HOMOGENEOUS FLOW FIELDS
OF DEGREE GREATER THAN ZERO

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This paper is dedicated to my wife, Shirley Ruth, whose patience cannot go without mention, and to the memory of my father, Max Lampert.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF FIGURES

(cont'd)

SUMMARY

Solutions to the Prandtl-Glauert differential equation expressed in terms of polynomial type Lamé functions can be applied to the problem of the thin delta wings with subsonic leading edges in a supersonic flow field. It is demonstrated how these functions of different species and degrees of homogeneity may be employed to obtain previously known results for certain lifting cases. For the non-lifting or thickness case which is treated in detail in this paper it is shown that a large class of thickness distributions with blunt leading edges may be obtained by systematically studying the Lamé functions of the first species. In particular these functions have been investigated up to, and including, $n = 5$. It is further shown by the methods of this paper that the prescription of the pressure distribution in problems of this sort is not always sufficient to determine the thickness distribution uniquely.

I. INTRODUCTION

It has been shown by various authors that the supersonic flow fields associated with thin triangular wings may be determined to first approximation by means of linearized compressible-flow theory. These wing problems may be further specialized and subdivided into two general types. The first of these is the so-called "Lifting Case" where the wing may be thought of as having zero thickness, and it is required to find the lift distribution associated with a specified angle of attack distribution, or to find an angle of attack distribution (Camber) to support a given lift distribution. The second type problem is the so-called "Non-lifting" or thickness case usually associated with the form drag of a wing. In this case the boundary conditions given for a wing with a thin symmetric profile are either the pressure distribution at the surface or the thickness distribution, and the problem is to find respectively the thickness distribution or the pressure distribution. Other boundary value problems of the so-called mixed type involving both lifting and non-lifting regions on a wing (say) as in the case of control surfaces on a wing at zero incidence may be handled by a superposition of solutions to the problems of the type outlined above.

This paper is primarily concerned with the second of these boundary value problems, namely the thickness case associated with triangular wings whose leading edges are swept behind the Mach cone and whose section profiles give blunt leading edges. Thickness problems of this type have been treated by Squire (Ref. 1) and Lomax and Heazlet (Ref. 2). The approach used by Squire is an application of the method developed by Robinson for expressing the solutions of the linearized equations of motion in terms of Lamé Functions. In this way Squire has demonstrated how to obtain the pressure distribution over thin triangular wings having elliptic cross-sections normal to the flow direction. Lomax and Heazlet, on the other hand, have used integral equation techniques relating the pressure distribution to the thickness distribution to solve problems of a similar nature in which they may prescribe the pressure to obtain thickness or thickness to obtain the pressure. In this paper, however, it will be shown that the former method may be applied and extended to provide a direct approach for relating the pressure distribution to the thickness distribution and may be used to obtain a large variety of continuous thickness distributions over triangular wings with blunt subsonic leading edges. Further it will be indicated how, by superposition, wings with closed trailing edges may be obtained.

II. ANALYSIS

A. General Theory

Incorporating the usual assumptions of zero viscosity and heat conduction, irrotationality, isentropy, no body forces, and no discontinuous shock phenomena, it can be shown that the steady-state compressible potential flow for a perfect gas may be described by the equation

$$2a^2(\nabla^2\Phi) = \nabla\Phi\cdot\nabla(\nabla\Phi^2) \tag{1}$$

where $a^2 = \frac{dP}{d\rho}$ is the local speed of sound and Φ is the velocity potential where

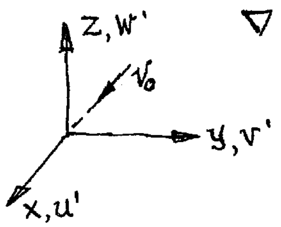
$$\nabla\Phi = \bar{A} \qquad |A| = \sqrt{u^2+v^2+w^2} \tag{2}$$

Using the well known concepts of small perturbation theory (see Ref. 4) wherein it is assumed that the velocity components vary slightly from the uniform flow of velocity, V_0, equation (1) may be linearized and expressed as

$$(M_0^2-1)\frac{\partial^2\varphi}{\partial x^2} - \frac{\partial^2\varphi}{\partial y^2} - \frac{\partial^2\varphi}{\partial z^2} = 0 \tag{3}$$

the Prandtl-Glauert equation. Here $M_0 = \frac{V_0}{a_0}$ is the Mach number associated with the conditions in the undisturbed free-stream. The velocity potential φ in equation (3) is now that potential which is associated with the disturbance (perturbation) velocities and may be considered additive to the uniform flow potential $V_0 x$; that is

$$\Phi = V_0 x + \varphi \tag{4}$$



$$\nabla\varphi = \bar{A}' \qquad |A'| = \sqrt{u'^2+v'^2+w'^2} \tag{4a}$$

The perturbation pressure, the difference between the local and free-stream pressure, is

$$P - P_o = -\rho_o V_o u' = -\rho_o V_o \frac{\partial \varphi}{\partial x} \tag{5}$$

The boundary condition governing the local flow inclination is

$$\frac{dz}{dx} = \frac{w'}{V_o} = \frac{1}{V_o}\frac{\partial \varphi}{\partial z} \tag{6}$$

which is prescribed in the x-y plane.

B. Hyperboloido-Conal Coordinates

It has been shown by A. Robinson (Ref. 5) to be of advantage when treating problems dealing with triangular wings with subsonic leading edges in supersonic flow to re-express equation (3) in terms of the so-called Hyperboloido-Conal Coordinate system which is characteristic of the geometry of these wings (see Appendix A-I). This system of coordinates μ, ν, r is defined by three families of quasi-orthogonal quadric surfaces whose equations may be written in general as

$$\frac{x^2}{\beta^2\theta^2} - \frac{y^2}{\theta^2 - k^2} - \frac{z^2}{\theta^2 - 1} = \frac{r^2(\mu^2-\theta^2)(\nu^2-\theta^2)}{\theta^2(\theta^2-k^2)(\theta^2-1)} \tag{7}$$

where $0 \leq r \leq \infty$ and $k \leq \nu \leq 1 \leq \mu \leq \infty$. Taking θ in equation (7) equal to μ, ν and ∞, respectively, the equations for these surfaces become

$$\frac{x^2}{\beta^2\mu^2} - \frac{y^2}{\mu^2-k^2} - \frac{z^2}{\mu^2-1} = 0 \tag{8}$$

$$\frac{x^2}{\beta^2\nu^2} - \frac{y^2}{\nu^2-k^2} + \frac{z^2}{\nu^2-1} = 0 \tag{9}$$

$$\frac{x^2}{\beta^2} - y^2 - z^2 = r^2 \tag{10}$$

Equations (8) and (9) represent families of elliptic cones while equation (10) represents a hyperboloid of two sheets. While μ, ν, and r may not be expressed simply in terms of x, y, z, these Cartesian coordinates may be readily expressed in terms of μ , ν , r , as follows:

$$x = \frac{\beta r \mu \nu}{k}$$
$$y = \pm \frac{r\sqrt{(\mu^2-k^2)(\nu^2-k^2)}}{k\sqrt{1-k^2}} \qquad (11)$$
$$z = \pm \frac{r\sqrt{(\mu^2-1)(1-\nu^2)}}{\sqrt{1-k^2}}$$

In Fig. 1 it may be observed that traces of the families of cones in the planes x const >0 are ellipses for the surfaces μ const., and rectangular hyperbolas for the surfaces ν const. For each point (μ, ν, r), (See Fig. 1) there correspond four points (x , y , z), hence, it is not clear what sign is associated with each of the quantities under the square roots in equation (11). This difficulty may be resolved by re-expressing these relations in terms of the Jacobi elliptic functions. In Appendix A-II it is shown that over the range of variation of μ and ν , the square roots involving the variable ν only change sign. That is, $\sqrt{\nu^2-k^2}$ takes the same sign as y and $\sqrt{1-\nu^2}$ takes the same sign as z .

The limit surfaces of the families of elliptic cones are of special interest. It may be noted from equations (8) and (10) that the surface $\mu = \infty$ and $r=0$ correspond to a circular cone emanating from the origin and for the cases considered here it represents the Mach cone

$$\frac{x^2}{\beta^2} - y^2 - z^2 = 0 \qquad (12)$$

where $\frac{1}{\beta}$ is the slope of the Mach cone in a meridian plane through the origin (See Fig. 1). It may be further noted in Fig. 1 and from equations (8), (9), and (10) that $\mu = 1$ corresponds to $z = 0$ and that part of the x - y plane bounded by the rays from the origin

$$\frac{y}{x} = \pm \frac{\sqrt{\nu_0^2-k^2}}{\beta \nu_0} \qquad (13)$$

where for

$$\nu_0 = k \ , \ t = 0 \qquad \text{and} \qquad \nu_0 = 1 \ , \ t = \pm\frac{\sqrt{1-k^2}}{\beta} = \pm m_0$$

Here the ray described by $t = 0$ corresponds to the positive x-axis and the one given for $\nu_0 = 1$ is taken as the slope of the leading edge, m_0, and where $k' = \sqrt{1-k^2} = \beta m_0$ that is; the ratio of the slope of the leading edge to the slope of the Mach lines. It should further be noted from equation (12) that the traces of the Mach cone in the $Z = 0$ plane corresponds to

$$\frac{y}{x} = \pm\frac{1}{\beta}$$

hence, the leading edges of the wings in question are always behind or, at most, lie on the Mach cone. The latter case will be omitted from this analysis and only the cases where $|\beta m_0| < 1$, namely $k > 0$, are treated.

In a similar manner if ν is taken const. equal to one, it will be noted in Fig. 1 and equations (8), (9) and (10) that the surfaces degenerate into that part of the x-y plane described by the rays from the origin

$$\left(\frac{y}{x}\right)_{\substack{\nu\to 1\\ z\to 0}} = \pm\frac{\sqrt{\mu^2-k^2}}{\beta\mu} \qquad 1 \leq \mu \leq \infty \tag{14}$$

where for

$$\mu = 1 \qquad \frac{y}{x} = \pm\frac{\sqrt{1-k^2}}{\beta} = \pm m_0$$

the slope of the leading edge and for

$$\mu = \infty \qquad \frac{y}{x} = \pm\frac{1}{\beta}$$

corresponds to the Mach lines in the x-y plane. This region in the x-y plane therefore is that part between the leading edge and the Mach cone (see Fig. 1).

C. Lamé Solutions

Equation (3) transformed into the Hyperboloido-Conal System of Coordinates becomes (See Appendix A-I for Development).

$$(\mu^2-\nu^2)\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)-\sqrt{(\mu^2-1)(\mu^2-k^2)}\frac{\partial}{\partial\mu}\left(\sqrt{(\mu^2-1)(\mu^2-k^2)}\frac{\partial\phi}{\partial\mu}\right) \tag{15}$$
$$-\sqrt{(\nu^2-k^2)(1-\nu^2)}\frac{\partial}{\partial\nu}\sqrt{(\nu^2-k^2)(1-\nu^2)}\frac{\partial\phi}{\partial\nu}\Big)=0$$

If a solution of equation (15) is chosen as

$$\phi = r^n\,\psi(\mu,\nu)$$

a partial separation of variables may be accomplished such that satisfies the following equation

$$n(n+1)(\mu^2-\nu^2)\psi-\sqrt{(\mu^2-1)(\mu^2-k^2)}\frac{\partial}{\partial\mu}\left(\sqrt{(\mu^2-1)(\mu^2-k^2)}\frac{\partial\psi}{\partial\mu}\right) \tag{16}$$
$$-\sqrt{(\nu^2-k^2)(1-\nu^2)}\frac{\partial}{\partial\nu}\left(\sqrt{(\nu^2-k^2)(1-\nu^2)}\frac{\partial\psi}{\partial\nu}\right)=0$$

A further separation of variables is possible if ψ is chosen as

$$\psi = S(\mu)\,T(\nu)$$

Substituting this solution into equation (16) leads to the normal form of Lamé equation as given in Hobson (Ref. 6); namely

$$\frac{1}{S}\left\{n(n+1)\mu^2 S-\mu\left[(\mu^2-k^2)+(\mu^2-1)\right]\frac{dS}{d\mu}\right.$$
$$\left.-(\mu^2-k^2)(\mu^2-1)\frac{d^2S}{d\mu^2}\right\}=p(k^2+1) \tag{17}$$

$$\frac{1}{T}\left\{n(n+1)\nu^2 T-\nu\left[(\nu^2-k^2)-(1-\nu^2)\right]\frac{dT}{d\nu}\right.$$
$$\left.+(\nu^2-k^2)(1-\nu^2)\frac{d^2T}{d\nu^2}\right\}=p(k^2+1)$$

In equation (17) above n is not necessarily an integer and it is sufficient to take $n \geqslant -1/2$ (see Ref. 62). Periodic solutions exist for certain characteristic values of p. For integral values of n there exist periodic Lamé functions which may be called Lamé Polynomials; however, it should be pointed out that there are also periodic solutions for integer values which are not polynomials. In this paper, however, only those solutions of integral n which are polynomials are studied. These polynomials are of degree n in one of the Jacobian elliptic functions or in two or more of them together. Furthermore these polynomials are doubly periodic of periods 2K or 4K (see Ref. 6a). In this paper it has been found that those polynomials which are doubly periodic of period 2K in μ and 2iK' in ν have led to certain usable physical results. It should be pointed out however that although solutions exist for non-integral values they have not been studied here. The integral values for which the Lamé polynomials exist of period 4K do not exhibit the symmetry conditions required for the particular boundary value problems under consideration here and hence have not been studied.

In particular the polynomial solutions which have been used in this paper may be determined in 2n + 1 different ways (see Refs. 6 or 6a). These functions are doubly periodic of 2K or 2iK' and single valued functions of the variables α and γ which are the arguments of the Jacobian elliptic functions introduced by the relations $\mu = \mathrm{ns}(\alpha, k)$ and $\nu/k = \mathrm{nd}(\gamma, k')$. (See Appendix A-II). The solutions which are functions of ν are finite over the range $k \leqslant \nu \leqslant 1$, while the solutions which are functions of μ over the range of variations $1 \leqslant \mu \leqslant \infty$ are finite for $1 \leqslant \mu < \infty$, and approach infinity like μ^n.

The notation of Ref. 6 has been employed here to define the Lamé functions although alternative definitions may be used (see Refs. 6a, 6b, and 6c). These solutions for $S(\mu)$ in equation (17) are characterized as follows:*

$$\mathbb{K} = \sum_{0}^{j_n} a_{jm}\,\mu^{n-2j} = \sum_{0}^{j_n} a_{jm}(ns\alpha)^{n-2j} = \mathbb{P}_K$$

$$j_n = \begin{Bmatrix} \frac{n}{2} & n \text{ even} \\ \frac{n-1}{2} & n \text{ odd} \end{Bmatrix} \qquad \text{No. of Sol.} = \begin{Bmatrix} \frac{n}{2}+1 & n \text{ even} \\ \frac{n+1}{2} & n \text{ odd} \end{Bmatrix}$$

$$\mathbb{L} = \sqrt{\mu^2-k^2}\,\sum_{0}^{j_n} a_{jm}\,\mu^{n-j-1} = ds\alpha \sum_{0}^{j_n} a_{jm}(ns\alpha)^{n-j-1} = ds\alpha\,\mathbb{P}_L$$

$$j_n = \begin{Bmatrix} \frac{n}{2}-1 & n \text{ even} \\ \frac{n-1}{2} & n \text{ odd} \end{Bmatrix} \qquad \text{No of Sol.} = \begin{Bmatrix} \frac{n}{2} & n \text{ even} \\ \frac{n+1}{2} & n \text{ odd} \end{Bmatrix}$$

$$\mathbb{M} = \sqrt{\mu^2-1}\,\sum a_{jm}\,\mu^{n-j-1} = cs\alpha \sum a_{jm}(ns\alpha)^{n-j-1} = cs\alpha\,\mathbb{P}_M$$

$$j_n = \begin{Bmatrix} \frac{n}{2}-1 & n \text{ even} \\ \frac{n-1}{2} & n \text{ odd} \end{Bmatrix} \qquad \text{No of Sol.} = \begin{Bmatrix} \frac{n}{2} & n \text{ even} \\ \frac{n+1}{2} & n \text{ odd} \end{Bmatrix}$$

$$\mathbb{N} = \sqrt{(\mu^2-1)(\mu^2-k^2)}\,\sum a_{jm}\,\mu^{n-2(j+1)} = cs\alpha\, ds\alpha \sum a_{jm}(ns\alpha)^{n-2(j+1)} = cs\alpha\,\mathbb{P}_N$$

$$j_n = \begin{Bmatrix} \frac{n}{2}-1 & n \text{ even} \\ \frac{n-3}{2} & n \text{ odd} \end{Bmatrix} \qquad \text{No of Sol.} = \begin{Bmatrix} \frac{n}{2} & n \text{ even} \\ \frac{n-1}{2} & n \text{ odd} \end{Bmatrix}$$

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* Note: The symbols $\mathbb{P}_K$, $\mathbb{P}_L$, and $\mathbb{P}_M$ are polynomials while $\mathbb{P}_N$ is a polynomial multiplied by $\sqrt{\mu^2-k^2} = ds\alpha$.

In Table II may be found some examples of these functions up to the degree n = 3. In particular, Appendix B illustrates the method for determining the coefficients for the $\mathbb{K}$ type Lamé functions which are polynomials of degree n. These polynomials will be referred to by the symbol $S_n^{(m)}(\mu)$ where n is the degree of homogeneity and where (m) is associated with the number of solutions. That is for the $\mathbb{K}$ species, (m) = 1, 2, $\cdots\cdots\cdots \frac{n}{2}+1$ or $\frac{n+1}{2}$ depending on whether n is odd or even. In addition there exists a second solution to equations (17), Lamé's Equation, which is usually described as a Lamé Function of the second kind which may be given in terms of the first kind as

$$\frac{\mathcal{S}_n^{(m)}(\mu)}{S_n^{(m)}(\mu)} = \int_{\mu}^{\infty} \frac{d\xi}{[S_n^{(m)}(\xi)]^2\sqrt{(\xi^2-1)(\xi^2-k^2)}} \qquad 1 \leq \mu \leq \infty \tag{19}$$

where the upper limit is chosen such that $\mathcal{S}_{(n)}^{(m)}$ vanishes at $\mu = \infty$. Equation (19) in general will be expressed in terms of the elliptic integrals of the first and second kind and it is useful to introduce the Jacobi elliptic functions, that is, if $\mu = ns(\alpha, k)$ then equation (19) may be re-written as

$$\frac{\mathcal{S}_n^{(m)}(\mu)}{S_n^{(m)}(\mu)} = \int_0^{\alpha = sn^{-1}\frac{1}{\mu}} \frac{d\alpha}{[\;S_{(n)}^{(m)}(ns\alpha)]^2} \tag{20}$$

From equations (19) or (20) it may be seen that there will be the same number of solutions for the second kind of Lamé function as there are for the first kind, namely (2n + 1). (See Appendix C.) The Lamé functions of the second kind, defined in equation (20), are finite for all space external to the elliptic cone μ const $\geqslant$ 1 and vanish on the Mach cone $\mu = \infty$. Hence it is possible to define a perturbation velocity potential in Hyperboloido-Conal Coordinates in terms of the Lamé functions of the first and second kind as

$$\varphi_n^{(m)} = C_n^{(m)} r^n \mathcal{S}_n^{(m)}(\mu)\, T_n^{(m)}(\nu) \tag{21}$$

where $C_{(n)}^{(m)}$ is a constant of proportionality and n is the degree of homogeneity.

D. Boundary Conditions

1. Thickness Case (Symmetrical Potential $\mathbb{K}$ type Lamé Function Symmetry with Respect to y and z)

The vertical velocity in the x-y plane is

$$V_0 \frac{dz}{dx} = \underset{\substack{\mu \to 1 \\ z \to 0}}{\frac{\partial \varphi}{\partial z}} = w' = \underset{\substack{\xi_3 \to 1 \\ z \to 0}}{\sum_{j=1}^{3} \frac{\partial \varphi}{\partial \xi_j} \frac{\partial \xi_j}{\partial z}} \qquad \begin{array}{l} \xi_1 = \lambda \\ \xi_2 = \nu \\ \xi_3 = \mu \end{array} \tag{22}$$

where the values of $\frac{\partial \xi_j}{\partial z}$ are given in Appendix A-I, equations A23) which when substituted into the above expression becomes

$$V_0 \frac{dz}{dx} = \frac{C_n^{(m)} r^{n-1}}{\sqrt{1-\nu_0^2}} \frac{\mathbb{P}_K(\nu_0)}{\mathbb{P}_K(1)}$$

$$\mathbb{P}_K(\nu_0) = \sum_0^{j_n} a_{jm} \nu_0^{n-2j} \tag{23}$$

$$\nu_0 = k\left[1-\beta^2 t^2\right]^{-1/2} \qquad r = \frac{x}{\beta}\left[1+\beta^2 t^2\right]^{1/2}$$

The thickness distribution then may be obtained by integrating equation (23) with respect $x = y/t$ as

$$V_o(z-z_o) = \frac{C_n^{(m)} y \beta}{\mathbb{P}_K(1)} \int_{m_o}^{t} r^{n-1} \mathbb{P}_K(\nu_o) \sqrt{\frac{1-\beta^2 t^2}{m_o^2 - t^2}} \frac{dt}{t^2} \tag{24}$$

or

$$V_o(z-z_o) = \frac{(y)^n k^{n-1} \beta}{\mathbb{P}_K(1)} \int_{\substack{\text{leading}\\ \text{edge}}}^{\frac{y\beta\nu_o}{\sqrt{\nu_o^2-k^2}}} \frac{\mathbb{P}_K(\xi)(\xi^2+k^2)\,d\xi}{(\xi^2-k^2)^{\frac{n}{2}+1}\sqrt{1-\xi^2}}$$

The pressure associated with this thickness distribution may be expressed as

$$\frac{\Delta p}{q} = -\frac{2}{V_o} \frac{\partial \Phi}{\partial x}\bigg|_{\substack{\mu\to 1\\ z\to 0}} = -\frac{2u'}{V_o} = \sum_{j=1}^{3} \frac{\partial \Phi}{\partial \xi_j} \frac{\partial \xi_j}{\partial x} \frac{1}{V_o} \tag{25}$$

where the values for $\frac{\partial \xi_j}{\partial x}$ are given in Appendix A1, equation (A23). This expression becomes upon substituting $\frac{\partial \xi_j}{\partial x}$

$$\frac{\Delta p}{q} = \frac{2C_n^{(m)} r^{n-1}}{V_o k} \mathbb{P}_K(1)\left[n\nu_o \mathbb{P}_K(\nu_o) - (\nu_o^2 - k^2)\frac{\partial}{\partial \nu_o}\mathbb{P}_K(\nu_o)\right] * \int_1^{\infty} \frac{d\xi}{[\mathbb{P}_K(\xi)]^2 \sqrt{(\xi^2-1)(\xi^2-k^2)}} \tag{26}$$

or, by putting in the values for ν_o, and inverting the elliptic integral, equation (26) becomes*

$$\frac{\Delta p}{q} = -\frac{2C_n^{(m)}}{k}\left(\frac{xk}{\beta}\right)^{n-1} \mathbb{P}_K(1) \sum_{j=0}^{j_n} \left(\frac{k}{\sqrt{1-\beta^2 t^2}}\right)^{2(1-2j)} \left[\beta^2 t^2 (2j-n) + n\right] \int_0^{K(k)} \frac{d\xi}{[\mathbb{P}_K(n s \alpha)]^2} \tag{27}$$

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* The upper limit is $K(k)$, the complete elliptic integral of first kind, and should not be confused with Lamé Function notation here.

2. Lifting Case (Asymetric Potential M type Lamé Functions Assymetric With Respect to Z)

The vertical velocity distribution in the x - y plane is

$$V_o \frac{dz}{dx} = \underset{\substack{\mu \to 1 \\ Z \to 0}}{\frac{\partial \Phi}{\partial Z}} = C_{(n)}^{(m)} r^{n-1} k' \mathbb{P}_M(\nu_o) \mathbb{P}_M(1) \int_1^{\infty} \frac{d}{d\xi}\left[\frac{1}{\xi \, \mathbb{P}_M(\xi)^2 \sqrt{\xi^2 - k^2}}\right] \frac{d\xi}{\sqrt{\xi^2 - 1}} \tag{28}$$

or substituting values for $\mathbb{P}$, r and inverting the integral

$$V_o \frac{dz}{dx} = C_{(n)}^{(m)} \left(\frac{xk}{\beta}\right)^{n-1} \mathbb{P}_M(1) \sum_{j=0}^{j_n} a_{jm} \left(\frac{\sqrt{1-\beta^2 t^2}}{k}\right)^j \int_0^{K(k)} \frac{d}{d\xi}\left[\frac{sn^2 \xi}{dn \xi \; \mathbb{P}(nS\xi)^2}\right] sc\,\xi \, d\xi \tag{29}$$

The above expression leads to the Camber shape expressed as follows:

$$V_o (z - z_o) = C_{(n)}^{(m)} \mathbb{P}_M(1) \, k' \int_0^{K(k)} \frac{d}{d\xi}\left[\frac{sn^2 \xi}{dn \xi \; \mathbb{P}(nS\xi)^2}\right] sc\,\xi \, d\xi \left[\int_{m_o}^{t} \mathbb{P}_M(\nu_o) \, r^{n-1} \frac{dt}{t^2}\right] \tag{30}$$

The lifting pressures, that is, coefficient of local lift c_ℓ associated with the above lifting surface, may be written as

$$\underset{\substack{\mu \to 1 \\ Z \to 0}}{\frac{2\Delta p}{q}} = c_\ell = \frac{-4 C_{(n)}^{(m)} r^{n-1}}{V_o \beta k' \mathbb{P}_M(1) \sqrt{m_o^2 - t^2}} \left\{ \frac{n k'^2 - \beta^2 t^2 (n - k^2)}{1 - \beta^2 t^2} \mathbb{P}_M(\nu_o) \right. \tag{31}$$

$$\left. - \beta^2 t^2 (m_o^2 - t^2) \frac{\partial \mathbb{P}_M(\nu_o)}{\partial t} \right.$$

E. Wave Drag For Wings Generated By Lamé Functions

1. Thickness Case

The Wave drag of the wings which are considered (blunt leading edges) may be divided into two parts, that is: 1. drag due to the surface pressures which contribute to the normal force and, 2. the drag due to the pressure exerted at the leading edge in the stream direction (finite force acting over the area at the leading edge). The pressure drag described in the former category may be determined simply by integrating the component of the pressure in the stream direction over the wing surface or

$$C_{D_o} = \frac{2}{S}\iint_{\Delta}\left(\frac{dz}{dx}\right)\frac{\Delta p}{q}dA = -\frac{4}{V_o^2 S}\iint_{\Delta}\left(\frac{\partial\varphi}{\partial x}\right)\left(\frac{\partial\varphi}{\partial z}\right)dA \qquad (32)$$

Substituting the values φ_z and φ_x given in equations (23) and (26) respectively, this drag may be expressed in coefficient form as

$$C_{D_o} = -\frac{2C^2}{\beta S k V_o^2}\int_0^{K(k)}\frac{d\xi}{[\mathbb{P}(ns\xi)]^2}\left[\iint_{\Delta}\overline{\mathbb{P}_k(\nu_o)}^2\left[nk + \frac{2\beta^4 t^3}{(1-\beta^2 t^2)}\frac{\partial}{\partial t}\mathbb{P}_k(\nu_o)\right]\left[y\frac{(1-\beta^2 t^2)}{\beta t}\right]^{n-1}\frac{y\,dy\,dt}{t^2\sqrt{m_o^2-t^2}}\right. \qquad (33)$$

The drag due to so-called leading edge "push" was computed by R. T. Jones in Ref. 7 for an elliptic leading edge, and it is given here in a more general form as

$$C_{D_\sigma} = \frac{2\pi m_o}{C_o^3\sqrt{1-\beta^2 m_o^2}}\left(\frac{\tau}{2C_o m_o}\right)^2\int_0^{C_o}H(x)dx \qquad (34)$$

where $\frac{\tau}{2m_o}$ as shown in Fig. 2 may be considered a thickness ratio of the cross-section of an elliptic cone. The development of equation (34) and values for $H(x)$ are given in Appendix D.

The total drag coefficient is therefore

$$C_D = C_{D_0} + C_{D_\sigma} \tag{35}$$

2. Lifting Case

Just as for the thickness case, the drag of a lifting wing in supersonic flow may be determined by integrating the lifting pressures over the wing surface and by subtracting from these forces the leading edge "thrust" due to the suction at the leading edge. For the infinitesimally thin wing at angle of attack the drag due to lift expressed in coefficient form is $C_D = C_D - C_{D_\sigma}$ where $C_{D_0} = C_L \alpha$ and where C_{D_σ} is the "coefficient of drag" due to leading edge suction which for a flat triangular wing is

$$C_{D_\sigma} = \pi k(1+m_0^2)\frac{\alpha^2}{E(k)^2} \tag{36}$$

(See Ref. 7) E(k) is the normalized complete elliptic integral of the second kind.

