THE INFLUENCE OF VARIABLE AIR DENSITY AND OF NON-LINEAR AERODYNAMIC CHARACTERISTICS ON DYNAMIC BEHAVIOR AT SUPERSONIC SPEEDS

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Telford Wilbert Oswald

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ABSTRACT

The effect of the variable density of the Standard Atmosphere on the dynamic stability of a missile in vertical flight is considered. The analysis is restricted to small disturbances from steady rectilinear flight. The exponential decrease of density with altitude characteristic of the Standard Stratosphere is introduced into the equations of motion and a stability criterion for the dynamic behavior immediately following a small disturbance is found. Alternatively, a hyperbolic variation of density with altitude is used to approximate the Standard Atmosphere and the identical stability criterion is obtained.

The effect of non-linear pitching moment and lift variations with angle of attack on the dynamic response to a sudden change in angle of attack is considered. An approximate solution to the non-linear equation of motion is developed. Several numerical examples are considered, and the results of the approximate solution are compared with the very accurate results of numerical integration as well as the classical linearized solution. The effect of a non-linear moment curve on the determination of stability derivatives from flight test data is discussed in the light of these examples.

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I. INTRODUCTION

The solutions to the equations of motion of an airplane when subjected to small disturbances from steady flight have comprised a standard part of the curriculum for students of aeronautics during the past several years. The classical treatment of the problem is due largely to G. H. Bryan who applied the theory of small oscillations previously developed by Routh. Bryan assumed that the forces and moments due to a slight disturbance from a state of equilibrium depend linearly on the disturbance. The solution is then shown to depend on a number of constants called stability derivatives. For many years, the development of experimental techniques for determining these derivatives has been the principal interest of investigators in the field of dynamic stability as applied to aeronautics. Until very recently there has been little or no change in the theory and it is in use today essentially in the form developed by Bryan.

It should be noted that the classical theory applies strictly only in the case of infinitesimal disturbances. However, the results have been found to apply with reasonable accuracy to finite disturbances of the magnitude experienced in the flight of conventional subsonic airplanes. In other words, the second order terms neglected in the theory do not have any appreciable effect on the motion.

With the advent of the high speed jet airplane and missiles, the assumptions of the classical theory are violated to such an extent that the applicability of the theory is questionable. In the case of supersonic rocket-powered missiles these violations are particularly flagrant. In the first place, such missiles rarely fly at constant speed; on the contrary, the usual flight comprises a period of very high acceleration followed by a less rapid deceleration. Secondly, the complete fuel supply is often consumed in a very short time, giving rise to appreciable variations of mass and center of gravity location. Thirdly, a portion of the flight is sometimes vertical in direction, hence the atmospheric density is rapidly varying. For these reasons, the aerodynamic forces and moments - the stability derivatives - are subject to appreciable variation during the period of the oscillation being studied.

But perhaps the most serious deviation from the conditions of the classical theory occurs when one attempts to extend the results to the case of finite disturbances. In most missile configurations, small aspect ratio wings are mounted in tandem on proportionately large diameter bodies. This exaggerates the aerodynamic interference effects and such missiles exhibit markedly non-linear dependence of the aerodynamic forces and moments on the angle of attack. Consequently, the amplitude of disturbances for which one would expect the classical theory to be valid is considerably reduced. But at the same time, in the normal operation of such missiles, large sudden changes in the angle of attack are frequently required.

Hence it would appear that, in the case of guided missiles, the extension of the theory of small oscillations to finite disturbances is very questionable.

The effect of variable speed on dynamic stability has been investigated by H. J. Stewart (Reference 1). He considered the case of a missile in coasting flight, decelerating under the influence of drag. His results show that although the effective damping is decreased in this case, the drag does not lead to instability.

The effect of variation of mass on the dynamic stability of jet propelled missiles has been investigated by N. V. Barton (Reference 2). His analysis shows that the disturbance of a missile with decreasing mass damps out more rapidly than it does for a constant mass missile, indicating that variable decreasing mass is more stable. Also the oscillation frequency of the variable mass missile increases with time and is greater than the oscillation frequency of the constant mass missile.

In Part IV of the present paper, the effect on the longitudinal dynamic stability of the variation of atmospheric density in vertical flight is considered. The analysis is restricted to the case of infinitesimal disturbances so that the forces and moments depend linearly on the angle of attack. The stability derivatives are, however, functions of the independent variable, and the problem is reduced to the solution of a linear differential equation with non-constant coefficients, for which the exact solution can be obtained by the methods of mathematical analysis.

In Part V of the present paper, the effect of non-linear variation of lift and pitching moment with angle of attack on the dynamic response to a step function input in angle of attack is investigated. The missile is assumed to be flying at constant speed and altitude. The problem is reduced to an ordinary second order non-linear differential equation which cannot be solved exactly by methods known today. However, approximate solutions may be obtained, and by the method of numerical integration a solution to any desired degree of accuracy can be obtained.

The stability of the non-linear system is determined from considerations of the solution in the neighborhood of the critical points of the differential equation, and the investigation of the dynamic response is confined to systems which have been previously determined to be stable.

In its application to a guided missile the problem of dynamic response properly involves the combination of the missile as an aerodynamic body and the control system. But since ultimately any motion of the missile must be brought about by aerodynamic forces, it is desirable to know the dynamic behavior of the missile alone. In particular, the "over-shoot" characteristic of the response to a sudden change in trim position is important in the design of the control system as it indicates the amount of "feed-back" necessary to provide satisfactory steering of the missile.

In the present analysis an attempt is made to approximate this overshoot without resorting to the tedious method of numerical integration. A comparison of this approximation with the numerical solution and the linearized solution is presented.

A very important problem confronting the aerodynamicist working in the field of guided missiles is the reduction of flight test data in order to evaluate the stability derivatives. One of the procedures followed is to analyse the transient response of the missile to a step function input in the control surface deflection. This is equivalent to a step function input in the angle of attack. In Appendix III of this paper the effect of a non-linear moment curve on this procedure is discussed.

II. COORDINATES, NOTATION, SYMBOLS

The equations of motion for a missile can be referred to a set of body axes which are fixed in the missile and move with it. An orthogonal set of principal body axes with origin in the missile center of gravity, the x-axis the longitudinal centerline, the y-axis in the normally horizontal plane and the z-axis in the normally vertical plane, is selected. This coordinate system is shown in Figure 1. The velocities, moments, angular velocities, displacements and angular displacements are all defined in accordance with the right hand rule.

Throughout the text, the usual convention of placing a dot over a quantity to denote differentiation with respect to time is used. Whenever a fractional power of a quantity is involved, the positive real branch of the multivalued function is taken. For the inverse trigonometric functions, the principal branch is taken. In the following list the principal quantities used in this paper are defined.

A,B,C,D,E,F = coefficients of non-linear equation of motion - defined in text

a = speed of sound, ft/sec

c = reference chord for moment coefficients, ft

C_o = damping constant for classical theory evaluated at sea level, lb/sec

C_L = $\frac{L}{\frac{1}{2}\rho V^2 S}$ = lift coefficient

C_M = $\frac{M}{\frac{1}{2}\rho V^2 Sc}$ = pitching moment coefficient

$C_{L_{\alpha_t}}$ = $\frac{\partial C_L}{\partial \alpha}$ evaluated at trim position

$C_{M_{\alpha_t}}$ = $\frac{\partial C_M}{\partial \alpha}$ evaluated at trim position

$C_{M_{\dot{\alpha}}}$ = $\frac{\partial C_M}{\partial (\dot{\alpha} c/2V)}$ evaluated at $\dot{\alpha} = 0$

C_{M_q} = $\frac{\partial C_M}{\partial (qc/2V)}$ evaluated at q = 0

$\mathcal{F}(\psi, \phi)$ = incomplete elliptic integral of first kind

g = acceleration of gravity, ft/sec^2

I_x = $\int (y^2 + z^2) dm$ = moment of inertia about x-axis (roll), lb sec^2/ft

I_y = $\int (x^2 + z^2) dm$ = moment of inertia about y-axis (pitch), lb sec^2/ft

I_z = $\int (x^2 + y^2) dm$ = moment of inertia about z-axis (yaw), lb sec^2/ft

k = parameter defining variation of relative density with altitude. 1/ft.

$k_{1,2,3}$ = constants used to define non-linear lift curve

$K_{1,2,3}$ = constants used to define non-linear moment curve

K (ψ) = complete elliptic integral of first kind

L = lift, lbs; rolling moment, lb/ft.

m	= mass, lb-sec^2/ft
M	= pitching moment, lb-ft; Mach number
n	= accelerometer reading
N	= yawing moment, lb-ft
p	$= \dot{\phi} = \frac{\partial \phi}{\partial t}$ = perturbation in roll rate (angular velocity about x-axis) rad/sec
P	$= \dot{\Phi} = \frac{\partial \Phi}{\partial t}$ = roll rate rad/sec
P_1	= steady state of value of P
q	$= \dot{\theta} = \frac{\partial \theta}{\partial t}$ = perturbation in pitch rate (angular velocity about y-axis) rad/sec
Q	$= \dot{\Theta} = \frac{\partial \Theta}{\partial t}$ = pitch rate rad/sec
Q_1	= steady state value of Q
q'	$= \frac{1}{2}\rho V^2$ = dynamic pressure, lb/ft^2
r	$= \dot{\psi} = \frac{\partial \psi}{\partial t}$ perturbation in yaw rate (angular velocity about z-axis) rad/sec
R	$= \dot{\Psi} = \frac{\partial \Psi}{\partial t}$ = yaw rate rad/sec
R_1	= steady state value of R, rad/sec
R_o	= radius of curvature of flight path, ft.
S	= reference area for aerodynamic coefficients, sq. ft.
t	= time, sec
u	= velocity perturbation in x-direction ft/sec
U	= velocity component in x-direction ft/sec
U_1	= steady state value of U ft/sec
v	= velocity perturbation in y-direction ft/sec
V	= velocity component in y-direction ft/sec

V_1 = steady state value of V ft/sec

V = missile velocity ft/sec

w = velocity perturbation in z-direction ft/sec

W = velocity component in z-direction ft/sec

W_1 = steady state value of W

x, y, z = coordinate axis, right handed principal body axes (roll, pitch, yaw axis respectively)

X = force in x-direction, lb

Y = force in y-direction, lb

Z = force in z-direction

X_u = $\frac{\partial X}{\partial u}$ = Resistance derivative, rate of change of force in x-direction with velocity in x-direction, lb-sec/ft

$(X,Z)_{u,w,\dot{w},q}$; $(Y)_{p,v,\dot{v},r}$ = resistance derivatives(1), lb-sec/ft

$(M)_{u,w,\dot{w}q}$; $(L,N)_{p,v,\dot{v},r}$ = rotary derivatives(2), lb-sec

$(M')_{w,\dot{w},q}$ = $\frac{(M)_{w,\dot{w},q}}{I_y}$

α = angle of attack, rad.

η = damping constant 1/time

ϑ = $\frac{w}{U_1}$ = incremental angle of attack rad.

θ = perturbation angular displacement in pitch, rad

Θ = angular displacement in pitch measured from horizontal, rad

Θ_1 = steady state value of Θ

λ = logarithmic decrement of relative density variation with altitude

ρ = mass density of air, lb-sec^2/ft

ρ_0 = sea level value of ρ

σ $= \frac{\rho}{\rho_o}$ = relative density

σ_s = relative density at 35,000 ft in the standard atmosphere

σ_1 = relative density at altitude corresponding to x

τ = time, sec.

ω = circular frequency, rad/sec

ω_n = natural undamped circular frequency, rad/sec

Ω = phase angle, rad

It has been necessary to use some of the symbols listed above with a different definition than the one indicated. Also, some additional quantities have been introduced. In both cases, the definition of the quantity is given at the time it is introduced.

III. EQUATIONS OF MOTION

The equations of motion are referred to a set of principal body axes; hence all moment of inertia terms vanish. It is assumed that the missile is initially in a state of steady flight, that the linear and angular velocities and displacements can be represented as:

$$
\begin{aligned}
U &= U_1 + u & P &= P_1 + p & \Theta &= \Theta_1 + \theta \\
V &= V_1 + v & Q &= Q_1 + q & \Phi &= \Phi_1 + \phi \\
W &= W_1 + w & R &= R_1 + r & \Psi &= \Psi_1 + \psi
\end{aligned}
\qquad (1)
$$

and that $P_1 = R_1 = \Phi_1 = \Psi = 0$. It is also assumed that θ is sufficiently small that $\sin\theta$ may be taken equal to θ and $\cos\theta$ may be taken equal to one; and that u, v, w, p, q, r are sufficiently small that their squares and products may be neglected. The equations of motion as given in Reference 3 are then:

$$
\begin{aligned}
m(\dot{u} - V_1 r + W_1[Q_1 + q]) &= X - mg\sin\Theta_1 - mg\cos\Theta_1\,\theta \\
m(\dot{v} - W_1 p + U_1 r) &= Y + mg\cos\Theta_1\,\phi \\
m(\dot{w} - U_1[Q_1 + q] + V_1 p) &= Z + mg\cos\Theta_1 \\
I_x\dot{p} &= L \\
I_y\dot{q} &= M \\
I_z\dot{r} &= N
\end{aligned}
\qquad (2)
$$

If the motion is confined to infinitesimals, each of the air reactions can be expressed in the form:

$$
X = X_1 + \frac{\partial X}{\partial U}dU + \frac{\partial X}{\partial W}dW + \frac{\partial X}{\partial Q}dQ + \frac{\partial X}{\partial P}dP + \frac{\partial X}{\partial R}dR
$$

where the second order terms are neglected. The partial derivatives are then written in the form $\frac{\partial X}{\partial U} = X_u$ and are known as "stability derivatives". In the analysis of the effect of the density gradient on longitudinal dynamic stability, this assumption is made. But in the analysis of the effect of the non-linear variation of pitching moment and lift with angle of attack on the dynamic behavior of a missile, the second and third order terms in dW are included in the expressions for M and Z. Accordingly, instead of the usual notation, $M_w w$, $Z_w w$ these terms are written $M(w)$, $Z(w)$ respectively.

If the missile has a longitudinal plane of symmetry (which is usually the case) and if it is flying at zero yaw, many of the stability derivatives are zero and the set of six simultaneous equations (2) reduce to two sets of three equations each:

$$m(\dot{u} + W_1 q + g\cos\Theta_1\,\theta) = X_u u + X_w w + X_q q + X_1 - mg\sin\Theta_1 - mW_1 Q_1 \quad (3.1)$$

$$m(\dot{w} - U_1 q + g\sin\Theta_1\,\theta) = Z_u u + Z(w) + Z_q q + Z_{\dot{w}}\dot{w} + Z_1 + mg\cos\Theta_1 + mU_1 Q_1 \quad (3.2)$$

$$I_y\dot{q} = M_u u + M(w) + M_q q + M_{\dot{w}}\dot{w} + M_1 \quad (3.3)$$

$$m(\dot{v} + U_1 r - W_1 p - g\cos\Theta_1\,\phi) = Y_v v + Y_p p + Y_r r + Y_{\dot{v}}\dot{v} + Y_1 \quad (4.1)$$

$$I_x\dot{p} = L_v v + L_p p + L_r r + L_1 \quad (4.2)$$

$$I_z\dot{r} = N_v v + N_p p + N_r r + N_{\dot{v}}\dot{v} + N_1 \quad (4.3)$$

It is to be emphasized that these equations are referred to principal body axes instead of the more commonly used "wind" axes or so-called "stability" axes, and that the appropriate expressions for the stability derivatives must be used.

Considering the longitudinal equations (3), it is observed that if $Z_u = M_u = 0$, Equations (3.2) and (3.3) become independent of Equation (3.1) and the pitch motion can be described by a two degree of freedom system. For a missile which is symmetrical with respect to the yaw plane and which is trimmed at zero angle of attack, it is apparent that the lift and moment on the missile are zero and their derivatives with respect to u will be small and can be taken equal to zero. Hence it is to be expected that at small angles of attack the coupling of u with w and θ is small and that the system can be adequately treated as a two degree of freedom system. This means that the influence of the long period motion on the short period motion is negligible. Calculations for representative missiles have been carried out and this is indeed found to be the case. The results of these calculations are presented in Table I.

It is noted that the long period for a supersonic missile is of the order of 100 seconds whereas the short period is of the order of 1 second. Therefore it is to be expected that the long period should have no appreciable effect on the short period. Furthermore, it is obvious that the short period motion is the one which is important for steering and control. The long period motion would appear to be of little practical importance since the assumptions made will certainly be violated during the course of the long period.

If it is assumed that the missile is stabilized in roll, the lateral equations (4) reduce to exactly the same form as the longitudinal equations except for the gravity terms.

Returning to the longitudinal equations, dividing Equation (3.2) by m and (3.3) by I_y and designating by primes the quantities so divided, the equations become:

$$(1 - Z'_{\dot{w}})\dot{w} - Z'(w) - (U_1 + Z'_q)q = -g \sin\Theta_1 \theta + g\cos\theta_1 + U_1 Q_1 \qquad (5.1)$$

$$-M'_{\dot{w}}\dot{w} - M'(w) - M'_q q + \dot{q} = M'_1 \qquad (5.2)$$

There remains a bothersome term in these equations, namely $g \sin\Theta_1 \theta$. If the missile is flying at a small angle of climb, $\sin\Theta_1 \doteq 0$ and this term may be neglected. However, as is often the case, if the missile is flying in a nearly vertical direction, it is not apparent that this term may properly be neglected. Furthermore, its presence gives rise to an additional root to the characteristic equation, and this additional root is divergent. The divergence is slow however, and it can be shown that it represents the curvature of the flight path due to gravity. To neglect this term would seem to be a reasonable assumption since the time interval for which the motion is being studied is of the order of five to ten seconds during which the change in the flight path due to gravity will be slight. Computations for representative cases have been made and the results are shown in Table I. These results indicate that there is no appreciable error involved in neglecting gravity.

Noting that $Z'_q \ll U_1$ and $Z'_{\dot{w}} \ll 1$, hence Z'_q and $Z'_{\dot{w}}$ can be neglected and Equations (5) are written:

$$\dot{w} - Z'(w) - U_1 q = Z_1' + U_1 Q_1 + g \cos \Theta_1 \tag{6.1}$$

$$-M'_{\dot{w}} \dot{w} - M'(w) - M'_q q + \dot{q} = M_1' \tag{6.2}$$

Solving (6.1) for q:

$$q = \frac{\dot{w}}{U_1} - \frac{Z'(w)}{U_1} - \left(\frac{Z_1' + U_1 Q_1 + g \cos \Theta_1}{U_1} \right) \tag{7}$$

and differentiating with respect to time:

$$\dot{q} = \frac{\ddot{w}}{U_1} - \frac{\dot{Z}'(w)}{U_1} \tag{8}$$

Substituting (7) and (8) into (6.2), the system is reduced to the one equation of motion:

$$\frac{\ddot{w}}{U_1} - \frac{\dot{Z}'(w)}{U_1} - M'(w) - M'_q \left(\frac{\dot{w}}{U_1} - \frac{Z'(w)}{U_1} \right) - M'_{\dot{w}} \dot{w} = M_1' - \frac{M'_q}{U_1} \left(Z_1' + U_1 Q_1 + g \cos \Theta_1 \right) \tag{9}$$

If it is desired to compute the response to a step function input in control surface deflection, the forcing functions, M_1' and Z_1' can be written as step functions and the response computed directly from Equation (9). The incremental velocity, w, is measured from the original trim position and Q_1 and Θ_1 correspond to the steady state values at the original trim position. Alternatively, a linear transformation in w can be made such that $w = 0$ corresponds to the trim position at the new control surface deflection, whence Q_1 and Θ_1 are referred to the new trim position. The solution of the transformed equation is found subject to the initial conditions, $w(0) = w_0$ and $\dot{w}(0) = 0$, where w_0 corresponds to the old trim position measured in the new coordinate system. If this latter viewpoint is adopted, $M_1' = 0$, Z_1' is the aerodynamic force in the z-direction and Q_1 is the angular velocity for the steady state motion in the new trim position. For the steady state motion of the missile in a vertical plane, equilibrium of forces in the z-direction, neglecting squares of the angle of attack (Reference Figure 2), requires:

$$-Z_1 - mg\cos\Theta_1 = mR_0 Q_1^2$$

where R_0 is the radius of curvature of the flight path.

But $U_1 = Q_1 R_0$, hence $-Z_1 - mg\cos\Theta_1 = mU_1Q_1$ or $-Z' - g\cos\Theta_1 = U_1Q_1$ and the right hand side of Equation (9) is zero. The problem is now reduced to finding a solution to the homogeneous equation:

$$\frac{\ddot{w}}{U_1} - \frac{\dot{Z}'(w)}{U_1} - M'(w) - M'_q\left(\frac{\dot{w}}{U_1} - \frac{Z'(w)}{U_1}\right) - M'_{\dot{w}}\,\dot{w} = 0 \tag{10}$$

with the initial conditions:

$$W(0) = W_0 \quad , \quad \dot{W}(0) = 0 \tag{10.1}$$

IV. EFFECT OF VARIABLE DENSITY ON LONGITUDINAL DYNAMIC STABILITY

To investigate the effect of the atmospheric density gradient on the dynamic stability of a missile, the analysis is restricted to infinitesimal disturbances from steady flight at constant speed. Accordingly in Equation (10), $M'(w)$ and $Z'(w)$ are written in the usual form, $M'_w w$ and $Z'_w w$ respectively. The assumption of a two degree of freedom system is still valid and the effect of gravity may still be neglected. The only difference from the classical theory is that the stability derivatives are not pure constants but are now functions of the density. This dependence on density is exhibited by writing the derivatives in the form:

$$\begin{aligned} Z'_w &= Z'_{w_o}\sigma \\ M'_w &= M'_{w_o}\sigma \\ M'_q &= M'_{q_o}\sigma \\ M'_{\dot{w}} &= M'_{\dot{w}_o}\sigma \end{aligned} \qquad (11)$$

where the subscript, o , refers to sea level conditions, and $\sigma = \sigma(x)$ x being the distance along the flight path.

With this notation, letting $\frac{w}{U_1} = \vartheta$

$$\frac{\dot{Z}'(w)}{U_1} = \frac{1}{U_1}\frac{\partial}{\partial t}\left(Z'_{w_o} w \sigma\right) = \frac{Z'_{w_o}}{U_1}\left(\sigma\dot{w} + w\dot{\sigma}\right) = Z'_{w_o}\left(\sigma\dot{\vartheta} + \vartheta\dot{\sigma}\right)$$

and the equation of motion is written:

$$\ddot{\vartheta} + \left(-M'_{q_o} - U_1 M'_{\dot{w}_o} - Z'_{w_o}\right)\sigma\dot{\vartheta} + \left(-U_1 M'_{w_o}\sigma + Z'_{w_o} M'_{q_o}\sigma^2 - Z'_{w_o}\dot{\sigma}\right)\vartheta = 0 \qquad (12)$$

Writing

$$\dot{\vartheta} = \frac{d\vartheta}{dt} = \frac{d\vartheta}{dx}\,\frac{dx}{dt} = U_1 \frac{d\vartheta}{dx}$$

$$\ddot{\vartheta} = U_1^2 \frac{d^2\vartheta}{dx^2} \quad , \quad \dot{\sigma} = U_1 \frac{d\sigma}{dx}$$

Equation (12) becomes:

$$U_1^2 \frac{d^2\vartheta}{dx^2} + \left(-M'_{q_0} - U_1 M'_{\dot{w}_0} - Z'_{w_0}\right) U_1 \sigma \frac{d\vartheta}{dx} + \left(U_1 M'_{w_0}\sigma + Z'_{w_0} M'_{q_0}\sigma^2 - U Z'_{w_0}\frac{d\sigma}{dx}\right)\vartheta = 0 \qquad (13)$$

Assuming vertical flight and letting x be the vertical distance above 35,000 ft, the standard atmospheric density variation with altitude for altitudes > 35,000 ft. may be written:

$$\sigma = \sigma_s e^{-\lambda x}$$

where σ_s is the relative density at 35,000 ft. (14)

hence

$$\frac{d\sigma}{dx} = -\lambda \sigma_s e^{-\lambda x} = -\lambda\sigma$$

$$\frac{d^2\sigma}{dx^2} = \lambda^2\sigma$$

and

$$\frac{d\vartheta}{dx} = \frac{d\vartheta}{d\sigma}\,\frac{d\sigma}{dx} \quad , \quad \frac{d^2\vartheta}{dx^2} = \lambda^2\sigma^2 \frac{d^2\vartheta}{d\sigma^2} + \lambda^2\sigma \frac{d\vartheta}{d\sigma}$$

Now the equation of motion becomes:

$$U_1^2\lambda^2\sigma^2 \frac{d^2\vartheta}{d\sigma^2} + U_1^2\lambda^2\sigma \frac{d\vartheta}{d\sigma} - C_0 U_1 \lambda\sigma^2 \frac{d\vartheta}{d\sigma} + \left(-U_1 M'_{w_0}\sigma + Z'_{w_0} M'_{q_0}\sigma^2 + U_1 Z'_{w_0}\lambda\sigma\right)\vartheta = 0 \qquad (15)$$

Where $C_0 = -M'_{q_0} - U_1 M'_{\dot{w}_0} - Z'_{w_0}$. Dividing through by $U_1^2\lambda^2\sigma^2$

$$\frac{d^2\vartheta}{d\sigma^2} + \left(\frac{1}{\sigma} - \frac{C_0}{U_1\lambda}\right)\frac{d\vartheta}{d\sigma} + \left(\frac{-M'_{w_0}}{\lambda^2 U_1 \sigma} + \frac{Z'_{w_0} M'_{q_0}}{U_1^2\lambda^2} + \frac{Z'_{w_0}}{U_1\lambda\sigma}\right)\vartheta = 0 \qquad (16)$$

Making a change of variable, $z = \frac{C_o \sigma}{U_1 \lambda}$ and dividing through by $\left(\frac{C_o}{U_1 \lambda}\right)^2$ Equation (16) becomes:

$$\frac{d^2\vartheta}{dz^2} + \left(\frac{1}{z} - 1\right)\frac{d\vartheta}{dz} + \left(\left[\frac{-M'_{w_o}}{C_o \lambda} + \frac{Z'_{w_o}}{C_o}\right]\frac{1}{z} + \frac{Z'_{w_o} M_{q_o}}{U_1^2 \lambda^2}\right)\vartheta = 0 \qquad (17)$$

Equation (17) has an exact solution in terms of the confluent hypergeometric function.

