INVARIANT MEASURES ON GROUPS

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ABSTRACT

A generalization of A. Weil's notion of measurable group is formulated and various properties are developed. The properties obtained for the generalization are applied to a problem in the theory of group representations and to the characterization of separable topological groups possessing a measure.

TABLE OF CONTENTS

INTRODUCTION

This thesis deals with a generalization of A. Weil's notion of "measurable group" ("groupe mesuré"). (See (1) and (7).) In the first section various properties of this generalized structure are obtained, and it is shown how one can strengthen Weil's principal result on measurable groups. In the second section our generalized concept is used to find an extension of a result of Loomis in the theory of representations of locally compact groups. In the third and last section we apply the results of the first section to show how local compactness is a combination of measure-theoretic properties and completeness of the group uniform structure.

Throughout this dissertation there is presumed a familiarity with the elementary topological notions and the approach to measure theory expounded in Halmos' Measure Theory[(1)]. Of course, a certain number of terms which are either not well known or are taken here in a sense different from the usual have been defined in the body of the papaer. Other notions and theorems are assumed known: e.g., the definition of an integral (section 25 of (1)), the notion of a σ-finite measure space (section 17 of (1)), the definition and existence of product measure spaces, the version of the Fubini theorem in (1).

Finally, I wish to express my gratitude to those who made this thesis possible. First, to Doctors Dye and Bohnenblust for their advice, encouragement, and patience while my research was in progress. Then, to my sister Maria for her constant understanding and moral support.

I. THEORY OF GENERALIZED MEASURABLE GROUPS

1. Definitions

Definition 0: Given $\bar{S}$ and $\bar{T}$, σ-rings of subsets of the abstract sets S and T, respectively, we define $\bar{S} \times \bar{T}$ to be the σ-ring generated by $\{M \times N \mid M \text{ in } \bar{S},\ N \text{ in } \bar{T}\}$. It is clear that given a finite collection of measurable spaces $\{(S_i, \bar{S}_i)\}_{i=1}^{n}$ one may define inductively a σ-ring $\bigtimes_{i=1}^{n} \bar{S}_i$, by the relation

$$\bigtimes_{i=1}^{m+1} \bar{S}_i = \left(\bigtimes_{i=1}^{m} \bar{S}_i\right) \times \bar{S}_{m+1} .$$

If we make all identifications necessary to insure the associativity of the process of forming Cartesian products of sets, the process of forming products of σ-rings is associative. This will be formulated and proved as a lemma:

Lemma: Let $(S, \bar{S})$, $(T, \bar{T})$, and $(V, \bar{V})$ be measurable spaces. Then

$$(\bar{S} \times \bar{T}) \times \bar{V} = \bar{S} \times (\bar{T} \times \bar{V}).$$

Proof: Defining $\bar{R}(\bar{E})$ to be the σ-ring generated by the family of sets $\bar{E}$ and $\bar{D} \times A$ to be the family of sets $\{F \times A \mid F \in D\}$, we make two preliminary observations.

First,

$$\bar{R}(\bar{E} \times A) = (\bar{R}(\bar{E})) \times A.$$

For, on the one hand,

$$\bar{R}(\bar{E} \times A) \subset (\bar{R}(\bar{E})) \times A.$$

And, on the other hand, since

$$\bar{R}(\bar{E}) \subset \{K \mid K \times A \in \bar{R}(\bar{E} \times A)\},$$

$$(\bar{R}(\bar{E})) \times A \subset \bar{R}(\bar{E} \times A) .$$

Second,

$$\overline{R}\Big(\bigcup_{\alpha\in I} \overline{E}_\alpha\Big) = \overline{R}\Big(\bigcup_{\alpha\in I} \overline{R}(E_\alpha)\Big).$$

For,

$$\begin{aligned}\overline{R}\Big(\bigcup_{\alpha\in I} \overline{E}_\alpha\Big) &\subset \overline{R}\Big(\bigcup_{\alpha\in I} \overline{R}(\overline{E}_\alpha)\Big) \\ &\subset \overline{R}\Big(\bigcup_{\alpha\in I} \overline{R}\Big(\bigcup_{\alpha\in I} \overline{E}_\alpha\Big)\Big) \\ &= \overline{R}\Big(\bigcup_{\alpha\in I} \overline{E}_\alpha\Big).\end{aligned}$$

We can now prove the lemma. If $\overline{V} = \{\overline{E}_\alpha\}_{\alpha\in I}$,

$$\begin{aligned}(\overline{S} \times \overline{T}) \times \overline{V} &= \overline{R}\Big[\bigcup_{\alpha\in I} \overline{R}\left(\{A \times B \mid A\in\overline{S}, B\in\overline{T}\}\right) \times E_\alpha\Big] \\ &=, \text{ by the first observation,} \\ &\quad \overline{R}\Big[\bigcup_{\alpha\in I} \overline{R}\left(\{A \times B \times E_\alpha \mid A\in\overline{S}, B\in\overline{T}\}\right)\Big] \\ &=, \text{ by the second observation,} \\ &\quad \overline{R}\Big(\bigcup_{\alpha\in I}\{A \times B \times E_\alpha \mid A\in\overline{S}, B\in\overline{T}\}\Big) \\ &= \overline{R}\left(\{A \times B \times C \mid A\in\overline{S}, B\in\overline{T}, C\in\overline{V}\}\right).\end{aligned}$$

By a similar argument,

$$\overline{S} \times (\overline{T} \times \overline{V}) = \overline{R}\left(\{A \times B \times C \mid A\in\overline{S}, B\in\overline{T}, C\in\overline{V}\}\right).$$

Hence, we have the desired result,

$$(\overline{S} \times \overline{T}) \times \overline{V} = \overline{S} \times (\overline{T} \times \overline{V}).$$

Definition 1: Given a measurable space $(X, \overline{S})$ we say that a real-valued function f on X (on $M\in\overline{S}$) is $\overline{S}$-measurable ($\overline{S}$-measurable on M) if for every α such that $-\infty \leq \alpha \leq +\infty$ $\{x \mid f(x) \neq 0,\ f(x) \leq \alpha\} \in \overline{S}$. If it is clear in any context that only one measurable space is being considered then an $\overline{S}$-measurable function will merely be referred to as measurable.

Definition 2: Given a measurable space $(X, \overline{S})$ we define a measurability-preserving transformation of $(X, \overline{S})$ to be any one-one mapping, T, of X onto itself such that $T(\overline{S}) = \overline{S}$. Thus for any E in $\overline{S}$ both T(E) and $T^{-1}(E)$ belong to $\overline{S}$.

Definition 3: We say that a measurability-preserving point-transformation, T, of a measure space $(X, \bar{S}, m)$ is non-singular if $m(E) = 0$ implies $m(T(E)) = 0$ for all E in $\bar{S}$.

An important example of a non-singular transformation on a measure space is of course that of left-translation on the measure space consisting of a locally compact topological group, its σ-ring of Borel sets $\bar{S}$, and the Haar measure on $\bar{S}$.

Definition 4: Given two-measure spaces $(X, \bar{S}, \mu)$, $(X, \bar{S}, \nu)$, μ is said to be equivalent to ν ($\mu \sim \nu$) if μ is absolutely continuous with respect to ν, and vice versa.

Definition 5: A σ-ring $\bar{G}$ of subsets of a group G is called invariant if the transformation

$$S : (x, y) \rightarrow (x, xy)$$

on G x G is measurability-preserving relative to the product measurable space $(G \times G, \bar{G} \times \bar{G})$.

Definition 6: By a generalized measurable group [= GMG] $(G, \bar{G}, m)$ we mean the structure composed of an abstract group G, an invariant σ-ring $\bar{G}$ of subsets of G, and a non-zero, σ-finite measure m on $\bar{G}$, where left-translation by any member of G is a non-singular, measurability-preserving transformation of $(G, \bar{G}, m)$.

We call $(G, \bar{G}, m)$ bounded if G is in $\bar{G}$ (namely, if $(G, \bar{G}, m)$ is totally σ-finite) and left-invariant if m is invariant under left-translation.

The definition of a GMG is of course suggested by that of a measurable group (a concept due to A. Weil), which in our terminology is simply a left-invariant GMG. (The reader is referred to (1) and (6) for a discussion of measurable groups.) And as with a measurable group an example of a GMG is afforded by a well known structure, namely, an arbitrary locally compact topological group G with the Haar measure on its σ-ring of Baire sets, $\overline{B}$. The proof that $\overline{B}$ is invariant in the sense of Definition 4 follows from the fact that the transformation

$$S : (x, y) \rightarrow (x, xy)$$

is a homeomorphism on G x G in the product topology, together with the fact that the family of Baire sets of G x G coincides with $\overline{B} \times \overline{B}$. (See Theorem 51.C of (1).) The use here of the σ-ring of Baire sets of G in preference to the σ-ring of Borel sets is partially motivated by the

Lemma: Let G be a compact topological group, $\overline{G}$ the Borel sets of G, and m Haar measure on G. If $(G, \overline{G}, m)$ is a measurable group, then the power of G is at most that of the continuum. (See p. 261, ex. 2 of (1).)

Definition 7: Two GMG's $(G, \overline{G}, m)$,$(H, \overline{H}, n)$ are called (weakly or strongly) equivalent if there exists a measurability-preserving group isomorphism T of G onto H such that (weak case) $m \circ T^{-1} \sim n$ or (strong case) $m \circ T^{-1} = \alpha n$, α a positive real number.

Thus each bounded GMG is weakly equivalent to a GMG in which m is finite.

Definition 8: The transformation R on G x G shall be defined by the relation:

$$R(x, y) = (y, x) .$$

R is measure-preserving on $(G \times G,\ \overline{G} \times \overline{G},\ m \times m)$ as is clear from the definition of this space.

Definition 9: Two functions, f and g, on a measure space $(S, \overline{S}, m)$ shall be said to be equal nearly everywhere (m) [= n.e. (m)] if they are equal except on a subset of S which intersects every measurable set in a set of m-measure zero. A statement which is a point-function on $(S, \overline{S}, m)$ is said to be true for nearly all x (m) if it is true except for x in a set whose characteristic function equals zero n.e. (m).

Proposition 0: If A and B are any two sets, then

$$\left[(S^{-1}RS)(A \times B)\right]_x = x^{-1}A \cap B^{-1}$$

and

$$\left[(S^{-1}RS)(A \times B)\right]^y = \begin{cases} Ay^{-1}, & y \in B^{-1} \\ \phi, & y \in (B^{-1})^c. \end{cases}$$

Proof: See Theorem 59.C of (1).

Proposition 1: Let G be a group and $\overline{G}$ an invariant σ-ring of subsets of G. If $(G, \overline{G}, m)$ and $(G, \overline{G}, n)$ are GMG's, then S and S^{-1} are non-singular measurability-preserving transformations on $(G \times G, \overline{G} \times \overline{G}, m \times n)$.

Proof: For any M in $\overline{G} \times \overline{G}$ one has, by the Fubini Theorem, $(m \times n)(M) = 0$ if and only if $\int n(M_x)\, dm(x) = 0$ and $(m \times n)(S(M)) = 0$ if and only if $\int n((S(M))_x)\, dm(x) = 0$. But $(S(M))_x = xM_x$ and, $(G, \overline{G}, n)$ being a GMG, $n(xM_x) = 0$ if and only if $n(M_x) = 0$. Hence $(m \times n)(S(M)) = 0$ if and only if $(m \times n)(M) = 0$.

If S is a measurability-preserving transformation on $(G \times G, \overline{G} \times \overline{G}, m \times n)$ then S^{-1} has the same property; thus the conclusion of the preceding paragraph yields the result that S and S^{-1} are non-singular measurability-preserving transformations on $(G \times G, \overline{G} \times \overline{G}, m \times n)$.

Proposition 2: On any GMG, the transformations U and R_y, defined by the relations

$$U = x \rightarrow x^{-1} \text{ and } R_y : x \rightarrow xy,$$

are measurability-preserving and non-singular.

Proof: Theorems 59.A, C, and D of (1), with an obvious modification in the proof of the latter theorem, carry over to the case of a GMG. The first two theorems are needed to prove the third, and the third is equivalent to the proposition under consideration.

Corollary 1: The function $f(x) = m(xE \cap F)$, E and F arbitrary members of G, is measurable for any GMG.

Proof: By Proposition 0,

$$\left[(S^{-1}RS)(E \times F^{-1})\right]_x = x^{-1}E \cap F .$$

Since, by Proposition 1 above, S is a measurability-preserving transformation on any GMG and, by Proposition 2, F^{-1} in G,

$$g(x) = m(\left[(S^{-1}RS)(E \times F^{-1})\right]_x) = m(x^{-1}E \cap F)$$

is a measurable function on (G, G, m) by Proposition 2. Hence, in view of Proposition 2, $f(x) = g(x^{-1}) = m(xE \cap F)$ is measurable on (G, G, m).

Corollary 2: Any GMG, (G, G, m), has the property that if E and F are in G and $m(E) > 0$ and $m(F) > 0$, then $m(\{x | m(xE \cap F) > 0\}) > 0$.

Proof: $\{x | m(xE \cap F) > 0\} \in G$ by Corollary 1. Since S and S^{-1} are non-singular on (G x G, G x G, m x m) by Proposition 1 and R has this property also,

$$(m \times m)((S^{-1}RS)(E \times F^{-1})) > 0.$$

As a result,

$$0 < (m \times m)((S^{-1}RS)(E \times F^{-1}))$$
$$= \int m(\left[(S^{-1}RS)(E \times F^{-1})\right]_x) \, dm(x)$$

$$= \int m(x^{-1}E \cap F)\, dm(x)$$
$$= \int_{\{x | m(x^{-1}E \cap F) > 0\}} m(x^{-1}E \cap F)\, dm(x)$$
$$= \int_{\{x | m(xE \cap F) > 0\}^{-1}} m(x^{-1}E \cap F)\, dm(x) ;$$

whence, in view of Proposition 2, $m(\{x | m(xE \cap F) > 0\}) > 0$.

Corollary 3: Let $(G, \bar{G}, m)$ be a bounded GMG. Then if h is a measurable function on every member of $\bar{G}$ and if

$$h(xy) = h(y) \text{ n.e. } (m)$$

for each x in G, h is constant n.e. (m) on G.

Proof: Observe that $g(x, y) = h(y)$ is measurable on $(G \times G, \bar{G} \times \bar{G}, m \times m)$ as is also $g(S(x,y))$, in view of Proposition 1. Since the hypothesis of this corollary implies that

$$g(S(x, y)) = g(x, y) \text{ n.e. } (m)$$

for each x in G, one can apply the Fubini theorem as follows:

$$0 = \int_E \left(\int_F |g(x, y) - g(S(x, y))| \, dm(y) \right) dm(x)$$
$$= \int_F \left(\int_E |g(x, y) - g(S(x, y))| \, dm(x) \right) dm(y),$$

where E and F are arbitrary members of G. From the arbitrariness possible in the choice of E and F, it can be concluded that for nearly all y in G,

$$g(x, y) = g(S(x, y)) \text{ n.e. } (m), \text{ i.e.,}$$

remembering the definitions of g and S,

$$h(y) = h(xy) \text{ n.e. } (m).$$

Thus there exists y_o in G such that

$$h(xy_o) = h(y_o) \text{ n.e. } (m),$$

so that by Proposition 2

$$h(x) = h(y_o) \text{ n.e. } (m) .$$

The remaining results of this section, suggested by the well-known decomposition of an arbitrary locally compact topological group into the family of left cosets of a certain σ-compact open subgroup, will be of fundamental importance in the remaining sections. These results will allow us to prove a Radon-Nikodym type of theorem for a GMG and thereby make it possible to obtain results about an arbitrary GMG which would otherwise be true only for a bounded GMG.

Lemma: Given a GMG, $(G, \overline{G}, m)$, G is a topological space if the defining family, $\overline{N}$, of neighborhoods of the identity is taken to be the collection of all sets of the form $\{x | m(xE \cap E) > 0 \ , \ 0 < m(E) < \infty\}$, and if the defining family of neighborhoods of an arbitrary element x of G is taken to be $x\overline{N}$. Further, taking the defining family of neighborhoods of an arbitrary x in G to be $\overline{N}x$ yields the same topology. For any N in $\overline{N}$, $N = N^{-1}$, and N is measurable and of positive measure.

Proof: To show that we have a valid topology it is necessary only to prove that the neighborhood axioms are satisfied relative to the members of $\overline{N}$, since for each x in G xN is the family of neighborhoods of x.

First, we show that given M, N in $\overline{N}$ there exists K in $\overline{N}$ such that $K \subset M \cap N$. Let $M = \{x | m(xE \cap E) > 0, \ 0 < m(E) < \infty\}$, and $N = \{x | m(xF \cap F) > 0, \ 0 < m(F) < \infty\}$. Observe that by Corollary 2 to Proposition 2 there exists y_o in G such that $y_o^{-1}F^{-1} \cap E^{-1}$ has positive measure. ($y_o^{-1}F^{-1} \cap E^{-1}$ is measurable by Proposition 2 and the non-singularity of left-translation.)

As a result,

$$0 < m(E \cap Fy_o) < \infty .$$

But then, letting $L = E \cap Fy_o$, $K = \{x | m(xL \cap L) > 0\} \in \overline{N}$. Also remembering the non-singularity of right-translation asserted in Proposition 2,

$$\begin{aligned} K &\subset \{x | m(xE \cap E) > 0\} \cap \{x | m(xFy_o \cap Fy_o) > 0\} \\ &= \{x | m(xE \cap E) > 0\} \cap \{x | m(xF \cap F) > 0\} \\ &= M \cap N. \end{aligned}$$

In order to show that given any N in $\overline{N}$ and any x_o in N there exists in N a neighborhood of x_o, we prove first that given N in $\overline{N}$ and z_o in N there exists B in $\overline{N}$ such that $Bz_o \subset N$, and we prove second that given N in $\overline{N}$ and y_o in G there exists M in $\overline{N}$ such that $M \subset y_o^{-1} N y_o$.

Let $N = \{x | m(xE \cap E) > 0\} \in \overline{N}$ and z_o in N be given. Then define

$$B = \{x | m(x(z_oE \cap E) \cap (z_oE \cap E)) > 0\} .$$

Since $z_o \in N$ and $0 < m(E) < \infty$, $B \in \overline{N}$. And if $x \in B$,

$$\begin{aligned} m((xz_o)E \cap E) &\geq m((xz_o)E \cap E \cap xE \cap z_oE) \\ &= m[x(z_oE \cap E) \cap (z_oE \cap E)] > 0. \end{aligned}$$

Then, by definition of N,

$$Bz_o \subset N.$$

Let $N = \{x | m(xE \cap E) > 0\} \in \overline{N}$ and $y_o \in N$ be given. Note that

$$\begin{aligned} y_o^{-1} N y_o &= y_o^{-1} \{x | m(xE \cap E) > 0\} y_o \\ &= \{x | m((y_o x y_o^{-1})E \cap E) > 0\} \\ &= \{x | m(x(y_o^{-1}E) \cap (y_o^{-1}E)) > 0\} , \end{aligned}$$

by the non-singularity of left-translation. By the σ-finiteness of m there exists $D \subset y_o^{-1}E$ such that $0 < m(D) < \infty$. Consequently, if

$$M = \{x | m(xD \cap D) > 0\} \in \overline{N},$$

then

$$M \subset \{x | m(x(y_o^{-1}E) \cap (y_o^{-1}E)) > 0\} = y_o^{-1} N y_o .$$

The two facts just obtained not only complete the proof that $\overline{N}$ is a true family of neighborhoods of the identity but also show that the same topology is formed regardless of whether a family of neighborhood at x is taken to be $x\overline{N}$ or $\overline{N}x$.

By the first two corollaries to Proposition 2, any set of the form $N = \{x | m(xE \cap E) > 0, \ 0 < m(E) < \infty\}$, is measurable and of positive measure.

Finally, if $N = \{x | m(xE \cap E) > 0\} \in \overline{N}$, then $N = N^{-1}$; for $xE \cap E = x(x^{-1}E \cap E)$ and left translation is a non-singular measurability-preserving transformation on any GMG.

It might be noted at this point that the topology constructed does not necessarily satisfy any separation axioms nor does this topology have, in general, the property that the group operation is continuous in it.

The topology constructed on an arbitrary GMG in the preceding lemma will be called the <u>m-topology</u> on that GMG. It should be noted that if $(G, \overline{G}, m)$ and $(G, \overline{G}, n)$ are weakly equivalent, then the m-topology on G is equivalent to the n-topology.

<u>Proposition 3</u>: In any GMG, $(G, \overline{G}, m)$, there exists a subgroup H of G with the properties that

1) $H \in \overline{G}$,

2) $m(H) > 0$,

3) H is open in the m-topology on $(G, \overline{G}, m)$.

<u>Proof</u>: Let $N = \{x | m(xE \cap E) > 0 , \ 0 < m(E) < \infty\}$. N is open in the m-topology, being a neighborhood of the identity in the m-topology, and,

as such, is measurable and of positive measure by the preceding lemma.

Define the set N^k for any positive integer k by the following relations:

1) $N^1 = N$;

2) $N^k = N^{k-1}N, \ k \geqslant 1$.

We shall prove by induction on k that for all k N^k is open in the m-topology, measurable, and of positive measure. From the remarks already made about $N = N^1$ we see that it is only necessary to show the truth of our assertion for $N^{(\ell+1)}$ to be implied by its truth for N^ℓ, for any $\ell \geqslant 1$.

Assume N^ℓ is open in the m-topology, measurable, and of positive measure. Then it is immediate that $N^{(\ell+1)} = N^\ell N$ is open in the m-topology and, if measurable, of positive measure by Proposition 2. It remains to prove that $N^{(\ell+1)}$ is measurable.

Note that $xN \cap N^\ell \neq \phi$ implies that there exists x_o in G such that for some $M = \{x | m(xF \cap F) > 0, \ 0 < m(F) < \infty\}$

$$x_o M \subset xN \cap N^\ell,$$

since xN is a neighborhood of x in the m-topology and N^ℓ is assumed to be open in the m-topology. From the last lemma proved and the non-singularity of left-translation we conclude that $x_o M$ has positive measure. xN is measurable and N^ℓ is assumed to be measurable. Thus $xN \cap N^\ell$ is measurable and of positive measure for all x such that $xN \cap N^\ell \neq \phi$, or,

$$\{x | m(xN \cap N^\ell) > 0\} = \{x | xN \cap N^\ell \neq \phi\}.$$

Hence, the last lemma proved having shown $N = N^{-1}$,

$$\{x \mid m(xN \cap N^{\ell}) > 0\} = N^{\ell}N^{-1} = N^{\ell}N = N^{(\ell+1)}.$$

Since N and, by the inductive hypothesis, N^{ℓ} are in $\overline{G}$, then Corollary 2 to Proposition 2 applies, yielding the result that $\{x \mid m(xN \cap N^{\ell}) > 0\}$, and thus $N^{\ell+1}$, is in $\overline{G}$.

This completes the proof that N^k is open in the m-topology, measurable, and of positive measure for all positive integers k.

Consider $H = \bigcup_{k=1}^{\infty} N^k$. On the one hand, we have, from what we have just shown, that H is measurable, of positive measure, and open in the m-topology.

On the other hand, H is a subgroup of G. First, by the last lemma proved above, we see that $N = N^{-1}$ and so $N^k = (N^k)^{-1}$. Next, $N^k N^{\ell} = N^{k+\ell}$. Finally, N contains the identity of G.

Thus we have exhibited a subgroup of G which is measurable, of positive measure, and open in the m-topology, thereby proving this proposition.

<u>Proposition 4</u>: Given a GMG, $(G, \overline{G}, m)$, let H be a subgroup, measurable and of positive measure. Let E in $\overline{G}$ have positive measure. Then E intersects in a set of positive measure at most a countable number of cosets of H, say $\{x_i H\}_{i=1}^{n \leq \infty}$, where $i \neq j \Rightarrow x_i H \cap x_j H = \phi$. Further, $m(E - \bigcup_{i=1}^{n} (E \cap x_i H)) = 0$.

<u>Proof</u>: Let $\{C_\alpha\}_{\alpha \in I}$ be the family of all left cosets of H. By σ-finiteness $E = \bigcup_{i=1}^{\infty} A_i$ where $0 < m(A_i) < \infty$. Since the members of $\{C_\alpha\}_{\alpha \in I}$ are mutually disjoint measurable sets, so also are the members $\{A_i \cap C_\alpha\}_{\alpha \in I}$. If $m(A_i \cap C_\alpha) > 0$ for a more than countable collec-

tion of α's, there exists $\delta > 0$ and an infinite collection of α's for which $m(A_i \cap C_\alpha) > \delta$, which in view of the fact that $C_\alpha \cap C_\beta = \phi$ for $\alpha \neq \beta$ would imply $m(A_i) = \infty$, a contradiction. So let $A = \{\alpha_j\}_{j=1}^{\infty}$ be the countable collection of α's such that α in $\bar{A}$ implies $m(C_\alpha \cap A_i) > 0$ for at least one i. Then $m(E \cap C_\alpha) > 0$ if and only if α is in $\bar{A}$, proving the first part of this proposition.

Now consider the measurable set

$$F = E - \bigcup_{j=1}^{\infty} (E \cap C_{\alpha_j}).$$

This set, by the properties of $\{C_{\alpha_j}\}_{j=1}^{\infty}$ just obtained, intersects every C_α in a set of measure zero. Observe that, by Proposition 0,

$$x^{-1}H \cap F = \left[(S^{-1}RS)(H \times F^{-1})\right]_x,$$

for each x in G. On the basis of this one sees that every set of the form

$$(S^{-1}RS)(H \times F^{-1})_x$$

is measurable and of measure zero. Then, as

1) $H \times F^{-1}$ is $\bar{G} \times \bar{G}$ — measurable,

2) $(S^{-1}RS)^{-1} = S^{-1}RS$ is by Proposition 1 a non-singular measurability-preserving transformation on $(G \times G, \bar{G} \times \bar{G}, m \times m)$,

3) A $\bar{G} \times \bar{G}$-measurable set S has $(m \times m)$-measure zero if and only if

$$m(S_x) = 0 \text{ n.e. } (m),$$

or

$$m(S^y) = 0 \text{ n.e. } (m),$$

we have that $(m \times m)(H \times F^{-1}) = 0$. It follows by Proposition 2 that $m(F) = 0$, which proves the remainder of the present proposition.

Definition 10: A measure space $(S, \bar{S}, m)$ is said to be complete if and only if for every $F \in \bar{S}$ such that $m(F) = 0$ $E \in \bar{S}$ whenever $E \subset F$.

Definition 11: Let $(S, \bar{S}, m)$ be an arbitrary measure space. Then $\bar{\bar{S}}^{(m)} \equiv \{K \subset S | K = (E - N_1) \cup N_2,\ E \in \bar{S}, \exists M \ni m(M) = 0 \text{ and } N_1 \cup N_2 \subset M\}$ is a σ-ring containing $\bar{S}$, and the measure $\bar{m}$ on $\bar{\bar{S}}^{(m)}$ defined by the equation

$$\bar{m}((E - N_1) \cup N_2) = m(E)$$

agrees with m on $\bar{S}$.* (The complete measure space $(S, \bar{\bar{S}}^{(m)}, m)$ is called the completion of $(S, \bar{S}, m)$.)