III. APPLICATIONS

A. Lifting Cases

It will be of interest, before progressing to the thickness and thickness drag considerations, to demonstrate the utility of this approach to obtain certain well known solutions for the infinitesimally thin, asymmetric potential case, triangular wings.

1. Lift of a Flat Triangular Wing $(n = 1)$
(Direct Problem Where Wing Geometry is Given)

B. C.

$$\frac{w'}{V_0} = \frac{1}{V_0}\left(\frac{\partial\phi}{\partial z}\right)_{\substack{\mu\to 1\\ z\to 0}} = -\alpha \qquad \nu_0^2 \leq 1$$

$$u' = \left(\frac{\partial\phi}{\partial x}\right)_{\substack{\mu\to 1\\ z\to 0}} = 0 \qquad \nu_0^2 > 1$$

or from Equation (29)

$$-V_0\alpha = k' C_1 r^{n-1} \mathbb{P}_M(\nu_0)\mathbb{P}_M(1)\int_0^{K(k)} \frac{d}{d\xi}\left[\frac{sn^2\xi}{dn\xi\,\mathbb{P}_M(ns\xi)^2}\right] sc\xi\, d\xi \tag{37}$$

$$C_1 = -\frac{V\alpha\, r^{1-n}}{k'\mathbb{P}_M(\nu_0)\mathbb{P}_M(1)\int_0^{K(k)} G(\xi)d\xi} \tag{38}$$

It will be noted from Table II that in order that the vertical velocity be a constant in the x - y plane n = 1. Hence $\mathbb{P}_M(\nu_0) = \mathbb{P}_M(1) = 1$ and the integral $\int_0^{K(k)} G(\xi)d\xi$ may be evaluated as follows:

$$\int_0^{K(k)} G(\xi)d\xi = \int_0^{K(k)} \frac{d}{d\xi}\left(\frac{sn^2\xi}{dn\xi}\right) sc\xi\, d\xi = \int_0^{K(k)} [sd^2\xi + sn^2\xi]d\xi$$

$$= -\frac{E(k)}{k'^2} \qquad k'^2 = 1 - k^2$$

and

$$C_1 = \frac{V_0\alpha k'}{E(k)} \tag{39}$$

Substituting the appropriate values for the $\mathbb{P}_M$ and C_1 for $n = 1$, in equation (31), the local lift may be written as

$$\frac{2\Delta p}{q} = c_\ell = \frac{-4\alpha k'^2}{\beta^2 E(k)\sqrt{m_o^2 - t^2}} = \frac{-4\alpha m_o^2}{E(k)\sqrt{m_o^2 - t^2}} \qquad (40)$$

this is the result obtained by Stewart in Ref. 8. The total lift may be obtained by integrating the local lift equation (40) over the wing surface. Expressed in coefficient form

$$C_L = \frac{2\pi m_o}{E(k)} \quad \text{or} \quad \frac{dC_L}{d\alpha} = \frac{2\pi m_o}{E(k)} \qquad (41)$$

2. Damping in Pitch Derivative C_{M_Q} $(n = 2)$
(Direct Problem)

B. C.
$$w' = \left(\frac{\partial\varphi}{\partial z}\right)_{\substack{\mu\to 1\\ z\to 0}} = -Qx \qquad \nu_o^2 \leq 1$$

$$u' = \left(\frac{\partial\varphi}{\partial x}\right)_{\substack{\mu\to 1\\ z\to 0}} = 0 \qquad \nu_o^2 > 1$$

Here Q is the pitching velocity in radians/sec.

Substituting this condition in equation (29) and choosing from Table II, the appropriate values for the $\mathbb{P}_M$'s for n = 2, namely $\mathbb{P}(\nu_o) = \nu_o$ $\mathbb{P}(1) = 1$

then

$$-Qx = C_2 h k' \nu_o \int_0^{K(k)} \frac{d}{d\xi}\left(\frac{\mathrm{sn}^4\xi}{\mathrm{dn}\,\xi}\right)\mathrm{sc}\,\xi\, d\xi = \frac{C_2 x k k'}{\beta} I_2\left(\frac{\pi}{2}, k\right)$$

(for evaluation of $I_2(\frac{\pi}{2}, k)$ see Appendix E, Part I)

or

$$C_2 = -\frac{Q\beta}{k k' I_2(\frac{\pi}{2}, k)} = -\frac{k k' Q \beta}{[(1-2k^2)E(k) + k^2 K(k)]} \qquad (42)$$

The longitudinal velocity distribution which is proportional to the lift distribution may be obtained from equation (31) and becomes

$$\left(\frac{\partial\varphi}{\partial x}\right)_{\mu\to 1} = \frac{xQ[1-\beta^2 m_o^2][2m_o^2 - t^2][m_o^2 - t^2]^{-\frac{1}{2}}}{[1-2\beta^2 m_o^2]E(k) + \beta^2 m_o^2 K(k)} \qquad (43)$$

where K(k) and E(k) are complete elliptic integrals of the first and second kind respectively for $k = \sqrt{1-\beta^2 m_o^2}$. This solution agrees with that obtained by Brown and Adams, Ref. 9. The Pitching Moment due to the steady pitching velocity Q may be obtained by integrating the local lift

$$\ell_l = \frac{2\Delta p}{q} = -\frac{4xQ[1-\beta^2 m_o^2][2m_o^2-t^2][m_o^2-t^2]^{-\frac{1}{2}}}{V_o[1-2\beta^2 m_o^2]E(k)+\beta^2 m_o^2 K(k)} \tag{44}$$

about the wing apex, hence the pitching moment becomes

$$\frac{M}{q} = -\frac{3\pi Q}{2V_o}\frac{S c_o^2[1-\beta^2 m_o^2]}{[1-2\beta^2 m_o^2]E(k)+\beta^2 m_o^2 K(k)} \tag{45}$$

where $C_M = \frac{M}{qSc_o}$ and the damping in pitch derivative may be written as

$$C_{M_Q} = \frac{\partial C_M}{\partial\left(\frac{Qc_o}{2V_o}\right)} = -\frac{3\pi[1-\beta^2 m_o^2]}{[1-2\beta^2 m_o^2]E(k)+\beta^2 m_o^2 K(k)} \tag{46}$$

3. Damping in Roll Derivative (n = 2)
(Direct Problem)

B. C.
$$w' = \left(\frac{\partial\phi}{\partial z}\right)_{\substack{\mu\to 1\\ z\to 0}} = Py \qquad \nu_o^2 \leq 1$$

$$u' = \left(\frac{\partial\phi}{\partial x}\right)_{\substack{\mu\to 1\\ z\to 0}} = 0 \qquad \nu_o^2 > 1$$

Here P corresponds to the rolling velocity in radians/sec.

Substituting these conditions in equation (29) and noting from Table II, Appendix B, that the appropriate functions are $\mathbb{P}_N$'s (n = 2) where

$$\mathbb{P}_N(\nu_o) = \sqrt{\nu_o^2 - k^2}, \quad \mathbb{P}_N(1) = k'$$

then the boundary condition on w' may be written as

$$Py = C_2 r k'^2 \sqrt{\nu_o^2-k^2}\int_0^{K(k)} \frac{d}{d\xi}\left[\frac{sn^4\xi}{dn^3\xi}\right] sc\,\xi\, d\xi \tag{47}$$

or

$$Py = C_2 r k'^2 k y I_3(\pi/2, k) \qquad I_3(\tfrac{\pi}{2}k) \text{ EVALUATED IN APP. E.}$$

$$C_2 = \frac{P}{k'^2 k^2 I_3(\frac{\pi}{2},k)} = \frac{P k'^2 k}{(2-\beta^2 m_o^2)E(k) - \beta^2 m_o^2 K(k)} \tag{48}$$

The longitudinal velocity distribution which is proportional to the local lift may be obtained from equation (31) and is

$$\left(\frac{\partial\varphi}{\partial x}\right)_{\mu\to 1} = -\frac{P[1-\beta^2 m_o^2]m_o y[m_o^2-t_o^2]^{-\frac{1}{2}}}{[(2-\beta^2 m_o^2)E(k)-\beta^2 m_o^2 K(k)]} \tag{49}$$

This solution was obtained by Brown and Adams, Ref. 9, by other methods. The rolling moment due to the steady roll velocity P may be obtained readily by integrating the local lift

$$c_\ell = \frac{2\Delta p}{q} = -\frac{4yPm_o^2[1-\beta^2 m_o^2][m_o^2-t^2]^{-\frac{1}{2}}}{V_o[(2-\beta^2 m_o^2)E(k)-\beta^2 m_o^2 K(k)]} \tag{50}$$

about the longitudinal axis of the wing (root chord axis). The Rolling Moment may be written as

$$\frac{L'}{q} = -\frac{\pi}{4}\frac{PSb^2 m_o[1-\beta^2 m_o^2]}{[(2-\beta^2 m_o^2)E(k)-\beta^2 m_o^2 K(k)]} \tag{51}$$

or $C_{L'} = \dfrac{L'}{qSb}$ where the damping in roll derivative is written as

$$C_{L'_p} = \frac{\partial C_{L'}}{\partial\left(\frac{Pb}{2V_o}\right)} = -\frac{\pi}{2}\frac{m_o[1-\beta^2 m_o^2]}{[(2-\beta^2 m_o^2)E(k)-\beta^2 m_o^2 K(k)]} \tag{52}$$

B. Thickness Cases

1. Uniform Pressure Distribution - Elliptic Cone (n = 1)

It can be shown that the functions $\mathbb{K}$ are those functions found to give a symmetric potential, and hence are associated with wing thickness.

(a) Direct Problem

If in equation (23) n = 1, then the local slope may be expressed as

$$V_o\frac{dz}{dx} = \frac{C_1}{\beta}\sqrt{\frac{1-\beta^2 m_o^2}{m_o^2-t^2}} \tag{53}$$

Upon comparing equation (53) with the local slope of the elliptic cone

$$z = \frac{\tau x}{2m_o}\sqrt{m_o^2-t^2} \tag{54}$$

which is

$$\frac{dz}{dx} = \frac{\tau}{2m_o} \frac{m_o^2}{\sqrt{m_o^2 - t^2}} \tag{55}$$

it may be noted that the constant

$$C_1 = \beta V_o \tau \frac{m_o}{2} \left[1 - \beta_o^2 m_o^2\right]^{-\frac{1}{2}} \tag{56}$$

τ in the above equations is the thickness ratio of the cone, that is where τC_o is the minor axis of the elliptic cone at $x = C_o$ (see Fig. 3).

The pressure distribution for such a wing may be obtained directly from equation (26) by substituting the value for C and $n = 1$, that is

$$\frac{\Delta p}{q} = 2\beta m_o^2 \frac{\tau}{2m_o} \frac{\left[K(k) - E(k)\right]}{k^2} \tag{57}$$

Here the pressure distribution is uniform all over the wing planform.

(b) Inverse Problem

The pressure distribution could have been prescribed (a priori) as uniform and the constant C determined. In any event the resultant thickness distribution would be that of the elliptic cone.

(c) Drag of Elliptic Cone

The drag due to the surface pressures is

$$C_{D_o} = -\frac{4\beta m_o^4}{S}\left(\frac{\tau}{2m_o}\right)^2 \frac{\left[K(k) - E(k)\right]}{k^2} \iint_{\Delta} \frac{1}{\sqrt{m_o^2 - t^2}} \frac{\partial(x,y)}{\partial(x,t)}\, dy\, dt \tag{58}$$

or

$$C_{D_o} = \frac{6\beta m_o}{C_o^3}\left(\frac{\tau}{2m_o}\right)\frac{\left[K(k) - E(k)\right]}{k^2}(\text{Volume of Cone})$$

The total drag is therefore

$$C_D = C_{D_o} + C_{D_\sigma}$$

(C_{D_σ} is evaluated in Appendix D for the elliptic cone)

or

$$C_D = \frac{\pi \beta m_o^3}{\sqrt{1-\beta^2 m_o^2}} \left(\frac{\tau}{2m_o}\right)^2 \left[1 + 2\,\frac{K(k)-E(k)}{\sqrt{1-\beta^2 m_o^2}}\right] \tag{59}$$

2. Linearly Increasing Pressure Distribution ($n = 2$) (Hypercone)

(a) Direct Problem

From equation (23) and Table II, it will be noted that the local slope may be written as

$$V_o\left(\frac{dz}{dx}\right)_{(2)}^{(m)} = \frac{C_{(2)}^{(m)}\, r}{\sqrt{1-\nu_o^2}}\; \frac{\nu_o^2 + a_{1m}}{1 + a_{1m}} \qquad (m) = 1, 2 \tag{60}$$

where the equation relating the coefficients is

$$3a_{1m}^2 + 2(k^2+1)a_{1m} + k^2 = 0$$

or

$$a_{11}, a_{12} = -\frac{1}{3}\left[(k^2+1) \pm \sqrt{k^4-k^2+1}\right] \tag{61}$$

These coefficients are plotted against k over the range 0 to 1 in Fig. (3). Equation (60) above may be written as

$$V_o\left(\frac{dz}{dx}\right)_{(2)}^{(m)} = \frac{C_{(2)}^{(m)}}{\beta^2}\,\frac{1}{\sqrt{m_o^2-t^2}}\left[\frac{1-\beta^2 m_o^2 + a_{1m}(1-\beta^2 t^2)}{1+a_{1m}}\right] \tag{62}$$

Comparing equation (62) with the equation for the slope of the surface

$$z = \frac{\tau x^2}{2C_o m_o}\sqrt{m_o^2-t^2} \tag{63}$$

which is

$$\frac{dz}{dx} = \frac{\tau x}{2C_o m_o}\,\frac{[2m_o^2-t^2]}{\sqrt{m_o^2-t^2}} \tag{63a}$$

it will be noted that they are of the same general form. Equation (63) is the equation of the surface whose thickness distribution or cross-section normal to the flow direction is elliptic and whose semi-minor axis increases parabolically (see Fig. 7). It has been found that if the two solutions, (m) = 1, 2 are added in the correct proportions equation (62) for the slope will be proportional to equation (63a). The required superposition is

$$V_o \frac{dz}{dx} = V_o \left[\left(\frac{dz}{dx}\right)^{(1)}_{(2)} - \left(\frac{dz}{dx}\right)^{(2)}_{(2)} \right] \tag{64}$$

which becomes

$$\frac{dz}{dx} = \frac{Cx}{V_o\sqrt{m_o^2-t^2}} \frac{3(a_{12}-a_{11})(2m_o^2-t^2)}{\beta^2 m_o^2} \tag{65}$$

where for this case

$$C = a_{11} C^{(1)}_{(2)} = a_{12} C^{(2)}_{(2)}$$

Comparing this equation with (63a) it may be seen that the value of C is

$$C = \frac{\tau V_o}{2C_o m_o} \frac{\beta^2 m_o^2}{3(a_{12}-a_{11})} \tag{66}$$

In Fig. 4 is shown the geometry of the surface described by equation (64).

The pressure distribution over such a body may be obtained by the following superposition of solutions of equation (26) for the appropriate values of n and $\mathbb{P}_k(\nu_o)$

$$\left(\frac{\Delta p}{q}\right)_a = \left(\frac{\Delta p}{q}\right)^{(1)}_{(2)} - \left(\frac{\Delta p}{q}\right)^{(2)}_{(2)} \tag{67}$$

where

$$\left(\frac{\Delta p}{q}\right)^{(m)}_{(2)} = \frac{4x C^{(m)}_{(2)}[1+a_{1m}][k^2+a_{1m}]}{V_o\beta} \int_0^{K(k)} \frac{sn^4\xi\, d\xi}{[1+a_{1m}\, sn^2\xi]^2} \tag{68}$$

The elliptic integral in the above equation is evaluated in Appendix E and if its value is substituted into (68), the equation for the pressure is

$$\left(\frac{\Delta p}{q}\right)^{(m)}_{(2)} = -\frac{2x}{a_{1m}} \frac{C^{(m)}_{(2)}}{\beta} \left[(1+a_{1m})K(k) - E(k)\right] \tag{69}$$

After substituting the appropriate values into the superposition (67) above, the form for the pressure distribution becomes

$$\left(\frac{\Delta p}{q}\right)_a = -\frac{2x\beta^2 m_o}{\beta[1-\beta^2 m_o^2]}\left(\frac{\tau_a}{2c_o m_o}\right)\left[2(2-\beta^2 m_o^2)E(k)-(3-\beta^2 m_o^2)K(k)\right] \tag{70}$$

This pressure distribution is seen to be linearly increasing in the stream-wise (x direction) direction and has parallel isobars normal to the stream direction.

(b) Inverse Problem (n = 2)

If now in this problem the pressure distribution is prescribed as a linearly increasing distribution x-wise with parallel isobars normal to the mean flow direction, namely

$$\left(\frac{\Delta p}{q}\right)^{(m)}_{(2)} = A_{1m}x = -\frac{4x}{V_o\beta}C^{(m)}_{(2)}[1+a_{1m}][k^2+a_{1m}]\int_0^{K(k)}\frac{sn^4\xi\, d\xi}{[1+a_{1m}sn^2\xi]^2} \tag{71}$$

where

$$A_{1m} = \frac{-2C^{(m)}_{(2)}[(1+a_{1m})K(k)-E(k)]}{V_o\beta a_{1m}}$$

and where $C^{(m)}_{(2)}$ is taken as

$$C^{(m)}_{(2)} = \frac{V_o\beta^2 m_o^2}{3(a_{12}-a_{11})}\left(\frac{1+a_{1m}}{k^2+a_{1m}}\right)\left(\frac{\tau}{2m_oC_o}\right)$$

then the local slope associated with the above pressure distribution is

$$V_o\left(\frac{dz}{dx}\right)^{(m)}_{(2)} = -\frac{C^{(m)}_{(2)}x}{\beta^2}\left[\frac{k^2+a_{1m}(1-\beta^2t^2)}{1+a_{1m}}\right]\frac{1}{\sqrt{m_o^2-t^2}} \tag{72}$$

or substituting $C^{(m)}_{(2)}$ into equation (72)

$$\left(\frac{dz}{dx}\right)^{(m)}_{(2)} = \frac{m_o^2x}{3(a_{12}-a_{11})(k^2+a_{1m})}\left(\frac{\tau}{2m_oC_o}\right)\frac{[k^2+a_{1m}(1-\beta^2t^2)]}{\sqrt{m_o^2-t^2}} \tag{72a}$$

The thickness distribution is obtained upon integrating (72a) with respect to x, or substituting y-t for x the equation for the thickness distribution is

$$Z^{(m)}_{(2)} = \frac{m_o^2y^2}{3(a_{12}-a_{11})(k^2+a_{1m})}\left(\frac{\tau}{2m_oC_o}\right)\int_{m_o}^{t}\frac{[k^2+a_{1m}-a_{1m}\beta^2t^2]}{\sqrt{m_o^2-t^2}}\frac{dt}{t^3}$$

or

$$Z_{(2)}^{(m)} = \frac{x^2}{3(a_{12}-a_{11})}\left(\frac{\tau}{2m_oC_o}\right)\left\{\sqrt{m_o^2-t^2} + \right.$$

$$\left.\frac{t^2(1+a_{1m})(1-\beta^2m_o^2)}{m_o^2\,(k^2+a_{1m})}\left[1-\frac{a_{1m}\,\beta^2m_o^2}{(1+a_{1m})(1-\beta^2m_o^2)}\right]\cosh^{-1}\left|\frac{m_o}{t}\right|\right\} \tag{73}$$

$$-m \leq t \leq m \qquad 0 \leq k^2 \leq 1$$

For the range of variation of a_{1m} with k, see Fig. 3. The two solutions for $Z_2^{(m)}$ may be written as

$$Z_{(2)}^{(1)} = \frac{x^2\,\tau_b}{3(a_{12}-a_{11})m_oC_o}\left\{\sqrt{m_o^2-t^2} + \frac{t^2}{m_o}\frac{k^2(1+a_{11})}{(k^2+a_{11})}\left[1-\frac{a_{11}\,\beta^2m_o^2}{k^2(1+a_{11})}\right]\cosh^{-1}\left|\frac{m_o}{t}\right|\right\} \tag{74}$$

$$Z_{(2)}^{(2)} = \frac{x^2\,\tau_b}{3(a_{12}-a_{11})m_oC_o}\left\{\sqrt{m_o^2-t^2} + \frac{t^2k^2}{m_o}\frac{(1+a_{12})}{(k^2+a_{12})}\left[1-\frac{a_{12}\,\beta^2m_o^2}{k^2(1+a_{12})}\right]\cosh^{-1}\left|\frac{m_o}{t}\right|\right\} \tag{74a}$$

It may be noted that by the proper superposition of equations (74) and (74a) that the terms involving the $\cosh^{-1}\left|\frac{m_o}{t}\right|$ may be eliminated such that the thickness distribution which is obtained will correspond to the previous result given in equation (63). Further it may be seen that by taking the difference between equations (74) and (74a), the terms involving $\sqrt{m_o^2-t^2}$ may be eliminated so that

$$Z_b = Z_{(2)}^{(1)} - Z_{(2)}^{(2)} = \frac{2y^2}{m_o}\left(\frac{\tau_b}{2C_om_o}\right)\cosh^{-1}\left|\frac{m_o}{t}\right| \tag{75}$$

This surface (see Fig. 6) has zero thickness along the root chord, and has its maximum thickness located at $\frac{m_o}{t}$ satisfying the equation

$$\frac{m_o}{t} = \cosh\frac{\frac{m_o}{t}}{2\sqrt{\left(\frac{m_o}{t}\right)^2-1}}$$

where

$$\frac{m_o}{t} \doteq 1.3 \qquad \text{or} \qquad t \doteq \frac{m_o}{1.3}$$

The pressure distribution associated with this thickness distribution is

$$\left(\frac{\Delta p}{q}\right)_b = \left(\frac{\Delta p}{q}\right)^{(1)}_{(2)} - \left(\frac{\Delta p}{q}\right)^{(2)}_{(2)} = \frac{2x\beta m_o^2}{k^4}\left(\frac{\tau_b}{2m_o c_o}\right)\left[(2-k^2)K(k) - 2E(k)\right] \quad (76)$$

Although it is possible to obtain many other thickness distributions by superposing equations (74) and (74a) in various ways, the thickness distributions given by equations (63) and (75) will be used here as examples to demonstrate the non-uniqueness of specifying a linearly increasing pressure distribution.

It may be shown that the linear pressure distribution sustained by the elliptic hypercone of equation (63) may be of the same magnitude as the pressure distribution given in equation (76); that is

$$\left(\frac{\Delta p}{q}\right)_a = \left(\frac{\Delta p}{q}\right)_b \quad (77)$$

if

$$\frac{\tau_a}{\tau_b} = \frac{[1+\beta^2 m_o^2]K(k) - 2E(k)}{[3-\beta^2 m_o^2]K(k) - 2[2-\beta^2 m_o^2]E(k)} \quad (77a)$$

It is of interest here to determine whether it will be possible to obtain a thickness distribution which corresponds to a zero pressure distribution making use of the relationship indicated in equations (77) and (77a). That is, if the pressure distribution is zero, the thickness distribution may be written as

$$z_u = \frac{\tau_a x^2}{2c_o m_o}\left[\sqrt{m_o^2 - t^2} - \frac{2t^2}{m_o}\frac{\tau_b}{\tau_a}\cosh^{-1}\left|\frac{m_o}{t}\right|\right] \quad (78)$$

It follows therefore, that if Z_u is to be non-negative (i.e., the upper and lower surfaces do not cross) then the terms in brackets must be greater than, or equal to zero, or

$$\lambda = \frac{2\cosh^{-1}\left|\frac{m_o}{t}\right|}{|m_o/t|\sqrt{(m_o/t)^2-1}} \leqslant \tau_a/\tau_b \tag{79}$$

It may be shown readily that $0 \leqslant \lambda \leqslant 2$ for $\infty \geqslant \frac{m_o}{t} \geqslant 1$ therefore, $\frac{\tau_a}{\tau_b} \geqslant 2$ however, $0.12 \leqslant \frac{\tau_a}{\tau_b} \leqslant 1$ for $0 \leqslant \beta m_o \leqslant 1$ (see Fig. 7 for the range of $\frac{\tau_a}{\tau_b}$).

Hence, for any given βm_o, it will be noted that there will always be a range of m_o/t for which the inequality in (79) is in contradiction. Therefore it follows that a zero pressure distribution obtained in the manner prescribed will always lead to non-physical thickness distributions.

It is possible, however, to obtain a multiplicity of thickness distributions which may have the same pressure distribution. If h is an arbitrary multiplicative factor then these surfaces may be characterized as

$$z = [h+1]^{-1}[z_a + h z_b] \tag{80}$$

The values for h are taken small enough so that z is always greater than zero.

It is indicated, therefore, that the prescription of the pressure distribution in this case will not result in a unique thickness distribution.

This same result was observed by Heazlet and Lomax in Ref. 2, using integral equation techniques.

(c) Drag for Surfaces Generated for n = 2

The drag for the surface described by equation (63) is

$$C_D = C_{D_o} + C_{D_\sigma}$$

where

$$C_D = -\beta m_o c_o^2 \left(\frac{\tau}{c_o m_o}\right)^2 \frac{[(3-\beta^2 m_o^2)K(k) - 2(2-\beta^2 m_o^2)E(k)]}{[1-\beta^2 m_o^2]^2} \int_{-m_o}^{m_o} \frac{[2m_o^2 - t^2]\,dt}{\sqrt{m_o^2 - t^2}} \tag{81}$$

$$C_D = 6\beta m_o^3 c_o^2 \pi \left(\frac{\tau}{c_o m_o 2}\right)^2 \frac{[2(2-\beta m_o^2)E(k) - (3-\beta^2 m_o^2)K(k)]}{[1-\beta^2 m_o^2]^2} \tag{82}$$

The total drag is obtained by adding the coefficient C_{D_σ} equation (34) to C_{D_o} that is

$$C_D = \pi\left(\frac{\tau}{2C_o m_o}\right)^2 \frac{\beta m_o^3}{\sqrt{1-\beta^2 m_o^2}}\left\{\frac{C_o^2}{4} + \frac{6C_o^2\left[2(2-\beta^2 m_o^2)E(k)-(3-\beta^2 m_o^2)K(k)\right]}{\left[1-\beta^2 m_o^2\right]^{3/2}}\right\} \tag{83}$$

The pressure drag for the general surface $Z_{(2)}^{(m)}$ is

$$C_{D_{(2)}}^{(m)} = \frac{4\beta m_o^4}{9S(a_{12}-a_{11})^2}\left(\frac{\tau}{2m_o C_o}\right)^2\left(\frac{1+a_{1m}}{k^2+a_{1m}}\right)\frac{\left[(1+a_{1m})K(k)-E(k)\right]}{a_{1m}}\iint_\Delta\left[1-\frac{a_{1m}\beta^2 t^2}{k^2+a_{1m}}\right]\frac{y^3}{t^4}\frac{dy\,dt}{\sqrt{m_o^2-t^2}}$$

$$C_{D_{(2)}}^{(m)} = \frac{4\beta m_o^3 C_o^2 \pi}{9a_{1m}(a_{12}-a_{11})^2}\left(\frac{\tau}{2m_o C_o}\right)^2\left(\frac{1+a_{1m}}{k^2+a_{1m}}\right)\left[(1+a_{1m})K(k)-E(k)\right]\left[1-\frac{1}{2}\frac{a_{1m}\beta^2 m_o^2}{k^2+a_{1m}}\right] \tag{84}$$

$$C_D = C_{D_o} + C_{D_\sigma} \tag{85}$$

See Appendix D for C_{D_σ}

3. Surface for n = 3

(a) Direct Problem

In this case and in succeeding cases the direct problem will be that problem for which the spanwise cross-section of the body will be kept elliptic, while the inverse problem will be the case where the isobars will be maintained parallel, normal to the flow direction.

