The $R + \varepsilon R^2$ Cosmology

Thesis by

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In memory of my mother

Acknowledgments

This longer work, much like the apprenticed time of my life it represents, owes its fulfillment to so many people and experiences that I would be foolish here to think of them each and all. In a delightful story by Italo Calvino (in the *Difficult Loves* collection, I think), the author tracks the growing obsession of a photographer with his art. The absurd endpoint of the tale is photography of photographs. I both like this story and call it to mind for two reasons. First, I married into an extended family of shutterbugs. Aunts, cousins, parents, every one of them is compelled to photograph every possible grouping and pose of a family get-together. Martha assures me that this behavior extends to trips to Mt. Rushmore and the like. I, of course, smugly superior in my disdain for this sort of thing, feel vindicated by Calvino's comic story. ''Photography!— Bah, humbug!'' Unfortunately (when I am at my most smug), Martha reminds me that I am the first to reach for our photograph album to remember a face or scenic view — a face or scenic view that she has had to lug our camera about to record. That brings me to the second reason I recall the short story. Even after I have nobly renounced cameras, I am left too morally weak to do the same for photographs. There are always photographs I will wish I had taken and others still I will wish I could better keep from decay. Not film photographs, these others, but memory photographs:

I will want to keep a photograph of Kip — not so much to recall his bearded, tall look or his soft voice, but to recall the kind way he taught and helped me;

I will want to keep a photograph of this place — not so much to recall its sun and smog and flowers, but to recall the closeness of the Caltech community, the approachableness of the faculty, and the common dedication to learning;

I will want to keep a photograph of TAPIR — faculty, students, and staff who made Theoretical Astrophysics such a friendly group in which to learn. I will want to remember, too, TAPIR's Interaction Room, in which Wai-Mo and Milan generated the topic on which this thesis is based, and began to guide me into a fruitful collaboration;

I will want to keep a photograph of the time — not so much to record a chronology of the last five years, but to collect in one warm memory the love and wholeness that friends have brought to Martha and me in our life here;

Finally, I will want to keep a photograph of my life here, and the way Martha has enthused this love and wholeness into me.

Abstract

This thesis presents the study of a model cosmology based on the $R+\varepsilon R^2$ gravitational Lagrangian. It may be roughly divided into two distinct parts. First, the classical inflationary scenario is developed. Then, the formalism of quantum cosmology is employed to determine initial conditions for the classical model.

In the work on the classical model, the evolution equations for an isotropic and homogeneous universe are solved to exhibit both early-time inflation and a smooth transition to subsequent radiation-dominated behavior. Then perturbations on this isotropic background are evolved through the model to provide constraints on the model parameters from the observational limits on anisotropy today. This study concludes that such an inflationary model will prove a viable description for our universe if the initial Hubble parameter H_i is bounded from below, $H_i > 10^{-5}\, l_{\rm Pl}{}^{-1}$, and if $\varepsilon > 10^{11}\, l_{\rm Pl}{}^{2}$.

In the work on the wave function, the two boundary conditions of Vilenkin ("tunneling from nothing") and Hartle and Hawking ("no boundary") are compared. The wave functions obtained are restricted to the initial edge of classical Lorentzian inflationary trajectories as distributions over initial conditions for the classical inflationary model. It is found that Vilenkin's wave function prefers the universe to undergo a great deal of inflation, whereas Hartle and Hawking's wave function prefers the universe to undergo little inflation. Finally, both boundary conditions are shown to require that inhomogeneous perturbative modes start out in their ground states.

Table of Contents

CHAPTER 1

Introduction to the $R + \varepsilon R^2$ Cosmology

In recent years, a very-early-time period of exponential expansion has become a standard feature in cosmology. This initial rapid expansion, known as inflation, provides the theoretical solution to a number of problems that arise out of the standard hot Big Bang model of our universe. The inflation, in turn, is theoretically based on corrections to the laws of physics that underlie the standard model. This thesis delves into one such inflationary scenario: a scenario in which the inflationary period of expansion is driven by a pure-gravitational correction term to the standard Einstein Lagrangian, R. This correction term, denoted εR^2, is immeasurably small today. It might arise as the low-energy effective residual from some more complete quantum-gravitational theory.

The three papers that follow as succeeding chapters in this thesis explore a cosmological model based on the $R+\varepsilon R^2$ Lagrangian. Roughly speaking, Chapter 2 details the early-time evolution of the classical $R+\varepsilon R^2$ model to constrain the parameter regime in which this evolution successfully meets observational requirements, and Chapters 3 and 4 show how quantum cosmology can deliver, to the start of the classical evolution, initial conditions that meet these constraints.

In this introduction, I will discuss a handful of interlacing contexts within which I can place the following chapters. I will try to convey some necessary background to the non-specialist, explain in more detail the terms and assertions made in these opening paragraphs, introduce notational conventions, and provide a historical frame.

THE STANDARD MODEL OF OUR UNIVERSE

The standard Einstein field equations, and the standard equations of momentum and energy conservation for matter in the curved spacetime of general relativity theory, are derivable by extremizing the action

$$S=\int_V d^4x\sqrt{-g}\left[\frac{R}{16\pi G}+L_m\right]+\frac{1}{8\pi G}\int_{\partial V} d^3x\sqrt{h}K \ . \tag{1}$$

Here g is the determinant of the spacetime 4-metric, R is the scalar curvature of spacetime (the Hilbert action), L_m is the matter Lagrangian (density), h is the determinant of the spatial 3-metric on the boundary ∂V of the integration region V, K is the extrinsic curvature on the boundary, and I will employ units throughout where $\hbar=c=1$ and $G=1/l_{\rm Pl}^2$ (sign conventions will be those of Misner, Thorne, and Wheeler[1]). For a perfect cosmological fluid, the Lagrangian density is

$$L_m=-\frac{1}{2}\left[(\rho+p)u_\mu u_\nu g^{\mu\nu}+\rho-p\right] \ , \tag{2}$$

where ρ is the energy density, p is the pressure, and u_μ is the fluid 4-velocity. Variation of the action (1) with respect to the metric, $g^{\mu\nu}$, then yields the Einstein field equations with fluid source. It is this action (1), (2) and the resulting field equations that govern the *standard (Friedmann-type) models* for the cosmological structure and evolution of our universe.

That portion of our universe which lies within our cosmological horizon (i.e., is observable today) is seen to be homogeneous and isotropic on large scales. Correspondingly, the standard model is based on the homogeneous, isotropic Friedmann-Robertson-Walker line element,

$$ds^2=-dt^2+a^2(t)\left[\frac{dr^2}{1-kr^2}+r^2(d\theta^2+\sin^2\theta d\phi^2)\right] , \tag{3}$$

where $k=+1$, -1, or 0 for a closed, open, or flat universe, respectively, and $a(t)$ is the scale factor (or radius). The Hubble parameter is defined by $H(t)\equiv\dot{a}/a$, the scalar curvature is $R=12H^2+6\dot{H}+6k/a^2$, the extrinsic curvature of the boundary is $K=-3H$, and I write $\sqrt{-g}=\sqrt{h}=a^3(t)f(r,\theta,\phi;k)$, letting $M(k)\equiv(3/8\pi G)\int d^3xf(r,\theta,\phi;k)$. The action (1) may then be written

$$S=M(k)\left\{\int dt\left[-a\dot{a}^2+ka+\frac{d}{dt}(a^2\dot{a})+\frac{8\pi G}{3}L_m\right]-(a^2\dot{a})_{\text{boundary}}\right\} . \tag{4}$$

The merit of the boundary term in the original action (1) is here openly displayed — it will cancel out any variations that depend on *derivatives* of the metric on the boundary.

The Friedmann equations, resulting from variation of the action (1), are

$$\frac{\ddot{a}}{a}=-\frac{4\pi}{3}G(\rho+3p) \tag{5a}$$

and

$$\left(\frac{\dot{a}}{a}\right)^2+\frac{k}{a^2}=\frac{8\pi}{3}G\rho , \tag{5b}$$

which together imply energy conservation,

$$\frac{\dot{a}}{a}=-\frac{\dot{\rho}}{3(\rho+p)} . \tag{5c}$$

The detailed evolution of such a model will depend on the detailed behavior of the equation of state for the perfect fluid and the detailed response of the matter to various stages of thermal excitation. The complete scenario is outlined in many texts (see, for example, Chs. 27 and 28 of Misner, Thorne, and Wheeler[1], Zel'dovich and Novikov[2], or the review by Linde[3]).

Despite its tremendous phenomenological success, a number of problems with the scenario remain; see, e.g., the review by Linde. I wish to discuss only two, known as the horizon and singularity problems. For simplicity I will focus on the epoch when the age of the universe was $\lesssim$100,000 years and its energy content was dominated by radiation. In this epoch the equation of state is $p=\rho/3$ and Eq. (5c) gives

$$a^4\rho=\text{const.} \,. \tag{6}$$

Equation (5b), then, with the k term negligible (as it is for early-time, radiation-dominated evolution) is easily integrated to give

$$a(t)\propto t^{1/2} \,. \tag{7}$$

To understand the horizon problem it is illustrative to compare two time-dependent distance scales in the evolving Universe. The first of these is the physical distance between fixed coordinate locations, which is just the scale factor, $a(t)$. The second of these is the "horizon size" $\simeq 1/H$, i.e., the distance that light could have traveled since $t=0$. The ratio of the physical comoving coordinate distance to the horizon size is $aH=\dot{a}$. This is a decreasing function of time in the radiation-dominated epoch [cf. Eq. (7) or Eq. (5a) with $\rho+3p>0$] (and also in today's matter dominated epoch). That is, the light cones associated with the horizon are catching up to the physical radius. A key consequence of this is that the roughly homogeneous region, which is our present horizon volume, came from causally disjoint regions in the past.

One might expect that the evolution of causally disjoint regions into one another would tend to homogenize the spacetime. But observationally, the microwave isotropy provides direct evidence of the homogeneity at $t \sim 10^5$yr., when regions with angular separations $\theta \gtrsim$ a few degrees on our sky were not yet in causal contact with each other. The problem is: Why should one expect initial conditions to provide such homogeneity to many causally disjoint regions at once?

The singularity problem is even easier to understand. From Eq. (7) as $t \rightarrow 0$, the radius of the universe goes to zero. From Eq. (6) as $t \rightarrow 0$, the energy density diverges. That is, the evolution equations demand a singularity at the origin, near which the evolution equations themselves can no longer be valid. That the field equations predict their own early-time demise is a general result for matter sources of the type in Eq. (2) [cf. Hawking and Ellis[4] for a thorough discussion of the singularity theorems].

A possible solution to the horizon and related problems is inflation ($R + \varepsilon R^2$ being one particular kind of inflation); and a possible solution of the singularity problem is to combine inflation with a program of quantum cosmology. The fundamental-ness of these problems within the standard model demands a certain fundamental-ness to their solution. Nevertheless, the successes of the standard model request little or no change to the resulting phenomenology. It is desirable to marry any very-early-time innovations as smoothly as possible into the subsequent evolution of the standard picture.

INFLATION

The general relativity action augmented by a cosmological constant, Λ, and with no matter present is written

$$S = \int_V d^4x \sqrt{-g} \left[\frac{R}{16\pi G} - \frac{2\Lambda}{16\pi G} \right] + \frac{1}{8\pi G} \int_{\partial V} d^3x \sqrt{h} K \ . \tag{8}$$

In a Robertson-Walker spacetime, this is

$$S = M(k)\int dt\left[-a\dot{a}^2 + ka - \frac{\Lambda a^3}{3}\right] . \tag{9}$$

The field equations are

$$\frac{\ddot{a}}{a} = \frac{\Lambda}{3} \tag{10a}$$

and

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{\Lambda}{3} . \tag{10b}$$

The solution to these equations (for the term in k neglectable) is de Sitter spacetime, for which $a = a_i \exp(+\sqrt{\Lambda/3}t)$. Here $\dot{a}$, in contrast to the evolution of the standard model, is an increasing function of time, meaning that the expansion is superluminal; i.e., light cones are being caught and passed by expansion of the physical comoving distances. An early epoch of this kind of evolution followed by standard-model evolution at later times would allow a single, initially causally connected region to end up as the present horizon volume (or typically even a much larger volume). This initial period of exponential expansion has been named inflation by Guth,[5] who first recognized its importance in solving the horizon and related problems.

The short history of research on inflationary cosmology has centered on various attempts to model a $\Lambda(t)$ [a non-constant cosmological constant] that would dominate the cosmological fluid at early times (for a long enough period to achieve a sufficient number of e-foldings to solve the horizon and related problems) and yet would become zero in some natural way, allowing the standard model to take over at later times.

The first such models attempted to achieve inflation by the use of symmetry-breaking phase transitions.[6] Here, some ''inflaton'' field, initially sitting at the minimum of its potential, would be left in an excited state after the symmetry breaking had altered its potential. The false vacuum potential sustained by this excited state would then act like a cosmological constant until the field could ''roll down'' to its new minimum. In the ''old inflation'', this transition was too fast, cutting off the inflation too quickly and filling our universe with many inflated bubble regions (and monopoles and domain walls between them) that would contradict the observed homogeneity. ''New inflation''[7] solved this problem by shaping the effective potential to make the roll-down much slower. This lengthened the inflationary era and placed the whole observable universe inside one phase-transition bubble. The idea, though, was still that an inflaton field, initially in the minimum of its potential, would be given an excited field value by a symmetry-breaking phase transition. This phase transition mechanism was removed by Linde[8] in his scheme for ''chaotic inflation''. In fact, there is no reason to assume that the inflaton field need start off in the minimum of its potential. At very early times, one would expect the inflaton field values to be chaotically distributed throughout the quantum soup. Regions, then, with random field values a few times above the Planck scale would be driven to inflate sufficiently to produce our presently homogeneous horizon volume.

During inflation, because of the exponential increase of the physical comoving distances, any initial matter content will get exponentially diluted [cf. Eq. (6)]. So, at the end of an inflationary phase, one would expect the homogeneous bubble to be cold and matterless, except for the kinetic energy gained by the inflaton field in its roll-down. This kinetic energy must then rapidly decay into particles and reheat the universe to some new matter-filled state at the reheating temperature, T_r, from which subsequent radiation-dominated evolution of the now inflated homogeneous region can take over. This reheating temperature parameterizes the transition from the inflationary era to the radiation-dominated phase and is one benchmark for any inflationary scenario. In addition, in inflationary models, perturbations around the Robertson-Walker background are assumed to evolve from quantum zero-point fluctuations. The amplitude and spectrum of the perturbations delivered to the present horizon volume provide another inflationary benchmark — a benchmark constrained by observational limits on the microwave anisotropy.[9]

HIGHER DERIVATIVE GRAVITY

The model that is the focus of this thesis has its action written

$$S=\int_V d^4x\sqrt{-g}\left[\frac{R}{16\pi G}+\frac{\varepsilon R^2}{16\pi G}\right]+\frac{1}{8\pi G}\int_{\partial V} d^3x\sqrt{h}\,K(1+2\varepsilon R)\,. \tag{11}$$

The boundary term here is different from that for Einstein gravity, as is discussed in the Appendix to Chapter 3. In a Friedmann-Robertson-Walker spacetime, this action becomes

$$S=M(k)\int dt\left\{-a\dot{a}^2+ka\right.$$
$$\left.+6\varepsilon a^3\left[-\frac{\dot{a}}{a^2}\frac{d^3a}{dt^3}-2\left(\frac{\dot{a}}{a}\right)^2\left(\frac{\ddot{a}}{a}\right)+\left(\frac{\dot{a}}{a}\right)^4+\frac{2k}{a^2}\left(\frac{\dot{a}}{a}\right)^2+\frac{k}{a^2}\left(\frac{\ddot{a}}{a}\right)+\frac{k^2}{a^4}\right]\right\}, \tag{12}$$

and the field equations are

$$\frac{\ddot{a}}{a}=-6\varepsilon\left[\frac{1}{a}\frac{d^4a}{dt^4}+\frac{\dot{a}}{a^2}\frac{d^3a}{dt^3}+3\left(\frac{\dot{a}}{a}\right)^4+2\left(\frac{\ddot{a}}{a}\right)^2\right.$$

$$\left.-7\left(\frac{\dot{a}}{a}\right)^2\left(\frac{\ddot{a}}{a}\right)+2\frac{k}{a^2}\left(\frac{\dot{a}}{a}\right)^2-2\frac{k}{a^2}\left(\frac{\ddot{a}}{a}\right)-\frac{k^2}{a^4}\right] \tag{13a}$$

and

$$\frac{\ddot{a}}{a}=6\varepsilon\left[-2\frac{\dot{a}}{a^2}\frac{d^3a}{dt^3}+3\left(\frac{\dot{a}}{a}\right)^4+\left(\frac{\ddot{a}}{a}\right)^2-2\left(\frac{\dot{a}}{a}\right)^2\left(\frac{\ddot{a}}{a}\right)+2\frac{k}{a^2}\left(\frac{\dot{a}}{a}\right)^2-\frac{k^2}{a^4}\right]. \tag{13b}$$

I present the forms of these equations not for a virtuosic display of their algebraic content, but rather in contrast to the field equations for a cosmological fluid source (5a,b) and in contrast to the field equations with a cosmological constant (10a,b). My first observation is that the structure of these equations, being much more complex than the structure of the analog Equations (5,10), is correspondingly richer. (The leading derivative term on the right-hand side of Eq. (13a) is responsible for the sometime labels ''fourth-order gravity'' or ''higher derivative gravity'' attached to this model — although models involving other terms nonlinear in the curvature added to the action (11) would earn much the same titles.) Second, I assert that this structure naturally divides into two distinct evolution regimes — one regime where the ε terms are dynamically important, another where they are neglectable. This behavior, in which the order of a differential equation can be reduced in some parameter regimes is familiar, e.g., from the singular perturbation boundary layer theory of the Navier-Stokes equation. In the sense that the higher derivative terms here are important only during early-time inflation, it is pleasant to think of the inflationary stage in this model as an evolutionary boundary layer for our universe.

That this pure gravity model exhibits inflation for a wide range of parameters and then smoothly shuts its inflation down, is the subject under principal investigation in Chapter 2 below. Here, I will just note that the terms on the right-hand side of Eqs. (13a,b) mimic a slowly decaying cosmological constant [cf. Eqs. (10a,b)].

It is appropriate to append some historical context to this motivational introduction to higher derivative gravity. Curvature-squared terms have had a surprisingly long history — even going back to the 20s, when such a term in the action was suggested in the attempt at unification of electromagnetism and gravity by Weyl and by Eddington.[10] This longevity is a consequence of both a remarkable resilience of the theory and the fact that εR^2 is often the simplest amendment to Einstein gravity and therefore might be easily posited to solve any number of problems that arise in general relativity theory.

After the work in the 20s, Pais and Uhlenbeck studied higher derivative gravity in quantum field theory in 1950.[11] In the 1960s, it was hoped that higher derivative gravity terms could help renormalize divergences from matter terms.[12] Then, in the 70s it was shown that the addition of higher derivative terms could make gravity itself renormalizable.[13] (A debate still rages as to whether or not higher derivative gravity can be unitary.[14]) Also in the late 60s and early 70s there were hopes that the richness in the higher derivative field Equations (13a,b) might get around the singularity problem.[15] Finally, in the 80s, alongside the development of inflation, it has been realized that the higher derivative terms drive a de Sitter-like expansion at early times.[16]

The classical evolution detailed in Chapter 2 below treats $R + \varepsilon R^2$ gravity as a full inflationary model. Its inflation mechanism is described, its reheating temperature is calculated, the join to Friedmann behavior is displayed, and the evolution of perturbations in the model is used to constrain the model parameters. Similar investigations of the inflation from higher derivative gravity have appeared at the same time as this work and have approached the classical model from differing points of view.[17]

QUANTUM COSMOLOGY

I return to the field Equation (10b) for a spacetime with cosmological constant,

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{\Lambda}{3} . \tag{10b}$$

For the $R + \varepsilon R^2$ Lagrangian, the early-time evolution equation can be written in this form. A model based on this Lagrangian possesses an effective cosmological constant at early times. For a closed Universe ($k=+1$), this equation (10b) predicts that the expansion factor a (''radius of the Universe'') can never be less than $a_{min} = \sqrt{3/\Lambda}$ (this is discussed in more detail in Chapter 3, Section III). More specifically, for $a > a_{min}$, $\dot{a}^2$ will be positive and there will be a Lorentzian-signatured metric of the Friedmann-Robertson-Walker form (3), describing a classical evolutionary trajectory. If $a < a_{min}$, then $\dot{a}^2$ is negative. This can be interpreted as corresponding to a metric with Euclidean time signature ($t_E = it$) along any solution to the equation of motion (10b). The presence of a classical turning point at a_{min} (near which any semiclassical description would presumably break down) naturally invites the notion of a quantum amplitude connecting semiclassical domains inside and outside the barrier. Then each classical trajectory outside the barrier can be thought of as the path of a Lorentzian Universe spontaneously born near a_{min}. I should stress that, depending on the potential generating the effective Λ in Eq. (10b), this a_{min} can be far from Planck-scale, bounding the whole quantum cosmological scheme away from a parameter regime in which one would not be likely to trust the model.

It has been the hope of enlisting an inflationary model into a realistic quantum cosmological program that has spurred attempts to obtain the wave function describing our universe (or the ensemble of possible universes from which ours was born). Canonical quantization yields a partial differential wave equation, second order in each degree of freedom present in the model (the Wheeler-DeWitt equation in superspace). This canonical quantization procedure is fraught with unresolved controversy (operator-ordering problems, non-renormalizability, the unsoundness of not beginning with a fundamental theory). But even after these issues have been brushed aside (in the hope that a zeroth-order approach will deliver some results that will survive refinement), there remains the fundamental question of what boundary condition to choose for each degree of freedom present in the wave function. There are two definite proposals for this boundary condition. The first is the proposal by Hartle and Hawking,[18] that the boundary condition is that there be ''no boundary'' (mathematically this is expressed as the condition that the wave function be given by the Euclidean path integral over all compact 4-geometries and regular matter-field configurations that have a specified 3-geometry and field strength on a fixed 3-surface). The second proposal is the one made by Vilenkin,[19] that the wave function should correspond to a tunneling amplitude from ''nothing'' (mathematically he requests only outgoing modes at the singular boundaries of superspace and regularity at zero radius). The comparison and contrast of these two proposals (for the homogeneous $R+\varepsilon R^2$ model, which possesses an effective cosmological constant at early times) is the project of Chapter 3. The treatment there always tries to stay physically simple and interpretive. The main conclusion can be summed up as follows: The two boundary conditions differ only in how they treat the expansion degree of freedom — all other degrees of freedom are required to be regular at the origin by both boundary conditions. In terms of the resulting wave function in the expansion degree of freedom, restricted to the initial edge of classical Lorentzian inflationary trajectories, the Hartle-Hawking boundary condition prefers universes to be born large and spend only a short time in inflation, whereas the Vilenkin boundary condition prefers universes to be born small and undergo a large amount of inflation.

In Chapter 4 this comparison and contrast of the boundary conditions is carried over into (the infinite-dimensional) perturbative superspace. The inhomogeneous scalar and tensor modes are added perturbatively to the wave function. For these modes it is shown that both boundary conditions require the same conclusion from the wave function — the inhomogeneous modes must start out in their ground states. This quantum prediction of the initial conditions for perturbations then ties back into the ground-state assumption as the starting point for the perturbation analysis of the classical evolution in Chapter 2.

SUMMARY OF CHAPTERS 2, 3, AND 4

Chapter 2 is a paper written by Milan Mijić, Wai-Mo Suen, and me that originally appeared as "The R^2 cosmology: Inflation without a phase transition" in Phys. Rev. D **34**, 2934 (1986). In Section I, we provide an overall introduction to the classical $R+\epsilon R^2$ Lagrangian and field equations and we introduce Whitt's "conformal picture"[20] — a pretty way to display the theory as Einstein gravity plus a scalar field that will be exploited many times throughout this work. In Section II, we investigate the classical Friedmann-Robertson-Walker evolution of the model. In particular, we display its "linear phase" inflationary behavior and summarize the results from a numerical investigation of the parameter space for its initial conditions. In Section III, we obtain the reheating behavior for the model and present the join of the inflationary phase to subsequent standard-model evolution. In Section IV, we evolve gravitational-wave perturbations on the background of this inflationary and postinflationary scenario to determine the strength and spectrum of cosmological gravitational radiation today in terms of the model parameters. In Section V, we do the same for scalar perturbations, to find that this produces our most stringent constraints on ε and the initial Hubble parameter, H_i. In Section VI, we discuss possible theoretical origins of the εR^2 term and in the concluding Section VII, we find that classical $R+\varepsilon R^2$ scenario is a viable inflationary model for $\varepsilon > 10^{11}\, l_{\rm Pl}{}^2$ and for $H_i > 10^{-5}\, l_{\rm Pl}{}^{-1}$.

Chapter 3 is a paper being submitted at the time of this thesis. Its title for Physical Review D will be ''Initial conditions for the $R+\varepsilon R^2$ cosmology'' [Caltech Goldenrod Preprint, GRP160, 1988], and it is written by the same three authors as in Chapter 2. In Section I of Chapter 3, we present a general historical introduction to the wave function, the boundary conditions, and the methods we will use. In Section II, we summarize the classical behavior from Chapter 2 and derive the Wheeler-DeWitt equation for the homogeneous degrees of freedom (minisuperspace). In Section III, we discuss spontaneous birth of a Lorentzian Universe and introduce the two boundary condition proposals. Then, in Section IV, we solve the Wheeler-DeWitt equation in minisuperspace in the semiclassical approximation (and in the strongly inflationary regime) to obtain wave functions for both boundary conditions. In Section V, we compare predictions from the two resulting wave functions and find that Vilenkin's boundary condition[19] prefers universes that inflate a great deal, and Hartle and Hawking's boundary condition[18] prefers universes that undergo little inflation. We finally discuss the boundary term for $R+\varepsilon R^2$ gravity in an appendix.

In Chapter 4, I then move on to the wave function for perturbations in the $R+\varepsilon R^2$ model. I am submitting this paper to Physical Review D at the time of this thesis as

"Initial conditions for perturbations in the $R+\varepsilon R^2$ cosmology" [Caltech Goldenrod Preprint, GRP172, 1988]. In Section I, again, I present an introduction. In Section II, I derive the Wheeler-DeWitt equation for the perturbative superspace in all the inhomogeneous modes. In Section III, I perturbatively solve this equation, verifying that, for both boundary condition proposals, all inhomogeneous scalar- and tensor-mode perturbations begin in their ground states.

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CHAPTER 2

The Classical $R + \varepsilon R^2$ Cosmology

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ABSTRACT

A pure gravity inflationary model for the universe is examined, which is based on adding an εR^2 term to the usual gravitational Lagrangian. The classical evolution is worked out, including eventual particle production and the subsequent join to radiation-dominated Friedmann behavior. We show that this model gives significant inflation essentially independent of initial conditions. The model has only one free parameter, which is bounded from above by observational constraints on scalar and tensorial perturbations and from below by both the need for standard baryogenesis and the need for galaxy formation. This requires 10^{11} GeV $< \varepsilon^{-1/2} < 10^{13}$ GeV.

I. INTRODUCTION

The inflationary universe model,[1,2] in which the universe has undergone a long period of exponential expansion, has successfully explained many problems in the standard Friedmann cosmology. A particularly attractive feature is that the model provides a mechanism to generate the small-scale density fluctuations in the universe that are needed as seed for galaxy formation.[3,4] They are the zero-point fluctuations of the quantum fields which get pushed into the classical regime by the large expansion.