Consider for a moment Equation (10). It is noted that $Z'_w M'_q << U M'_w$, hence $Z'_w M'_q$ may be neglected. Letting $Z'_{w_o} M'_{q_o} = 0$ Equation (17) reduces exactly to the confluent hypergeometric equation, which written in standard form is:

$$z \frac{d^2\vartheta}{dz^2} + (\rho - z)\frac{d\vartheta}{dz} + \eta\vartheta = 0 \qquad (18)$$

where now

$$\rho = 1$$

$$\eta = \frac{-M'_{w_o}}{C_o \lambda} + \frac{Z'_{w_o}}{C_o}$$

This equation has an exact solution valid for all finite z which is expressed in standard form:

$$\vartheta = {}_1F_1(-\eta, 1; z) \qquad (19)$$

The values of the parameter, η, for two representative missiles at two Mach numbers and altitudes, together with the values of the independent variable, are presented in Table II. The function ${}_1F_1(-\eta, 1; z)$ has the expansion in terms of Bessel functions for large values of η

given in Reference 4

$$ {}_1F_1(-\eta,1,z) = e^{\frac{z}{2}} \sum_{k=0}^{\infty} A_k \left(\frac{z}{\eta}\right)^{\frac{k}{2}} J_k(2\sqrt{\eta z}) \tag{20} $$

the series being uniformly and absolutely convergent for z, η real and $0 \leq z \leq X$. The A_k are given by the generating function

$$ e^{\eta\zeta} \frac{\left(1+\frac{\zeta}{2}\right)^{\eta}}{\left(1+\frac{\zeta}{2}\right)^{\eta+1}} = \sum_{k=0}^{\infty} A_k \zeta^k \tag{21} $$

Substituting into Equation (20)

$$ {}_1F_1(-\eta,1;z) = e^{\frac{z}{2}} \left\{ J_0(2\sqrt{\eta z}) - \frac{1}{2}\left(\frac{z}{\eta}\right)^{\frac{1}{2}} J_1(2\sqrt{\eta z}) + \frac{1}{4}\left(\frac{z}{\eta}\right) J_2(2\sqrt{\eta z}) + \cdots \right\} \tag{22} $$

For large values of z the Bessel functions have the asymptotic expansion: (Reference 5)

$$ J_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \left\{ \cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \sum_{r=0}^{\infty} \frac{(-1)^r(\nu,2r)}{(2z)^{2r}} - \sin\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \sum_{r=0}^{\infty} \frac{(-1)^r(\nu,2r+1)}{(2z)^{2r+1}} \right\} \tag{23} $$

where $$ (\nu,r) = \frac{(4\nu^2-1^2)(4\nu^2-3^2)\cdots(4\nu^2-[2r-1]^2)}{2^{2r}\, r!} $$

Substituting Equation (23) and neglecting $\frac{z}{\eta}$ and higher powers of $\frac{z}{\eta}$, Equation (22) becomes:

$$ {}_1F_1(-\eta,1;z) \sim \frac{e^{\frac{z}{2}}}{\sqrt{\pi}(\eta z)^{\frac{1}{4}}} \left\{ \cos\left(2\sqrt{\eta z} - \frac{\pi}{4}\right) + \frac{1}{16z}\sqrt{\frac{z}{\eta}} \sin\left(2\sqrt{\eta z} - \frac{\pi}{4}\right) - \frac{1}{2}\sqrt{\frac{z}{\eta}} \cos\left(2\sqrt{\eta z} - \frac{3\pi}{4}\right) \right\} \tag{24} $$

which can be written

$$ {}_1F_1(-\eta,1;z) \sim \frac{e^{\frac{z}{2}}}{\sqrt{\pi}(\eta z)^{\frac{1}{4}}}\sqrt{1+\left(\frac{1}{16z}-\frac{1}{2}\right)^2\frac{z}{\eta}}\ \sin\left(2\sqrt{\eta z}+\tan^{-1}\left[\frac{1}{\left(\frac{1}{16z}-\frac{1}{2}\right)\sqrt{\frac{z}{\eta}}}\right]-\frac{\pi}{4}\right) \qquad (25) $$

But $\sqrt{1+\left(\frac{1}{16z}-\frac{1}{2}\right)^2\frac{z}{\eta}} \doteq 1+\frac{1}{2}\left(\frac{1}{16z}-\frac{1}{2}\right)^2\frac{z}{\eta}$ and since $\frac{z}{\eta}$ is neglected compared to 1, the quantity $\sqrt{1+\left(\frac{1}{16z}-\frac{1}{2}\right)^2\frac{z}{\eta}}$ is to be replaced by 1 and Equation (25) becomes:

$$ {}_1F_1(-\eta,1;z) \sim \frac{-e^{\frac{z}{2}}}{\sqrt{\pi}(\eta z)^{\frac{1}{4}}}\ \cos\left(2\sqrt{\eta z}+\epsilon(z)+\frac{\pi}{4}\right) \qquad (26) $$

With these substitutions the solution to Equation (18) is then given by:

$$ \vartheta = C_1 e^{\frac{z}{2}}(\eta z)^{-\frac{1}{4}} \cos\left(2\sqrt{\eta z}+\epsilon(z)+\varphi_1\right) \qquad (27) $$

where $\epsilon(z) = \tan^{-1}\left[\frac{1}{\left(\frac{1}{16z}-\frac{1}{2}\right)\sqrt{\frac{z}{\eta}}}\right]$ and C_1 and φ_1 are constants to be determined by the initial conditions.

To get the solution in terms of the distance along the flight path note that $z = \frac{C_o\sigma_s}{U_1\lambda}e^{-\lambda x}$ where x is the altitude measured from 35,000 feet. Introduce the new variable, $y = x - x_1$ where x_1 is the value of x when the disturbance occurs. With this notation:

$$ z = \frac{C_o\sigma_1}{U_1\lambda}e^{-\lambda y} \qquad (28) $$

$$ \eta z \doteq \frac{M'_{w_o}\sigma_1}{U_1\lambda^2}e^{-\lambda y} \qquad (28.1) $$

neglecting Z'_{w_o} compared to $\frac{-M_{w_o}}{\lambda}$

where $\sigma_1 = \sigma_s e^{-\lambda x_1}$ is the value of the relative density when the disturbance occurs. Substituting these relations into Equation (27):

$$\vartheta = C_1 \frac{\exp\left\{\frac{C_0\sigma_1}{2U_1\lambda} e^{-\lambda y}\right\}}{\left(\frac{-M_{w_0}\sigma_1}{U_1\lambda^2}\right)^{\frac{1}{4}} e^{-\frac{\lambda y}{4}}} \cos\left(\sqrt{\frac{-4M_{w_0}\sigma_1}{U_1\lambda^2}}\, e^{-\frac{\lambda y}{2}} + \epsilon(z) + \varphi_1\right)$$

$$= \vartheta_0 \exp\left\{\frac{C_0\sigma_1}{2U_1\lambda} e^{-\lambda y} + \frac{\lambda y}{4}\right\} \cos\left(\sqrt{\frac{-4M_{w_0}\sigma_1}{U_1\lambda^2}}\, e^{-\frac{\lambda y}{2}} + \epsilon(z) + \varphi_1\right) \qquad (29)$$

From Equation (29) it is observed that as y becomes very large the amplitude of the oscillation, though it may decrease initially depending on the value of the parameter $\frac{C_0\sigma_1}{2U_1\lambda}$, will eventually become and remain very large. This indicates that a missile which is stable in a constant density atmosphere will eventually become unstable in vertical flight through the standard atmosphere.

To investigate the initial stability of the oscillation, $e^{-\lambda y}$ is expanded in powers of λy. The amplitude of the oscillation is then given by:

$$|\vartheta| = |\vartheta_0|\, e^{\left\{\frac{C_0\sigma_1}{2U_1\lambda}\left(1 - \lambda y + \frac{(\lambda y)^2}{2} + \cdots\right) + \frac{\lambda y}{4}\right\}}$$

$$= \vartheta_0\, e^{-\lambda\left(\left[\frac{C_0\sigma_1}{2U_1\lambda} - \frac{1}{4}\right] y - \frac{C_0\sigma_1}{2U_1} y^2\right)} \qquad (30)$$

which for small values of λy may be written:

$$|\vartheta| = \vartheta_0\, e^{-\lambda\left(\frac{C_0\sigma_1}{2U_1\lambda} - \frac{1}{4}\right) y} \qquad (31)$$

From this it is seen that the initial character of the oscillation will be stable if the parameter $\frac{C_0\sigma_1}{2U_1\lambda} > \frac{1}{4}$ and unstable if $\frac{C_0\sigma_1}{2U_1\lambda} < \frac{1}{4}$

The preceding analysis is valid for η large compared to z. This occurs, as is seen from Table II, at high altitude. A different asymptotic expansion for the confluent hypergeometric function must be used for low altitude cases where η and z are of the same order. However, an alternate procedure can be followed which will apply equally well for all altitudes. The fact that the asymptotic expansion for the confluent hypergeometric function is expressed in terms of Bessel functions suggests that a different approximation to the relative density gradient be made in order to reduce the differential equation of motion to Bessel's equation.

It is observed that the density variation with altitude for the standard atmosphere can be approximated for limited increments in altitude by a hyperbola of the form:

$$\sigma(x) = \frac{\sigma_1}{1 + k(x - x_1)}$$

where x = altitude measured in feet from sea level

x_1 = altitude where the disturbance from steady flight occurs

σ_1 = the value of $\sigma(x)$ at $x = x_1$

Such an approximation with $k = \left(\frac{\sigma_1}{\sigma(x_1 + 10{,}000)} - 1\right) \times 10^{-4}$ is found to be reasonable for increments of altitude of the order of 10,000 feet. For altitudes x_1 greater than 35,000 ft. the value of $k = .61 \times 10^{-4}$ is used. This approximation is compared with the standard atmosphere in Figure 3.

Substituting this expression for $\sigma(x)$ in Equation (13) the equation of motion becomes:

$$\frac{d^2\vartheta}{dx^2} + \frac{\frac{C_o\sigma_1}{U_1}}{1+k(x-x_1)}\frac{d\vartheta}{dx} + \left[\frac{-M'_{w_o}\sigma_1}{U_1(1+k[x-x_1])} + \frac{\frac{Z'_{w_o}M'_{q_o}}{U_1^2}\sigma_1^2 + kZ'_{w_o}\sigma_1}{(1+k[x-x_1])^2}\right]\vartheta = 0 \qquad (32)$$

Now let $x - x_1 = y$, then $\frac{d}{dx} = \frac{d}{dy}$

and $$\frac{d^2\vartheta}{dy^2} + \frac{\frac{C_o\sigma_1}{U_1}}{1+ky}\frac{d\vartheta}{dy} + \left[\frac{-M'_{w_o}\sigma_1}{U_1(1+ky)} + \frac{\frac{Z'_{w_o}M'_{q_o}}{U_1^2}\sigma_1^2 + kZ'_{w_o}\sigma_1}{(1+ky)^2}\right]\vartheta = 0 \qquad (33)$$

Substitute $\frac{1}{k} + y = z$ then $\frac{d}{dy} = \frac{d}{dz}$ and the equation of motion is:

$$\frac{d^2\vartheta}{dz^2} + \frac{\frac{C_o\sigma_1}{kU_1}}{z}\frac{d\vartheta}{dz} + \left(\frac{\frac{-M'_{w_o}\sigma_1}{kU_1}}{z} + \frac{\frac{Z'_{w_o}M'_{q_o}\sigma_1^2}{k^2U_1^2} + \frac{Z'_{w_o}\sigma_1}{kU_1}}{z^2}\right)\vartheta = 0 \qquad (34)$$

which is a form of Bessel's equation and its solution is given in Reference 6 as:

$$\vartheta = C_1 z^{\delta} Z_p(\beta z^{\frac{1}{2}})$$

where

$$\beta = \left(\frac{-4M'_{w_o}\sigma_1}{kU_1}\right)^{\frac{1}{2}}$$

$$\delta = \frac{1}{2} - \frac{C_o\sigma_1}{2kU_1} \qquad (35.1)$$

$$p^2 = 4\left\{\delta^2 - \frac{Z'_{w_o}M'_{q_o}\sigma_1^2}{k^2U_1^2} + \frac{Z'_{w_o}\sigma_1}{kU}\right\}$$

$$z = \frac{1}{k} + (x - x_1)$$

Z_p = Bessel function or order p

Since the value of z in this expression is always large, an asymptotic expression is possible and ϑ is asymptotically represented as:

$$\vartheta \sim C_2 \, z^{-\left(\frac{C_0 \sigma_1}{2kU_1} - \frac{1}{4}\right)} \cos(\beta z^{\frac{1}{2}} + \phi_2) \tag{36}$$

where C_2 and ϕ_2 are constants to be determined by the initial conditions.

Now it is clear that the damping of the system is expressed by the exponent of z:

$$-\left(\frac{C_0 \sigma_1}{2kU_1} - \frac{1}{4}\right) = -\mu \tag{37}$$

where C_0 is the sea level value of the logarithmic decrement as defined for the classical theory and given by:

$$C_0 = -M'_{q_0} - U_1 M'_{\dot{w}_0} - Z'_{w_0}$$

Rewriting the damping parameter in a form which more clearly exhibits its characteristics:

$$\mu = \varepsilon - \frac{1}{4} \qquad \text{where} \quad \varepsilon = \left(\frac{C_0}{2kM}\right)\left(\frac{\sigma_1}{a}\right) \tag{37.1}$$

M = Mach number, a = speed of sound in ft/sec

the first factor, $\left(\frac{C_0}{2kM}\right)$ in the expression for ε is a function of the missile configuration and Mach number only, while the second factor, $\left(\frac{\sigma_1}{a}\right)$, contains the dependence on altitude. From the expression (36) above, it is clear that the missile is stable if $\varepsilon > \frac{1}{4}$ and unstable if $\varepsilon < \frac{1}{4}$. The unstable case can occur even for positive values of the damping factor, C_0, which in the classical theory would mean stability. Hence again the result is obtained that a missile which is stable in a constant density atmosphere, such as experienced in level flight, can become unstable in vertical flight due to the density variation. It is seen from (37.1) above that, since the damping factor C_o decreases with increasing Mach number at supersonic speeds, and since Mach number appears explicitly in the denominator of the first factor of $\mathcal{E}$, and since the relative density of the initial altitude appears in the numerator of the second factor, the tendency toward instability increases with both increasing initial altitude and speed.

It is to be emphasized that the hyperbolic variation of density with altitude is a reasonable approximation only for limited intervals of altitude, hence the stability criterion pertains to the initial stability of the system. The result is then in exact agreement with that obtained previously (Equation (31)) if k is replaced by λ .

This instability in vertical flight can be explained physically. If the missile is undamped, the total energy of oscillation remains constant and varies sinusoidally between all potential energy and all kinetic energy. The potential energy is stored against the "aerodynamic spring", and if the spring becomes weakened during the oscillation, which is the case in vertical flight, the amplitude of vibration must increase. On the other hand, if the missile is damped and flying at constant altitude, some of the energy will be dissipated in work done against the damping forces, and the sum of the potential plus kinetic energies will decrease, hence the amplitude of oscillation will decrease. Combining these two effects for a damped missile in vertical flight, if the damping is light enough, the increase in amplitude due to the weakening spring will outweigh the decrease due to damping with the net result that the amplitude increases; and as the missile speed increases, the aerodynamic spring weakens more rapidly and this effect becomes more pronounced.

In Figures 4 and 5 the response of a disturbed missile in vertical flight computed from the above analysis is presented in comparison with the constant density case for high and low altitude. It is noted that the effect of variable density is negligible at low altitude but quite pronounced at high altitude.

V. EFFECT OF NON-LINEAR PITCHING MOMENT AND LIFT VARIATION WITH ANGLE OF ATTACK ON THE RESPONSE TO A STEP FUNCTION INPUT IN ANGLE OF ATTACK

In order to formulate the problem in a manner amenable to solution by analytical means, the pitching moment and normal force as functions of the angle of attack are approximated by polynomials. It is convenient to use cubic polynomials of the form:

$$M(\vartheta) = M_{\alpha_t}\vartheta + K_1\vartheta^2 + K_2\vartheta^3 \tag{38}$$

$$Z(\vartheta) = Z_{\alpha_t}\vartheta + k_1\vartheta^2 + k_2\vartheta^3 \tag{38.1}$$

where ϑ is the increment in angle of attack measured from the trim position. Since w is a small quantity compared with U_1, $\vartheta = \frac{w}{U_1}$ neglecting higher order terms. Substituting these relations into Equation (10) and collecting terms:

$$\ddot{\vartheta} + (A + B\vartheta + E\vartheta^2)\dot{\vartheta} + (C + D\vartheta + F\vartheta^2)\vartheta = 0 \tag{39}$$

And the initial conditions (10.1) are:

$$\vartheta(0) = \alpha_0 = \frac{w_0}{U_1} \qquad \dot{\vartheta}(0) = 0 \tag{39.1}$$

where:

$$A = -M_q' - \frac{Z_{\alpha_t}'}{U_1} - U_1 M_{\dot{w}}' = \frac{q' S c^2}{2 I_Y U_1}\left(-C_{M_q} - C_{M_{\dot{\alpha}}} + \frac{2 I_Y}{m c^2} C_{L_{\alpha_t}}\right) \tag{40}$$

$$B = \frac{-2k_1'}{U_1} = \frac{q' S}{m U_1} \frac{\partial^2 C_L}{\partial \alpha^2_t} \tag{40.1}$$

$$E = \frac{-3k_2'}{U_1} = \frac{q'S}{2mU_1}\frac{\partial^3 C_L}{\partial \alpha^3_t} \qquad (40.2)$$

$$C = -M'_{\alpha_t} + \frac{M'_q Z'_{\alpha_t}}{U_1} = \frac{q'Sc}{I_Y}(-C_{M\alpha_t}) + \left(\frac{q'Sc}{U_1}\right)^2 \frac{C_{L\alpha_t}(-C_{M_q})}{2 I_Y m} \qquad (40.3)$$

$$D = -K_1' + \frac{k_1' M_q'}{U_1} = \frac{q'Sc}{2 I_Y}\left(-\frac{\partial^2 C_M}{\partial \alpha^2_t}\right) + \left(\frac{q'Sc}{U_1}\right)^2 \frac{\frac{1}{2}\left(\frac{\partial^2 C_L}{\partial \alpha^2_t}\right)(-C_{M_q})}{2 I_Y m} \qquad (40.4)$$

$$F = -K_2' + \frac{k_2' M_q'}{U_1} = \frac{q'Sc}{I_Y}\left(\frac{-1}{3!}\frac{\partial^3 C_M}{\partial \alpha^3_t}\right) + \left(\frac{q'Sc}{U_1}\right)^2 \frac{\left(\frac{1}{3!}\frac{\partial^3 C_L}{\partial \alpha^3_t}\right)(-C_{M_q})}{2 I_Y m} \qquad (40.5)$$

where:

$$C_{M\dot{\alpha}} = \frac{\partial C_M}{\partial\left(\frac{\dot{\alpha} c}{2U_1}\right)} \qquad (40.6)$$

$$C_{M_q} = \frac{\partial C_M}{\partial\left(\frac{q c}{2U_1}\right)} \qquad (40.7)$$

$$C_{L\alpha_t} = \left.\frac{\partial C_L}{\partial \alpha}\right|_{\alpha = \alpha_{trim}} \qquad (40.8)$$

$$C_{M\alpha_t} = \left.\frac{\partial C_M}{\partial \alpha}\right|_{\alpha = \alpha_{trim}} \qquad (40.9)$$

Before proceeding with the solution of Equation (39) it is desirable to establish the stability of the system. This is done in the usual manner by considering the solution in the neighborhood of the critical points of the equation and identifying the type of the critical points. This is carried out in detail for the seven numerical examples considered in this paper and is presented in Appendix I.

Assuming a stable system, it is desired to find a solution to the equation:

$$\ddot{\vartheta} + f(\vartheta)\dot{\vartheta} + g(\vartheta)\vartheta = 0 \tag{41}$$

subject to the initial conditions:

$$\vartheta(0) = \alpha_0$$
$$\dot{\vartheta}(0) = 0 \tag{41.1}$$

where in particular,

$$f(\vartheta) = A + B\vartheta + E\vartheta^2$$
$$g(\vartheta) = C + D\vartheta + F\vartheta^2 \tag{42}$$

and without loss of generality α_0 is taken < 0. Unfortunately the exact solution to this equation cannot be expressed in terms of the known functions of mathematics (except in the cases A = B = E = 0 or B = E = D = F = 0). However, a solution to any desired degree of accuracy can be obtained by the methods of numerical integration. This procedure is tedious when done by hand and is best accomplished by means of automatic computing machines. The latter method is expensive and the additional cost can not always be justified. It is therefore desirable to find an approximate solution in terms of the tabulated functions of mathematics.

An iteration or perturbation procedure would appear to be in order. However, since $g(\vartheta)\vartheta$ is of the third power in ϑ and $f(\vartheta)$ is of the second power in ϑ, the differential equations for the second and higher approximations become increasingly complex and the numerical application of the results to any specific example becomes even more tedious than the numerical integration, while the results are less accurate. Also, there is the question of how many terms must be carried. Since the solution is essentially a Fourier expansion, and terms of the form $t \sin\omega t$ and $t \cos\omega t$ occur, which become large as t becomes large. These terms do not constitute a divergence if the entire series is taken, since they are canceled out by subsequent terms; but the question of how many terms must be taken to insure this canceling is not easily answered. For these reasons, this approach has been abandoned.

A step by step approximating procedure has been resorted to. The equation is linearized over a closed interval of the amplitude of ϑ, the linearization being based on a time average of the response over the interval. The constants used in the linearization are changed for each step. In the following section, this method is developed and explained. The analysis is of necessity lacking in mathematical rigor and relies heavily on physical intuition and analogy to the well-known linear oscillator.

The solution to the equation:

$$\ddot{\vartheta} + k\dot{\vartheta} + \omega_n^2 \vartheta = 0 \qquad (42)$$

where η and ω_n are constants, subject to the initial conditions:

$$\vartheta(0) = \alpha_0$$

$$\dot{\vartheta}(0) = 0$$

is well known to be:

$$\vartheta = \alpha_0 \frac{\omega_n}{\omega} e^{-\frac{\eta}{2}t} \cos(\omega t - \Omega) \qquad (43)$$

where

$$\omega = \sqrt{\omega_n^2 - \left(\frac{\eta}{2}\right)^2}$$

$$\Omega = \tan^{-1}\left(\frac{\eta/2}{\omega}\right)$$

and the first derivative is given by:

$$\dot{\vartheta} = -\alpha_0 \omega_n \frac{\omega_n}{\omega} e^{-\frac{\eta}{2}t} \sin \omega t \qquad (43.1)$$

If values of ω_n^2 and η are selected to replace $g(\vartheta)$ and $f(\dot{\vartheta})$ respectively in Equation (41), the approximate solution can be written as Equation (42) which will be valid for sufficiently small values of time. It is noted that in the theory of small oscillations, $\omega_n^2 = C$ and $\eta = A$, but this approximation is unsatisfactory for the magnitude of values of ϑ under consideration here.

Before selecting a value of ω_n^2, it is observed that in the linearized case, the damping factor, η has a second order effect on the frequency, ω . For example, if $\frac{\eta/2}{\omega_n} = 0.6$, i.e. the system is 60% critically damped, $\omega = 0.80\,\omega_n$, i.e. the frequency is 80% of the undamped natural frequency; and if $\frac{\eta/2}{\omega_n} = 0.2$, i.e. the system is 20% critically damped, $\omega \doteq 0.98\,\omega_n$, i.e. the frequency is approximately 98% of the undamped natural frequency. In the cases which arise in practice, the missile is considerably less than 60% critically damped; in fact, it is usually of the order of 25% or less. Since it is desired to find ϑ as a function of time it would seem reasonable to select a value of ω_n corresponding to the frequency of the undamped system.

Accordingly, consider for the moment $f(\vartheta) \equiv k \equiv 0$. Now Equations (41) and (42) become respectively:

$$\ddot{\vartheta} + g(\vartheta)\vartheta = 0 \qquad (44)$$

$$\vartheta(0) = \alpha_0 \ , \ \dot{\vartheta}(0) = 0$$

$$\ddot{\vartheta} + \omega_n^2 \vartheta = 0 \qquad (45)$$

$$\vartheta(0) = \alpha_0 \ , \ \dot{\vartheta}(0) = 0$$

Equation (44) can be solved as follows:

$$\ddot{\vartheta} = \frac{d}{d\vartheta}\left(\frac{\dot{\vartheta}^2}{2}\right) = -g(\vartheta)\vartheta \qquad (46)$$

Integrating with respect to ϑ :

$$\frac{\dot{\vartheta}^2}{2} = -\int_{\alpha_0}^{\vartheta} g(x)x\,dx = G(\alpha_0) - G(\vartheta) \qquad (47)$$

If $g(\vartheta)$ is a polynomial in ϑ , $G(\alpha_0) - G(\vartheta)$ will also be a polynomial in ϑ , and hence, $\dot{\vartheta}^2 = P(\vartheta)$ where $P(\vartheta)$ is again a polynomial in ϑ . Writing

$$\dot{\vartheta} = \frac{d\vartheta}{dt} = \sqrt{P(\vartheta)} \qquad (48)$$

and integrating:

$$t = \int_{\alpha_o}^{\vartheta} \frac{dx}{\sqrt{P(x)}} \tag{49}$$

which defines ϑ as a function of t.

Similarly, Equation (45) can be solved and its solution is:

$$t = \int_{\alpha_o}^{\vartheta} \frac{dx}{\omega_n \sqrt{\alpha_o^2 - x^2}} \tag{50}$$

Letting the interval under consideration be $\alpha_o \leqslant \vartheta \leqslant \vartheta_1$ where $\alpha_o < 0$,

then from Equation (49)

$$t_1 = \int_{\alpha_o}^{\vartheta_1} \frac{dx}{\sqrt{P(x)}} \tag{51}$$

and from Equation (50)

$$t_1' = \int_{\alpha_o}^{\vartheta_1} \frac{dx}{\omega_n \sqrt{\alpha_o^2 - x^2}} = \frac{\frac{\pi}{2} - \sin^{-1}\left(\frac{\vartheta_1}{\alpha_o}\right)}{\omega_n} \tag{52}$$

Equating $t_1 = t_1'$ and solving for ω_n :

$$\omega_n = \frac{\frac{\pi}{2} - \sin^{-1}\left(\frac{\vartheta_1}{\alpha_o}\right)}{\int_{\alpha_o}^{\vartheta_1} \frac{dx}{\sqrt{P(x)}}} \tag{53}$$

and the approximate solution to Equation (44) can be written:

$$\vartheta = \alpha_o \cos \omega_n t \tag{54}$$

from which: $$\dot{\vartheta} = -\alpha_o \omega_n \sin \omega_n t \tag{54.1}$$

Consider now Equation (41) with $f(\vartheta) \neq 0$. Forming an energy balance:

$$\frac{\dot{\vartheta}^2}{2} + \int_{\alpha_0}^{\vartheta} g(x)x\,dx = -\int_{\alpha_0}^{\vartheta} f(\vartheta)\dot{\vartheta}\,d\vartheta \tag{55}$$

where the integral on the right hand side represents the energy dissipated during the time ϑ varies from α_0 to ϑ. Letting E_{DIS} represent this integral:

$$E_{DIS} = \int_{\alpha_0}^{\vartheta} f(\vartheta)\dot{\vartheta}\,d\vartheta = \int_0^t f(\vartheta(\tau))\,\dot{\vartheta}^2(\tau)\,d\tau \tag{56}$$

Similarly for the linearized system:

$$\frac{\dot{\vartheta}^2}{2} + \frac{\omega_n^2}{2}(\vartheta^2 - \alpha_0^2) = -\eta\int_{\alpha_0}^{\vartheta} \dot{\vartheta}\,d\vartheta \tag{57}$$

and
$$E'_{DIS} = \eta\int_0^t \dot{\vartheta}^2(\tau)\,d\tau \tag{58}$$

Substituting the approximate solution to Equation (44) as given by Equations (54) and (54.1) into Equations (56) and (58):

$$E_{DIS} = \alpha_0^2\omega_n \int_0^{\omega_n t} f(\alpha_0 \cos x)\sin^2 x\,dx$$
$$E'_{DIS} = \alpha_0^2\omega_n\eta\left(\frac{\omega_n t}{2} - \frac{\sin 2\omega_n t}{4}\right) \tag{59}$$

Now at the end of time t_1, equating the energy dissipated in the two systems, i.e. $E_{DIS} = E'_{DIS}$ and solving for η:

$$\eta = \frac{\int_0^{\omega_n t_1} f(\alpha_0 \cos x)\sin^2 x\,dx}{\frac{\omega_n t_1}{2} - \frac{\sin 2\omega_n t_1}{4}} \tag{60}$$

The approximate solution to Equation (41) valid in the time interval $0 \leq t \leq t_1$ for which $\alpha_0 \leq \vartheta \leq \vartheta_1$ is then:

$$\vartheta = \alpha_0\frac{\omega_n}{\omega}e^{-\frac{\eta}{2}t}\cos(\omega t - \Omega) \tag{61}$$

where

$$\omega_n = \frac{\sin^{-1}\left(\frac{\vartheta_1}{\alpha_o}\right)}{\int_{\alpha_o}^{\vartheta_1} \frac{dx}{\sqrt{P(x)}}}$$

$$t_1 = \frac{\sin^{-1}\left(\frac{\vartheta_1}{\alpha_o}\right)}{\omega_n}$$

$$\eta = \frac{\int_0^{\omega_n t_1} f(\alpha_o \cos x) \sin^2 x \, dx}{\frac{\omega_n t_1}{2} - \frac{\sin 2\omega_n t_1}{4}} \qquad (62)$$

$$\omega = \sqrt{\omega_n^2 - (\eta/2)^2}$$

$$\Omega = \tan^{-1}\left(\frac{\eta/2}{\omega}\right)$$

Differentiating Equation (61), the value of $\dot{\vartheta}$ is found to be:

$$\dot{\vartheta} = -\alpha_o \omega_n \frac{\omega_n}{\omega} e^{-\frac{\eta}{2}t} \sin \omega t \qquad (63)$$

It is observed from Equations (43) for the linearized system that as $\eta \to 0$, $\omega \to \omega_n$ and $\Omega \to 0$ hence $\dot{\vartheta} \to -\alpha_o \omega_n \sin \omega_n t$ which agrees identically with Equation (54.1). The quantity $|\alpha_o \omega_n|$ represents the maximum velocity of the undamped system.