It will be noted from Table II that there are two solutions for the $\mathbb{K}$ type Lamé function of the form

$$\mathbb{P}_k = \mathbb{K}(\xi) = \xi^3 + a_{1m}\xi \tag{86}$$

hence substituting in equation (23) it is observed that the local slope distribution is given as

$$V_o\left(\frac{dz}{dx}\right)_{(3)}^{(m)} = \frac{C_{(3)}^{(m)}\pi^2}{\sqrt{1-\nu_o^2}}\left(\frac{\nu_o^3 + a_{1m}\nu_o}{1+a_{1m}}\right) \tag{87}$$

where the coefficients a_{1m} are related by the following equation

$$5a_{1m}^2 + 4(k^2+1)a_{1m} + 3k^2 = 0$$

where

$$a_{11}, a_{12} = -\frac{1}{5}\left[2(k^2+1) \pm \sqrt{4k^4-7k^2+4}\right] \tag{88}$$

these coefficients are plotted against k in Fig. 8. Equation (87) may be rewritten as

$$V_o\left(\frac{dz}{dx}\right)_{(3)}^{(m)} = \frac{C_{(3)}^{(m)} x^2 k}{\beta^2}\left(\frac{k^2+a_{1m}}{1+a_{1m}}\right)\left[1-\frac{\beta^2 t^2 a_{1m}}{k^2+a_{1m}}\right]\frac{1}{\sqrt{m_o^2-t^2}} \tag{89}$$

It is interesting to note that the form of equation (89) is similar to equation (62) multiplied by x. However, the constants a_{1m} are as defined in (88) above. As previously it may be noted further that equation (89) is of the same general form as the local slope of the surface

$$z = \frac{\tau x^3}{2C_o^2 m_o}\frac{[3m_o^2-2t^2]}{\sqrt{m_o^2-t^2}}\frac{[m_o^2-t^2]}{[3m_o^2-2t^2]} = \frac{\tau x^3}{2C_o^2 m_o}\sqrt{m_o^2-t^2} \tag{90}$$

which is

$$\frac{dz}{dx} = \frac{\tau x^2[3m_o^2-2t^2]}{2C_o^2 m_o\sqrt{m_o^2-t^2}} \tag{91}$$

This surface or thickness distribution is elliptic in lateral cross-section, the semi-minor axis of which increases as x^3. See Fig. 9. It may be noted that for the following superposition

$$\left(\frac{dz}{dx}\right)_{3a} = \left(\frac{dz}{dx}\right)_{(3)}^{(2)} - \left(\frac{dz}{dx}\right)_{(3)}^{(1)} \tag{92}$$

where

$$C = \left(1+\frac{3}{a_{11}}\right)^{-1} C_{(3)}^{(1)} = \left(1+\frac{3}{a_{12}}\right)^{-1} C_{(3)}^{(2)} \tag{93}$$

the local slope is

$$\left(\frac{dz}{dx}\right)_{3a} = \frac{5Cx^2k(a_{12}-a_{11})(3m_o^2-2t^2)}{V_o\beta^3 m_o^2\sqrt{m_o^2-t^2}}$$

If this equation is compared with equation (91) it will be noted that the constant of proportionality is

$$C = \frac{1}{5}\left(\frac{\tau}{2C_o^2 m_o}\right)\frac{V_o\beta^3 m_o^2}{(a_{12}-a_{11})k}$$

The pressure coefficient for n = 3 may be written as

$$\left(\frac{\Delta p}{q}\right)_{(3)}^{(m)} = \frac{2C_{(3)}^{(m)}\pi^2}{k}(1+a_{1m})[3\nu_o^2(a_{1m}+k^2)-a_{1m}(\nu_o^2-k^2)]\int_0^{K(k)}\frac{sn^6\xi\, d\xi}{[1+a_{1m}sn^2\xi]^2} \tag{94}$$

Evaluating the elliptic integral in equation (94) (see Appendix E) the pressure distribution is

$$\left(\frac{\Delta p}{q}\right)_{(3)}^{(m)} = \frac{C_{(3)}^{(m)}\pi^2}{k}\left[\frac{(5a_{1m}+4k^2+2)\nu_o^2-k^2}{a_{1m}+k^2}\right]\left[\frac{(3a_{1m}+2+k^2)}{k^2}(K-E)-E\right] \tag{95}$$

Superposing the pressures in the same manner as indicated in equation (92), that is

$$\left(\frac{\Delta p}{q}\right)_{3a} = \left(\frac{\Delta p}{q}\right)_{(3)}^{(2)} - \left(\frac{\Delta p}{q}\right)_{(3)}^{(1)} \tag{96}$$

after some manipulation one gets for this superposition

$$\left(\frac{\Delta p}{q}\right)_{3a} = \left(\frac{\tau}{2C_o^2 m_o}\right)\frac{x^2}{k^6}\left\{ m_o^2\left[(12-7\beta^2m_o^2+3\beta^4m_o^4)(K-E) - (1-\beta^2m_o^2)(7-3\beta^2m_o^2)E\right] + t^2\left[(7+\beta^2m_o^2)\beta^2m_o^2(K-E) - 2(1-\beta^4m_o^4)E\right]\right\} \tag{97}$$

The pressure distribution increases as x^2 in the stream direction and has hyperbolic isobars which have the asymptotes

$$0 = x^2m_o^2\left[(12-7\beta^2m_o^2+3\beta^4m_o^4)(K-E) - (1-\beta^2m_o^2)(7-3\beta^2m_o^2)E\right] - y^2\left[(7+\beta^2m_o^2)\beta^2m_o^2(K-E) - 2(1-\beta^4m_o^4)E\right] \tag{98}$$

in Fig. 9 the pressure distribution as given in equation (97) is characterized by the "isobaric" distribution for a triangular wing whose semi-apex angle is 31° when the Mach No. is $\sqrt{2}$.

(b) Inverse Problem (n = 3)

For the inverse problem n = 3 it is proposed that the pressure distribution be proportional to the second power of x and have isobars normal to the stream direction. In part (a) equations (95) and (89) give the pressure distribution and the associated local slope respectively for any value of a_{1m}, hence by superposing the equation for the pressure, one may obtain the desired distribution and the accompanying equation for computing the thickness. The desired pressure distribution is

$$\left(\frac{\Delta p}{q}\right)_{3b} = Bx^2 = -\sum_{m=1}^{2}\frac{(a_{1m}+k^2)(-1)^m}{\left[(3a_{1m}+2+k^2)(K-E)-k^2E\right]}\left(\frac{\Delta p}{q}\right)_{(3)}^{(m)} \tag{99}$$

where

$$\left(\frac{\Delta p}{q}\right)_{(3)}^{(m)} = \frac{4C_{(3)}^{(m)}x^2}{kV_o\beta}\left[1+\frac{a_{1m}+1+\beta^2t^2}{4(a_{1m}+k^2)}\right]\left[(3a_{1m}+2+k^2)(K-E)-k^2K\right] \tag{100}$$

hence $\qquad \dfrac{\Delta p}{q} = \dfrac{-5Cx^2(a_{12}-a_{11})}{kV_o\beta}$ and $\qquad C = \dfrac{BkV_o\beta}{5(a_{11}-a_{12})} \qquad (101)$

The local slope associated with this pressure distribution may be written as

$$\left(\frac{dz}{dx}\right)_{3b} = -\sum_{m=1}^{2} \frac{(-1)^m (a_{1m}+k^2)}{[(3a_{1m}+2+k^2)(K-E)]}\left(\frac{dz}{dx}\right)_{(3)}^{(m)} \tag{102}$$

where

$$\left(\frac{dz}{dx}\right)_{(3)}^{(m)} = -\frac{Bk^2x^2}{5\beta^2(a_{12}-a_{11})}\,\frac{[k^2+a_{1m}-\beta^2t^2a_{1m}]}{[1+a_{1m}]\sqrt{m_o^2-t^2}}$$

Performing the operations in equation (102) the slope becomes

$$\left(\frac{dz}{dx}\right)_{3b} = \frac{Bk^2}{\beta}\frac{x^2}{\sqrt{m_o^2-t^2}}\left\{(7k^4-6k^2+8)(K-E)+(5k^2-4)k^2E - \beta^2t^2[2(k^2-5)(K-E)+5k^2E\right\} \tag{103}$$

where

$$B^{-1} = \frac{2m_oC_o}{\tau}\left[k^2(7k^4-4)(K-E)^2-2k^2(4-k^2)E(K-E)+5k^4E^2\right] \tag{104}$$

Integrating equation (103) with respect to x the thickness distribution associated with the "parallel isobaric" distribution n = 3 is then

$$Z = \frac{Bk^2}{3\beta^2}\frac{x^3}{m_o^4}\sqrt{m_o^2-t^2}\left\{m_o^2[(K-E)(7k^4-6k^2+8)+(5k^2-4)k^2E] + t^2[2(8k^4-12k^2+13)(K-E)+(15k^2-13)k^2E]\right\} \tag{105}$$

This thickness distribution is plotted in Fig. 10 for a representative wing (ω=31°) at a Mach No. $\sqrt{2}$.

In general it is possible to generate a large number of thickness distributions by superposing pressure distributions represented by equation (100) and obtaining the associated thickness distribution by integrating the appropriate superposition of equation (89). In Fig. 11, the pressure and thickness distribution are computed for the two solutions to the Lamé equation for n = 3. The Lamé coefficients for this case are plotted against $k = \sqrt{1-\beta^2 m_o^2}$ in Fig. 8 and the coefficients for k = 0.8 (m_o = 0.6 β = 1, ω=31°) are selected to demonstrate this and other examples in Section III. The appropriate values for this example as determined in Fig. 8 are, a_{11} = -0.875 and a_{12} = +0.450. The pressure distributions are therefore

$$\left(\frac{\Delta p}{q}\right)_{(3)}^{(1)} = \frac{12\,C_{(3)}^{(1)}x^2[1-1.90t^2]}{V_o}$$

$$\left(\frac{\Delta p}{q}\right)_{(3)}^{(2)} = -\frac{13.9\,C_{(3)}^{(2)}x^2[1+1.72t^2]}{V_o} \tag{106}$$

and the associated slopes and thickness distributions are

$$\left(\frac{dz}{dx}\right)^{(1)}_{(3)} = 1.5\ \frac{C^{(1)}_{(3)}x^2}{V_o}\ \frac{\left[1-3.72t^2\right]}{\sqrt{m_o^2-t^2}} \quad ; \quad m_o = .6$$

$$\left(\frac{dz}{dx}\right)^{(2)}_{(3)} = -.55\ \frac{C^{(2)}_{(3)}x^2}{V_o}\ \frac{\left[1-.412t^2\right]}{\sqrt{m_o^2-t^2}} \tag{107}$$

$$z^{(1)}_{(3)} = 4.13\ \frac{C^{(1)}_{(3)}x^3}{V_o}\sqrt{m_o^2-t^2}\left[1+1.84t^2\right]$$

$$z^{(2)}_{(3)} = -1.53\ \frac{C^{(2)}_{(3)}x^3}{V_o}\sqrt{m_o^2-t^2}\left[1+5.138t^2\right] \tag{108}$$

(c) Drag for Surfaces Generated for n 3

The drag coefficients associated with the various surfaces of triangular planform which were derived in the foregoing section may be tabulated as follows

(1) Elliptic Hypercone

$$C_{D_{3a}} = \frac{\pi}{6}\left(\frac{\tau}{2C_o^2}\right)^2\frac{C_o^4}{k^6}\left\{(4-7\beta^2m_o^2+7\beta^4m_o^4)(K-E)-(9-\beta^2m_o^2)(1-\beta^2m_o^2)E\right\} + C_{D\sigma} \tag{109}$$

C_{D_σ} — drag coefficient due to leading edge push given in equation (34).

(2) Pressure Distribution with Parallel Isobars

$$C_{D_{3b}} = \frac{\pi}{6}\ \frac{B^2k^2C_o^4}{m_o\beta^2}\left\{2(9-7\beta^2m_o^2+2\beta^4m_o^4)(K-E)+(1-\beta^2m_o^2)(2-15\beta^2m_o^2)E\right\} + C_{D\sigma} \tag{110}$$

(3) Arbitrary Thickness or Pressure Distribution

$$C^{(m)}_{D_{(3)}} = \frac{\pi}{6}\left[C^{(m)}_{(3)}\right]^2 C_o^6\left[(3a_{1m}+2+k^2)(K-E)-k^2E\right]\times \left\{8(1+k^2+2a_{1m})+\beta^2m_o^2\left[1-4a_{1m}-a_{1m}\frac{(1+a_{1m})}{(k^2+a_{1m})}-\frac{3}{4}\ \frac{\beta^4a_{1m}}{k^2+a_{1m}}\right\} + C_{D\sigma} \tag{111}$$

4. Surfaces for n = 4

(a) Direct Problem

For this case there exist three real solutions to Lamé's equation which may be represented by the polynomials

$$\mathbb{P}_k = \mathbb{K}(\xi) = \xi^4 + a_{1m}\xi^2 + a_{2m} \qquad m = 1, 2, 3\ . \tag{112}$$

The values for the 6 coefficients a_{jm} are plotted in Fig. 12, as a function of $k=\sqrt{1-\beta^2 m_o^2}$ over the range of applicability of this modulus, for the cases considered here ($o \leq k \leq 1$).

Substituting the polynomial of equation (112) into equation (23) the local slope distribution in the plane for $n=4$ may be represented as

$$V_o\left(\frac{dz}{dx}\right)_{(4)}^{(m)} = \frac{C_{(4)}^{(m)} \pi^2}{\sqrt{1-\nu_o^2}} \; \frac{\nu_o^4 + a_{1m}\nu_o^2 + a_{2m}}{[1+a_{1m}+a_{2m}]} \tag{113}$$

or

$$V_o\left(\frac{dz}{dx}\right)_{(4)}^{(m)} = \frac{C_{(4)}^{(m)} x^3\left[(1-\beta^2 m_o^2)^2 + a_{1m}(1-\beta^2 m_o^2)(1-\beta^2 t^2) + a_{2m}(1-\beta^2 t^2)^2\right]}{\beta^4 \sqrt{m_o^2-t^2}\,[1+a_{1m}+a_{2m}]} \tag{114}$$

The thickness distribution associated with this local slope distribution is

$$\begin{aligned} Z_{(4)}^{(m)} = \frac{C_{(4)}^{(m)} x^4}{4\beta^4 m_o^2 V_o(1+a_{1m}+a_{2m})} \Big\{ & \sqrt{m_o^2-t^2}\left[(1-\beta^2 m_o^2)^2 + a_{1m}(1-\beta^2 m_o^2) + a_{2m}\right] \\ & + \frac{t^2}{2m_o^2}\sqrt{m_o^2-t^2}\left[3(1-\beta^2 m_o^2)^2 + (1-\beta^2 m_o^2)(3+4\beta^2 m_o^2)a_{1m} + (3+8\beta^2 m_o^2)a_{2m}\right] \\ & + \frac{t^4}{2m_o^3}\cosh^{-1}\left|\frac{m_o}{t}\right|\left[3(1-\beta^2 m_o^2) + a_{1m}(1-\beta^2 m_o^2)(3+4\beta^2 m_o^2) + a_{2m}(3-8\beta^2 m_o^2+8\beta^4 m_o^4)\right]\Big\} \end{aligned} \tag{115}$$

For this and higher order cases it has been found more convenient to examine the thickness distributions for the three solutions independently and then to determine what distributions might be obtained by superposition. It is not as apparent in this case what the combinations of equation(114) should be to produce the elliptic hypercone. However, by choosing a specific case, i.e., where the apex angle and Mach number are known, and computing the thickness distributions associated with each solution of the Lame equation for $n=4$, $(m)=1, 2, 3$, see Figs. 13, 13a, and 13b; it is possible to superpose these solutions in the proper proportions to give the elliptic contour. That is

$$C_o^{n-1}\frac{Z_{an}}{x^n} = \frac{\tau_a}{2m_o}\sqrt{m_o^2-t^2} = A\, C_o^{n-1}\sum_{\lambda=1}^{(m)} \frac{Z_n^{\lambda}}{x^n} \tag{116}$$

where the values for $Z_{(4)}^{(m)}$ are given in Figs. 13, 13a, and 13b. For the elliptic hypercone case where $\omega = 31^{\circ}$, $k = 0.8$, and $\beta = 1$ the coefficients $C_{(4)}^{(\lambda)}$ in equations (115) and (116) were determined as

$$C_{(4)}^{(1)} = \frac{2}{25} V_o \beta^4 \quad C_{(4)}^{(2)} = \frac{4}{35} V_o \beta^4 \quad C_{(4)}^{(3)} = V_o \beta^4$$

$$A = \frac{3\tau}{2C_o^3 m_o} \qquad m_o = .6 \tag{117}$$

The elliptic cone resulting from this superposition is shown in Fig. (13C). The pressure distribution is determined from the superposition of the pressures associated with equation (115). *That is if*

$$\left(\frac{\Delta p}{q}\right)_{(4)}^{(m)} = \frac{4\nu_o \kappa^3}{V_o k} C_{(4)}^{(m)} [1+a_{1m}+a_{2m}][\nu_o^2(a_{1m}+2k^2)+2a_{2m}+k^2 a_{1m}] \int_0^{K(k)} \frac{sn^8 \xi \, d\xi}{[1+a_{1m} sn^2 \xi + a_{2m} sn^4 \xi]^2}$$

or

$$\left(\frac{\Delta p}{q}\right)_{(4)}^{(m)} = -\frac{4x^3}{\beta^3 V_o} C_{(4)}^{(m)} [1+a_{1m}+a_{2m}][k^2(a_{1m}+2k^2)+(2a_{2m}+k^2 a_{1m})(1-\beta^2 t^2)] I_{(4)}^{(m)}(k) \tag{118}$$

where $I_{(4)}^{(m)}(k)$ is evaluated in Appendix E, Part II, then the pressure distribution associated with the elliptic hypercone is

$$\frac{1}{x^3}\left(\frac{\Delta p}{q}\right)_{4a} = A \sum_{m=1}^{3} \frac{1}{x^3}\left(\frac{\Delta p}{q}\right)_{(4)}^{(m)} \qquad \text{For the case considered the } m_o \text{ is given in (117)} \tag{119}$$

The above pressure distribution is shown in Fib. 13c. Other superpositions will be demonstrated in a later section, hence any discussion of this topic will be limited to obtaining the elliptic hypercones or the parallel isobars normal to the flow direction.

(b) Inverse Problem (n = 4)

In this case a superposition may be effected whereby the "isobaric" distribution will be such that the constant pressure lines will be parallel and normal to the flow direction. This superposition may be characterized as

$$\frac{1}{x^3}\left(\frac{\Delta p}{q}\right)_{4b} = \frac{K}{x^3} = B \sum_{m=1}^{3} \frac{1}{x^3}\left(\frac{\Delta p}{q}\right)_{(4)}^{(m)} \tag{120}$$

for the case under consideration, namely the wing with apex angle $\omega = 31^\circ$ and $\beta = 1$. The constants $C_{(n)}^{(m)}$ may be determined with the help of the pressure distributions given in Figs. 13, 13a, and 13b, and in this case were found to be

$$C_{(4)}^{(1)} = -0.805\,\beta^3 V_0 \quad C_{(4)}^{(2)} = 0.207\,\beta^3 V_0 \quad C_{(4)}^{(3)} = -.125\,\beta^3 V_0 \tag{121}$$

where

$$\mathbb{K} = B = \frac{\tau}{2C_0^3 m_0}$$

This superposition requires that the thickness distribution to support such a pressure be

$$x^{-4} Z_{4b} = \frac{\tau}{2C_0^3 m_0} = \sum_{m=1}^{3} \frac{Z_4^{(m)}}{x^4} \tag{122}$$

where the form for $Z_4^{(m)}$ is given in equation (115) and the appropriate coefficients are given in (119). This thickness distribution is plotted in Fig. 14.

It should be pointed out here that it is possible to superpose the three solutions for n = 4 in such a way that the zero pressure case on the wing planform in the x-y plane may be realized. Hence, the prescription of the pressure in this case may lead to multiple solutions for the thickness distribution just as for the n = 2 case.

In general it may be readily shown that for any even degree of homogeneity there exists a sufficient number, $(\frac{n}{2}+1)$, of solutions to Lamé's equation of the type $\mathbb{K}$ which when superposed in the proper way will result in a potential function φ which will be independent of x in the x-y plane on the wing. Hence, the derivative $\frac{\partial \varphi}{\partial x} \sim \frac{\Delta p}{q}$ in the x-y plane will be zero on the wing.

(c) Drag for Surfaces Generated for n = 4.

The drag coefficient given here is calculated for the arbitrary surface and pressure distribution which may superposed in the proper amounts for the shape under consideration. The values for the a_{jm}'s are given in Fig. 12, and the total drag is:

$$\begin{aligned} C_{D_{(4)}}^{(m)} = \pi C_0^6 \left[C_{(4)}^{(m)}\right]^2 I_{(4)}^{(m)}(k) \Big\{ & 2k^2(1+a_{1m}+a_{2m})(a_{2m}+a_{1m}k^2+k^4) \\ & - \frac{\beta^2 m_0^2}{2}(2a_{2m}+k^2 a_{1m})\left[k^2(1+a_{1m}+a_{2m})+2(a_{2m}+a_{1m}k^2+k^4)\right] \\ & + \frac{3}{8}\beta^4 m_0^4\left[2a_{2m}(a_{2m}+a_{1m}k^2+k^4)+(2a_{2m}+k^2 a_{1m})^2\right] \\ & - \frac{5}{16}\beta^6 m_0^6\left[a_{2m}(2a_{2m}+k^2 a_{1m})\right]\Big\} + C_{D_\sigma} \end{aligned} \tag{123}$$

5. Surfaces for $n > 4$ may be obtained in the same way as outlined in the preceding sections. Although no specific shape has been computed, the values for the Lamé coefficients for the $n = 5$ case have been evaluated and are presented in Fig. 15.

IV. SUPERPOSITIONS

In general the solutions in the preceding sections are classed as wings with fixed cross-sections or with fixed pressures.

It is of interest to superpose those solutions (Direct Problem) where the wing geometry is prescribed, in particular, for the solutions which have elliptic cross-sections. This class of solutions may be expressed in general as

$$Z_n = \frac{\tau}{2m_o} \frac{x^n}{C_o^{n-1}} \sqrt{m_o^2 - t^2} \tag{124}$$

By combining the solutions $n = 1, 2, 3, 4$ in various proportions, it is possible to obtain a large number of wings of different airfoil sections. In the superpositions that are demonstrated in Figure 16, the root chord airfoil section is chosen as the parameter governing the wing thickness distribution. Hence, it may be noted that

$$\underset{y\to 0}{Z_n} = \frac{\tau\, x^n}{2C_o^{n-1}} = \frac{\tau}{2} x^n \qquad C_o = 1 \tag{125}$$

At $x = 1$ all Z_n have the same elliptic cross-section. Hence, the direct subtraction of any two thickness distributions gives a straight closed trailing edge wing. Any disturbances which occur downstream of $x = 1$ are neglected since the Kutta condition is automatically satisfied in supersonic flow at a supersonic trailing edge and no disturbances introduced beyond the trailing edge may affect conditions on the wing.

The method of superposition is illustrated in Figure 16 and the resultant wing sections are as indicated. The corresponding pressure distributions are shown in Figure 16a. For all cases considered in this section, the leading edge slope was $m_o = 0.5$, $\beta = 1$, $k = .866$, $\omega = 26.56°$.

The foregoing example was inserted to demonstrate the case where the geometry is prescribed (Direct Problem). The same technique may be employed where the pressure is prescribed; however, the closed trailing edges that may be obtained will no longer be straight. These solutions, however, are admissible as long as the edge remains supersonic (slope at trailing edge $> \frac{1}{\beta}$).

TABLE I.

NOMENCLATURE

a - local speed of sound, subscript o free stream conditions

b - span at trailing edge

c_o - root chord of triangular wings

$cn\,\xi$ - Jacobi functions described in Appendix E

C_{D_o} - drag coefficient due to normal force $\frac{\text{drag}}{qS}$

C_{D_σ} - drag coefficient due to leading edge pressure (See Appendix D)

C_D - $C_{D_o} + C_{D_\sigma}$

c_l - local lift coefficient

C_L - lift coefficient $\frac{\text{lift}}{qS}$

$C_{L'}$ - rolling moment coefficient $\frac{\text{rolling moment}}{qSb}$

$C_{L'_p}$ - damping in roll derivative $\partial C_{L'} / \partial\left(\frac{pb}{2V_o}\right)$

C_M - pitching moment coefficient $\frac{\text{pitching moment}}{qSc_o}$

C_{M_q} - damping in pitch derivative $\partial C_M / \partial\left(\frac{qc_o}{2V_o}\right)$

$-C_{D_\sigma}$ - drag due to leading edge suction (negative drag or thrust)

$dn\,\xi$ - Jacobi function described in Appendix E

$E, E(k)$ - complete normalized elliptic integral of the second kind of modulus k

$E(\varphi, k)$ - incomplete normalized elliptic integral of the second kind of modulus k

$F(\varphi, k)$ - incomplete normalized elliptic integral of the first kind of modulus k

k - $\sqrt{1-\beta^2 m_o^2}$ modulus of elliptic integrals

k' - $\sqrt{1-k^2}$ conjugate modulus

$K, K(k)$ - complete normalized elliptic integral of the first kind of modulus k

$\mathbb{K}$, $\mathbb{L}$, $\mathbb{M}$, $\mathbb{N}$ - Lamé Functions of the first kind described in equation (18) of the text

m_o - slope of the leading edge of triangular wings

(m) - superscript

M_o - free stream Mach number V_o/a_o

n - degree of homogeneity

p - local static pressure on wing, also used as rolling velocity radians/sec

p_o - free stream pressure

$\mathbb{P}_K$ - $\mathbb{K}$

$\mathbb{P}_L$ - $\mathbb{L}[\mu^2-k^2]^{-\frac{1}{2}}$

$\mathbb{P}_M$ - $\mathbb{M}[\mu^2-1]^{-\frac{1}{2}}$

$\mathbb{P}_N$ - $\mathbb{N}[\mu^2-1]^{-\frac{1}{2}}$

Q - pitching velocity, radians/sec

q - dynamic pressure $\frac{1}{2}\rho V_o^2$

r - hyperbolic distance $\sqrt{\frac{x^2}{\beta^2}-y^2-z^2}$

S - wing area

S_n^m - Lamé function of first kind

$\mathcal{S}_n^m$ - Lamé function of second kind

$\operatorname{sn}\xi$ - Jacobi function described in detail in Appendix E.

t - Ray through the origin (z = 0) y/x

u' - perturbation velocity in the stream direction (positive x direction)

V_o - free stream velocity directed along positive x-axis

v' - perturbation velocity in cross-stream direction

w' - perturbation velocity in the vertical direction

x, y, z - Cartesian coordinates

α - angle of attack, also used as argument of the Jacobi functions

β – $\sqrt{M^2-1}$

γ - argument of Jacobi functions

$\frac{\Delta p}{q}$ - local pressure coefficient

ρ_o - free stream density

μ, ν - Hyperboloido-Conal coordinates, also used as superscripts in Appendix A-I

ν_o - $k^2[1-\beta^2 t^2]^{-1/2}$ $\quad \nu_{\mu\to 1} = \nu_o$

τ - thickness ratio of wings and shapes at the trailing edge, $x = c_o$.

