SOME APPROXIMATIONS IN THE DYNAMIC SHELL EQUATIONS

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ABSTRACT

A theoretical analysis was performed on the linear dynamic equations of thin cylindrical shells to find the error committed by the Donnell assumption and the neglect of inplane inertia.

The Donnell approximation was found to be valid at high frequencies as compared to the ring frequencies, for all admissible sets of boundary conditions for finite length thin shells.

The error from neglecting tangential inertia is appreciable for long circumferential and axial wave lengths, independent of shell thickness.

The effect of boundary conditions was investigated from an exact solution of the linear eigenvalue problem. The inplane boundary conditions proved to be very influential in the neighborhood of the minimum frequency. An approximate technique which accounts for the inplane boundary conditions was then developed and shown to be in good agreement with the exact solution.

Finally, the effect of an elastic end ring on the eigenfrequencies was studied. The out-of-plane and torsional rigidities of the ring were found to govern the overall shell stiffness. Considerable mode interaction was noticed at low circumferential wave numbers for low values of ring stiffness. The computed eigenfrequencies were found to be in good agreement with the experimental results.

NOMENCLATURE

A_r	Area of ring cross section
a	Shell mean radius
a_r	Ring mean radius
$\tilde{a} = \frac{a_r}{a}$	Mean radii ratio
$D = \frac{Eh^3}{12(1-\nu^2)}$	Shell flexural rigidity
E	Young's modulus of elasticity
G_r, G_r', H_r	Moment resultants of ring
h	Shell thickness
h_r	Ring thickness
I_{xx}, I_{yy}, I_p	Principal moments of inertia of ring cross section
$J_{eff.}$	Effective moment of inertia in torsion
$k_{xx}, k_{\theta\theta}, k_{x\theta}$	Nondimensional change of curvature of shell
ℓ^*	Shell length
$\ell = \frac{\ell^*}{a}$	Nondimensional shell length
$m_{xx}, m_{\theta\theta}, m_{x\theta}$	Shell nondimensional moment resultants
N_r, N_r', T_r	Stress resultants of ring
$n_{xx}, n_{\theta\theta}, n_{x\theta}, n_{\theta x}$	Shell nondimensional normal stress resultants
m	Number of half axial waves
n	Number of circumferential full waves
$q_{xx}, q_{\theta\theta}$	Shell nondimensional shear resultants

$r = \frac{h}{\sqrt{12}\, a}$	Nondimensional radius of gyration of shell cross section
$r_x = \frac{1}{a_r}\sqrt{\frac{I_{xx}}{A_r}}$	Nondimensional out-of-plane radius of gyration of ring cross section
$r_y = \frac{1}{a_r}\sqrt{\frac{I_{yy}}{A_r}}$	Nondimensional inplane radius of gyration of ring cross section
$r_p = \frac{1}{a_r}\sqrt{\frac{I_p}{A_r}}$	Nondimensional polar radius of gyration of ring cross section
$r_t = \frac{1}{a_r}\sqrt{\frac{J_{eff.}}{A_r}}$	Nondimensional radius of gyration in torsion of ring cross section
$S_{tx} = \frac{r_x}{r}\sqrt{W_r \tilde{a}}$	Out-of-plane ring stiffness factor
$S_{ty} = \frac{r_y}{r}\sqrt{W_r \tilde{a}}$	Inplane ring stiffness factor
$s = \frac{m\pi}{\ell}$	Nondimensional axial wave length parameter
t	Time
u^*, v^*, w^*	Shell displacements
$u=\frac{u^*}{a},\ v=\frac{v^*}{a},\ w=\frac{w^*}{a}$	Nondimensional shell displacements
$u_r^*, v_r^*, w_r^*, \beta_r$	Ring displacements and rotation about its center line (Z axis of ring)
$u_r=\frac{u_r^*}{a_r},\ v_r=\frac{v_r^*}{a_r},\ w_r=\frac{w_r^*}{a_r}$	Nondimensional ring displacements
W_r	(weight of ring/weight of shell) Weight ratio

X, Y, Z	Orthogonal coordinate system for ring
x^*	Running coordinate in axial direction for shell
$x = \frac{x^*}{a}$	Nondimensional axial running coordinate
β_r	Rotation of ring cross section about its center line
$\epsilon_{xx}, \epsilon_{\theta\theta}, \epsilon_{x\theta}$	Membrane strains of shell
θ	Running coordinate in circumferential direction
ν	Poisson's ratio
ρ	Density of shell material
$\tau = \omega_o t$	Nondimensional time
$\omega_o = \sqrt{\frac{E}{\rho a^2(1-\nu^2)}}$	Axisymmetric ring frequency
$\tilde{\omega} = \frac{\omega}{\omega_o}$	Nondimensional frequency

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I. APPROXIMATIONS IN THE EQUATIONS OF MOTION OF THIN RINGS AND SHELLS

1. INTRODUCTION

The problem of determining the stresses and displacements of shell structures subjected to time dependent loading, has been treated by various methods. Techniques, such as finite elements or finite differences and direct integration of the time dependent equations are very popular. These techniques are not very well suited for parameter studies due to the computer time involved. However, they become essential when dealing with complex flight structures. For detailed studies concerned with the effect of boundary conditions, isolated rings etc., a direct attack on the differential equations is preferable as long as the structure studied is simple in geometry. Unfortunately, when this is done, one immediately is faced with the difficulty of deciding which differential equations are to be used. This same problem occurs with the use of finite element approach, but does not add greatly to the difficulties. If the differential equations are to be used, one would like to simplify consistently these equations as much as possible and yet retain their accuracy. Such a desire was the motivation for this study.

Many investigators have discussed the consistency of various shell equations governing the static analysis (Refs. 1 to 12). Most of this work was based on a relative order of magnitude estimate of terms in the strain energy expression. A few studies have been carried out for the dynamic shell equations. However, most of these have been concerned with derivations of new sets of equations, and have not been a consistent study on accuracy (Refs. 13, 14 and 15).

Forsberg (Ref. 16) has discussed the natural frequencies of cylindrical shells and the effect of various parameters on these frequencies. However some of his general conclusions are somewhat in doubt. In addition, sufficient information is not given in his work to estimate the error made in neglect of certain terms in the governing equations.

This work has as one of its major goals the determination of the actual errors involved when one or more terms in the governing equations has been eliminated. In order to present a complete picture of the cylindrical shell frequency spectrum, the analysis is divided into three portions. In the first portion the ring is studied. In the second portion the cylindrical shell with classical simply supported edges is studied. In the final portion the influence of boundary conditions is considered. The work is divided in this manner since it has been possible to provide an accurate estimate of the error introduced in the approximate cylindrical shell equations, by comparing the resulting frequency to the ring frequency. This will be discussed further. The analysis will cover a range of frequencies with mode shapes having thirty or less full waves in the circumferential direction and six or less half waves in the axial direction. This is the range of interest for response modes resulting from smoothly varying external loads.

2. EXACT EQUATIONS AND SOLUTION FOR FREE VIBRATION

A. Method of Comparison

The exact solution of all linear elastic response problems can be represented in the form of an eigenfunction expansion. The generalized coordinates in this expansion are functions of the eigen-frequencies. It seems then logical to compare the approximate equations of motion with the exact ones, by looking at the difference between their corresponding eigenfrequencies for the free vibration problem. An error is defined as follows:

$$e = \frac{\omega_{appr.}}{\omega_{exact}} - 1$$

B. "Exact" Equilibrium and Constitutive Equations

The equations of equilibrium and the corresponding set of constitutive relations, derived by Koiter (Ref. 9) will be referred to throughout this work, as the "exact" equations of motion. Koiter's derivation is based upon Kirchhoff's hypothesis and the uncoupling of the membrane and bending energies. These dynamic equations include the following quantities which are sometimes neglected in the simplified equations:

1. Inplane displacements in the curvature relations
2. Transverse shear force in the inplane equilibrium equations
3. Order $(\frac{h}{a})^2$ in the inplane shear force $n_{x\theta}$ (i.e., $n_{x\theta} \neq n_{\theta x}$)

4. Tangential inertia

5. Rotary inertia

In terms of nondimensional quantities, the "exact" equilibrium equations and constitutive relations can be written as follows:

$$\frac{\partial n_{xx}}{\partial x} + \frac{\partial \bar{n}_{x\theta}}{\partial \theta} = \frac{\partial^2 u}{\partial \tau^2}$$

$$\frac{\partial \bar{n}_{x\theta}}{\partial x} + \frac{\partial n_{\theta\theta}}{\partial \theta} = \frac{\partial^2 v}{\partial \tau^2} + q_\theta \qquad (1)$$

$$\frac{\partial^2 m_{xx}}{\partial x^2} - 2\frac{\partial^2 m_{x\theta}}{\partial x \partial \theta} + \frac{\partial^2 m_{\theta\theta}}{\partial \theta^2} + n_\theta = \frac{\partial^2 w}{\partial \tau^2} - r^2 \frac{\partial^2}{\partial \tau^2}(\nabla^2 w)$$

$$n_{xx} = \epsilon_{xx} + \nu\epsilon_{\theta\theta}$$

$$n_{\theta\theta} = \epsilon_{\theta\theta} + \nu\epsilon_{xx} \qquad (2)$$

$$\bar{n}_{x\theta} = \frac{1-\nu}{2}\epsilon_{x\theta}$$

$$n_{x\theta} = \bar{n}_{x\theta} + m_{x\theta}$$

$$n_{\theta x} = \bar{n}_{x\theta}$$

$$m_{xx} = r^2(k_{xx} + \nu k_{\theta\theta})$$

$$m_{\theta\theta} = r^2(k_{\theta\theta} + \nu k_{xx})$$

$$m_{x\theta} = r^2(1-\nu)\,k_{x\theta}$$

$$q_\theta = \frac{\partial m_{\theta\theta}}{\partial \theta} - 2\frac{\partial m_{x\theta}}{\partial x}$$

The coordinate system and resultant forces and moments are shown in Figure 1. The geometric parameter "r" is defined as:

$$r = \frac{h}{\sqrt{12}\,a}$$

The strain-displacement and nondimensional curvature-displacement relations are given below:

$$\begin{aligned}
\epsilon_{xx} &= \frac{\partial u}{\partial x} \\
\epsilon_{\theta\theta} &= \frac{\partial v}{\partial \theta} - w \\
\epsilon_{x\theta} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial \theta} \\
k_{xx} &= -\frac{\partial^2 w}{\partial x^2} \\
k_{\theta\theta} &= -\left(\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial v}{\partial \theta}\right) \\
k_{x\theta} &= -\left(\frac{\partial^2 w}{\partial x \partial \theta} + \frac{\partial v}{\partial x}\right)
\end{aligned} \qquad (3)$$

Substituting (2) and (3) in (1) we obtain:

$$\begin{bmatrix} D_{ij} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = 0 \qquad (i, j = 1, 2, 3) \qquad (4)$$

$\begin{bmatrix} D_{ij} \end{bmatrix}$ is a symmetric matrix of linear differential operators given in Appendix Ia.

The above operator matrix can be diagonalized in "u" and "v" and the resulting uncoupled differential equations are given in Appendix Ib.

C. Exact Solution for Free Vibration

For simplicity, we assume both edges to have simple support of the following type:

$$n_{xx} = v = w = m_{xx} = 0 \text{ at } x = 0 \text{ and } x = \ell \tag{5}$$

Such boundary condition will be referred to as "SS1/SS1". As is well known, system (4) with boundary conditions (5) admits an exact solution of the form:

$$\begin{aligned} u(x,\tau) &= u_{mo} \cos sx \sin \tilde{\omega}\tau \\ w(x,\tau) &= w_{mo} \sin sx \sin \tilde{\omega}\tau \end{aligned} \qquad (n = 0)$$

$$\begin{aligned} u(x,\theta,\tau) &= u_{mn} \cos sx \sin n\theta \;\; \sin \tilde{\omega}\tau \\ v(x,\theta,\tau) &= v_{mn} \sin sx \cos n\theta \;\; \sin \tilde{\omega}\tau \qquad (n \geq 1) \\ w(x,\theta,\tau) &= w_{mn} \sin sx \sin n\theta \;\; \sin \tilde{\omega}\tau \end{aligned} \tag{6}$$

Upon substitution of (6) in (4), the following set of homogeneous algebraic equations is obtained:

$$\left[L_{ij}\right] \begin{Bmatrix} u_{mn} \\ v_{mn} \\ w_{mn} \end{Bmatrix} = 0 \qquad \begin{matrix} (i, j = 1, 3 \text{ for } n=0) \\ (i, j = 1, 2, 3 \text{ for } n \geq 1) \end{matrix} \tag{7}$$

$\left[L_{ij}\right]$ is a constant symmetric matrix given in Appendix Ic. System (7) has a nontrivial solution if and only if:

$$\det \left[L_{ij}\right] = 0 \tag{8}$$

3. THE RING

A. Asymptotic Expressions of $\tilde{\omega}$ and $\tilde{v}_n$ for Small (rn)

For the special case of the ring (or an infinite length shell), we put $u = \frac{\partial}{\partial x} = 0$ in (1), (2), and (3). This leads to an eigendeterminant of the form:

$$\det \left[L_{ij}\right] = 0 \qquad (i, j = 2, 3) \tag{9}$$

$\left[L_{ij}\right]$ in (9) is a symmetric matrix with the same coefficients as in (8) when "s" is taken equal to zero.

For fixed values of the geometric parameter "r", the solution of (9) leads to the following results for $n = 0, 1$.

For $n = 0$, $\tilde{\omega}^2 = 1 \rightarrow \omega = \omega_o$ ("w" motion)

For $n = 1$, $\tilde{\omega}^2 = 2 \rightarrow \omega = \sqrt{2}\, \omega_o$ ("v" motion)

For $n \geq 2$, two distinct eigenfrequencies are determined;

a. $\tilde{\omega}_{HF}$ (high frequency root, "v" predominant motion)

b. $\tilde{\omega}_{LF}$ (low frequency root, "w" predominant motion)

The expression for $\tilde{\omega}$ from equation (9) is as follows

$$\tilde{\omega}^2 = \frac{b \pm \sqrt{c+d}}{e} \tag{10}$$

where b, c, d, e (functions of r and n) are given in Appendix Id.

For thin rings ($r = 0(10^{-3})$) and for $n \leq 30$, the right hand side of (10) can be expanded in Taylor series in powers of "rn". This leads to the following expressions for the eigenfrequencies:

$$\tilde{\omega}^2_{LF} = \frac{(rn)^2(n^2-1)^2(1-(rn)^2)}{(n^2+1)} + 0(rn)^4$$

$$\tilde{\omega}^2_{HF} = (n^2+1)\left(1 + \frac{(rn)^2(7n^2-1)}{(n^2+1)^2}\right) + 0(rn)^4 \qquad (11)$$

The displacement ratio $\tilde{v}_n$ can be calculated from (9):

$$\frac{v_n}{w_n} = \tilde{v}_n = \frac{n(1+(rn)^2)}{\tilde{\omega}^2 - n^2(1+r^2)} , \qquad n \geq 0 \qquad (12)$$

Expanding this ratio in powers of "rn" gives the following results:

$$(\tilde{v}_n)_{LF} = -\frac{1}{n}\left(1 + \frac{2(rn)^2(n^2-1)}{n^2+1}\right) + 0(rn)^4$$

$$(\tilde{v}_n)_{HF} = n\left(1 - \frac{(rn)^2(n^2-3)}{n^2+1}\right) + 0(rn)^4 \qquad (13)$$

B. <u>Error in $\tilde{\omega}$ and $\tilde{v}_n$ from Simplifications in the Equations</u>

The expressions for the frequency and displacement ratio will now be calculated when certain terms are neglected in the differential equations. The objective is to determine the errors in these quantities that result from these simplifications. The terms that will be neglected are shown in the table below.

Notation	Referring to
"v" in k_{ij}	"v" terms in curvature
q_θ	Transverse shear
$\frac{\partial^2 u}{\partial \tau^2}, \frac{\partial^2 v}{\partial \tau^2}$	Tangential inertia
$r^2 \frac{\partial^2}{\partial \tau^2} (\nabla^2 w)$	Rotary inertia

Neglection of the first two terms in the table, are the simplifications that lead to the Donnell equations for a cylindrical shell. The third term is the inplane inertia which is commonly neglected in dynamic shell analysis. The last term, the rotary inertia, is neglected in most analyses and will also be shown here not to contribute for small wave numbers.

The complete results from a systematic study of these effects are given in Table I and II. The conclusions are summarized in the next section.

C. Conclusions

1. The high frequency root and its corresponding displacement ratio are unaffected by all approximations except tangential inertia which when neglected, will delete this frequency.

2. The error in the low frequency from all approximations except rotary inertia, is of order (n^{-2}) for relatively large "n". It can thus be considered a high wave number or a high frequency approximation. The per cent error in the ring eigenfrequencies is shown in Figure 2.

3. $(\tilde{v}_n)_{LF}$ is nearly unaffected by all approximations; the error involved is of the order $(rn)^4$. Consequently, we conclude that the Donnell type equations are inaccurate when calculating the low frequencies for a ring at low circumferential wave number.

4. THE CYLINDRICAL SHELL WITH BOUNDARY CONDITION SS1/SS1

The effect of the various approximations on the eigenfrequencies of a cylindrical shell with boundary conditions SS1/SS1 will now be discussed in detail. The governing differential equations are given in matrix form by equation (4). The eigenvalue problem for free vibration reduces to the set of homogeneous algebraic equations given by (7). The eigenmatrix $\left[L_{ij}\right]$ can be written in the form:

$$\left[L_{ij}\right] = \left[c_{ij} - \epsilon_{ij} - \tilde{\omega}^2 \delta_{ij}\right] - \left[I\right]\begin{Bmatrix} 0 \\ 0 \\ \bar{\epsilon}_{33} \end{Bmatrix} \tilde{\omega}^2 \qquad (14)$$

$\left[I\right]$ is the unit matrix. $\left[c_{ij}\right]$, $\left[\epsilon_{ij}\right]$ and $\bar{\epsilon}_{33}$ are given in Appendix Ic. ϵ_{ij} represent corrections to c_{ij} when allowing for "v in k_{ij} and q_θ". $\bar{\epsilon}_{33}$ is the correction in $\tilde{\omega}^2$ when allowing for rotary inertia. For a nontrivial solution of system (7), the determinant of $\left[L_{ij}\right]$ must vanish. Since $\left[L_{ij}\right]$ is a real symmetric matrix, its eigenvalues are all real, and the eigenvectors corresponding to distinct eigenvalues, are orthogonal. Thus, for each combination of $n > 0$ and m, i.e., for each mode shape, there exist three distinct eigenfrequencies:

1. $\tilde{\omega}_{LF}$ (Low frequency, "w" predominant mode)
2. $\tilde{\omega}_{IF}$ (Intermediate frequency, "u" predominant mode)
3. $\tilde{\omega}_{HF}$ (High frequency, "v" predominant mode).

A. Low Frequency Root

Asymptotic expressions for the error in $\tilde{\omega}_{LF}$ that results from the different approximations, will be derived in this section. It is shown in Appendix II that the error in $\tilde{\omega}_{LF}$ caused by the neglect of "v in k_{ij} and q_θ" does not depend on whether or not tangential and rotary inertias are included, and vice versa. As a consequence, it is also shown in the same Appendix that the error in the low frequency root, resulting from all approximations, is nearly equal to the sum of the errors resulting from each approximation when performed separately. Using the subscripted variables as in Table III, we can express the above statement in symbolic form as follows:

$$\begin{aligned} e_t &\simeq e_1^* + e_2^* + e_3^* \\ &\simeq e_1 + e_2 + e_3 \end{aligned} \tag{15}$$

Thus for simplicity, two of the effects will be neglected when discussing the third effect.

i. Error in $\widetilde{\omega}_{LF}$ by Neglecting "v in k_{ij} and q_θ" and its Relation to e_{ring}

From Appendix II equation (5):

$$e_1^* \simeq e_1 \simeq \frac{\mathcal{E}}{2C_o} = \frac{r^2}{2}\left[\frac{(2s^2+n^2)^2(2n^2-1)-2\nu^2 s^4(n^2-2)}{r^2(s^2+n^2)^4+(1-\nu^2)s^4}\right] \quad (16)$$

In the limit when $(s/\sqrt{rn}\,4) \rightarrow 0$:

$$e_1 \simeq \frac{n^2}{n^2-1} - 1 = e_{ring} \quad (17)$$

If $n^2 \gg s^2$ such that $\frac{\sqrt{(1-\nu^2)}s^2}{r} = 0(n^4)$, then:

$$e_1 \simeq \frac{n^6}{n^8 + \frac{(1-\nu^2)s^4}{r^2}} \quad (18)$$

e_1 as given by (18), has a maximum at $n = \widetilde{n}$ where:

$$\widetilde{n}^8 = \frac{3(1-\nu^2)s^4}{r^2}$$

$$(e_1)_{max} \simeq \frac{3}{4\widetilde{n}^2} = \frac{3^{3/4}\ r^{1/2}}{4(1-\nu^2)^{1/4}s} \quad (19)$$

This maximum error in frequency calculated in the above equations does not necessarily occur at low circumferential wave numbers as has been found for static problems. In fact, it will be shown below that this maximum error occurs very near to the minimum frequency for the shell. In addition, this maximum error can be related to the error for an infinitely long shell or a ring, at a frequency equal to the minimum shell frequency. Let "$\hat{n}$" be the value of "n" at which $\tilde{\omega} = (\tilde{\omega})_{min}$, then from equation (3) of Appendix III:

$$\hat{n}^2 \simeq \frac{(1-\nu^2)^{1/4} s}{r^{1/2}} \tag{20}$$

From (19) and (20) and equation (5) of Appendix III we obtain:

$$(e_1)_{max} \simeq \frac{3^{3/4}}{4\hat{n}^2} \simeq \left(\frac{3^{1/2}}{2}\right)^{\frac{3}{2}} \frac{r}{\tilde{\omega}_{min}} = 0.805 \ \frac{r}{\tilde{\omega}_{min}} \tag{21}$$

But $e_{ring} \simeq \frac{r}{\tilde{\omega}}$ for $n > 2$, thus:

$$(e_1)_{max} \simeq 0.805 \quad e_{ring} \tag{22}$$

where e_{ring} is evaluated at $\frac{\tilde{\omega}}{r} = \frac{\tilde{\omega}_{min}}{r}$. A master chart for the maximum error derived above is shown in Figure 3. A summary of the above results is given in Table IV.

In further studies to be carried out in this work, it is convenient to neglect "v in k_{ij}" and still study consistently the effect of neglecting both "v in k_{ij}" and "q_θ". Such a simplification is justified by the following argument.

Since $\left[L_{ij}\right]$ in (7) is symmetric, then it follows that the effect of neglecting "v in k_{ij}" only, is exactly equivalent to the effect of neglecting "q_θ" only. Also since $e_t \sim e_1 + e_2 + e_3$ as shown in Appendix II, then neglecting any of the above terms will contribute to nearly half the total error from both effects.

ii. Error in $\tilde{\omega}_{LF}$ by Neglecting Tangential and Rotary Inertia

The first two differential equations in (4) can be uncoupled in "u" and "v" to give the following two fourth order differential equations (Appendix Ib):

$$\mathcal{L}(u) = \mathcal{E}_1(w)$$

$$\mathcal{L}(v) = \mathcal{E}_2(w) \tag{23}$$

$$\mathcal{L} = (\nabla^2 - \frac{\partial^2}{\partial\tau^2})(\nabla^2 - \frac{2}{1-\nu}\frac{\partial^2}{\partial\tau^2})$$

For free vibration, these operators take the form of equation (5b) of Appendix Ib. From the nature of the operator "$\mathcal{L}$", we note that the effect of tangential inertia is noticeable if:

$$0(\frac{2}{1-\nu}\tilde{\omega}^2) \sim 0(n^2) \sim 0(s^2) \tag{23a}$$

i.e., at low circumferential and axial wave numbers

$$n \leqq 3, \quad \frac{\ell}{m} \geq 2 \tag{23b}$$

For boundary condition SS1/SS1, the characteristic equation (dispersion relation) is given by:

$$C_3\,\tilde{\omega}_2^6 - C_2\tilde{\omega}_2^4 + C_1\tilde{\omega}_2^2 - C_o = 0 \qquad (n \geq 1) \tag{24}$$

$$\tilde{\omega}_2^4 - \overline{C}_1 \tilde{\omega}_2^2 + \overline{C}_o = 0 \qquad (n = 0)$$

$$\overline{C}_1 = 1 + s^2 + r^2 s^4 \tag{25}$$

$$\overline{C}_o = s^2(1 - \nu^2 + r^2 s^4)$$

C_i (i = 1, 2, 3, 0) are given in Appendix II equation (2). Asymptotic expressions for $\tilde{\omega}_2$ and e_2 are now derived for small and large values of "n". For n = 0, $\tilde{\omega}_2$ is obtained by solving (25). For n = 1, 2, $\frac{\partial^2 u}{\partial \tau^2}$ can be neglected as compared to $\frac{\partial^2 v}{\partial \tau^2}$, thus (24) reduces to:

$$\tilde{C}_2 \tilde{\omega}_2^4 - \tilde{C}_1 \tilde{\omega}_2^2 + \tilde{C}_o = 0$$

$$\tilde{C}_2 = n^2 + \frac{2s^2}{1 - \nu}$$

$$\tilde{C}_1 = (s^2 + n^2)^2 + 2(1 + \nu)s^2 + n^2 + \tilde{C}_2 r^2 (s^2+n^2)^2 \tag{26}$$

$$\tilde{C}_o = (1 - \nu^2)s^4 + r^2(s^2+n^2)^4$$

For $n \geq 3$, we have from Appendix II equation (4):

$$\tilde{\omega}_2^2 \simeq \frac{C_1 - \sqrt{C_1^2 - 4C_o C_2}}{2C_2} \tag{27}$$

Since $\left(\frac{4C_o C_2}{C_1^2}\right) \ll 0(1)$, (27) can be expanded to $0\left(\frac{4C_o C_2}{C_1^2}\right)$ in powers of the same quantity. The results are summarized in Table V. The per cent error in a cylindrical shell with boundary condition SS1/SS1 by neglecting tangential inertia is shown in Figure 4.

The error in $\tilde{\omega}_{LF}$ by neglecting rotary inertia is given by:

$$e_3 \simeq \frac{r^2(s^2 + n^2)}{2} \tag{28}$$

e_3 is negligible in the range of m, n and r for which this work is valid, i.e., $0 \leq n \leq 30$, $1 \leq m \leq 6$, $r < 10^{-2}$.

B. Intermediate and High Frequency Roots

In the characteristic equation (24), since $\tilde{\omega}^2_{IF}$ and $\tilde{\omega}^2_{HF}$ are much greater than order unity and, since

$$C_o = (s^2 + n^2)^2 \; \tilde{\omega}_4^2 \left(\frac{1-\nu}{2}\right)$$

$$\tilde{\omega}_4^2 = r^2(s^2 + n^2)^2 + \frac{(1-\nu^2)s^4}{(s^2 + n^2)^2} \ll 0(1) \tag{29}$$

thus C_o can be neglected as compared to the remaining terms in (24) which are of $0(s^2 + n^2)^2$. The characteristic equation then reduces to:

$$C_3 \tilde{\omega}^4 - C_2 \tilde{\omega}^2 + C_1 = 0 \tag{30}$$

Asymptotic expressions for $\tilde{\omega}_{IF}$ and $\tilde{\omega}_{HF}$ can be obtained by solving for $\tilde{\omega}$ in (25) and (30). The results are tabulated in Table VI.

C. Displacement Ratios

The displacement ratios $\tilde{u}_{mn}$ and $\tilde{v}_{mn}$ defined by:

$$\tilde{u}_{mn} = \frac{u_{mn}}{w_{mn}} , \qquad \tilde{v}_{mn} = \frac{v_{mn}}{w_{mn}}$$

are obtained by solving for u and v from the first two equations in (7). The resulting expressions are tabulated in Table VII.

D. Conclusions

1. The upper bound of the error committed in the eigen-frequencies of a cylindrical shell with boundary conditions SS1/SS1 by neglecting "v" terms in curvature and transverse shear, was related to the simple expression for the error in the ring, derived previously

$$(e_1)_{max} \simeq 0.805 \quad e_{ring}$$

The maximum error occurs nearly at the minimum frequency, showing that the neglect of the above terms is actually a high frequency approximation. Asymptotically when $\tilde{\omega} \gg \tilde{\omega}_{min}$, the error e_1 approaches the error in the ring $e_{ring} \simeq \frac{1}{n^2} \simeq (\frac{r}{\tilde{\omega}})$. From equation (19):

$$(e_1)_{max} \simeq \frac{3^{3/4}\, r^{1/2}}{4(1-\nu^2)^{1/4}\, s} , \qquad s = \frac{m\pi}{\ell} \qquad r = \frac{h}{\sqrt{12}a}$$

it is clear that the maximum error is small for thin and short shells. In fact the maximum error is less than 6 per cent for an SS1/SS1 shell with $\frac{\ell}{m} = 6$ and $\frac{a}{h} = 144$. Consequently, transverse shear and inplane displacements in curvature can be consistently neglected for finite length thin shells with boundary condition SS1/SS1.

2. Tangential inertia has a noticeable effect on the low eigen-frequencies $\tilde{\omega}_{LF}$ at low values of circumferential and axial wave numbers, i.e., $n \leq 3$ and $\frac{\ell}{m} \geq 2$. The error e_2 increases with $\frac{\ell}{m}$ i.e., increases with length for fixed values of the axial wave number m; but it is nearly independent of the thickness parameter $\frac{h}{a}$. For large values of "n", the error e_1 is asymptotic to the error in the ring $e_{ring} \simeq \frac{1}{2n^2}$.

3. $\tilde{u}_{mn}$ and $\tilde{v}_{mn}$ corresponding to $\tilde{\omega}_{LF}$ are unaffected by all approximations, except tangential inertia for $n < 5$ and $\frac{\ell}{m} > 1$.

4. $\tilde{\omega}_{IF}$ and $\tilde{\omega}_{HF}$ and their corresponding displacement ratios are unaffected by all approximations except tangential inertia, which when neglected, will delete these frequencies.

II. EFFECT OF BOUNDARY CONDITIONS ON THE EIGEN-FREQUENCIES OF FINITE LENGTH CYLINDRICAL SHELLS

1. INTRODUCTION

The analysis of linear and nonlinear response of shells of revolution due to external loading, has been studied previously without much consideration of the boundary conditions (Ref. 17). Most of the work was carried out for infinitely long shells thus deleting the boundary effect, or by assuming the classical simply supported boundary condition (SS1/SS1) at both ends. It is thus of interest to find how close these approximate calculations are to some exact analysis which includes the effect of more realistic boundary conditions.

Before going into the complicated problem of the response, the boundary condition effect is at first studied on the small amplitude low eigenfrequencies of a cylindrical shell. This gives an estimate of the effect of boundary conditions on the linear response problem, since the low eigenfrequencies enter in the evaluation of the generalized coordinates, when an eigenfunction expansion type solution is assumed. Previous work in this subject was done by Arnold and Warburton (Refs. 18 and 19), and by Weingarten (Ref. 20), but their analysis was approximate. Forsberg (Refs. 21 and 22) used the same technique as in this work, to obtain an exact solution of the equations of motion. He considered the effect of boundary conditions on the low frequency spectrum envelope (i.e., the minimum frequency for all modes with one axial half wave) without giving insight to the effect for other modes, and the range of circumferential and axial wave numbers for which this effect is noticeable.

Unfortunately, the exact solution is convenient only for the special case of a cylindrical shell, since the governing differential equations have constant coefficients. Also, a closed form relation for the error in the eigenfrequencies by neglecting "v" terms in curvature, transverse shear and tangential inertia cannot be obtained from the exact solution for boundary conditions other than SS1/SS1. Consequently, an approximate method is suggested, which takes into consideration the inplane boundary conditions that proved to be most influential. Based on such an approximate solution, an accurate estimate of the error in the low frequencies by neglecting inplane displacements in curvature, transverse shear and tangential inertia is obtained for boundary conditions other than SS1/SS1.

Finally, a more practical boundary condition of a shell with an elastic ring at one end and with complete fixity at the other, is considered. The effect of the ring rigidities on the eigenfrequencies of the ring-shell system is studied.

