EFFECTIVE LAGRANGIANS

AND INFINITY CANCELLATIONS

FOR OPEN STRING THEORIES

Thesis by
James M. Cline

In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

California Institute of Technology
Pasadena, California

1988
(Submitted May 20, 1988)

ACKNOWLEDGMENTS

I wish to thank my wife, Rebecca, for her remarkable patience, support, and encouragement during my years of graduate study. I am grateful to John Preskill for invaluable advice, criticism, and the physics he patiently explained to me. I have also benefited from conversations with many of the members of the Caltech high energy theory group: Ted Allen, Mike Douglas, Jiyu Feng, Gerry Gilbert, Elias Kiritsis, Ben Grinstein, Desmond Johnston, David Politzer, Soo-Jong Rey, Ryan Rohm, John Schwarz, Sandip Trivedi, Mark Wise, Osamu Yasuda.

ABSTRACT

The covariant path integral formalism for theories of open and closed strings is used to study the first order of string perturbation theory beyond tree level for the closed-string states, in which the string world sheet has the topology of the disk or the real projective plane. We find that scattering amplitudes (in flat spacetime) confirm these surfaces' contribution to the low-energy effective action for the bosonic string theory, as derived by another method, demanding consistency of string propagation in background gravitational and dilaton fields (the "sigma model approach"). However, we are not able to obtain results consistent with this effective action by demanding that amplitudes in a curved background be finite; this is an unresolved puzzle. Decoupling of spurious tachyon states from the superstring S-matrix is discussed, and finiteness of amplitudes for the disk plus projective plane is demonstrated for a large class of external states, when the gauge group is SO(32).

Contents

Chapter 1. Introduction and Summary

This thesis is about the type I string—a theory of interacting open and closed strings whose world sheets can be orientable or nonorientable surfaces. Four years ago Green and Schwarz revolutionized the course of elementary particle theory by showing that gauge anomalies in the low-energy effective field theory of type I superstrings canceled precisely for the gauge group, $SO(32)$, which was singled out by consistency of the string theory [1]. Since then, open strings have been superseded by the phenomenologically more promising heterotic string [2]. So why study type I strings? Our motivation is to gain a better understanding of some general features of string theory. String theories are so much more complex than quantum field theories, and require such greater mathematical and "technological" sophistication, that we would like a relatively simple example or toy model on which to test some of the emerging principles that are hoped to be common to all string theories. The restricted Virasoro-Shapiro model [3], which we shall for convenience call the "type I bosonic string," is such an example, for it behaves very analogously to its supersymmetric successor. For instance, its dilaton tadpole amplitude vanishes for the gauge group $SO(2^{D/2})$ (where $D = 26$ is the critical dimension) at lowest nontrivial order in the string loop expansion [4, 5].

The perturbation series in powers of the string coupling, λ, consists of a sum of world sheets whose topologies are of increasingly negative Euler characteristic, χ. For purely closed strings, this just means more and more handles on a sphere, but for open strings there exist many other surfaces. Midway between tree level (the sphere, S_2) and one loop (the torus, T_2), there is the disk (D_2), and for unoriented strings, the real projective plane (P_2). These are similar to loop contributions of closed strings in the way that they modify the low-energy effective field theory; for example, they generate a nonzero cosmological constant, unless there is supersymmetry. However, amplitudes on D_2 and P_2 are easier to evaluate than loop diagrams because the former are integrals over familiar functions, whereas the latter involve Jacobi theta functions. Furthermore D_2 and P_2 are distinguished from the torus and higher loop surfaces in that any disk can be transformed into any other disk by a local rescaling

(Weyl transformation) of the two-dimensional world-sheet metric; similarly for P_2 [6]. Higher-genus surfaces do not have this property: to transform a torus of one geometry to another, it is in general necessary not only to Weyl transform the metric but also to adjust the relative sizes of the two kinds of noncontractible loops. The latter is called a Teichmüller deformation [7], and the parameters in the metric that effect this are called moduli. The path integral evaluation of string scattering amplitudes due to Polyakov [8] requires integrating over all possible geometries of the world sheet, which includes an integral over the moduli. D_2 and P_2 have no moduli, and so are also simpler in this respect than loop contributions. In what follows we will restrict our attention to D_2 and P_2 and concentrate mainly on the bosonic string. Also we use the Polyakov path integral approach throughout. Although the path integral gives results equivalent to those of the operator formalism, in which string theory was originally cast, it is possible to compute some quantities in the former that would be very awkward or impossible to calculate directly in the latter, such as tadpole amplitudes. The fixing of gauge symmetries of the string action also has a more intuitive, geometric meaning in the path integral.

Chapter 2 deals with a subtlety of gauge fixing in the Polyakov path integral [32]. The string action is

$$S = \frac{T}{2} \int d^2 z \sqrt{h} h^{ab} \partial_a x^\mu \partial_b x^\mu, \tag{1.1}$$

where T is the tension, z is a coordinate on the world sheet, h_{ab} is a two-dimensional metric, and $x^\mu(z)$ is the spacetime coordinate of the string. The path integral is $\int \mathcal{D}h\mathcal{D}x e^{-S}$. Since S is invariant under world-sheet reparametrizations $z^a \to z^{a'}$, $h_{ab}(z) \to h'_{ab}(z')$ and Weyl transformations $h_{ab} \to e^{\sigma(z)} h_{ab}$, this is a two-dimensional gauge theory. The gauge is almost fixed on low-genus surfaces by fixing $h_{ab} = \delta_{ab}$, but for S_2 and D_2 there is still a residual symmetry in the remaining integral $\int \mathcal{D}x e^{-S}$, called the conformal Killing group. It consists of reparametrizations $x(z) \to x(z')$, which change the world sheet metric only by a Weyl transformation. For S_2 the group is $SL(2,C)$ and for D_2 it is $SL(2,R)$, both of which are noncompact. Scattering amplitudes are obtained by inserting into the path integral a vertex operator for each external line, whose position is integrated over the string world sheet (e.g., the integrated vertex operator for a tachyon of momentum p is $\int d^2z\sqrt{h}e^{ip\cdot x(z)}$). The residual $SL(2,...)$ symmetry can be fixed by fixing the positions of three (one) vertex operators for S_2 (D_2). One also expects that the gauge could be fixed by introducing a constraint in the functional integral over x^μ using the method of Faddeev and Popov. In chapter 2 it is shown that this is not the case: the formal $SL(2,...)$ symmetry of the path integral is broken when the latter is defined by introducing an ultraviolet regulator. Therefore the $SL(2,...)$ symmetry that appears in scattering amplitudes can only be connected to an underlying symmetry of the path integral in a formal, nonrigorous way. Although this observation has little practical consequence (since it is much easier to fix positions of vertex operators than to constrain the $\int \mathcal{D}x$ integral), it seems to indicate that the $SL(2,...)$ symmetry is a less fundamental feature of strings at tree level than one might have thought. This suspicion was subsequently borne out by Liu and Polchinski [9], who showed that the infinite volume of $SL(2,R)$ should be renormalized to a finite value in order to obtain a nonvanishing vacuum energy for the D_2 contribution to the low-energy effective action of the string.

The following three chapters are about the precise form of higher-genus contributions to the effective action of the string's massless modes: the graviton, dilaton, antisymmetric tensor, and vector gauge boson. There are two methods of deducing the effective Lagrangian. The traditional approach has been to compute scattering amplitudes in the limit of large string tension, T, and find a field theory that produces the same amplitudes [10, 11]. This is in fact how strings were first observed to contain gravity. The closed-string massless sector is described at tree level by the action

$$S_{\text{tree}} = \int d^Dx\sqrt{g}\left(-\frac{2}{\kappa^2}R + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{6}e^{-2(\kappa/\sqrt{D-2})\phi}H^2\right) + O(T^{-1}) \tag{1.2}$$

where $\kappa^2/32\pi$ is Newton's constant, ϕ is the dilaton field, $H_{\alpha\beta\gamma} = B_{\alpha\beta,\gamma} + B_{\gamma\alpha,\beta} + B_{\beta\gamma,\alpha}$ is the field strength of the antisymmetric tensor $B_{\alpha\beta}$, and $D = 26$. The terms of $O(T^{-1})$ involve higher derivatives and powers of R and are suppressed by powers of the Planck mass, which is proportional to T.

More recently a rather powerful and completely different way of obtaining (1.2) was developed by Fradkin and Tseytlin [12], and by Callan et al. [13-16]. The idea was to quantize the string in a nontrivial classical configuration of the massless fields, similar to the background field method for quantizing gauge theories. The flat spacetime string action (1.1) is replaced by

$$S_{bg} = \frac{T}{2}\int d^2z \left(\sqrt{h}h^{ab}\partial_a x^\mu \partial_b x^\nu g_{\mu\nu}(x) + \epsilon^{ab}\partial_a x^\mu \partial_b x^\nu B_{\mu\nu}(x) - \frac{1}{4\pi T}\phi(x)\sqrt{h}R\right). \tag{1.3}$$

(Here R is the world-sheet curvature, not to be confused with the spacetime curvature in (1.2).) Since the original action is Weyl-invariant, its stress-energy tensor has a vanishing trace. Upon quantization this symmetry is anomalously broken, except in the critical dimension, $D = 26$. It is necessary to maintain scale invariance; otherwise the Weyl factor $\sigma(z)$ (which appears in $h \to e^\sigma h$) would be integrated over in the sum over surfaces [17, 12], which would vastly complicate the dynamics of the string and destroy its most physically appealing features as a fundamental theory. When tracelessness of the stress energy tensor is imposed for the theory (1.3), including effects of loops of the $x^\mu(z)$ field, it is found that the background fields must satisfy equations of motion corresponding to an action [13]

$$S = \int d^Dx\sqrt{g}e^\phi \left(-R + (\nabla\phi)^2 + 2\nabla^2\phi + \frac{1}{12}H^2\right) + O(T^{-1}). \tag{1.4}$$

This is identical to the action (1.2) after making the field redefinitions

$$g_{\mu\nu} \to e^{-2\phi/(D-2)}g_{\mu\nu}, \quad \phi \to \frac{1}{2}\kappa\sqrt{D-2}\phi, \quad \text{and } B_{\mu\nu} \to \kappa B_{\mu\nu}. \tag{1.5}$$

At first it may seem surprising that such unrelated methods should lead to the same result. Prior to [13], however, Lovelace [18] had given some general arguments for why this should be the case, which we shall not try to explain here.

In the above treatment the world sheet was taken to be the complex plane, which for a Weyl-invariant theory is equivalent to the sphere. The procedure was also applied to the upper half-plane [14], which is topologically a disk, with the action (1.3) supplemented by a background abelian gauge field coupling to the boundary,

$$\oint d\vec{s}\cdot(\vec{\partial}x^{\mu})A_{\mu}(x),$$

as well as a coupling of the dilaton to the extrinsic curvature of the boundary. The requirement of scale invariance thus also gives a field equation for A_{μ}, which was found to correspond to the effective action

$$\Lambda\int d^{D}x e^{\phi/2}\left(\det(g_{\mu\nu}+B_{\mu\nu}+F_{\mu\nu})\right)^{\frac{1}{2}}+O(T^{-1}), \tag{1.6}$$

where $F_{\mu\nu}$ is the field strength of A_{μ}, and Λ is an undetermined constant, the cosmological constant.*

The factor of $e^{\phi/2}$ in (1.6) is easy to explain. A scattering amplitude for massless particles in the background (1.3) at a given order in the topological expansion is given by

$$A_{N}\sim\int\mathcal{D}x e^{-S_{bg}}V_{1}\cdots V_{N},$$

where the V_i are vertex operators for the external states and the path integral is over a surface of Euler characteristic χ. If the dilaton condensate is a constant, $<\phi>$, then

$$A_{N}\sim e^{\frac{1}{2}\chi<\phi>}$$

because ϕ couples to $-\frac{1}{2}$ the Euler density in S_{bg}. Since the effective Lagrangian $\mathcal{L}_{eff}$ must reproduce A_N, the contribution to $\mathcal{L}_{eff}$ from this topology must also contain a factor of $e^{\frac{1}{2}\chi\phi}$. Therefore the cosmological term is of the form $\Lambda_{\chi}\sqrt{g}e^{\frac{1}{2}\chi\phi}$. Redefining the fields as in (1.5) to eliminate mixing of the graviton and dilaton in their kinetic

* The natural generalization of (1.6) to the nonabelian case is to take the determinant over spacetime indices and include a trace over gauge group indices.

terms, we see that the correction to $\mathcal{L}_{eff}$ to first order ($\chi = 1$) in the string coupling λ and lowest order in T^{-1} is

$$\mathcal{L}_{D_2(P_2)} = \Lambda_{D_2(P_2)}\sqrt{g}\exp\left(-\frac{D+2}{4\sqrt{D-2}}\kappa\phi\right). \tag{1.7}$$

Eq. (1.7) says, among other things, that the amplitude A_D for dilaton emission into the vacuum on D_2 (P_2) should be exactly $\frac{7}{\sqrt{24}}\kappa$ times the corresponding vacuum energy density $\Lambda_{D_2(P_2)}$. A_D and Λ for P_2, as computed from the Polyakov path integral by Grinstein and Wise [19] do not, however satisfy this relation; also Douglas and Grinstein did similar calculations for D_2 [4], finding $\Lambda = 0$ but $A_D \neq 0$. Furthermore Fischler, Klebanov, and Susskind [20] found a discrepancy between N-graviton amplitudes on D_2 computed string-theoretically and those determined by the effective action (1.2)+(1.7). It seemed that the background field method effective Lagrangian was incorrect, and more evidence to that effect was presented by this author in [21].

It now appears that (1.7) *is* correct, due to a number of recent developments [9, 22, 23]. The first two of these references indicated that previous calculations of the dilaton tadpole were wrong and that a different dilaton vertex operator than previously used gives the correct answer. Ref. [9] also showed how one can obtain a nonvanishing value for the D_2 cosmological constant that is consistent with (1.7). In chapter 3 it is shown that the proposed dilaton vertex operator is not really Weyl-invariant and must be supplemented by a prescription to give sensible results. Using the same prescription, a new vertex operator for gravitons can also be constructed in which the the graviton polarization tensor is not traceless. This enables one to compute the graviton trace tadpole, which appears in field theory by expanding $g_{\mu\nu} = \delta_{\mu\nu} + \kappa h_{\mu\nu}$ in the cosmological term, and thus the cosmological constant for D_2 and P_2 can be directly computed. In chapter 3 Λ and A_D are found to be in the correct ratio to agree with (1.7) in this way.

Chapter 4 is a reanalysis of the paradox found by Fischler, Klebanov, and Susskind (FKS). It can be formulated as follows. An N-graviton amplitude at tree level, $A_{NG}(S_2)$, consists of terms with two powers of external momenta from the Einstein

Lagrangian, $\sqrt{g}R$, plus terms of order p^4/T, p^6/T^2, etc., which come from string corrections like $\sqrt{g}R^2$, $\sqrt{g}R^3$, etc. On the disk, the N-graviton amplitude $A_{NG}(D_2)$ will have divergences because of on-shell massless propagators in dilaton and graviton-trace tadpoles, as in fig. 1.1.

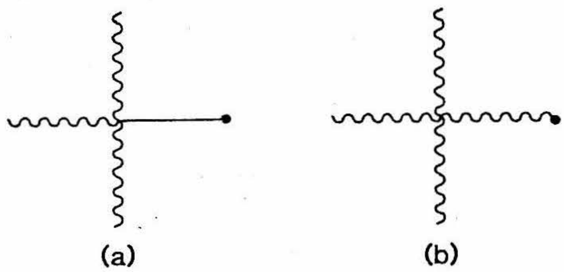

Figure 1.1. (a) Dilaton and (b) graviton tadpole contributions to the three-graviton coupling.

FKS found that in the effective field theory, (1.1)+(1.7), the divergent part of A_{NG} could be expressed as

$$A_{NG}(D_2) \propto \Lambda_{D_2}\left(T^{-1}\frac{\partial}{\partial(T^{-1})}+1\right)A_{NG}(S_2), \tag{1.8}$$

whereas a direct string theory calculation gives

$$A_{NG}(D_2) \propto T^{-1}\frac{\partial}{\partial(T^{-1})}A_{NG}(S_2). \tag{1.9}$$

The paradox was resolved by Polchinski [23], who found that a very careful factorization of tadpole divergences gives a term of +1 in (1.9). His solution required a great deal of machinery—conformal field theory, inclusion of ghost fields, Teichmüller theory, BRST quantization, local frame dependence of fields, etc.—so he also gave a simpler alternative explanation for this problem, which was uncovered using very simple techniques. In chapter 4 we confirm the result of FKS and extend it to the case of tachyon scattering amplitudes. We go a step further than they did by replacing the proportionalities in (1.8) and (1.9) with equalities, and equating (1.8) with the corrected version of (1.9), to determine Λ in terms of stringy quantities. It is found that Λ has the correct value relative to the dilaton tadpole, providing a confirmation of the effective action (1.7). We point out, however that this consistency hinges upon using Feynman gauge for the graviton propagator in the field theory calculation, an assumption that seems inadequately justified so far. Furthermore flaws in the heuristic resolution of the FKS paradox given in [23] are pointed out, and we conclude that if a simple explanation does exist, it has not yet been fully discovered.

There ought to be yet a third method of verifying the D_2/P_2–corrected effective action. In field theory the tadpole divergences in diagrams such as fig. 1.1 can be removed by shifting the vacuum so that the classical equations of motion are satisfied (hence there will be no tadpoles). Similarly Fischler and Susskind [24] noted that one should be able to cancel divergences in string loop diagrams by shifting the background for the lower-order surfaces. For example, the divergences on D_2 quantized in a flat background are canceled by tree-level (S_2) contributions in a curved background which differs from the trivial background by a configuration of order λ. The background needed to cancel the divergences will not satisfy the field equations of the tree-level effective Lagrangian (1.2); rather, it should satisfy the field equations of (1.2) plus a cosmological term and a dilaton tadpole term. By seeing how much background is needed to cancel the divergences and comparing with the equations of motion, one can determine the values of Λ and the dilaton tadpole and see whether they agree with direct determinations. In [24, 25] this was asserted but not demonstrated. In chapter 5 we pursue this idea and find that it does not give the correct value of Λ for D_2 (or P_2). The resolution of this potentially serious problem is left for future inquiries, since it is not clear to us how to fix it.

In chapter 6 we leave behind the subject of effective Lagrangians to examine the finiteness of type I strings. As a warm-up a simple proof is given that the tadpole divergences cancel between D_2 and P_2 for arbitrary amplitudes of the $SO(2^{D/2})$ bosonic string. Of greater interest is an analogous demonstration for the superstring. This was done for amplitudes with N gravitons by Itoyama and Moxhay [26]. It was not clear, however whether several technical points in their work were handled quite correctly. The first problem concerns quadratic divergences that appear in amplitudes on D_2 and P_2. It is surprising that these occur because they are associated with tachyon tadpoles, and tachyons should not be present in the superstring. [26] relates the difficulty to the fixing of residual gauge symmetries in the amplitude. As noted earlier, this entails fixing the positions of vertex operators in the case of the bosonic string. For superstrings the world sheet can be thought of as being replaced by a supermanifold having Grassmann coordinates θ^a as well as the usual z^a. Vertex operators become integrals over the supermanifold, and amplitudes on $S_2(D_2)$ are invariant under the supersymmetric extension of $SL(2,C)(SL(2,R))$. (We shall refer to these graded Lie groups as super-$SL(2,...)$.) Thus one can fix not only the values of z^a for several of the vertex operators but also some of the θ^a. Ref. [26] notes that the quadratic divergences disappear if this is done for D_2, P_2. On the other hand, they observe that one is free to fix or not fix θ^a's on S_2, and also on D_2 if only open-string external states are considered: either way gives the same result. Moreover fixing vertex operator positions corresponds to extracting the group volume. For a graded Lie group this will be an integral over real and Grassmann variables. But the integral over the latter is always finite,* so it is hard to see how fixing or not fixing the odd part of the symmetry can affect the finiteness of an amplitude.

We clarify this puzzle by showing that extraction of the full group volume requires making a change of variables in the integrals over the supermanifold; however, superspace integrals are in general unaltered by a change of variables only if the manifold has no boundary. For an open surface like D_2, a change of variables can induce a surface term on the boundary. We show that in the present case the surface term is infinite; thus fixing the odd part of the super-$SL(2,R)$ symmetry amounts to a particular way of rewriting quadratic divergences as integrals over the boundary and discarding these.