Φ - velocity potential

φ - perturbation velocity potential

TABLE II

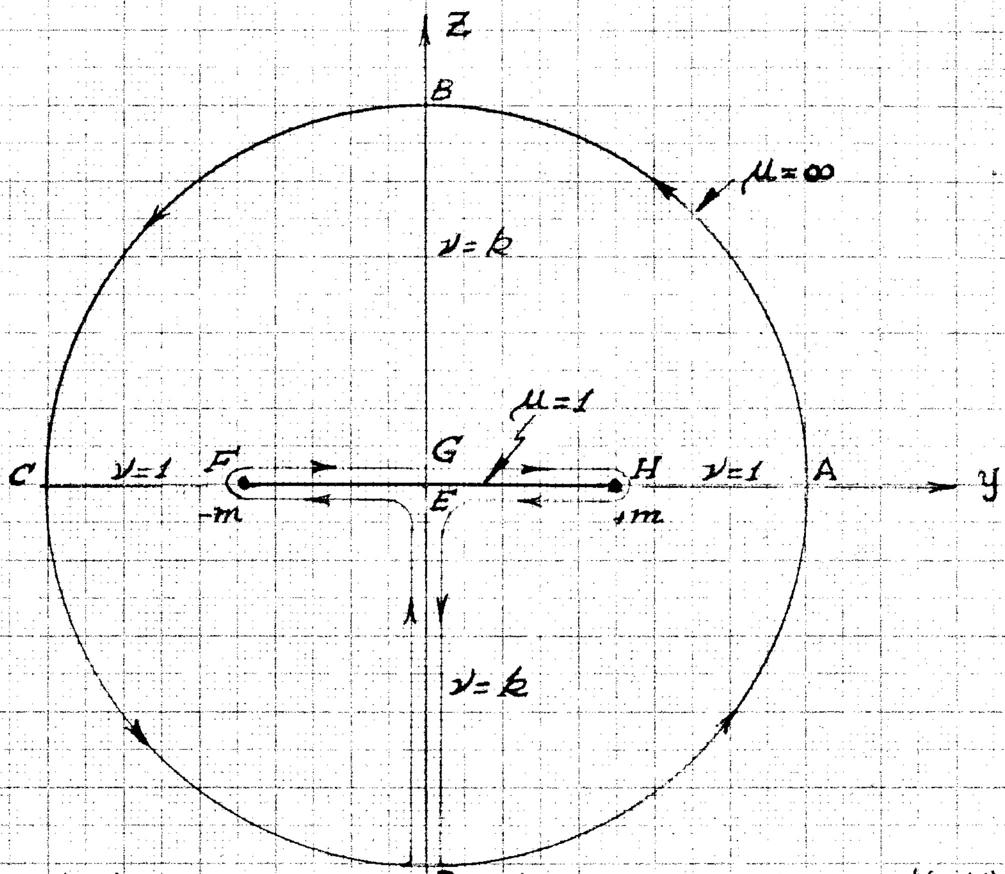
SOME EXAMPLES OF LAMÉ FUNCTIONS OF VARIOUS SPECIES

n	$^{+}\mathbb{K}$	$^{+}\mathbb{L}$	$^{+}\mathbb{M}$	$^{+}\mathbb{N}$
0	1			
1	μ	$\sqrt{\mu^2-k^2}$	$\sqrt{\mu^2-1}$	
2	μ^2+a_{jm} *	$\mu\sqrt{\mu^2-k^2}$	$\mu\sqrt{\mu^2-1}$	$\sqrt{(\mu^2-k^2)(\mu^2-1)}$
3	$\mu^3+a_{jm}\mu$ *	$[\mu^2+a_{jm}]\sqrt{\mu^2-k^2}$ *	$[\mu^2+a_{jm}]\sqrt{\mu^2-1}$ *	$\mu\sqrt{(\mu^2-k^2)(\mu^2-1)}$

* TWO SOLUTIONS

+ NOTE:
$$\mathbb{K} = \mathbb{P}_K$$
$$\mathbb{L} = [\mu^2-k^2]^{1/2}\,\mathbb{P}_L$$
$$\mathbb{M} = [\mu^2-1]^{1/2}\,\mathbb{P}_M$$
$$\mathbb{N} = [\mu^2-1]^{1/2}\,\mathbb{P}_N$$

FIGURES



$$\mu = k\,\mathrm{sn}(\alpha + iK', k) = \mathrm{ns}(\alpha, k)$$
$$[\mu^2-1]^{1/2} = i\,\mathrm{dn}(\alpha+iK', k) = \mathrm{cs}(\alpha, k)$$
$$[\mu^2-k]^{1/2} = i\,k\,\mathrm{cn}(\alpha+iK', k) = \mathrm{ds}(\alpha, k)$$

$$\nu = k\,\mathrm{sn}(K+i\gamma, k) = k\,\mathrm{nd}(\gamma, k')$$
$$[1-\nu^2]^{1/2} = i\,\mathrm{dn}(K+i\gamma, k) = k'\mathrm{cd}(\gamma, k')$$
$$[\nu^2-k^2]^{1/2} = ik\,\mathrm{cn}(K+i\gamma, k) = kk'\mathrm{sd}(\gamma, k')$$

$$\mathrm{cd}(\alpha+i\gamma) = \frac{\nu\sqrt{(\mu^2-1)(\mu^2-k^2)}}{k(\mu^2-\nu^2)} + \frac{i\mu\sqrt{(1-\nu^2)(\nu^2-k^2)}}{k(\mu^2-\nu^2)}$$

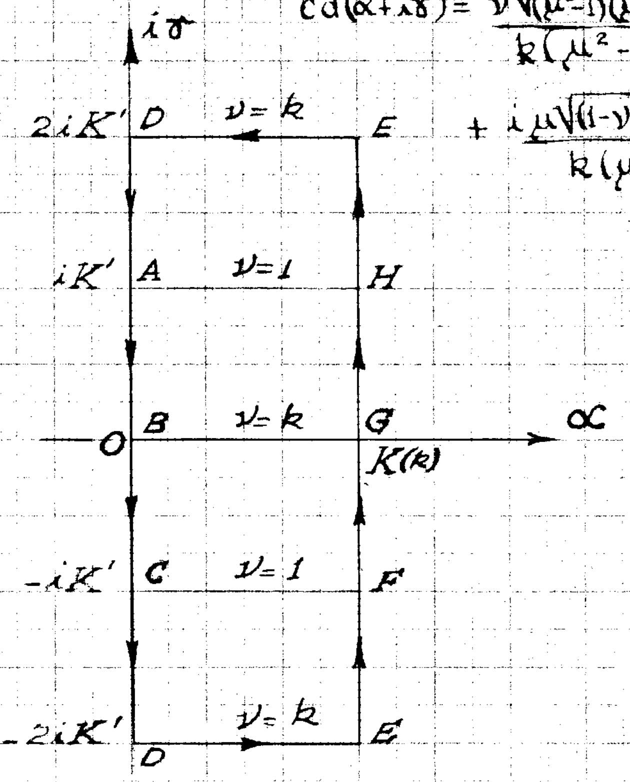


FIGURE A - SYMMETRY CONSIDERATIONS

$$\frac{x^2}{\beta^2\mu^2} - \frac{y^2}{\mu^2-k^2} - \frac{z^2}{\mu^2-1} = 0$$

$$\frac{x^2}{\beta^2\nu^2} - \frac{y^2}{\nu^2-k^2} + \frac{z^2}{1-\nu^2} = 0$$

$$\frac{x^2}{\beta^2} - y^2 - z^2 = r^2$$

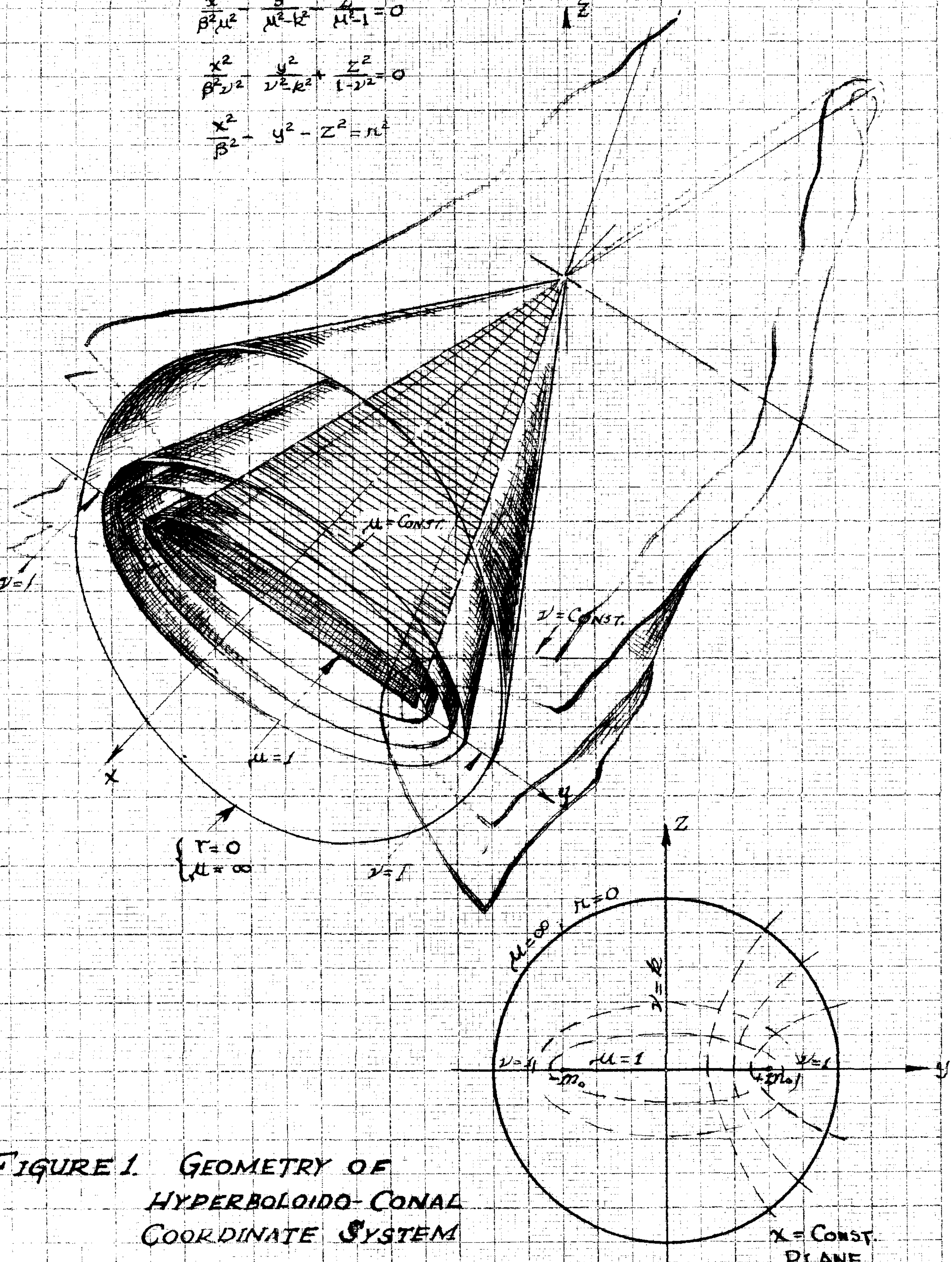


FIGURE 1. GEOMETRY OF HYPERBOLOIDO-CONAL COORDINATE SYSTEM

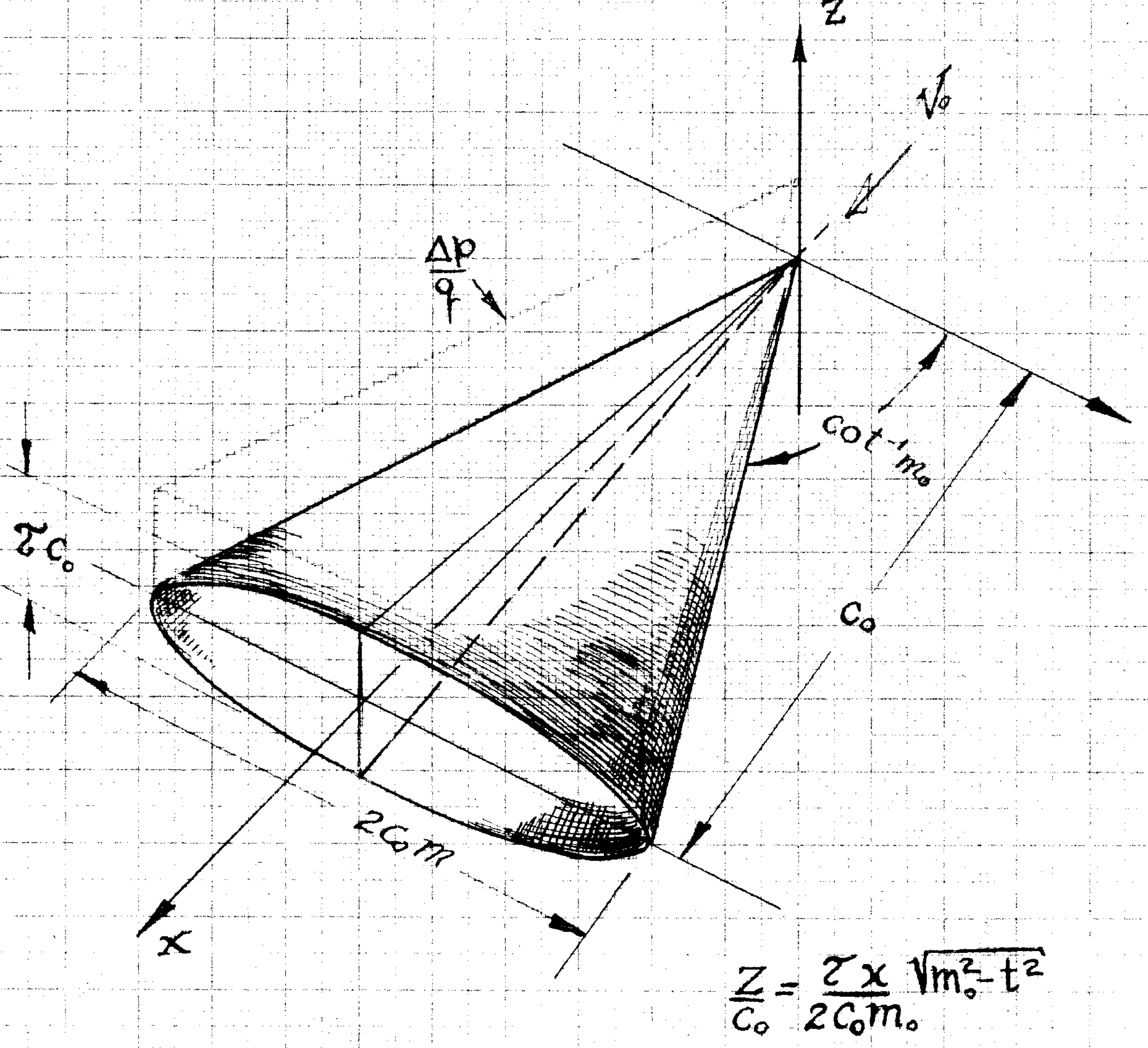


FIGURE 2.- ELLIPTIC CONE (n=1) MAINTAINS A UNIFORM PRESSURE DISTRIBUTION

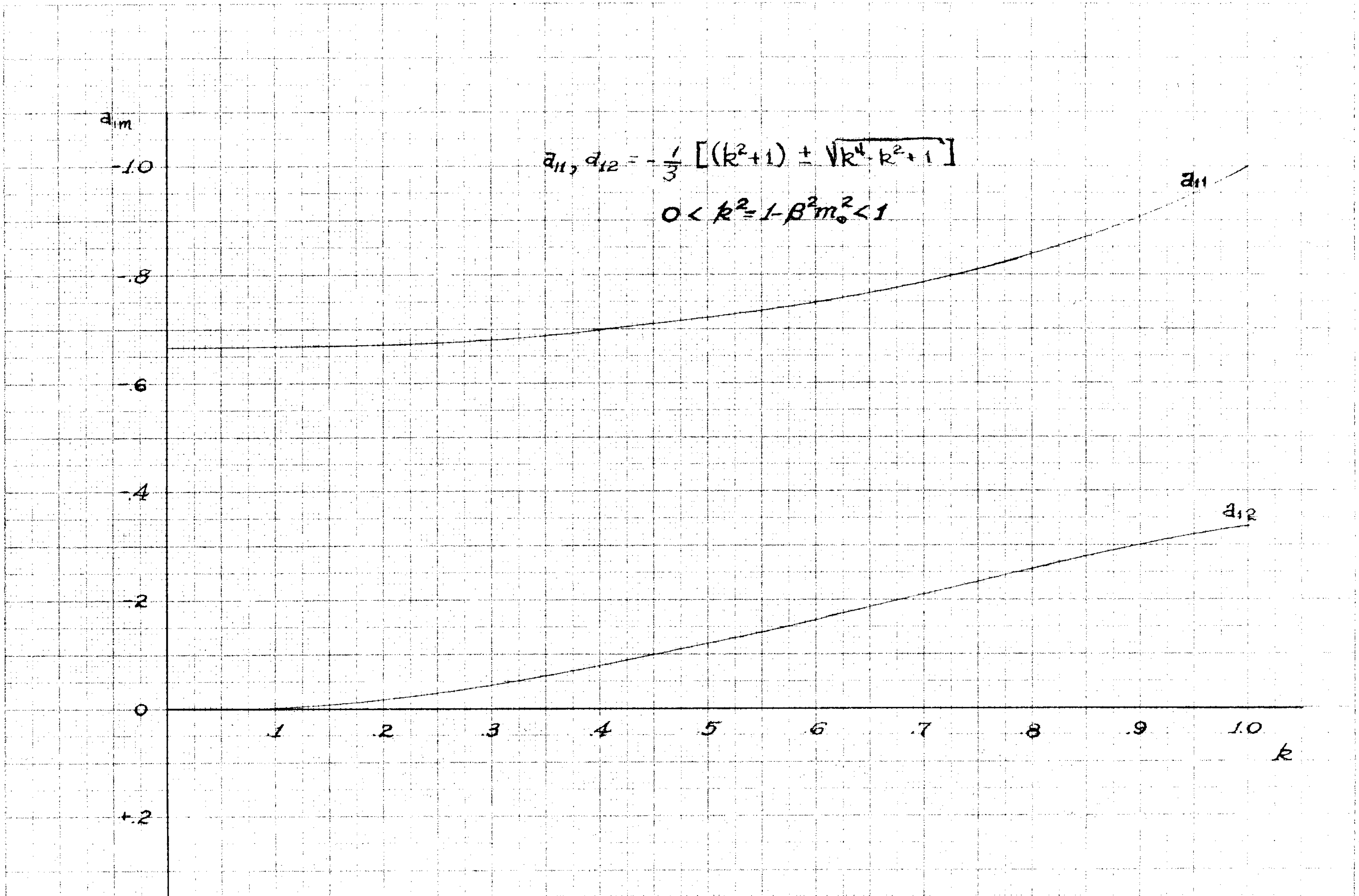


FIGURE 3. VALUES FOR THE COEF a_{jm} FOR LAMÉ FUNCTION OF THE FIRST SPECIES ($n=2$) FOR $0 < k \leq 1$

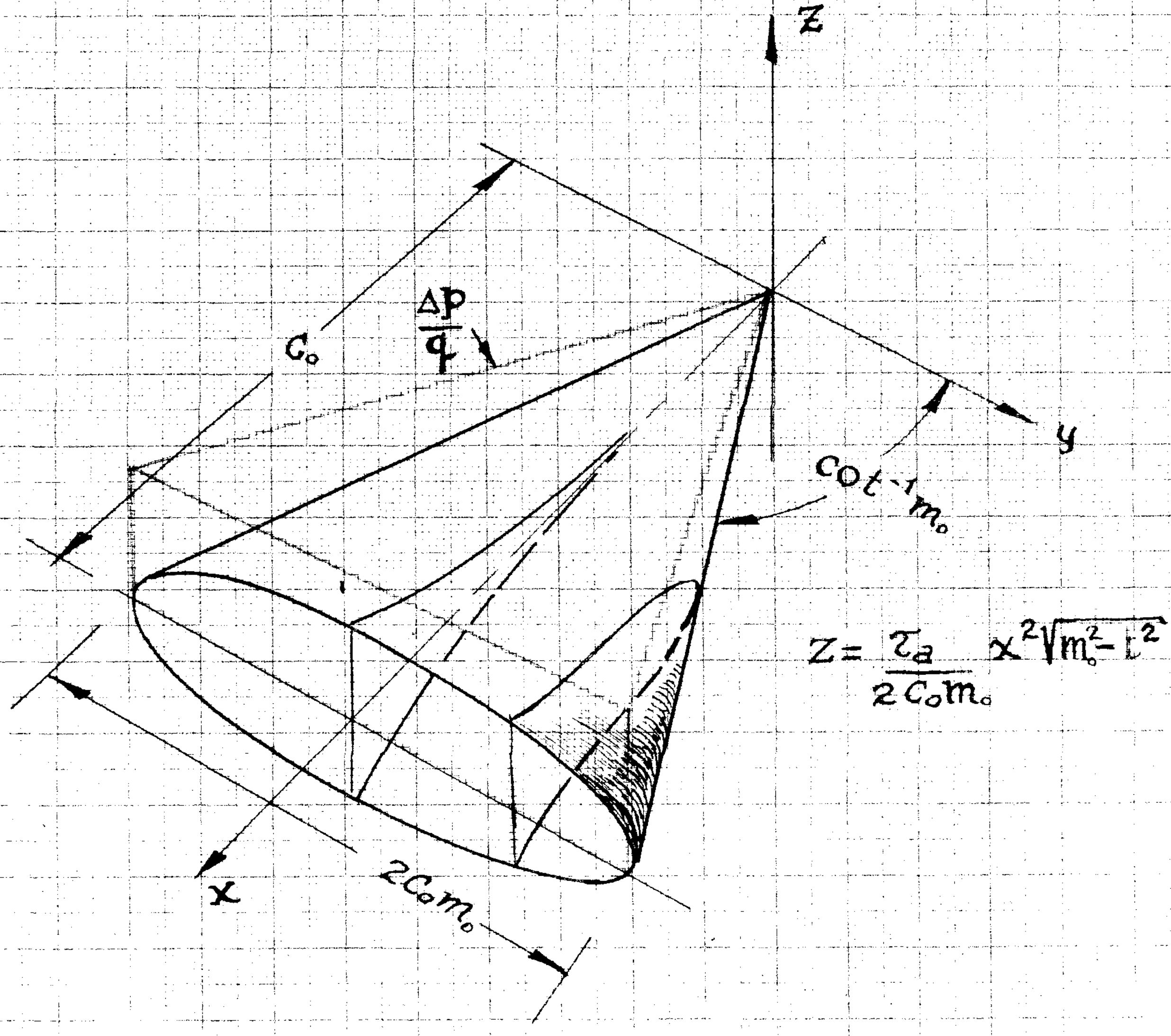


FIGURE 4 - ELLIPTIC HYPERCONE (n=2) WHICH MAINTAINS A LINEARLY INCREASING PRESSURE DISTRIBUTION

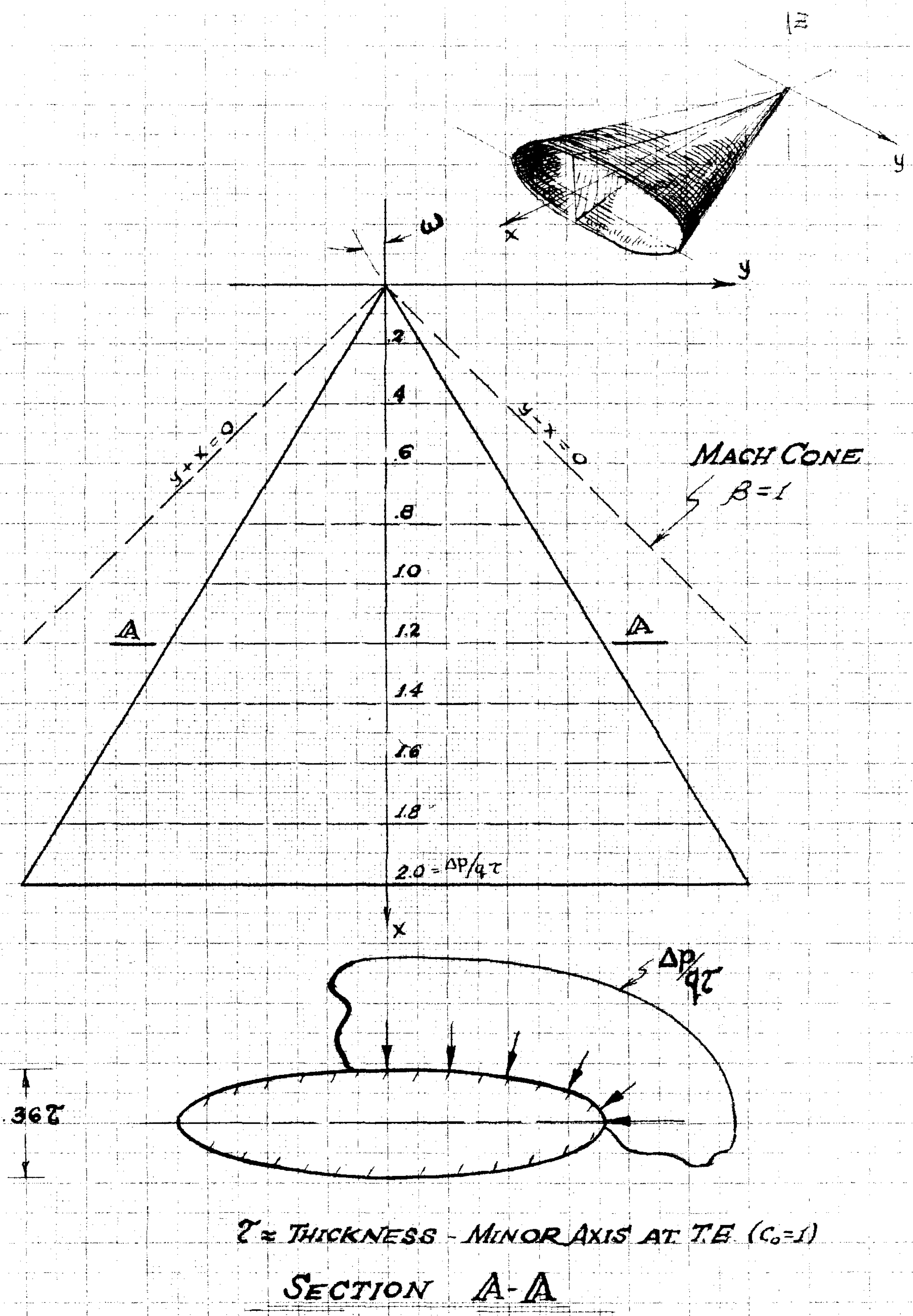


FIGURE 5.- ELLIPTIC HYPERCONE SURFACE ($n=2$) SHOWING THE ISOBARS FOR WING WITH APEX ANGLE $\omega = 31°$ AND $M = \sqrt{2}$ $C_0 = 1$