For the non-linear case with zero damping, the angular velocity, is given by Equation (48) as:

$$\dot{\vartheta} = \sqrt{P(\vartheta)}$$

However, letting $\eta \to 0$ in Equation (63) the angular velocity is:

$$\dot{\vartheta} = -\alpha_o \omega_n \sin \omega_n t$$

which in general will not equal $\sqrt{P(\vartheta)}$. The maximum value of $\dot{\vartheta}$ for the undamped non-linear system is obtained when ϑ is equal to zero in Equation (48):

$$\dot{\vartheta}_{MAX} = \sqrt{G(\alpha_o)}$$

For the linear system, the maximum value of $\dot{\vartheta}$ is:

$$\dot{\vartheta}_{MAX} = |\alpha_o \omega_n|$$

Hence, if the quantity $-\alpha_o \omega_n$ in Equation (63) is replaced by $\sqrt{G(\alpha_o)}$ the approximate expression for $\dot{\vartheta}$ becomes:

$$\dot{\vartheta} = \sqrt{G(\alpha_o)}\ \frac{\omega_n}{\omega}\ e^{-\frac{\eta}{2}t} \sin \omega t \qquad (64)$$

which in the limit as $\eta \to 0$ becomes:

$$\dot{\vartheta} = \sqrt{G(\alpha_o)}\ \sin \omega_n t \qquad (64.1)$$

It should be noted that the expression for $\dot{\vartheta}$ given by Equation (64) is not the time derivative of the approximate solution for ϑ given by Equation (61) but it has the same general character and gives the proper value of $\dot{\vartheta}_{MAX}$ in the limit as $\eta \to 0$. Because the approximate value of $\dot{\vartheta}(t)$ obtained in this way is not the time derivative of the approximate value of $\vartheta(t)$, the functions $\dot{\vartheta}(t)$ and $\vartheta(t)$ should be treated as unrelated functions.

Since $g(\vartheta)$ is in general non-constant, it is clear that the appropriate value of ω_n will be a function of the amplitude of the oscillation. If the system is not conservative, i.e. $f(\vartheta) \neq 0$, this amplitude will change with time and after time t_2 its magnitude is given by:

$$|\vartheta_1'| = \left|\alpha_0 \frac{\omega_n}{\omega}\right| e^{-\frac{\eta}{2}t_2} \tag{65}$$

The general procedure outlined above can be continued for another interval $\vartheta_1' \leq \vartheta \leq \vartheta_2$, using the reduced value of the oscillation amplitude to determine the value of ω_n, and modifying the procedure to account for the different initial conditions for the second interval, since $\dot{\vartheta}(t_2)$ will in general not be zero. This solution will be valid for $t_2 \leq t \leq t_3$. Following this step by step process, the solution can be continued, and by selecting the intervals of time sufficiently small, the approximate solution can presumably be made to differ from the exact solution by as little as desired. Furthermore, the approximate solution for the undamped case given by Equations (61) and (64) by letting $\eta \rightarrow 0$ and the exact solution in the undamped case given by Equations (48) and (49) agree at $t = 0$ and at $t = t_2$ irrespective of the magnitude of t_2. This suggests, by analogy to the damped linear system, that in the case of small damping, the approximate solutions should be reasonably good for fairly large values of t_2.

It is convenient to select t_2 such that $\dot{\vartheta}_1 = 0$, since this simplifies the initial conditions for the next interval. With this choice of t_2 , the approximate solution can be written:

for $o \leq t \leq t_2$, $\alpha_o \leq \vartheta \leq o$

$$\vartheta = \alpha_o \left(\frac{\omega_n}{\omega}\right) e^{-\frac{\eta}{2}t} \cos(\omega t - \Omega) \tag{66}$$

$$\dot{\vartheta} = \sqrt{G(\alpha_o)} \left(\frac{\omega_n}{\omega}\right) e^{-\frac{\eta}{2}t} \sin \omega t \tag{67}$$

where
$$\omega_n = \frac{\frac{\pi}{2}}{\int_{\alpha_o}^{o} \frac{dx}{\sqrt{P(x)}}} \tag{68}$$

$$\eta = \frac{4}{\pi} \int_o^{\frac{\pi}{2}} f(\alpha_o \cos y) \sin^2 y \, dy \tag{69}$$

$$\omega = \sqrt{\omega_n^2 - (\eta/2)^2}$$

$$\Omega = \tan^{-1}\left(\frac{\eta/2}{\omega}\right) \tag{70}$$

And when $t = t_2 = \dfrac{\frac{\pi}{2} + \Omega}{\omega}$

$$\dot{\vartheta}(t_2) = \sqrt{G(\alpha_o)} \left(\frac{\omega_n}{\omega}\right) e^{-\frac{\eta}{2}\left(\frac{\frac{\pi}{2}+\Omega}{\omega}\right)} \sin\left(\frac{\pi}{2} + \Omega\right) \tag{71}$$

$$\alpha_1 = \alpha_o e^{-\frac{\eta}{2}\left(\frac{\frac{\pi}{2}+\Omega}{\omega}\right)} \tag{72}$$

For the next interval, it is convenient to choose t_3 such that $\dot{\vartheta}$ varies from $\dot{\vartheta}(t_2)$ to 0 . This gives the new values of ω_n and η :

$$\omega_{n_1} = \frac{\frac{\pi}{2}}{\int_0^{\delta_1} \frac{dx}{\sqrt{P(x)}}}$$

$$\eta_1 = \frac{4}{\pi} \int_0^{\frac{\pi}{2}} f(\alpha_1 \cos y) \sin^2 y \, dy$$

where δ_1 is the amplitude of the peak of the undamped oscillation on the opposite side of zero from α_1. Hence $G(\delta_1) = G(\alpha_1)$. If $g(\vartheta)\vartheta$ is an odd function of ϑ then $\delta_1 = \alpha_1$.

Letting $\tau = t - t_2$ where $t = 0$ corresponds to $\vartheta = \alpha_0, \dot{\vartheta} = 0$, the second interval of the oscillation is given by:

$$\vartheta = \delta_1 \left(\frac{\omega_{n_1}}{\omega_1}\right) e^{-\frac{\eta_1}{2}\tau} \sin \omega_1 \tau$$

$$\dot{\vartheta} = \dot{\vartheta}(t_2) \left(\frac{\omega_{n_1}}{\omega_1}\right) e^{-\frac{\eta_1}{2}\tau} \cos(\omega_1 \tau + \Omega_1)$$

where

$$\omega_1 = \sqrt{\omega_{n_1}^2 - (\eta_1/2)^2}$$

$$\Omega_1 = \tan^{-1} \left(\frac{\eta_1/2}{\omega_1}\right)$$

The process is then continued for a third interval such that ϑ varies from $\vartheta(t_3)$ to 0; hence this interval is exactly similar to the first. The fourth interval is similar to the second, et cetera.

The procedure outlined above can be followed for any reasonable functions $f(\vartheta)$ and $g(\vartheta)$ provided the amplitude of the oscillations considered is such that the system is stable. It is found in practice that the pitching moment vs. angle of attack, which determines the function $g(\vartheta)\vartheta$ can be satisfactorily represented by a cubic polynomial. This has the advantage of leading to elliptic integrals of the first kind for the determination of ω_n and these functions are tabulated. If higher order polynomials are used, the integrals will have to be computed.

Seven representative numerical examples have been worked out by the method outlined above. These were taken from actual missile configurations and include high and low altitude operation at different Mach numbers and angles of attack. The initial conditions were chosen to correspond to step function control surface deflections necessary for turns of the order of from one to five g's. The general character of the moment and lift curves vs. angle of attack together with their mathematical approximations by cubic polynomials are presented in Figures 6 and 7.

The seven examples can be described by the values of the coefficients A, B, C, D, E, and F, and the initial displacement, α_o. These are presented in Table III. The details of the calculations for these cases are presented in Appendix II. The results of the analysis in comparison with numerical integrations obtained by I.B.M. punched card techniques and the classical, linearized solution are presented in two ways. The responses in both ϑ and $\dot{\vartheta}$ are plotted against time and presented alternately with the appropriate plots of the restoring moment, $-g(\vartheta)\vartheta$ and the damping function, $f(\vartheta)$ against displacement in Figures 9a to 15b. Phase space diagrams on which the trajectories of the solutions are plotted are presented in Figures 16 to 22.

VI. CONCLUSION

The variable density analysis indicates that a missile which is stable at constant speed in level flight can become unstable in vertical flight through the standard atmosphere, the tendency toward instability increasing with both altitude and Mach number. The analysis assumes constant speed and the result is due entirely to the density gradient. This effect together with Stewart's result, that the effective damping is decreased when a missile is decelerating under the influence of drag, would indicate that some form of control should be incorporated in any missile which is intended for vertical flight at high altitudes.

The response to a sudden angle of attack in the case of a non-linear moment curve is found to be qualitatively similar to that for a linear moment curve, though quantitatively, the difference is very marked. This difference in the examples treated, which are considered to be representative, indicates that the classical, linearized solution in such cases leads to large errors. The approximate solution derived gives results which are considerably more representative of the actual behavior, and appears to be adequate for design purposes. If extreme accuracy is required, the methods of numerical integration should be used.

APPENDIX I

Stability Considerations

To investigate the stability of the system described by the equation:

$$\ddot{\vartheta} + f(\vartheta)\dot{\vartheta} + g(\vartheta)\vartheta = 0$$

with initial conditions: $\vartheta(0) = \alpha_0$, $\dot{\vartheta}(0) = 0$

it is convenient to let $\dot{\vartheta} = \varphi$ and replace the equation by the equivalent autonomous pair of equations:

$$\dot{\vartheta} = F(\vartheta, \varphi) = \varphi$$

$$\dot{\varphi} = G(\vartheta, \varphi) = -g(\vartheta)\vartheta - f(\vartheta)\varphi$$

The critical points of this set of equations are defined by:

$$F(\vartheta_c, \varphi_c) = G(\vartheta_c, \varphi_c) = 0$$

This occurs when $\varphi = 0$ and either $\vartheta = 0$ or $g(\vartheta) = 0$.

In the cases to be discussed,

$$f(\vartheta) = A + B\vartheta + E\vartheta^2$$

$$g(\vartheta) = C + D\vartheta + F\vartheta^2$$

Consider first the critical point, $\vartheta = 0$, $\varphi = 0$.

In the neighborhood of $\vartheta = 0$, $\varphi = 0$, neglecting higher order terms:

$$\dot{\vartheta} = \varphi$$

$$\dot{\varphi} = -C\vartheta - A\varphi$$

This can be written in matrix notation as

$$z = \begin{pmatrix} \vartheta \\ \varphi \end{pmatrix}$$

$$\dot{z} = \beta z = \begin{pmatrix} 0 & 1 \\ -C & -A \end{pmatrix} \begin{pmatrix} \vartheta \\ \varphi \end{pmatrix}$$

for which the determinant

$$\Delta = \begin{vmatrix} 0 & 1 \\ -C & -A \end{vmatrix} = C$$

The matrix is reduced to canonical form by applying a complex transformation, and the secular equation is found to be: $\lambda^2 + A\lambda + C = 0$ for which the characteristic roots are:

$$\lambda_1 = -\frac{A}{2} + \sqrt{\left(\frac{A}{2}\right)^2 - C}$$

$$\lambda_2 = -\frac{A}{2} - \sqrt{\left(\frac{A}{2}\right)^2 - C}$$

If $A > 0$ and $C > \left(\frac{A}{2}\right)^2$, λ_1 and λ_2 are complex with negative real parts; hence the origin is a stable spiral point. For the seven numerical cases considered, these conditions are satisfied. Hence in all cases, the system is stable in the neighborhood of $\vartheta = \varphi = 0$.

The other critical points are more conveniently discussed separately for the various examples.

Examples I and II

For these cases $B = D = 0$ hence $g(\vartheta) = C + F\vartheta^2$

For $g(\vartheta) = 0$

$$\vartheta_1 = \sqrt{\frac{-C}{F}}$$

$$\vartheta_2 = -\sqrt{\frac{-C}{F}}$$

Since $C > 0, F > 0$, ϑ_1, ϑ_2 are imaginary, and the origin is therefore the only real critical point.

Examples III and IV

For these cases, $g(\vartheta) = C + D\vartheta + F\vartheta^2$ where $D > C > 0$, $F < 0$

Hence there are two critical points other than the origin. They are $(\vartheta_1, 0)$ and $(\vartheta_2, 0)$ where: $\vartheta_{1,2} = -\frac{D}{2F} \pm \sqrt{\left(\frac{D}{2F}\right)^2 - \frac{C}{F}}$

Now $-\frac{C}{F} > 0$ hence ϑ_1, ϑ_2 are real and of opposite sign and $\vartheta_1 > -\vartheta_2 > 0$

Expanding $g(\vartheta)\vartheta$ and $f(\vartheta)\varphi$ in Taylor series about the point $\vartheta = \vartheta_1$, $\varphi = 0$ and neglecting second and higher order terms:

$$\dot{\vartheta} = \varphi$$

$$\dot{\varphi} = F\vartheta_1(\vartheta_2 - \vartheta_1)(\vartheta - \vartheta_1) - (A + B\vartheta_1 + E\vartheta_1^2)\varphi$$

for which the characteristic roots are:

$$\lambda_1 = \frac{-\eta_1}{2} + \sqrt{\left(\frac{\eta_1}{2}\right)^2 + K_1}$$

$$\lambda_2 = \frac{-\eta_1}{2} - \sqrt{\left(\frac{\eta_1}{2}\right)^2 + K_1}$$

where $K_1 = F\vartheta_1(\vartheta_2 - \vartheta_1) > 0$, $\eta_1 = A + B\vartheta_1 + E\vartheta_1^2$

Hence λ_1 and λ_2 are real and of different sign, and the point $\vartheta = \vartheta_1$, $\varphi = 0$ is a saddle point.

Similarly, in the neighborhood of $\vartheta = \vartheta_2$, $\varphi = 0$

$$\dot{\vartheta} = \varphi$$

$$\dot{\varphi} = F\vartheta_2(\vartheta_1 - \vartheta_2)(\vartheta - \vartheta_2) - (A + B\vartheta_2 + E\vartheta_2^2)\varphi$$

for which the characteristic roots are:

$$\lambda_1 = -\frac{\eta_2}{2} + \sqrt{\left(\frac{\eta_2}{2}\right)^2 + K_2}$$

$$\lambda_2 = -\frac{\eta_2}{2} - \sqrt{\left(\frac{\eta_2}{2}\right)^2 + K_2}$$

where $K_2 = F\vartheta_2(\vartheta_1 - \vartheta_2) > 0$, $\eta_2 = A + B\vartheta_2 + E\vartheta_2^2$

Hence λ_1, λ_2 are real and of different sign and the point $\vartheta = \vartheta_2$, $\varphi = 0$ is also a saddle point.

Examples V, VI and VII

For these examples $g(\vartheta) = C + D\vartheta$ and $g(\vartheta) = 0$ for $\vartheta_1 = \frac{-C}{D}$

In the neighborhood of $\vartheta = \vartheta_1$, $\varphi = 0$,

$$\dot{\vartheta} = \varphi$$

$$\dot{\varphi} = C(\vartheta - \vartheta_1) - A\varphi$$

The characteristic roots are

$$\lambda_1 = \frac{-A}{2} + \sqrt{\left(\frac{A}{2}\right)^2 + C}$$

$$\lambda_2 = \frac{-A}{2} - \sqrt{\left(\frac{A}{2}\right)^2 + C}$$

Now since $C > 0$, λ_1 and λ_2 are real and of different sign, the point $\vartheta = \vartheta_1$, $\varphi = 0$ is a saddle point.

The nature of the critical points is established and a discussion of the stability for finite initial conditions is possible. In Examples I and II, since the only real critical point is the origin which is a stable spiral point, and since the damping term, $f(\vartheta) = A + E\vartheta^2$ is always > 0 for all ϑ, the system is stable for any initial conditions.

In Examples III through VII, the trajectories through the saddle points can be computed in each case and the separatrix found. The system is then stable for any initial conditions which lie within the separatrix. Because of the nature of the equations, this is a tedious computation, and for the initial conditions used in the seven examples, a simpler procedure is possible.

It is noted that the undamped system, i.e. $A = B = E = 0$, has the same critical points as the damped system, with the exception that origin is a center instead of a stable spiral point. For the former, the trajectories through the saddle points are easily computed, and the separatrices are found. Now if the damping term does not change sign over the range of ϑ included within the separatrix, the damped system will be stable for any initial conditions contained within the separatrix of the undamped system. This is found to be the case for the examples considered, as is shown graphically in Figures 23 to 27.

APPENDIX II

Application of Approximate Method of Solution to Specific Examples

In the application of the approximate method developed in Part V to actual cases it is assumed that $f(\vartheta) = A + B\vartheta + E\vartheta^2$ and that $g(\vartheta) = C + D\vartheta + F\vartheta^2$. The determination of the effective damping constant is straightforward by application of Equation (69), and the result is:

$$\eta = A + \frac{4B}{3\pi}\alpha_o + \frac{E}{4}\alpha_o^2$$

The algebra in the determination of ω_n varies somewhat depending on the values of C, D, and F. The different cases encountered in the examples cited in this paper are treated separately below.

Case I. (Examples I and II) $D = 0\,;\ C, F > 0, \alpha_o < 0$, hence $g(\vartheta) = C + F\vartheta^2$ (II.1)

From Equation (47):

$$P(\vartheta) = \dot{\vartheta}^2 = -2\int_{\alpha_o}^{\vartheta}(Cx + Fx^3)\,dx$$

$$P(\vartheta) = \frac{F}{2}(\alpha_o^2 - \vartheta^2)\left(\frac{2C}{F} + \alpha_o^2 + \vartheta^2\right) \qquad \text{(II.2)}$$

Let $\zeta = \frac{\vartheta}{\alpha_o}$ and $a^2 = \frac{\alpha_o^2}{\alpha_o^2 + \frac{2C}{F}}$, then $0 < a^2 < 1$

Now Equation (II.2) becomes:

$$P(\vartheta) = \alpha_o^2 P_1(\zeta) = \frac{\alpha_o^2 F}{2}\left(\alpha_o^2 + \frac{2C}{F}\right)(1-\zeta^2)(1+a^2\zeta^2)$$

and

$$\int_{\alpha_o}^{0}\frac{dx}{\sqrt{P(x)}} = \frac{1}{\sqrt{\frac{F}{2}\alpha_o^2 + C}}\int_0^1 \frac{d\zeta}{\sqrt{(1-\zeta^2)(1+a^2\zeta^2)}}$$

Let $\zeta = \cos\varphi$

$$\int_{\alpha_o}^{0}\frac{dx}{\sqrt{P(x)}} = \frac{1}{\sqrt{C + F\alpha_o^2}}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{1 - k^2\sin^2\varphi}} = \frac{K(\psi)}{\sqrt{C + F\alpha_o^2}}$$

where $\psi = \sin^{-1}k = \sin^{-1}\sqrt{\frac{F\alpha_o^2}{2F\alpha_o^2 + 2C}}$ and $K(\psi)$ is the complete elliptic integral of the first kind.

From Equation (68):

$$\omega_n = \frac{\frac{\pi}{2}\sqrt{C + F\alpha_o^2}}{K(\psi)}$$

The approximate solution for the first interval: $0 \leq t \leq T_1$, $\alpha_o \leq \vartheta \leq 0$ is given by:

$$\vartheta = \alpha_o\left(\frac{\omega_{n_o}}{\omega_o}\right) e^{-\frac{\eta_o}{2}t} \cos(\omega_o t - \Omega_o) \tag{II.3}$$

$$\dot{\vartheta} = \dot{\vartheta}_{M_o}\left(\frac{\omega_{n_o}}{\omega_o}\right) e^{-\frac{\eta_o}{2}t} \sin \omega_o t \tag{II.4}$$

where the constants are given by the following equations on setting $i = 0$:

$$\omega_{n_i} = \frac{\frac{\pi}{2}\sqrt{C + F\alpha_i^2}}{K(\psi_i)} \tag{II.5}$$

$$\psi_i = \sin^{-1}\sqrt{\frac{F\alpha_i^2}{2F\alpha_i^2 + 2C}} \tag{II.6}$$

$$\eta_i = A + \frac{4B}{3\pi}\alpha_i + \frac{E}{4}\alpha_i^2 \tag{II.7}$$

$$\omega_i = \sqrt{\omega_{n_i}^2 - (\eta_i/2)^2} \tag{II.8}$$

$$\Omega_i = \tan^{-1}\left(\frac{\eta_i/2}{\omega_i}\right) \tag{II.9}$$

$$\dot{\vartheta}_{M_i} = \sqrt{C\alpha_i^2 + \frac{2D}{3}\alpha_i^3 + \frac{F}{2}\alpha_i^4} \tag{II.10}$$

Now T_1 is the value of t when $\vartheta = 0$, hence:

$$T_1 = \frac{\frac{\pi}{2} + \Omega_0}{\omega_0} \tag{II.11}$$

$$\dot{\vartheta}(T_1) = \dot{\vartheta}_{M_0}\, e^{-\frac{\eta_0}{2}T_1} \tag{II.12}$$

$$\alpha_1 = -\alpha_0 e^{-\frac{\eta_0}{2}T_1} \tag{II.13}$$

Since $g(\vartheta)\vartheta = C\vartheta + F\vartheta^3$ which is an odd function of ϑ, the amplitude of oscillation for the undamped system is symmetrical with respect to $\vartheta = 0$. The solution for the second interval, $T_1 \leq t \leq T_2$, $\dot{\vartheta}(T_1) \geq \dot{\vartheta} \geq 0$, $0 \leq \vartheta \leq \alpha_2$, is given by the following equations where $\tau = t - T_1$

$$\vartheta = \alpha_1\left(\frac{\omega_{n_1}}{\omega_1}\right) e^{-\frac{\eta_1}{2}\tau} \sin \omega_1\tau \tag{II.14}$$

$$\dot{\vartheta} = \dot{\vartheta}(T_1)\left(\frac{\omega_{n_1}}{\omega_1}\right) e^{-\frac{\eta_1}{2}\tau} \cos(\omega_1\tau + \Omega_1) \tag{II.15}$$

The constants are given by Equations (II.5) through (II.9) on setting $i = 1$. Now T_2 is the value of t when $\dot{\vartheta} = 0$, hence:

$$T_2 = T_1 + \frac{\frac{\pi}{2} - \Omega_1}{\omega_1} \tag{II.16}$$

$$\alpha_2 = \alpha_1 e^{-\frac{\eta_1}{2}(T_2 - T_1)} \qquad \text{(II.17)}$$

For the third interval, $T_2 \leq t \leq T_3$, $\alpha_2 \geq \vartheta \geq 0$ the solution is given by the following equations where $\tau = t - T_2$

$$\vartheta = \alpha_2 \left(\frac{\omega_{n_2}}{\omega_2}\right) e^{-\frac{\eta_2}{2}\tau} \cos(\omega_2 \tau - \Omega_2) \qquad \text{(II.18)}$$

$$\dot{\vartheta} = \dot{\vartheta}_{M_2} \left(\frac{\omega_{n_2}}{\omega_2}\right) e^{-\frac{\eta_2}{2}\tau} \sin \omega_2 \tau \qquad \text{(II.19)}$$

the constants being given by Equations (II.5) through (II.10) on setting i = 2. Now T_3 is the value of t when $\vartheta = 0$, hence:

$$T_3 = T_2 + \frac{\frac{\pi}{2} + \Omega_2}{\omega_2} \qquad \text{(II.20)}$$

$$\dot{\vartheta}(T_3) = \dot{\vartheta}_{M_2} e^{-\frac{\eta_2}{2}(T_3 - T_2)} \qquad \text{(II.21)}$$

$$\alpha_3 = -\alpha_2 e^{-\frac{\eta_2}{2}(T_3 - T_2)} \qquad \text{(II.22)}$$

For the fourth interval, $T_3 \leq t \leq T_4$, $\dot{\vartheta}(T_3) \leq \dot{\vartheta} \leq 0$, $0 \geq \vartheta \geq \alpha_4$ the solution is given by the following equations where $\tau = t - T_3$

$$\vartheta = \alpha_3 \left(\frac{\omega_{n_3}}{\omega_3}\right) e^{-\frac{\eta_3}{2}\tau} \sin \omega_3 \tau \qquad \text{(II.23)}$$

$$\dot{\vartheta} = \dot{\vartheta}(T_3) \left(\frac{\omega_{n_3}}{\omega_3}\right) e^{-\frac{\eta_3}{2}\tau} \cos(\omega_3 \tau + \Omega_3) \qquad \text{(II.24)}$$

The constants are given by Equations (II.5) through (II.9) on setting i = 3. Now T_4 is the value of t when $\dot{\vartheta} = 0$, hence

$$T_4 = T_3 + \frac{\frac{\pi}{2} + \Omega_3}{\omega_3} \tag{II.25}$$

$$\alpha_4 = \alpha_3 \, e^{-\frac{\eta_3}{2}(T_4 - T_3)} \tag{II.26}$$

The solution for the next four intervals is obtained in exactly the same way as for the first four.

Case II. (Examples III and IV) $C, D > 0 ; F, \alpha_0 < 0$ hence $g(\vartheta) = C + D\vartheta + F\vartheta^2$

From Equation (47):

$$P(\vartheta) = -2 \int_{\alpha_0}^{\vartheta} (Cx + Dx^2 + Fx^3)\,dx$$

$$P(\vartheta) = \frac{-F}{2}(\vartheta - \alpha_0)(\vartheta^3 + b\vartheta^2 + c\vartheta + d) \tag{II.27}$$

where

$$b = \alpha_0 + \frac{4D}{3F}$$

$$c = \alpha_0^2 + \frac{4D}{3F}\alpha_0 + \frac{2C}{F} \tag{II.28}$$

$$d = \alpha_0^3 + \frac{4D}{3F}\alpha_0^2 + \frac{2C}{F}\alpha_0$$

Factoring, $P(\vartheta) = \frac{-F}{2}(\vartheta - \alpha_0)(\vartheta - \beta_0)(\vartheta - \gamma_0)(\vartheta - \delta_0)$ where $\beta_0, \gamma_0, \delta_0$ are the roots of the cubic: $\vartheta^3 + b\vartheta^2 + c\vartheta + d = 0$

Since the system is stable under the given initial conditions $\beta_0, \gamma_0, \delta_0$ are real and $\beta_0 < \alpha_0 < 0 < \delta_0 < \gamma_0$ and the oscillation in ϑ is in the interval $\alpha_0 \leq \vartheta \leq \delta_0$

Now $$\int_{\alpha_0}^{0}\frac{dx}{\sqrt{P(x)}} = \frac{1}{\sqrt{\frac{-F}{2}}}\int_{\alpha_0}^{0}\frac{dx}{\sqrt{(x-\alpha_0)(x-\beta_0)(x-\gamma_0)(x-\delta_0)}} \tag{II.29}$$

$$= \frac{2}{\sqrt{\frac{-F}{2}(\alpha_0-\gamma_0)(\beta_0-\delta_0)}}\int_{0}^{\phi_0}\frac{d\phi}{\sqrt{1-k_0^2\sin^2\phi}}$$

$$= \frac{2\,\mathcal{F}\{\psi_0,\phi_0\}}{\sqrt{\frac{-F}{2}(\alpha_0-\gamma_0)(\beta_0-\delta_0)}} \tag{II.30}$$

where: $$\psi_0 = \sin^{-1}k_0 = \sin^{-1}\sqrt{\frac{(\alpha_0-\delta_0)(\beta_0-\gamma_0)}{(\beta_0-\delta_0)(\alpha_0-\gamma_0)}} \tag{II.31}$$

$$\phi_0 = \sin^{-1}\sqrt{\frac{\alpha_0(\beta_0-\delta_0)}{\beta_0(\alpha_0-\delta_0)}} \tag{II.32}$$

and $\mathcal{F}\{\psi_0,\phi_0\}$ is the incomplete elliptic integral of the first kind of amplitude ϕ_0 .