We further relate this phenomenon to recent work of Green and Seiberg [27], who noted that the operator product expansion (O.P.E.) of certain superstring vertex operators contains spurious states for some values of the momenta of the two particles. Among these spurious states is the bosonic string tachyon, and we show that this state is failing to decouple on D_2 amplitudes in which the θ variables are not fixed. Fixing

---

* One could imagine that the result of the Grassmann integration is an integral for the even part that is more divergent than the usual $SL(2,C)$ volume, but this does not occur.

the θ's corresponds to rewriting the vertex operators in a different form, known as the F_2 picture, as opposed to the original form, the F_1 picture. In [27] it was shown that the spurious tachyon state does not occur in the O.P.E. of vertex operators in the F_2 picture, but only for F_1. It is the equivalence of the two pictures that allows one either to fix or not fix θ's on S_2. This equivalence breaks down on D_2, where the existence of a tachyon tadpole puts the spurious state in a restricted kinematic region ($p = 0$), so that analytic continuation in the momenta of the external particles cannot remove its effects, as was possible for S_2.

The second problem in [26] is that the fermionic partners of the x^μ fields in the supersymmetrized version of the string action (1.1) are Majorana spinors ψ^μ, but Majorana spinors cannot be globally defined on P_2 with a Euclidean metric, and the results of Riemannian geometry needed to define the Polyakov path integral for a general topology require that the world sheet have a Euclidean signature. Actually a surface with N external closed-string states can be thought of as having N punctures, or boundaries, and it was shown by Grinstein and Rohm [28] that Majorana spinors can be consistently defined on such a surface, but that the spinor field must suffer a discontinuity in its sign around an odd number of the punctures if the number of punctures is odd. This corresponds to an amplitude with an odd number of external Ramond-Ramond particles,* if we take all the spinor fields to be antiperiodic around the same punctures, regardless of their chirality. Therefore an amplitude on P_2 with an odd number of external NS-NS particles and no other kinds is ill defined because of the nonexistence of Majorana fermions with the correct boundary conditions. If a problem was to occur in the infinity cancellations between D_2 and P_2, we might expect it to appear for this class of amplitudes. The nonexistence of Majorana spinors on Euclidean P_2 manifests itself as the nonexistence of a graded extension of the conformal Killing group $SO(3)$ in Euclidean signature. Ref. [26] deals with this by temporarily continuing amplitudes to Lorentzian signature where the super-$SO(3)$

* R-R boundary conditions mean that both the left- and right-moving spinor fields are antiperiodic about a puncture on S_2, whereas NS-NS (Neveu-Schwarz) boundary conditions are periodic [29]. Both kinds correspond to spacetime bosons; the antisymmetric tensor is a R-R particle, whereas gravitons and dilatons are NS-NS.

exists, fixing the θ's and continuing the result back to Euclidean P_2. In the text several weaknesses of their procedure are pointed out, and a more careful treatment is presented. In the end we confirm the result of [26], giving a more lucid (in our opinion) proof that the divergences of the P_2 amplitudes have the correct sign and magnitude for canceling those of D_2. Therefore, if type I superstrings are to be ruled out on the basis of self-consistency rather than phenomenology, as is the hope of some, it seems likely that the problem will have to come from some other quarter than the nonexistence of Majorana spinors on the closed unoriented surfaces of odd Euler number, of which P_2 is only the simplest example.

Chapter 2. The Conformal Killing Anomaly of the Polyakov String

2.1. Introduction

An essential ingredient of the Polyakov formalism for string theory is the fixing of gauge symmetries of the classical action. For the bosonic string these are the general coordinate transformations and the Weyl rescalings of the metric. Demanding that the quantum theory retain these symmetries leads to the critical dimension, $D = 26$ [8]. It also leads to a peculiar prescription for computing on-shell amplitudes, that one should divide by the volume of the group generated by the conformal Killing vectors, Vol(CKG) [7]. Because this volume is infinite for the topologies of the sphere and disk, the n-point amplitudes on these surfaces vanish for $n < 3\,(S_2)$ and $n < 1\,(D_2)$ [4, 21, 30]. For larger n the SL(2,C) or SL(2,R) symmetries of the world-sheet integrals appear to exactly cancel the infinite factor, leaving a finite answer.

The factor 1/Vol(CKG) is supposed to compensate for gauge-equivalent configurations in the functional integral. It was expected that one could avoid explicitly mentioning the factor by fully fixing the gauge [30]. The usual procedure is to fix the world-sheet coordinates of several vertex operators, which means performing the Polyakov path integral, obtaining an infinite result if it is on the sphere or the disk, and then extracting the infinite factor of Vol(CKG). One might wonder whether the resulting finite value is really unique, since it comes from something of the form ∞/∞. It would be preferable, as suggested in [30], to fix this symmetry in the functional integral at the outset by choosing a slice in the space of embeddings, rather than fixing positions of vertex operators, and then to show that the two procedures are really equivalent. The present chapter demonstrates that this is not possible: the classical CKG symmetry of the string action is broken by any regulator, even in the critical dimension. The standard prescription does not correspond to *fixing* a gauge, but rather to *averaging* over gauges that are actually inequivalent due to this anomaly. The major result of Moore and Nelson is untouched by this difficulty however, because of the absence of moduli for S_2 and D_2 (see sect. 2.4).

2.2. Gauge dependence of amplitudes

The Polyakov partition function is an integral over metrics and embeddings,

$$Z = \sum_{topologies} \frac{\int [dh][dx] \exp(-S)}{\mathrm{Vol}(\mathcal{W} \times \mathcal{D})}. \tag{2.1}$$

$\mathcal{W}$ is the group of Weyl transformations and $\mathcal{D}$ is the group of diffeomorphisms,

$$\begin{aligned} &\mathcal{W}\colon h_{ab} \rightarrow e^{\sigma} h_{ab} \\ &\mathcal{D}\colon x \rightarrow f^* x \equiv x(f(\xi));\ h \rightarrow f^* h \equiv \frac{\partial f^a}{\partial \xi^b} \frac{\partial f^c}{\partial \xi^d} h_{ac}(f(\xi))\,, \end{aligned} \tag{2.2}$$

where $f(\xi)$ is a coordinate transformation on the world sheet. The conformal Killing group (CKG) is the subgroup of $\mathcal{D}$ such that

$$j^* h_{ab}(\xi) = \exp(p[j]) h_{ab}(\xi) \tag{2.3}$$

for some scalar function $p[j]$. The usual procedure for evaluating (2.1) is to express g in terms of a gauge fixed metric $\hat{h}$, via $h = f^*(e^{\sigma}\hat{h})$.* Then $[dh]$ is just a Jacobian $(\det P^{\dagger}P)^{1/2}$, times $[df][d\sigma]$. The latter is canceled by $(\mathrm{Vol}(\mathcal{W} \times \mathcal{D}))^{-1}$ because of the gauge symmetry $\mathcal{W} \times \mathcal{D}$. This is not quite right if there is a nontrivial CKG, for then we can always find a σ that compensates for $j \in$ CKG, giving $j^*(e^{\sigma}\hat{h}) = \hat{h}$. To avoid overcounting each metric—supposedly—we should take $\int [df]$ not over $\mathcal{D}$ but rather $\mathcal{D}/$CKG. Hence the factor of $1/\mathrm{Vol(CKG)}$ comes in.

Alternatively [30] one should be able to choose a slice $\mathcal{S}$ in the space of embeddings, which is transverse to the action of the CKG, i.e., if $\tilde{x} \in \mathcal{S}$ and $j \in$ CKG, then $j^*\tilde{x} \notin \mathcal{S}$. Since $\tilde{x}$ transforms nontrivially, the parametrization $h = f^*(e^{\sigma}\hat{h})$, $x = f^*\tilde{x}$ is a good coordinate system on the space of embeddings and metrics. The measure becomes $[dh][dx] = [df][d\sigma][d\tilde{x}](\det P^{\dagger}P)^{1/2} J$, where J is the additional Jacobian that comes from gauge fixing in the x sector.

---

* We ignore modular parameters since these do not exist for the sphere or disk, which are the interesting topologies for our purposes.

A problem arises when one tries to carry this out. We shall see that CKG gauge-fixed quantities depend on the choice of slice $\mathcal{S}$, showing that the CKG is not a good symmetry of the quantized theory. It is sufficient to focus on the embedding part of the partition function,

$$\tilde{Z} = \int [dx]_{\hat{h}} \exp(-S[\hat{h}, x]), \tag{2.4a}$$

$$S[\hat{h}, x] = \frac{1}{2} \int d^2\xi \sqrt{\hat{h}} x^\mu \Delta_{\hat{h}} x^\mu, \tag{2.4b}$$

using the curved space Laplacian $\Delta_{\hat{h}} x = -(1/\sqrt{\hat{g}})\partial_a(\sqrt{\hat{h}}\hat{h}^{ab}\partial_b x)$. The measure in (2.4a) depends on the gauge-fixed metric because it is defined in terms of an inner product $\langle \delta x(\xi), \delta x(\xi) \rangle = \int d^2\xi \sqrt{\hat{h}} \delta x(\xi)^2$. If there is a CKG, (2.4) has a further gauge symmetry, even though h has already been fixed to $\hat{h}$:

$$\begin{aligned} x &\to j^* x, \qquad j \in \mathrm{CKG} \\ \hat{h} &\to \hat{h}. \end{aligned} \tag{2.5}$$

The action $S[\hat{h}, x]$ is unchanged under (2.5) because it amounts to a diffeomorphism $x \to j^* x$, $\hat{h} \to j^* \hat{h}$, followed by a Weyl rescaling $j^* \hat{h} \to \exp(-p) j^* \hat{g} = \hat{h}$, using (3). Moreover the measure $[dx]_{\hat{h}}$ is invariant under (2.5), as was argued in ref. [30]. Henceforth we shall redefine CKG to mean the group of transformations (2.5), under which $\hat{h}$ does not change. (This is isomorphic to its original meaning.) To fix the gauge in (2.4a), let x be constrained to lie in a slice $\vec{F}(x) = 0$ through the CKG orbits. $\vec{F}$ has three components for D_2 because SL(2,R) has three generators. The Faddeev-Popov procedure gives

$$\begin{aligned} \tilde{Z} &= \int dj \int [dx]_{\hat{h}} \delta(\vec{F}(x)) J_F(x) \exp(-S[\hat{h}, x]) \\ &\equiv \mathrm{Vol(CKG)} Z_{g.f.}, \end{aligned} \tag{2.6}$$

where dj is the invariant group measure and J_F is the F-P determinant corresponding to the gauge condition $\vec{F}(x) = 0$. Even though dj is a finite-dimensional measure, the group volume is infinite, since SL(2,R) is not compact.

Although the following analysis could also be carried out for spherical (S_2) world sheets, it is simpler to illustrate the problem for the disk (D_2), which may be obtained by stereographic projection of a hemisphere onto the region of the complex plane $|z| \leq 1$. Let $\hat{g}$ be the standard round metric for the hemisphere

$$\hat{h}_{z\bar{z}} = 4(1+|z|^2)^{-2}, \tag{2.7}$$

which has constant bulk curvature and zero extrinsic curvature on the boundary. The eigenfunctions of $\Delta_{\hat{h}}$ are just the spherical harmonics, with eigenvalues $l(l+1)$, except that they are restricted to have $l-m$ even so that no momentum flows off the boundary [4]. In this basis

$$x^\mu = \sum c^\mu_{lm} Y_{lm}, \quad \text{and} \quad [dx]_{\hat{g}} = \prod dc^\mu_{lm}. \tag{2.8}$$

Since x is real, the c's are also restricted by $\bar{c}^\mu_{lm} = (-1)^m c^\mu_{l,-m}$. We omit the zero-mode integrals throughout, as this just gives a factor of $(\text{Vol(spacetime)})^{26}$.

On D_2 there are three normalizable conformal Killing vectors (CKV's) $\vec{V}_a$,

$$\begin{pmatrix} V^z \\ V^{\bar{z}} \end{pmatrix}_{1,2,3} = \sqrt{\frac{3}{16\pi}}\begin{pmatrix} 1-z^2 \\ 1-\bar{z}^2 \end{pmatrix}, \quad i\sqrt{\frac{3}{16\pi}}\begin{pmatrix} 1+z^2 \\ -1-\bar{z}^2 \end{pmatrix}, \quad i\sqrt{\frac{3}{4\pi}}\begin{pmatrix} z \\ -\bar{z} \end{pmatrix} \tag{2.9}$$

satisfying

$$\partial_{\bar{z}} V^z = \partial_z V^{\bar{z}} = 0 \tag{2.10a}$$

and the boundary condition

$$\vec{n}\cdot\vec{V}|_{|z|=1} \;=\; (\bar{z}V^z + zV^{\bar{z}})_{|z|=1} \;=\; 0, \tag{2.10b}$$

which insures that the boundary is preserved by reparametrizations. The CKV's generate infinitesimal conformal isometries $j^* x^\mu = (1+\epsilon_a \vec{V}_a\cdot\vec{\partial})x^\mu \equiv \sum (c^\mu_{lm})^\epsilon Y_{lm}$.

The transformed c's are

$$(c_{lm})^{\epsilon} = c_{lm} + \epsilon(A_{l-1,-m+1}c_{l-1,m-1} + B_{l+1,-m+1}c_{l+1,m-1})$$

$$-\bar{\epsilon}(A_{l-1,m+1}c_{l-1,m+1} + B_{l+1,m+1}c_{l+1,m+1}) - \epsilon_3 \mathrm{im} c_{lm}, \qquad (2.11)$$

where $\epsilon = \epsilon_1 + i\epsilon_2$ and A, B are Clebsch-Gordan-like coefficients,

$$A_{lm} = l\left(\frac{(l-m+1)(l-m+2)}{(2l+1)(2l+3)}\right)^{1/2}, \qquad B_{lm} = (l+1)\left(\frac{(l+m)(l+m-1)}{(2l-1)(2l+1)}\right)^{1/2}. \qquad (2.12)$$

We now compute the gauge-fixed partition function, $Z_{g.f.}$. The CKG symmetry is fixed by imposing the gauge condition $\vec{F}(c) = 0$. To expose the gauge dependence of $Z_{g.f.}$, we consider a family of slices

$$\vec{F}(c) = (c^1_{11}, c^1_{1,-1}, c^1_{2,-2} - 2\kappa c^1_{20}) \qquad (2.13)$$

that depend on a real parameter κ. Then J_F is the modulus of the determinant

$$\left.\frac{\partial(\vec{F}(c^{\epsilon}))}{\partial(\epsilon, \bar{\epsilon}, \epsilon_3)}\right|_{\vec{F}(c)=0} \propto \left|\begin{matrix} B_{20}c^1_{20} & -B_{22}c^1_{22} & 0 \\ B_{22}c^1_{2,-2} & -B_{20}c^1_{20} & 0 \\ * & * & -2i(c_{22} - c_{2,-2}) \end{matrix}\right|. \qquad (2.14)$$

(Note that $dj \propto d^3\epsilon$ near the identity.) Omitting a finite normalization factor, the gauge-fixed partition function in (2.6) is then

$$Z_{g.f.} = (\det \Delta_{\hat{h}})^{-D/2} \int_{-\infty}^{\infty} d\alpha d\beta \left|\alpha^3 + (\kappa^2 - \frac{1}{6})\alpha\beta^2\right| \exp(-3(\alpha^2 + (1+\kappa^2)\beta^2)) \qquad (2.15)$$

(where $\alpha = \mathrm{Im}\ c^1_{22}$ and $\beta = c^1_{20}$), which clearly depends on κ, hence the gauge. A more glaring inconsistency is that $\tilde{Z} = (\det \Delta_{\hat{h}})^{-D/2}$ is finite when regulated, whereas $\int dj$ is infinite, since Vol(SL(2,R))$= \infty$. Therefore $Z_{g.f.}$ should vanish, according to (2.6), and in disagreement with (2.15).

When vertex operators are inserted in the path integral, there is a divergence associated with the symmetry that allows one of their positions to be fixed on the world sheet. (On the sphere three may be fixed). Since this symmetry group turns out to be identically the CKG,* one might wonder whether the gauge dependence goes away for higher point functions computed in the same manner as (2.6), that is, not fixing the vertex operator positions, but fixing several modes of $x^\mu(\xi)$. We investigated this for the dilaton tadpole on D_2, denoted A'_D, which can be calculated by expanding the usual dilaton vertex operator in modes of $\Delta_{\hat{h}}$ and integrating over the c's. That is,

$$A'_D = \int \prod dc\delta(\vec{F}(c))J_F(c)\exp(-S_{reg}(c))V_D(c;p=0), \tag{2.16}$$

where S_{reg} is the regulated action (defined below), and the vertex operator for a dilaton of momentum p is

$$V_D(c;p) = \int d^2z \left(\sum_{\substack{lm \\ l'm'}} c^\mu_{lm} c^\nu_{l'm'} \partial Y_{lm} \bar{\partial} Y_{l'm'} \delta^\perp_{\mu\nu}(p) + c.t. \right) \exp(i \sum_{ab} p^\alpha \hat{c}^\alpha_{ab} Y_{ab}). \tag{2.17}$$

In (2.17), $\delta^\perp_{\mu\nu}$ is a transverse polarization tensor, "*c.t.*" are Weyl anomaly counterterms [31,38], and $\hat{c}^\alpha_{ab}$ is given by (for example)

$$\hat{c}^\alpha = \Lambda^{1/2}_{reg} e^{-\eta\Lambda/2} \Lambda^{-1/2} c^\alpha, \tag{2.18}$$

where Λ is the matrix of unregulated eigenvalues of $\Delta_{\hat{h}}$, and Λ_{reg} is regulated so as to have a finite determinant. The reason we need $\hat{c}$ in the exponent rather than c is that even though the action is regulated to give a finite value for $\det \Delta_{\hat{h}}$, this does not give a regulated Green's function $\langle x^\mu(z) x^\nu(z') \rangle$ at short distances. If we define

---

* I say "turns out to be" because the formal proof that the CKG-invariance of the functional integral implies the CKG-invariance of the world-sheet integrals breaks down, since the former is not really a symmetry.

$x^{\mu}_{reg} = \hat{c}^{\mu}_{lm} Y_{lm}$ however, the Green's function $\langle x^{\mu}_{reg}(z) x^{\nu}_{reg}(z') \rangle$ *is* finite as $z \rightarrow z'$. Thus, η is the cutoff for the Green's functions in (2.18), and eq. (2.16) would give the usual expression for the dilaton tadpole if we removed the gauge-fixing apparatus. Call the latter A_D. It is straightforward to show that

$$A'_D = (Z_{g.f.}(\kappa)/\tilde{Z}) A_D + (\kappa\text{—dependent term}), \tag{2.19}$$

where $Z_{g.f.}(\kappa)$ is given in (2.15). Thus A'_D is gauge dependent.

It seems clear that all the amplitudes will be afflicted in this manner. Also it is obvious that the same problems will occur for S_2, which has three extra CKV's in addition to those for D_2, eq. (2.9). Other topologies are discussed below.

3. Regulator effects

Our unsuccessful attempt to fix the gauge symmetry (2.5) shows that it is not really a symmetry of $\tilde{Z}$, eq. (2.4a). How is it broken? One might suspect the functional measure, since this is the source of chiral anomalies in theories with fermions. Using (2.11), however, the Jacobian for $\prod dc$ can be explicitly computed for small CKG transformations on D_2; it is

$$\left| 1 + i\epsilon_3 \sum_{\substack{l-m \\ even}} m + O(\epsilon^2) \right| . \tag{2.20}$$

Since any reparametrization-invariant regulator must weight every m within an l multiplet equally, this is always unity to $O(\epsilon^2)$.

That leaves the action. Indeed we already know that regulating the action so as to make $\det \Delta_{\hat{h}}$ finite breaks Weyl invariance (which is partly responsible for the usual conformal anomaly), and then $S[\hat{h}, x]$ is no longer invariant under $x \rightarrow j^* x$ for

$j \in$ CKG. The correct version of (2.6) should accordingly read

$$\begin{aligned}\tilde{Z} &= \int dj \int [dx]_{\hat{h}} \delta(\vec{F}(x)) J_F(x) \exp(-S_{reg}[j^*\hat{h}, x]) \\ &\equiv \int dj Z_{g.f.}[j].\end{aligned} \tag{2.21}$$

Since $Z_{g.f.}$ is now seen to depend on j, the volume $\int dj$ no longer factors out.

One may at first be puzzled as to why the breaking of $\mathcal{W}$-invariance by the regulated action should lead to any problem, since the counterterms

$$S_{c.t.} = A \int d^2\xi \sqrt{h} + B \oint ds \tag{2.22}$$

and space-time dimension are carefully chosen to preserve $\mathcal{W}$ -invariance of the full partition function [7, 8]. But there is no paradox. Even though

$$Z = (\det P^\dagger P)_{\hat{h}}^{1/2} \exp(-S_{c.t.}[\hat{h}]) \int dj Z_{g.f.}[j, \hat{h}] \tag{2.23}$$

is $\mathcal{W}$-invariant, this is not sufficient for $Z_{g.f.}$ to be j-invariant. The latter obtains only if $Z_{g.f.}$ by itself is $\mathcal{W}$-invariant, which is clearly not the case.