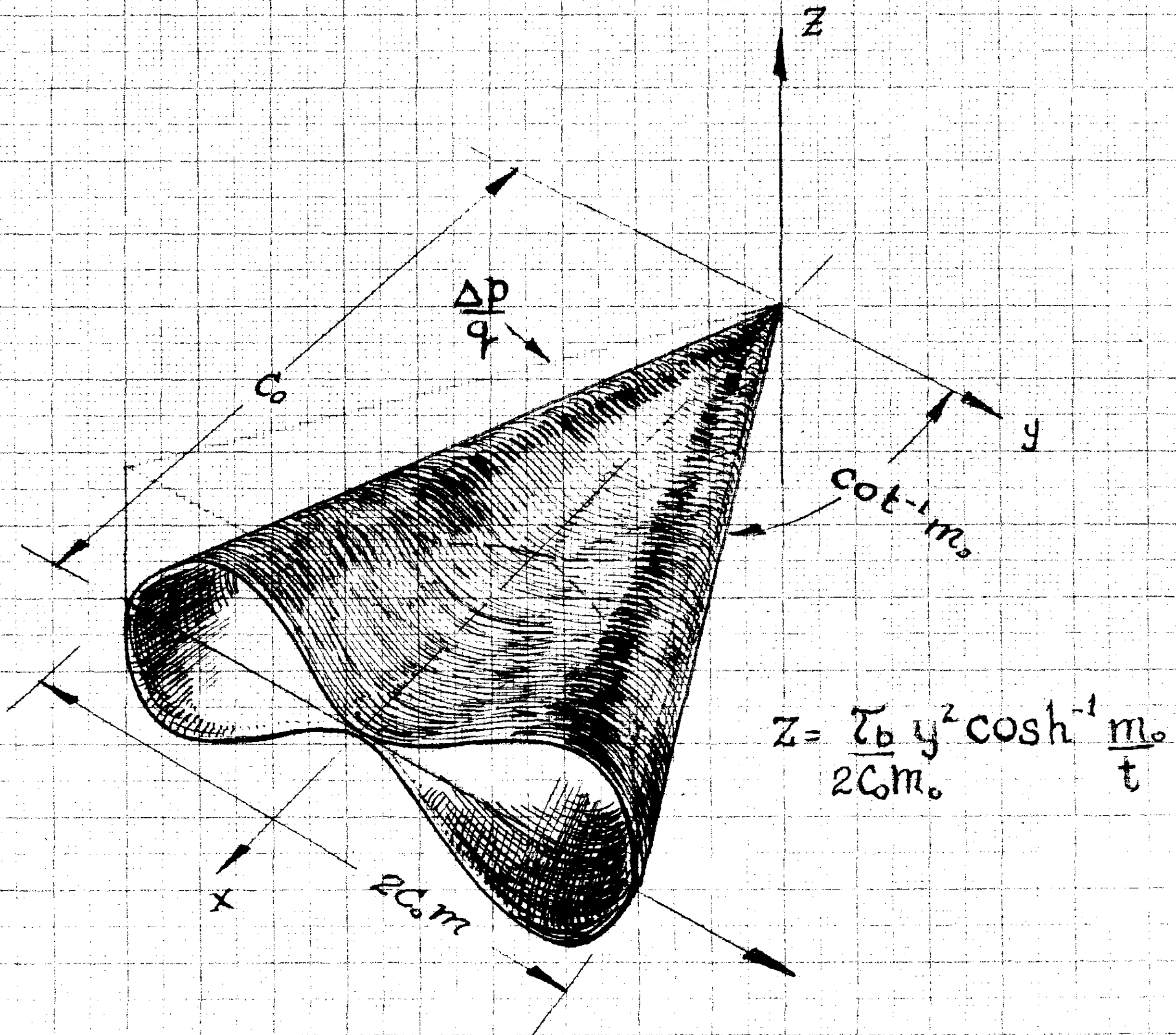


FIGURE 6.- SURFACE OBTAINED FOR $n = 2$ WHICH MAINTAINS A LINEARLY INCREASING PRESSURE DISTRIBUTION

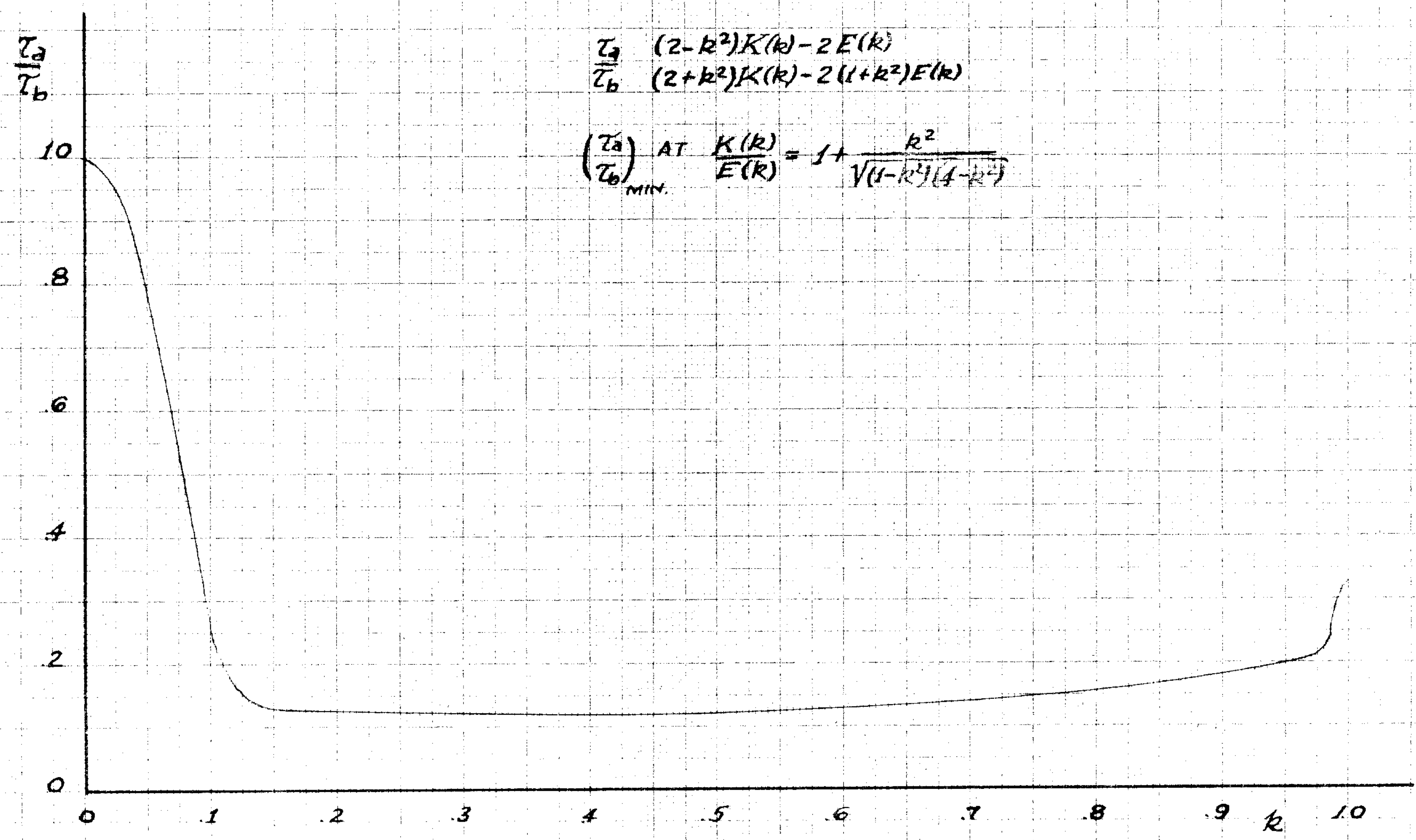


FIGURE 7.- RANGE OF τ_a/τ_b FOR $0 \le k \le 1$

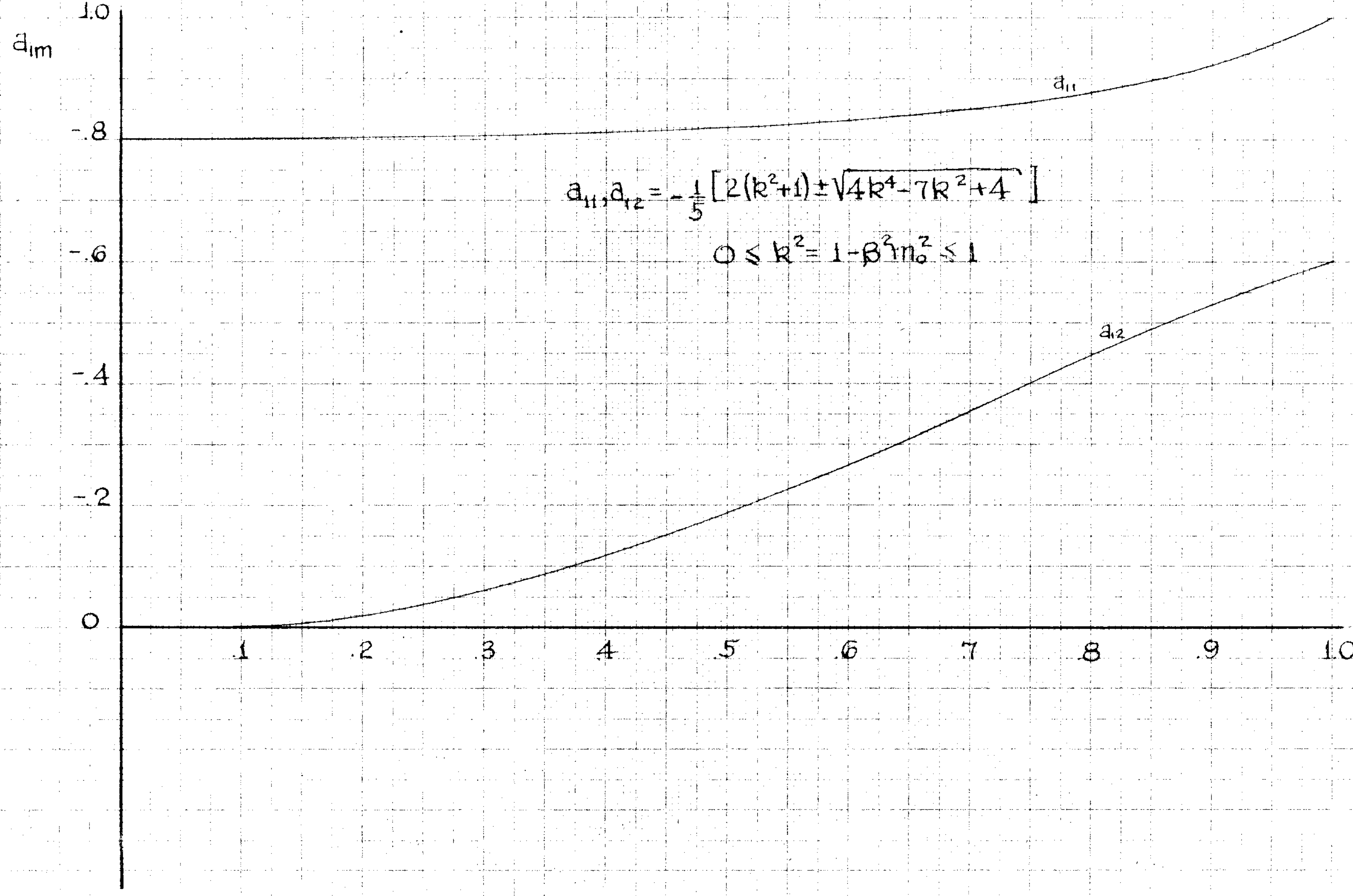


FIGURE 8 - VALUES OF THE COEF. a_{1m} FOR LAMÉ FUNCTION OF THE FIRST SPECIES ($n=3$) FOR VALUES OF $0 \leq k \leq 1$

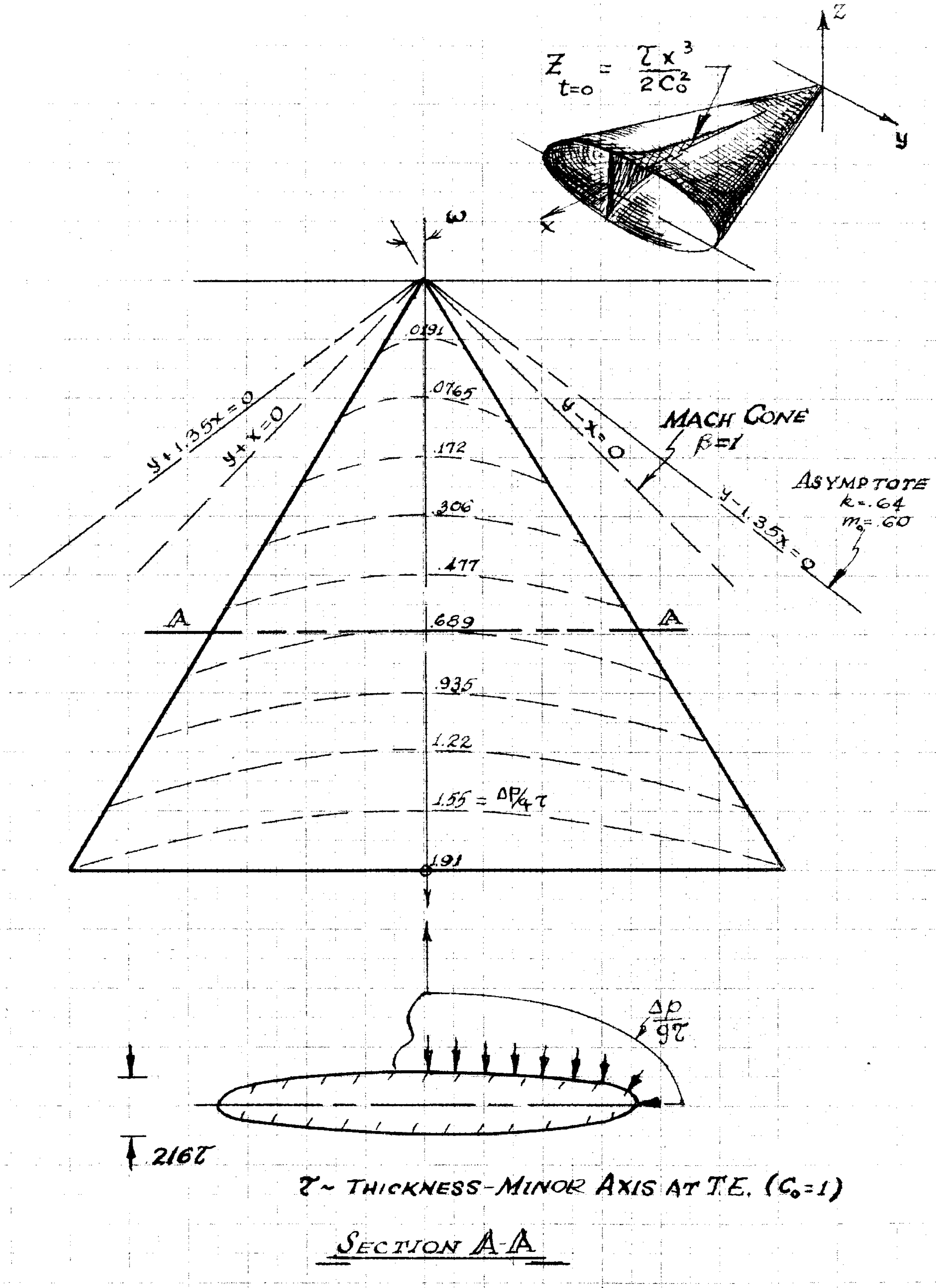


FIGURE 9.- ELLIPTIC HYPERCONE SURFACE ($n=3$) SHOWING THE ISOBARS FOR APEX ANGLE $\omega = 31°$ AT A MACH No. $\sqrt{2}$ $C_0 = 1$

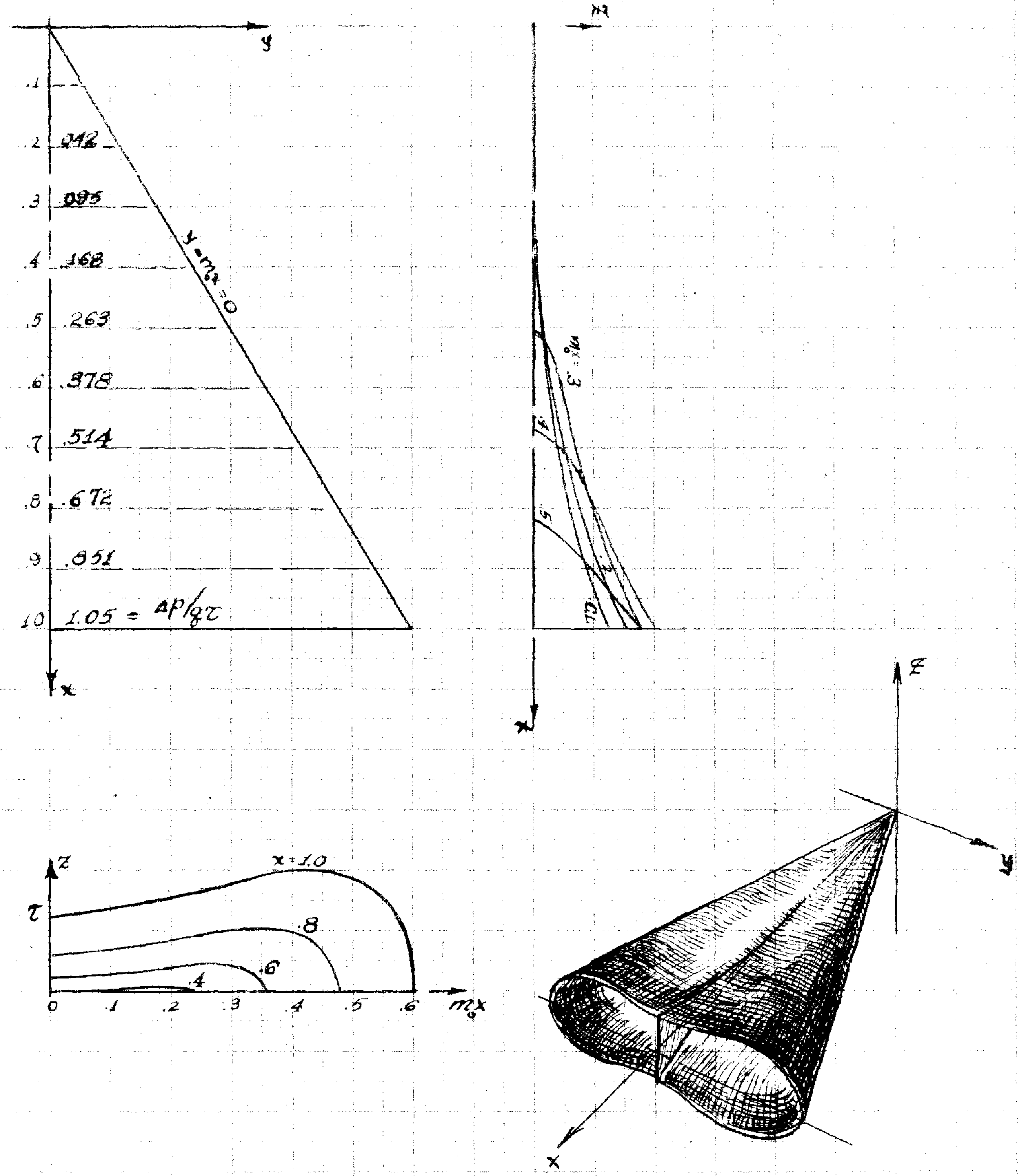


FIGURE 10. - THICKNESS DISTRIBUTION FOR PRESSURE DISTRIBUTION WHICH INCREASES AS x^2 WITH ISOBARS NORMAL TO THE FLOW DIRECTION $n = 3$, $\omega = 31°$, $\beta = 1$ $C_0 = 1$

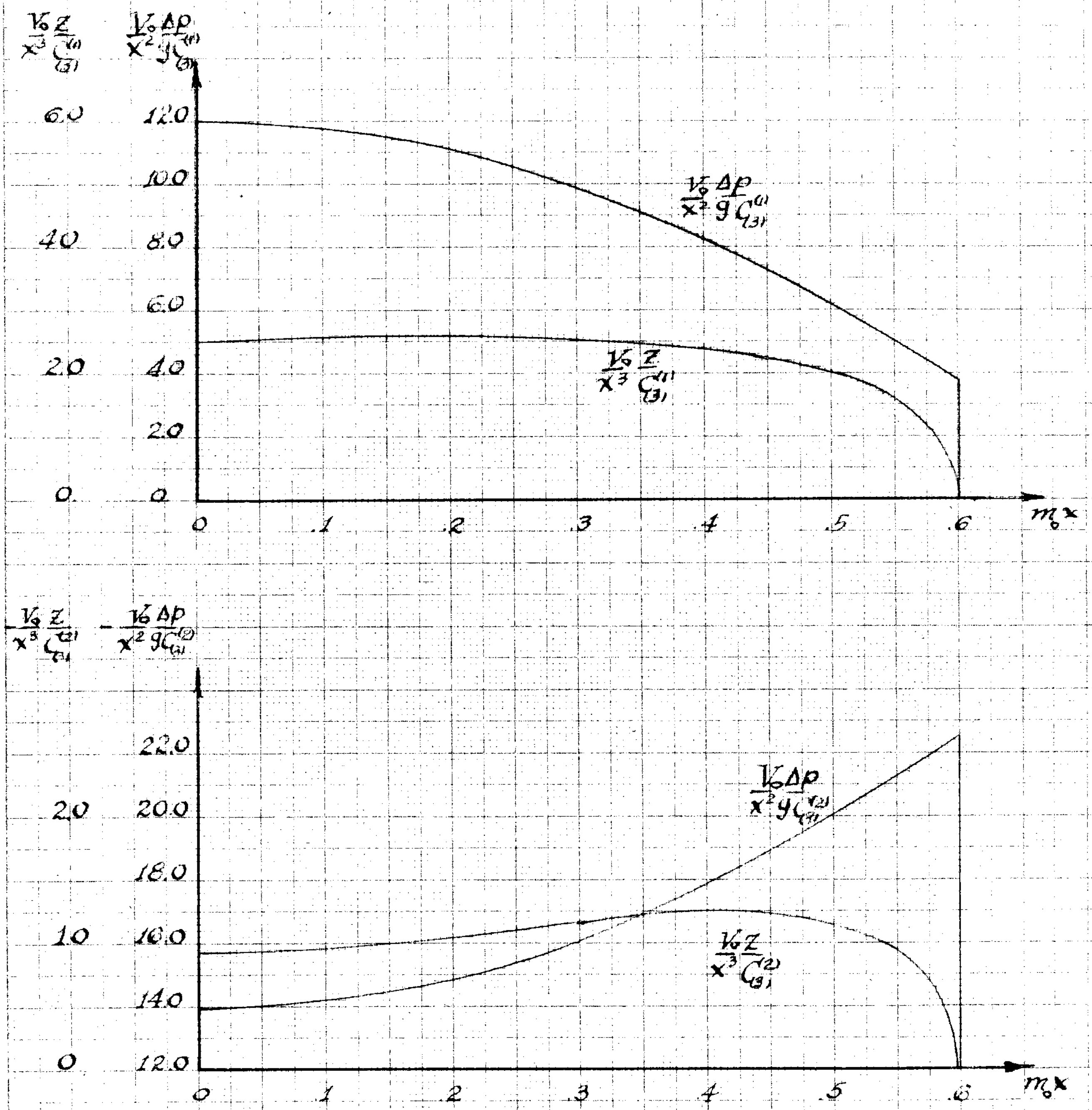


FIGURE 11 – THICKNESS AND PRESSURE DISTRIBUTIONS FOR THE TWO SOLUTIONS TO LAMÉ EQUATION ($n=3$) FOR A TRIANGULAR WING $\omega = 31°$ AT $M=\sqrt{2}$

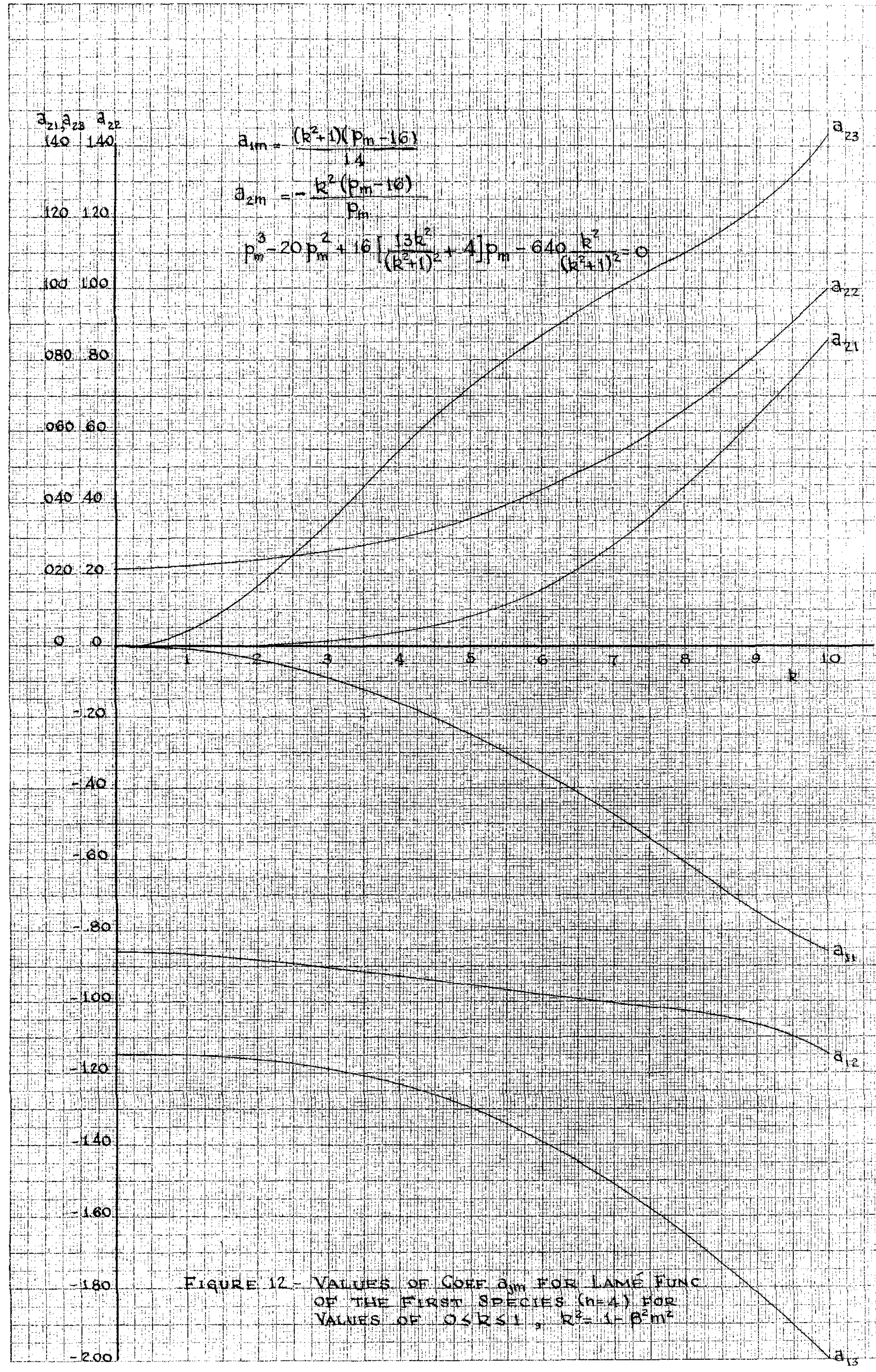


FIGURE 12 - VALUES OF COEF a_{jm} FOR LAMÉ FUNC OF THE FIRST SPECIES (n=4) FOR VALUES OF $0 \leq k \leq 1$, $k^2 = 1 - \beta^2 m^2$

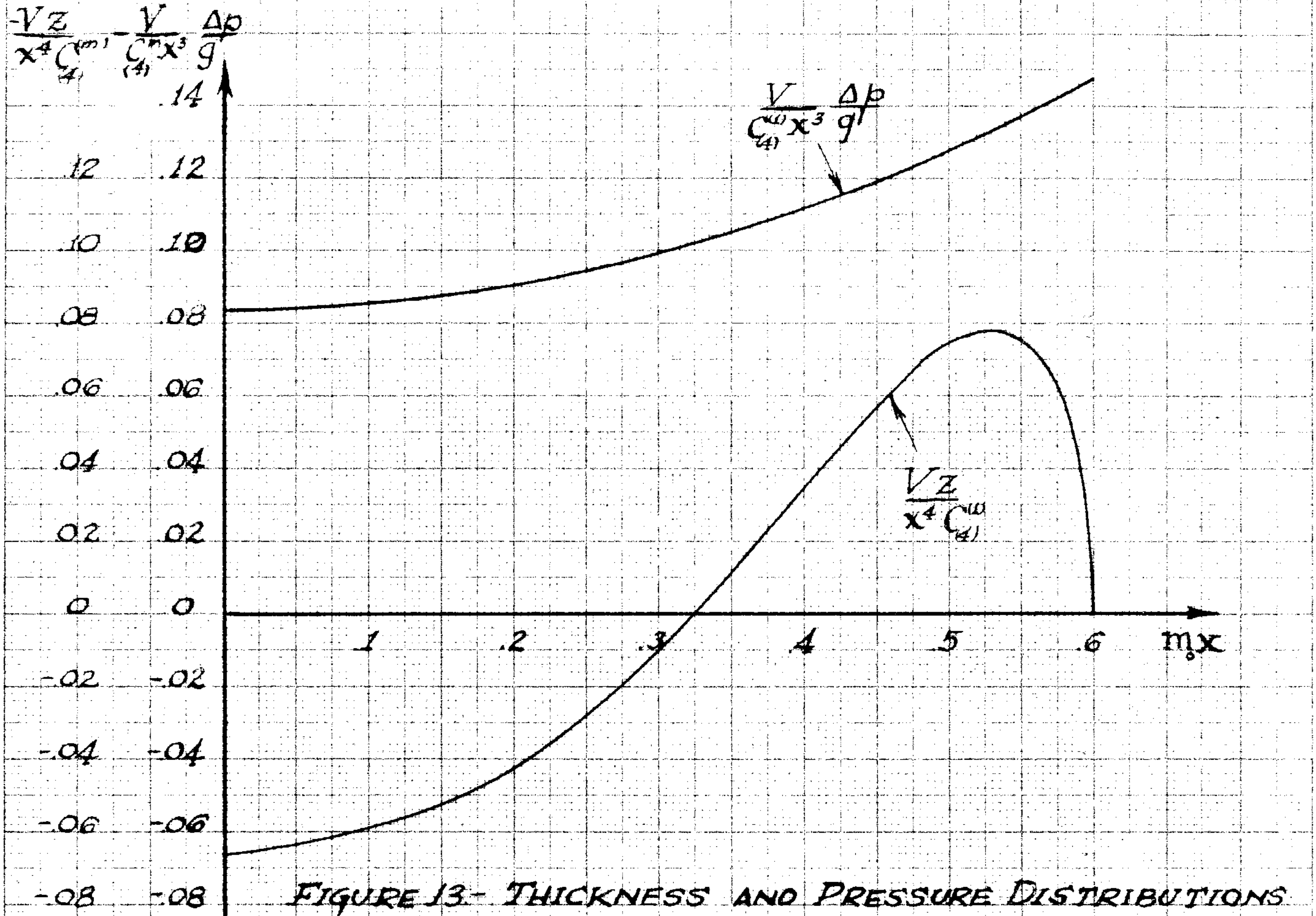


FIGURE 13- THICKNESS AND PRESSURE DISTRIBUTIONS FOR $n=4$ $(m)=1$ $k=.8$ $\beta=1$ $\omega=31°$