By Equation (68): $$\omega_{n_0} = \frac{\frac{\pi}{4}\sqrt{\frac{-F}{2}(\alpha_0-\gamma_0)(\beta_0-\delta_0)}}{\mathcal{F}\{\psi_0,\phi_0\}} \tag{II.33}$$

For the first interval, $0 \leq t \leq T_1$, $\alpha_0 \leq \vartheta \leq 0$ the solution is given by Equations (II.3) and (II.4); the constants are determined by Equations (II.7) through (II.10) on setting i = 0, and Equations (II.31) through (II.33). T_1 and $\dot{\vartheta}(T_1)$ are obtained from Equations (II.11) and (II.12) respectively. Since $g(\vartheta)\vartheta$ is not an odd function of ϑ , the amplitude of oscillation of the undamped system is not symmetrical with respect to $\vartheta = 0$. Hence for the second interval $0 \leq \vartheta \leq \delta_1$ where δ_1 is the lesser of the two positive roots of the equation: $\vartheta^3 + b_1\vartheta^2 + c_1\vartheta + d_1 = 0 = (\vartheta - \beta_1)(\vartheta - \delta_1)(\vartheta - \gamma_1)$

$$\beta_1 < \alpha_1 < 0 < \delta_1 < \gamma_1$$

The quantities b_1, c_1, d_1 are obtained by substituting $\alpha_1 = \alpha_0 e^{-\frac{\eta_1}{2}T_1}$ for α_0 in Equations (II.28). Hence for the second interval, $T_1 \leq t \leq T_2$, $\dot{\vartheta}(T_1) \geq \dot{\vartheta} \geq 0$, $0 \leq \vartheta \leq \delta_2$, letting $\tau = t - T_1$ the solution is given by Equations (II.14) and (II.15) where now:

$$\omega_{n_1} = \frac{\frac{\pi}{4}\sqrt{\frac{-f}{2}(\alpha_1 - \gamma_1)(\beta_1 - \delta_1)}}{K(\psi_1) - \mathcal{F}(\psi_1, \varphi_1)} \tag{II.34}$$

ψ_1, φ_1 are determined from Equations (II.31) and (II.32) upon replacing $\alpha_0, \beta_0, \gamma_0, \delta_0$ by $\alpha_1, \beta_1, \gamma_1, \delta_1$ respectively and Equations (II.7) through (II.10) on setting i = 1 apply. T_2 is obtained from Equation (II.16) and δ_2 is now given by

$$\delta_2 = \delta_1 e^{-\frac{\eta_1}{2}(T_2 - T_1)} \tag{II.35}$$

For the third interval, $T_2 \leq t \leq T_3$, $\delta_2 \geq \vartheta \geq 0$, the solution is given by Equation (II.18) on replacing α_2 by δ_2 and Equation (II.19). The constants are obtained on setting i = 2 and replacing α_2 with δ_2 in Equations (II.7) through (II.10); ω_{n_2} is obtained from Equation (II.34) on replacing $\alpha_1, \beta_1, \gamma_1, \delta_1, \psi_1$ and φ_1 with $\alpha_2, \beta_2, \gamma_2, \delta_2, \psi_2$, and φ_2 respectively, ψ_2 and φ_2 being determined from Equations (II.31) and (II.32) upon replacing $\alpha_0, \beta_0, \gamma_0, \delta_0$ with $\alpha_2, \beta_2, \gamma_2, \delta_2$ respectively. T_3 and $\dot{\vartheta}(T_3)$ are given by Equations (II.20) and (II.21).

For the fourth interval, $T_3 \leqslant t \leqslant T_4$, $\dot{\vartheta}(T_3) \leqslant \dot{\vartheta} \leqslant 0$, $0 \geqslant \vartheta \geqslant \alpha_4$ the solution is given by Equations (II.23) and (II.24). α_3 is the numerically smaller of the two negative roots of the cubic

$\vartheta^3 + b_3\vartheta^2 + c_3\vartheta + d_3 = 0 = (\vartheta - \alpha_3)(\vartheta - \beta_3)(\vartheta - \gamma_3)$

where b_3 , c_3 , d_3 are obtained from Equation (II.28) on replacing α_o by $\delta_3 = \delta_2 e^{-\frac{\gamma_2}{2}(T_3 - T_2)}$. The constants are determined by setting i = 3 in Equations (II.7) through (II.10) and by replacing $\alpha_o, \beta_o, \delta_o, \gamma_o$ with $\alpha_3, \beta_3, \delta_3, \gamma_3$ respectively in Equations (II.31) through (II.33). T_4 and α_4 are given by Equations (II.25) and (II.26).

Case III. $F = 0$, $g(\vartheta) = C + D\vartheta$

Now

$$P(\vartheta) = -2\int_{\alpha_o}^{\vartheta}(Cx + Dx^2)dx$$

$$= \frac{-2D}{3}(\vartheta - \alpha_o)(\vartheta - \beta_o)(\vartheta - \delta_o) \qquad \text{(II.36)}$$

where

$$\beta_o = \frac{-\left(\alpha_o + \frac{3C}{2D}\right)}{2}\left[1 + \sqrt{1 - \frac{4\alpha_o}{\alpha_o + \frac{3C}{2D}}}\right]$$

$$\delta_o = \frac{-\left(\alpha_o + \frac{3C}{2D}\right)}{2}\left[1 - \sqrt{1 - \frac{4\alpha_o}{\alpha_o + \frac{3C}{2D}}}\right] \qquad \text{(II.37)}$$

Since the system is stable under the chosen initial conditions, are real.

IIIa. (Example V) If $C, D > 0$, $\beta_o < \alpha_o < 0$, $\delta_o > 0$, the solution is obtained in the same way as in Case II with the exceptions that: Equation (II.33) is replaced by:

$$\omega_{n_o} = \frac{\frac{\pi}{4}\sqrt{\frac{2D}{3}(\delta_o - \beta_o)}}{\mathcal{F}\{\psi_o, \phi_o\}} \qquad \text{(II.38)}$$

Equation (II.31) is replaced by: $\psi_o = \sin^{-1}\sqrt{\frac{\delta_o - \alpha_o}{\delta_o - \beta_o}}$ (II.39)

and Equation (II.34) is replaced by: $\omega_{n_1} = \frac{\frac{\pi}{4}\sqrt{\frac{2D}{3}(\delta_1 - \beta_1)}}{K(\psi_1) - \mathcal{F}\{\psi_1, \varphi_1\}}$ (II.40)

<u>IIIb</u>. (Examples VI and VII) $C > 0,\ D < 0,\ \beta_o > \delta_o > 0,\ \alpha_o < 0$

The solution is obtained in the same way as Case II with the exceptions:

Equation (II.33) is replaced by: $\omega_{n_o} = \frac{\frac{\pi}{4}\sqrt{\frac{-2D}{3}(\beta_o - \alpha_o)}}{\mathcal{F}\{\psi_o, \varphi_o\}}$ (II.41)

Equation (II.31) is replaced by $\psi_o = \sin^{-1}\sqrt{\frac{\delta_o - \alpha_o}{\beta_o - \alpha_o}}$ (II.42)

Equation (II.32) is replaced by $\varphi_o = \sin^{-1}\sqrt{\frac{-\alpha_o}{\delta_o - \alpha_o}}$ (II.43)

Equation (II.34) is replaced by $\omega_{n_1} = \frac{\frac{\pi}{4}\sqrt{\frac{-2D}{3}(\beta_1 - \alpha_1)}}{K(\psi_1) - \mathcal{F}(\psi_1, \varphi_1)}$ (II.44)

APPENDIX III

Effect of Non-Linear Pitching Moment Curves on the Determination of Stability Derivatives from Flight Test Data

One of the techniques used for the determination of stability derivatives from flight test data is the analysis of the transient response to a step function input in control surface deflection. It is customary to assume a linear system. From a measurement of the amplitude of the oscillation vs. time, the logarithmic decrement is determined. The zeros of the oscillation determine the damped natural frequency. The combination of these two then defines the undamped natural frequency which to all practical purposes determines the quantity

$$M_w' = \frac{q' S c}{I_y} \frac{d C_M}{d \alpha}$$

If the moment curve is a non-linear function of the angle of attack, the "frequency" of the oscillation varies with the amplitude. In this case, the usual procedure of measuring the zeros of the oscillation and averaging them will not determine the value of $\frac{d C_M}{d \alpha}$ at the trim position, but some sort of average value. The comparison of the classical, linearized solution with the numerical integration for the seven examples considered in this paper clearly indicates that the value of $\frac{d C_M}{d \alpha}$ obtained in the usual way will not be very meaningful if the non-linearity is at all severe.

Alternatively, an attempt to determine the non-linear moment curve, using the approximate solution for the response may be made. The procedure to be outlined requires that angle of attack, normal acceleration in the pitch plane, and pitch rate be measured as functions of time; and assumes, of course, a knowledge of Mach number, velocity, and dynamic pressure, in order that the moment curve may be reduced to coefficient form. In addition, it is necessary to assume the general shape of the moment curve. (This is analogous to the assumption of linearity usually made).

Consider first a moment curve of the form:

$$-g(\vartheta)\vartheta = -(C + F\vartheta^2)\vartheta \quad , \quad C, F > 0$$

Assume also that the damping function is constant

$$f(\vartheta) = \eta$$

From the records of angle of attack vs. time, normal acceleration vs. time, and pitch rate vs. time, the damping constant can be determined. The initial amplitude, α_0, of the oscillation in angle of attack is also measured. Designating T_1 as the first zero of ϑ and using Equation (II.11)

$$\omega_0 = \frac{\frac{\pi}{2} + \Omega_0}{T_1}$$

where

$$\Omega_0 = \tan^{-1}\left(\frac{\eta/2}{\omega_0}\right)$$

If the damping is small compared to critical damping, Ω_o may be determined as:

$$\Omega_o = \frac{r/2}{\omega_{n_o}}$$

and ω_{n_o} can then be determined from:

$$\omega_{n_o} = \frac{\pi}{4T_1} + \sqrt{\left(\frac{\pi}{4T_1}\right)^2 + \frac{r/2}{T_1} + \frac{(r/2)^2}{2}}$$

r, ω_{n_o} and α_o are determined and can be substituted in Equation (II.5):

$$\omega_{n_o} = \frac{\frac{\pi}{2}\sqrt{F\alpha_o^2 + C}}{K(\psi_o)} \qquad \text{where} \quad \psi_o = \sin^{-1}\sqrt{\frac{F\alpha_o^2}{2F\alpha_o^2 + 2C}}$$

which determines a relation between F and C.

Now the normal accelerometer measures the quantity:

$$\ddot{n} = \dot{w} - U_1\dot{\theta} - g\cos\Theta_1$$

where $\dot{\theta}$ is the pitch rate. Hence, from the normal acceleration and pitch rate records, the quantity:

$$\dot{\vartheta} = \frac{\ddot{n}}{U_1} + \dot{\theta} + \frac{g}{U_1}\cos\Theta_1$$

is determined as a function of time. Designating T_2 as the first zero of $\dot{\vartheta}$ and proceeding in the same manner as before using now Equation (II.16) instead of (II.11):

$$\omega_{n_1} = \frac{\pi}{4(T_2 - T_1)} + \sqrt{\left(\frac{\pi}{4(T_2 - T_1)}\right)^2 - \frac{r/2}{(T_2 - T_1)} + \frac{(r/2)^2}{2}}$$

and a second relation between F and C is determined from the equations:

$$\omega_{n_1} = \frac{\frac{\pi}{2}\sqrt{F\alpha_1^2 + C}}{K(\psi_1)}$$

$$\psi_1 = \sin^{-1}\sqrt{\frac{F\alpha_1^2}{2F\alpha_1^2+2C}}$$

$$\alpha_1 = \alpha_o e^{-\frac{n}{2}T_1}$$

Following this same procedure for the subsequent zeros of the oscillations in ϑ and $\dot{\vartheta}$, additional relations between C and F can be obtained. If C is plotted against F, these various curves will intersect each other. Since the equations used to obtain the relations between C and F are only approximate, it cannot be expected that all of the curves will intersect in a common point. However, depending on the accuracy of the data, they should define a small, closed region in the F,C plane, and any point (F,C) in this region may be used for the approximate values of F and C.

As a second example of the application of the approximate solution to the determination of the moment curve, assume the curve to be of the form:

$$-g(\vartheta)\vartheta = -(C\vartheta + D\vartheta^2) \quad , \quad C,D > 0$$

and assume that the damping is constant.

From the envelope of the angle of attack vs. time record, α_o and δ_o are determined as the initial values of the amplitude on opposite sides of $\vartheta = 0$. Proceeding as before, using now Equation (II.38)

$$\omega_{n_o} = \frac{\frac{\pi}{4}\sqrt{\frac{2D}{3}(\delta_o - \beta_o)}}{\mathcal{F}\{\psi_o, \phi_o\}}$$

$$\psi_o = \sin^{-1}\sqrt{\frac{\delta_o - \alpha_o}{\delta_o - \beta_o}}$$

$$\phi_o = \sin^{-1}\sqrt{\frac{\alpha_o(\beta_o - \delta_o)}{\beta_o(\alpha_o - \delta_o)}}$$

Observing that $P(\vartheta) = \frac{-2D}{3}(\vartheta - \alpha_o)(\vartheta - \beta_o)(\vartheta - \delta_o) = \frac{-2D}{3}(\vartheta^3 + \frac{3C}{2D}\vartheta^2 - \alpha_o^3 - \frac{3C}{2D}\alpha_o^2)$

and comparing coefficients: $\beta_o = \frac{\alpha_o \delta_o}{\alpha_o - \delta_o}$ and $\frac{3C}{2D} = \alpha_o + \beta_o - \delta_o$

This determines D and C. Repeating the procedure for the other zeros of ϑ and $\dot{\vartheta}$, additional determination of C and D may be made. An average of the different values may then be used as the correct values of C and D.

Obviously a similar procedure can be used for other assumptions of the moment curve shape.

It is to be emphasized that the above procedure applied to a non-linear system, since it is based on an approximate solution, cannot be expected to give results as accurate as those obtained for a linear system where the exact solution is known. However, it should yield a more useful result for a non-linear system than that obtained from an assumption of linearity.

In the application of the proposed method, there are several sources of error which are to be guarded against. The first case discussed, where the moment curve was assumed to be symmetric with respect to the origin, applies to the case of a symmetric missile trimmed at zero angle of attack. Consequently, the abscissa of the various instrument records is accurately known (after correction of $\dot{\theta}$ and $\dot{w}$ records for effect of gravity, which is small). Furthermore, after η is determined, the envelope can be faired through the peaks of the oscillation and the zero should bisect the envelope. This affords a check. Hence the major source of error will be in the amplitudes of oscillation. Since the records from all three instruments will exhibit the same damping, all three should be used in the determination of η. The determination of $\dot{\vartheta}(t)$ requires the sum of the two instrument readings, $\ddot{n}(t)$ and $\dot{\theta}(t)$; hence it cannot be expected to be as accurate as that of $\vartheta(t)$.

The case of a moment curve which is not symmetrical about the origin arises where a symmetrical missile is trimmed at a finite angle of attack. In this case, the oscillation has different amplitudes on the opposite sides from zero. The abscissa of the oscillations cannot be measured in this case and must be obtained from an extrapolation back from the steady state following the transient. This is of course a very inaccurate procedure and will undoubtedly introduce large errors.

An attempt has been made to apply the above method to the numerical solution of a case with a moment curve of the type $g(\vartheta)\vartheta = C\vartheta + D\vartheta^2$. Errors of ten percent were assumed in the measured quantities, and it was found that such errors could reverse the curvature of the moment curve. Hence it would appear that in order to determine a non-linear moment curve from flight test data, extremely accurate instrumentation will be necessary.

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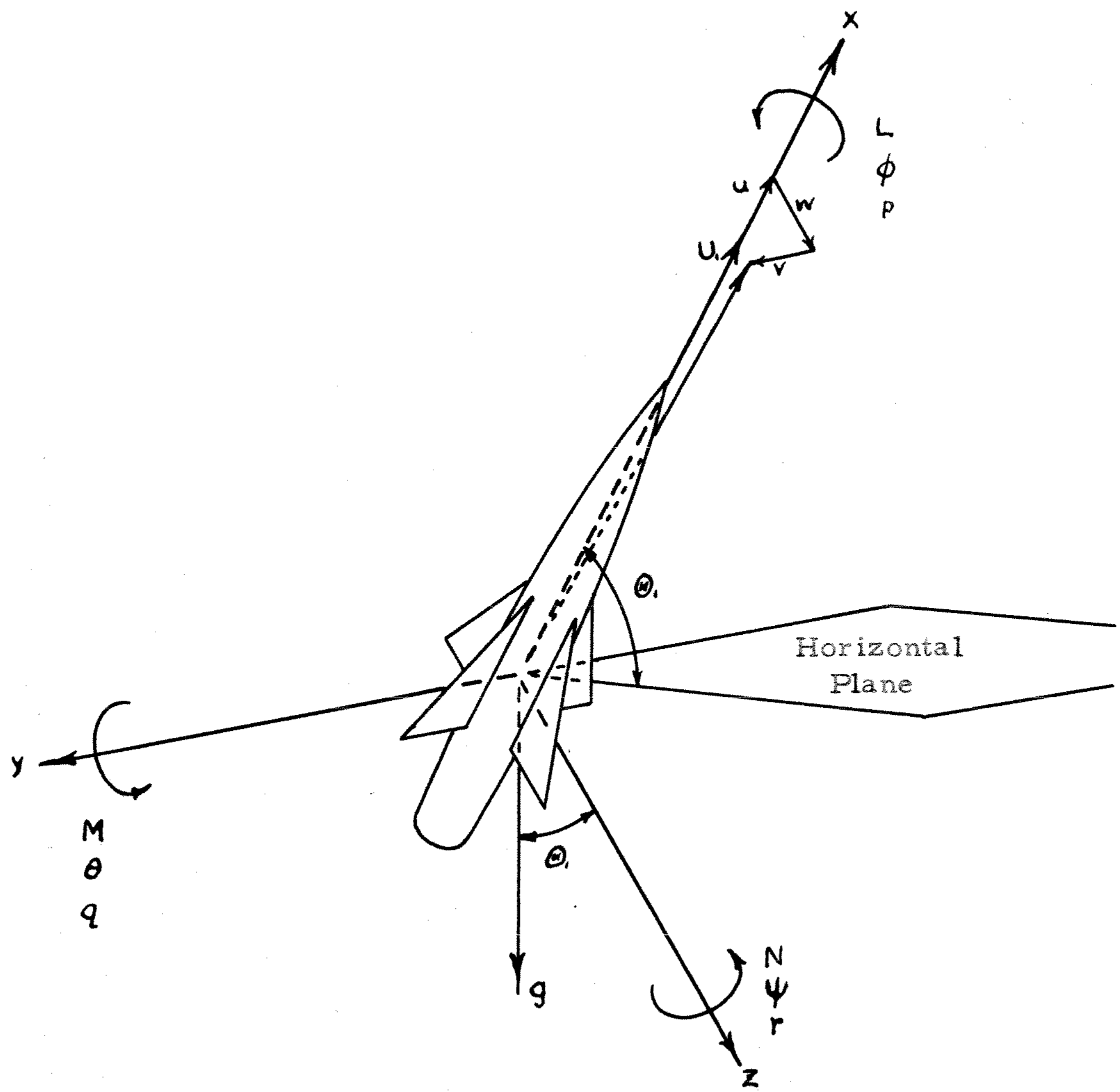
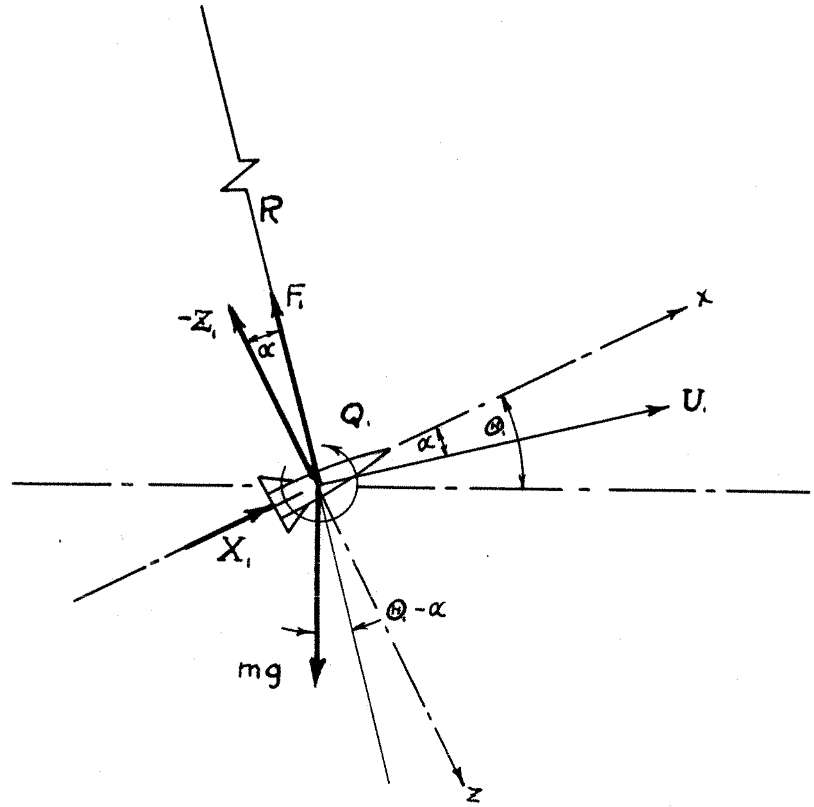


Figure 1. Coordinate System



For Equilibrium: $F_1 = -Z_1 \cos\alpha + X_1 \sin\alpha - mg \cos(\Theta_1 - \alpha)$ (a)

$X_1 \cos\alpha = mg \sin(\Theta_1 - \alpha)$ (b)

From equation (a): $F_1 = -(Z_1 + mg\cos\Theta_1)\cos\alpha + (X_1 - mg\sin\Theta_1)\sin\alpha$ (c)

From equation (b): $X_1 - mg \sin\Theta_1 = mg\cos\Theta_1 \tan\alpha$ (d)

Combining equations (c) & (d):

$$F_1 = -(Z_1 + mg\cos\Theta_1)\cos\alpha + mg\cos\Theta_1 \tan\alpha \sin\alpha \quad \text{(e)}$$

Expanding in powers of α : $\cos\alpha = 1 - \frac{\alpha^2}{2} + \cdots$; $\tan\alpha \sin\alpha = \alpha^2 + \frac{\alpha^4}{3!} + \cdots$

Hence, for small values of α , neglecting squares

$$F_1 = -Z_1 - mg \cos\Theta_1 \quad \text{(f)}$$

Figure 2. Forces Acting on Missile Turning in Vertical Plane

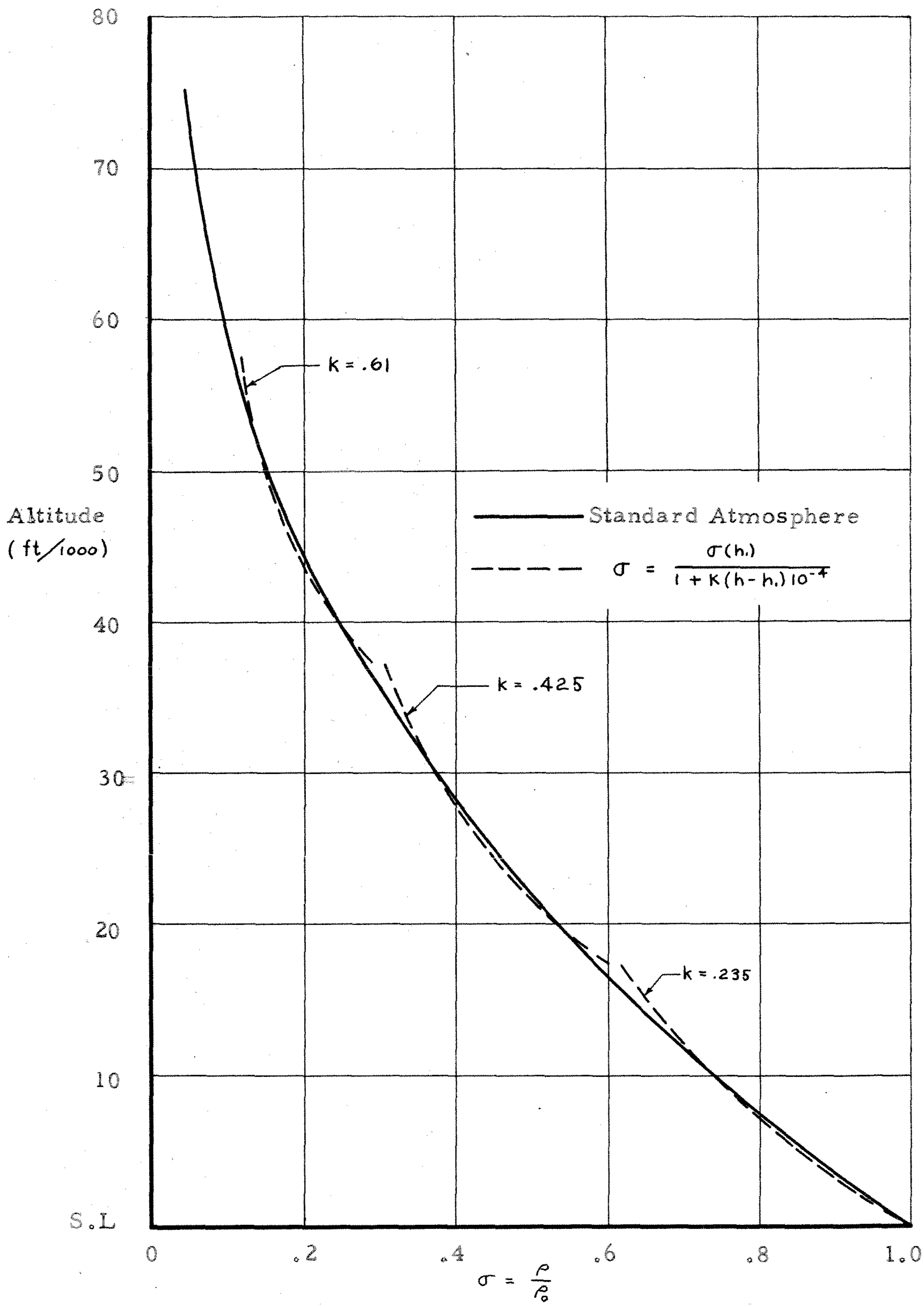


Figure 3a. Comparison of Approximate Density Variation with Altitude and the Standard Atmosphere

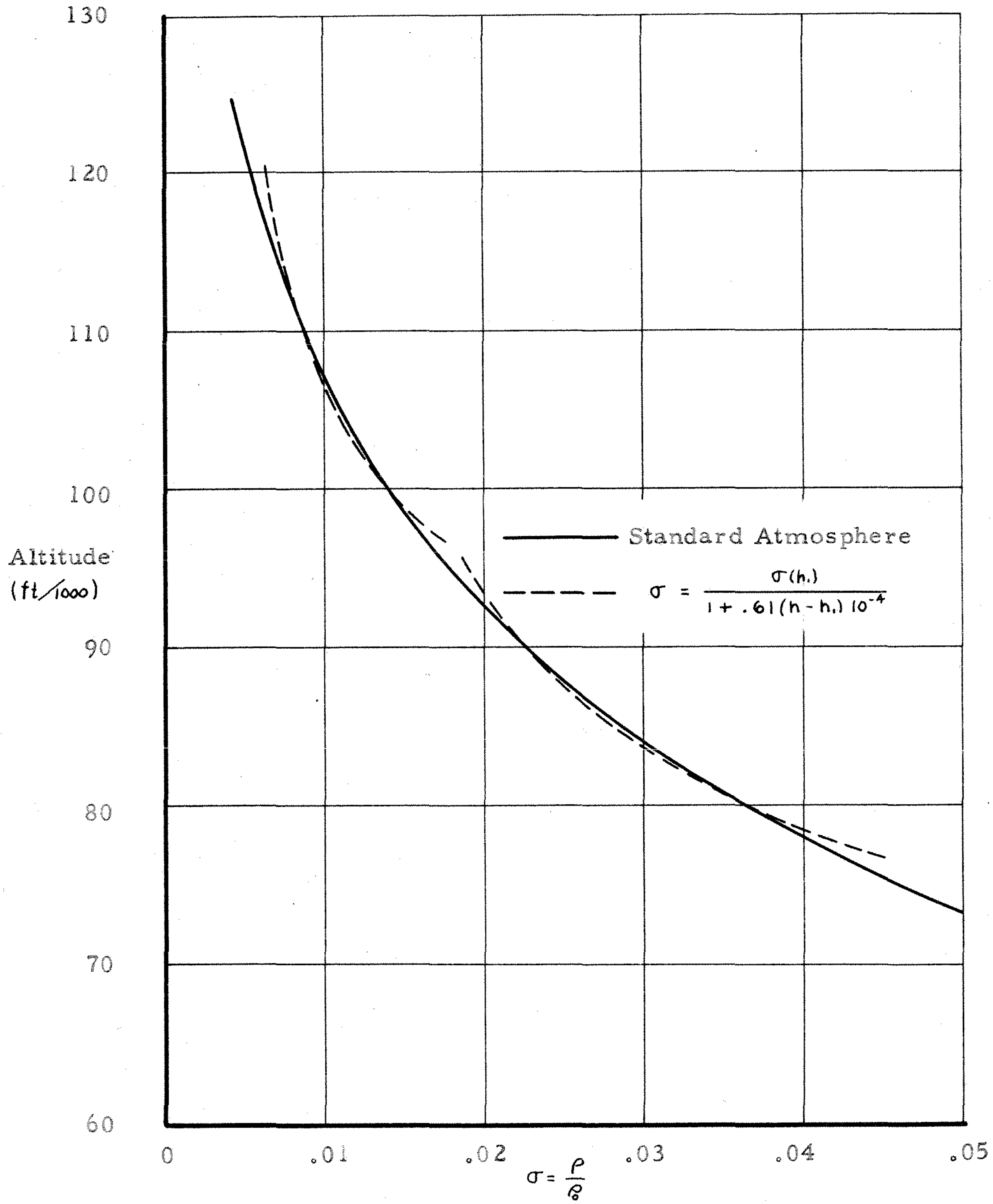


Figure 3b. Comparison of Approximate Density Variation with Altitude and the Standard Atmosphere

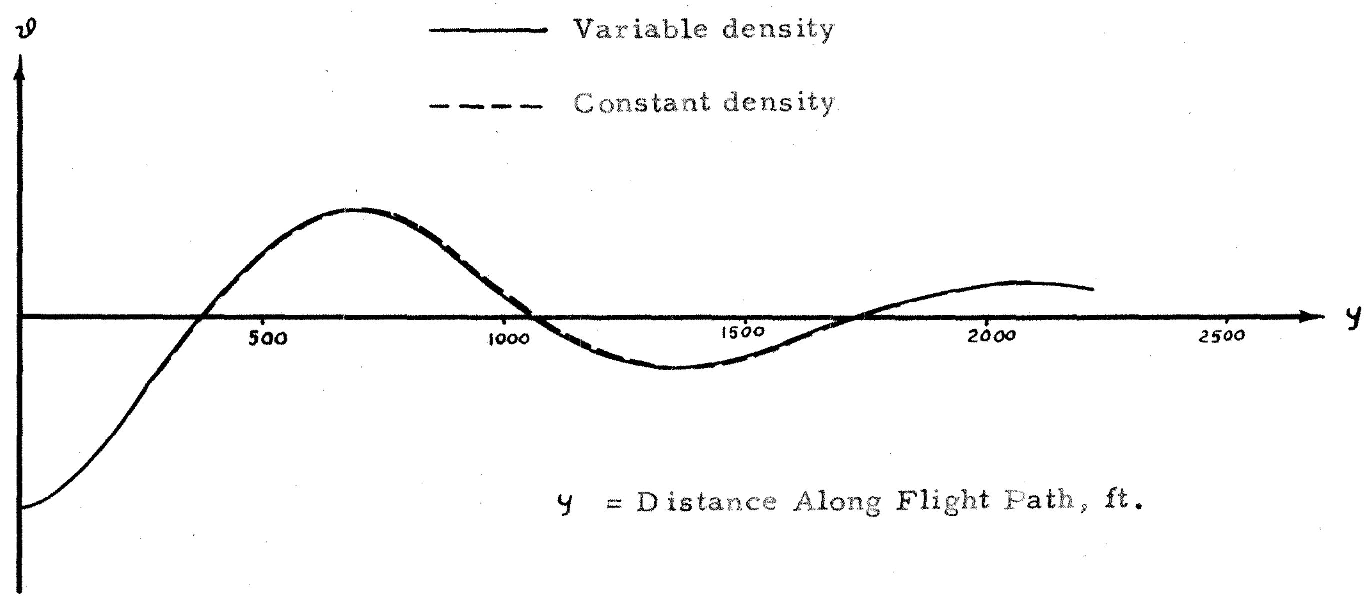


Figure 4. Effect of Variable Density on the Response of Disturbed Missile In Vertical Flight. Missile "B", Initial Altitude = 10,000 ft.