It is instructive to see just how the breaking of the CKG symmetry occurs for the infinitesimal transformations (2.11). Let Λ_{reg} be the diagonal matrix of regulated eigenvalues λ_l of $\Delta_{\hat{h}}$, as before. For the variation of $S_{reg} = \frac{1}{2} c^\dagger \Lambda_{reg} c$ to vanish under (2.11) to first order in ϵ, it is necessary that

$$\lambda_l B_{l+1,-m} = \lambda_{l+1} A_{l,m+1}. \tag{2.24}$$

This reflects the fact that some of the infinitesimal CKG transformations (those which are not isometries) mix modes whose values of l differ by one unit. Eq. (2.24) is satisfied only if

$$\lambda_{l+1}/\lambda_l = 1 + 2/l. \tag{2.25}$$

Now any regulator must make $\det \Lambda$ finite, which means the regulated eigenvalues go to 1 as $l \to \infty$. But suppose $\lambda_l = 1 + \delta$ for some l. Then (2.25) implies

$$\lambda_N = \frac{(N+1)N}{(l+1)l}(1+\delta), \tag{2.26}$$

which diverges as $N \to \infty$! Therefore no regulator respects (2.25).

Eq. (2.26) demonstrates that the CKG symmetry is badly broken even for arbitrarily small values of the cutoff for $\det \Lambda$—call it s. For any $s > 0$, $\lambda_\infty = 1$, whereas the condition (2.25) for maintaining CKG invariance implies $\lambda_\infty = \infty$. Therefore $Z_{g.f.}[j, s]$ does not continuously become j-independent as $s \to 0$. This can also be seen by looking at the expression

$$\int dj\, Z_{g.f.}(j,s)/Z_{g.f.}(0,s) = \tilde{Z}(s)/Z_{g.f.}(0,s). \tag{2.27}$$

Using eq. (2.15) (which is $Z_{g.f.}$ evaluated at j = identity) and the fact that $\tilde{Z} = (\det \Delta_{\hat{h}})^{-D/2}$, the r.h.s. of (2.27) is manifestly cutoff-independent and finite, whereas the l.h.s. would diverge like $\int dj$ as $s \to 0$, if $Z_{g.f.}$ became j-independent in a smooth way. This behavior can be qualitatively described by the Ansatz

$$Z_{g.f.}(j,s) \sim \exp(-N_F |j|^{1/s}) Z_{g.f.}(0,s), \tag{2.28}$$

where the constant N_F depends on the gauge condition $\vec{F}$ in just such a way as to make the integral $\int dj$ of (2.28) independent of $\vec{F}$, as $\tilde{Z}$ must be. Note that at $s = 0$ the r.h.s. of (2.28) is invariant under infinitesimal CKG transformations but does depend on large ones, $|j| > 1$.

The reason for saying that the CKG symmetry is *anomalously* broken is that it cannot be preserved by adding local counterterms to the action, such as (2.22).

This is because all such counterterms are already determined by the requirement of $\mathcal{W}$-invariance of the partition function [7]. These are due to local effects and are the same for all genera in the topological expansion.

4. Discussion

We have shown that the CKG gauge symmetry of the Polyakov partition function is anomalous, but not necessarily in the traditional sense of having nonconserved charges. The Nöther currents for the CKG generators are found to be

$$\begin{pmatrix} J^z \\ J^{\bar{z}} \end{pmatrix} = \begin{pmatrix} V^{\bar{z}}\bar{\partial}x^\mu\bar{\partial}x^\mu \\ V^z\partial x^\mu\partial x^\mu \end{pmatrix} \tag{2.29}$$

in complex conformal coordinates. Then one can show that for S_2, for example, whose CKV's are

$$\vec{V}_{1,\dots,6} \propto \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} i \\ -i \end{pmatrix}, \begin{pmatrix} z \\ \bar{z} \end{pmatrix}, \begin{pmatrix} iz \\ -i\bar{z} \end{pmatrix}, \begin{pmatrix} z^2 \\ \bar{z}^2 \end{pmatrix}, \begin{pmatrix} iz^2 \\ -i\bar{z}^2 \end{pmatrix}, \tag{2.30}$$

the integrated "anomaly" is

$$A_a \equiv \int d^2z \left\langle \vec{\partial} \cdot \vec{J}_a \right\rangle \propto \int d^2z\,\partial(V_a^{\bar{z}}\bar{\partial}\bar{\partial}\ln h_{z\bar{z}}) + c.c., \tag{2.31}$$

where we have used the conformal anomaly of $\langle\bar{\partial}x(z)\bar{\partial}x(z)\rangle \sim \bar{\partial}\bar{\partial}\ln h_{z\bar{z}}$. For the metric (2.7), the only nonvanishing "anomaly" is A_3, yet the CKG symmetry is broken in the regulated action for all but three linear combinations of the generators, those three being

$$\vec{V}_1 + \vec{V}_5, \qquad \vec{V}_2 - \vec{V}_6, \qquad \vec{V}_4. \tag{2.32}$$

These generate pure rotations, which do not change the metric at all and hence do not depend on having unbroken $\mathcal{W}$-invariance in the action. Therefore the A_a's are not a reliable indication of how the CKG symmetry is broken.

A difference between the present situation and ordinary quantum field theory applications is that in the latter we often ignore the divergent functional determinants as being uninteresting normalization factors for physical amplitudes, and anomalies arise from symmetry breaking due to regulating Green's functions rather than determinants. But in string theory these determinants have direct physical significance: they determine how much a given order in string perturbation theory contributes to the vacuum energy of the effective field theory, as well as the gauge group for which tadpole cancellations occur in open string theories [4, 26].

The significance of the above result is that one cannot define an infinite dimensional integral for scattering amplitudes on S_2 or D_2 that is not of the form ∞/∞. The usual prescription is simply to define an N-particle amplitude to be

$$A_N \sim (\det P^\dagger P)^{1/2} (\det \Delta_{\hat{h}})^{-D/2} \langle V_1 \cdots V_N \rangle / \mathrm{Vol}(CKG), \tag{2.33}$$

by using ζ-function or proper time regularization for the determinants, a world-sheet cutoff for the vertex operator correlations, and fixing positions of vertex operators to compensate for Vol(CKG). It would be satisfying if (2.33) could be derived from an actual integral, with no ∞/∞ factors, such as eq. (2.16). Since this cannot be done, it is, strictly speaking, incorrect to speak of the Polyakov path integral for S_2 or D_2 amplitudes. The expression (2.33) is only formally equal to the result of performing an integral. This is in contrast to the situation for lattice gauge theories, for example. Admittedly there may be no real physics at stake: eq. (2.33) seems to give physically sensible results. But from the standpoint of mathematical rigor, it is annoying that we cannot carry out the gauge-fixing construction that was so elegantly outlined in [30]. Thus, as has already been emphasized, the usual procedure of integrating over all embeddings and dividing by Vol(CKG) is an average over all CKG-gauge slices, each one of which generally gives a different contribution.

Fortunately the proof of $\mathcal{W}$-invariance of the functional integral given in [30] is still valid despite this problem. There it was pointed out that $\mathcal{W}$-invariance was apparently spoiled by the presence in the functional measure of the determinant of the inner products of the Teichmüller deformations, which depends on the Weyl factor σ. By carefully defining the measure for moduli it was shown that this source of Weyl dependence is canceled out. Fixing the CKG gauge symmetry was an intermediate step in the proof, but since there are no moduli on S_2 or D_2, it is not invalidated by the breakdown of CKG invariance for these topologies. The only thing affected is the claim that no special formalism (fixing vertex operator positions) is needed for the tree-level topologies; we have shown that it is needed.

There are several other topologies with CKG's. These are the projective plane, torus, cylinder, Klein bottle, and Möbius strip. This list is exhaustive because one can prove that no CKV's exist for surfaces with negative Euler characteristic [7]. For all of these topologies the CKG's are compact, in contrast to S_2 and D_2. For example, the CKG is SO(3) for the projective plane [7] and $\mathrm{U}(1) \times \mathrm{U}(1)$ for the torus. Since the remaining topologies can be obtained from the torus by identifying pairs of points under the action of some involutive functions [6], their CKG's turn out to be a U(1) subgroup of that for the torus. Since the CKG is compact for all these surfaces, the Polyakov path integral is perfectly well defined on them without fixing the CKG gauge. It is curious that in all these cases the CKV's are actually *true* Killing vectors, i.e., $j^*\hat{h} = \hat{h}$. For example, the projective plane has the CKV's in (2.32). It has been shown explicitly that the CKG can be gauge-fixed in that case [33], giving the same result as when it is not fixed. Therefore the CKG symmetry is anomalous only for the sphere and the disk.*

* The difficulty of trying to do CKG-gauge fixing on D_2 was first discovered by M. Douglas, B. Grinstein, and M. Wise, who collaborated in the early stages of this work.

Chapter 3. Tadpole Amplitudes

3.1. The dilaton tadpole problem

Scattering amplitudes involving an external dilaton of momentum p are computed by inserting the vertex operator [10]

$$V_D(p) = \frac{\kappa_D}{\sqrt{24}} \int d^2z\sqrt{h}h^{ab}{:}\,\partial_a x^\mu \partial_b x^\nu{:}\,\delta^\perp_{\mu\nu} e^{ip\cdot x} \tag{3.1}$$

into the Polyakov path integral Z, where the polarization tensor

$$\delta^\perp_{\mu\nu} = \delta_{\mu\nu} - p_\mu \bar{p}_\nu - p_\nu \bar{p}_\mu; \qquad p\cdot\bar{p} = 1; \quad p^2 = \bar{p}^2 = 0, \tag{3.2}$$

is designed to be transverse and κ_D is a normalization factor that can be determined by unitarity of tree amplitudes [34] to be

$$\kappa_D = \kappa T/2 \tag{3.3}$$

(κ is the gravitational constant appearing in the effective action (1.2)).

The normal-ordering symbol in (3.1) means omitting all singular self-contractions in the operator $\partial x \partial x$. For example, introducing a short-distance regulator ϵ into the Green's function $\langle x(z_1)x(z_2)\rangle \sim \ln(|z_1 - z_2|^2 + \epsilon^2)$ gives $\langle \partial x \partial x\rangle \sim 1/\epsilon^2$. We must regulate in a reparametrization-invariant way, however, since physical quantities must not depend on what coordinate system is used. The cutoff ϵ does not have this property, but in conformal coordinates, where the metric is $h_{ab} = e^{\sigma(z)}\delta_{ab}$, the quantity $ds^2 = e^\sigma |dz|^2$ is invariant; thus one should replace ϵ^2 by $e^{-\sigma}\epsilon^2$ in the regulated Green's function [35]. One then finds that $\langle \partial x \partial x\rangle \sim 1/\epsilon^2 + \partial^2\sigma$. The last term has geometrical significance [7]: it is related to the world-sheet curvature,

$$\partial_a \partial_a \sigma = -\sqrt{h}R. \tag{3.4}$$

This is the conformal, or Weyl, anomaly of $\langle \partial x \partial x\rangle$. It is clearly not invariant under Weyl transformations, $\sigma \to \sigma + \sigma'$, and so we take ${:}\,\partial x \partial x{:}$ to mean that the Weyl anomaly is also subtracted out, in addition to the divergent part. Similarly the operator $x\partial x$ has a Weyl anomaly $\langle x\partial x\rangle \sim \partial\sigma$, which would appear in contractions between ∂x and $e^{ip\cdot x}$, but the transversality of $\delta^{\perp}_{\mu\nu}$ eliminates these.

The dilaton tadpole amplitude for a string world sheet of arbitrary topology χ is $Z_\chi \langle V_D(p=0)\rangle_\chi$, where Z_χ is the path integral on the surface and $\langle\ \ \rangle_\chi$ is normalized so that $\langle 1\rangle_\chi = 1$. For S_2 this vanishes doubly since, as discussed in the previous chapter, $Z_{S_2} = 0$ and also $\langle :\partial x\partial x:\rangle_{S_2} = 0$. At the next order, the expectation values $\langle :\partial x\partial x:\rangle_{D_2,P_2}$ are no longer zero because the Green's functions have contributions that are not singular at short separations: in complex coordinates,

$$\langle x^\mu(z_1)x^\nu(z_2)\rangle = -\frac{\delta_{\mu\nu}}{4\pi T}\left(\ln|z_1-z_2|^2 + \ln|1 \pm z_1\bar{z}_2|^2\right), \tag{3.5}$$

where the upper (lower) sign applies for P_2 (D_2). The extra contribution can be thought of as being due to an image charge at the point $\tilde{z} = \pm 1/\bar{z}$. One finds that the dilaton tadpole has the value [19,4]

$$A_D = \frac{\sqrt{24}\kappa}{2\pi}\begin{cases} Z_{D_2}\mathrm{Vol}(\mathrm{SL}(2,\mathrm{R})), & D_2 \\ -Z_{P_2}\frac{\pi}{2}, & P_2 \end{cases}. \tag{3.6}$$

In chapter 1 it was explained that quantization of the string in a condensate of massless background fields led to a prediction about the ratio of A_D to the vacuum energy density Λ_χ contributed by topology χ,

$$A_D = \frac{7}{\sqrt{24}}\kappa\Lambda. \tag{3.7}$$

But Λ_χ is supposed to be given by $-Z_\chi$, so (3.7) can be checked using (3.6). Let us first review why $\Lambda_\chi = -Z_\chi$. The relation is most easily understood for the torus, where Z is just the one loop string path integral with no external lines. This is analogous to the one loop vacuum bubble diagram in a free field theory, which is responsible for generating the cosmological constant, to lowest order in $\hbar$. In fact, Polchinski's evaluation of Z_{T_2} [36] showed that it was exactly the same as minus the sum of vacuum energies for the constituent point particle field theories corresponding to the string excitations, except for a natural ultraviolet cutoff provided by the string that makes Z_{T_2} UV-convergent. More generally, the correspondence between string theory and effective field theory amplitudes for massless modes is

$$\sum_{\text{topologies}} \int \mathcal{D}x e^{-S[x]} V_1 \cdots V_M = \prod_{i=1}^{M} \int d^D x f(x, p_i) K\left[\partial/\partial x\right] \frac{\delta}{\delta J(x)} \cdot \ln\left(\int \mathcal{D}\phi \exp(-S_{eff}[\phi] + \int d^D x J\phi) \right), \tag{3.8}$$

where ϕ is a generic massless field, f is the wave function of the external state, and K is the inverse propagator. When there are no external particles ($M = 0$) (3.8) reduces to

$$\sum_{\text{top.}} Z_i = Z_{S_2} + Z_{D_2} + Z_{P_2} + \cdots = -\Lambda, \tag{3.9}$$

provided that the field theory path integral is normalized to unity when $\Lambda = 0$. (For example, the Euclidean path integral for a point particle with $\mathcal{L} = \frac{1}{2}m\dot{x}^2 + V(x)$ on a lattice of length L goes like $(2\pi/m)^{L/2}e^{-LE_0}$ as $L \to \infty$, where E_0 is the vacuum energy.) Λ should be thought of as the sum of contributions from the various topologies. Using $Z = -\Lambda$ in (3.6), one sees that eq. (3.7) is not satisfied. For D_2 the disagreement seems to be drastic since Vol(SL(2,R)) is divergent. However it was argued in [9] that the volume should be renormalized in a certain way, which gives $\text{Vol}(\text{SL}(2,\text{R})) = -\pi/2$. We will not repeat their argument here but will accept their result. Thus (3.6) is off by the same factor of $(D-2)/(D+2)$ for both topologies.

It was claimed by Klebanov (see [9]) that (3.6) is actually the sum of two tadpoles, the dilaton and the graviton trace tadpole, which comes from the cosmological term $\Lambda\sqrt{g}$ when $g_{\mu\nu}$ is expanded around flat space, $g_{\mu\nu} = \delta_{\mu\nu} + \kappa h_{\mu\nu}$. For example, the tachyon mass shift is divergent because of the massless propagators at $p = 0$ in the two diagrams of fig. 3.1.

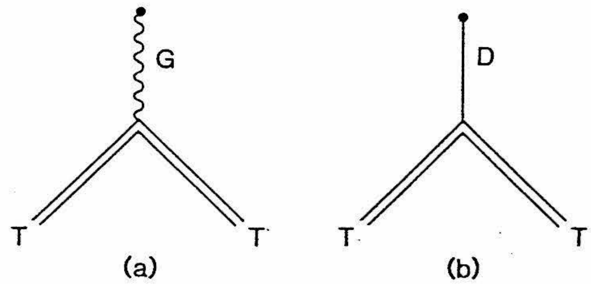

Figure 3.1. (a) Graviton trace tadpole and (b) dilaton tadpole contribution to the tachyon two-point function.

We have verified this claim by showing that the divergent part of the tachyon two-point function $A_{2T}(D_2)$ on the disk factorizes into the dilaton tadpole (3.6) and the two-tachyon, one-dilaton coupling on S_2,

$$A_{2T}(D_2) = \lim_{p \to 0} A_{2T,D}(S_2) \frac{1}{p^2} A_D(D_2) + O(p^0) \tag{3.10}$$

(the details of this calculation are given in chapter 4). If A_D was the correct dilaton tadpole, we would expect the right-hand side of (3.10) to make up only that part of the mass shift corresponding to fig. 3.1(b), not the whole thing. It is in this sense that A_D in (3.6) includes the graviton tadpole. The plausible appearance of (3.10), if one forgets the graviton tadpole, explains why Weinberg was able to corroborate the incorrect value of A_D using factorization of loop amplitudes [5].

3.2 A covariant dilaton vertex operator

From the above discussion we see that the correct dilaton tadpole should be a factor of $(D+2)/(D-2)$ greater than that computed with the vertex operator (3.1). Rey [22] verified this using a method of Cohen et al. [37] which uses punctures on the surface rather than vertex operators and is supposed to be more fundamental. But it is also far less convenient than using vertex operators, and one would like to find a vertex operator that gives the correct tadpole. A natural variant of (3.1) is the covariant expression

$$V_D(p) = \frac{\kappa T}{2\sqrt{24}} \int d^2z\sqrt{h}\left(h^{ab}\!:\partial_a x^\mu \partial_b x^\nu\!:\delta_{\mu\nu} - \frac{1}{4\pi T}R\right)e^{ip\cdot x}. \tag{3.11}$$

Since the trace of $\delta_{\mu\nu}$ is D instead of $D-2$ and there is the extra term $\int d^2z\sqrt{h}R$, the tadpoles computed from (3.11) are $(D+2)/(D-2)$ times (3.6), as desired.

Let us now derive the vertex operator (3.11). Because $\delta_{\mu\nu}$ is not transverse, a counterterm is needed to cancel the Weyl anomaly of $\langle \partial x^\mu e^{ip\cdot x}\rangle = ip^\mu e^{ip\cdot x}\langle \partial x x\rangle$. Using

$$\langle x(z_1)x(z_2)\rangle \sim -\frac{1}{4\pi T}\ln\left(|z_1-z_2|^2 + \epsilon^2 e^{-\sigma((z_1+z_2)/2)}\right) \tag{3.12}$$

for short distances, one finds that

$$\langle \partial_a x(z) x(z)\rangle = \frac{1}{8\pi T}\partial_a\sigma(z). \tag{3.13}$$

Therefore the Weyl anomaly of $T\!:\partial_a x^\mu \partial_a x^\mu\!:e^{ip\cdot x}$ is

$$\frac{1}{4\pi}\partial_a\sigma i p^\mu \partial_a x^\mu = \frac{1}{4\pi}\partial_a\sigma\partial_a e^{ip\cdot x}.$$

Integrating by parts gives $-\frac{1}{4\pi}\partial^2\sigma e^{ip\cdot x} = \frac{1}{4\pi}\sqrt{h}Re^{ip\cdot x}$, which is canceled by the counterterm in (3.11).

Unfortunately the derivation just given contains mistakes. First of all, the integration by parts has ignored a surface term, as is clear from the fact that $\partial\sigma\partial e^{ip\cdot x}$ is zero when $p=0$, whereas $\partial^2\sigma e^{ip\cdot x}$ is not. For definiteness consider the case of S_2 in which z is integrated over the entire plane. Although S_2 has no boundary, a surface term still arises because $\sigma(z)$ is not globally well defined, but diverges as $z\to\infty$. For the homogeneous, round sphere,

$$\sigma = -2\ln(1+|z|^2), \tag{3.14}$$

which at $p=0$ gives a surface term on the contour at ∞ of the form $\oint |dz|/|z|$. For spheres with different geometry, σ must still have the same asymptotic behavior as (3.14) in order to have nonsingular curvature at $z=\infty$.