2. GOVERNING EQUATIONS AND EXACT SOLUTION FOR ANY BOUNDARY CONDITION

A. Method of Comparison

The classical simply supported boundary condition (SS1/SS1), as defined in equation (5), will be taken as a basis for comparison. For some boundary condition and for a particular mode shape, i.e., for some "m" and "n", the boundary effect will be measured by looking at the difference between the eigenfrequency with such a boundary condition and the corresponding eigenfrequency of SS1/SS1. A "difference" is now defined as follows:

$$d = \frac{\tilde{\omega}}{\tilde{\omega}_{SS1}} - 1 \quad \text{(for the same m and n)}$$

B. Equilibrium Equations

Based on the results of Part I, the following terms will be neglected:

1. "v" terms in curvature and transverse shear
2. Rotary inertia

However, tangential inertia will be included since the error caused by neglecting it is large at low circumferential wave number ($n \leq 3$) and for relatively long shells ($\frac{\ell}{m} \geq 3$). For free vibration, the resulting uncoupled system of equilibrium equations in terms of displacements is as follows:

$$\begin{aligned} \mathscr{L}_o(u_o) &= \frac{dw_o}{dx} \\ & \qquad\qquad (n = 0) \\ \mathscr{R}_o(w_o) &= 0 \end{aligned} \tag{31}$$

$$\begin{aligned} \mathscr{L}_n(u_n) &= \tilde{\mathscr{E}}_{1_n}(w_n) \\ \mathscr{L}_n(v_n) &= \tilde{\mathscr{E}}_{2_n}(w_n) \qquad (n \geq 1) \\ \mathscr{R}_n(w_n) &= -\tilde{\omega}^2 \mathscr{H}_n(w_n), \quad x \ni \left[-\frac{\ell}{2}, \frac{\ell}{2}\right] \end{aligned} \tag{32}$$

The operators in the above differential equations are linear and ordinary with constant coefficients (see equation (5d) of Appendix Ib) where $o(r^2)$ was neglected in $\mathscr{C}_{1_n}$ and $\mathscr{C}_{2_n}$. Since the above operators are self-adjoint, the solution of the eigenvalue problem in (31) and (32) leads to two and three real eigenvalues respectively for each particular eigenfunction. This is similar to the special case of SS1/SS1.

C. Exact Solution of the Differential Equations

The last of equations (31) and (32) admits an exact solution of the form:

$$w_n(x) = \sum_{j=1}^{j^*} A_{nj} \; e^{\lambda_{nj} x} \qquad \begin{matrix} (j^* = 6 \text{ for } n = 0) \\ (j^* = 8 \text{ for } n \geq 1) \end{matrix} \qquad (33)$$

Upon substitution of (33) in (31) and (32), we obtain the following "characteristic equations" (dispersion relations):

$$C_o(1) \;\; \lambda_{oj}^6 + C_o(2)\; \lambda_{oj}^4 + C_o(3)\; \lambda_{oj}^2 + C_o(4) = 0 \qquad (n=0)$$

$$C_n(1) \;\; \lambda_{nj}^8 + C_n(2)\; \lambda_{nj}^6 + C_n(3)\; \lambda_{nj}^4 + C_n(4)\; \lambda_{nj}^2 + C_n(5) = 0 \qquad (n \geq 1) \qquad (34)$$

The coefficients $C_o(i)$ and $C_n(i)$ (function of $n, r, \tilde{\omega}, \nu$) are given in Appendix IV. The type of roots of (34) changes with $\tilde{\omega}$ and n. This is shown in Table VIII. The solution of (31) and (32) can also be written in the following real form:

$$w_n(x) = \sum_{j=1}^{j^*} B_{nj} F_{nj}(x)$$

$$u_n(x) = \mathcal{L}_n^{-1} \left[\tilde{\mathcal{E}}_{1n}(w_n) \right] = \sum_{j=1}^{j^*} B_{nj} G_{nj}(x) \tag{35}$$

$$v_n(x) = \mathcal{L}_n^{-1} \left[\tilde{\mathcal{E}}_{2n}(w_n) \right] = \sum_{j=1}^{j^*} B_{nj} H_{nj}(x)$$

$$(j^* = 6 \text{ for } n = 0 \text{ and } j^* = 8 \text{ for } n \geq 1)$$

$F_{nj}(x)$, $G_{nj}(x)$ and $H_{nj}(x)$ are real independent functions of "x" (combinations of trigonometric and hyperbolic functions) which change form with the type of roots of (34), and B_{nj} are real independent constants of integration. Substituting the expressions given by (35) in any of the homogeneous boundary conditions of Table IX, leads to a system of homogeneous simultaneous equations in B_{nj}:

$$\left[\mathcal{F}_{nij} \right] \left\{ B_{nj} \right\} = 0 \qquad \begin{matrix} (i, j = 1 \text{ to } 6, \; n = 0) \\ (i, j = 1 \text{ to } 8, \; n \geq 1) \end{matrix} \tag{36}$$

System (36) has a nontrivial solution if and only if:

$$\det \left[\mathcal{F}_{nij} \right] = 0 \tag{37}$$

All boundary conditions, except SS1/SS1, lead to an eigendeterminant (37) which depends on both geometrical parameters $\frac{\ell^*}{a}$ and $\frac{a}{h}$, and which is singular only for values of $\tilde{\omega} = (\tilde{\omega})_{resonance}$. For the special case of boundary condition SS1/SS1, the eigendeterminant (37) can be expressed as a product:

$$\det \left[\mathcal{F}_{nij} \right] = F_n^*(\frac{\ell}{2}) \cdot \det \left[\mathcal{D}_{ij} \right]$$

$F_n^*(\frac{\ell}{2})$ is formed of the independent functions $F_{nj}(x)$ given in (35), evaluated at $x = \frac{\ell}{2}$; and $\det \left[\mathcal{D}_{ij} \right]$ is independent of $\frac{\ell}{a}^*$. The dependence of $\tilde{\omega}$ on $\frac{\ell}{a}^*$ implies that $\det \left[\mathcal{D}_{ij} \right]$ cannot vanish at $\tilde{\omega} = (\tilde{\omega})_{resonance}$. As a consequence, $F_n^*(\frac{\ell}{2})$ is equal to zero. This gives rise to the well known one term trigonometric solution for boundary condition SS1/SS1. Details of the proof, are given in Appendix V.

The following boundary effects were studied:

1. Axial displacement restraint (SS2/SS2)
2. Circumferential displacement restraint (SS3/SS3)
3. Slope restraint (FX1/FX1)
4. Combined effect of slope and axial restraint (i.e., total fixity) (FX2/FX2)
5. One free boundary (the cantilever modes) (FX2/FR)

The symbols used in the above boundary conditions are given in detail in Table IX.

The computations of the exact low, intermediate and high frequencies and their corresponding eigenvectors and stress field, were performed on an IBM 360-75 machine. Figure 5 and 6 show the low frequency spectrum and corresponding per cent difference (^{o}/o d) for a shell with boundary condition SS2/SS2, where

$$^o/od = 100 \left(\frac{\tilde{\omega}\ \text{boundary condition}}{\tilde{\omega}_{SS1}} - 1 \right)$$

We note that the maximum ^{o}/od occurs in the neighborhood of the minimum frequency. The range of circumferential wave number "n" in which the boundary condition is effective, becomes wider the larger the axial wave number "m", although the $^o/od_{max}$ decreases with "m". Figures 7 and 8 show the low frequency spectrum and corresponding ^{o}/od for a shell with boundary condition FX2/FR. In this case we note that the boundary condition is effective even at n = 0 and 1 for the cantilever mode (m = 0). This is in contrast to boundary condition SS2/SS2. In both cases, the ^{o}/od decreases with "m" for small "n" and the effect is reversed for large "n". The steps necessary to find the eigenvalues and eigenfunctions are as follows. First the frequency is assumed. The characteristic equation (34) is solved numerically to find the eight roots (for $n \geq 1$) or six roots (for n = 0). Depending on the type of the roots (see Table VIII), the form of the solution is determined. Next the coefficients of the eigendeterminant are calculated and its value found. The frequency is then increased by a predetermined amount and the procedure is repeated to find the new value of the eigendeterminant. A change of sign of the eigendeterminant indicates that an eigenfrequency lies between the two assumed frequencies. An iterative procedure is then used to obtain the exact eigenfrequency to any degree of accuracy (taken as 10^{-8}). Once the eigenfrequency has been determined, the eigenvector is found by solving a set of linear simultaneous equations.

D. Conclusions

1. The inplane boundary conditions, i.e. n_{xx}, $n_{x\theta}$, u and v, have an appreciable effect on the low eigenfrequencies in the neighborhood of their minimum. The difference "d" as defined in Part II, Section 2A, is large even for axial wave numbers greater than one. The range of circumferential wave number "n" for which the inplane boundary conditions are influential decreases with the increase of $\frac{\ell^*}{a}$ and $\frac{h}{a}$, although the maximum difference "d" increases with the increase of $\frac{\ell^*}{a}$ and $\frac{a}{h}$.

2. Out of plane boundary conditions, i.e. q_{xx}, m_{xx}, w and w', are effective only for thick and short shells, $(r > 0(10^{-2}), \frac{\ell}{m} < 1)$.

3. Inplane boundary conditions have little influence on the intermediate and high eigenfrequencies.

4. Boundary conditions with axial displacement restraint at both ends, have the effect of deleting the mode with one axial half wave in the intermediate frequencies for all "n", as well as the low frequencies for axisymmetric modes with n = 0.

3. APPROXIMATE SOLUTION OF THE DIFFERENTIAL EQUATIONS WITH CONSIDERATION OF THE INPLANE BOUNDARY CONDITIONS

The exact solution of the eigenvalue problem as described in Section 2C of this part, necessitates the use of a digitial computer to calculate the roots of the characteristic equation (34), since these roots are complicated functions of the eigenfrequencies. Consequently, a "closed-form" relation for the error in the eigenfrequencies caused by neglecting "v in k_{ij}" and "q_θ" or tangential inertia, is rather impossible to obtain when considering the exact solution, except for the special case of boundary condition SS1/SS1 as shown in Part I. An alternative is thus to establish the general trends using a different kind of solution that retains the accuracy of the exact solution.

The method described in this section applies only to geometries for which uncoupling of the inplane displacement equations in "u" and "v" is possible. For simplicity, "v" terms will be neglected in curvature. For free vibration, the governing sets of ordinary differential equations are:

$$\left.\begin{aligned} \mathcal{L}_o\left[u_o(x)\right] &= \mathcal{C}_{1_o}\left[w_o(x)\right] = \nu\,\frac{dw_o(x)}{dx} \\ \left[r^2\,\overline{\nabla}_o^{\,4} + 1 - \tilde{\omega}^2\right] w_o(x) &= \nu\,\frac{du_o(x)}{dx} \end{aligned}\right\} \quad (n=o) \qquad (38)$$

$$\mathcal{L}_n \left[u_n(x)\right] = \mathcal{C}_{1n} \left[w_n(x)\right]$$

$$\mathcal{L}_n \left[v_n(x)\right] = \mathcal{C}_{2n} \left[w_n(x)\right] \qquad (n \geq 1) \tag{39}$$

$$\left[r^2 \nabla_n^4 + 1 - \tilde{\omega}^2\right] w_n(x) = \nu \frac{du_n(x)}{dx} - nv_n(x)\, , \quad x \ni \left[o, \ell\right]$$

Above operators are given by equation (5d) of Appendix Ib. If the right hand side of the inplane equations in (38) and (39) is considered as a known forcing function, then the exact solution of these inplane equations can be obtained by the method of "Variation of Parameters". This solution has the form:

$$u_o(x) = \sum_{j=1}^{2} B_{oj} G_{oj}(x) + \int^x \mathcal{C}_{1_o} \left[w_o(\xi)\right] H_o(x-\xi)\, d\xi \qquad (n=0) \tag{40a}$$

$$\left.\begin{aligned} u_n(x) &= \sum_{j=1}^{4} B_{nj} G_{nj}(x) + \int^x \mathcal{C}_{1n} \left[w_n(\xi)\right] H_n(x-\xi)\, d\xi \\ v_n(x) &= \sum_{j=1}^{4} \bar{B}_{nj} G_{nj}(x) + \int^x \mathcal{C}_{2n} \left[w_n(\xi)\right] H_n(x-\xi)\, d\xi \end{aligned}\right\} \qquad (n \geq 1) \tag{40b}$$

B_{nj}, $\bar{B}_{nj}$ are independent constants of integration; $G_{nj}(x)$ are known independent complementary solutions, and $H_n(x)$ is a known kernel whose form depends on $G_{nj}(x)$. $\bar{B}_{nj}$ can be expressed in terms of B_{nj} (for $n \geq 1$) by substituting the complementary function of (40b) in the homogeneous part of either of the inplane equilibrium equations before uncoupling.

Let us choose the first equilibrium equation:

$$\frac{n(1+\nu)}{2} \quad \frac{dv_n(x)}{dx} = \left(\frac{d^2}{dx^2} - \frac{(1-\nu)}{2}\beta_2^2\right)u_n(x), \; (\beta_2^2 = n^2 - \frac{2}{1-\nu}\tilde{\omega}^2) \tag{41}$$

We now have sufficient constants (B_{nj}) which can be used to satisfy exactly all inplane boundary conditions. Eliminating $u_n(x)$ and $v_n(x)$ from the last of equations (38) and (39), by the use of (40a) and (40b) respectively, we obtain an integro-differential equation in the unknown "$w_n(x)$":

$$(r^2\bar{\nabla}_n^4 + 1 - \tilde{\omega}^2)\, w_n(x) + K(w_n(x)) = 0 \qquad (n \geq 0) \tag{42}$$

Equation (42) is exact and the previous steps in its derivation do not involve any approximation. Finally, $w_n(x)$ is expressed in the form of a complete orthogonal set, chosen preferably so as to satisfy all out-of-plane boundary conditions:

$$w_n(x) = \sum_{j=1}^{j^*} a_{nj} \; W_{nj}(x) \; (j^* \text{ finite}) \tag{43}$$

The generalized coordinates "a_{nj}" are determined by applying an averaging method (Galerkin method) on (42). This leads to a homogeneous set of algebraic equations in "a_{nj}":

$$\left[\tilde{\mathcal{F}}_{nij}\right]\left\{a_{nj}\right\} = 0 \qquad (i, j = 1, 2, \ldots, j^*) \tag{44a}$$

which has a nontrivial solution if and only if:

$$\det\left[\widetilde{\mathcal{F}}_{nij}\right] = 0 \tag{44b}$$

Details of the above approximate method, when applied to boundary condition SS2/SS2, are given in Appendix VI.

The low eigenfrequencies for a shell with $\frac{\ell^*}{a} = 3$ and $r = 5 \cdot 10^{-4}$, were computed with the approximate method for boundary condition SS2/SS2 and compared with the exact solution in Table X. The error in the approximate frequencies was always less than 0.1 °/o when an eight term series was taken.

4. EFFECT OF APPROXIMATIONS ON $\widetilde{\omega}_{LF}$ FOR BOUNDARY CONDITIONS OTHER THAN SS1/SS1

A. Error in $\widetilde{\omega}_{LF}$ by neglecting "v in k_{ij}" and "q_θ" for Boundary Conditions other than SS1/SS1

The approximate solution developed in the previous section will now be used to determine the error in the low eigenfrequency resulting from the neglect of "v in k_{ij}" and "q_θ" in the governing equations. This error can be studied consistently by looking at the effect of "q_θ" only, since it has been shown previously in this work (Part I, Section 4A)that the error committed when neglecting both "v in k_{ij}" and "q_θ" is nearly twice the error resulting from the neglect of either "v in k_{ij}" or "q_θ". Furthermore, tangential inertia will be neglected, based on the results of Appendix II.

Consider the boundary condition SS2/SS2. The governing differential equations, solutions and corresponding eigenmatrix are given in Appendix VI. First, we make the following change of variables:

$$\hat{\omega}^2 = \tilde{\omega}^2 + r^2 n^2 \tag{45}$$

The eigenmatrix $\left[\tilde{\mathcal{F}}_{nij}\right]$ can then be written in the following form:

$$\left[\tilde{\mathcal{F}}_{nij}\right] = \left[M_{ij} + \varepsilon \tilde{M}_{ij} - \hat{\omega}^2 \ \delta_{ij}\right] \tag{46a}$$

$$M = \left[M_{ij}\right] = \left[\tilde{\omega}_{i4}^2 \ \delta_{ij} + P_{ij} \ s_j (n^2 - \nu s_j^2)\right]$$

$$\varepsilon = r^2 n^2 \ll 0(1)$$

$$\tilde{M} = \left[\tilde{M}_{ij}\right] = \left[-\frac{(1+\nu)s_i^2}{(s_i^2 + n^2)} \ \delta_{ij} + \frac{1+\nu}{1-\nu} \ s_j \ (n^2 + s_j^2) \ P_{ij}\right] \tag{46b}$$

$$\delta_{ij} = 1 \quad \text{for } i = j, \qquad \delta_{ij} = o \quad \text{for } i \neq j$$

P_{ij}, s_j and $\tilde{\omega}_{i4}^2$ are given in Appendix VI, equation (20b). For boundary condition SS1/SS1, the maximum error $(e_1)_{max}$ from the above approximations occurs at $n = \tilde{n}$ as given by equation (19):

$$\tilde{n} = (3(1-\nu^2))^{1/8} \frac{s^{1/2}}{r^{1/4}}$$

$$\tilde{n}\ell = (3(1-\nu^2))^{1/8} \frac{(m\pi\ell)^{1/2}}{r^{1/4}} \geq \frac{2\ell^{1/2}}{r^{1/4}}$$

$$(\sim 24 \text{ for } \ell = 3,\ r = 5.10^{-4})$$

It is expected that "$\tilde{n}\ell$" will not change its order of magnitude from the above estimate for boundary condition SS2/SS2. Therefore, P_{ij} can be expanded in powers of $(\frac{1}{n\ell})$ for large $(n\ell)$. This leads to:

$$\frac{1+\nu}{1-\nu} s_j(n^2+s_j^2)\, P_{ij} \simeq \frac{4\, s_i\, s_j\, n}{\ell(s_i^2+n^2)(s_j^2+n^2)} \tag{47a}$$

$\left[\tilde{M}_{ij}\right]$ then becomes:

$$\left[\tilde{M}_{ij}\right] \simeq \left[-\frac{(1+\nu)s_i^2}{(s_i^2+n^2)}\, \delta_{ij} + \frac{4\, s_i\, s_j\, n}{\ell(s_i^2+n^2)(s_j^2+n^2)} \right]$$

$$\simeq \left[-\frac{(1+\nu)s_j^2}{(s_j^2+n^2)} \left(\delta_{ij} + 0(\frac{3}{n\ell}) \right) \right] \tag{47b}$$

i.e., the nondiagonal terms in $\left[\tilde{M}_{ij}\right]$ can be neglected to a first approximation. The eigenvalues $\hat{\omega}^2$ corresponding to the eigen-matrix (46a), can be expressed in the form of an asymptotic expansion as follows:

$$(\hat{\omega}^{(k)})^2 = (\hat{\omega}_1^{(k)})^2 + \varepsilon\,(\hat{\omega}_2^{(k)})^2 + o(\varepsilon^2) \qquad (48a)$$

where

$$(\hat{\omega}_2^{(k)})^2 = (\bar{B}_1^{(k)},\ \widetilde{M}\bar{B}_1^{(k)}) \quad \text{(inner product)} \qquad (48b)$$

$\bar{B}_1^{(k)}$ is the eigenvector corresponding to the kth eigenvalue, evaluated from the solution to 0(1), i.e., from the solution of

$$\left[M - (\hat{\omega}_1^{(k)})^2 \delta \right] \bar{B}_1^{(k)} = 0 \qquad (48c)$$

and $(\hat{\omega}_1^{(k)})^2$ is the corresponding eigenvalue. $\hat{\omega}_1^{(k)}$ represents the approximate eigenfrequency when "v in k_{ij}" and "q_θ" are neglected. Let

$$\bar{B}_1^{(1)} = \left\{ B_{1,i}^{(1)} \right\}$$

For the first mode with one axial half wave (m = 1),

$$B_{1,1}^{(1)} > B_{1,i}^{(1)} > B_{1,(i+1)}^{(1)} \qquad \text{for all "i"} \qquad (49)$$

(49) was verified from calculations. Substituting (47b) in (48b):

$$(\hat{\omega}_2^{(1)})^2 = \widetilde{M}_{11} + (B_{1,2}^{(1)})^2\ \widetilde{M}_{22} + \ldots \quad ([\widetilde{M}] \text{ is diagonal}) \qquad (50a)$$

From (47b) we note that

$$\widetilde{M}_{ii} \rightarrow o \text{ as } s_i \rightarrow o \text{ and } \widetilde{M}_{ii} \rightarrow (1+\nu) \text{ as } s_i \rightarrow \infty$$
$$\text{where } s_i = \frac{i\pi}{\ell} \qquad (50b)$$

hence,

$$(B^{(1)}_{1,2})^2 \quad \widetilde{M}_{22} < (B^{(1)}_{1,2})^2 \,(1+\nu) < \widetilde{M}_{11} \tag{50b}$$

Using (50b), (50a) simplifies to the following:

$$(\hat{\omega}^{(1)}_{2})^2 \simeq \widetilde{M}_{11} \,(1 + o(B^{(1)}_{1,2})^2) \tag{50c}$$

Substituting (50c) and (48a) in (45) we obtain:

$$\widetilde{\omega}^2 \simeq -(rn)^2 \left(1 + \frac{(1+\nu)s_i^2}{(s_i^2 + n^2)}\right) + (\hat{\omega}^{(1)}_{1})^2$$

$$\Delta(\widetilde{\omega}^2)_{ss2} = ((\hat{\omega}^{(1)}_{1})^2 - \widetilde{\omega}^2) \simeq (rn)^2 \cdot \left(1 + \frac{(1+\nu)s_i^2}{(s_i^2 + n^2)}\right) = \hat{\mathcal{E}} \tag{51}$$

From Appendix II, equation (10), we have:

$$\Delta(\widetilde{\omega}^2)_{ss1} \simeq \hat{\mathcal{E}} \quad \text{(as defined in (51))} \tag{51a}$$

From (51) and (51a) we have that:

$$\Delta(\widetilde{\omega}^2)_{ss2} \simeq \Delta(\widetilde{\omega}^2)_{ss1}$$

i.e.,

$$\frac{\Delta(\widetilde{\omega})_{ss2}}{\widetilde{\omega}_{ss2}} \simeq \frac{\Delta(\widetilde{\omega})_{ss1}}{\widetilde{\omega}_{ss1}} \left(\frac{\widetilde{\omega}_{ss1}}{\widetilde{\omega}_{ss2}}\right)^2 . \tag{52a}$$

Using the definition of the "error" "e", (52a) can be written as:

$$(e_1)_{ss2} \simeq \frac{(e_1)_{ss1}}{(1+d)^2} \quad \text{(for some m and n)} \tag{52b}$$

where "d" is the "difference" as defined in Part II, Section 2A

$$d = \frac{\tilde{\omega}_{ss2}}{\tilde{\omega}_{ss1}} - 1$$

Relation (52b) was checked with numerical calculations and was found to be accurate to 0.05 °/o. As an example, if a shell with boundary condition SS2/SS2 is taken, having $a/h = 577.35$ and $\frac{\ell^*}{a} = 3$, the error in $\tilde{\omega}_{LF}$ for the mode with $n = 8$ and $m = 1$ was found to be 0.73 °/o from the exact solution as compared to 0.76 °/o when estimated from expression (52b).

B. Error in $\tilde{\omega}_{LF}$ by Neglecting Tangential Inertia for Boundary Conditions Other than SS1

Based on the results of Appendix II, "v in k_{ij}" and "q_θ" will be neglected. Once more, the approximate solution will be used when applied to boundary condition SS2/SS2. The governing differential equations are given by equations (9a, b, and c) of Appendix VI.

For $n \geq 3$ and $\tilde{\omega} = \tilde{\omega}_{LF}$, $\left(\frac{\tilde{\omega}^2}{n^2}\right) \ll 0(1)$. Accordingly, we can expand the coefficients of the eigenmatrix $\left[\tilde{\mathcal{F}}_{nij}\right]$ (equation (13b), Appendix VI) in powers of $(\frac{\tilde{\omega}}{n})^2$ to $0(\frac{\tilde{\omega}}{n})^4$. This leads to the following:

$$\left[\tilde{\mathcal{F}}_{nij}\right] = \left[M_{ij}^{(t)} - \mathcal{E}_t \tilde{M}_{ij}^{(t)} - \tilde{\omega}^2 \delta_{ij}\right] \tag{53a}$$

$$\left[M_{ij}^{(t)}\right] = \left[P_{ij} s_j(n^2 - \nu s_j^2) + \tilde{\omega}_{4i}^2 \delta_{ij}\right] \tag{53b}$$

$$\mathcal{E}_t = (\frac{\tilde{\omega}}{n})^2 \ll 0(1), \qquad s_j = \frac{j\pi}{\ell}$$

$$\left[\widetilde{M}_{ij}^{(t)}\right] = \left[\left(\frac{n^2}{s_j^2 + n^2} + \frac{2(1+\nu)s_j^4\ n^2}{(s_j^2 + n^2)^2}\right)\delta_{ij} + o(\mathcal{E}_t)\right] \tag{53c}$$

P_{ij}, $\tilde{\omega}_4^2$ and δ_{ij} are given in Appendix VI, equation (20b). The form of $\left[\widetilde{M}_{ij}^{(t)}\right]$ as given above, suggests that its nondiagonal terms can be neglected to a first approximation. Following exactly the same line of proof as in the previous section, we finally obtain:

$$\tilde{\omega}^2 \simeq -\tilde{\omega}^2\left(\frac{1}{s_i^2 + n^2} + \frac{2(1+\nu)s_i^4}{(s_i^2 + n^2)^2}\right) + \tilde{\omega}_p^2 \tag{54}$$

$\tilde{\omega}_p^2$ is the eigenvalue of the eigenmatrix (53a) in the limit when $\mathcal{E}_t$ tends to zero. From the definition of the error (e_2) given in Table III, (54) can be reduced to the following expression:

$$(e_2) = \frac{\Delta\tilde{\omega}}{\tilde{\omega}} = \frac{\tilde{\omega}_p}{\tilde{\omega}} - 1 \simeq \sqrt{1 + \frac{1}{s_i^2 + n^2}\ \frac{2(1+\nu)}{(s_i^2 + n^2)^2} \cdot s_i^4} - 1 \tag{55}$$

Comparing (55) with the expression of $(e_2)_{ss1}$ given in Table V for $n \geq 3$, we conclude that:

$$(e_2)_{ss2} \simeq (e_2)_{ss1} \tag{56}$$

Thus, the effect of tangenial inertia on $\tilde{\omega}_{LF}$ is nearly the same for all boundary conditions. Relation (56) was compared with numerical calculations and was found to be accurate to 0.01 °/o.

C. Conclusion

1. An approximate method has been developed to compute the low, intermediate and high eigenfrequencies of thin cylindrical shells with consideration of the inplane boundary conditions. It was found to be in good agreement with the exact solution (error $\leq 0.1\,^o/o$).

2. Based on the above approximate solution, a relation was derived between the error (e_1) in the eigenfrequencies by neglecting "v in k_{ij}" and "q_θ", for any boundary condition, and the corresponding error in boundary condition SS1/SS1

$$(e_1)_{b.c.} \simeq \frac{(e_1)_{ss1}}{(1+d)^2}$$

where "d" is the "difference defined by:

$$d = \frac{\tilde{\omega}_{b.c.}}{\tilde{\omega}_{ss1}} - 1$$

This relation justifies once more that the neglect of "v in k_{ij}" and "q_θ", is a high frequency approximation. Furthermore, it also shows that these terms can be consistently neglected for a finite length shell with any boundary condition.

3. The error (e_2) in the eigenfrequencies by neglecting tangential inertia remains nearly unchanged for all boundary conditions for $n \geq 3$, i.e.,

$$(e_2)_{b.c.} \simeq (e_2)_{ss1}$$

III. EFFECT OF ELASTIC END RINGS ON THE EIGEN-FREQUENCIES

1. INTRODUCTION

Total fixity as well as classical simple supports are seldom found in practical applications. An intermediate state with elastic supports is more commonly encountered. A clàssical example is a shell with elastic rings at the ends.

The effect of the ring elastitity on the static buckling load of cylindrical shells was studied by Cohen, G. (Ref. 23). He assumed a hypothetical ring with no out-of-plane stiffness and concluded that total fixity can be realized when the stiffness factor $S_t = \frac{2EI_y}{D\ell^*}$ reaches the optimum value of 90. 5, where I_y is the inplane moment of inertia of the ring cross section (E, D and ℓ* are defined in the nomenclature). Such an analysis is inconsistent when applied to dynamics since, as shown previously in this work, the inplane boundary conditions of the shell are the most effective. Therefore, the out-of-plane stiffness of the ring cannot be neglected. Forsberg (Ref. 24) noticed a discrepency between the calculated frequencies of a cylindrical shell with elastic rings and the frequencies determined experimentally, at low circumferential wave number, when his calculations were based on the assumption that the shell was totally fixed at both ends. However, no further study was done concerning the boundary effect due to the ring elasticity or the stiffness required to reach a state of total fixity.

In this part, the effect of the ring elasticity is studied on a cylindrical shell fixed at one end and with a ring at the other. In the limit as the ring stiffness goes to zero, the shell approaches the fixed-free condition. Also as the ring stiffness goes to very large values, the shell reaches the condition of total fixity. Thus the above two limiting conditions constitute the lower and upper bounds respectively of the boundary condition in question. The ring cross section is taken to be a symmetric "H" section as shown in figure 10. This type of section was chosen since one can study the effect of eccentricity between the loading point and the center of gravity of the section, on the rigidity and efficiency of the ring. Such eccentricity induces a radial as well as an axial displacement when the section rotates in torsion about an axis normal to its plane. The axial displacement is responsible for the change in the inplane boundary conditions of the shell. The symmetry of the section also enables the uncoupling of the inplane and out-of-plane motions of the ring. Due to the number of ring geometrical parameters involved, a detailed analysis of the effect of each parameter is rather impossible. As an alternative, all geometrical parameters are fixed with the exception of the thickness ratio "$\tilde{h}$" (thickness of ring/thickness of shell). The increase in shell stiffness resulting from an increase in "$\tilde{h}$" can now be studied by looking at the increase in the eigenfrequencies corresponding to a particular mode shape.

2. EQUATIONS OF MOTION AND EXACT SOLUTION

The effectiveness of the ring as a flexible boundary will be measured by comparing the low frequencies corresponding to boundary condition FX2/FXR to those of FX2/FX2 for the same mode shape i.e., for the same "m" and "n". A difference is now defined as:

$$d_f = \frac{\tilde{\omega}_{fxr}}{\tilde{\omega}_{fx2}} - 1$$

The difference "d_f" gives a quantitative measure of how close the boundary with the ring is to its upper bound, the totally fixed case. The stiffness of the ring depends on the principal moments of inertia of its cross section (I_{xx}, I_{yy}) as well as its mean radius "a_r". A stiffening end-ring is efficient if the boundary to which it is attached, can reach the totally fixed condition with a minimum weight ratio "W_r" (weight of ring/weight of shell). These factors affecting stiffness and efficiency, therefore suggest the definition of stiffness factors of the form:

$$S_{tx} = \frac{r_x}{r}\sqrt{W_r \tilde{a}}, \qquad S_{ty} = \frac{r_y}{r}\sqrt{W_r \tilde{a}}$$

where,

$$r_x = \frac{1}{a_r}\sqrt{\frac{I_{xx}}{A_r}}, \quad r_y = \frac{1}{a_r}\sqrt{\frac{I_{yy}}{A_r}}, \quad r = \frac{h}{\sqrt{12}\, a}, \quad \tilde{a} = \frac{a_r}{a}$$

A_r is the cross sectional area of the ring. The out-of-plane stiffness factor "S_{tx}" will be taken to represent the stiffness of the ring since it controls the axial boundary conditions of the shell.