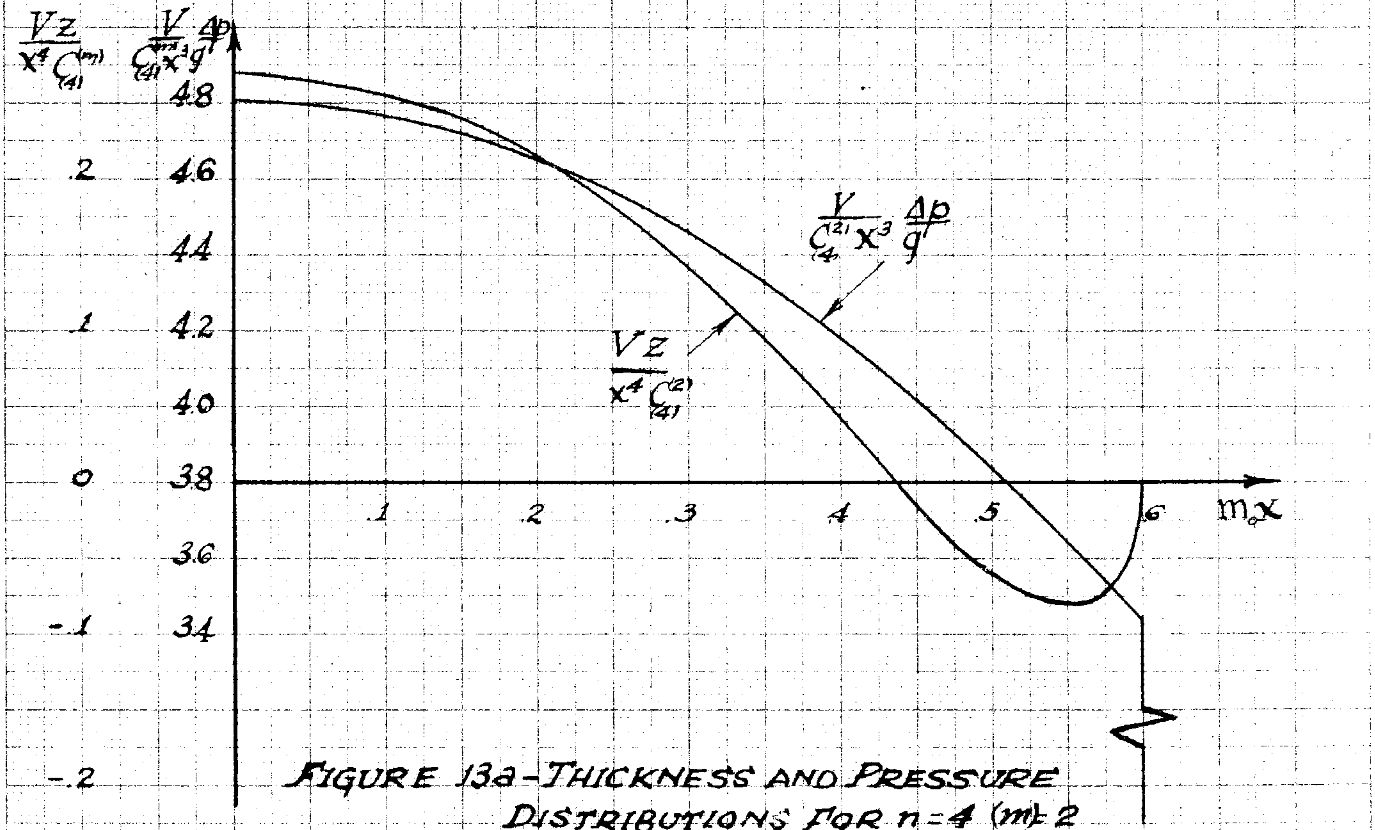


FIGURE 13a - THICKNESS AND PRESSURE DISTRIBUTIONS FOR $n=4$ $(m)=2$ $\beta=1$ $\omega=31°$

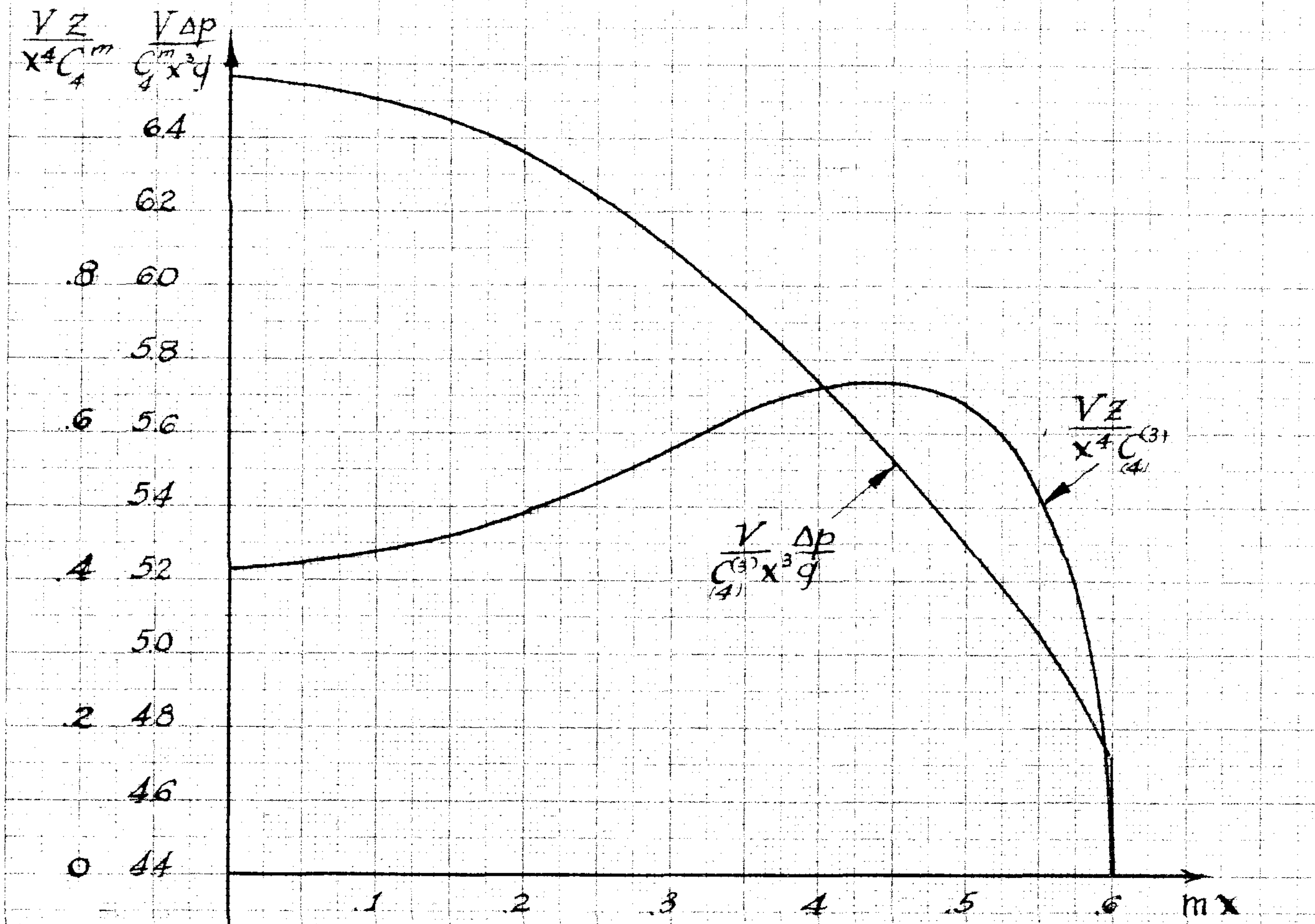


FIGURE 13b - THICKNESS AND PRESSURE DISTRIBUTION FOR $n = 4$, $(m) = 3$ $\beta = 1$ $k = .8$ $\omega = 31°$

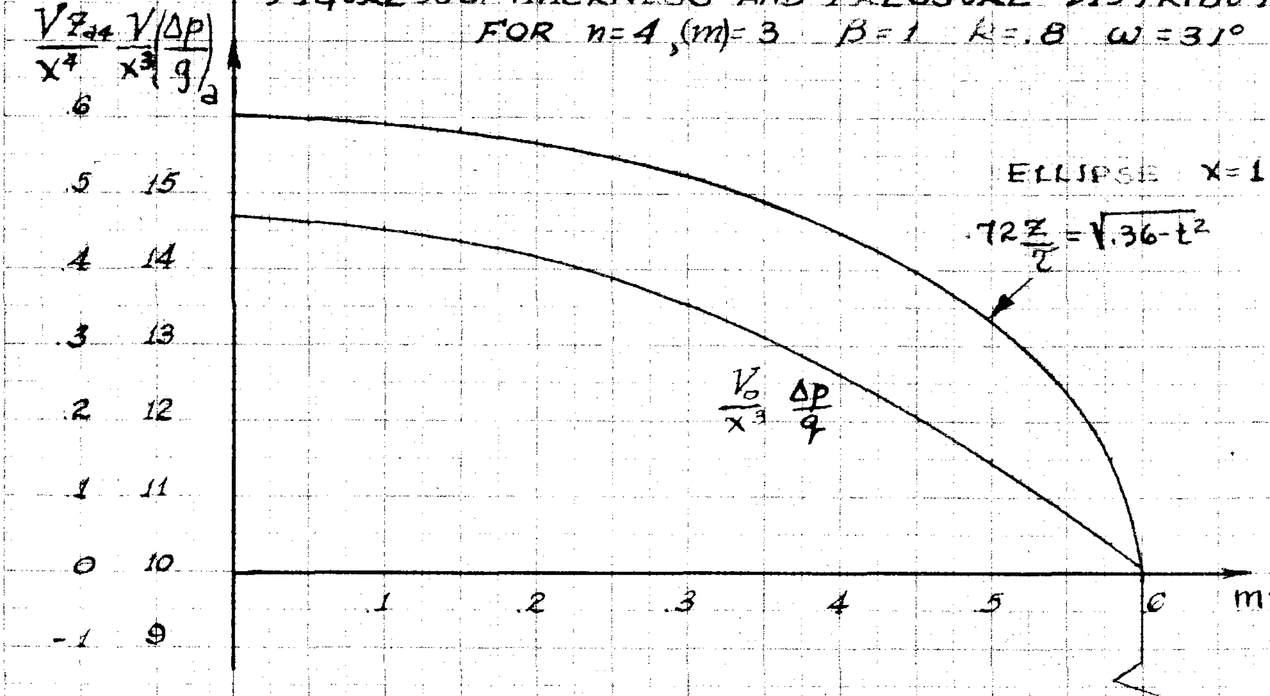


FIGURE 13c - THICKNESS AND PRESSURE DIST. DETERMINED BY SUPERPOSITION.
$C_o = 1$

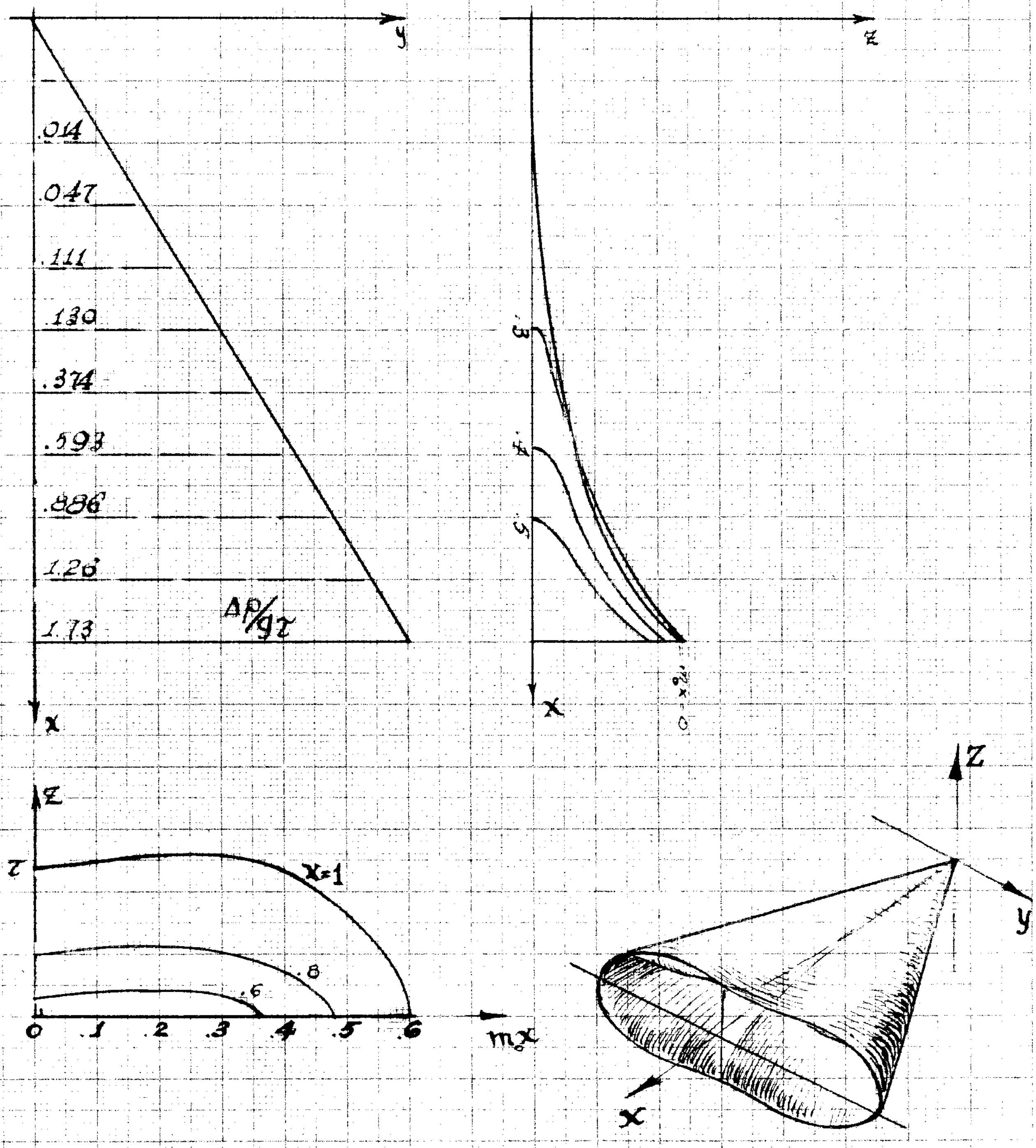


FIGURE 14 - THICKNES DISTRIBUTION FOR PRESSURE DISTRIBUTION WHICH INCREASES AS x^3 WITH ISOBARS NORMAL TO THE FLOW DIRECTION

$n=4$ $\omega = 31°$ $\beta = 1$ $C_o = 1$

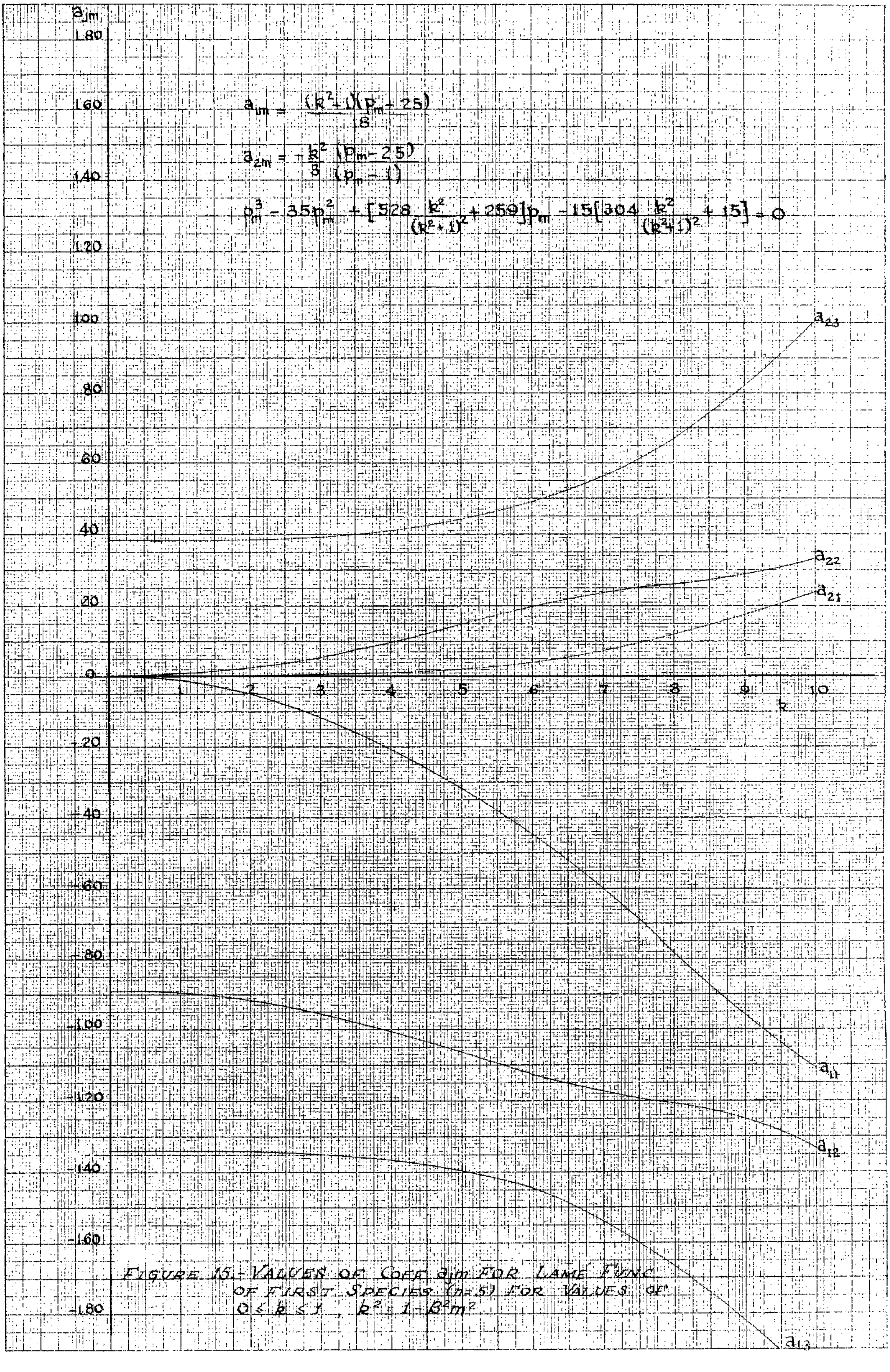


FIGURE 15.- VALUES OF COEF a_{jm} FOR LAME FUNC OF FIRST SPECIES ($n=5$) FOR VALUES OF $0 < k \leq 1$, $k^2 = 1 - B^2m^2$

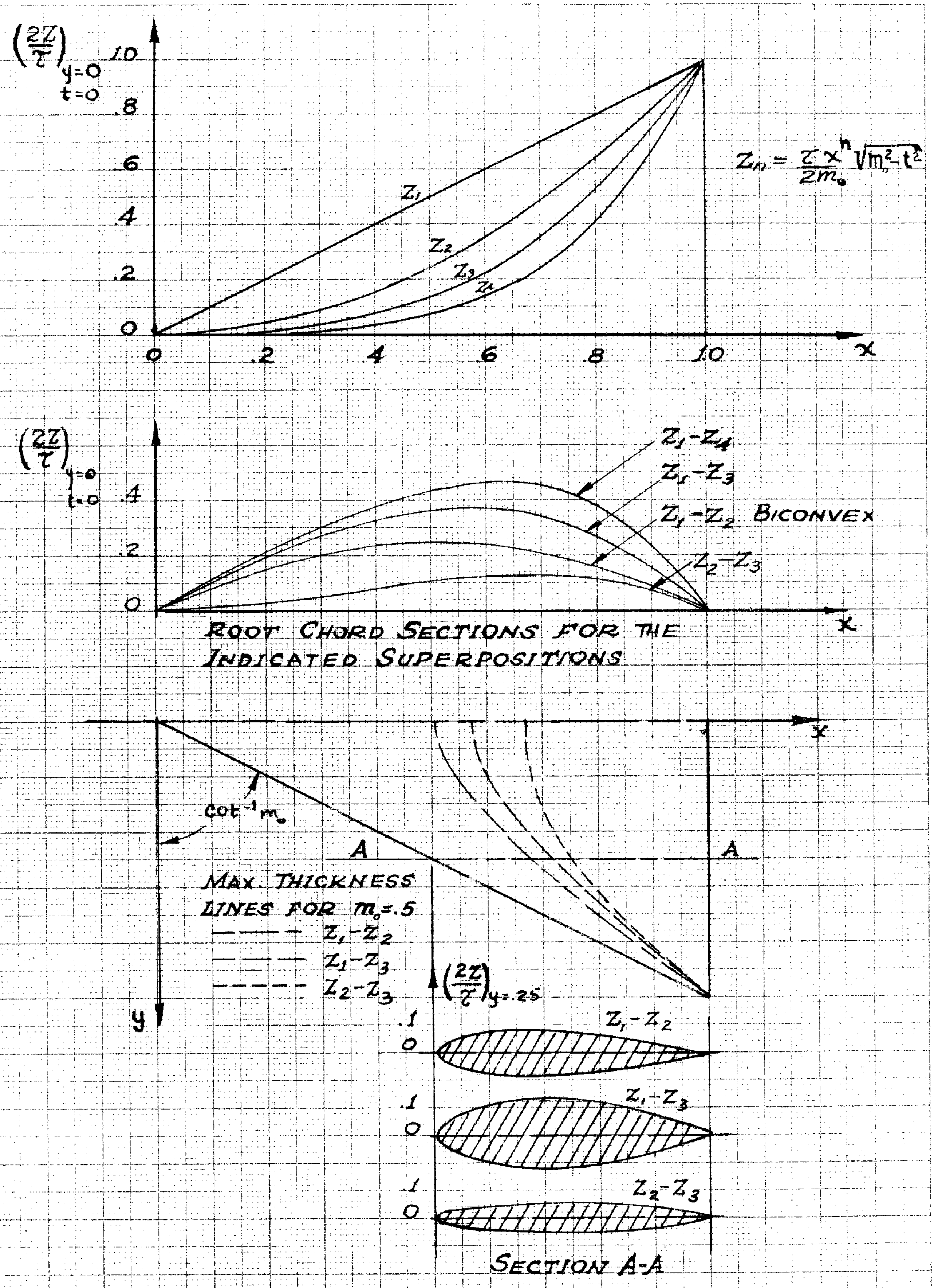
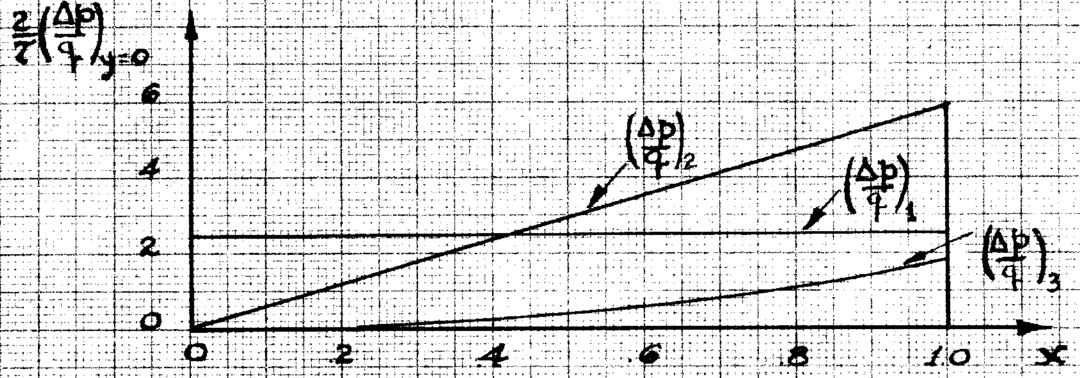
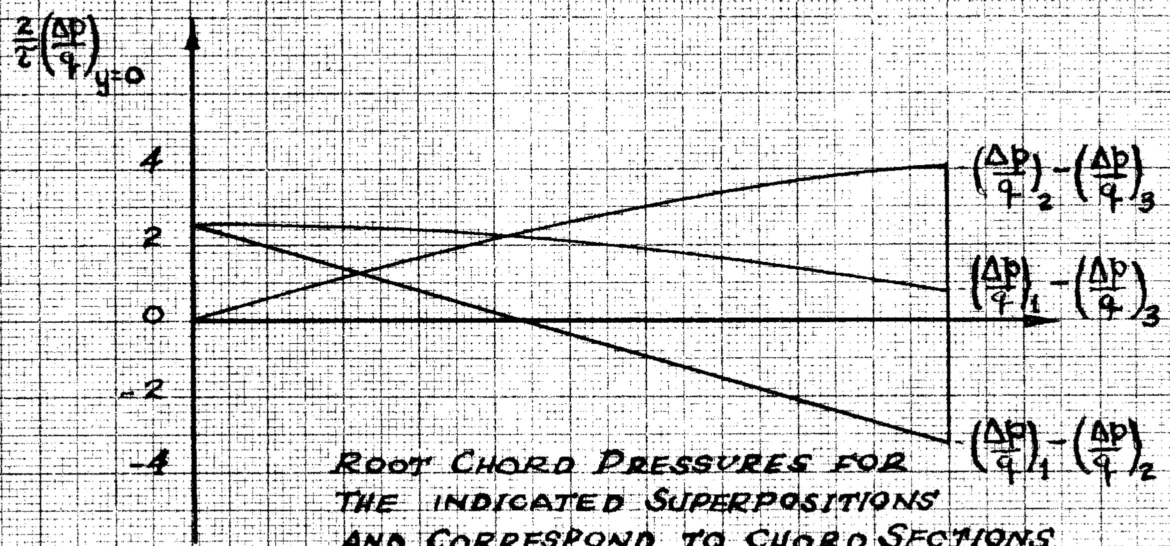
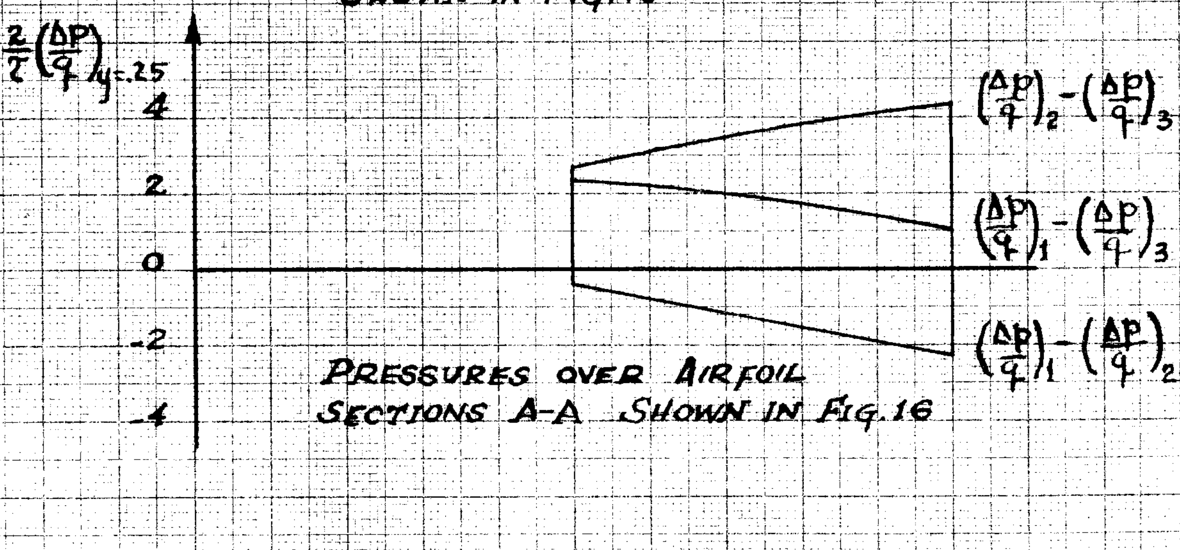


FIGURE 16.- METHOD FOR OBTAINING WINGS WITH STRAIGHT CLOSED TRAILING EDGES BY SUPERPOSITION





ROOT CHORD PRESSURES FOR THE INDICATED SUPERPOSITIONS AND CORRESPOND TO CHORD SECTIONS SHOWN IN FIG. 16



PRESSURES OVER AIRFOIL SECTIONS A-A SHOWN IN FIG. 16

FIGURE 16a.- PRESSURE DISTRIBUTIONS ASSOCIATED WITH THE WINGS OBTAINED BY SUPERPOSITION IN FIG. 16 $\beta = 1$ $m = .5$

APPENDIX A I.

It is often convenient to express the differential equation:

$$\Box^2\varphi = \beta^2\varphi_{x^1x^1} - \varphi_{x^2x^2} - \varphi_{x^3x^3} \tag{A1}$$

in terms of a coordinate system which is more appropriate for the boundaries of the particular problem under consideration. In this paper it has been found useful to re-express $\Box^2\varphi$ in terms of the Hyperboloido-Conal coordinates described by Robinson in Ref. 5.

Equation (A1) alone may be expressed in terms of any desired new set of coordinates ξ^1, ξ^2, ξ^3, that is:

$$\xi^\mu = \xi^\mu(x^1, x^2, x^3) \tag{A2}$$

where the differentials of the transformation are:

$$dx^\alpha = \frac{\partial x^\alpha}{\partial \xi^\mu} d\xi^\mu \qquad \alpha = 1, 2, 3 \tag{A3}$$

where the repeated superscripts indicate summation over μ. (tensor summation convention).