The second flaw is that (3.13), although identical to the result of other authors [38], is missing a piece. Eq. (3.12) describes only the short-distance behavior of the Green's function. For S_2 the complete (unregulated) expression is

$$G(z_1, z_2) = -\frac{1}{4\pi T} \ln\left(\frac{|z_1 - z_2|^2}{F(z_1)F(z_2)}\right), \tag{3.15a}$$

where

$$F(z) = \exp\left(\frac{1}{A}\int d^2u\sqrt{h(u)}\ln|u-z|^2\right) \tag{3.15b}$$

and $A = \int d^2u\sqrt{g}$ is the area of the world sheet. This can be understood by writing G in terms of the orthonormal eigenfunctions and eigenvalues of the world-sheet Laplacian $\Delta = -\frac{1}{\sqrt{h}}\partial_a h^{ab}\sqrt{h}\partial_b$,

$$G(z_1, z_2) = \sum_{\lambda_n > 0} \frac{\rho_n(z_1)\rho_n(z_2)}{\lambda_n}. \tag{3.16}$$

Since the sum excludes the zero mode, operating on G with $T\Delta$ gives

$$T\Delta G(z_1, z_2) = \frac{\delta(z_1 - z_2)}{\sqrt{h}} - \frac{1}{A} \tag{3.17}$$

(since $\rho_0^2 = 1/A$). The function of the F factors in (3.15a) is to give the necessary $-1/A$ in (3.17). Using (3.15) with a cutoff, the anomaly (3.13) becomes*

$$\langle \partial_a x(z) x(z)\rangle = \frac{1}{8\pi T}\partial_a\left(\sigma(z) + \ln F^2(z)\right). \tag{3.18}$$

This solves the previously mentioned problem; since $\ln F \sim \ln|z|^2$ for large z, (3.18) is finite as $|z| \to \infty$ and gives rise to no surface term in the partial integration, but

* Similar observations have also been made in [39] and [40].

there is now an extra contribution proportional to $\partial^2 \ln F \propto \sqrt{h}/A$. To eliminate it, we must add another counterterm to the vertex operator, which becomes

$$V_D(p) = \frac{\kappa}{2\sqrt{24}} \int d^2z\sqrt{h} \left(Th^{ab} \colon \partial_a x^\mu \partial_b x^\nu \colon \delta_{\mu\nu} - \frac{1}{4\pi T} R + \frac{2}{A} \right) e^{ip\cdot x}. \tag{3.19}$$

We have checked that this operator gives tree-level amplitudes entirely equivalent to those constructed from the noncovariant operator (3.1). However, it is disturbing that the extra counterterm is nonlocal (since A is an integral over the world sheet). This new term also shifts the value of the dilaton tadpole—curiously it gives the same result on P_2 as the noncovariant vertex gave. Therefore if we wish to verify the sigma model prediction for A_D using the covariant vertex operator, we must not include the $2/A$ counterterm in (3.19). This in turn means ignoring surface terms at ∞ if we use the covariant vertex on S_2.

In retrospect it is not surprising that we must ignore such global aspects of the world sheet in order to corroborate the sigma model results, for the latter were obtained on a world sheet of infinite area, using dimensional regularization. In dimensional regularization, Weyl anomalies are computed by treating σ as a perturbation to $1 \cong e^\sigma$, and it is always assumed that integration by parts is valid. It would be interesting to see whether the sigma model effective action is the same if a regulator that respects global properties of the world sheet is employed, such as the heat kernel method (*i.e.*, multiplying (3.16) by $e^{-\epsilon\lambda_n}$). This was checked in [40] for the tree-level-type contributions (1.4), but not for the more interesting case of the cosmological terms from higher-genus surfaces.

3.3. The graviton tadpole and the cosmological constant

Given that the prescription described above for the covariant dilaton vertex operator makes sense, we can construct a new version of the graviton vertex operator, usually written as*

$$V_G = -\frac{\kappa T}{2}\int d^2z\sqrt{h}h^{ab}\partial_a x^\mu \partial_b x^\nu \epsilon_{\mu\nu} e^{ip\cdot x}, \tag{3.20}$$

where $\epsilon_{\mu\nu}$ is transverse and traceless (TT). However, taking $\epsilon_{\mu\nu}$ to be TT is just a choice of gauge for the external state; under a gauge transformation $\epsilon_{\mu\nu}$ goes to

$$\epsilon'_{\mu\nu} = \epsilon_{\mu\nu} + \zeta_\mu p_\nu + \zeta_\nu p_\mu, \tag{3.21}$$

where ζ is an arbitrary vector. $\epsilon'_{\mu\nu}$ is neither transverse nor traceless, but satisfies the harmonic coordinate condition [10]

$$(p\cdot\epsilon')_\mu = \frac{1}{2}p_\mu \mathrm{tr}\epsilon', \tag{3.22}$$

which is nothing more than the linearized equation of motion for the graviton. It should be possible to make a Weyl-invariant vertex operator for gravitons satisfying this more general condition. The lack of tracelessness and transversality will, as for the dilaton, lead to Weyl anomalies that must be subtracted by counterterms. The result, analogous to (3.9), is

$$V_G = -\frac{\kappa T}{2}\int d^2z\sqrt{h}\left(h^{ab}{:}\,\partial_a x^\mu \partial_b x^\nu{:}\,\epsilon_{\mu\nu} - \frac{1}{8\pi T}\epsilon_{\mu\mu}R\right)e^{ip\cdot x}. \tag{3.23}$$

This of course reduces to (3.18) in the TT gauge.

If we now compute $\langle V_G\rangle$ on P_2 or D_2, it gives a nonzero graviton tadpole proportional to $\epsilon_{\mu\mu}$. To avoid complications with renormalizing Vol(SL(2,R)), we can

---

* the sign of V_G is chosen so that it gives amplitudes in agreement with the effective action (1.2) when $g_{\mu\nu} = \delta_{\mu\nu} + \kappa h_{\mu\nu}$.

restrict our attention to P_2, for which the one-graviton amplitude is found to be

$$A_G(P_2) = \frac{\kappa}{2} Z_{P_2} \mathrm{tr}\epsilon. \tag{3.24}$$

Clearly this is a rather unphysical quantity since $\mathrm{tr}\epsilon$ is gauge-dependent. Nevertheless it can be compared to the field theory tadpole that comes from the cosmological term

$$\Lambda\sqrt{g} = \Lambda\left(1 + \frac{\kappa}{2} h_{\mu\mu} + O(h^2)\right) \tag{3.25}$$

when $g_{\mu\nu}$ is expanded around flat space,

$$g_{\mu\nu} = \delta_{\mu\nu} + \kappa h_{\mu\nu}. \tag{3.26}$$

Using the usual LSZ procedure, the one-point amplitude in Euclidean space (which we use throughout) is

$$-\frac{\kappa}{2}\Lambda \mathrm{tr}\epsilon. \tag{3.29}$$

Comparison with the string theory result implies

$$\Lambda = -Z_{P_2},$$

in agreement with the more general (but perhaps less convincing) argument given earlier, eq. (3.8). This provides another check on the consistency of the prescription described in the previous section. We have also computed the graviton tadpole for D_2 using the method of ref. [9], obtaining $\frac{\kappa}{2} Z_{D_2} \mathrm{tr}\epsilon$ as in eq. (3.24).

The reader may wonder how it is that gauge-noninvariant amplitudes have arisen here, since the field theory, and presumably string theory, are generally covariant. The problem is that the background $g_{\mu\nu} = \delta_{\mu\nu}$ is not a minimum of the action when a cosmological term is present. In general a gauge theory quantized about a classical background gives gauge-invariant amplitudes only if the background *satisfies* the equations of motion [41]. A more satisfying procedure would be to work in a background such as deSitter space to eliminate this problem, but computing string amplitudes (as opposed to just the partition function) in curved backgrounds introduces a whole new set of difficulties, to be described in chapter 5. The philosophy taken here and also by other authors [20, 42] is that although amplitudes in a flat background in field theory or in string theory are not meaningful by themselves (if $\Lambda \neq 0$), the comparison of the two is meaningful. This comparison is explored in greater depth in the next chapter, leading to further corroboration of the relation between the dilaton tadpole and cosmological constant, as well as the unitarity of string theory scattering amplitudes on D_2 and P_2.

Chapter 4. Factorization of divergent amplitudes

4.1. Introduction

Because there are no tadpoles at tree level in string theory, all amplitudes on S_2 are finite, but at the next order, tadpoles are responsible for infinities in the amplitudes [43]. They can be illustrated using the coupling of three closed-string tachyons on the disc, for example, as shown in fig. 4.1.

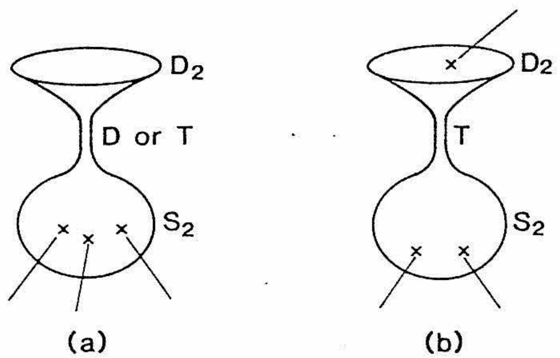

Figure 4.1. Factorization of $A_{3T}(D_2)$ into: (a) $A_{3T,D}(S_2)$ and $A_D(D_2)$ or $A_T(D_2)$; (b) $A_{3T}(S_2)$ and $A_{2T}(D_2)$.

Configurations in which all three vertex operators are close together are conformally equivalent to the degeneration of the disk into a sphere (S_2) and a disk connected by a narrow tube, representing a dilaton or tachyon propagator at zero momentum. These give rise to logarithmic and quadratic divergences, respectively, as can be understood by the way propagators are represented in the operator formalism of string theory. For states satisfying the $L_0 = \bar{L}_0$ constraint, in Euclidean space, it is

$$(p^2 + m^2)^{-1} = (4\pi T)^{-1} \int_0^1 d\tau \, \tau^{(4\pi T)^{-1}(p^2+m^2)-1}, \tag{4.1}$$

which at $p = 0$ diverges quadratically for a tachyon with $m^2 = -8\pi T$ and logarithmically for a massless particle. There is also a logarithmic divergence when only two of the vertex operators come together (fig. 4.1b) because the intermediate tachyon is on shell rather than at zero momentum. The quadratic divergences are *fictitious* [5]: we know that (4.1) should give $-1/8\pi T$ for tachyons at $p = 0$, not infinity, and indeed the problem can be circumvented by analytically continuing (4.1) in the variable p^2 to obtain the right answer. However, the logarithmic divergences are physical and cannot be removed.

One way of isolating these divergences is to use a short-distance cutoff such as (3.1) in the world-sheet integrals and express amplitudes as $a\epsilon^{-2} + b\ln\epsilon + c\epsilon^0$; the $\ln\epsilon$ term is interpreted as an on-shell propagator. One wants to equate this term with field theory amplitudes in which the divergence is expressed in the form $(1/p^2)|_{p^2=0}$, however, and it is not clear what the exact correspondence is between this expression and $\ln\epsilon|_{\epsilon=0}$. The connection can be found using the ideas of the previous paragraph. Fig. 4.1 and eq. (4.1) suggest that if some extra momentum q is forced through the intermediate propagator, with q going to zero at the end of the calculation, tachyon divergences would be rendered finite, and log divergences would appear as poles in q^2.* This can be accomplished by inserting

$$\exp\left(i\int d^2z\sqrt{g}q\cdot x(z)\right) \tag{4.2}$$

into the Polyakov path integral, so that the integral over the zero mode gives $(2\pi)^D\delta(q + \sum_{i=1}^N p_i)$ rather than $(2\pi)^D\delta(\sum p_i)$ in an N-particle amplitude. Using this approach, we will show that the connection between the world-sheet cutoff and the divergent propagator is

$$-\ln\epsilon \longleftrightarrow \frac{4\pi T}{q^2}. \tag{4.3}$$

To demonstrate (4.3) and our conventions, we factorize the dilaton tadpole divergence in the two-tachyon amplitude on the disk,

$$A_{2T}(D_2) = Z_{D_2}\left\langle V_T(p_1)V_T(p_2)\right\rangle, \tag{4.4}$$

where we have omitted the factor $(2\pi)^D\delta(p_1 + p_2)$, and the tachyon vertex operator

* This idea was used by Weinberg in [5].

is

$$V_T(p) = \frac{\kappa}{2\pi} \int\limits_{|z|\leq 1} d^2 z \sqrt{h(z)} \colon e^{ip\cdot x(z)} \colon . \tag{4.5}$$

In complex conformal coordinates, the string action is

$$S = 2T \int\limits_{|z|<1} d^2 z \partial x^\mu \bar{\partial} x^\mu , \tag{4.6}$$

where $\partial = \partial/\partial z$ (and $\bar{\partial} = \partial/\partial \bar{z}$), and the Neumann Green's function is

$$\begin{aligned} G(z_1, z_2) &= \langle x(z_1) x(z_2) \rangle \\ &= -\frac{1}{4\pi T} \ln \left(\frac{|z_1 - z_2|^2 |1 - z_1 \bar{z}_2|^2}{F(z_1) F(z_2)} \right) . \end{aligned} \tag{4.7}$$

The function $F(z)$ in (4.8) is defined analogously to eq. (3.15b) for S_2 but drops out of scattering amplitudes due to momentum conservation* and can be ignored henceforth. The amplitude (4.4) is invariant under SL(2,R) transformations [4]

$$z^\Omega \equiv \frac{az + b}{\bar{b}z + \bar{a}}, \qquad |a|^2 - |b|^2 = 1. \tag{4.8}$$

The invariant volume element is

$$d\Omega \propto d^2 b d(\arg(a)), \tag{4.9}$$

which for convenience is assumed to be normalized so that the Faddeev-Popov determinant for fixing one of the vertex operators at $z = 0$ is unity. The volume $\int d\Omega = \mathrm{Vol(SL(2,R))}$ is canceled by a factor of $1/\ \mathrm{Vol(SL(2,R))}$ in Z_{D_2} [7]. In what follows Z_{D_2} will always appear together with $\int d\Omega$. Since this combination is independent of the value of $\int d\Omega$, it will not be necessary to renormalize it as in [9], although one could presumably do so.

* Inclusion of (4.2) in the path integral spoils momentum conservation, but the F's will only appear in the form $(F)^{q^2} = 1 + q^2 \ln F \ldots$, which will not affect the divergent part of the amplitude. (4.2) also spoils SL(2,R) invariance, but similar remarks apply.

With these conventions the two-tachyon function is

$$A_{2T}(D_2) = \left(\frac{\kappa}{2\pi}\right)^2 Z_{D_2} \int d\Omega \int\limits_{|z|\le 1} d^2z (1-|z|^2)^{2-\alpha} |z|^{-4+\alpha} \left(1 + O(\alpha)\right), \qquad (4.10)$$

where $\alpha = q^2/4\pi T$. The pole part of (4.10) comes from expanding the integrand to obtain the term $\int d^2z |z|^{-2+\alpha}$. Alternatively, at $q^2 = 0$ this term can be defined by cutting off the integral using $|z|^2 > \epsilon^2$; whether we use a covariant cutoff $\epsilon^2 e^{-\sigma}$ is irrelevant since this does not affect the divergent part. Evaluating $\int d^2z|z|^{-2}$ in these two ways gives the correspondence (4.3). Using the momentum source regulator, the divergent part of A_{2T} is

$$A_{2T}(D_2)_{\text{pole}} = -\kappa^2 Z_{D_2} \int d\Omega \frac{4T}{q^2}. \qquad (4.11)$$

On the other hand, the two-tachyon, one-dilaton coupling on S_2 is found to be

$$A_{2T,D}(S_2) = -\frac{1}{\sqrt{24}} 8\pi\kappa T. \qquad (4.12)$$

Combining this with the tadpole of the noncovariant dilaton vertex operator, (3.6), and comparing with (4.11), we see that

$$A_{2T}(D_2)_{\text{pole}} = A_{2T,D}(S_2) \frac{1}{q^2} A_D(D_2)\Big|_{q\to 0}, \qquad (4.13)$$

as was quoted in chapter 3.

Eq. (4.13) is what one would expect for the divergent part of $A_{2T}(D_2)$ based on unitarity, if there was only a dilaton tadpole. But we also expect a graviton tadpole contribution, fig. 3.1. This is one way of seeing that the value of the dilaton tadpole used above is incorrect. If we use the corrected value of $A_D(D_2)$, eq. (4.13) becomes

$$A_{2T}(D_2)_{\text{pole}} = A_{2T,D}(S_2) \frac{1}{q^2} A_D(D_2) - \frac{4}{D-2} \left(-\frac{4\kappa^2 T}{q^2} Z_{D_2} \int d\Omega \right). \qquad (4.14)$$

The last term in (4.14) is the graviton tadpole contribution, whose consistency can be checked in a nontrivial way by comparing it with the Feynman diagram fig. 3.1(a) in the effective field theory. Although in general it does not make much sense to construct effective field theories for the massive modes of the string (since their masses are of the order M_{Planck}), experience shows that at least the coupling of a graviton to two tachyons derived from the kinetic term

$$\frac{1}{2}\sqrt{g}\left(g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi + M^2\Phi^2\right) \tag{4.15}$$

agrees with the string theoretic determination of $A_{2T,G}(S_2)$, so we can use (4.15)+$\Lambda\sqrt{g}$ with some confidence to evaluate the graviton tadpole graph. Using the graviton propagator in Feynman gauge,

$$P_{\alpha\beta,\mu\nu}(p) = \frac{1}{p^2}\left(\frac{1}{2}(\delta_{\alpha\mu}\delta_{\beta\nu} + \delta_{\alpha\nu}\delta_{\beta\mu}) - \frac{\delta_{\alpha\beta}\delta_{\mu\nu}}{D-2}\right), \tag{4.16}$$

one finds that

$$\text{fig. 3.1(a)} = \frac{\Lambda\kappa^2}{(D-2)q^2}8\pi T, \tag{4.17}$$

where $-8\pi T$ is the $(\text{mass})^2$ of the tachyon. Equating this with the last term of (4.14) gives

$$\Lambda_{D_2} = \frac{2}{\pi}Z_{D_2}\textstyle\int d\Omega. \tag{4.18}$$

This is in fact the value of Λ_{D_2} needed to confirm the sigma model prediction (3.7) for A_D/Λ. The actual value of Λ_{D_2} is independent of $\int d\Omega$ because of the implicit factor of $1/\int d\Omega$ in Z_{D_2}, but if we use [9]'s result for the renormalized value of $\int d\Omega = -\pi/2$, then $\Lambda_{D_2} = -Z_{D_2}$, in agreement with the argument based on eq. (3.8). Completely analogous calculations for P_2 also verify that $\Lambda_{P_2} = -Z_{P_2}$.

4.2. The FKS paradox

The simple factorization example of the preceding section has the basic features of a more extensive study done by Fischler, Klebanov, and Susskind [20]. They looked at divergences of graviton rather than tachyon amplitudes on D_2, since a low-energy effective field theory exists for the former. Taking this field theory to be the sigma model result (1.2)+(1.7), they obtained the contradictory expressions (1.8) and (1.9) for the field theory versus string theory value of the divergent part of graviton amplitudes on D_2. We have recalculated these expressions, keeping track of the overall normalization and using the correspondence (4.3) so that the magnitude as well as the form of the two equations can be compared. The string-theoretic result for the N-point amplitude is

$$A_{NG}(D_2) = \frac{\kappa^2}{2\pi q^2} Z_{D_2} \int d\Omega \left(T^{-1} \frac{\partial}{\partial(T^{-1})} \right) A_{NG}(S_2) + \text{finite} \tag{4.19}$$

and its field-theoretic counterpart is

$$A_{NG}(D_2) = \frac{\kappa^2 \Lambda}{4q^2} \left(T^{-1} \frac{\partial}{\partial(T^{-1})} + 1 \right) A_{NG}(S_2) + \text{finite}. \tag{4.20}$$

Assuming the discrepancy in form between these results has been resolved as in [23], they can be equated, giving

$$\Lambda_{D_2} = \frac{2}{\pi} Z_{D_2} \int d\Omega, \tag{4.18}$$

the expected value for the disk vacuum energy.

Actually there are a number of hidden assumptions in eq. (4.20). A term linear in N, the number of external legs, has been suppressed. This N-dependent term comes partly from external leg divergences as in fig. 4.2,

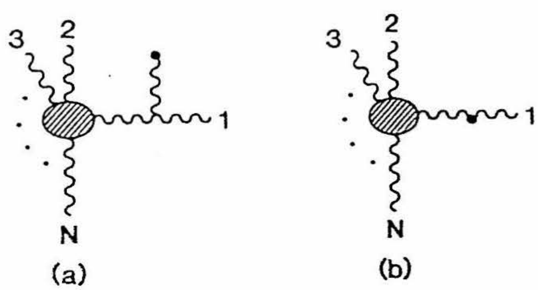

Figure 4.2. N-graviton scattering amplitudes with divergences due to on-shell propagators on the external legs.

