RENORMALIZATION CORRECTIONS IN HEAVY COLORED SCALAR EFFECTIVE FIELD THEORY

Thesis by

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Abstract

Recently, QCD processes involving a heavy quark at energies much smaller than its mass have been examined in an effective field theory approach. In this 'heavy quark theory', the mass of the quark is taken to infinity while its four velocity is held fixed. The effective theory has a large set of symmetries because of the decoupling of the flavor (when the kinematic dependence on masses is removed) and spin of the heavy quark from its interactions with the light degrees of freedom. As a consequence, several matrix elements of the theory are determined in terms of a single function, the Isgur-Wise function. Being nonperturbative in character, this function is not fully calculable. However, it has a calculable logarithmic dependence on the masses of the heavy particles, arising from QCD effects in the full theory.

Some extensions of the standard model contain heavy color triplet scalars. It is instructive therefore to consider the analogous effective field theory for scalars. In processes where pair production does not occur, the statistics of the heavy particles are irrelevant, and their interactions are identical with those of quarks. Thus there is a 'superflavor symmetry' that interchanges quarks and scalars, and a flavor symmetry between scalars. Again, these symmetries determine several matrix elements involving scalars up to the same Isgur-Wise function. In this thesis, the logarithmic mass dependence of the operators $\phi_2^{\dagger}\phi_1$, $\phi_2^{\dagger}(i\partial^{\mu}\phi_1)$, and $(i\partial^{\mu}\phi_2)^{\dagger}\phi_1$ is calculated. The latter two operators mix under renormalization.

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1. Introduction

Quantum Chromodynamics (QCD) is now widely accepted to be the theory behind what were historically known as the 'strong interactions', the interactions between the nucleons in the nucleus of an atom. QCD is a quantum field theory of interactions between quarks, the elementary constituents of the nucleons, the charge responsible for the interaction being called 'color'. The strong interactions between hadrons are now viewed as a Van-der-Waals type remnant of the interactions between these quarks. QCD is characterized by a mass scale, Λ_{QCD}, at which the coupling constant diverges. The theory is asymptotically free,[1)] and conventional perturbation theory works for high energies, but fails for energies comparable to the QCD scale. One is thus in the strange situation of having a theory but not being able to calculate all the consequences of interest. As a result, one is forced to resort to the symmetries of the theory, rather than dynamical calculations, to make predictions.

The simplest symmetry relevant to the strong interactions, isospin, was introduced in the thirties. The masses of the proton and the neutron being almost identical, the two particles were related by this isospin symmetry, which would then be exact in the limit that the two masses coincide. The symmetry was enlarged in the sixties to the 'flavor $SU(3)$' of the baryons. Empirically, the predictions of the latter symmetry are less accurate than those of pure isospin. With the coming of the quark model and of QCD, these facts are understood to be due to the differences between the masses of the up, down and strange quarks being small relative to the QCD scale, rather than due to the proximity of their absolute values. Isospin is a good symmetry because $(m_d - m_u)/\Lambda_{QCD}$ is small, while flavor $SU(3)$ is not quite as good because $(m_s - m_u)/\Lambda_{QCD}$ is not as small. Corrections to the predictions of these symmetries could then be studied as an expansion in these parameters (there are also corrections arising from the electromagnetic interactions).

Another symmetry arises in the QCD Lagrangian when the quarks are taken to be massless. The two chiral components of the quark flavors independently exhibit flavor symmetries that result in the conservation of the vector and axial vector currents. The symmetry is manifest in the massless theory, but not in QCD with massive quarks. Thus, when the absolute values of the masses of the quarks are much smaller than the QCD scale, one expects this 'chiral symmetry' to be approximately true. 'Effective field theories' with manifest chiral symmetry can be built to study the interactions of (possibly composite) particles in this limit. Again, $SU(2)_L \times SU(2)_R$ is experimentally a better symmetry than $SU(3)_L \times SU(3)_R$ because the strange quark mass is not as small relative to the QCD scale as the up and down quark masses. Chiral symmetry is spontaneously broken in nature; however, the existence of this symmetry enables a perturbative expansion that relates several low energy processes. Corrections to the predictions of the effective theory with massless quarks are computed as an expansion in m_q / Λ_{QCD}.

The three quarks, up, down, and strange, are all lighter than the QCD scale. In the 'Standard Model' of elementary particles, there are in addition three other quarks, the charm, bottom, and top quarks (the top is yet to be experimentally confirmed) that are heavier than this scale. Again, one of these, the charm, is not much heavier, while the other two are significantly heavier. It is natural, therefore to examine the consequences of two kinds of symmetries: where the difference in the masses of these heavy quarks is *insignificant* relative to the QCD scale, and where their absolute magnitudes are much larger than the QCD scale. In the first case, one expects a flavor symmetry among the heavy flavors; while in the latter, the QCD Lagrangian must be examined in the appropriate limit of infinitely massive quarks for new symmetries that are not manifest in QCD itself. Corrections to the effective theory must be computable as an expansion in Λ_{QCD}/m_Q*.

Intuitively, the picture is familiar.[2)] In atomic physics, the nucleus is assumed to be infinitely heavy relative to the electrons, and is therefore unaffected by the interactions of the electrons. The interactions of the electrons, in turn, are identical for different flavors of the nucleus ('isotopes'), provided only that the nucleus has the same charge. Now consider a hadronic bound state. The asymptotic freedom of QCD implies that if all the quarks were very heavy, then since the coupling constant would be small on the scale of their Compton wavelengths, it would be possible to calculate properties of hadrons from first principles. However, there are also light quarks, and their QCD interactions are very complicated. Consider the bound state of a very heavy quark with light quarks. However complicated the inter-

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*. For reviews, see references [2-4] and the references therein.

actions of the light quarks, they do not affect the motion of a sufficiently heavy quark. Further, the light quarks are unaffected by the actual flavor of the heavy quark. This becomes especially clear in the rest frame of the heavy quark, where the light quarks simply see a static color triplet source at the origin. Different flavors of the heavy quark with the same color would have the same strong interaction effects on the light ones. In the limit of infinite heavy quark mass, there are no relativistic effects such as color magnetism. Thus the spin of the heavy quark decouples from the gluon field, and thence from the interactions of the light quarks. There is therefore a $SU(2N_h)$ symmetry, where N_h is the number of heavy flavors. The decoupling of the spin was first understood in a nonrelativistic model of the heavy quark[5)] and is of particular relevance to calculations on the lattice.[6)] In fact, one can go further. The light quarks are unaffected by the total spin of the heavy object that is the source of the color flux. If the heavy object were a color triplet scalar rather than a quark, the light quark interactions would not be different. Thus, there is also a symmetry between heavy scalars and heavy fermions. Such a symmetry is called a 'superflavor' symmetry.[7)]

The Standard Model does not include any colored scalar objects. There is also no experimental indication that such objects exist at currently accessible energies. However, although the Standard Model successfully correlates all current observations in terms of a few parameters, it is widely believed that there must be an extension of the model that resolves some of its unsatisfactory features.[10] These include the gauge hierarchy problem (the existence of disparate scales among the gauge interactions), the problem of the origin of the fermion masses and mixing angles, and the lack of unification with gravity. Perhaps the problem that is most significant is that of the Higgs Boson that implements the mechanism of spontaneous symmetry breakdown to give masses to the fermions. It is yet to be observed, and arguments suggest that there should be new physics at the TeV scale, the expected mass scale of the Higgs.[11] Extensions of the standard model that address these issues, such as supersymmetric or technicolor models, often involve heavy color triplet scalars.[12] It is of interest therefore to include such triplets in the formalism, and examine the consequences of the superflavor symmetry.

To make the symmetries manifest, one must construct from QCD an effective theory where the quarks are infinitely massive. Since an infinity is involved, one must consider carefully how this limit is to be taken. An analogy is helpful.[4] A heavy object thrown in the air is to a good approximation unaffected by the air resistance, in the sense that its world line remains unaltered. Thus the appropriate limit is one where the heavy quark has a straight world line, that is, the four-velocity v_μ of the quark is held fixed.[8,9] It is important to realize however that the four-velocity here can be relativistic. The effective theory thus differs from nonrelativistic models. The four-velocity satisfies the usual constraint $v_\mu v^\mu = 1$. In this effective theory, the $SU(2N_h)$ and the Super-flavor symmetry (when colored scalar particles are included) are manifest. Various amplitudes can be related in this effective theory using these symmetries. The most significant predictions are for the semileptonic B-meson decays, the results being of relevance to the determination of the Cabibbo-Kobayashi-Maskawa mixing angles V_{cb} and V_{ub}. It is important that these predictions are model-independent, being simply predictions of an effective theory that is a limit of QCD. The superflavor symmetry relates these amplitudes to those involving heavy colored scalars, and would therefore relate the interactions of bound states of such scalar particles to those of the heavy mesons.[7)]

There are two kinds of corrections to the effective theory that can be computed. First, physical particles have finite masses, and one can classify the corrections due to having set them to infinity, as an expansion in Λ_{QCD}/m_Q. The second kind arises from QCD processes. Virtual gluons in the full theory can have an arbitrary momentum, greater or smaller than that of the quarks. In the effective theory however, since the quarks are infinitely massive, the virtual gluons have momenta that are always smaller. Thus there are corrections to the effective theory amplitudes that arise from that region of phase space where the virtual gluons have momenta that are of the order of or larger than the quark momenta. The asymptotic freedom of QCD enables these to be accessible to perturbation theory, and the renormalization group can then be used to scale down to low energies.

In this thesis, QCD corrections are calculated to the relations predicted by superflavor symmetry.[13)] In chapter 2, the effective field theory is laid out, followed in chapter 3 by the predictions of the Super-flavor symmetry. The QCD corrections are calculated in chapter 4, and the final chapter makes some concluding remarks.