Proposition 5: Let $(G, \bar{G}, m)$ and $(G, \bar{G}, n)$ be a pair of GMG's such that n is absolutely continuous with respect to m and $(G, \bar{G}, m)$ is a complete measure space. Then there is a non-negative function f on G, measurable on every set in $\bar{G}$, such that for every E in $\bar{G}$

$$n(E) = \int_E f(x)\, dm(x)$$

If there exists a function g on G with the same properties as f, then

$$f = g \text{ n.e. } (m).$$

Proof: By Proposition 4 we can dissect any E in $\bar{G}$ into a set N of m-measure zero and the collection $\{E_i\}_{i=1}^{\infty}$ of the intersections of E with a countable collection of left cosets of a subgroup H in $\bar{G}$ such

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*That $\bar{m}$ is uniquely defined is shown as follows. Let K in $\bar{\bar{S}}^{(m)}$ be such that

$$K = (E - N_1) \cup N_2$$
$$= (D - R_1) \cup R_2,$$

where D and E belong to $\bar{S}$ and $N_1 \cup N_2 \cup R_1 \cup R_2$ is contained in a member N of $\bar{S}$ having zero m-measure. Then,

$$E - N \subset (E - N_1) \cup N_2 \subset E \cup N,$$

and

$$D - N \subset (D - R_1) \cup R_2 \subset D \cup N.$$

Since $m(N) = 0$ and $(E - N_1) \cup N_2 = (D - R_1) \cup R_2$, $m(E) \leq m(D)$, and $m(D) \leq m(E)$, i.e. $m(D) = m(E)$. Thus, $\bar{m}(K)$ is uniquely defined.

that $m(H) > 0$.

Let $\{C_\alpha\}_{\alpha \in I}$ be the family of all mutually disjoint left cosets of the subgroup in question. Then we know by the Radon-Nikodym Theorem (valid for totally σ-finite measure spaces) that for every there is a non-negative function h_α on G (vanishing outside C_α) such that 1)

$$n(F \cap C_\alpha) = \int_{F \cap C_\alpha} h_\alpha(x)\, dm(x)$$
$$= \int_F h_\alpha(x)\, dm(x)$$

for every measurable F and 2) if k_α has the same properties as h_α

$$k_\alpha = h_\alpha \text{ n.e. } (m).$$

Thus, because of the already-mentioned decomposition of the arbitrary measurable set E,

$$n(E) = \sum_{i=1}^{\infty} n(E_i)$$
$$= \sum_{i=1}^{\infty} n(E \cap C_{\alpha_i})$$
$$= \sum_{i=1}^{\infty} \int_{E-N} h_{\alpha_i}(x)\, dm(x),$$

and, since $\left\{\sum_{i=1}^{n} h_{\alpha_i}\right\}_{n=1}^{\infty}$ is an increasing sequence of non-negative measurable functions,

$$\sum_{i=1}^{\infty} \int_{E-N} h_{\alpha_i}(x)\, dm(x) = \int_{E-N} \left(\sum_{i=1}^{\infty} h_{\alpha_i}(x)\right) dm(x).$$

So let us define f on G by the relation:

$$f(x) = h_\alpha(x)$$

for x in C_α . Then, since

$$h_{\alpha_i}(x) = \chi_{C_{\alpha_i}}(x)\; h_{\alpha_i}(x)$$

for all x and

$$m\left[E - \bigcup_{i=1}^{\infty} (E \cap C_{\alpha_i})\right] = m(N)$$

$$= 0,$$

f is measurable on E by the completeness of $(G, \overline{G}, m)$, and the formula just obtained becomes:

$$n(E) = \int_{E-N} f(x)\, dm(x).$$

Inasmuch as $m(N) = 0$, we have that

$$n(E) = \int_{E} f(x)\, dm(x) .$$

Also, if g is a function on G with the same properties as f, $g\chi_{C_\alpha}$ is a function having the same properties as h_α so that

$$g\chi_{C_\alpha} = h_\alpha \text{ n.e. } (m).$$

But, by definition,

$$h_\alpha = f\chi_{C_\alpha} ,$$

so that

$$g\chi_{C_\alpha} = f\chi_{C_\alpha} \text{ n.e. } (m) .$$

Since α was arbitrarily chosen and since every set is contained in a countable collection of C_α's if we delete a set of measure zero,

$$g = f \text{ n.e. } (m).$$

Proposition 6: If $\{(G_i, \overline{G}_i, m_i)\}_{i=1}^{n}$ is a collection of GMG's, then the assertion of Proposition 5 is valid for the completion of measure spaces *of the form*

$$\left(\bigtimes_{i=1}^{n} G_i, \bigtimes_{i=1}^{n} \overline{G}_i, m_1 x \ldots\ldots x\, m_n\right).$$

Proof: We can prove by induction that $\bigtimes_{i=1}^{n} \overline{G}_i$ is the same as the σ-ring generated by $\{A_1 \times A_2 \ldots\ldots \times A_n \subset \bigtimes_{i=1}^{n} G_i \mid A_i \in G_i\}$. This combined with Proposition 4 yields the result that for every $K \in \bigtimes_{i=1}^{n} \overline{G}_i$ there exists $N \in \bigtimes_{i=1}^{n} \overline{G}_i$ for which $(m_1 \times m_2 \times \ldots\ldots \times m_n)(N) = 0$ and $K - N \subset \bigcup_{j=1}^{\infty} (a_j^{(1)}H_1 \times \ldots\ldots \times a_j^{(n)}H_n)$; here H_i is a subgroup of G_i of the sort constructed by Proposition 4. Regarding $\bigtimes_{i=1}^{n} G_i$ as a direct-product group, we see that $H_1 \times \ldots\ldots \times H_n$ is a measurable subgroup of $\bigtimes_{i=1}^{n} G_i$ having positive measure and that $a_1H_1 \times \ldots\ldots \times a_nH_n$ is always a left coset of $H_1 \times \ldots\ldots \times H_n$ and a set of positive measure. Hence, by the definition of $\overline{\left(\bigtimes_{i=1}^{n} \overline{G}_i\right)}^{(m_1 \times \ldots\ldots \times m_n)}$, for every $E \in \overline{\left(\bigtimes_{i=1}^{n} \overline{G}_i\right)}^{(m_1 \times \ldots \times m_n)}$ there is an N in the same σ-ring such that $(m_1 \times \ldots\ldots \times m_n)(N) = 0$ and $E - N \subset \bigcup_{j=1}^{\infty} (a_j^{(1)}H_1 \times \ldots\ldots \times a_j^{(n)}H_n)$.

This last conclusion makes it possible to repeat the argument of Proposition 5; so, the assertion of the present proposition must follow.

<u>Lemma:</u> Let $(S, \overline{S})$ and $(T, \overline{T})$ be measurable spaces. Suppose $A \subset S$ and $B \subset T$. Then

$$(A \times B) \cap \overline{S} \times \overline{T} = (A \cap \overline{S}) \times (B \cap \overline{T}).$$

<u>Proof:</u> We observe first that $(A \times B) \cap \overline{S} \times \overline{T}$ contains all of the collection of sets which generates $(A \cap \overline{S}) \times (B \cap \overline{T})$, namely, $\{E \times F \subset S \times T \mid E \in (A \cap \overline{S}),\ F \in (B \cap \overline{T})\}$. This shows that

$$\overline{R}_1 = (A \cap \overline{S}) \times (B \cap \overline{T}) \subset (A \times B) \cap \overline{S} \times \overline{T}.$$

Now, by the same reasoning as was used to get the preceding,

$$\overline{R}_2 = (A \cap \overline{S}) \times (B^c \cap \overline{T}) \subset (A \times B^c) \cap \overline{S} \times \overline{T},$$

$$\overline{R}_3 = (A^c \cap \overline{S}) \times (B \cap \overline{T}) \subset (A^c \times B) \cap \overline{S} \times \overline{T},$$

and

$$\overline{R}_4 = (A^c \cap \overline{S}) \times (B^c \cap \overline{T}) \subset (A^c \times B^c) \cap \overline{S} \times \overline{T}.$$

Thus,

$$\overline{R} = \left\{ \bigcup_{i=1}^{4} E_i \subset S \times T \,\middle|\, E_i \in \overline{R}_i,\ i = 1, \ldots, 4 \right\} \subset \overline{S} \times \overline{T}.$$

Since all the $\bar{R}_i$'s are σ-rings, since no two $\bar{R}_i$'s have any set in common, and since $\bar{R}$ contains every member of the family of sets generating $\bar{S} \times \bar{T}$, $\bar{R}$ is a σ-ring containing the generators of $\bar{S} \times \bar{T}$, which means that

$$\bar{R} \supset \bar{S} \times \bar{T}.$$

As a result,

$$\bar{R} = \bar{S} \times \bar{T}.$$

We see now that any member E of $(A \times B) \cap (\bar{S} \times \bar{T})$ not belonging to $\bar{R}_1 = (A \cap \bar{S}) \times (B \cap \bar{T})$ must, being a member of $(A \times B) \cap \bar{R}$, contain a non-void subset of $(A \times B)^c$. But this contradicts the assumption that E is a subset of $A \times B$. Thus

$$(A \times B) \cap (\bar{S} \times \bar{T}) \supset (A \cap \bar{S}) \times (B \cap \bar{T}),$$

which, in view of the fact that the latter family contains the former, proves that

$$(A \times B) \cap (\bar{S} \times \bar{T}) = (A \cap \bar{S}) \times (B \cap \bar{T}).$$

<u>Proposition 7:</u> Given two σ-finite measure spaces, $(S, \bar{S}, m)$ and $(T, \bar{T}, n)$, consider $(S \times T, \overline{(\bar{S} \times \bar{T})}^{(m \times n)}, \overline{m \times n})$, the completion of $(S \times T, \bar{S} \times \bar{T}, m \times n)$. Let f be a non-negative, $\overline{(\bar{S} \times \bar{T})}^{(m \times n)}$-measurable function on $S \times T$. Then, except for x in a set of m-measure zero, $f(x, y)$ is a $\bar{\bar{T}}^{(n)}$-measurable function of y, and, except for y in a set of n-measure zero, $f(x, y)$ is $\bar{\bar{S}}^{(m)}$-measurable function of x. Further,

$$\int_T f(x, y)\, dn(y)$$

is $\bar{\bar{S}}^{(m)}$-measurable, if we define it in some arbitrary manner at all x for which $f(x, y)$ is not $\bar{\bar{T}}^{(n)}$-measurable, and, with a similar qualification,

$$\int_S f(x,y)\, dm(x)$$

is $\overline{\overline{S}}^{(m)}$-measurable. Finally,

$$\int_{S \times T} f(x, y)\ d(\overline{m \times n})(x, y) = \int_S \left(\int_T f(x, y)\ d\overline{n}(y) \right) d\overline{m}(x)$$

$$= \int_T \left(\int_S f(x, y)\ d\overline{m}(x) \right) d\overline{n}(y).$$

Proof: Let us note first that by the assumption of $(\overline{\overline{S} \times \overline{T}})^{(m \times n)}$-measurability for f and the definition of $(\overline{\overline{S} \times \overline{T}})^{(m \times n)}$ there must be a set E in $\overline{S} \times \overline{T}$ such that

$$\{(x, y) \in S \times T \mid f(x, y) > 0\} \subset E.$$

Then, since the σ-ring $\{K \subset S \times T \mid \exists F \in \overline{S}, G \in \overline{T} \ni K \subset F \times G\}$ contains $\overline{S} \times \overline{T}$, there must be an $A \in \overline{S}$ and a $B \in \overline{T}$ for which

$$\{(x, y) \in S \times T \mid f(x, y) > 0\} \subset A \times B.$$

In addition, f, considered as a function on A x B, is a non-negative function measurable relative to $(A \times B) \cap (\overline{\overline{S} \times \overline{T}})^{(m \times n)}$, which, using the lemma just proved, is identical with $\overline{(A \cap \overline{S}) \times (B \cap \overline{T})}^{(m \times n)}$. But, A and B each being the union of a sequence of sets of finite measure, we can apply Theorem 9.10 of (5) to f considered as a non-negative, measurable function on $\left(A \times B, \left(\overline{(A \cap \overline{S}) \times (B \cap \overline{T})}\right)^{(m \times n)}, \overline{m \times n}\right)$. Hence, remembering that $\overline{(A \cap \overline{S}) \times (B \cap \overline{T})}^{(m \times n)} = (A \times B) \cap (\overline{\overline{S} \times \overline{T}})^{(m \times n)}$, $(\overline{A \cap \overline{S}})^{(m)} = A \cap \overline{\overline{S}}^{(m)}$, and $(\overline{B \cap \overline{T}})^{(n)} = B \cap \overline{\overline{T}}^{(n)}$ and that f vanishes on $(A \times B)^c$, the assertion of this proposition follows.

We now have the tools necessary to prove some uniqueness properties of GMG's and to discuss the existence of invariant measures on the σ-ring of a GMG.

3. Uniqueness Theorems.

In this section we shall discuss certain uniqueness properties of GMG's. We show that the family of null sets in a GMG is dependent only on the underlying measurable space and that the left-invariant GMG corresponding to the measurable space $(G, \overline{G})$ is essentially unique if it exists.

Proposition 8: If two GMG's have the same underlying measurable space $(G, \overline{G})$, they are weakly equivalent.

Proof: It follows from the definition of the transformation R and the assumptions on S that $S^{-1}RS$ is a measurability-preserving transformation on $(G \times G, \overline{G} \times \overline{G}, m \times n)$, if we consider two GMG's of the form $(G, \overline{G}, m)$, $(G, \overline{G}, n)$. Further, by Proposition 0

$$\left[(S^{-1}RS)(A \times B)\right]_x = x^{-1} A \cap B^{-1}$$

and

$$\left[(S^{-1}RS)(A \times B)\right]^y = \begin{cases} Ay^{-1}, & y \in B^{-1} \\ \phi, & y \in (B^{-1})^c \end{cases},$$

where A and B are arbitrary members of G. As a result the following formula is true:

$$\int n(x^{-1}A \cap B^{-1})\, dm(x) = \int_{B^{-1}} m(Ay^{-1})\, dn(y) .$$

Thus, since m and n are assumed to be left non-singular and since by Proposition 2 the inverse of set of m-measure zero has m-measure zero, we can see that if $n(A) = 0$, then $m(A) = 0$.

Interchange of the roles of m and n in the preceding argument yields the converse result and completes this proposition.

Theorem 1: Two left-invariant GMG's having the same underlying measurable space $(G, \overline{G})$ are strongly equivalent.

Proof: Let the two left-invariant GMG's be denoted by $(G, \overline{G}, m)$, $(G, \overline{G}, n)$. By Proposition 8, they and their completions are weakly equivalent. Also, $\overline{\overline{G}}^{(m)} = \overline{\overline{G}}^{(n)}$. By Proposition 5 there exists a non-negative function h on G such that h is $\overline{\overline{G}}^{(n)}$-measurable on every $\overline{\overline{G}}^{(n)}$-measurable set and

$$\overline{m}(E) = \int_E h(y)\, d\overline{n}(y)$$

for each E in $\overline{\overline{G}}^{(n)}$. As a result, we have for any E in $\overline{\overline{G}}^{(n)}$ and any x in G,

$$\int_E h(y)\, d\overline{n}(y) = \overline{m}(E) = \overline{m}(xE) = \int_{xE} h(y)\, d\overline{n}(y) = \int_E h(xy)\, d\overline{n}(y).$$

Therefore $h(xy) = h(y)$ n.e. $(\overline{n})$, so that by Proposition 4 and Corollary 3 to Proposition 2 $h(y) = \text{constant}$ n.e. (n). Further, the assumptions that $h \geqslant 0$ and that m is not identically zero make the constant positive. So we have that for any E in $\overline{G}$

$$m(E) = \overline{m}(E) = \int_E h(x)\, d\overline{n}(x) = \int_E (k)\, d\overline{n}(x) =$$

$$k \cdot \int_E \chi_E(x)\, d\overline{n}(x) = k \cdot \overline{n}(E) = k \cdot n(E)\,,$$

where $k > 0$. This proves the strong equivalence of $(G, \overline{G}, m)$ and $(G, \overline{G}, n)$.

It may be remarked that the preceding provides a new proof of the uniqueness of the Haar measure on a locally compact topological group. For if G is a locally compact group, $\overline{G}$ its family of Baire sets, and m the Haar measure on $\overline{G}$, then as shown in Section 1 above $(G, \overline{G}, m)$ is a left-invariant GMG. Theorem 1 now enables us to say that m is unique up to a multiplicative constant. Since, finally, Theorem 52.H of (1) states that for a locally compact space agreement of two regular Borel measures on all Baire sets implies agreement on all Borel sets, the Haar measure on a locally compact group is seen to be unique to within a multiplicative constant.

3. Existence Theorems

The present section is devoted to showing that any GMG is weakly equivalent to a left-invariant GMG. Since left-translation is a non-singular, measurability-preserving transformation on a GMG, $(G, \overline{G}, m)$, we may apply Proposition 5 to $(G, \overline{\overline{G}}^{(m)}, m)$, the completion of $(G, \overline{G}, m)$, to obtain a non-negative function $f(x, y)$, measurable in y on every set in $\overline{\overline{G}}^{(m)}$ for each x and having the property that

$$\overline{m}(xE) = \int_E f(x, y)\, d\overline{m}(y)$$

for every E in $\overline{\overline{G}}^{(m)}$. Speaking intuitively, $f(x, y)$ is the density of the measure $\overline{m}$ at x relative to y. So, if for some fixed y, say y_o, $f(x, y)$ were measurable on every set in $\overline{\overline{G}}^{(m)}$, the measure on $\overline{\overline{G}}^{(m)}$ defined for each E in $\overline{\overline{G}}^{(m)}$ by the formula,

$$n(E) = \int_E \frac{1}{f(x, y_o)}\, d\overline{m}(xy_o),$$

would appear to be a left-invariant measure equivalent to $\overline{m}$. n is a fortiori equivalent to m on $\overline{G}$. We shall show that this argument, together with the assumption about the measurability properties of $f(x, y)$, is essentially correct.

We begin with some definitions. Consider a GMG, $(G, \overline{G}, m)$, and the complete measure space $(G, \overline{\overline{G}}^{(m)}, m)$. If

$$\overline{m}_x(E) = \overline{m}(xE), \text{ then}$$

$$f(x, y) = \frac{d(\overline{m}_x)}{d\overline{m}} .$$

If $(\overline{m \times m})_S(F) = (\overline{m \times m})(S(F))$, then

$$g(x, y) = \frac{d\,(\overline{m \times m})_S}{d(\overline{m \times m})}$$

Lemma: Consider $(G, \overline{G}, m)$, a GMG. If $f(x_1, \ldots, x_n)$ is $\left(\overline{\bigtimes_{i=1}^{n} G}\right)^{(m\times\ldots\times m)}$ - measurable on every set in $\left(\overline{\bigtimes_{i=1}^{n} G}\right)^{(m\times\ldots\times m)}$ so also are $f(x_1, \ldots, x_i x_j, \ldots, x_n)$ and $f(x_1, \ldots, x_i^{-1} x_j, \ldots, x_n)$, $i < j$.

Proof: Since the transformation S is non-singular (by Proposition 1) and measurability-preserving on $(G \times G, \overline{G} \times \overline{G})$ and since the process of forming products of σ-rings is associative, the transformation on $(\bigtimes_{i=1}^{n} G, \bigtimes_{i=1}^{n} \overline{G})$ defined by:

$$U: (x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) \to (x_1, \ldots, x_i, x_j, x_{i+2}, \ldots, x_{j-1}, x_{i+1}, \ldots, x_n)$$
$$\to (x_1, \ldots, S(x_i, x_j), x_{i+2}, \ldots, x_{j-1}, x_{i+1}, x_{j+1}, \ldots, x_n)$$
$$\to (x_1, \ldots, x_i, \ldots, x_i x_j, \ldots, x_n)$$

is 1-1, non-singular, and measurability-preserving. Because of the nonsingularity of U, it is then a measurability-preserving transformation on $(\bigtimes_{i=1}^{n} G_i, (\bigtimes_{i=1}^{n} G_i)^{(m\times\ldots\times m)}, \overline{m\times\ldots\times m})$. $f(U(x_1, \ldots, x_n)) = f(x_1, \ldots, x_i x_j, \ldots, x_n)$ and $f(U^{-1}(x, \ldots, x_n)) = f(x_1, \ldots, x_i^{-1} x_j, \ldots, x_n)$. Hence the lemma.

Lemma: Let $(G, \overline{G}, m)$ be a GMG. Then there exists a function h on $G \times G \times G$ such that

a) for every x in G

$f(x, z) = h(x, y, z)$ n.e. $(\overline{m \times m})$,

b) h is $(\overline{\overline{G} \times \overline{G} \times \overline{G}})^{(m \times m \times m)}$-measurable on every member of $(\overline{\overline{G} \times \overline{G} \times \overline{G}})^{(m \times m \times m)}$,

c) for every x in G $h(x, y, z)$ is $(\overline{\overline{G} \times \overline{G}})^{(m \times m)}$-measurable on $(\overline{\overline{G} \times \overline{G}})^{(m \times m)}$.

Proof: Let x be a fixed, arbitrary element of G, and let $E, F \subset G$ be arbitrary measurable subsets of G.

First, note that $m(yF) = m((S(E \times F))_y)$; hence, $m(xyF)$ is a $\widehat{G}$-measurable function of y on E, left-translation being measurability-preserving. Then, keeping Proposition 5 in mind,

$$\int_E m(xyF)\, dm(y) = \int_E \left(\int_{yF} f(x, z)\, d\bar{m}(z) \right) d\bar{m}(y)$$

$$= \int_G \left(\int_G \chi_{S(E \times F)}(y, z)\, f(x, z)\, d\bar{m}(z) \right) d\bar{m}(y)$$

$$= \int_{S(E \times F)} f(x, z)\, d(\overline{m \times m})(y, z) = ,$$

if $F_x(y, z) \equiv f(x, z)$, a function $\overline{\overline{G} \times \overline{G}}^{(m \times m)}$-measurable on every set in $\overline{\overline{G} \times \overline{G}}^{(m \times m)}$ for each x,

$$= \int_{S(E \times F)} F_x(y, z)\, d(\overline{m \times m})(y, z)$$

$$= \int_{E \times F} F_x(S(y, z))\, d(m \times m)(S(y, z))$$

$$= \int_{E \times F} f(x, yz)\, d(\overline{m \times m})(S(y, z))$$

$$= \int_{E \times F} f(x, yz) g(y, z)\, d(\overline{m \times m})(y, z).$$

Second, we have that for any K in G, remembering Proposition 7,

$$\int_K m(yF)\, d\bar{m}(y) = \int_G m\left(\left[S(K \times F) \right]_y \right)\, dm(y)$$

$$= (m \times m)(S(K \times F))$$

$$= (\overline{m \times m})(S(K \times F))$$

$$= \int_{K \times F} g(y, z) \, d(\overline{m \times m})(y, z)$$

$$= \int_K \left(\int_F g(y, z) \, d\overline{m}(z) \right) d\overline{m}(y).$$

Thus, since

$$\left\{ y | m(xyF) \neq \int_F g(xy, z) \, d\overline{m}(z) \right\}$$

$$= x^{-1} \left\{ y \; m(yF) \neq \int_F g(y, z) \, d\overline{m}(z) \right\},$$

we have for each x, $m(xyF) = \int_F g(xy, z) \, d\overline{m}(z)$ n.e. $(\overline{m})$.

Consequently, for each x

$$\int_E m(xyF) \, d\overline{m}(y) = \int_E \left(\int_F g(xy, z) \, d\overline{m}(z) \right) d\overline{m}(y).$$

Then, since the assumption that left-translation is a measurability-preserving, non-singular transformation of $(G, \overline{G}, m)$ makes $G_x(y, z) = g(xy, z)$ $(\overline{\overline{G} \times \overline{G}})^{(m \times m)}$-measurable on every set of $(\overline{\overline{G} \times \overline{G}})^{(m \times m)}$, Proposition 7 applies to $G_x(y, z)$. Hence,

$$\int_E m(xyF) \, d\overline{m}(y) = \int_E \left(\int_F g(xy, z) \, d\overline{m}(z) \right) d\overline{m}(y)$$

$$= \int_{E \times F} g(xy, z) \, d(\overline{m \times m})(y, z)$$

The preceding paragraphs have shown that the measures defined on $(\overline{\overline{G} \times \overline{G}})^{(m \times m)}$ for each x in G by

$$\kappa_x(M) = \int_M f(x, yz) g(y, z) \, d(\overline{m \times m})(y, z)$$

and

$$\lambda_x(M) = \int_M g(xy, z) \, d(\overline{m \times m})(y, z)$$

will agree on the ring $\bar{R}$ of all unions of finite, mutually disjoint collections of measurable rectangles in $(\overline{\bar{G} \times \bar{G}})^{(m \times m)}$.

We note now that $\bar{R}$ generates $\bar{G} \times \bar{G}$ and that, by Theorem 13.A of (1), that two measures coinciding on a ring and defined on the σ-ring generated by that wing coincide on the σ-ring. Hence for every x in G and every M in $\bar{G} \times \bar{G}$, $\lambda_x(M) = \kappa_x(M)$. But it is immediate that for every x in G $\lambda_x = \kappa_x$ on $(\overline{\bar{G} \times \bar{G}})^{(m \times m)}$. So, for each x in G

$$f(x, yz) = g(xy, z)/g(y, z) \text{ n.e. } (\overline{m \times m}).$$

Here $\kappa(x, y, z) = g(xy, z)/g(y, z)$ is $(\overline{\bar{G} \times \bar{G} \times \bar{G}})^{(m \times m \times m)}$-measurable on each member of $(\overline{\bar{G} \times \bar{G} \times \bar{G}})^{(m \times m \times m)}$ by the definition of g(y, z) and the preceding lemma, and for each x $\kappa(x, y, z)$ is $(\overline{\bar{G} \times \bar{G}})^{(m \times m)}$-measurable on each member of $(\overline{\bar{G} \times \bar{G}})^{(m \times m)}$.

Now for each fixed x in G

$$S^{-1} \{(y, z) | f(x, yz) \neq \kappa(x, y, z)\} = \{(y, z) | f(x, z) \neq \kappa(x, y, y^{-1}z)\}$$

and S^{-1} is a non-singular measurability-preserving transformation on $(G \times G, (\overline{\bar{G} \times \bar{G}})^{(m \times m)}, \overline{m \times m})$. So, for each x in G

$$f(x, z) = \kappa(x, y, y^{-1}z) \text{ n.e. } (\overline{m \times m}),$$

and for each x $\kappa(x, y, y^{-1}z)$ is $(\overline{\bar{G} \times \bar{G}})^{(m \times m)}$-measurable on every set in $(\overline{\bar{G} \times \bar{G}})^{(m \times m)}$ by the $(\overline{\bar{G} \times \bar{G}})^{(m \times m)}$-measurability property of $\kappa(x, y, z)$ and the preceding lemma. Also, $\kappa(x, y, y^{-1}z)$ is measurable.

<u>Lemma:</u> If h is the function of the preceding lemma, then

a) for every b and x in G

$$\frac{1}{h(x, y, z)} = \frac{h(b, y, xz)}{h(bx, y, z)} \text{ n.e. } (\overline{m \times m}).$$

b) for every b in G h(b, y, xz) and h(bx, y, z) are $(\overline{\bar{G} \times \bar{G} \times \bar{G}})^{(m \times m \times m)}$-measurable functions of (x, y, z) on every member of $\overline{\overline{(\bar{G} \times \bar{G} \times \bar{G})}}^{(m \times m \times m)}$.

Proof: Using Proposition 7, the fact that left-translation preserves $\overline{\overline{G}}^{(m)}$-measurability, and the last two lemmas, we can perform the following computation. If E and F are arbitrary members of $\bar{G}$, then

$$\int_{E \times F} h(bx, y, z)\, d(\overline{m \times m})(y, z) = \int_{E \times F} f(bx, z)\, d(\overline{m \times m})(y, z)$$

$$= \int_E \left(\int_F f(bx, z)\, d\bar{m}(z) \right) d\bar{m}(y)$$

$$= \int_E m(bxF)\, d\bar{m}(y)$$

$$= \int_E \left(\int_{xF} f(b, z)\, d\bar{m}(z) \right) d\bar{m}(y)$$

$$= \int_E \left(\int_F f(b, xz)\, f(x, z)\, d\bar{m}(z) \right) d\bar{m}(y)$$

$$= \int_{E \times F} f(b, xz)\, f(x, z)\, d(\overline{m \times m})(y, z)$$

$$= \int_{E \times F} h(b, y, xz)\, h(x, y, z)\, d(\overline{m \times m})(y, z).$$

But this means that if for every V in $\bar{G} \times \bar{G}$ we define

$$r(V) = \int_V h(bx, y, z)\, d(\overline{m \times m})(y, z)$$

and

$$s(V) = \int_V h(b, y, xz)\, h(x, y, z)\, d(\overline{m \times m})(y, z)$$

then r and s are measures of $\bar{G} \times \bar{G}$ which agree on all measurable rectangles in $\bar{G} \times \bar{G}$. Then, using the same reasoning as in the preceding lemma, we can conclude that r = s on all of $\bar{G} \times \bar{G}$ and, indeed, on all of $\overline{\overline{(\bar{G} \times \bar{G})}}^{(m \times m)}$. Hence, for every b and x in G

$$h(bx, y, z) = h(b, y, xz)\, h(x, y, z) \text{ n.e. } (\overline{m \times m}),$$

or, what is the same, for every b and x in G

$$\frac{1}{h(x, y, z)} = \frac{h(b, y, xz)}{h(bx, y, z)} \quad \text{n.e. } (\overline{m \times m}).$$

It follows from the last two lemmas and the fact that left-translation is non-singular and $\bar{G}$-measurability-preserving that h(b, y, xz) and h(bx, y, z) have the asserted measurability properties.