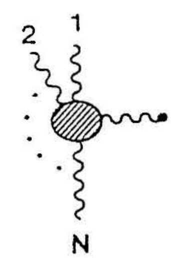

Figure 4.3. Tadpole contribution to N-graviton scattering.

where the dots represent tadpole and mass terms from expanding $\Lambda\sqrt{g}$ around flat space. In addition, there is an N-dependent piece in the contact term, fig. 4.3.

FKS argue that in the string calculation the N-dependent divergences cancel each other, and so they should be ignored in the field theory calculation. This seems plausible to us because the diagrams represented by fig. 4.2(a) depend sensitively on how one regulates the infrared divergence of the internal on-shell propagator, and this makes the value of the N-dependent part rather ambiguous. Since these contributions are strictly proportional to N, they cannot account for the discrepancy between (4.19) and (4.20).

Another assumption, not discussed by FKS, is that the graviton propagator is in a particular gauge, often called Feynman gauge because, in analogy to electrodynamics, it is the covariant gauge where the propagator is most simple. As was discussed in sect. 3.3, graviton amplitudes in a flat background are not gauge-invariant when the cosmological constant is not zero. In a general covariant gauge, with the gauge fixing term

$$\mathcal{L}_{g.f.} = \frac{1}{\xi}(h_{\mu\nu,\nu} - \zeta h_{\nu\nu,\mu})^2, \tag{4.21}$$

the propagator is

$$P_{\alpha\beta,\mu\nu}(p) = \frac{1}{p^2}\left(\delta^{(\mu}_{(\alpha}\delta^{\nu)}_{\beta)} - \frac{\delta_{\alpha\beta}\delta_{\mu\nu}}{D-2}\right) + \frac{2(\xi-1)}{p^4}p_{(\alpha}\delta^{(\mu}_{\beta)}p^{\nu)}$$

$$+\frac{2\zeta-1}{(\zeta-1)(D-2)}\left(\delta_{\mu\nu}\frac{p_\alpha p_\beta}{p^4} + \frac{p_\mu p_\nu}{p^4}\delta_{\alpha\beta}\right)$$

$$-\frac{2\zeta-1}{(\zeta-1)^2}\left(\frac{D+2\zeta-3}{D-2}+\frac{1}{2}\xi(2\zeta-3)\right)\frac{p_\mu p_\nu p_\alpha p_\beta}{p^6}, \tag{4.22}$$

where $M_{(\alpha\beta)}=\frac{1}{2}(M_{\alpha\beta}+M_{\beta\alpha})$. Feynman gauge, (4.16), corresponds to $\xi=1$, $\zeta=1/2$. Graviton tadpole diagrams involve a trace of the propagator, which is a complicated function of ξ, ζ and D (the dimension of spacetime), and this is where the gauge dependence comes in. For other values of ξ and ζ, eq. (4.20) would be different; in fact, the simple derivation used to evaluate the graphs of fig. 4.3 in Feynman gauge is no longer applicable in other gauges (see appendix to this chapter). One might wonder if this could be the cause of the FKS paradox, but ref. [23] shows that the problem is with the string theory equation (4.19), not the field theory result. Why should string amplitudes be equal to field theory amplitudes in a particular gauge? There exists an argument [44] that string amplitudes computed in the usual first quantized approach can be derived from string field theory with the Siegel-Feynman gauge fixing condition, in which the string propagator has a simple form that is supposed to yield the graviton propagator in Feynman gauge. Perhaps this idea is correct, but it has not yet been demonstrated in detail, as far as we are aware.

Even though no effective field theory exists for tachyons, we can use unitarity to derive a (nonrigorous) soft graviton/dilaton emission theorem for N-tachyon functions, analogous to (4.2):

$$\begin{aligned}A_{NT}(D_2)_{\text{pole}} &= A_{NT,D}(S_2)\frac{1}{q^2}A_D(D_2)\\ &+\sum_i A_{NT,G}(S_2)\bigg|_{\epsilon_{\mu\nu}=\epsilon^{(i)}_{\mu\nu}}\frac{1}{q^2}\left(-\frac{\kappa\Lambda}{2}\text{tr}\epsilon^{(i)}\right),\end{aligned} \tag{4.23}$$

where the graviton and dilaton in the S_2 amplitudes are at zero momentum. The sum in the second term is over a complete set of polarization tensors $\epsilon^{(i)}_{\mu\nu}$, and $-\frac{1}{2}\kappa\Lambda\text{tr}\epsilon$ is the value of the graviton tadpole. This sum, with the factor of $1/q^2$, clearly corresponds to the graviton propagator. To imitate the analogous calculation for

A_{NG} (see appendix) as closely as possible, we therefore make the replacement

$$\sum_i \frac{\epsilon^{(i)}_{\alpha\beta}\epsilon^{(i)}_{\mu\nu}}{q^2} \longrightarrow \frac{1}{q^2}\left(\delta^{(\mu}_{(\alpha}\delta^{\nu)}_{\beta)} - \frac{\delta_{\alpha\beta}\delta_{\mu\nu}}{D-2}\right). \tag{4.24}$$

Furthermore the amplitudes $A_{NT,G}$ and $A_{NT,D}$, with G and D at zero momentum, should be obtained from $A_{NT}(S_2)$ by insertion of the operators

$$V_D = \frac{2\kappa T}{\sqrt{24}}\int d^2z\left(:\partial x^\mu\bar{\partial}x^\mu: -\frac{\sqrt{h}R}{16\pi T}\right)$$

and

$$V_G = -2\kappa T\int d^2z\left(:\partial x^\mu\bar{\partial}x^\nu: \epsilon^{(i)}_{\mu\nu} - \frac{\epsilon^{(i)}_{\mu\mu}\sqrt{h}R}{32\pi T}\right),$$

respectively, according to the discussion of vertex operators in chapter 3. Using $A_D(D_2) = \frac{D+2}{4\sqrt{24}}\kappa\Lambda$, the result is

$$A_{NT}(D_2)_{\text{pole}} = \frac{\kappa^2\Lambda T}{2q^2}\left\{A_{NT}(S_2) \text{ with} \int d^2z\left(:\partial x^\mu\bar{\partial}x^\mu: +\frac{\sqrt{h}R}{16\pi T}\right) \text{inserted}\right\}. \tag{4.25}$$

On the other hand, a direct examination of the string-theoretic amplitude as in [1b] and [25] gives

$$A_{NT}(D_2)_{\text{pole}} = \frac{\kappa^2 T Z_{D_2}\int d\Omega}{\pi q^2}\left\{A_{NT}(S_2) \text{ with} \int d^2z :\partial x^\mu\bar{\partial}x^\mu: \text{inserted}\right\}. \tag{4.26}$$

Eqs. (4.25) and (4.26) are directly analogous to (4.20) and (4.19), respectively, for graviton scattering. FKS showed that the operator insertion $\int d^2z : \partial x^\mu\bar{\partial}x^\mu :$ gives rise to the term $T^{-1}\partial/\partial(T^{-1})$ for the graviton case; $\int d^2z\sqrt{h}R$ corresponds to the term 1 in (4.20). The particular combination of these operators in (4.25) is exactly what Polchinski found for factorization of D_2 tadpoles using a much more complicated (and rigorous) method [23]. As for the prefactors in (4.25, 26), comparison gives the usual result, $\Lambda_{D_2} = \frac{2}{\pi}Z_{D_2}\int d\Omega$.

For the divergence in A_{2T}, sect. 4.1, the reader will have noticed that there was no disagreement between the string theory and field theory results. This is because an insertion of $\int d^2z\sqrt{h}R$ in an N-particle amplitude on S_2 merely multiplies it by $4\pi\chi = 8\pi$, and all two-particle amplitudes on S_2 are zero: it takes at least three vertex operators to saturate the SL(2,C) symmetry.

4.3 Resolution of the FKS paradox?

In sect. 4.1 the disk was represented by the region $|z| \leq 1$, but since it is obtained from the sphere by identifying z with $1/\bar{z}$, one could equally well take the disk to be the region $|z| \geq 1$, a sphere with a hole cut out. The SL(2,R) symmetry of D_2 amplitudes allows one vertex operator position to be fixed. In order to make the D_2 amplitude resemble that on S_2 as closely as possible, FKS fixed three vertex operator positions, and integrated over the radius a and position z of the hole. Their expression for the amplitude was

$$A_{NG}(D_2) = \int \frac{da}{a^3} \int d^2z \prod_{i<j\leq 3} |z_i - z_j|^2 \int \prod_{k\geq 4}^{N} d^2z_k \, \langle V_G(z_1) \cdots V_G(z_N) \rangle_{D_2}, \quad (4.27)$$

and correlations were computed using the Green's function

$$G(z_1, z_2) = -\ln |z_1 - z_2|^2 \left|a^2 - (z_1 - z)(\bar{z}_2 - \bar{z})\right|^2, \quad (4.28)$$

in units where $4\pi T = 1$ (cf. eq. (4.7)). They found that the logarithmically divergent part could be expressed by inserting the operator $a^2 \int d^2z : \partial x^\mu \bar{\partial} x^\mu :$ and evaluating the correlations using the S_2 Green's function. The log divergence comes from the integral $\int da/a$. In order to resolve the paradox, one would like to find that the divergence is given by an insertion of

$$a^2 \int d^2z \left(: \partial x^\mu \bar{\partial} x^\mu : + \frac{1}{4}\sqrt{h}R \right),$$

rather than just the first term.

The simplified resolution offered in [23] is that $\int da/a^3 \int d^2z$ should be rewritten in terms of the *invariant* radius and position of the hole, which generates the extra term of order a^2. The position of the hole's geometric center z' differs from z by an amount

$$\delta z = \frac{\int\limits_{|u|<a} d^2u\sqrt{h(z+u)}u}{\int\limits_{|u|<a} d^2u\sqrt{h(z+u)}} = \frac{a^2}{2}\bar{\partial}\sigma(z) + O(a^4) \tag{4.29}$$

(using $h_{ab} = e^{\sigma}\delta_{ab}$). The invariant radius can be defined as an average over angles of the integrated radial line element,

$$\begin{aligned} a' &= \frac{1}{2\pi}\int d\theta \int_0^a du \exp\left(\sigma(z+ue^{i\theta})/2\right) \\ &= ae^{\sigma(z)/2}\left(1 + a^2(\frac{1}{6}\partial\bar{\partial}\sigma + \frac{1}{12}|\partial\sigma|^2) + O(a^4)\right). \end{aligned} \tag{4.30}$$

One then finds that

$$\frac{da}{a^3}d^2z = \frac{da'd^2z'}{(a')^3}\left(e^{\sigma} + (a')^2(\frac{1}{4}\sqrt{h}R + |\partial\sigma|^2)\right). \tag{4.31}$$

Although this has the desired $\frac{1}{4}\sqrt{h}R$ term, it also has an undesired noncovariant term $|\partial\sigma|^2$, which was missed in [23] because the $O(a^3)$ correction to a' was ignored there. Thus the resolution given here does not seem to be quite correct.

Appendix. Derivation of (4.20)

Here we derive the field theory result for massless tadpole divergences, eq. (4.20). The evaluation of graviton tadpole insertions in tree graphs is simplified in Feynman gauge by observing that they are equivalent to shifting the background by

$$\begin{aligned} \delta_{\mu\nu} &\longrightarrow \delta_{\mu\nu} + \kappa\left\langle\tilde{h}_{\mu\nu}(p)\right\rangle_{p=0} \\ &= \delta_{\mu\nu} + \left.\frac{\kappa^2\Lambda\delta_{\mu\nu}}{(D-2)p^2}\right|_{p=0}. \end{aligned} \tag{4.32}$$

(In other gauges there are terms of the form $p_\mu p_\nu/p^4$ whose meaning is ambiguous.)

Let

$$\frac{\kappa^2 \Lambda \delta_{\mu\nu}}{(D-2)p^2} \equiv \beta.$$

Then the metric with shifted background is

$$g_{\mu\nu} = (1+\beta)\left(\delta_{\mu\nu} + \kappa(1-\beta)h_{\mu\nu}\right) \equiv (1+\beta)\hat{g}_{\mu\nu} \tag{4.33}$$

to first order in β. Now consider terms in the effective Lagrangian proportional to T^{-n}, such as

$$f_n(g,R) = T^{-n}\sqrt{g}R^n, \qquad T^{-n}\sqrt{g}R^{\mu\nu}\nabla_\mu\nabla_\nu R^{n-2}, \ldots \tag{4.34}$$

Under the scale change (4.33), such terms transform like [45]

$$f_n(g,R) \longrightarrow (1+\beta)^{\frac{D}{2}-n} f_n(\hat{g},\hat{R}), \tag{4.35}$$

where $\hat{R}$ is the curvature formed from $\hat{g}$. The only difference between amplitudes from $f_n(g,R)$ and $f_n(\hat{g},\hat{R})$ is that each external leg in the latter will have a factor of $(1-\beta)$, but we are ignoring the correction proportional to the number of external lines, as explained in sect. 4.2. Linearizing (4.35) in β shows that the graviton tadpole contribution to such an amplitude is

$$\frac{\left(\frac{D}{2}-n\right)}{(D-2)}\frac{\kappa^2\Lambda}{p^2} A_N(S_2). \tag{4.36}$$

Similarly the coupling of one dilaton to the terms $e^\phi f_n(g,R)$ (this is the factor of e^ϕ that multiplies the whole tree-level Lagrangian in (1.4)) comes from the scale transformation and field redefinition (1.5):

$$e^\phi f_n(g,R) \longrightarrow e^{(n-1)\kappa\phi/\sqrt{D-2}} f_n(g,R). \tag{4.37}$$

Linearizing this in ϕ and combining it with the dilaton tadpole and zero-momentum propagator gives

$$\frac{(n-1)(D+2)}{4(D-2)p^2}\kappa^2 \Lambda A_N(S_2). \tag{4.38}$$

Using $n = T^{-1}\partial/\partial(T^{-1})$, the sum of the graviton and dilaton tadpole parts, (4.36) and (4.38) gives the desired result, eq. (4.20). Although this derivation was for one-particle irreducible (1PI) diagrams, it is also true for one-particle reducible (1PR) graphs because the tadpole divergence of a 1PR graph is the sum of the divergences for the 1PI parts, and $T^{-1}\partial/\partial(T^{-1})$ for the graph is also the sum of $T^{-1}\partial/\partial(T^{-1})$ for the 1PI parts.

Chapter 5. String Amplitudes in Curved Backgrounds

5.1. Introduction

In chapter 4 it was shown that divergences in the N-tachyon function on D_2 should be equal to an insertion of

$$-\frac{\ln\epsilon}{4\pi}\kappa^2\Lambda\int d^2z\left(:\partial x^\mu\bar{\partial}x^\mu:+\frac{\sqrt{h}R}{16\pi T}\right) \tag{5.1}$$

in the same amplitude on S_2, assuming the FKS paradox has been correctly cleared up by [23]. The idea of Fischler and Susskind [24] is to cancel (5.1) against a contribution to $A_{NT}(S_2)$ from background graviton and dilaton fields in the string action,

$$A_{NT}(S_2)\propto\int\mathcal{D}x(z)\prod_z\sqrt{g(x(z))}e^{-S_{\rm bg}}V_1\cdots V_N, \tag{5.2}$$

where $S_{\rm bg}$ is given by eq. (1.3), except we omit the antisymmetric tensor field $B_{\mu\nu}$. The $\sqrt{g}$ factor in (5.2) insures that the functional measure transforms covariantly under spacetime coordinate changes $x^\mu(z)\to y^\mu(x(z))$. It is often useful to go to a Riemann normal coordinate system about a point x_0 [46], so that $S_{\rm bg}$ can be expanded around the free string action S_0 as $S_0+\delta S$, where

$$S_0=2T\int d^2z\partial\xi^a\bar{\partial}\xi^a, \tag{5.3a}$$

$$\begin{aligned}\delta S=&\frac{2T}{3}R_{abcd}(x_0)\int d^2z\xi^b\xi^c\partial\xi^a\bar{\partial}\xi^d\\&-\frac{1}{4\pi}\int d^2z\left(\phi(x_0)+\xi^a\nabla_a\phi(x_0)+\frac{1}{2}\xi^a\xi^b\nabla_a\nabla_b\phi(x_0)\right)\sqrt{h}R^{(2)}+\cdots.\end{aligned} \tag{5.3b}$$

The ellipsis represents terms higher order in R_{ab} and ∇^2, which are suppressed by powers of $1/T$, and Latin indices for the flat tangent space have been obtained from the Greek (curved space) indices using the vielbein, $e^a_\mu(x_0)$. Treating δS as a perturbation, we can obtain an expression that looks like (5.1) by doing the $\langle\xi\xi\rangle$ contractions in (5.3b), using

$$\left\langle \xi^b(z)\xi^c(z)\right\rangle = -\frac{\ln\epsilon^2}{4\pi T}\delta_{bc} + \text{finite}, \tag{5.4}$$

as in chapter 3. The result is that

$$-\delta S = \frac{\ln\epsilon}{4\pi}\left(\frac{4}{3}R_{ad}(x_0)\int d^2z\partial\xi^a\bar\partial\xi^d - \frac{1}{4\pi T}\nabla^2\phi(x_0)\int d^2z\sqrt{h}R^{(2)}\right). \tag{5.5}$$

Apart from the action S_{bg}, log divergences can also come from the functional measure,

$$\prod_z\sqrt{\det g_{\mu\nu}(x(z))} = 1 + \frac{1}{6}R_{ab}(x_0)\int d^2z\,\text{“}\delta(0)\text{”}\xi^a(z)\xi^b(z) + \cdots \tag{5.6}$$

by contracting $\langle\xi^a\xi^b\rangle$. The symbol "$\delta(0)$" means the world sheet δ function at zero separation, which can be written as $-4T\left\langle\partial\bar\partial\xi\xi\right\rangle$. Thus the log divergence of (5.6) is

$$\left(\frac{4}{3}\right)\frac{\ln\epsilon}{4\pi}R(x_0)\int d^2z\left\langle\partial\bar\partial\xi\xi\right\rangle.$$

This exactly cancels against the self-contraction of the first term in (5.5) by doing a partial integration. So the net effect of the measure contribution (5.6) is to normal order $\partial\xi^a\bar\partial\xi^d$ in (5.5).

Comparing (5.1) and (5.5), one sees that the background field contribution to $A_{NT}(S_2)$ cancels the divergence of $A_{NT}(D_2)$ if

$$R_{\alpha\delta} = \frac{3}{4}\kappa^2\Lambda g_{\alpha\delta} \tag{5.7a}$$

and

$$\nabla^2\phi = -\frac{1}{4}\kappa^2\Lambda. \tag{5.7b}$$

Eqs. (5.7) should be a linear combination of the equations of motion for the graviton and dilaton. To check this, recall that the spacetime effective action corresponding to the background field string action we have used was shown to be eqs. (1.4) plus (1.6),

$$S = \frac{2}{\kappa^2}\int d^Dx\sqrt{g}e^{\phi}\left(-R-(\nabla\phi)^2\right) + \Lambda\int d^Dx\sqrt{g}e^{\phi/2}, \tag{5.8}$$

in the sigma model approach. It has the field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}(\nabla\phi)^2 g_{\mu\nu} = -\frac{1}{4}\kappa^2\Lambda e^{-\phi/2}g_{\mu\nu}, \tag{5.9a}$$

$$2\nabla^2\phi + (\nabla\phi)^2 - R = -\frac{1}{4}\kappa^2\Lambda e^{-\phi/2}, \tag{5.9b}$$

so that R and $\nabla^2\phi$ are of order $\kappa^2\Lambda$. Treating Λ as a perturbation, $\nabla\phi$ will also be $O(\Lambda)$, so we can drop terms of $O((\nabla\phi)^2)$. Taking traces and linear combinations of (5.7a, b), we obtain

$$R = \frac{D}{2(D-2)}\kappa^2\Lambda e^{-\phi/2}, \tag{5.10a}$$

$$\nabla^2\phi = \frac{D+2}{8(D-2)}\kappa^2\Lambda e^{-\phi/2}, \tag{5.10b}$$

which clearly disagrees with (5.7).

A possible source of error in the above treatment is the strength of the $\sqrt{h}R^{(2)}$ divergence from the background fields. In dimensional regularization (D.R.) the value of "$\delta(0)$" $= -4T\left\langle\partial\bar{\partial}\xi\xi\right\rangle$ is strictly zero, even for the finite part, which a priori might have had a Weyl anomaly. On the other hand,

$$\left\langle\partial\xi\bar{\partial}\xi\right\rangle = -\frac{1}{32\pi T}\sqrt{h}R \tag{5.11}$$

in D.R. [38]. It is very cumbersome to compute the divergent part of D_2 amplitudes using the dimensionally regularized Green's function, and our attempt to do so yielded divergences of the form $1/\sqrt{\epsilon}$, (where $\epsilon = d-2$) instead of the $1/\epsilon$ pole that comes from the $\langle\xi\xi\rangle$ factors of the background field contributions. Thus it does not seem that D.R. will fix the problem, although it does make one wonder whether the $\sqrt{h}R^{(2)}$ divergences of the sigma model are unambiguously defined.