2. The Effective Theory

The physical situation of interest is a bound state of a heavy particle and light degrees of freedom, where the momenta carried by the latter are small compared to the mass of the heavy particle. The heavy particle moves at a constant velocity, unaffected by the QCD processes involving the light particle. An external electroweak current can change either its flavor and/or velocity, but this does not have an effect on the strong interactions with the light degrees of freedom. Thus, the effective theory is built by considering the limit where the mass of the quark is taken to infinity while keeping its four velocity fixed. Clearly, a theory where the particle has a fixed velocity breaks Lorentz invariance, so one must consider a whole spectrum of effective theories corresponding to all possible velocities to retain relativistic invariance. Formally, the independence of heavy particles with different velocities corresponds to a velocity superselection rule.[9)]

The Velocity Super-Selection Rule

In the limit that the heavy particle is infinitely massive, one can take its mass to be the mass of the bound state, $m_{heavy} \cong m_{boundstate}$. The heavy particle momentum is the difference between the momentum of the bound state and the small momentum of the light degrees of freedom[3)]

$$P^{\mu}_{heavy} = P^{\mu}_{bound} - q^{\mu}. \tag{1}$$

The residual momentum is defined as the difference between this heavy particle momentum, and the momentum it would have if it were on-shell,

$$k^{\mu} = P^{\mu}_{heavy} - m_{heavy} v^{\mu} . \tag{2}$$

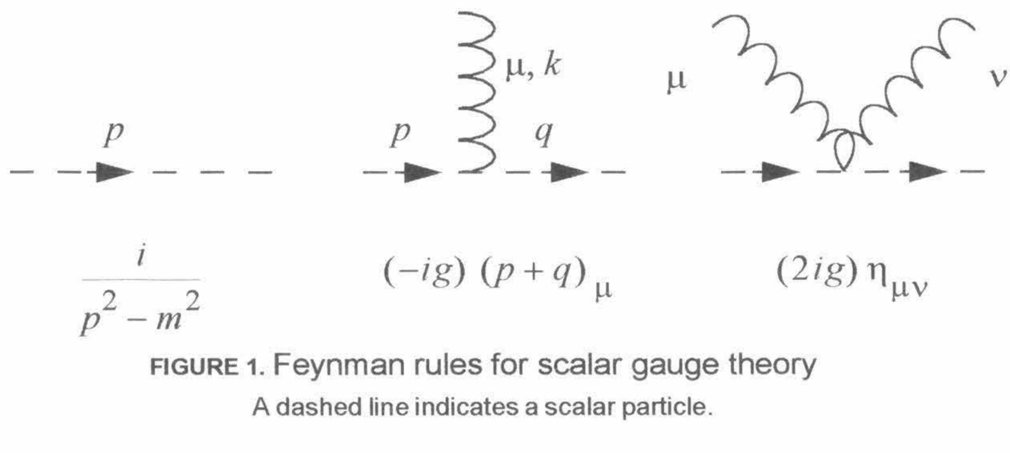
The velocity of the heavy particle,

$$v^{\mu}_{heavy} = \frac{P^{\mu}_{heavy}}{m_{heavy}} = v^{\mu}_{bound} + \frac{k^{\mu}}{m_{heavy}} , \tag{3}$$

approaches the velocity of the bound state as its mass becomes larger. In the effective theory the two velocities are equal. Thus, the velocity of the heavy particle is unchanged by the QCD interactions with the light degrees of freedom, and one can tag these particles by their velocities. In the effective theory therefore, there is a field for each velocity v.

The Feynman Rules

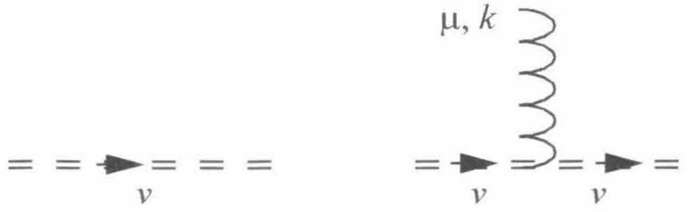
There are several ways to build the Lagrangian of the effective theory. The simplest approach perhaps is to directly derive its Feynman rules from the QCD Feynman rules.[6,14] Consider first the scalar case. In scalar gauge theory, we have the following Feynman rules for the propagator and the vertices. The gauge propagator is unchanged in the effective theory and is not shown below.



$$\frac{i}{p^2 - m^2} \qquad (-ig)\,(p+q)_\mu \qquad (2ig)\,\eta_{\mu\nu}$$

FIGURE 1. Feynman rules for scalar gauge theory
A dashed line indicates a scalar particle.

Substituting $P^\mu_{heavy} \to m_{heavy} v^\mu + k^\mu$ and retaining terms of leading order in m_{heavy}, one gets the Feynman rules shown below. Note that the 'seagull' vertex is suppressed relative to the other vertex by a power of the heavy mass, and hence need not be considered. The factors of $2m$ in the vertex and



$$\frac{i}{2m\,(v\cdot k)} \qquad (-2ig)\,(mv)_{\mu}$$

FIGURE 2. Feynman rules for a heavy scalar particle.
The double dashed line indicates a heavy scalar particle The seagull vertex is suppressed and is not shown.

the propagator cancel in calculations and may be dropped. The lines are labeled by the velocity v, and this velocity is unchanged by the interaction with the gluon.

The same procedure can be carried out for a heavy quark. The usual propagator

$$\frac{i\,(P_Q\cdot\gamma + m_Q)}{P_Q^2 - m_Q^2}\,, \tag{4}$$

with the substitution $P^{\mu}_{heavy} \rightarrow m_{heavy} v^{\mu} + k^{\mu}$ gives, upon retaining the leading terms,

$$\frac{i\,(v \cdot \gamma + 1)}{2\,(v \cdot k)} \quad . \tag{5}$$

Since the velocity of the heavy quark is unchanged by the interaction with the gluon, the quark-gluon vertex $-ig\gamma_{\mu}T^{a}$, with g the strong interaction coupling, and T^{a} the color $SU(3)$ generator, is always sandwiched between two factors of $(1 + v \cdot \gamma)/2$. Using

$$\frac{(1 + v \cdot \gamma)}{2}\gamma^{\mu}\frac{(1 + v \cdot \gamma)}{2} = v^{\mu} \tag{6}$$

it is clear then that the heavy quark-gluon interaction is $-igv_{\mu}T^{a}$. The absence of a Dirac structure to the vertex is an indication of the decoupling of the spin of the heavy quark from the QCD interactions. The expression for the propagator may then be simplified. The factor $(1 + v \cdot \gamma)/2$ may be commuted past the vertex, until it hits an external on-shell spinor, on which it is unity, since it is a projection operator on the heavy field (see Equation (16) below). Thus, one gets the Feynman rules for the heavy quark shown below.

$$\frac{i}{v \cdot k} \qquad (-ig)\, v_\mu T^a$$

FIGURE 3. Feynman rules for a heavy quark.
The double line indicates a heavy quark. Compare these rules with those for the heavy scalar particle (see Figure 2 on page 13).

The Super-flavor symmetry is now apparent. The scalar and the spinor heavy particles have identical Feynman rules in the effective theory.

The Lagrangian

The effective theory Lagrangian can be derived from the QCD Lagrangian, with appropriate field redefinitions.[9)] To do this, one needs the connection between the field in the effective theory that corresponds to a particle of a given velocity, and the field in the full theory that corresponds to all momenta of the particle. First, the kinematic dependence on the mass in the full field is removed, and then the spin structure is accounted for. For scalar particles, the definition

$$\chi_v = e^{im(v \cdot x)} \phi \tag{7}$$

implies that the mass dependence is purely in the commutator of the momentum operator with the field. That is,

$$[\hat{P},\chi] = (mv_\mu)\chi + i\partial_\mu\chi \tag{8}$$

and the derivatives on the field involve only the residual momentum. With this definition, the usual Lagrangian for colored scalars

$$(D\phi)^\dagger \cdot D\phi - m^2\phi^\dagger\phi \tag{9}$$

becomes

$$L_v^{scalar} = 2im\chi_v^\dagger v \cdot D\chi_v \tag{10}$$

after neglecting the kinetic term, since it is suppressed relative to the interaction term by a power of the mass. The color gauge symmetry has been imposed by replacing the ordinary derivative with the covariant derivative. Clearly this Lagrangian results in the Feynman rules of Figure 2 on page 13.

Pair creation does not occur in the effective theory while it does in the full theory. Therefore, the number of degrees of freedom of the heavy quark field must be less than that of the usual fermion field. That is, the heavy quark field must satisfy a constraint equation. To deduce the constraint, it is probably easiest to look at the form of the functional integral in momentum space.[3)] The functional integral gets significant contributions from the stationary points of the action. The kinetic term for the quark in the action is of the form

$$\int \frac{d^4 p}{(2\pi)^4} \bar{q}(-p)\,(\gamma \cdot p - m)\,q(p)\ . \qquad (11)$$

It becomes obvious, upon substituting $P^{\mu}_{heavy} \rightarrow m_{heavy} v^{\mu} + k^{\mu}$, that for a heavy quark of velocity v the relevant region is the solution to

$$(v \cdot \gamma - 1)\,q(p) \approx -\frac{k \cdot \gamma}{m} \approx 0\ . \qquad (12)$$

To leading order in the heavy mass therefore, the heavy quark field h_v of velocity v may be taken to obey exactly

$$(v \cdot \gamma)\,h_v = h_v\ . \qquad (13)$$

The relationship between the fields in the effective theory and the full theory is then

$$h_v = e^{im(v \cdot x)} Q \,. \tag{14}$$

This equation cannot be exact, because of the constraint equation (13). $v \cdot \gamma$ squares to unity, and thus has two doubly degenerate eigenvalues ± 1. It is a straightforward mathematical result that for any such idempotent operator, the vector space may be written as a direct sum of two vector spaces, corresponding to the two eigenvalues ± 1. The projection operators on to these two subspaces are then $(1 \pm v \cdot \gamma)/2$. Thus, more generally,

$$Q = e^{-im(v \cdot x)} (h_v + \delta_v) \tag{15}$$

where besides projecting on to the subspaces with eigenvalues ± 1, so that

$$\begin{aligned} (v \cdot \gamma)\, \delta_v &= -\delta_v \\ (v \cdot \gamma)\, h_v &= h_v \end{aligned} \,, \tag{16}$$

we have removed the kinematic dependence on the large mass. The δ_v part is of order Λ_{QCD}/m and can be ignored. This is obvious upon considering the equation of motion

$$\{ i (D \cdot \gamma) - m \} Q = 0 \,. \tag{17}$$

In terms of the fields h_v and δ_v this equation becomes

$$[m(v\cdot\gamma-1)+i(\gamma\cdot D)]\,(h_v+\delta_v) = 0. \tag{18}$$

Thus, for large m, this results in

$$\delta_v = \frac{i(\gamma\cdot D)}{m}h_v\,. \tag{19}$$

This is analogous to the two components of the Dirac spinor being suppressed in the nonrelativistic approximation. Thus, the connection between the heavy quark field and the full field is, up to terms suppressed by powers of the mass,

$$h_v = \frac{(1+\gamma\cdot v)}{2}e^{imv\cdot x}Q\,. \tag{20}$$

The antiquark has been ignored so far. Since in the effective theory there is no pair creation, only the heavy quark field need be considered if the process of interest has no heavy antiquarks. However, antiquarks can be easily included in the same way. The antiquark field for a given velocity is defined by

$$h_{-v} = \frac{(1-\gamma\cdot v)}{2}e^{-imv\cdot x}Q\,. \tag{21}$$

The projection is on to the other subspace, and the sign in the exponent is changed to reflect the antiparticle nature.