The differential form of the line element associated with equation (1) is:

$$ds^2 = \frac{1}{\beta^2}(dx^1)^2 - (dx^2)^2 - (dx^3)^2 \tag{A4}$$

or in terms of the covariant metric tensor it becomes:

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu \qquad \mu, \nu = 1, 2, 3 \tag{A5}$$

where

$$g_{11} = \frac{1}{\beta^2}, \quad g_{22} = g_{33} = -1$$

$$g_{\mu\nu} = 0 \qquad \mu \neq \nu$$

Since the interval ds^2 is invariant (independent of choice of coordinates) then for any pair of coordinate systems:

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = \bar{g}_{\alpha\beta}\, d\xi^\alpha d\xi^\beta \tag{A6}$$

where

$$ds^2 = \left(\frac{dx^1}{\beta}\right)^2 - (dx^2)^2 - (dx^3)^2 = \left(\frac{h_1}{\beta} d\xi^1\right)^2 - (h_2 d\xi^2)^2 - (h_3 d\xi^3)^2$$

From equation (A3) it may be seen that

$$\bar{g}_{\alpha\beta} = \frac{\partial x^{\mu}}{\partial \xi^{\alpha}} \frac{\partial x^{\nu}}{\partial \xi^{\beta}} g_{\mu\nu} \tag{A7}$$

hence in the new system of coordinates

$$\begin{aligned}
\bar{g}_{11} &= \frac{1}{\beta^2}\left(\frac{\partial x^1}{\partial \xi^1}\right)^2 - \left(\frac{\partial x^2}{\partial \xi^1}\right)^2 - \left(\frac{\partial x^3}{\partial \xi^1}\right)^2 = \left(\frac{h_1}{\beta}\right)^2 \\
\bar{g}_{22} &= \frac{1}{\beta^2}\left(\frac{\partial x^1}{\partial \xi^2}\right)^2 - \left(\frac{\partial x^2}{\partial \xi^2}\right)^2 - \left(\frac{\partial x^3}{\partial \xi^2}\right)^2 = -h_2^2 \\
\bar{g}_{33} &= \frac{1}{\beta^2}\left(\frac{\partial x^1}{\partial \xi^3}\right)^2 - \left(\frac{\partial x^2}{\partial \xi^3}\right)^2 - \left(\frac{\partial x^3}{\partial \xi^3}\right)^2 = -h_3^2
\end{aligned} \tag{A8}$$

In Ref. 13, page 117 Michal derives Laplace's equation in terms of a general coordinate system as:

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^{\alpha}}\left(\sqrt{g}\, g^{\alpha\beta} \frac{\partial \varphi}{\partial x^{\beta}}\right) = 0 \tag{A9}$$

where ϕ is a scaler field. This equation may be seen to apply for equation (A1) of the proper values for $g^{\alpha\beta}$ and g are inserted. That is:

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^{\alpha}}\left(\sqrt{g}\, g^{\alpha\beta} \frac{\partial \varphi}{\partial x^{\beta}}\right) = \frac{1}{\sqrt{\bar{g}}} \frac{\partial}{\partial \xi^{\mu}}\left(\sqrt{\bar{g}}\, \bar{g}^{\mu\nu} \frac{\partial \varphi}{\partial \xi^{\nu}}\right) = 0 \tag{A10}$$

where

$$\bar{g}^{\mu\nu} = \frac{\text{cofactor } \bar{g}_{\mu\nu}}{|g|} \qquad |\bar{g}| = \det.\, g_{\mu\nu}$$

making use of relations (A8)

$$\bar{g} = \bar{g}_{11}\bar{g}_{22}\bar{g}_{33} = \frac{h_1^2 h_2^2 h_3^2}{\beta^2} \tag{A11}$$

and

$$[\bar{g}^{\mu\nu}] = \begin{bmatrix} \frac{1}{\bar{g}_{11}} & 0 & 0 \\ 0 & \frac{1}{\bar{g}_{22}} & 0 \\ 0 & 0 & \frac{1}{\bar{g}_{33}} \end{bmatrix} = \begin{bmatrix} \frac{\beta^2}{h_1^2} & 0 & 0 \\ 0 & -\frac{1}{h_2^2} & 0 \\ 0 & 0 & -\frac{1}{h_3^2} \end{bmatrix} \tag{A12}$$

substituting (A11) and (A12) into (A10)

$$\frac{\beta}{h_1h_2h_3}\left\{\frac{\partial}{\partial\xi^1}\left[\left(\frac{h_2h_3}{h_1}\right)\frac{\partial\varphi}{\partial\xi^1}\right]-\frac{\partial}{\partial\xi^2}\left[\left(\frac{h_1h_3}{h_2\beta}\right)\frac{\partial\varphi}{\partial\xi^2}\right]-\frac{\partial}{\partial\xi^3}\left[\left(\frac{h_1h_2}{h_3\beta}\right)\frac{\partial\varphi}{\partial\xi^3}\right]\right\}=0 \tag{A13}$$

In the Hyperboloido-Conal System where

$$\xi^1 = r \qquad \xi^2 = \nu \qquad \xi^3 = \mu$$

$$h_1 = \beta \qquad h_2 = \frac{r\sqrt{\mu^2-\nu^2}}{\sqrt{(\nu^2-k^2)(1-\nu^2)}} \qquad h_3 = \frac{r\sqrt{\mu^2-\nu^2}}{\sqrt{(\mu^2-k^2)(\mu^2-1)}}$$

equation (A13) becomes

$$\begin{aligned}(\mu^2-\nu^2)\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right)&-\sqrt{(\nu^2-k^2)(1-\nu^2)}\,\frac{\partial}{\partial\nu}\sqrt{(\nu^2-k^2)(1-\nu^2)}\,\frac{\partial\varphi}{\partial\nu}\\&-\sqrt{(\mu^2-k^2)(\mu^2-1)}\,\frac{\partial}{\partial\mu}\sqrt{(\mu^2-k^2)(\mu^2-1)}\,\frac{\partial\varphi}{\partial\mu}=0\end{aligned} \tag{A14}$$

If a solution to equation (A1) is known that is where

$$\varphi = \varphi(\xi^1, \xi^2, \xi^3) \tag{A15}$$

and it is desired to obtain $\frac{\partial\varphi}{\partial x^\alpha}$ (see equation (25) text) which may be written as

$$\frac{\partial\varphi}{\partial x^\alpha} = \frac{\partial\varphi}{\partial\xi^\mu}\frac{\partial\xi^\mu}{\partial x^\alpha} \qquad \begin{array}{l}\mu = 1,2,3\\ \alpha = 1,2,3\end{array} \tag{A16}$$

then it is necessary to determine the derivatives $\frac{\partial\xi^\mu}{\partial x^\alpha}$ in terms of the known values for $\frac{\partial x^\alpha}{\partial\xi^\mu}$.

Consider the differentials given in equation (A3)

$$dx^\alpha = \frac{\partial x^\alpha}{\partial\xi^\mu}\,d\xi^\mu \tag{A17}$$

and let $\mathbf{a}$ represent the three by three matrix defined by

$$a = [a_{ji}] = \left[\frac{\partial x^\alpha}{\partial\xi^\mu}\right] \tag{A18}$$

and the inverse matrix defined by

$$a^{-1} = \frac{\text{Cofactor } a_{ij}}{|a|} = \left[A_{ji}\right] = A \qquad \text{(A19)}$$

where

$$a^{-1}a = Aa = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{(A20)}$$

Solving (A17) for $d\xi^{\mu}$ it is observed that

$$\left[A_{ji}\right] = \left[\frac{\partial \xi^{\mu}}{\partial x^{\alpha}}\right]$$

It may be noted that equations (A8) may be written in the following matrix form

$$\begin{pmatrix} \frac{1}{h_1^2}\frac{\partial x^1}{\partial \xi^1} & -\frac{\beta^2}{h_1^2}\frac{\partial x^2}{\partial \xi^1} & -\frac{\beta^2}{h_1^2}\frac{\partial x^3}{\partial \xi^1} \\ \frac{-1}{h_2^2\beta^2}\frac{\partial x^1}{\partial \xi^2} & \frac{1}{h_2^2}\frac{\partial x^2}{\partial \xi^2} & \frac{1}{h_2^2}\frac{\partial x^3}{\partial \xi^2} \\ -\frac{1}{\beta^2 h_3^2}\frac{\partial x^1}{\partial \xi^3} & \frac{1}{h_3^2}\frac{\partial x^2}{\partial \xi^3} & \frac{1}{h_3^2}\frac{\partial x^3}{\partial \xi^3} \end{pmatrix} \begin{pmatrix} \frac{\partial x^1}{\partial \xi^1} & \frac{\partial x^1}{\partial \xi^2} & \frac{\partial x^1}{\partial \xi^3} \\ \frac{\partial x^2}{\partial \xi^1} & \frac{\partial x^2}{\partial \xi^2} & \frac{\partial x^2}{\partial \xi^3} \\ \frac{\partial x^3}{\partial \xi^1} & \frac{\partial x^3}{\partial \xi^2} & \frac{\partial x^3}{\partial \xi^3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{(A21)}$$

where the matrix just to left of equal sign is seen to be $[a_{ij}]$ and from equation (A20) it will be noted that the left matrix is $[A_{ji}]$.
Therefore, the following relations may be established

$$\begin{aligned}
&\frac{\partial \xi^1}{\partial x^1} = \frac{1}{h_1^2}\frac{\partial x^1}{\partial \xi^1} \qquad &&\frac{\partial \xi^1}{\partial x^2} = -\frac{\beta^2}{h_1^2}\frac{\partial x^2}{\partial \xi^1} \qquad &&\frac{\partial \xi^1}{\partial x^3} = -\frac{\beta^2}{h_1^2}\frac{\partial x^3}{\partial \xi^1} \\
&\frac{\partial \xi^2}{\partial x^1} = \frac{-1}{\beta^2 h_2^2}\frac{\partial x^1}{\partial \xi^2} \qquad &&\frac{\partial \xi^2}{\partial x^2} = \frac{1}{h_2^2}\frac{\partial x^2}{\partial \xi^2} \qquad &&\frac{\partial \xi^2}{\partial x^3} = \frac{1}{h_2^2}\frac{\partial x^3}{\partial \xi^2} \\
&\frac{\partial \xi^3}{\partial x^1} = \frac{-1}{\beta^2 h_3^2}\frac{\partial x^1}{\partial \xi^3} \qquad &&\frac{\partial \xi^3}{\partial x^2} = \frac{1}{h_3^2}\frac{\partial x^2}{\partial \xi^3} \qquad &&\frac{\partial \xi^3}{\partial x^3} = \frac{1}{h_3^2}\frac{\partial x^3}{\partial \xi^3}
\end{aligned} \qquad \text{(A22)}$$

In the Hyperboloido-Conal System where

$$\xi^1 = r \qquad \xi^2 = \nu \qquad \xi^3 = \mu$$

$$h_1 = \beta \qquad h_2 = \frac{r\sqrt{\mu^2-\nu^2}}{\sqrt{(\nu^2-k^2)(1-\nu^2)}} \qquad h_3 = \frac{r\sqrt{\mu^2-\nu^2}}{\sqrt{(\mu^2-k^2)(\mu^2-1)}}$$

and

$$x^1 = x = \frac{\beta r \mu\nu}{k}, \quad x^2 = y = \frac{r\sqrt{(\mu^2-k^2)(\nu^2-k^2)}}{k\sqrt{1-k^2}}, \quad x^3 = z = \frac{r\sqrt{(\mu^2-1)(1-\nu^2)}}{\sqrt{1-k^2}}$$

then

$$\frac{\partial r}{\partial x} = \frac{1}{\beta}\frac{\mu\nu}{k} \qquad \frac{\partial \nu}{\partial x} = -\frac{\mu(\nu^2-k^2)(1-\nu^2)}{\beta r k(\mu^2-\nu^2)}$$

$$\frac{\partial r}{\partial y} = -\frac{\sqrt{(\mu^2-k^2)(\nu^2-k^2)}}{k\sqrt{1-k^2}} \qquad \frac{\partial \nu}{\partial y} = \frac{\nu(1-\nu^2)\sqrt{(\nu^2-k^2)(\mu^2-k^2)}}{rk\sqrt{1-k^2}(\mu^2-\nu^2)}$$

$$\frac{\partial r}{\partial z} = -\frac{\sqrt{(\mu^2-1)(1-\nu^2)}}{\sqrt{1-k^2}} \qquad \frac{\partial \nu}{\partial z} = -\frac{\nu(\nu^2-k^2)\sqrt{(1-\nu^2)(\mu^2-1)}}{r\sqrt{1-k^2}(\mu^2-\nu^2)}$$

(A23)

$$\frac{\partial \mu}{\partial x} = -\frac{\nu(\mu^2-k^2)(\mu^2-1)}{\beta k r(\mu^2-\nu^2)}$$

$$\frac{\partial \mu}{\partial y} = \frac{\mu(\mu^2-1)\sqrt{(\mu^2-k^2)(\nu^2-k^2)}}{rk\sqrt{1-k^2}(\mu^2-\nu^2)}$$

$$\frac{\partial \mu}{\partial z} = \frac{\mu(\mu^2-k^2)\sqrt{(\mu^2-1)(1-\nu^2)}}{r\sqrt{1-k^2}(\mu^2-\nu^2)}$$

APPENDIX-II

If in equation (11), p. 5

$$\mu = ns(\alpha, k) = k\,sn(\alpha + iK', k)$$
$$\nu = k\,nd(\gamma, k') = k\,sn(K + i\gamma, k)$$
$$0 \leqslant \alpha \leqslant K \qquad -2iK' \leqslant i\gamma \leqslant 2iK'$$

then

$$x = -\beta r k\,sn(\alpha + iK', k)\,sn(K + i\gamma, k)$$
$$y = -r\frac{k}{k'}\,cn(\alpha + iK', k)\,cn(K + i\gamma, k)$$
$$z = -r\frac{1}{k'}\,dn(\alpha + iK', k)\,dn(K + i\gamma, k)$$

In Fig. A, the region inside the Mach cone $\mu = \infty$ and exterior to the plate $\mu = 1$ in the μ, ν, r plane may be transformed into the rectangle, $-2iK'$, $K + 2iK'$, $K - 2iK'$, and $-2iK'$ in the $\alpha + i\gamma$ plane by the mapping

$$cd(\alpha + i\gamma) = \frac{\nu\sqrt{(\mu^2-1)(\mu^2-k^2)}}{k(\mu^2-\nu^2)} + \frac{i\mu\sqrt{(1-\nu^2)(\nu^2-k^2)}}{k(\mu^2-\nu^2)}$$

It may be seen that the upper surface of the plate FGH maps into the strip between $K \pm iK'$ and the lower surface into the strips between $K \pm 2iK'$ and $K \pm iK'$. The function $cn(K + i\gamma)$ is positive for $2iK' \leqslant i\gamma < 0$ (y negative) and negative for $0 < i\gamma \leqslant 2iK'$ (y positive). As indicated in Fig. A, this Jacobi function corresponds to $\sqrt{\nu^2 - k^2}$. Going from the top to the bottom of the plate the function $dn(K + i\gamma)$ is positive for $-iK' < i\gamma < 2iK'$ (z positive) and negative for $iK' < i\gamma < 2iK'$, $-2iK' < i\gamma < iK'$. From Fig. A, it will be noted that $dn(K + i\gamma)$ corresponds to $\sqrt{1-\nu^2}$. The functions involving α, i.e. μ, do not change signs over the range of variation of α.

$$sn(\alpha + iK') \geqslant 0$$
$$dn(\alpha + iK') \leqslant 0$$
$$cn(\alpha + iK') < 0$$

(see Ref. 10, p. 22)

APPENDIX B

Evaluation of coefficients of Lamé Functions

It is often convenient to re-express the equations given in (11) in terms of the Jacobi Functions, as discussed in Whittaker and Watson, Chapter XXII. Hence if:

$$\mu = ns(\alpha, k) \qquad \text{where} \qquad \alpha, k = \int_{\mu}^{\infty} \frac{d\xi}{\sqrt{(\xi^2-1)(\xi^2-k^2)}}$$

equation (11) becomes:

$$\frac{d^2 S_n^{(m)}}{d\alpha^2} - \left[n(n+1)(ns\alpha)^2 - p(k^2+1)\right] S_{(n)}^{(m)} = 0 \qquad \text{(B1)}$$

$$S_{(n)}^{(m)} = \mathbb{K} = \sum_0^{j_n} a_{jm} (sn\alpha)^{2j-n} \qquad j_n = \begin{Bmatrix} \frac{n}{2} & n \text{ even} \\ \frac{n-1}{2} & n \text{ odd} \end{Bmatrix}$$

$$\frac{d\mathbb{K}}{d\alpha} = \sum_0^{j_n} a_{jm}(2j-n)(sn\alpha)^{2j-n-1} cn\alpha\, dn\alpha \qquad \text{(B2)}$$

$$\frac{d^2\mathbb{K}}{d\alpha^2} = \sum_0^{j_n} a_{jm}(2j-n)\Big\{(2j-n-1)\, sn^{2j-n-2} - (k^2+1)(2j-n)\, sn\alpha^{2j-n} + k^2(2j-n+1)\, sn\alpha^{2j-n+2}\Big\}$$

Substituting (B2) in (B1), the equation relating the coefficients a_{jm} becomes:

$$\sum_{j=0}^{n} a_{jm}\Big\{2j(2j-2n-1)\, sn\alpha^{2j-n-2} + (k^2+1)\left[p-(2j-n)^2\right] sn\,\alpha^{2j-n} + k^2(2j-n)(2j-n+1)\, sn\alpha^{2j-n+2}\Big\} = 0 \qquad \text{(B3)}$$

Equating coefficients of like powers of $S_n\alpha$ to zero, one obtains the following set of relations:

$$
\begin{aligned}
2j(2n+1-2j)a_{jm} &= a_{(j-1)m}(k^2+1)[p-(n-2j+2)^2]+a_{(j-2)m}(n-2j+4)(n-2j+3)\\
(2j+2)(2n-1-2j)a_{(j+1)m} &= a_{jm}(k^2+1)[p-(n-2j)^2]+a_{(j-1)m}k^2(n-2j+2)(n-2j+1)\\
(2j+3)(2n-3-2j)a_{(j+2)m} &= a_{(j+1)m}(k^2+1)[p-(n-2j-2)^2]
\end{aligned}
\qquad \text{(B4)}
$$

Here, as in equation (B2) $j=\frac{n}{2}$ or $\frac{n-1}{2}$ as n is odd or even. The value of p in equation (B4) is determined such that $a_{(j+1)m}=a_{(j+2)m}\cdots a_{(j+k)m}=0$ Therefore, the function $S_n^{(m)}=\mathbb{K}$ is an algebraic polynomial of $\mu=nS\alpha,k$ of degree n. The values of p may be obtained by solving the $j+1$ simultaneous equations which are obtained relating the coefficients and p (see (B4). However, as stated above, p is determined such that $a_{(j+1)m}=a_{(j+2)m}=0$; therefore, p may be obtained by equating to zero the determinate. (See Ref. 6, pp. 460-461)

$$
\begin{vmatrix}
0, & \cdots\cdots & 0, & 2(2n-1), & -(k^2+1)(p-n^2) \\
 & & 4(2n-3), & -(k^2+1)[p-(n-2)^2], & -k^2n(n-1) \\
 & & & & 0 \\
 & & & & 0 \\
 & & & & \vdots \\
-(k^2+1)[p-(n-2j)^2], & k^2(n-2j-2)(n-2j+1), & \cdots & & 0
\end{vmatrix} = 0 \qquad \text{(B5)}
$$

which is of degree $j+1$ in p; hence, the resulting equation will be one of the same degree having $j+1$ values for p.

In order to demonstrate the method for obtaining the Lamé Functions $S_n^m = \mathbb{K}$, the case where n = 3 is chosen, as an example.

For n = 3

$$S_3^{(m)} = \mathbb{K} = \sum_0^1 a_{jm} (sn\,\alpha)^{2j-3} = a_{om}(sn\,\alpha)^{-3} + a_{1m}(sn\,\alpha)^{-1}$$

however, from equation (18) there are 2 solutions, or

$$\begin{aligned} S_3^{(1)} &= a_{01}\, ns^3\alpha + a_{11}\, ns\alpha \\ S_3^{(2)} &= a_{02}\, ns^3\alpha + a_{12}\, ns\alpha \end{aligned} \qquad \text{(B6)}$$

From equation (B4) there are two equations involving a_{om}, a_{1m}, and p namely:

$$\begin{aligned} 10a_{1m} &= a_{om}(k^2+1)(p-9) \\ 0 &= 6a_{om}k^2 + a_{1m}(k^2+1)(p-1) \end{aligned}$$

evaluating p by use of equation (B5)

$$\begin{vmatrix} 10, & -(k^2+1)(p-9) \\ -(k^2+1)(p-1), & -6k^2 \end{vmatrix} = 0$$

the determinate becomes

$$60k^2 + (k^2+1)^2(p-9)(p-1) = 0$$

or,

$$p^2 - 10p + 9 + 60k^2[k^2+1]^{-2} = 0$$

for which the roots are

$$p_1, p_2 = \frac{5(k^2+1) \pm 2\sqrt{4k^4 - 7k^2 + 4}}{k^2+1}$$

hence

$$\frac{a_{11}}{a_{01}} = -\frac{1}{5}\left[2(k^2+1) + \sqrt{4k^4 - 7k^2 + 4}\right.$$

$$\frac{a_{12}}{a_{02}} = -\frac{1}{5}\left[2(k^2+1) - \sqrt{4k^4 - 7k^2 + 4}\right.$$

If $a_{01} = a_{02} = 1$ then the two $\mathbb{K}$ species Lamé Functions of degree 3 are

$$S^{(1)}_{(3)} = ns^3\alpha + a_{11}\, ns\alpha$$
$$S^{(2)}_{(3)} = ns^3\alpha + a_{12}\, ns\alpha$$

(See Table II)

In general the solutions for the $\mathbb{K}$ type functions in this paper will be evaluated for $\mu = 1$ where $\mathbb{K} = \underset{\mu\to 1}{\mathbb{K}(\nu)}$. The variable ν may be represented by the following Jacobi function

$$\nu = \frac{k}{dn\,\gamma, k'} \qquad \gamma, k' = \int_k^{\nu} \frac{d\xi}{\sqrt{(1-\xi^2)(\xi^2-k^2)}} \tag{B7}$$

Equation (17) becomes

$$\frac{d^2 T_n^m}{d\gamma^2} - \left[n(n+1)k^2 nd^2\gamma - p(k^2+1)\right] T_n^m = 0 \tag{B8}$$

$$T_n^m = \mathbb{K}(\nu) = \sum_0^{J_n} a_{jm} \left(\frac{dn\,\gamma}{k}\right)^{2j-n} \tag{B9}$$

APPENDIX C

Lamé Functions of the Second Kind

If, as was shown in Appendix A, that S_n^m satisfies Lamé's equation, namely (B1)

$$\frac{d^2 S_n^m}{d\alpha^2} - [n(n+1)(ns^2\alpha) - p(k^2+1)]\, S_n^m = 0$$

then there may exist other solutions $\mathcal{S}_n^m$ for which the equation is also satisfied, that is:

$$\frac{d^2 \mathcal{S}_n^m}{d\alpha^2} - [n(n+1)(ns^2\alpha) - p(k^2+1)]\, \mathcal{S}_n^m = 0 \qquad \text{(C1)}$$

The form for S_n^m is known therefore, and it remains to find $\mathcal{S}_n^{(m)}$ in terms of $S_n^{(m)}$. If equations (B1) and (C1) are multiplied by $\mathcal{S}_n^m(ns\alpha)$ and $S_n^m(ns\alpha)$ respectively, the difference between these two equations is:

$$S_n^m \frac{d^2 \mathcal{S}_n^m}{d\alpha^2} - \mathcal{S}_n^m \frac{d^2 S_n^m}{d\alpha^2} = 0 \qquad \text{(C2)}$$

which may be restated as:

$$\frac{d}{d\alpha}\left[S_n^m \frac{d\mathcal{S}_n^m}{d\alpha} - \mathcal{S}_n^m \frac{dS_n^m}{d\alpha}\right] = 0 \qquad \text{(C3)}$$

or

$$S_n^m \frac{d\mathcal{S}_n^m}{d\alpha} - \mathcal{S}_n^m \frac{dS_n^m}{d\alpha} = \text{Const.} \qquad \text{(C4)}$$

If both sides of equation (C4) are divided by $[S_n^m]^2$ the left hand side is seen to be the derivative of the fraction $\mathcal{S}_n^m / S_n^m$,or more precisely, equation (C4) becomes:

$$\frac{d}{d\alpha}\left(\frac{\mathcal{S}_n^m}{S_n^m}\right) = \frac{\text{Const.}}{(S_n^m)^2}$$

or

$$\frac{\mathcal{S}_n^m(ns\alpha)}{S_n^m(ns\alpha)} - \frac{\mathcal{S}_n^m(ns\alpha_0)}{S_n^m(ns\alpha_0)} = \text{Const.} \int_{\alpha_0}^{\alpha} \frac{d\alpha}{[S_n^m(ns\alpha)]^2} \qquad \text{(C5)}$$

$\mathcal{S}_n^m(nS\alpha)$ is chosen such that it goes to zero on the Mach cone, $\mu=\infty$, therefore; the lower limit of equation (C5) $\alpha_0 = Sn^{-1}\frac{1}{\mu} = Sn^{-1}0 = 0$;

hence:

$$\frac{\mathcal{S}_n^m(nS\alpha)}{S_n^m(nS\alpha)} = \text{Const.}\int_0^{\alpha = Sn^{-1}\left(\frac{1}{\mu}\right)} \frac{d\alpha}{\left[S_n^m(nS\alpha)\right]^2} \tag{C6}$$

or

$$\mathcal{S}_n^m(nS\alpha) = S_n^m(nS\alpha)\int_0^{\alpha = Sn^{-1}\frac{1}{\mu}} \frac{d\alpha}{\left[S_n^m(nS\alpha)\right]^2}$$

This second solution of Lamé's equation is always in terms of the first and second kinds of elliptic integals.

APPENDIX D

In Ref. (7) Jones has indicated that the force due to impact pressure per unit length normal to an oblique leading edge could be expressed as:

$$\frac{F_n'}{q} = \frac{\pi R m_o^2}{\sqrt{1+m_o^2}} \frac{1}{\sqrt{1-\beta^2 m_o^2}} \tag{D-1}$$

where R is the radius of curvature of the leading edge, that is,

$$\underset{y\to mx}{\mathrm{L}} R = \frac{\left[1+\left(\frac{dz}{dx}\right)^2\right]^{3/2}}{\frac{d^2z}{dx}} \tag{D-2}$$

In the cases considered in this paper the local slope of the surfaces generated are of the form:

$$\frac{dz}{dx} = \frac{w'}{V_o} = \frac{\tau}{2C_o^{n-1} m_o} \frac{g(x,y)}{\sqrt{m_o^2 x^2 - y^2}} \tag{D-3}$$

$$g(x,y) < \infty$$

and

$$\frac{d^2z}{dx^2} = \frac{\tau}{2C_o^{n-1} m_o} \frac{G(x,y)}{\sqrt{m_o^2 x^2 - y^2}} \tag{D-4}$$

$$G(x,y) < \infty$$

Substituting equation (D-3) and (D-4) in (D-5) and passing to the limit the radius of curvature at the leading edge is:

$$R = \left(\frac{\tau}{2C_o^{n-1} m_o}\right)^2 \frac{\overline{g(x)}^3}{G(x)} = \left(\frac{\tau}{2C_o^{n-1} m_o}\right)^2 H(x) \tag{D-5}$$

The force exerted by the pressure per unit length on the leading edge becomes:

$$\frac{F_n'}{q} = \frac{\pi m_o^2}{\sqrt{1+m_o^2}} \left(\frac{\tau}{2C_o^{n-1} m_o}\right)^2 \frac{H(x)}{\sqrt{1-\beta^2 m_o^2}} \tag{D-6}$$

and the total drag in coefficient form may be written as:

$$C_{D_\sigma} = \frac{2\pi m_o}{C_o^3\sqrt{1-\beta^2 m_o^2}}\left(\frac{\tau}{2C_o^{n-1} m_o}\right)^2 \int_0^{C_o} H(x)\,dx \qquad \text{(D-7)}$$

For the case of the elliptic cone (n = 1), worked out by Jones in Ref. 7,

where
$$Z = \frac{\tau}{2m_o}\sqrt{m_o^2x^2-y^2}$$

and
$$H(x) = m_o^2 x$$

the drag due to leading edge pressure or push is:

$$C'_{D_\sigma} = \frac{\pi}{4}\frac{m_o\tau^2}{\sqrt{1-\beta^2 m_o^2}}$$

APPENDIX E

Evaluation of Certain Elliptic Integrals

Part I.

It has been found to be of definite advantage when solving elliptic integrals to restate these integals in terms of the Jacobi elliptic functions (see Ref. 10 or Ref. 12 chapt. XXII). The standard forms for elliptic integrals in this notation are easier to recognize.