Another possible source of error is that the effective action (5.6) may be modified by field redefinitions that depend on the method of regulating divergences in the string calculations. This is closely related to the ambiguity in the $\sqrt{h}R$ terms mentioned above and is just a conjecture based on the fact that ϕ couples to $\sqrt{h}R$ in S_{bg}.

One way to eliminate these ambiguities is to restrict our attention to two-point functions and shift only the gravitational background. As explained in sect. 4.2, the $\sqrt{h}R$ terms do not contribute in a two-point amplitude on S_2. By shifting only the metric and not the dilaton field, we should be able to consistently cancel only part of the divergence in D_2 amplitudes, the part due to the graviton tadpole. In the case of $A_{2T}(D_2)$, we know just what fraction of the total divergence this is: from (4.14) and (4.11),

$$A_{2T}(D_2)_{\text{grav.tadpole}} = -\frac{4}{D-2} A_{2T}(D_2)_{\text{pole}}. \tag{5.12}$$

Thus the l.h.s. of (5.7a) should be multiplied by $-4/(D-2)$. This still does not agree with the field theory equation (5.10a) where $\phi = 0$.

5.2. Vertex operators in curved space

There is one more source of log divergences in the background field amplitude that we have neglected—modifications to the vertex operators due to the background curvature. These were completely ignored in [24]. In [25] it was argued that these divergences are canceled by modifying the tachyon mass shell condition to

$$\left(-\nabla^2 + M_T^2\right) V(x) = 0, \tag{5.13}$$

which is the natural generalization of the flat space version,

$$\left(-(\partial/\partial x)^2 + M_T^2\right) e^{ip\cdot x} = 0, \tag{5.14}$$

and also is necessary for maintaining conformal invariance of the amplitude [47]. But ref. [25] neglected some logarithmically divergent contractions that cannot be absorbed into the mass shell condition. Using a very simple exercise, we can show that these divergences must in general combine with divergences from the action $S_{\rm bg}$ to give results that do not depend on the choice of coordinates for spacetime.

Consider the two-tachyon function on the sphere in a flat background, and perform the change of variables $x^\mu \to (1+Ax^2)x^\mu$ in the path integral, which is a coordinate transformation in spacetime. To lowest order in A, the change in the amplitude is

$$\delta A_{2T}(S_2) = \frac{AN}{\Omega}\int \mathcal{D}x \int \prod_{i=1}^{2} d^2z_i \colon e^{ip_i\cdot x(z_i)}\colon \exp\left(-2T\int d^2z\,\partial x^\mu \bar\partial x^\mu\right)$$

$$\times\left\{(D+2)\int d^2z\,\text{“}\delta(0)\text{”}x^2(z) + i\sum_{i=1}^{2} p_i\cdot x(z_i)x^2(z_i)\right.$$

$$\left. -4T\int d^2z\,\partial x^\mu \bar\partial x^\nu(\delta_{\mu\nu}x^2 + 2x^\mu x^\nu)\right\}, \tag{5.15}$$

where $N = 8\pi^2 T$ and $\Omega = \mathrm{Vol(SL(2,C))}$. The first term in brackets is from the Jacobian of the functional measure, the second term is from the vertex operators, and the third term is from the action. The log divergence of the latter is

$$\frac{D+2}{\pi}\ln\epsilon^2\int d^2z\,\partial x^\mu\bar\partial x^\mu \tag{5.16}$$

after contracting $\langle\delta_{\mu\nu}x^2 + 2x^\mu x^\nu\rangle$. Then inserting (5.16) back into (5.15) we find that the $\ln\epsilon^2$ contribution to δA_{2T} from the variation of the action is

$$\ln\epsilon^2\frac{AN}{\Omega}\left(\frac{D+2}{\pi}\right)\int\frac{d^2z_1d^2z_2d^2z_3}{|\Delta_{12}|^4}\left(\langle\partial x^\mu(z_3)\bar\partial x^\mu(z_3)\rangle - \frac{1}{2\pi T}\frac{|\Delta_{12}|^2}{|\Delta_{13}|^2|\Delta_{23}|^2}\right), \tag{5.17}$$

where $\Delta_{ij} = z_i - z_j$. The $\langle\partial x\bar\partial x\rangle$ term cancels with the "$\delta(0)$" term in (5.15), and the remaining integral is SL(2,C)-invariant so that (5.17) is nonvanishing.

Now δA_{2T} must be zero, since it is just the effect of a coordinate transformation, but the first and third terms in (5.15) give a net nonzero result. The second term is an integral over only two world-sheet coordinates, and ordinarily we would need at least three to saturate the factor of $1/\Omega$, so we have a puzzle. It can be resolved as follows. Integrate (5.16) by parts; then $\left\langle \Pi : e^{ip_i \cdot x(z_i)} : \int d^2 z_3 x^\mu(z_3) \partial\bar{\partial} x^\mu(z_3) \right\rangle$ no longer gives $|\Delta_{12}|^2 |\Delta_{13}|^{-2} |\Delta_{23}|^{-2}$ as in (5.17), but rather

$$\ln \frac{|\Delta_{13}|^2}{|\Delta_{23}|^2} \partial_3 \bar{\partial}_3 \ln \frac{|\Delta_{13}|^2}{|\Delta_{23}|^2} = \pi \ln \frac{|\Delta_{13}|^2}{|\Delta_{23}|^2} \left\{ \delta(z_1 - z_3) - \delta(z_2 - z_3) \right\}, \tag{5.18}$$

where $|\Delta_{ij}|^2 + \epsilon^2$ is to be understood for $|\Delta_{ij}|^2$ when $z_i \to z_j$. In this form it is easily seen that the piece in question is exactly canceled by the middle term in brackets in (5.15).

We therefore see that: (1) terms that do not look invariant under SL(2,C) can be changed to an invariant form using partial integration; and (2) it is not correct to separate the log divergences of the action from those of the vertex operators even for flat spacetime (in curved coordinates). It is clear that in a curved background, coordinate transformations will also shift part of the divergence between the two, so that only their sum has an invariant meaning. Let the metric be flat space $\delta_{\mu\nu}$ plus a perturbation,

$$\delta g_{\mu\nu} = A'(x^2 \delta_{\mu\nu} + 2x^\mu x^\nu) + A x^\mu x^\nu . \tag{5.19}$$

It is convenient to split $\delta g_{\mu\nu}$ as in (7) because the A' part corresponds to flat space in the coordinate system used previously. Since it has just been shown that such contributions cancel against each other, as they must, we might as well set $A' = 0$ immediately.* It has no effect on the curvature, given by

$$R_{\alpha\beta\mu\nu} = A(\delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu}) . \tag{5.20}$$

The tachyon vertex operator should satisfy eq. (5.13), using covariant derivatives

* Other choices of A' correspond to different coordinate systems. For example, choosing $A' = -\frac{1}{3}A$ would give Riemann normal coordinates.

constructed from the above metric. A class of solutions (with $A' = 0$) is

$$V(x) = e^{ip\cdot x}\left(1 + A\left\{\frac{i(x\cdot p)^3}{6M^2} + (D-1)\left(\frac{x^2}{4}(1-a) + \frac{(x\cdot p)^2}{4M^2}a\right)\right.\right.$$

$$\left.\left. + (D-1)(D(1-a)+a)\frac{ip\cdot x}{4M^2}\right\}\right) + O(A^2), \qquad (5.21)$$

where $p^2 = M^2 = 8\pi T$ and a is an undetermined constant. It is a nice consistency check to show that the divergent parts of amplitudes do not depend on a, which we have done. Therefore we are free to let a take some convenient value, say $a = 1$.

Using the vertex operator (5.21), the $O(A)$ part of the tachyon N-point function is*

$$\delta A_{NT}(S_2) = \frac{AN}{\Omega}\int \prod_{i=1}^{N} d^2 z_i \prod_{i<j} |\Delta_{ij}|^{p_i\cdot p_j/2\pi T}$$

$$\times\left\{2T\int d^2 z (G_0 \partial J_z \cdot \bar{\partial} J_z - |J_z\cdot \partial J_z|^2)\right.$$

$$\left. + \sum_{i=1}^{N}\left(-\frac{1}{2}G_0 J_i\cdot p_i + \frac{1}{6M^2}(J_i\cdot p_i)^3 + \frac{D-1}{4M^2}(M^2 G_0 - (p_i\cdot J_i)^2 - p_i\cdot J_i)\right)\right\}, \qquad (5.22)$$

where $G_0 = -\frac{1}{4\pi T}\ln\epsilon^2$, $G_{ij} = -\frac{1}{4\pi T}\ln(|\Delta_{ij}|^2 + \epsilon^2)$, and $J_i^\mu = \sum_j p_j^\mu G_{ij}$. The $G_0\partial J_z \cdot \bar{\partial} J_z$ term can be shown to cancel the $-\frac{1}{2}G_0 J_i \cdot p_i$ term by partial integration, as in eq. (5.18). For $N = 2$, the $|J_z\cdot\partial J_z|^2$ term can also be integrated by parts, since

* In eq. (5.22) we have suppressed momentum-nonconserving contributions that come from the integral over the zero mode of the string coordinate, due to the fact that the integrand contains factors of x and not just ∂x, apart from the usual $e^{ip_j\cdot x(z_j)}$ factor. The divergent parts of these contributions cannot mix with the momentum-conserving ones; they must cancel among themselves.

$J_z = p_1(G_{z1} - G_{z2}) \equiv p_1 \Delta G$ by momentum conservation, and then

$$-2T|J_z \cdot \partial J_z|^2 = \frac{-2TM^4}{3}\partial(\Delta G)^3 \bar{\partial}\Delta G \qquad \rightarrow -\frac{M^4}{6}(\Delta G)^3(-4T\partial\bar{\partial})\Delta G$$
$$= -\frac{M^4}{6}(\Delta G)^3(\delta(z-z_1) - \delta(z-z_2)) \rightarrow \frac{M^4}{3}(G_0 - G_{12})^3 \; .$$

Therefore this term cancels the $(J_i \cdot p_i)^3$ term in (5.22), and we are left with

$$\delta A_{2T}(S_2) = \frac{AN}{\Omega}\left(\frac{D-1}{2}\right)\int \frac{d^2z_1 d^2z_2}{|\Delta_{12}|^4}\left\{G_{12} - M^2(G_0 - G_{12})^2\right\} \; . \qquad (5.23)$$

Once again we are faced with an integral that does not have manifest SL(2,C) invariance, but it can be changed to an invariant form using the same trick as in (5.18), namely,

$$G_0 - G_{12} = \frac{1}{8\pi^2 T}\int d^2z_3 \frac{|\Delta_{12}|^2}{|\Delta_{13}|^2|\Delta_{23}|^2}. \qquad (5.24)$$

The resulting integral after SL(2,C) fixing is logarithmically divergent. With the usual cutoff, we find that

$$\delta A_{2T}(S_2) = 2(D-1)A \ln \epsilon;$$

using (5.20), A is related to the spacetime curvature by $A = R/(D(D-1))$. Canceling $\delta A_{2T}(S_2)$ against the disk contribution (5.12) gives

$$R = \frac{D}{D-2}\kappa^2 \Lambda. \qquad (5.25)$$

Eq. (5.25) differs from the field equation (5.10a) by a factor of 2. Previously we were off by factors of $3(D-2)/2$ (eq. (5.7) versus eq. (5.10)), and -6 (using eq. (5.12) but leaving out vertex operator divergences). So inclusion of the vertex operator divergences appears to be a step in the right direction; as seen from the flat space calculation, it is necessary to include them so that results are independent of the choice of spacetime coordinates.

In ref. [23] it was found that the Fischler-Susskind mechanism, which is what we have been attempting to use, gives correct results for the tadpole and vacuum energy, but the method used there was quite different from ours. It involved the cancellation of BRST anomalies on a degenerating Riemann surface rather than a straightforward evaluation of amplitudes such as we have attempted. Undoubtedly, discovering a connection between the two approaches would shed light on the difficulty found here.

Chapter 6. Infinity cancellations for SO($2^{D/2}$)

6.1. The bosonic string

The similarities between the bosonic and supersymmetric type I strings are most evident in the cancellation of tadpole divergences for amplitudes on D_2 and P_2 [1]. Because of the relative simplicity of the bosonic case, it is possible to show that the cancellation occurs for arbitrary external states. The reason that cancellations occur between these topologies is that their Green's functions are the same except for a sign,

$$G_{ij} = -\ln |z_i - z_j|^2 \, |1 - k z_i \bar{z}_j|^2 ,$$

$$k = \begin{cases} +1, & D_2 \\ -1, & P_2 \end{cases} \tag{6.1}$$

when P_2 and D_2 are both represented by the unit disk (with antipodal points identified for P_2).

Scattering amplitudes are correlation functions of vertex operators using (6.1) for $\langle x(z_i)x(z_j)\rangle$. A vertex operator has the general form

$$\begin{aligned} V_M(z) &= V'_M(z) e^{ip\cdot x} \\ V'_M(z) &= \left|(\partial x)^{n_1} (\partial^2 x)^{n_2} \cdots (\partial^N x)^{n_N}\right|^2 , \end{aligned} \tag{6.2}$$

omitting spacetime indices and the polarization tensor. For our purposes the important characteristic of a vertex operator is the number of derivatives, $2M$, which determines the (mass)^2 of the associated particle to be

$$m^2 = 2(M-1), \tag{6.3}$$

in units where $4\pi T = 1$.

In an amplitude the conformal Killing symmetry can be used to set z_1 (the position of the first vertex opertor), say, to zero, as well as the phase of z_2. Let $z_2 = a$ and $z_n \to az_n$ for $n > 2$. Performing the contractions between the exponentials in the vertex operators gives

$$A_N = N_{D_2(P_2)} \int_0^1 da a^{-3+2\Sigma_i M_i} \int_{|z_i|<1/a} \prod_{i=3}^{N} d^2 z_i {\prod_{i\leq j}}' |\Delta_{ij} D_{ij}|^{2p_i \cdot p_j}$$

$$\left\langle :e^{i\Sigma_i p_i \cdot x(z_i)}: V'_{M_1}(0) V'_{M_2}(a) V'_{M_3}(az_3) \cdots V'_{M_N}(az_N) \right\rangle_{D_2(P_2)}, \tag{6.4}$$

where $N_{D_2(P_2)} = (Z\mathrm{Vol(CKG)})_{D_2(P_2)}$,

$$\Delta_{ij} = z_i - z_j, \qquad D_{ij} = 1 - a^2 k z_i \bar{z}_j, \tag{6.5}$$

and $z_1 = 0$, $z_2 = 1$. The prime on $\prod'$ means to omit the Δ_{ij}'s when $i = j$. Because V'_M is a dimension $2M$ operator, it will go like a^{-2M} after the above change of variables. This cancels the factor of $a^{2\Sigma_i M_i}$ in (6.4) so that the integral over a is $\int da/a^3$ to leading order. These quadratic divergences are associated with the tachyon tadpole and are unphysical, as was argued in chapter 4. The massless tadpole divergences are obtained by expanding the integral to $O(a^2)$. We will show that these terms are always linear in the variable k in (6.1); therefore they have the same magnitude but opposite sign for the two topologies. To see this, note that each term in the correlation function in (6.4) consists of products of derivatives of the Green's function. After $z_i \to az_i$ we have, for example,

$$\partial_i G_{ij} = -\frac{1}{a}\left(\frac{1}{\Delta_{ij}} - \frac{ka^2 \bar{z}_j}{D_{ij}}\right),$$

$$\partial_i^2 G_{ij} = -\frac{1}{a^2}\left(\frac{1}{\Delta_{ij}^2} - \frac{k^2 a^4 \bar{z}_j^2}{D_{ij}^2}\right),$$

$$\partial_i \bar{\partial}_j G_{ij} = -\frac{1}{a^2}\left(\frac{ka^2}{D_{ij}} - \frac{k^2 a^4 z_i \bar{z}_j^2}{D_{ij}^2}\right),$$

and in general,

$$\partial^n \bar{\partial}^m G_{ij} = \frac{1}{a^{n+m}} \sum_{l=0} (ka^2)^l f_l(z_i, z_j).$$

From this it follows that the integrand in (6.4) is of the form

$$\prod{}' |\cdots|^{2p_i \cdot p_j} \langle \cdots \rangle = \frac{1}{a^{2M}} \sum_{l=0} (ka^2)^l g_l(z_i). \tag{6.6}$$

The functions $g_l(z_i)$ must be the same for D_2 and P_2 since the value of k is all that distinguishes between them. The $l = 1$ term in the sum gives the logarithmic divergence and is linear in k. Therefore if the normalization factor $N_{D_2(P_2)}$ is such that $A_N(D_2)+A_N(P_2) = 0$ for one amplitude, this log divergence will vanish in the sum for an arbitrary amplitude. As we have already mentioned, the dilaton tadpole vanishes to this order when the gauge group is SO(8192) [4]. (The gauge group dependence is through a trace for each boundary of a product of group generators, one for each open-string external state [48]. For amplitudes with no open-string external states and gauge group SO(N), this is a factor of $\mathrm{Tr}(1) = N$ for each boundary. Thus Z_{D_2} [and N_{D_2}] are multiplied by N.)

This completes the proof of tadpole divergence cancellations except for a few technicalities. One is a problem with fixing the SO(3) conformal Killing symmetry for P_2 (*i.e.*, setting $z_1 = 0$). The SO(3) transformations are [19]

$$z^\Omega = \frac{Az + B}{-\bar{B}z + \bar{A}}, \qquad |A|^2 + |B|^2 = 1. \tag{6.7}$$

Although the integrands of P_2 amplitudes transform as a density under (6.7) (such that the change in the measure $\prod d^2 z_i$ is compensated), the integration region $|z^\Omega| \leq 1$ is not preserved. This means the integral over the group volume will not factor out when the integration over z_1 is traded for $\int d\Omega$. This can be solved by using the equivalence of the P_2 represented by $|z| > 1$ with the original P_2, $|z| < 1$. Replacing

$$\int_{|z|<1} d^2 z \longrightarrow \frac{1}{2} \int_{\mathbb{R}^2} d^2 z \tag{6.8}$$

just means to integrate over P_2 twice and divide by two. The region $\mathbb{R}^2$ is invariant under (6.7). After setting $z_1 = 0$, the remaining integrals are invariant under $z_i \to -k/\bar{z}_i$, so the operation (6.8) can be undone.

A more serious problem is that in expanding the integrand to $O(a^2)$, we have ignored the a-dependence coming from the boundary of the integration regions $|z_i| < 1/a$. Formally they should give $-\frac{1}{2}$ of the nonboundary terms, since if we leave the integration regions in the form $\frac{1}{2}\int_{I\!R^2} d^2z$, then $|z_i| < \infty$ in (6.4). But a direct evaluation of the boundary contributions for various amplitudes does not give this result, and moreover the boundary contributions do not have opposite signs for P_2 and D_2.* Physically there is some justification for leaving them out: they contribute only as z_i approaches the boundary, whereas the tadpole divergences should occur when all the vertex operators are close together. The boundary contributions should be external line divergences (ELD's), as in fig. 4.2(b). The fact that they do not cancel between P_2 and D_2 is mysterious, since the dilaton and graviton mass shifts should be zero whenever the dilaton tadpole and vacuum energy cancel, due to the form of the effective Lagrangian $\Lambda\sqrt{g}e^{\phi}$. There are other ELD's that can be calculated by rescaling $z_i \to az_i$ for all but one of the z_i. It should be that inclusion of these gives a net result of zero for P_2 plus D_2 when $N = 8192$.

6.2. Decoupling of spurious states of the superstring

The transition from bosonic string to superstring is most naturally accomplished in the superspace formalism. The gauge-fixed action is

$$\begin{aligned} S &= \frac{1}{2\pi}\int d^2z d^2\theta DX^\mu \bar{D}X^\mu \\ &= \frac{1}{2\pi}\int d^2z \left(\partial x^\mu \bar{\partial} x^\mu - \bar{\psi}^\mu \partial \bar{\psi}^\mu - \psi^\mu \bar{\partial}\psi^\mu + F^\mu F^\mu\right), \end{aligned} \tag{6.9}$$

where X is the superfield,

$$X^\mu = x^\mu + \theta\psi^\mu + \bar{\theta}\bar{\psi}^\mu + \theta\bar{\theta}F^\mu,$$

written in terms of the string coordinate x^μ, its superpartner, a two-component Majo-

* The ambiguity is due to manipulating divergent integrals before introducing a cutoff.

rana spinor $(\psi, \bar{\psi})$, and an auxiliary field F^μ, which allows the supersymmetry algebra to close without using the equation of motion, $\bar{D}DX = 0$. θ is a complex Grassmann coordinate for the supermanifold, and the supercovariant derivatives are

$$D = \partial_\theta + \theta\partial, \qquad \bar{D} = \partial_{\bar{\theta}} + \bar{\theta}\bar{\partial}, \tag{6.10}$$

where $\partial_\theta = \partial/\partial\theta$. D and $\bar{D}$ anticommute with the supersymmetry generators

$$Q = \partial_\theta - \theta\partial, \qquad \bar{Q} = \partial_{\bar{\theta}} - \bar{\theta}\bar{\partial}. \tag{6.11}$$

Since F^μ has the trivial equation of motion $F^\mu = 0$, we eliminate it from X^μ. More will be said about this later.