The coordinate system, displacements, rotation and resultant forces and mements are shown in figure 9. The uncoupled differential equations in terms of displacements for the shell are given by equations (1) to (4) in Appendix Ib. The governing differential equations for the ring, derived by Love (Ref. 1) on pages 397 and 451, will be used in this analysis. These equations include the following quantities:

1. Inplane displacements in curvature
2. Transverse shear in the inplane equilibrium equations
3. Inplane inertia
4. Rotary inertia

It is shown in Part I earlier in this work that the first three quantities have a large effect on the eigenfrequencies of the ring at low circumferential wave number "n" and that their effect is independent of the radii of gyration "r_x" and "r_y". However the error by neglecting rotary inertia was shown to be of $o(\frac{r^2 n^2}{2})$ and since, in our case, r_x and r_y can reach $o(10^{-1})$, rotary inertia must be included for $n \geq 3$.

The inplane and out-of-plane motions of the ring can be uncoupled due to the symmetry of the cross section leading to the following two systems of differential equations:

$$\left[D^{(i)}_{r,\,ij}\right] \begin{Bmatrix} v_r \\ w_r \end{Bmatrix} = \begin{Bmatrix} F^{(i)}_{r,\,i} \end{Bmatrix}$$

$$\left[D^{(o)}_{r,\,ij}\right] \begin{Bmatrix} u_r \\ \beta_r \end{Bmatrix} = \begin{Bmatrix} F^{(o)}_{r,\,i} \end{Bmatrix} \qquad (i,j = 1,2) \qquad (57)$$

$\left\{F^{(i)}_{r,i}\right\}$ and $\left\{F^{(o)}_{r,i}\right\}$ are forcing functions that depend on the non-dimensional stress resultants of the shell at the boundary where the ring is connected. The differential operators and forcing functions in (57) are given in Appendix VII.

For free vibration, the exact solution and resulting ordinary differential equations for the shell, are given in Appendix Ib. For the ring, (57) has an exact solution of the form:

$$v_r = v_{rn} \cos (n\theta) \sin (\tilde{\omega} \tau)$$

$$\begin{Bmatrix} w_r \\ u_r \\ \beta_r \end{Bmatrix} = \begin{Bmatrix} w_{rn} \\ u_{rn} \\ \beta_{rn} \end{Bmatrix} \sin (n\theta) \sin (\tilde{\omega} \tau) \tag{58}$$

$$\left\{F^{(i)}_{r,i}\right\} = \left\{F^{(i)}_{r,ni}\right\} \sin (n\theta) \sin (\tilde{\omega} \tau)$$

$$\left\{F^{(o)}_{r,i}\right\} = \left\{F^{(o)}_{r,ni}\right\} \sin (n\theta) \sin (\tilde{\omega} \tau)$$

Upon substitution of (58) in (57) we obtain the following system of linear algebraic equations:

$$\begin{aligned} \left[L^{(i)}_{r,ij}\right] \begin{Bmatrix} v_{rn} \\ w_{rn} \end{Bmatrix} &= \left\{F^{(i)}_{r,ni}\right\} \\ \left[L^{(o)}_{r,ij}\right] \begin{Bmatrix} u_{rn} \\ \beta_{rn} \end{Bmatrix} &= \left\{F^{(o)}_{r,ni}\right\} \end{aligned} \qquad (i, j = 1, 2) \tag{59}$$

The constant matricies in (59) are given in Appendix VII.

Solving for v_{rn}, w_{rn}, u_{rn} and β_{rn} in (59) and rearranging terms:

$$\begin{bmatrix} w_{rn} \\ u_{rn} \\ v_{rn} \\ \beta_{rn} \end{bmatrix} = \begin{bmatrix} M_{d,ij} \end{bmatrix} \begin{bmatrix} \tilde{q}_{xx}(\ell/2) \\ \tilde{n}_{xx}(\ell/2) \\ \tilde{n}_{x\theta}(\ell/2) \\ \tilde{m}_{xx}(\ell/2) \end{bmatrix} \quad (i, j = 1 \text{ to } 4) \qquad (60)$$

The right hand vector is composed of the nondimensional stress and moment resultants of the shell evaluated at the boundary where the ring is connected. $\begin{bmatrix} M_{d,ij} \end{bmatrix}$ is given in Appendix VII. The final step in the analysis is the matching of the displacements and slope at the boundary of the shell with the displacements and rotation of the ring. This is given by the following relations:

$$\begin{bmatrix} w_{rn} \\ u_{rn} \\ v_{rn} \\ \beta_{rn} \end{bmatrix} = \begin{bmatrix} M_{T,ij} \end{bmatrix} \begin{bmatrix} w_n(\ell/2) \\ u_n(\ell/2) \\ v_n(\ell/2) \\ w_n'(\ell/2) \end{bmatrix} \quad (i, j = 1 \text{ to } 4) \qquad (61)$$

$\begin{bmatrix} M_{T,ij} \end{bmatrix}$ is given in Appendix VII. Eliminating the ring displacement vector from the left hand sides of (60) and (61) we obtain four homogeneous boundary conditions relating the displacements and slope of the shell at the ring boundary to the nondimensional normal stress and moment resultants of the shell at the same boundary:

$$\begin{bmatrix} M_{d,ij} \end{bmatrix} \begin{bmatrix} w_n(\ell/2) \\ u_n(\ell/2) \\ v_n(\ell/2) \\ w'_n(\ell/2) \end{bmatrix} = \begin{bmatrix} M_{T,ij} \end{bmatrix} \begin{bmatrix} \tilde{q}_{xx}(\ell/2) \\ \tilde{n}_{xx}(\ell/2) \\ \tilde{n}_{x\theta}(\ell/2) \\ \tilde{m}_{xx}(\ell/2) \end{bmatrix} \quad (62)$$

The procedure for the determination of the exact eigenfrequencies is then similar to the one adopted in Section 2 of Part II earlier in this work.

The eigenfrequencies for a shell with $\frac{a}{h}$ = 577.35 were calculated for two different lengths, $\frac{\ell^*}{a}$ = 1 and 2, and for a ring stiffness factor S_{tx} in the range:

$$5 \leq S_{tx} \leq 160$$

corresponding to a thickness ratio $\tilde{h}$ in the range: $3 \leq \tilde{h} \leq 16$. The dimensions of the "H" type ring cross section are shown in figure 10. The above computations were performed for two boundary conditions:

1. FX2/FXR: fixed totally/fixed with flexible ring
2. FXR/FXR: fixed with flexible ring at both ends.

For each eigenfrequency, the $^o/o\ d_f$

$$^o/o\ d_f = \left(\frac{\tilde{\omega}_{fxr}}{\tilde{\omega}_{fx2}} - 1\right) 100$$

was plotted against S_{tx} for boundary condition FX2/FXR as shown in figures 11 and 12 for modes with $n \geq 4$ and with one half axial wave (m = 1). For low circumferential wave number with n = 2 and n = 3, the $\tilde{\omega}_{LF}$ corresponding to the first three modes was plotted against S_{tx} as shown in figures 13 and 14.

3. CONCLUSIONS

The influence of an elastic ring on the eigenfrequencies of the ring-shell system is different for various values of the circumferential wave number "n". This results since the ring frequencies are monotonously increasing with "n" whereas the shell low frequencies exhibit a minimum. We distinguish three different ranges of circumferential wave number "n":

A. For small values of "n" (n = 2), both the "w" and "u" predominant ring frequencies are lower than the fixed/free shell frequencies for the range of stiffness parameters (S_{tx}) of interest and there is no mode interaction except for very large values of "S_{tx}".

B. For intermediate "n" (n = 3, 4), the "w" and "u" ring frequencies are nearly of the same magnitude as the fixed/free shell frequencies for the range of "S_{tx}" of interest and considerable mode interaction is evidenced.

C. For large values of "n" ($n \geq 5$), the ring frequencies are considerably higher than the shell frequencies even for small but finite ring stiffness parameter "S_{tx}". The region of interaction is confined to very small values of "S_{tx}" below the range of interest.

The most interesting range is the one with intermediate "n" and will now be discussed in more detail. The variation of $\tilde{\omega}_{LF}$

with "S_{tx}" is shown in figure 14a for the first three modes with $n = 3$. The ring itself has four eigenfrequencies:

i. "w" predominant inplane eigenfrequency

$$\tilde{\omega}_{r,1} = 0(r_y n^2) \longrightarrow 0 \text{ as } S_{tx} \longrightarrow 0$$

ii. "v" predominant inplane eigenfrequency

$$\tilde{\omega}_{r,2} = 0(n^2 + 1)^{1/2} \text{ (independent of } S_{tx})$$

iii. "u" predominant out-of-plane eigenfrequency

$$\tilde{\omega}_{r,3} = 0(r_x n^2) \longrightarrow 0 \text{ as } S_{tx} \longrightarrow 0$$

iv. "β" predominant out-of-plane eigenfrequency

$$\tilde{\omega}_{r,4} = 0\left(\frac{r_t}{r_p}\left(\frac{n^2}{2(1+\nu)} + \frac{r_x^2}{r_t^2}\right)^{1/2}\right)$$

$$\rightarrow \text{finite value as } S_{tx} \longrightarrow 0$$

where $\left(\frac{S_{tx}}{S_{ty}}\right)$ is held constant. The asymptotic behavior of the first three eigenfrequencies of the shell-ring system is now described in detail. Let regions "I" and "II" be the regions in the left and right sides of the "$\tilde{\omega}_{r,1}$" line respectively (see figure 14a).

A. The first system eigenfrequency "$\tilde{\omega}_{s,1}$" approaches the first cantilever frequency corresponding to boundary condition FX2/FR (fixed/free) as "S_{tx}" goes to zero in region "I". It becomes asymptotic to "$\tilde{\omega}_{r,1}$" for some range of S_{tx} then changes asymptote to the first frequency of FX2/FX2 (totally fixed at both edges) in region II.

B. The second system eigenfrequency "$\tilde{\omega}_{s,2}$" approaches the second frequency of FX2/FR as "S_{tx}" goes to zero in region "I", then becomes asymptotic to "$\tilde{\omega}_{r,1}$" in the neighborhood where "$\tilde{\omega}_{s,1}$" changes asymptote. After some range of "S_{tx}", "$\tilde{\omega}_{s,2}$" changes asymptote to "$\tilde{\omega}_{r,3}$" line in region "II".

C. The third system eigenfrequency "$\tilde{\omega}_{s,3}$" approaches the second frequency of FX2/FR as "S_{tx}" goes to zero in region "I". It then follows the first FX2/FX2 frequency very closely and in the neighborhood where "$\tilde{\omega}_{s,2}$" changes asymptote, the "$\tilde{\omega}_{s,3}$" shifts asymptote to "$\tilde{\omega}_{r,1}$". A second change of asymptote occurs at some "S_{tx}" to the second frequency of FX2/FX2 in region "II".

The change of asymptotes for each of the above three modes can be explained in the following way. Since the system is linear elastic, governed by formally self-adjoint differential equations having unmixed natural boundary conditions, each system eigenvalue must have only one independent eigenfunction (see Appendix VIII). This implies that crossing of the three lines representing the first three system eigenfrequencies is impossible.

At first one might expect that four additional system frequencies will be induced due to the introduction of the four ring eigenfrequencies in the fixed/free shell spectrum. However, only two additional system modes are excited. The first is "$\tilde{\omega}_{s,2}$" and is induced by the "β" predominant ring frequency "$\tilde{\omega}_{r,4}$", and the second is a "v" predominant system frequency induced by the "v" predominant ring frequency "$\tilde{\omega}_{r,2}$". The following argument shows that no system frequency can be induced by either the "w" or the "u" predominant ring frequencies, "$\tilde{\omega}_{r,1}$" and "$\tilde{\omega}_{r,3}$". Assume at first that one of these ring frequencies induces a system frequency, "$\tilde{\omega}_{s,1}$". This system frequency cannot follow "$\tilde{\omega}_{r,1}$" nor "$\tilde{\omega}_{r,3}$" to zero as "S_{tx}" goes to zero since as this limit is approached, the system frequency must converge to a fixed/free shell eigenfrequency in a continuous manner. The only possible ring frequency that can induce "$\tilde{\omega}_{s,2}$" is then "$\tilde{\omega}_{r,4}$" the "β" predominant frequency, which has a finite value at $S_{tx} = 0$. The "w" motion in "$\tilde{\omega}_{s,2}$" is induced by the torsion and eccentricity of the ring cross section. Following the same argument, we expect that the "v" predominant ring frequency "$\tilde{\omega}_{r,2}$" induces an additional system frequency which approaches the "$\tilde{\omega}_{HF}$" corresponding to the first cantilever mode for boundary condition (fixed/free).

One can see from figure 14a that the third system frequency "$\tilde{\omega}_{s,3}$" is higher than the FX2/FX2 frequency for low values of "S_{tx}". This can be explained as follows. The "$\tilde{\omega}_{s,3}$" is out-of-phase with "$\tilde{\omega}_{r,1}$" in that range of "S_{tx}". The ring being very weak, will exhibit relatively large amplitudes and due to the change in phase, the corresponding mode shape will have one circumferential node near the ring boundary as is shown in figure 14b. The axial wave number "m" will lie between 1 and 2. And since "$\tilde{\omega}_{LF}$" is monotonously increasing with "m", it follows that "$\tilde{\omega}_{s,3}$" will be higher than

" $\tilde{\omega}_{FX2}$", noting that the axial restraint has little effect on the frequency at $n = 3$, As S_{tx} increases, the ring becomes stiffer and consequently its amplitude becomes smaller, thus shifting the circumferential node towards the boundary. This shift tends to decrease "m" hence a decrease in " $\tilde{\omega}_{s,3}$".

The same conclusions apply for $n = 2$ with the only difference that the change in asymptotes occurs outside the "S_{tx}" range of interest (see figures 13a and 13b). As "n" becomes higher ($n \geq 4$), the " $\tilde{\omega}_{r,1}$" line becomes steeper, thus confining region "I" to very small values of "S_{tx}" below the range of interest.

For large values of circumferential wave number "n" ($n \geq 5$), the FX2/FXR (fixed/free with elastic ring) eigenfrequencies are smaller than the corresponding ones for FX2/FX2 (totally fixed at both edges), as shown in figures 11 and 12. The overall shell stiffness, which is measured by the magnitude of the eigenfrequency for some mode shape, increases slowly and monotonously with the stiffness factor "S_{tx}". The "$^o/o\ d_f$" defined by:

$$^o/od_f = 100 \left(\frac{\tilde{\omega}_{FXR}}{\tilde{\omega}_{FX2}} - 1 \right)$$

increases with "n" for fixed values of "S_{tx}" until it reaches a maximum at some $n = n^*$, after which the "$^o/od_f$" decreases with "n". This can be explained as follows. It is seen from figure 6 that the

$$(^o/od)_{FX2} = 100 \left(\frac{\tilde{\omega}_{FX2}}{\tilde{\omega}_{SS1}} - 1 \right)$$

exhibits a maximum at some value of "n" which is smaller than "$\hat{n}$" at which $\tilde{\omega} = (\tilde{\omega})_{min}$. This shows that at this value of "n", the axial restraint in the boundaries is the most effective. The "$^o/od_f$" variation with "S_{tx}" is expected to behave in a similar fashion.

The $^o/od_f$ increases with the wave length parameter $(\frac{\ell}{m})$ as can be seen by comparing figures 11 and 12. Consequently, a stiffer ring is required to reach a certain "$^o/od_f$" from FX2/FX2 the larger the axial wave length and the nearer the circumferential wave number "n" is to "n^*" at which the "$^o/od$" of the corresponding FX2/FX2 shell is maximum. It follows then that the out-of-plane stiffness of the ring as well as its torsional rigidity (for the case of an eccentric ring) are responsible for the stiffening of the shell. All open sections are weak in torsion; a closed section is preferable if it does not present practical difficulties.

For $n \geq 5$, the "$^o/od_f$" for boundary condition FXR/FXR (ring at both edges) is nearly double the "$^o/od_f$" for boundary condition FX2/FXR (ring at one edge). This property shows that each ring is responsible for the stiffening of nearly half the shell length.

IV. EXPERIMENTAL DETERMINATION OF THE FREQUENCIES

1. INTRODUCTION

An experiment was carried out to check the validity of the previous theoretical analysis and demonstrate the effect of elastic rings on the eigenfrequencies of a cylindrical shell.

In Part III of this work, it was shown that the mode shapes with predominantly membrane energy (i.e., low circumferential wave number modes are greatly influenced by the ring stiffness. The eigenfrequencies associated with these modes differ greatly from those for the totally fixed shell, even for rings that are considered stiff in practical applications. It was also noticed that at those low circumferential wave numbers, two additional low frequency modes were induced due to the interaction of the ring frequencies with the free shell frequencies.

The experiment consists of finding the eigenfrequencies and corresponding mode shapes of a cylindrical shell with integral end rings of rectangular cross section, located on the outside of the shell. The details of the experiment are given in the next section.

2. EXPERIMENTAL SETUP

The shell used in the experiment was machined from a seamless tube of Al 6061-T6 material. The tube was first turned on the inside and then placed on a steel mandral by heating the aluminum tube. The outside machining process was carried out on the mandral and the final shell removed by heating. The end rings were an integral part of the shell. The final nominal dimensions are shown in figure 15. The wall thickness variation from the nominal was $\pm$ 0.0009 inches corresponding to $\pm$ 6 ^{o}/o variation.

The support conditions desired for the test were a free support for the shell-ring system. However, it was necessary to constrain the rigid body motion of the shell in order that the mode shapes could be readily measured. This was accomplished by mounting one end of the shell on the base plate but making only a flexible line contact with the end ring. For this purpose, the end plate was fitted with two "O rings", one to support the axial motion and the other to support the radial motion. The "O ring" for the radial support was mounted so as to allow a few thousands of an inch clearance. It was felt that this provided sufficient play so as not to restrict the radial motion and in the same time provide a reasonably good support.

The shell was oscillated using an acoustical driver, powered by a 75 watt amplifier and variable frequency oscillator. The driver was fitted with a conical nose which had a one quarter inch circular opening. The outlet of the driver was positioned approximately a tenth of an inch from the shell surface normal to it. This type of excitation forces only in the radial direction and therefore it is difficult to excite modes which are not predominantly radial motion. This will be seen in the results of the next section.

The response of the shell was measured using a reluctance type pickup which can traverse in the axial and circumferential directions. This equipment is described in reference 26. The signal from the pickup was displayed on an oscilloscope along with the input signal to the driver. It was necessary to run the pickup signal through a narrow band pass filter to remove the noise caused by the electric motors of the scaning system. This was necessary only for modes with very low amplitude (high frequencies), which in some cases was as small as 10 micro-inches. The mode shapes were determined by plotting the root mean square of the displacement against the circumferential distance on an x-y plotter. A count of the nodes gave the circumferential wave number "n". The axial wave number was determined by just observing the signal on the oscilloscope during an axial traverse.

3. CONCLUSIONS

The theoretical frequencies $(\tilde{\omega}_{th.})_{FXR}$ and the experimental frequencies $(\tilde{\omega}_{exp.})_{FXR}$ for the boundary condition in the test (FXR/FXR), as well as the theoretical frequencies for the fully fixed shell $(\tilde{\omega}_{th.})_{FX2}$ with the same geometrical parameters are listed in Table XII, for the following range of circumferential and axial wave numbers:

$$2 \leq n \leq 18, \qquad 1 \leq m \leq 4$$

The above three frequency spectrums are plotted in figure 16.

For low circumferential wave number (i. e., $n < \hat{n}$ at which $\tilde{\omega} = \tilde{\omega}_{min}$), the $(\tilde{\omega}_{exp.})_{FXR}$ are lower than the $(\tilde{\omega}_{th.})_{FXR}$. The per cent difference '$^{o}/o\tilde{d}$' defined as:

$$o/o\ \widetilde{d} = 100 \left(\frac{(\widetilde{\omega}_{th.})_{FXR}}{(\widetilde{\omega}_{exp.})_{FXR}} - 1 \right)$$

reaches + 6 ^{o}/o at n = 4 and m = 1. A possible explanation for the positiveness of the ^{o}/o $\widetilde{d}$ could be the fact that the idealized ring cross section used in the analysis is assumed to keep the same shape after deformation of the ring. However, the actual section will distort under stress thus inducing more out-of-plane displacement which softens the inplane boundary conditions of the shell and hence a decrease in frequency.

In the neighborhood of the minimum frequency ($n \simeq \hat{n}$), both $(\widetilde{\omega}_{exp.})_{FXR}$ and $(\widetilde{\omega}_{th.})_{FXR}$ differ by less than 0.5 ^{o}/o. For $n > \hat{n}$, the $(\widetilde{\omega}_{exp.})_{FXR}$ become higher than $(\widetilde{\omega}_{th.})_{FXR}$ where we notice that the ^{o}/o $\widetilde{d} \simeq -3$ ^{o}/o for all $n \geq 11$. Such a constant ^{o}/o $\widetilde{d}$ suggests the following explanation for this discrepancy. The frequency "$\widetilde{\omega}$" can be approximated by the following relation (see Appendix III):

$$\widetilde{\omega}^2 = \frac{(1-\nu^2)s^4}{(s^2+n^2)^2} + r^2(s^2+n^2)^2$$

where $s = \frac{m\pi}{\ell}$, m is the number of axial half waves and n is the number of circumferential full waves. The above estimate is good only for qualitative analysis for boundary conditions other than SS1/SS1. The first term in this relation is the membrane contribution to the frequency and is predominant at low "n" ($n < \hat{n}$). The second term accounts for the bending effect and becomes predominant at large "n" ($n > \hat{n}$). Both terms have the same order of magnitude in the neighborhood of the minimum frequency ($n \simeq n$). Thus for large "n", $\tilde{\omega} \simeq r(s^2 + n^2)$. The possibility for a constant error in $\tilde{\omega}$ for all $n > n$ can exist from an error in $r = \frac{h}{\sqrt{12}\, a}$. The above argument implies that the negative $^o/o\ \tilde{d}$ for $n > \hat{n}$ is caused by an underestimation of the shell thickness. The distribution of this thickness was measured by a micrometer during manufacture and by volume considerations after manufacture. Both methods involve unavaoidable errors that lead to an average thickness which was relatively inaccurate.

The $(\tilde{\omega}_{exp.})_{FXR}$ differ largely from $(\tilde{\omega}_{th.})_{FX2}$ at low "n" ($3 \leq n \leq 6 < \hat{n}$) as predicted by theory, in spite of the fact that the rings were sufficiently stiff (weight of a ring/weight of shell = W_r = 1.77). This is explained by the fact that at low "n", the inplane motion which governs the inplane boundary conditions of the shell, is comparable to the radial motion. This boundary effect dies out at large values of "n" ($n > \hat{n}$).

Difficulties were encountered in determining the frequencies for $n \leq 3$. This was probably caused by the method of excitation. Since the speaker was normal to the shell surface, mainly radial displacements were induced. Therefore, the forcing method was unable to generate inplane displacements of sufficient magnitude, relevant with these membrane predominant modes. Much distortion was noticed in the mode shape scans at these low circumferential wave numbers due to the effect of initial imperfection and nonuniform thickness distribution.

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APPENDIX Ia

DIFFERENTIAL OPERATOR MATRIX FOR CYLINDRICAL SHELL EQUATIONS

$$\left[D_{ij}\right]\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = 0 \qquad (i, j=1, 2, 3)$$

$$D_{11} = \frac{\partial^2}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2}{\partial \theta^2} - \frac{\partial^2}{\partial \tau^2}$$

$$D_{12} = \frac{1-\nu}{2}\frac{\partial^2}{\partial x \partial \theta} = D_{21}$$

$$D_{13} = -\nu\frac{\partial}{\partial x} = D_{31}$$

$$D_{22} = \frac{1-\nu}{2}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \theta^2} + r^2\left\{2(1-\nu)\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \theta^2}\right\} - \frac{\partial^2}{\partial \tau^2}$$

$$D_{23} = -\frac{\partial}{\partial \theta} + r^2\left\{(2-\nu)\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \theta^2}\right\}\frac{\partial}{\partial \theta} = D_{32}$$

$$D_{33} = 1 + r^2\,\nabla^4 + \frac{\partial^2}{\partial \tau^2}\,(1 - r^2\,\nabla^2)$$

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial \theta^2} + \frac{\partial^4}{\partial \theta^4}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \theta^2}$$

APPENDIX Ib

UNCOUPLING OF THE INPLANE DISPLACEMENTS EQUATIONS IN "u" AND "v"

It is shown later in this work (Part I, Section 4A), that the effect of neglecting the inplane displacement "v" in the curvature expressions, is equivalent to the effect of neglecting the transverse shear "q_θ" in the inplane equilibrium equation. Each effect contributes nearly half the total error from both effects. It is now clear that the combined effect (v in k_{ij} and q_θ) can be studied consistently by using simplified equations obtained by neglecting "v" terms in the curvature relations. The uncoupled system can then be written as follows:

$$\mathcal{L}(u) = \mathcal{E}_1(w) \tag{1}$$

$$\mathcal{L}(v) = \mathcal{E}_2(w) \tag{2}$$

$$\mathcal{R}(w) = \frac{\partial^2}{\partial\tau^2}\mathcal{H}(w) + r^2\, S(w) \tag{3}$$

where,

$$\mathcal{L} = (\nabla^2 - \frac{\partial^2}{\partial\tau^2})(\nabla^2 - \frac{2}{1-\nu}\frac{\partial^2}{\partial\tau^2})$$

$$\mathcal{E}_1 = \frac{\partial}{\partial x}\left\{\nu\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial\theta^2} - \frac{2\nu}{1-\nu}\frac{\partial^2}{\partial\tau^2} + r^2\,(\frac{1+\nu}{1-\nu})\frac{\partial^2}{\partial\theta^2}\,\nabla^2\right\}$$

$$\mathcal{E}_2 = \frac{\partial}{\partial\theta}\left\{(2+\nu)\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial\theta^2} - \frac{2}{1-\nu}\frac{\partial^2}{\partial\tau^2} - r^2\left(\frac{2}{1-\nu}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial\theta^2} - \frac{2}{1-\nu}\frac{\partial^2}{\partial\tau^2}\right)\nabla^2\right\}$$

$$\mathcal{R} = r^2\nabla^8 + \frac{\partial^2}{\partial\tau^2}\nabla^4 + (1-\nu^2)\frac{\partial^4}{\partial x^4}$$

$$\mathcal{H} = \left(\frac{3-\nu}{1-\nu}\nabla^2 - \frac{2}{1-\nu}\frac{\partial^2}{\partial\tau^2}\right)\left(r^2\nabla^4 + \frac{\partial^2}{\partial\tau^2}\right) + \nabla^2 - \frac{2}{1-\nu}\frac{\partial^2}{\partial\tau^2} + 2(1+\nu)\frac{\partial^2}{\partial x^2}$$

$$S = -\frac{\partial^2}{\partial\theta^2}\left\{(2+\nu)\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial\theta^2} - \frac{2}{1-\nu}\frac{\partial^2}{\partial\tau^2}\right\}\nabla^2$$

If "q_θ" is neglected then terms of $0(r^2)$ in $\mathcal{E}_1$, $\mathcal{E}_2$, and $S(w)$ will vanish. If tangential inertia is neglected, $\mathcal{H}(w) = 0$. The system of differential equations given by (1), (2) and (3) is valid for "asymmetric motion" i.e., $\frac{\partial}{\partial\theta} \neq 0$. However for axisymmetric motion, i.e., $\frac{\partial}{\partial\theta} = 0$, $v = 0$, the governing set of equations is:

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial\tau^2}\right)u = \nu\frac{\partial w}{\partial x}$$

$$\left\{\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial\tau^2}\right)\left(r^2\frac{\partial^4}{\partial x^4} + 1 + \frac{\partial^2}{\partial\tau^2}\right) - \nu^2\frac{\partial^2}{\partial x^2}\right\}w = 0 \tag{4}$$

For free vibration, an exact solution to system (1), (2), (3) can be written as follows:

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} u_n(x)\ \sin(n\theta) \\ v_n(x)\ \cos(n\theta) \\ w_n(x)\ \sin(n\theta) \end{Bmatrix} \sin(\tilde{\omega}\tau) \tag{5a}$$

Upon substitution of (5a) in (1), (2), (3) and (4), we obtain the following two systems of linear ordinary differential equations:

$$\mathcal{L}_n(u_n) = \mathcal{C}_{1n}(w_n)$$

$$\mathcal{L}_n(v_n) = \mathcal{C}_{2n}(w_n) \qquad (n \geqslant 1) \qquad \text{(5b)}$$

$$\mathcal{R}_n(w_n) = -\varpi^2 \mathcal{H}_n(w_n) + r^2 S_n(w_n)$$

$$\mathcal{L}_o(u_o) = \nu \frac{dw_o}{dx}$$

$$(n = 0) \qquad \text{(5c)}$$

$$\mathcal{R}_o(w_o) = 0$$

where

$$\mathcal{L}_n = (\frac{d^2}{dx^2} - \beta_1^2)(\frac{d^2}{dx^2} - \beta_2^2)$$

$$\mathcal{C}_{1n} = \frac{d}{dx}\left\{\nu \frac{d^2}{dx^2} + (1+\nu)n^2 - \nu\beta_2^2 - r^2 n^2 (\frac{1+\nu}{1-\nu}) \bar{\nabla}_n^2\right\}$$

$$\mathcal{C}_{2n} = n\left\{(2+\nu)\frac{d^2}{dx^2} - \beta_2^2 - r^2(\frac{2}{1-\nu}\frac{d^2}{dx^2} - \beta_2^2)\bar{\nabla}_n^2\right\}$$

$$\mathcal{R}_n = r^2 \bar{\nabla}_n^8 - \varpi^2 \bar{\nabla}_n^4 + (1-\nu^2)\frac{d^4}{dx^4}$$

$$\mathcal{H}_n = (\frac{3-\nu}{1-\nu}\bar{\nabla}_n^2 + \frac{2\varpi^2}{1-\nu})(r^2 \bar{\nabla}_n^4 - \varpi^2) + (3+2\nu)\frac{d^2}{dx^2} - \beta_2^2$$

$$S_n = n^2\left\{(2+\nu)\frac{d^2}{dx^2} - \beta_2^2\right\}\bar{\nabla}_n^2$$

(5d)

$$\mathscr{L}_o = \frac{d^2}{dx^2} + \tilde{\omega}^2$$

$$\mathscr{R}_o = \left(\frac{d^2}{dx^2} + \tilde{\omega}^2\right)\left\{r^2 \frac{d^4}{dx^4} + 1 - \tilde{\omega}^2\right\} - \nu^2 \frac{d^2}{dx^2}$$

(5d)

$$\beta_1^2 = n^2 - \tilde{\omega}^2; \quad \beta_2^2 = n^2 - \frac{2}{1-\nu}\tilde{\omega}^2$$

$$\bar{\nabla}_n^2 = \frac{d^2}{dx^2} - n^2$$

$$\bar{\bar{\nabla}}_n^4 = \bar{\nabla}_n^2(\bar{\nabla}_n^2) = \frac{d^4}{dx^4} - 2n^2\frac{d^2}{dx^2} + n^4$$

APPENDIX Ic

EIGENMATRIX COEFFICIENTS FOR A CYLINDRICAL SHELL WITH BOUNDARY CONDITION SS1/SS1

$$\left[L_{ij}\right] \begin{Bmatrix} u_{mn} \\ v_{mn} \\ w_{mn} \end{Bmatrix} = 0$$

$$\left[L_{ij}\right] = \left[(c_{ij} + \epsilon_{ij})\right] - \left[\bar{I}\right] \bar{\omega}^2$$

$$c_{11} = s^2 + \frac{(1-\nu)}{2} n^2$$

$$c_{12} = \frac{(1+\nu)}{2} s\, n = c_{21}$$

$$c_{13} = \nu s = c_{31}$$

$$c_{22} = \frac{(1-\nu)}{2} s^2 + n^2$$

$$c_{23} = n = c_{23}$$

$$c_{33} = 1 + r^2(s^2 + n^2)^2$$

$$\epsilon_{ij} = 0 \text{ except for the following:}$$

$$\epsilon_{22} = r^2(2(1-\nu)s^2 + n^2)$$

$$\epsilon_{23} = r^2 n((2-\nu)s^2 + n^2) = \epsilon_{32}$$

$$\left[\bar{I}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (1+\bar{\epsilon}_{33}) \end{bmatrix}$$

$$\bar{\epsilon}_{33} = r^2(s^2 + n^2)$$

APPENDIX Id

CHARACTERISTIC EQUATION FOR THIN RINGS

For a thin ring, the characteristic equation is:

$$C_{2r}\,\varpi^4 - C_{1r}\,\varpi^2 + C_{or} = 0$$

$$C_{2r} = 1 + r^2 n^2$$

$$C_{1r} = 1 + r^2 n^4 + (1 + r^2)(1 + r^2 n^2) n^2$$

$$C_{or} = -\, n^2 \left\{ (1 + r^2 n^2)^2 - (1 + r^2)(1 + r^2 n^4) \right\}$$

Solving for ϖ^2 we obtain:

$$\varpi^2 = \frac{b \pm \sqrt{c + d}}{e}$$

where

$$b = 1 + r^2 n^4 + (1 + r^2)(1 + r^2 n^2) n^2$$

$$c = \left\{ 1 + r^2 n^4 - (1 + r^2)(1 + r^2 n^2) n^2 \right\}^2$$

$$d = 4n^2 (1 + r^2 n^2)^3$$

$$e = 2(1 + r^2 n^2)$$

APPENDIX II

RELATIONSHIP BETWEEN THE ERROR FROM THE DIFFERENT APPROXIMATIONS AND THE TOTAL ERROR FROM THE COMBINED EFFECT

The error "e" in the low frequency "$\tilde{\omega}_{LF}$" of a cylindrical shell with boundary condition SS1/SS1, from all approximations (v in k_{ij} and q_θ, tangential inertia, rotary inertia), is nearly equal to the sum of the errors from each approximation when performed separately, i.e., $e_t \simeq e_1^* + e_2^* + e_3^* \simeq e_1 + e_2 + e_3$.