The heavy quark Lagrangian is now easily derived. The QCD Lagrangian

$$L_{QCD} = \bar{\psi}(i\gamma \cdot D - m)\psi \quad (22)$$

gives in this case

$$L_v = \bar{h}_v [m(\gamma \cdot v - 1) + i(\gamma \cdot D)] h_v \, . \quad (23)$$

Here, $\bar{h}_v$ is the Dirac adjoint, and D_μ the $SU(3)$ color covariant derivative. Using the constraint equation (16), this simplifies to

$$L_v = \bar{h}_v (i\gamma \cdot D) h_v \, . \quad (24)$$

This equation can be further simplified by introducing on either side of the derivative the projection operators $(1 + v \cdot \gamma)/2$, and using equation (6). The result is then

$$L_v^{quark} = \bar{h}_v i(v \cdot D) h_v \, . \quad (25)$$

Clearly, this Lagrangian results in the Feynman rules of Figure 3 on page 15. The Lagrangian for the antiquark can be derived in the same way. It is

$$L_v^{antiquark} = -\bar{h}_{-v}\, i\,(v \cdot D)\, h_{-v} \,. \tag{26}$$

Thus, the propagator is unchanged, while the color charges have changed their sign, as expected.

There are several unusual features in the effective theory. The Lagrangian is not even Lorentz invariant, a reflection of the fact that in the effective theory, the QCD interactions do not change the velocity of the heavy particle. Heavy particles moving along with different velocities are completely independent. To get a Lorentz invariant Lagrangian, Lagrangians corresponding to various velocities have to be added.

$$L = \sum_v L_v \,. \tag{27}$$

However, this is not a nonrelativistic model. The velocities involved are not constrained in any way, and when there is more than one heavy quark involved, the velocity of one need not be small in the rest frame of the other.

The kinematic mass dependence has been removed from the heavy fields. Thus, in their mode expansions, only the residual momenta are involved, and derivatives on the fields result in a factor of these momenta. There is no pair creation in the effective theory, so only the heavy particle fields are involved. The Lagrangians (10) and (25) have a large set of symmetries that are not manifest in the QCD Lagrangians. The symmetries and their consequences will be explored in the next chapter.

3. Symmetries and Consequences

The Lagrangians of the effective theory have a large set of symmetries that are not manifest in the full theory of QCD. The various flavors (masses) of the heavy particles have the same interactions. The spin components of the heavy particle decouple from the interactions with the light degrees of freedom, and indeed the total spin itself is not relevant: the scalar and spinor particles have identical interactions. While these symmetries were apparent on inspection of the Feynman rules, it is useful to construct them formally, to enable rigorous calculation of the consequences. There are many equivalent ways to derive these. Several matrix elements of currents between the observable bound states of these heavy particles may all be calculated up to a common function, the 'Isgur-Wise' function.

Spin Symmetry

The heavy scalar Lagrangian has the usual $U(1)$ symmetry of charged scalar particles,

$$\delta\chi_v = i\varepsilon_s\chi_v \,. \tag{28}$$

A more interesting symmetry however, is the spin symmetry[8)] of the quark, since it has implications for heavy meson decay amplitudes. The Lagrangian of equation (25) has an $SU(2)$ symmetry corresponding to the two spin components of the heavy quark. This is most easily seen in the rest frame of the quark. In this frame, the Lagrangian (25) reduces to

$$L_0^{quark} = \overline{h}_0 i D^0 h_0 \quad , \tag{29}$$

while the constraint equation (16) becomes

$$\gamma^0 h_0 = h_0 \, . \tag{30}$$

This equation is invariant under the $SU(2)$ symmetry on the two non-zero components of the spinor. (There still is of course a $U(1)$ symmetry corresponding to the number of heavy quarks). The generators of the symmetry can be constructed explicitly. Suppose that one is in the standard representation where

$$\gamma^0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} , \tag{31}$$

then the generators can be taken to be

$$\underset{\sim}{S}_0 = \begin{bmatrix} \underset{\sim}{\sigma} & 0 \\ 0 & 0 \end{bmatrix} . \tag{32}$$

This can be generalized to an arbitrary velocity. First choose an orthonormal set of space-like vectors ('polarization vectors') that are orthogonal to the velocity:

$$e_j \cdot e_k = -\delta_{jk} \qquad v \cdot e_j = 0 \,. \tag{33}$$

Using the fact that $\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^{\mu},\gamma^{\nu}]$ satisfies the Lorentz algebra

$$[\Sigma^{\mu\nu},\Sigma^{\rho\sigma}] = ig^{\nu\rho}\Sigma^{\mu\sigma} - ig^{\mu\rho}\Sigma^{\nu\sigma} + ig^{\mu\sigma}\Sigma^{\nu\rho} - ig^{\mu\sigma}\Sigma^{\nu\rho} \,, \tag{34}$$

the three generators defined by

$$S_j^v = i\varepsilon_{jpq}[\gamma \cdot e_p,\gamma \cdot e_q]\left(\frac{1 + v \cdot \gamma}{2}\right) \tag{35}$$

can be shown to satisfy the $SU(2)$ algebra

$$[S_p,S_q] = i\varepsilon_{pqr}S_r \,. \tag{36}$$

The projection operator commutes with the commutator in the definition because of the orthogonality of the velocity and the polarization vectors, and it is sufficient to check that the commutator term satisfies the required relation. These generators are not Hermitian, but are Dirac self-adjoint,

$$\bar{S}_j^v = \gamma^0 S^{\dagger v}_j \gamma^0 = S_j^v \,. \tag{37}$$

As a consequence, the transformation

$$\delta h_v = i\,(\underset{\sim}{\varepsilon} \cdot \underset{\sim}{S}^v)\,h_v \tag{38}$$

implies the transformation for the adjoint

$$\delta \bar{h}_v = -i\bar{h}_v\,(\underset{\sim}{\varepsilon} \cdot \underset{\sim}{S})\ , \tag{39}$$

and the effective field theory Lagrangian is then invariant under this transformation.

This symmetry has an unusual structure.[3)] For every velocity, there is a $SU(2)$ symmetry, the different velocities being related by Lorentz transformations. This is somewhat analogous to gauge symmetries where there is an independent symmetry at each point in spacetime, the different spacetime points being related by Lorentz transformations. Unlike gauge symmetries however, here the Lorentz transformations also rotate the polarization vectors, and thus mix the spin symmetries.

Flavor and Super-Flavor Symmetry

When several flavors of the heavy particle are present, the effective theory Lagrangian becomes, (using equations (10) and (25)),

$$L_v^{quark} = \sum_{j=1}^{N_h} \bar{h}_v^{(j)} \, i\,(v \cdot D)\, h_v^{(j)} \tag{40}$$

in the case of quarks, and

$$L_v^{scalar} = \sum_{j=1}^{N_\chi} 2m\chi_v^{(j)\dagger} \, i\,(v \cdot D)\, \chi_v^{(j)} \tag{41}$$

in the case of scalars. While the latter seems to have a dependence on the masses of the scalar particles, this is a trivial normalization dependence that can be eliminated by rescaling the field. It is important to note that all the heavy fields involved have the same velocity. The spin symmetry of the heavy quark Lagrangian is now enlarged to a $SU(2N_h)$ symmetry corresponding to unitary rotations of the flavors and the individual spin components. Similarly, the flavors of the scalar particles result in a $SU(N_\chi)$ symmetry.

The interactions with the gluons with the heavy scalar and spinor fields being identical, the two heavy fields may be combined into a five component heavy field, on which the super-flavor symmetry can be defined:[7)]

$$\Psi_v = \begin{pmatrix} h_v \\ \chi_v \end{pmatrix} . \qquad (42)$$

The Lagrangian in terms of the superfield is

$$L_v = i\overline{\Psi}_v M (v \cdot D) \Psi_v , \qquad (43)$$

where the mass matrix is

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 2m_\chi \end{bmatrix} , \qquad (44)$$

and the adjoint of the superfield is defined by

$$\overline{\Psi}_v = \Psi^\dagger{}_v \begin{bmatrix} \gamma^0 & 0 \\ 0 & 1 \end{bmatrix} . \qquad (45)$$

The mass of the scalar could be scaled into the field, but it is convenient to retain the conventional dimensions.

The symmetry is generated by the following transformation:

$$\delta\Psi_v = i \begin{bmatrix} \varepsilon_0 S_0 + \underset{\sim}{\varepsilon} \cdot \underset{\sim}{S} & s\sqrt{2m_\chi} \\ \frac{\bar{s}}{\sqrt{2m_\chi}} & \varepsilon_s \end{bmatrix} \Psi_v . \tag{46}$$

Here, s is an infinitesimal spinor that satisfies $(v \cdot \gamma)\, s = s$ and allows the heavy scalar to mix with the heavy spinor. The usual $SU(2)$ symmetry on the heavy quark is generated by $\underset{\sim}{S}$, while the new generator is the projection operator

$$S_0 = \frac{(1 + v \cdot \gamma)}{2} . \tag{47}$$

The super-flavor part implies

$$\begin{aligned} 2m_\chi \delta\,(i\chi^\dagger D\chi) &= \sqrt{2m_\chi}\,\{\bar{h}sD\chi - \chi^\dagger D\,(\chi s)\,\} \\ \delta\,(i\bar{h}Dh) &= \sqrt{2m_\chi}\,\{\bar{s}\chi^\dagger Dh - \bar{h}D\,(\chi s)\,\} \end{aligned} , \tag{48}$$

and is clearly a symmetry of the Lagrangian since $Ds = 0$. It is obvious that this is a $SU(3) \times U(1)$ symmetry on the three non-zero components of the five component super-field. Thus, in the rest frame of the heavy objects, we have generators of the type

$$\begin{bmatrix} x & x & 0 & 0 & z \\ x & x & 0 & 0 & z \\ 0 & 0 & x & x & 0 \\ 0 & 0 & x & x & 0 \\ z & z & 0 & 0 & y \end{bmatrix}, \qquad (49)$$

where the zeros prevent mixing with the anti-quark, (the third and fourth columns are irrelevant because the corresponding entries in the superfield are vanishing in this frame) and the other entries are simply the corresponding entries in the identity matrix and the Gell-Mann matrices

$$\begin{bmatrix} x & x & z \\ x & x & z \\ z & z & y \end{bmatrix}. \qquad (50)$$

Similarly, the generators of the flavor symmetries may also be constructed by putting the appropriate fields in a super-field.