Most of the formalism for scattering amplitudes in the bosonic string goes over for the superstring by making the replacements

$$x^\mu \to X^\mu, \quad \partial \to D, \quad \bar{\partial} \to \bar{D}, \quad d^2z \to d^2z d^2\theta$$

in the vertex operators for bosons in the NS-NS sector. The superstring also has vertex operators for fermions and R-R sector bosons that have no analog in the bosonic string, but these will not be considered here. Correlations of vertex operators on S_2 are constructed from the superfield Green's function,

$$G_{12} = G(z_1, \theta_1; z_2, \theta_2) = -\ln\left|\Delta_{12} - \theta_1\theta_2\right|^2. \tag{6.12}$$

We will also need the Green's function for the disk,

$$G_{12} = -\ln\left|\Delta_{12} - \theta_1\theta_2\right|^2 \left|1 - z_1\bar{z}_2 + i\theta_1\bar{\theta}_2\right|^2. \tag{6.13}$$

One way to obtain this is to add to (6.12) the contribution from an "image charge"

[26] at

$$(\tilde{z},\, \tilde{\theta}) = \left(\frac{1}{\bar{z}},\, i\frac{\bar{\theta}}{\bar{z}}\right) \tag{6.14}$$

on the super S_2 manifold. The Grassmann part of (6.14) follows from the requirement that under a supersymmetry transformation

$$\delta z = \theta\epsilon, \qquad \delta\theta = \epsilon, \tag{6.15}$$

$\delta\tilde{z}$, $\delta\tilde{\theta}$ should also be a SUSY transformation. A more concrete way to get the D_2 Green's function is to expand $\langle X_1 X_2\rangle$ in component fields,

$$\begin{aligned}\langle X_1 X_2\rangle =& \langle x_1 x_2\rangle - \theta_1\theta_2 \langle \psi_1\psi_2\rangle - \bar{\theta}_1\bar{\theta}_2 \langle \bar{\psi}_1\bar{\psi}_2\rangle \\ &- \theta_1\bar{\theta}_2 \langle \psi_1\bar{\psi}_2\rangle - \bar{\theta}_1\theta_2 \langle \bar{\psi}_1\psi_2\rangle .\end{aligned} \tag{6.16}$$

The top line is identical to the Green's function for S_2. The second line vanishes on S_2 because ψ and $\bar{\psi}$ are not coupled in the action, but on D_2 they are coupled through boundary conditions: in deriving the equations of motion for ψ, a surface term appears,

$$\oint d\bar{z}\bar{\psi}\delta\bar{\psi} - \oint dz\psi\delta\psi,$$

which is eliminated by the condition

$$\bar{\psi} = \pm iz\psi \quad \text{at} \quad |z| = 1. \tag{6.17}$$

The choice of sign corresponds to the two spin structures for fermions on D_2 [49], but on D_2 they give identical results for bosonic scattering amplitudes [50, 26], so we can choose the upper sign. Since $\langle\psi_1\psi_2\rangle = -1/(z_1 - z_2)$, (6.17) implies

$$\langle \psi_1\bar{\psi}_2\rangle = \frac{iz_2}{z_1 - z_2} \tag{6.18a}$$

$$= \frac{i}{1 - z_1\bar{z}_2}, \tag{6.18b}$$

for $|z_2| = 1$. Because of the equations of motion $\bar{\partial}\psi = \partial\bar{\psi} = 0$, one expects $\langle\psi_1\bar{\psi}_2\rangle$ to be a function of z_1 and $\bar{z}_2$, so (6.18b) should be the correct form for $|z_2| < 1$. It agrees with the expression for G_{12} on D_2 in (6.13), using (6.16).

Tree-level amplitudes for massless states are correlations of vertex operators like

$$\int d^2z d^2\theta \bar{D}X^\mu DX^\nu \epsilon_{\mu\nu} e^{ip\cdot X} \tag{6.19}$$

for gravitons (on S_2 or D_2) or

$$\oint dz d\theta DX^\mu \zeta_\mu e^{ip\cdot X} \tag{6.20}$$

for vector bosons (on D_2). They have an enlarged conformal Killing symmetry relative to bosonic string amplitudes. For S_2 it is the graded extension of SL(2,C), whose generators are

$$\begin{aligned} L_1 &= -\partial & G_{\frac{1}{2}} &= \partial_\theta - \theta\partial \\ L_0 &= -(z\partial + \tfrac{1}{2}\theta\partial_\theta) & G_{-\frac{1}{2}} &= z(\partial_\theta - \theta\partial) \\ L_{-1} &= -(z^2\partial + z\theta\partial_\theta) \end{aligned} \tag{6.21}$$

plus the complex conjugate generators. By exponentiating them, one obtains the super–SL(2,C) transformations

$$z^\Omega = \frac{az+b}{cz+d} + \theta\frac{\eta z + \epsilon}{(cz+d)^2}, \qquad \theta^\Omega = \frac{\theta + \eta z + \epsilon}{cz+d}; \tag{6.22a}$$

$$ad - bc = 1 + \eta\epsilon, \tag{6.22b}$$

where η and ϵ are complex Grassmann parameters.* Similarly, amplitudes on D_2 are invariant under the super–SL(2,R) group obtained from (6.22) by making the restrictions

$$d = \bar{a}, \qquad c = \bar{b}, \qquad \eta = i\epsilon. \tag{6.23}$$

Because of these enlarged symmetries, there is the freedom to fix not only several of the z_i, but also some of the θ_i in the world-sheet integral over vertex operator

* Similar expressions have been written by other authors [26, 51] who rescaled $a, b, c, d \to (1 + \frac{1}{2}\eta\epsilon)(a, b, c, d)$ to get the simpler constraint $ad - bc = 1$. But the product of two such transformations will not be another such transformation with $ad - bc = 1$, so (6.22) is the more correct form.

locations. For example, one can fix $\theta_1 = \theta_2 = 0$ and z_1, z_2, z_3 to arbitrary values $\hat{z}_1$, $\hat{z}_2$, $\hat{z}_3$ on the boundary of D_2 by inserting

$$1 = \int d\Omega \Delta_{FP} \delta(z_1^\Omega - \hat{z}_1) \ldots \delta(z_3^\Omega - \hat{z}_3) \delta(\theta_1^\Omega) \delta(\theta_2^\Omega) \tag{6.24}$$

in the amplitude. The invariant volume element can be shown to be

$$d\Omega = d^2 a d^2 b d^2 \eta \delta \left(|a|^2 - |b|^2 - 1 - i\eta\bar{\eta} \right) , \tag{6.25}$$

and the Faddeev-Popov determinant is a superdeterminant [52],

$$\begin{aligned} \Delta_{FP} &= \text{sdet} \begin{vmatrix} A & * \\ 0 & B \end{vmatrix} \\ &= \left| \det A / \det B \right| , \end{aligned} \tag{6.26}$$

where A is the ordinary SL(2,R) Jacobian,

$$A = \begin{pmatrix} 1 & \hat{z}_1 & \hat{z}_1^2 \\ 1 & \hat{z}_2 & \hat{z}_2^2 \\ 1 & \hat{z}_3 & \hat{z}_3^2 \end{pmatrix} = \prod_{i<j} |\hat{z}_i - \hat{z}_j|^2 \tag{6.27}$$

and B is the Grassmann part,

$$B = \frac{\partial(\theta_1^\Omega, \, \theta_2^\Omega)}{\partial(\eta, \, \bar{\eta})} = \begin{pmatrix} z_1 & 1 \\ z_2 & 1 \end{pmatrix} . \tag{6.28}$$

For super–SL(2,C), Δ_{FP} is just $|\text{eq. (6.26)}|^2$.

It was observed by Itoyama and Moxhay [26] that equivalent results are obtained for S_2 if instead of fixing θ_1 and θ_2, all the Grassmann integrals are performed, and the regular SL(2,C) is subsequently fixed. Similarly on D_2 with only open-string external states, one can either fix θ_1 and θ_2 or integrate over them with the same result. We show this for the three-vector coupling,

$$A_{3V} \sim \oint \prod_{i=1}^{3} dz_i d\theta_i \left(\frac{\theta_1 - \theta_2}{\Delta_{12}} - \frac{\theta_1 - \theta_3}{\Delta_{13}}\right)\left(\frac{1}{\Delta_{32}} - \frac{\theta_2\theta_3}{\Delta_{32}^2}\right). \tag{6.29}$$

Setting $\theta_1 = \theta_2 = 0$ and fixing the z_i gives

$$A_{3V} \sim \int d\theta_3\theta_3 \frac{1}{\Delta_{13}\Delta_{32}}, \tag{6.30}$$

which is just the reciprocal of Δ_{FP}, (6.26), so that A_{3V} is independent of the z_i, as is necessary. Alternatively, integrating over all the θ's and fixing the z_i gives

$$A_{3V} \sim \frac{1}{\Delta_{12}\Delta_{23}\Delta_{31}}, \tag{6.31}$$

which is the inverse of the regular SL(2,R) Jacobian, (6.27).

Ref. [26] noted that for closed-string amplitudes on D_2 (and P_2), however, the two methods do not give equivalent answers: if the θ's are not fixed, the amplitude is quadratically divergent. An example is the graviton two-point function,

$$A_{2G} = \mathrm{tr}\epsilon_1\epsilon_2 \int d^2z_1 d^2z_2 d^2\theta_1 d^2\theta_2 \left(\frac{1}{|z_1 - z_2 - \theta_1\theta_2|^2} - \frac{1}{|1 - z_1\bar{z}_2 + i\theta_1\bar{\theta}_2|^2}\right). \tag{6.32}$$

Using super–SL(2,R) to fix $\theta_1 = \bar{\theta}_1 = 0$ makes the integral independent of θ_2, hence $A_{2G} = 0$. Alternatively, integrating over θ_1 and θ_2 gives

$$\int\limits_{|z_i|<1} d^2z_1 d^2z_2 \left(\frac{1}{|z_1 - z_2|^4} + \frac{1}{|1 - z_1\bar{z}_2|^4}\right), \tag{6.34}$$

which is quadratically divergent even after SL(2,R) has been used to set $z_1 = 0$.

Formally it should not matter which method is used, inserting (6.24) or the ordinary SL(2,R)-fixing factor, since both are equal to unity, so the inequivalence is at first surprising. Mathematically the problem is that integrals on superspace are in general not invariant under a change of variables when the manifold has a boundary. A simple illustration is

$$\int dx d\theta \theta f(x) = \int dx f(x). \tag{6.35}$$

Let $x \longrightarrow x + \alpha\theta$, $\theta \longrightarrow \theta + \beta x$, (where α and β are Grassmann numbers), which has the super-Jacobian [52]

$$\mathrm{sdet} \begin{vmatrix} \frac{\partial x'}{\partial x} & x' \overleftarrow{\frac{\partial}{\partial \theta}} \\ \frac{\partial \theta'}{\partial x} & \frac{\partial \theta'}{\partial \theta} \end{vmatrix} = \mathrm{sdet} \begin{vmatrix} 1 & +\alpha \\ \beta & 1 \end{vmatrix} = 1 - \alpha\beta. \tag{6.36}$$

Thus (6.35) becomes

$$\begin{aligned} (1-\alpha\beta) \int & dx d\theta (\theta + \beta x)(f(x) + \alpha\theta x f'(x)) \\ &= (1-\alpha\beta) \int dx (f + \beta\alpha x f'). \end{aligned} \tag{6.37}$$

Integrating the xf' term by parts gives

$$(1-\alpha\beta)(1+\alpha\beta) \int dx f(x) = \int dx f(x), \tag{6.38}$$

provided there are no surface terms, but if the integration region is bounded, they will in general be present. Applying similar reasoning to the two-graviton function, we have found that a surface term arises in the change of variables needed to fix $\theta_1 = 0$; it is

$$\frac{1}{2} \oint \frac{dz_1}{z_1} \int_{|z_2|<1} \left(\frac{1}{|z_1 - z_2|^4} + \frac{1}{|1 - z_1 \bar{z}_2|^4} \right). \tag{6.39}$$

Although this does not look the same as (6.34), using the change of variables $z \longrightarrow 1/\bar{z}$, the z_2 integral can be written as

$$\int_{|z_2|<1} \left(\frac{1}{|z_1 - z_2|^4} + \frac{1}{|1 - z_1 \bar{z}_2|^4} \right) = \int_{|z_2|<\infty} \frac{1}{|z_1 - z_2|^4}, \tag{6.40}$$

which is independent of z_1. Since

$$\frac{1}{2} \oint \frac{dz_1}{z_1} = \int_{|z_1|<1} d^2 z_1,$$

the surface term indeed accounts for the entire difference between the two forms of the amplitude. Nevertheless it is clear that the former answer (zero) is better than the divergent one, and since fixing θ_1 works similarly for any amplitude on D_2, we should accept this as the correct way to proceed.

Still one would like to have a deeper understanding of this phenomenon and why the discrepancy occurs only for closed-string amplitudes on D_2 (or P_2). Experience with the bosonic string shows that nonlogarithmic divergences are due to tachyon tadpoles. The argument based on eq. (4.1) in fact shows that a quadratic divergence is due to a tachyon of $(\text{mass})^2 = -2$ (where $4\pi T = 1$). This is the closed bosonic string tachyon, which has twice the $(\text{mass})^2$ of the closed-string tachyon present in the superstring before making the GSO projection.

The key to understanding these observations is to realize that the two ways of evaluating amplitudes (fixing θ's or integrating over them) correspond precisely to the two "pictures," or formalisms, called F_1 and F_2, which have long been known to be equivalent at tree level [53]. Integrating over θ's gives vertex operators in the F_1 picture; for example, the vector boson vertex operator is

$$V_1 = \zeta^\mu (\partial x^\mu + ik \cdot \psi \psi^\mu) e^{ik \cdot x} \tag{6.41}$$

in component fields. Setting $\theta = 0$ and including a bosonized superconformal ghost field* $e^{-\phi(z)}$ [29] yields an F_2 vertex operator,

$$V_2 = e^{-\phi(z)} \zeta^\mu \psi^\mu e^{ik\cdot x}. \tag{6.42}$$

In the F_1 picture an amplitude is made entirely of F_1 vertex operators, whereas in the F_2 picture two of them must be F_2 and the rest F_1:

$$\langle V_2 V_2 V_1 \cdots V_1 \rangle .$$

This is the same as fixing the odd part of super-SL(2,R), since two of the θ's are set to zero, and the ghost fields provide a factor of

$$\left\langle e^{-\phi(z_1)} e^{-\phi(z_2)} \right\rangle = \frac{1}{z_1 - z_2}, \tag{6.43}$$

needed to reproduce the odd part of the FP superdeterminant, $1/\det B$ in (6.26, 28).

Why does the equivalence of the F_1 and F_2 pictures break down for closed-string amplitudes on D_2? This has to do with the tachyon tadpole. In the F_1 picture there are many "spurious states" that do not occur in the F_2 picture. A spurious state is a mass eigenstate of the string that is orthogonal to all the physical states.† One way to demonstrate these spurious states is through the operator product expansion (OPE) of two vertex operators. Green and Seiberg [27] showed that the OPE of two V_1's has the form

$$\begin{aligned} V_1(z;k)V_1(z';k') \sim & \frac{1-4k\cdot k'}{(z-z')^{2-4k\cdot k'}} e^{i(k+k')\cdot x(z')} \\ & + \frac{1}{(z-z')^{1-4k\cdot k'}} V_1(z';k+k') + \cdots \end{aligned} \tag{6.44}$$

The first term is the vertex operator for a bosonic string tachyon; the second term is another vector boson vertex, and there are also vertex operators for massive states.

* The superconformal ghosts are Faddeev-Popov ghosts associated with fixing the local supersymmetry of the original world-sheet action to get the gauge–fixed form (6.9).

† Physical states are those that vanish under the action of the positive-frequency modes of the stress-energy tensor, T_{ab}. $T_{ab} = 0$ is the equation of motion for the world-sheet metric from the action (1.1).

In contrast the OPE of V_1 with V_2 goes like

$$V_1(z;k)V_2(z';k') \sim \frac{1}{(z-z')^{1-4k\cdot k'}} V_2(z';k+k') + \cdots, \tag{6.45}$$

with no tachyons appearing. Physically the OPE tells what kinds of states can be produced by the two V's, and (6.44) implies that two vector bosons in the F_1 picture can produce a tachyon. The reason this tachyon is spurious is that its coefficient is a total derivative,

$$\frac{1-4k\cdot k'}{(z-z')^{2-4k\cdot k'}} = -\partial(z-z')^{-1+4k\cdot k'}, \tag{6.46}$$

which will vanish in the world-sheet integral under suitable conditions: the spurious state decouples from the S-matrix. The "suitable condition" is that one can define the amplitude using values of $k\cdot k'$ for which possible surface terms vanish. Ref. [27] shows this for D_2 represented by the upper half-plane, so that $z=\tau$ on the boundary, by cutting off the integral near $\tau=\tau'$:

$$\begin{aligned} \int^{\tau'-\epsilon} d\tau\,\partial(\tau-\tau')^{-1+4k\cdot k'} &\sim \epsilon^{-1+4k\cdot k'} \\ &= \epsilon^{-1+2(k+k')^2}. \end{aligned} \tag{6.47}$$

For $k\cdot k' > \frac{1}{4}$, the limit $\epsilon \to 0$ exists and is zero. To get sensible results, we should take this to be the value of the surface term for $k\cdot k' > \frac{1}{4}$ also, by analytic continuation. If the spurious tachyon created disappears into the vacuum, however, it has exactly zero momentum, and there is no freedom to remove divergent surface terms like (6.47) by varying the momenta of the external particles. Since there is no tachyon tadpole amplitude on S_2, this will never occur there. Similarly, there is no open-string tachyon tadpole on D_2, and since it is the open-string tachyon that is the spurious state in the OPE of two open string vertex operators, the problem will not affect amplitudes for open-string states on D_2. But there is a closed-string tachyon tadpole on D_2, so the spurious state can fail to decouple for amplitudes of closed-string states. This is the origin of quadratic divergences in F_1 picture amplitudes on D_2 and also P_2.

The above remarks can be demonstrated by taking successive operator products of vertex operators in the two pictures. The OPE of N V_1 operators will contain the tachyon vertex

$$\left[(z_1-z_2)^{1-4k_1\cdot k_2}(z_1-z_3)^{1-4k_1\cdot k_3}\cdots(z_1-z_{N-1})^{1-4k_1\cdot k_{N-1}}(z_1-z_N)^{2-4k_1\cdot k_N}\right]^{-1}$$

$$\exp\left(i\sum k_i\cdot x(z_i)\right), \tag{6.48}$$

obtained by keeping the vector part of the first $N-2$ OPE's and the tachyon part of the last OPE. A graviton vertex operator is the product of two vectors, $V_1\bar{V}_1$, so (6.48) can be converted to the OPE for N gravitons by squaring the first factor. Using SL(2,R) to fix $z_1=0$ and letting $z_2=a$, $z_3\rightarrow az_3$, etc., as in sect. 6.1, the z_2 integral becomes

$$\int\frac{da}{a^3}a^{8k_1\cdot\Sigma_{i=2}^N k_i}=\int\frac{da}{a^3}, \tag{6.49}$$

using momentum conservation and the mass shell condition for gravitons. Thus the quadratic divergence is inevitable in this case. For S_2, not only z_1 but z_2 and z_3 can be fixed. Letting $z_4=a$ and $z_i\rightarrow az_i$ for $i>4$ gives

$$\int\frac{da}{a^3}a^{f(k_i)}$$

instead of (6.49), where $f(k)$ is a nontrivial function of the momenta that can make the integral converge: the SL(2,C) symmetry effectively forbids a tachyon tadpole.

In contrast the OPE of $N-2$ V_1 operators and one V_2 gives only another V_2 to leading order, since $V_1V_2\sim V_2$ in eq. (6.45). No spurious states like the tachyon appear in the F_2 picture.