PROOF

We use the same notation as in A, B and C of Table III. At first we prove that:

$$e_{12}^* \simeq e_1^* + e_2^* \tag{1a}$$

e_{12}^* can be written in the form:

$$e_{12}^* = \frac{\tilde{\omega}_3 - \tilde{\omega}^*}{\tilde{\omega}^*} = \frac{\tilde{\omega}_3 - (\tilde{\omega}_2^* - \Delta\,\tilde{\omega}_2^*)}{\tilde{\omega}^*}$$

$$e_{12}^* = e_{1a} + e_2^* \tag{1b}$$

$$e_{1a} = \frac{\tilde{\omega}_3 - \tilde{\omega}_2^*}{\tilde{\omega}^*}\,, \qquad e_2^* = \frac{\Delta\,\tilde{\omega}_2^*}{\tilde{\omega}^*}$$

Thus we would like to prove that $e_{1a} \simeq e_1^*$

A. <u>Determination of e_1^*</u>

Expanding the eigendeterminant (8) on page 7 , we obtain the characteristic equation (dispersion relation):

$$C_3 \tilde{\omega}^{*6} - C_2 \tilde{\omega}^{*4} + C_1 \tilde{\omega}^{*2} - (C_o - \mathcal{E}) = 0, \quad n \geq 1 \tag{2}$$

$$C_3 = 1 + 0(rn)^2$$

$$C_2 = 1 + \frac{(3-\nu)}{2}(s^2 + n^2) + r^2(s^2 + n^2)^2 + 0(rn)^2$$

$$C_1 = \frac{(1-\nu)}{2}\Bigg((s^2 + n^2)^2 + (s^2 + n^2) + 2(1+\nu)s^2$$

$$+ \frac{(3-\nu)}{(1-\nu)} r^2(s^2 + n^2)^3 + 0(r^2n^4)\Bigg)$$

$$C_o = \frac{(1-\nu)}{2}\left(r^2(s^2 + n^2)^4 + (1-\nu^2)s^4\right)$$

$$\mathcal{E} = \frac{(1-\nu)}{2} r^2\left((2s^2 + n^2)^2(2n^2 - 1) - 2\nu^2 s^4(n^2 - 2)\right)$$

We note that $C_3 = 0(1)$ compared to $C_2 = 0(n^2)$. Since we are interested in e_1^* for $n > 2$ at which values $\tilde{\omega}^{*2} << 0(1)$, we conclude that:

$$C_3 \tilde{\omega}^{*6} << C_2 \tilde{\omega}^{*4} \tag{2a}$$

Equation (2) then reduces to:

$$C_2 \tilde{\omega}^{*4} - C_1 \tilde{\omega}^{*2} + (C_o - \mathcal{E}) \simeq 0 \tag{3}$$

Solving for $\tilde{\omega}^{*2}$ in (3) and expanding in powers of $\mathcal{E}$ to $0(\mathcal{E})$

$$\tilde{\omega}^{*2} = \frac{C_1 - \sqrt{C_1^2 - 4C_o C_2}}{2 C_2} - \frac{\mathcal{E}}{C_1^2 - 4C_o C_2} + o(\mathcal{E}^2)$$

$$\tilde{\omega}^{*2} = \tilde{\omega}_1^{*2} - \frac{\mathcal{E}}{C_1^2 - 4C_o C_2} + 0(\mathcal{E}^2) \tag{4}$$

Substituting (4) in the expression for e_1^* (Table III), and expanding in powers of $\mathcal{E}$ to $0(\mathcal{E})$:

$$e_1^* \simeq \frac{\mathcal{E}}{2C_o} = \frac{r^2}{2} \left(\frac{(2s^2 + n^2)^2 (2n^2 - 1) - 2\nu^2 s^4 (n^2 - 2)}{r^2 (s^2 + n^2)^4 + (1 - \nu^2) s^4} \right) \tag{5}$$

B. Determination of e_{1a}

Neglecting tangential inertia and solving for $\tilde{\omega}^{*2}_2$:

$$\tilde{\omega}^{*2}_2 = \frac{C_o - \mathcal{E}}{C_1^*}, \quad C_1^* = \frac{(1-\nu)}{2} (s^2 + n^2)^2 (1 + r^2 (s^2 + n^2))$$

$$\simeq \tilde{\omega}_3^2 - \frac{\mathcal{E}}{C_1^*} \tag{6}$$

Substituting (6) in the expression for e_{1a} in (1a) and expanding in powers of $\mathcal{E}$ to $0(\mathcal{E})$:

$$e_{1a} = \frac{\tilde{\omega}_3 - \tilde{\omega}^*_2}{\tilde{\omega}^*_2} \simeq \frac{\tilde{\omega}_3 - \tilde{\omega}^*_2}{\tilde{\omega}^*} \simeq \frac{\mathcal{E}}{2C_o} \tag{7}$$

Comparing (5) and (7), we deduce that:

$$e_1^* \simeq e_{1a} \simeq e_1 \tag{8}$$

Thus, relation (1a) is proven.

In a similar manner, one can prove that:

$$\begin{aligned} e_{13} &\simeq e_1^* + e_3^* \\ e_{23} &\simeq e_2^* + e_3^* \end{aligned} \tag{9}$$

From (1a) and (9), we conclude that:

$$\begin{aligned} e_t &\simeq e_1^* + e_2^* + e_3^* \\ &\simeq e_1 + e_2 + e_3 \end{aligned}$$

Let $\tilde{\omega}_4^*$ be the $\tilde{\omega}_4$ (as defined in Table III) when "q_θ" is included, then:

$$\begin{aligned} \tilde{\omega}_4^{*2} &\simeq \tilde{\omega}_4^2 - r^2 n^2 \left(1 + \frac{2s^2}{(s^2+n^2)} + 0(\frac{1}{n^4})\right) \\ \Delta(\tilde{\omega}_4^2)_{SS1} &= \left[\tilde{\omega}_4^2 - \tilde{\omega}_4^{*2}\right] \simeq r^2 n^2 \left(1 + \frac{2s^2}{(s^2+n^2)}\right) \end{aligned} \tag{10}$$

APPENDIX III

SOME PROPERTIES OF $\tilde{\omega}_{LF}$ FOR $n > 2$ FOR BOUNDARY CONDITION SS1/SS1

It is shown in this work (Part I, section 4A) that tangential inertia has a noticeable effect on $\tilde{\omega}_{LF}$ only for low circumferential wave number, $n \leqslant 3$. Consequently, the $\tilde{\omega}_{LF}$ spectrum can be studied with sufficient accuracy for $n > 2$ by neglecting tangential inertia. From Appendix II and with notations as given in Table III

$$\tilde{\omega}_4^2 = r^2(s^2 + n^2)^2 + \frac{(1-\nu^2)s^4}{(s^2+n^2)^2} \tag{1}$$

For fixed "s", $\tilde{\omega}_4$ has a minimum at $n = \hat{n}$:

$$\hat{n}^2 = s\left(\frac{(1-\nu^2)^{1/4}}{r^{1/2}} - s\right) \tag{2}$$

This minimum always exists if:

$$(1-\nu^2)^{1/4} \geqslant r^{1/2}\, s$$

which is always the case for thin shell ($r = 0(10^{-2})$) and $s < 6$. If $(r^{1/2}\, s) \ll 0(1)$, (2) simplifies to the following:

$$\hat{n}^2 \simeq \frac{s(1-\nu^2)^{1/4}}{r^{1/2}} \tag{3}$$

Substituting (2) in (1) we obtain:

$$(\tilde{\omega}_4)_{min} = \sqrt{2}\; r(\hat{n}^2 + s^2) = (2r)^{1/2}(1-\nu^2)^{1/4}\, s \tag{4}$$

Solving for $\hat{n}^2$ in (4):

$$\hat{n}^2 \simeq \frac{(\tilde{\omega}_4)_{min}}{\sqrt{2}\, r} + 0(s^2) \tag{5}$$

The locus of minima has itself a maximum at $s = s^*$:

$$s^* = \frac{(1-\nu^2)^{1/4}}{2\, r^{1/2}} \tag{6}$$

For $n^2 \gg \hat{n}^2$

$$\tilde{\omega}_4^2 \simeq r^2 n^4 + \frac{(1-\nu^2) s^4}{n^4} \tag{7}$$

For $n^2 \gg s^2$ and $n^2 \gg \hat{n}^2$

$$\tilde{\omega}_4 \simeq r n^2 \simeq (\tilde{\omega})_{ring} \tag{8}$$

APPENDIX IV

DISPERSION RELATION FOR CYLINDRICAL SHELLS

Characteristic equation (dispersion relation) for n = 0:

$$C_o(1)\,\lambda_{oj}^6 + C_o(2)\,\lambda_{oj}^4 + C_o(3)\,\lambda_{oj}^2 + C_o(4) = 0$$

$$C_o(1) = r^2$$

$$C_o(2) = r^2\tilde{\omega}^2$$

$$C_o(3) = 1 - \nu^2 - \tilde{\omega}^2$$

$$C_o(4) = \tilde{\omega}^2(1 - \tilde{\omega}^2)$$

Characteristic equation (dispersion relation) for $n \geq 1$:

$$C_n(1)\,\lambda_{nj}^8 + C_n(2)\,\lambda_{nj}^6 + C_n(3)\,\lambda_{nj}^4 + C_n(4)\,\lambda_{nj}^2 + C_n(5) = 0$$

$$C_n(1) = r^2$$

$$C_n(2) = r^2\left[\frac{3-\nu}{1-\nu}\tilde{\omega}^2 - 4n^2\right]$$

$$C_n(3) = r^2\left(6n^4 - \frac{3(3-\nu)}{(1-\nu)}\tilde{\omega}^2 n^2 + \frac{2\tilde{\omega}^4}{(1-\nu)}\right) + 1 - \nu^2 - \tilde{\omega}^2$$

$$C_n(4) = r^2 n^2\left(-4n^4 + \frac{3(3-\nu)}{(1-\nu)}\tilde{\omega}^2 n^2 - \frac{4\tilde{\omega}^4}{(1-\nu)}\right) + {}^2\left[2n^2 - \frac{3-\nu}{1-\nu}\tilde{\omega}^2 + 3 + 2\right]$$

$$C_n(5) = r^2 n^4\left(n^4 - \frac{3-\nu}{1-\nu}\tilde{\omega}^2 n^2 + \frac{2\tilde{\omega}^4}{1-\nu}\right) + \tilde{\omega}^2\left(n^2(-n^2 - 1 + \tilde{\omega}^2\ \frac{3-\nu}{1-\nu}) + \frac{2\tilde{\omega}^2}{1-\nu}\ (1 - \tilde{\omega}^2)\right)$$

APPENDIX V

CLASSICAL BOUNDARY CONDITION "SS1/SS1" AS THE LIMIT OF GENERAL BOUNDARY CONDITIONS

Boundary condition SS1/SS1 is given by:

$$w(\pm \tfrac{\ell}{2}) = n_{xx}(\pm \tfrac{\ell}{2}) = v(\pm \tfrac{\ell}{2}) = m_{xx}(\pm \tfrac{\ell}{2}) = 0 \qquad (1)$$

As an example, consider roots of the characteristic equation (34) for $n \geqslant 1$, having the following form: (see Table VIII)

$$\pm\, iC1 \;;\; \pm C2 \;;\; \pm(C3 \pm iC4) \qquad (C1, C2, C3, C4, \text{ real}) \qquad (2)$$

The eigendeterminant (37) reduces to a four by four determinant due to the symmetry of the boundary condition (1):

$$\det\left[\mathcal{F}_{nij}\right] = \begin{vmatrix} F_{n1} & F_{n3} & F_{n6} & F_{n7} \\ K_{11}F_{n1} & K_{12}F_{n3} & K_{13}F_{n6} + K_{14}F_{n7} & K_{13}F_{7n} - K_{14}F_{n6} \\ K_{21}F_{n1} & K_{22}F_{n3} & K_{23}F_{n6} + K_{24}F_{n7} & K_{23}F_{n7} - K_{24}F_{n6} \\ K_{31}F_{n1} & K_{32}F_{n3} & K_{33}F_{n6} + K_{34}F_{n7} & K_{33}F_{n7} - K_{34}F_{n6} \end{vmatrix}$$

$$(i, j = 1 \text{ to } 4) \qquad (3)$$

K_{sp} are constants which depend on $(r, n, \tilde{\omega}^2, \nu)$ only, and F_{nj} are the independent functions of "x" evaluated at $x = \frac{\ell}{2}$, as defined by (35). For antisymmetric modes, i.e., $m = 2, 4, 6, \ldots$, and with roots in the form of (2), $F_{nj}(\frac{\ell}{2})$ are given as follows:

$$F_{n1}(\tfrac{\ell}{2}) = \sin(C1\,\tfrac{\ell}{2})$$

$$F_{n3}(\tfrac{\ell}{2}) = \sinh(C2\,\tfrac{\ell}{2})$$

$$F_{n6}(\tfrac{\ell}{2}) = \sin(C4\,\tfrac{\ell}{2})\cosh(C3\,\tfrac{\ell}{2}) \tag{3a}$$

$$F_{n7}(\tfrac{\ell}{2}) = \cos(C4\,\tfrac{\ell}{2})\sinh(C3\,\tfrac{\ell}{2})$$

For symmetric modes, i.e., m = 1, 3, 5, . . . , the "sin" and "sinh" in (3a) are substituted by the "cos" and "cosh" respectively and vice versa. The same line of proof applies for m odd as for m even. Take "m" to be even, i.e., antisymmetric modes.

After some algebra, (3) reduces to the following simple form:

$$\det\left[\mathcal{F}_{nij}\right] = -F_{n1}F_{n3}(F_{n6}^2 + F_{n7}^2)\det\left[\mathcal{D}_{ij}\right]$$

$$= -\sin(C1\,\tfrac{\ell}{2})\sinh(C2\,\tfrac{\ell}{2})\left\{\sin^2(C4\,\tfrac{\ell}{2}) + \sinh^2(C3\tfrac{\ell}{2})\right\}\det\left[\mathcal{D}_{ij}\right] \tag{4a}$$

$$\det\left[\mathcal{D}_{ij}\right] = \begin{vmatrix} 1 & 1 & 1 & 0 \\ K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \end{vmatrix} \tag{4b}$$

(i, j = 1 to 4)

We note that $\left[\mathscr{D}_{ij}\right]$ is independent of the parameter $\frac{\ell}{a}^*$, thus if it vanishes, this will imply that the eigenfrequencies $\tilde{\omega}$ are independent of $\frac{\ell}{a}^*$, which is a contradiction. Thus:

$$\det\left[\mathscr{D}_{ij}\right] \neq 0 \qquad\qquad (i, j=1 \text{ to } 4) \tag{5}$$

We are now left with the following three possibilities:

1. $F_{n1}(\frac{\ell}{2}) = 0 \implies \frac{C1\,\ell}{2} = k\pi \quad (k = 1, 2, \ldots)$ (6a)

2. $F_{n3}(\frac{\ell}{2}) = 0 \implies C2 = 0$ (6b)

3. $(F_{n6}^2 + F_{n7}^2) = 0 \implies C3 = 0$ and $\frac{C4\,\ell}{2} = k\pi \; (k = 1, 2, \ldots)$ (6c)

Possibilities 2 and 3 must be ruled out since $C2 = 0$ or $C3 = 0$ contradict the assumption that the characteristic equation (34) has roots of the form (2). Thus:

$$F_{n3}(\tfrac{\ell}{2}) \neq 0 \qquad \text{and} \qquad (F_{n6}^2 + F_{n7}^2) \neq 0 \tag{7a}$$

Consequently, we are left with:

$$C1 = \frac{2k\pi}{\ell} \rightarrow F_{n1}(\tfrac{\ell}{2}) = \sin(C1\,\tfrac{\ell}{2}) = \sin(k\pi) = 0 \tag{7b}$$

The displacement function can be written as in equation (35):

$$w_n(x) = \sum_{j=1}^{4} B_{nj}\, F_{nj}(x) \tag{8}$$

Let the B_{nj}'s be normalized by B_{n1}, i.e., take $B_{n1} = 1$. The remaining B_{nj}'s can now be determined from the following set of inhomogeneous algebraic equations:

$$\left[\mathcal{F}_{nrs}\right]\left\{B_{ns}\right\} = -\begin{Bmatrix} 1 \\ K_{11} \\ K_{21} \end{Bmatrix} F_{n1}\left(\frac{\ell}{2}\right) \qquad \begin{matrix}(r = 1, 2, 3) \\ (s = 2, 3, 4)\end{matrix} \tag{9}$$

The matrix $\left[\mathcal{F}_{nrs}\right]$ is the modified $\left[\mathcal{F}_{nij}\right]$ in (3), by excluding the first column and fourth row. Using (7b), we get:

$$\left[\mathcal{F}_{nrs}\right]\left\{B_{ns}\right\} = 0 \qquad (r = 1, 2, 3 \text{ and } s = 2, 3, 4) \tag{10}$$

System (10) has a nontrivial solution if and only if:

$$\det\left[\mathcal{F}_{nrs}\right] = -F_{n3}\left(F_{n6}^2 + F_{n7}^2\right)\det\left[\mathcal{D}_{rs}\right] = 0 \tag{11}$$

The matrix $\left[\mathcal{D}_{rs}\right]$ is the modified $\left[\mathcal{D}_{ij}\right]$ in (4b), by excluding the first column and fourth row. Alternatively, we can take:

$$\left.\begin{matrix} r = 1, 2, 4 \\ \text{or } \; r = 1, 3, 4 \\ \text{or } \; r = 2, 3, 4 \end{matrix}\right\} \quad \text{with } s = 2, 3, 4$$

Each of the above cases will reduce to a system similar to (10). The condition of a nontrivial solution for the B_{ns}'s will still be represented by (11). If $\det\left[\mathcal{D}_{rs}\right]$ vanishes in any of the above systems, it must vanish in all other systems since the B_{ns}'s are uniquely determined for some ϖ. If this happens it will imply that $\det\left[\mathcal{D}_{ij}\right] = 0$

(i, j = 1, 2, 3, 4) since all its co-factors vanish. However this contradicts the result given by (5). Consequently,

$$\det \left[\mathscr{D}_{rs} \right] \neq 0 \qquad \text{(for all combinations of r)} \qquad (11a)$$

Using the results given by (7a) and (11a) we conclude that:

$$\det \left[\mathscr{F}_{nrs} \right] \neq 0 \qquad (r = 1, 2, 3 \text{ and } s = 2, 3, 4) \qquad (12)$$

Thus, system (10) admits only the trivial solution:

$$\left\{ B_{ns} \right\} = 0, \qquad (s = 2, 3, 4) \qquad (13)$$

The exact solution for boundary condition SS1/SS1 is then:

$$w_n(x) = B_{n1} \sin(C1 \; x) = \sin(\frac{m\pi x}{\ell}) \qquad \text{(m even)}$$

for antisymmetric modes. Similarly, the solution is:

$$w_n(x) = \cos(\frac{m\pi x}{\ell}) \qquad \text{(m odd)}$$

for symmetric modes. In fact, these functions are admissible functions that satisfy both differential equations and boundary conditions.

APPENDIX VI

APPROXIMATE SOLUTION OF THE DIFFERENTIAL EQUATIONS FOR BOUNDARY CONDITION SS2/SS2

The following cases will be considered:

(I) Tangential inertia included

A - Axisymmetric vibration (n = 0) (valid for $\tilde{\omega}_{LF}$ and $\tilde{\omega}_{IF}$)

B - Asymmetric vibration ($n \geqslant 1$)

B1 - $n > \sqrt{\frac{2}{1-\nu}}\,\tilde{\omega}$ (valid for $\tilde{\omega}_{LF}$ for $n \geqslant 2$)

B2 - $\sqrt{\frac{2}{1-\nu}}\,\tilde{\omega} > n > \tilde{\omega}$ (valid for $\tilde{\omega}_{LF}$ for n = 1, and $\tilde{\omega}_{IF}$ for $n \geqslant 2$)

B3 - $n < \tilde{\omega}$ (valid for $\tilde{\omega}_{IF}$ for n = 1, and $\tilde{\omega}_{HF}$ for $n \geqslant 1$)

(II) Tangential inertia neglected (valid for $n \geqslant 3$ for $\tilde{\omega}_{LF}$ only)

"Case (II)" was developed due to computational difficulties encountered when "Case (I)" was used to compute $\tilde{\omega}_{LF}$ for $n^2 \gg \frac{2\tilde{\omega}^2_{LF}}{1-\nu}$. Fortunately, it was found that $\tilde{\omega}_{LF}$ as calculated in (II), could be corrected to take into account tangential inertia. This was possible since the error by neglecting tengential inertia is nearly the same for SS2/SS2 as it is for SS1/SS1, i.e.,

$$\left\{\frac{\tilde{\omega}_4}{\tilde{\omega}}\right\}_{SS2} \simeq \left\{\frac{\tilde{\omega}_4}{\tilde{\omega}}\right\}_{SS1}$$

where $\tilde{\omega}_4$ is $\tilde{\omega}_{LF}$ when tangential inertia is neglected. This is shown later in Part II, section 4B.

CASE (I): TANGENTIAL INERTIA INCLUDED

A - Axisymmetric vibration (n = 0)

The equations of motion are:

$$\left(\frac{d^2}{dx^2} + \tilde{\omega}^2\right) u_o(x) = \nu \frac{dw_o(x)}{dx} \tag{1a}$$

$$\left(r^2 \frac{d^4}{dx^4} + 1 - \tilde{\omega}^2\right) w_o(x) = \nu \frac{du_o(x)}{dx} \tag{1b}$$

$$x \ni [0, \ell]$$

Solving for $u_o(x)$ in (1a):

$$u_o(x) = B_{0_1} \sin(\tilde{\omega}x) + B_{0_2} \cos(\tilde{\omega}x) + \frac{\nu}{\tilde{\omega}} \int^x \sin\tilde{\omega}(x-\xi) \frac{dw_o(\xi)}{d\xi} d\xi \tag{2}$$

The inplane boundary conditions for SS2/SS2 are:

$$u_o(0) = u_o(\ell) = 0 \tag{3}$$

Substituting (2) in (3) and solving for B_{01} and B_{02}:

$$u_o(x) = \frac{\nu}{\sin(\tilde{\omega}\ell)} \left\{ \int_0^x \sin\tilde{\omega}(\ell - x) \cos(\tilde{\omega}\xi) w_o(\xi)\, d\xi - \int_x^\ell \sin(\tilde{\omega}x) \cos\tilde{\omega}(\ell-\xi) w_o(\xi) d\xi \right\} \tag{4}$$

Eliminating $u_o(x)$ from (1b) using (4):

$$\left(r^2 \frac{d^4}{dx^4} + 1 - \nu^2 - \tilde{\omega}^2\right) w_o(x) = \frac{\nu^2 \tilde{\omega}}{\sin(\tilde{\omega}\ell)} \left\{ \int_0^x \cos\tilde{\omega}(\ell - x) \cos(\tilde{\omega}\xi) w_o(\xi) d\xi + \int_x^\ell \cos(\tilde{\omega}x) \cos\tilde{\omega}(\ell-\xi) w_o(\xi) d\xi \right\} \tag{5}$$

Assume an expansion for $w_o(x)$ that satisfies all out-of-plane boundary conditions i.e., $w_o(0) = w_o(\ell) = 0$ and $w''_o(0) = w''_o(\ell) = 0$:

$$w_o(x) = \sum_{j=1}^{j^*} a_{oj} \sin(\frac{j\pi x}{\ell}) \qquad (j^* \text{ finite}) \tag{6}$$

Substituting (6) in (5) and applying Galerkin's method, we obtain the following set of homogeneous algebraic equations in a_{oj}:

$$\left[\tilde{\mathcal{F}}_{oij}\right]\left\{a_{oj}\right\} = 0 \qquad (i, j = 1, 2, .., j^*) \tag{7a}$$

$$\tilde{\mathcal{F}}_{oij} = \left[\frac{-\nu^2\varphi^2}{i^2\pi^2-\varphi^2} + (1-\nu^2 - \frac{\varphi^2}{\ell^2}) + r^2\frac{i^4\pi^4}{\ell^4}\right]\delta_{ij} + \frac{4ij\pi^2\nu^2\varphi(\cos\varphi - (-1)^j)}{(i^2\pi^2-\varphi^2)(j^2\pi^2-\varphi^2)\sin\varphi} \tag{7b}$$

$$\varphi = \tilde{\omega}\ell \qquad (i, j = 1, 2, ..., j^*)$$

System (7a) has a nontrivial solution if and only if:

$$\det\left[\tilde{\mathcal{F}}_{oij}\right] = 0 \tag{8}$$

The eigenfrequencies of (8) are obtained by an iterative method.

B <u>Asymmetric vibration ($n \geqslant 1$)</u>

The equations of motion are:

$$\mathcal{L}_n[u_n(x)] = \mathcal{C}_{1n}[w_n(x)] \tag{9a}$$

$$\mathcal{L}_n[v_n(x)] = \mathcal{C}_{2n}[w_n(x)] \tag{9b}$$

$$(r^2\bar{\nabla}_n^4 + 1-\tilde{\omega}^2)\, w_n(x) = \nu\frac{du_n(x)}{dx} - n\, v_n(x) \tag{9c}$$

$$x \ni [0, \ell]$$

The operators in (1a, b, c) are given in equation (5d) of Appendix Ib.

B1. $n > \sqrt{\frac{2}{1-\nu}}\,\tilde{\omega}$ (Valid for $\tilde{\omega}_{LF}$ for $n \geq 2$)

Solving for $u_n(x)$ and $v_n(x)$ in (9a) and (9b), then substituting the homogeneous solutions in equation (41) of the main text to find the relation between the constants of integration of $u_n(x)$ and $v_n(x)$, we finally obtain:

$$u_n(x) = B_{n1} \sinh(\beta_1 x) + B_{n2} \cosh(\beta_1 x) + B_{n3} \sinh(\beta_2 x) + B_{n4} \cosh(\beta_2 x) + \int^x \frac{\mathscr{E}_{1n}[w_n(\xi)]}{(\beta_1^2 - \beta_2^2)} \left(\frac{\sinh\beta_1(x-\xi)}{\beta_1} - \frac{\sinh\beta_2(x-\xi)}{\beta_2} \right) d\xi$$

$$v_n(x) = \frac{n}{\beta_1}\left\{ B_{n1} \cosh(\beta_1 x) + B_{n2} \sinh(\beta_1 x)\right\} + \frac{\beta_2}{n} B_{n3} \cosh(\beta_2 x) + \frac{\beta_2}{n} B_{n4} \sinh(\beta_2 x) + \int^x \frac{\mathscr{E}_{2n}[w_n(\xi)]}{(\beta_1^2 - \beta_2^2)} \left(\frac{\sinh\beta_1(x-\xi)}{\beta_1} - \frac{\sinh\beta_2(x-\xi)}{\beta_2} \right) d\xi$$

$$\beta_1^2 = n^2 - \tilde{\omega}^2, \qquad \beta_2^2 = n^2 - \frac{2}{1-\nu}\,\tilde{\omega}^2 \tag{10}$$

The inplane boundary conditions for SS2/SS2 are:

$$u_n(0) = u_n(\ell) = v_n(0) = v_n(\ell) = 0 \tag{11}$$

We can solve for B_{nj} by substituting (10) in (11).