The Isgur-Wise Function

The consequences of these symmetries can easily be explored qualitatively.[3)] Consider the bound states of the heavy particles with antiquarks. For definiteness, let these be the D and D* (the pseudo-scalar and vector meson bound states of the charm quark), the B and B* mesons (bound states of the bottom quark) and a hypothetical X particle (the bound state of a heavy scalar particle with an anti-quark). In the effective field theory, one can assume that the wave functions of these bound states factorize into the product of a free heavy particle wave function and a wave function of the remaining light degrees of freedom,

$$|Bound\rangle \approx |heavy\rangle\,|light\rangle\ . \qquad (51)$$

Transitions between the bound states being considered are caused by currents that involve changes in the flavor, spin, or velocity of the heavy particle. In the effective theory, these currents leave the light degrees of freedom unaffected. Thus one expects the transition amplitudes to be factorizable as

$$\begin{aligned} &\langle Bound_1(v_1)|\,J^h\,|Bound_2(v_2)\rangle \\ &= \langle heavy_1(v_1)|\,J^h\,|heavy_2(v_2)\rangle\langle light_1(v_1)|light_2(v_2)\rangle \end{aligned} \qquad (52)$$

The overlap integral between the initial and final light states involves non-perturbative QCD, and is not calculable. However, general arguments simplify its structure. In the brick wall frame where $v_1 = -v_2$, the angular momentum of the light degrees of freedom must be conserved, because they are unaffected by the current. Therefore the incoming and outgoing helicities of the light particles must be equal and opposite. Further parity invariance of QCD implies that the overlap integral is identical for both helicities. Consequently, it can depend only on the velocities. Lorentz invariance, and the fact that the velocities square to unity imply that the dependence is on the single scalar kinematic invariant $v_1 \cdot v_2$. Thus, one can write

$$\langle light_1 (v_1) | light_2 (v_2) \rangle = \xi (v_1 \cdot v_2) \ . \tag{53}$$

This function incorporating information about the single independent helicity amplitude is called the Isgur-Wise function.

The symmetries in the effective theory relate the matrix elements of the current between heavy particles of different flavor or spin. These relations can be derived by explicit application of the generators, or by using group theoretic techniques. In this way, all the transition amplitudes between the bound states considered above are determined up to the Isgur-Wise function. The function arises from non-perturbative QCD and cannot be calculated explicitly. However, its value is known at the kinematic point $v_1 = v_2$, corresponding to 'zero-recoil' (in the rest frame of the initial heavy particle, the final heavy particle is also at rest). Consider the transition between mesons of two different flavors of the heavy particle. The flavor symmetry of the effective theory indicates that the mediating current is a symmetry current. Its matrix element is therefore known in terms of the normalization of the states. Choosing appropriate normalizations for the meson and heavy particle states, the value of the Isgur-Wise function can be fixed at this threshold point. (Evaluating in the rest frame of the first particle, $v_1 \cdot v_2 = v_2^0 = \sqrt{1 + \underset{\sim}{v}_2^2} \geq 1$).

The Interpolating Field Method

The straightforward way to explore the consequences of the symmetries is to use the action of the generators of these symmetries on the states to relate various transition amplitudes. A simpler way is to use 'interpolating fields' that create and destroy the bound states to keep track of the transformation properties under the various symmetries, and then to construct the general form of the matrix elements consistent with these symmetries.[4)] To compare the two methods, the most instructive example is the transition between mesons, which is relevant for semileptonic B and D decays.

Consider the definitions of the transition matrix elements below.

$$\frac{\langle P_j(v')|\,\bar{h}^j_{v'}\gamma^\mu h^i_v\,|P_i(v)\rangle}{\sqrt{m_{P_j}m_{P_i}}} = \tilde{f}_+(v+v')_\mu + \tilde{f}_-(v-v')_\mu$$

$$\frac{\langle P^*_j(v',\varepsilon)|\,\bar{h}^j_{v'}\gamma^\mu\gamma^5 h^i_v\,|P_i(v)\rangle}{\sqrt{m_{P^*_j}m_{P_i}}} \qquad (54)$$

$$= \tilde{f}\varepsilon^*_\mu + (\varepsilon^*\cdot v)\,\{\tilde{a}_+(v+v')_\mu + \tilde{a}_-(v-v')_\mu\}$$

$$\frac{\langle P^*_j(v',\varepsilon)|\,\bar{h}^j_{v'}\Upsilon^\mu h^i_v\,|P_i(v)\rangle}{\sqrt{m_{P^*_j}m_{P_i}}} = i\tilde{g}\varepsilon_{\mu\rho\lambda\sigma}\varepsilon^{*\rho}v'^\lambda v^\sigma$$

Here P_i is a pseudoscalar meson and P^*_i a vector meson with the heavy quark Q_i, and ε the polarization vector of the vector meson. The factors of masses appear in the denominator to make the quantities on the right independent of the heavy masses in the effective theory. This follows from the normalization of meson states in QCD:

$$\langle B(p,s)|B(p',s')\rangle = 2E\delta_{s,s'}(2\pi)^3\delta^3(\underset{\sim}{p}-\underset{\sim}{p}') . \qquad (55)$$

These forms of the matrix elements are the most general consistent with Lorentz invariance and parity invariance of the strong interactions. By using the spin and flavor symmetry generators to relate various states, and the commutation relations of the symmetries, all the form factors here may be expressed in terms of the Isgur-Wise function as shown below.[8)]

$$\begin{gathered} \tilde{f}_+ = \xi \qquad \tilde{f}_- = 0 \\ \tilde{f} = (1 + v \cdot v')\,\xi \\ (\tilde{a}_+ - \tilde{a}_-) = -\xi \qquad (\tilde{a}_+ + \tilde{a}_-) = 0 \\ \tilde{g} = \xi \end{gathered} \tag{56}$$

The first of these relations follows straightforwardly from contracting the first definition in (54) with $(v - v')_\mu$ and using the constraint equations for the heavy fields.

It is simpler to derive these relations using the idea of interpolating fields.[4)] These are fields carrying the right quantum numbers to create or destroy the initial or final states. Of course, these are not elementary states, and such fields do not rigorously exist. However, they do provide a way to keep track of the appropriate transformation properties. Consider for example the interpolating fields

$$\begin{aligned} P_j(v') &= \bar{l}_{v'}\gamma_5 h^j_{v'}\sqrt{m_{P_j}} \\ P^*_j(v', \varepsilon) &= \bar{l}_{v'}\gamma_5\,(\varepsilon^* \cdot \gamma)\,h^j_{v'}\sqrt{m_{P^*_j}} \end{aligned}, \tag{57}$$

which destroy the mesons. Then, the matrix elements become

$$\frac{\langle P_j(v')|\bar{h}^j_{v'}\Gamma h^i_v|P_i(v)\rangle}{\sqrt{m_{P_j}m_{P_i}}} = -\langle 0|\bar{l}_{v'}\gamma^5 h^j_{v'}\bar{h}^j_{v'}\Gamma h^i_v \bar{h}^i_v \gamma^5 l_v|0\rangle$$
$$\frac{\langle P^*_j(v',\varepsilon)|\bar{h}^j_{v'}\Gamma h^i_v|P_i(v)\rangle}{\sqrt{m_{P^*_j}m_{P_i}}} = -\langle 0|\bar{l}_{v'}\varepsilon^*\cdot\gamma h^j_v\bar{h}^j_{v'}\Gamma h^i_v\bar{h}^i_v\gamma^5 l_v|0\rangle \quad . \tag{58}$$

The vacuum expectation values on the right may be evaluated as if this was a physical Green function, by Wick's theorem and the associated contractions. For the heavy quark, the contraction is just the spin structure of the heavy quark propagator (since only the transformation properties rather than the magnitudes are tracked this way)

$$\langle 0|h^i_v\bar{h}^i_v|0\rangle = \frac{1+v\cdot\gamma}{2}, \tag{59}$$

while for the light particles we have some general function of the velocities

$$\langle 0|l_v\bar{l}_{v'}|0\rangle = M\,. \tag{60}$$

Note that the velocity of the light particles has changed, while it is unaltered for the heavy ones. Being a four dimensional matrix, M may be expanded in terms of the sixteen Dirac matrices. Parity conservation precludes the terms with γ^5, while the Lorentz transformation property precludes $\sigma^{\mu\nu}$. Thus, the general form of M is

$$M = A(v \cdot v')I + B(v \cdot v')v \cdot \gamma + C(v \cdot v')v' \cdot \gamma \, . \qquad (61)$$

The matrix elements above are then evaluated as traces.

$$\frac{\langle P_j(v') | \bar{h}^j_{v'} \Gamma h^i_v | P_i(v) \rangle}{\sqrt{m_{P_j} m_{P_i}}} = Tr\left[\gamma^5 \left(\frac{1 + v' \cdot \gamma}{2}\right) \Gamma \left(\frac{1 + v \cdot \gamma}{2}\right) \gamma^5 M\right]$$

$$\frac{\langle P^*_j(v', \varepsilon) | \bar{h}^j_{v'} \Gamma h^i_v | P_i(v) \rangle}{\sqrt{m_{P^*_j} m_{P_i}}} = Tr\left[(\varepsilon^* \cdot \gamma) \left(\frac{1 + v' \cdot \gamma}{2}\right) \Gamma \left(\frac{1 + v \cdot \gamma}{2}\right) \gamma^5 M\right] \qquad (62)$$

Although there seem to be three independent functions in M, only the combination $M = (A - B - C)I$ is significant since the terms with $v \cdot \gamma$ and $v' \cdot \gamma$ anticommute with the γ^5 and leave the projection operators invariant. This combination may therefore be identified with the Isgur-Wise function,[4)]

$$\xi(v \cdot v') = (A - B - C) \, . \qquad (63)$$

The transitions between spinorial bound states of the heavy scalar are far simpler because of the absence of the Dirac structure. If the spinor wave function for the bound state is u, satisfying $(v \cdot \gamma)u = u$ and $\bar{u}u = 2m_\chi$, transitions between such mesons are clearly proportional to

$\xi(v \cdot v')\,\bar{u}'u$. This procedure of using interpolating fields is equivalent to the usual tensor methods of group theory. While this approach is more intuitive, the tensor methods are perhaps more efficient. Tensor methods are spelt out in the next section, and the results written down.

Tensor Methods

The matrix elements to be evaluated are of the type

$$\langle B(v') | O(v, v') | B(v) \rangle . \tag{64}$$

The operator destroys the initial heavy particle and creates the final one, and is therefore of the form

$$O = \bar{\Psi}_{v'} \Gamma \Psi_{v} . \tag{65}$$

In the effective theory, Γ is simply a matrix, all derivatives on the heavy fields are replaced by the momenta. The Wigner-Eckart theorem implies that if the states and operators are represented by matrices that transform in the appropriate way under the symmetries, then the amplitude must be given by traces of products of these matrices. This technique is widely used in flavor $SU(3)$ for instance.[17)] Here, we have two $SU(3)$ symmetries corresponding to the two different velocities v and v'. The initial state transforms as

$(3, 1)$, the final as $(1, \bar{3})$ and the operator as $(3, \bar{3})$. There is only one possible way to take traces to get a singlet.