At the beginning of this section the auxiliary field F^μ was eliminated because of its trivial equation of motion $F^\mu=0$ and hence its trivial Green's function,

$$\langle F^\mu(z)F^\nu(z')\rangle=2\pi\delta^{\mu\nu}\delta^{(2)}(z-z'). \tag{6.50}$$

Usually the F^μ field would not be expected to have any role in scattering amplitudes. For simplicity consider an unphysical amplitude, that for N tachyons on S_2. Inclusion of F in the superfield would give the delta function contributions in

$$A_{NT}(S_2) \sim \int \prod d^2 z_i d^2 \theta_i \prod_{i<j} |z_i - z_j - \theta_i \theta_j|^{2k_i \cdot k_j} \left(1 - \theta_i \bar{\theta}_i \theta_j \bar{\theta}_j \sum_{i<j} \pi k_i \cdot k_j \delta(z_i - z_j)\right). \tag{6.51}$$

Since the rest of the integral has only factors like $|z_i - z_j|^{2k_i \cdot k_j - n}$, the delta function contributions can only be zero or divergent. The amplitude should be defined using values of $k_i \cdot k_j$ where they give zero and analytically continued. However, [27] showed that in some cases these delta functions could be used to cancel divergent surface terms like (6.47) if one preferred to evaluate the amplitude for arbitrary values of the momenta without analytic continuation. If this was also true for the case we have been considering, it should eliminate the divergences of the F_1 amplitudes on D_2. Apparently it is not true. Reevaluating the graviton two-point function (6.32) including the auxiliary field gives

$$\int \prod d^2 z_i d^2 \theta_i \left(\left| \frac{1}{z_1 - z_2 - \theta_1 \theta_2} - 2\pi \bar{\theta}_1 \bar{\theta}_2 \delta(z_1 - z_2) \right|^2 - \left| \frac{i}{1 - z_1 \bar{z}_2 + i\theta_1 \bar{\theta}_2} - 2\pi \bar{\theta}_1 \theta_2 \delta(z_1 - z_2) \right|^2 \right). \tag{6.52}$$

The cross terms vanish identically; only the terms with $|\delta(z_1 - z_2)|^2$ survive in addition to the original part. Although they are divergent, they do not have the right form or sign to cancel the quadratic divergence. The only correct way to calculate amplitudes like $A_{NG}(D_2)$ seems to be in the F_2 picture.

So far we have discussed the amplitudes with all open- or closed-string external states. In mixed amplitudes having at least one open-string external state, there will never be any problem with tachyon tadpoles, since the states on the boundary can cause any amount of momentum to flow through the neck of a disk degenerating into D_2 plus S_2, as in fig. 4.1(a). In other words, the string propagator represented by the neck (and eq. (4.1)) can always be made finite by an appropriate choice of the external momenta.

6.3. Tadpole cancellations for the superstring

To demonstrate the cancellation of divergences between D_2 and P_2 for superstring amplitudes, we must first construct the Green's function for super–P_2. One way is to find a condition between $\psi(z)$ and ψ at the antipodal point, $\psi(-1/\bar{z})$, and proceed analogously to the D_2 case. Such a condition exists because vector fields on P_2, represented as the sphere with antipodal points identified, must satisfy [19]

$$\frac{1}{z^2} v^z(z) = v^{\bar{z}}(-1/\bar{z}),$$

and vectors can be made from bilinears of fermions. It turns out to be easier to use the method of images [26]. Given that $\tilde{z} = -1/\bar{z}$ is the even part of the image point position, the Grassmann part can be determined to be

$$\tilde{\theta} = \mp \frac{\bar{\theta}}{\bar{z}} \tag{6.53a}$$

by requiring $\delta\tilde{z}$, $\delta\tilde{\theta}$ to be a supersymmetry transformation when δz, $\delta\theta$ is. The sign choice is a convention; we take the upper sign. Since the image of super–P_2 should be an equivalent copy of super–P_2, it is reasonable to require that $(\tilde{\tilde{z}},\ \tilde{\tilde{\theta}}) = (z,\ \theta)$. This implies

$$\tilde{\tilde{\theta}} = \frac{\theta}{z}. \tag{6.53b}$$

The fact (6.53a) and (6.53b) are inconsistent (using complex conjugation) is a manifestation of the nonexistence of Majorana spinors on Euclidean P_2 [26]. Ignoring this inconsistency, (6.53) can still be used to determine a Green's function, *similar* enough to that for D_2 that both can be written in one formula,

$$G_{12} = -\ln|z_1 - z_2 - \theta_1\theta_2|^2 - \ln(1 + \zeta^2 z_1\bar{z}_2 + \zeta\theta_1\bar{\theta}_2)(1 + \zeta^2\bar{z}_1 z_2 - \zeta\bar{\theta}_1\theta_2),$$

$$\zeta = \begin{cases} i, & D_2 \\ 1, & P_2 \end{cases}. \tag{6.54}$$

For brevity we will write the argument of the second logarithm in (6.54) as

$$\left|1 + \zeta^2 z_1\bar{z}_2 + \zeta\theta_1\bar{\theta}_2\right|^2,$$

even though it is a holomorphic square only for D_2.

Amplitudes on P_2 will be invariant under a super conformal Killing group (super–SO(3)) only if a subgroup of super–SL(2,C) can be found under which the above quantity transforms as a density. We find (as did [26]) that the only such subgroup is eq. (6.22) with the restrictions

$$\bar{c} = -b, \quad \bar{d} = a, \quad (\text{and } c = -\bar{b}, \quad d = \bar{a}); \tag{6.55a}$$

$$\epsilon = -\bar{\eta}, \quad \bar{\epsilon} = +\eta. \tag{6.55b}$$

The second line is, like (6.53), inconsistent if ϵ, $\bar{\epsilon}$ and η, $\bar{\eta}$ are complex conjugates: the super–CKG does not exist for Euclidean P_2. In Minkowski space, however, continuing via $\tau \rightarrow i\tau$,

$$\begin{aligned} z = \tau + i\sigma &\longrightarrow i(\tau + \sigma) = iz_E, \\ \bar{z} = \tau - i\sigma &\longrightarrow i(\tau - \sigma) = i\bar{z}_E, \end{aligned} \tag{6.56}$$

and z_E, $\bar{z}_E$ are independent real variables. (Of course, θ, $\bar{\theta}$ can also be continued to θ_E, $\bar{\theta}_E$.) Instead of SL(2,C), the CKG for Lorentzian S_2 is SL(2,R)×SL(2,R), whose parameters a, $\bar{a}$, ϵ, $\bar{\epsilon}$, etc., are likewise independent real variables. Then the restriction (6.55) and the image point formula (6.53) make sense. In [26] this procedure was used to continue P_2 amplitudes to Minkowski space, fix the super–CKG symmetry, and then continue back to Euclidean signature.

The above procedure, as carried out in [26], has some disturbing features. Under $z \longrightarrow iz_E$, $\bar{z} \longrightarrow i\bar{z}_E$, the image point equations become

$$\tilde{z} = -\frac{1}{\bar{z}} \longrightarrow \tilde{z}_E = +\frac{1}{\bar{z}_E}, \tag{6.57a}$$

$$\tilde{\theta} = -\frac{\bar{\theta}}{\bar{z}} \longrightarrow \tilde{\theta}_E = i\frac{\bar{\theta}_E}{\bar{z}_E}. \tag{6.57b}$$

But if (6.57) is used, the Green's function looks identical to that for super–D_2, (6.13), as does the super–CKG. Surely it would be too trivial a cancellation if the amplitude on Lorentzian P_2 looked identical to Euclidean D_2. Apparently because of this, [26] does not use (6.57), but rather the Euclidean conditions, with $z \longrightarrow z_E$, and $\bar{z} \longrightarrow \bar{z}_E$. Then the image charge contribution to the Green's function is

$$-\ln|1 + z_1\bar{z}_2 + \theta_1\bar{\theta}_2|^2 \tag{6.58}$$

in Minkowski space. But in continuing back to Euclidean space, [26] once again ignores the factors of i in $z_E \longrightarrow -iz$, $\bar{z}_E \longrightarrow -i\bar{z}$, giving (6.58) as the Euclidean version as well.

Thus one wonders whether some serious difficulty may have been hidden by the above manipulations, especially in light of the argument given in chapter 1 that an amplitude with an odd number of external bosons should be more fundamentally affected by the lack of Majorana spinors on Euclidean P_2. But in fact the difficulty discussed in the previous paragraph can be overcome trivially by realizing that the continuation to Minkowski space can also be accomplished via $\sigma \longrightarrow -i\sigma$. Then $z \longrightarrow z_E$, $\bar{z} \longrightarrow \bar{z}_E$, with no troublesome factors of i. Alternatively one could start with the F_2 picture and avoid the whole issue of super–CKG fixing entirely, although it is more satisfying to see that either way can give the same answer. We have explicitly computed dilaton tadpoles, and logarithmic divergences of A_{3G} and A_{4G} on $P_2 + D_2$ to leading order in $1/T$ in this way. Using the results of [26] for the overall normalization (which involves various functional determinants), we verified their claim that the above quantities vanish for the gauge group SO(32). We also find that the finite part of A_{3G} vanishes at order p^2/T. This means the $\sqrt{g}R$ term in the tree-level effective Lagrangian will receive no finite renormalizations from D_2 plus P_2.

Rather than show particular amplitudes, it is more interesting to exhibit the finiteness of a class of amplitudes. The argument of sect. 6.1 for bosonic strings can easily be adapted to the superstring, giving a simpler proof of finiteness than in [26], and one that more readily generalizes to other external states. The N-graviton function will be considered for simplicity, but is is obvious how to make the argument for other NS-NS states. After fixing $z_1 = \theta_1 = 0$, let $z_2 = a$ and $z_n \to az_n$ for $n > 2$ as before, but now also rescale $\theta_i \to \sqrt{a}\theta_i$. (Note that $d\theta_i \to d\theta_i/\sqrt{a}$.) Then

$$\begin{aligned} z_i - z_j - \theta_i\theta_j &\longrightarrow a(z_i - z_j - \theta_i\theta_j) \equiv az_{ij}, \\ 1 + \zeta^2 z_i\bar{z}_j + \zeta\theta_i\bar{\theta}_j &\longrightarrow 1 + a^2\zeta^2 z_i\bar{z}_j + a\zeta\theta_i\bar{\theta}_j \equiv d_{ij}, \end{aligned} \tag{6.59}$$

and the amplitude on D_2 or P_2 becomes, similarly to (6.4),

$$A_{NG} = N_{D_2(P_2)} \int_0^1 daa^{-2+N} \int d^2\theta_2 \int_{|z_i|<1/a} \prod_{i=3}^{N} d^2z_i d^2\theta_i \prod_{i<j} |z_{ij}d_{ij}|^{2p_i\cdot p_j}$$

$$\left\langle :e^{i\sum p_i\cdot X(z_i,\theta_i)}: V'(0,0)V'(a,\sqrt{a}\theta_1)\cdots V'(az_N,\sqrt{a}\theta_N)\right\rangle_{D_2(P_2)}, \tag{6.60}$$

where

$$V' = \bar{D}X^\mu DX^\nu \epsilon_{\mu\nu}. \tag{6.61}$$

The crucial property of $\langle\cdots\rangle$ in (6.60) is that it consists of products of supercovariant derivatives acting on Green's functions. A few examples are

$$D_1G_{12} = -\frac{1}{\sqrt{a}}\left(\frac{\theta_1 - \theta_2}{\Delta_{12}} + \zeta a\frac{\bar{\theta}_2 + \zeta a\theta_1\bar{z}_2}{1 + \zeta^2 a^2 z_1\bar{z}_2}\right),$$

$$D_1^2 G_{12} = -\frac{1}{a}\left(\frac{1}{z_{12}} + \frac{\zeta^2 a^2 \bar{z}_2}{d_{12}}\right),$$

$$D_1 \bar{D}_2 G_{12} = \frac{1}{a}\left(\frac{\zeta a}{d_{12}}\right).$$

In general,

$$D^n \bar{D}^m G_{12} = \frac{1}{a^{(n+m)/2}} \sum_{l=0} (\zeta a)^l f_l(z_1, \theta_1; z_2, \theta_2),$$

and the integrand of (6.60) has the form

$$\frac{1}{a^N} \sum_{l=0} (\zeta a)^l g_l(z_i, \theta_i), \tag{6.62}$$

since there are N D's and N $\bar{D}$'s in $\langle \cdots \rangle$. The log divergence comes from the $l = 1$ term in (6.62), which is linear in ζ; otherwise the divergent parts are the same for D_2 and P_2. So if $N_{D_2}\zeta_{D_2} = -N_{P_2}\zeta_{P_2}$, *i.e.*,

$$iN_{D_2} = -N_{P_2}, \tag{6.63}$$

the logarithmic divergence will cancel. The factor of i may look strange, but it is compensated by another factor of i hidden in N_{D_2}. This is because N_{D_2} and N_{P_2} contain the invariant group volume elements

$$d\Omega_\zeta = d^2 a d^2 b d^2 \eta \delta \left(|a|^2 + \zeta^2 |b|^2 - 1 - \zeta \eta \bar{\eta}\right), \tag{6.64}$$

and for P_2 it was extracted in Minkowski space, whereas for D_2 we always worked in Euclidean space. In either case the volume elements are defined so that

$$\int d^2 \eta \eta \bar{\eta} = 1, \qquad \eta = \begin{cases} \eta_1 + i\eta_2, & D_2 \\ \eta_1 + \eta_2, & P_2 \end{cases}, \tag{6.65}$$

and $d^2 a = da_1 da_2$. Working out eq. (6.65) in terms of η_1 and η_2 gives

$$d^2 \eta = \frac{1}{2} d\eta_1 d\eta_2 \begin{cases} i, & D_2 \\ -1, & P_2 \end{cases}. \tag{6.66}$$

Since the CKG has this factor of i for D_2, the cancellation takes place for real values of the partition functions Z_{D_2} and Z_{P_2}.

Epilogue

If one were to describe the results of this work in a word, "ambivalent" might come to mind. For although we have largely affirmed the status quo, *i.e.*, the sigma model effective Lagrangian for D_2 and P_2, and finiteness of the type I superstring at the $\chi = 1$ level of the string loop expansion, several questions have been raised: What is the justification for ignoring extra contributions to the Weyl anomaly of the covariant dilaton vertex? Is there a simple way to see where the authors of [20] erred in deriving the paradox of sect. 4.2? Why does our study of tachyon scattering in curved spacetime disagree with the sigma model effective Lagrangian?

All of the above issues have one thing in common—they hinge upon just how the short-distance singularities of the two-dimensional quantum field theory on the world sheet are handled. We have tried to regulate the divergences as carefully and consistently as possible, but there may be a better way that clears up the difficulties. As has been emphasized in the text, the more sophisticated techniques of BRST quantization [15, 23] seem to circumvent some of these problems at the expense of greater abstractness. To understand the derivation of low-energy effective actions and finiteness of string theories in the most intuitive and concrete way, it is desirable to have a path integral formulation of string perturbation theory that is fully consistent with the other methods. What we have presented is a step in that direction.

References

1a. M. B. Green and J. H. Schwarz, Phys. Lett. **149B** (1984) 117;

Nucl. Phys **B255** (1985) 93.

1b. M. B. Green and J. H. Schwarz, Phys. Lett. **151B** (1985) 21.

2. D. J. Gross, J. A. Harvey, E. Martinec, and R. Rohm, Nucl. Phys B256 (1985) 253;

Nucl. Phys **B267** (1986) 75.

3. M. A. Virasoro, Phys. Rev. **177** (1969) 2309;

J. A. Shapiro, Phys. Lett. **33B** (1970) 361.

4. M. Douglas and B. Grinstein, Phys. Lett. **183B** (1987) 52;

Erratum: Phys. Lett. **187B** (1987) 442.

5. S. Weinberg, Phys. Lett. **187B** (1987) 278.

6. C. P. Burgess and T. R. Morris, Nucl. Phys. **B291** (1987) 256.

7. O. Alvarez, Nucl. Phys. **B216** (1983) 125.

8. A. M. Polyakov, Phys. Lett. **103B** (1981) 207.

9. J. Liu and J. Polchinski, Texas preprint UTTG-26-87 (1987).

10. J. Scherk and J. H. Schwarz, Nucl. Phys. **B81** (1974) 118;

Phys. Lett. **52B** (1974) 347.

11. T. Yoneya, Prog. Theor. Phys. **51** (74) 1907

12. E. S. Fradkin and A. A. Tseytlin, Phys. Lett. **158B** (1985) 316;

Phys. Lett. **163B** (1985) 123;

Nucl. Phys. **B261** (1985) 1.

13. C. G. Callan, D. Friedan, E. J. Martinec, and M. J. Perry, Nucl. Phys. **B262** (1985) 593;

C. G. Callan, I. R. Klebanov, and M. J. Perry, Nucl. Phys. **B278** (1986) 78.

14. A. Abouelsaood, C. G. Callan, C. R. Nappi, and S. A. Yost, Nucl. Phys. **B280** (1987) 599;

    C. G. Callan, C. Lovelace, C. R. Nappi, and S. A. Yost, Nucl. Phys. **B288** (1987) 525.

15. C. G. Callan, C. Lovelace, C. R. Nappi, and S. A. Yost, Nucl. Phys. **B293** (1987) 83.
16. C. G. Callan, C. Lovelace, C. R. Nappi, and S. A. Yost, Princeton preprints PUPT-1070, 1083 (1988).
17. A. M. Polyakov, Mod. Phys. Lett. **A2** (1987) 893.
18. C. Lovelace, Nucl. Phys. **B273** (1986) 413.
19. B. Grinstein and M. Wise, Phys. Rev. **D35** (1987) 655.
20. W. Fischler, I. Klebanov, and L. Susskind, SLAC preprint PUB-4298, (1987).
21. J. Cline, Caltech preprint 68-1458 (1987).
22. S.-J. Rey, Caltech preprint 68-1464 (1987).
23. J. Polchinski, Texas preprint UTTG-31-87 (1987).
24. W. Fischler and L. Susskind, Phys. Lett. **B173** (1986) 262;

    Phys. Lett. **B173** (1986) 383.

25. S. Das and S.-J. Rey, Phys. Lett. **186B** (1987) 328.
26. H. Itoyama and P. Moxhay, Nucl. Phys. **B293** (1987) 685.
27. M. B. Green and N. Seiberg, IAS preprint IASSNS-HEP-87/38 (1987).
28. B. Grinstein and R. Rohm, Commun. Math. Phys. **111** (1987) 667.
29. D. Friedan, E. Martinec, and S. Shenker, Nucl. Phys. **B271** (1986) 93.
30. G. Moore and P. Nelson, Nucl. Phys. **B266** (1986) 58.

31. J. Polchinski, Nucl. Phys. **B289** (1987) 465.

32. J. Cline, Phys. Lett. **B201** 447 (1988).

33. B. Grinstein, unpublished.

34. S. Weinberg, Phys. Lett. **156B** (1985) 309.

35. W. Weisberger, Nucl. Phys. **B284** (1987) 171

36. J. Polchinski, Comm. Math. Phys. **104** (1986) 37.

37. A. Cohen, G. Moore, P. Nelson, and J. Polchinski, Nucl. Phys. **B281** (1987) 127;

W. Weisberger, Nucl. Phys. **B294** (1987) 113-124.

38. S. P. de Alwis, Phys. Lett. **168B** (1986) 59.

39. E. Onofri and M. A. Virasoro, Nucl. Phys. **B201** (1982) 159.

40. S. Randjbar-Daemi, A. Salam, and J. A. Strathdee, Int. J. Mod. Phys. **A2** (1987) 667.

41. N. K. Nielsen, Nucl. Phys **B101** (1975) 173;

I. J. R. Aitchison and C. M. Fraser, Ann. Phys. **156** (1985) 1.

42. J. Polchinski and Y. Cai, Nucl. Phys. **B296** (1988) 91.

43. M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory*, vol. 2, ch. 8 (Cambridge University Press, 1987).

44. J. Polchinski, private communication.

45. L. P. Eisenhart, *Riemannian Geometry* (Princeton University Press, 1926).

46. L. Alvàrez-Gaumé, D. Z. Freedman, and S. Mukhi, Ann. Phys. (N.Y.) **134**, 85;

D. Friedan, Phys. Rev. Lett. **45** (1980) 1057.

47. C. Callan and Z. Gan, Nucl. Phys. **B272** (1986) 647.

48. J. E. Paton and H. M. Chan, Nucl. Phys. **B10** (1969) 516.

49. N. Seiberg and E. Witten, Nucl. Phys. **B276** (1986) 272.

50. C. P. Burgess and T. R. Morris, Nucl. Phys. **B291** (1987) 285.

51. L. Crane and J. M. Rabin, Commun. Math. Phys. **113** (1988) 601.

52. B. DeWitt, *Supermanifolds* (Cambridge University Press, 1984).

53. A. Neveu, J. Schwarz, and C. Thorn, Phys. Lett. **35B** (1971) 551.