In the standard picture of inflation this exponential expansion of the universe is driven by the false vacuum energy density of a Higgs field, which acts like an effective cosmological constant in the Einstein equations. Many different underlying particle physics theories have been proposed. The most popular of these are the Coleman-Weinberg model,[5] Witten's model with a logarithmic potential,[6] and the N=1 supergravity version of Nanopoulos et al. and Linde.[7]

These proposals, though, are not without their problems. First, one has to typically introduce a scalar ''inflaton'' field, which is postulated especially for the purpose. This makes the whole scenario less plausible in that it is less natural. Second, to achieve a large enough inflation, suitable reheating after the inflation, and to make the material fluctuations small enough to be consistent with observation, relevant couplings or masses in the suggested models all have to be fine-tuned in one way or another. An even more serious problem has been pointed out by Mazenko, Unruh, and Wald.[8] A quantum field that is violently fluctuating in its high temperature symmetric state may not settle into the false vacuum state as the universe cools. This then may invalidate the whole picture of vacuum-energy-driven inflation. Although the problem might be circumvented again by fine-tuning the parameters involved,[9] it is reasonable to assert that the idea of inflation is very attractive, whereas the "standard" models that generate the inflationary phase by a false vacuum energy density are less satisfying.

Is it possible to inflate the universe by a different mechanism? Linde[10] has proposed in his chaotic inflation scenario that the inflation may be a direct result of large fluctuations of quantum fields in the very hot primordial universe. In the Planck regime, a scalar field ϕ will tend to be excited to large values so that its energy density inside some domain will be of order Planck. If ϕ has a very flat potential, i.e., a small "restoring force", it will remain roughly at the fluctuated value for a comparatively long time and hence will drive an essentially exponential expansion. Linde has shown that in a $\lambda\phi^4$ theory, there will be a classically tractable sufficient inflation when $\lambda < 10^{-2}$ (for more details see Linde[2]). However, two new questions immediately appear, which a cosmology based on chaotic inflation must answer: What is the underlying particle model and what determines the initial fluctuations? Without these, one has neither a complete nor a realistic model of chaotic inflation. This is one thread leading to the present work.

A second thread leads from the fact that within different frameworks one is repeatedly led to consider an action containing terms of quadratic or higher order in the curvature tensor. We will discuss this point more fully in Section 6. It is nonetheless important to understand the implication of these higher derivative terms on the evolution of the early universe. In this work we will restrict our attention to terms that are quadratic. They can be written $\alpha R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}+\beta R_{\mu\nu}R^{\mu\nu}+\gamma R^2=\varepsilon R^2+\zeta C_{\mu\nu\lambda\sigma}C^{\mu\nu\lambda\sigma}$ $+\eta\chi_E$ (where χ_E is the density of the Euler number for the manifold and C is the Weyl tensor). When we consider a Robertson-Walker metric (homogeneous and isotropic universe),[11]

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{1.1}$$

(here $\kappa = +1, -1$, or 0 — although, unless otherwise indicated, we will be studying the case $\kappa = 0$). This metric is conformally flat so that the C^2 term vanishes. The effective gravitational Lagrangian density yielding the evolution of the universe is then given by

$$L = R + \varepsilon R^2. \tag{1.2}$$

The evolution equation for R determined by (1.2) can be written as

$$\ddot{R} + 3H\dot{R} + \frac{1}{6\varepsilon}R = 0, \tag{1.3}$$

where the dot denotes a coordinate time derivative ($= d/dt$) and H is the Hubble parameter ($H = \dot{a}/a$).

Thus, R behaves like a damped harmonic oscillator with the restoring force given by $1/6\varepsilon$. If ε is large, the potential is flat and R takes up the role of the inflation-driving field. The aim of this paper is to study the cosmology based on this model. We show the range of initial data and the allowed value of ε so that inflation can be realized in this curvature-squared model in a manner consistent with observational constraints. We consider now the generic evolution of the universe to be divided into four regimes: (i) There may be a quantum phase in which the universe begins its Lorentzian life — as described in the wave function picture[15] — with some expectation values for the initial conditions, but continues with strong fluctuations for some time. The classical evolution becomes meaningful only after fluctuations around the average trajectory have become small. Whether this subsequent classical evolution is applicable to the universe as a whole or just to an homogeneous "bubble" part of it (as in Linde's chaotic inflation picture) we expect to be answered by a proper quantum treatment at very early times; (ii) At the start of the classical evolution there will quite generally be an inflationary phase of superluminal expansion in which the Hubble parameter decays linearly in time with small slope; (iii) When the Hubble parameter hits zero and bounces back the universe goes into an oscillation phase in which it is reheated as material fields are excited by the oscillating geometry; and (iv) There will be a final Friedmann phase in which our now matter-content-dominated model is joined to standard cosmology. We will exhibit and explain the inflationary solution, and will discuss reheating of the Friedmann universe and the generation and evolution of scalar and tensor perturbations. These considerations all place constraints on the parameters of the model.

The effect of higher derivative terms on the evolution of the early universe has been studied by many authors. Zeldovich and Pitaevskii[12] have discussed the possibility of avoiding the initial singularity by including the higher-order term. Starobinsky[13] has shown that the quantum corrections for a conformally invariant free field will modify the Einstein equations with higher order terms such that an unstable de Sitter solution will result. Whitt[14] points out that the evolution equation for an $R + \varepsilon R^2$ Lagrangian admits primordial inflation. Hawking and Luttrell[15] have also shown that the wave function of the universe for this Lagrangian is peaked about classical trajectories that exhibit an exponential expansion. In fact, the initial motivation for our work comes from the desire to understand and investigate in detail the inflationary phase displayed in the numerical solution of Hawking and Luttrell's wave function.

Parallel to conducting our discussion directly in the physical spacetime, we will make use of the fact that this theory can be rewritten as pure Einstein gravity plus matter in a conformal spacetime. Whitt[14] has shown that by a transformation, $\tilde{g}_{\mu\nu} = (1+2\varepsilon R)g_{\mu\nu}$, we can discuss the theory as Einstein gravity described by $\tilde{g}_{\mu\nu}$ plus a scalar field, R (which is the scalar curvature in the physical space), with minimal coupling to gravity by means of the equation

$$\tilde{R}_{\tilde{\mu}\tilde{\nu}} - \frac{1}{2}\tilde{g}_{\tilde{\mu}\tilde{\nu}}\tilde{R} = 8\pi G \tilde{T}_{\tilde{\mu}\tilde{\nu}}(R), \tag{1.4a}$$

where

$$\tilde{T}_{\tilde{\mu}\tilde{\nu}} = \frac{6\varepsilon^2}{8\pi G(1+2\varepsilon R)^2}\left[\partial_{\tilde{\mu}}R\,\partial_{\tilde{\nu}}R - \tilde{g}_{\tilde{\mu}\tilde{\nu}}\left(\frac{1}{2}\partial^{\tilde{\sigma}}R\,\partial_{\tilde{\sigma}}R + \frac{R^2}{12\varepsilon}\right)\right]. \tag{1.4b}$$

Here, the scalar field, R, can be given an action

$$S[R] = \int d^4x\; 6\varepsilon^2\sqrt{-\tilde{g}}\;[1+2\varepsilon R]^{-2}\left[\partial_{\tilde{\sigma}}R\,\partial^{\tilde{\sigma}}R + \frac{R^2}{6\varepsilon}\right]. \tag{1.5}$$

In this conformal picture — as we are working with standard Einstein gravity — we already have some known tools that provide for us both insight and a good check on the less familiar behavior of the full fourth-order model. We will appreciate its full power in evaluating scalar and tensor perturbations.

In Section 2, we consider the classical evolution of a flat ($\kappa = 0$) Robertson-Walker universe under the influence of an R^2 term in the effective Lagrangian. In Section 3, we then treat in greater detail the exit from the inflationary phase, the reheating of the universe, and the subsequent join to Friedmann behavior. Next, in Sections 4 and 5, we estimate the generation of gravitational wave and scalar perturbations in the model. In Section 6, we display some present constraints on, and possible origins for, ε. Finally, conclusions are presented in Section 7.

Throughout this work we use units in which $\hbar = c = k_B = 1$. We measure all quantities in Planck units so that the gravitational constant, G, is equal to $1\ l_{Pl}{}^2$ (where l_{Pl} denotes the Planck length).

II. CLASSICAL EVOLUTION

We begin discussion of the universe and its evolution at the time when it emerges from the Planck era. The universe would then be filled with relativistic particles of violently fluctuating energy density and its spacetime geometry, too, would be violently fluctuating. However, a region not too big compared to the Planck size could be approximately isotropic and homogeneous and could then be described by the Robertson-Walker metric (1.1). For simplicity, we consider only the case $\kappa = 0$. We follow the evolution of this small region with the classical equations of motion derived from the Lagrangian density (1.2).

It is straightforward to write down the field equation for the effective gravitational Lagrangian density (1.2) with a cosmological constant term and matter field terms added:[14,15]

$$R_{\mu\nu} - 1/2 g_{\mu\nu} R + \Lambda g_{\mu\nu} + 2\varepsilon[R(R_{\mu\nu} - 1/4 g_{\mu\nu} R) + R_{;\kappa\lambda}(g^{\kappa\lambda} g_{\mu\nu} - \delta^{\kappa}{}_{\mu}\delta^{\lambda}{}_{\nu})] \tag{2.1}$$

$$= 8\pi G T_{\mu\nu}.$$

For the most part in this paper, we will set $\Lambda = 0$ (except briefly in Section 6) and we will always use a perfect cosmological fluid expression for $T_{\mu\nu}$,

$$T_{\mu\nu} = (\rho + p) u_{\mu} u_{\nu} + p g_{\mu\nu}, \tag{2.2}$$

where $p = \rho/3$ (a relativistic equation of state) and $\vec{u} = \partial/\partial t$ (comoving 4-velocity). It is simple to verify that the left-hand side of (2.2) is divergence-free so that energy-momentum conservation is still given by

$$T^{\mu\nu}{}_{;\nu}=0, \tag{2.3a}$$

which implies

$$\rho \sim \frac{1}{a^4}, \tag{2.3b}$$

as in the standard Einstein cosmology.

There are only two nonvacuous field equations. The $t-t$ component of (2.1) can be written as

$$\dot{R}=\frac{1}{12}\frac{R^2}{H}-RH-\frac{H}{2\varepsilon}+\frac{4\pi}{3}\frac{G}{\varepsilon}\frac{\rho}{H}, \tag{2.4}$$

and the contraction of (2.1) gives

$$\ddot{R}+3H\dot{R}+\frac{1}{6\varepsilon}R=0. \tag{2.5}$$

The relations of R and H to the scale factor $a(t)$ are given by

$$R=6\dot{H}+12H^2 \tag{2.6}$$

and

$$H=\dot{a}/a. \tag{2.7}$$

Equations (2.4), (2.5), (2.6), and (2.7) are then a complete set for describing the classical evolution of the universe.

Next, we notice that with ρ given by (2.3b), Eq. (2.4) is the first integral of (2.5). Therefore, the system we have left is equivalent to a third-order differential equation in the scale factor $a(t)$. We set the time coordinate origin so that our analysis begins at t = 0, which is the time the classical evolution begins to make sense. A complete set of initial conditions for the system is then given by ρ_i, a_i, H_i, and R_i (the subscript i will be used to denote quantities at t = 0). We first assume for simplicity the matter term on the right-hand side of Eq. (2.4) to be negligible (that is, $\rho_i \approx 0$) — we shall insert its contribution at a later point. Now the initial size, a_i, of the small homogeneous domain does not enter the dynamical equations and it relates coordinate length to physically measured length at $t = 0$ (the equation for $a(t)$ is trivially integrated in terms of $H(t)$). We will take ε to be a free parameter, since before appeal to a higher theory it can be regarded as a new fundamental constant subject to experimental verification. So, one way to phrase the question that this paper addresses is: What are the allowed ranges of ε and the initial data, H_i and R_i, so that the non-Einstein term will produce a sensible inflation, give sufficient expansion to solve the horizon and flatness problems, command an exit from the inflationary phase, yield a reheating temperature high enough not to thwart standard baryogenesis but low enough to avoid the GUT phase transition and its associated monopole problem, and finally deliver the correct material and gravitational perturbation spectrum and magnitude?

We study first the classical evolution by means of Equations (2.4)-(2.7). To ensure the classical validity of the evolution we will think of H_i and R_i to be both less than or of order the Planck scale. We may combine Equations (2.4) and (2.6) to derive a master equation for the classical evolution with zero matter content:

$$\ddot{H} - \frac{1}{2}\frac{1}{H}\dot{H}^2 + 3H\dot{H} + \frac{1}{12\varepsilon}H = 0. \tag{2.8}$$

The remaining dependence on the parameters H_i, R_i, and ε can then be discussed as follows:

(A) $\varepsilon > 0, R_i > 0$, and $H_i > 0$

We will show that this is the only case that will be of interest, so that we will consider it in detail:

(i) First, we look at the case where R starts at roughly its maximum value; that is, $\dot{R}(t=0) = 0$. Then Eq. (2.4) relates R_i and H_i by

$$R_i = 6H_i^2\left[1+\sqrt{1+\frac{1}{6\varepsilon H_i^2}}\right]. \tag{2.9}$$

The typical behavior of $H(t)$ for this case is shown in Fig. 1. There is a long phase in which H decreases linearly in time with a small slope. This slope may be estimated from Eq. (2.8). For $\varepsilon \geq 1$ and $H \geq 1/6\sqrt{6\varepsilon}$, we have

$$\dot{H} \approx -\frac{1}{36\varepsilon}. \tag{2.10}$$

Hence, the total expansion in the scale factor of the universe after this linear near-de Sitter phase is given by

$$a(t_{H\approx 0}) = a_i\, e^{18\varepsilon H_i^2}. \tag{2.11}$$

To obtain a cosmologically significant expansion — say a factor of e^{75} (cf. Linde[2]) — we see that we need only to have $\varepsilon H_i^2 \geq 4.2$, a perfectly natural value in our picture. This is explicitly the sought-for inflation in the model. When H finally gets small, as shown in Fig. 1, it switches from the linearly decaying phase into a damped oscillation. This oscillation will be seen to reheat the universe.

(ii) What if $R_i \gg 6H_i^2(1+\sqrt{1+(1/6\varepsilon H_i^2)})$? From Equations (2.4) and (2.6) it is clear that both R and H will increase rapidly:

$$R \approx \frac{1}{12}\frac{R_i^2}{H_i}t \tag{2.12a}$$

and

$$H \approx \frac{1}{6}\int R dt \approx \frac{1}{144}\frac{R_i^2}{H_i}t^2. \tag{2.12b}$$

Therefore, $12H^2$ will catch up with R at t_m:

$$t_m \approx 5.2\left(\frac{H_i}{R_i^2}\right)^{\frac{1}{3}} \tag{2.13}$$

and

$$H_m \equiv H(t_m) \approx 0.2\left(\frac{R_i^2}{H_i}\right)^{\frac{1}{3}}. \tag{2.14}$$

Then by Eq. (2.6), $\dot{H}$ will change sign and then go into the linear decaying phase of the previous case (i). The total expansion accumulated during the initial rapidly rising period is negligible:

$$\int H dt \approx \frac{1}{400}.$$

We can thus perfectly well regard $H(t_m)$ and $R(t_m)$ as the initial values from which the linear phase begins.

(iii) If $R_i \ll 6H_i^2(1+\sqrt{1+(1/6\varepsilon H_i^2)})$, then

$$\dot{R} \approx -\frac{H}{2\varepsilon} \tag{2.15a}$$

and

$$\dot{H} \approx -2H^2. \tag{2.15b}$$

Both H and R will fall rapidly. For a typical value of H, there will not be sufficient inflation before it bounces at zero. The universe will go into the oscillation phase without having been inflated.

(B) $H_i < 0$

From Eq. (2.8) we can see that as $H \to 0$, $\dot{H}$ must also go to zero so that $\dot{H}^2/H$ is finite. Therefore, $\ddot{H}$ is negative if H approaches zero on the negative side. Thus, when H hits zero it will bounce back and remain negative (on the other hand, a positive H will remain positive for the same reason). For the case $H_i < 0$ the domain in consideration will always be contracting until it collapses back to the Planck regime.

(C) $R_i < 0, H_i > 0$

From Eq. (2.6), H will be decreasing rapidly as long as R is negative. Since H has to remain positive as argued in case (B), R will have to cross zero and become positive. Again, typically the total expansion in the initial period will be negligible and we arrive back at case (A).

(D) $\varepsilon < 0$

From Eq. (2.5), we see that when ε is negative we have an antirestoring force. Indeed, it is easy to see that when H_i is positive, the solution will go into a linearly increasing form asymptotic to a slope

$$\dot{H}=-\frac{1}{36\varepsilon}>0,$$

which is physically unacceptable. When H_i is negative, $H(t)$ will be decreasing and will not be interesting as described under case (B).

We conclude that (i) ε has to be positive to give a finite period of inflation (note that tachyonic solutions would also exist if ε were negative[16]). (ii) To study the inflation, we have only to study the case with positive H_i. The inflation occurs during a period when H decreases linearly with a slope $-1/36\varepsilon$. The total expansion factor in this phase is given by Eq. (2.11) (with H_i replaced by H_m in the case of (Aii) or (Aiii)). (iii) The linearly decaying $H(t)$ will bounce into an oscillation phase when it approaches zero. These descriptions of the evolution have been verified numerically.

Now we return to consider the contribution of the material term which we neglected in Eq. (2.4). By Eq. (2.3b), the energy density ρ of the relativistic particles evolves inversely proportional to a^4. It is then clear that once the inflationary era begins, ρ will be quickly red-shifted away. Thus, by Eq. (2.4), the effect of ρ on the evolution is just to give R an initial kick. That is, if ρ_i is large while H_i and R_i are of order 1, then R will quickly rise to

$$\left[\frac{16\pi}{\varepsilon}\rho_i\right]^{\frac{1}{2}}$$

in a short time. The subsequent evolution is then given by case (Aii).

It is nice to see the inflationary solution also by considering the conformal picture. In the conformal picture, the classical background consists of gravity described by a scale factor $\tilde{a}(\tilde{t})$ and a spatially homogeneous scalar field $R(\tilde{t})$. They evolve according to

$$\frac{d^2R}{d\tilde{t}^{\,2}} - \frac{2\varepsilon}{1+2\varepsilon R}\left[\frac{dR}{d\tilde{t}}\right]^2 + 3\tilde{H}\frac{dR}{d\tilde{t}} + \frac{R(\tilde{t})}{6\varepsilon(1+2\varepsilon R)} = 0 \tag{2.16}$$

and

$$\tilde{H}^2 = \frac{\varepsilon^2}{(1+2\varepsilon R)^2}\left[(\frac{dR}{d\tilde{t}})^2 + \frac{R^2}{6\varepsilon}\right], \tag{2.17}$$

where $d\tilde{t} = (1+2\varepsilon R)^{1/2}dt$. It is easy to see that there is a consistent solution for $\varepsilon R \gg 1$:

$$\tilde{H} = \frac{1}{2\sqrt{6\varepsilon}} \tag{2.18}$$

and

$$R(\tilde{t}) = R_i - \frac{\tilde{t}}{3\varepsilon\sqrt{6\varepsilon}}. \tag{2.19}$$

Transforming back, we find a linearly decreasing Hubble parameter as discussed above. The fact that in the conformal picture one has a solution as nice as de Sitter makes the prospect for further analysis very promising.

From now on we consider only the case (A) above, since the other cases either lead back to it or are uninteresting, and we will refer to the inflated region as ''the universe''. In the linear phase, we have by comparing terms in Eq. (2.8)

$$|\frac{1}{2}\frac{1}{H}\dot{H}^2| \ll |3H\dot{H}|. \tag{2.20}$$

As H decreases and becomes small, the inequality sign will eventually flip and we will go over to the oscillatory phase. Equation (2.8) then becomes

$$\ddot{H}-\frac{1}{2}\frac{1}{H}\dot{H}^2+\frac{1}{12\varepsilon}H=-3H\dot{H}\approx 0. \tag{2.21}$$

If one neglects the $3H\dot{H}$ term in Eq. (2.21), the solution is given easily by

$$H(t)=\text{Const.}\times\cos^2\omega t, \tag{2.22}$$

where

$$\omega\equiv\frac{1}{\sqrt{24\varepsilon}}.$$

To do better in approximation and in particular to obtain the damping for the amplitude, we have to include the presently neglected term. We do this by substituting a form for $H(t)$, which is $H=f(t)\cos^2\omega t$ and then finding $f(t)$ in the approximation that the damping is slow $\dot{f}^2/f\approx 0$, $f\dot{f}\approx 0$. The initial value of f is determined by matching on to the linear phase — that is, requiring the two terms in (2.20) to be equal at $t=t_{os}$, the time the oscillation phase begins. When this has been accomplished, we determine the following approximate analytic form for the whole classical evolution of the universe in the absence of matter fields:

$$H(t)\approx\begin{cases} H_m-\dfrac{1}{36\varepsilon}(t-t_m) & t_m<t\leq t_{os} \\ \left(\dfrac{3}{\omega}+\dfrac{3}{4}(t-t_{os})+\dfrac{3}{8\omega}\sin 2\omega(t-t_{os})\right)^{-1}\cos^2\omega(t-t_{os}) & t_{os}\leq t, \end{cases} \tag{2.23}$$

where $\dot{R}(t_m)=0$, $\omega\equiv(1/\sqrt{24\varepsilon})$, and $t_{os}=36\varepsilon H_m+t_m-(1/(2\omega))\approx 36\varepsilon H_m$. A simple approximate solution for $a(t)$ in the oscillatory phase can be obtained by integrating the H averaged over a few cycles:

$$a(t) \approx \begin{cases} a_m e^{H_m(t-t_m)+\frac{t_m}{72\varepsilon}(2t-t_m)-\frac{t^2}{72\varepsilon}} & t_m < t \le t_{os} \\ a_{os}\left[1+\frac{\omega(t-t_{os})}{4}\right]^{2/3} & t_{os} \le t\,, \end{cases} \tag{2.24}$$

where $a_{os} \equiv a_m \exp(18\varepsilon H_m{}^2 - 1/12)$.

In the oscillation phase, R is essentially $6\dot{H}$ (cf. Eq. (2.6)) so that we have

$$R(t) \approx \begin{cases} 6\left[2H_m{}^2 - 1/36\varepsilon - \frac{H_m}{9\varepsilon}(t-t_m) + \frac{2}{36\varepsilon^2}(t-t_m)^2\right] & t_m < t < t_{os} \\ -6\left[\frac{3}{\omega}+\frac{3}{4}(t-t_{os})+\frac{3}{8\omega}\sin 2\omega(t-t_{os})\right]^{-1} \omega \sin 2\omega(t-t_{os}) & t_{os} < t. \end{cases} \tag{2.25}$$

Notice that $a(t)$ and $H(t)$ are matched at $t = t_{os}$ whereas $R(t)$ is not — otherwise, we would have had an exact solution. It is important that the oscillatory phase depends only on the parameter ε for size and shape — the oscillatory solution has no dependence on the initial conditions except in the time the phase begins (at $t_{os} \approx 36\varepsilon H_m$). Eq. (2.24) shows that the scale factor expands like a matter-dominated universe: $a(t) \propto t^{2/3}$ — as in the postinflationary phase of the Starobinsky model,[13] where it is known as the ''scalaron'' phase.

III. REHEATING OF THE UNIVERSE

These oscillations will excite the material fields and reheat the universe. To estimate the reheating, we consider the simple case of a scalar field ϕ satisfying

$$g^{\mu\nu}\phi_{;\mu\nu} = 0\,. \tag{3.1}$$

The energy density of the scalar particles produced can be easily determined. Let

$$\phi=\int d^3k\,(\hat{a}_k u_k+\hat{a}_k^{\,+}u_k^{\,*}) \tag{3.2a}$$

and

$$u_k(x,t)=\frac{1}{(2\pi)^{3/2}}\frac{1}{a}\chi_k(t)e^{ikx}, \tag{3.2b}$$

where $\hat{a}_k$ and $\hat{a}_k^{\,+}$ are the usual annihilation and creation operators. In terms of the conformal time $\eta\equiv\int_0^t a^{-1}dt$, χ_k satisfies[17]

$$\frac{d^2\chi_k}{d\eta^2}+k^2\chi_k=V\chi_k, \tag{3.3a}$$

where

$$V\equiv\frac{1}{6}a^2R. \tag{3.3b}$$

As we shall see, the typical wavenumber k that enters our calculation is much bigger than one, whereas V is of order one at early times ($\eta\sim 0$). Therefore, the wave is essentially living on a flat background at early times, and the positive frequency mode is then given by

$$\chi_k^{(i)}\simeq\left[\frac{1}{\sqrt{2k}}e^{-ik\eta}\right]. \tag{3.4}$$

Now we follow Zeldovich and Starobinsky[18] and rewrite (3.3) as an integral equation:

$$\chi_k(\eta)=\chi_k^{(i)}+\frac{1}{k}\int_0^\eta V(\eta')\sin(k\eta-k\eta')\chi_k(\eta')d\eta'. \tag{3.5}$$

For a first-order iteration, we substitute $\chi_k^{(i)}$ in the integrand of (3.5) for $\chi_k(\eta')$. At asymptotically late times the universe will be flat again and the positive frequency mode function is again given by (3.4). Hence, the Bogoliubov coefficient describing the particle production is given by

$$\beta_{kk'}=\delta_{kk'}\frac{-i}{2k}\int_0^\infty V(\eta')e^{-2ik\eta'}d\eta'. \tag{3.6}$$

And the coordinate energy density $\vec{p}\cdot(\partial/\partial\eta)$ (where $p \equiv$ momentum per unit comoving volume) is given by

$$p_\eta=\frac{\pi}{(2\pi)^3}\int_0^\infty d\eta\int_0^\infty d\eta' V(\eta)V(\eta')\int_0^\infty dk\left[ke^{2ik(\eta'-\eta)}\right]. \tag{3.7}$$

Note that prior to the inflation $V=(1/6)a^2R$ is many orders of magnitude less than its value during the oscillating phase. Also V becomes small after the universe goes into the radiation-dominated Friedmann phase (cf. Eq. (3.17) below). Thus, we can drop the surface terms in evaluating (3.7) and arrive at

$$p_\eta=\frac{1}{8}\frac{1}{(2\pi)^2}\int_0^\infty d\eta\frac{dV}{d\eta}\int_0^\infty d\eta'\frac{V(\eta')}{\eta'-\eta}. \tag{3.8}$$

We restrict attention to a case where $V(\eta)=F(\eta)\sin(k'\eta)$ and the amplitude $F(\eta)$ for the oscillation is only slowly varying in time, which is the case for our present model. Then with $k'\eta \gg 1$, Equation (3.8) gives approximately the energy production rate

$$\frac{dp_\eta}{d\eta}\approx\frac{1}{32\pi}k'F^2(\eta)\cos^2k'\eta \tag{3.9}$$

$$\approx\frac{k'a^4}{1152\pi}\bar{R}^2. \tag{3.10}$$

Here, $\bar{R}$ denotes the scalar curvature (2.25) with a $\pi/2$ phase shift in the oscillating factor, and the scale factor $a(t)$ is given by (2.24). The proper energy density,

$\rho \equiv \frac{1}{a^3}\vec{p}\cdot(\partial/\partial t)$, is determined by

$$\frac{d\rho}{dt} = -4\rho H + \frac{1}{a^5}\frac{dp_\eta}{d\eta} = -4\rho H + \frac{\omega\bar{R}^2}{1152\pi}, \tag{3.11}$$

where $\omega = k'/a = 1/\sqrt{24\varepsilon}$ is the angular frequency of the oscillation in proper time and is given by Eq. (2.23).

When the final term in (3.11) vanishes at late times we have $d(\rho a^4)/dt = 0$ as radiation with an equation of state $p = (1/3)\rho$ should give. When the $\bar{R}^2$ term is nonzero, the equation of state is modified. The pressure of the particles is determined by Equations (2.3a) and (3.11) to be

$$p = \frac{1}{3}\rho - \frac{\omega}{1152\pi}\frac{\bar{R}^2}{H}. \tag{3.12}$$

The complete field equations with the back reaction of the particle generation included can be estimated by putting this p and ρ (Eq. (3.11) and (3.12)) back into the field equations.[19] The $t-t$ part of Eq. (2.1) becomes

$$H^2 + 2\varepsilon\left[H\dot{R} - \frac{1}{12}R^2 + RH^2\right] = \frac{8\pi}{3}G\frac{N}{a^4}\int_{t_{os}}^{t}\frac{\omega}{1152\pi}\bar{R}^2 a^4 dt \tag{3.13}$$

$$= \frac{8\pi}{3}G\,\rho_{\text{matter}}(t),$$

and the trace of Eq. (2.1) gives

$$\ddot{R} + 3H\dot{R} + \frac{1}{6\varepsilon}R = \frac{4\pi GN}{\varepsilon}\left(\frac{\omega\bar{R}^2}{1152\pi H}\right), \tag{3.14}$$

where we have inserted a factor N, which denotes the number of fields that can be excited by the cosmological oscillation (since massless conformal fields will not be excited, this N will be less than the total number of particles in the theory).