The choice of matrices is not unique, but they must transform appropriately under the Lorentz transformations. The 5×4 matrices are defined by[7)]

$$\tilde{\Psi}_P(v) = \begin{bmatrix} \sqrt{m_P}\left(\frac{1 + v \cdot \gamma}{2}\right)\gamma_5 \\ 0 \end{bmatrix} . \qquad (66)$$

$$\tilde{\Psi}_{P^*}(v) = \begin{bmatrix} \sqrt{m_{P^*}}\left(\frac{1 + v \cdot \gamma}{2}\right)(\varepsilon \cdot \gamma) \\ 0 \end{bmatrix} . \qquad (67)$$

$$\tilde{\Psi}_X(v) = \begin{bmatrix} 0 \\ \frac{u^T C}{\sqrt{2m_\chi}} \end{bmatrix} . \qquad (68)$$

Here C is the charge conjugation matrix arising because the antiquark is responsible for the spinorial nature. Here it is required to satisfy $C\gamma_\mu^T C = -\gamma_\mu$. The adjoints are defined by

$$\bar{\Psi} = \gamma_0 \Psi^\dagger \begin{bmatrix} \gamma_0 & 0 \\ 0 & 1 \end{bmatrix} . \qquad (69)$$

The matrix elements are then computed as

$$\langle B'(v')|\, O \,|B(v)\rangle = -\xi(v\cdot v')\, Tr\left[\bar{\Psi}_{B'}\Gamma\Psi_B\right] . \quad (70)$$

It is clear from the structure of the matrices that essentially the same traces are taken in both the interpolating field and the tensorial approaches. Using either method, one gets the following results for the relevant matrix elements in the effective theory.[7)]

$$\langle P_j(v')|\, \bar{h}_j\gamma^\mu h_i \,|P_i(v)\rangle = \xi(v\cdot v')\sqrt{m_j m_i}\,(v^\mu + v'^\mu) . \quad (71)$$

$$\begin{aligned}\langle P^*_j(\varepsilon', v')|\, \bar{h}_j\gamma^\mu h_i \,|P^*_i(\varepsilon, v)\rangle &= -\xi(v\cdot v')\sqrt{m_j m_i} \\ &\left[(\varepsilon'^*\cdot\varepsilon)(v^\mu + v'^\mu) - (\varepsilon'^*\cdot v)\varepsilon^\mu - (\varepsilon\cdot v')\varepsilon'^{\mu^*}\right] .\end{aligned} \quad (72)$$

$$\langle P^*_j(\varepsilon', v')|\, \bar{h}_j\gamma^\mu h_i \,|P_i(v)\rangle = i\xi(v\cdot v')\sqrt{m_j m_i}\,\varepsilon^{\mu\nu\alpha\beta}\varepsilon'^*{}_\nu v_\alpha v'_\beta . \quad (73)$$

$$\begin{aligned}\langle P^*_j(\varepsilon', v')|\, \bar{h}_j\gamma^\mu\gamma^5 h_i \,|P_i(v)\rangle &= \xi(v\cdot v') \\ &\sqrt{m_j m_i}\left[(1 + v\cdot v')\varepsilon'^{*\mu} - v'^\mu(\varepsilon^*\cdot v)\right] .\end{aligned} \quad (74)$$

$$\langle X_j(v')|\, \chi_j^\dagger\chi_k \,|X_k(v)\rangle = \frac{1}{2}\xi(v\cdot v')\sqrt{\frac{1}{m_j m_i}}\,\bar{u}'u . \quad (75)$$

$$\langle X_j(v')|\, \chi_j^\dagger(i\partial^\mu\chi_k) \,|X_k(v)\rangle = \frac{v^\mu}{2}\xi(v\cdot v')\sqrt{\frac{m_k}{m_j}}\,\bar{u}'u . \quad (76)$$

$$\langle X_j(v')|\,(i\partial^{\mu}\chi_j)^{\dagger}\chi_k\,|X_k(v)\rangle = \frac{-v'^{\mu}}{2}\xi(v\cdot v')\sqrt{\frac{m_j}{m_k}}\,\bar{u}'u\,. \qquad (77)$$

In the limit $v \rightarrow v'$, the currents here become symmetry currents generating the flavor symmetry, and to give consistent normalization, $\xi(1)$ must be unity.

In the effective theory, the transition between the mesons are related to the spinorial bound states of the scalar particle by the super-flavor symmetry. If heavy colored scalar triplets were indeed to exist, their decays would thus be related to the decays of the Bottom and Charm mesons. Corrections to this effective field theory picture are dealt with in the next chapter.

4. Running Corrections

In the early days of field theory, the need for renormalization was viewed as a serious deficiency. With the advent of the modern view point,[18] however, it is regarded as both essential and useful.[19] Field theories are viewed as descriptions of physics at a certain energy scale, rather than as universally true. At energies much lower than the heavy particle masses, an effective field theory description, where these particles are completely absent, can be built. Renormalization then provides a way of summing some of their effects. An example is the removal of the heavy quarks from consideration in Kaon decays.[20]

There are two steps in this procedure.[3] First, the particle is 'integrated out', giving a nonlocal action. The coefficients at this stage are influenced by all the energy scales. To remove the dependence of the coefficients on long distance physics, so that this is incorporated explicitly, one must match the two theories at the boundary. Physical quantities are calculated using both the full and the effective theory, and the parameters of the effective theory chosen so that the results are identical at the boundary. This way, only the short distance effects are present in the coefficients at the boundary. The renormalization group can be used to scale these coefficients down to the energy scales of interest.

Renormalization in QCD

In the full theory of QCD, the Lagrangian density in terms of the bare fields (bare quantities are 'hatted') is

$$L = -\frac{1}{4}\hat{G}_{\mu\nu}\hat{G}^{\mu\nu} + \sum_{j=1}^{N_q} i\bar{\hat{q}}_j\gamma^{\mu}D_{\mu}\hat{q}_j + \sum_{j=1}^{N_s} \left(D_{\mu}\hat{\phi}_j^{\dagger}\right)(D^{\mu}\hat{\phi}_j) , \tag{78}$$

where hypothetical colored scalar particles are included, and the covariant derivative is

$$D_{\mu} = (\partial_{\mu} + i\hat{g}\hat{A}_{\mu}T^{a}) , \tag{79}$$

T^a being the color $SU(3)$ generators with the conventional normalization $Tr\left[T^aT^b\right] = \frac{1}{2}\delta^{ab}$. The bare Lagrangian does not lead to finite Green functions in perturbation theory. The procedure of renormalization introduces renormalized fields (unhatted) related to the bare fields by renormalization constants:

$$A^a_\mu = \frac{\hat{A}^a_\mu}{\sqrt{Z_A}} \qquad q_j = \frac{\hat{q}_j}{\sqrt{Z_q}} \qquad \phi_j = \frac{\hat{\phi}_j}{\sqrt{Z_\phi}} \qquad g = \frac{\mu^{-\varepsilon/2}\hat{g}}{Z_g}. \tag{80}$$

Here $\varepsilon = 4 - n$, n being the dimension of space-time. μ is a quantity with dimensions of mass that is introduced to keep the coupling constant dimensionless in all space-time dimensions. The Lagrangian is then written in terms of the renormalized fields and 'counterterms' CT.

$$L = -\frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \sum_{j=1}^{N_q} i\bar{q}_j\gamma^\mu D_\mu q_j + \sum_{j=1}^{N_s} (D_\mu\phi_j^\dagger)(D^\mu\phi_j) + CT. \tag{81}$$

Renormalizability requires the counterterms to be of the same form as the terms in the original Lagrangian.

The counterterms are chosen according to a 'renormalization prescription' (often conditions on the renormalized Green functions) after the infinities in the calculations are controlled using a 'regularization scheme' (that usually modifies the physics at short distances). In QCD, 'dimensional regularization' with the 'minimal subtraction' prescription is used.[22)] In this scheme, the infinities are isolated as poles in ε, and the counterterms are constructed by removing just these poles (as opposed to subtracting further finite quantities). In this scheme, the renormalization constants are of the form

$$Z(g, \varepsilon) = \sum_{p=1}^{\infty} \frac{Z^{(p)}(g)}{\varepsilon^p} . \qquad (82)$$

An important advantage of this scheme is that it is independent of the masses.[19] This is clear by dimensional analysis. Since the renormalization constants are dimensionless, they can depend on the masses only via μ/m. But μ enters only through logarithms, $\mu^\varepsilon = 1 + \varepsilon \log \mu$, and hence the dependence must be on $\log(\mu/m)$. But the constants are regular for vanishing mass, and therefore must not depend on the masses at all.

As an example, look at the renormalization of the colored scalar field in QCD. We need to evaluate the two-point function, with the external legs amputated. The graph is infrared convergent, but has a quadratic ultraviolet divergence.

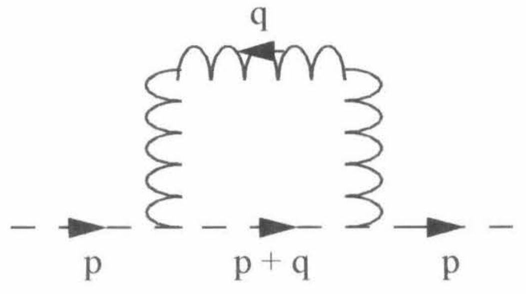


FIGURE 4. Self energy of a colored scalar.
Infrared convergent, but quadratically ultraviolet divergent.

Using the Feynman rules, the off-shell self-energy is

$$\Sigma = \left(-\frac{4g^2}{3}\right)\int \frac{d^n q}{(2\pi)^n} \frac{(2p+q)^2}{q^2 (p+q)^2} . \tag{83}$$

Using standard techniques, (only the pole term need be evaluated) this evaluates to

$$\Sigma = -i\frac{g^2}{3\pi^2}\frac{p^2}{\varepsilon} . \tag{84}$$

Equating this to $ip^2 (Z_\phi - 1)$, we get the field renormalization constant

$$Z_\phi = 1 + \frac{g^2}{3\pi^2}\frac{1}{\varepsilon} . \tag{85}$$

By calculating the renormalization of the scalar, quark and gluon fields, the dependence of the renormalized coupling on the subtraction point can be calculated. This is expressed in terms of the β function defined by

$$\beta(g, \varepsilon) = \frac{\partial g}{\partial(\log \mu)} . \quad (86)$$

This has been calculated to be[1,23]

$$\beta(g) \equiv \lim_{\varepsilon \to 0} \beta(g, \varepsilon) = -\left(33 - 2N_q - \frac{N_s}{2}\right)\frac{g^3}{48\pi^2} . \quad (87)$$

Instead of specifying a subtraction point, and the value of the coupling at that subtraction point, it is conventional to integrate this equation and write

$$\alpha_s(q^2) \equiv \frac{g^2}{4\pi} = \frac{12\pi}{\left(33 - 2N_q - \frac{N_s}{2}\right)\log\left(\frac{q^2}{\Lambda_{QCD}^2}\right)} , \quad (88)$$

where the integration constant Λ_{QCD} is the only mass scale of the theory. The actual value of this scale depends on the renormalization scheme, and is of the order of $200\,MeV$. This is completely independent of the masses of the quarks and scalars, or indeed whether these particles have masses at all.