Lemma: The transformation T on G x G x G defined by the equation

$$T(x, y, z) = (xz, y, z)$$

is a $(\overline{\bar{G} \times \bar{G} \times \bar{G}})^{(m \times m \times m)}$-measurability-preserving transformation. Both T and T^{-1} are non-singular transformations on $(G \times G \times G, (\overline{\bar{G} \times \bar{G} \times \bar{G}})^{(m \times m \times m)}, \overline{m \times m \times m})$.

Proof: We note as a preliminary that if we define the transformations L, M and N as follows they are $(\overline{\bar{G} \times \bar{G} \times \bar{G}})^{(m \times m \times m)}$-measurability-preserving and they, together with their inverses, are non-singular on $(G \times G \times G, (\overline{\bar{G} \times \bar{G} \times \bar{G}})^{(m \times m \times m)}, \overline{m \times m \times m})$:

a) $L(x, y, z) = (x^{-1}, y, z^{-1})$,

b) $M(x, y, z) = (z, y, x)$,

c) $N(x, y, z) = (x, y, xz)$.

To begin with, the mapping which sends x onto x^{-1} for every x in G is non-singular and measurability-preserving on $(G, \bar{G}, m)$; whence, by the Fubini Theorem and the definition of $\bar{G} \times \bar{G} \times \bar{G}$, L and L^{-1} are non-singular and measurability-preserving on $(G \times G \times G, \bar{G} \times \bar{G} \times \bar{G}, m \times m \times m)$. It follows that they have the same properties relative to $(G \times G \times G, (\overline{\bar{G} \times \bar{G} \times \bar{G}})^{(m \times m \times m)}, \overline{m \times m \times m})$.

As noted after Definition 8, the mapping R on $(G \times G, \bar{G} \times \bar{G}, m \times m)$ such that $R(x, y) = (y, x)$ is a measure-preserving transformation; thus it follows by definition of $(\overline{\overline{G} \times \overline{G} \times \overline{G}})^{(m \times m \times m)}$ and $\overline{m \times m \times m}$ that M and M^{-1} are measure-preserving transformations of $(G \times G \times G, (\overline{\overline{G} \times \overline{G} \times \overline{G}})^{(m \times m \times m)}, \overline{m \times m \times m})$.

Finally, let us use the fact, noted in the last paragraph, that the transformation

$$(x, y) \to (y, x)$$

is a measure-preserving transformation on $(G \times G, \overline{G} \times \overline{G}, m \times m)$, and let us remember that the transformation

$$(x, y) \to (x, xy),$$

as well as its inverse, is non-singular and measurability-preserving on $(G \times G, \overline{G} \times \overline{G}, m \times m)$. Then, by the definition of $(\overline{\overline{G} \times \overline{G} \times \overline{G}})^{(m \times m \times m)}$, the transformation N is $(\overline{\overline{G} \times \overline{G} \times \overline{G}})^{(m \times m \times m)}$-measurability-preserving, and both N and N^{-1} are non-singular. A computation shows us that the transformation T is expressible by the formula

$$T = LMNM^{-1}L^{-1} ;$$

hence, T is $(\overline{\overline{G} \times \overline{G} \times \overline{G}})^{(m \times m \times m)}$-measurability-preserving, and both T and T^{-1} are non-singular.

<u>Theorem 2:</u> Given a GMG, $(G, \overline{G}, m)$, there exists a measure μ on G such that $(G, \overline{G}, \mu)$ is a left-invariant GMG and $(G, \overline{G}, \mu)$ is weakly equivalent to $(G, \overline{G}, m)$. If $(G \times G \times G, (\overline{\overline{G} \times \overline{G} \times \overline{G}}), \overline{m \times m \times m})$ is the completion of $(G \times G \times G, \overline{G} \times \overline{G} \times \overline{G}, m \times m \times m)$ considered as a measure space and if h and T are, respectively, the function and transformation of the preceding lemmas, then for every E in $\overline{G}$

$$\mu(E) = \int_{E \times F_o \times K_o} \frac{1}{h(T^{-1}(x, y, z))} d(\overline{m \times m \times m})(x, y, z).$$

(Here F_o and K_o are fixed sets of positive, finite measure belonging to

$\overline{G}$.)

Proof: By the preceding three lemmas and Proposition 7 $\frac{1}{h(T^{-1}(x,y,z))}$ is positive and finite n.e. $(\overline{m \times m \times m})$ and is $\widetilde{(\overline{G} \times \overline{G} \times \overline{G})}^{(m \times m \times m)}$-measurable on any member of $\widetilde{(\overline{G} \times \overline{G} \times \overline{G})}^{(m \times m \times m)}$. Thus μ is well defined for all E in $\overline{G}$ and determines a σ-finite measure on $\overline{G}$. Thus $(G, \overline{G}, \mu)$ is a GMG weakly equivalent to $(G, \overline{G}, m)$, as a result of the definition of a GMG and Proposition 8.

We now complete the proof by showing μ to be left-translation invariant.

Let a be an arbitrary member of G and E be any member of G. Then,

$$\begin{aligned}
\mu(aE) &= \int_{(aE) \times F_o \times K_o} \frac{1}{h(T^{-1}(x,y,z))} \, d(\overline{m \times m \times m})(x,y,z) \\
&= \int_{(aE) \times F_o \times K_o} \frac{1}{h(xz^{-1}, y, z)} \, d(\overline{m \times m \times m})(x,y,z) \\
&= \int_{K_o} \left(\int_{F_o} \left(\int_{aE} \frac{1}{h(xz^{-1}, y, z)} \, d\overline{m}(x) \right) d\overline{m}(y) \right) d\overline{m}(z) \\
&= \int_{K_o} \left(\int_{F_o} \left(\int_{E} \frac{f(a,x)}{h(axz^{-1}, y, z)} \, d\overline{m}(x) \right) d\overline{m}(y) \right) d\overline{m}(z) \\
&= \int_{K_o} \left(\int_{E \times F_o} \frac{f(a,x)}{h(axz^{-1}, y, z)} \, d(\overline{m \times m})(x,y) \right) d\overline{m}(z) \\
&= \int_{K_o} \left(\int_{E \times F_o} \frac{h(a,y,x)}{h(axz^{-1}, y, z)} \, d(\overline{m \times m})(x,y) \right) d\overline{m}(z) \\
&= \int_{E \times F_o \times K_o} \frac{h(a,y,x)}{h(axz^{-1}, y, z)} \, d(\overline{m \times m \times m})(x,y,z) \, .
\end{aligned}$$

The steps of the preceding computation are justified by the results of the last three lemmas plus Propositions 5 and 7, except for the implicit assertion that $h(axz^{-1}, y, z)$ is $\widetilde{(\overline{G} \times \overline{G} \times \overline{G})}^{(m \times m \times m)}$-measurable on $E \times F_o \times K_o$ since $h(xz^{-1}, y, z)$ has the same property on

$(aE) \times F_o \times K_o$. But by Proposition 2 the transformations

$$x \to ax$$

and

$$x \to a^{-1}x$$

are non-singular and measurability-preserving on $(G, \overline{G}, m)$; whence,

$$(x, y, z) \to (ax, y, z)$$

and

$$(x, y, z) \to (a^{-1}x, y, z)$$

are non-singular and measurability-preserving on $(G \times G \times G, \overline{G} \times \overline{G} \times \overline{G}, m \times m \times m)$, which means that

$$(x, y, z) \to (ax, y, z)$$

is measurability-preserving on $(G \times G \times G, (\overline{\overline{G} \times \overline{G} \times \overline{G}})^{(m \times m \times m)}, \overline{m \times m \times m})$.

Now let us examine the formula just obtained for $\mu(aE)$. If

$$q_a(x, y, z) = \frac{h(a, y, xz)}{h(ax, y, z)} ,$$

we know on the one hand that for all x

$$\frac{1}{h(x, y, z)} = q_a(x, y, z) \text{ n.e. } (\overline{m \times m})$$

and we know on the other hand that

$$q_a(T^{-1}(x, y, z)) = \frac{h(a, y, x)}{h(axz^{-1}, y, z)} .$$

Further, we know by the next to last lemma above that $\frac{1}{h(x, y, z)}$ and $q_a(x, y, z)$ are $(\overline{\overline{G} \times \overline{G} \times \overline{G}})^{(m \times m \times m)}$-measurable on every member of $(\overline{\overline{G} \times \overline{G} \times \overline{G}})^{(m \times m \times m)}$, so that by Proposition 7

$$\frac{1}{h(x, y, z)} = q_a(x, y, z) \text{ n.e. } (\overline{m \times m \times m}) .$$

But these facts and the fact that T and T^{-1} are non-singular, measurability-preserving transformations on $(G \times G \times G, (G \times G \times G)^{(m \times m \times m)}, m \times m \times m)$ yield the following:

$$\frac{h(a, y, x)}{h(axz^{-1}, y, z)} = q_a(T^{-1}(x, y, z))$$

$$= \frac{1}{h(T^{-1}(x, y, z))} \text{ n.e. } (\overline{m \times m \times m})$$

As a result,

$$\int_{E \times F_o \times K_o} \frac{h(a, y, z)}{h((axz^{-1}), y, z)} \, d(\overline{m \times m \times m})(x, y, z)$$

$$= \int_{E \times F_o \times K_o} \frac{1}{h(T^{-1}(x, y, z))} \, d(\overline{m \times m \times m})(x, y, z)$$

$= \mu(E)$.

Thus, remembering our formula above for $\mu(aE)$,

$$\mu(aE) = \mu(E).$$

This establishes the left-translation invariance of μ and concludes our proof.

Finally, we wish to show that our results yield a generalization of a theorem of A. Weil on left-invariant GMG's. In order to discuss this theorem, and, for later use in Section III of this thesis, we must state certain definitions from the theory of uniform spaces. (See (8).)

Definition A: Given an abstract set E, let C and D be any two subsets of E x E, the cartesian product of E with itself. Then we mean by CD the set of all (p, r) in E x E such that there exists a q in E for which we have both (p, q) in C and (q, r) in D. Further, by C^{-1} we mean the set of all (p, q) in E x E such that (q, p) belongs to C.

Definition B: Given E x E, the cartesian product of an abstract set E with itself, we shall call a family $\{V_\alpha\}_{\alpha\in I}$ of subsets of E x E a system of neighborhoods of the diagonal if and only if the following properties hold:

1) $\bigcap_{\alpha\in I} V = \Delta \equiv \{(p, q)\in E \times E \; p = q\}$,

2) For every pair, α and β, in I, there exists a γ in I for which

$$V_\gamma \subset V_\alpha \cap V_\beta ,$$

3) For every α in I there is a β in I such that

$$V_\beta (V_\beta{}^{-1}) \subset V_\alpha .$$

Every member of $\{V_\alpha\}_{\alpha\in I}$ shall be called a neighborhood of the diagonal. An example is afforded by any topological group G. Let $\{U_\gamma\}_{\gamma\in J}$ be a defining system of neighborhoods of the identity for G. Then $\{W_\lambda\}_{\lambda\in J}$ is a system of neighborhoods of the diagonal if we define for every λ :

$$W_\lambda = \{(p, q)\in G \times G \; q\in pU_\lambda\}.$$

Definition C: Given a set E, let there be defined a system of neighborhoods of the diagonal $\{V_\alpha\}_{\alpha\in I}$ for E x E. Then E is a Hausdorff space if we assign to every p in E the family of neighborhoods $\{(V_\alpha)_p\}_{\alpha\in I}$. (See p. 8 of (8).) By $(V_\alpha)_p$ we mean, in accordance with the definition of Section 34 of (1), $\{q\in E \; (p, q)\in V_\alpha\}$. We define the couple (E, $\{V_\alpha\}_{\alpha\in I}$) to be a uniform space, and, for the sake of simplicity, we make the convention of referring to (E, $\{V_\alpha\}_{\alpha\in I}$) simply as "E".

Definition D: If (E, $\{V_\alpha\}_{\alpha\in I}$) is a uniform space, a family $\{C_\delta\}_{\delta\in k}$ of subsets of E is said to be a Cauchy filter if

1) every subset of E containing a C_δ is a member of $\{C_\delta\}_{\delta\epsilon k}$,

2) the intersection of every finite subfamily of $\{C_\delta\}_{\delta\epsilon k}$ is non-void and belongs to $\{C_\delta\}_{\delta\epsilon k}$,

3) for every α in I there is a C_δ such that

$$C_\delta \times C_\delta \subset V_\alpha .$$

$\{C_\delta\}_{\delta\epsilon k}$ is said to <u>converge</u> if there is a point p in E such that the family of neighborhoods $\{(V_\alpha)_p\}_{\alpha\epsilon I}$ is a subfamily of $\{C_\delta\}_{\delta\epsilon k}$. $\{C_\delta\}_{\delta\epsilon k}$ is then said to <u>converge to p.</u>

<u>Definition E:</u> A uniform space E is said to be <u>complete</u> if every Cauchy filter made up of subsets of E converges.

<u>Lemma:</u> Given a uniform space $(E, \{V_\alpha\}_{\alpha\epsilon I})$ there exists a complete uniform space $(\overline{E}, \{\overline{V}_\alpha\}_{\alpha\epsilon I})$ such that E is equivalent to a dense subset E' of $\overline{E}$ in the following way: there is a biunique mapping f of E onto E' such that $(f \times f)(V_\alpha) = (E' \times E') \cap \overline{V}_\alpha$ for all α. $\overline{E}$ is unique in the sense that any other uniform space with the same properties has the form $(g(\overline{E}), \{(g \times g)(\overline{V}_\alpha)\}_{\alpha\epsilon I})$, where g is a biunique mapping of $\overline{E}$. (See Theorem 2 of (8).)

We can now state the theorem which we desire to generalize:

<u>Theorem 3:</u> Let $(G, \overline{G}, m)$ be a left-invariant GMG. Then G can be given a topology $\overline{T}$ which can be defined by requiring a family of neighborhoods of an arbitrary element x to be the family $x\overline{U}$ of all sets of the form:

$$x\{y \mid m((yE - E) \cup (E - yE)) < \epsilon,\ 0 < \epsilon < 2m(E) < \infty\};$$

all members of $x\overline{U}$ have positive measure and, when x is the identity, are symmetric. The subcollection $x\overline{N}$, consisting of the members of $x\overline{U}$ for which $m(E^{-1})$ is finite, has the properties that all its members have finite measure and that it is an equivalent family of neighborhoods of x. Special properties of $\overline{T}$ are:

a) for any neighborhood U of the identity there exists a neighborhood **V** of the identity such that

$$VV^{-1} \subset U;$$

b) given any neighborhood V of the identity and any a in G there is a neighborhood W such that

$$aWa^{-1} \subset V;$$

c) there exists a neighborhood of the identity coverable by a finite number of left-translates of any other neighborhood of the identity.

Suppose there is made an additional hypothesis on $(G, \overline{G}, m)$, namely, that for every a in G there exists an E in $\overline{G}$ such that $m(aE \cap E) < m(E)$. Then the topology $\overline{T}$ has the Hausdorff property, so that G is a topological group in $\overline{T}$. Finally, the completion $\widehat{G}$ of G is a locally compact topological group such that if $\hat{G}$ is the family of Baire sets and $\hat{m}$ is the Haar measure on $\hat{G}$ then $\hat{G} \cap G \subset \overline{G}$ and $\hat{m}(E) = m(E \cap G)$ for all E in $\hat{G}$.

It might be noted that this is a combination of the results of Section 62 of (1) and the remark following Theorem 10 of (7).

Our generalization is:

<u>Theorem 4</u>: If $(G, \overline{G}, m)$ is an arbitrary GMG, then there exists a unique equivalent left-invariant measure n such that the first paragraph of conclusions of Theorem 3 holds for $(G, \overline{G}, n)$. Further, if for every a in G there exists an E in $\overline{G}$ such that $m(aE \cap E) < m(E)$, then the second paragraph of conclusions of Theorem 3 holds for G.

Proof: The first assertion of this theorem follows from Theorems 1 and 2 together with the first paragraph of conclusions of Theorem 3. The second assertion follows from the equivalence of m to n and the second paragraph of conclusions of Theorem 3.

It can be seen that this last theorem is a true generalization of A. Weil's theorem, for the uniqueness of n means that when m is left invariant m(E) is positive and finite if and only if n(E) has the same two properties; whence, if the members of a base at the identity for the topology $\bar{T}$ all have positive, finite n-measure, then they all have positive finite m-measure. This takes care of the only point in which the generalization is not apparent.

II. APPLICATIONS TO REPRESENTATION THEORY

The purpose of the present section is to prove a generalization of the succeeding theorem of L. H. Loomis:

THEOREM: Let $\rho \to U_\rho$ be a strongly continuous representation of alocally compact group G by unitary operators on a hilbert space H, and let $A \to P_A$ be a non-zero Boolean σ-homomorphism from the Baire sets of G onto an Abelian family of projections on H, such that

$$U_\rho P_A = P_{\rho A} U_\rho$$

for every ρ in G and every Baire set A. Suppose first that H is irreducible under the combined families $\{U_\rho\}$ and $\{P_A\}$. Then there exists a unique unitary mapping of $L^2(G)$ onto H such that U_ρ corresponds to left translation through ρ, and P_A corresponds to multiplication by the characteristic function ϕ_A of A. That is,

$$T^{-1}U_\rho Tf \;(s) = f(\rho^{-1}s),$$

and

$$T^{-1}P_A Tf \;(s) = \phi_A(s)\, f(s) .$$

In general H is a direct sum of subspaces each of which is identifiable with $L^2(G)$ in the above way.

Specifically our results are as follows. If H is an abstract group with an invariant (in the sense of Section I) σ-ring of subsets $\bar{S}$ and if there exist a representation of H and a σ-homomorphism π of $\bar{S}$ which satisfy certain assumptions, $\bar{S}$ supports a unique translation-invariant σ-finite measure μ such that the conclusion of Loomis's theorem holds with $L^2(H, \bar{S}, \mu)$ substituted for $L^2(G)$. Further, we have a result which adds to the interest of the preceding. If to our hypothesis on H and $\bar{S}$ we add the requirement that for every a in H there exists an E in $\bar{S}$

such that $\pi(E - (aE)) \neq 0$ then H can be given a topology of the sort that a) H is a dense subgroup of a locally compact group, b) the representation is a strongly continuous function on H, and c) the neighborhood system defining the topology is equivalent to a neighborhood system composed of members of $\bar{S}$ which have positive, finite μ-measure.

Loomis's theorem is obtained from our results by showing that $\pi^{-1}(0)$ is identical with the family of Baire sets having zero Haar measure. This fact, combined with Loomis's assumption that the given representation of G is strongly continuous, then yields our assumptions on the representation and the σ-homomorphism as simple consequences.

We begin the discussion by making a number of definitions.

Definition 1: An Abelian ring of projections $\bar{A}$ on a hilbert space $\mathcal{h}$ is a family of projection operators on $\mathcal{h}$ which is commutative with respect to operator multiplication and has the following additional properties:

a) If P, Q belong to $\bar{A}$ then PQ and P - PQ belong to $\bar{A}$.

b) Given any sequence $\{P_i\}_{i=1}^{\infty}$ of elements of $\bar{A}$ such that $P_iP_j = 0$ for $i \neq j$ there exists a member P of A such that $\left\{\sum_{i=1}^{n} P_i\right\}_{n=1}^{\infty}$ converges weakly to P, i.e., $\sum_{i=1}^{\infty} (P_i, \xi, \eta) = (P\xi, n)$ for all ξ and η in $\mathcal{h}$.

Definition 2: Consider a non-zero element B of an Abelian ring of projections $\bar{A}$ and any collection $\{A_\alpha\}_{\alpha \in I}$ of elements of $\bar{A}$ such that

$$A_\alpha B = A_\alpha$$

and, if $\alpha \neq \beta$

$$A_\alpha A_\beta = 0.$$

If for every such collection at most a countable number of A_α's are non-zero, then B is said to be countably decomposable relative to $\bar{A}$. If all elements of $\bar{A}$ are countably decomposable relative to $\bar{A}$ then $\bar{A}$ is said to be countably decomposable. A system which, as we shall see later, is a very appropriate example of a countably decomposable Abelian ring of projections is the following. (As can be perceived from (5) this example is in most respects typical of the general Abelian ring.) This is that subset of the L^∞ of a measure space which consists of projections of the L^2 of that measure space, namely, the collection of all members of L^∞ corresponding to characteristic functions of measurable sets.

Definition 3: An Abelian ring of projections on a hilbert space $\mathcal{h}$ is said to be full if no non-zero vector is mapped into 0 by all members of the ring.

As we see from the fact that the set on which a measurable function (if not equal to 0 n.e.) is non-zero has positive measure, our example of a countably decomposable Abelian ring also provides us with an example of a full Abelian ring.

Definition 4: The union of a family $\{P_\alpha\}_{\alpha\in I}$ of elements of an Abelian ring of projections $\bar{A}$ is a projection P such that

1) $P_\alpha P = P_\alpha$ for all α,

2) If for any Q in $\bar{A}$ $P_\alpha Q = P_\alpha$ for all α then $PQ = P$.

The union of $\{P_\alpha\}_{\alpha\in I}$ shall be denoted by $\bigcup_{\alpha\in I} P_\alpha$.

It is a consequence of the second defining property of a union that it is unique.

Lemma 1: Let $\bar{A}$ be an Abelian ring of projections on a hilbert space $\mathcal{h}$. If $\{P_i\}_{i=1}^\infty$ has the property that $P_iP_j = 0$ if $i \neq j$, then

$\bigcup_{i=1}^{\infty} P_i$ exists and $\left\{\sum_{i=1}^{n} P_i\right\}_{n=1}^{\infty}$ converges weakly to $\bigcup_{i=1}^{\infty} P_i$.

Proof: Suppose we have $\{P_i\}_{i=1}^{\infty}$ such that $P_i P_j = 0$ if $i \neq j$. We know from the definition of an Abelian ring of projections that there exists P in $\bar{A}$ with the property that $\left\{\sum_{i=1}^{n} P_i\right\}_{n=1}^{\infty}$ converges weakly to P. Thus, to prove this lemma, we shall simply verify that P has the two properties defining $\bigcup_{i=1}^{\infty} P_i$, since they define it uniquely.

Let ξ and η be any two vectors in $\mathfrak{h}$. Now we see first that for all i

$$
\begin{aligned}
(P_i P\xi, \eta) &= (P\xi, P_i\eta) \\
&= \lim_{n\to\infty} \left(\left(\sum_{j=1}^{n} P_j\right)\xi, P_i\eta\right) \\
&= \lim_{n\to\infty} \left(\left(\sum_{j=1}^{n} P_i P_j\right)\xi, \eta\right) \\
&= (P_i\xi, \eta),
\end{aligned}
$$

which from the arbitrariness of ξ and η proves $P_i P = P_i$. Next if Q in $\bar{A}$ has the property that $P_i Q = P_i$ for all i then

$$
\begin{aligned}
(PQ\xi, \eta) &= (P\xi, Q\eta) \\
&= \lim_{n\to\infty} \left(\left(\sum_{i=1}^{n} P_i\right)\xi, Q\xi\right) \\
&= \lim_{n\to\infty} \left(\left(\sum_{i=1}^{n} P_i Q\right)\xi, \eta\right) \\
&= \lim_{n\to\infty} \left(\left(\sum_{i=1}^{n} P_i\right)\xi, \eta\right) \\
&= (P\xi, \eta) ,
\end{aligned}
$$

so that $PQ = P$ from the arbitrariness of ξ and η. So we see that P has the defining properties of the unique element $\bigcup_{i=1}^{\infty} P_i$ and that, as a result, $\sum_{i=1}^{n} P_i$ converges weakly to $\bigcup_{i=1}^{\infty} P_i$.

(The next lemma is essentially Lemma 2.5 of Part II of (5), with appropriate modifications.)

Definition 5: A separating vector for an Abelian ring of projections on a hilbert space $\mathcal{H}$ is a vector ξ such that if for any P in $\overline{A}$ $P\xi = 0$ then $P = 0$.

Lemma 2: Let $\overline{A}$ be a countably decomposable Abelian ring of projections with identity on a hilbert space. Then there is a separating vector for $\overline{A}$ on $\mathcal{H}$.

Proof: Let $\overline{F} = \{P_\alpha\}_{\alpha \in K}$ be a family of non-zero projections in $\overline{A}$ maximal with respect to the following properties: 1) the members of $\overline{F}$ are mutually orthogonal, 2) $P_\alpha \overline{A}$ has a separating vector for every P_α in $\overline{F}$. The existence of such a collection follows on application of the Maximal Principle. We see that because of the countable decomposability of the identity projection in $\overline{A}$ $\{P_\alpha\}_{\alpha \in K}$ must be countable or is the empty set.

In order to construct a separating vector for $\overline{A}$ we first prove that $\bigcup_{\alpha \in K} P_\alpha$ exists and

$$\bigcup_{\alpha \in K} P_\alpha = I.$$

The existence of $\bigcup_{\alpha \in K} P_\alpha$ follows from the countability of $\{P_\alpha\}_{\alpha \in K}$ and from Lemma 1. The remainder is obtained thus.

Letting ζ be an arbitrary non-zero vector in $(I - \bigcup_{\alpha \in K} P_\alpha)$ not annihilated by all members of $(I - \bigcup_{\alpha \in K} P_\alpha)\overline{A}$, we show the existence in $(I - \bigcup_{\alpha \in K} P_\alpha)\overline{A}$ of a maximal projection annihilating ζ. To this end, we first apply the Maximal Principle to show the existence of $\{Q_\delta\}_{\delta \in L}$, a family of non-zero projections in $(I - \bigcup_{\alpha \in K} P_\alpha)\overline{A}$ maximal with respect to the succeeding properties: 1) the members are mutually orthogonal,

2) $Q_\delta \zeta = 0$ for every δ in L. Then we note that $\bigcup_{\delta \in L} Q_\delta$ exists and annihilates ζ by the countable decomposibility of $\bar{A}$ and Lemma 1. Finally, if S in $(I - \bigcup_{\alpha \in K} P_\alpha)\bar{A}$ annihilates ζ, then $S(\bigcup_{\delta \in L} Q_\delta) = S$, for, inasmuch as

$$\|S(I - \bigcup_{\alpha \in K} P_\alpha - \bigcup_{\delta \in L} Q_\delta)\zeta\|^2 = ((I - \bigcup_{\alpha \in K} P_\alpha - \bigcup_{\delta \in L} Q_\delta)\zeta, S\zeta) = 0,$$

$S(I - \bigcup_{\alpha \in K} P_\alpha - \bigcup_{\delta \in L} Q_\delta)$ must vanish due to the maximality properties of $\{Q_\delta\}_{\delta \in L}$.