The right-hand side of Eq. (3.13), $8\pi GN\rho_{\text{matter}}/3$, can be estimated using (2.24) and (2.25). Not too long after the universe has come into the oscillation phase, say at $t-t_{os} \sim 10/\omega \approx 10\sqrt{24\varepsilon}$, we have $\rho \approx 6\times10^{-7}N/\varepsilon^2$, which corresponds to a reheating temperature of

$$T_r \approx 3\times10^{-2}/\sqrt{\varepsilon} = 4\times10^{17}\,\text{GeV}\left(\frac{\varepsilon}{1l_{Pl}{}^2}\right)^{-1/2}. \tag{3.15}$$

If ε is not too much bigger than one, this particle production timescale may be shorter than the thermalization of the particle content. Still, the reheating temperature, T_r, is a useful characterization of the reheating energy (we will, however, show that ε must be indeed large). If this temperature were higher than the GUT phase transition temperature, we would be left with the monopole problem. If this temperature were too cool, then baryogenesis may no longer go through. We will return to this point shortly.

When $t-t_{os} \gg 1/\omega$, the time dependence of ρ_{matter} is given by

$$\rho_{\text{matter}}(t) \approx \frac{3}{5}\frac{32}{1152\pi}\frac{N\omega^3}{(t-t_{os})}.$$

If we now neglect the back-reaction, H^2 at late times is given by (2.23) to be

$$H^2 \sim \frac{4}{9}\frac{1}{(t-t_{os})^2}. \tag{3.16}$$

Hence, at $(t-t_{os}) \approx 1200\,\varepsilon^{3/2}/GN$, the term on the right-hand side of (3.11) will be comparable with H^2 and the matter produced will begin to have a significant dynamical effect on the evolution of the universe. The solution of Eq. (3.13) gradually goes over to a radiation-dominated Friedmann expansion with

$$H \propto \frac{1}{2t}\ ,\ R=0\ ,\ a \propto t^{1/2}\ ,\ \text{and}\ \rho \propto 1/t^2. \tag{3.17}$$

However, the transition from the oscillation phase to the radiation-dominated phase will be slow even after $8\pi G\,\rho_{\rm matter}/3$ is comparable to H^2 as a numerical integration of Eq. (3.13) shows. We estimate the time it takes for the Friedmann phase to begin by taking roughly 10 times this value so that the time the Friedmann phase begins is given by $t_F \geq t_{os} + 12000\ \varepsilon^{3/2}/GN$. The energy density will then be

$$\rho(t_F) \leq 4\times10^{-9} GN^2/\varepsilon^3. \tag{3.18}$$

And the Friedmann Universe thus begins with the temperature

$$T_F \leq 1\times10^{17}\ \text{GeV}\left(\frac{\varepsilon}{1\ l_{Pl}^{\ 2}}\right)^{-3/4} N^{1/4}. \tag{3.19}$$

Notice that the ways T_r and T_F depend on ε are different. It is clear that any constraint on T_F will not be significant. There are important constraints on T_r, however. It must be higher than 10^{10-12} GeV so that gauge and Higgs bosons can be created and baryogenesis can proceed in the usual way, but lower than any GUT phase transition temperature $\sim 10^{16}$ GeV, so that the monopole problem can be avoided.[2] Eq. (3.15) then requires ε to be in the range

$$10^3\ l_{Pl}^{\ 2} < \varepsilon < 10^{15-12}\ l_{Pl}^{\ 2}. \tag{3.20}$$

These bounds will be tightened when we consider perturbations generated in the inflationary phase. We summarize the classical evolution of the universe as follows:

(i) A homogeneous and isotropic region near the Planck time with a Hubble parameter H_m will expand with a linearly decreasing H for a total expansion factor $\sim \exp(18\varepsilon H_m{}^2)$.

(ii) Particles will be created during the oscillation phase. The total expansion factor during this time will be

$$\sim \exp\left[\int_{t_{os}}^{t_{os}+12000\varepsilon^{3/2}/GN} H dt\right] \approx 70\left[\frac{\varepsilon}{NG}\right]^{2/3} .$$

(iii) The universe will then go over to a radiation-dominated Friedmann phase with the temperature T_F given by Eq. (3.19). To red-shift this to the present value of $3\,^oK$, we must have an expansion factor

$$\left[\frac{T_F}{3\,^oK}\right] \approx \left[5\times10^{29}\,(\frac{G}{\varepsilon})^{\frac{3}{4}} N^{1/4}\right] .$$

Therefore, the total expansion since the Planck era is obtained by multiplying the expansion factors under (i), (ii), and (iii), and it should be greater than the present horizon size, where $1/H_0 \sim 10^{55}\, l_{Pl}$. This requires in terms of the expansion factor

$$e^{18\varepsilon H_m{}^2} \geq 2\times10^{25}\left[\frac{H_m}{1\, l_{Pl}{}^{-1}}\right] \qquad (3.21)$$

(the dependence on N is very weak, so we have set $N \approx 100$ as a typical value). The expansion factor is very sensitively depending on $\varepsilon H_m{}^2$, so that unless the initial parameter, H_m, is fine-tuned, the left-hand side of Eq. (3.21) is likely to be very much bigger than 10^{25}. We thus expect to have much more inflation than is necessary.

IV. GRAVITATIONAL WAVE GENERATION

It is crucial to study the generation of gravitational waves in the model, since it is well known that inflation close to the Planck time tends to yield excessive gravitational wave generation.[20] In the transverse-traceless gauge, a gravitational wave can be expressed in terms of a scalar amplitude, h. For a wave with wavenumber k the metric can be written as

$$ds^2=-dt^2+a^2(t)[\delta_{ij}+he_{ij}]dx^i dx^j, \tag{4.1}$$

where $i,j=1,2,3$ and e_{ij} is the polarization tensor satisfying both the transverse condition, $e_{ij}k^j=0$ and the traceless condition, $e_i{}^i=0$. The field Equation (2.1) then reduces to

$$\ddot{h}+\left[3H+\frac{1}{6}\frac{\varepsilon R^2}{(1+2\varepsilon R)H}\right]\dot{h}-\frac{1}{a^2}\partial_i{}^2h=0. \tag{4.2}$$

The second term in the bracket is due to the presence of the εR^2 term in the gravitational Lagrangian. Other than this term, $h(t)$ satisfies the same equation as an ordinary scalar field in a Robertson-Walker background. Since Eq. (4.2) is second order in the spacetime derivatives, the quantization can proceed in the usual way. We construct an action S from which (4.2) can be derived:

$$S=\int d^4x\sqrt{-g}\,L, \tag{4.3a}$$

where

$$L=(1+2\varepsilon R)g^{\mu\nu}\partial_\mu h\,\partial_\nu h \tag{4.3b}$$

(here we use the background metric of equation (1.1) (with $\kappa=0$) to compute the quantities $\sqrt{-g}$, $g_{\mu\nu}$, and R). The quantization condition is then

$$\left[h(t,x), \frac{\partial L}{\partial \dot{h}}(t,y) \right] = iG \frac{\delta^3(x-y)}{a^3}. \tag{4.4}$$

For L given by Eq. (4.3b), we have

$$\left[h(t,x), \dot{h}(t,y) \right] = iG \frac{\delta^3(x-y)}{a^3(1+2\varepsilon R)} \tag{4.5}$$

(note that the additional factor of $1/(1+2\varepsilon R)$ in the normalization enters because of the εR^2 term). It is straightforward to check that the evolution equation preserves this commutation relation.

If h is composed of modes of more than one wave vector, it can be written as

$$h(t,x) = \int d^3k \left[\hat{a}_k h_k e^{ikx} + \hat{a}_k{}^{+} h_k{}^{*} e^{-ikx} \right], \tag{4.6}$$

with the creation and annihilation operators satisfying the usual relations:

$$\left[\hat{a}_k, \hat{a}_{k'}{}^{+} \right] = \delta^3(k-k'), \text{ etc..} \tag{4.7}$$

Then Equations (4.4) and (4.5) determine the normalization for (4.6):

$$h_k \dot{h}_k{}^{*} - h_k{}^{*} \dot{h}_k = \frac{iG}{(2\pi)^3 a^3 (1+2\varepsilon R)}. \tag{4.8}$$

The evolution equation for h_k is then

$$\ddot{h}_k + \left[3H + \frac{2\varepsilon \dot{R}^2}{H(1+2\varepsilon R)} \right] \dot{h}_k + \frac{k^2}{a^2} h_k = 0. \tag{4.9}$$

Now we consider a wave with wavelength equal to or smaller than the present horizon size, $1/H_0$. If the expansion factor in the linear phase is much greater than the minimum requirement (3.21) (cf. also text following (3.21)), the wavenumber k of these waves will be much greater than 1. On the other hand, the term inside the brackets in Equation (4.9) is of order 1 as $t \to 0$ and is thus negligible compared to k/a. Once again we are considering a wave evolving on an essentially flat background. Thus, the initial mode function can be chosen as

$$h_k = h_k^{(i)} e^{-ik\int \frac{dt}{a}}. \tag{4.10a}$$

And the normalization $h_k^{(i)}$ is determined by Eq. (4.5) to be

$$|h_k^{(i)}| = \frac{\sqrt{G}}{\sqrt{2k}\,(2\pi)^{3/2} a\,(1+2\varepsilon R)^{1/2}}. \tag{4.10b}$$

In the linear phase, $a(t)$ is rapidly increasing so that the wave is soon well outside the horizon (i.e., $k \ll aH$) and the third term in Eq. (4.9) becomes negligible, so that h_k approaches a constant. This constant can be estimated by extrapolating (4.10) to the horizon crossing time. h_k then remains at this value until it finally reenters the horizon in the Friedmann phase. This "freezing out" of the gravitational waves often goes by the name of amplification,[21] since it is amplification above the adiabatic behavior (Eq. (4.10)). The amplitude of the gravitational wave of wave number k at reentry is thus given by

$$A_k = (2\pi k)^{3/2} |h_k(t_{hc})| = \frac{\sqrt{G}H(t_{hc})}{\sqrt{2}(1+2\varepsilon R(t_{hc}))^{1/2}}, \tag{4.11}$$

where t_{hc} denotes the initial horizon crossing time in the linear phase. At that time, $\dot{H} \sim -1/36\varepsilon$, so we have

$$\varepsilon R(t_{hc}) \sim 12\varepsilon H^2(t_{hc}). \tag{4.12}$$

We assume that waves which reenter the horizon at late times have left the horizon during the inflationary epoch, so that $2\varepsilon R(t_{hc}) \gg 1$ and

$$A_k \approx \frac{1}{\sqrt{2}} \frac{\sqrt{G}}{\sqrt{24\varepsilon}}. \tag{4.13}$$

Notice that the spectrum is flat. Comparing to the $\Delta T/T$ limit for the microwave anisotropy,[20] we have

$$A_k \approx \left(\frac{\Delta T}{T}\right) \leq \sqrt{7}\times 10^{-4} \tag{4.14}$$

or

$$\varepsilon \geq 3\times 10^5 \, l_{Pl}{}^2, \tag{4.15}$$

which somewhat tightens up the bound (3.20). Unlike usual inflationary models, it turns out that the microwave measurements constrain not the value of $H(t_{hc})$ but rather the value of ε. This is due to the fact that the quantization condition (4.5) is modified by the curvature-squared coupling.

In the conformal picture we arrive at the result quite easily because the background is de Sitter. Note that the conformal transformation maps backgrounds, but leaves the perturbations unchanged: $A = \tilde{A}$, so we have by conventional means

$$A_k = \tilde{A}_k \approx \sqrt{4\pi G}\,\tilde{H} , \tag{4.16}$$

which leads to $\varepsilon > 7\times 10^6 \, l_{Pl}{}^2$, agreeing with the above limit (4.15) to the order of approximation we are using.

Note that in this picture one matches the amplitude at $\tilde{a}\tilde{H} = k$, while the true perturbation crosses the physical horizon at $aH = k$. However, the difference between the two is $O(\dot{R}/R)$, so that with the same accuracy by which we have obtained the de Sitter solution, we can safely evaluate the perturbation at $\tilde{a}\tilde{H} = k$.

A comparison between the two pictures sheds more light in understanding why the final result does not depend on H_{hc} as in the usual case. In the standard calculation, we can estimate the amplitude of the wave by requiring that the expectation value of the total energy of waves within the horizon equals the zero point energy of quantum fluctuations[21], $E = (1/2)\omega = (1/2)(k/a)$:

$$\frac{1}{H^3} <\rho> = E. \tag{4.17}$$

The amplitude of the wave at the horizon crossing is obtained by extrapolating this relation to t_{hc}, which gives $A \propto H_{hc}$. Now in conformal space where the gravity is pure Einstein and the stress tensor for gravitational waves has the usual form, we require

$$\frac{1}{\tilde{H}^3} <\tilde{\rho}> = \tilde{E}. \tag{4.18}$$

However, this relation is not conformally covariant, as $\tilde{H} \approx \Omega^{-1/2} H$, $\tilde{E} = \Omega^{-1/2} E$, and $\tilde{\rho} = \Omega^{-1}\rho$ (here Ω is the conformal factor $= (1+2\varepsilon R)$). So, in terms of the physical H and R, this relation reads

$$\frac{1}{H^3} <\rho> = \frac{E}{\Omega}. \tag{4.19}$$

Since $\Omega = (1+2\varepsilon R) \approx 24\varepsilon H_{hc}{}^2$, we have that the Hubble parameter drops from the final answer.

V. SCALAR PERTURBATIONS

As is usual in inflationary models, rather stringent constraints on the model parameters arise from present observational limits on scalar perturbations. In our model, scalar perturbations are generated by quantum fluctuations in the scalar curvature around background values. A major obstacle to evaluating these fluctuations is that we are dealing with a fourth-order gravity in which the quantization is not easy. We thus avoid the problem by working in the conformal picture. In the conformal picture there is a neat separation of the degrees of freedom, and the background is de Sitter, so that our result is easily obtained. From the action (1.5), we obtain a field equation for δR, which is full of nonlinearities. However, we may make use of the fact that during the inflationary epoch, εR is large ($\varepsilon R \approx 12\varepsilon H^2 \geq 20$, where physical quantities are without tildes, conformal quantities have tildes) and the field equation reduces in this exponential expansion phase to

$$\frac{d^2(\delta R)}{d\tilde{t}^2}+3\tilde{H}\frac{d(\delta R)}{d\tilde{t}}-\tilde{a}^{-2}(\partial_i{}^2\delta R)=0. \tag{5.1}$$

That is, δR evolves like a minimally coupled scalar field. However, it is not really one, as can be seen by its stress tensor. We may use the stress-energy tensor given by Eq. (1.4b) to find the background energy density and pressure during this expansion phase (when the matter content is negligible):

$$\tilde{\rho}=\tilde{T}_{\tilde{t}\tilde{t}}=\frac{1}{64\pi G\,\varepsilon}\frac{1}{(1+\frac{1}{2\varepsilon R})^2}\left[1+6\varepsilon\left(\frac{1}{R}\frac{dR}{d\tilde{t}}\right)^2\right] \tag{5.2a}$$

and

$$\tilde{p}=\frac{\tilde{T}_{\tilde{x}\tilde{x}}}{\tilde{a}^2}=\frac{-1}{64\pi G\,\varepsilon}\frac{1}{(1+\frac{1}{2\varepsilon R})^2}\left[1-6\varepsilon\left(\frac{1}{R}\frac{dR}{d\tilde{t}}\right)^2\right]. \tag{5.2b}$$

For a scalar wave perturbation of wavenumber k, we can find the linear and quadratic corrections to the energy density:

$$\delta\tilde{\rho}=\delta\tilde{\rho}^{(1)}+\delta\tilde{\rho}^{(2)}, \tag{5.3a}$$

where, in particular, to leading order in $\frac{1}{\varepsilon R}$ we have

$$\delta\tilde{\rho}^{(2)}=\frac{3}{16\pi G}\left(\frac{k}{\tilde{a}}\right)^2\left(\frac{\delta R}{R}\right)^2. \tag{5.3b}$$

Now we proceed to determine the mean-square quantum fluctuations of δR (i.e., for waves much shorter than horizon) from the fact that their energy is just the zero-point energy. That is,

$$\frac{1}{\tilde{H}^3}<\delta\tilde{\rho}^{(1)}+\delta\tilde{\rho}^{(2)}>=\tilde{E}=\frac{1}{2}\frac{k}{\tilde{a}}. \tag{5.4}$$

We evaluate Eq. (5.4) using (5.3) for scales much shorter than the horizon. The expectation value $<\delta\tilde{\rho}^{(1)}>$ is zero and we obtain

$$<\delta R^2>\doteq\frac{8\pi G}{3}\left(\frac{k}{\tilde{a}}\right)^2 R^2. \tag{5.5}$$

Finally, we extrapolate this to the horizon crossing of the fluctuation, where it is physically matched to the classical post-horizon-crossing amplitude by $|\delta R_{hc}|^2 = 2<\delta R^2>$, so

$$|\delta R_{hc}|\doteq\frac{1}{3}\left(\frac{2\pi G}{\varepsilon}\right)^{1/2} R_{hc}\doteq 4\left(\frac{2\pi G}{\varepsilon}\right)^{1/2} H_{hc}^{\ 2}. \tag{5.6}$$

Now we may determine the metric potential, $\tilde{A}$, due to a classical wave of amplitude $|\delta R_{hc}|$ using the "time-lag" method:[3]

$$\tilde{A} \sim \frac{\delta(\tilde{a}^2)}{\tilde{a}^2} = 2\tilde{H}\,\delta\tilde{t} = 2\tilde{H}\,\left|\frac{dR}{d\tilde{t}}\right|^{-1} |\delta R_{hc}|. \tag{5.7}$$

If we now plug (5.6) into (5.7), we obtain

$$\tilde{A} \sim \frac{2}{3}\left(\frac{2\pi G}{\varepsilon}\right)^{1/2} (18\varepsilon H_{hc}{}^2). \tag{5.8}$$

We stress that this is the asymptotic value of the metric perturbation at the end of the inflationary phase and therefore gives the magnitude of the inhomogeneities in the subsequent Friedmann evolution.

Alternatively, we proceed more cautiously, using the gauge-invariant formalism of Brandenberger and Kahn.[4] We neglect the effect of sources outside the horizon so that we may use a quantity, ζ, as a conserved gauge-invariant expression between horizon crossings:

$$\zeta = 2/3\,\frac{\left[\tilde{\Phi}_H + \tilde{H}^{-1}\dfrac{d\tilde{\Phi}_H}{d\tilde{t}}\right]}{(1+\tilde{p}/\tilde{\rho})} + \tilde{\Phi}_H\left[1+\frac{2}{9}\left(\frac{k}{\tilde{a}\tilde{H}}\right)^2 \frac{1}{(1+\tilde{p}/\tilde{\rho})}\right], \tag{5.9}$$

where $\tilde{\Phi}_H$ is now a gauge-invariant metric potential given by

$$\tilde{\Phi}_H = 4\pi G\tilde{a}^2\nabla^{-2}\left[\tilde{T}_{\tilde{t}\tilde{t}}{}^{(1)} - 3\tilde{a}\frac{d\tilde{a}}{d\tilde{t}}\nabla^{-2}\tilde{T}_{\tilde{t}}{}^{\tilde{j}}{}_{,\tilde{j}}{}^{(1)}\right]. \tag{5.10}$$

Here, ∇^{-2} is the inverse Laplacian and $\tilde{T}_{\tilde{\mu}\tilde{\nu}}{}^{(1)}$ is the first-order perturbation in the stress-energy. We may calculate from (1.4b) to leading order in $\frac{1}{\varepsilon R}$ (that is, during the inflationary epoch after the horizon crossing so that the wave is fully classical)

$$\tilde{T}_{rr}{}^{(1)}=\delta\tilde{\rho}^{(1)}\approx\frac{1}{64\pi G\varepsilon}\frac{1}{\varepsilon R}\frac{\delta R}{R}. \tag{5.11a}$$

And from the stress-energy (1.4b) we find, again to leading order (this term is the same order as the first, contrary to Brandenberger and Kahn[4])

$$\tilde{T}_{\tilde{r}}{}^{\tilde{j}}{}_{\tilde{j}}{}^{(1)}\approx\left(\frac{k}{\tilde{a}}\right)^2\left(\frac{dR}{d\tilde{t}}\right)\delta R. \tag{5.11b}$$

We have then at the horizon crossing of Eq. (5.9)

$$\zeta_{hc}=\left[\frac{20}{3}\frac{\delta\tilde{\rho}^{(1)}}{(\tilde{\rho}+\tilde{p})}+\frac{2\tilde{H}^{-1}\frac{d\,\delta\tilde{\rho}^{(1)}}{d\tilde{t}}}{(\tilde{\rho}+\tilde{p})}+\frac{3\delta\tilde{\rho}^{(1)}}{\tilde{\rho}}\right], \tag{5.12}$$

where $\delta\tilde{\rho}^{(1)}$ is now calculated in (5.11a) from the classical amplitude $|\delta R_{hc}|$ in Eq. (5.6). And we may find ζ_{hc} by putting (5.11a,b) into (5.12):

$$\zeta_{hc}=39\varepsilon|\delta R_{hc}|=\frac{26}{3}\left(\frac{2\pi G}{\varepsilon}\right)^{1/2}(18\varepsilon H_{hc}{}^2). \tag{5.13}$$

This fixes ζ at the initial horizon crossing, which quantity is roughly conserved until reentry. At the reentry of the scale of interest, the universe will be in a matter-dominated Friedmann phase ($\tilde{p}=0$), and we may use the Friedmann equation at reentry, $\tilde{H}^2=(8/3)\pi G\tilde{\rho}$, to find

$$\tilde{\Phi}_H(\tilde{t}_{\text{reentry}})=\frac{3}{2}\frac{\delta\tilde{\rho}^{(1)}}{\tilde{\rho}}. \tag{5.14}$$

We may now drop the tildes at reentry, since during this late phase the conformal factor is ≈ 1. We have

$$\zeta_{\text{reentry}} \approx \frac{35}{9} \Phi_H (t_{\text{reentry}}) \approx \zeta_{hc} . \tag{5.15}$$

And finally, the metric potential after reentry is

$$\Phi_H (t_{\text{reentry}}) \approx \frac{78}{35} (\frac{2\pi G}{\varepsilon})^{1/2} (18 \varepsilon H_{hc}{}^2). \tag{5.16}$$

We see that $\Phi_H (t_{\text{reentry}}) \sim \tilde{A}$ (here $\tilde{A}$ is given by Eq. (5.8)) to within numerical factors. In the $H_{hc}{}^2$ factor, we have some weak scale dependence in the perturbation spectrum. In fact, the spectrum is scale-invariant up to a logarithmic term as in the case of standard inflation. We calculate this dependence in the following way — at both the initial and final horizon crossings, we have in the physical space

$$a_{hc} H_{hc} = k = a_{\text{reentry}} H_{\text{reentry}}. \tag{5.17}$$

We plug into this our evolution law (2.23)-(2.24), assuming, of course, that the initial horizon crossing occurs during the linear inflationary phase of the model and we obtain

$$H_{hc} = H_0 \left(\frac{k_{\text{reentry}}}{k_0} \right) e^{18 \varepsilon H_{hc}{}^2} (3.3 \times 10^{31}) \left(\frac{\varepsilon}{G} \right)^{-1/12} N^{-5/12}, \tag{5.18}$$

where, H_0 is the Hubble parameter today (we use $H_0 = 50$ km/sec $\text{Mpc}^{-1} = 9 \times 10^{-56}\, l_{Pl}{}^{-1}$, and k_0 is the scale which crosses the horizon today). From this equation we may directly exhibit the logarithmic scale dependence of the perturbations:

$$\frac{\tilde{A}_2}{\tilde{A}_1} \doteq 1 - \frac{1}{18 \varepsilon H_{hc}{}^2} \ln \left(\frac{k_2}{k_1} \right) . \tag{5.19}$$

We note that Eq. (5.18) for a given scale of observational interest completely fixes the horizon crossing Hubble parameter in terms of the model parameter ε. That is, the metric potential, $\tilde{A}$, given by Eq. (5.8), again for a given scale, is dependent only on ε. Scales that are inside the horizon today are bounded by the microwave anisotropy limit[20] so that $\tilde{A} \leq \sqrt{7}\times10^{-4}$ and $k_{\rm reentry}/k_0 = 1$. We have

$$H_{hc}(k_0) \simeq 5\times10^{-6}\, l_{\rm Pl}^{-1} \text{ and } \varepsilon > 8\times10^{10}\, l_{\rm Pl}^{2}. \qquad (5.20)$$

If we want this primordial spectrum of density fluctuations to be a successful seed for galaxy formation, and we use a standard value for the scalar perturbation amplitude of $\sim 10^{-4}$, then essentially our bound in (5.20) would change into an equality. If, however, we choose a different scenario,[22] that is less constraining in which $\tilde{A} > 10^{-6}$ for scales $k_{\rm reentry}/k_0 \simeq 150$, we have

$$H_{hc}(k_{\rm cluster}) \simeq 2\times10^{-8}\, l_{\rm Pl}^{-1} \text{ and } \varepsilon < 5\times10^{15}\, l_{\rm Pl}^{2}. \qquad (5.21)$$

The bound (5.20) tightens up (3.20) considerably — although this number is to be taken only as very rough. Notice also that $18\varepsilon(H_{hc}(k_0))^2 \simeq 52$, so that the early evolution for $H(t) > H_{hc}(k_0) \sim 5\times10^{-6}\, l_{\rm Pl}^{-1}$ is irrelevant to all present observation. Putting it another way, with initial conditions of order Planck, the model predicts that our universe has been expanded something like 2×10^{12} e-foldings, so that the observable part of the universe will be the same for many future generations.

The scales that cross the horizon at $H_{hc} > H_b \equiv 1/(12\sqrt{2\pi G\,\varepsilon})$ have perturbations bigger than one today. From Eq. (5.20),

$$H_b \leq 10^{-3}\, l_{\rm Pl}^{-1} .$$

If $H_m > H_b$, that simply means that one has at scales much larger than the present horizon fluctuations, which cannot be treated in linear theory. Of course, H_m can as well be less than H_b — it is bounded below only by $H_{hc}(k_0)$. The requirement that the perturbations are small at the initial horizon crossing so that the use of perturbation theory is justified leads to only a very weak constraint on ε — well within our other bounds.

Interestingly, all these numbers tell us that there is one characteristic mass scale present in the theory as $H_{hc}(k_0) \sim \varepsilon^{-1/2} \sim 10^{-6}\, l_{\rm Pl}^{-1}$. Perturbations in an inflationary model with a massive scalar inflaton have been considered by Halliwell and Hawking,[23] using the full wave function formalism. They found that compatibility with observation restricts this mass to be less than 10^{14} GeV. As we have seen, the scalar curvature does obey an equation for a massive scalar field of mass $\sim 1/\sqrt{6\varepsilon}$. So we see that despite the unusual self-couplings present in the εR^2 theory, the physical analogy works remarkably well.

Finally, from Equations (4.13) and (5.8), the neat result follows that the contribution to the microwave anisotropy of the scalar fluctuations overpowers that from gravitational waves by a factor of $18\varepsilon(H_{hc}(k_0))^2 \sim 52$. This is the reason that the bound on ε is much tighter from considering scalar perturbations.

VI. PRESENT BOUNDS ON ε AND POSSIBLE ORIGINS

It may seem that the condition $\varepsilon > 10^{11}\, l_{\rm Pl}^2$ places a very large unnatural limit on ε, which in terms of Planck units it does. We would like to point out that in terms of any presently measured curvature, this is really quite small.