Assume an expansion for $w_n(x)$ that satisfies all out-of-plane boundary conditions:

$$w_n(x) = \sum_{j=1}^{j*} a_{nj} \sin(s_j x), \quad s_j = \frac{j\pi}{\ell} \tag{12}$$

Eliminating $u_n(x)$ and $v_n(x)$ from (9c), substituting for $w_n(x)$ the series given by (12), then applying Galerkin's method, we obtain:

$$\left[\tilde{\mathcal{F}}_{nij}\right] \{a_{nj}\} = 0 \qquad (i, j = 1, 2, \ldots, j*) \tag{13a}$$

$$\begin{aligned}
\tilde{\mathcal{F}}_{nij} = {} & \left[r^2(s_i^2+n^2)^2 + 1 - \tilde{\omega}^2 + \frac{\nu s_i f_{1i} + n f_{2i}}{(s_i^2+\beta_1^2)(s_i^2+\beta_2^2)} \right] \delta_{ij} \\
& \frac{+2(-)^i}{\ell} \left\{ n b_{1j} \left(\frac{(1-\nu)\gamma_{2i}}{\beta_2(1+\gamma_{2i}^2)} (\cosh(\beta_2 \ell) - (-)^i) \right.\right. \\
& \qquad \left. - \frac{(1-\nu \frac{\beta_1^2}{n^2})\gamma_{1i}}{\beta_1(1+\gamma_{1i}^2)} (\cosh(\beta_1 \ell) - (-)^i) \right) \\
& + b_{2j} \left(\frac{(1-\nu)\gamma_{2i}}{(1+\gamma_{2i}^2)} \sinh(\beta_2 \ell) - \frac{(\frac{n^2}{\beta_1^2} - \nu)\gamma_{1i}}{(1+\gamma_{1i}^2)} \sinh(\beta_1 \ell) \right) \\
& \left. + \frac{(1-\nu)\gamma_{2i} f_{1j} \sinh(\beta_2 \ell)}{(1+\gamma_{2i}^2)(s_j^2+\beta_1^2)(s_j^2+\beta_2^2)} \right\} \qquad \begin{matrix} (\delta_{ij} = 0, \; i \neq j) \\ (\delta_{ij} = 1, \; i = j) \end{matrix}
\end{aligned} \tag{13b}$$

$$\gamma_{1i} = \frac{s_i}{\beta_1}, \qquad \gamma_{2i} = \frac{s_i}{\beta_2}$$

$$\begin{aligned}
b_{1j} = & \left\{ g_{1j} \left(\frac{n}{\beta_1} \sinh(\beta_1 \ell) - \frac{\beta_2}{n} \sinh(\beta_2 \ell) \right) \right. \\
& \left. - g_{2j} \left(\cosh(\beta_1 \ell) - \cosh(\beta_2 \ell) \right) \right\} \cdot D_s
\end{aligned}$$

$$b_{2j} = \left\{ g_{2j}\left(\frac{\beta_1}{n}\sinh(\beta_1\ell) - \frac{n}{\beta_2}\sinh(\beta_2\ell)\right) - g_{1j}\left(\cosh(\beta_1\ell) - \cosh(\beta_2\ell)\right)\right\} \cdot D_s$$

$$g_{1j} = \frac{f_{1j}(\cosh(\beta_2\ell) - (-)^j)}{(s_j^2+\beta_1^2)(s_j^2+\beta_2^2)}$$

$$g_{2j} = \frac{f_{1j}\frac{\beta_2}{n}\sinh(\beta_2\ell)}{(s_j^2+\beta_1^2)(s_j^2+\beta_2^2)}$$

$$D_s^{-1} = 2\left(\cosh(\beta_1\ell)\cosh(\beta_2\ell) - \frac{1}{2}\left(\frac{\beta_1\beta_2}{n^2} + \frac{n^2}{\beta_1\beta_2}\right) \times \sinh(\beta_1\ell)\sinh(\beta_2\ell) - 1\right)$$

$$f_{1j} = \left(n^2(1+\nu) - \nu(\beta_2^2 + s_j^2) + \frac{1+\nu}{1-\nu} r^2 n^2 (s_j^2 + n^2)\right) s_j$$

$$f_{2j} = -n\left(\beta_2^2 + (2+\nu)s_j^2 + r^2\left(\beta_2^2 + \frac{2}{1-\nu}s_j^2\right)(s_j^2+n^2)\right) \qquad (13.c)$$

B2 - $\sqrt{\frac{2}{1-\nu}}\,\tilde{\omega} > n > \tilde{\omega}$ (Valid for $\tilde{\omega}_{LF}$ for n = 1, and $\tilde{\omega}_{IF}$ for $n \geq 2$)

Since all steps are similar to case B1, only the final equations will be mentioned. Solving for $u_n(x)$ and $v_n(x)$ in (9a) and (9b):

$$u_n(x) = B_{n_1}\sinh(\beta_1 x) + B_{n_2}\cosh(\beta_1 x) + B_{n_3}\sin(\beta_2 x) + B_{n_4}\cos(\beta_2 x)$$

$$+ \frac{1}{\beta_1^2+\beta_2^2}\int^{x} \mathcal{E}_{1n}[w_n(\xi)]\left(\frac{\sinh(\beta_1(x-\xi))}{\beta_1} - \frac{\sin(\beta_2(x-\xi))}{\beta_2}\right)d\xi$$

$$v_n(x) = \frac{n}{\beta_1}\left\{B_{n1}\cosh(\beta_1 x) + B_{n2}\sinh(\beta_1 x)\right\} + \frac{\beta_2}{n} B_{n3}\cos(\beta_2 x) + B_{n4}\sin(\beta_2 x)\frac{\beta_2}{n} + \int^{x} \frac{\mathcal{E}_{2n}\left[w_n(\xi)\right]}{\beta_1^2+\beta_2^2}\left(\frac{\sinh(\beta_1(x-\xi))}{\beta_1} - \frac{\sin(\beta_2(x-\xi))}{\beta_2}\right) d\xi$$

$$\beta_1^2 = n^2 - \tilde{\omega}^2 \quad , \quad \beta_2^2 = \frac{2}{1-\nu}\tilde{\omega}^2 - n^2 \tag{14}$$

The eigenmatrix $\left[\tilde{\mathcal{F}}_{nij}\right]$ is as follows :

$$\tilde{\mathcal{F}}_{nij} = \left[r^2(s_i^2+n^2)^2 + 1 - \tilde{\omega}^2 + \frac{\nu s_i f_{1i} + n f_{2i}}{(s_i^2+\beta_1^2)(s_i^2-\beta_2^2)}\right]\delta_{ij}$$
$$+\frac{2(-)^i}{\ell}\left\{ n\, b_{1j}\left(\frac{-(1-\nu)\gamma_{2i}}{\beta_2(1-\gamma_{2i}^2)}(\cos(\beta_2\ell) - (-)^i) - \frac{(1-\frac{\nu\beta_1^2}{n^2})\gamma_{1i}}{\beta_1(1+\gamma_{1i}^2)}(\cosh(\beta_1\ell) - (-)^i)\right)\right.$$
$$+ b_{2j}\left(\frac{(1-\nu)\gamma_{2i}}{(1-\gamma_{2i}^2)}\sin(\beta_2\ell) - \frac{(\frac{n^2}{\beta_1^2}-\nu)\gamma_{1i}}{(1+\gamma_{1i}^2)}\sinh(\beta_1\ell)\right)$$
$$\left. + \frac{(1-\nu)\gamma_{2i} f_{1j}\sin(\beta_2\ell)}{(1-\gamma_{2i}^2)(s_j^2+\beta_1^2)(s_j^2-\beta_2^2)}\right\} \qquad \begin{matrix}(\delta_{ij}=0,\ i\neq j)\\ (\delta_{ij}=1,\ i=j)\end{matrix} \tag{15a}$$

$$\gamma_{1i} = \frac{s_i}{\beta_1} \quad , \quad \gamma_{2i} = \frac{s_i}{\beta_2}$$

$$b_{1j} = \left\{ g_{1j}\left(\frac{n}{\beta_1} \sinh(\beta_1 \ell) + \frac{\beta_2}{n} \sin(\beta_2 \ell)\right) - g_{2j}\left(\cosh(\beta_1 \ell) - \cos(\beta_2 \ell)\right)\right\} D_s$$

$$b_{2j} = \left\{ g_{2j}\left(\frac{\beta_1}{n} \sinh(\beta_1 \ell) - \frac{n}{\beta_2} \sin(\beta_2 \ell)\right) - g_{1j}\left(\cosh(\beta_1 \ell) - \cos(\beta_2 \ell)\right)\right\} D_s$$

$$g_{1j} = \frac{-f_{1j}(\cos(\beta_2 \ell) - (-)^j)}{(s_j^2 + \beta_1^2)(s_j^2 - \beta_2^2)}$$

$$g_{2j} = \frac{-\beta_2 \; f_{1j} \sin(\beta_2 \ell)}{n\,(s_j^2 + \beta_1^2)(s_j^2 - \beta_2^2)}$$

$$D_s^{-1} = 2\left(\cosh(\beta_1 \ell)\cos(\beta_2 \ell) - \frac{1}{2}\left(\frac{n^2}{\beta_1 \beta_2} - \frac{\beta_1 \beta_2}{n^2}\right) \times \sinh(\beta_1 \ell)\sin(\beta_2 \ell) - 1\right)$$

$$f_{1j} = s_j\left(n^2(1+\nu) + (\beta_2^2 - s_j^2) + \frac{1+\nu}{1-\nu} r^2 n^2 (s_j^2 + n^2)\right)$$

$$f_{2j} = n\left(\beta_2^2 - (2+\nu)\, s_j^2 + r^2(\beta_2^2 - \frac{2}{1-\nu}\, s_j^2)(s_j^2+n^2)\right) \qquad (15b)$$

B3 - $n < \tilde{\omega}$ (Valid for $\tilde{\omega}_{IF}$ for $n = 1$, and $\tilde{\omega}_{HF}$ for $n \geqslant 1$)

The displacement functions $u_n(x)$ and $v_n(x)$ are given by :

$$u_n(x) = B_{n1} \sin(\beta_1 x) + B_{n2} \cos(\beta_1 x) + B_{n3} \sin(\beta_2 x) + B_{n4} \cos(\beta_2 x)$$

$$+ \frac{1}{\beta_2^2 - \beta_1^2} \int^x \mathcal{E}_{1n}\,[w_n(\xi)] \left(\frac{\sin(\beta_1(x-\xi))}{\beta_1} - \frac{\sin(\beta_2(x-\xi))}{\beta_2}\right) d\xi$$

$$v_n(x) = \frac{n}{\beta_1}\left\{-B_{n1}\cos(\beta_1 x) + B_{n2}\sin(\beta_1 x)\right\}$$

$$+ \frac{\beta_2}{n}\left\{B_{n3}\cos(\beta_2 x) + B_{n4}\sin(\beta_2 x)\right\} +$$

$$\int^{x} \frac{\mathcal{E}_{2n}[w_n(\xi)]}{\beta_2^2 - \beta_1^2}\left(\frac{\sin(\beta_1(x-\xi))}{\beta_1} - \frac{\sin(\beta_2(x-\xi))}{\beta_2}\right) d\xi$$

$$\beta_1^2 = \tilde{\omega}^2 - n^2 \quad , \quad \beta_2^2 = \frac{2}{1-\nu}\tilde{\omega}^2 - n^2 \qquad (16)$$

The eigenmatrix $\left[\tilde{\mathcal{F}}_{nij}\right]$ is as follows :

$$\tilde{\mathcal{F}}_{nij} = \left[r^2(s_i^2+n^2)^2 + 1 - \tilde{\omega}^2 + \frac{\nu s_i f_{1i} + n f_{2i}}{(s_i^2-\beta_1^2)(s_i^2-\beta_2^2)}\right] \delta_{ij}$$

$$+ \frac{2(-)^i}{\ell}\left\{ n b_{1j}\left(\frac{(1-\nu)\gamma_{2i}}{\beta_2(1-\gamma_{2i}^2)}(\cos(\beta_2\ell) - (-)^i) - \right.\right.$$

$$\left. - \frac{(1+\frac{\nu\beta_1^2}{2})\gamma_{1i}}{\frac{n}{\beta_1(1-\gamma_{1i}^2)}}(\cos(\beta_1\ell) - (-)^i)\right)$$

$$+ b_{2j}\left(\frac{(1-\nu)\gamma_{2i}}{\beta_2(1-\gamma_{2i}^2)}\sin(\beta_2\ell) + \frac{(\frac{n^2}{\beta_1^2} - \nu)\gamma_{1i}}{(1-\gamma_{1i}^2)}\sin(\beta_1\ell)\right)$$

$$\left. + \frac{(1-\nu)\gamma_{2i} f_{1j}\sin(\beta_2\ell)}{(1-\gamma_{2i}^2)(s_j^2-\beta_1^2)(s_j^2-\beta_2^2)}\right\} \qquad \begin{matrix}(\delta_{ij} = 0,\ i \neq j)\\ (\delta_{ij} = 0,\ i = j)\end{matrix} \qquad (17a)$$

$$\gamma_{1i} = \frac{s_i}{\beta_1} \quad , \quad \gamma_{2i} = \frac{s_i}{\beta_2}$$

$$b_{1j} = \left\{ g_{1j}\left(\frac{n}{\beta_1}\sin(\beta_1\ell) + \frac{\beta_2}{n}\sin(\beta_2\ell)\right) - g_{2j}\left(\cos(\beta_1\ell) - \cos(\beta_2\ell)\right)\right\} D_s$$

$$b_{2j} = \left\{ g_{2j}\left(\frac{\beta_1}{n}\sin(\beta_1\ell) + \frac{n}{\beta_2}\sin(\beta_2\ell)\right) + g_{1j}\left(\cos(\beta_1\ell) - \cos(\beta_2\ell)\right)\right\} D_s$$

$$g_{1j} = \frac{f_{1j}(\cos(\beta_2\ell) - (-)^j)}{(s_j^2-\beta_1^2)(s_j^2-\beta_2^2)}$$

$$g_{2j} = \frac{-\beta_2 \ f_{1j} \sin(\beta_2\ell)}{n(s_j^2-\beta_1^2)(s_j^2-\beta_2^2)} \tag{17b}$$

$$D_s^{-1} = -2\left(\cos(\beta_1\ell)\cos(\beta_2\ell) - \frac{1}{2}\left(\frac{n^2}{\beta_1\beta_2} + \frac{\beta_1\beta_2}{n^2}\right)\sin(\beta_1\ell)\sin(\beta_2\ell) - 1\right)$$

f_{1j} and f_{2j} have the same form as given in equation (15b).

CASE (II): TANGENTIAL INERTIA NEGLECTED

(VALID FOR $n \geq 3$ FOR $\tilde{\omega}_{LF}$ ONLY)

The governing ordinary differential equations are :

$$\bar{\nabla}_n^4[u_n(x)] = T_{1n}[w_n(x)] \tag{18a}$$

$$\bar{\nabla}_n^4[v_n(x)] = T_{2n}[w_n(x)] \tag{18b}$$

$$\left\{r^2\bar{\nabla}_n^4 + 1 - \tilde{\omega}^2\right\} w_n(x) = \nu\frac{du_n(x)}{dx} - n\,v_n(x) \tag{18c}$$

T_{1n} and T_{2n} are the modified $\mathcal{E}_{1n}$ and $\mathcal{E}_{2n}$ respectively, given by equation (5d) of Appendix Ib, when $\tilde{\omega}$ is taken equal to zero in these operators. Solving for $u_n(x)$ and $v_n(x)$ in (18a) and (18b) :

$$u_n(x) = \left[B_{n1} + (nx)\, B_{n2}\right] \sinh(nx) + \left[B_{n3} + (nx)\, B_{n4}\right] \cosh(nx)$$
$$+ \frac{1}{2n^3} \int^x T_{1n}\left[w_n(\xi)\right] \left\{ n(x-\xi) \cosh(n(x-\xi)) - \sinh(n(x-\xi)) \right\} d\xi$$

$$v_n(x) = B_{n1} \cosh(nx) + B_{n2} \left[(nx) \cosh(nx) + \frac{3-\nu}{1+\nu} \sinh(nx)\right]$$
$$+ B_{n3} \sinh(nx) + B_{n4} \left[(nx) \sinh(nx) + \frac{3-\nu}{1+\nu} \cosh(nx)\right]$$
$$+ \int^x \frac{T_{2n}\left[w_n(\xi)\right]}{2n^3} \left\{ n(x-\xi) \cosh(n(x-\xi)) - \sinh(n(x-\xi)) \right\} d\xi \quad (19)$$

The eigenmatrix $\left[\tilde{\mathcal{F}}_{nij}\right]$ can then be written as follows :

$$\tilde{\mathcal{F}}_{nij} = \left\{ \tilde{\omega}^2_{i4} - \tilde{\omega}^2 - r^2 n^2 \left(1 + \frac{(1+\nu)\, s_i^2}{(s_i^2 + n^2)}\right) \right\} \delta_{ij} + P_{ij}\, f_{1j} \quad (20a)$$

$$P_{ij} = \frac{8(1-\nu)\gamma_i \left(\frac{1}{1+\gamma_i^2} - \frac{\nu}{1+\nu} \right)\left(\cosh(n\ell) - (-)^i\right)}{\ell\,(1+\gamma_i^2) \left[\frac{3-\nu}{1+\nu} \sinh(n\ell) + (-)^i (n\ell)\right] (s_j^2 + n^2)^2}$$

$$f_{1j} = s_j (n^2 - \nu s_j^2) + \frac{1+\nu}{1-\nu}\, r^2 n^2\, s_j (n^2 + s_j^2)\,, \quad \left(s_j = \frac{j\pi}{\ell}\right)$$

$$\tilde{\omega}^2_{i4} = r^2 (s_i^2 + n^2)^2 + \frac{(1-\nu^2)\, s_i^4}{(s_i^2 + n^2)^2}$$

$\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i=j$

$$\gamma_i = \frac{s_i}{n}\,, \quad s_i = \frac{i\pi}{\ell} \quad (20b)$$

$\tilde{\omega}_{i4}$ is actually the low eigenfrequency corresponding to a mode shape with "n" circumferential full waves and "i" axial half waves, of a cylindrical shell with boundary condition SS1/SS1, when tangential inertia is neglected (see Appendix III).

APPENDIX VII

DIFFERENTIAL OPERATORS AND INFLUENCE MATRICES FOR THE END-RING

The uncoupled differential equations for the ring are:

$$\begin{bmatrix} D^{(i)}_{r,ij} \end{bmatrix} \begin{Bmatrix} v_r \\ w_r \end{Bmatrix} = \begin{Bmatrix} F^{(i)}_{r,i} \end{Bmatrix}$$

$$\begin{bmatrix} D^{(o)}_{r,ij} \end{bmatrix} \begin{Bmatrix} u_r \\ \beta_r \end{Bmatrix} = \begin{Bmatrix} F^{(o)}_{r,i} \end{Bmatrix} \qquad (i, j = 1, 2) \qquad (1)$$

where,

$$D^{(i)}_{r,11} = -\frac{\partial}{\partial\theta}\left[\frac{\partial^2}{\partial\theta^2} + 1 - \frac{a^2}{(1-\nu^2)}\frac{\partial^2}{\partial\tau^2}\right]$$

$$D^{(i)}_{r,12} = \frac{\partial^2}{\partial\theta^2} + 1 + \frac{\tilde{a}^2}{(1-\nu^2)}\frac{\partial^2}{\partial\tau^2}$$

$$D^{(i)}_{r,21} = \frac{\partial}{\partial\theta}\left[r_y^2\frac{\partial^2}{\partial\theta^2} - 1 - r_y^2\frac{\tilde{a}^2}{(1-\nu^2)}\frac{\partial^2}{\partial\tau^2}\right]$$

$$D^{(i)}_{r,22} = r_y^2\frac{\partial^4}{\partial\theta^4} + 1 + \frac{\tilde{a}^2}{1-\nu^2}\frac{\partial^2}{\partial\tau^2}\left(1 - r_y^2\frac{\partial^2}{\partial\theta^2}\right)$$

$$D^{(o)}_{r,11} = r_x^2\frac{\partial^4}{\partial\theta^4} - \frac{r_t^2}{2(1+\nu)}\frac{\partial^2}{\partial\theta^2} + \frac{\tilde{a}^2}{1-\nu^2}\frac{\partial^2}{\partial\tau^2}\left(1 - r_x^2\frac{\partial^2}{\partial\theta^2}\right)$$

$$D^{(o)}_{r,12} = -\left[r_x^2 + \frac{r_t^2}{2(1+\nu)}\right]\frac{\partial^2}{\partial\theta^2} = D^{(o)}_{r,21}$$

$$D^{(o)}_{r,22} = -\left[\frac{r_t^2}{2(1+\nu)}\frac{\partial^2}{\partial\theta^2} - r_x^2 - r_p^2\frac{\tilde{a}^2}{(1-\nu^2)}\frac{\partial^2}{\partial\tau^2}\right]$$

$$\left\{F^{(i)}_{r,i}\right\} = \begin{bmatrix} f_{c1} & f_{c2} \\ f_{c1} & -f_{c3} \end{bmatrix} \begin{Bmatrix} q_{xx}(\ell/2) \\ \dfrac{\partial n_{x\theta}(\ell/2)}{\partial\theta} \end{Bmatrix}$$

$$\left\{F^{(o)}_{r,i}\right\} = \begin{bmatrix} -f_{c4} & f_{c7} & 0 & 0 \\ f_{c5} & 0 & -f_{c6} & f_{c1} \end{bmatrix} \begin{bmatrix} n_{xx}(\ell/2) \\ \dfrac{\partial n_{x\theta}(\ell/2)}{\partial\theta} \\ q_{xx}(\ell/2) \\ m_{xx}(\ell/2) \end{bmatrix}$$

f_{ck} (k = 1 to 7) are constant factors that depend on the cross section properties of the ring and are given in Table XI. For free vibration, system (1) has an exact solution as given by equation (58) of this work. Such a solution leads to a system of algebraic linear equations of the form:

$$\left[L^{(i)}_{r,ij}\right] \begin{Bmatrix} v_{rn} \\ w_{rn} \end{Bmatrix} = \left\{F^{(i)}_{r,ni}\right\}$$
$$\left[L^{(o)}_{r,ij}\right] \begin{Bmatrix} u_{rn} \\ \beta_{rn} \end{Bmatrix} = \left\{F^{(o)}_{r,ni}\right\} \qquad (i,j = 1,2) \qquad (2)$$

where,

$$L^{(i)}_{r,11} = -n(n^2 - 1 - \tilde{\omega}^2)$$

$$L^{(i)}_{r,12} = -(n^2 - 1 + \tilde{\omega}^2)$$

$$L^{(i)}_{r,21} = n(r_y^2\, n^2 + 1 - \tilde{\omega}^2\, r_y^2)$$

$$L^{(i)}_{r,22} = r_y^2 n^4 + 1 - \bar{\omega}^2 (r_y^2 n^2 + 1)$$

$$L^{(o)}_{r,11} = n^2 \left(r_x^2 n^2 + \frac{r_t^2}{2(1+\nu)}\right) - \bar{\omega}^2 (1 + r_x^2 n^2)$$

$$L^{(o)}_{r,12} = n^2 \left(r_x^2 + \frac{r_t^2}{2(1+\nu)}\right) = L^{(o)}_{r,21}$$

$$L^{(o)}_{r,22} = r_x^2 + \frac{r_t^2 n^2}{2(1+\nu)} - r_p^2 \bar{\omega}^2$$

$$\bar{\omega}^2 = \tilde{\omega}^2 \frac{\tilde{a}^2}{(1-\nu^2)} \; , \quad \tilde{a} = \frac{a_r}{a}$$

r_x, r_y, r_t and r_p are the nondimensional radii of gyration of the ring cross section.

$$\left\{ F^{(i)}_{r,ni} \right\} = \begin{bmatrix} f_{c1} & -n f_{c2} \\ f_{c1} & n f_{c3} \end{bmatrix} \begin{Bmatrix} \tilde{q}_{xx}(\ell/2) \\ \tilde{n}_{x\theta}(\ell/2) \end{Bmatrix}$$

$$\left\{ F^{(o)}_{r,ni} \right\} = \begin{bmatrix} -f_{c4} & -n f_{c7} & 0 & 0 \\ f_{c5} & 0 & -f_{c6} & f_{c1} \end{bmatrix} \begin{bmatrix} \tilde{n}_{xx}(\ell/2) \\ \tilde{n}_{x\theta}(\ell/2) \\ \tilde{q}_{xx}(\ell/2) \\ \tilde{m}_{xx}(\ell/2) \end{bmatrix}$$

$$\begin{Bmatrix} n_{xx}(\ell/2) \\ q_{xx}(\ell/2) \\ m_{xx}(\ell/2) \end{Bmatrix} = \begin{Bmatrix} \tilde{n}_{xx}(\ell/2) \\ \tilde{q}_{xx}(\ell/2) \\ \tilde{m}_{xx}(\ell/2) \end{Bmatrix} \quad \sin(n\theta) \quad \sin(\tilde{\omega}\tau)$$

$$n_{x\theta}(\ell/2) = \tilde{n}_{x\theta}(\ell/2) \quad \cos(n\theta) \quad \sin(\tilde{\omega}\tau)$$

Solving for w_{rn}, u_{rn}, v_{rn} and β_{rn} in (2) we obtain:

$$\begin{bmatrix} w_{rn} \\ u_{rn} \\ v_{rn} \\ \beta_{rn} \end{bmatrix} = \begin{bmatrix} M_{d,ij} \end{bmatrix} \begin{bmatrix} \tilde{q}_{xx}(\ell/2) \\ \tilde{n}_{xx}(\ell/2) \\ \tilde{n}_{x\theta}(\ell/2) \\ \tilde{m}_{xx}(\ell/2) \end{bmatrix} \qquad (i, j = 1 \text{ to } 4) \qquad (3)$$

$$\begin{bmatrix} M_{d,ij} \end{bmatrix} = \begin{bmatrix} f_{c1}\dfrac{(L^{(i)}_{r,21} - L^{(i)}_{r,11})}{D^{(1)}_{tr}} & 0 & -n\dfrac{(f_{c2}L^{(i)}_{r,21} + f_{c3}L^{(i)}_{r,11})}{D^{(1)}_{tr}} & 0 \\ \dfrac{f_{c6}L^{(o)}_{r,12}}{D^{(2)}_{tr}} & \dfrac{-(f_{c4}L^{(o)}_{r,22} + f_{c5}L^{(o)}_{r,12})}{D^{(2)}_{tr}} & -n\dfrac{L^{(o)}_{r,22}f_{c7}}{D^{(2)}_{tr}} & \dfrac{+f_{c1}L^{(o)}_{r,12}}{D^{(2)}_{tr}} \\ -f_{c1}\dfrac{(L^{(i)}_{r,12} - L^{(i)}_{r,22})}{D^{(1)}_{tr}} & 0 & n\dfrac{(f_{c2}L^{(i)}_{r,22} + f_{c3}L^{(i)}_{r,12})}{D^{(1)}_{tr}} & 0 \\ \dfrac{-f_{c6}L^{(o)}_{r,11}}{D^{(2)}_{tr}} & \dfrac{(f_{c4}L^{(o)}_{r,21} + f_{c5}L^{(o)}_{r,11})}{D^{(2)}_{tr}} & n\dfrac{L^{(o)}_{r,21}f_{c7}}{D^{(2)}_{tr}} & \dfrac{-f_{c1}L^{(o)}_{r,11}}{D^{(2)}_{tr}} \end{bmatrix}$$

$$D^{(1)}_{tr} = \det\begin{bmatrix} L^{(i)}_{r,ij} \end{bmatrix}, \quad D^{(2)}_{tr} = \det\begin{bmatrix} L^{(o)}_{r,ij} \end{bmatrix}$$

Finally, matching of the displacements and slope of the shell at the boundary where the ring is attached, with the displacements and rotation of the ring leads to the following relations:

$$\begin{bmatrix} w_{rn} \\ u_{rn} \\ v_{rn} \\ \beta_{rn} \end{bmatrix} = \begin{bmatrix} M_{T,ij} \end{bmatrix} \begin{bmatrix} w_n(\ell/2) \\ u_n(\ell/2) \\ v_n(\ell/2) \\ w'_n(\ell/2) \end{bmatrix} \tag{4}$$

$$\begin{bmatrix} M_{T,ij} \end{bmatrix} = \begin{bmatrix} \tilde{a} & 0 & 0 & \frac{\tilde{d}\,\tilde{h}^2}{2(a/h)} \\ 0 & \tilde{a} & 0 & -\frac{\tilde{h}_e\tilde{h}}{(a/h)} \\ -\frac{n\,\tilde{h}_e\tilde{h}}{(a/h)} & 0 & -\frac{\tilde{h}_e\tilde{h}}{(a/h)}+1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad (i, j = 1 \text{ to } 4)$$

The nondimensional ring parameters in the above matrix are given in Table XI. These parameters correspond to a symmetric section (symetric "H" type section) that is attached in the interior of the shell (see fig. 10). Combining (3) and (4) leads to four homogeneous boundary conditions relating the displacements and slope of the shell at the boundary of the ring, to the nondimensional stress and moment resultants of the shell at the same end:

$$\begin{bmatrix} M_{d,ij} \end{bmatrix} \begin{bmatrix} w_n(\ell/2) \\ u_n(\ell/2) \\ v_n(\ell/2) \\ w'_n(\ell/2) \end{bmatrix} = \begin{bmatrix} M_{T,ij} \end{bmatrix} \begin{bmatrix} \tilde{q}_{xx}(\ell/2) \\ \tilde{n}_{xx}(\ell/2) \\ \tilde{n}_{x\theta}(\ell/2) \\ \tilde{m}_{xx}(\ell/2) \end{bmatrix} \tag{5}$$

$$(i, j=1 \text{ to } 4)$$

APPENDIX VIII

SIMPLICITY OF THE EIGENVALUES FOR THE LINEAR DYNAMIC SHELL EQUATIONS

It is proved in this Appendix that for fixed values of the circumferential wave number "n", there exists only one independent eigenfunction for each eigenfrequency "$\tilde{\omega}$" for all admissible homogeneous natural boundary conditions.

PROOF:

We will first show that the problem:

$$L\,[X] = 0 \tag{1a}$$

$$U\,[X] = 0 \tag{1b}$$

$$U\,[X] = [h_{ij}] \begin{Bmatrix} X(0) \\ X'(0) \\ \vdots \\ X^{(n_d-1)}(0) \end{Bmatrix} + [g_{ij}] \begin{Bmatrix} X(\ell) \\ X'(\ell) \\ \vdots \\ X^{(n_d-1)}(\ell) \end{Bmatrix} \tag{1c}$$

$i, j \quad = 1 \text{ to } n_d$

$h_{ij} \quad = 0 \text{ for } i > \frac{n_d}{2} \text{ and all } j$

$g_{ij} \quad = 0 \text{ for } i < \frac{n_d}{2} \text{ and all } j$

$$\det\left[[h_{ij}] + [g_{ij}] \right] \neq 0$$

has only one independent solution if and only if the corresponding eigenmatrix

$$\mathscr{S} = U\left[\tilde{\Phi}\right] \tag{2}$$

has rank (n_d - 1), where "n_d" is the order of the homogeneous ordinary differential equation (1a) and " $\tilde{\Phi}$ " is the Wronskian formed by the foundamental set of independent solutions corresponding to (1a). "$U\left[X\right]$" is a concise form of the homogeneous boundary conditions imposed on (1a) and is often referred to as "the vector boundary form". Let

$$\phi_1(x), \quad \phi_2(x), \; \ldots\ldots, \; \phi_{n_d}(x) \tag{3}$$

be the "n_d" linearly independent solutions forming the fundamental set for (1a), then

$$X = \sum_{i=1}^{n_d} C_i \quad \phi_i(x) \tag{4}$$

Substituting (4) into (1c), we get

$$U\left[\tilde{\Phi}\right]\begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_{n_d} \end{pmatrix} = 0 \tag{5}$$

$$\tilde{\Phi} = \begin{bmatrix} \phi_1 & \phi_2 & \cdot & \cdot & \cdot & \phi_{n_d} \\ \phi_1' & \phi_2' & \cdot & \cdot & \cdot & \phi_{n_d}' \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \phi_1^{(n_d-1)} & \cdot & \cdot & \cdot & \cdot & \phi_{n_d}^{(n_d-1)} \end{bmatrix}$$

System (5) has a nontrivial solution if and only if

$$\det\left[\mathscr{A}\right] = \det\left[U\left[\tilde{\Phi}\right]\right] = 0 \tag{6}$$

If the rank of "$\mathscr{A}$" is $(n_d - 1)$, this will imply that system (1) has only one linearly independent solution. This is based on the theorem which states that problem (1) has exactly "k" $(0 \leqslant k \leqslant n_d)$ linearly independent solutions if and only if the corresponding vector boundary form "$\mathscr{A}$" has rank $(n_d - k)$, where "n_d" is the order of "$L[X]$" (see reference 25 page 291).

We now proceed in applying the above result to the linear eigenvalue problem for a cylindrical shell. For $\tilde{\omega} = \tilde{\omega}_{LF}$ and $n \geqslant 2$, the resulting characteristic equation is given by

$$C_n(1)\tilde{\lambda}^4_{nj*} + C_n(2)\tilde{\lambda}^3_{nj*} + C_n(3)\tilde{\lambda}^2_{nj*} + C_n(4)\tilde{\lambda}_{nj*} + C_n(5) = 0 \tag{7}$$

where $\tilde{\lambda}_{nj*} = \lambda^2_{nj}$. The above coefficients "$C_n(k)$" are given in Appendix IV. (j*=1 to 4 and j=1 to 8)

From previous results we know that for $\tilde{\omega} = \tilde{\omega}_{LF}$ and $n \geqslant 2$:

$$1 > \tilde{\omega}^2 > r^2 n^4 \simeq (\tilde{\omega}^2)_{ring} \tag{8}$$

The bounds on "$\tilde{\omega}_{LF}$" in (8) require that for positive $\tilde{\lambda}$, we must have

$$C_n(1) \rightarrow (+), \quad C_n(2) \rightarrow (-), \quad C_n(3) \rightarrow (+), \quad C_n(4) \rightarrow (\pm), \quad C_n(5) \rightarrow (-)$$

We see that the number of sign alternations in the "$C_n(k)$" is "three" implying that (7) has either "three" or "one" positive real root. Similarly one can show that (7) has at most "one" negative real root. For the above range of "$\tilde{\omega}$" and "n", (7) has one complex pair of roots thus deleting the possibility of "three" real positive roots. The form of "λ_{nj}" is then:

$$\lambda_{n1,2} = \pm \alpha_1, \quad \lambda_{n3,4} = \pm i\alpha_2, \quad \lambda_{n5,6,7,8} = \pm(\alpha_3 \pm i\alpha_4) \tag{9}$$

The roots in (9) are distinct leading to a foundamental set of eight distinct and independent solutions. This result implies that the eigen-matrix columns will be linearly independent. And since the eigen-determinant itself must vanish for a nontrivial solution to exist, thus the largest submatrix with nonvanishing determinant will be a seven-by-seven. We conclude that the eigenmatrix has rank "seven" and thus there exists only one linearly independent eigenfunction for each eigenfrequency.