Since $(I - \bigcup_{\alpha \in K} P_\alpha)\zeta \neq 0$ by the choice of ζ, then $R = I - \bigcup_{\alpha \in K} P_\alpha - \bigcup_{\delta \in L} Q_\delta$ is a non-zero projection in $\bar{A}$. R is orthogonal to $\bigcup_{\alpha \in K} P_\alpha$, and so by Lemma 1 orthogonal to every P_α. Further, ${}_{\delta \in K} Q_\delta$ being maximal among those elements of $(I - {}_{\in K} P)A$ which annihilate ζ, $R\bar{A}$ has ζ as a separating vector. But then $\{P_\alpha\}_{\alpha \in K} \cup \{R\}$ properly contains $\{P_\alpha\}_{\alpha \in K}$ and has the same properties, a contradiction of the maximality of $\{P_\alpha\}_{\alpha \in K}$. Hence, $I - \bigcup_{\alpha \in K} P_\alpha$ is the zero-projection, or,

$$\bigcup_{\alpha \in K} P_\alpha = I.$$

We can now construct a separating vector for $\bar{A}$. From the definition of $\{P_\alpha\}_{\alpha \in K}$ and the countability of K, there exists a countable collection of vectors $\{\xi_\alpha\}_{\alpha \in K}$ such that 1) ξ_α is a separating vector for $P_\alpha \bar{A}$ for all α, 2) $P_\alpha \xi_\alpha = \xi_\alpha$, and 3) $\xi = \sum_{\alpha \in K} \xi_\alpha$ exists as a limit in the norm. Then, if we recall that $\bigcup_{\alpha \in K} P = I$ and if we remember that by Lemma 1 for any countable disjoint collection $\{Q_i\}_{i=1}^{\infty}$ of projections in $\bar{A}$ $\{\sum_{i=1}^{n} Q_i\}_{n=1}^{\infty}$ converges weakly to $\bigcup_{i=1}^{\infty} Q_i$, then ξ is actually a vector such as we desire. For if $P\xi = 0$, we can conclude that

$$\sum_{\alpha \in K} \|(PP_\alpha)\xi_\alpha\|^2 = \sum_{\alpha \in K} \|PP_\alpha \xi\|^2$$
$$= \sum_{\alpha \in K} (P_\alpha \xi, P\xi)$$
$$= (\xi, P\xi)$$
$$= \|P\xi\|^2$$
$$= 0,$$

so that $PP_\alpha = 0$ for all α. But this means that for every η in $\mathcal{H}$

$$\|P\eta\|^2 = (\eta, P\eta)$$
$$= \sum_{\alpha \in K} (P_\alpha \eta, P\eta)$$
$$= \sum_{\alpha \in K} ((PP_\alpha)\eta, \eta)$$
$$= 0,$$

i.e., that

$$P = 0.$$

Thus, by definition, ξ is a separating vector for $\bar{A}$.

Lemma 3: A full, countably-decomposable Abelian ring of projections $\bar{A}$ of a hilbert space $\mathcal{H}$ contains a collection of non-zero projections $\{P_\alpha\}_{\alpha \in I}$ such that 1) $P_\alpha P_\beta = 0$ if $\alpha \neq \beta$ and 2) given any P in $\bar{A}$

$$P = \bigcup_{i=1}^{\infty} (PP_{\alpha_i})$$

for some sequence of α's.

Proof: We can prove this lemma by taking $\{P_\alpha\}_{\alpha \in I}$ to be any maximal collection of mutually disjoint projections in $\bar{A}$. (This exists by the Maximal Principle.) Then, since for all β in I and every non-zero Q in $\bar{A}$

$$P_\beta \Big[Q - \bigcup_{\substack{\alpha \in I \\ QP_\alpha \neq 0}} (QP_\alpha)\Big] = P_\beta \Big[Q - Q \bigcup_{\substack{\alpha \in I \\ QP_\alpha \neq 0}} (QP_\alpha)\Big]$$

$$= P_\beta Q - \Big[(P_\beta Q) \bigcup_{\substack{\alpha \in I \\ QP_\alpha \neq 0}} (QP_\alpha)\Big]$$

$$= P_\beta Q - P_\beta Q$$

$$= 0,$$

the maximality of $\{P_\alpha\}_{\alpha \in I}$ is contradicted if $Q - \bigcup_{\substack{\alpha \in I \\ QP_\alpha \neq 0}} (QP_\alpha) = 0$.

The collection of α's for which $QP_\alpha \neq 0$ is countable in virtue of our assumption of countable decomposability for $\overline{A}$. The indicated unions thus all exist if we invoke Lemma 1.

<u>Definition 6</u>: A <u>σ-homomorphism</u> of a σ-ring $\overline{S}$ of subsets of an abstract set onto an Abelian ring of projections $\overline{A}$ is a mapping π of $\overline{S}$ onto $\overline{A}$ such that

1) $\pi(A \cap B) = \pi(A)\pi(B)$

2) $\pi(A \cup B) = (\pi(A) - \pi(A)\pi(B)) + \pi(B)$

3) If $\{A_i\}_{i=1}^{\infty}$ is a collection of mutually disjoint elements of $\overline{S}$ then $\left\{\sum_{i=1}^{n} \pi(A_i)\right\}_{n=1}^{\infty}$ converges weakly to $\pi(\bigcup_{i=1}^{\infty} A_i)$.

<u>Definition 7:</u> A function f on a measure space to a hilbert space $\mathcal{h}$ is called a <u>countably-valued step-function</u> if there is a countable, mutually disjoint collection of measurable sets $\{E_i\}_{i=1}^{\infty}$ and a sequence of vectors in $\mathcal{h}$, $\{\xi_i\}_{i=1}^{\infty}$, such that for every x

$$f(x) = \sum_{i=1}^{\infty} \xi_i \chi_{E_i}(x) .$$

Definition 8: Let f be a function on a measurable space $(M, \bar{S})$ to a hilbert space $\mathfrak{K}$, and let π be a σ-homomorphism on $\bar{S}$ to a family of projections on $\mathfrak{K}$. Then f is said to be strongly measurable relative to π if given any member E of $\bar{S}$ one can find a sequence $\{f_n\}_{n=1}^{\infty}$ of countably-valued step-functions with the following special property: there exists an increasing sequence $\{E_n\}_{n=1}^{\infty}$ of measurable subsets of E such that $\pi(E - \bigcup_{n=1}^{\infty} E_n) = 0$ and such that on each $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f.

As an example of the above sort of function let us consider $\phi(x)\xi$, where ξ is an arbitrary element of $L^2(G, \bar{B}, \mu)$ and where ϕ is the "left regular representation" of G, i.e., the representation in which each element of G is mapped into the corresponding left-translation operator on $L^2(G, \bar{B}, \mu)$. ($\bar{B}$ is the σ-ring of Baire sets of G and μ is the Haar measure on $\bar{B}$.) Our measurable space is $(G, \bar{B})$ and π is any σ-homomorphism of B onto an Abelian ring of projections on $L^2(G, \bar{B}, \mu)$. As shown on p. 41 of (8) $\phi(x)\xi$ is continuous at the identity. Clearly $\phi(x)\xi$ is uniformly continuous in the following sense: if N is a neighborhood of the identity e such that $\|\phi(x)\xi - \phi(e)\xi\| < \epsilon$ for x in N then for y = ax in aN

$$\begin{aligned} \|\phi(y)\xi - \phi(a)\xi\| &= \|\phi(ax)\xi - \phi(a)\xi\| \\ &= \|\phi(a)\phi(x)\xi - \phi(a)\phi(e)\xi\| \\ &= \|\phi(a)\| \cdot \|\phi(x)\xi - \phi(e)\xi\| \\ &= \|\phi(x)\xi - \phi(e)\xi\| \\ &< \epsilon. \end{aligned}$$

Also, $\bar{B}$ being generated by a family of compact sets, every member of $\bar{B}$ is contained in the union of a countable collection of compact sets, so that any E in $\bar{B}$ is contained in the union of a countable number of translates of any open member of $\bar{B}$. Adding to this the fact that any neighborhood of the identity contains an open Baire set which in turn contains the identity, the "uniform continuity" of $\phi(x)\xi$ makes it a uniform limit of countably-valued step functions on E, and so, a fortiori, $\phi(x)\xi$ is a strongly measurable relative to π.

Definition 9: By a projection-measurable group we shall mean the ordered quintuple composed of:

1) An abstract group G,

2) A σ-ring $\bar{S}$ of subsets of G which is invariant in the sense of Definition 5 of Section I,

3) A hilbert space $\mathcal{H}$,

4) A σ-homomorphism π of S onto a full, countably decomposable Abelian ring of projections on $\mathcal{H}$,

5) A representation ϕ on G to a group of unitary operators on $\mathcal{H}$ such that:

a) $\phi(x)\xi$ is strongly measurable relative to π for every ξ in $\mathcal{H}$,

b) For every x in G and E in $\bar{S}$ $\pi(xE) = \phi(x^{-1})\pi(E)\phi(x)$.

We shall denote a projection-measurable group by the symbol $(G, \phi; \bar{S}, \pi; \mathcal{H})$.

Proposition 1: Let $(G, \phi; \bar{S}, \pi; \mathcal{H})$ be a projection-measurable group. Then the following assertions are true.

First, there exists on $\bar{S}$ a left-invariant, σ-finite measure μ unique up to a multiplicative constant, with the property that it vanishes on precisely those sets of $\bar{S}$ which belong to $\pi^{-1}(0)$.

Second, G can be given a topology $\bar{T}$ in which a family of neighborhoods of the arbitrary element x is given by the family $x\bar{U}$ of all sets of the form:

$$x\left\{y \mid \mu((yE - E) \cup (E - yE)) < \epsilon,\ 0 < \epsilon < 2\mu(E) < \infty\right\};$$

all such sets have positive measure and, when x is the identity, are symmetric. The subcollection $x\bar{N}$, consisting of those sets for which $\mu(E^{-1})$ is finite, has the properties that it is a family of neighborhoods equivalent to the defining family $x\bar{U}$, and that each member has finite measure. Special properties of $\bar{T}$ are:

a) for any neighborhood U of the identity there exists a neighborhood V of the identity such that

$$VV^{-1} \subset U;$$

b) given any neighborhood V of the identity and any a in G there is a neighborhood W such that

$$aWa^{-1} \subset V;$$

c) there exists a neighborhood of the identity coverable by a finite number of left translates of every other neighborhood of the identity.

<u>Proof:</u> We obtain our first conclusion as follows: 1) we construct on $\bar{S}$ a σ-finite measure m such that left-translation is a non-singular transformation and such that $m(E) = 0$ if and only if $\pi(E) = 0$; 2) we invoke Theorems 1 and 2 of Section I to assert the existence and uniqueness (up to a multiplicative constant) of an equivalent left-invariant, σ-finite measure.

Let $\{P_\alpha\}_{\alpha \in I}$ be a collection of mutually disjoint elements of $\pi(\bar{S})$ such that for any other element Q of $\pi(\bar{S})$

$$Q = \bigcup_{i=1}^{\infty} (QP_{\alpha_i}).$$

(Such a collection exists by Lemma 3.) Then if for each α in I we choose for $P_\alpha \pi(\bar{S})$ a separating vector ξ_α (existing by Lemma 2), we can define a real-valued function m on $\bar{S}$ as follows: for every E in $\bar{S}$

$$m(E) = \begin{cases} \sum_{\substack{\alpha \in I \\ \pi(E)P_\alpha \neq 0}} (\pi(E)P_\alpha \xi_\alpha, \xi_\alpha), & \pi(E) \neq 0 \\ 0 & , \pi(E) = 0. \end{cases}$$

Since if $\pi(E) \neq 0$ then $\pi(E)P_\alpha \neq 0$ for some α in I, we can see that $m(E) = 0$ if and only if $\pi(E) = 0$.

That left-translation carries a set on which m is zero onto another set on which m is zero is seen by combining two facts. The first is that we have assumed for all a in G and E in $\bar{S}$ the formula:

$$\pi(aE) = \phi(a^{-1})\pi(E)\phi(a),$$

from which we see $\pi(aE) = 0$ if and only if $\pi(E) = 0$. The second fact, established in the preceding paragraph, is that $m(E) = 0$ if and only if $\pi(E) = 0$.

Next we show σ-finiteness. In view of the properties of the collection $\{P_\alpha\}_{\alpha \in I}$, we have for any E in $\bar{S}$ and a suitable sequence of α's, $\{\alpha_i\}_{i=1}^{\infty}$:

1) $(\pi(E)P_{\alpha_i})(\pi(E)P_{\alpha_j}) = 0, \; i \neq j,$

2) $\pi(E)P_{\alpha_i} \neq 0,$

3) $\pi(E) = \bigcup_{i=1}^{\infty} \pi(E)P_{\alpha_i}$.

We choose a sequence of sets in $\bar{S}$, $\{E_i\}_{i=1}^{\infty}$, such that

$$E_i \cap E_j = \phi, \ i \neq j,$$

and

$$\pi(E_i) = P_{\alpha_i}.$$

Then, keeping in mind the definition of π and the fact that Lemma 1 makes $\left\{\sum_{i=1}^{n} \pi(E \cap E_i)\right\}_{n=1}^{\infty}$ converge weakly to $\bigcup_{i=1}^{\infty} \pi(E \cap E_i)$,

$$\begin{aligned} \pi(E) &= \bigcup_{i=1}^{\infty} \pi(E)P_{\alpha_i} \\ &= \bigcup_{i=1}^{\infty} \pi(E \cap E_i) \\ &= \pi(\bigcup_{i=1}^{\infty} E \cap E_i) \\ &= \pi(E)\pi(\bigcup_{i=1}^{\infty} E_i); \end{aligned}$$

whence,

$$\begin{aligned} \pi(E - (E \cap \bigcup_{i=1}^{\infty} E_i)) &= \pi(E) - \pi(E)\pi(\bigcup_{i=1}^{\infty} E_i) \\ &= 0, \end{aligned}$$

or

$$m(E - (E \cap \bigcup_{i=1}^{\infty} E_i)) = 0.$$

Further, for every i

$$\begin{aligned} m(E_i) &= \sum_{\substack{\alpha \in I \\ \pi(E_i)P_\alpha \neq 0}} (\pi(E_i)P_\alpha \xi_\alpha, \xi_\alpha) \\ &= \sum_{\substack{\alpha \in I \\ P_{\alpha_i} P_\alpha \neq 0}} (P_{\alpha_i} P_\alpha \xi_\alpha, \xi_\alpha) \\ &= (P_{\alpha_i} P_{\alpha_i} \xi_{\alpha_i}, \xi_{\alpha_i}). \end{aligned}$$

Hence, since $E = (E \cap (\bigcup_{i=1}^{\infty} E_i)) \cup (E - \bigcup_{i=1}^{\infty} E_i)$, E is the union of a countable collection of sets for which m is finite.

Finally, we prove the countable additivity of m. Consider any countable, disjoint collection of sets, $\{E_i\}_{i=1}^{\infty}$. In the case that $m(E_i) = 0$ for all i, we know that $\pi(E_i) = 0$ for all i, and this, combined with the property of π requiring that $\left\{\sum_{i=1}^{n} \pi(E_i)\right\}_{n=1}^{\infty}$ converge weakly to $\pi(\bigcup_{i=1}^{\infty} E_i)$, shows $\pi(\bigcup_{i=1}^{\infty} E_i) = 0$ and shows countable additivity. In the remaining case where $m(E_i) > 0$ for at least one i, let

$$\{\alpha_j\}_{j=1}^{\infty} = \{\alpha \in I \mid \exists i \ni \pi(E_i)P_\alpha \neq 0\},$$

and observe that

$$\begin{aligned}\sum_{i=1}^{\infty} m(E_i) &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} (\pi(E_i)P_{\alpha_j}\xi_{\alpha_j}, \xi_{\alpha_j}) \\ &= \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} (\pi(E_i)P_{\alpha_j}\xi_{\alpha_j}, \xi_{\alpha_j}) \\ &= \sum_{j=1}^{\infty} (\pi(\bigcup_{i=1}^{\infty} E_i)\, P_{\alpha_j}\xi_{\alpha_j}, \xi_{\alpha_j})\end{aligned}$$

and, since $\pi(\bigcup_{i=1}^{\infty} E_i)\, P_\alpha \neq 0$ if and only if $\pi(E_i)P_\alpha \neq 0$ for some α,

$$\begin{aligned}\sum_{j=1}^{\infty} (\pi(\bigcup_{i=1}^{\infty} E_i)P_{\alpha_j}\xi_{\alpha_j}, \xi_{\alpha_j}) &= \sum_{\substack{\alpha\in I \\ \pi(\bigcup_{i=1}^{\infty} E_i)P_\alpha \neq 0}} (\pi(\bigcup_{i=1}^{\infty} E_i)P_\alpha \xi_\alpha, \xi_\alpha) \\ &= m(\bigcup_{i=1}^{\infty} E_i).\end{aligned}$$

Thus, we have proved the existence of a left non-singular, σ-finite measure on $\widetilde{S}$ for which $m(E) = 0$ if and only if $\pi(E) = 0$. Theorems 1 and 2 of Section 1 now assert the existence and uniqueness of an equivalent, left-invariant, σ-finite measure.

We have now shown, by proving the preceding, that the hypotheses permitting the first conclusion of Theorem 3 of Section I are satisfied; hence, the second conclusion of the present proposition, which is precisely the same as the above must hold.

Definition 10: A set E in a projection-measurable group $(G, \phi; \overline{S}, \pi; \mathcal{K})$ shall be called measurable if and only if E belongs to $\overline{S}$. By the measure of a measurable set E in $\overline{S}$ we shall mean $\mu(E)$.

Definition 11: By $\overline{N}$ we shall mean the family of neighborhoods $\overline{N}$ at the identity of Proposition 1.

Definition 12: We shall say that a function on G is continuous if and only if it is continuous in the topology $\overline{T}$.

Definition 13: We shall say that a subset of G is a neighborhood of a point x in G if it is a neighborhood of the point x in the topology $\overline{T}$.

Remark: Of the following sequence of results, Propositions 2, 3, and 4, as well as Theorem 1, constitute what is essentially a reproduction of Loomis' main arguments in (3), while Lemmas 10, 11, and 12 are known in the literature; the others are original and arise from the necessity of using the not quite locally compact topology $\overline{T}$ to prove results obtained by Loomis from local compactness.

Lemma 4: Given a projection-measurable group $(G, \phi; \overline{S}, \pi; \mathcal{K})$, then for every ξ in $\mathcal{K}$ $\phi(x)\xi$ is a continuous function on G to $\mathcal{K}$.

Proof: Proposition 1 of this section states that an arbitrary member of N has the form

$$\{x \mid \mu((xE - E) \cup (E - xE)) < \epsilon,\ 0 < \epsilon < 2\mu(E) < \infty\}.$$

Accordingly, it will suffice to show that for every $\eta > 0$ and every element ξ of $\mathcal{K}$ there exists a set F in $\overline{S}$ such that

$$\mu(F) > 0$$

and

$$\|\phi(x)\xi - \xi\| < \eta$$

for every x in FF^{-1}.

Let $\eta > 0$ and $\xi \in \mathcal{H}$ be given. Then using the definition of strong measurability with respect to π we see that there is a K in $\overline{S}$ for which

$$\mu(K) > 0$$

and that there is a ζ in $\mathcal{H}$ for which

$$\|\phi(x)\xi - \zeta\| < \eta/4,$$

when x is in K.

Then for some fixed x_o in K we have

$$\|\phi(x)\xi - \phi(x_o)\xi\| \leq \|\phi(x)\xi - \zeta\| + \|\phi(x_o)\xi - \zeta\|$$
$$< \eta/2$$

for every x in K. Or, since $\phi(x_o)$ is unitary,

$$\|\phi(x)\xi - \xi\| < \eta/2$$

for every x in $F = x_o^{-1}K$. As a result, if r and s are any two elements of F,

$$\begin{aligned} \|\phi(rs^{-1})\xi - \xi\| &\leq \|\phi(rs^{-1})\xi - \phi(r)\xi\| + \|\phi(r)\xi - \xi\| \\ &= \|\phi(rs^{-1})(\xi - \phi(s)\xi\| + \|\phi(r)\xi - \xi\| \\ &= \|\phi(s)\xi - \xi\| + \|\phi(r)\xi - \xi\| \\ &< \eta. \end{aligned}$$

But this is the same as saying that for every x in FF^{-1}

$$\|\phi(x)\xi - \xi\| < \eta.$$

Lemma 5: Given a projection-measurable group $(G, \phi; \overline{S}, \pi; \mathcal{H})$, there exists a measurable set E having positive, finite measure, a non-zero vector ξ_o in $\mathcal{H}$, and a positive constant k with the following properties:

1) E is coverable by a finite number of right translates of each neighborhood of the identity,

2) For every measurable subset A of E

$$(\pi(A)\xi_o, \xi_o) \leq k\mu(A).$$

Proof: By Proposition 1 we can choose an F from the neighborhood base $\bar{N}$ such that $\mu(F)$ is positive and finite and F is coverable by a finite number of left translates of every member of $\bar{N}$. By Proposition 1 the fact that $\mu(F) > 0$ means that $\pi(F) \neq 0$, so that, $\pi(\bar{S})$ being countably decomposable, then on $\pi(F)\mathfrak{h}$, $\pi(F)\pi(\bar{S})(= \pi(F \cap \bar{S}))$ must be a countably decomposable Abelian ring of projections with identity. It follows by Lemma 2 that $\pi(F \cap \bar{S})$ possesses a separating vector in $\pi(F)\mathfrak{h}$, a vector we shall call ξ_o. Hence, the function n on $F \cap \bar{S}$ defined for every $E \in F \cap \bar{S}$ by the equation

$$n(E) = (\pi(E)\xi_o, \xi_o)$$

is a measure on $F \cap \bar{S}$, and n is equivalent to μ on $F \cap \bar{S}$.

The proof of our last assertion is a result of the following two remarks. For $\{E_i\}_{i=1}^{\infty}$, an arbitrary disjoint sequence of members of S, the definition of π requires that $\pi(\bigcup_{i=1}^{\infty} E_i)$ be the weak limit of $\left\{\sum_{i=1}^{n} \pi(E_i)\right\}_{n=1}^{\infty}$. Further, from the definition of n and the fact that ξ_o is a separating vector for $\pi(F \cap \bar{S})$ we have that $n(E) = 0$ if and only if $\pi(E) = 0$, while we know from the first conclusion of Proposition 1 that $\mu(E) = 0$ if and only if $\pi(E) = 0$.

Now by the Radon-Nikodym Theorem there exists a non-negative function f integrable on the measure space $(F, F \cap \bar{S}, \mu)$ such that

$$n(E) = \int_E f(x)\, d\mu(x)$$

for every $E \epsilon F \cap \bar{S}$.

Then, inasmuch as

$$n(F) = (\pi(F)\xi_o, \xi_o),$$

we know that

$$n(F - \bigcup_{n=1}^{\infty} \{x \epsilon F | f(x) \leq n\}) = 0.$$

It follows that for some sufficiently large n_o $E = \{x \epsilon F | f(x) \leq n_o\}$ has positive, finite μ-measure. So, we see that for every K in E S

$$\begin{aligned}(\pi(K)\xi_o, \xi_o) &= n(K) \\ &= \int_K f(x)\, d\mu(x) \\ &\leq \int_K n_o\, d\mu(x) \\ &= n_o \int_K \chi_K(x)\, d\mu(x) \\ &= n_o \mu(K).\end{aligned}$$

This is the second conclusion.

To obtain the first conclusion, let us consider E, F, and V, an arbitrary member of $\bar{N}$. By the construction of E and the choice of F

$$E \subset F$$

and

$$F \subset \bigcup_{i=1}^{n} b_i V.$$

Since by Proposition 1 F and V, as members of $\bar{N}$, must be symmetric, we have

$$E \subset \bigcup_{i=1}^{n} Va_i ,$$

which gives us the last conclusion of this lemma.

Lemma 6: Given a projection-measurable group $(G, \phi; \bar{S}, \pi; \mathcal{R})$, the product of any finite collection of measurable sets open in the topology $\bar{T}$

is itself measurable.

Proof: Let R and S both be measurable and open. Then, inasmuch as Proposition 1 shows that every member of $\bar{N}$ is symmetric and that for every x $\bar{N}x$ is a base of neighborhoods at x for the topology T, S^{-1} is open as well as measurable; thus,

$$RS = \{x \mid xS^{-1} \cap R = \phi\}$$
$$= \{x \mid \mu(xS^{-1} \cap R) > 0\} .$$

But the latter set is measurable by Corollary 1 to Proposition 2 of Section I; whence, RS is measurable.

Having proved the present lemma for the case of two open, measurable sets, we see that the method used is applicable to the case of n + 1 sets, n arbitrary, if the lemma is assumed in the case of n sets. But this means that we have found an induction proof of our lemma.

Lemma 7: Let $(G, \phi; \bar{S}, \pi; \mathcal{L})$ be a projection-measurable group. Then for any N in $\bar{N}$ and any $\epsilon > 0$ there is an M in $\bar{N}$ and a measurable set K, open in the topology $\bar{T}$, such that

$$K \subset KMM^{-1} \subset N$$

and

$$\mu(KMM^{-1} - K) < \epsilon\mu(KMM^{-1}).$$

Proof: Given N in $\bar{N}$, then, as asserted by Proposition 1, there is a W in $\bar{N}$ for which

$$WW^{-1} \subset N$$

and

$$W = W^{-1}.$$

As a result, we have that there is W in N such that

$$W^2 \subset N.$$

By induction we can construct the sequence $\{N_i\}_{i=1}^{\infty}$ of members of N such that for all i: 1) N_i belongs to $\bar{\bar{N}}_W$ (= $\{N \epsilon \bar{\bar{N}} | N \subset W\}$), 2)

$$N_{i+1}^3 \subset N_i .$$

The proof is based on the following assertions of Proposition 1: 1) for every N in N there is an M in $\bar{\bar{N}}$ such that $MM^{-1} \subset N$, 2) $N = N^{-1}$ for every N in $\bar{\bar{N}}$.

We can see that for all i

$$\lim_{j \to \infty} WN_j \subset WN_{i+1}$$

$$= (WN_{i+1})N_{i+1}^2$$

$$\subset WN_i .$$

And since for all i

$$WN_i \subset W^2 \subset N$$

and since Lemma 6 holds, the following sequence of inequalities results:

$$\ell = \lim_{j \to \infty} \mu(WN_j)$$

$$\leq \mu(WN_{i+1})$$

$$\leq \mu((WN_{i+1})N_{i+1}^2)$$

$$\leq \mu(WN_i).$$

Thus, there is an i_o for which

$$\ell \leq \mu(WN_{i_o})$$

$$\leq \mu(WN_{i_o})N_{i_o}^2)$$

$$< \ell + \epsilon \ell.$$

And so,

$$\mu\left[(WN_{i_o})N_{i_o}^2 - (WN_{i_o})\right] < \epsilon \mu(WN_{i_o})N_{i_o}^2)$$

Now, let

$$K = WN_{i_o}$$

and

$$M = N_{i_o} ,$$

and let us remember, on the one hand, that WN_{i_o} is open and measurable by Lemma 6 and, on the other hand, that $N_{i_o} = N_{i_o}^{-1}$. Then we see that we have found an open, measurable set K and a set M in $\overline{N}$ such that

$$K \subset KMM^{-1}$$

and

$$\mu(KMM^{-1} - K) < \epsilon\mu(KMM^{-1})$$

Since, further, Lemma 6 makes it clear that KMM^{-1} is measurable, and since

$$\begin{aligned} KMM^{-1} &= (WN_{i_o})N_{i_o}(N_{i_o})^{-1} \\ &= (WN_{i_o})N_{i_o}^{2} \\ &\subset WN_{(i_o - 1)} \\ &\subset WW \\ &\subset N, \end{aligned}$$

our proof is complete.