We can manipulate the field Equation (2.1) in the usual way to get

$$\Lambda - 4\pi G\,(\rho_S + 3p_S) = 3H_0^2(\sigma_0 - q_0)\,, \tag{6.1}$$

where

$$\rho_S \equiv \frac{3\varepsilon}{4\pi G}\left(\frac{R^2}{12} - \dot{R}H - RH^2\right) \tag{6.2a}$$

and

$$p_S \equiv \frac{\varepsilon}{4\pi G}\left(\ddot{R} + 2\dot{R}H + \frac{R^2}{12} - RH^2 - \frac{R\kappa}{a^2}\right). \tag{6.2b}$$

This is the usual equation that is used to set a limit on the cosmological constant Λ in terms of the presently observed H_0 (the Hubble parameter), σ_0 (density parameter), and q_0 (deceleration parameter). If we assume $\Lambda = 0$, we thus obtain a cosmological limit on ε:

$$\varepsilon \leq 10^{120}\, l_{\mathrm{Pl}}{}^2. \tag{6.3}$$

Similarly, one can consider a limit on ε by asserting that εR is small in all horizon-exterior curvatures encountered presently in our universe. We may use for R typically M/r^3 and go to the gravitational radius of a black hole. Then $\varepsilon R \ll 1$ requires only

$$\varepsilon \ll 10^{77}\left(\frac{M}{M_\odot}\right)^2 l_{\mathrm{Pl}}{}^2. \tag{6.4}$$

This, of course, is a bit of a swindle, because a black hole is also a solution of εR^2 gravity[14] so that $R = 0$ and ε will have no effect. We conclude, though, that $\varepsilon = 10^{11}\, l_{Pl}{}^2$ in terms of any presently encounterable curvature is very small.

We have not as yet addressed the question of the origin of the ε term. Basically, there are three ways that one might imagine it arising. First, it may be that the full fourth-order theory should be postulated as fundamental. Such a form is naturally suggested if one thinks about gravity as the gauge theory of the Poincare group.[24]

Furthermore, the εR^2 terms in the field equations violate the strong energy condition so that the initial singularity might be avoided.[12] It has also been shown that such a theory is renormalizable.[24] And the long-standing objection that it is nonunitary might not be true.[25] Secondly, it may be a remnant from some more fundamental theory. For instance, in the superstring theory the Lagrangian of the point-particle limit of the 10-dimensional full string theory contains the following terms:[26]

$$R^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho}+aR^{\mu\nu}R_{\mu\nu}+bR^2,$$

where a and b are some constants, and these are 10-dimensional curvature tensors. After compactification this leads to

$$L=R+(\frac{a+1}{3}+b)\frac{GV_6}{\phi}R^2, \tag{6.5}$$

where V_6 is the compactified volume of the six "other" dimensions and ϕ is the vacuum expectation value of a scalar field known as the dilaton. We see that this might directly give us an εR^2 behavior even classically in the Lagrangian with a completely determined ε. However, the highly preferred values[27] for a and b are $a=-4$, $b=1$ and then $\varepsilon=0$ at the classical level, and there is no R^2 term in superstring theory.

Nevertheless, ε should also be expected to arise in a third way — as a quantum effective action correction to the bare theory. Here, the specific fields will contribute to its value. Indeed, this is the approach of Starobinsky.[13] As a quantum correction term ε would be given by

$$\varepsilon \sim G\ln(\frac{\Lambda_{\text{high cutoff}}}{\Lambda_{\text{low cutoff}}}) \tag{6.7}$$

and we would again be forced to consider a higher, more complete theory to fix ε.

VII. CONCLUSION

We thus conclude that at a classical level a cosmology based on the $R + \varepsilon R^2$ Lagrangian generically has an inflationary phase with a linearly decreasing Hubble parameter. The total number of expansion e-foldings during this phase is $\sim 18\varepsilon H_m{}^2$ (if $\dot{R}_i = 0$, then $t_i = 0 = t_m$). After the linear decaying phase, $H(t)$ bounces off zero and the universe goes into an oscillatory phase. The total expansion is sufficient to solve the horizon and flatness problems if $18\varepsilon H_m{}^2 > 75$. At the classical level, this is a natural and consistent model that relies solely on a modified gravity for its dynamics. Here, the quadratic correction to the Hilbert-Einstein action would be expected to be present somewhat independently of the specific form of the matter Lagrangian (although a value for ε must necessarily come from a higher theory).

The postinflation oscillatory phase yields a maximal reheating temperature which is small:

$$T_r \approx 1.2\times10^{12}\,\mathrm{GeV}\left(\frac{\varepsilon}{10^{11} l_{\mathrm{Pl}}{}^2}\right)^{-1/2},$$

in any case very much below any expected GUT phase transition, so that the monopole problem is avoided by the εR^2-driven expansion. Standard baryogenesis still may go through at this temperature, but the details of this on the non-standard background will require further attention. Finally, there is a join to a Friedmann phase at a temperature

$$T_F \le 6\times10^{8}\,\mathrm{GeV}\left(\frac{\varepsilon}{10^{11} l_{\mathrm{Pl}}{}^2}\right)^{-3/4} N^{1/4},$$

when the evolution goes over to a radiation-dominated expansion.

Gravitational waves and scalar perturbations both yield bounds on the parameters of the model when we must set them small so as not to disturb the isotropy of the microwave background. The bound from gravitational waves is $\varepsilon > 10^6\, l_{Pl}^{\,2}$ with no restriction on H_{hc} as would occur for the standard inflationary scenario. This spectrum of gravitational waves is scale-invariant. However, the scalar perturbations give the much tighter bound of $\varepsilon \geq 10^{11}\, l_{Pl}^{\,2}$, and this in turn implies that the perturbation scale, that reenters the horizon today, must cross the horizon at $H_{hc}(k_0) \sim 10^{-6}\, l_{Pl}^{\,-1}$ — that is, at a late stage of the extremely long linear phase. The spectrum of scalar perturbations has only logarithmic dependence on the scale. If one wants baryogenesis to proceed in the usual way, there is an upper bound $\varepsilon < 10^{15}\, l_{Pl}^{\,2}$. A similar bound follows from a comparison between galaxy formation and the microwave anisotropy in models of galaxy formation with cold dark matter.[22] However, both considerations carry their own difficulties, so that we place somewhat less emphasis here on the upper bound. The condition of sufficient inflation requires that $H_m > 10^{-5} l_{Pl}^{\,-1}$ — that is, we find that our model would work for essentially all reasonable initial conditions. We thus conclude that the εR^2 model satisfies all requirements for a realistic inflationary model as long as ε is large enough.

To investigate the very early phase, we have attempted a preliminary wave function calculation by solving the Wheeler-DeWitt equation to WKB approximation subject to a tunneling boundary condition in the manner of Vilenkin.[28] We thus obtain peak values for the wave function, assuming a closed ($\kappa = +1$) universe of $\langle a \rangle \sim .056\, l_{Pl}$, $\langle R \rangle \sim 3800\, l_{Pl}^{\,-2}$, and $\langle H \rangle \sim 18\, l_{Pl}^{\,-1}$ independent of ε (the details of that calculation will be reported in subsequent work). We interpret these as typical of the tunneling values for the universe into the Lorentzian/classically allowed regime.

Also, the peak is not very strong, so that these numbers end up only as bounds. That is, we might say

$$R_i \gtrsim 4000\, l_{\rm Pl}^{-2},$$

$$a_i \sim .06\, l_{\rm Pl} \left(\frac{R_i}{4000\, l_{\rm Pl}^{-2}} \right)^{-1/2},$$

and

$$H_i \sim 20\, l_{\rm Pl}^{-1} \left(\frac{R_i}{4000\, l_{\rm Pl}^{-2}} \right)^{1/2}$$

(and $t_i = t_m = 0$). These numbers are sufficiently distant from the horizon crossing of interesting perturbations that the wave function offers no conflict with our lower bound on H_m. We thus find the classical evolution to be generally independent of initial conditions. The one remaining question is whether or not there will be a long quantum gap separating the tunneling point from the onset of the classical model. That is, are quantum fluctuations large for an extended period during early times? This, of course, must be answered by the wave function itself. Also, after doing this further calculation, we can determine whether the inflated portion of our present universe is the whole universe or only a fluctuated bubble part of it as in Linde's chaotic inflation picture. We note now only that the initial parameters preferred above indicate that the tunneled universe is strongly quantum.

The model we are considering has a lot in common with the Starobinsky model.[13] While our work was carried on, papers by Starobinsky,[29] Kofman, Linde, and Starobinsky,[31] and Vilenkin[28] appeared, from which we also learned about earlier work.[30] All of these papers treat the Starobinsky model in considerable detail, so that we would like to comment here about similarities and differences between cosmologies based on Eq. (1.2) and the Starobinsky model and also to discuss our results in relation to this other work.

Starobinsky considers a model in which the one-loop quantum corrections to the matter stress-energy tensor of a conformally coupled scalar field are used as source terms for the Einstein equations. At the Lagrangian level, this introduces a new parameter, H_S, and a new term to Eq. (1.2), $(1/H_S{}^2)R^2\ln(R/\mu)$, where μ is some renormalization scale. The important point is that H_S is completely fixed by the number of degrees of freedom that give quantum corrections:[28] for example $H_S \sim 0.7\, l_{\text{Pl}}{}^{-1}$ for minimal SU_5. There is an exact de Sitter solution in this case, with H_S being the initial Hubble parameter. This solution is shown to be unstable— offering an exit from the inflationary phase. Vilenkin[28] has shown by a wave function calculation that there will be sufficient inflation in the de Sitter phase. For an initial H_i not bigger than H_S, $H(t)$ will decrease in time. When $H(t) \ll H_S$, the decrease will be linear with time,[30] and the subsequent evolution should be the same as in the present R^2 model. In comparison to the Starobinsky model, our work shows that the initial de Sitter phase is not necessary. We have shown that a generic solution of the field equations will have sufficient inflation based solely on the R^2 term. We have analyzed the reheating in the oscillation phase, showing that it is characterized by two different temperatures. The reheating temperature, T_r, is much higher than the temperature, T_F, when the Friedmann phase begins. We have analyzed the metric perturbations both in the conformal picture and in the direct approach. The results obtained essentially agree with those obtained in the Starobinsky model.[28,30] These results indicate that the part of the expansion that is relevant for present observation happens at $H(t) < 10^{-5}\, l_{\text{Pl}}{}^{-1}$ and cannot be due to the de Sitter phase of the Starobinsky model. Finally, we note that as the two models have very different early stage evolution, the wave function calculation yields very different initial parameters.

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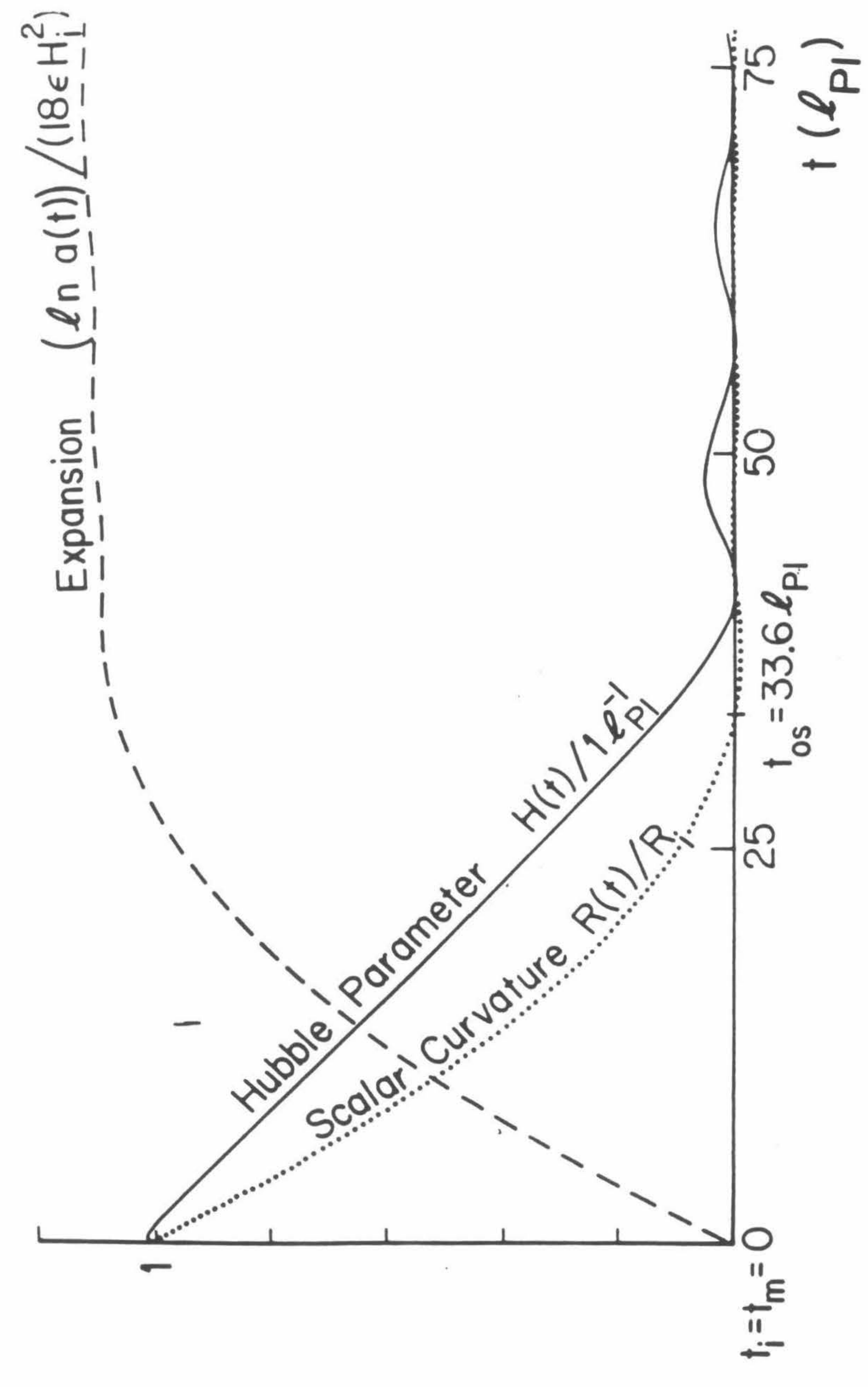
FIGURE CAPTION FOR FIGURE 1, CHAPTER 2

FIG. 1. A model cosmology for $\varepsilon = 1$, $H_i = 1\ l_{Pl}^{-1}$, and $\dot{R}_i = 0$ (corresponding to the case (Ai) of the text so that $R_i \approx 12.5\ l_{Pl}^{-2}$)— showing typical behavior of the Hubble parameter ($H(t)/1\ l_{Pl}^{-1}$), the normalized scalar curvature ($R(t)/R_i$), and the inflation-normalized number of expansion e-foldings ($\ln a(t)/18\varepsilon H_i^2$). This plot has been generated from a numerical integration of the field Equations (2.4)-(2.7) with zero initial matter content. The Hubble parameter displays a clean separation between the linear inflationary phase and the subsequent oscillation phase at $t_{os} = 36\varepsilon H_m - (1/(2\omega)) \approx 33.6\ l_{Pl}$ (cf. Eq. (2.23)). The slight initial rise in $H(t)$ is real, since at the start, $\dot{H} = (1/6)(R - 12H^2) > 0$. For models with a much higher value of the parameter ε (we are observationally constrained to $\varepsilon > 10^{11} l_{Pl}^2$), the linear phase is stretched out to a shallow slope, and the subsequent oscillations are correspondingly reduced in both amplitude and frequency.


Expansion (ℓn a(t))/(18εH_i^2)
Hubble Parameter H(t)/1ℓ_Pl^-1
Scalar Curvature R(t)/R_i
t_os = 33.6 ℓ_Pl
t (ℓ_Pl)
0
25
50
75
1
t_i = t_m = 0

CHAPTER 3

The Quantum $R + \varepsilon R^2$ Cosmology

(by Milan B. Mijić, Michael S. Morris, and Wai-Mo Suen)

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ABSTRACT

A pure gravity cosmology based on the $R+\epsilon R^2$ Lagrangian is known to exhibit inflation for a wide range of initial conditions. In this paper we use the wave function from quantum cosmology to describe this inflation as a chaotic inflationary phase immediately following the quantum creation of the universe. We evaluate, compare, and discuss the distributions over initial conditions that are fixed by the two boundary condition proposals of Hartle-Hawking (''no boundary'') and Vilenkin (''tunneling from nothing''). We find that among all classical inflationary trajectories that begin on the Classical/Quantum boundary, those that lead to an inflation of at least 70 e-foldings make up a fraction of $\sim\exp[-10^{12}]$ in the former case and $\sim 1-\exp[-8\times 10^{10}]$ in the latter. Thus, in the simplest interpretation, the observable universe would be the outcome of a rare event for the first boundary condition proposal and a typical event for the second.

I. INTRODUCTION

Inflation has become standard in modern cosmology.[1] It can explain basic features of our present universe and can occur rather generally in particle physics models.[2]

The question that naturally follows the development of inflation is to ask what came before, a question that has been embedded in a broader context by the advent of chaotic inflation.[3,2] We shall understand by chaotic inflation that phase of the inflationary expansion during which some scalar field relaxes to the minimum of its potential, with the provision that: (i) this relaxation need not be accompanied by any kind of phase transition; (ii) the scalar field potential can be of classical origin as well as due to quantum corrections; (iii) the preinflationary phase need not be a hot, radiation-dominated Robertson-Walker Universe (as all models of old and new inflation have assumed, explicitly or implicitly); and (iv) the field's initial conditions have been assigned in some ''random way''.

As analysis has shown,[2] the typical initial conditions for chaotic inflation are Planck-scale, so we might expect quantum gravity to come directly to play. In fact, as we argue hereafter, the idea of chaotic inflation can be joined with the older concept of quantum creation of the universe.[4]

Two things are done in this paper: (i) We suggest a physical context, in which semiclassical wave functions (fixed by boundary conditions in quantum cosmology) can be used to compute the distribution of initial conditions for the classical inflationary expansion; and (ii) we apply this method to an inflationary model based on higher derivative gravity where, in particular, we compare predictions resulting from different proposals for the boundary conditions.

The model we study is $R + \varepsilon R^2$ gravity.[5–9] We provide the classical details necessary to our present work in Section II. One can think of this model as the relevant, dominant part of a renormalizable [and so far (perturbatively) nonunitary] higher derivative gravity.[10] Perhaps a more promising context is that of an effective theory that describes the first short-distance corrections to General Relativity, for example, the low energy limit of superstrings.[11] Thus, we have an inflationary model without introducing an additional inflaton field especially for the purpose. This point lends the model some advantage over others.

The wave function for higher derivative gravity with Hartle and Hawking's boundary condition has been studied analytically by Hawking and Luttrell,[6] and numerically by Hawking and Wu.[7] They have shown that the wave function is oscillatory in a certain regime of superspace, corresponding to Lorentzian spacetime. We extend their discussion of the Hartle-Hawking boundary condition, compute the wave function in more detail, compare it to the wave function that satisfies Vilenkin's boundary condition,[12] and make contact with the scenario of chaotic inflation. We have earlier reported some preliminary results in the discussion following our own analysis of the classical model.[8]

If (in this model or in any other) we follow a classical trajectory backward in time to when the curvature approaches values $\sim l_{\rm Pl}^{-2}$, this trajectory will hit a highly quantum region. Classical equations of motion cannot be used any more. In fact, as we shall see, there remains a substantial range of initial curvatures ($\varepsilon^{-1} < R_i < l_{\rm Pl}^{-2}$) in this model for which quantum creation of the inflationary universe may take place, and classical inflationary trajectories may start. This range is large because ε is constrained to be large, $\varepsilon \approx 10^{11}\, l_{\rm Pl}^2$, by a tiny observational bound on the anisotropy of the microwave background.

As shown in the literature,[5–9] the classical scenario of $R+\varepsilon R^2$ cosmology is sensible and attractive within this parameter regime — it can give more inflation than minimally required and then lead smoothly to reheating of the Universe and initiation of a Friedmann phase. It is plausible (especially at the lower acceptable curvatures), though by no means necessary, that still higher order correction terms to the Lagrangian would not change these results. In this paper we assume that whatever the true quantum theory of the world is, it should be well approximated in this regime by the quantum mechanics of the $R+\varepsilon R^2$ model. Since we are interested in the phase when the Universe emerges as a classical object, the semiclassical limit of the quantum theory is sufficient. Hence, we do not worry about the (un)calculability of loop corrections. We do not consider initial curvatures below ε^{-1} because we confine ourselves to analysis of the inflationary phase, which does not extend to such low curvatures. And, moreover, for such low curvatures, the εR^2 term will not be important and the evolution of the Universe will be strongly affected by other terms (e.g., matter fields) in the Lagrangian, terms that we have not taken into account in our present calculation. We reject consideration of curvatures above l_{Pl}^{-2} because our quantum model is presumably not the fundamental theory. We take the Planck scale to be the scale above which a full theory must come into account. That is, we limit our attention and our analysis to the initial edge of the region of semiclassical inflationary trajectories.

Such an approach, as restricted as it is, still has the power to yield important information. It might also survive modifications that the development of a more fundamental theory would bring. There are, however, two obvious shortcomings of this work. First, in this truncated use of quantum cosmology, we have dodged the problem of interpretation. In particular, we have not addressed the analog of the measurement problem from quantum mechanics. Second, a more technical weakness of our work is that, in order to carry out explicit calculation, we have resorted to calculating the wave function for Robertson-Walker models only. We reduce superspace to minisuperspace: all that is ''chaotic'' now in this inflation is the stochastic choice of the initial values of the scale factor, the homogeneous curvature, and their initial time derivatives. These four initial values for subsequent classical evolution will be determined by the wave function. We have ignored any distribution over inhomogeneity, anisotropy, etc.. We hope that both of these weaknesses will be amended by further development.

In Section II, we summarize the classical behavior of the $R+\varepsilon R^2$ model and derive the Wheeler-DeWitt equation appropriate to the two homogeneous and isotropic variables of superspace. We spell out two competing boundary condition proposals and explore their connection with quantum creation in Section III. In Section IV, we then implement both proposals for the boundary conditions and fix their respective wave functions for our model. We finally explore consequences of these wave functions in their regime of validity and state our conclusions in Section V. An appendix is provided to support material in Section II.

II. ACTION AND THE WHEELER-DEWITT EQUATION

We study a model governed by the action,

$$S=\frac{1}{16\pi G}\int d^4x\sqrt{-g}\,(R+\varepsilon R^2)+\frac{1}{8\pi G}\int d^3x\sqrt{h}\,K(1+2\varepsilon R)\,, \tag{2.1}$$

that represents Einstein gravity with an additional quadratic gravitational correction term. Here R is the scalar curvature, g is the determinant of the spacetime 4-metric, h is the determinant of the induced spatial 3-metric on the boundary, and K is the trace of the extrinsic curvature. Our sign conventions are those of Ref. 13 and we choose units in which $\hbar = c = 1$ and $G = 1\ l_{Pl}^2$. The parameter ε will then have dimensions of l^2. The boundary term [displayed as the surface integral in Eq. (2.1)] is the expression needed to cancel out arbitrary variations of metric derivatives at the boundaries of the 4-dimensional action integral. It is thus dependent on the form of the local Lagrangian density and in the appendix we provide details of its derivation.

For tractability we focus attention on a cosmological model described by the Robertson-Walker metric,

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] , \tag{2.2}$$

where $a(t)$ is the scale factor, k is the sign of the spatial curvature, R is given by $R = 12H^2 + 6\dot{H} + 6k/a^2$ where H is the Hubble parameter $H = \dot{a}/a$, and the extrinsic curvature is $K = -3H$.

The present authors[8] and others[5,6,7,9] have analyzed the classical behavior of such a model and we summarize it here. The R^2 term drives inflation. Any initial matter content will be rapidly redshifted, and the evolution goes over to a pure gravity near-de Sitter expansion. The equation of motion can be written

$$\ddot{H} + 3\dot{H}H + \frac{H}{12\varepsilon} = \frac{\dot{H}^2}{2H} + \frac{kH}{a}\left(1 - \frac{1}{12\varepsilon H^2} - \frac{k}{2H^2a^2}\right) . \tag{2.3}$$

For a wide range of initial data, there will be a ''linear'' phase, during which the terms on the right hand side of Eq. (2.3) are neglectable and we have the solution,

$$H(t) = H_i - \frac{t}{36\varepsilon} . \tag{2.4}$$

This solution will be very nearly de Sitter if the linear decay is slow — which is the same requirement as for the solution's validity. The linear decay of the Hubble parameter self-regulates its own inflationary epoch. The near-de Sitter behavior ends after $t_e \approx 36\varepsilon H_i$ and the universe then goes into an oscillation phase in which the scale factor increases on average as $\propto t^{2/3}$ — as in a matter-dominated Friedmann expansion. This "scalaron" dominated phase is unstable to particle creation. The universe will reheat to a temperature constrained below monopole production and above baryogenesis. The total number of expansion e-foldings, e, during the inflationary epoch is

$$e \approx 18\varepsilon H_i^{\,2} \,. \tag{2.5}$$

By analyzing perturbations (most importantly scalar perturbations), applying to our model their known observational limits, and requiring the inflationary period to be of sufficient duration, we have found the following parameter constraints:[8]

$$10^{11}\, l_{\rm Pl}^{\;2} < \varepsilon < 10^{15}\, l_{\rm Pl}^{\;2} \,; \tag{2.6a}$$

$$H_i > 10^{-5}\, l_{\rm Pl}^{\;-1} \quad \Longrightarrow \quad R_i > R_h = 1.2\times 10^{-9}\, l_{\rm Pl}^{\;-2} \,. \tag{2.6b}$$

Here, R_h is the curvature at which the perturbation, whose wavelength today is equal to the horizon size, crossed the horizon during inflation.

The inflation exhibited by this model is not substantially different from the inflation exhibited by any other chaotic inflationary model. The quadratic gravitational term lends to Einstein gravity an additional scalar degree of freedom. There is an explicit and very useful way to display the structure of this extra degree of freedom that is due to Whitt,[14] which is to perform the conformal transformation,

$$\tilde{g}_{\mu\nu} = (1+2\varepsilon R)\, g_{\mu\nu} \,, \quad d\tilde{s}^{\,2} = -d\tilde{t}^{\,2} + \tilde{a}^{\,2}(\tilde{t}\,)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right] \,. \tag{2.7}$$

The action (2.1) can then be rewritten (with geometric quantities in conformal space denoted by a tilde) as

$$S=\frac{1}{16\pi G}\int d^4x\,(-\tilde{g}\,)^{1/2}(\tilde{R}-\frac{1}{4\varepsilon})+\frac{1}{8\pi G}\int d^3x\,(\tilde{h}\,)^{1/2}\tilde{K}$$

$$-\int d^4x\,(-\tilde{g}\,)^{1/2}\left[\frac{3}{8\pi G}\phi_{,\mu}\phi_{,\nu}\tilde{g}^{\,\mu\nu}+\frac{e^{-2\phi}}{64\pi G\,\varepsilon}(e^{-2\phi}-2)\right]\ , \tag{2.8}$$

with $\phi=(1/2)\ln(1+2\varepsilon R)$. An effective cosmological constant, $1/(8\varepsilon)$, has been generated by the quadratic term. For $\varepsilon R \gg 1$, the potential for ϕ is negligible and this effective cosmological constant dominates. As in the scalar inflaton case, if the kinetic term were to dominate initially, it would decay away quickly as $\tilde{a}^{\,-6}$.

