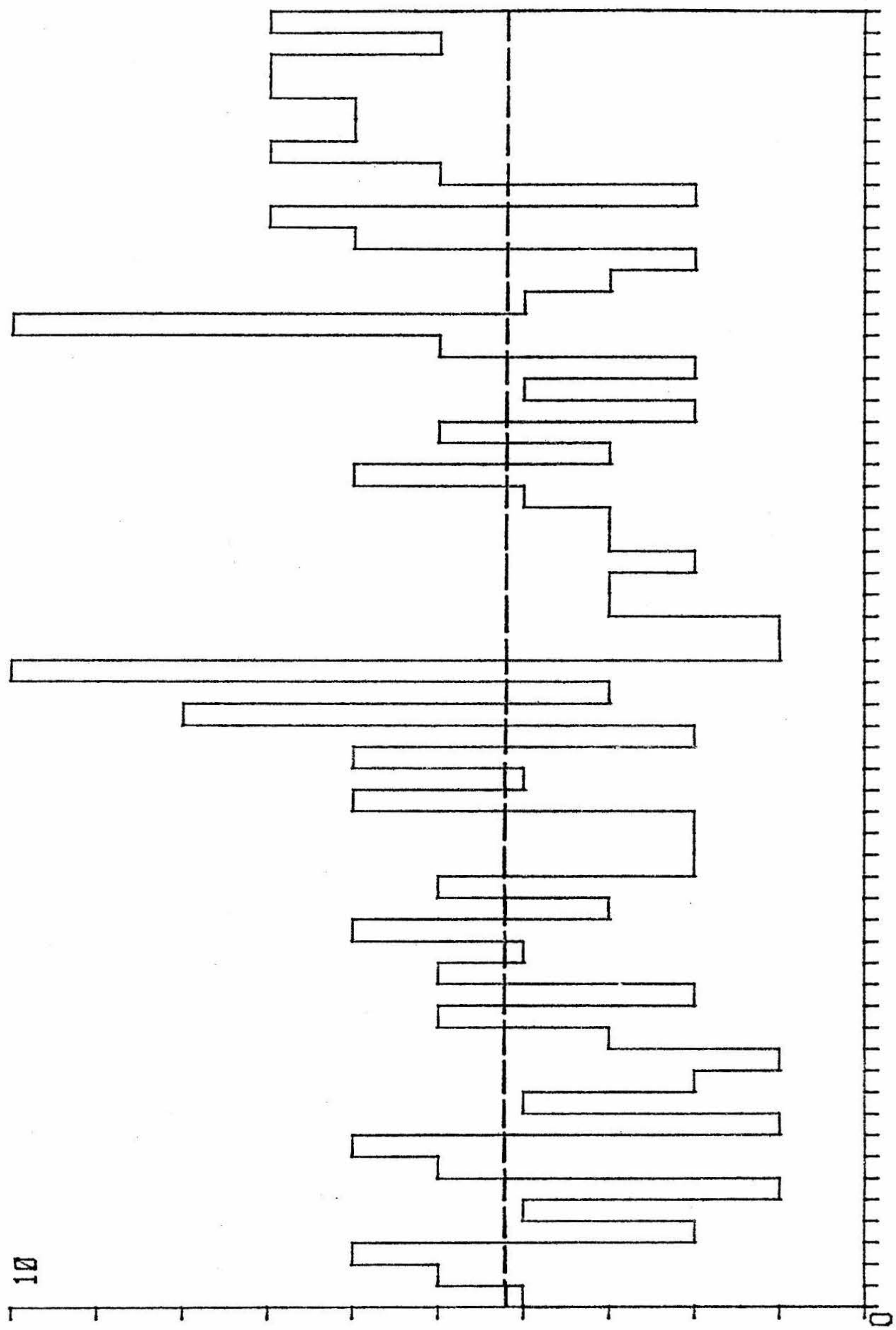


Figure 6.3.3



HISTOGRAM OF EXIT POSITIONS

Figure 6.3.4



HISTOGRAM OF EXIT POSITIONS

Figure 6.3.5

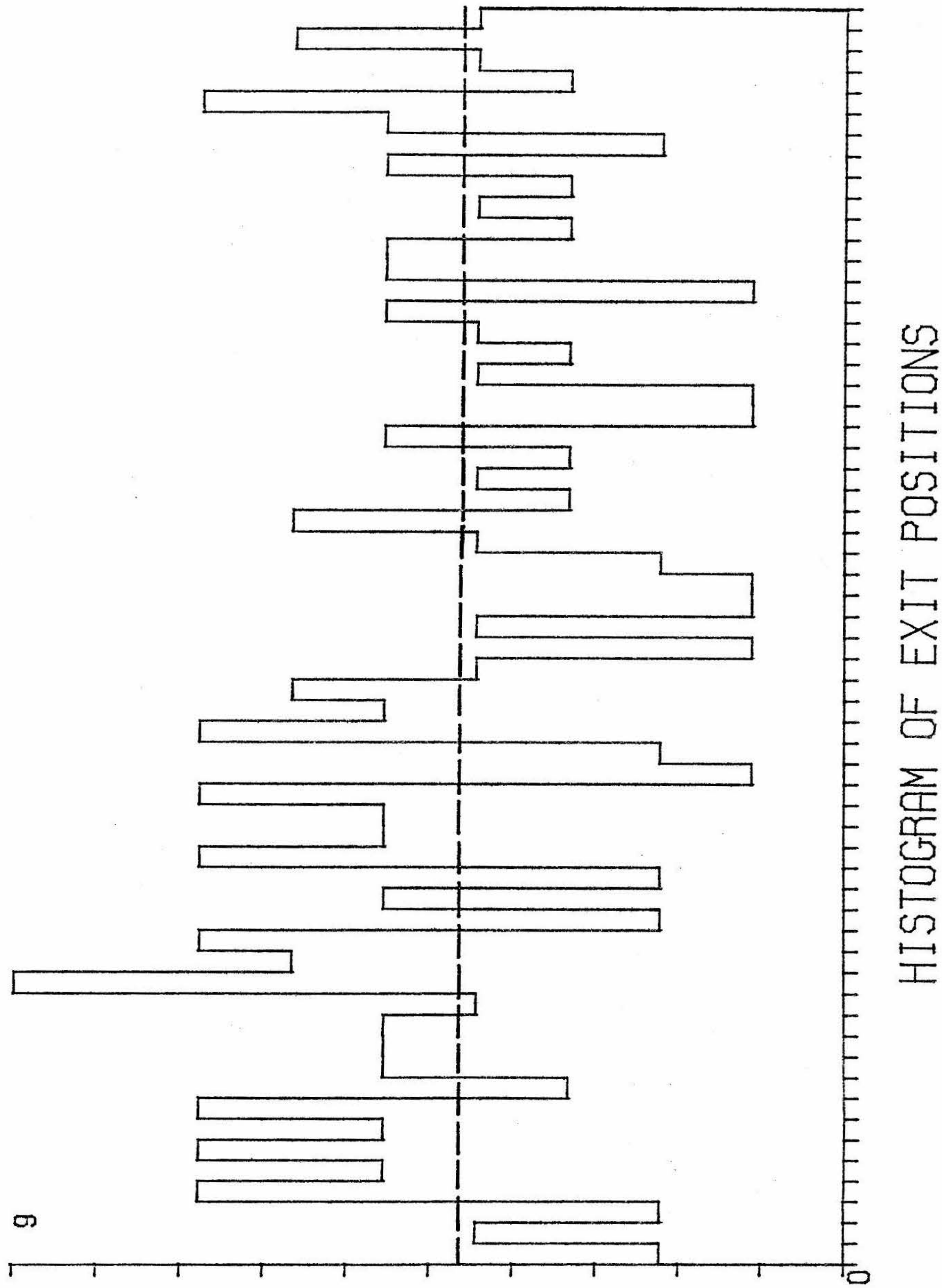


Figure 6.3.6

Figure 6.4.1

ε	β	Δt	Mean	Std. Dev.	Predicted Value
0.707	0.25	0.05	3.86	0.27	3.44
0.707	0.25	0.05	3.36	0.30	3.44
0.707	0.25	0.03	3.13	0.26	3.44
0.707	0.25	0.03	3.33	0.28	3.44
0.707	0.25	0.01	2.72	0.24	3.44
0.707	0.25	0.01	2.65	0.20	3.44
0.707	0.5	0.05	4.52	0.43	3.20
0.707	0.5	0.05	4.93	0.44	3.20
0.707	0.5	0.03	5.04	0.50	3.20
0.707	0.5	0.03	4.79	0.37	3.20
0.707	0.5	0.01	4.23	0.38	3.20
0.707	0.5	0.01	4.82	0.39	3.20
0.5	0.25	0.05	8.52	0.73	6.39
0.5	0.25	0.05	8.00	0.66	6.39
0.5	0.25	0.03	7.38	0.58	6.39
0.5	0.25	0.03	7.53	0.71	6.39
0.5	0.25	0.01	6.63	0.50	6.39
0.5	0.25	0.01	8.10	0.63	6.39
0.5	0.5	0.05	21.96	1.66	13.40
0.5	0.5	0.05	19.31	1.63	13.40
0.5	0.5	0.03	21.61	2.04	13.40
0.5	0.5	0.03	20.73	1.81	13.40
0.5	0.5	0.01	20.40	2.06	13.40
0.3535	0.25	0.05	30.04	2.67	26.80
0.3535	0.25	0.03	35.51	2.61	26.80
0.3535	0.25	0.01	43.87	3.83	26.80
0.3535	0.5	0.05	452.58	43.94	372.50
0.3535	0.5	0.05	481.72	47.51	372.50
0.3535	0.5	0.03	628.48	53.30	372.50
0.3535	0.5	0.03	541.57	58.21	372.50