TABLE I

ERROR IN THE LOW FREQUENCY AND ITS CORRESPONDING

DISPLACEMENT RATIO FOR A RING

k_θ	Neglecting	$\tilde{\omega}_{LF}(n \geq 2)$	e $(n \geq 2)$	e $n \gg 2$	$\% e_{max}$	$(\tilde{v}_n)_{LF}$ $(n \geq 2)$	e
$k_\theta = \frac{\partial^2 w}{\partial \theta^2} + \frac{\partial v}{\partial \theta}$	$r^2 \frac{\partial^2}{\partial \tau^2} (\nabla^2 w)$	$\frac{rn(n^2-1)}{\sqrt{n^2+1}}$	$\frac{(rn)^2}{2}$	$\frac{(rn)^2}{2}$		$-\frac{1}{n}\left(1 + \frac{2r^2n^2(n^2-1)}{n^2+1}\right)$	$(rn)^2$
	q_θ	$rn^2\sqrt{\frac{n^2-1}{n^2+1}}$	$\frac{n}{\sqrt{n^2-1}} - 1$	$\frac{1}{2n^2}$	15.47 $n = 2$	$-\frac{1}{n}\left(1 + \frac{r^2n^2(n^2-1)}{n^2+1}\right)$	$o(rn)^4$
	$\frac{\partial^2 v}{\partial \tau^2}$	$r(n^2-1)$	$\frac{\sqrt{n^2+1}}{n} - 1$	$\frac{1}{2n^2}$	11.805 $n = 2$	$-\frac{1}{n}(1 + r^2n^2)$	$o(rn)^4$

TABLE I (Cont'd)

ERROR IN THE LOW FREQUENCY AND ITS CORRESPONDING

DISPLACEMENT RATIO FOR A RING

k_θ	Neglecting	$\tilde{\omega}_{LF} (n \geq 2)$	$e\ (n \geq 2)$	e $n \gg 2$	$\% e_{max}$	$(\tilde{v}_n)_{LF}\ (n \geq 2)$	e
$k_\theta = \frac{\partial^2 w}{\partial \theta^2}$		$rn^2 \sqrt{\frac{n^2-1}{n^2+1}}$	$\frac{n}{\sqrt{n^2-1}} - 1$	$\frac{1}{2n^2}$	15.47 $n = 2$	$-\frac{1}{n}\left(1 + \frac{r^2 n^2 (n^2-1)}{n^2+1}\right)$	$o(rn)^4$
	q_θ	$\frac{rn^3}{\sqrt{n^2+1}}$	$\frac{n^2}{n^2-1} - 1$	$\frac{1}{n^2}$	33.33 $n = 2$	$-\frac{1}{n}\left(1 + \frac{r^2 n^4}{n^2+1}\right)$	$o(rn)^4$
	$q_\theta, \frac{\partial^2 v}{\partial \tau^2}$	rn^2	$\frac{n\sqrt{n^2+1}}{n^2-1} - 1$	$\frac{3}{2n^2}$	49.08 $n = 2$	$-\frac{1}{n}$	$o(rn)^4$

TABLE II

ERROR IN THE HIGH FREQUENCY AND ITS CORRESPONDING

DISPLACEMENT RATIO FOR A RING

k_θ	Neglecting	$\tilde{\omega}_{HF}$ $(n \geq 2)$	e	$(\tilde{v}_n)_{HF}$ $(n \geq 2)$	e
$k_\theta = \frac{\partial^2 w}{\partial \theta^2} + \frac{\partial v}{\partial \theta}$	$r^2 \frac{\partial^2}{\partial \tau^2} (\nabla^2 w)$	$\sqrt{(n^2+1)\left(1 + \frac{r^2(n^6+6n^4+1)}{(n^2+1)^2}\right)}$	$\frac{(rn)^2}{2}$	$n\left(1 - \frac{r^2 n^2(n^4+4n^2-2)}{n^2+1}\right)$	$o(r^2 n^4)$
	q_θ	$\sqrt{(n^2+1)\left(1 + \frac{r^2 n^2(n^2-1)}{(n^2+1)^2}\right)}$	$o(rn)^4$	$n\left(1 + \frac{2r^2 n^2}{n^2+1}\right)$	$o(rn)^2$
	$\frac{\partial^2 v}{\partial \tau^2}$	Deleted		Deleted	

TABLE II (Cont'd)

ERROR IN THE HIGH FREQUENCY AND ITS CORRESPONDING

DISPLACEMENT RATIO FOR A RING

k_θ	Neglecting	$\tilde{\omega}_{HF}$ $(n \geq 2)$	e	$(\tilde{v}_n)_{HF}$ $(n \geq 2)$	e
$k_\theta = \frac{\partial^2 w}{\partial \theta^2}$		$\sqrt{(n^2+1)\left(1 + \frac{r^2n^2(n^2-1)}{(n^2+1)^2}\right)}$	$o(rn)^4$	$n\left(1 + \frac{2r^2n^2}{n^2+1}\right)$	$o(rn)^2$
	q_θ	$\sqrt{(n^2+1)\left(1 - \frac{r^2n^2}{(n^2+1)^2}\right)}$	$o(rn)^4$	$n\left(1 + \frac{r^2n^2}{n^2+1}\right)$	$o(rn)^2$
	$q_\theta, \frac{\partial^2 v}{\partial \tau^2}$	Deleted		Deleted	

TABLE III

NOTATION

	$\tilde{\omega}$	Terms Neglected: v in k_{ij} and q_θ	Terms Neglected: $\left[\frac{\partial^2 u}{\partial \tau^2}, \frac{\partial^2 v}{\partial \tau^2}\right]$	Terms Neglected: $r^2 \frac{\partial^2}{\partial \tau^2}(\nabla^2 w)$	Error Notation
	$\tilde{\omega}^*$				
	$\tilde{\omega}_1^*$	X			$e_1^* = \frac{\Delta \tilde{\omega}_1^*}{\tilde{\omega}^*} = \frac{\tilde{\omega}_1^*}{\tilde{\omega}^*} - 1$
A					
	$\tilde{\omega}_2^*$		X		$e_2^* = \frac{\Delta \tilde{\omega}_2^*}{\tilde{\omega}^*} = \frac{\tilde{\omega}_2^*}{\tilde{\omega}^*} - 1$
	$\tilde{\omega}_3^*$			X	$e_3^* = \frac{\Delta \tilde{\omega}_3^*}{\tilde{\omega}^*} = \frac{\tilde{\omega}_3^*}{\tilde{\omega}^*} - 1$

TABLE III (Cont'd)

NOTATION

	$\tilde{\omega}$	Terms Neglected: v in k_{ij} and q_θ	$\left[\frac{\partial^2 u}{\partial \tau^2}, \frac{\partial^2 v}{\partial \tau^2}\right]$	$r^2 \frac{\partial^2}{\partial \tau^2}(\nabla^2 w)$	Error Notation
	$\tilde{\omega}_4$	X	X	X	$e_t = \frac{\tilde{\omega}_4}{\tilde{\omega}^*} - 1$
	$\tilde{\omega}_1$		X	X	$e_1 = \frac{\tilde{\omega}_4}{\tilde{\omega}_1} - 1$
B					
	$\tilde{\omega}_2$	X		X	$e_2 = \frac{\tilde{\omega}_4}{\tilde{\omega}_2} - 1$
	$\tilde{\omega}_3$	X	X		$e_3 = \frac{\tilde{\omega}_4}{\tilde{\omega}_3} - 1$

TABLE III (Cont'd)

NOTATION

	Terms Neglected			Error Notation
$\tilde{\varpi}$	v in k_{ij} and q_θ	$\left[\frac{\partial^2 u}{\partial \tau^2}, \frac{\partial^2 v}{\partial \tau^2}\right]$	$r^2 \frac{\partial^2}{\partial \tau^2} (\nabla^2 w)$	
$\tilde{\varpi}_1$		X	X	$e_{23}^* = \frac{\tilde{\varpi}_1}{\tilde{\varpi}^*} - 1$
C $\tilde{\varpi}_2$	X		X	$e_{13}^* = \frac{\tilde{\varpi}_2}{\tilde{\varpi}^*} - 1$
$\tilde{\varpi}_3$	X	X		$e_{12}^* = \frac{\tilde{\varpi}_3}{\tilde{\varpi}^*} - 1$

TABLE IV

ERROR IN $\tilde{\omega}_{LF}$ FOR BOUNDARY CONDITION SS1/SS1 WHEN NEGLECTING "ν in k_{ij}" AND "q_θ"

Valid for	"e_1" (error when neglecting "ν in k_{ij}" and "q_θ"
All $n, s \neq 0$	$\frac{r^2}{2}\left(\frac{(2s^2+n^2)(2n^2-1) - 2\nu^2 s^4(n^2-2)}{r^2(s^2+n^2)^4 + (1-\nu^2)s^4}\right)$
$\tilde{\omega} \simeq \tilde{\omega}_{min}$	$\frac{n^6}{n^8 + \frac{(1-\nu^2)s^4}{r^2}}$
$(e_1)_{max}$	$\left(\frac{3}{2}^{\frac{1}{2}}\right)^{\frac{3}{2}} \frac{r}{\tilde{\omega}_{min}} = 0.805\ e_{ring}$
$\tilde{\omega} \gg \tilde{\omega}_{min}$	$\frac{1}{n^2} = e_{ring}$

TABLE V

ERROR IN $\tilde{\omega}_{LF}$ FOR BOUNDARY CONDITION SS1/SS1

WHEN NEGLECTING TANGENTIAL INERTIA

n	$\tilde{\omega}_2^2$	e_2
0	$\frac{1}{2}\left(1 + s^2 - \sqrt{(1-s^2)^2 + 4\nu^2 s^2}\right)$	$\left(\frac{2(1-\nu^2)}{1 + s^2 - \sqrt{(1-s^2)^2 + 4\nu^2 s^2}}\right)^{\frac{1}{2}} - 1$
1, 2	$\frac{\mathcal{F} - \sqrt{\mathcal{F}^2 - 8(1+\nu)s^4(s^2 + \frac{1-\nu}{2} n^2)}}{2(n^2 + \frac{2s^2}{1-\nu})}$	$\frac{s^2\sqrt{(1-\nu^2)\, 2(n^2 + \frac{2s^2}{1-\nu})}}{(s^2+n^2)\left(\mathcal{F} - \sqrt{\mathcal{F}^2 - 8(1+\nu)s^4(n^2\frac{1-\nu}{2} + s^2)}\right)^{\frac{1}{2}}} - 1$

$\mathcal{F} = (s^2+n^2)^2 + 2(1+\nu)s^2 + n^2$

TABLE V (Cont'd)

ERROR IN $\tilde{\omega}_{LF}$ FOR BOUNDARY CONDITION SS1/SS1

WHEN NEGLECTING TANGENTIAL INERTIA

n	$\tilde{\omega}_2^2$	e_2
≥ 3	$\tilde{\omega}_4^2 \left(\frac{s^2+n^2}{1+s^2+n^2+\frac{2(1+\nu)s^2}{s^2+n^2}} \right)$ $\tilde{\omega}_4^2 = r^2(s^2+n^2)^2 + \frac{(1-\nu^2)s^2}{(s^2+n^2)^2}$	$\sqrt{1+\frac{1}{s^2+n^2}+\frac{2(1+\nu)s^2}{(s^2+n^2)^2}} -1$
$\gg 3$	$\tilde{\omega}_4^2 \; (1 - \frac{1}{n^2})$	$\frac{n}{\sqrt{n^2-1}} -1 \simeq \frac{1}{2n^2}$

TABLE VI

INTERMEDIATE AND HIGH FREQUENCIES

FOR BOUNDARY CONDITION SS1/SS1

n	ϖ^2_{IF}	ϖ^2_{HF}
0	$\frac{1}{2}\left(1+s^2+\sqrt{(1-s^2)^2+4\nu^2 s^2}\right)$	0
≥ 1	$\frac{(1-\nu)}{2}(s^2+n^2)+\frac{(1-\nu^2)s^2}{1+\frac{(1+\nu)}{2}(s^2+n^2)}$ $N = (s^2+n^2)$	$1+s^2+n^2-\frac{(1-\nu^2)s^2}{1+\frac{(1+\nu)}{2}(s^2+n^2)}$
$\gg 3$	$\frac{(1-\nu)}{2}(s^2+n^2)$	$1+s^2+n^2$
> 10 $n \gg s$	$\frac{(1-\nu)}{2}n^2$	$n^2 \quad (\simeq \tilde{\omega}^2_{HF})_{ring}$

TABLE VII

DISPLACEMENT RATIOS FOR THE LOW, INTERMEDIATE AND HIGH FREQUENCIES FOR BOUNDARY CONDITION SS1/SS1

	n	$\tilde{u}_{mn}$	$\tilde{v}_{mn}$
	0	$\dfrac{2\nu s}{1-\sqrt{(1-s^2)^2+4\nu^2 s^2}}$	0
Low Frequency	≥ 1	$\dfrac{s(n^2-\nu s^2+\frac{2}{1-\nu}\,\tilde{\omega}_2^2)}{(s^2+n^2)^2-\frac{3-\nu}{1-\nu}(s^2+n^2)\,\tilde{\omega}_2^2}$	$\dfrac{-n(n^2+(2+\nu)s^2-\frac{2}{1-\nu}\,\tilde{\omega}_2^2)}{(s^2+n^2)^2-\frac{3-\nu}{1-\nu}(s^2+n^2)\,\tilde{\omega}_2^2}$
Low Frequency	>5	$\dfrac{s(n^2-\nu s^2)}{(s^2+n^2)^2}$	$\dfrac{-n(n^2+(2+\nu)s^2)}{(s^2+n^2)^2}$
	$\gg s$	$\dfrac{s}{(n^2)}$	$\dfrac{1}{n}\ (\simeq \tilde{v}_n)_{ring}$

TABLE VII (Cont'd)

DISPLACEMENT RATIOS FOR THE LOW, INTERMEDIATE AND HIGH FREQUENCIES FOR BOUNDARY CONDITION SS1/SS1

	n	$\tilde{u}_{mn}$	$\tilde{v}_{mn}$
	0	$\frac{2\nu s}{1+\sqrt{(1-s^2)^2+4\nu^2 s^2}}$	0
Intermediate Frequency	≥ 1	$\frac{s\left(n^2-\nu s^2+\frac{2}{1-\nu}\tilde{\omega}_{if}^2\right)}{\frac{2}{1-\nu}\tilde{\omega}_{if}^2-\left((3+2\nu)s^2+n^2\right)}$	$\frac{-n\left(n^2+(2+\nu)s^2-\frac{2}{1-\nu}\tilde{\omega}_{if}^2\right)}{\frac{2}{1-\nu}\tilde{\omega}_{if}^2-\left((3+2\nu)s^2+n^2\right)}$
Intermediate Frequency	>5	$-\frac{n^2}{2s}\left(1+\frac{2\left(1+\frac{2\nu s^2}{n^2}\right)}{(1+\nu)(s^2+n^2)}\right)$	$\frac{n}{2}\left(1-\frac{2}{(1+\nu)(s^2+n^2)}\right)$
	$\gg s$	$\frac{-n^2}{(2s)}$	$\frac{n}{2}$

TABLE VII (Cont'd)

DISPLACEMENT RATIOS FOR THE LOW, INTERMEDIATE AND HIGH FREQUENCIES FOR BOUNDARY CONDITION SS1/SS1

	n	$\tilde{u}_{mn}$	$\tilde{v}_{mn}$
High Frequency	0	0	0
	>1	$\dfrac{s(n^2-\nu s^2+\frac{2}{1-\nu}\tilde{\omega}_{hf}^2)}{\frac{2}{1-\nu}\tilde{\omega}_{hf}^2-((3+2\nu)s^2+n^2)}$	$\dfrac{-n\,(n^2+(2+\nu)s^2-\frac{2}{1-\nu}\tilde{\omega}_{hf}^2)}{\frac{2}{1-\nu}\tilde{\omega}_{hf}^2-((3+2\nu)s^2+n^2)}$
	>5	$s\left(1-\dfrac{2-(1-\nu)s^2}{(s^2+n^2)+\frac{2}{1+\nu}-2(1-\nu)s^2}\right)$	$n\left(1+\dfrac{(1-\nu)s^2}{(s^2+n^2)+\frac{2}{1+\nu}-2(1-\nu)s^2}\right)$
	$\gg s$	s	$n\ (\simeq \tilde{v}_n)_{ring}$

TABLE VIII

CHANGE OF ROOTS OF THE CHARACTERISTIC EQUATION

n	Range of $\tilde{\omega}$		Type of Roots		
	Greater	Less	No. of Imaginary	No. of Real	No. of Complex
	0	$(1-\nu^2)^{\frac{1}{2}}$	2		4
0	$(1-\nu^2)^{\frac{1}{2}}$	1. 0	2	4	
	1. 0		4	2	
	0	$\left(\frac{1-\nu}{2}\right)^{\frac{1}{2}}$	2	2	4
1	$\left(\frac{1-\nu}{2}\right)^{\frac{1}{2}}$	0. 962	4		4
	0. 962	$(n^2+1)^{\frac{1}{2}}$	4	4	
	$(n^2+1)^{\frac{1}{2}}$		6	2	

TABLE VIII (Cont'd)

CHANGE OF ROOTS OF THE CHARACTERISTIC EQUATION

n	Range of $\tilde{\omega}$		Type of Roots		
	Greater	Less	No. of Imaginary	No. of Real	No. of Complex
2	rn^2	0.978	2	2	4
	0.978	$\left(\frac{1-\nu}{2}\right)^{\frac{1}{2}} n$	2	6	
	$\left(\frac{1-\nu}{2}\right)^{\frac{1}{2}} n$	$(n^2+1)^{\frac{1}{2}}$	4	4	
	$(n^2+1)^{\frac{1}{2}}$		6	2	
≥ 3	rn^2	1.0	2	2	4
	1.0	$\left(\frac{1-\nu}{2}\right)^{\frac{1}{2}} n$	2	6	
	$\left(\frac{1-\nu}{2}\right)^{\frac{1}{2}} n$	$(n^2+1)^{\frac{1}{2}}$	4	4	
	$(n^2+1)^{\frac{1}{2}}$		6	2	

TABLE IX

NOTATION FOR BOUNDARY CONDITIONS AND MOTIVATION

Boundary Condition	$u = 0$	$v = 0$	$w = 0$	$w' = 0$	$n_{xx} = 0$	$n_{x\theta} = 0$	$q_{xx} = 0$	$m_{xx} = 0$	Motivation for choice
SS1/SS1 (both ends)		X	X		X			X	Basis for comparison
SS2/SS2 (both ends)	X	X	X					X	Effect of axial restraint
SS3/SS3 (both ends)			X		X	X		X	Effect of circumferential restraint
FX1/FX1 (both ends)		X	X	X	X				Effect of slope restraint

TABLE IX (Cont'd)

NOTATION FOR BOUNDARY CONDITIONS AND MOTIVATION

Boundary Condition	$u = 0$	$v = 0$	$w = 0$	$w' = 0$	$n_{xx} = 0$	$n_{x\theta} = 0$	$q_{xx} = 0$	$m_{xx} = 0$	Motivation for choice
FX2/FX2 (both ends)	X	X	X	X					Upper bound for FX2/FXR
FX2/FR	X	X	X	X	X	X	X	X	Lower bound for FX2/FXR and effect of free end
FX2/FXR	X	X	X	X	Elastic ring				Effect of elastic ring in one end
FXR/FXR	Elastic ring at both ends								Effect of elastic ring at both ends

TABLE X

COMPARISON OF EXACT AND APPROXIMATE LOW FREQUENCIES

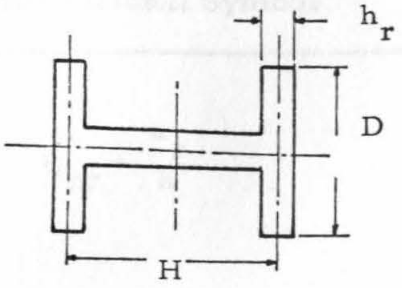
FOR BOUNDARY CONDITION SS2/SS2

($\ell = 3$, $\frac{a}{h} = 577.35$ i. e., $r = 0.0005$)

m	1		2		3	
n	Exact	Approx.	Exact	Approx.	Exact	Approx.
0	not existent	not existent	0.856654	0.857717	0.942562	0.942644
1	0.408208	0.408263	0.711681	0.711681	0.845446	0.845447
2	0.243312	0.243436	0.469722	0.469743	0.652761	0.652763
3	0.155055	0.155192	0.322102	0.322173	0.487844	0.487871
4	0.105212	0.105326	0.231436	0.231536	0.369015	0.369077
5	0.075709	0.075753	0.172978	0.172848	0.285661	0.285349
6	0.058269	0.058315	0.133945	0.133945	0.226530	0.226413
7	0.049151	0.049182	0.107670	0.107701	0.184006	0.183980
8	0.046531	0.046548	0.090625	0.090656	0.153402	0.153409

TABLE XI

RING GEOMETRICAL PARAMETERS AND RELATED COEFFICIENTS



$$\tilde{d} = \frac{D}{H}$$

$$\tilde{h} = \frac{h_r}{h}$$

$$\tilde{a} = \frac{a_r}{a}$$

$$\tilde{H} = \frac{H}{h_r}$$

Nondimensional Symbol	Equivalent Relation
$\tilde{A}_r = \frac{A_r}{h_r^2}$	$(1 + 2\tilde{d})\tilde{H}$
$\tilde{h}_e = \frac{h_e}{h_r}$	$\frac{\tilde{H} + 1 + \frac{1}{\tilde{h}}}{2}$
$r_t = \frac{1}{a_r}\sqrt{\frac{J_{eff}}{A_r}}$	$\frac{h\tilde{h}}{a_r}\left(\frac{1}{3} - \frac{0.6301875}{\tilde{H}\,(1 + 2\tilde{d})}\right)^{\frac{1}{2}}$
$r_x = \frac{1}{a_r}\sqrt{\frac{I_{xx}}{A_r}}$	$\frac{h\tilde{h}}{a_r}\left(\frac{1 + 2\tilde{H}^2\tilde{d}^3}{12\,(1 + 2\tilde{d})}\right)^{\frac{1}{2}}$
$r_y = \frac{1}{a_r}\sqrt{\frac{I_{yy}}{A_r}}$	$\frac{h\tilde{h}}{a_r}\left(\frac{\tilde{H}^2 + 2\tilde{d}(1+3\tilde{H}^2)}{12\,(1 + 2\tilde{d})}\right)^{\frac{1}{2}}$

TABLE XI (Cont'd)

RING GEOMETRICAL PARAMETERS AND RELATED COEFFICIENTS

Nondimensional Symbol	Equivalent Relation
$R_{hr} = \frac{a_r}{h}$	$\frac{a}{h} - \frac{1+\tilde{h}(1+\tilde{H})}{2}$
$W_r = \frac{\text{(weight ring)}}{\text{(weight shell)}}$	$\frac{a_r}{h} \frac{\tilde{A}_r \tilde{h}}{\left[\frac{a}{h}\right]^2}$
S_{tx} , S_{ty}	$\frac{r_x}{r} \sqrt{W_r \tilde{a}}$, $\frac{r_y}{r} \sqrt{W_r \tilde{a}}$
f_{c1}	$\left[12(1-\nu^2)\left(\frac{a}{h}\right) \tilde{A}_r \tilde{h}^2\right]^{-1}$
f_{c2}	$\frac{\frac{a}{h}}{2(1+\nu)\tilde{A}_r \tilde{h}^2}$

TABLE XI (Cont'd)

RING GEOMETRICAL PARAMETERS AND RELATED COEFFICIENTS

Nondimensional Symbol	Equivalent Relation
f_{c3}	$\dfrac{\widetilde{h}_e}{2(1+\nu)\,\widetilde{A}_r\,\widetilde{h}}$
f_{c4}	$\dfrac{2}{(1-\nu)}\ f_{c2}$
f_{c5}	$\dfrac{2}{(1-\nu)}\ f_{c3}$
f_{c6}	$\dfrac{\widetilde{d}\,\widetilde{h}}{24(1-\nu^2)\left(\frac{a}{h}\right)^2\widetilde{A}_r\,\widetilde{h}}$
f_{c7}	$\dfrac{\widetilde{d}\,\widetilde{h}}{2\widetilde{h}_e}\ f_{c3}$

TABLE XII

COMPARISON OF THEORETICAL AND EXPERIMENTAL EIGENFREQUENCIES FOR BOUNDARY CONDITION FXR/FXR

($\ell = 1.75$, $\frac{a}{h} = 284.93$, $\omega_o = 8308.4$ cps)

	m = 1			m = 2		
	FXR/FXR		FX2/FX2	FXR/FXR		FX2/FX2
n	Exper.	Theory	Theory	Exper.	Theory	Theory
2		3080	3522		5992	5862
3		2240	2475		4369	4566
4	1213	1286	1826		3664	3582
5	1094	1150	1405		2504	2871
6	967	1004	1131	2123	2194	2358
7	887	904	969	1854	1897	1990
8	861	868	900	1656	1680	1735
9	906	897	912	1540	1545	1578
10	1003	983	991	1485	1488	1507
11	1145	1114	1118	1524	1503	1515
12	1321	1280	1282	1615	1581	1588
13		1473	1474	1763	1711	1716
14	1747	1690	1691	1942	1884	1886
16	2265	2183	2184	2409	2326	2327
18	2860	2739	2739	2982	2868	2868

TABLE XII (Cont'd)

COMPARISON OF THEORETICAL AND EXPERIMENTAL EIGENFREQUENCIES FOR BOUNDARY CONDITION FXR/FXR

(ℓ = 1.75, $\frac{a}{h}$ = 284.93, ω_o = 8308.4 cps)

	m = 3			m = 4		
	FXR/FXR		FX2/FX2	FXR/FXR		FX2/FX2
n	Exper.	Theory	Theory	Exper.	Theory	Theory
2		6931	6899		7349	7335
3		6146	5974		6773	6726
4		5091	5067		6033	6035
5		4580	4282		5472	5352
6	3330	3444	3641		4454	4729
7	2972	3037	3136	4008	4098	4190
8	2647	2687	2747	3630	3682	3741
9	2402	2423	2461	3314	3344	3382
10	2241	2242	2267	3070	3082	3109
11	2160	2141	2157	2909	2898	2916
12		2115	2125	2810	2789	2801
13	2201	2156	2163		2751	2759
14	2313	2256	2261		2779	2785
16	2696	2600	2602	3062	3006	3009
18	3196	3097	3088	3490	3416	3417

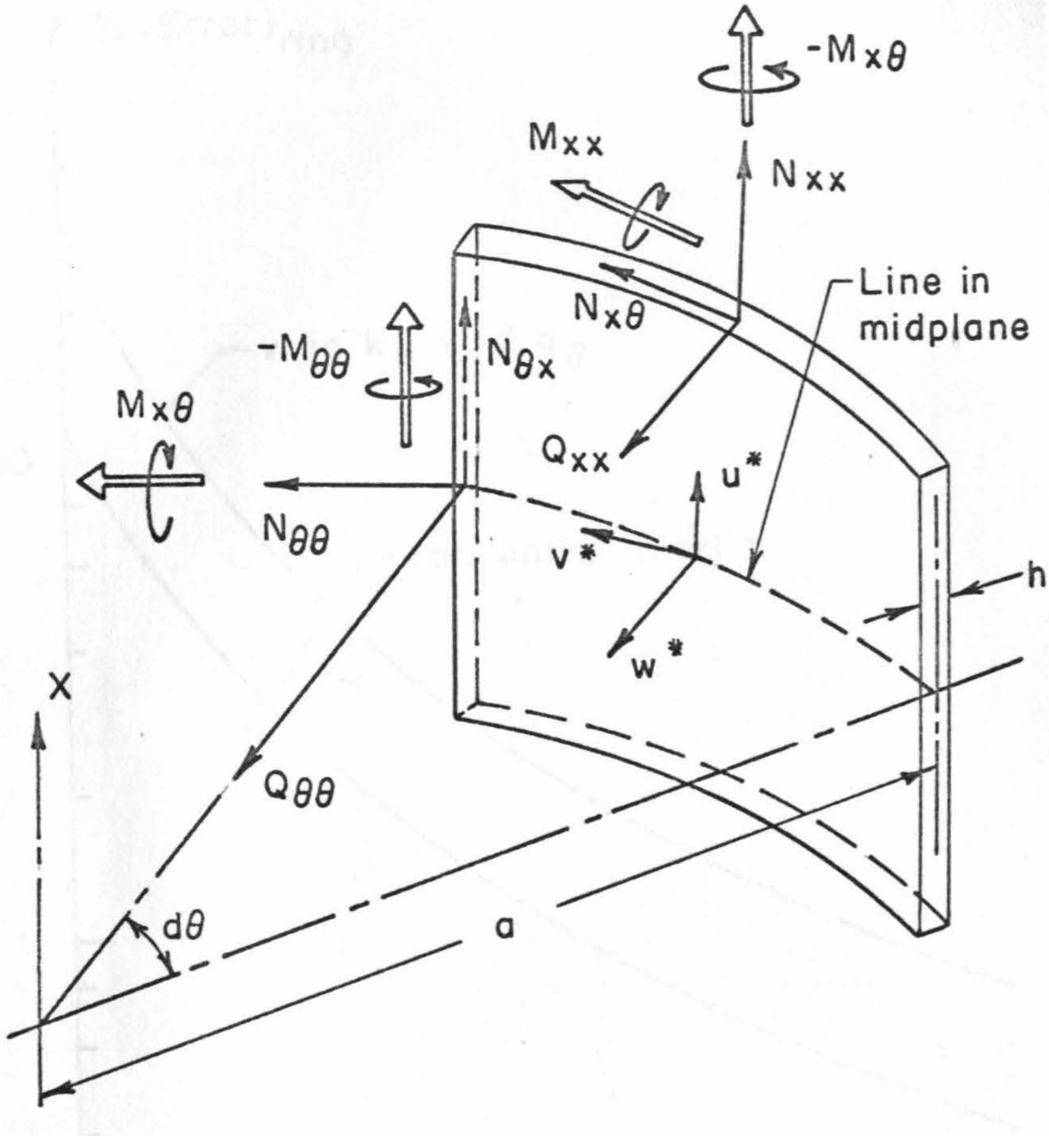


FIG. I ELEMENT OF CYLINDRICAL SHELL

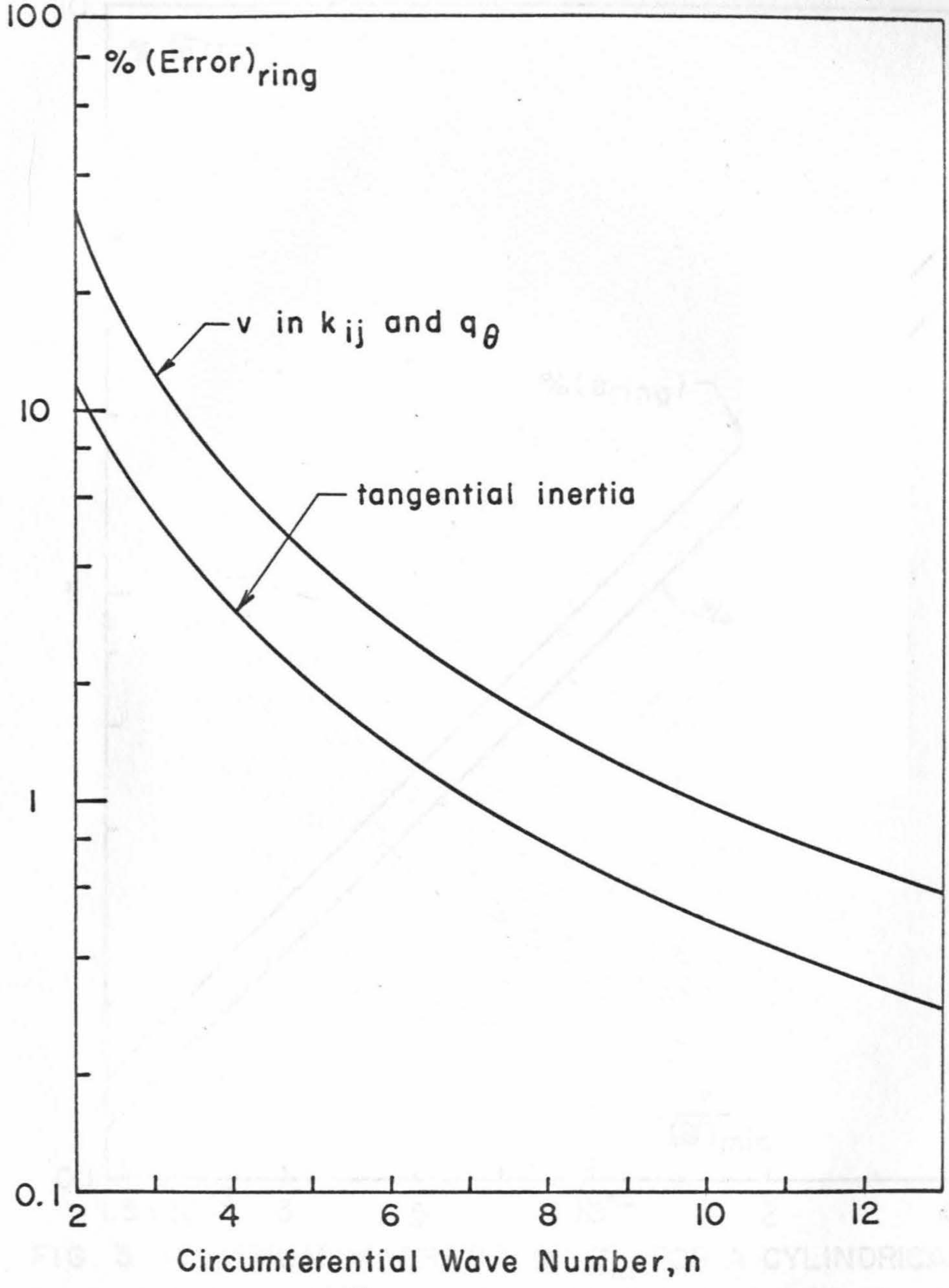


FIG. 2 %ERROR IN $\tilde{\omega}_{LF}$ FOR A RING WHEN NEGLECTING " v IN k_{ij}" AND "q_θ" AND TANGENTIAL INERTIA