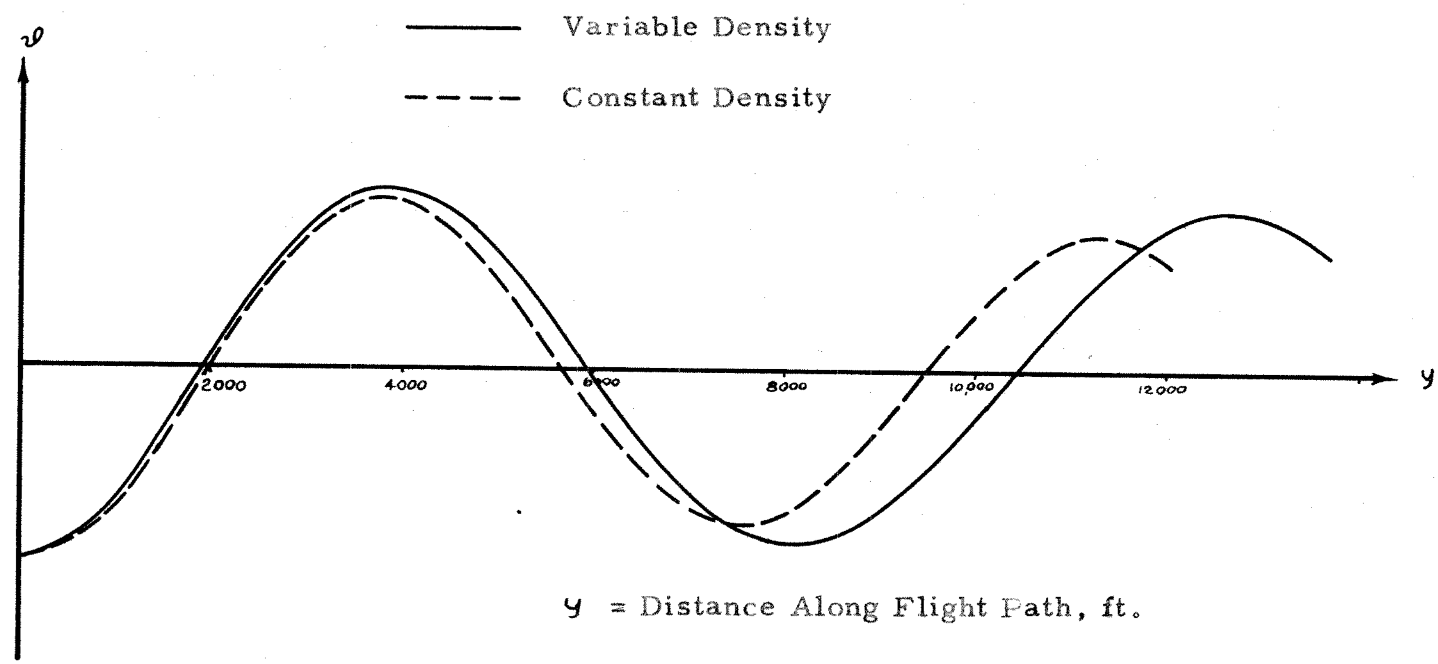


Figure 5. Effect of Variable Density on the Response of a Disturbed Missile in Vertical Flight. Missile "B", Initial Altitude = 90,000 ft.

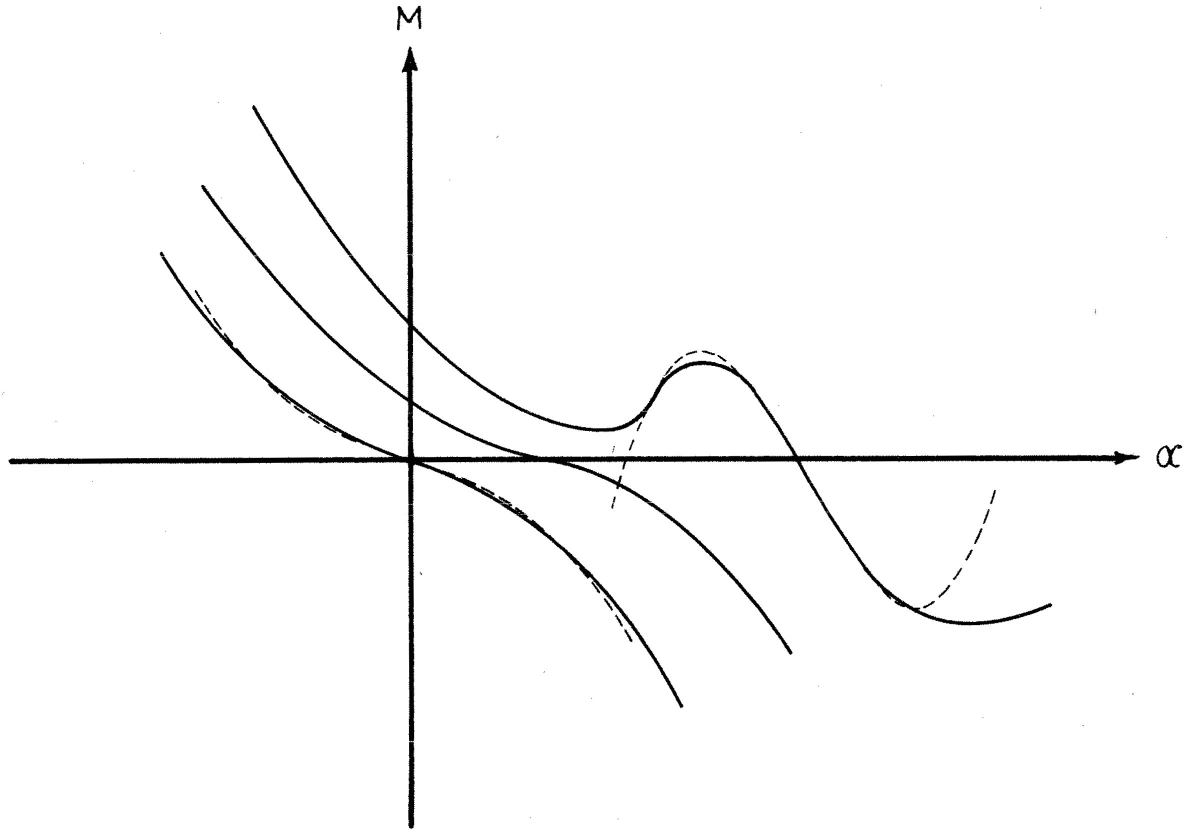


Figure 6. General Character of Pitching Moment vs. Angle of Attack for Missile "A"

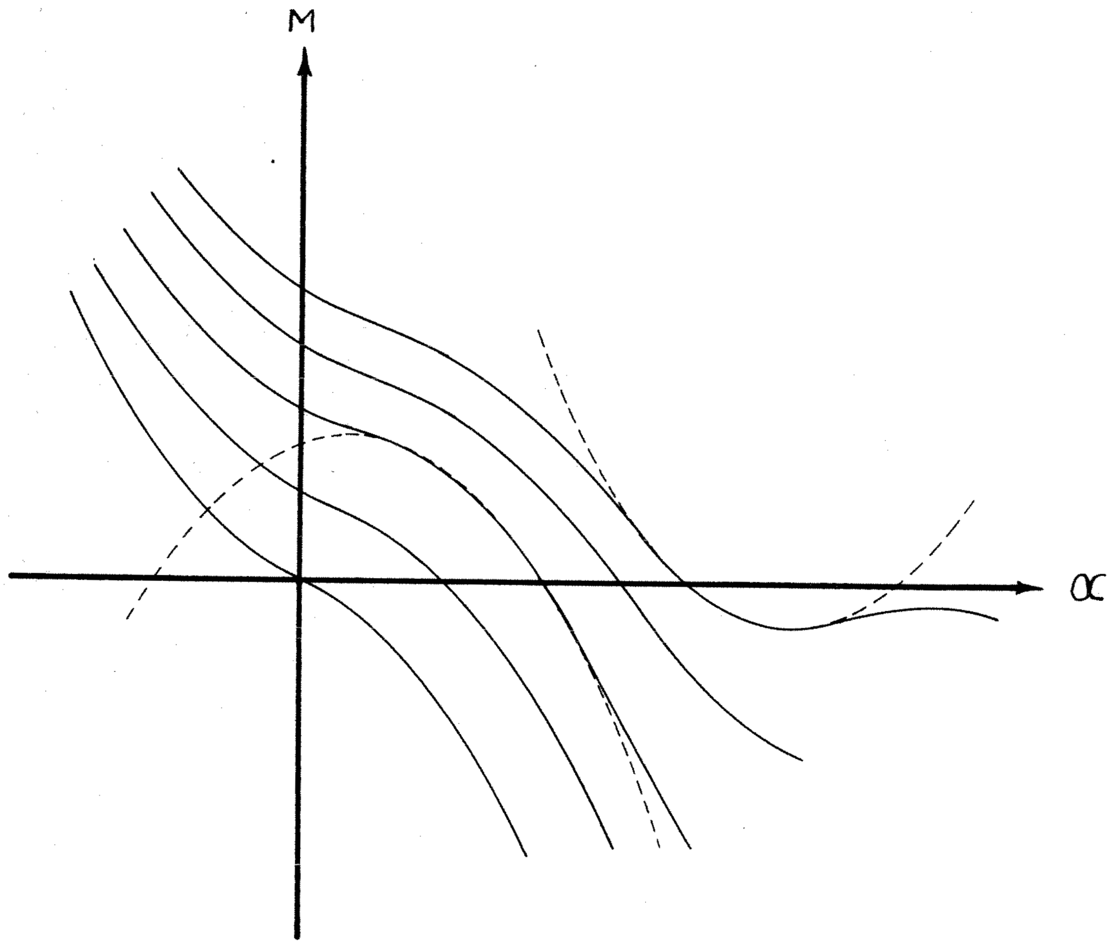


Figure 7. General Character of Pitching Moment vs. Angle of Attack for Missile "B"

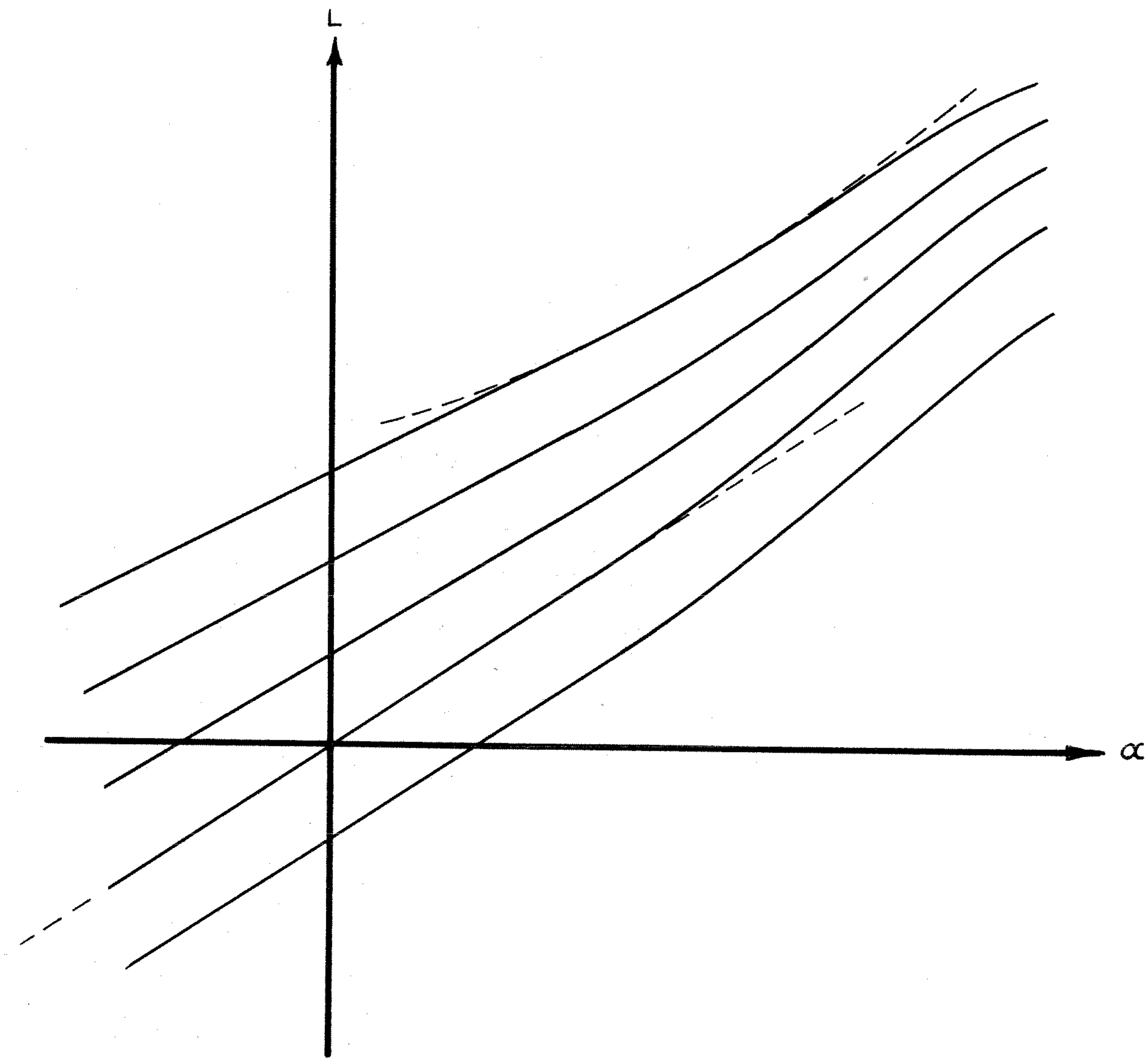


Figure 8. General Character of Lift vs. Angle of Attack for Representative Missiles.

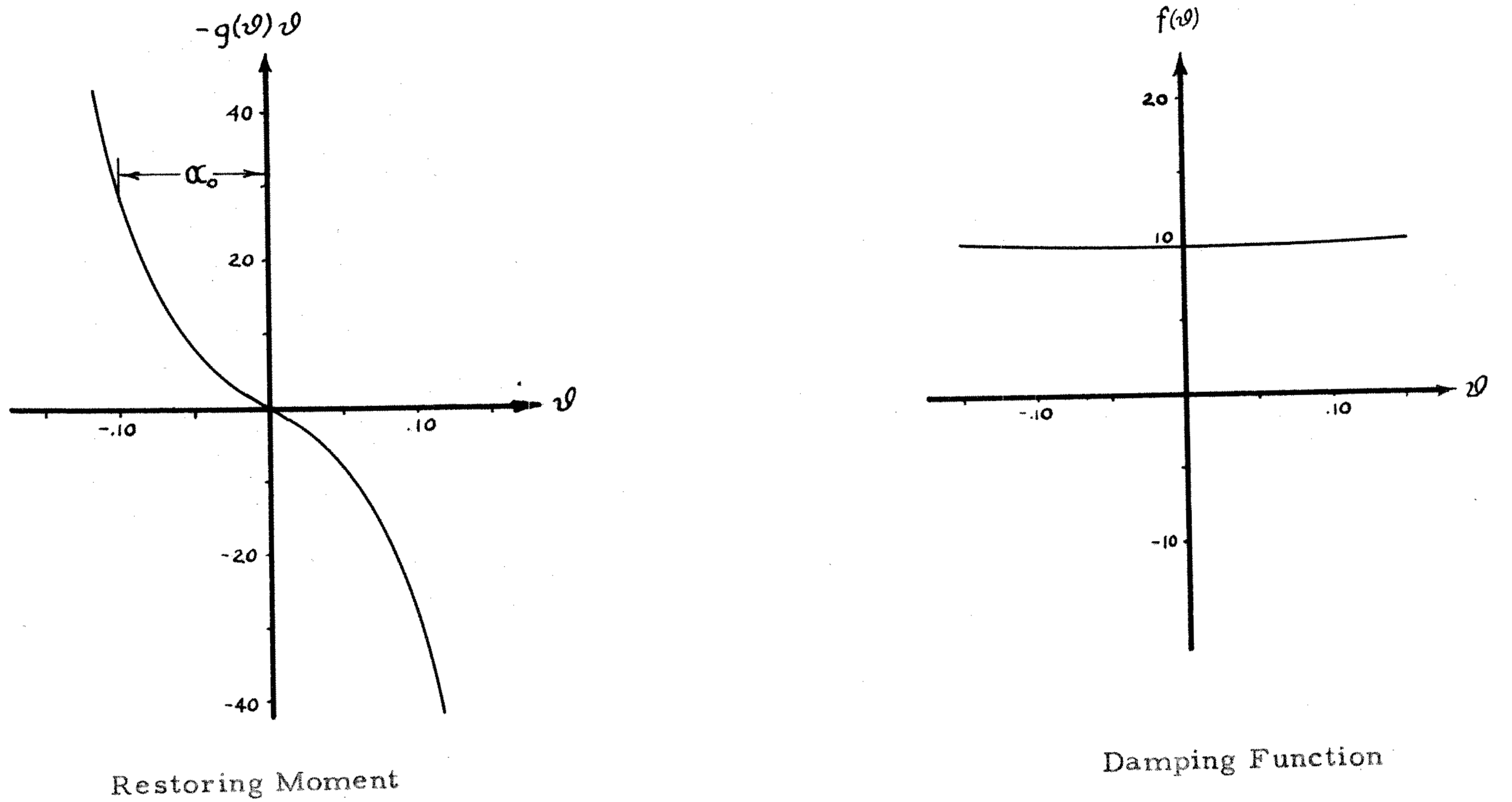


Figure 9a. Restoring Moment and Damping Function vs. Incremental Angle of Attack. Example I

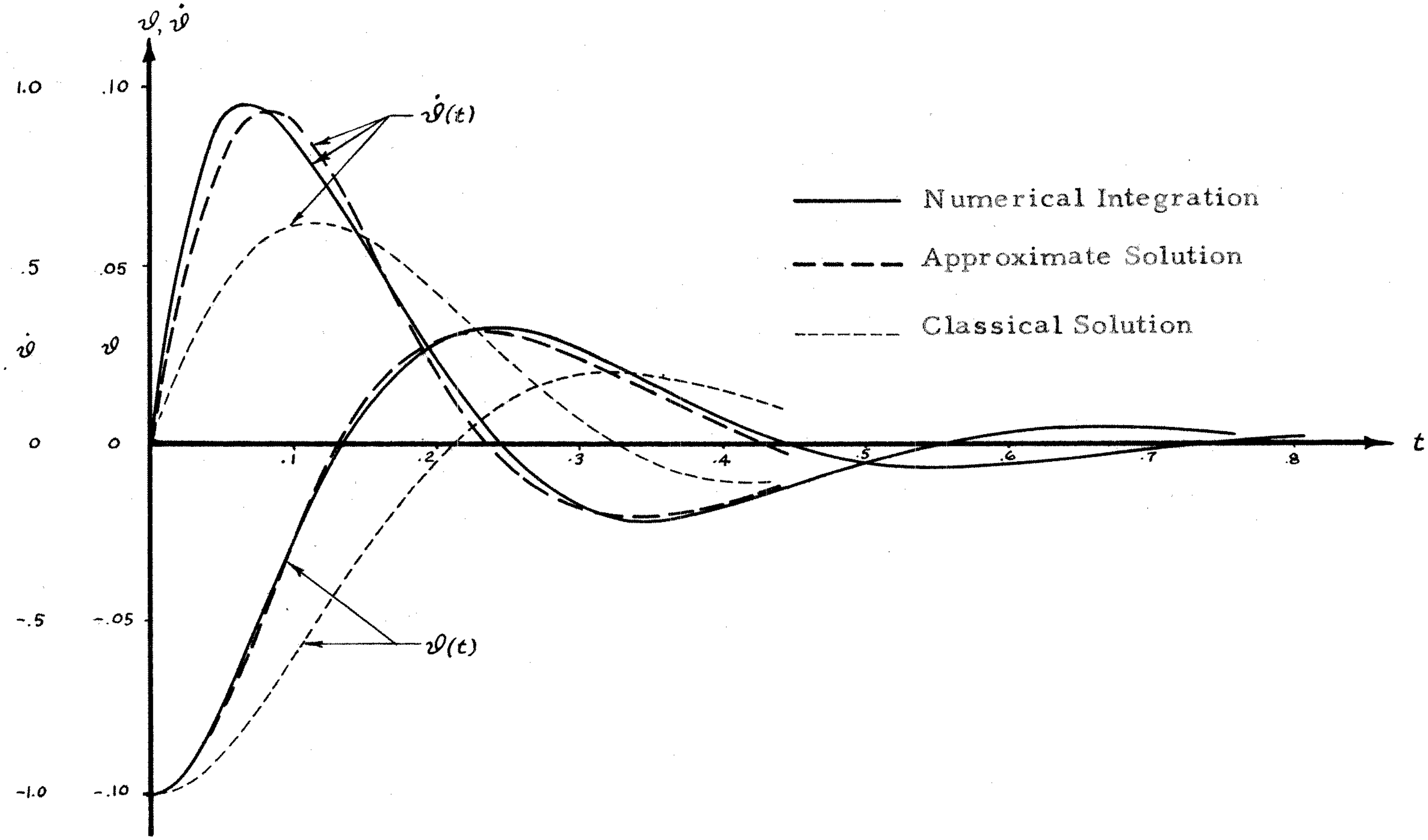
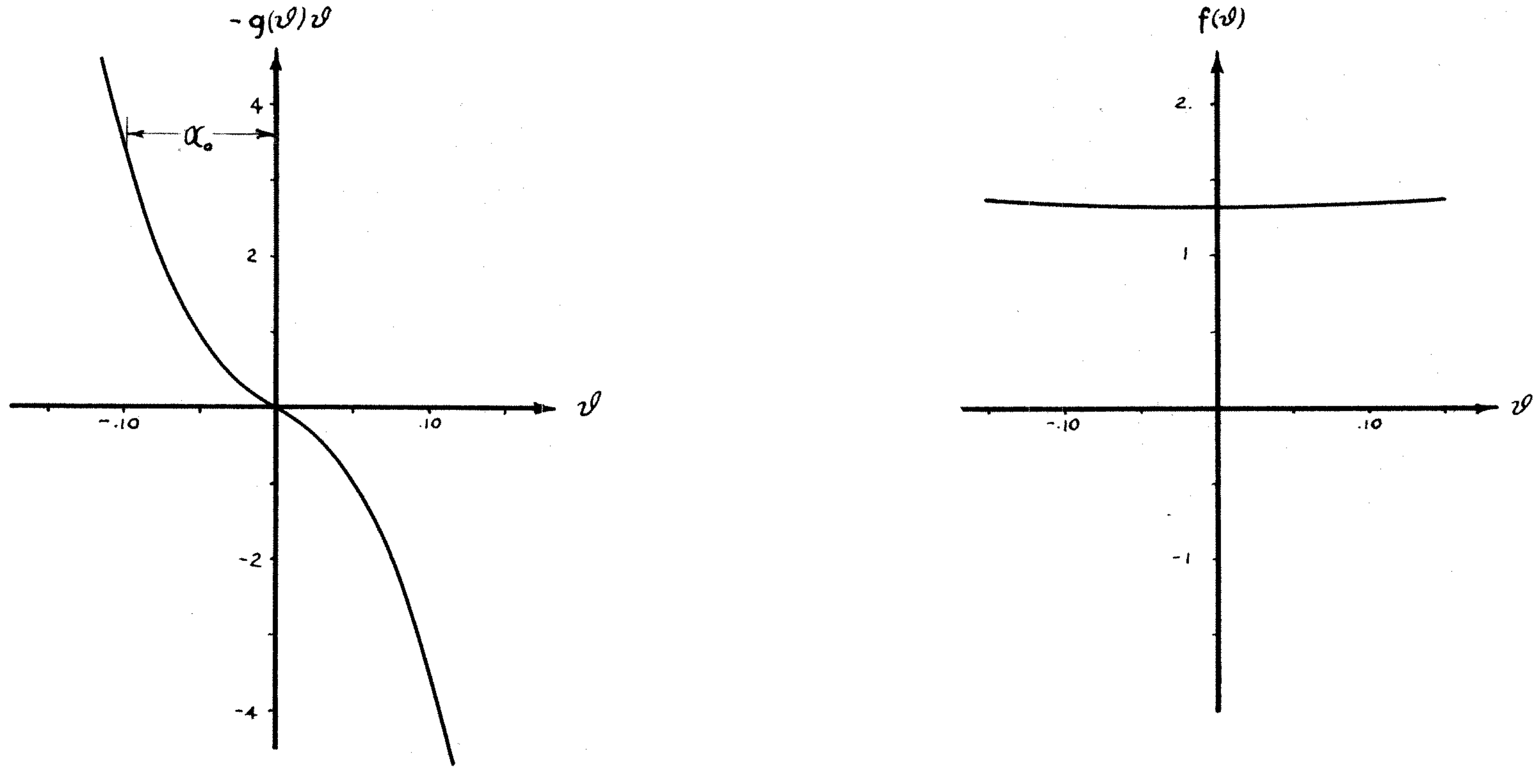


Figure 9b. Dynamic Response to Step Function Input in Angle of Attack. Example I.