The vanishing of the coupling at large energies ('asymptotic freedom') together with the renormalization group enables improvement of perturbative calculations. Thus, a perturbative calculation that yields a result of the form

$$\Gamma_R^{(N)} = g(\mu)^{N/2-1} f(p_{ext}) \sum_{n=0}^{\infty} g^n \sum_{l=0}^{n} a_{n,p} \left(\log \frac{p_{ext}}{\mu} \right)^l \quad (89)$$

is improved by choosing the subtraction point to be of the order of the external momenta. Formally, the Callan-Symanzik equations, which express the invariance of such physical quantities under a change in the subtraction point, sum the 'leading logarithms' in such an expansion.

Renormalization in the Effective Theory

In the effective theory, in addition to the QCD Lagrangian for the light particles (78), one must include the Lagrangians (10) and (25) for the heavy particles. Once again, one can write these as sums of renormalized Lagrangians and counterterms.

$$L_v = \sum_{j=0}^{N_S} 2im\chi_v^{j\dagger} v \cdot D\chi_v^j + \sum_{k=0}^{N_Q} \bar{h}_v^k i (v \cdot D) h_v^k + CT . \quad (90)$$

In the full theory, the renormalization constants are mass independent. But the finite parts of Green functions can have logarithmic dependence on the masses. In the effective theory where these masses are taken to infinity, these terms can cause divergences. Thus, the renormalization constants must be calculated afresh in the effective theory.

The first quantity to evaluate is the field renormalization constant. The superflavor symmetry implies that this constant is identical for both the heavy quark and the heavy scalar. (This is also clear from the Feynman rules). The relevant graph is shown below.

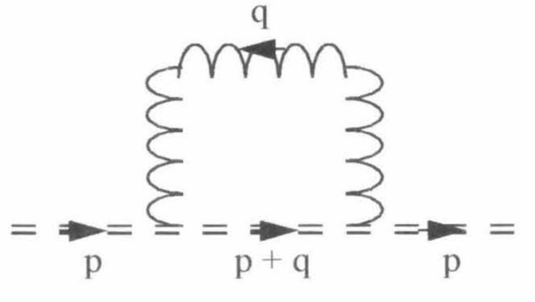


FIGURE 5. Self energy of a heavy colored particle.
Infrared convergent, but linearly ultraviolet divergent.

Using the Feynman rules for the effective theory (Figure 2 on page 13), the self energy in this case is

$$\Sigma = \left(-\frac{4g^2}{3}\right)\int \frac{d^n q}{(2\pi)^n} \frac{1}{q^2 [v \cdot (p+q)]} . \tag{91}$$

There is a slight subtlety that arises when this is evaluated using the usual Feynman method for combining denominators. To begin with, there is no infrared divergence. However, when the denominators are combined using the usual Feynman parametrization

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[ax + b(1-x)]^2} , \tag{92}$$

an infrared divergence is introduced. The final integral over the Feynman parameter reads

$$\int_0^1 dx\,(1-x)^{3-n}\,(x)^{n-5}\,. \qquad (93)$$

The divergence that occurs at $x = 0$ is an infrared divergence since it occurs for $n \leq 4$. This can be isolated by putting $x = 1$ in the second factor. Another approach is to combine the denominators differently. The Feynman method seems to work best when the denominators have equal powers of the momenta. Here, combining denominators using the identity[3)]

$$\frac{1}{ab} = \int_0^\infty \frac{d\lambda}{[a+b\lambda]^2} \qquad (94)$$

eliminates the problem. The result is

$$\Sigma = -(2im)\,\frac{g^2}{3\pi^2}\,\frac{(v\cdot k)}{\varepsilon}\,. \qquad (95)$$

In the scalar case, the counterterm is $2im\,(Z_\chi - 1)\,\chi^\dagger\,(v\cdot D)\,\chi$, and we get the renormalization constant to be

$$Z_\chi = 1 + \frac{g^2}{3\pi^2}\,\frac{1}{\varepsilon}\,. \qquad (96)$$

It seems to be a coincidence that the renormalization constants are identical for the full scalar and the heavy scalar.

The problem encountered here is in fact a general problem with dimensional regularization. First, since there is no explicit cutoff, the degree of divergence is not apparent. Next, the divergence can equally well arise from short distance or long distance effects. This can be seen using the following argument.[3)] A general integral

$$I = \int d^4 l \frac{1}{(l^2 + A^2)^{\alpha}} \tag{97}$$

with l a typical loop momentum and A a function of external momenta, is split up in $n = 4 + \delta$ dimensions as

$$I_{\delta} = \int d^4 l \int d^{\delta} l \frac{1}{(l_{\delta}^2 + l^2 + A^2)^{\alpha}} . \tag{98}$$

The integral over the δ dimensions can be done, giving

$$I_{\delta} = \int d^4 l \frac{1}{(l^2 + A^2)^{\alpha}} r(\delta) \left(\frac{l^2 + A^2}{\mu^2} \right)^{\delta/2} , \tag{99}$$

where $\lim_{\delta \to 0} r(\delta) = 1$. Thus, I_δ differs considerably from I if the momenta are significantly larger than the subtraction point (the ultraviolet divergence), and also if they are significantly smaller (the infrared). Keeping the external momenta nonzero is one way to avoid the latter in anomalous dimension calculations (in matching calculations, the infrared divergences are common to both theories, and are automatically removed).

At first sight, it might seem that the gluon field renormalization needs to be recalculated in the effective theory. However, in the effective theory, heavy loops do not occur, because there is no pair production. Therefore there is no heavy particle contribution to the gluon field renormalization. As a consequence, the β function is still given by (87), with only the light quarks and scalars in the theory being included in it.

Composite Operators

In the previous chapter, the matrix elements of several composite operators (those which involve a product of elementary fields at a point) were related in the effective theory. In general, renormalizing the component fields is *not sufficient* to renormalize the operator, indeed, the very definition of a product of fields at a point is subtle in a continuum field theory; further, different operators may mix under renormalization. In the case of quarks, the vector and axial vector currents do not get renormalized in QCD. This in fact follows from general considerations. These are conserved currents associated with the chiral symmetry of the Lagrangian, and thus are related to the charges that generate this symmetry. The normalization of charges is fixed however (since the commutation rules are nonlinear), and related to physically measurable observables. Thus, conserved currents do not get renormalized. For chiral currents, this is in fact true even if the chiral symmetry is broken by mass terms, since the renormalization constants are mass independent. The operator $\bar{q}_j q_k$ is an example of an operator that requires renormalization in QCD.

An operator that does not require renormalization in QCD might still have terms of the type $\log (m_{heavy}/\Lambda_{QCD})$ in its matrix elements. In the effective theory, where the heavy masses are considered infinite, such terms result in divergences. Thus, the corresponding operator in the effective theory will require renormalization. The renormalization of the operator is defined in general by*

$$O_{bare} = Z_O O_{ren} . \qquad (100)$$

The operator renormalization constant Z_O is chosen so that Green functions with the insertion of the renormalized operator satisfy appropriate conditions. In the dimensional regularization with minimal subtraction scheme, they are required to be made finite by removal of poles in ε. This would give a series of the same form as for the other renormalization constants,

$$Z_O(g, \varepsilon) = \sum_{p=1}^{\infty} \frac{Z_O^{(p)}(g)}{\varepsilon^p} . \qquad (101)$$

A bilinear operator O involving fields ϕ_1 and ϕ_2 therefore satisfies

$$\begin{aligned} O_{bare}(\phi_1^{bare}, \phi_2^{bare}) &= \sqrt{Z_1}\sqrt{Z_2} O_{bare}(\phi_1^{ren}, \phi_2^{ren}) \\ &= Z_O O_{ren} \end{aligned} . \qquad (102)$$

*. A different convention, with the inverse of Z in the same position as Z in the defining equation, is used in ref. [13]. This changes the sign of the anomalous dimension term in the Callan-Symanzik equation (108), but does not alter the results.

Note that after renormalization, insertion of a O_{ren} is not equivalent to inserting the appropriate product of fields. For Green functions with L insertions of the operator, and N fields, we have

$$G_{ren}^{(N,L)} = \left(\prod_{i=1}^{N} \frac{1}{\sqrt{Z_i}}\right) (Z_O)^{-L} G_{bare}^{(N,L)} , \qquad (103)$$

while for the one particle irreducible functions we have

$$\Gamma_{ren}^{(N,L)} = \left(\prod_{i=1}^{N} \sqrt{Z_i}\right) (Z_O)^{-L} \Gamma_{bare}^{(N,L)} . \qquad (104)$$

These facts have implications for the symmetry relations derived in the previous chapter. There we computed the form of matrix elements of composite operators built of heavy fields. The physical quantities however involve matrix elements in the full theory of QCD. Consider for instance the operator $O(\phi_1, \phi_2)$ in QCD. Now, in an effective theory where the first particle is considered infinitely heavy, it is logical to assume that the relevant operator is $O^1(h_1, \phi_2)$, that is, of the same functional form but with the heavy field instead of the full field. The latter operator however requires renormalization, and hence acquires a subtraction point dependence. Thus, the two operators may be identified at at most one energy scale. Typically, this is chosen to be the mass of the particle. Thus,

$$O_{QCD} = C(\mu)\, O^{1}_{eff}(\mu) \ , \tag{105}$$

where the coefficient function satisfies the boundary condition

$$C(m_1) = 1 \tag{106}$$

up to terms of the order of $\alpha_s(m_1)$. At this boundary, the matrix elements of the operator have large logarithms of the form $\log(m_1/\Lambda_{QCD})$. The renormalization group allows these logarithms to be transferred to the coefficient functions from the operator. Using the connection between the renormalized and bare operators, we have

$$O_{QCD} = C(\mu)\, \frac{O^{bare}_{eff}}{Z_O} \ . \tag{107}$$

The subtraction point independence of the bare operator in the effective theory and the operator in QCD implies the renormalization group equation

$$\mu \frac{d}{d\mu} C(\mu) - \gamma_O C(\mu) = 0 \ , \tag{108}$$

where the anomalous dimension of the operator is defined by

$$\gamma_O \equiv \mu \frac{d}{d\mu} (\log Z_O) \ . \tag{109}$$

The anomalous dimension is thus calculable in perturbation theory as a function of the coupling constant, and the solution to the equation is

$$C(\mu) = \left[exp(\int_{g(m_1)}^{g(\mu)} dg \frac{\gamma_O(g)}{\beta(g)}) \right] C(m_1) \ . \qquad (110)$$