Definition 14: Let $\mathfrak{h}$ be a hilbert space, $(M, \overline{S}, m)$ be a measure space, and f a function on M to $\mathfrak{h}$ such that $(\eta, f(x))$ is integrable for every η in $\mathfrak{h}$. Suppose that $g(\eta) = \int_M (\eta, f(x))\, dm(x)$ is a continuous linear functional on $\mathfrak{h}$. This means (see p. 9 of (4)) that there is a unique ζ in $\mathfrak{h}$ such that

$$g(\eta) = (\eta, \zeta).$$

We may thus define the integral of f(x) over M ($\int_M f(x)\, dm(x)$) by the equation

$$\int_M f(x)\, dm(x) = \zeta.$$

By the very definition of $\int_M f(x)\, dm(x)$ we have the following formula holding for every η in $\mathcal{K}$:

$$(\eta, \int_M f(x)\, dm(x)) = \int_M (\eta, f(x))\, dm(x).$$

Lemma 8: The following are properties of the integral defined in Definition 14:

1) For every integrable f on M to

$$(\eta, \int_M f(x)\, dm(x)) = \int_M (\eta, f(x))\, dm(x),$$

2) The family of integrable functions I is a linear space over the field of scalars belonging to $\mathcal{K}$ and the integral is a linear operation on I,

3) If T is a bounded linear operator on $\mathcal{K}$ and f(x) is integrable then T(f(x)) is integrable and

$$\int_M T(f(x))\, dm(x) = T \int_M f(x)\, dm(x),$$

4) If

$$M = \bigcup_{i=1}^{\infty} A_i ,$$

where the A_i's are measurable and $i \neq j$ $A_i \cap A_j = \phi$, and if f is integrable on M and on all A_i, then we have for all η in $\mathcal{K}$

$$(\eta, \int_M f(x)\, dm(x)) = \sum_{i=1}^{\infty} (\eta, \int_{A_i} f(x)\, dm(x)).$$

Proof: The first property is immediate from our definition of integral (Definition 10), while property 4 is the result of combining the first property with the fact that the indefinite integral of a real-valued integrable function (integrable in the standard sense) is a countably additive set function.

We can now dispose of properties 2 and 3 by comparing our definition of integral with Pettis' definition for the same situation. Because of the standard representation for continuous linear functionals on a hilbert space (mentioned in Definition 10) property 1 of our integral implies that the two definitions coincide. But properties 2 and 3 are properties of the Pettis definition, and, hence, of our definition. (See (2). Here the underlying measure space is assumed to be totally σ-finite, but this does not affect the properties under consideration which only require those properties of the abstract Lebesgue integral that are true in general.)

Lemma 9: Let R be a set of finite measure contained in a neighborhood N of the identity which is coverable by a finite number of left-translates of any other neighborhood. Then there exists $\int_R (\phi(\rho)\xi)\, dm(\rho)$ for every ξ in $\mathfrak{K}$. If $R = \bigcup_{i=1}^{n} R_i$, where the R_i are measurable, then for all i $\int_{R_i} (\phi(\rho)\xi)\, dm(\rho)$ exists and

$$\int_R (\phi(\rho)\xi)\, dm(\rho) = \sum_{i=1}^{n} \int_{R_i} (\phi(\rho)\xi)\, dm(\rho) .$$

Proof: Since by definition of $\phi(\rho)$ $\phi(\rho)\xi$ is strongly measurable relative to π on R for every ξ in $\mathfrak{K}$, then by Lemma 4 $\phi(\rho)\xi$ is continuous. Also, in view of Proposition 1, an N of the sort hypothesized actually exists. Thus, $\phi(\rho)\xi$ is the uniform limit of countable-valued step-functions on R, and $(\eta, \phi(\rho)\xi)$ is measurable on R. Combining this with the fact that

$$|(\eta, \phi(\rho)\xi)| \leq \|\eta\| \cdot \|\phi(\rho)\xi\| = \|\eta\| \cdot \|\xi\|,$$

it follows that $\int_R (\eta, \phi(\rho)\xi)\, dm(\rho)$ exists for all η and ξ in $\mathcal{h}$ and defines for each fixed ξ a continuous linear functional on $\mathcal{h}$. Hence, the requirements of Definition 10 are satisfied so that $\int_R (\phi(\rho)\xi)\, dm(\rho)$ must exist for every ξ in $\mathcal{h}$.

If $R = \sum_{i=1}^{n} R_i$, where the R_i's are mutually disjoint, then precisely the argument given above to show the existence of $\int_R (\phi(\rho)\xi)\, dm(\rho)$ will suffice to show the existence of $\int_{R_i} (\phi(\rho)\xi)\, dm(\rho)$ for all i. Now we see from property 3 of the integral defined in Definition 10 that

$$\int_R (\phi(\rho)\xi)\, dm(\rho) = \sum_{i=1}^{n} \int_{R_i} (\phi(\rho)\xi)\, dm(\rho)$$

Proposition 2: In a projection-measurable group, $(G, \phi; \overline{S}, \pi; \mathcal{h})$, there exist R, D, and V, three measurable sets of positive, finite measure, with these properties:

1) $RD^{-1} \subset V$,

2) There is a vector ξ in $\mathcal{h}$ such that if

$$\eta = \frac{1}{\mu(V)} \int_V \phi(\rho)\pi(D)\xi \;\; d\mu(\rho)$$

then

$$\pi(R)\eta \neq 0.$$

Proof: We know by Lemma 5 that there exists a measurable set F of positive, finite measure, a positive constant k, and a non-zero vector ξ in $\pi(F)\mathcal{h}$ such that 1) F is coverable by a finite number of right translates of each neighborhood of the identity, 2)

$$(\pi(E)\xi, \xi) \leq k\mu(E)$$

for every measurable $E \subset F$. We can assume, in addition, that

$$\|\xi\|^2 = \mu(F).$$

And, since we know by Lemma 4 that $\phi(x)\xi$ is continuous, we can choose by Proposition 1 a neighborhood W of the identity such that $W = W^{-1}$, W has positive, finite measure, and for every x in W

$$\|\phi(x)\xi - \xi\| < \frac{1}{2}\|\xi\| .$$

Then by Lemma 7 we can find a measurable, open set K and a member M of $\bar{N}$, both having positive measure, such that

$$K \subset KMM^{-1}$$

$$\subset W,$$

KMM^{-1} is measurable, and

$$\mu(KMM^{-1} - K) < (\frac{1}{2k})\mu(KMM^{-1}).$$

Now suppose we define the vector δ by the equation

$$\delta = \frac{1}{\mu(KMM^{-1})}\int_{KMM^{-1}} \phi(x)\xi \, dm(x).$$

(The integral exists by Lemma 9, by the fact that $KMM^{-1} \subset W$, and by the inequality

$$\mu(KMM^{-1}) \leq \mu(W) < \infty.)$$

Then, observe that ξ is integrable over KMM^{-1} and remember that our integral is a linear operation; we have, if $\|\delta - \xi\|$ is positive,

$$\|\delta - \xi\|^2 = \left[\delta - \xi, \frac{1}{\mu(KMM^{-1})}\int_{KMM^{-1}} \phi(\rho)\xi \, d\mu(\rho) - \frac{1}{\mu(KMM^{-1})}\int_{KMM^{-1}} \xi \, d\mu(\rho)\right]$$

$$= \left[\delta - \xi, \frac{1}{\mu(KMM^{-1})}\int_{KMM^{-1}} (\phi(\rho)\xi - \xi) \, d\mu(\rho)\right]$$

$$= \frac{1}{\mu(KMM^{-1})}\int_{KMM^{-1}} (\delta - \xi, \phi(\rho)\xi - \xi) \, d\mu(\rho)$$

$$< \frac{1}{\mu(KMM^{-1})} \int_{KMM^{-1}} \|\delta - \xi\| \left(\frac{1}{2} \|\xi\|\right) d\mu(\rho)$$

$$= \|\delta - \xi\| \frac{\|\xi\|}{2},$$

so that in every case

$$\|\delta - \xi\| < \frac{1}{2} \|\xi\|.$$

Next, expanding $(\delta - \xi, \delta - \xi)$ shows that

$$\text{Re } (\delta,\xi) = \frac{1}{2} \left(\|\xi\|^2 + \|\delta\|^2 - \|\delta - \xi\|^2 \right).$$

Hence, combining this with the fact that

$$\|\delta - \xi\| < \frac{1}{2} \|\xi\|,$$

$$\begin{aligned} |(\delta, \xi)| &\geq \text{Re}[(\delta, \xi)] \\ &= \frac{1}{2} \left(\|\xi\|^2 + \|\delta\|^2 - \|\delta - \xi\|^2 \right) \\ &> \frac{1}{2} \left(\|\xi\|^2 + \|\delta\|^2 - \frac{1}{4}\|\xi\|^2 \right) \\ &> \frac{1}{2} \left(\|\xi\|^2 + \frac{1}{4}\|\xi\|^2 - \frac{1}{4}\|\xi\|^2 \right) \\ &= \frac{1}{2} \|\xi\|^2 = \frac{1}{2} \mu(F). \end{aligned}$$

Now since F was chosen so as to be coverable by the union of a finite number of right translates of every neighborhood of the identity,

$$F \subset \bigcup_{i=1}^{n} Wa_i .$$

As a result,

$$F = \bigcup_{i=1}^{n} C_i,$$

where the C_i's are measurable, mutually disjoint, and for all i

$$C_i \subset Wa_i.$$

Further, by Lemma 9, $\zeta_i = \frac{1}{\mu(KMM^{-1})} \int_{KMM^{-1}} \phi(\rho)\pi(C_i)\xi \, d\mu(\rho)$ exists for all i, so that by the linearity property of the integral of Definition 14 and the fact that $\pi(F)\xi = \xi$

$$\delta = \sum_{i=1}^{n} \zeta_i.$$

Thus, as we know

$$|(\delta,\xi)| > \frac{1}{2}\mu(F),$$

it must be that for some i_o

$$|(\zeta_{i_o},\xi)| > \frac{1}{2}\mu(C_{i_o}).$$

So, setting $\zeta = \zeta_{i_o}$, $C = C_{i_o}$, $a_{i_o} = a$, we can say: 1)

$$|(\zeta,\xi)| > \frac{1}{2}\mu(C),$$

where

$$\zeta = \frac{1}{\mu(KMM^{-1})}\int_{KMM^{-1}} \phi(\rho)\pi(C)\xi \, d\mu(\rho)$$

and

$$C \subset F \cap Ma;$$

2)

$$L = KMa$$

is a well-defined measurable set, where the measurability of L comes from Lemma 6. But this last assertion proves the present proposition, if, also, $\pi(L)\zeta \neq 0$ and LC^{-1} is a subset of KMM^{-1}.

We can show from the preceding that $\pi(L)\zeta \neq 0$. Suppose $\pi(L)\zeta = 0$. Then, using properties 1) and 2) of the integral of Definition 14, the commutation relative between $\phi(\rho)$ and $\pi(E)$, and the identity $\pi(F)\xi = \xi$,

$$(\xi, \zeta) = (\xi, \zeta - \pi(L)\zeta)$$

$$= (\xi, \frac{1}{\mu(KMM^{-1})}\int_{KMM^{-1}} \phi(\rho)\pi(C)\xi \, d\mu(\rho) - \frac{1}{\mu(KMM^{-1})}\pi(L)\int_{KMM^{-1}} \phi(\rho)\pi(C)\xi \, d\mu(\rho))$$

$$= (\xi, \frac{1}{\mu(KMM^{-1})}\int_{KMM^{-1}} [\phi(\rho)\pi(C)\xi - \pi(L)\phi(\rho)\pi(C)\xi] \, d\mu(\rho))$$

$$= \frac{1}{\mu(KMM^{-1})}\int_{KMM^{-1}} (\xi, [\phi(\rho)\pi(C) - \pi(L)\phi(\rho)\pi(C)]\, \xi) \, d\mu(\rho)$$

$$= \frac{1}{\mu(KMM^{-1})}\int_{KMM^{-1}} (\pi(F)\xi, \pi(\rho C \cap L^c)\phi(\rho)\pi(C)\xi) \, d\mu(\rho)$$

$$= \frac{1}{\mu(KMM^{-1})}\int_{KMM^{-1}} (\pi(\rho C \cap F)\xi, \pi(\rho C \cap L^c)\phi(\rho)\pi(C)\xi) \, d\mu(\rho)$$

(Here L^c is the complement of L.) But $L^c \cap \rho C$ is void unless

$$\begin{aligned} \rho \in L^c C^{-1} &= (KMa)^c C^{-1} \\ &\subset (KMa)^c a^{-1} M^{-1} \\ &= (KM)^c M^{-1} \\ &\subset K^c \;, \end{aligned}$$

so that the last integral need only to be taken over $KMM^{-1} - K$. Further we have

$$\|\pi(F \cap \rho C)\xi\| \leq \sqrt{k\mu(F \cap \rho C)} \leq \sqrt{k\mu(\rho C)} = \sqrt{k\mu(C)}$$

and

$$\|\pi(L^c \cap \rho C)\,\phi(\rho)\pi(C)\xi\| \leq \|\pi(C)\xi\| \leq \sqrt{k\mu(C)}\,,$$

where we use the special properties of F and ξ as well as the left invariance of μ; this shows that

$$|(\pi(F \cap \rho C)\xi, \pi(L^c \cap \rho C)\phi(\rho)\pi(C)\xi)| \leq k\mu(C)\;.$$

Hence, we see that

$$|(\zeta, \xi)| = |(\xi, \zeta)| = \frac{1}{\mu(KMM^{-1})} \int_{KMM^{-1}-K} \left[\pi(F \cap \rho C)\xi, \pi(L^c \cap \rho C)\phi(\rho)\pi(C)\xi\right] d\mu(\rho)$$

$$\frac{1}{\mu(KMM^{-1})} \int_{KMM^{-1}-K} (k\mu(C))\, d\mu(\rho)$$

$$= \left(\frac{\mu(KMM^{-1}-K)}{\mu(KMM^{-1})}\right) (k\mu(C)),$$

and, remembering the choice of K and M,

$$\left(\frac{\mu(KMM^{-1}-K)}{\mu(KMM^{-1})}\right) k\mu(C) < \left(\frac{1}{2k}\right) (k\mu(C))$$

$$= \frac{1}{2}\mu(C).$$

But since we have shown already that

$$|(\zeta, \xi)| > \frac{1}{2}\mu(C),$$

we see that assuming $\pi(L)\zeta = 0$ leads to a contradiction.

If we combine this result with the observation that

$$LC^{-1} = (KMa)C^{-1}$$

$$\subset (KMa)(Ma)^{-1}$$

$$= KMM^{-1},$$

we see that L, C, and KMM^{-1} are sets of the sort proved to exist by this proposition.

Proposition 3: Let R, D, and V be the three sets asserted to exist in the preceding proposition. Then if E is a measurable set such that

$$E \subset (aR) \cap (bR)$$

for some pair of elements, a and b, of G, and if

$$\eta = \frac{1}{\mu(V)} \int_V \phi(\rho)\pi(D)\xi \, d\mu(\rho),$$

then

$$\pi(E)\phi(a)\eta = \pi(E)\phi(b)\eta.$$

Proof: Since by our hypotheses that

$$E \subset aR \cap bR$$

and

$$RD^{-1} \subset V$$

we have

$$ED^{-1} \subset (aV) \cap (bV),$$

it results that

$$\int_{aV} \pi(E \cap \rho D)\phi(\rho)\xi \, d\mu(\rho) = \int_{bV} \pi(E \cap \rho D)\phi(\rho)\xi \, d\mu(\rho),$$

$\pi(E \cap \rho D)$ being 0 except when ρ belongs to ED^{-1}. Thus for all η in $\mathcal{h}$,

$$(\eta, \int_{aV} \pi(E \cap \rho D)\phi(\rho)\xi \, d\mu(\rho)) = (\eta, \int_{bV} \pi(E \cap \rho D)\phi(\rho)\xi \, d\mu(\rho)),$$

or

$$\int_{aV} (\eta, \pi(E \cap \rho D)\phi(\rho)\xi) \, d\mu(\rho) = \int_{bV} (\eta, \pi(E \cap \rho D)\phi(\rho)\xi) \, d\mu(\rho).$$

Then, by the left-invariance of μ,

$$\int_V (\eta, \pi(E \cap a\rho D)\phi(a\rho)\xi) \, d\mu(\rho) = \int_V (\eta, \pi(E \cap b\rho D)\phi(b\rho)\xi) \, d\mu(\rho),$$

so that

$$\int_V \pi(E \cap a\rho D)\phi(a\rho)\xi \, d\mu(\rho) = \int_V (E \cap b\rho D)\phi(b\rho)\xi \, d\mu(\rho).$$

Next, since the definition of a projection-measurable group requires the identity

$$\phi(x)\pi(S) = \pi(xS)\phi(x),$$

$$\int_V \pi(E)\phi(a\rho)\pi(D)\xi \, d\mu(\rho) = \int_V \pi(E)\phi(b\rho)\pi(D)\xi \, d\mu(\rho),$$

Finally, we know that ϕ is a representation of G, and we know, by conclusion 2 of Lemma 8 that bounded linear operators commute with the operation of integration; hence,

$$\pi(E)\phi(a)\int_V \phi(\rho)\pi(D)\xi \, d\mu(\rho) = \pi(E)\phi(b)\int_V \phi(\rho)\pi(D)\xi \, d\mu(\rho),$$

or, by the definition of η,

$$\pi(E)\phi(a)\eta = \pi(E)\phi(b)\eta.$$

Lemma 10: Let E and F be measurable subsets of G, both of positive measure. Then there is an at most countable disjoint collection of mutually disjoint, measurable subsets of F, $\{A_\alpha\}_{\alpha\in I}$, such that

$$A_\alpha \subset a_\alpha E$$

for every α in I and

$$\mu(F - \bigcup_{\alpha\in I} A_\alpha) = 0.$$

Proof: Since μ is a σ-finite measure, E and F are each the union of an at most countable disjoint collection of sets of positive, finite measure. Thus we need only prove the lemma in the case where F is of finite measure.

Consider the family Φ of all collections of disjoint measurable subsets of F such that each subset is contained in a left-translate of E. This family is non-empty since we know by Corollary 2 to Proposition 2 of Section 1 that

$$\mu(xE \cap F) > 0$$

for some x in G.

Let $\{F_\gamma\}_{\gamma\in J}$ be a subfamily of the above totally ordered with respect to the ordering relation of inclusion. If we define $\overline{F}$ to be the union of all sets occurring in at least one of the $\overline{F}_\gamma$'s we see that $\overline{F}$ is an upper bound for $\{F_\gamma\}_{\gamma \in J}$. Thus, we may apply the Maximal Principle to conclude that Φ contains an element $\overline{H}$ which is maximal with respect to the inclusion ordering.

We shall conclude our proof by showing that $\overline{H}$ is at most countable and that, if

$$\overline{H} = \{H_\alpha\}_{\alpha \in I},$$
$$\mu(F - \bigcup_{\alpha \in I} H_\alpha) = 0.$$

The fact that $\overline{H}$ is at most countable comes from the fact that it is a collection of mutually disjoint sets of positive measure all contained in a set of finite measure, for the latter type of collection is known to be at most countable. And, since we can now say that $F - \bigcup_{\alpha \in I} H_\alpha$ is measurable, the second property of $\overline{H}$ follows: if

$$\mu(F - \bigcup_{\alpha \in I} H_\alpha) > 0$$

then for some x

$$\mu\left[xE \cap (F - \bigcup_{\alpha \in I} H_\alpha)\right] > 0$$

and

$$\{xE \cap (F - \bigcup_{\alpha \in I} H_\alpha)\} \cup \{H_\alpha\}_{\alpha \in I}$$

is a member of Φ properly containing $\{H_\alpha\}_{\alpha \in I} \equiv \overline{H}$, contradicting the maximality of H.

Definition 13: If $(G, \phi; \pi, \overline{S}; \hbar)$ is a projection-measurable group, let real (complex $L^2(G, \overline{S}, \mu)$ be the real (complex hilbert space corresponding to the family of all real (complex)-valued functions square-integrable on $(G, \overline{S}, \mu)$. In the following, the symbol "$L^2(G, \overline{S}, \mu)$" shall be prefixed by the adjective "real" or the adjective "complex" only if the context does not make it clear which is appropriate.

Definition 14: Let $(G, \overline{S}, \mu)$ be a left-invariant GMG. If f is the element of $L^2(G, \overline{S}, \mu)$ corresponding to a square-integrable f, then for every a in G and every E in $\overline{S}$ we can define the operators T_a and P_E by the relations:

$$1) \quad T_a(f) = f_a,$$

where

$$f_a(x) = f(a^{-1}x) \text{ n.e.},$$

$$2) \quad P_E(f) = f_E,$$

where

$$f_E(x) = \chi_E(x)f(x). \text{ n.e.}$$

Since μ is a left-invariant measure, T_a and P_E are both bounded operators on $L^2(G, \overline{S}, \mu)$.

Lemma 11: The family of all real (complex) linear combinations of characteristic functions of sets of positive, finite measure is dense in $L^2(G, \overline{S}, \mu)$.

Proof: (See Exercise 1 of section 42 of (1).)

Proposition 4: Given a projection-measurable group $(G, \phi; \overline{S}, \pi; \mathcal{L})$, there exists $\overline{\overline{R}}$, a subfamily of $\overline{S}$, with the following properties:

1) for all A in $\overline{S}$

$$A \cap \overline{\overline{R}} \subset \overline{\overline{R}};$$

2) for all a in G

$$a\overline{\overline{R}} \subset \overline{\overline{R}}$$

3) the collection of all real (complex) linear combinations of characteristic functions of members of $\overline{\overline{R}}$ is a dense subset of the real (complex) $L^2(G, \overline{S}, \mu)$;

4) $\overline{R}$ can be mapped into $\mathcal{H}$ by the function λ in such a manner that for all E in $\overline{R}$

a) If ρ is an arbitrary member of G

$$\lambda(\rho E) = \phi(\rho)[\lambda(E)],$$

b) If A is an arbitrary member of $\overline{S}$

$$\lambda(A \cap E) = \pi(A)[\lambda(E)],$$

c) $$\|\lambda(E)\|^2 = \mu(E).$$

Proof: Let $\overline{R}$ the family of all measurable subsets of left translates of the measurable set R of Proposition 2 of this section. We proceed to verify that it has the four asserted properties.

Property 1) of $\overline{R}$ is a consequence of the fact that $\overline{R}$ is a subfamily of $\overline{S}$ and the fact that $\overline{S}$ is closed under the operation of intersection.

Property 2) follows from observing that if

$$E \subset bR$$

then

$$aE \subset (ab)R$$

and that translation is a measurability-preserving transformation on a generalized measurable group.

Property 3) results from combining the preceding two lemmas. For Lemma 11 shows linear combinations of characteristic functions of sets of positive, finite measure to form a dense subset of $L^2(G, \overline{S}, \mu)$.

And we observe that Lemma 10 shows any measurable set in G to be, up to a set of measure zero, a disjoint union of sets in R. Hence, combining the preceding remarks, we see that any f in $L^2(G, \overline{S}, \mu)$ can be approximated arbitrarily well in the L^2-norm by a linear combinations of characteristic functions of subsets of $\bar{R}$.

Proof that $\bar{R}$ possesses the fourth property asserted will be obtained by first defining a function λ on $\bar{R}$ to $\mathcal{K}$ and then showing it to have the desired characteristics.

We define λ for an arbitrary member E of $\bar{R}$ by the equation,

$$\lambda(E) = \pi(E)\phi(a)\eta,$$

where a is any element of G such that

$$E \subset aR$$

and η is the element of $\mathcal{K}$ defined and discussed in Propositions 2 and 3 above. We observe first that by Proposition 3 λ is uniquely defined for every E in $\bar{R}$. Further, since by Proposition 2 $\pi(R)\eta \neq 0$ and $\mu(R) > 0$, η can be assumed to be such that

$$\|\pi(R)\eta\|^2 = \mu(R);$$

thus, by definition of λ,

$$\|\lambda(R)\|^2 = \mu(R).$$

We see that λ has the first two desired properties by some computations. For, in view of the definition of λ and the commutation relation

$$\phi(\rho)\pi(E) = \pi(\rho E)\phi(\rho),$$

we have

$$\begin{aligned}\phi(\rho)[\lambda(E)] &= \phi(\rho)[\pi(E)\phi(a)\eta] \\ &= \pi(\rho E)\phi(\rho)\phi(a)\eta \\ &= \pi(\rho E)\phi(\rho a)\eta \\ &= \lambda(\rho E),\end{aligned}$$

and

$$\pi(A)[\lambda(E)] = \pi(A)[\pi(E)\phi(a)\eta]$$

$$= \pi(A \cap E)\phi(a)\eta$$
$$= \lambda(A \cap E).$$

That

$$\|\lambda(E)\|^2 = \mu(E)$$

for every E in $\bar{R}$ will be shown as follows. We shall construct a left-invariant measure ν on $\bar{S}$ such that

$$\nu(E) = \|\lambda(E)\|^2$$

for every E in $\bar{R}$. Since an invariant measure on $\bar{S}$ is unique up to a multiplicative constant by Theorem 1 of Section 1, since we know that for the set of positive measure R

$$\mu(R) = \|\lambda(R)\|^2$$
$$= \nu(R),$$

then μ and ν are the same measure. Hence, for every E in $\bar{R}$

$$\|\lambda(E)\|^2 = \nu(E)$$
$$= \mu(E).$$

We construct the measure ν using Lemma 10. By Lemma 10, any measurable set E has the form

$$E = P \cup \left(\bigcup_{i=1}^{\infty} E_i \right)$$

where $\mu(P) = 0$ (so that by Proposition 2 $\pi(P) = 0$) and $\{E_i\}_{i=1}^{\infty}$ is a mutually disjoint collection of subsets of translates of R. Then, by definition of $\bar{R}$, $\lambda(E_i)$ exists for all E_i, and for an appropriate sequence of elements of G, $\{a_i\}_{i=1}^{\infty}$,

$$\lambda(E_i) = \pi(E_i)\phi(a_i)\eta.$$

We define ν for E by setting

$$\nu(E) = \sum_{i=1}^{\infty} \|\lambda(E_i)\|^2$$

First of all, ν is uniquely defined. Let the arbitrary measurable E of the preceding paragraph have a second decomposition:

$$E = Q \cup \left(\bigcup_{i=1}^{\infty} F_i \right),$$

Here, of course, Q and $\{F_i\}_{i=1}^{\infty}$ have the same properties as, respectively, P and $\{E_i\}_{i=1}^{\infty}$, and $\lambda(F_i) = \pi(F_i)\phi(b_i)\eta$. So for all i

$$\begin{aligned} \|\lambda(E_i)\|^2 &= \|\pi(E_i)\phi(a_i)\eta\|^2 \\ &= \left\|\pi\left[\bigcup_{j=1}^{\infty} (E_i \quad F_j)\right]\phi(a_i)\eta\right\|^2 \\ &= \left(\pi\left[\bigcup_{j=1}^{\infty} (E_i \cap F_j)\right]\phi(a_i)\eta,\ \phi(a_i)\eta\right) \\ &= \sum_{j=1}^{\infty} (\pi(E_i \cap F_j)\phi(a_i)\eta, \phi(a_i)\eta) \\ &= \sum_{j=1}^{\infty} \|\lambda(E_i \cap F_j)\|^2. \end{aligned}$$

Hence,
$$\begin{aligned} \sum_{j=1}^{\infty} \|\lambda(F_j)\|^2 &= \sum_{j=1}^{\infty} (\pi(F_j)\phi(b_j)\eta,\ \phi(b_j)\eta) \\ &= \sum_{j=1}^{\infty} \left(\pi\left[F_j \cap \bigcup_{i=1}^{\infty} E_i\right]\phi(b_j)\eta,\ \phi(b_j)\eta\right) \\ &= \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} (\pi(E_i \cap F_j)\phi(b_j)\eta,\ \phi(b_j)\eta) \\ &= \sum_{j=1}^{\infty}\ \sum_{i=1}^{\infty} \|\pi(E_i)\lambda(F_j)\|^2 \\ &= \sum_{j=1}^{\infty}\ \sum_{i=1}^{\infty} \|\lambda(E_i \cap F_j)\|^2 \\ &= \sum_{j=1}^{\infty} \|\lambda(E_i)\|^2 . \end{aligned}$$

Thus one could have defined $\nu(E)$ to be $\sum_{i=1}^{\infty} \|\lambda(F_i)\|^2$ without changing the value of the former, which shows ν to be a uniquely defined function on $\bar{S}$.