We study the distribution of possible initial conditions, using the wave function. We specialize to the line element with k=+1. This is a tremendous winnowing of many possible variables down to the two homogeneous degrees of freedom, $a(t)$ and $R(t)$, or $\tilde{a}\,(\tilde{t}\,)$ and $\phi(\tilde{t}\,)$. We complete the spatial integrals in the action (2.8), $\int(-\tilde{g}\,)^{1/2}d^4x=2\pi\int\tilde{a}^{\,3}d\tilde{t}$ and $\int(\tilde{h}\,)^{1/2}d^3x=2\pi^2\tilde{a}^{\,3}$, to get the action in the simple form

$$S=\int L\left(\tilde{a}\,,\frac{d\tilde{a}}{d\tilde{t}},\phi,\frac{d\phi}{d\tilde{t}}\right)d\tilde{t}$$

$$=\frac{3\pi}{4G}\int d\tilde{t}\left\{-\left(\frac{d\tilde{a}}{d\tilde{t}}\right)^2\tilde{a}+\tilde{a}^{\,3}\left(\frac{d\phi}{d\tilde{t}}\right)^2+\tilde{a}\left[1-\frac{\tilde{a}^{\,2}}{24\varepsilon}(e^{-2\phi}-1)^2\right]\right\}\ . \tag{2.9}$$

From this action we can read off the Hamiltonian. First, however, we interpose one last change of variables to make dimensionless the scale factor and time: $\alpha\equiv\tilde{a}\,/(2G/3\pi)^{1/2}$ and $\tau\equiv\tilde{t}\,/(2G/3\pi)^{1/2}$, which give us

$$\pi_\alpha = \frac{\partial \mathcal{L}}{\partial \left(\dfrac{d\alpha}{d\tau} \right)} = -\alpha \frac{d\alpha}{d\tau} \,, \tag{2.10a}$$

$$\pi_\phi = \frac{\partial \mathcal{L}}{\partial \left(\dfrac{d\phi}{d\tau} \right)} = \alpha^3 \frac{d\phi}{d\tau} \,, \tag{2.10b}$$

$$S = \frac{1}{2} \int d\tau \left\{ -\left(\frac{d\alpha}{d\tau} \right)^2 \alpha + \alpha^3 \left(\frac{d\phi}{d\tau} \right)^2 + \alpha \left[1 - \frac{G\alpha^2}{36\pi\varepsilon} (e^{-2\phi} - 1)^2 \right] \right\} \,, \tag{2.10c}$$

and the classical Hamiltonian,

$$\mathcal{H} = \frac{1}{2} \left\{ -\frac{{\pi_\alpha}^2}{\alpha} + \frac{{\pi_\phi}^2}{\alpha^3} - \alpha \left[1 - \frac{G}{36\pi\varepsilon} \alpha^2 (e^{-2\phi} - 1)^2 \right] \right\} . \tag{2.11}$$

To quantize, we canonically substitute $\pi_\alpha \to \hat{\pi}_\alpha = -i\,\partial/\partial\alpha$ and $\pi_\phi \to \hat{\pi}_\phi = -i\,\partial/\partial\phi$. The Wheeler-DeWitt equation[15,16] in the minisuperspace governing the wave function Ψ over the two variables $\alpha = a\,(1+2\varepsilon R)^{1/2} \sqrt{3\pi/2G}$ and $\phi = (1/2)\ln(1+2\varepsilon R)$ is:

$$\left\{ \frac{\partial^2}{\partial\alpha^2} - \frac{1}{\alpha^2} \frac{\partial^2}{\partial\phi^2} - \alpha^2 \left[1 - \frac{G}{36\pi\varepsilon} \alpha^2 (e^{-2\phi} - 1)^2 \right] \right\} \Psi(\alpha,\phi) = 0 \,. \tag{2.12}$$

Here we have chosen a simple factor ordering because our solution will be independent of factor ordering to the order of accuracy we demand. We display the restricted region of our analysis in minisuperspace in Fig. 1.

III. THE BOUNDARY CONDITIONS

The wave function of the universe is a solution of the Wheeler-DeWitt equation for some specified boundary condition. We discuss here the motivation for two of the most definite proposals for the boundary condition — the proposal of "no boundary" that is due to Hartle and Hawking[16,17] and the proposal of "tunneling from nothing" that is due to Vilenkin.[12] We will then (in Section IV) examine the specific solutions of the Wheeler-DeWitt equation for our specific model — which will in turn help us to further our physical understanding of the boundary conditions.

Both proposals apply to a spatially closed universe (k=+1) and are therefore intimately connected with the concept of quantum creation of the universe. First, we illustrate the basic idea by considering the case of $\varepsilon R \gg 1$ in Eq. (2.8), where the potential of the ϕ field is dominated by a cosmological constant, $\tilde{\Lambda}_{\rm eff}=1/(8\varepsilon)$. We display this effective cosmological constant as function of the ϕ field in Fig. 2. For a spatially closed Robertson-Walker spacetime, the classical equation of motion is

$$\tilde{H}^2=\frac{\tilde{\Lambda}_{\rm eff}}{3}-\frac{1}{\tilde{a}^2} . \tag{3.1}$$

There will be a classical Lorentzian trajectory for $\tilde{a}(\tilde{t})>(3/\tilde{\Lambda}_{\rm eff})^{1/2}$ only. We can think[4] of an initial configuration of finite size $\tilde{a}\sim(3/\tilde{\Lambda}_{\rm eff})^{1/2}$ being "spontaneously" born from the vacuum. If $\tilde{H}>0$, an expansion will follow. After quantization, Eq. (3.1) becomes a Wheeler-DeWitt equation

$$\left[-\frac{\hat{\pi}_{\tilde{a}}^{\,2}}{\tilde{a}}-\left(\frac{3\pi}{2G}\right)^2\tilde{a}\,(1-\tilde{a}^2\frac{\tilde{\Lambda}_{\rm eff}}{3})\right]\Psi(\tilde{a})=0 . \tag{3.2}$$

For $\tilde{a}>(3/\tilde{\Lambda}_{\rm eff})^{1/2}$, corresponding to the Lorentzian signature classical solution, the potential term is positive and the wave function will be oscillatory. For $\tilde{a} <(3/\tilde{\Lambda}_{\rm eff})^{1/2}$, we can introduce a classical solution with Euclidean signature, the potential term is negative, and the wave function will have exponential mode solutions. We observe that at both zero size for the universe, $\tilde{a} \sim 0$, and at the Euclidean/Lorentzian boundary, $\tilde{a} \sim (3/\tilde{\Lambda}_{\rm eff})^{1/2}$, the potential term in the Wheeler-DeWitt equation vanishes. Therefore, the creation of a Lorentzian universe is not a process that can be described in classical terms, since the semiclassical approximation to the wave function breaks down in the boundary regions.

There is a semiclassical regime at large $\tilde{a}$ and, depending on the value of $\tilde{\Lambda}_{\rm eff}$, there might be another semiclassical regime for $l_{\rm Pl} \ll \tilde{a} \ll (3/\tilde{\Lambda}_{\rm eff})^{1/2}$, where a classical Euclidean solution might be introduced. The oscillatory/exponential character of the wave function is intimately tied to the Lorentzian/Euclidean character of the classical trajectory. And it is the semiclassical regime on the Lorentzian side of the boundary (for large $\tilde{a}$) that provides initial conditions for the Lorentzian classical universe as we know it.

In the Lorentzian domain, the two oscillatory modes can be chosen to correspond to expanding and contracting universes. Vilenkin's tunneling boundary condition proposal, in this one-dimensional example, is to fix the wave function by demanding that it describe the expanding universe only. By the WKB matching conditions across the boundary, such a solution would contain both exponentially growing and exponentially decaying modes in the Euclidean domain. Near $\tilde{a} = 0$, of course, it is the exponentially growing mode that dominates. Thus, we can think of this boundary condition as physically ascribing the origin of the universe to the result of quantum tunneling within the Euclidean domain away from $\tilde{a} = 0$. This is the main physical idea behind Vilenkin's proposal, and this wave function is said to describe tunneling from nothing.[12]

Next we turn to Hartle and Hawking's boundary condition. Their proposal[16,17] is that the wave function of the universe is given by the Euclidean path integral over all compact 4-geometries and regular matter fields that induce a given 3-geometry and matter-field configuration on a given 3-boundary. The physical motivation for the Hartle-Hawking proposal comes from the path integral representation for a ground state. In a compact spacetime, there is no preferred notion of energy, and one can interpret their wave function as that for the state of minimal excitation. It is practically impossible, however, to compute this path integral in a closed form for any realistic model. In this paper, we determine the Hartle-Hawking wave function in the semiclassical regime by a semiclassical approximation to the path integral. This procedure fixes the solution that obeys the Hartle-Hawking boundary condition.

The structure of the minisuperspace with its Euclidean and Lorentzian domains is naturally independent of the boundary conditions for the wave function. Both boundary conditions for the wave function describe quantum creation of the universe. The form of the wave function is different for different boundary conditions, however, and the quantum mechanical probability for creation of a universe of a certain size and certain matter configuration, etc., will also be different for different boundary conditions. It is a main goal of this work to compare the predictions that depend on the choice of boundary condition for the quantum cosmology of the $R+\varepsilon R^2$ model.

Now we can state the boundary conditions for the wave function of the universe in the $R+\varepsilon R^2$ model. We will use the variables α and ϕ, with $\Psi=\Psi(\alpha,\phi)$ in Eq. (2.12). Following the usage of Ref. 12, we implement Vilenkin's boundary condition (''tunneling of the Universe from nothing'') in our present model by choosing

(i) the outgoing mode in the Lorentzian regime, along the classical trajectory; and

(ii) the wave function ϕ-independent as $\alpha \rightarrow 0$, ϕ finite. (3.3)

We implement Hartle and Hawking's boundary condition ("the universe is without boundary") by fixing

$$\Psi^{\text{semiclass.}}(\alpha,\phi)=\left\{ \int [\mathcal{D}\alpha][\mathcal{D}\phi] e^{-S_E[\alpha,\phi]} \right\}_{\text{semiclass.}} . \tag{3.4}$$

Here, the path integral is to be evaluated over all compact 4-geometries and regular field configurations that induce α and ϕ on the fixed 3-surface, and the subscript "E" means that the action is Euclidean.

When the full potential is kept in Eq. (2.8) we have,

$$\tilde{\Lambda}_{\text{eff}}(\phi)=\frac{1}{8\varepsilon}(1-e^{-2\phi})^2 . \tag{3.5}$$

The Wheeler-DeWitt Equation (2.12) includes a kinetic term for the ϕ field as well. At the classical level, the effect of such a term is well understood.[18–20] If the kinetic term were to dominate initially, it would quickly decay away and leave the effective cosmological constant dominant. That is, the $\tilde{\Lambda}_{\text{eff}}(\phi)$-driven phase is an attractor,[20] and the kinetic term becomes unimportant for classical inflationary evolution. We shall see that, for the quantum distribution over initial conditions as determined by the Hartle-Hawking or Vilenkin boundary conditions, the kinetic term is also unimportant. We simply extend the discussion with which this section began. The Euclidean/Lorentzian boundary in (α,ϕ) minisuperspace is the curve on which the potential for the Wheeler-DeWitt equation vanishes,

$$1=\frac{G}{36\pi\varepsilon}\alpha^2(1-e^{-2\phi})^2 . \tag{3.6}$$

We might say that a universe of given scalar field strength becomes Lorentzian when its size exceeds the horizon size determined by that field strength.

The evolution of the universe will follow classical equations of motion after the universe crosses the Classical/Quantum boundary in the Lorentzian regime of superspace. Creation of the universe then takes place on the Lorentzian side of the $V_{WDW}(\alpha,\phi)=0$ curve, but spaced away from it at the boundary of the semiclassical regime. The solution of the Wheeler-DeWitt equation, evaluated on this Classical/Quantum boundary (the "$t=0$" curve in Fig. 1), is the amplitude for creation of a classical Lorentzian universe with given (α,ϕ). The square modulus of this amplitude is the probability distribution over initial conditions for an inflationary universe. Since the values of α and ϕ are related to each other on the Classical/Quantum boundary (a curve near the Lorentzian/Euclidean curve, $V_{WDW}=0$), this distribution will depend on only one of the variables, and we shall take it to be ϕ. We thus interpret its value on the Classical/Quantum boundary, ϕ_i, as the initial "displacement" of this scalar field. In this sense we arrive at the chaotic inflationary picture.

IV. SOLUTIONS OF THE WHEELER-DEWITT EQUATION

We wish to find a distribution over initial conditions for the classical inflationary expansion described in Section II. This inflation takes place in the regime $\varepsilon R \gg 1$, and we will limit ourselves to exploring semiclassical quantum cosmology during this early epoch.

In principle, the Schroedinger equation and the Feynman path integral are two equivalent ways of computing quantum amplitudes and, again in principle, we can use either of the two methods in quantum cosmology. However, for purposes of computation and comparison of differing boundary condition proposals we prefer to adopt the point of view of the Wheeler-DeWitt equation. We accordingly first solve for the (approximate) general solution of the Wheeler-DeWitt equation. We then implement Vilenkin's boundary condition by a straightforward imposition of (3.3). And finally, we evaluate the path integral (3.4) in semiclassical approximation to determine Hartle and Hawking's wave function, but we shall think of this last procedure as fixing the specific Hartle-Hawking component of the general solution to the Wheeler-DeWitt equation.

In general, it is not possible to solve the partial differential Wheeler-DeWitt equation (2.12) in closed form — even to WKB approximation. We shall study the general solution only for small α. For larger values of α, we treat the ϕ degree of freedom perturbatively by expanding in $\exp(-2\phi)\sim 1/(\varepsilon R)$ on top of the semiclassical approximation.

A. The General Solution for Small α

For small α, it is possible to solve the Wheeler-DeWitt Equation (2.12) to all orders in $1/(\varepsilon R)$ [with our factor-ordering choice]. For $\alpha^2 \ll \alpha_*^2 \equiv 36\pi\varepsilon/G$, we can drop the α^4 term in Eq. (2.12),

$$\left\{ \frac{\partial^2}{\partial\alpha^2} - \frac{1}{\alpha^2}\frac{\partial^2}{\partial\phi^2} - \alpha^2 \right\} \Psi(\alpha,\phi) = 0 \; . \tag{4.1}$$

We stress that this equation is, in fact, correct to arbitrary order in $1/(\varepsilon R)$, since $(e^{-2\phi}-1)^2 < 1$ for the whole range of ϕ. Now, the equation separates. We write

$$\Psi(\alpha,\phi) = \mathcal{A}(\alpha)\Phi(\phi) \; . \tag{4.2}$$

The α part of the wave function, $\mathcal{A}(\alpha)$, satisfies

$$\left[\frac{d^2}{d\alpha^2}+\frac{\lambda^2}{\alpha^2}-\alpha^2\right]A(\alpha)=0\ , \tag{4.3}$$

where λ^2 is the separation constant. This has the general solution,

$$A(\alpha)=c_{\rm D}A_{\rm D}(\alpha)+c_{\rm G}A_{\rm G}(\alpha)\ , \tag{4.4}$$

where

$$A_{\rm D}\propto\alpha^{1/2}K_\nu(\alpha^2/2)\ \text{and}\ A_{\rm G}(\alpha)\propto\alpha^{1/2}I_\nu(\alpha^2/2)\ . \tag{4.5}$$

Here, $c_{\rm D}$ and $c_{\rm G}$ are constants, D and G subscripts denote decaying and growing modes respectively, and K and I are modified Bessel functions with index, $\nu=(1/4)(1-4\lambda^2)^{1/2}$ (cf. Ref. 21). For $\alpha\ll1$, we find

$$A(\alpha)=\begin{cases} c_{\rm G}\alpha^{(1/2)(1+4|\nu|)}+c_{\rm D}\alpha^{(1/2)(1-4|\nu|)} & \text{for}\quad 1-4\lambda^2>0 \\ c_{\rm G}\sqrt{\alpha}+c_{\rm D}\sqrt{\alpha}\ln\alpha & \text{for}\quad 1-4\lambda^2=0 \\ c_{\rm G}\sqrt{\alpha}\cos(s\ln\alpha)+c_{\rm D}\sqrt{\alpha}\sin(s\ln\alpha) & \text{for}\quad 1-4\lambda^2<0\ , \end{cases} \tag{4.6}$$

where $s=(1/2)\sqrt{4\lambda^2-1}$ and the constants $c_{\rm G}$ and $c_{\rm D}$ are proportional to those of Eq. (4.4). And for $1\ll\alpha\ll\alpha_*$, we have

$$A(\alpha)=c_{\rm D}\left(\frac{\pi}{\alpha}\right)^{1/2}e^{-\alpha^2/2}+c_{\rm G}\left(\frac{1}{\pi\alpha}\right)^{1/2}e^{\alpha^2/2}\ , \tag{4.7}$$

independent of ν as long as $\nu\ll\alpha^2/2$. This form best displays the merit of designating the two modes decaying and growing. The corresponding solution for the ϕ wave function is

$$\Phi(\phi)=\begin{cases} c_1 e^{i\lambda\phi}+c_2 e^{-i\lambda\phi} & \lambda\neq 0 \\ c_1\phi+c_2 & \lambda=0\,. \end{cases} \tag{4.8}$$

In this paper, we choose to apply the Vilenkin boundary condition in the Lorentzian regime $\alpha>\alpha_*$, and this will necessitate WKB expansion combined with expanding to successive orders in $e^{-2\phi}\sim 1/(\varepsilon R)$.

B. The Zeroth-Order Equation

At zeroth-order in $1/(\varepsilon R)$, we can write Eq. (2.12) as

$$\left[\frac{\partial^2}{\partial\alpha^2}-\frac{1}{\alpha^2}\frac{\partial^2}{\partial\phi^2}-\alpha^2\left(1-\frac{G}{36\pi\varepsilon}\alpha^2\right)\right]\Psi_{(0)}(\alpha,\phi)=0\,. \tag{4.9}$$

This equation also separates. We write, as before,

$$\Psi_{(0)}(\alpha,\phi)=\mathcal{A}(\alpha)\Phi(\phi) \tag{4.10a}$$

and obtain

$$\left[\frac{d^2}{d\alpha^2}+\frac{\lambda^2}{\alpha^2}-\alpha^2\left(1-\frac{G}{36\pi\varepsilon}\alpha^2\right)\right]\mathcal{A}(\alpha)=0 \tag{4.10b}$$

and

$$\left[\frac{d^2}{d\phi^2}+\lambda^2\right]\Phi(\phi)=0\,, \tag{4.10c}$$

where λ^2 is the separation constant.

We focus attention on the equation for α-dependence of the wave function (4.10b). This may be rewritten in the form of a Schroedinger equation,

$$-A''(\alpha)+U(\alpha)A(\alpha)=0 \ , \ \text{where} \ \ U(\alpha)=-\frac{\lambda^2}{\alpha^2}+\alpha^2\left[1-\frac{G}{36\pi\varepsilon}\alpha^2\right] \ . \qquad (4.11)$$

Note that the sign of the potential in Eq. (4.11) differs from the one in the Wheeler-DeWitt equation [Eq. (2.12)], so that here the contribution of ''matter'' terms is negative, while for ''gravity'' terms it is positive.

The qualitative features of the potential, $U(\alpha)$, especially near $\alpha=0$, will depend on the sign of the separation constant, λ^2. The three possible cases are displayed in Fig. 3 and we discuss them as follows:

(i) $\lambda^2>0$

Here, there are two qualitatively different subcases, both depicted in Fig. 3(a). If $\lambda^2>(1/3)(24\pi\varepsilon/G)^2$, then the Schroedinger potential, $U(\alpha)$, remains negative for all α, and the wave function, $A(\alpha)$, is oscillatory. For essentially the whole range of α, the semiclassical approximation is valid, and the corresponding classical solution describes (for small α) a Lorentzian universe expanding (*or* contracting) in a power law manner ($\sim\tau^{1/3}$) out of the big bang (or into the big crunch) singularity. If, on the other hand, we have $0<\lambda^2<(1/3)(24\pi\varepsilon/G)^2$, the potential is negative near $\alpha=0$, has a positive region between the two roots of the equation, $U(\alpha,\lambda^2>0)=0$, and then is negative again for large α. This corresponds to an oscillatory wave function near $\alpha=0$, exponential modes under the barrier, and oscillatory modes again outside. When $\lambda^2\gg1$ (recall $\varepsilon/G\simeq10^{11}$), the semiclassical approximation will be valid even in the region near $\alpha=0$. This would represent expanding (or contracting) Lorentzian classical behavior near $\alpha=0$, Euclidean trajectories through the barrier, and a Lorentzian universe outside for large α. When $\lambda^2\leq1$, the semiclassical approximation breaks down near $\alpha\simeq0$, and there will be a highly quantum mechanical region in which the evolution cannot be described in classical terms. In any event, for $\lambda^2>0$, the potential diverges near $\alpha=0$. The field energies,

$$-\frac{\hat{\pi}_\alpha^{\;2}}{\alpha}=\frac{1}{\alpha}\frac{d^2}{d\alpha^2}=\frac{U(\alpha)}{\alpha}=-\frac{\lambda^2}{\alpha^3}+\alpha\left[1-\frac{G}{36\pi\varepsilon}\alpha^2\right] \tag{4.12a}$$

and

$$\frac{\hat{\pi}_\phi^{\;2}}{\alpha^3}=-\frac{1}{\alpha^3}\frac{d^2}{d\phi^2}=\frac{\lambda^2}{\alpha^3}\,, \tag{4.12b}$$

both diverge at $\alpha\rightarrow 0$. We may think of this $\lambda^2>0$ case as describing expansion of a universe out of (or contraction of a universe into) a region of singular field energy (or a highly quantum mechanical region) around $\alpha=0$, perhaps followed (or preceded) by a tunneling through a barrier at finite size into continued expansion (or from continued contraction).

(ii) $\lambda^2<0$

In this case [shown in Fig. 3(b)], $U(\alpha)$ is positive near $\alpha=0$ and crosses over to negative at the one real root of the equation, $U(\alpha,\lambda^2<0)=0$. The wave function is exponential under the barrier, switching to oscillatory for α outside the barrier. This corresponds to a tunneling trajectory near $\alpha=0$ for a universe to appear (or disappear) as Lorentzian at finite size. In parallel to case (i) above, if $-\lambda^2<(1/3)(24\pi\varepsilon/G)^2$, there will be both a metastable minimum and a local maximum to the potential. And, as in the $\lambda^2>0$ case, both energies (4.12) diverge at $\alpha=0$. Again, the semiclassical approximation is or is not valid near $\alpha\approx 0$, depending on whether or not $-\lambda^2\gg 1$. Here, though, the singular region is hidden under the barrier.

(iii) $\lambda^2=0$

This special case avoids a divergence of the potential $U(\alpha)$ at $\alpha=0$, and it is displayed in Fig. 3(c). There are now two zeros of the potential, a root of $U(\alpha,\lambda^2=0)=0$ at the origin and another at $\alpha_*^2=(36\pi\varepsilon/G)$. There are two extrema, at $\alpha=0$ and $\alpha^2=\alpha_*^2/2$. The first is a minimum and the second, a maximum. This case allows exponential solutions under the barrier connecting to oscillatory modes outside (denoting Lorentzian expansion or contraction). It is rather remarkable that this case is the only one that avoids a singularity in the potential at $\alpha=0$. By itself, this divergence does not imply a singular wave function — just as the divergence of the effective potential for the radial equation in the elementary quantum mechanics central force problem [$l(l+1)$ plays the role of λ^2 here] does not imply a divergence of the wave function at the origin. However, in both the hydrogen atom and our model cosmology, the case without this divergence is special: it corresponds to the ground state. We shall see that both the Vilenkin and Hartle-Hawking requirements pick the $\lambda^2=0$ case.

C. Qualitative Discussion of Solutions

From Eq. (4.10c), we see that Vilenkin's boundary condition (3.3) picks out $\lambda^2=0$. And, we can see that the words, ''tunneling from nothing'', physically correspond to this case from Fig. 3. Only then do we avoid the region of separately singular field energies near $\alpha=0$ and can we associate the wave function with a tunneling amplitude from a classically stable minimum at the origin. Other choices for λ correspond to wave functions that describe an origin out of the region of separately singular field energies followed either by ''tumbling'' from some finite α, or a short-lived pass through the barrier. Occasionally,[12] regularity at the origin has been invoked as an explicit demand, but taken literally, the ''tunneling from nothing'' proposal (in the sense of tunneling from a state peaked at $\alpha \approx 0$) physically suggests $\lambda^2=0$. In our model, with our factor-ordering choice, this case (iii) corresponds to a nonsingular wave function at $\alpha=0$.

We shall see (in Section IV.D) that $\lambda^2=0$ is a mathematical consequence of the path integral formulation of the Hartle-Hawking proposal. By looking at the potential for the expansion degree of freedom, this assumes a special physical significance: Recall first that the true potential has opposite sign from the one used for visual display in Fig. 3. The higher the maximum on Fig. 3(a), the lower the total potential energy is for the expansion degree of freedom, α (the total energy, summed over both the α and ϕ degrees of freedom, is always zero). That is, case (iii), with $\lambda^2 \to 0$, is the minimum limit of case (i) in the potential for the α degree of freedom. In case (ii), for λ^2 not too large, the minimum is even lower, but it is then only a local minimum. Therefore, the top of the potential on Fig. 3(c), corresponding to case (iii) above, is the lowest *global* minimum for all gravitational and matter field configurations. Now, the value of the potential in Fig. 3(c) at maximum is $U_{max}={\alpha_*}^2/4$. Thus, when we restore to α_* its ϕ dependence, a state of lower energy will correspond to larger α_* and we might expect the ground state wave function to prefer a larger size for the start of the Lorentzian evolution, and consequently lower initial curvature.

As we shall see, the Hartle-Hawking wave function has this property. Understood in this way, Vilenkin's and Hartle and Hawking's boundary condition proposals are complementary: the former avoids the Euclidean connection between $\alpha \approx 0$ and the Lorentzian domain, while the latter is built around it (in the sense of being built from the path integral around trajectories in the Euclidean regime). This complementarity is reflected in the sign difference in the exponential of the square modulus of the wave function on the Classical/Quantum boundary, as we shall derive below [cf. Eqs. (5.8), (5.9)].

One should also keep in mind that the model could be excluded from some small but nonzero lengthscale, α_{cutoff} [pictured in Fig. 3(d)]. Then some nonzero λ^2s might be seen to be indistinguishable from $\lambda^2=0$. Other choices of λ might be used to mimic unknown physical effects from inside the cutoff region. Indeed, even without such a cutoff, there is no physical necessity, other than the physical motivations for the particular boundary condition proposals and their mathematical consequences, that the cases of nonzero λ^2 be excluded.

D. WKB Solutions to Zeroth Order in $1/(\varepsilon R)$

The preceding discussion has shown the general behavior of the solutions, how they differ in describing quantum creation, and where we might find the solutions that obey the two boundary condition proposals. We shall now work out the $\lambda^2=0$ case in detail and find the solutions explicitly.

Far under the barrier, for $\alpha^2 \ll 36\pi\varepsilon/G$, Eq. (4.10b) becomes (with $\lambda^2=0$)

$$\left[\frac{d^2}{d\alpha^2}-\alpha^2\right] A(\alpha)=0 \, . \tag{4.13}$$

The general solution for small α has been studied in Eqs. (4.4)—(4.7).