Next, an effective theory where the second particle is also infinitely heavy can be considered. Once again, the operators are matched at the boundary, the mass of the particle,

$$O^1_{eff}(m_2) = C(\mu) O^{1,2}_{eff}(\mu) \ , \qquad (111)$$

where now the operator $O^{1,2}_{eff}$ is got by using the same functional form with both the fields heavy. Note that the boundary condition for the coefficient is now got from equation (110). Thus, the coefficient at low energies is

$$C(\mu) = C(m_1) \mathrm{x}$$

$$\left[exp(\int_{g(m_2)}^{g(\mu)} dg \frac{\gamma_{O^{1,2}}(g)}{\beta(g)}) \right] \left[exp(\int_{g(m_1)}^{g(m_2)} dg \frac{\gamma_{O^1}(g)}{\beta(g)}) \right] . \qquad (112)$$

The corrections calculated here can be combined with the symmetry relations of the previous chapter, to make predictions about physically measurable transitions. Naively, the effective field theory approach suggests a relationship of the form

$$\langle B_1 | O_{QCD} | B_2 \rangle = \xi (v_1 \cdot v_2) \, Tr\left[\bar{B}_1 \Gamma B_2 \right] \tag{113}$$

if both bound states have heavy quarks, with the trace being of appropriate matrices in the formalism. The corrections here imply that the Isgur-Wise function can be factorized into two bits, one of which has no dependence on the actual masses at all, while the other has a calculable logarithmic dependence on the masses. That is,

$$\langle B_1 | O_{QCD} | B_2 \rangle = C\left(v_1 \cdot v_2, m_1, m_2, \mu \right) \xi_0 (v_1 \cdot v_2, \mu) \, Tr\left[\bar{B}_1 \Gamma B_2 \right] . \tag{114}$$

The function ξ_0 is truly universal since it has no dependence on the masses.

There is an additional subtlety that can occur in operator renormalization. It is possible that the operators mix under renormalization, thus requiring the operator renormalization constant to be a matrix. A particular choice of an operator basis can then ensure independent multiplicative renormalization. This in fact occurs in the case of the heavy scalar. The coefficient function has been evaluated for heavy quarks[21]. In this thesis, it is evaluated for hypothetical heavy scalars[13].

Low Energy Running

The low energy (that is, lower than both the masses) running is particularly interesting because it turns out to be velocity dependent. We need to evaluate the following graph for insertions of the operators $\chi^{\dagger}{}_2\chi_1$, $\chi^{\dagger}{}_2(i\partial^{\mu}\chi_1)$, and $(i\partial^{\mu}\chi^{\dagger}{}_2)\chi_1$; these are the natural ones to consider. .

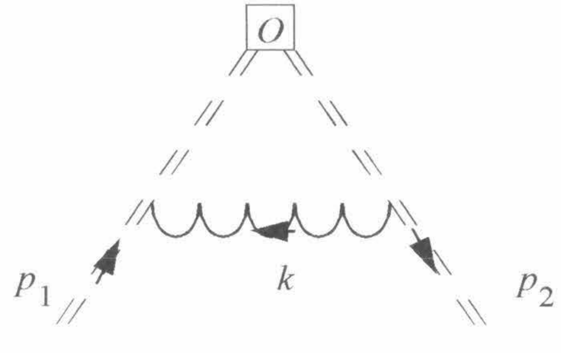


FIGURE 6. Low energy running for operator O.
Both fields are considered heavy. All the three operators have proportional matrix elements.

When a derivative acts on a heavy field, it is equivalent to inserting the operator without the derivative, but multiplying by the momentum. Thus, the latter two operators give the same result as the first, but multiplied by $m_1 v_1^{\mu}$ and $-m_2 v_2^{\mu}$ respectively.

Using the Feynman rules (Figure 2 on page 13), the value of the graph for an insertion of $\chi^{\dagger}{}_2\chi_1$ is

$$\left(-i\frac{4}{3}g^2\right)(v_1 \cdot v_2)\int\frac{d^4k}{(2\pi)^4}\frac{1}{(v_1 \cdot k)\,(v_2 \cdot k)\,k^2}\,. \tag{115}$$

This has both an ultraviolet and an infrared logarithmic divergence. Combining the denominators using

$$\frac{1}{abc} = 2\int_0^\infty d\lambda\int_0^\infty d\kappa\frac{1}{[a + b\lambda + c\kappa]^3}\,, \tag{116}$$

this becomes

$$\left(-i\frac{4}{3}g^2\right)(v_1 \cdot v_2)\,2\int_0^\infty d\lambda\int_0^\infty d\kappa\int\frac{d^4k}{(2\pi)^4}\frac{1}{\left[k^2 + (v_1 \cdot k)\,\lambda + (v_2 \cdot k)\,\kappa\right]^3}\,. \tag{117}$$

The integral over the momentum can be done in the usual way, using the formula

$$\int \frac{d^D k}{(2\pi)^D} \frac{(k^2)^m}{(k^2 - a^2)^n}$$
$$= \frac{i(-1)^{m-n}}{(16\pi^2)^{D/4}} \frac{\Gamma(m+\frac{D}{2})}{\Gamma(\frac{D}{2})} \frac{\Gamma(n-m-\frac{D}{2})}{\Gamma(n)} \left(\frac{1}{a^2}\right)^{n-m-\frac{D}{2}} \quad (118)$$

to give

$$\left(-i\frac{4}{3}g^2\right)(4v_1 \cdot v_2)\left(\frac{-i}{16\pi^2}\right)\int_0^\infty d\lambda \int_0^\infty d\kappa \frac{1}{\left[\kappa^2 + \lambda^2 + (v_2 \cdot v_1)\,2\kappa\lambda\right]^{1+\varepsilon}} . \quad (119)$$

It is convenient to change to polar coordinates, $\kappa = \rho\sin\theta$, and $\lambda = \rho\cos\theta$, so that we can use

$$\int_0^{\pi/2} \frac{d\theta}{(1 + p\sin 2\theta)} = \frac{\log\left(p + \sqrt{p^2 - 1}\right)}{\sqrt{p^2 - 1}} \equiv r(p) . \quad (120)$$

Then the integral evaluates to

$$\left(-i\frac{4}{3}g^2\right)(4v_1 \cdot v_2)\left(\frac{-i}{16\pi^2}\right) r(v_1 \cdot v_2)\int_0^\infty d\rho\rho^{D-5} . \quad (121)$$

The divergence at infinity is the ultraviolet divergence (it occurs for $D \geq 4$), the one at zero the infrared (occurs for $D \leq 4$). The infrared divergence could have been avoided by putting a fictitious mass. Thus, the Feynman graph evaluates to

$$-\left(\frac{16}{3}\right)\frac{g^2}{16\pi^2}(v_1 \cdot v_2)\, r(v_1 \cdot v_2)\,. \qquad (122)$$

The superflavor symmetry implies that the same result is got for heavy quarks. Any spin structure matrix simply multiplies this result because of the spin symmetry. It follows from this that

$$\begin{aligned} Z_O &= \frac{Z_\chi}{1 + \dfrac{g^2}{3\pi^2\varepsilon}(v_1 \cdot v_2)\, r(v_1 \cdot v_2)} \\ &= 1 - \frac{g^2}{3\pi^2\varepsilon}\left[v_1 \cdot v_2 r(v_1 \cdot v_2) - 1\right] \end{aligned} \qquad (123)$$

Suppose that $Z_O = 1 + \dfrac{ag^2}{\varepsilon}$, the anomalous dimension is $\gamma_O = -ag^2$, since $\beta(g, \varepsilon) = [-\varepsilon g/2 + \beta(g)]$. With $\beta(g) = -bg^3$, the renormalization group equation (110) is easily solved. The factors that appear are of the form $\left[\alpha_s(\mu)/\alpha_s(m)\right]^{a/(2b)}$. Thus, in this case, the running is of the form

$$\left[\frac{\alpha_s(m_2)}{\alpha_s(\mu)}\right]^{\frac{8\,[v_1\cdot v_2 r(v_1\cdot v_2)-1]}{33-2N_q-N_s/2}} . \qquad (124)$$

Since $\lim_{x\to 1} r(x) = 1$, there is no running in the limit that the velocities are equal. This is to be expected; in this limit, the current is a flavor symmetry current. All the operators in the scalar and spinor cases exhibit the same running at low energies.

Intermediate Energy

For energies between the two masses, the first particle is considered to be infinitely heavy, while the second is treated as usual. To calculate the running in this region, the following graph needs to be evaluated.

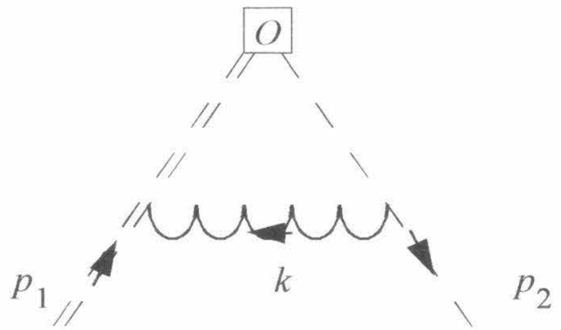


FIGURE 7. Intermediate energy running for operator O. The first field is treated in the effective theory, while the second is an usual QCD field.