Now, to prove that ν is invariant under left-translation, we must observe first that if

$$E = P \cup \left(\bigcup_{i=1}^{\infty} E_i \right),$$

where P has measure 0 and the E_i's are mutually disjoint subsets of R, then

$$aE = (aP) \cup \left(\bigcup_{i=1}^{\infty} (aE_i) \right),$$

where the left invariance of μ gives aP measure zero and the second property of $\bar{R}$ makes the aE_i mutually disjoint members of $\bar{R}$. As a result,

$$\begin{aligned}
\nu(aE) &= \sum_{i=1}^{\infty} \|\lambda(aE_i)\|^2 \\
&= \sum_{i=1}^{\infty} \|U_a \lambda(E_i)\|^2 \\
&= \sum_{i=1}^{\infty} \|\lambda(E_i)\|^2 \\
&= \nu(E).
\end{aligned}$$

Now only the countable additivity of ν remains to be proved. Let $\{F_k\}_{k=1}^{\infty}$ be a collection of mutually disjoint sets in $\bar{S}$, and suppose that $\{K_i\}_{i=1}^{\infty}$ is a collection of mutually disjoint subsets of $\bigcup_{k=1}^{\infty} F_k$, all belonging to $\bar{R}$, such that

$$\mu\left(\bigcup_{k=1}^{\infty} F_k - \bigcup_{i=1}^{\infty} K_i \right) = 0.$$

Then for every k, $\{F_k \cap K_i\}_{i=1}^{\infty}$ has the same properties relative to F_k as has $\{K_i\}_{i=1}^{\infty}$ relative to $\bigcup_{k=1}^{\infty} F_k$. But this enables us to say, remembering the definition of ν and assuming $\lambda(K_i) = \pi(K_i)\phi(a_i)\eta$, that

$$\begin{aligned}\sum_{k=1}^{\infty} \nu(F_k) &= \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \|\lambda(F_k \cap K_i)\|^2 \\ &= \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \|\pi(F_k)\lambda(K_i)\|^2 \\ &= \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \|\pi(F_k)\pi(K_i)\phi(a_i)\eta\|^2 \\ &= \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} (\pi(F_k \cap K_i)\phi(a_i)\eta,\ \phi(a_i)\eta) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} (\pi(F_k \cap K_i)\phi(a_i)\eta,\ \phi(a_i)\eta) \\ &= \sum_{i=1}^{\infty} (\pi(\bigcup_{k=1}^{\infty} F_k \cap K_i)\phi(a_i)\eta,\ \phi(a_i)\eta) \\ &= \sum_{i=1}^{\infty} (\pi(K_i)\phi(a_i)\eta,\ \phi(a_i)\eta) \\ &= \sum_{i=1}^{\infty} \|\lambda(K_i)\|^2 \\ &= \nu(\bigcup_{k=1}^{\infty} F_k).\end{aligned}$$

Thus ν must be countably additive.

Having proved that ν is a left translation invariant measure on S, it follows in the manner already described above that

$$\|\lambda(E)\|^2 = \mu(E)$$

for every E in M.

This concludes the proof of the properties asserted for λ and, hence, of those asserted for $\overline{M}$, ending the proof of this proposition.

<u>Definition 15</u>: By a <u>unitary mapping</u> of a hilbert space $\mathfrak{h}_1$, onto (into) a hilbert space $\mathfrak{h}_2$ we shall mean a linear, homogeneous mapping T on $\mathfrak{h}_1$, onto (into) $\mathfrak{h}_2$ such that if ξ and η are any two vectors of $\mathfrak{h}_1$, then

$$(\xi, \eta) = (T(\xi), T(\eta)).$$

It is clear that $\mathfrak{h}_1$ and $T(\mathfrak{h}_1)$ are either both real or both complex.

A familiar mapping provides a simple example of the notion just defined. This is the well-known mapping of the complex $L^2([-\pi, \pi])$ onto the following complex hilbert sequence space,

$$\left\{\{s_n\}_{n=1}^{\infty} \,\middle|\, s_n \text{ complex}, \sum_{n=1}^{\infty} |s_n|^2 < \infty\right\}.$$

Proposition 5: Let $(G, \phi; \bar{S}, \pi; \mathfrak{h})$ be a projection-measurable group. Let V be a unitary mapping of $L^2(G, \bar{S}, \mu)$ onto $\mathfrak{k}$, a closed hilbert subspace of $\mathfrak{h}$, and let V have following properties:

1) for every ρ in G and f in $L^2(G, \bar{S}, \mu)$

$$(V^{-1}\phi(\rho)V)f = T_\rho f$$

2) for every A in $\bar{S}$ and f in $L^2(G, \bar{S}, \mu)$

$$(V^{-1}\pi(A)V)f = P_A f.$$

Then the set of all unitary mappings of $L^2(G, \bar{S}, \mu)$ which send it onto $\mathfrak{k}$ and have properties 1) and 2) has the form

$$\{\alpha V \mid \alpha \text{ a scalar}, \ |\alpha| = 1\}.$$

Proof: (In the following we shall use the symbol for the characteristic function of a set of finite measure as the symbol for the corresponding element of $L^2(G, \bar{S}, \mu)$.)

We can verify directly that if α is a scalar of absolute value 1, then the mapping αV is unitary and has properties 1) and 2) of V. Thus, we can restrict ourselves to showing that for any unitary mapping W of $L^2(G, \bar{S}, \mu)$ onto $\mathfrak{k}$ having properties 1) and 2) the unitary operator $Y(=W^{-1}V)$ on $L^2(G, \bar{S}, \mu)$ has the form:

$$Y = \alpha I,$$

α a scalar. $|\alpha|$ must certainly be 1, W^{-1} being unitary.

First, it is useful to observe two relations satisfied by Y. For all a in G and all E in $\widetilde{S}$ we see from properties 1) and 2) satisfied by V and W that

$$T_a = V^{-1}U_aV = W^{-1}U_aW,$$

whence

$$VT_aV^{-1} = WT_aW^{-1},$$

or

$$T_aY = YT_a.$$

Similarly,

$$P_EY = YP_E .$$

Using these relations, we can obtain our result once we know that for every E of positive, finite measure

$$Y(\chi_E) = \beta(\chi_E),$$

β being a scalar possibly depending on E. For it follows first the β is independent of E, since the contrary would contradict the fact that for any two sets of positive, finite measure, A and B,

$$\begin{aligned} Y(\chi_A) &= Y\,P_A(\chi_{A\cup B}) \\ &= P_A(Y(\chi_{A\cup B})) \\ &= P_A(\gamma\chi_{A\cup B}) \\ &= \chi_A(\gamma\chi_{A\cup B}) \\ &= \gamma\chi_A \end{aligned}$$

and, by similar reasoning,

$$Y(\chi_B) = \gamma\chi_B.$$

Then, inasmuch as

$$Y = \alpha I$$

on all characteristic functions of sets of positive finite measure, we see that the set on which this equality holds can be extended to the set all linear combinations of such functions. But by Lemma 11 the latter set is dense in $L^2(G, \overline{S}, \mu)$, which, combined with the continuity of Y and I, means that

$$Y = \alpha I$$

on all of $L^2(G, \overline{S}, \mu)$. Thus we have proved our result knowing that corresponding to every E of positive, finite measure there is a scalar β such that

$$Y(\chi_E) = \beta \chi_E.$$

Proving the above-described preliminary result is as follows. Consider any set B of positive, finite measure. If $Y(\chi_B)$ is not a constant multiple of χ_B, it must be that either $\mathrm{Re}\,[Y(\chi_B)]$ or $\mathrm{Im}\,[Y(\chi_B)]$ has the property that its infimum on one subset of positive measure C (contained in B) is greater than its supremum on another subset of positive measure D. This means, in view of the fact that some translate of C intersects D in a set of positive measure, that there exist in B subsets of positive measure, A and aA, for which $T_a(P_A Y \chi_B) \neq P_{aA} Y \chi_B$. But this yields a contradiction, since the following computation shows that $T_a(P_A Y \chi_B) = P_{aA} Y \chi_B$:

$$\begin{aligned} T_a(P_A Y \chi_B) &= T_a Y P_A \chi_B \\ &= T_a Y(\chi_A \chi_B) \\ &= T_a Y \chi_A \\ &= Y T_a \chi_A \\ &= Y \chi_{aA} \end{aligned}$$

$$= Y(\chi_{aA}\chi_B)$$

$$= Y(P_{aA}\chi_B)$$

$$= P_{aA}Y\chi_B.$$

This proves that for every set E of positive, finite measure there is a scalar β such that

$$Y\chi_E = \beta\chi_E,$$

which, as already shown, proves the present proposition.

Definition 16: Given a collection $\{h_\alpha\}_{\alpha\in I}$ of real (or complex) hilbert spaces we define the direct sum ($\bigoplus_{\alpha\in I} h_\alpha$) of $\{h_\alpha\}_{\alpha\in I}$ to be the family of all functions f on the index set I such that

1) $f(\alpha)$ is a member of h_α for every α in I,

2) $f(\alpha)$ is the zero vector of h_α for all but an at most countable collection of α's $\{\alpha_i\}_{i=1}^{\infty}$, and $\sum_{i=1}^{\infty} |f(\alpha_i)|^2$ converges.

$\bigoplus_{\alpha\in I} h_\alpha$ is a hilbert space if we define the inner product of any two members, f and g, by

$$(f, g) = \sum_{\alpha\in I}(f(\alpha), g(\alpha))$$

This provides a frequently useful method of expressing a given hilbert space in terms of a sufficiently exhaustive collection of closed, mutually orthogonal subspaces, if no vector of the space is orthogonal to every member of the collection. For instance, the complex $L^2([0,1])$ can be mapped unitarily onto the direct sum of a countable collection of mutually orthogonal one-dimensional subspaces corresponding to the family of functions

$$\{e^{inx}\}_{n=-\infty}^{n=+\infty}.$$

Also, hilbert spaces with various unusual properties are constructible by forming direct sums of suitably chosen collections of familiar hilbert spaces.

Lemma 12: $L^2(G, \bar{S}, \mu)$ is irreducible under the combined family of operators $\{T_a\}_{a\in G} \cup \{P_E\}_{E\in\bar{S}}$.

Proof: This result is known, but the proof is included for completeness.

It will suffice to prove the lemma for the case where $L^2(G, \bar{S}, \mu)$ is a real hilbert space. For in the case where $L^2(G, \bar{S}, \mu)$ is complex we can verify the lemma from its validity for L_R^2, the real hilbert space made up of those elements of $L^2(G, \bar{S}, \mu)$ which correspond to real-valued square-integrable functions. If we should have

$$L^2(G, \bar{S}, \mu) = h_1 \oplus h_2,$$

where each of the two spaces h_1 and h_2 is carried into itself by the family in question, L_R^2 would be contained in one of these two subspaces, say h_1. But, h_1 being a complex hilbert space,

$$h_1 \supset L_R^2 + iL_R^2 = L^2(G, \bar{S}, \mu),$$

showing the irreducibility of $L^2(G, \bar{S}, \mu)$.

So let us suppose that $L^2(G, \bar{S}, \mu)$ is real and that $L^2(G, \bar{S}, \mu) = h_1 \oplus h_2$, where neither h_1 nor h_2 is a trivial space and each is carried into itself by $\{T_a\}_{a\in G} \cup \{P_E\}_{E\in S}$. Thus we may find f in h_1 and g in h_2 such that both $\{x|f(x)>0\}$ (= E) and $\{x|g(x)>0\}$ are sets of positive measure. Further, by Corollary 2 to Proposition 2 of Section 1, there is an s in G for which $\mu(sE \cap F)$ is positive. Thus

$$\begin{aligned}(T_sP_Ef, P_Fg) &= \int \chi_E(s^{-1}x)f(s^{-1}x)\chi_F(x)g(x)\, d\mu(x) \\ &= \int_{sE\cap F} f(s^{-1}x)g(x)\, d\mu(x) \\ &> 0.\end{aligned}$$

But, since f is in $\mathfrak{h}_1$ and g is in $\mathfrak{h}_2$, we have by definition of $\mathfrak{h}_1$ and $\mathfrak{h}_2$ that $T_s P_E f \in \mathfrak{h}_1$ and $P_F g \in \mathfrak{h}_2$, so that from the assumption that $\mathfrak{h}_1$ is orthogonal to $\mathfrak{h}_2$

$$(T_s P_E f, P_F g) = 0,$$

a contradiction. This concludes the proof.

We are now in a position to prove our generalization of that theorem of Loomis which we stated at the beginning of this section.

Theorem 1: Let $(G, \phi; \bar{S}, \pi; \mathfrak{h})$ be a projection-measurable group. Then $\bar{S}$ supports a left-translation invariant measure μ, unique up to a scalar factor, such that for all E in $\bar{S}$ $\mu(E) = 0$ if and only if $\pi(E) = 0$. Further, $\mathfrak{h}$ is expressible as the direct sum of a family of mutually orthogonal, closed subspaces $\{\mathfrak{h}_\alpha\}_{\alpha \in I}$ with the following property: for every α there exists a unitary mapping V_α (unique up to a unimodular constant factor) of $L^2(G, \bar{S}, \mu)$ onto $\mathfrak{h}_\alpha$ for which we have

$$V_\alpha^{-1} \phi(a) V_\alpha = T_a$$

and

$$V_\alpha^{-1} \pi(E) V_\alpha = P_E$$

for every a in G and E in $\bar{S}$. Assuming all of the other defining properties of $(G, \phi; \bar{S}, \pi; \mathfrak{h})$, the following three properties form necessary and sufficient conditions for the truth of the present assertion:

1) $\pi(\bar{S})$ is full,

2) $\pi(\bar{S})$ is countably decomposable,

3) $\phi(x)\xi$ is strongly measurable relative to π for every ξ in $\mathfrak{h}$.

Proof: We give the sufficiency proof first. We shall devote ourselves completely to the proof of the assertions about the structure of ϕ, π, and h, since our assertion about measure is just a restatement of part of Proposition 1.

Using the terminology of Proposition 4 let us define the following linear mapping T on a linear subspace of $L^2(G, \overline{S}, \mu)$ to a linear subspace of h. For every finite sequence of scalars $\{\alpha_i\}_{i=1}^n$ and every finite collection of mutually disjoint sets of positive measure $\{E_i\}_{i=1}^n$ belonging to $\overline{\overline{R}}$ let

$$T\left(\sum_{i=1}^{n} \alpha_i \chi_{E_i}\right) = \sum_{i=1}^{n} \alpha_i \lambda(E_i).$$

First, we shall verify that T is uniquely defined by showing that if E, F, and $E \cup F$ are in $\overline{\overline{R}}$ and

$$E \cap F = \phi$$

then

$$\lambda(E \cup F) = \lambda(E) + \lambda(F) .$$

Remembering, from Proposition 4 that for A in $\overline{S}$ and B in $\overline{\overline{M}}$

$$\lambda(A \cap B) = \pi(A)\lambda(B),$$

we have

$$\begin{aligned} \lambda(E \cup F) &= \pi(E \cup F)\lambda(E \cup F) \\ &= (\pi(E) + \pi(F))\lambda(E \cup F) \\ &= \pi(E)\lambda(E \cup F) + \pi(F)\lambda(E \cup F) \\ &= \lambda(E) + \lambda(F). \end{aligned}$$

Next T is unitary on its domain of definition. Note that any particular pair of members of the domain of T may be regarded as two linear combinations of the same finite collection of mutually disjoint elements of $\overline{\overline{R}}$; hence, if

$$\|\lambda(E)\|^2 = \mu(E)$$

for all E in $\bar{R}$ and if, whenever

$$E \cap F = \emptyset$$

$$(\lambda(E), \lambda(F)) = 0,$$

it must be that

$$(Tf, Tg) = (f, g)$$

for any f and g in the domain of definition of T. But we know by Proposition 4 that

$$\|\lambda(E)\|^2 = \mu(E)$$

for every E in $\bar{R}$. And if

$$E \cap F = \emptyset$$

for E and F both in $\bar{R}$ the following computation shows that $(\lambda(E), \lambda(F))$ vanishes:

$$\begin{aligned}(\lambda(E), \lambda(F)) &= (\pi(E)\lambda(E), \pi(F)\lambda(F)) \\ &= ([\pi(E)\pi(F)]\lambda(E), \lambda(F)) \\ &= (\pi(E \cap F)\lambda(E), \lambda(F)) \\ &= 0.\end{aligned}$$

We can see finally that for all a in G and A in S

$$T^{-1}\phi(a)T = T_a$$

and

$$T^{-1}\pi(A)T = P_E.$$

For in view of Proposition 4, we have

$$\begin{aligned}T_a \chi_E = \chi_{aE} \\ &= T^{-1}\lambda(aE) \\ &= T^{-1}\phi(a)\lambda(E) \\ &= T^{-1}\phi(a)T\chi_E ,\end{aligned}$$

and

$$P_A \chi_E = \chi_{A \cap E}$$
$$= T^{-1}\lambda(A \cap E)$$
$$= T^{-1}\pi(A)\lambda(E)$$
$$= T^{-1}\pi(A)T\chi_E$$

for all a in G, A in $\overline{S}$, and E in $\overline{R}$.

Finally, since T is a unitary mapping into h of its domain of definition and the domain of definition of T is, by Proposition 4, dense in $L^2(G, \overline{S}, \mu)$, T can be uniquely extended to a unitary mapping (which we continue to call T) of all $L^2(G, \overline{S}, \mu)$ onto a closed subspace of h. And, in addition, the continuity of all operators involved being assured, the following identities verified in the last paragraph for a dense subset can be extended to the whole of $L^2(G, \overline{S}, \mu)$:

1) $T^{-1}\phi(a)T = T_a$

2) $T^{-1}\pi(A)T = P_A$,

for every a in G and A in $\overline{S}$.

Now, it can be seen from these identities that $T[L^2(G, \overline{S}, \mu)]$ is invariant and irreducible under the combined families $\{\phi(a)\}_{a \in G}$ and $\{\pi(A)\}_{A \in \overline{S}}$. The invariance property comes from rewriting the two identities just obtained as:

1') $\phi(a) = TT_aT^{-1}$

2') $\pi(A) = TP_AT^{-1}$

and remembering that T is a unitary mapping of $L^2(G, \overline{S}, \mu)$ onto $T[L^2(G, \overline{S}, \mu)]$. The irreducibility property comes from combining Lemma 12 with identities 1) and 2) of the previous paragraph.

We can now show that $\mathfrak{h}$ is the direct sum of a family of closed, mutually orthogonal subspaces, $\{\mathfrak{h}_\alpha\}_{\alpha \in I}$, for each of which the results of the preceding paragraphs are true.

For, consider a maximal collection of closed, mutually orthogonal subspaces of $\mathfrak{h}$ all of which have the same properties as $T\,[L^2(G, \overline{S}, \mu)]$ (Such a collection exists by the arguments of the preceding paragraphs and by the Maximal Principle), and call the direct sum of its members $\mathfrak{k}$.

If $\mathfrak{h} \ominus \mathfrak{k}$ is not the zero vector, then, first of all, we know from the definition of a projection-measurable group that $\pi(\overline{S})$ is full and countably decomposable on $\mathfrak{h}$ and so, a fortiori, is full and countably decomposable if contracted to $\mathfrak{k}$. In addition, because $\pi(E)\mathfrak{k}$ $[\phi(a)\mathfrak{k}]$ is a subset of $\mathfrak{k}$ for every E in $\overline{S}$ [a in G], we have for every E in $\overline{S}$ (a in G), ξ in $\mathfrak{h} \ominus \mathfrak{k}$, and η in $\mathfrak{k}$ that

$$(\pi(E)\xi, \eta) = (\xi, \pi(E)\eta)$$
$$= 0$$
$$[(\phi(a)\xi, \eta) = (\xi, [\phi(a)]^* \eta)$$
$$= (\xi, \phi(a^{-1})\eta)$$
$$= 0],$$

i.e., $\pi(E)(\mathfrak{h} \ominus \mathfrak{k})$ $[\phi(a)(\mathfrak{h} \ominus \mathfrak{k})]$ is a subset of $\mathfrak{h} \ominus \mathfrak{k}$ for all E in S (a in G). Hence, if we contract all members of $\{\phi(g)\}_{g \in G}$ and $\pi(\overline{S})$ to $\mathfrak{h} \ominus \mathfrak{k}$, $(G, \phi; \overline{S}, \pi; \mathfrak{h} \ominus \mathfrak{k})$ is a true projection-measurable group, the other properties being clear from the definition of $(G, \phi; \overline{S}, \pi; \mathfrak{h})$.

The methods of preceding paragraphs of this proof are clearly applicable to $(G, \phi; \overline{S}, \pi; \mathfrak{h} \ominus \mathfrak{k})$, and they yield the result that there exists a closed subspace with the same properties relative to ϕ and π as those possessed by $T\,[L^2(G, \overline{S}, \mu)]$. But this contradicts the fact that $\mathfrak{k}$ is the direct sum of our maximal, mutually orthogonal collection of closed subspaces of $\mathcal{h}$ each having the same properties as $T\left[L^2(G, \overline{S}, \mu)\right]$. Thus it is false to assume that $\mathcal{h} \ominus \mathcal{k}$ is not the zero vector, which means that $\mathcal{h}$ possesses the desired decomposition. This completes the sufficiency proof for the present theorem.

Now let us give the necessity proof. We are assuming the existence of a σ-finite, left-invariant measure μ on $\overline{S}$ such that $\mu(E)$ vanishes if and only if $\pi(E)$ vanishes. Also, we assume the decomposition of $\mathcal{h}$ into a family of mutually orthogonal, closed subspaces $\{\mathcal{h}_\alpha\}_{\alpha \in I}$ with the property that

1) $\mathcal{h}_\alpha = V_\alpha\left[L^2(G, \overline{S}, \mu)\right]$

and

2) V_α is a unitary mapping of $L^2(G, S, \mu)$ into $\mathcal{h}$ such that on $\mathcal{h}_\alpha$

$$V_\alpha T_a V_\alpha^{-1} = \phi(a)$$

and

$$V_\alpha P_E V_\alpha^{-1} = \pi(E).$$

The other conclusions of the present theorem will not be needed for our necessity arguments.

The fact that μ is a σ-finite measure on $\overline{S}$ causes $\{P_E\}_{E \in \overline{S}}$, considered as an Abelian algebra of projections on $L^2(G, \overline{S}, \mu)$, to be full and countably decomposable. (These facts were pointed out in the process of defining fullness and countable decomposability.) Hence, in virtue of the relation between $\{P_E\}_{E \in \overline{S}}$ and $\pi(\overline{S})$, it must be that $\pi(\overline{S})$ is full and countably decomposable.

Next, we wish to show that $\phi(x)\xi$ is strongly measurable relative to π for every ξ in $\mathcal{h}$. Because of the relation between T_x and $\phi(x)$ and the assumption that $\pi(E)$ vanishes if and only if $\mu(E)$ vanishes, it follows that this will be accomplished if we show that on each member of $\bar{S}$ (minus a subset of zero measure) $T_x f$ is a uniform limit of countably-valued step-functions for every f in $L^2(G, \bar{S}, \mu)$. Observe that by Proposition 1 an arbitrary neighborhood of the identity in the topology $\bar{T}$ has the form

$$\{x \mid [\mu(xE) + \mu(E) - 2\mu(xE \quad E)] < \epsilon, \ 0 < \epsilon < 2\mu(E) < \infty\}$$

and that

$$\mu(xE) + \mu(E) - 2\mu(xE \cap E) = \int |\chi_{xE}(y) - \chi_E(y)|^2 \, d\mu(y).$$

As a result $T_x(\chi_E)$ is continuous in the topology $\bar{T}$. Then, linear combinations of such characteristic functions being dense in $L^2(G, \bar{S}, \mu)$ by Lemma 11 and T_x being a unitary operator on this space, $T_x f$ is continuous in $\bar{T}$ for all f in $L^2(G, \bar{S}, \mu)$. Then if we remember that Lemma 10 decomposes any given E in $\bar{S}$ into an at most countable collection of measurable subsets of left translates of any given neighborhood of the identity plus a set of measure zero, it follows that if we remove a subset of measure zero from E $T_x f$ is a uniform limit of countably-valued step-functions on the subset remaining. As remarked, this shows that $\phi(a)\xi$ is strongly measurable relative to π for every ξ in $\mathcal{h}$.

This completes the necessity proof and, hence, completes the proof of Theorem 1.

<u>Theorem 2:</u> Let $(G, \phi; \bar{S}, \pi; \mathcal{h})$ be a projection-measurable group such that for any E in $\bar{S}$ there is an a in G such that $\pi(E \cap (aE)^c) = 0$. Then G possesses a topology $\bar{T}$ such that

1) there exists a family of neighborhoods of the identity

equivalent to the defining family and possessing the property that its members belong to $\bar{S}$ and have positive, finite μ-measure;

2) G is a topological group in the topology $\bar{T}$ such that each neighborhood of the identity is contained in the union of a finite number of left translates of any other neighborhood of the identity;

3) $\phi(x)\xi$ is continuous in $\bar{T}$ for every ξ in $\mathfrak{h}$;

4) G is a dense subgroup of a locally compact group.

Proof: Conclusions 1, 2, and 4 of this theorem are a rewording of the conclusions of Theorem 3 of Section 1. But, since by definition of a projection-measurable group $\bar{S}$ is an invariant σ-ring and by Proposition 1 μ is a left invariant, σ-finite measure on $\bar{S}$, $(G, \bar{S}, \mu)$ is a left-invariant GMG, the hypotheses of Theorem 3 of Section 1 are satisfied, with one exception. The hypothesis which is unverified is that if E is any member of $\bar{S}$ for which $\mu(E)$ is positive then there is an a in G for which $\mu(aE \cap E) < \mu(E)$. But the new hypothesis made on a projection-measurable group in the statement of the present theorem is just precisely equivalent to the desired hypothesis, since we have that for every E in $\bar{S}$ $\mu(E)$ vanishes if and only if $\pi(E)$ vanishes. So all but conclusion 3 of the theorem in hand may be considered proved.

Now if we compare the statements of Theorem 4 of Section 1 and of Proposition 1 we see that under the present hypothesis they impose the same topology in G. This, combined with the fact that Lemma 4 shows $\phi(x)\xi$ to be continuous in the topology of Proposition 1, proves the third conclusion of our theorem.

Remark 1: The somewhat tortuous methods used here to obtain Theorems 1 and 2 might appear avoidable if the following approach is taken. First, restrict the discussion to projection-measurable groups of the type considered by Theorem 2, which, after all, is not really much of a restriction. Then construct on $\tilde{S}$ the left-invariant measure μ such that $\mu(E) = 0$ if and only if $\pi(E) = 0$. Next, apply Theorem 3 of Section I to the resulting left-invariant GMG, so that one can consider the locally compact group which is the completion of the GMG. (Theorem 2 of the present section also falls out at this point.) Finally, define π on the Baire sets of the locally compact group just constructed, extend ϕ to all of the locally compact group, and apply Loomis's theorem, which would seem to make Theorem 1 of this section true, *a fortiori*. But there is one difficulty: there is no way of knowing whether all of $\tilde{S}$ or merely a proper σ-subring of $\tilde{S}$ corresponds to the Baire sets of our locally compact group, so that in applying Loomis's theorem as above we may end up with the representation of a proper subalgebra of $\pi(\tilde{S})$ rather than of the whole of $\pi(\tilde{S})$. The author could see no way around this, with the result that he took the present approach, namely, that of generalizing Loomis's arguments.

Remark 2: It is necessary and sufficient for the truth of Theorem 1 that $\pi(\tilde{S})$ be full and countably decomposable and that $\phi(x)\xi$ be strongly measurable relative to π for every ξ in $\mathcal{K}$, yet the mutual independence of these three conditions, in the context of the other properties which define a projection-measurable group, may seem dubious. This independence actually exists, for we shall now give three examples of structures each one of which has all properties of a projection-measurable group excepting exactly one of our group of necessary and sufficient conditions for Theorem 1. A different condition will be violated in each example, of course.