Restoring Moment Damping Function

Figure 10a. Restoring Moment and Damping Function vs. Incremental Angle of Attack. Example II

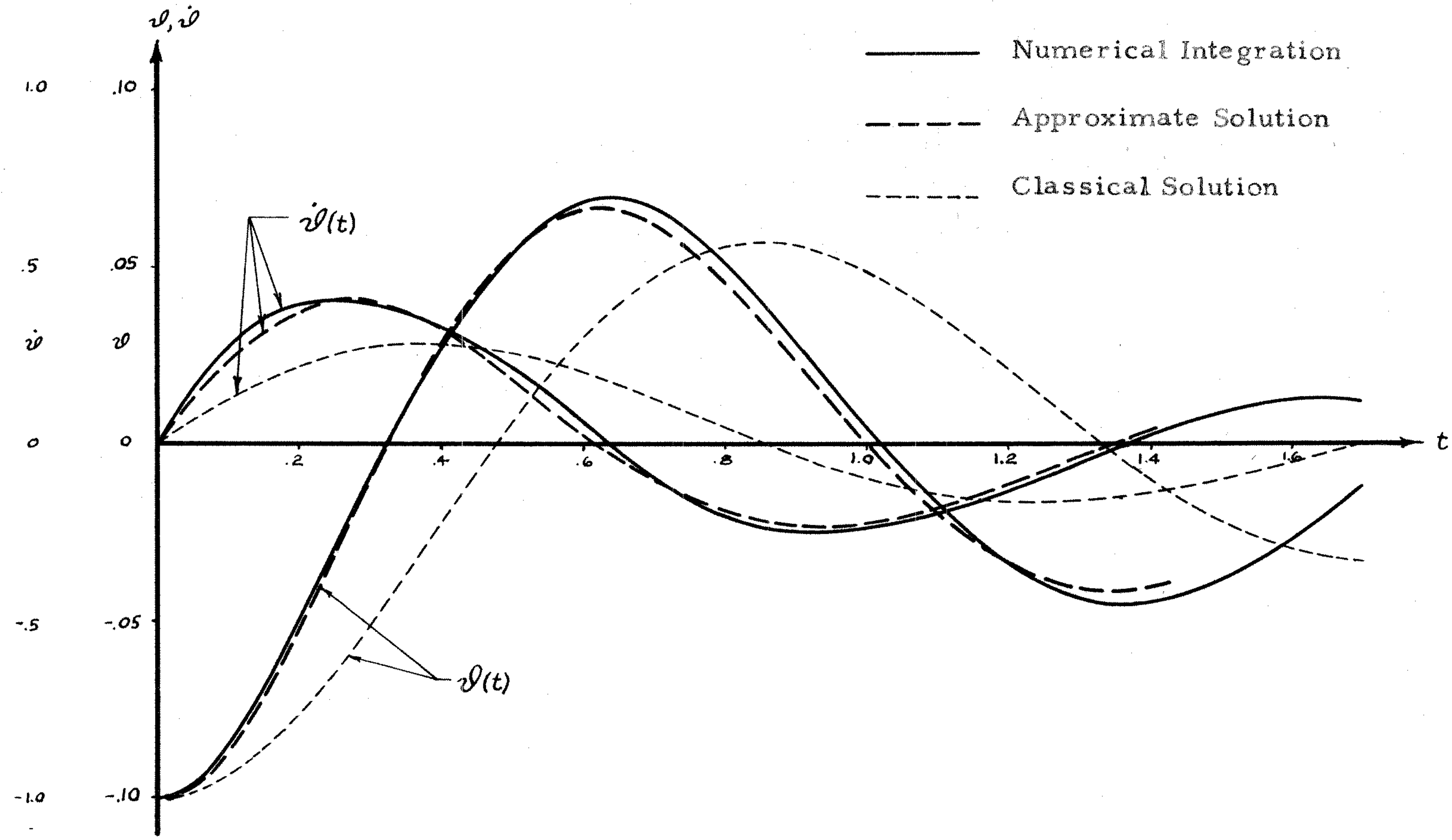
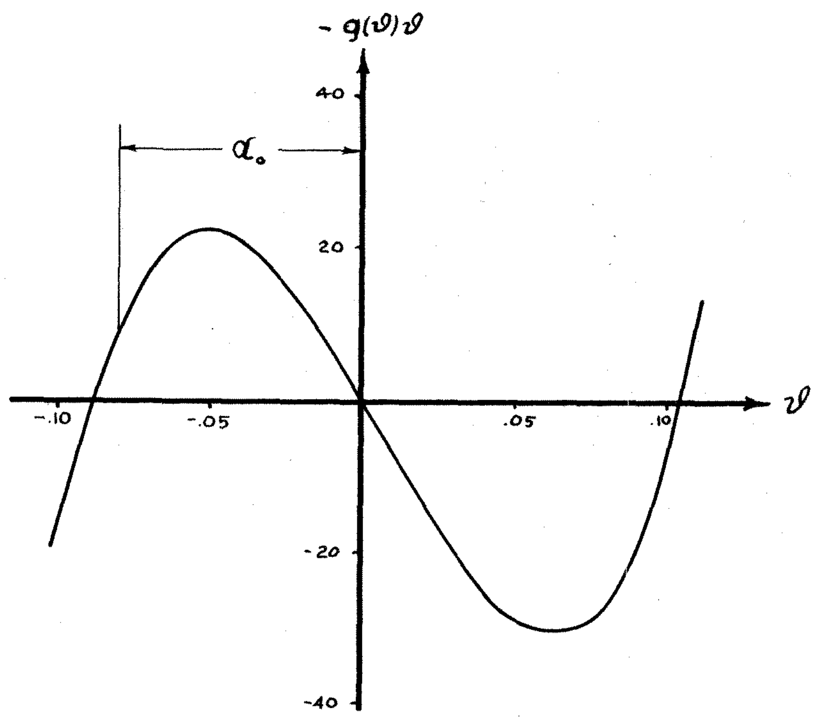
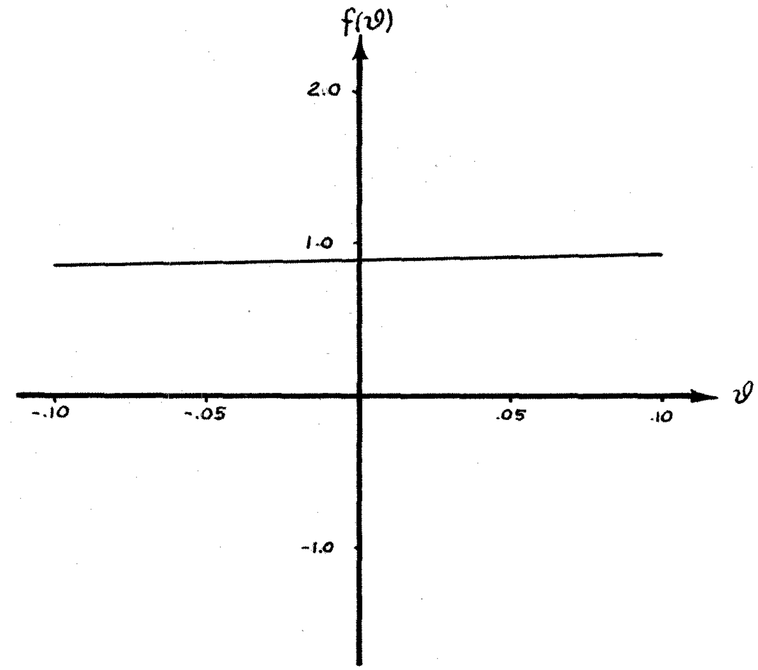


Figure 10b. Dynamic Response to Step Function Input in Angle of Attack. Example II



Restoring Moment



Damping Function

Figure 11a. Restoring Moment and Damping Function vs. Incremental Angle of Attack, Example III

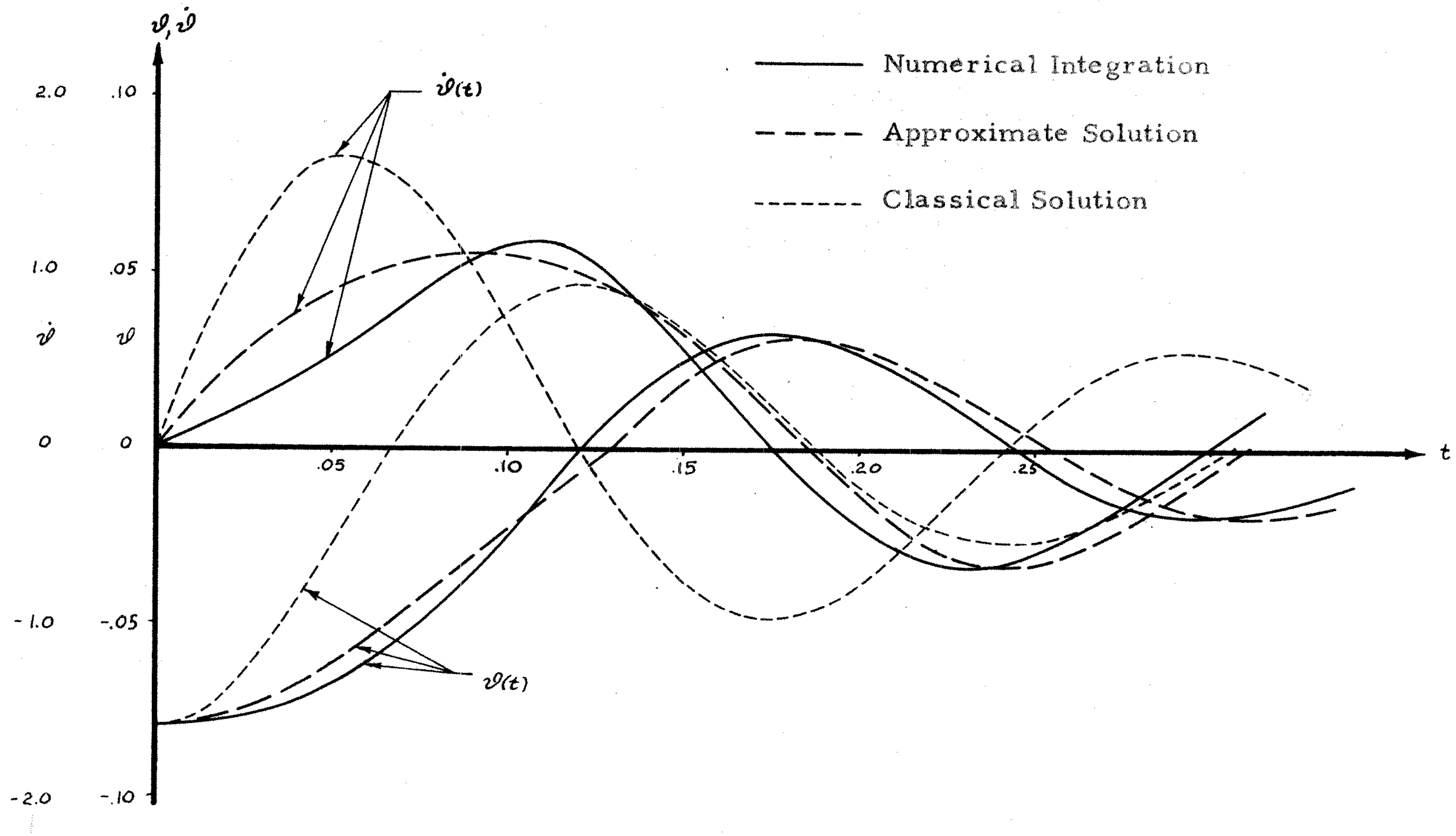
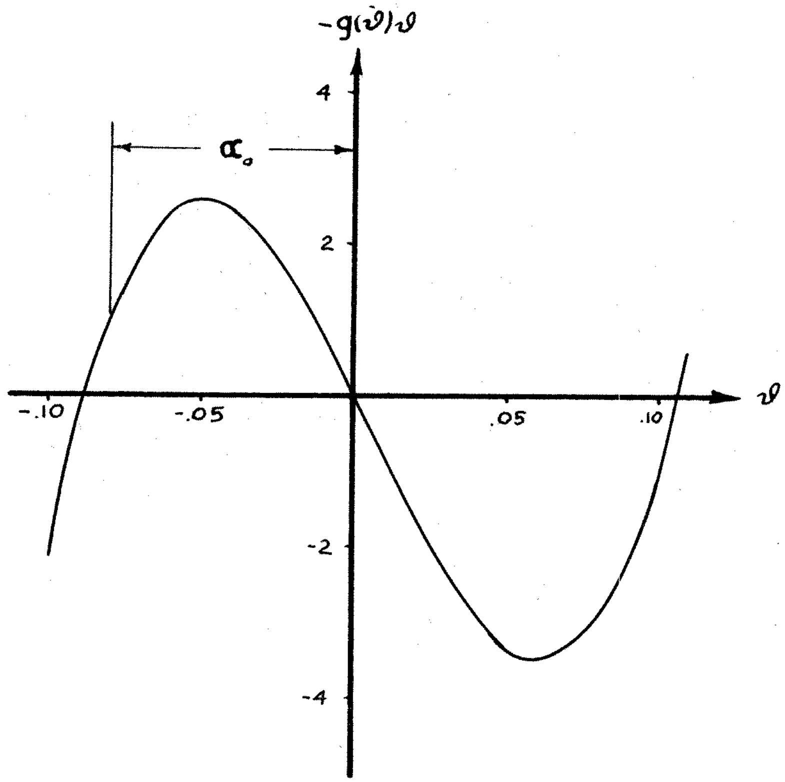
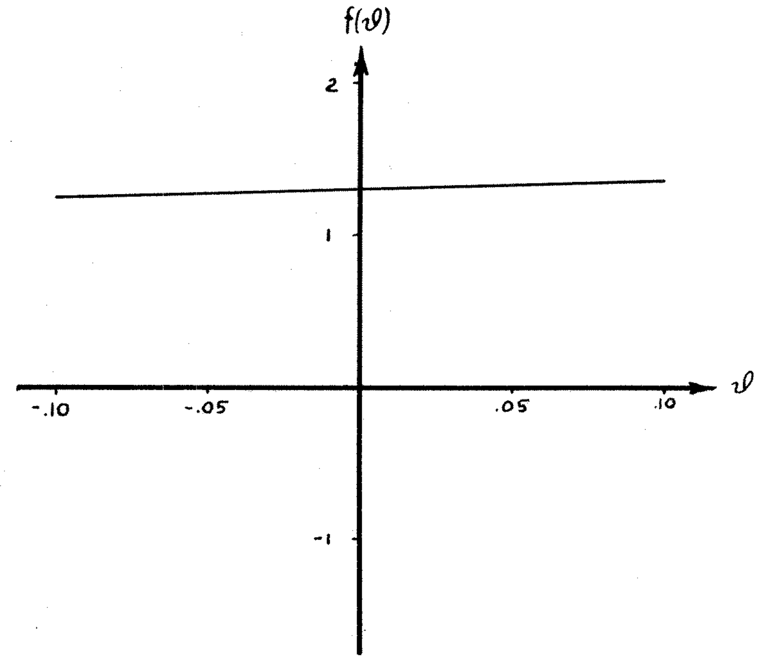


Figure 11b. Dynamic Response to Step Function Input in Angle of Attack. Example III



Restoring Moment



Damping Function

Figure 12a. Restoring Moment and Damping Function vs. Incremental Angle of Attack. Example IV

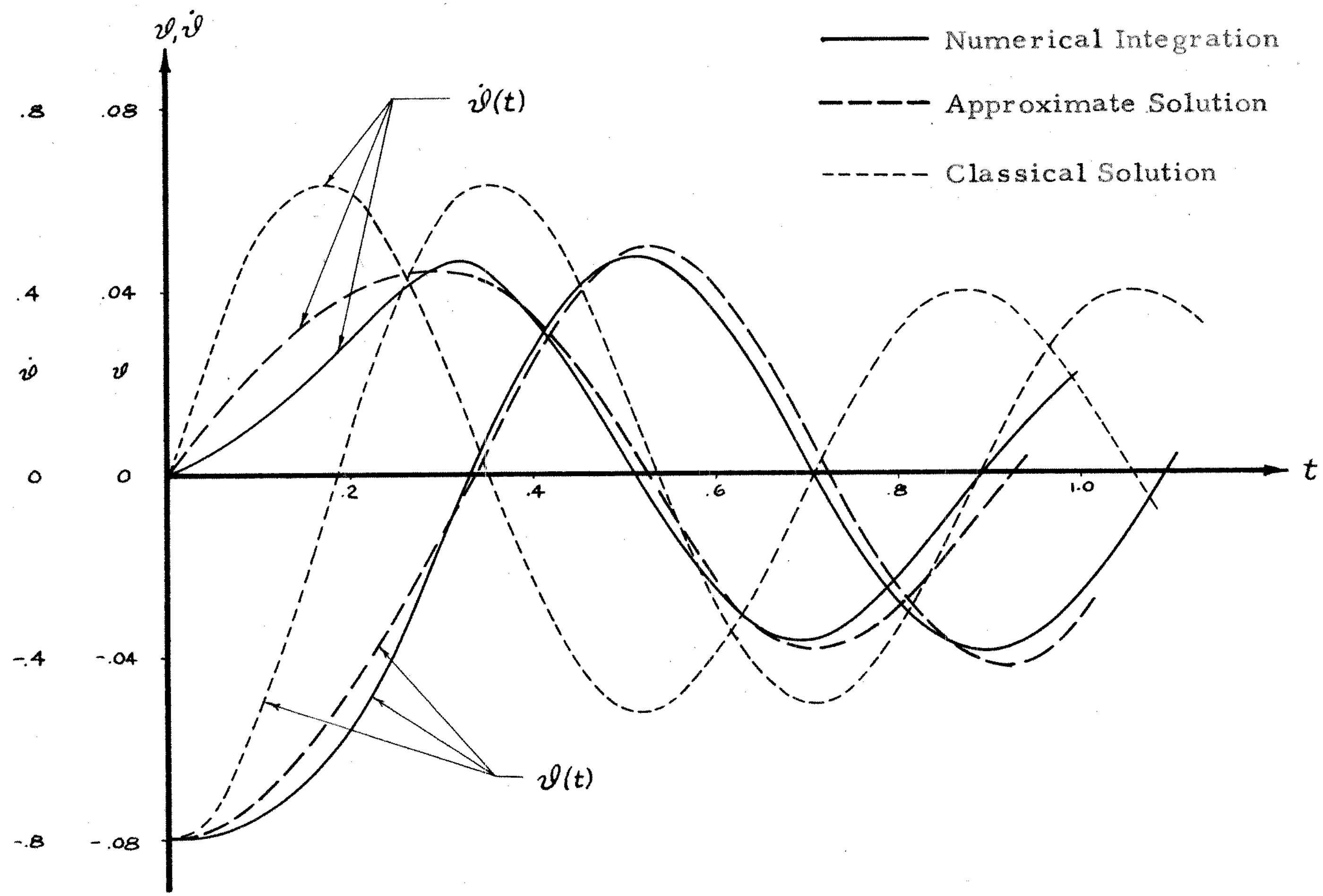
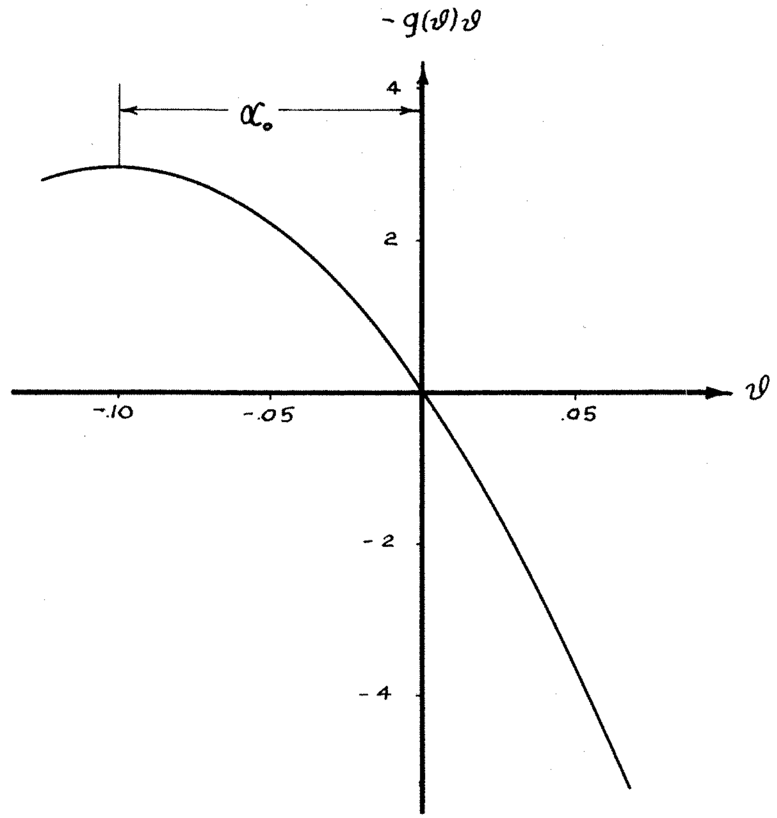
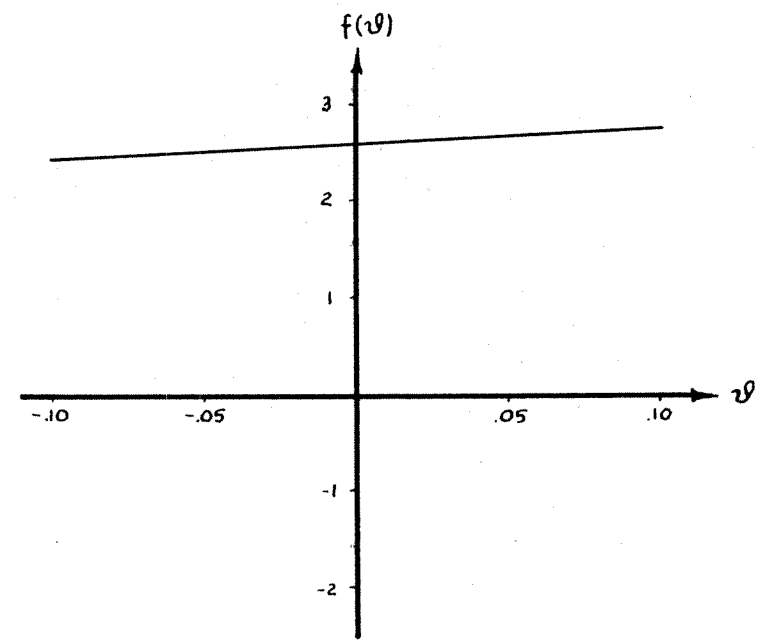


Figure 12b. Dynamic Response to Step Function
Input in Angle of Attack. Example IV



Restoring Moment



Damping Function

Figure 13a. Restoring Moment and Damping Function vs. Incremental Angle of Attack. Example V

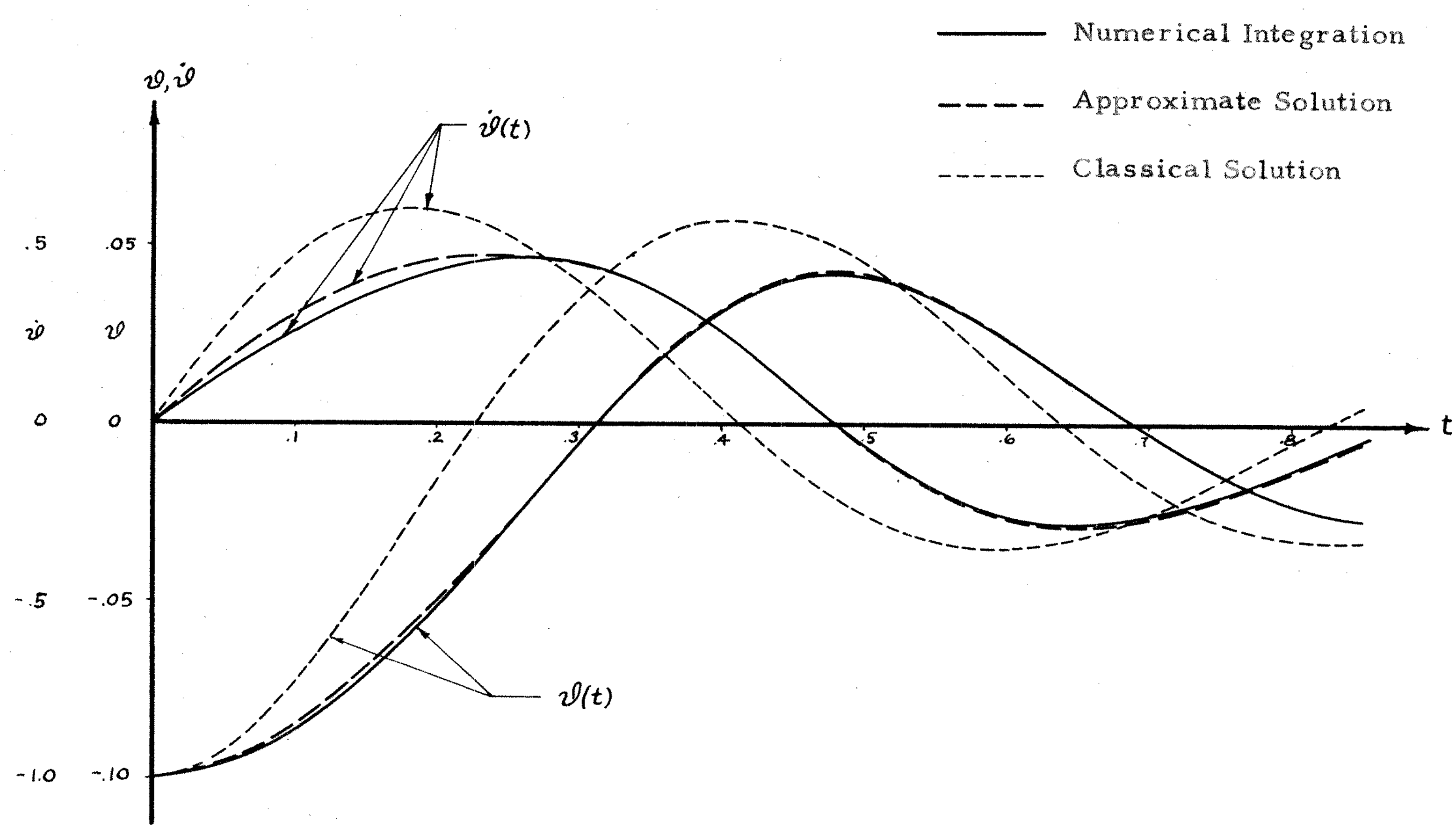
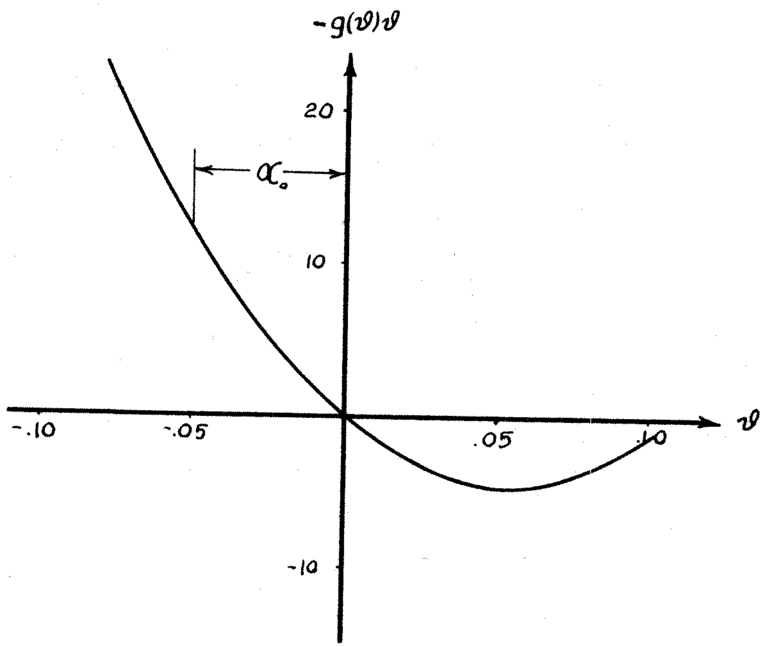
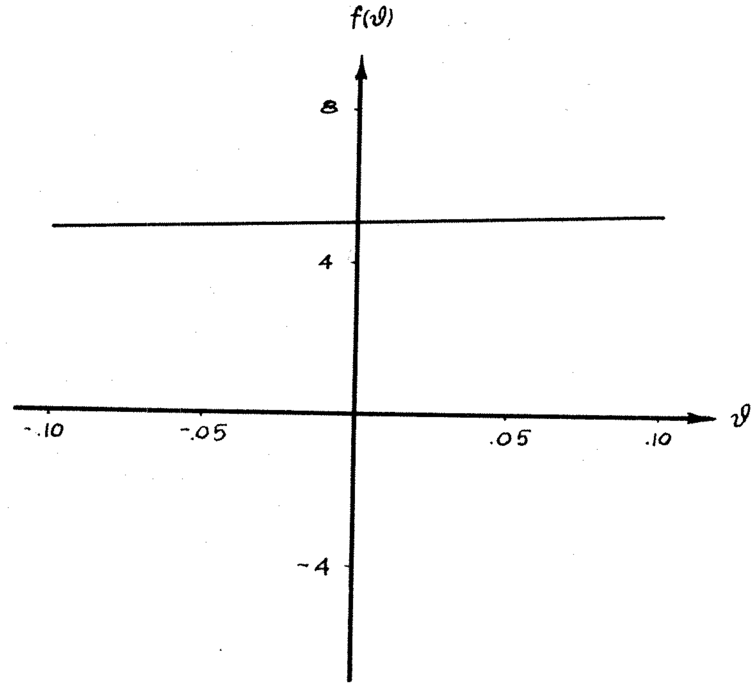


Figure 13b. Dynamic Response to Step Function Input in Angle of Attack. Example V.