First consider the operator $\phi_2^{\dagger}\chi_1$. The Feynman rules give us

$$\left(-i\frac{4}{3}g^2\right)\int\frac{d^4k}{(2\pi)^4}\frac{v_1\cdot(2m_2v_2+k)}{(v_1\cdot k)\,(k^2+2m_2v_2\cdot k)\,k^2}\ . \qquad (125)$$

Once again, there is both an ultraviolet and an infrared logarithmic divergence, but here they arise from different terms. Ignoring the ultraviolet finite part, and combining the denominators we have

$$\left(-i\frac{4}{3}g^2\right)\int_0^1 dx\int\frac{d^4k}{(2\pi)^4}\frac{1}{\left[k^2+2xm_2v_2\cdot k\right]^2}. \tag{126}$$

This turns out to be

$$\left(-i\frac{4}{3}g^2\right)\left(\frac{i}{16\pi^2}\right)\left(\frac{2}{\varepsilon}\right). \tag{127}$$

Thus, using equations (85) and (96), we get

$$\begin{aligned} Z_O &= \frac{\sqrt{Z_\phi Z_\chi}}{1-\dfrac{g^2}{6\pi^2\varepsilon}} \\ &= 1+\frac{g^2}{2\pi^2\varepsilon} \end{aligned} \tag{128}$$

The intermediate energy running of this operator is

$$\left[\frac{\alpha_s(m_1)}{\alpha_s(m_2)}\right]^{\frac{-12}{33-2N_q-N_s/2}}. \tag{129}$$

Now there is a distinction between the theories with quarks and scalars. The superflavor symmetry relates only the first particle. Thus, the graph evaluates differently, and the result for the intermediate energy running has a 6 instead of the 12 here.[21)]

The graph with an insertion of $\phi_2^\dagger (i\partial^\mu \chi_1)$ is simply $m_1 v_1^\mu$ times that with an insertion of $\phi_2^\dagger \chi_1$. The remaining graph, with an insertion of $(i\partial^\mu \phi^\dagger{}_2)\,\chi_1$ is

$$\left(-i\frac{4}{3}g^2\right)\int\frac{d^4k}{(2\pi)^4}\frac{v_1\cdot(2m_2v_2+k)\,(m_2v_2+k)^\mu}{(v_1\cdot k)\,(k^2+2m_2v_2\cdot k)\,k^2}\,. \tag{130}$$

This has an ultraviolet divergent part proportional to

$$(p_1\cdot p_2)\int\frac{d^4k}{(2\pi)^4}\frac{k^\mu}{(p_1\cdot k)\,(k^2+2p_2\cdot k)\,k^2}\,. \tag{131}$$

Since the ultraviolet divergence in this integral is unaltered by equating p_2 to zero, the integral must be proportional to the momentum p_1, which is the only momentum left. The renormalization of this operator therefore requires a counterterm proportional to $\phi_2^\dagger (i\partial^\mu \chi_1)$.

The linear ultraviolet divergence proportional to

$$\int \frac{d^4k}{(2\pi)^4} \frac{k^\mu}{(k^2 + 2p_2 \cdot k)\, k^2} \tag{132}$$

may be evaluated using the standard methods:

$$\int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{k^\mu}{(k^2 + 2xp_2 \cdot k)} = \frac{-i}{16\pi^2 \varepsilon} (m_2 v_2)^\mu . \tag{133}$$

One term proportional to $m_2 v_2^\mu$ has been evaluated in the previous case (equation (127)). The ultraviolet divergence of the mixing term (131) may be evaluated as

$$\begin{aligned} &\left(4m_2 v_1 \cdot v_2\right) \int \frac{d^4k}{(2\pi)^4} \frac{k^\mu}{(2v_1 \cdot k)\,(k^2 + 2p_2 \cdot k)\, k^2} \\ &= \left(8m_2 v_1 \cdot v_2\right) \int_0^1 dx \int_0^\infty d\lambda \int \frac{d^4k}{(2\pi)^4} \frac{k^\mu}{(k^2 + 2xm_2 v_2 \cdot k + 2\lambda v_1 \cdot k)^3} . \end{aligned} \tag{134}$$

Setting $m_2 = 0$ in the denominator to isolate the ultraviolet divergence gives

$$\frac{i}{16\pi^2 \varepsilon} (v_1 \cdot v_2)\, 4m_2 v_1^\mu . \tag{135}$$

It is significant that the mixing is proportional to m_2 rather than m_1. The total ultraviolet divergence of the graph is

$$i\frac{4g^2}{3}\frac{i}{16\pi^2\varepsilon}\{m_2 v_2^{\mu} + 4m_2 v_1^{\mu}(v_1 \cdot v_2)\}. \quad (136)$$

The mixing requires a matrix of renormalization constants. Thus, we have

$$\begin{bmatrix} O^1_{bare} \\ O^2_{bare} \end{bmatrix} = Z \begin{bmatrix} O^1_{ren} \\ O^2_{ren} \end{bmatrix}, \quad (137)$$

where Z is a two dimensional matrix of the form

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}. \quad (138)$$

Similarly, we choose a coefficient matrix

$$\begin{bmatrix} O_1^{QCD} \\ O_2^{QCD} \end{bmatrix} = \underset{\sim}{C}(\mu) \begin{bmatrix} O_1^{eff} \\ O_2^{eff} \end{bmatrix}. \quad (139)$$

Using the same procedure (all entries are of the same order, being one loop results)

$$\underset{\sim}{Z}^{-1} = \underset{\sim}{I}_2 - \frac{g^2}{12\pi^2\varepsilon}\begin{bmatrix} 6 & 0 \\ -r & 5 \end{bmatrix}, \tag{140}$$

where the dimensionless ratio r is

$$r = 4\,(v_1 \cdot v_2)\,\frac{m_2}{m_1}\,. \tag{141}$$

As a consequence, we have the matrix equation

$$\mu\frac{d}{d\mu}\underset{\sim}{C}(\mu) = \frac{g^2}{12\pi^2}\underset{\sim}{C}(\mu)\begin{bmatrix} 6 & 0 \\ -p & 5 \end{bmatrix} \tag{142}$$

with the boundary condition

$$\underset{\sim}{C}(m_1) = \underset{\sim}{I}_2\,. \tag{143}$$

This is most conveniently solved by going to a basis in which the anomalous dimension matrix is diagonal, and then reverting to the original basis. Then, we get the result

$$\underset{\sim}{C}(\mu) = \alpha_\mu^{e_1}\begin{bmatrix} \alpha_r^{e_2} & 0 \\ r\left(\alpha_r^{e_3} - \alpha_r^{e_2}\right) & \alpha_r^{e_3} \end{bmatrix}. \tag{144}$$

the coupling constant ratios are

$$\alpha_\mu = \left(\frac{\alpha_s(m_2)}{\alpha_s(\mu)}\right)$$
$$\alpha_r = \left(\frac{\alpha_s(m_1)}{\alpha_s(m_2)}\right) \tag{145}$$

and the exponents

$$e_1 = \frac{8\{v_1 \cdot v_2 r(v_1 \cdot v_2) - 1\}}{33 - 2N_q - \frac{N_s}{2}}$$
$$e_2 = \frac{-12}{33 - 2N_q - \frac{N_s}{2}} \tag{146}$$
$$e_3 = \frac{-10}{33 - 2N_q - \frac{N_s}{2}}$$

Thus, the mixing is not necessarily small. The factor of r in the off-diagonal term ensures that the operator $(i\partial^\mu \phi_2)\phi_1$ has a coefficient of order m_2 rather than m_1.

5. Conclusion

The effective field theory approach provides a model independent way of making predictions based on symmetry in the case of heavy quarks and scalars. Further, corrections to these relations can also be treated systematically. Experimentally, this approach is most significant for semileptonic decays. Thus, using the results of the previous chapters, the relevant relations for the decays $\bar{B} \rightarrow D e \bar{\nu}_e$ and $\bar{B} \rightarrow D^* e \bar{\nu}_e$ are

$$\frac{\langle D(v')|\, \bar{c}\gamma_\mu b\, |\bar{B}(v)\rangle}{\sqrt{m_B m_D}} = C_{cb}\xi(v \cdot v')\,(v + v')_\mu$$

$$\frac{\langle D^*(v',\varepsilon)|\, \bar{c}\gamma_\mu\gamma_5 b\, |\bar{B}(v)\rangle}{\sqrt{m_B m_{D^*}}} = C_{cb}\xi(v \cdot v')\mathrm{x} \qquad (147)$$

$$\left[\, (1 + v \cdot v')\,\varepsilon_\mu^{\;*} - (\varepsilon^* \cdot v)\, v_\mu{}' \,\right]$$

$$\frac{\langle D^*(v',\varepsilon)|\, \bar{c}\gamma_\mu b\, |\bar{B}(v)\rangle}{\sqrt{m_B m_{D^*}}} = iC_{cb}\xi(v \cdot v')\varepsilon_{\mu\rho\alpha\beta}\varepsilon^{*\rho}v'^{\alpha}v^{\beta}$$

The subtraction point dependence of the Isgur-Wise function is cancelled by that of the coefficient function, which here is,

$$C_{cb} = \left(\frac{\alpha_s(m_b)}{\alpha_s(m_c)}\right)^{-\frac{6}{25}} \left(\frac{\alpha_s(m_c)}{\alpha_s(\mu)}\right)^{\frac{8\,[v \cdot v' r(v \cdot v') - 1]}{27}} \qquad (148)$$

The analogous relations for the case of scalars have been calculated here. Consider two elementary colored scalar particles of masses m_1 and m_2, the first being more massive. At low energies, the matrix element of the simplest current is

$$\langle X_2(v')|\,\chi_2^{\dagger}\chi_1\,|X_1(v)\rangle = C(\mu)\frac{1}{2}\xi(v\cdot v')\sqrt{\frac{1}{m_2 m_1}}\bar{u}'u\,, \qquad (149)$$

where the function is now*

$$C = \left(\frac{\alpha_s(m_1)}{\alpha_s(m_2)}\right)^{-\frac{12}{33-2N_q-\frac{N_s}{2}}}\left(\frac{\alpha_s(m_2)}{\alpha_s(\mu)}\right)^{\frac{8[v\cdot v' r(v\cdot v')-1]}{33-2N_q-\frac{N_s}{2}}}\,. \qquad (150)$$

The same Isgur-Wise function appears, and is normalized to unity at the threshold $v = v'$. The derivative operators mix, as demonstrated in the last chapter. That is,

$$\begin{bmatrix}\langle X_2(v')|\,\chi_2^{\dagger}(i\partial^{\mu}\chi_1)\,|X_1(v)\rangle \\ \langle X_2(v')|\,(i\partial^{\mu}\chi_2)^{\dagger}\chi_1\,|X_1(v)\rangle\end{bmatrix} = \underset{\sim}{C}(\mu)\begin{bmatrix}\frac{v^{\mu}}{2}\xi(v\cdot v')\sqrt{\frac{m_1}{m_2}}\bar{u}'u \\ \frac{-v'^{\mu}}{2}\xi(v\cdot v')\sqrt{\frac{m_2}{m_1}}\bar{u}'u\end{bmatrix}, \qquad (151)$$

---

*. There are minor errors in the equations corresponding to (150) and (144) in ref. [13]; the exponents have wrong factors there.

where $\underset{\sim}{C}(\mu)$ is now the matrix in equation (144). Thus, if heavy colored scalar particles do exist, the decays of their bound states are related to the decays of the heavy mesons.

These results can be further improved by including corrections of higher order in α_s (accessible to perturbation theory) as well terms suppressed by powers of Λ_{QCD}/m_h that have been ignored in the effective theory. The latter corrections are not accessible to perturbation theory, and in general may have complicated effects. Data thus far seems to indicate that such corrections are often not large. Further, in some special situations they are not relevant, say at zero recoil[24,25)] or for decays of the Λ_b or the Λ_c baryons.[26)] Another possible consideration is the ratio of the masses of the heavy particles. There are arguments[27)] that suggest that when this ratio is not very large, it is more accurate not to have an intermediate energy theory. Instead, the low energy theory has matching corrections that depend on this ratio.

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