Example I (Violation of countable decomposability):

Let our abstract group G be the direct product with itself of the group R of real members under addition, i.e.,

$$G = R \times R$$

Let $\overline{S}$ be the family of Baire sets in G relative to the Euclidean topology on G. To define the hilbert space $\mathcal{h}$, we observe that G underlies the topological group formed by giving one copy R_1 of R the discrete topology, giving a second copy R_2 of R the Euclidean topology, and constructing the product of the two topologies on the abstract product-group $R_1 \times R_2$ (=G); then, letting $\overline{T}$ and ν be, respectively, the family of Baire sets of the latter topological group and the Haar measure on T, we define $\mathcal{h}$ to be the real $L^2(G, \overline{T}, \nu)$. As might be expected now, we define the representation ϕ of G on a group of unitary operators on $\mathcal{h}$ by the relation:

$$\phi(x) = T_x$$

for all x in G. (T_x is the unitary translation operator on $L^2(G, \overline{T}, \nu)$ defined in Definition 14.) But, we define a σ-homomorphism π of $\overline{S}$ on an Abelian ring of projections on $\mathcal{h}$ as follows: for every E in $\overline{S}$

$$\pi(E) = P_E,$$

where P_E is the projection operator on $\mathcal{h}$ (=$L^2(G, \overline{T}, \nu)$) corresponding to the multiplication of a function square-integrable on $(G, \overline{T}, \nu)$ by the characteristic function of E.*

The structure just exhibited certainly has all the properties of a projection-measurable group other than those three with whose independence we are concerned. A discussion of the last three follows.

We can show $\phi(x)\xi$ has the right measurability property for every ξ in $\mathfrak{h}$. In fact, let us consider G as the abstract group underlying the product topological group $R_1 \times R_2$, R_1 being the discrete reals and R_2, the Euclidean reals. We see that since $\phi(x) = T_x \phi$ is representation of G on a group of unitary operators on the hilbert space $L^2(G, \overline{\overline{T}}, \nu)$. As shown in the example following the definition (Definition 8) of strong measurability relative to a σ-homomorphism, $\phi(x)\xi$ is strongly measurable relative to any σ-homomorphism of $\overline{\overline{T}}$ onto an Abelian ring of projections on $L^2(G, \overline{\overline{T}}, \nu)$. Since the domain of π contains $\overline{\overline{T}}$, it is now clear that $\phi(x)\xi$ is a strongly measurable function relative to π.

Next, $\pi(\overline{\overline{S}})$ is full since it contains all the projections on $L^2(G, \overline{\overline{T}}, \nu)$ generated by characteristic functions of sets of positive, finite ν-measure.

---

*Since E is an arbitrary member of S, we must justify the last definition by showing that

$$E \cap \overline{\overline{T}} \subset \overline{\overline{T}}$$

for every E in $\overline{\overline{S}}$. We obtain our result by applying the following assertion first to $\overline{\overline{T}}$ and then to $\overline{\overline{S}}$: the family of Baire sets of the product group $G_1 \times G_2$ generated by two locally compact topological groups, G_1 and G_2, is identical with the σ-ring generated by the collection of all cartesian products of the form $B_1 \times B_2$, where B_i is an arbitrary Baire set of G_i. (See Theorem 51.C of (1).) Applying this assertion to $\overline{\overline{T}}$, we see that $\overline{\overline{T}}$ consists of all sets of the form $\bigcup_{i=1}^{\infty} \{x_i\} \times E_i$, where for all i x_i is an element of R_1 and E_i is a Baire subset of R_2 relative to the Euclidean topology. And, applying the quoted result to $\overline{\overline{S}}$, it follows that for every F in $\overline{\overline{S}}$ every section of the form F_x is a Baire subset of R_2 relative to the Euclidean topology. Thus for all E in $\overline{\overline{S}}$

$$E \cap \overline{\overline{T}} \subset \overline{\overline{T}}.$$

But $\pi(\widetilde{S})$ does not have the desired property of countable decomposability. For any set in $\overline{S}$ having positive Haar measure, considered as a Baire set in the Euclidean topology, contains an uncountable collection of mutually disjoint sets having positive ν-measure and belonging to S, since positive ν-measure is assigned to any set in G of the form $\{x\} \times E$, x being any member of R and E being any Baire set in the Euclidean topology having positive Haar measure. $\pi(E) \neq 0$ for every E in S such that E belongs to T and $\nu(E)$ is positive, so it follows that there exists an F in S such that $\pi(F)$ contains uncountably many disjoint, non-zero projections of $\pi(\widetilde{S})$.

Example II (Violation of strong measurability property): Let us take as G the additive group of the reals, R, and let us take as $\overline{S}$ the family of Baire sets relative to the Euclidean topology, or, what is the same thing, the family $\overline{B}$ of all Borel sets relative to the Euclidean topology. Defining m to be the Haar measure on $\overline{B}$, i.e., Borel measure, we shall take $\mathcal{h}$ to be the direct sum of a collection of continuum many copies of the real $L^2(R, \overline{B}, m)$. If we write

$$\mathcal{h} = \bigoplus_{\alpha \in R} L_\alpha^2$$

and define for any ξ in $\mathcal{h}$ the projection of ξ on L_α^2 to be ξ_α, then for every ξ in $\mathcal{h}$, E in $\overline{S}(=\overline{B})$, and x in G(=R)

$$(\pi(E)\xi)_\alpha = P_E \xi_\alpha$$

and

$$(\phi(x)\xi)_\alpha = T_x \xi_{(\alpha - x)}.$$

(P_E and T_a are the operators defined in Definition 14 above, $(G, \overline{S}, m)(=(R, \overline{B}, m))$ being after all a left-invariant GMG.)

Noting, among other things, that $(R, \bar{B}, m)$ is a left-invariant GMG, that $\{P_E\}_{E \epsilon \bar{B}}$ is a full, countably decomposable Abelian algebra of projections on $L^2(R, \bar{B}, m)$, and that $a \rightarrow T_a$ is a representation of G on a group of unitary operators on $L^2(R, \bar{B}, m)$, we see that the structure just described has all the properties of a projection-measurable group. Properties that may be lacking are the truth of the commutation relation:

$$\phi(a)\pi(E) = \pi(aE)\phi(a)$$

for all a in G and all E in $\bar{S}$ and the strong measurability property of $\phi(x)\xi$.

That the commutation relation holds is seen as follows: for every a in G, E in $\bar{S}$, ξ in $\mathcal{h}$, and α in R

$$\begin{aligned}(\pi(aE)\xi)_\alpha &= P_{aE}\xi_\alpha \\ &= T_{(-a)}P_E T_a \xi_{((\alpha+a)-a)} \\ &= T_{(-a)}P_E(\phi(a)\xi)_{(\alpha+a)} \\ &= T_{(-a)}(\pi(E)\phi(a)\xi)_{(\alpha+a)} \\ &= T_{(-a)}(\quad (E)\phi(a)\xi)_{(\alpha-(-a))} \\ &= (\phi(-a)\pi(E)\phi(a)\xi)_\alpha .\end{aligned}$$

But there exists an element $\bar{\xi}$ of $\mathcal{h}$ such that $\phi(x)\bar{\xi}$ is not strongly measurable relative to π. $\bar{\xi}$ is defined by the condition that for every in R

$$\bar{\xi}_\alpha = \begin{cases} 0, & \alpha \neq 1 \\ f, & \alpha = 1 \end{cases}$$

where f is any element of $L^2(G, \bar{S}, m)$ such that $\|f\| = 1$. We observe from the definition of ϕ that

$$(\phi(a)\bar{\xi}, \phi(b)\bar{\xi}) = \delta_{ab} ,$$

so that

$$\|\phi(a)\bar{\xi} - \phi(b)\bar{\xi}\| = \sqrt{2}$$

as long as $a \neq b$. Now let us look at the definition of strong measurability relative to π (Definition 8) and note that by our present definition of π if $\pi(E) = 0$ then E has zero Borel measure. If $\phi(x)\xi$ is strongly measurable relative to π, then on a set A having positive Borel measure (hence an uncountable set) $\phi(x)\xi$ is the uniform limit of a countable sequence of countably-valued step-functions. Thus, there is a countable collection, F, of vectors of $\mathfrak{h}$ such that $\{\phi(a)\bar{\xi} \mid a \in A\}$ belongs to the closure of F. But this means that for every a in A there is an η_a in F such that

$$\|\phi(a)\bar{\xi} - \eta_a\| < \frac{\sqrt{2}}{3}$$

But the fact that

$$\|\phi(a)\bar{\xi} - \phi(b)\bar{\xi}\| = \sqrt{2}$$

when $a \neq b$ means that no two η_a's can be the same vector; so $\{\eta_a\}_{a\in A}$ must be countable, A being an uncountable set. This conclusion contradicts the deduction that $\{\eta_a\}_{a\in A}$ is a subset of the countable set F, which shows the falsity of the assumption that $\phi(a)\xi$ is strongly measurable relative to π.

<u>Example III (Violation of fullness property)</u>: Let G be the real numbers under addition, $\bar{S}$ be the family of Baire sets in G relative to the Euclidean topology, and $\mathfrak{h}$ be the hilbert space which is the direct sum of two copies of $L^2(G, \bar{S}, m)$, where m is the Haar measure on $\bar{S}$ (Borel measure). Further, letting (ξ, η) be a generic member of $\mathfrak{h}$, we can define a

σ-homomorphism π on S to projections on $\mathfrak{h}$ by the equation

$$\pi(E)(\xi, \eta) = (P_E \xi, 0)$$

and a representation on G to unitary operators on $\mathfrak{h}$ by the equation

$$\phi(x)(\xi, \eta) = (T_x \xi, \eta).$$

(P_E and T_x are the operators on $L^2(G, \overline{S}, m)$ defined in Definition 14.)

Now to describe the structure that has been set up. To begin with, the fact that $T_x P_x = P_{xE} T_x$ makes it clear that $\phi(x)\pi(E) = \pi(xE)\phi(x)$. Since by reasoning used in Example 1 $T_x \xi$ is strongly measurable relative to π for every ξ in $L^2(G, \overline{S}, m)$ and $\{P_E\}_{E \epsilon \overline{S}}$ is countably decomposable, then $\phi(x)(\xi, \eta) (= (T_x \xi, \eta))$ must be strongly measurable relative to π for every (ξ, η) in $\mathfrak{h}$, and $\pi(\overline{S})$ is a countably decomposable Abelian ring of projections. However, $\pi(\overline{S})$ is not full, for any element of $\mathfrak{h}$ having the form $(0, \eta)$, $\eta \neq 0$, is annihilated by every member of $\pi(\overline{S})$.

Thus we have shown for a projection-measurable group the mutual independence of the hypotheses that a) $\pi(\overline{S})$ be countably decomposable, b) $\pi(\overline{S})$ be full, c) $\phi(a)\xi$ be strongly measurable relative to π for every ξ in $\mathfrak{h}$.

Having derived and discussed our generalization of Loomis' theorem, we propose now to give the actual proof that the latter is a consequence of our purported generalization. To accomplish this we need the following lemma and proposition.

<u>Lemma 13:</u> Let G be a locally compact topological group and $\overline{S}$ be the family of all Baire sets in G. Suppose there exists a non-zero σ-homomorphism π of $\overline{S}$ onto an Abelian ring of projections on a hilbert space $\mathfrak{h}$, and suppose there is a representation ϕ of G on a group of unitary operators on $\mathfrak{h}$ such that

$$\pi(aE) = \phi(a^{-1})\pi(E)\phi(a)$$

for all a in G and all E in $\bar{S}$. If $\phi(x)\xi$ is continuous on G for all ξ in $\mathfrak{h}$, and if H is an open Baire subgroup of G, then for every A in $H \cap \bar{S}$ $\pi(A) = 0$ if and only if A has Haar measure zero.

Proof: For every ξ in $\mathfrak{h}$ let us define the measure m_ξ on $\bar{S}$ by the equation

$$m_\xi(E) = (\pi(E)\xi, \xi).$$

We see that $(G, \bar{S}, m_\xi)$ is a σ-finite measure space, so that $(G \times G, \bar{S} \times \bar{S}, \mu \times m_\xi)$ is a measure space for which the Fubini theorem is valid. (μ is the Haar measure on $\bar{S}$.) Further, for every A in $H \cap \bar{S}$, $(S^{-1}RS)(A \times H)$ is a member of $\bar{S} \times \bar{S}$ by Proposition 1 of Section I. (Let us remember from Section I that

$$S(x, y) = (x, xy)$$

and

$$R(x, y) = (y, x).)$$

As a result, we have by the Fubini theorem and Proposition 0 of Section I that

$$\begin{aligned}
\int_G m_\xi(x^{-1}A \cap H^{-1})\, d\mu(x) &= \int_G m_\xi\left([(S^{-1}RS)(A \times H)]_x\right) d\mu(x) \\
&= \int_G \left(\int_G \chi_{(S^{-1}RS)(A \times H)}(x, y)\, dm_\xi(y)\right) d\mu(x) \\
&= \int_{G \times G} \chi_{(S^{-1}RS)(A \times H)}(x, y)\, d(\mu \times m_\xi)(x, y) \\
&= \int_G \left(\int_G \chi_{(S^{-1}RS)(A \times H)}(x, y)\, d\mu(x)\right) dm_\xi(y) \\
&= \int_G \mu\left([(S^{-1}RS)(A \times H)]^y\right) dm_\xi(y) \\
&= \int_{H^{-1}} \mu(Ay^{-1})\, dm_\xi(y)
\end{aligned}$$

$$= \int_H \mu(Ay^{-1})\, dm_\xi(y).$$

And, since the fact that A belongs to $H \cap \bar{S}$ means

$$\int_G m_\xi(x^{-1}A \cap H^{-1})\, d\mu(x) = \int_H m_\xi(x^{-1}A)\, d\mu(x),$$

we have the following identity true for all A in $H \cap \bar{S}$ and all ξ in $\mathfrak{h}$.

$$\int_H m_\xi(x^{-1}A)\, d\mu(x) = \int_H \mu(Ay^{-1})\, dm_\xi(y).$$

Using this identity, we show first that if A belongs to $H \cap \bar{S}$ and

$$\pi(A) = 0$$

then

$$\mu(A) = 0.$$

$\bar{S}$ being the σ-ring of Baire sets, it follows from the definition of $\bar{S}$ that each is contained in the union of an at most countable number of compact sets, so that every member of $\bar{S}$ is contained in the union of at most countably many left cosets of H. This means, in particular, that there must exist an E in $\bar{S}$ such that

$$E \cap x_o H = E$$

for some x_o and

$$\pi(E) = 0.$$

But since

$$\pi(x_o^{-1}E) = \phi(x_o)\pi(E)\phi(x_o^{-1}),$$

$x_o^{-1}E$ is a member of $H \cap \bar{S}$ on which π does not vanish. As a result, we know that for some ξ_o in $\mathfrak{h}$ $m_{\xi_o}(M) = (\pi(M)\xi_o, \xi_o))$ does not vanish for every M in $H \cap \bar{S}$. Yet if $\pi(A) = 0$ for some A in $H \cap \bar{S}$, then

$$m_{\xi_o}(x^{-1}A) = (\pi(x^{-1}A)\xi_o, \xi_o)$$

$$= (\phi(x)\pi(A)\phi(x^{-1})\xi_o, \xi_o)$$
$$= 0$$

for all x in H, so that

$$\int_H \mu(Ay^{-1})\, dm_{\xi_o}(y) = \int_H m_{\xi_o}(x^{-1}A)\, d\mu(x)$$
$$= 0.$$

The fact that m_{ξ_o} is not identically zero on $H \cap \bar{S}$ can now be brought to bear with the consequence that

$$\mu(Ay^{-1}) = 0$$

for at least one y in H. Thus, since right-translation maps sets of Haar measure zero onto sets of Haar measure zero,

$$\mu(A) = 0.$$

Now suppose that for some A in $H \cap \bar{S}$

$$\mu(A) = 0;$$

then we can show that

$$\pi(A) = 0.$$

If $\mu(A) = 0$, then $\mu(Ay^{-1}) = 0$ for all y, so that for all ξ in

$$\int_H m_\xi(x^{-1}A)\, d\mu(x) = \int_H \mu(Ay^{-1})\, dm_\xi(y)$$
$$= \int_H 0 \cdot dm_\xi(y)$$
$$= 0.$$

Since we know that this holds, that every neighborhood of any point in G contains an open Baire set of positive measure, and that H is open, we can conclude that for every ξ in

$$f_\xi(x) = m_\xi(x^{-1}A) = 0$$

for x on a dense subset of H. Consequently, if we can show that $f_\xi(x)$

is continuous at $x = e$, $m_\xi(A) = 0$. But the identity $\pi(xA) = \phi(x^{-1})\pi(A)\phi(x)$ and the continuity of $\phi(x)\xi$ prove the continuity of f at e:

$$
\begin{aligned}
|f_\xi(x) - f_\xi(e)| &= |(\pi(xA)\xi, \xi) - (\pi(eA)\xi, \xi)| \\
&= |(\pi(A)\phi(x)\xi, \phi(x)\xi) - (\pi(A)\xi, \xi)| \\
&\leq |(\pi(A)(\phi(x)\xi - \xi), \phi(x)\xi)| \\
&\quad + |(\pi(A)\xi, \phi(x)\xi - \xi)| \\
&\leq \|\pi(A)(\phi(x)\xi - \phi(e)\xi)\| \cdot \|\phi(x)\xi\| \\
&\quad + \|\pi(A)\xi\| \cdot \|\phi(x)\xi - \phi(e)\xi\| \\
&\leq 2\|\xi\| \cdot \|\phi(x)\xi - \phi(e)\xi\| .
\end{aligned}
$$

So

$$m_\xi(A) = 0$$

for all ξ in h, which in view of the identity

$$
\begin{aligned}
m_\xi(A) &= (\pi(A)\,\xi, \xi) \\
&= \|\pi(A)\xi\|^2
\end{aligned}
$$

means that

$$\pi(A) = 0.$$

Thus we have shown that for every A in $H \cap \bar{S}$

$$\mu(A) = 0$$

if and only if

$$\pi(A) = 0,$$

which is the desired result.

Proposition 6: Let G be a locally compact topological group and $\bar{S}$ be the family of all Baire sets in G. Suppose there exists a non-zero σ-homomorphism π of $\bar{S}$ onto an Abelian ring of projections on a hilbert space h, and suppose there is a representation ϕ of G on a group of unitary operators on h such that

$$\pi(aE) = \phi(a^{-1})\pi(E)\phi(a)$$

for all a in G and all E in $\bar{S}$. Then if $\phi(a)\xi$ is continuous on G to for every ξ in $\mathfrak{H}$, $\pi(\bar{S})$ is countably decomposable.

Proof: (Unless there is a statement to the contrary, we shall understand "set" to mean "Baire set" wherever it occurs in the following.) The proof can be reduced to showing that, under the present hypotheses, for every E in $\bar{S}$

$$\pi(E) = 0$$

if and only if the Haar measure on E is zero. The reduction is possible s since the truth of the preceding assertion permits us to set up a biunique union- and intersection-preserving mapping $\pi(\bar{S})$ onto a countably decomposable Abelian ring of projections on $L^2(G, \bar{S}, \mu)$, μ being Haar measure. For, to begin with, mapping every element of $\bar{S}$ into the projection on $L^2(G, \bar{S}, \mu)$ generated by its characteristic function is a σ-homomorphism of $\bar{S}$ onto an Abelian ring of projections $\bar{A}$ on $L^2(G, \bar{S}, \mu)$. Combining this with the hypothesis on π, it follows that $\pi(\bar{S})$ and the last-mentioned Abelian ring of projections are maps of $\bar{S}$ under σ-homomorphisms relative to each of which zero has the same inverse image. This fact, since the common inverse image of zero is the family of all Baire sets of Haar measure zero, makes the mapping of $\pi(\bar{S})$ onto $\bar{A}$ which sends $\pi(E)$ into the projection on $L^2(G, \bar{S}, \mu)$ corresponding to the characteristic function of E a biunique, as well as operation-preserving, mapping. And, as pointed out in the definition of countable decomposability, Abelian rings of the sort of $\bar{A}$ are countably decomposable. Thus, because of the relation it permits us to set up between $\pi(\bar{S})$ and a countably decomposable Abelian ring, proof that

$$\pi(E) = 0$$

if and only if

$$\mu(E) = 0$$

yields the proof of the present lemma.

To implement our argument, it will be useful to construct an open subgroup H of G which is a Baire set and, hence, a set of positive μ-measure. If O is an open Baire set, then the sequence of sets $\{O^n\}_{n=1}^{\infty}$ defined by the recursion relation

$$O^n = (O^{n-1})O$$

is an increasing sequence of open sets of positive measure and

$$H = \bigcup_{n=1}^{\infty} O^n$$

is a subgroup of G which is a set of positive measure. We prove that O^n (and so H) is a Baire set by induction. Since $(G, \bar{S}, \mu)$ is a generalized measurable group, we know by Corollary 1 of Proposition 2 of Section I that $\mu(x^{-1}O \cap O^{-1})$ is an $\bar{S}$-measurable function on G, whence

$$O^2 = \{x \mid \mu(x^{-1}O \cap O^{-1}) > 0\}$$

is an $\bar{S}$-measurable set, i.e., a Baire set. Further, $\mu(x^{-1}O^n \quad O^{-1})$ is an $\bar{S}$-measurable function and

$$O^{n+1} = (O^n)O = \{x \mid \mu(x^{-1}O^n \cap O^{-1}) > 0\}$$

if O^n is $\bar{S}$-measurable, i.e. a Baire set, and if we again use Corollary 1 of Proposition 2. Hence, O^{n+1} is a Baire set. Using the Principle of Finite Induction, every O^n, and so H, is a Baire set.

Now, since the family of all distinct left cosets of H constitutes a disjoint dissection of G into open sets and since every Baire set is contained in the union of at most countably many compact sets, every member of $\bar{S}$ intersects at most countably many left cosets of H. This means, in view of the invariance of μ and the fact that

$$\pi(aE) = \phi(a^{-1})\pi(E)\phi(a),$$

that proving

$$\pi(E) = 0$$

if and only if

$$\mu(E) = 0$$

for all E in S is equivalent to proving the same assertion for all E in $H \cap \bar{S}$. But now the lemma just proved (Lemma 13) applies so that the present proposition can be considered proved.

We can now derive a theorem which is essentially Loomis' theorem. The only differences are: 1) a correction in his statement about the uniqueness properties of the mapping described and 2) the addition of the necessary requirement that the Abelian ring of projections involved be full.

Theorem 3: Let G be a locally compact topological group, $\bar{S}$ be its family of Baire sets, and μ be the left Haar measure on $\bar{S}$. Suppose there exists a non-zero σ-homomorphism π on $\bar{S}$ to a full Abelian ring of projections on a hilbert space $\mathcal{h}$, and suppose that there exists a strongly continuous representation ϕ of G on a group of unitary operators on $\mathcal{h}$ having the property that

$$\pi(aE) = \phi(a^{-1})\pi(E)\phi(a).$$

Then $\mathcal{h}$ is expressible as the direct sum of a family of closed subspaces $\{\mathcal{h}_\alpha\}_{\alpha \in I}$ such that:

1) For every α there exists a unitary mapping V_α of $L^2(G, \bar{S}, \mu)$ onto $\mathcal{h}_\alpha$ (unique up to a unimodular constant multiple) with the property that for every a in G and every A in $\bar{S}$

$$V_\alpha^{-1}\phi(a)V_\alpha = T_a$$

and

$$V_\alpha^{-1}\pi(A)V_\alpha = P_A,$$

2) Every $\mathfrak{h}_\alpha$ is invariant and irreducible under the combined families $\{\phi(a)\}_{a\in G}$ and $\{\pi(A)\}_{A\in\bar{S}}$.

Proof: In the light of Theorem 1 of this section and the fact that the left-invariant measure on $\bar{S}$ is unique up to a constant multiple, we need only verify that $(G, \phi; \bar{S}, \pi; \mathfrak{h})$ is a projection-measurable group.

But this quickly reduces to showing that for every ξ in $\mathfrak{h}$ $\phi(x)\xi$ is strongly measurable on $\bar{S}$ relative to π as we can see. For, to begin with, $\bar{S}$, the family of Baire sets of G, is shown in the remarks following the definition of a generalized measurable group to be invariant. Also, the definition of a strongly continuous representation and Proposition 6 of this section yield the result that $\pi(\bar{S})$ is countably decomposable. And the truth of all other properties, with the exception of the strong measurability property desired for ϕ, are explicit in the hypotheses of the present theorem.

To prove that ϕ has the desired strong measurability property, we shall remember that every set of positive measure is the union of an at most countable collection of mutually disjoint sets of positive, finite measure. It follows that we need only consider sets of the latter type in proving that $\phi(x)\xi$ is strongly measurable relative to π for all ξ. Now, since the Baire sets are generated by a family of compact sets, every Baire set is contained in the union of a countable collection of compact sets. Also, $\phi(x)\xi$ is continuous and so uniformly continuous on each such compact set. But this means that $\phi(x)\xi$ is the uniform limit of a sequence of countably-valued step-functions on any given Baire set. Thus for all ξ $\phi(x)\xi$ is by definition strongly measurable relative to π.

This concludes the proof of the last result of this section.

III. CHARACTERIZATION OF SEPARABLE TOPOLOGICAL GROUPS POSSESSING A MEASURE

In the present section we wish first to show that any separable topological group G possessing a certain kind of measure has as its completion a locally compact group G' such that G' - G contains no sets of positive Haar measure, and conversely. More specifically, we shall prove the following theorem:

Theorem 1: Let G be a separable topological group. Let m be a σ-finite measure on the σ-ring generated by the family of all open sets in G such that:

1) $m(aE) = 0$ if and only if $m(E) = 0$,
2) $\{x \mid m(xE \cap E) > 0\}$ contains a neighborhood of the identity for every E such that $m(E) > 0$.

Then the existence of m is necessary and sufficient for the existence of a locally compact topological group G' with the property that the group uniform space of G' is the completion of the group uniform space of G and G' - G contains no sets of positive Haar measure.

The other purpose of this section is to determine necessary and sufficient conditions that a separable topological group shall be locally compact. In fact, the following will be proved:

Theorem 2: Let G be a separable topological group. Then necessary and sufficient conditions that G be locally compact are that:

1) there be defined on the σ-ring $\overline{S}$ generated by the open sets of G a σ-finite measure m such that all left-translates of sets of measure zero are sets of measure zero,
2) the group uniform space of G be complete.

We shall proceed to the proof of Theorem 1, beginning with several topological lemmas.

Lemma 1: Given any family $\{O_\alpha\}_{\alpha \in I}$ of open subsets of a separable topological group G there is an at most countable subfamily

$$\left\{O_{\alpha_i}\right\}_{i=1}^{n \leq \infty}$$

such that

$$\bigcup_{\alpha \in I} O_\alpha = \bigcup_{i=1}^{n \leq \infty} O_{\alpha_i}.$$

Proof: Since G satisfies the Second Axiom of Countability, there is a sequence of open sets

$$\left\{N_i\right\}_{i=1}^{n \leq \infty}$$

such that for each i there is an α with the property that

$$N_i \subset O_\alpha$$

and for each α

$$\bigcup_{j=1}^{m \leq \infty} N_{i_j} = O_\alpha .$$

It follows that if O_{α_i} is any O_α containing N_i, then

$$\left\{O_{\alpha_i}\right\}_{i=1}^{n \quad \infty}$$

is the desired subfamily of the family $\left\{O_\alpha\right\}_{\alpha \in I}$.

Lemma 2: If G x G is the cartesian product of two copies of a topological group G, each endowed with the same separable topology, then the resulting product topology $\overline{T}_{G \times G}$ on G x G is separable.

Proof: By definition $\{M \times N \subset G \times G | M, N \text{ are open}\}$ is a basis for $\overline{T}_{G \times G}$. If

$$\{O_i\}_{i=1}^{n \leq \infty}$$

is an at most countable basis for the topology on G, then, clearly,

$$\left\{\{O_i \times O_j\}_{i=1}^{n}\right\}_{j=1}^{n}$$

is an at most countable basis for $\overline{T}_{G \times G}$.

Lemma 3: If $\overline{S}$ is the σ-ring generated by the family of open sets of a separable topological group G, then $\overline{S} \times \overline{S}$ is an invariant σ-ring (in the sense of Section I).