Figure 6.4.2

ε	β	Δt	Mean	Std. Dev.	Predicted Value
0.707	0.125	0.05	15.32	1.38	11.60
0.707	0.125	0.05	15.45	1.27	11.60
0.707	0.125	0.03	15.47	1.31	11.60
0.707	0.125	0.03	13.42	1.00	11.60
0.707	0.125	0.01	16.10	1.29	11.60
0.707	0.125	0.01	14.46	1.22	11.60
0.707	0.25	0.06	38.67	3.18	20.43
0.707	0.25	0.05	36.75	3.36	20.43
0.707	0.25	0.03	32.20	3.22	20.43
0.707	0.25	0.03	34.91	3.01	20.43
0.707	0.25	0.01	40.53	3.71	20.43
0.707	0.25	0.01	37.62	3.57	20.43
0.50	0.125	0.05	59.65	5.08	42.90
0.50	0.125	0.05	57.60	5.05	42.90
0.50	0.125	0.03	59.58	5.42	42.90
0.50	0.125	0.03	62.85	5.42	42.90
0.50	0.125	0.01	72.44	6.32	42.90
0.50	0.125	0.01	65.62	5.93	42.90
0.50	0.25	0.05	692.79	66.44	585.32
0.50	0.25	0.05	728.35	68.22	585.32
0.50	0.25	0.03	685.22	62.41	585.32
0.50	0.25	0.03	913.53	97.73	585.32

REFERENCES

[1] Abramowitz, M. and Stegun, I., Handbook of Mathematical Functions, (1965), Dover, New York

[2] Davis, H.T., Introduction to Nonlinear Differential and Integral Equations, (1962), Dover, New York

[3] Franklin, J.N., Difference Methods for Stochastic Ordinary Differential Equations, Math. Comp., 19 (1965), 552-561

[4] Gartner, J., On the Asymptotic Behavior of the Time of First Exit from a Region, Theor. Probability Appl., 20 (1975), no. 1, 169-174

[5] Grasman, J., and Matkowsky, B.J., A Variational Approach to Singularly Perturbed Boundary Value Problems for Ordinary and Partial Differential Equations, to appear

[6] Khasminskii, R.Z., On Diffusion Processes with a Small Parameter, Izv. Nauk, S.S.S.R. Ser. Mat., 27 (1963), 1281-1300 (in Russian)

[7] Kifer, Y.I., Certain Results Concerning Small Random Perturbations of Dynamical Systems, Theor. Probability Appl., 19 (1974), no. 3, 487-505

[8] Lagerstrom, P.A., personal communication

[9] Levinson, N., The First Boundary Value Problem for $\varepsilon \Delta u + Au_x + Bu_y + Cu = D$ for small ε, Ann. of Math., 51 (1950) no. 2, 428-445

[10] Ludwig, D., Persistence of Dynamical Systems Under Random Perturbations, SIAM Rev., 17 (1975) no. 4, 605-640

[11] Matkowsky, B.J., and Schuss, Z., On the Problem of Exit, Bull. Amer. Math. Soc., 82 (1976), no. 2, 321-324

[12] Matkowsky, B.J., and Schuss, Z., The Exit Problem for Randomly Perturbed Dynamical Systems, to appear

[13] McKean, H.P., Stochastic Integrals, (1969), Academic Press, N.Y.

[14] Miller, G.F., The Evaluation of Eigenvalues of a Differential Equation Arising in a Problem in Genetics, Proc. Cambridge Philos. Soc., 58 (1962), 588-593

[15] Ventsel', A.D. and Freidlin, M.I., On Small Random Perturbations of Dynamical Systems, Uspehi Mat. Nauk, 25 (1970), 3-55, (translated as Russian Math. Surveys, 25 (1970), 1-55

References (Cont'd.)

[16] Ventsel', A.D. and Freidlin, M.I., Some Problems Concerning Stability Under Small Random Perturbations, Theor. Probability Appl., 17 (1972), no.2, 269-283