Restoring Moment



Damping Function

Figure 14a. Restoring Moment and Damping Function vs. Incremental Angle of Attack. Example VI.

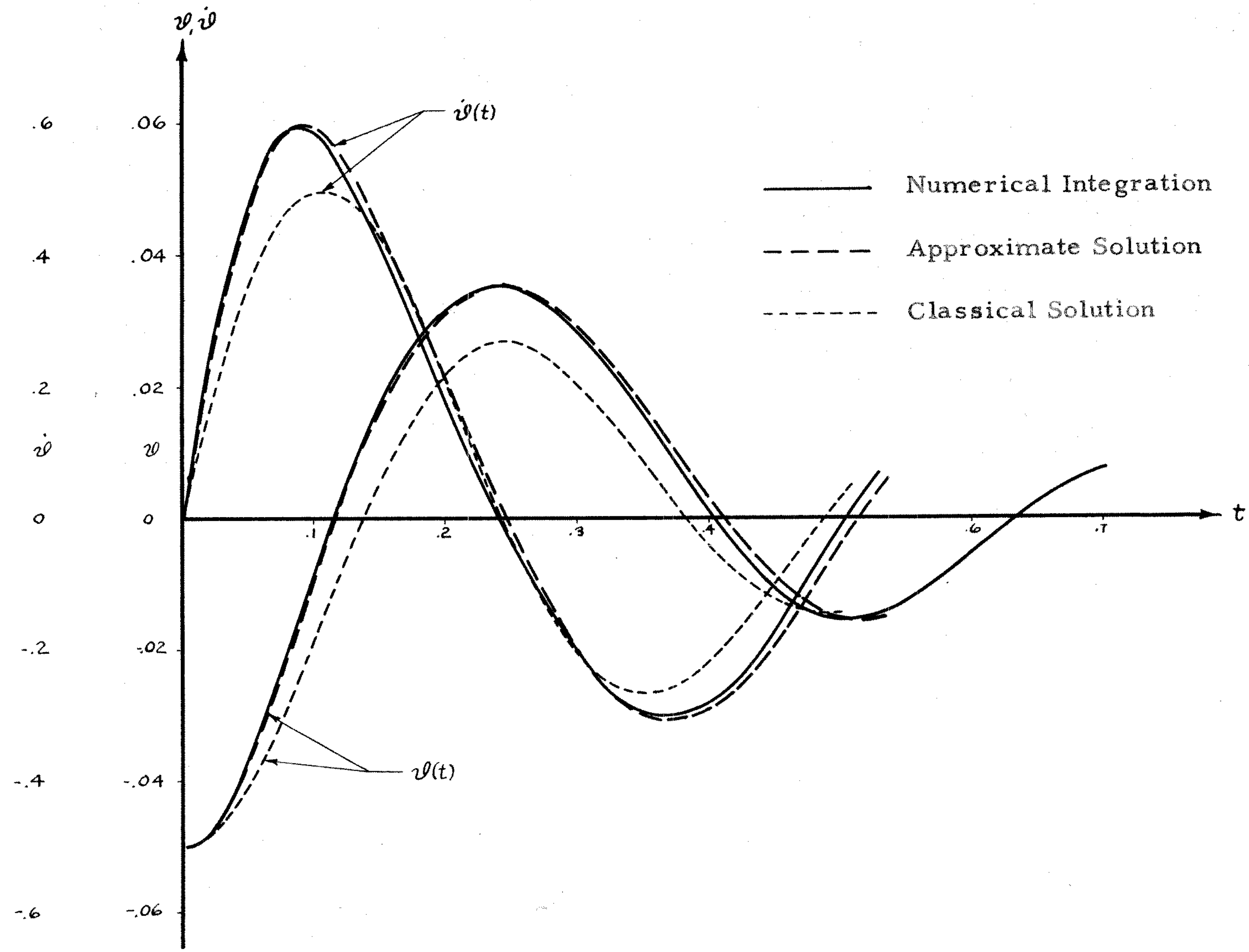
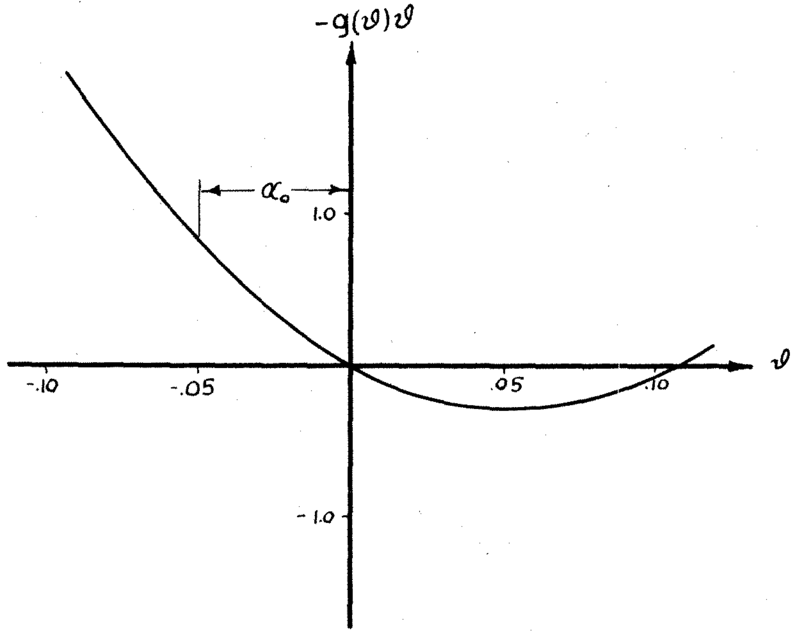
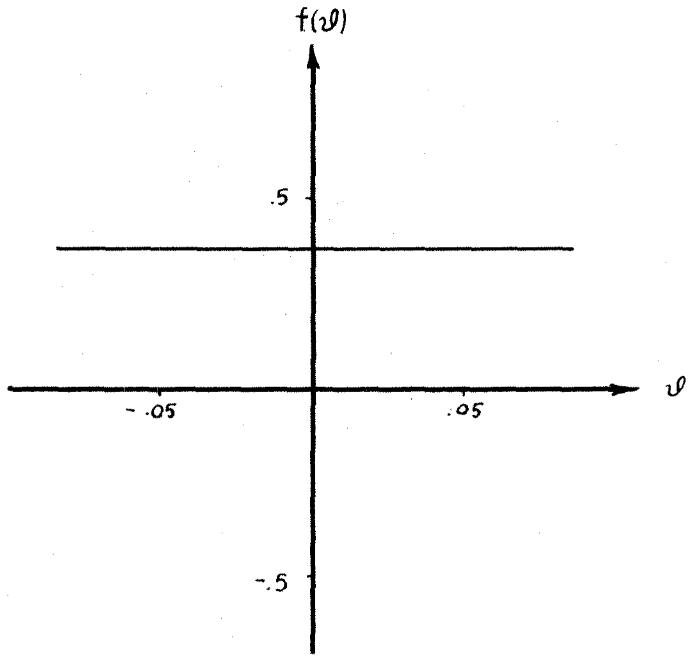


Figure 14b. Dynamic Response to Step Function Input in Angle of Attack. Example VI



Restoring Moment



Damping Function

Figure 15a. Restoring Moment and Damping Function vs. Incremental Angle of Attack. Example VII.

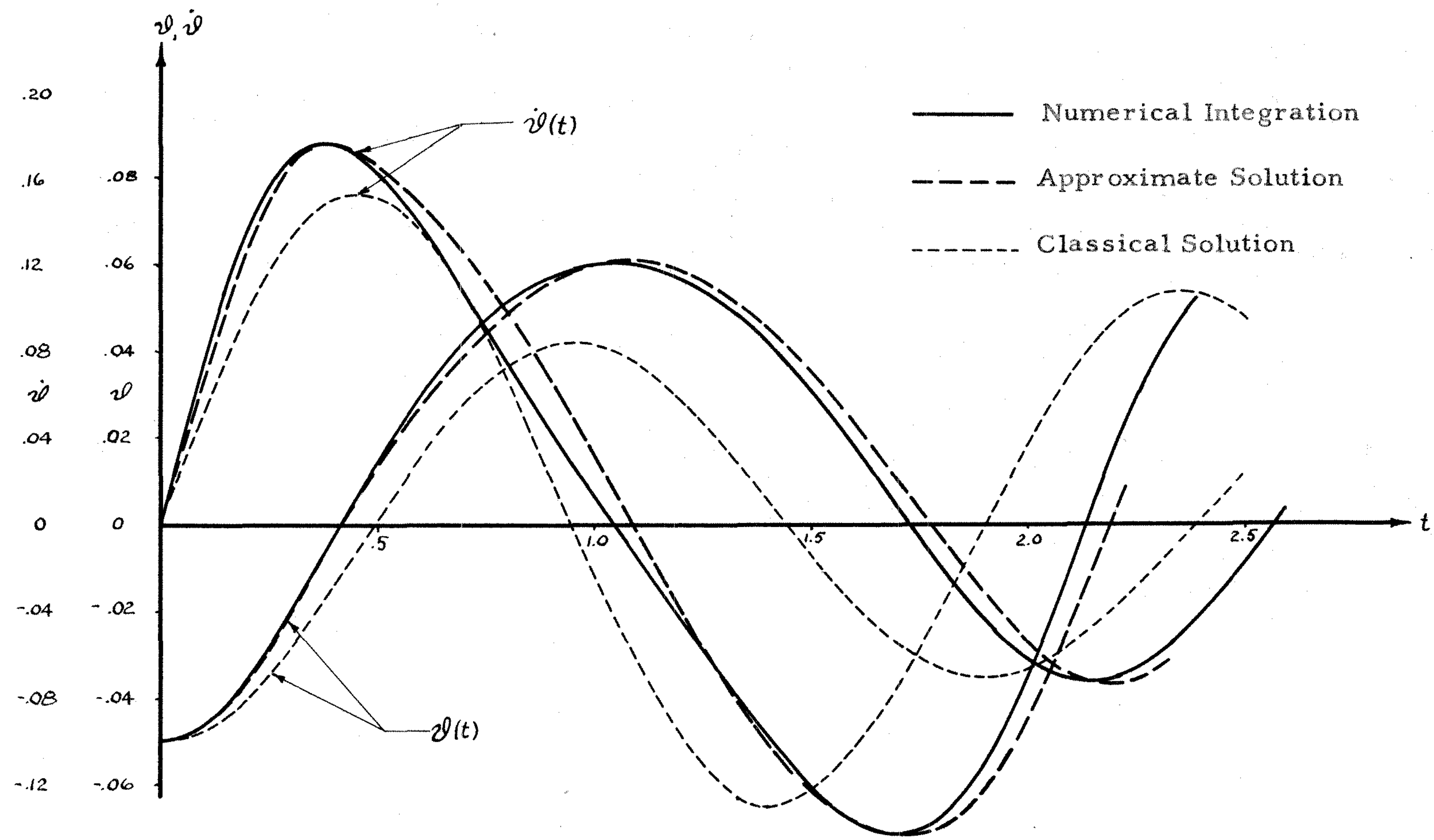


Figure 15b. Dynamic Response to Step Function Input in Angle of Attack. Example VII.