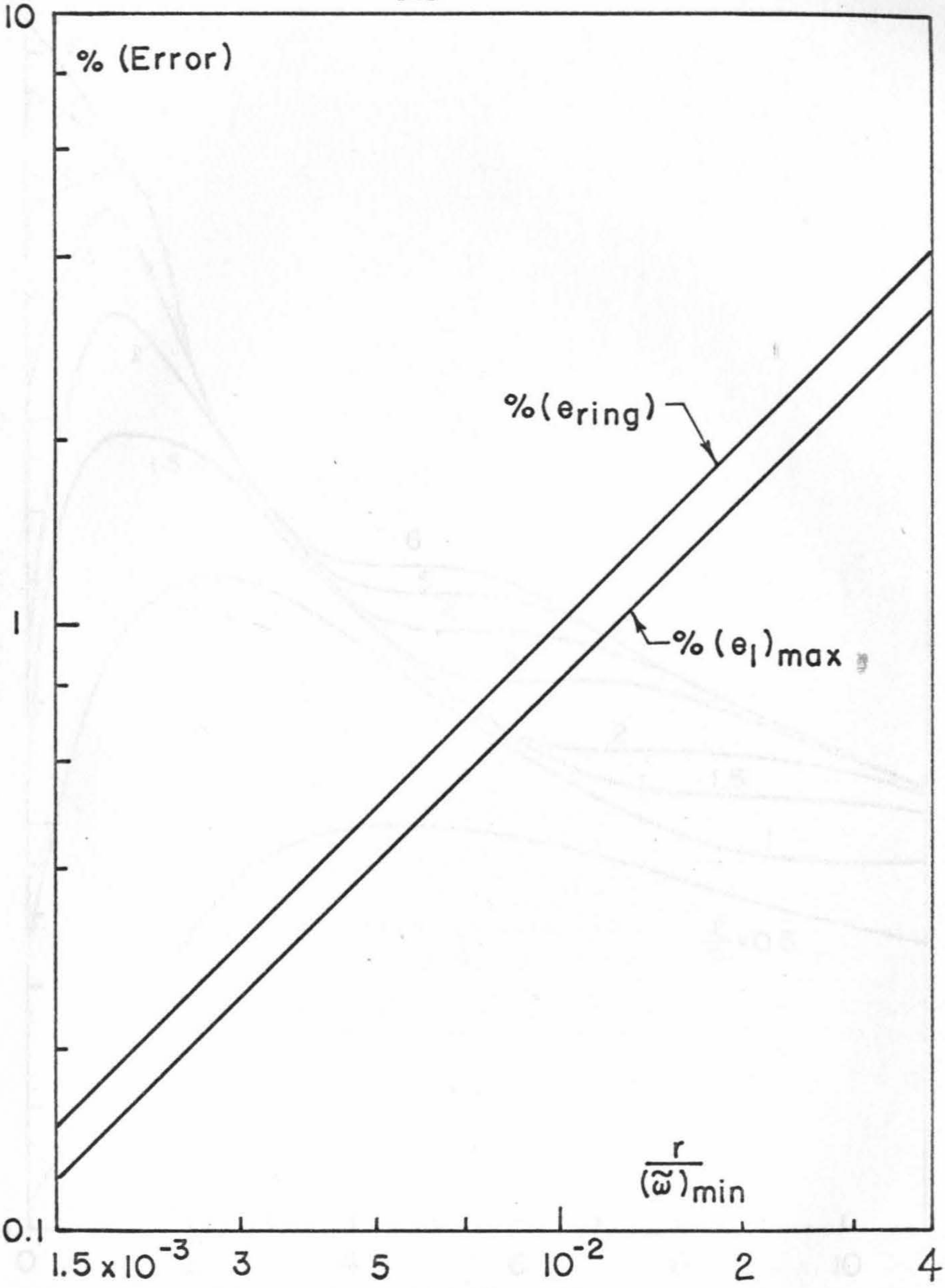


FIG. 3 MAXIMUM % ERROR IN $\tilde{\omega}_{LF}$ FOR A CYLINDRICAL SHELL WITH BOUNDARY CONDITION SSI/SSI WHEN NEGLECTING "ν IN k_{ij}" AND "q_θ"

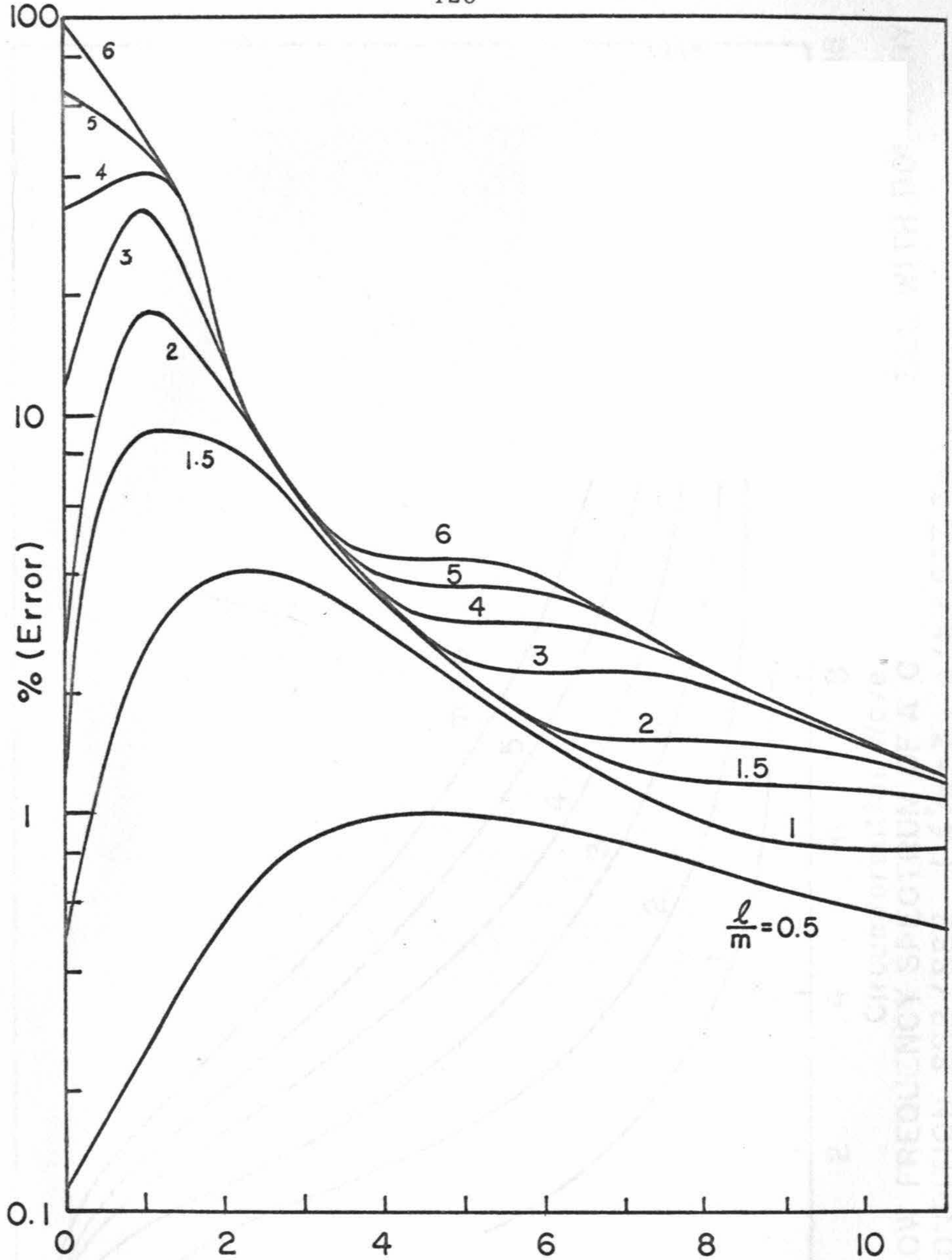


FIG. 4 % ERROR IN $\tilde{\omega}_{LF}$ FOR A CYLINDRICAL SHELL WITH BOUNDARY CONDITION SSI/SSI WHEN NEGLECTING TANGENTIAL INERTIA

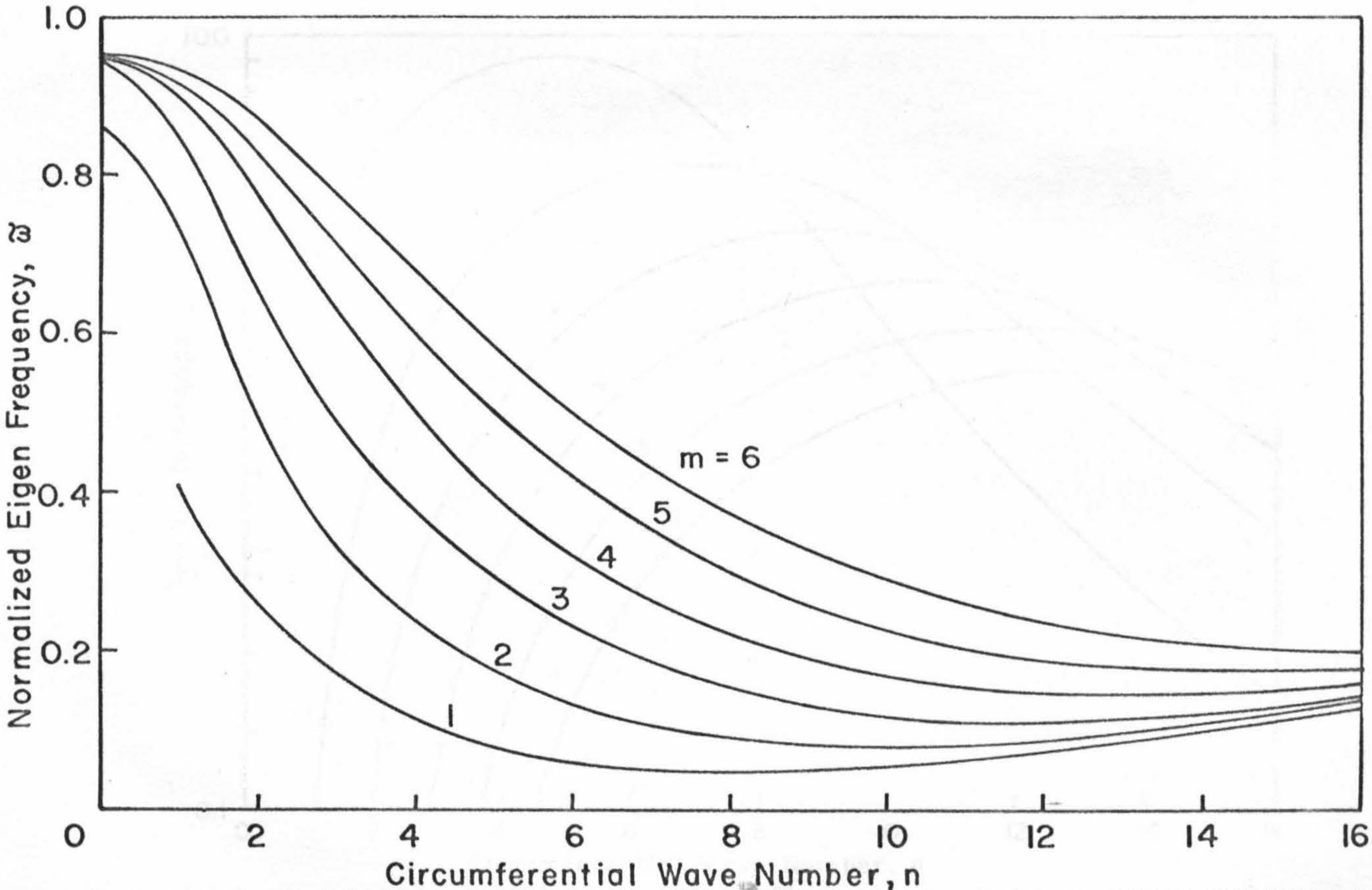


FIG. 5 LOW FREQUENCY SPECTRUM OF A CYLINDRICAL SHELL WITH BOUNDARY CONDITION SS2 / SS2 ($\ell^*/a = 3$, $a/h = 577.35$)

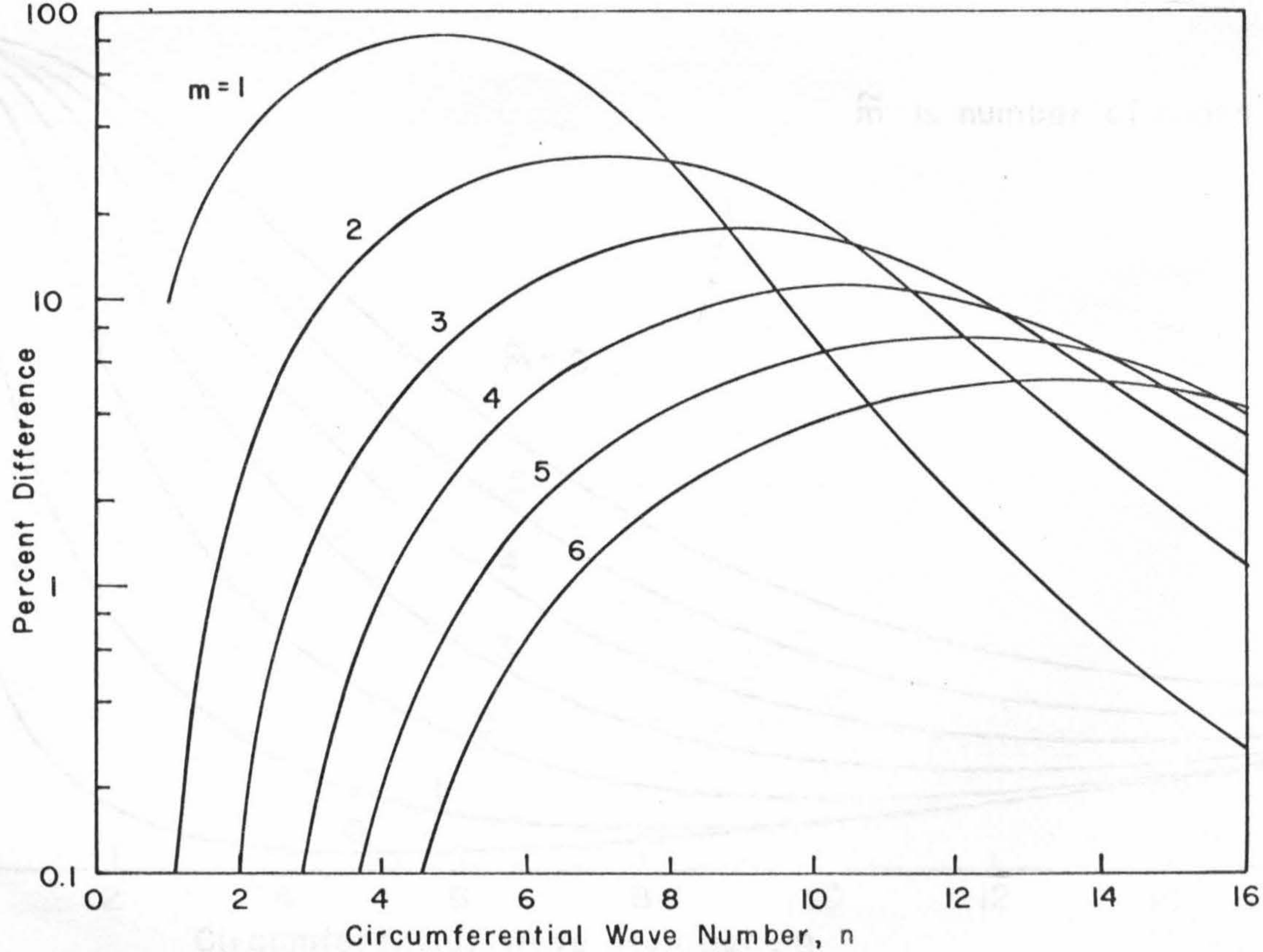


FIG. 6 % DIFFERENCE FROM SS1/SS1 OF $\tilde{\omega}_{LF}$ FOR A CYLINDRICAL SHELL WITH BOUNDARY CONDITION SS2/SS2 ($\ell^*/a = 3$, $a/h = 577.35$)

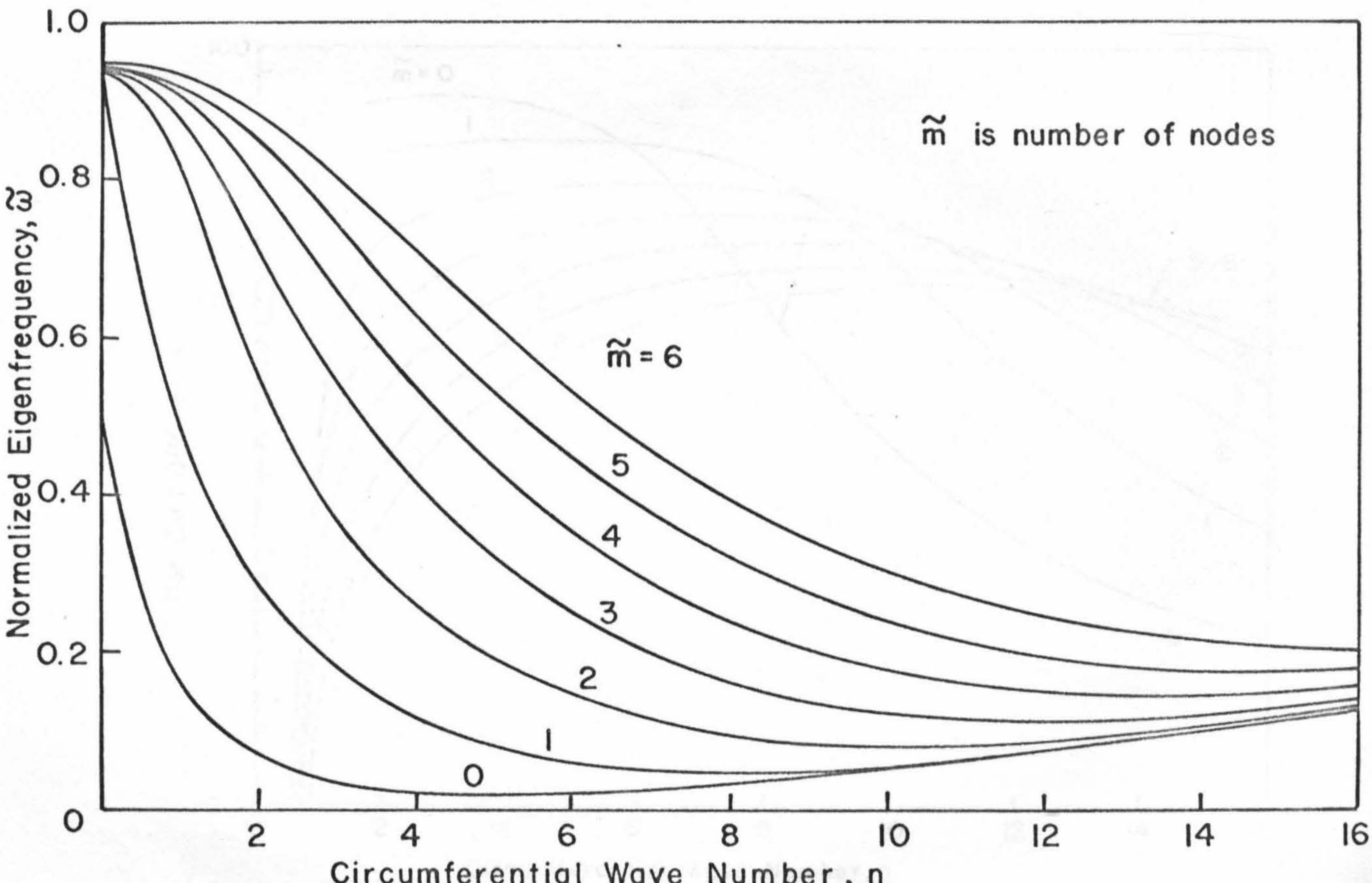


FIG. 7 LOW FREQUENCY SPECTRUM OF A CYLINDRICAL SHELL WITH BOUNDARY CONDITION FX2/FR ($\ell^*/a = 3$, $a/h = 577.35$)

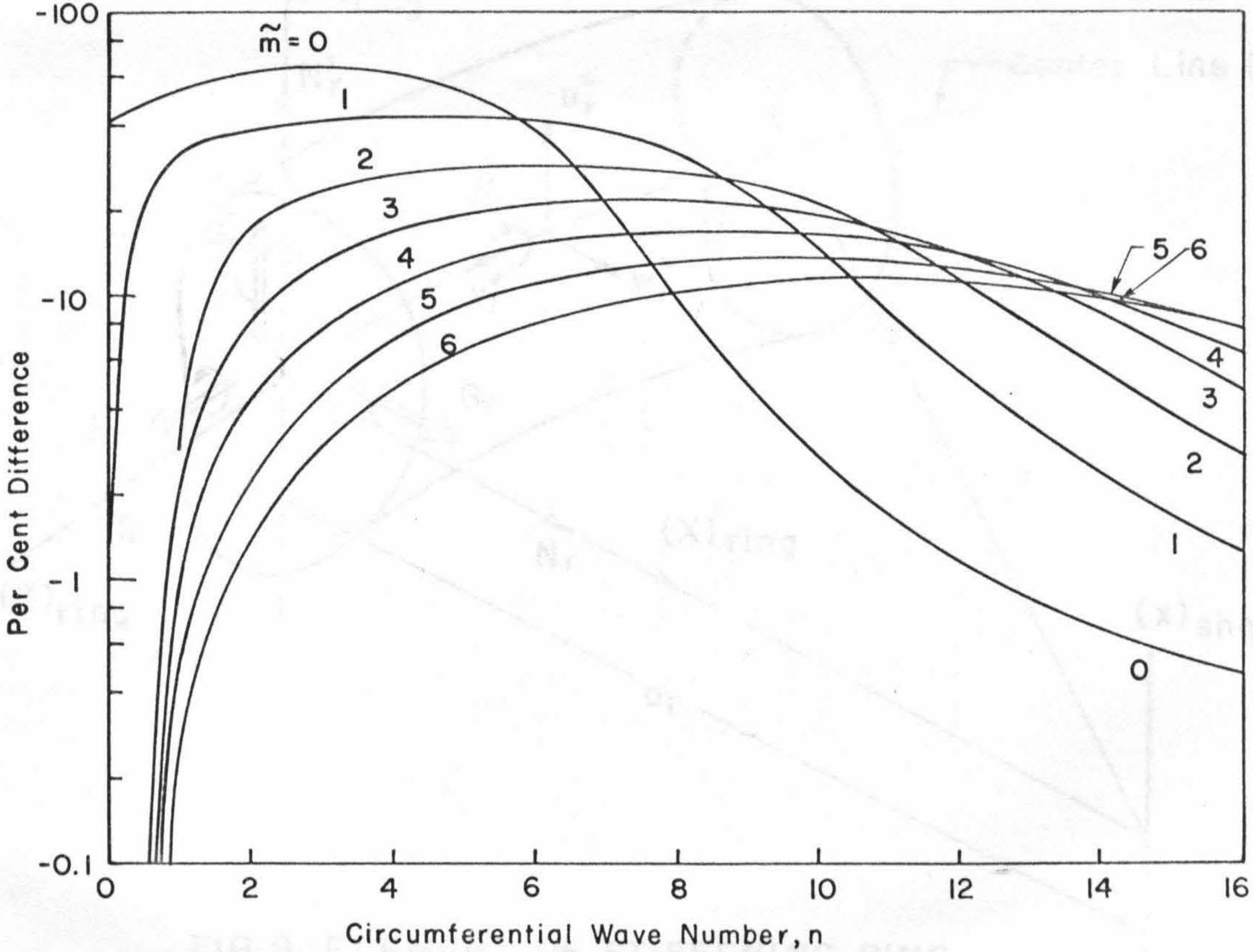


FIG. 8 % DIFFERENCE FROM SSI/SSI OF $\tilde{\omega}_{LF}$ FOR A CYLINDRICAL SHELL WITH BOUNDARY CONDITION FX2/FR ($\ell^{*}/a = 3$, $a/h = 577.35$)