Proof: From the definition we see that it is required to show that the transformation

$$\sigma\ (x, y) \rightarrow (x, xy)$$

simply permutes the members of $\overline{S} \times \overline{S}$ among themselves. This will be clear if we show that $\overline{S} \times \overline{S}$ is the σ-ring generated by the family of all sets open in the product topology, for σ is a homeomorphism relative to the product topology.

For, if $\overline{O}_{G \times G}$ is the σ-ring generated the family of all sets open in the product topology on G x G, then, first,

$$\overline{O}_{G \times G} \supset \overline{P},$$

where $\overline{P}$ is the σ-ring generated by all cartesian products of pairs of open sets in G. Next, since every set open in the product topology is the union of a collection generators of $\overline{P}$ (an at most countable collection by Lemma 1),

$$\overline{O}_{G \times G} \subset \overline{P}.$$

As a consequence,

$$\bar{O}_{G \times G} = \overline{P}.$$

But $\overline{P} = \overline{S} \times \overline{S}$, for, as shown by the lemma immediately succeeding Definition 1 of Section I, if $\overline{X}$ is the σ-ring generated by $\{C_\alpha\}_{\alpha \in I}$, then $\overline{X} \times \overline{X}$ is the σ-ring generated by

$$\{\{C_\alpha \times C_\beta\}_{\alpha \in I}\}_{\beta \in I}.$$

Hence,

$$\overline{S} \times \overline{S} = \bar{O}_{G \times G},$$

concluding the proof.

<u>Proposition 1:</u> Let m be a σ-finite measure on the σ-ring $\overline{S}$ generated by the open sets of a separable topological group. Further, let m be such that for every a in G and E in $\overline{S}$ m(aE) vanishes if and only if m(E) vanishes. Then there exists a left-translation invariant σ-finite measure μ on $\overline{S}$ which is equivalent to m. Further, G can be given a topology such that a base of neighborhoods at the point y is the family of all sets of the form:

$$y\{x \mid \mu((xE - E) \cup (E - xE)) < \epsilon,\ 0 < \epsilon < 2\mu(E) < \infty\}.$$

In this topology, G is a topological group, and the group uniform space of G has as its completion the group uniform space of a locally compact group G' such that G' - G contains no Baire sets of positive Haar measure. In fact, if $\hat{S}$ is the family of Baire sets of H and $\hat{\mu}$ is the Haar measure on $\hat{S}$, then

$$H \cap \hat{S} \subset \overline{S}$$

and

$$\mu(H \cap E) = \hat{\mu}(E)$$

for every E in $\overline{S}$.

Proof: We combine Lemma 3 of this section with Theorems 2 and 4 of Section I, assuming we know that for every a in G there is a set E of positive finite measure such that $m(aE \cap E) < m(E)$. But the latter is true because the topology on G is Hausdorff, sets open in the given topology have positive measure, and m is a σ-finite measure.

We can now show that if we strengthen the hypotheses on the measure m of Proposition 1 then the given topology on the group G of that proposition is equivalent to the constructed topology. However, we need the following preliminary result.

Proposition 2: Let m be a σ-finite measure on the σ-ring generated by the open sets of a separable topological group. Further let m have the following properties:

1) $m(aE) = 0$ if and only if $m(E) = 0$ for every a in G and every measurable set E,

2) $\left\{x \mid m(xE \cap E) > 0\right\}$ contains a neighborhood of the identity for all E of positive measure.

Then the topology in which a generic neighborhood of an element y is given by

$$y\left\{x \mid \mu\left[(xE - E) \cup (E - xE)\right] < \epsilon,\ 0 < \epsilon < 2\mu(E) < \infty\right\}$$

is equivalent to the given topology. (Here μ is the left-translation invariant σ-finite measure asserted to exist and to be equivalent to m by Proposition 1.)

Proof: As shown in the proof of Proposition 1, $(G, \overline{S}, m)$ is a GMG, if by $\overline{S}$ we mean the σ-ring generated by all open sets of G. Then, in the terminology of Section I, $(G, \overline{S}, \mu)$ is a left-invariant GMG, which is called a measurable group in (1). Now by Theorem 62.A of (1) for every set of the form

$$M = \left\{ x | \mu \left[(xE - E) \bigcup (E - xE) \right] < \epsilon, \ 0 < \epsilon < 2\mu(E) < \infty \right\}$$

there exists in every set of positive μ-measure a subset F of positive μ-measure such that

$$FF^{-1} \subset M,$$

so that

$$\left\{ x | \mu(xF \bigcap F) < 0 \right\} \subset M.$$

As a result, since we also know that

$$\left\{ x | \mu \left[(xE - E) \bigcup (E - xE) \right] < \epsilon, \ 0 < \epsilon < 2\mu(E) < \infty \right\} = \left\{ x | \mu(xE \bigcap E) > \mu(E) - \frac{\epsilon}{2} > 0 \right\}$$
$$\subset \left\{ x | \mu(xE \bigcap E) > 0 \right\}$$

and that μ is σ-finite, we can prove the present proposition by showing that every neighborhood of the identity contains a set of the form $\left\{ x | \mu(xE \bigcap E) > 0, \ 0 < \mu(E) \right\}$ and every set of the form $\left\{ x | \mu(xE \bigcap E) > 0, \ 0 < \mu(E) \right\}$ contains a neighborhood of the identity. (In fact, m being equivalent to μ, we can substitute m for μ in the preceding assertion.) But actually we need only demonstrate the former, since by hypothesis we have that for every E of positive measure $\left\{ x | m(xE \bigcap E) > 0 \right\}$ contains a neighborhood of the identity.

To show that every neighborhood of the identity contains a set of the form

$$\left\{ x | m(xE \bigcap E) > 0, \ 0 < m(E) \right\}$$

we must first note that every neighborhood of the identity has positive m-measure. If N is an arbitrary neighborhood of the identity and E is any set of positive measure then EN contains E and, by Lemma 1, is a countable union of left translates of N. Thus a countable union of left translates of N has positive measure, so that some particular left translate of N has positive measure. But, in view of the assumption that any left translate of a measurable set has zero m-measure only if the original set has zero m-measure, N has positive measure.

Now since for any neighborhood of the identity O there is a neighborhood N such that

$$NN^{-1} \subset O,$$
$$\{x \mid m(xN \cap N) > 0\} \subset NN^{-1}$$
$$\subset O .$$

Inasmuch as we know that m(N) is positive, we have shown that an arbitrary neighborhood of the identity contains a set of the form

$$\{x \mid m(xE \cap E) > 0, \ 0 < m(E)\} ,$$

thereby completing the proof of this proposition.

We can now see that Propositions 1 and 2, when taken together, yield the sufficiency proof for Theorem 1.

Next we obtain the results requisite to the necessity proof for Theorem 1.

Lemma 4: Let H be a dense sub-group of a locally compact topological group G such that G - H contains no sets of positive Haar measure. Let $\bar{B}$ be the family of Borel sets in G, and let μ be the Haar measure on $\bar{B}$. Then the function ν on the σ-ring $H \cap \bar{B}$ defined by the equation

$$\nu(H \cap E) = \mu(E)$$

for all E in $\bar{B}$ is a left-translation invariant measure on $H \cap \bar{B}$.

Proof: First of all, ν is a uniquely defined function on $H \cap \bar{B}$. For if any pair of sets, E and F, in $\bar{B}$ are such that

$$H \cap E = H \cap F$$

and

$$\mu(E) \neq \mu(F) \text{ ,}$$

then $(E \cap F^c) \cup (F \cap E^c)$ is a measurable subset of G - H and has positive measure, a contradiction of one of the assumptions on H.

Second, ν is completely additive. Suppose

$$\left\{E_i \cap H\right\}_{i=1}^{\infty}$$

is a mutually disjoint collection sequence of sets in $H \cap \overline{B}$, and E_i being members of $\overline{B}$. Then every $E_i \cap E_j$ has μ-measure zero, each being a subset of G - H. As a consequence,

$$\sum_{n=1}^{\infty} \mu(E_n) = \mu(\bigcup_{n=1}^{\infty} E_n),$$

so that, by definition of μ,

$$\sum_{n=1}^{\infty} (H \cap E_n) = (H \cap \bigcup_{n=1}^{\infty} (E_n)).$$

Hence,

$$\sum_{n=1}^{\infty} \nu(H \cap E_n) = \nu(\bigcup_{n=1}^{\infty} (H \cap E_n)),$$

proving the countable additivity of ν.

As to the left-invariance of ν, observe that for any element a of H and any E in $\overline{B}$,

$$\begin{aligned} \nu\left[a(H \cap E)\right] &= \nu(H \cap aE) \\ &= \mu(aE) \\ &= \mu(E) \\ &= \nu(H \cap E). \end{aligned}$$

Lemma 5: Given a locally compact topological group G, let $\overline{B}$ be the family of all Borel sets of G, and let H be a subgroup of G separable in the relative topology. Then $H \cap \overline{B}$ contains the σ-ring generated by the collection of all subsets of H open in the relative topology.

Proof: Observe that any x in H, considered as a point in the topological group G, possesses a family of neighborhoods which is made up of members of $\overline{B}$ and is equivalent to any other family of neighborhoods of that point. As a result, by the definition of the topology on H, x, considered as a point in the topological group H, has the same property, except that $H \cap \overline{B}$ plays the role played by $\overline{B}$ in the first case. Thus every open set in H is the union of a collection of open sets contained in $H \cap \overline{B}$. Since by Lemma 1 and the separability of the topology on H this collection can be taken to be countable, every open set in H must belong to $H \cap \overline{B}$. Thus the σ-ring generated by all open subsets of H open in the relative topology on H is a subcollection of $H \cap \overline{B}$.

Proposition 3: Let H be a dense subgroup of a locally compact topological group G such that H is separable in the relative topology and G - H contains no sets of positive Haar measure. Then there exists a measure ν on the σ-ring generated by the family of all open sets in H which satisfies the hypotheses made on the measure m of Theorem 1.

Proof: We shall prove this proposition by showing that the restriction to $\overline{S}$ of the measure ν defined by Lemma 4 has the desired properties. (This restriction is possible since, by Lemma 5,

$$\overline{S} \subset H \cap \overline{B}$$

and since $H \cap \bar{B}$ is the original domain of definition of ν.) But we see from Lemma 4 that ν is a left-translation invariant measure on $\bar{S}$, from which it is obvious that for all a in H and all E in $\bar{S}$ $\nu(aE) = 0$ if and only if $\nu(E) = 0$. Further, every set of the form

$$\{x \in H \mid \nu(xE \cap E) > 0\},$$

where $\nu(E) > 0$, contains a set which is a neighborhood of the identity in the relative topology on H, for we know from the local compactness of G and from the continuity on G of $\mu(xE \cap F)$ for every set F in $\bar{B}$ that if $\mu(F) > 0$ $\{x \in G \mid \mu(xF \cap F) > 0\}$ contains a neighborhood of the identity. (The continuity of $\mu(xF \cap F)$ is proved on p. 50 of (7).) Since, finally, it is clear from the definition of ν that it is a σ-finite, we see that ν is a measure on $\bar{S}$ having all the properties assumed for the measure m of Theorem 1.

Having proved Proposition 3, we see that we have the necessity proof for Theorem 1, which means that we have disposed of the latter completely since the sufficiency proof is contained in Proposition 2.

Remark: One of the assumptions on the measure m of Theorem 1 was that for every E for which $m(E) > 0$ the set $\{x \mid m(xE \cap E) > 0\}$ must contain a neighborhood of the identity. Now, it might appear that some separable topological group satisfying the conclusion of Theorem 1 could be such as to permit a measure having all the properties assumed with the exception of the one in question. But in actuality this seeming restrictiveness does not exist.

Suppose that on the σ-ring $\bar{S}$ generated by the family of open sets of a separable topological group H there exists a σ-finite measure m and that, in addition, H is a dense subgroup of a locally compact topological group G such that G - H contains no Borel sets of positive Haar measure. Then since Lemmas 4 and 5 apply to H we can use an argument occurring in the proof of Proposition 3 to show there is a σ-finite left-translation invariant measure ν on $\bar{S}$ such that $\{x \in H \mid \nu(xE \cap E) > 0\}$ contains a neighborhood of the identity whenever $\nu(E) > 0$. But, since $\bar{S}$ is an invariant σ-ring, $(H, \bar{S}, \nu)$ and $(H, \bar{S}, m)$ are both GMG's, so that by Proposition 7 of Section I m and ν are equivalent measures on $\bar{S}$. It is now obvious that whenever $m(E) > 0$ $\{x \in H \mid m(xE \cap E) > 0\}$ contains a neighborhood of the identity. Thus we see that the apparent restrictiveness of assuming this property for m in the statement of Theorem 1 is indeed nonexistent.

Now, we proceed to the proof of Theorem 2, which will characterize locally compact, separable groups.

Definition 1: A function g on a uniform space G to a uniform space G' is uniformly continuous if and only if the following condition is satisfied. If V_α' is any neighborhood of the diagonal in G' x G', then there exists a neighborhood V_β of the diagonal in G x G such that for every (x, y) in V_β (g(x), g(y)) is in V_α'.

As we see from the definition in Section I of a neighborhood of the diagonal (Definition B of Section I), g is a continuous function on G to G'.

Lemma 6: Let E be a uniform space, and let f be a uniformly continuous function from E onto a complete uniform space U. Then there exists one and only one function of $\bar{f}$ from the completion of E, $\bar{E}$, onto U such that

a) $\bar{f}$ is continuous on $\bar{E}$ to U,

b) $\bar{f} = f$ on E.

Further, $\bar{f}$ is uniformly continuous on $\bar{E}$.

Proof: This is a special case of Theorem III of (8).

Lemma 7: Let the hypotheses of Proposition 1 hold. Every neighborhood of the identity in H, considered as a separable group, contains a neighborhood of the identity in the topology defined on H by Proposition 1.

Proof: For one thing, by Lemma 1 of this section, H is the union of a countable number of left-translates of any neighborhood of the identity in the original topology on H; thus, μ being left-invariant and defined on all open sets, if μ is zero on any neighborhood of identity in the original topology, it is identically zero, contrary to the definition of a measure. Further, given any neighborhood N in the original topology there is a neighborhood M such that

$$MM^{-1} \subset N,$$

and, μ being σ-finite, there is a measurable subset F of M such that $0 < \mu(F) < \infty$. As a result of all this, it follows that N contains a neighborhood of the identity in the topology defined on H by Proposition 1

$$\begin{aligned} \left\{x \mid \mu\left[(xF - F) \cup (F - xF)\right] < \epsilon,\ 0 < \epsilon < 2\mu(F) < \infty\right\} &\subset \left\{x \mid \mu(xF \cap F) > 0\right\} \\ &\subset \left\{x \mid \mu(xM \cap M) > 0\right\} \\ &\subset \left\{x \mid xM \cap M = \emptyset\right\} \\ &= MM^{-1} \\ &\subset N. \end{aligned}$$

Lemma 8: Let the hypotheses of Proposition 1 hold. Let the group uniform space of H be complete. Then there is a uniformly continuous function $\bar{f}$ from the group uniform space of the locally compact group $\bar{H}$ of Proposition 1 onto the group uniform space of H. $\bar{f}$ has the further property that the restriction of it to H is the identity mapping on H to itself.

Proof: In view of the preceding lemma, the identity mapping I on the abstract group H to itself is a uniformly continuous function on H, considered as the group uniform space of the topological group formed from it by Proposition 1, to the complete group uniform space of H in its original group structure.

Further, by Proposition 1 the group uniform space of the locally compact group $\bar{H}$ of that lemma is the completion of the former of these two uniform spaces. But now we see that the hypotheses of Lemma 6 are satisfied, so that the assertion of the present lemma follows.

Lemma 9: Let $\bar{H}$ be the locally compact topological group constructed in Proposition 1. Then there is a sequence

$$\{K_i\}_{i=1}^{n \leq \infty}$$

of compact subsets of $\bar{H}$ such that

$$\bar{H} = \bigcup_{i=1}^{\infty} K_i$$

Proof: Let K be a compact Baire subset of $\bar{H}$ such that $K = K^{-1}$ and K has positive Haar measure. Then from the continuity of the mapping

$$f: (x, y) \to xy^{-1}$$

on $\bar{H} \times \bar{H}$ to $\bar{H}$ we see that

$$\{K^n\}_{n=1}^{\infty}$$

is a sequence of compact Baire sets of positive measure, so that

$$L = \bigcup_{n=1}^{\infty} K^n$$

is a subgroup of $\bar{H}$ which is a union of compact Baire sets. Now let us use the terminology of Proposition 1, and let us observe that

$$\overline{H} = \bigcup_{\alpha \in I} a_\alpha L,$$

where

$$\{a_\alpha L\}_{\alpha \in I}$$

is a family of mutually disjoint left cosets of the Baire subgroup L. It is apparent that

$$H = H \cap \overline{H}$$
$$= \bigcup_{\alpha \in I} (H \cap a_\alpha L)$$

and that

$$\mu(H \cap a_\alpha L) = \overline{\mu}(a_\alpha L)$$
$$= \overline{\mu}(L)$$
$$> 0 .$$

But since $\mu(H)$ is positive and μ is a σ-finite measure, we know that H can contain at most countably many measurable subsets of positive μ-measure, which caused I to be an at most countable set. Thus,

$$\overline{H} = \bigcup_{i=1}^{\infty} a_i L$$
$$= \bigcup_{i=1}^{\infty} a_i (\bigcup_{j=1}^{\infty} K^j)$$
$$= \bigcup_{i=1}^{\infty} (\bigcup_{j=1}^{\infty} a_i K^j)$$
$$= \bigcup_{n=1}^{\infty} C_n,$$

where all C_n's are compact.

<u>Lemma 10</u>: Let H be a separable topological group such that its group uniform space is complete. Then H is not expressible as a countable union of nowhere dense, closed sets, i.e., H is not of the second category.

Proof: Since H is separable, there is by definition a countable collection of open sets

$$\left\{O_i\right\}_{i=1}^{\infty},$$

such that every open set in H is the union of a collection of O_i's. As a result, any family of neighborhoods of the identity is equivalent to the collection of all O_i's containing the identity.

Now if

$$H = \bigcup_{i=1}^{\infty} C_i,$$

where the C_i's are all closed and nowhere dense, then, because of these properties of the C_i's and the remarks of the preceding paragraph, the following is true. There is a sequence of elements

$$\left\{x_i\right\}_{i=1}^{\infty}$$

and a sequence of open sets

$$\left\{O_i\right\}_{i=1}^{\infty}$$

such that every O_i contains the identity, every neighborhood of the identity contains some O_i, for every i

$$x_i O_i \supset x_{(i+1)} O_{(i+1)},$$

and

$$x_i O_i \subset \left(\bigcup_{j=1}^{i} C_j\right)^c.$$

Then the family of all sets each of which contains some $x_i O_i$ as a subset is a Cauchy filter and, H being assumed a complete uniform space, must converge; so, there is an x_o in H such that x_o belongs to every $x_i O_i$.

Hence,

$$
\begin{aligned}
x_o \in \bigcap_{i=1}^{\infty} x_i O_i
&\subset \bigcap_{i=1}^{\infty} \left(\bigcup_{j=1}^{i} C_i \right)^c \\
&= \left(\bigcup_{i=1}^{\infty} \left(\bigcup_{j=1}^{i} C_i \right) \right)^c \\
&= \left(\bigcup_{i=1}^{\infty} C_i \right)^c \\
&= H^c \\
&= \phi,
\end{aligned}
$$

a contradiction. It can now be seen that it is false to assume H to be a countable union of closed, nowhere dense sets.

Proposition 4: Let H be a separable topological group such that its group uniform space is complete and such that there exists a left-translation invariant measure of the σ-ring generated by the family of all open sets. Then H is locally compact.

Proof: Let us apply Proposition 1 to H and consider the locally compact group $\overline{H}$ whose group uniform space is the completion of the group uniform space of H in the topology imposed upon it by Proposition 1. Let us obtain by Lemma 6 a sequence of compact sets in $\overline{H}$,

$$\left\{K_i\right\}_{i=1}^{\infty},$$

such that

$$\overline{H} = \bigcup_{i=1}^{\infty} K_i.$$

Finally, let $\overline{f}$ be the function of Lemma 8 which maps $\overline{H}$ continuously onto H in its original topology.

Now, since

$$\bar{f}(\bar{H}) = H,$$

$$\bigcup_{i=1}^{\infty} \bar{f}(K_i) = H.$$

But each K_i is compact, and $\bar{f}$ is continuous, so that $\bar{f}(K_i)$ is compact for every i and, a fortiori, closed. H being the union of the $\bar{f}(K_i)$'s it now is clear by Lemma 9 that at least one has interior. Thus for some i_i there is a point y and a neighborhood M of the identity such that

$$yM \subset f(K_{i_o});$$

whence

$$M \subset y^{-1} f(K_{i_o}).$$

Finally, inasmuch as there is a neighborhood N of the identity such that

$$NN^{-1} \subset M,$$

we have

$$\begin{aligned} N &\subset NN^{-1} \\ &\subset M \\ &\subset (y^{-1})\bar{f}(K_{i_o}). \end{aligned}$$

Thus the closure of N is compact, being a closed subset of the compact set $(y^{-1})\bar{f}(K_{i_o})$.

Theorem 2: Let H be a separable topological group. Then necessary and sufficient conditions that H be locally compact are that

1) the group uniform space of H be complete,

2) there be defined a σ-finite measure m on the σ-ring $\bar{S}$ generated by the open sets of H such that m(aE) = 0 if and only if m(E) = 0 for every a in H and E in $\bar{S}$.

Further, m is equivalent to the Haar measure on $\overline{S}$, which is the family of Baire sets if H is locally compact.

Proof:

I. Sufficiency: As shown in Proposition 1 of this section there exists on $\overline{S}$ a left-translation invariant σ-finite measure μ which is equivalent to m. Combining the existence of μ with the completeness and separability of H we see that the hypotheses of Proposition 4 are satisfied, making H locally compact.

Because $\overline{S}$ is clearly identical with the family of Baire sets if H is locally compact, μ, a left-translation invariant Baire measure, must be a constant multiple of the Haar measure on $\overline{S}$.

II. Necessity: By the second form of Theorem 7 of (8) every compact uniform space is complete. Since the group uniform space of a locally compact group is uniformly locally compact, it must then be complete. Hence, combining this fact with the fact that for every locally compact group G there exists a left-translation invariant, σ-finite measure on the family of Baire sets of G (identical with the σ-ring generated by its open sets), we have the necessity proof for this theorem.

The preceding theorem possesses a corollary which may be of some interest, namely:

Corollary: A (real or complex) separable Banach space B is such that there exists a left-translation non-singular measure on the σ-ring generated by its open sets, if and only if it is of finite dimension, or, if and only if there exists a linear homeomorphism of B onto a (real or complex) Euclidean space.

Proof: We observe first that if B is finite-dimensional it is linearly homeomorphic to a Euclidean space, a well-known result.

Next, if B is linearly homeomorphic to a Euclidean space, then it supports a left-translation invariant σ-finite measure on the σ-ring generated by its open sets. For if E_n is real (complex), the n-dimension ((2n) - dimensional) Borel measure is well-known to be just such a measure on the σ-ring generated by the open sets of E_n.

Finally, suppose that the σ-ring generated by the open sets of B supports a σ-finite measure such that sets of measure zero are always translated onto sets of measure zero; then B must be finite dimensional. We shall prove this in two steps: first, by showing that B is locally compact and, second, by showing that a locally compact Banach space must be of finite dimension.

To show that B is locally compact we show that its group uniform space is a complete uniform structure and then invoke Theorem 2 to conclude that it is locally compact. But we can see that the group uniform space of B is complete, since in any metric space any Cauchy filter is the same as the filter of all sets each one of which contains some member of the family of sequences $\left\{\left\{a_k\right\}_{k=n}^{\infty}\right\}_{n=1}^{\infty}$, $\left\{a_k\right\}_{k=1}^{\infty}$ being a fixed Cauchy sequence.

Now, in order to prove that a locally compact Banach space has finite dimension and thereby finish the proof of our corollary, we shall show that Banach space which is not finite-dimensional is locally compact. To do this, we construct by induction an infinite sequence $\left\{x_n\right\}_{n=1}^{\infty}$ of elements of B such that $||x_n|| = 1$ for all n and $||x_m - x_n|| > \frac{1}{2}$ whenever m is unequal to n.

Suppose that we have constructed a finite sequence $\left\{x_n\right\}_{n=1}^{k}$ with these properties, and the further properties that it is linearly independent. Then the set of all linear combinations of the x_n's is closed. Next, denoting the set of all linear combinations of the x_n's by M_k, we observe that in virtue of M_k being closed and the fact that B is not finite-dimensional there exists a non-zero element $\bar{x}$ of B such that $\|\bar{x} - M_k\|$ is not zero. So, if m belongs to M_k and is such that

$$\|\bar{x} - m\| < 2 \|\bar{x} - M_k\| ,$$

then

$$\begin{aligned} \left\| \frac{\bar{x} - m}{\|\bar{x} - m\|} - M_k \right\| &= \left(\frac{1}{\|\bar{x} - M_k\|} \right) \|(\bar{x} - m) - M_k\| \\ &= \left(\frac{1}{\|\bar{x} - m\|}\right) \|\bar{x} - M_k\| \\ &> \left(\frac{1}{2\|\bar{x} - M_k\|}\right) \|\bar{x} - M_k\| \\ &= \frac{1}{2} . \end{aligned}$$

But now, combining the definition of M_k with the above inequality, we see that if we define

$$x_{k+1} = \frac{\bar{x} - m}{\|\bar{x} - m\|}$$

then $\|x_{k+1}\| = 1$, $\|x_{k+1} - x_\ell\| > \frac{1}{2}$ for $\ell < k + 1$. Thus the existence of the sequence $\{x_n\}_{n=1}^{k}$ implies the existence of an x_{k+1} such that $\{x_n\}_{n=1}^{k+1}$ has the same properties as $\{x_n\}_{n=1}^{k}$. Applying the Principle of Finite Induction, we see that there does indeed exist a sequence of $\{x_n\}_{n=1}^{\infty}$ of elements of B such that $\|x_n\| = 1$ for all n and $\|x_m - x_n\| > \frac{1}{2}$ whenever m is unequal to n.

As remarked above, this shows that if B is locally compact it must have finite dimension, which completes the proof of this corollary.

Remark: It might be of some interest to point out the logical independence of the two provisions of the criterion for local compactness given by Theorem 2.

For, to begin with, it is clear from the above corollary to Theorem 2 that there exists a complete separable topological group which does not permit the existence of the sort of measure required by Theorem 2.

On the other hand, there exists the desired kind of measure on the σ-ring generated by the open sets of a separable topological group which is not complete. In this case, our example is found in Exercise 6 at the end of Section 62 of (1); here there is constructed a proper subgroup K of the real Euclidean plane E_2 such that E_2 - K contains no sets of positive Borel measure.

Since all open sets have positive Borel measure, K is thus a dense subgroup of E_2. Consequently, being a proper subgroup of E_2, K cannot be complete relative to the Euclidean metric and so cannot be complete considered as a uniform space; this is a consequence of the fact that in a uniform space determined by a metric every Cauchy filter is the family of all sets each of which contains a member of the family of sequences $\left\{\{a_k\}_{k=n}^{\infty}\right\}_{n=1}^{\infty}$, $\{a_k\}_{k=1}^{\infty}$ being some suitably chosen Cauchy sequence.

But K is a separable topological group supporting a left-translation invariant measure on the σ-ring generated by its open sets. For K is clearly a separable topological group in the relative Euclidean topology.

Further, since the left-translation invariant Borel measure is defined on the σ-ring $\bar{S}$ generated by the open sets of E_2, Lemma 4 of this section shows that there is a left-translation invariant measure defined on the σ-ring $K \cap \bar{S}$. However, $K \cap \bar{S}$ contains the σ-ring generated by the subsets of K which are open in the relative topology.

This verifies all the properties asserted for K and shows the independence of the completeness condition in Theorem 2.

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