To take into account the other term in the potential of the Wheeler-DeWitt Equation (4.9), we use the WKB method (still requiring $\varepsilon R \gg 1$). The Wheeler-DeWitt potential, given by

$$V_{\mathrm{WDW}}=-\alpha^2\left[1-\frac{G}{36\pi\varepsilon}\alpha^2\right] , \tag{4.14}$$

vanishes at $\alpha=0$ and $\alpha=\alpha_*=(36\pi\varepsilon/G)^{1/2}$, so there are two WKB domains,

$$V_{\rm WDW}(\alpha)<0 \ , \ 0<\alpha<\alpha_* \ \text{and} \ V_{\rm WDW}(\alpha)>0 \ , \ \alpha>\alpha_* \ . \tag{4.15}$$

(Note that for ε large, the semiclassical approximation is valid very near the barrier.) The zeroth order WKB solution is obtained by

$$\mathcal{A}_{\rm WKB-(0)}(\alpha)=e^{-\mathcal{S}} \ \text{where} \ \left(\frac{d\,\mathcal{S}}{d\,\alpha}\right)^2=\alpha^2\left[1-\frac{G}{36\pi\varepsilon}\alpha^2\right] \ . \tag{4.16}$$

Its solutions are

$$\mathcal{A}_{\rm WKB-(0)}(\alpha)=\exp\left\{\pm\frac{{\alpha_*}^2}{3}\left[1-\left(1-\frac{\alpha^2}{{\alpha_*}^2}\right)^{3/2}\right]\right\} \ \text{for} \ 0<\alpha<\alpha_* \tag{4.17a}$$

and

$$\mathcal{A}_{\rm WKB-(0)}(\alpha)=\exp\left\{\pm\frac{{\alpha_*}^2}{3}\left[1+i\left(\frac{\alpha^2}{{\alpha_*}^2}-1\right)^{3/2}\right]\right\} \ \text{for} \ \alpha>\alpha_* \ . \tag{4.17b}$$

The first-order WKB corrections are then obtained by

$$\mathcal{A}_{\rm WKB-(1)}(\alpha)=\mathcal{C}(\alpha)e^{-\mathcal{S}(\alpha)} \ \text{where} \ \frac{d}{d\,\alpha}\left[\mathcal{C}^2(\alpha)\frac{d\,\mathcal{S}}{d\,\alpha}(\alpha)\right]=0 \ . \tag{4.18}$$

We solve to find

$$\mathcal{C}(\alpha)=\frac{C}{\left[\alpha^2\left(1-\frac{\alpha^2}{{\alpha_*}^2}\right)\right]^{1/4}} \ , \tag{4.19a}$$

where C is a constant. For a first-order WKB accuracy, we need keep only the leading part of this correction to find

$$\mathcal{C}(\alpha)=C\,\alpha^{-1/2} \qquad \text{for } 0<\alpha<\alpha_* \tag{4.19b}$$

$$\mathcal{C}(\alpha)=C\,\alpha_*^{\,1/2}\alpha^{-1}e^{\pm i\pi/4} \ \text{ for } \alpha>\alpha_* . \tag{4.19c}$$

We then summarize the behavior of the first-order WKB solutions for the case $\lambda^2=0$ by writing

$$\Psi_{(0),\mathrm{WKB}-(1)}(\alpha,\phi)=\mathcal{A}_{\mathrm{WKB}-(1)}(\alpha)\Phi(\phi) , \tag{4.20}$$

where we have the ϕ-dependence from Eq. (4.10c),

$$\Phi(\phi)=a\,\phi+b \ , \text{ and } a \text{ and } b \text{ are constants.} \tag{4.21}$$

For the α-dependence, we find

$$\mathcal{A}_{\mathrm{WKB}-(1)}(\alpha)=c_{\mathrm{D}}\mathcal{A}_{\mathrm{D,WKB}-(1)}(\alpha)+c_{\mathrm{G}}\mathcal{A}_{\mathrm{G,WKB}-(1)}(\alpha) \ \text{ for } 0<\alpha<\alpha_* \tag{4.22a}$$

and

$$\mathcal{A}_{\mathrm{WKB}-(1)}(\alpha)=c_{\mathrm{L}}\mathcal{A}_{\mathrm{L,WKB}-(1)}(\alpha)+c_{\mathrm{R}}\mathcal{A}_{\mathrm{R,WKB}-(1)}(\alpha) \ \text{ for } \alpha>\alpha_* . \tag{4.22b}$$

The modes themselves are (where constants have been fixed conveniently)

$$\mathcal{A}_{\mathrm{D,WKB}-(1)}(\alpha)=\alpha^{-1/2}\exp\left\{-\frac{\alpha_*^{\,2}}{3}\left[1-\left(1-\frac{\alpha^2}{\alpha_*^{\,2}}\right)^{3/2}\right]\right\} \tag{4.23a}$$

and

$$\mathcal{A}_{\mathrm{G,WKB}-(1)}(\alpha)=\alpha^{-1/2}\exp\left\{+\frac{\alpha_*^{\,2}}{3}\left[1-\left(1-\frac{\alpha^2}{\alpha_*^{\,2}}\right)^{3/2}\right]\right\} , \tag{4.23b}$$

both under the barrier, $0<\alpha<\alpha_*$. These forms (4.23a,b) then agree with the forms (4.7) to within multiplicative constants for $\alpha \gg 1$ and again, D and G subscripts denote decaying and growing modes, respectively. Outside the barrier we find

$$\mathcal{A}_{\mathrm{R,WKB-(1)}}(\alpha)=\frac{\alpha_*^{1/2}}{\alpha}\exp\left\{-\frac{\alpha_*^2}{3}-i\frac{\alpha_*^2}{3}\left[\frac{\alpha^2}{\alpha_*^2}-1\right]^{3/2}+i\frac{\pi}{4}\right\} \tag{4.23c}$$

and

$$\mathcal{A}_{\mathrm{L,WKB-(1)}}(\alpha)=\frac{\alpha_*^{1/2}}{\alpha}\exp\left\{+\frac{\alpha_*^2}{3}+i\frac{\alpha_*^2}{3}\left[\frac{\alpha^2}{\alpha_*^2}-1\right]^{3/2}-i\frac{\pi}{4}\right\} . \tag{4.23d}$$

The R and L subscripts denote right and left moving modes (keeping in mind that the energy of the gravitational expansion is negative) and correspond respectively to expanding and contracting classical trajectories — as can be seen by applying the operator $\hat{\pi}_\alpha=-i\,\partial/\partial\alpha$ $(=-\alpha(d\,\alpha/d\,\tau))$.

With the mode solutions in hand for the regime $\varepsilon R \gg 1$, we can proceed to implement the boundary condition proposals (3.3) and (3.4). Vilenkin's mode is just the one corresponding to the expanding classical trajectory. This is $\mathcal{A}_{\mathrm{R}}$ in the oscillatory domain and connects to $i\,\mathcal{A}_{\mathrm{D}}+\exp(-2\alpha_*^2/3)\mathcal{A}_{\mathrm{G}}/2$ under the barrier.[22] Since the wave function is ϕ-independent at $\alpha\to 0$, we have $\Phi(\phi)=b$ (a constant) and we determine Vilenkin's wave function,

$$\Psi_{\mathrm{V,(0),WKB-(1)}}(\alpha,\phi)\propto\begin{cases}\mathcal{A}_{\mathrm{R,WKB-(1)}}(\alpha) & \text{for } \alpha>\alpha_* \\ i\,\mathcal{A}_{\mathrm{D,WKB-(1)}}(\alpha)+\dfrac{e^{-\frac{2\alpha_*^2}{3}}}{2}\mathcal{A}_{\mathrm{G,WKB-(1)}}(\alpha) & \text{for } 0<\alpha<\alpha_* .\end{cases} \tag{4.24}$$

We see that for α small, the exponentially decaying mode dominates the other and hence justifies the tunneling interpretation of this solution.

From our qualitative discussion, we have some notion of the solution that obeys the Hartle-Hawking boundary condition, but to find it explicitly we have to evaluate the path integral. In the semiclassical approximation, we may write it (employing the conformal rotation procedure of Ref. 16 and Ref. 23) as

$$\Psi^{\text{semiclass.}}(\alpha,\phi)=\sum e^{-S_E} , \tag{4.25}$$

where the sum goes over all classical solutions that satisfy the imposed boundary condition on the path integral. The Euclidean action [with $\tau_E=i\tau$ denoting Euclidean time in Eq. (2.10c)] is

$$S_E=\frac{1}{2}\int d\tau_E\left\{-\left(\frac{d\alpha}{d\tau_E}\right)^2\alpha+\alpha^3\left(\frac{d\phi}{d\tau_E}\right)^2-\alpha\left[1-\left(\frac{\alpha}{\alpha_*}\right)^2\right]\right\} . \tag{4.26a}$$

This action, in turn, determines the Euclidean classical path,

$$\frac{d^2\phi}{d\tau_E{}^2}+\frac{3}{\alpha}\frac{d\alpha}{d\tau_E}\frac{d\phi}{d\tau_E}=0 \tag{4.26b}$$

and

$$2\alpha\frac{d^2\alpha}{d\tau_E{}^2}+\left(\frac{d\alpha}{d\tau_E}\right)^2+3\alpha^2\left(\frac{d\phi}{d\tau_E}\right)^2-1+3\frac{\alpha^2}{\alpha_*{}^2}=0 . \tag{4.26c}$$

We choose $\tau_E=0$ at $\alpha=0$ and consider paths that link $\alpha=0$ to the boundary α_0,ϕ_0 at $\tau_E>0$. The path that has finite ϕ and $d\alpha/d\tau_E$ at $\alpha=0$ is simply $\phi=\phi_0$, $\alpha=\alpha_*\sin(\tau_E/\alpha_*)$. (This is an expanding and contracting 3-sphere.) Inserting this into the action (4.26a), we find

$$S_E(\alpha,\phi)=-\frac{\alpha_*^{\,2}}{3}\left[1-\frac{\alpha^2}{\alpha_*^{\,2}}\right]^{3/2}, \tag{4.27}$$

and hence,

$$\Psi^{\text{semiclass.}}\propto\exp\left\{-\frac{\alpha_*^{\,2}}{3}\left[1-\frac{\alpha^2}{\alpha_*^{\,2}}\right]^{3/2}\right\}. \tag{4.28}$$

We see that the Hartle-Hawking boundary condition has automatically picked out the $\lambda^2=0$ case (because of the regularity of $d\phi/d\tau_E$ at $\alpha=0$).

We can now compare this expression with the semiclassical approximation to the general solution of the WDW equation, Eq.(4.23), to find

$$\Psi_{\text{H-H}}(\alpha,\phi)\propto \mathcal{A}_{\text{G,WKB}-(1)}(\alpha)\,. \tag{4.29}$$

A calculation of the Hartle-Hawking wave function without resorting to the semi-classical approximation is pursued in Ref. 24 for small α [Eq. (4.5)], where the same result is obtained. Thus, the growing mode, $\mathcal{A}_{\text{G}}$, is indeed the Hartle-Hawking wave function in the Euclidean regime, and we can connect this outside the barrier to $\exp(2\alpha_*^{\,2}/3)\mathcal{A}_{\text{R}}+\mathcal{A}_{\text{L}}$. We determine the wave function,

$$\Psi_{\text{H-H,(0),WKB}-(1)}(\alpha,\phi)\propto\begin{cases} e^{\frac{2\alpha_*^{\,2}}{3}}\mathcal{A}_{\text{R,WKB}-(1)}(\alpha)+\mathcal{A}_{\text{L,WKB}-(1)}(\alpha) & \text{for } \alpha>\alpha* \\ \mathcal{A}_{\text{G,WKB}-(1)}(\alpha) & \text{for } 0<\alpha<\alpha_*\,. \end{cases} \tag{4.30}$$

As explained in the literature,[16,25,26] one can as well compute the path integral directly in the Lorentzian regime to find that left and right moving components should contribute with equal weight. Our answer [keeping in mind our normalization of the modes, Eqs. (4.23)] confirms both this and our earlier qualitative guesses.

At this zeroth order of approximation in $1/(\varepsilon R)$ the amplitude of either solution is just a constant (apart from the power-law factor). Thus, we can discern the process of creation of an inflationary Lorentzian Universe, but we have not yet determined the ϕ or α dependent amplitude for its generation. To obtain this, we have to go to the next approximation in $1/(\varepsilon R)$.

E. Solutions to First Order in $1/(\varepsilon R)$

When terms of order $e^{-2\phi}$ are left in the potential, $V(\alpha,\phi)$, the Wheeler-DeWitt equation (2.12) is no longer separable and we are confronted with a two-dimensional WKB problem. Instead of working out the general two-dimensional WKB solution, we would like to restrict ourselves to its subclass that contains both the WKB approximation to Ψ_V and the WKB approximation to $\Psi_{H\text{-}H}$. We observe that both Ψ_V (4.24) and $\Psi_{H\text{-}H}$ (4.30) are ϕ-independent and yield a zero kinetic term for the ϕ field in Eq. (2.12). When terms of order $e^{-2\phi}$ are kept, this kinetic term will no longer be zero, but we expect that, for the Vilenkin and Hartle-Hawking wave functions, it will be of order $\alpha^{-2}e^{-2\phi}$. Hence, in the region of superspace where $\alpha \gg 1$ and $e^{-2\phi}$ is negligible, we can restrict ourselves to the subclass of wave functions that satisfy

$$\left\{\frac{\partial^2}{\partial\alpha^2}-\alpha^2\left[1-\frac{\alpha^2}{\alpha_*^{\,2}(\phi)}\right]\right\}\Psi(\alpha,\phi)=0\ , \tag{4.31}$$

where

$$\alpha_*^{\,2}(\phi)\equiv\alpha_*^{\,2}(1+2e^{-2\phi})\ . \tag{4.32}$$

As in Eq. (4.18), we write the WKB wave function,

$$\Psi_{(1),\mathrm{WKB}-(1)}(\alpha,\phi)=\mathcal{C}(\alpha,\phi)e^{-\mathcal{S}(\alpha,\phi)}\ . \tag{4.33}$$

The WKB equation for $\mathcal{S}(\alpha,\phi)$ is

$$\left(\frac{\partial \mathcal{S}}{\partial \alpha}\right)^2 - \alpha^2\left[1-\frac{G\alpha^2}{36\pi\varepsilon}(1-2e^{-2\phi})\right]=0 . \tag{4.34}$$

This equation for $\mathcal{S}$ is valid to first order in $e^{-2\phi}$ in the region of superspace where $1 \ll \alpha < \alpha_*(\phi)$ and $\alpha^4\alpha_*^{-2}e^{-2\phi} \gg 1$. In this region, the terms dropped from the equation are negligible compared to the terms included. Indeed, it is straightforward to evaluate the path integral with the Hartle-Hawking boundary condition in the semiclassical approximation in this region of superspace to see that it satisfies Eq. (4.34).

We solve (4.34) under the barrier to find

$$\mathcal{S}(\alpha,\phi)=\pm\frac{\alpha_*^2(\phi)}{3}\left(1-\frac{\alpha^2}{\alpha_*^2(\phi)}\right)^{3/2} \pm f(\phi) .$$

The integration constant, $f(\phi)$, is determined by matching this to the general solution of Eq. (2.12) in the limit $\alpha^2 \ll \alpha_*^2(\phi)$ [cf. Eq. (4.7)]. We see that requiring $\partial^2/\partial\phi^2 \sim e^{-2\phi}$ has restricted us to the case $\lambda^2=0$. From this we obtain

$$\mathcal{S}(\alpha,\phi)=\pm\frac{\alpha_*^2(\phi)}{3}\left[1-\left(1-\frac{\alpha^2}{\alpha_*^2(\phi)}\right)^{3/2}\right] . \tag{4.35}$$

In hindsight, we have merely performed the rather trivial substitution of the function, $\alpha_*(\phi)$, for the constant, α_*, in the zeroth-order solutions. The prefactor $\mathcal{C}$ in Eq. (4.32) can be obtained from this $\mathcal{S}$ and Eq. (4.31). We summarize the semiclassical wave functions to first order in $1/(\varepsilon R)$: Inside the barrier (for $1 \ll \alpha < \alpha_*(\phi)$) we have

$$\Psi_{D,(1),WKB-(1)}(\alpha,\phi)=\alpha^{-1/2}\exp\left\{-\frac{\alpha_*^2(\phi)}{3}\left[1-\left(1-\frac{\alpha^2}{\alpha_*^2(\phi)}\right)^{3/2}\right]\right\} \tag{4.36a}$$

and

$$\Psi_{G,(1),WKB-(1)}(\alpha,\phi)=\alpha^{-1/2}\exp\left\{+\frac{\alpha_*^2(\phi)}{3}\left[1-\left(1-\frac{\alpha^2}{\alpha_*^2(\phi)}\right)^{3/2}\right]\right\}. \tag{4.36b}$$

Outside the barrier (for $\alpha>\alpha_*(\phi)$), we have, by analytic continuation,

$$\Psi_{R,(1),WKB-(1)}(\alpha,\phi)=\frac{\alpha_*^{-1/2}}{\alpha}\exp\left\{-\frac{\alpha_*^2(\phi)}{3}\left[1+i\left(\frac{\alpha^2}{\alpha_*^2(\phi)}-1\right)^{3/2}\right]+i\frac{\pi}{4}\right\} \tag{4.36c}$$

and

$$\Psi_{L,(1),WKB-(1)}(\alpha,\phi)=\frac{\alpha_*^{-1/2}}{\alpha}\exp\left\{+\frac{\alpha_*^2(\phi)}{3}\left[1+i\left(\frac{\alpha^2}{\alpha_*^2(\phi)}-1\right)^{3/2}\right]-i\frac{\pi}{4}\right\}. \tag{4.36d}$$

The wave function that obeys Vilenkin's boundary condition (''tunneling from nothing'') is

$$\Psi_{V,(1),WKB-(1)}(\alpha,\phi)\propto\begin{cases} i\,\Psi_{D,(1),WKB-(1)}(\alpha,\phi)+\dfrac{e^{-\frac{2\alpha_*^2(\phi)}{3}}}{2}\Psi_{G,(1),WKB-(1)}(\alpha,\phi) & \text{for } 1\ll\alpha<\alpha_*(\phi) \\ \Psi_{R,(1),WKB-(1)}(\alpha,\phi) & \text{for } \alpha>\alpha_*(\phi)\,. \end{cases} \tag{4.37}$$

The wave function that obeys Hartle and Hawking's boundary condition (''no boundary'') is

$$\Psi_{\text{H-H},(1),\text{WKB}-(1)}(\alpha,\phi)\propto\begin{cases}\Psi_{\text{G},(1),\text{WKB}-(1)}(\alpha,\phi) \\ \qquad\qquad \text{for } 1\ll\alpha<\alpha_*(\phi) \\ e^{\frac{2\alpha_*^2(\phi)}{3}}\Psi_{\text{R},(1),\text{WKB}-(1)}(\alpha,\phi)+\Psi_{\text{L},(1),\text{WKB}-(1)}(\alpha,\phi) \\ \qquad\qquad \text{for } \alpha>\alpha_*(\phi)\ .\end{cases} \tag{4.38}$$

V. CONCLUSION

We are now in a position to find the distribution over the initial conditions for chaotic inflation. The classical evolution of the universe begins when the phase of the wave function is rapidly oscillating; i.e., $\mathcal{S}(\alpha,\phi)\gg1$. The Classical/Quantum boundary, $\alpha=\alpha_c$, is very close to the Euclidean/Lorentzian boundary $\alpha=\alpha_*(\phi)$ in our present model because of the large parameter, $\varepsilon/G\approx10^{11}$. The evolution is classical when α is slightly larger than $\alpha_*(\phi)$:

$$\frac{\alpha}{\alpha_*(\phi)}>1+\frac{G}{36\pi\varepsilon}(1-e^{-2\phi})\ . \tag{5.1}$$

It is straightforward to determine how the initial conditions of a classical trajectory (α, ϕ, $d\alpha/d\tau$, and $d\phi/d\tau$) are correlated. The rapidly oscillating phase, $\mathcal{S}(\alpha,\phi)$, of the wave function in the classical region is given by

$$\mathcal{S}_{\pm}(\alpha,\phi)=\mp\frac{\alpha_*^{\,2}(\phi)}{3}\left(\frac{\alpha^2}{\alpha_*^{\,2}(\phi)}-1\right)^{3/2}\ , \tag{5.2}$$

where the upper sign ($\mathcal{S}_+$) corresponds to Ψ_R and the lower sign ($\mathcal{S}_-$) corresponds to Ψ_L.

The initial "velocities" of a classical trajectory are correlated to α and ϕ as

$$\alpha^3 \frac{d\phi}{d\tau} = \left[\frac{\partial \mathcal{S}_\pm}{\partial \phi} \right]_{\alpha=\alpha_c} \approx \pm \left[\frac{\alpha^3}{6\alpha_*{}^3} \frac{\partial}{\partial \phi} (\alpha_*{}^2) \right]_{\alpha=\alpha_c} \tag{5.3}$$

and

$$-\alpha \frac{d\alpha}{d\tau} = \left[\frac{\partial \mathcal{S}_\pm}{\partial \alpha} \right]_{\alpha=\alpha_c} \approx \pm \left[\frac{\alpha}{\alpha_*} \right]_{\alpha=\alpha_c} , \tag{5.4}$$

where the right-hand sides are to be evaluated on the Classical/Quantum boundary $\alpha=\alpha_c \approx \alpha_*(\phi)$. These relations can be transformed back to the physical variables a, R, da/dt, and dR/dt: On the Classical/Quantum boundary,

$$a \approx \left[\frac{12}{R} \right]^{1/2} , \tag{5.5}$$

$$\frac{dR}{dt} \approx \mp \frac{2}{3\varepsilon} \left[\frac{R}{12} \right]^{1/2} , \tag{5.6}$$

and

$$H = \frac{1}{a} \frac{da}{dt} \approx \pm \left[\frac{R}{12} \right]^{1/2} , \tag{5.7}$$

where the upper (lower) sign corresponds to Ψ_R (Ψ_L). We see that the Ψ_L component in the Hartle-Hawking wave function (4.38) corresponds to a collapsing universe and is irrelevant to our cosmological observations.[26] The correlation (5.7) is particularly important: It indicates that the Ψ_R component corresponds to a universe that begins its classical evolution with $R_i \approx 12 H_i{}^2$, i.e., right at the beginning of the linear phase described by Eq. (2.4) (cf. Ref. 8 for detailed analysis of the classical evolution). If H

and R were not so correlated on the Classical/Quantum boundary, there would be some complicated dynamical evolution before the universe would follow an inflationary trajectory.

With the correlations, Eqs. (5.5)—(5.7), the distributions over initial conditions are determined by the distributions over R_i (or equivalently, ϕ_i), and we need just the square modulus of the wave function in the Lorentzian regime near the Classical/Quantum boundary. Unlike the correlations, (5.5)—(5.7), which take the same form for both Ψ_V and $\Psi_{H\text{-}H}$ (or the Ψ_R component of $\Psi_{H\text{-}H}$), the distributions over R_i differ for the two wave functions. For Vilenkin's boundary condition, it is

$$|\Psi_V(\alpha,\phi)|^2 \propto \frac{\alpha_*}{\alpha^2} e^{-\frac{2\alpha_*^2(\phi)}{3}} . \tag{5.8}$$

For Hartle and Hawking's boundary condition, it is (with the irrelevant component, Ψ_L, dropped)

$$|\Psi_{H\text{-}H}(\alpha,\phi)|^2 \propto \frac{\alpha_*}{\alpha^2} e^{+\frac{2\alpha_*^2(\phi)}{3}} . \tag{5.9}$$

We express this probability distribution near the Lorentzian boundary as a function of (initial) curvature, R_i. We have

$$dP(R_i) \propto d\mu(R_i)\,|\Psi(\alpha(R_i),R_i)|^2 . \tag{5.10}$$

The measure $d\mu(R_i)$ is just the line element along the Classical/Quantum boundary, given through the natural metric on minisuperspace,[27] which can be read off the kinetic terms in the Wheeler-DeWitt Hamiltonian (2.11). It is

$$\delta\sigma^2 = -\alpha\delta\alpha^2 + \alpha^3\delta\phi^2 . \tag{5.11}$$

For $\varepsilon R_i \gg 1$, the Classical/Quantum boundary is simply α=const., and we have

$$d\mu(R_i) \propto d\phi = \frac{dR_i}{R_i} \ . \tag{5.12}$$

We finally obtain the probability distributions [using $\alpha_*^2(\phi) \approx (36\pi\varepsilon/G)(1+1/(\varepsilon R_i))$],

$$dP_{\mathrm{V}} \propto \frac{dR_i}{R_i} e^{-\frac{24\pi}{GR_i}} \tag{5.13a}$$

and

$$dP_{\text{H-H}} \propto \frac{dR_i}{R_i} e^{+\frac{24\pi}{GR_i}} \ . \tag{5.13b}$$

We sketch the distributions in Fig. 4. Remarkably, they are complementary. This is not something that we have been able to expect from the formulation of the boundary conditions [Eqs. (3.3) and (3.4)], but we hope to have gleaned some qualitative understanding through the discussion of Sections III and IV.

We should note that although we have called the distribution (5.13a) Vilenkin's distribution, a distribution of just this type (for the specific model of a self-interacting scalar inflaton field) has been proposed on its own merit.[28] Namely, this distribution gives a preference for Planck-scale creation, which is what one might expect from quantum cosmology. Here, high values of the curvature are favored and the universe is more likely to be small after tunneling. The power-law factor that bends the distribution, thereby creating a maximum, could be modified with, say, a different factor ordering. However, the maximum will remain very roughly on the Planck scale. The value of this maximum is, in fact, somewhat above the Planck scale, $R_{i,\max} = 24\pi l_{\mathrm{Pl}}^{-2}$.

The Hartle-Hawking distribution prefers the universe to start out at low curvature. The universe starts out in the linear phase, and R can only decrease in the subsequent classical evolution. In light of our earlier qualitative discussion of their boundary condition (Section IV.C), this is not surprising — we expect the typical Hartle-Hawking Universe to be born large and spend not too many e-foldings in the inflationary phase.

In numbers, the difference between the two proposals is dramatic. We normalize both distributions in our target range (see Section I), $\varepsilon^{-1}<R_i<<l_{\mathrm{Pl}}^{-2}$, and find that the likelihood of an inflationary phase that would at least be sufficient for the current horizon volume (in one trial "universe") is

$$P_{\mathrm{V}}(R_i \geq R_h)=1-e^{-8\times10^{10}} \tag{5.14a}$$

or

$$P_{\mathrm{H\text{-}H}}(R_i \geq R_h)=e^{-10^{12}}, \tag{5.14b}$$

where $R_h \sim 10^{-9} l_{Pl}^{-2}$ is the value of the curvature at which the perturbation, whose wavelength today is equal to the horizon size, crossed the horizon during inflation [cf. Eq. (2.6b)].

The simplest interpretation of this result follows: Like every quantum probability, this distribution represents a set of classical outcomes. In this case, the set is the ensemble of "all possible universes" that could be created on the restricted Classical/Quantum boundary. Our universe is one such outcome, a particular result of a single process of quantum creation. As we understand it, our universe apparently attained its long age, remarkable flatness, homogeneity, and isotropy because of an initial inflationary phase of at least ~70 e-foldings. Thus, within this interpretation, and within the adopted restrictions of the model, we can say only that if the boundary condition is the one proposed by Vilenkin, our universe is a highly typical product of quantum creation. If the boundary condition is the one proposed by Hartle and Hawking, our universe is an atypical event.

Of course, we have dealt with a severely limited model, and a better statement will have to await more realistic analysis (treating other degrees of freedom, both in the gravitational and matter sectors). However, we do not expect the basic picture to become very different. Studies so far in this model[20] and other models (at the classical[29,30] and quantum[31] levels) have shown that the effects of a large initial kinetic term, a large initial anisotropy, and a small initial inhomogeneity all become rapidly unimportant. Thus, apart from the unexplored case of a large initial inhomogeneity, the dominant input from quantum dynamics is in the distribution over initial curvature (or size), which was the subject of this work. In another approach to this model,[32] the probability of $R+\varepsilon R^2$ inflation is studied, using the canonical measure of Ref. 33. It is shown that this canonical measure leaves open the question of the predominance of inflationary over non-inflationary trajectories.

As for the interpretation of (5.5)—(5.7) and (5.13a,b) advocated here, we confront what may be expected to become a general feature of quantum cosmology: Two (or more) competing hypotheses lead to predictions (probability statements) that include our universe as an outcome (where ''our universe'' means a classical model that agrees with observation as far as it goes). We are left with two possible criteria to judge such hypotheses. First, we might prefer the hypothesis that is more readily extendible to more and more realistic models. Only further refinement of quantum cosmology can explore this possibility. Second, we might prefer the hypothesis that shows that our universe is the more probable outcome. This arguably more observationally based criterion can be weakened when we allow the interjection of some form of an anthropic principle, which can exclude from our consideration cosmological outcomes that are not likely to be like our universe (not likely to evolve observers to observe them). We simply state the horns of this dilemma because our rather unrefined model makes any choice premature.