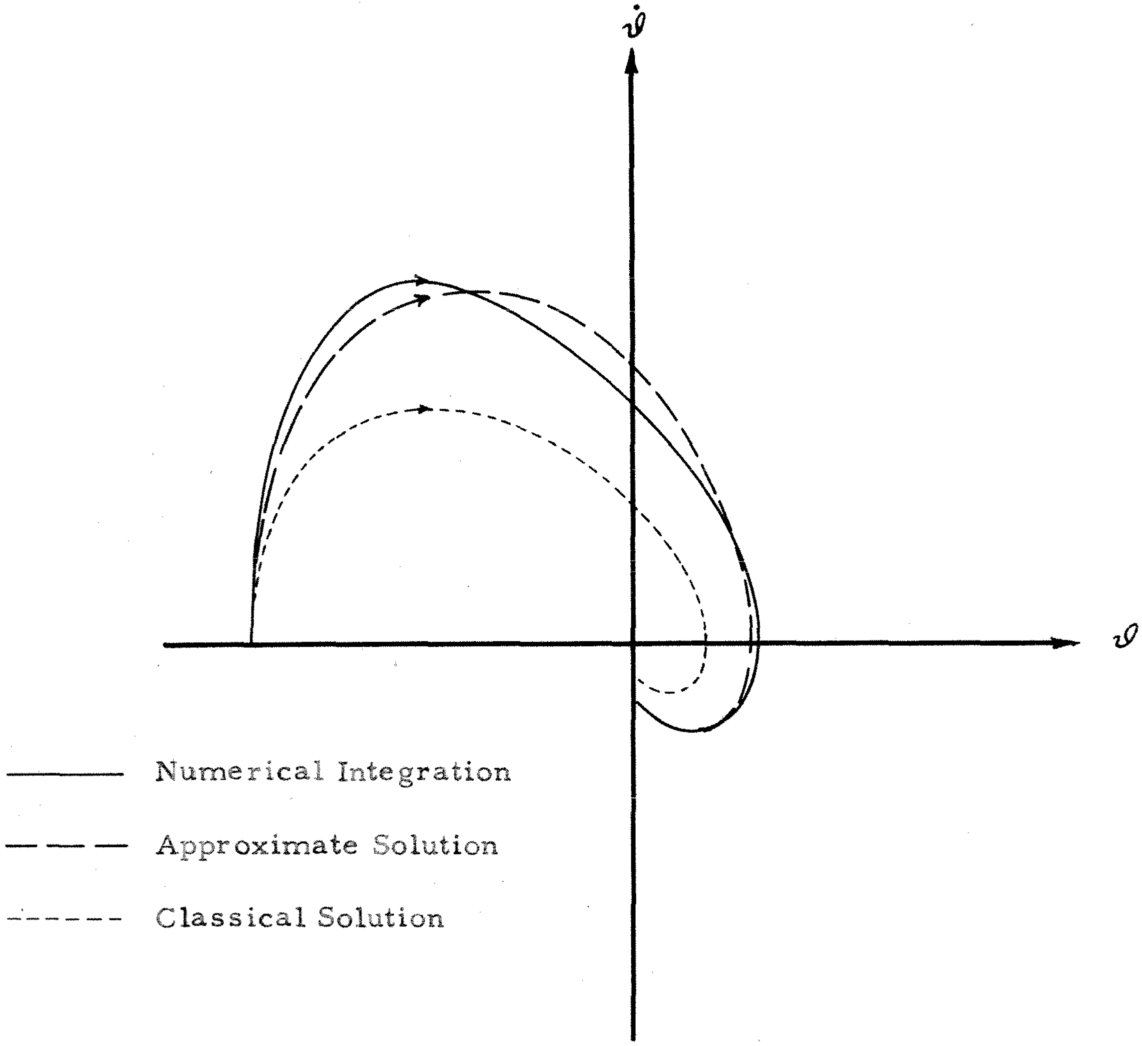


Figure 16. Phase Space Diagram for Example I

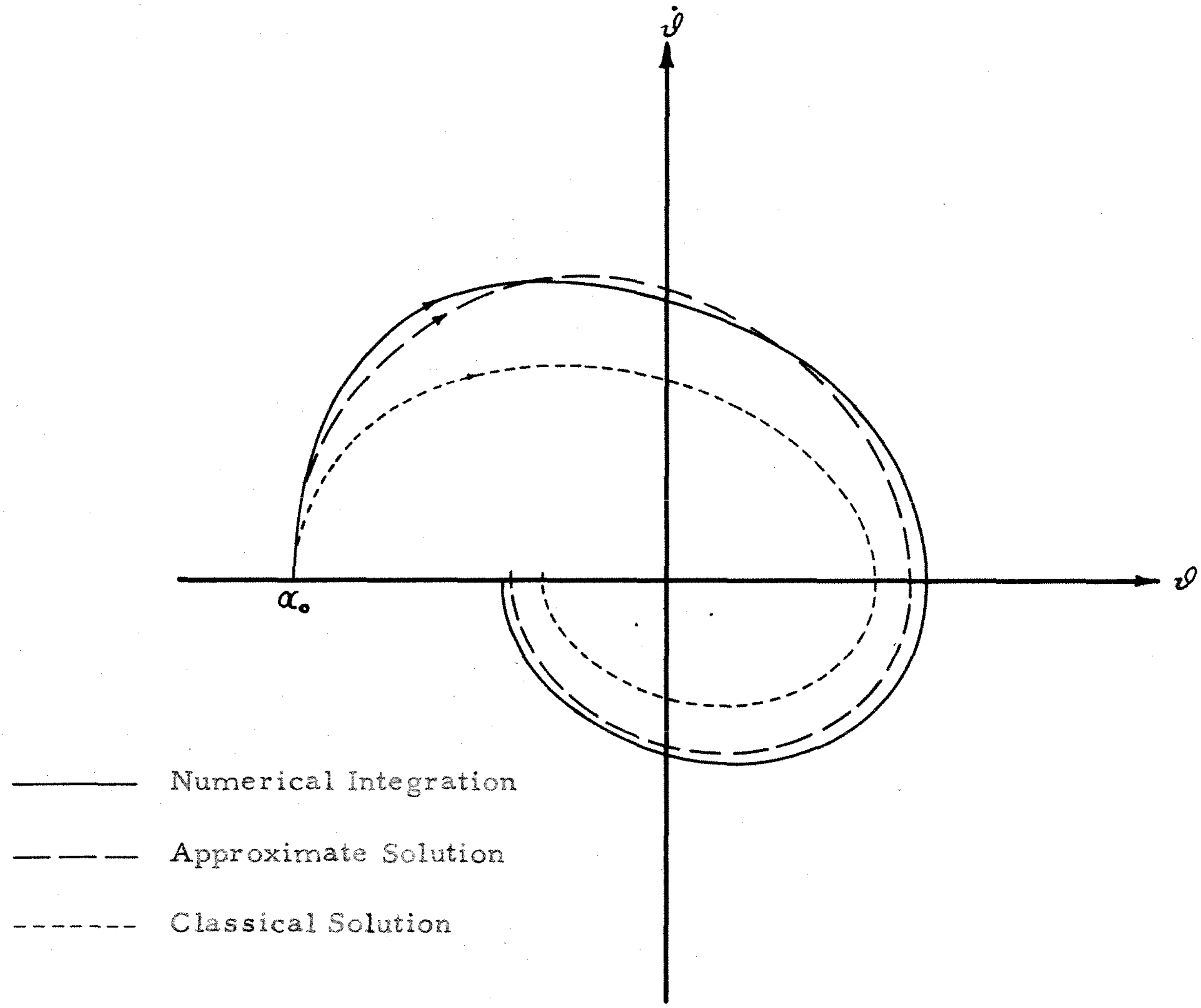


Figure 17. Phase Space Diagram for Example II

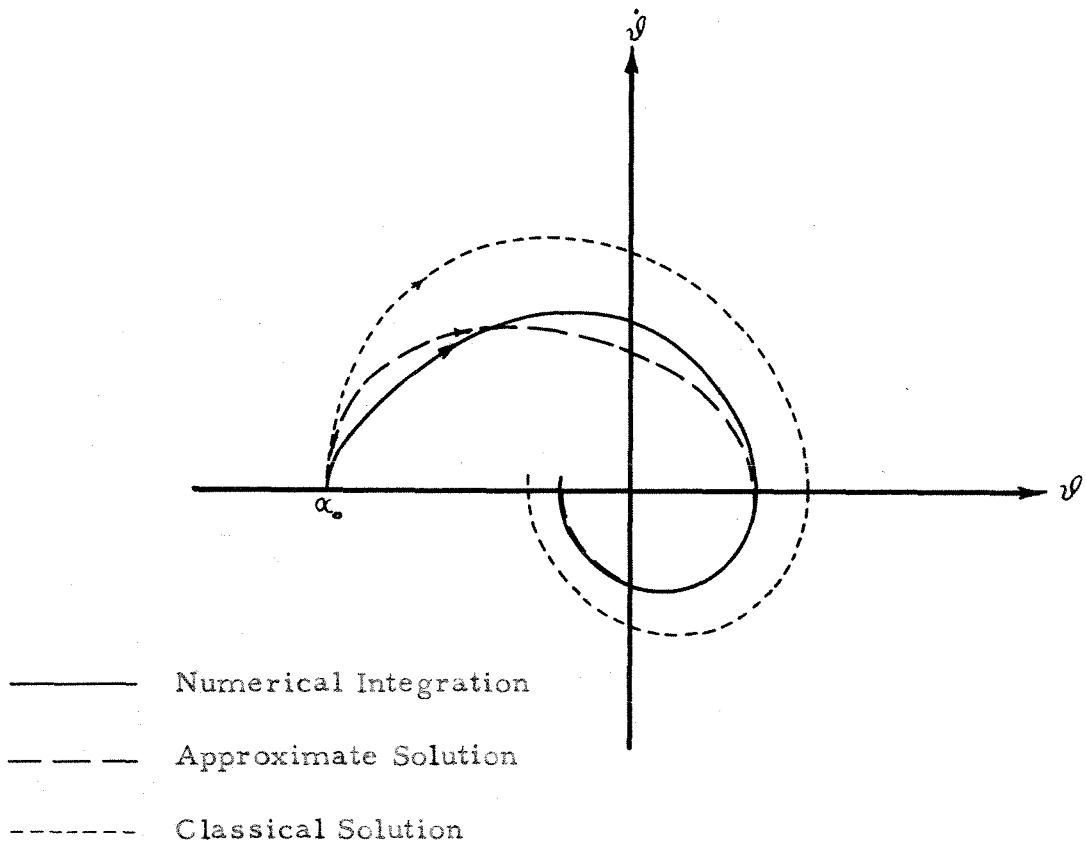


Figure 18. Phase Space Diagram for Example III

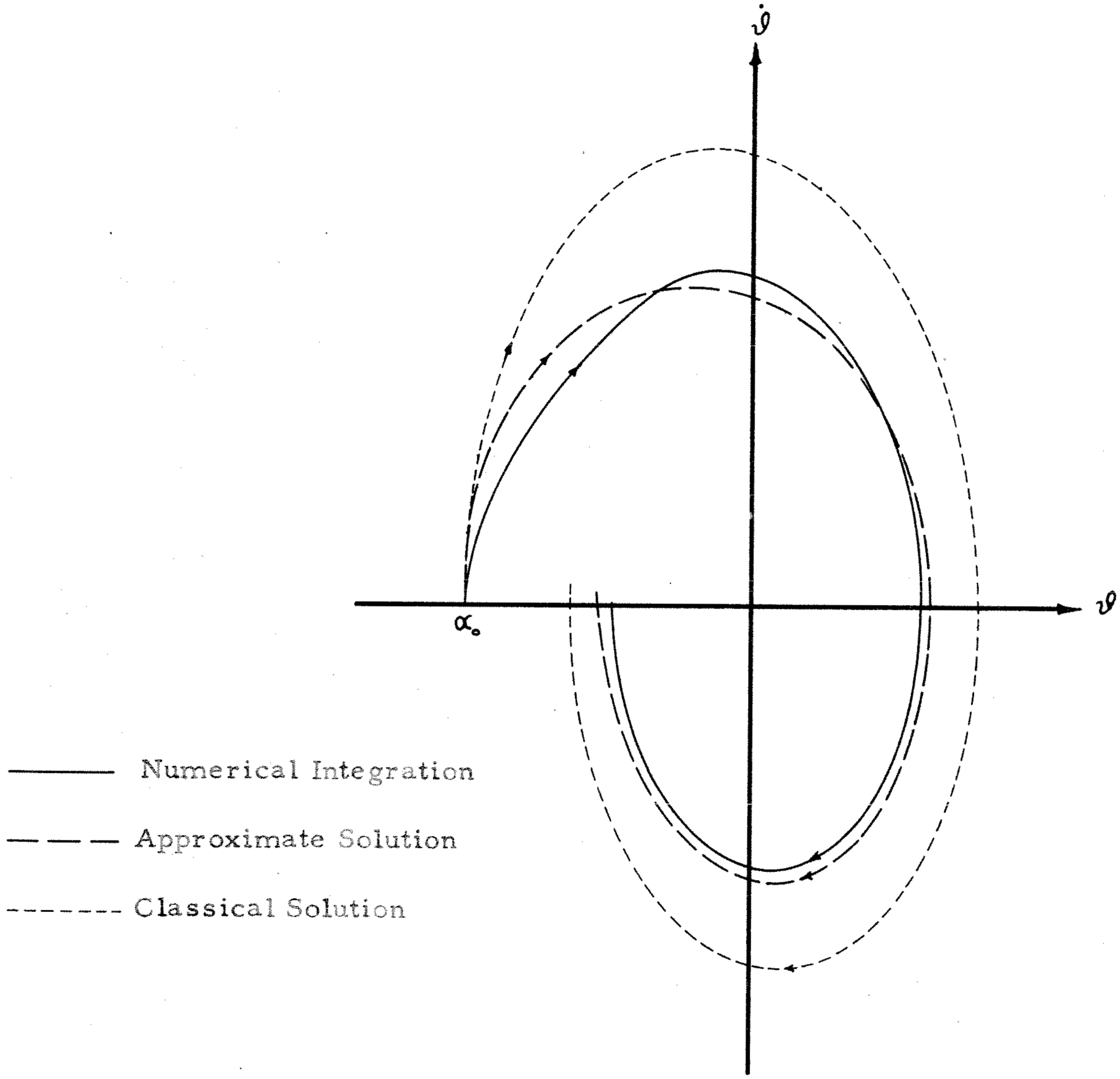


Figure 19. Phase Space Diagram for Example IV

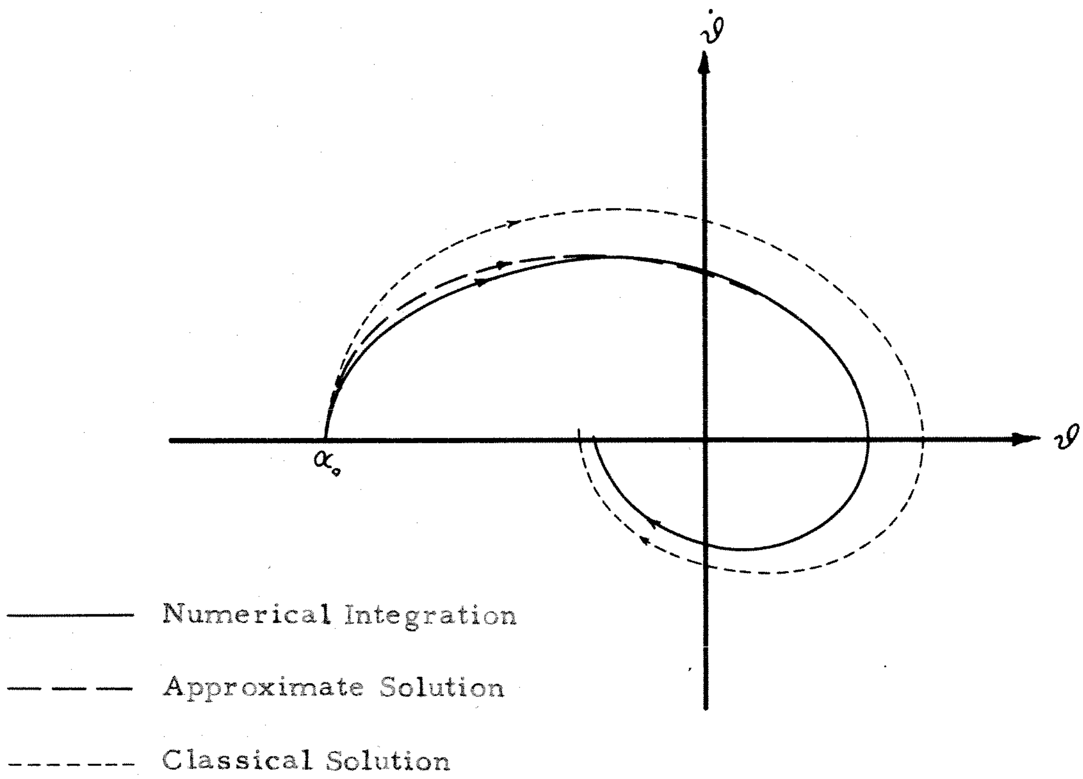


Figure 20. Phase Space Diagram for Example V

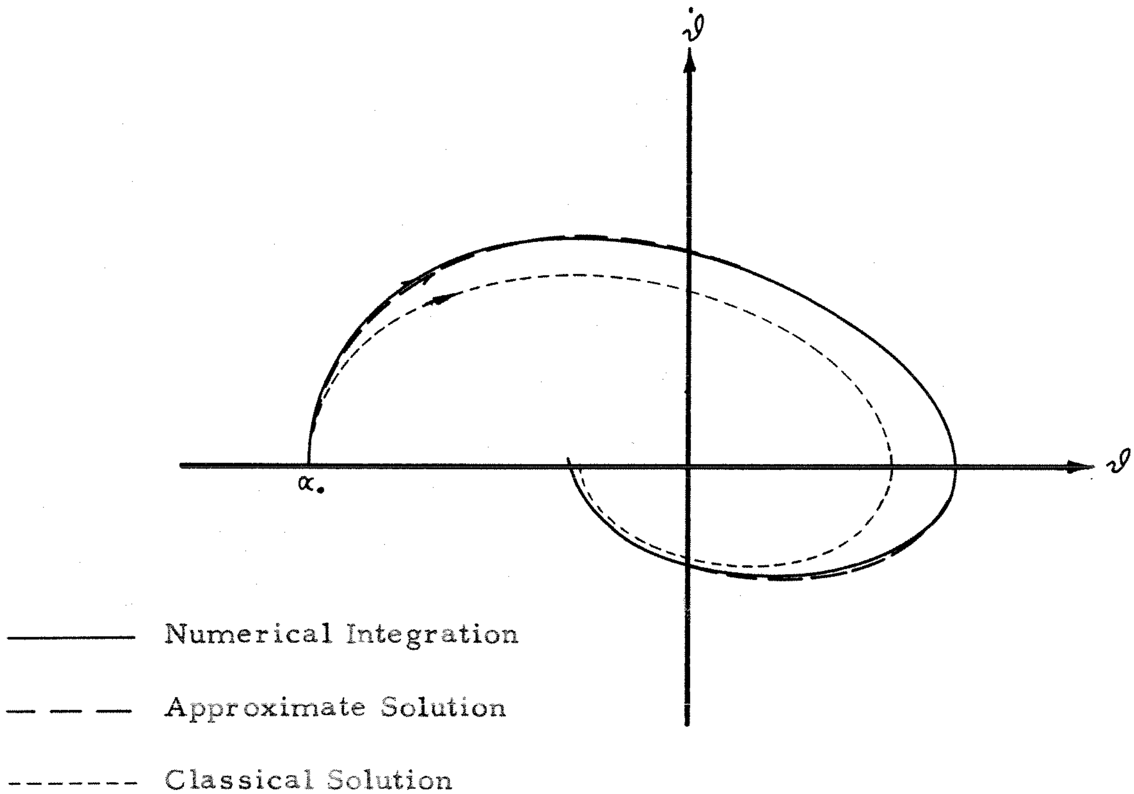


Figure 21. Phase Space Diagram for Example VI

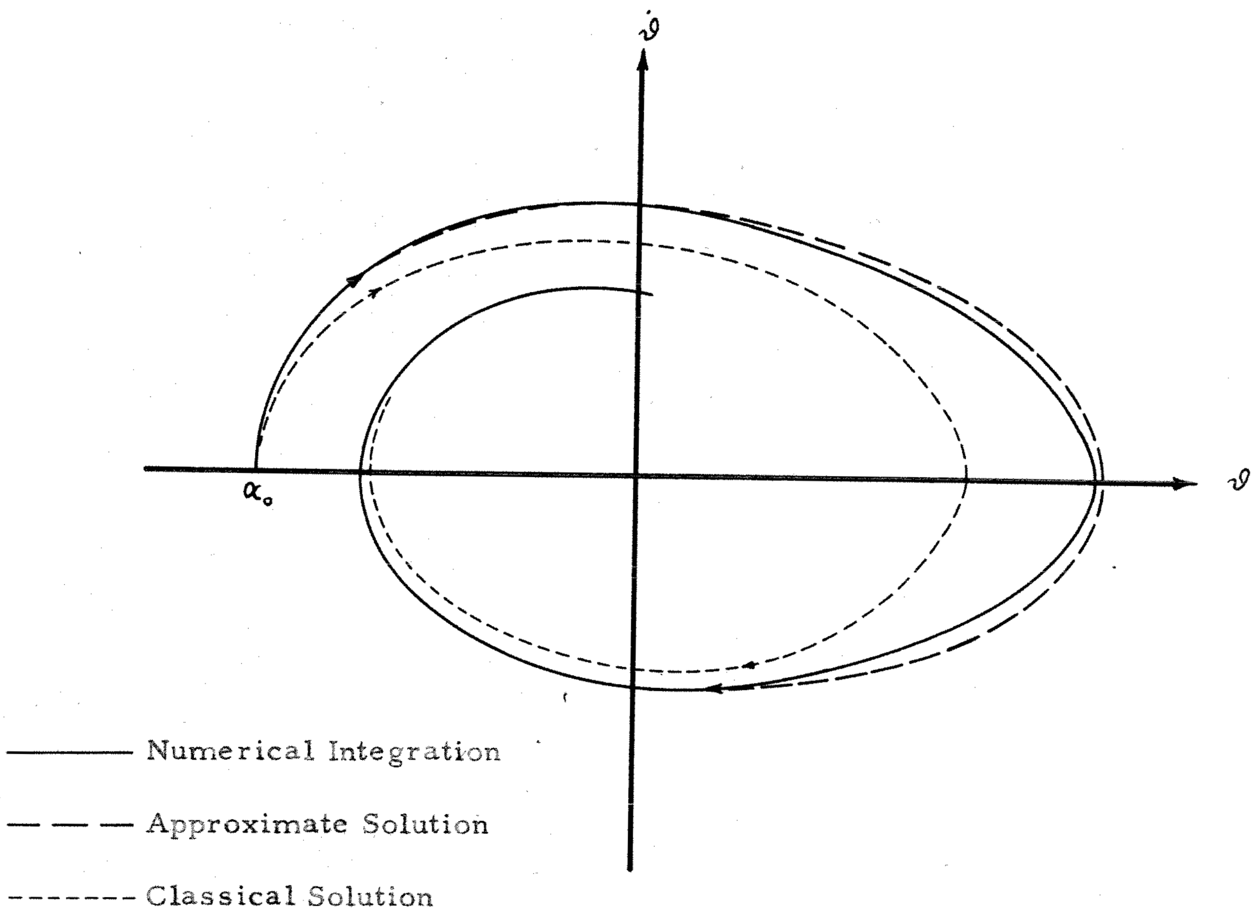


Figure 22. Phase Space Diagram for Example VII

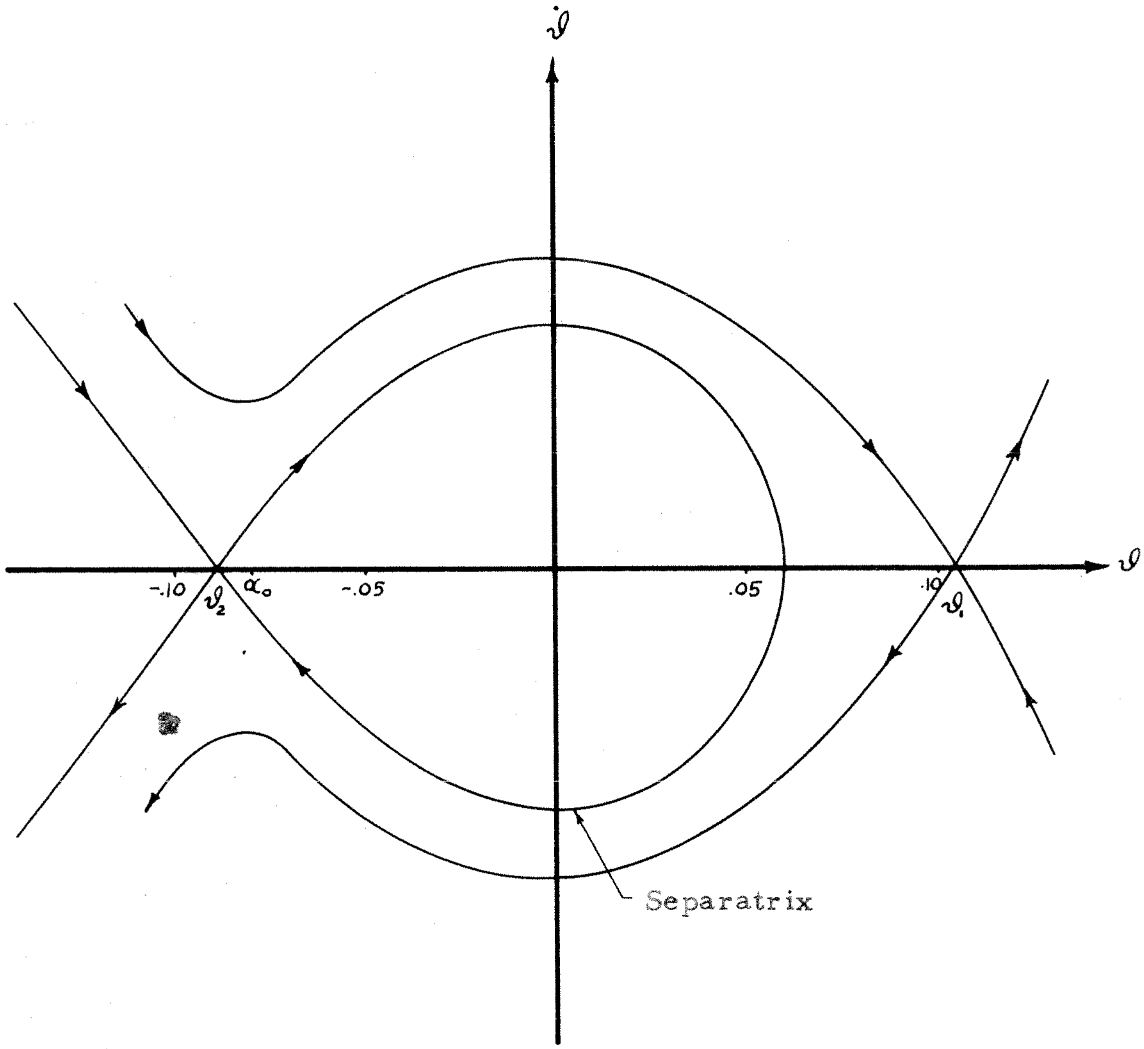


Figure 23. Trajectories Through Saddle Points for Undamped System Corresponding to Example III

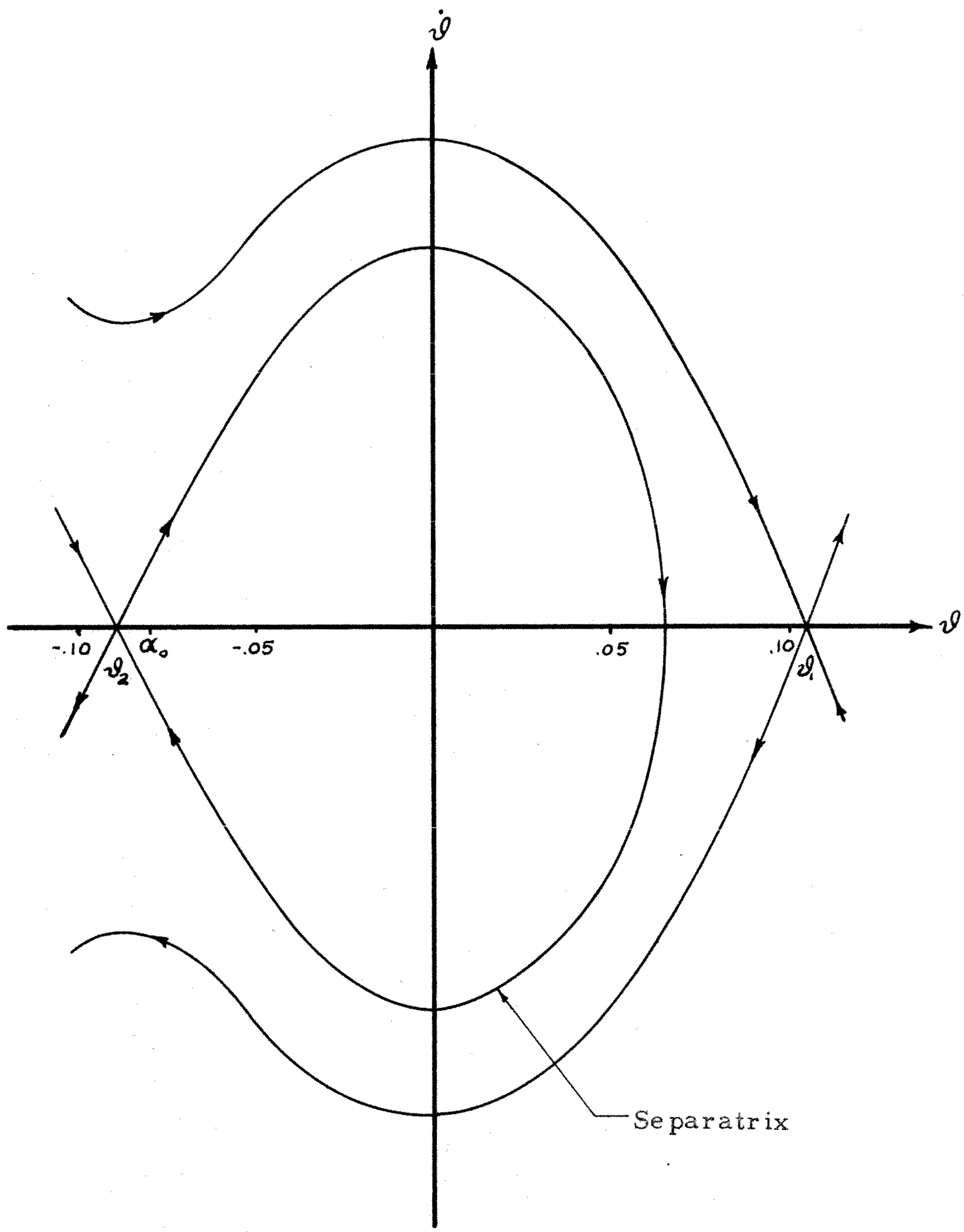


Figure 24. Trajectories Through Saddle Points for Undamped System Corresponding to Example IV

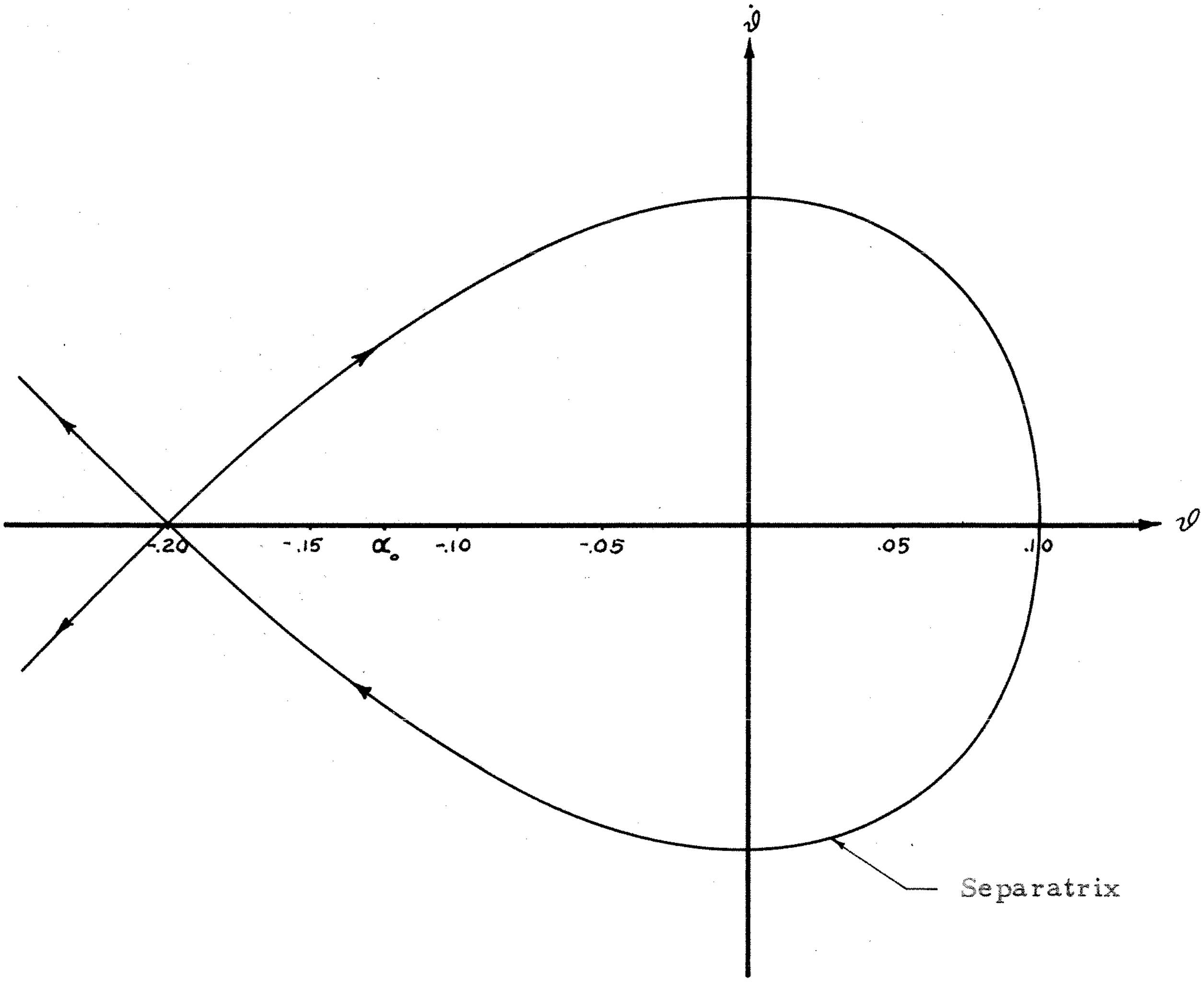


Figure 25. Trajectory Through Saddle Point for Undamped System Corresponding to Example V

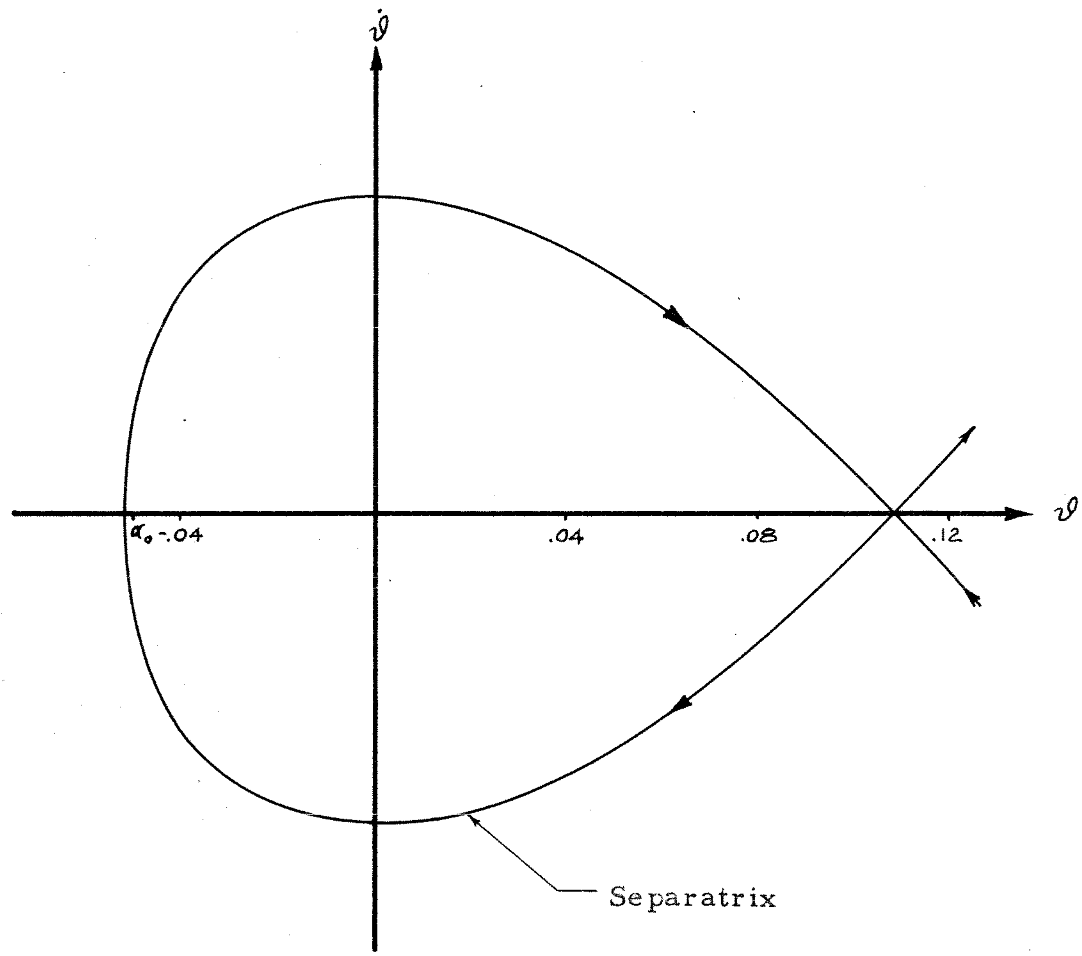


Figure 26. Trajectory Through Saddle Point for Undamped System Corresponding to Example VI

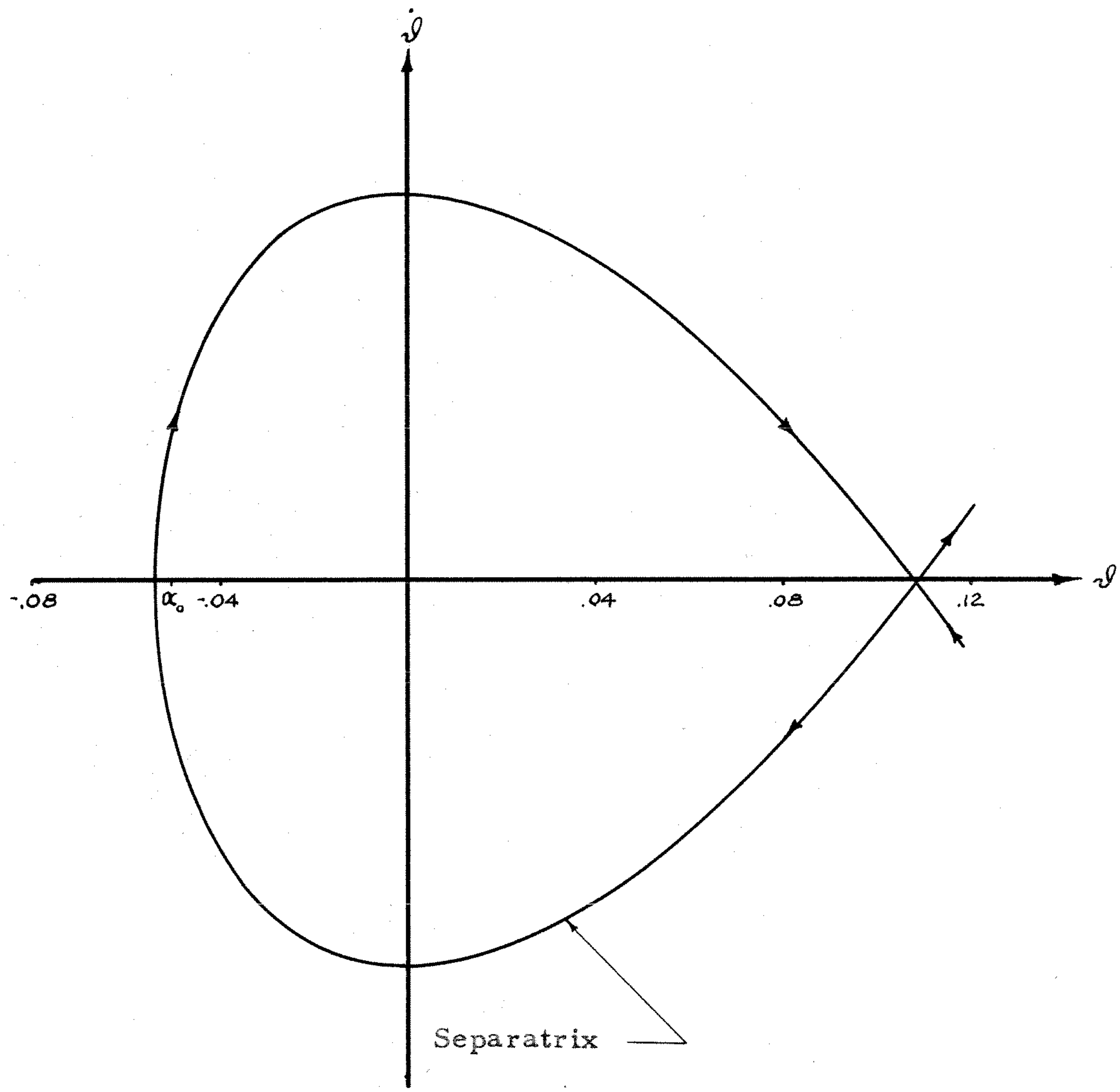


Figure 27. Trajectory Through Saddle Point for Undamped System Corresponding to Example VII

TABLE I

Effects of Long Period Oscillation and Gravity on the Characteristics of the Short Period Oscillation for a Representative Missile

Mach No.	2.45	1.42
Altitude (ft)	40,000	70,000
Angle of Climb	83°	15°
Three degrees of freedom including gravity		
Damping constant	.76	.12
Circular frequency	4.78	1.38
Two degrees of freedom including gravity		
Damping constant	.76	.12
Circular frequency	4.77	1.38
Two degrees of freedom neglecting gravity		
Damping constant	.75	.12
Circular frequency	4.78	1.38

TABLE II

Comparison of the Values of the Parameter, η , with the Independent Variable, z , in the Confluent Hypergeometric Function, ${}_1F_1(-\eta, 1; z)$ for Two Representative Missiles

	Missile "A"		Missile "B"	
Mach No.	1.2	1.5	1.2	2.6
η	-1400	-174	-410	-281
z at 35,000 ft	91	81	36	32
z at 70,000 ft	9.5	8.5	3.8	3.4

where $$\eta = \frac{M'_{w_0}}{C_0 \lambda} - \frac{Z'_{w_0}}{C_0}$$

$$z = \frac{C_0 \sigma}{U_1 \lambda}$$

TABLE III

Values of the Coefficients in the Non-Linear Equation of Motion (39) and Initial Displacement for the Seven Examples Considered

	Damping Function Coefficients			Restoring Moment Coefficients			Initial Displacement
	A	B	E	C	D	F	
Missile "A"							
Example I	10.0	0	17.0	120	0	17,000	-.10
Example II	1.32	0	2.24	14	0	1,950	-.10
Example III	9.0	3.5	0	700	1300	-74,800	-.08
Example IV	1.3	.5	0	80	150	- 8,600	-.08
Missile "B"							
Example V	2.65	1.0	0	60	300	0	-.10
Example VI	5.1	2.0	0	172	-1,580	0	-.05
Example VII	.37	0	0	11	-102	0	-.05