A. Definitions

$$sn^{-1}(\varphi,k) = F(\varphi,k) = \int_0^{\varphi} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \int_0^u d\xi \tag{E1}$$

$$\varphi = \sin^{-1} sn\,u = am\,u$$

$$sn^{-1}(\tfrac{\pi}{2},k) = K(k) = \int_0^{K(k)} d\xi \tag{E1a}$$

$$E(\varphi,k) = \int_0^u dn^2\xi\, d\xi \tag{E2}$$

$$E(\varphi,k) = \int_0^u k'^2\, nd^2\xi - k^2\, sn\,u\, cd\,u$$

$$E(k) = \int_0^{K(k)} dn^2\xi\, d\xi = k'^2 \int_0^{K(k)} nd^2\xi\, d\xi \tag{E2a}$$

B. Evaluation of $I_o(k)$

$$I_o(\varphi,k) = \int_0^u dn^4\xi\, d\xi = \frac{1}{3}\left[2(1+k'^2)E(\varphi,k) - k'^2F(\varphi,k) + k^2 dn\,u\, cn\,u\, sn\,u\right] \tag{E3}$$

(see equation 569 Ref. 11)

$$I_o(k) \equiv I_o(\tfrac{\pi}{2},k) = \frac{1}{3}\left[2(1+k'^2)E(k) - k'^2K(k)\right] \tag{E3a}$$

C. Evaluation of $I_1(k)$

$$I_1(\varphi,k) = \int_0^u nd^4\xi\, d\xi = \frac{1}{3k'^4}\Big[2(1+k'^2)E(\varphi,k) - k'^2F(\varphi,k) -$$

$$- 2(1+k'^2)k^2 sn\,u\, cn\,u\, dn\,u - k'^2k^2 sn\,u\, cn\,u\, nd^3u\Big] \tag{E4}$$

$$I_1(k) \equiv I_1(\tfrac{\pi}{2},k) = \frac{I_o(\frac{\pi}{2},k)}{k'^4} = \frac{1}{3k'^4}\left[2(1+k'^2)E(k) - k'^2K(k)\right]$$

D. Evaluation of $I_2(k)$

$$I_2(k) \equiv I_2\left(\tfrac{\pi}{2}, k\right) = \int_0^{K(k)} \frac{d}{d\xi}\left(\frac{sn^4\xi}{dn\xi}\right) sc\xi \, d\xi = \int_0^{K(k)} sn^4\xi \left[nd^2\xi + 3\right] d\xi$$

$$= \frac{1}{k^4} \int_0^{K(k)} \left[nd^2\xi - 2 + dn^2\xi + 3(1 - 2dn^2\xi + dn^4\xi)\right] d\xi$$

$$= \frac{1}{k^4}\left[\frac{1+k'^2}{k'^2} E(k) - 2K(k)\right] + \frac{3}{k^4}\left[K(k) - 2E(k) + I_0(k)\right]$$

$$= \frac{1}{k'^2 k^2}\left[(1-2k'^2)E(k) + k'^2 K(k)\right] \tag{E5}$$

E. Evaluation of $I_3(k)$

$$I_3(k) \equiv I_3\left(\tfrac{\pi}{2}, k\right) = \int_0^{K(k)} \frac{d}{d\xi}\left(\frac{sn^4\xi}{dn^4\xi}\right) sc\xi \, d\xi = \int_0^{K(k)} \frac{sn^4\xi}{dn^2\xi}\left(3nd^2\xi + 1\right) d\xi$$

$$= \frac{1}{k^4} \int_0^{K(k)} \left[3(nd^4\xi - 2nd^2\xi + 1) + (nd^2\xi - 2 + dn^2\xi)\right] d\xi$$

$$= \frac{3}{k^4}\left[\frac{I_0(k)}{k'^4} - 2\frac{E(k)}{k'^2} + K(k)\right] + \frac{1}{k^4}\left[\frac{(1+k'^2)}{k'^2} E(k) - 2K(k)\right]$$

$$= \frac{1}{k'^2 k^2}\left[(1+k^2)E(k) - k'^2 K(k)\right] \tag{E6}$$

APPENDIX E

Part II.

The elliptic integrals presented in Part I of this appendix have been manipulated so that they may be expressed in terms of one of the normalized forms for the first or second kind. In the following it has been found convenient to express the elliptic integrals in terms of the standardized third kind - (In all cases considered here the integrals are the complete third kind). It has been noted, however, that the Lamé functions of the second kind which involve these elliptic integrals may be expressed in terms of the elliptic integrals of the first and second kind only. This will be evident in the tables of the complete third kind included in this appendix pp.82-84 . It should be further noted, that the equations involving the Lamé coefficients will be the coefficients which are multiplied by the functions Λ_0 or Z which characterize the elliptic integrals of the third kind.

The following integrals have been evaluated with the help of the tables of integrals which are included toward the end of this appendix (pp.82-84).

A. Evaluation of $I_{(2)}^{(m)}(k)$

$$I_{(2)}^{(m)}{}_{(k)} = \int_0^{K(k)} \frac{sn^4\xi \, d\xi}{[1+a_{1m}\,sn^2\xi]^2} = -\frac{d}{da_{1m}} \int_0^{K(k)} \frac{sn^2\xi}{[1+a_{1m}\,sn^2\xi]} \qquad \text{(E7)}$$

$$0 < -a_{1m} < k^2$$

If $-a_{1m}=\alpha^2$ it may be observed that Case III p.84 applies here and equation (E7) may be written as

$$I_{(2)}^{(m)}{}_{(k)} = \frac{d}{d\alpha^2}\int_0^{K(k)} \frac{sn^2\xi\, d\xi}{[1-\alpha^2 sn^2\xi]} = \frac{d}{d\alpha^2} S(\alpha^2, k)$$

$$= \frac{d}{d\alpha^2} \frac{K(k)\, Z(B,k)}{\sqrt{\alpha^2(1-\alpha^2)(k^2-\alpha^2)}} \qquad B = \sin^{-1}\frac{\alpha}{k}$$

$$= K(k)\left\{-\frac{Z(B,k)}{2}\frac{[3\alpha^4 - 2\alpha^2(1+k^2)+k^2]}{[\alpha^2(1-\alpha^2)(k^2-\alpha^2)]^{3/2}} + \frac{\frac{d}{d\alpha^2}Z(B,k)}{\sqrt{\alpha^2(1-\alpha^2)(k^2-\alpha^2)}}\right\}$$

where

$$\frac{d\,Z(B,k)}{d\alpha^2} = \frac{1}{2}\left[\frac{(1-\alpha^2)K(k)-E(k)}{K\sqrt{\alpha^2(k^2-\alpha^2)(1-\alpha^2)}}\right]$$

hence

$$I_{(2)}^{(m)(k)} = -K(k)\frac{Z(B,k)}{2}\frac{[3\alpha^4-2\alpha^2(k^2+1)+k^2]}{[\alpha^2(1-\alpha^2)(k^2-\alpha^2)]^{3/2}} + \frac{(1-\alpha^2)K(k)-E(k)}{2[\alpha^2(1-\alpha^2)(k^2-\alpha^2)]}$$

or in terms of a_{1m}

$$I_{(2)}^{(m)(k)} = K(k)\frac{Z(B,k)}{2}\frac{[3a_{1m}^2+2a_{1m}(k^2+1)+k^2]}{[a_{1m}(1+a_{1m})(k^2+a_{1m})]^{3/2}} - \frac{(1+a_{1m})K(k)-E(k)}{2a_{1m}(1+a_{1m})(k^2+a_{1m})}$$

The equations relating the coefficients for n = 2 is $3a_{1m}^2+2a_{1m}(k^2+1)+k^2=0$

hence

$$I_{(2)}^{(m)} = -\frac{(1+a_{1m})K(k)-E(k)}{2a_{1m}(1+a_{1m})(k^2+a_{1m})} \tag{E7a}$$

B. Evaluation of $I_{(3)}^{(m)}(k)$

$$I_{(3)}^{(m)}(k) = \int_0^{K(k)}\frac{sn^6\xi\,d\xi}{[1+a_{1m}sn^2\xi]^2} = \frac{d}{d\alpha^2}\int_0^{K(k)}\frac{sn^4\xi\,d\xi}{[1-\alpha^2 sn^2\xi]} \tag{E8}$$

where

$$0<\alpha^2<k^2$$

$$I_{(3)}^{(m)}(k) = \frac{1}{\alpha^4}\left\{\frac{K(k)Z(B,k)}{\sqrt{\alpha^2(1-\alpha^2)(k^2-\alpha^2)}} - \frac{K-E}{k^2}\right\} + \frac{1}{2\alpha^4}\left\{\frac{(1-\alpha^2)K(k)-E}{(1-\alpha^2)(k^2-\alpha^2)} - \frac{K(k)Z(B,k)[3\alpha^4-2\alpha^2(k^2+1)+k^2]}{[1-\alpha^2][k^2-\alpha^2]\sqrt{\alpha^2(1-\alpha^2)(k^2-\alpha^2)}}\right\}$$

$$I_{(3)}^{(m)}(k) = -\frac{1}{\alpha^4}\frac{K(k)Z(B,k)}{\sqrt{\alpha^2(1-\alpha^2)(k^2-\alpha^2)}}\frac{[5\alpha^4-4\alpha^2(k^2+1)+3k^2]}{2(1-\alpha^2)(k^2-\alpha^2)} + \frac{1}{\alpha^2}\frac{1}{2(1-\alpha^2)(k^2-\alpha^2)}\left[\frac{(3k^2-2\alpha^2)(1-\alpha^2)}{k^2}(K-E)-E\right]$$

If now it is noted that for n = 3 , $5a_{1m}^2+4a_{1m}+3k^2=0$, $I_{(3)}^{(m)}(k)$

becomes

$$I_{(3)}^{(m)}(k) = -\frac{1}{2a_{1m}}\times\frac{1}{(1+a_{1m})(k^2+a_{1m})}\left\{\frac{(3k^2+a_{1m})(1+a_{1m})}{k^2}(K-E)-E\right\} \tag{E8a}$$

where α^2 is replaced by $-a_{1m}$.

C. Evaluation of $I_4^{(m)}(k)$

$$I_{(4)}^{(m)}(k) = \int_0^{K(k)} \frac{sn^8 \xi \, d\xi}{[1+a\,sn^2\xi]^2[1+b\,sn^2\xi]^2} \tag{E9}$$

where

$$a = 1/2\,[a_{1m} + \sqrt{a_{1m}^2 - 4a_{2m}}\,]$$
$$b = 1/2\,[a_{1m} - \sqrt{a_{1m}^2 - 4a_{2m}}\,]$$

Dividing through by the denominator of the integrand and putting this expression in normal form, (see Integral Tables pp.82-84).

$$\begin{aligned} I_{(4)}^{(m)}(k) = \frac{1}{a^2b^2}\Big\{ & K(k) + \Big[\frac{4(a-b)}{ab^2} - 2 - \frac{(a-b)^3}{(ab)^3}\Big(1-\frac{2}{b}\Big)\Big]\Pi(a,k) \\ + & \Big[-\frac{4(a-b)}{a^2b} - 2 - \Big(\frac{a-b}{ab}\Big)^3\Big(1+\frac{2}{a}\Big)\Big]\Pi(b,k) \\ + & \Big[1 + \Big(\frac{a-b}{ab}\Big)\Big(\frac{a-b}{ab^3} - 2\Big)\Big]\Big[\frac{d}{da}\Pi(a,k) + \Pi(a,k)\Big] \\ + & \Big[1 + \Big(\frac{a-b}{ab}\Big)\Big(\frac{a-b}{ba^3} + 2\Big)\Big]\Big[\frac{d}{db}\Pi(b,k) + \Pi(b,k)\Big]\Big\} \end{aligned} \tag{E10}$$

Where from the Lamé differential equation the relation between the coefficients for n = 4 is

$$\frac{a}{b}\,[3b^2 + 2(1+k^2)b + k^2] - [7b^2 + 6(1+k^2)b + 5k^2] = 0$$

$$\frac{b}{a}\,[3a^2 + 2(1+k^2)a + k^2] - [7a^2 + 6(1+k^2)a + 5k^2] = 0$$

After substituting these relationships into the above form and expressing the integrated expression in the original coeficients a_{1m} and a_{2m} one obtains

$$\begin{aligned} I_{(4)}^{(m)}(k) = \frac{1}{a_{2m}(a_{1m}^2 - 4a_{2m})(a_{2m} + k^2a_{1m} + k^4)}\Big\{ & E(k)\,[a_{1m}^2 - 3a_{1m}a_{2m} \\ & + (1+k^2)(a_{1m}^2 - 2a_{2m}) + k^2a_{1m}] \\ & - K(k)\,[(1 + a_{1m} + a_{2m})(a_{1m}^2 - 2a_{2m} + k^2a_{1m})]\Big\} \end{aligned} \tag{E11}$$

D. Complete Elliptic Integrals of the Third Kind

1. Method for evaluating

The elliptic integrals which are evaluated in these tables are of the following general type

$$\Pi(y,\alpha^2,k)=\int_0^y \frac{d\xi}{(1-\alpha^2\xi)\sqrt{(1-\xi^2)(1-k^2\xi)}} \tag{E12}$$

which when put into Jacobi notation becomes

$$\Pi(u,\alpha^2,k)=\int_0^u \frac{d\xi}{[1-\alpha^2 sn^2\xi]} \tag{E12a}$$

where $u = sn^{-1}y,k$; for the cases treated here $u = K(k) = sn^{-1}(1,k)$.

In the tables compiled here, equation (E12) has been evaluated for different ranges of the parameter α^2. As an example of the method used, equation (12a) will be evaluated for $k < \alpha^2 < 1$ (Case I).

The following substitution

$$\alpha = k^2 sn^2\alpha \qquad \frac{\alpha^2}{k^2} = sn^2\alpha \qquad 1 < \frac{\alpha^2}{k^2} < \frac{1}{k^2}$$

is introduced, and equation (12a) is now rewritten in a slightly different form as

$$\Pi(\alpha^2,k) = \int_0^{K(k)} \left[d\xi + \frac{k^2 sn^2\eta\, sn^2\xi\, d\xi}{1-k^2 sn^2\eta\, sn^2\xi}\right]$$

or

$$\Pi(\alpha^2,k) = K(k) + \frac{sn\eta}{cn\eta\, dn\eta}\int_0^{K(k)} \frac{k^2 sn\eta\, cn\eta\, dn\eta\, sn^2\xi\, d\xi}{1-k^2 sn^2\eta\, sn^2\xi} \tag{E13}$$

It may be noted that the second integal, which is designated as $\Pi(\eta,k)$, is the same as given in Ref. 12 p. 523 as the fundamental integral of the Third kind. This integral is evaluated in Ref. 12 in terms of Θ, Jacobi Theta function, and Z, Jacobi Zeta function. For the case under consideration

$$\Pi(\eta,k) = \frac{1}{2}\log\frac{\Theta(K-\eta)}{\Theta(K+\eta)} + K\,Z(\eta,k) \tag{E14}$$

However, Θ is doubly-periodic in 2K and 2iK' (see Ref. 12, sec. 22.732) hence

$$\Pi(\eta,k) = K\,Z(\eta,k) \quad , \quad \Theta(K-\eta)=\Theta(K+\eta)$$

or

$$\Pi(\eta,k) = K(k)E(\eta,k) - F(\eta,k)E(k) \tag{E14a}$$

(see Ref. 12 sec. 22.731)

Here

$$F(\eta,k) = \int_0^{\alpha/k} \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}} \qquad 1 < \alpha/k < \frac{1}{k}$$

$$= \int_0^1 \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}} + \int_1^{\alpha/k} \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}}$$

$$F(\eta,k) = K(k) - i\int_1^{\alpha/k} \frac{d\xi}{\sqrt{(\xi^2-1)(1-k^2\xi^2)}} \tag{E15}$$

If now the substitution $\zeta = [\alpha^2-k^2]^{1/2}/\alpha k'$ is, introduced, equation (E15) may be written as follows:

$$F(\eta,k) = K(k) - i\int_0^{\frac{\sqrt{\alpha^2-k^2}}{k'\alpha}} \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k'^2\zeta)}}$$

or

$$F(\eta,k) = K(k) - i\,F(\varphi,k) \qquad \varphi = \sin^{-1}\sqrt{\alpha^2-k^2}/k'\alpha \tag{E16}$$

The integral $E(\eta,k)$ may be written for $1 < \frac{\alpha}{k} < \frac{1}{k}$ as

$$E(\eta,k) = \int_0^1 \sqrt{\frac{1-k^2\xi^2}{1-\xi^2}}\,d\xi - i\int_1^{\alpha/k} \sqrt{(1-k^2\xi^2)}\,[\xi^2-1]^{-\frac{1}{2}}d\xi$$

$$E(\eta,k) = E(k) - i\,k'\int_0^{\frac{\sqrt{\alpha^2-k^2}}{k'\alpha}} \frac{\sqrt{1-\zeta^2}}{[1-k'^2\zeta^2]^{3/2}} \tag{E17}$$

Introducing the Jacobi notation $\zeta = \mathrm{sn}(u,k')$ equation (E17) becomes

$$E(\eta,k) = E(k) - i\,k'^2\int_0^{\mathrm{sn}^{-1}\sqrt{\alpha^2-k^2}/k'\alpha} \mathrm{cd}^2\zeta\, d\zeta \tag{E17a}$$

The integral involving $\mathrm{cd}^2\zeta$ is evaluated in Ref. 12 p. 516 as

$$\int_0^u \mathrm{cd}^2\zeta\, d\zeta = \frac{u - E(u,k') + k'^2\mathrm{sn}(u,k')\,\mathrm{cd}(u,k')}{k'^2}$$

Hence equation (17a) becomes

$$E(\eta,k) = E(k) - i\left[F(\varphi,k') - E(\varphi,k') + \frac{\sqrt{(1-\alpha^2)(\alpha^2-k^2)}}{\alpha}\right] \tag{E18}$$

Upon substituting equations (E16) and (E18) back into (E14a)

$$\Pi(\eta,k) = KE - i\left[F(\varphi,k') - E(\varphi,k') + \frac{\sqrt{(1-\alpha^2)(\alpha^2-k^2)}}{\alpha}\right]K(k) - E(k)\left[K(k) - iF(\varphi,k)\right]$$

or (E19)

$$\Pi(\eta,k) = i\left[\frac{\pi}{2}\Lambda_0(\varphi,k) + K(k)\sqrt{(1-\alpha^2)^{?}(\alpha^2-k^2)}\right.$$

where Λ_0 is defined in Ref. 14 as

$$\Lambda_0 = \frac{2}{\pi}\left[(E-K)F(\varphi,k') - K(k)E(\varphi,k')\right]$$

If $\Pi(\eta,k)$ as given in equation (E19) is put into (13), the integral representation given in CaseI is obtained; that is

$$\Pi(\alpha^2,k) = K(k) + \frac{\alpha}{\sqrt{(\alpha^2-1)(k^2-\alpha^2)}}\left[\frac{\pi}{2}\Lambda_0(\varphi,k) - \frac{\sqrt{(\alpha^2-1)(k^2-\alpha^2)}}{\alpha}K(k)\right]$$

or

$$\Pi(\alpha^2,k) = \frac{\pi}{2}\,\frac{\Lambda_0(\varphi,k)}{\sqrt{(\alpha^2-1)(k^2-\alpha^2)}} \qquad \varphi = \sin^{-1}\sqrt{\alpha^2-k^2}/k'\alpha \tag{E20}$$

It should be pointed out here that if the range of α^2 were $0<\alpha^2<k^2$ (Case III) then the evaluation of equation (E13) could be obtained readily by substituting the value for $\Pi(\eta)$ given in (14a) into (E13) or

$$\Pi(\alpha^2,k) = K(k) + \frac{\alpha K(k)Z(B,k)}{\sqrt{(1-\alpha^2)(k^2-\alpha^2)}} \tag{E21}$$

$$\eta = B = \sin^{-1}\frac{\alpha}{k}$$

E. Tables of Elliptic Integrals

Case I. $k^2 < \alpha^2 < 1$

$$\Lambda_0(\varphi, k) \equiv \frac{2}{\pi}\left[(E-K)F(\varphi, k') + K E(\varphi, k')\right]$$

This function may be found tabulated in Ref. 14 for various values of φ and k

$$\varphi = \sin^{-1}\frac{\sqrt{\alpha^2 - k^2}}{k'\alpha}$$

$$\Pi(\alpha^2, k) \equiv \int_0^{K(k)} \frac{d\xi}{[1-\alpha^2 \mathrm{sn}^2\xi]} = \frac{\pi\alpha^2 \Lambda_0(\varphi, k)}{2\sqrt{(\alpha^2-k^2)(1-\alpha^2)}} \tag{E22-1}$$

$$S(\alpha^2, k) \equiv \int_0^{K(k)} \frac{\mathrm{sn}^2\xi \, d\xi}{[1-\alpha^2 \mathrm{sn}^2\xi]} = \frac{1}{\alpha^2}\left[\Pi(\alpha^2, k) - K(k)\right] \tag{E22-2}$$

$$C(\alpha^2, k) \equiv \int_0^{K(k)} \frac{\mathrm{cn}^2\xi \, d\xi}{[1-\alpha^2 \mathrm{sn}^2\xi]} = \Pi(\alpha^2, k) - S(\alpha^2, k) \tag{E22-3}$$

$$D(\alpha^2, k) \equiv \int_0^{K(k)} \frac{\mathrm{dn}^2\xi \, d\xi}{[1-\alpha^2 \mathrm{sn}^2\xi]} = \Pi(\alpha^2, k) - k^2 S(\alpha^2, k) \tag{E22-4}$$

$$\int_0^{K(k)} \frac{1-\alpha_1 \mathrm{sn}^2\xi}{1-\alpha^2 \mathrm{sn}^2\xi} d\xi = \Pi(\alpha^2, k) - \alpha_1^2 S(\alpha^2, k) \tag{E22-5}$$

$$\int_0^{K(k)} \frac{d\xi}{[1-\alpha^2 \mathrm{sn}^2\xi]^2} = \alpha^2 \frac{d}{d\alpha^2}\Pi(\alpha^2, k) + \Pi(\alpha^2, k) \tag{E22-6}$$

$$= \frac{K}{2(1-\alpha^2)(\alpha^2-k^2)}\left[\alpha^2 E + (k^2-\alpha^2)K\right.$$

$$\left. + \frac{\pi\alpha(2\alpha^2 + 2\alpha^2 k^2 - 3k^2 - \alpha^4)\Lambda_0}{2\sqrt{(1-\alpha^2)(\alpha^2-k^2)}}\right]$$

In all succeeding cases equations (1) and (6) will be given only, since equations (2) - (5) may be obtained from (1).

Alternate Form for Case I. $\xi = \sin^{-1}\sqrt{\frac{1-\alpha^2}{k'}}$

$$\Lambda_0(\xi,k) = \frac{2}{\pi}\left[(E-K)F'(\xi,k') + K\,E(\xi,k')\right]$$

$$\Pi(\alpha^2,k) = K + \frac{\pi\left[1-\Lambda_0(\xi,k)\right]\alpha}{2\sqrt{(\alpha^2-k^2)(1-\alpha^2)}} \qquad \text{(E23-1)}$$

$$\int_0^{K(k)} \frac{du}{[1-\alpha^2 sn^2 u]^2} = \frac{1}{2(1-\alpha^2)(\alpha^2-k^2)}\left\{\alpha^2 E + (2\alpha^2k^2 - 2k^2 - \alpha^4)K + \frac{\pi\alpha(\alpha^4 - 2\alpha^2k^2 - 2\alpha^2 - 3k^2)(1-\Lambda_0(\xi,k))}{2\sqrt{(\alpha^2-k^2)(1-\alpha^2)}}\right\} \qquad \text{(E23-6)}$$

Case II. $0 < -\alpha^2 < \infty$

$$\psi = \sin^{-1}\sqrt{\frac{\alpha^2}{\alpha^2-k^2}}$$

$$\Lambda_0(\psi,k) = \frac{2}{\pi}\left[(E-K)F(\psi,k') + K\,E(\psi,k')\right]$$

$$\Pi(\alpha^2,k) = \frac{k^2K}{k^2-\alpha^2} - \frac{\pi\alpha^2\Lambda_0(\psi,k)}{2\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}} \qquad \text{(E24-1)}$$

$$\int_0^{K(k)} \frac{du}{[1-\alpha^2 sn^2 u]^2} = \frac{1}{2(\alpha^2-1)(k^2-\alpha^2)}\left\{\alpha^2 E + \frac{2k^4\alpha^2 - 2k^4 + \alpha k'^2}{k^2-\alpha^2}K - \frac{\pi}{2}\frac{(2\alpha^2k^2 + 2\alpha^2 - \alpha^4 - 3k^2)\alpha^2\Lambda_0(\psi,k)}{\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}}\right\} \qquad \text{(E24-6)}$$

Alternate Case II.

$$B = \sin^{-1}\frac{1}{\sqrt{1-\alpha^2}}$$

$$\Lambda_0(B,k) = \frac{2}{\pi}\left[(E-K)F'(B,k') + K\,E(B,k')\right]$$

$$\Pi(\alpha^2,k) = \frac{K}{1-\alpha^2} + \frac{\pi\alpha^2\left[\Lambda_0(B,k)-1\right]}{2\sqrt{\alpha^2(1-\alpha^2)(\alpha^2-k^2)}} \qquad \text{(E25-1)}$$

$$\int_0^{K(k)} \frac{du}{[1-\alpha^2 sn^2 u]^2} = \frac{1}{2(1-\alpha^2)(\alpha^2-k^2)}\left\{\pi\alpha^2(3k^2 - 2\alpha^2k^2 - 2\alpha^2 + \alpha^4)\Lambda_0(B,k) + \alpha^2E + \frac{2k^2-\alpha^2-\alpha^2k^2}{1-\alpha^2}K\right\} \qquad \text{(E25-6)}$$

Case III. $0 < \alpha^2 < k^2$ $\qquad B = \sin^{-1}\frac{\alpha}{k}$

$$Z(B,k) \equiv E(B,k) - \frac{E}{K} F(B,k)$$

$$\Pi(\alpha^2, k) = K + \frac{\alpha^2 K\, Z(B,k)}{\sqrt{\alpha^2(1-\alpha^2)(k^2-\alpha^2)}} \tag{E26-1}$$

$$\int_0^{K(k)} \frac{du}{[1-\alpha^2 \operatorname{sn}^2 u]^2} = \frac{1}{2(1-\alpha^2)(k^2-\alpha^2)} \Big\{ (\alpha^4 - 2\alpha^2 k^2 + 2k^2 - \alpha^2) K(k) - \alpha^2 E + \frac{(\alpha^4 - 2\alpha^2 k^2 - 2\alpha^2 + 3k^2) K Z(B,k)}{\sqrt{\alpha^2(1-\alpha^2)(k^2-\alpha^2)}} \Big\} \tag{E26-6}$$

Case IV. $\infty > \alpha^2 > 1$ $\qquad A = \sin^{-1}\left(\frac{1}{\alpha}\right)$

$$Z(A,k) \equiv E(A,k) - \frac{E}{K} F(A,k)$$

$$\Pi(\alpha^2, k) = -\frac{\alpha K\, Z(A,k)}{\sqrt{(\alpha^2-1)(\alpha^2-k^2)}} \tag{E27-1}$$

$$\int_0^{K(k)} \frac{du}{[1-\alpha^2 \operatorname{sn}^2 u]^2} = \frac{1}{2(\alpha^2-1)(\alpha^2-k^2)} \Big\{ (\alpha^2-k^2) K - \alpha^2 E + \frac{(2\alpha^2 + 2\alpha^2 k^2 - 3k^2 - \alpha^4) K Z(A,k)}{\sqrt{(\alpha^2-1)(\alpha^2-k^2)}} \Big\} \tag{E27-6}$$

REFERENCES

1. Squire, H. B., An Example in Wing Theory at Supersonic Speed, R and M 2549, 1951.

2. Lomax, H. and Heazlet, Generalized Conical-flow Fields in Supersonic Wing Theory, NACA TN 2497, September 1951.

3. Germain, P., La theorie des mouvements homogenes et son application au calcu de certaines ailes delta en regime supersonique, La Recherche Aeronautique No. 7, January - February 1949.

4. Lagerstrom, P. A., Linearized Supersonic Theory of Conical Wings, NACA TN 1685, 1948.

5. Robinson, A., Aerofoil Theory of a Flat Delta Wing at Supersonic Speeds, R and M, 2548, September 1946.

6. Hobson, E. W., The Theory of Spherical and Ellipsoidal Harmonics, The University Press, Cambridge, 1931.

6a. Ince, E. L., Periodic Lamé Functions, Mathematical Institute, University of Edinburgh, Royal Soc. of Edinburgh Proceedings, Vol. IX, pp. 47-63, 1939-40.

6b. Ince, E. L., Further Investigations into Periodic Lamé Functions, Royal Soc. of Edinburgh Proceedings, Vol. LX, pp. 83-99, 1939-40.

6c. Erdelyi, A., Expansions of Lamé Functions into Series of Legendre Functions, Mathematical Institute, University of Edinburgh, Royal Society of Edinburgh Proceedings, pp. 247-67, 1943-49.

7. Jones, R. T., Leading Edge Singularities in Thin-Airfoil Theory, Journal Aeronautical Sciences, Vol. 17, No. 5, p. 307. May 1950.

8. Stewart, H. J., The Lift of a Delta Wing at Supersonic Speeds, Quarterly Appl. Math., Vol. IV, No. 3, October 1946. pp. 246-254.

9. Brown, Clinton E. and Adams, Mac C., Damping in Pitch and Roll of Triangular Wings at Supersonic Speeds, NACA Rep. 892, 1948.

10. Oberhettinger, F. and Magnus, W., Anwendung der Elliptischen Funktionen in Physik und Technik, Springer-Verlag Gottingen, 1949.

11. Pierce, B. O., A Short Table of Integrals, New York, Ginn and Company, 1929.

12. Whittaker, E. T. and Watson, G. N., A Course of Modern Analysis, Cambridge Press, 1946.

13. Michal, A. D., Matrix and Tensor Calculus, John Wiley and Sons, Inc., New York, 1947.

14. Heuman, Carl, Tables of Complete Elliptic Integrals, Journal Math. Physics, Vol. 19-20 (1940-1941), pp. 127-49.