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APPENDIX: THE BOUNDARY TERM FOR $R+\varepsilon R^2$ GRAVITY

The action (2.1), just as is well known for Einstein gravity (i.e., ε=0), does not lead to a well-posed variational problem without a boundary term. That is, we wish to extremize the classical action under arbitrary variations of the metric, which vanish on the boundary. In general, however, the varied action can depend on variations of *derivatives* of the metric on the boundary which, indeed, need not vanish. The boundary term is required in order to cancel just these surface variations of metric derivatives. Further, as a quantitative piece of the action, the boundary term plays a necessary role in evaluation of the path integral for the Hartle-Hawking approach to the wave function. Therefore, we sketch here its derivation for $R+\varepsilon R^2$ gravity.

There are two methods of derivation we consider. The first is straightforward: If we stay within the full fourth-order theory, we have only to rework carefully a usual (ε=0) derivation of the boundary term. We start from the "bare" gravitational action (2.1) — the action without boundary term and without matter content (L_m=0) or cosmological constant (Λ=0). This bare action we write as

$$S'=\frac{1}{16\pi G}\int d^4x\,(-g)^{1/2}[R+\varepsilon R^2]\,. \tag{A1}$$

A head-on algebraic assault will find

$$\delta S'=\frac{1}{16\pi G}\int(-g)^{1/2}\Big\{R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R$$

$$+2\varepsilon[R\,(R_{\mu\nu}-\frac{1}{4}g_{\mu\nu}R\,)+R_{;\alpha\beta}(g^{\alpha\beta}g_{\mu\nu}-\delta^{\alpha}{}_{\mu}\delta^{\beta}{}_{\nu})]\Big\}\,\delta g^{\mu\nu}$$

$$-\frac{1}{16\pi G}\int d^4x\,(-g)^{1/2}[(1+2\varepsilon R\,)_{;\delta}(g^{\delta\lambda}g_{\sigma\tau}-\delta^{\delta}{}_{\sigma}\delta^{\lambda}{}_{\tau})\delta g^{\sigma\tau}]_{;\lambda}$$

$$+\frac{1}{16\pi G}\int d^4x\,(-g)^{1/2}[(1+2\varepsilon R\,)g^{\alpha\lambda}g^{\gamma\delta}(\delta g_{\alpha\gamma;\delta}-\delta g_{\gamma\delta;\alpha})]_{;\lambda}\,. \tag{A2}$$

Here we see three distinct terms emerge. The first term merely displays the $R+\varepsilon R^2$ field equations, the second term yields a surface integral, which will give only vanishing contributions on the boundary (because variations of the metric itself vanish on the boundary), and the last term gives also a surface integral which, however, need not vanish. We can, though, follow almost directly the argument of Wald[34] (given there for the ε=0 case) to rewrite the third term as

$$\frac{1}{16\pi G}\int d^4x\,(-g)^{1/2}[(1+2\varepsilon R\,)g^{\alpha\lambda}g^{\gamma\delta}(\delta g_{\alpha\gamma;\delta}-\delta g_{\gamma\delta;\alpha})]_{;\lambda}$$

$$=-\frac{1}{8\pi G}\int d^3x\sqrt{h}\,(1+2\varepsilon R)\delta K+\text{vanishing surface pieces.} \tag{A3}$$

Now ignoring the contributions that vanish by reason of vanishing metric variations on the boundary, we obtain the form,

$$\delta S'=\frac{1}{16\pi G}\int d^4x(-g)^{1/2}\left\{\text{ field equations }\right\}\delta g^{\mu\nu}-\frac{1}{8\pi G}\int d^3x\sqrt{h}\,(1+2\varepsilon R)\delta K\ . \tag{A4}$$

One final twist not present in the ε=0 case is readily verified:

$$-\frac{1}{8\pi G}\int d^3x\sqrt{h}\,(1+2\varepsilon R)\delta K=-\frac{1}{8\pi G}\delta\int d^3x\sqrt{h}\,(1+2\varepsilon R)K\ . \tag{A5}$$

We thus derive the boundary term for the action (2.1),

$$S=S'+\frac{1}{8\pi G}\int d^3x\sqrt{h}\,(1+2\varepsilon R)K\ . \tag{A6}$$

The conformal picture inspires another derivation. Whitt[14] originally transformed the equations of motion for the fourth-order theory into the Einstein equations and then read off an action for the scalar degree of freedom. Here, we shall transform directly the action (A1) and shall find the boundary term on the way. We are motivated in this approach because we know from the analysis at linearized level[10] that the theory (2.1) has in its spectrum a scalar degree of freedom with mass, m^2=1/(6ε) (with ε/G large). We also know that the scalar curvature itself obeys an equation of motion,

$$R_{;\alpha\beta}g^{\alpha\beta}-\frac{R}{6\varepsilon}=0\ . \tag{A7}$$

We can thus take the scalar curvature to be an interpolating field for that same scalar degree of freedom. With our conventions, a mass term will appear in the action with a (–) sign. Therefore, we begin with the action in the physical picture,

$$S=\frac{1}{16\pi G}\int d^4x(-g)^{1/2}[R+\varepsilon R^2]+\text{b. t.} \ , \tag{A8}$$

where *b.t.* stands for the unknown boundary term. We then split the quadratic term according to

$$R+\varepsilon R^2=R(1+p\,\varepsilon R)-(p-1)\varepsilon R^2 \ , \tag{A9}$$

to simulate a mass term (here p is some real number). The expression that multiplies the scalar curvature in the first term can be removed by a conformal transformation to leave a pure Einstein action. We need

$$\tilde{g}_{\alpha\beta}=\Omega g_{\alpha\beta} \, , \quad (-\tilde{g})^{1/2}=\Omega^2(-g)^{1/2} \, , \text{ and } \quad R=\Omega\tilde{R}+3\tilde{\Box}\Omega-\frac{9}{2}(\Omega)^{-1}\Omega_{,\rho}\Omega_{,\sigma}\tilde{g}^{\rho\sigma}. \tag{A10}$$

The special choice, $\Omega=1+p\,\varepsilon R$, does the trick. After one integration by parts of the second term, we obtain a kinetic term for the R field and a surface term is generated:

$$S=\frac{1}{16\pi G}\int d^4x(-\tilde{g})^{1/2}\left\{\tilde{R}-\frac{3}{2}\frac{p^2\varepsilon^2}{(1+p\,\varepsilon R)^2}\left[R_{,\beta}R_{,\delta}\tilde{g}^{\beta\delta}+\frac{2}{3}\frac{(p-1)}{p^2\varepsilon}R^2\right]\right\}$$

$$+\text{b. t.}-\frac{1}{8\pi G}\int d^3x(h)^{1/2}\frac{3}{2}(1+p\,\varepsilon R)_{,t} \ . \tag{A11}$$

We are led uniquely to the choice $p=2$ to secure the correct mass term at linear order. The kinetic term for the scalar degree of freedom can be brought into canonical form by the substitution $\Phi=(1/2)\ln(1+2\varepsilon R)$ [cf. Eq.(2.8)]. Since manifestly, we have just a scalar field plus Einstein gravity, the original boundary term and the surface term in Eq.(A11) should combine to give the standard boundary term for Einstein gravity.

In the 3+1 split of spacetime in which the boundary is a t=*const.* slice, we make use of the identity,

$$\tilde{K}\,(\tilde{h}\,)^{1/2} = \Omega K \sqrt{h} - \frac{3}{2}\sqrt{h}\,\Omega_{,t} \quad , \tag{A12}$$

and find

$$\text{b. t.} = \frac{1}{16\pi G}\int d^3x \sqrt{h}\; 2K\,(1+2\varepsilon R\,) \quad , \tag{A13}$$

as we found in Eq. (A6).

One remaining question is: Just what fields are to be held fixed on the boundary? In the conformal picture, the answer is straightforward: As the theory is only Einstein gravity plus a scalar field, we need only fix the field ϕ and the conformal 3-metric on the boundary to obtain a well-posed variational problem.[34] From the conformal transformation (A10), with $\Omega = 1+2\varepsilon R$, we see that this corresponds to fixing the physical 3-metric and scalar curvature on the boundary.

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FIGURE CAPTIONS FOR FIGURES 1, 2, 3, AND 4, CHAPTER 3

FIG. 1. Regions of minisuperspace. The thick line is the Euclidean/Lorentzian boundary, $a^2=24\varepsilon(1+2\varepsilon R)/(2\varepsilon R)^2$ [cf. Eq. (3.6) with $\alpha=(3\pi/2G)^{1/2}(1+2\varepsilon R)^{1/2}a$ and $\phi=(1/2)\ln(1+2\varepsilon R)$]. Shaded regions of the figure are excluded from consideration, either because they are too far into the quantum domain for us to have any confidence in our model (region to the top of the figure with $R>l_{Pl}^{-2}$), or because the classical solutions that do exist are not inflationary (region at the bottom of the figure with $R<\varepsilon^{-1}$). Classical inflationary trajectories start at "t=0" (the dashed line — Classical/Quantum boundary), which is slightly away from the Euclidean/Lorentzian boundary.

FIG. 2. Plot of the effective cosmological constant, $\tilde{\Lambda}_{eff}(\phi)=(1/8\varepsilon)(1-e^{-2\phi})^2$ [cf. Eq. (3.5)]. For ϕ large, this is a constant, $\approx 1/(8\varepsilon)$, and we are in the regime of validity of the analysis of the early part of Section III.

FIG. 3. The effective potential for the Schroedinger-like Eq. (4.11) for the expansion degree of freedom in the separation limit, $\phi\rightarrow\infty$ (cf. the qualitative analysis of Sections IV.B and IV.C). The subfigures display the potential for different values of the separation constant, λ^2 (where λ^2/α^3 corresponds to the kinetic energy in the ϕ field): (a) — Typical potential curves for the separation constant positive [$\lambda^2>0$, case (i) of Section IV.B]; (b) — Typical potential curves for the separation constant negative [$\lambda^2<0$, case (ii) of Section IV.B]; (c) — The heavy line displays the potential curve for the separation constant equal to zero [$\lambda^2=0$, case (iii) of Section IV.B]. Dashed curves display the potential curves for nonzero values of the separation parameter. Only $\lambda^2=0$ avoids the region of singular field energies at $\alpha=0$; (d) — Shows the near equivalence of potentials with different values of the separation constant, where a short-distance region has been excluded from the theory (shaded region excluded for $\alpha<\alpha_{cutoff}$, cf.

end of Section IV.C).

FIG. 4. The distributions over initial curvature, R_i, for classical inflationary trajectories are plotted — as determined by the boundary condition proposals of quantum cosmology. The distribution derived from the wave function obeying Vilenkin's boundary condition is peaked to the right, preferring large values of the initial curvature [cf. Eq. (5.7a)]. The distribution derived from the wave function obeying Hartle and Hawking's boundary condition is peaked to the left, preferring low values of the initial curvature [cf. Eq. (5.7b)]. We have restricted our analysis to $\varepsilon^{-1} \approx 10^{-11}\, l_{\mathrm{Pl}}^{-2} < R_i < l_{\mathrm{Pl}}^{-2}$ for reasons elaborated in the Introduction. An inflation sufficient that is consistent with observations has to start at $R_i > R_h \approx 10^{-9}\, l_{\mathrm{Pl}}^{-2}$.

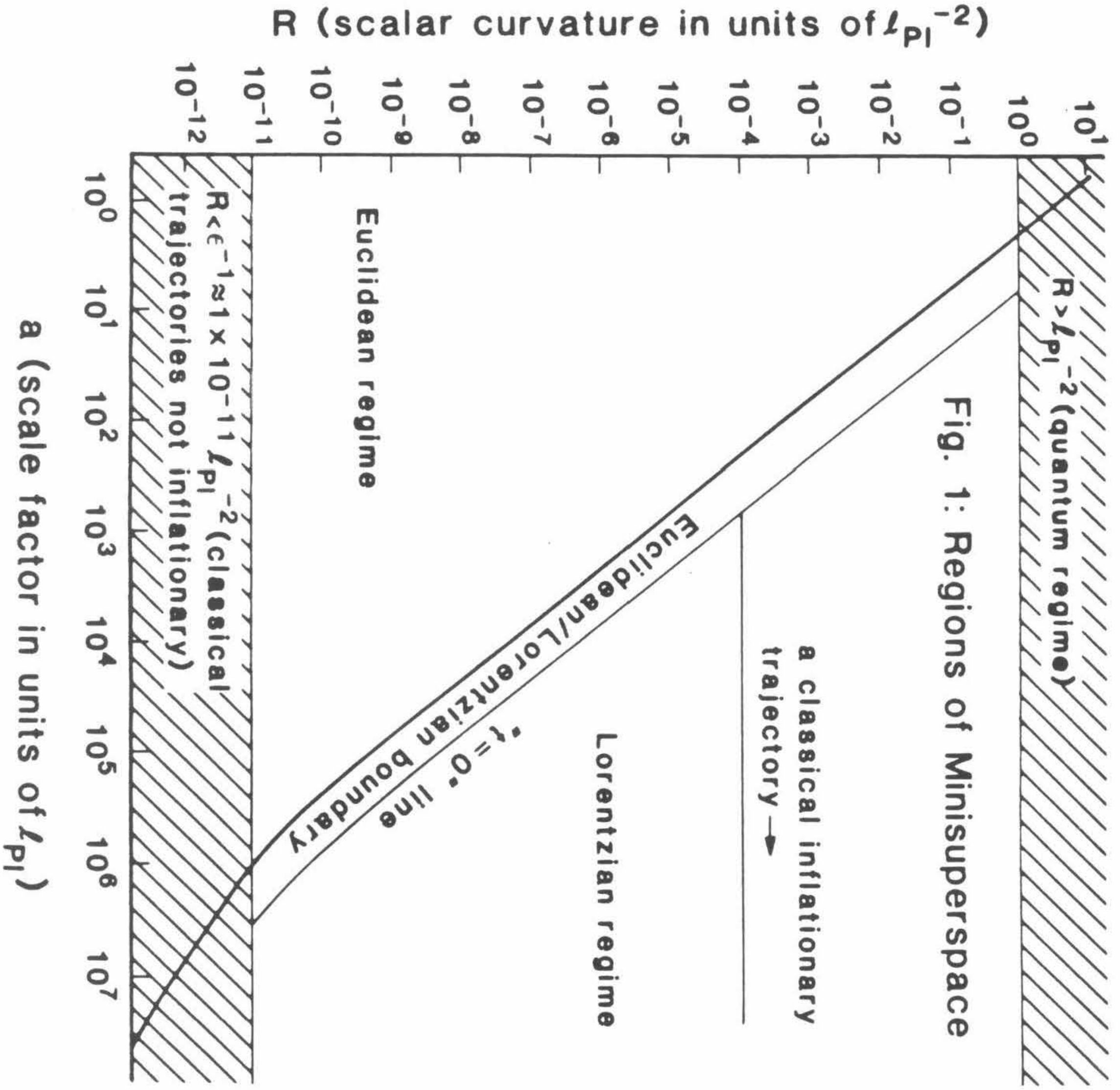

Fig. 1: Regions of Minisuperspace

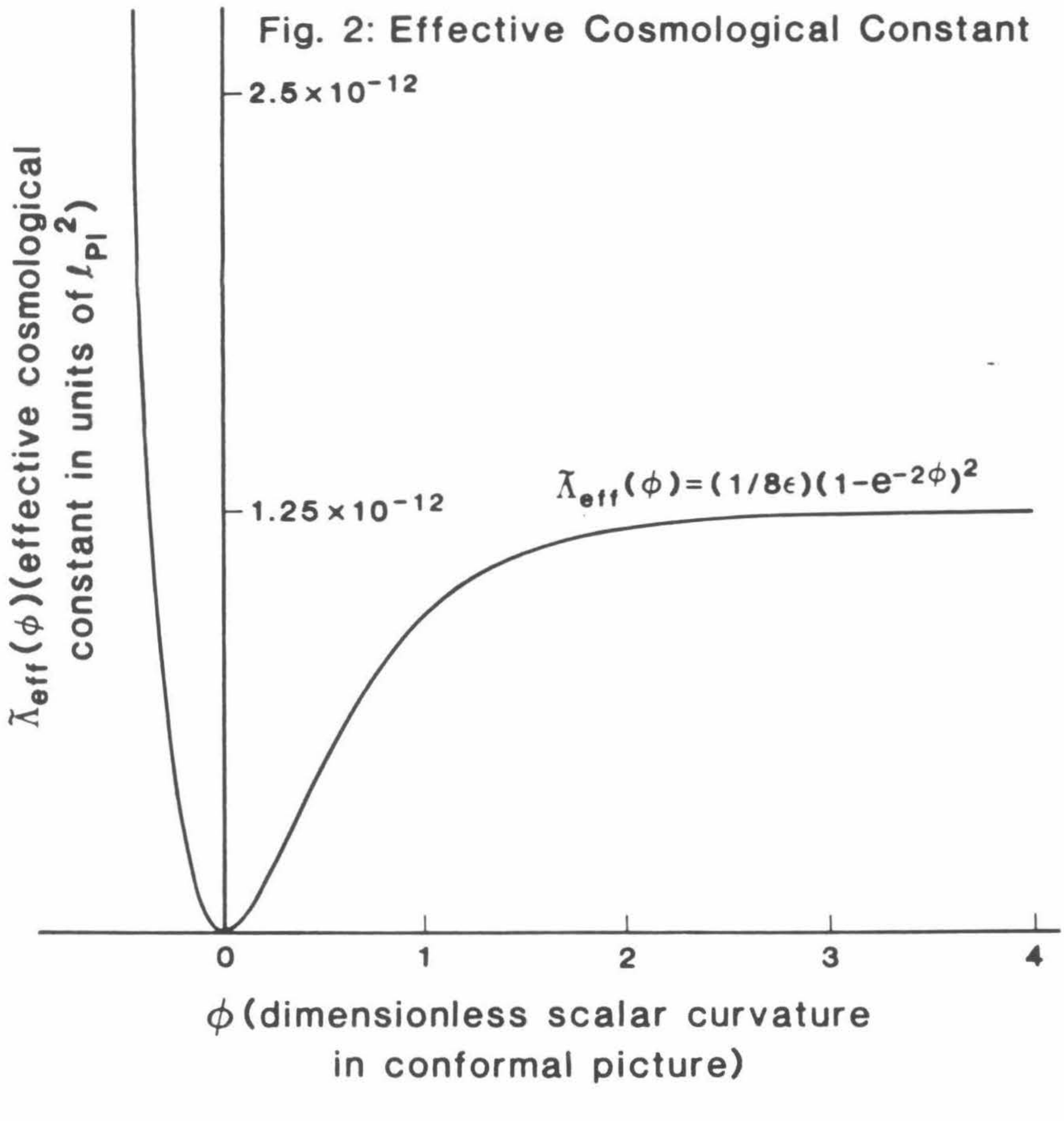
Fig. 2: Effective Cosmological Constant
2.5×10⁻¹²
1.25×10⁻¹²
$\bar{\Lambda}_{eff}(\phi)=(1/8\epsilon)(1-e^{-2\phi})^2$
$\bar{\Lambda}_{eff}(\phi)$(effective cosmological constant in units of ℓ_{Pl}^2)
0
1
2
3
4
ϕ(dimensionless scalar curvature in conformal picture)

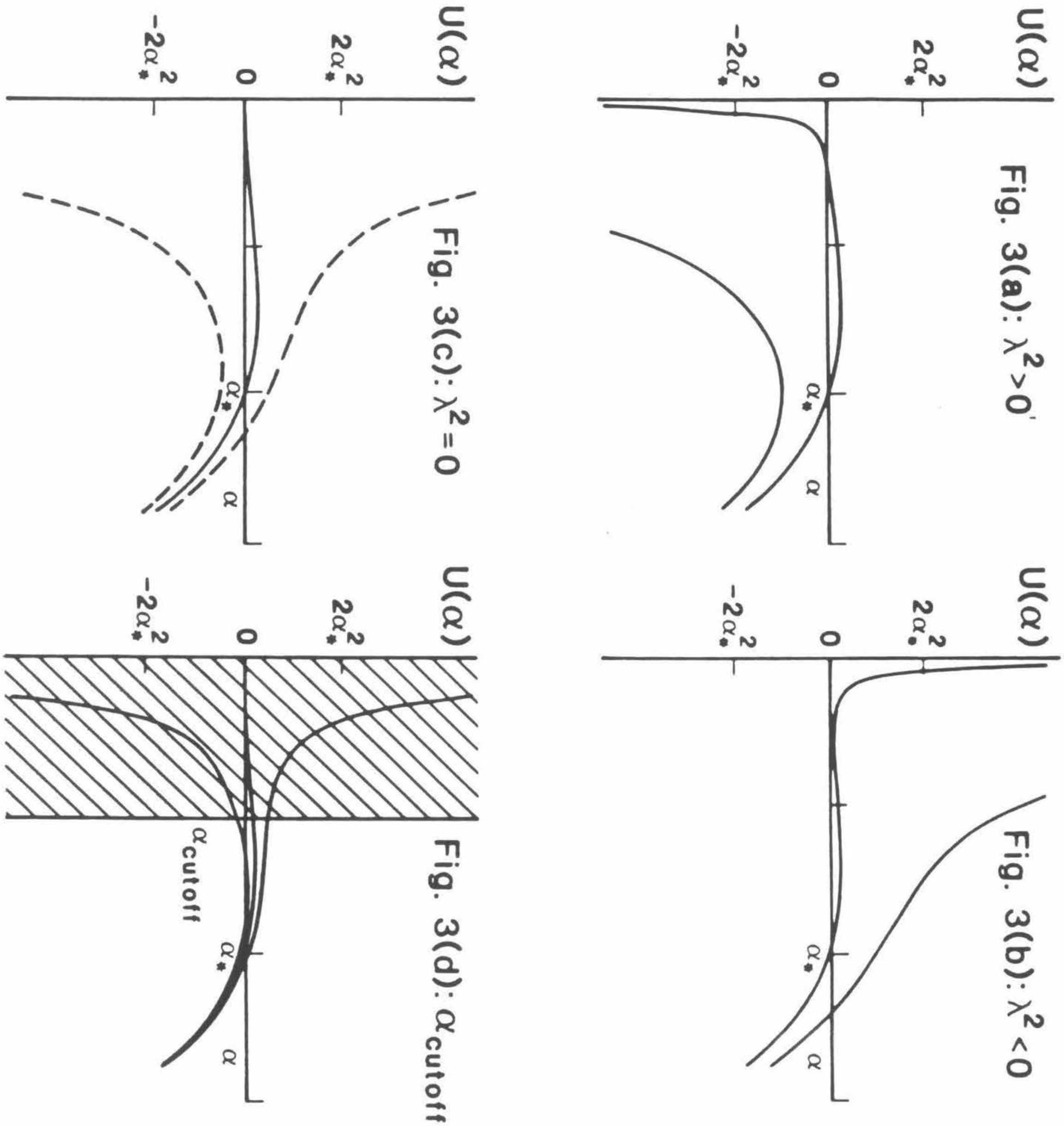

Fig. 3(a): $\lambda^2>0$

Fig. 3(b): $\lambda^2<0$

Fig. 3(c): $\lambda^2=0$

Fig. 3(d): α_{cutoff}

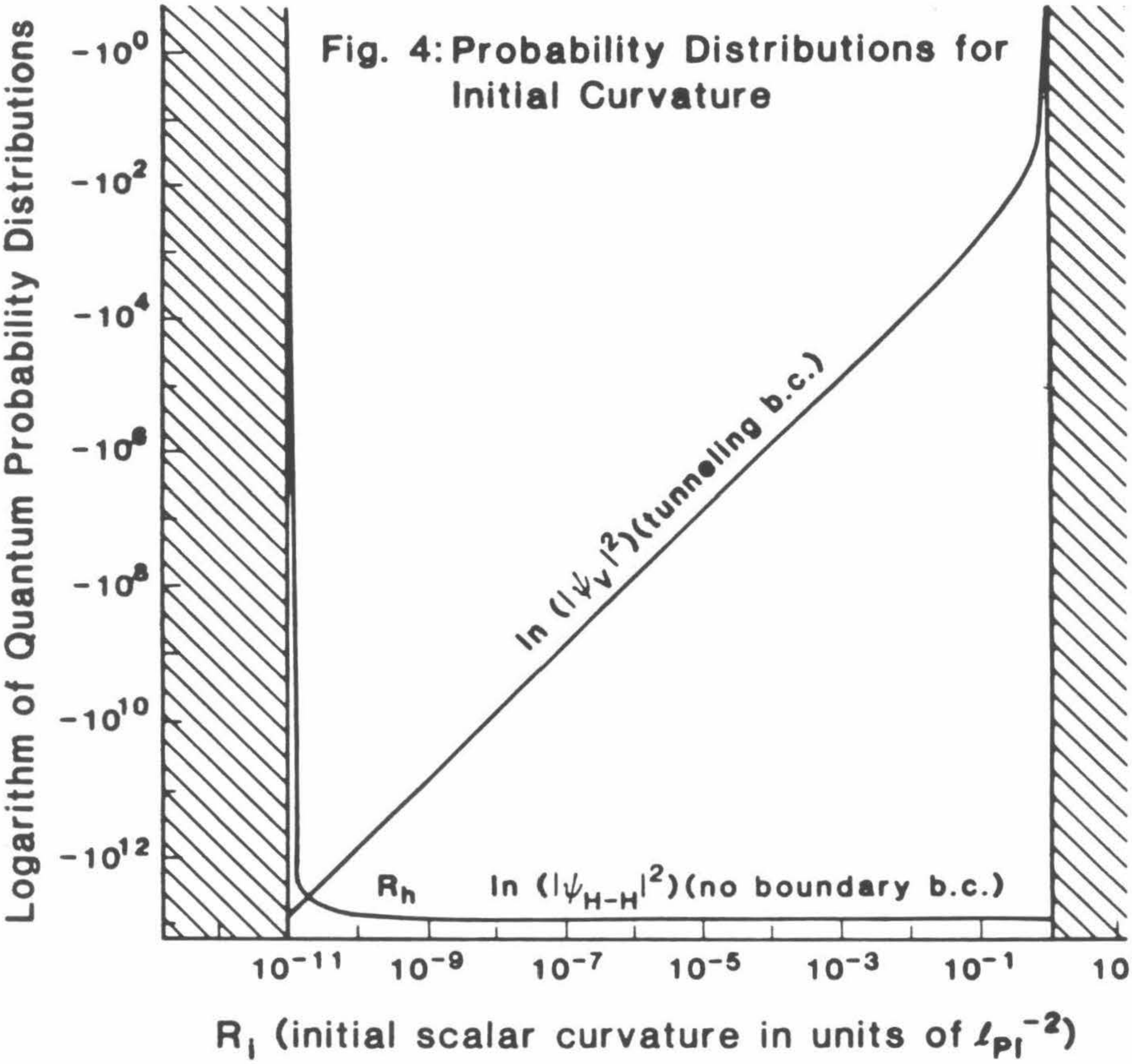
Fig. 4: Probability Distributions for Initial Curvature
Logarithm of Quantum Probability Distributions
-10^0
-10^2
-10^4
-10^6
-10^8
-10^{10}
-10^{12}
ln $(|\psi_V|^2)$(tunnelling b.c.)
R_h
ln $(|\psi_{H-H}|^2)$(no boundary b.c.)
10^{-11} 10^{-9} 10^{-7} 10^{-5} 10^{-3} 10^{-1} 10^1
R_i (initial scalar curvature in units of ℓ_{Pl}^{-2})

CHAPTER 4

Quantum Initial Conditions for Perturbations in the $R+\varepsilon R^2$ Cosmology

Submitted as ‘‘Initial conditions for perturbations in the $R+\varepsilon R^2$ cosmology’’ to Physical Review D.

ABSTRACT

In a classical inflationary cosmology based on the $R+\varepsilon R^2$ Lagrangian the parameters of the model (such as ε and the initial conditions for inflationary trajectories) are constrained by the observational requirement that any perturbations be delivered small to the present horizon volume. Previous calculations of the evolution of these perturbations (and hence, of the parameter constraints enforced by their evolution) have assumed that the modes begin in their ground state. In this paper, following the procedure of Halliwell and Hawking, the Wheeler-DeWitt equation is derived for this model's inhomogeneous modes in perturbative superspace. Then, the two boundary condition proposals of Hartle-Hawking (''no boundary'') and Vilenkin (''tunneling from nothing'') are implemented, verifying that both boundary conditions require the inhomogeneous modes to begin in their ground states.