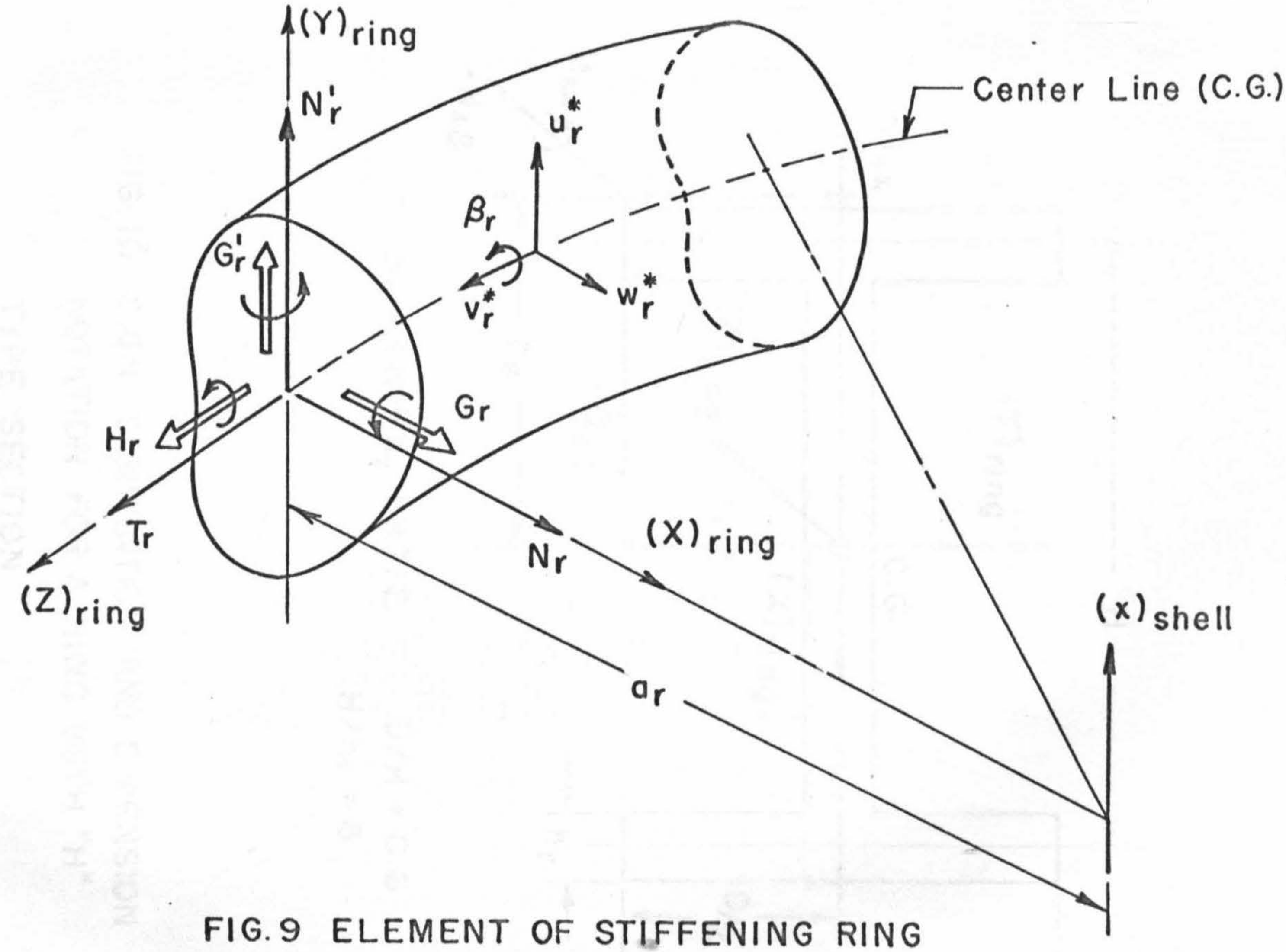


FIG. 9 ELEMENT OF STIFFENING RING

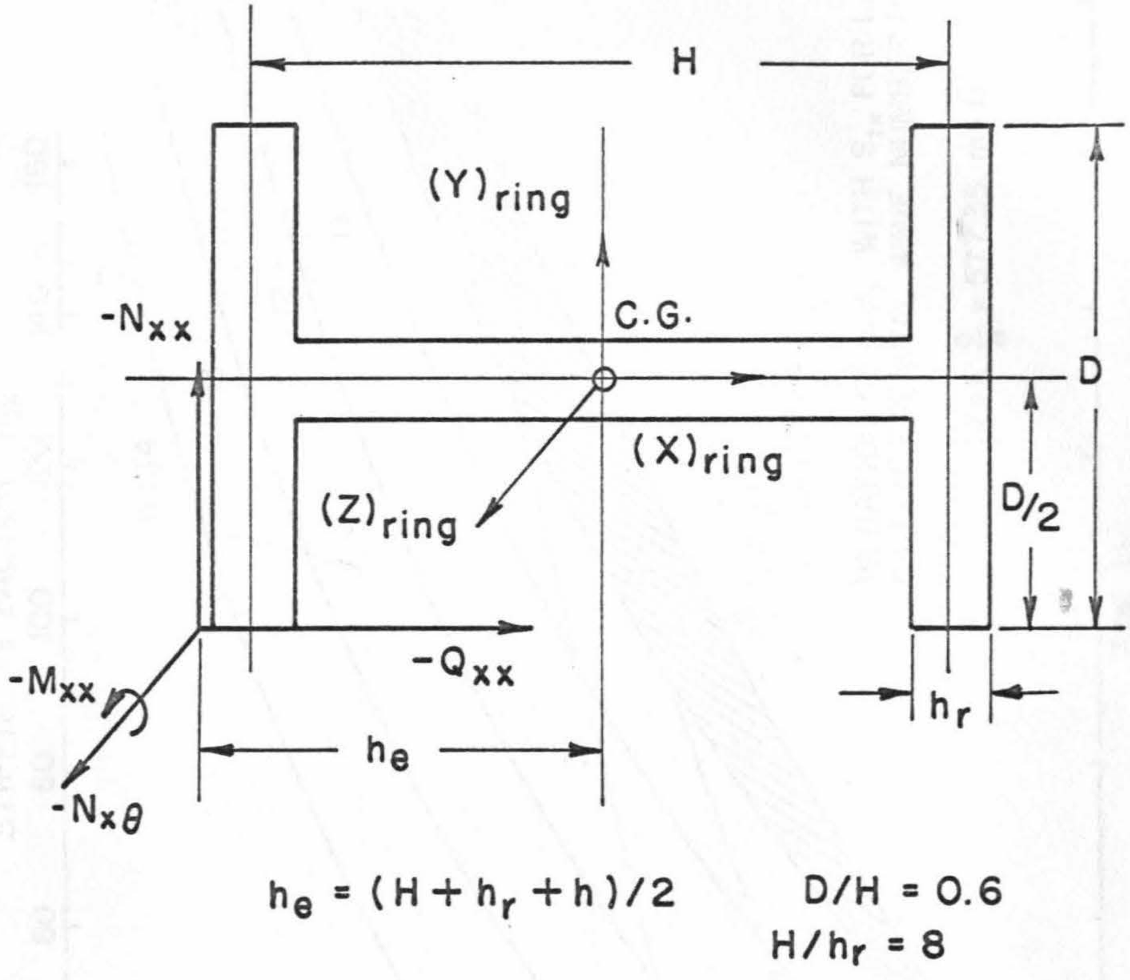


FIG. 10 SIGN CONVENTION AND DIMENSION NOTATION FOR A RING WITH "H" TYPE SECTION

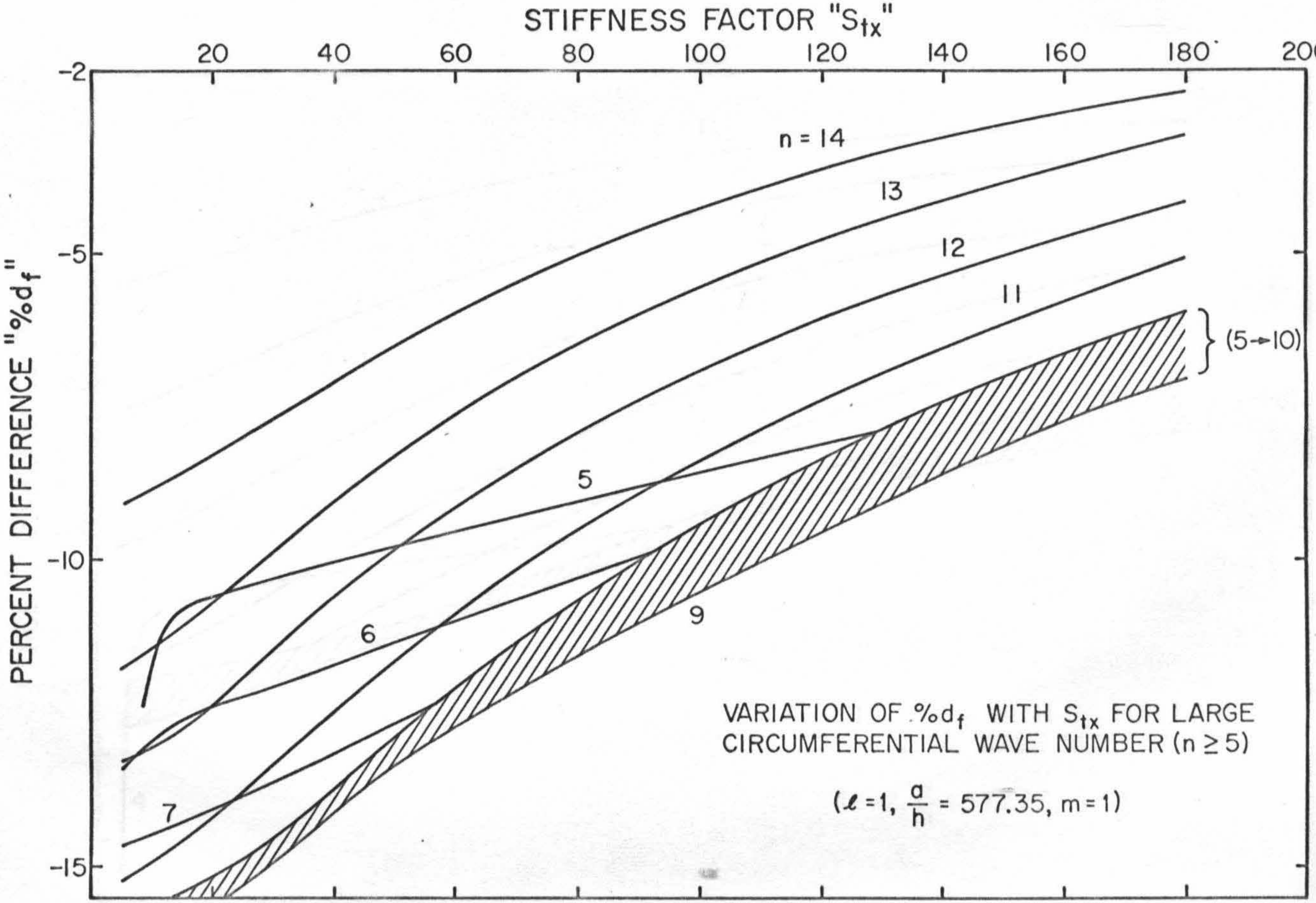
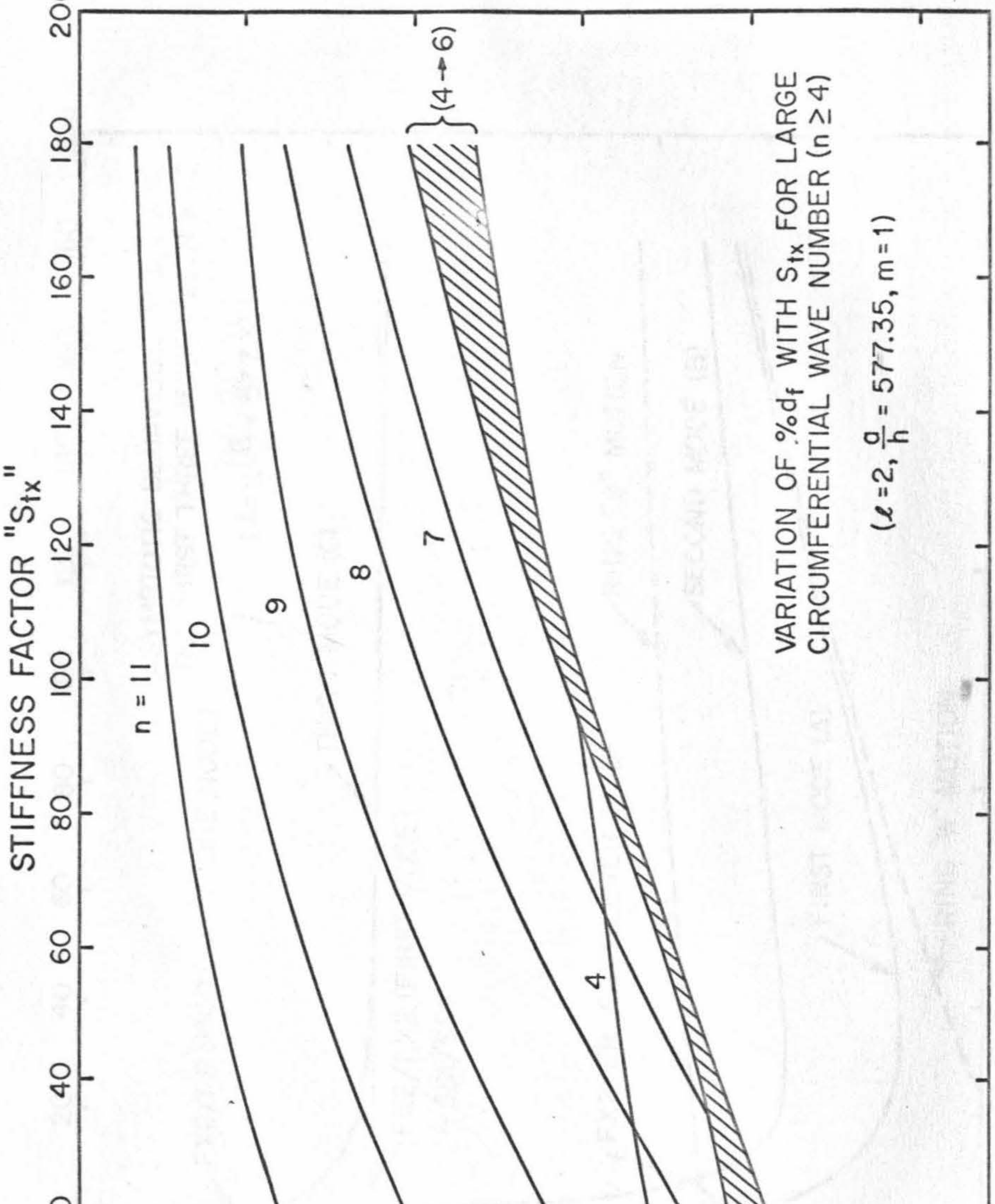

STIFFNESS FACTOR "S_{tx}"
20
40
60
80
100
120
140
160
180
200
PERCENT DIFFERENCE "$\%d_f$"
-2
-5
-10
-15
n = 14
13
12
11
(5→10)
5
6
7
9
VARIATION OF $\%d_f$ WITH S_{tx} FOR LARGE CIRCUMFERENTIAL WAVE NUMBER ($n \geq 5$)
($\ell = 1$, $\frac{a}{h} = 577.35$, $m = 1$)

Fig. 11



VARIATION OF $\%d_f$ WITH S_{tx} FOR LARGE CIRCUMFERENTIAL WAVE NUMBER ($n \geq 4$)

($\ell = 2$, $\frac{a}{h} = 577.35$, $m = 1$)

Fig. 12

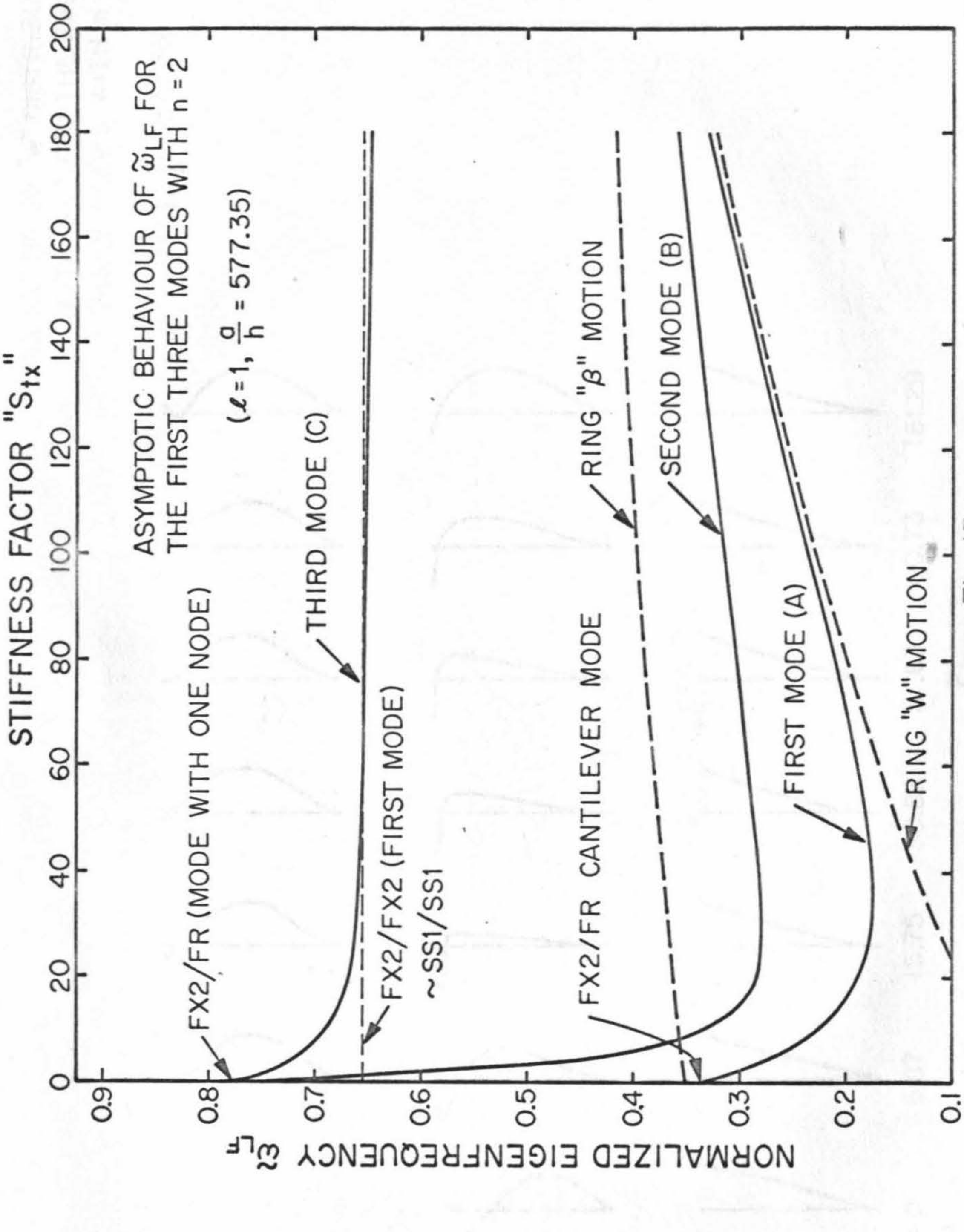

STIFFNESS FACTOR "S_{tx}"
0
20
40
60
80
100
120
140
160
180
200
ASYMPTOTIC BEHAVIOUR OF $\tilde{\omega}_{LF}$ FOR
THE FIRST THREE MODES WITH n = 2
(ℓ = 1, $\frac{a}{h}$ = 577.35)
FX2/FR (MODE WITH ONE NODE)
THIRD MODE (C)
FX2/FX2 (FIRST MODE)
~SS1/SS1
FX2/FR CANTILEVER MODE
RING "β" MOTION
SECOND MODE (B)
FIRST MODE (A)
RING "W" MOTION
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
NORMALIZED EIGENFREQUENCY $\tilde{\omega}_{LF}$

Fig. 13a

VARIATION OF "w" DISTRIBUTION WITH "S_{tx}" FOR THE FIRST THREE MODES WITH n = 2

($\ell = 1$, $\frac{a}{h} = 577.35$)

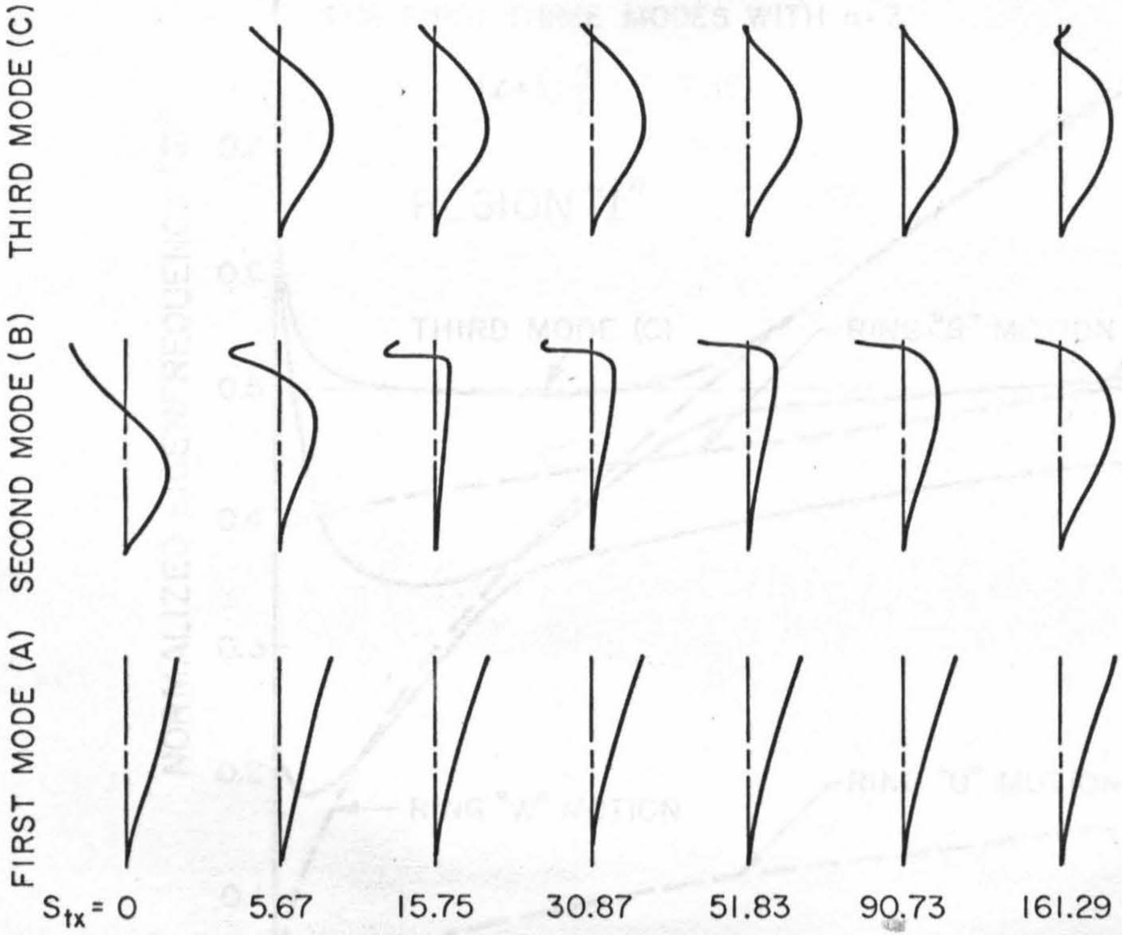


Fig. 13b

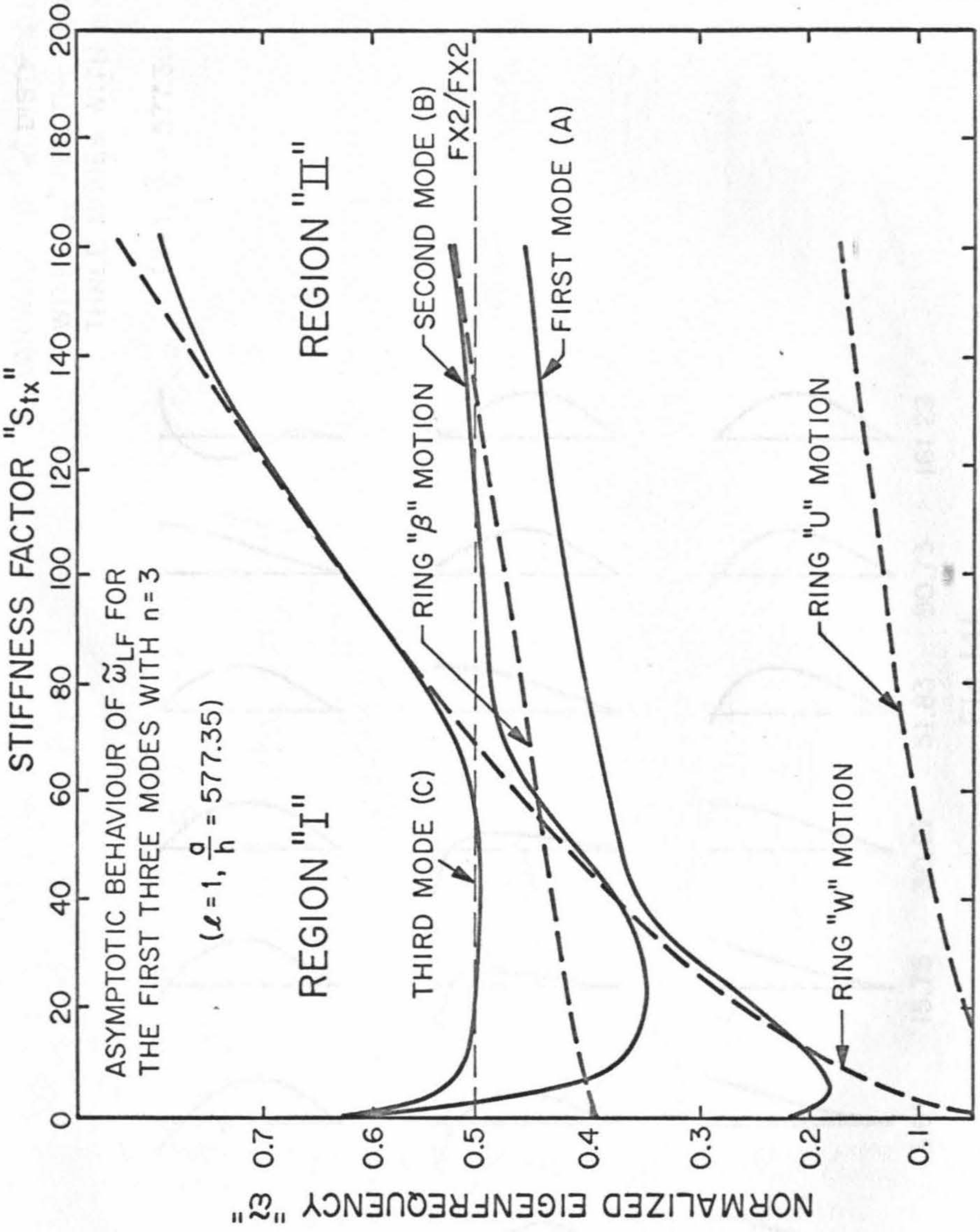

STIFFNESS FACTOR "S_{tx}"
NORMALIZED EIGENFREQUENCY "$\tilde{\omega}$"
ASYMPTOTIC BEHAVIOUR OF $\tilde{\omega}_{LF}$ FOR THE FIRST THREE MODES WITH n=3
(ℓ=1, $\frac{a}{h}$=577.35)
REGION "I"
REGION "II"
THIRD MODE (C)
SECOND MODE (B)
FIRST MODE (A)
FX2/FX2
RING "β" MOTION
RING "U" MOTION
RING "W" MOTION
0
20
40
60
80
100
120
140
160
180
200
0.1
0.2
0.3
0.4
0.5
0.6
0.7

Fig. 14a

VARIATION OF "w" DISTRIBUTION WITH "S_{tx}" FOR THE FIRST THREE MODES WITH n = 3

($\ell = 1$, $\frac{a}{h} = 577.35$)

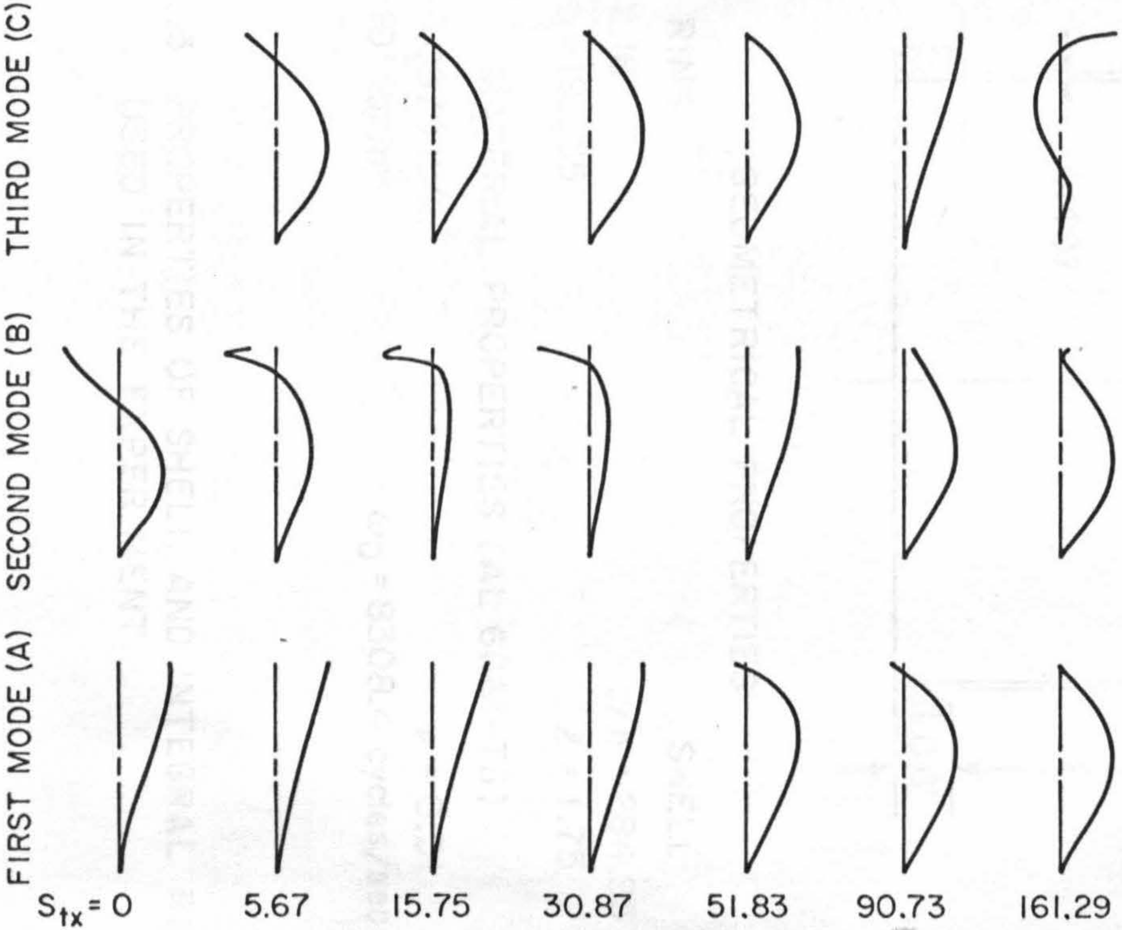
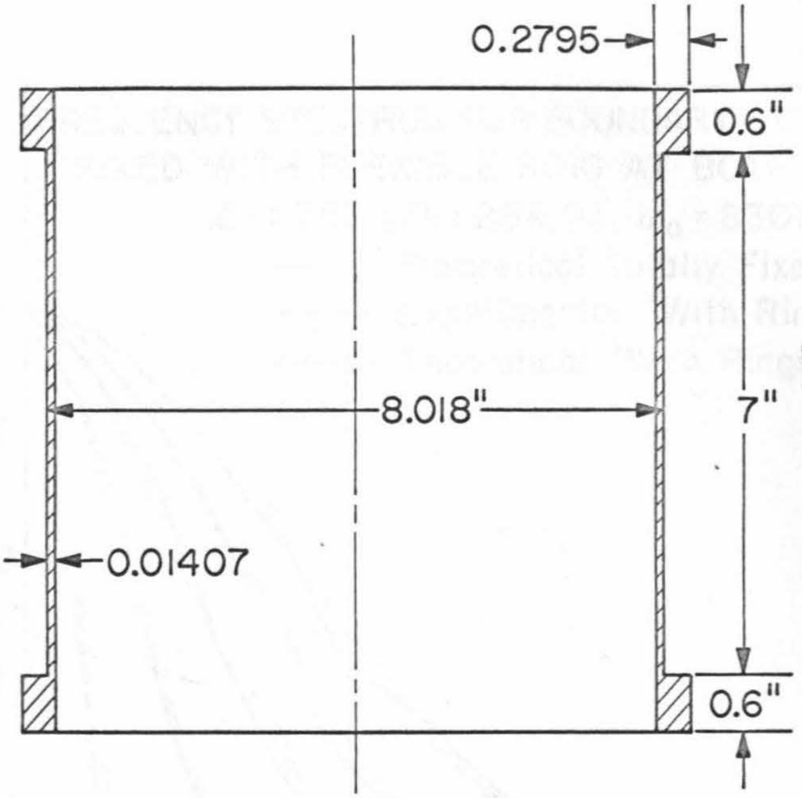


Fig. 14b



GEOMETRICAL PROPERTIES

RING	SHELL
$\tilde{d} = 2.167$	$a/h = 284.93$
$\tilde{H} = \tilde{h} = 19.865$	$\ell = 1.75$

MATERIAL PROPERTIES (AL 6061-T6)

$\rho = 0.0976\ \mathrm{lb/in^3}$	$\nu = 0.31$
$E = 10^7\ \mathrm{lb/in^2}$	$\omega_0 = 8308.4$ cycles/sec

Fig.15 PROPERTIES OF SHELL AND INTEGRAL RINGS USED IN THE EXPERIMENT.

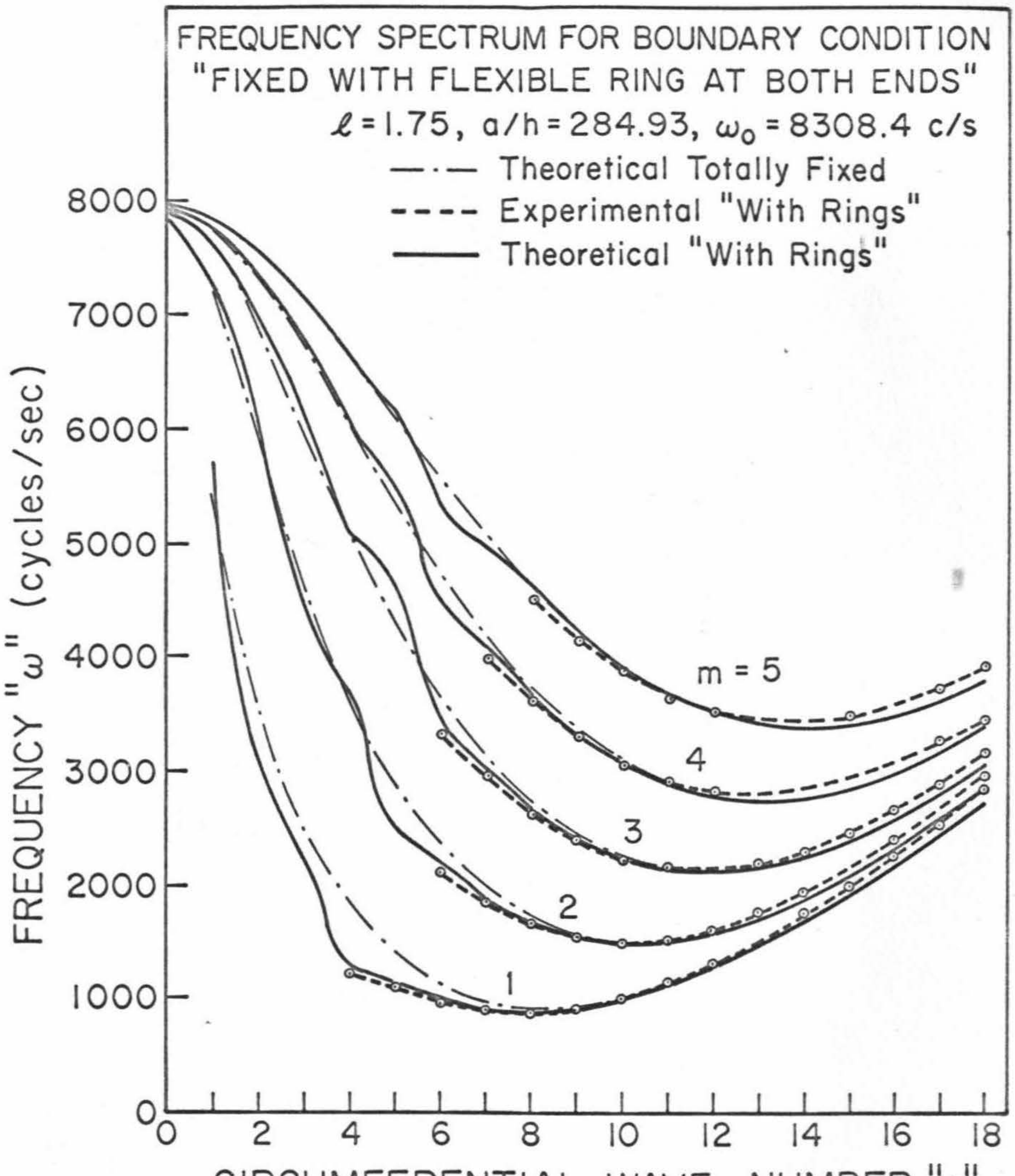

FREQUENCY SPECTRUM FOR BOUNDARY CONDITION
"FIXED WITH FLEXIBLE RING AT BOTH ENDS"
ℓ = 1.75, a/h = 284.93, ω₀ = 8308.4 c/s
Theoretical Totally Fixed
Experimental "With Rings"
Theoretical "With Rings"
m = 5
4
3
2
1
8000
7000
6000
5000
4000
3000
2000
1000
0
0
2
4
6
8
10
12
14
16
18
FREQUENCY "ω" (cycles/sec)
CIRCUMFERENTIAL WAVE NUMBER "n"

Fig. 16