I. INTRODUCTION

In Ref. 1, Milan Mijić, Wai-Mo Suen, and I explored a classical cosmological model based on the $R+\varepsilon R^2$ Lagrangian. We showed that Robertson-Walker domains would inflate for a wide range of initial conditions, that this pure gravity inflation would smoothly shut itself down, and that the evolution of perturbations on the background could be used to constrain ε and the initial parameters of the model. In Ref. 2, we turned to the wave function formalism and applied it to the same model to obtain distributions over initial conditions for the classical model. There, we derived approximately the general solution in minisuperspace to the Wheeler-DeWitt equation, we implemented the two boundary condition proposals of Vilenkin[3] (''tunneling from nothing'') and Hartle and Hawking[4] (''no boundary'') to obtain specific solutions, and we compared the resulting distributions by restricting these wave functions to the initial edge of the Lorentzian semiclassical domain of inflationary trajectories.

In Ref. 1, we showed that the classical inflation tends to smooth out scalar and tensor perturbations. We thus could convert the observational bound (that perturbations presently reentering the horizon be small) into a lower bound on ε, $\varepsilon > 10^{11}\, l_{\rm Pl}{}^2$. The only necessary input was the assumption that the inhomogeneous scalar and tensor modes begin in their ground states. In this paper, I obtain the wave function for these inhomogeneous modes in the perturbative superspace approximation that the mode strengths are small (this on top of the approximations already made in Ref. 2 to determine the wave function in minisuperspace). I then apply the boundary condition proposals to verify the ground-state assumption for both. This ground-state conclusion should not be surprising, since in work on perturbations in a model of Einstein gravity plus a scalar field Vilenkin[5] has found his boundary condition to fix the inhomogeneous parts of the wave function precisely the same as they are fixed in Halliwell and

Hawking.[6] There (Einstein gravity plus a scalar field) as here ($R+\varepsilon R^2$ cosmology), both boundary conditions start off perturbations in the ground state; the proposals differ (to semiclassical order) only in the initial state of the expansion degree of freedom.

I thus conceive of this paper as an application of the perturbation analysis of Ref. 6 — an application that supports and extends the work in Refs. 1 and 2, and verifies the intuition gleaned from the wave function formalism applied to the scalar field model (Refs. 5 and 6). My notation will accordingly follow closely that of Refs. 1 and 2. All three papers, though, should be viewed in the larger contexts of work on higher derivative gravity and the wave function formalism.[7]

My approach here becomes straightforward after I exploit one strategic fact: Whitt[8] has exhibited a conformal transformation that expresses $R+\varepsilon R^2$ as Einstein gravity plus a scalar field. This transformation, important to the calculation and insight of Ref. 1 and central to the method of Ref. 2, is no less key here. The potential for this ''conformal-picture'' scalar field (which, of course, carries the extra scalar degree of freedom present in the scalar curvature in higher derivative gravity) is zero for large values of the field (the inflationary regime) and approaches the ''scalaron mass'', $\sim 1/\sqrt{6\varepsilon}$, in the linearized limit.

Once in the conformal picture, I can borrow (almost) wholesale the formalism of Halliwell and Hawking[6] to set up and analyze the wave function for the perturbations. In their paper, Halliwell and Hawking present the mode expansion in detail for the perturbed Friedmann model. My application of their work requires me simply to consider the effect of the special form of the potential for the $R+\varepsilon R^2$ model. Halliwell and Hawking require regularity of the perturbative parts of the wave function in the Euclidean regime to match the Hartle-Hawking compact-manifold boundary condition and Vilenkin[5] requires the same to enforce literally his "tunneling from nothing" proposal; and as I shall show, this leads to the ground-state initial conditions. Wada[9] has analyzed the wave function tensor modes in some detail for a model of Einstein gravity plus a cosmological constant that is nearly equivalent to the $R+\varepsilon R^2$ conformal picture in the inflationary limit. His methods for solution of the perturbative superspace Wheeler-DeWitt equation will prove useful here.

The body of this paper is split into two sections: In Section II, I obtain the Wheeler-DeWitt equation, including all the mode-strength variables out to quadratic perturbative order. In Section III, I solve the perturbed Wheeler-DeWitt equation for the inhomogeneous-mode parts of the wave function (for high mode number) and apply the boundary condition(s), verifying that these modes begin in the ground state.

II. THE WHEELER-DEWITT EQUATION WITH PERTURBATIONS

I study a model cosmology governed by the action,[1,2]

$$S=\frac{1}{16\pi G}\int d^4x\sqrt{-g}\,(R+\varepsilon R^2)+\frac{1}{8\pi G}\int d^3x\sqrt{h}\,K\,(1+2\varepsilon R)\,, \tag{2.1}$$

which represents Einstein gravity with an additional quadratic gravitational term. Here, R is the scalar curvature, g is the determinant of the spacetime 4-metric, h is the determinant of the induced spatial 3-metric on the boundary of integration, and K is the trace of the extrinsic curvature. The sign conventions are those of Ref. 1, and I choose units in which $\hbar=c=1$ and $G=1\ l_{\rm Pl}{}^2$. The parameter ε will then have dimensions of l^2. Under the Whitt conformal transformation,[8]

$$\tilde{g}_{\mu\nu}=e^{2\phi}g_{\mu\nu}\,,\quad \phi\equiv(1/2)(1+2\varepsilon R)\,, \tag{2.2}$$

the action (2.1) can be reexpressed as Einstein gravity plus a scalar field,

$$S=\frac{1}{16\pi G}\int d^4x(-\tilde{g})^{1/2}\tilde{R}-\int d^4x(-\tilde{g})^{1/2}\left[\frac{3}{8\pi G}\phi_{,\mu}\phi_{,\nu}\tilde{g}^{\mu\nu}+\frac{1}{64\pi G\varepsilon}(e^{-2\phi}-1)^2\right]$$

$$+\frac{1}{8\pi G}\int d^3x(\tilde{h})^{1/2}\tilde{K}\ . \tag{2.3}$$

Geometric quantities in conformal space are here denoted by a tilde. During the classical inflationary epoch, the εR^2 term will dominate the action (2.1). This corresponds to large ϕ, generating in (2.3) an effective cosmological constant, $1/(8\varepsilon)$. The unperturbed Robertson-Walker line element is

$$ds^2=-dt^2+a^2(t)[d\chi^2+\sin^2\chi(d\theta^2+\sin^2\theta d\phi^2)]=-dt^2+a^2(t)(\Omega_{ij}dx^i dx^j)\ , \tag{2.4}$$

where $0\leq\chi\leq\pi$, $0\leq\theta\leq\pi$, and $0\leq\phi\leq 2\pi$. With a convenient choice of variables in the conformal picture, this can be rewritten

$$d\tilde{s}^2=\sigma^2[-d\tau^2+\alpha^2(\tau)(\Omega_{ij}dx^i dx^j)]\ , \tag{2.5}$$

where $\sigma^2=2G/3\pi$, $\alpha=a(1+2\varepsilon R_0)^{1/2}\sigma^{-1}$, $\tau=t(1+2\varepsilon R_0)^{1/2}\sigma^{-1}$, $\phi_0\equiv(1/2)\ln(1+2\varepsilon R_0)$, and the 0 subscript denotes the homogeneous part. Now, the action (2.3) in this unperturbed model can be written

$$S_0=\frac{1}{2}\int d\tau\left\{-\alpha\left[\frac{d\alpha}{d\tau}\right]^2+\alpha^3\left[\frac{d\phi_0}{d\tau}\right]^2+\alpha\left[1-\left[\frac{\alpha}{\alpha_*}\right]^2(1-e^{-2\phi_0})^2\right]\right\}\ , \tag{2.6}$$

where $\alpha_*^2\equiv 36\pi\varepsilon/G$. This action, and its corresponding Wheeler-DeWitt equation, have been studied in Ref. 2, where the 0 subscript distinction on the homogeneous variables was omitted. To include the effect of perturbations, I now explore the full action (2.3) in the manner of Halliwell and Hawking.[6] In the conformal picture, the

3+1 split is written

$$d\tilde{s}^2=-(\tilde{N}^2-\tilde{N}_i\tilde{N}^i)d\tau^2+2\tilde{N}_i dx^i d\tau+\tilde{h}_{ij}dx^i dx^j \ . \qquad (2.7)$$

This metric may then be expressed as a general expansion around the unperturbed metric (2.5),

$$\tilde{N}=\sigma(1+6^{-1/2}\sum_{nlm} g_{nlm} Q^n{}_{lm}) \ , \qquad (2.8a)$$

$$\tilde{N}_i=\sigma\alpha(\tau)\sum_{nlm}\left[6^{-1/2}k_{nlm}(P_i)^n{}_{lm}+2^{1/2}j_{nlm}(S_i)^n{}_{lm}\right] \ , \qquad (2.8b)$$

$$\phi=\phi_o(\tau)+2^{1/2}\pi\sum_{nlm} f_{nlm} Q^n{}_{lm} \ , \qquad (2.8c)$$

and

$$\tilde{h}_{ij}=\sigma^2\alpha^2(\tau)(\Omega_{ij}+\varepsilon_{ij}) \ , \qquad (2.8d)$$

where

$$\varepsilon_{ij}=\sum_{nlm}\left[6^{1/2}a_{nlm}\frac{1}{3}\Omega_{ij}Q^n{}_{lm}+6^{1/2}b_{nlm}(P_{ij})^n{}_{lm}\right. \qquad (2.8e)$$

$$\left.+2^{1/2}c^{(o,e)}{}_{nlm}(S_{ij}{}^{(o,e)})^n{}_{lm}+2d^{(o,e)}{}_{nlm}(G_{ij}{}^{(o,e)})^n{}_{lm}\right] . \qquad (2.8e)$$

The coefficients a_n, b_n, c_n, d_n, f_n, g_n, j_n, and k_n are all perturbatively small functions of time. I will henceforth follow the convention of denoting all the indices n, l, m and the odd-even parity designators o, e by the single index n. The Q^n are hyperspherical scalar harmonics; $P_i{}^n$ and $S_i{}^n$ are hyperspherical vector harmonics of the scalar and vector types; $P_{ij}{}^n$, $S_{ij}{}^n$, and $G_{ij}{}^n$ are hyperspherical tensor harmonics of the scalar, vector, and tensor types, respectively. All are defined and displayed (together with some of their most useful properties) in Ref. 6. The 3-metric $\tilde{h}_{ij}\sigma^{-2}\alpha^{-2}$ will be used to raise and lower spatial indices.

To simplify the calculation, I introduce the gauge choice, $a_n = b_n = c_n = j_n = 0$. Then, following the procedure of Ref. 6, I can expand the action in the conformal picture (2.3) out to quadratic order in the perturbations. The only wrinkle concerns the potential term,

$$-\int d^4x\,(-\tilde{g}\,)^{1/2}\frac{(e^{-2\phi}-1)^2}{64\pi G\,\varepsilon}\,. \tag{2.9}$$

The simplifying assumption I wish to make is to hold the homogeneous part of the scalar field, ϕ_0 in Eq. (2.8c), large, corresponding to the strongly inflationary regime. Indeed, in Ref. 2, the wave function is derived only up to first order in $e^{-2\phi_0}$ (first order in $1/(\varepsilon R_0)$). If I keep terms only to this order here and assume additionally that the perturbation mode strengths are small, I can rewrite the potential term as

$$-\int d^4x\,(-\tilde{g}\,)^{1/2}\frac{(1-2e^{-2\phi_0})}{64\pi G\,\varepsilon}\,. \tag{2.10}$$

Note that the only remaining coupling of the perturbations to the potential will come from perturbations of $(-\tilde{g}\,)^{1/2}$. The rest of the perturbed action is straightforward, if tedious, to obtain.

I should stress that this method of analysis consigns the wave function to three successive approximations: first to small inhomogeneous mode strengths, then to first order in $1/(\varepsilon R_0)$, and finally (below) to first order WKB. The latter two approximations already severely restrict the realm of validity in minisuperspace, and the wave functions here must then be held near the unperturbed wave functions of Ref. 2. But these approximations are valid on the Quantum/Classical boundary at the initial edge of inflationary trajectories in the Lorentzian domain of minisuperspace. I will thus stay with the interpretation of Ref. 2 and will consider the wave function to give the amplitude for branching to a classical trajectory in the expansion degree of freedom on this boundary (other degrees of freedom may remain in the highly quantum regime long after this branching).

The calculation sketched above gives for the action to quadratic perturbative order

$$S=S_0+\sum_n S_n \ , \tag{2.11a}$$

where the unperturbed action is now

$$S_0=\frac{1}{2}\int d\tau\left\{-\alpha\dot{\alpha}^2+\alpha^3\dot{\phi}_0^{\,2}+\alpha\left[1-\left(\frac{\alpha}{\alpha_*}\right)^2(1-2e^{-2\phi_0})\right]\right\} \tag{2.11b}$$

(the dot denotes $d/d\tau$), and

$$S_n=\int d\tau L_n \ , \tag{2.11c}$$

where

$$\begin{aligned} L_n=\frac{1}{2}\alpha^3\Bigg\{&\left[\left(\dot{d}_n+4\frac{\dot{\alpha}}{\alpha}d_n\right)^2-d_n^{\,2}\left[10\left(\frac{\dot{\alpha}}{\alpha}\right)^2+6\dot{\phi}_0^{\,2}\right]-\frac{d_n^{\,2}}{\alpha^2}\left[n^2+1-6\left(\frac{\alpha}{\alpha_*}\right)^2\right]\right] \\ &+\left[(\dot{f}_n-g_n\dot{\phi}_0)^2-(n^2-1)\frac{f_n^{\,2}}{\alpha^2}\right] \\ &+\left[\frac{2}{3}\left(\frac{\dot{\alpha}}{\alpha}\right)\frac{k_n g_n}{\alpha}-\left(\frac{\dot{\alpha}}{\alpha}\right)^2 g_n^{\,2}-\frac{k_n^{\,2}}{3\alpha^2(n^2-1)}-\frac{2}{\alpha}k_n f_n\dot{\phi}_0\right]\Bigg\} \ . \end{aligned} \tag{2.11d}$$

These equations (2.11) are now the $R+\varepsilon R^2$ version of Eqs. (B1)-(B5) of Ref. 6 with my choice of gauge and notational conventions. At this point, it is possible to achieve a vast simplification to the tensor-mode parts of these equations by choosing a new expansion variable in the manner of Wada[9],

$$\bar{\alpha}\equiv\alpha e^{-2d_n^2} \tag{2.12}$$

Then the action (2.11) can be reexpressed as

$$S=S_0+\sum_n S_n \ , \tag{2.13a}$$

where

$$S_0=\frac{1}{2}\int d\tau\left\{-\bar{\alpha}\dot{\bar{\alpha}}^2+\bar{\alpha}^3\dot{\phi}_0^2+\bar{\alpha}\left[1-\left(\frac{\bar{\alpha}}{\alpha_*}\right)^2(1-2e^{-2\phi_0})\right]\right\} \tag{2.13b}$$

and

$$S_n=\int d\tau L_n \ , \tag{2.13c}$$

where

$$L_n=\frac{1}{2}\bar{\alpha}^3\left\{\left[\dot{d}_n^{\,2}-\frac{(n^2-1)}{\bar{\alpha}^2}d_n^{\,2}\right]+\left[(\dot{f}_n-g_n\dot{\phi}_0)^2-(n^2-1)\frac{f_n^{\,2}}{\bar{\alpha}^2}\right]\right.$$

$$\left.+\left[\frac{2}{3}\left(\frac{\dot{\bar{\alpha}}}{\bar{\alpha}}\right)\frac{k_n g_n}{\bar{\alpha}}-\left(\frac{\dot{\bar{\alpha}}}{\bar{\alpha}}\right)^2 g_n^{\,2}-\frac{k_n^{\,2}}{3\bar{\alpha}^2(n^2-1)}-\frac{2}{\bar{\alpha}}k_n f_n\dot{\phi}_0\right]\right\} \ . \tag{2.13d}$$

Now, $\partial L_n/\partial g_n=0$ and $\partial L_n/\partial k_n=0$ provide the constraints,

$$k_n = 3(n^2-1)\bar{\alpha}\left[\frac{\frac{1}{3}\frac{\dot{\bar{\alpha}}}{\bar{\alpha}} f_n \dot{\phi}_0 - f_n(\dot{\phi}_0)^3 + f_n \dot{\phi}_0 \left(\frac{\dot{\bar{\alpha}}}{\bar{\alpha}}\right)^2}{\dot{\phi}_0^{\,2} + \frac{n^2-4}{3}\left(\frac{\dot{\bar{\alpha}}}{\bar{\alpha}}\right)^2}\right] \tag{2.14a}$$

and

$$g_n = \left[\frac{\left((n^2-1)\frac{\dot{\bar{\alpha}}}{\bar{\alpha}} f_n + \dot{f}_n\right)\dot{\phi}_0}{\dot{\phi}_0^{\,2} + \frac{n^2-4}{3}\left(\frac{\dot{\bar{\alpha}}}{\bar{\alpha}}\right)^2}\right] . \tag{2.14b}$$

The only perturbative degrees of freedom are the scalar modes, carried by the f_n, and the tensor modes, carried by the d_n.

Now, the canonical momenta are

$$\pi_{\bar{\alpha}} = \frac{\partial L_n}{\partial \dot{\bar{\alpha}}} = -\bar{\alpha}\dot{\bar{\alpha}} + \sum_n \left[-\bar{\alpha}\dot{\bar{\alpha}} g_n^{\,2} + \frac{1}{3} k_n g_n \bar{\alpha}\right] , \tag{2.15a}$$

$$\pi_{\phi_0} = \frac{\partial L_n}{\partial \dot{\phi}_0} = \bar{\alpha}^3 \dot{\phi}_0 + \sum_n \left[-g_n \bar{\alpha}^3 (\dot{f}_n - g_n \dot{\phi}_0) - \bar{\alpha}^2 k_n f_n\right] , \tag{2.15b}$$

$$\pi_{d_n} = \frac{\partial L_n}{\partial \dot{d}_n} = \bar{\alpha}^3 \dot{d}_n , \tag{2.15c}$$

and

$$\pi_{f_n} = \frac{\partial L_n}{\partial \dot{f}_n} = \bar{\alpha}^3 (\dot{f}_n - g_n \dot{\phi}_0) . \tag{2.15d}$$

The Hamiltonian is obtained by the usual prescription, "$H = \pi_x \dot{x} - L$":

$$H=\frac{1}{2}\left\{-\frac{\pi_{\bar{\alpha}}^{2}}{\bar{\alpha}}+\frac{\pi_{\phi_0}^{2}}{\bar{\alpha}^{3}}-\bar{\alpha}\left[1-\left(\frac{\bar{\alpha}}{\alpha_*}\right)^{2}(1-2e^{-2\phi_0})\right]\right.$$

$$+\sum_n\left(\frac{\pi_{f_n}^{2}}{\bar{\alpha}^{3}}+\bar{\alpha}(n^2-1)f_n^{2}\right)+\sum_n\left(\frac{\pi_{d_n}^{2}}{\bar{\alpha}^{3}}+\bar{\alpha}(n^2-1)d_n^{2}\right)$$

$$\left.+\left[\sum_n g_n\frac{\pi_{\phi_0}\pi_{f_n}}{\bar{\alpha}^{3}}+\sum_n k_n\frac{f_n\pi_{\phi_0}}{\bar{\alpha}}\right]\right\}. \tag{2.16}$$

The constraint parts of this Hamiltonian (the last two sums), corresponding to the independent constants g_n, k_n must be individually satisfied by the wave function (since the wave function is independent of g_n, k_n). They will be trivially satisfied at the order of approximation used here because they are of quadratic order in the perturbations and π_{ϕ_0} is small. Factor-ordering worries left out (again to this order of approximation), canonical quantization, "$\hat{\pi}_x=-i\,\partial/\partial x$", yields finally the Wheeler-DeWitt equation appropriate to this approximation and gauge,

$$\hat{H}\Psi(\bar{\alpha},\phi_0,\{d_n\},\{f_n\})=0\ , \tag{2.17a}$$

where

$$\hat{H}=\hat{H}_0+\sum_n\hat{H}_n\ , \tag{2.17b}$$

$$\hat{H}_0=\frac{1}{2\bar{\alpha}}\left\{\frac{\partial^2}{\partial\bar{\alpha}^2}-\frac{1}{\bar{\alpha}^2}\frac{\partial^2}{\partial\phi^2}-\bar{\alpha}^2\left[1-\left(\frac{\bar{\alpha}}{\alpha_*}\right)^{2}(1-2e^{-2\phi_0})\right]\right\}, \tag{2.17c}$$

and

$$\hat{H}_n=\frac{1}{2\bar{\alpha}}\left\{\left[-\frac{1}{\bar{\alpha}^2}\frac{\partial^2}{\partial f_n^{2}}+\bar{\alpha}^2(n^2-1)f_n^{2}\right]+\left[-\frac{1}{\bar{\alpha}^2}\frac{\partial^2}{\partial d_n^{2}}+\bar{\alpha}^2(n^2-1)d_n^{2}\right]\right\}. \tag{2.17d}$$

III. SOLUTION OF THE WHEELER-DEWITT EQUATION

To solve the infinite-dimensional Wheeler-DeWitt Equation (2.17), I follow Wada[9] and write the wave function as

$$\Psi=\exp(iS)\ , \tag{3.1}$$

and expand $S(\bar{\alpha},\phi_0,\{d_n\},\{f_n\})$ out to quadratic order in the perturbations,

$$S(\bar{\alpha},\phi_0,\{d_n\},\{f_n\})=S_0(\bar{\alpha},\phi_0)+\frac{1}{2}\sum_n S_n{}^d(\bar{\alpha})d_n{}^2+\frac{1}{2}\sum_n S_n{}^f(\bar{\alpha})f_n{}^2\ . \tag{3.2}$$

Separating the order of perturbation, and keeping terms to semiclassical order, I obtain three equations,

$$-\left(\frac{\partial S_0}{\partial\bar{\alpha}}\right)^2+\frac{1}{\bar{\alpha}^2}\left(\frac{\partial S_0}{\partial\phi_0}\right)^2-\bar{\alpha}^2\left[1-\left(\frac{\bar{\alpha}}{\alpha_*}\right)^2(1-2e^{-2\phi_0})\right]=0\ , \tag{3.3a}$$

$$-\left(\frac{\partial S_0}{\partial\bar{\alpha}}\right)\left(\frac{dS_n{}^d}{d\bar{\alpha}}\right)+\frac{(S_n{}^d)^2}{\bar{\alpha}^2}+\bar{\alpha}^2(n^2-1)=0\ , \tag{3.3b}$$

and

$$-\left(\frac{\partial S_0}{\partial\bar{\alpha}}\right)\left(\frac{dS_n{}^f}{d\bar{\alpha}}\right)+\frac{(S_n{}^f)^2}{\bar{\alpha}^2}+\bar{\alpha}^2(n^2-1)=0\ . \tag{3.3c}$$

Now, Eq. (3.3a) has been solved in Ref. 2 in the region of minisuperspace, where the kinetic term in ϕ_0 is ignorable (near the Hartle-Hawking and Vilenkin wave functions). For Eqs. (3.3b,c), I can make the adiabatic approximation for large mode number n. Writing $\partial S_0/\partial\bar{\alpha}=\pi_{\bar{\alpha}}=-\bar{\alpha}\dot{\bar{\alpha}}$ and assuming that $\bar{\alpha}$ is slowly varying, Eqs. (3.3b) and (3.3c) can be rewritten as

$$(S_n{}^d)^2+\bar{\alpha}^4(n^2-1)\approx 0 \tag{3.3d}$$

and

$$(S_n{}^f)^2+\bar{\alpha}^4(n^2-1)\approx 0\ . \tag{3.3e}$$

These equations (3.3d,e) are algebraically solvable to get

$$S_n{}^d \approx \pm i\sqrt{n^2-1}\,\bar{\alpha}^2 \text{ and } S_n{}^f \approx \pm i\sqrt{n^2-1}\,\bar{\alpha}^2\ . \tag{3.4}$$

Now, both boundary conditions require that the wave function is regular in the inhomogeneous modes as $\bar{\alpha}\rightarrow 0$ in the Euclidean regime. Though this procedure is admittedly not rigorous (as Vilenkin[5] has pointed out), because f_n and d_n have been assumed small in deriving (3.4), the regularity requirement demands the positive sign. A final expression for the wave function may now be written down. I introduce the notation [based on Eqs. (3.1) and (3.2)],

$$\Psi(\bar{\alpha},\phi_0,\{f_n\},\{d_n\})=\Psi_0(\bar{\alpha},\phi_0)\prod_n \Psi^n{}_{\text{scalar}}(\bar{\alpha},f_n)\Psi^n{}_{\text{tensor}}(\bar{\alpha},d_n)\ . \tag{3.5a}$$

The homogeneous part of this wave function $\Psi_0 = \exp(iS_0[\bar{\alpha},\phi_0])$ is given by Eq. (4.37) of Ref. 2 for Vilenkin's boundary condition and by Eq. (4.38) of Ref. 2 for Hartle and Hawking's. The wave functions for the inhomogeneous modes for large n in the adiabatic approximation, as inferred from (3.1), (3.2), and (3.4) with the + sign, can be written for both boundary conditions as

$$\Psi^n{}_{\text{scalar}}(\bar{\alpha},f_n)\approx e^{-\frac{1}{2}\sqrt{n^2-1}\,\bar{\alpha}^2 f_n{}^2} \tag{3.5b}$$

and

$$\Psi^n{}_{\text{tensor}}(\bar{\alpha},d_n)\approx e^{-\frac{1}{2}\sqrt{n^2-1}\,\bar{\alpha}^2 d_n{}^2} \quad . \tag{3.5c}$$

Both of these wave functions have the ground state form for harmonic oscillators of frequency $\omega_n{}^2 \approx \bar{\alpha}^4(n^2-1)$. Indeed, from Eq. (2.17), they satisfy Schroedinger equations in the form for such an oscillator in the semiclassical approximation,

$$i\frac{\partial}{\partial\tau}\Psi^n{}_{\text{scalar}}=-\frac{i}{\bar{\alpha}}\left(\frac{\partial S_0}{\partial\bar{\alpha}}\right)\frac{\partial}{\partial\bar{\alpha}}\Psi^n{}_{\text{scalar}}$$

$$=\frac{1}{2\bar{\alpha}}\left\{-\frac{1}{\bar{\alpha}^2}\frac{\partial^2}{\partial f_n{}^2}+\bar{\alpha}^2(n^2-1)f_n{}^2\right\}\Psi^n{}_{\text{scalar}}(\bar{\alpha},f_n) \tag{3.6a}$$

and

$$i\frac{\partial}{\partial\tau}\Psi^n{}_{\text{tensor}}=-\frac{i}{\bar{\alpha}}\left(\frac{\partial S_0}{\partial\bar{\alpha}}\right)\frac{\partial}{\partial\bar{\alpha}}\Psi^n{}_{\text{tensor}}$$

$$=\frac{1}{2\bar{\alpha}}\left\{-\frac{1}{\bar{\alpha}^2}\frac{\partial^2}{\partial d_n{}^2}+\bar{\alpha}^2(n^2-1)d_n{}^2\right\}\Psi^n{}_{\text{tensor}}(\bar{\alpha},d_n)\ . \tag{3.6b}$$

Here, time has been reintroduced in terms of the expansion of the classical background $\bar{\alpha}(\tau)$ in the Lorentzian semiclassical domain. That tensor modes should satisfy the same Schroedinger equation as scalar modes directly follows from the work of Ford and Parker,[10] who showed that odd- and even-parity gravitational perturbations are equivalent to massless minimally coupled scalar fields.

The modes remain in the ground state until the adiabatic approximation breaks down — until they cross out of the horizon.[11] This crossing was shown in Ref. 1 to occur during the inflationary epoch, where the approximation of large ϕ_0 still holds. The ground-state wave function (3.5) at the outgoing horizon crossing is the starting point of the evolution calculations in Sections 4 and 5 of Ref. 1.

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