A Theoretical and Experimental Investigation of Auctions in Multi-Unit Demand Environments

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To Miriam and Nabih Noussair

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ABSTRACT

In many existing markets demanders wish to buy more than one unit from a group of identical units of a commodity. Often, the units are sold simultaneously by auction. The vast majority of literature pertaining to the economics of auctions, however, considers environments in which demanders buy at most one object. In this dissertation we present a collection of results concerning the generalization of theoretical and experimental results from environments in which buyers have single-unit demands to environments with two-unit demands. We derive necessary and sufficient conditions for a set of bidding strategies to be a symmetric monotone equilibrium to a uniform price sealed bid auction. We prove that equilibrium bidding strategies converge to truthful revelation as the number of bidders gets large. We also prove that the uniform price sealed bid auction and the English clock are not isomorphic in the two-unit demand environment. Either type of auction may generate higher efficiency and either may generate higher revenue. Finally, we report a set of experimental results which demonstrates that the revenue generating properties of the two auctions are different in two-unit demand environments. In the experimental environment, more revenue is generated by the uniform price sealed bid auction than the English clock, and more revenue is generated per market period if the market is run only once than if it is repeated with the same participants.

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Chapter 1

Introduction

An auction is a "market institution with an explicit set of rules determining resource allocation and prices on the basis of bids from the market participants" (McAfee and McMillan (1987)). Auctions are used in every part of the world to transact trillions of dollars worth of objects every year. The omnipresence of auctions has certainly not gone unnoticed by economists who have generated a huge literature on the subject. The vast majority of the literature focuses on environments where a single seller has one or more indivisible object(s) to be sold to multiple bidders, each of whom wants to buy at most one of the objects.

In most markets however, it is common for buyers to wish to buy more than one unit of a commodity. U.S. treasury bills are an example; about 1.5 trillion dollars worth of U.S. treasury bills are sold per annum (Salwen (1992)). They have been sold by two types of sealed bid auction: discriminatory (currently in use, where buyers pay the amount of their bids), and uniform price (in which all buyers pay the same price.). In many of the world's stock markets, initial issues of common stock are sold through sealed bid auctions. The Burlington Northern Railroad holds an auction for leasing railroad cars using a type of uniform price sealed bid auction. BNR is the buyer and the owners of railroad cars are the sellers who submit bids. The railroad leases all cars for an amount equal to the highest accepted bid. In Great Britain, the National Grid Co., jointly owned by the UK's 12 privatized distributors, buys electric power using an auction. Generators bid per-unit prices for 30 minute periods the following day and National accepts the lowest bids. The price paid is equal to the highest accepted bid.

Economists have also advocated auctions as a means of resource allocation and price determination. Both the uniform price sealed bid auction (Furbush(1991)) and the open ascending bid auction (Salwen(1992)) have been proposed as alternative methods for selling treasury bills. Other examples of such proposals include those made by Grether, Isaac and Plott (1981), and Rassenti, Smith and Bulfin (1982), who advanced the idea of using sealed bid auctions to sell airport landing slots.

The properties of auctions in environments with multi-unit demands have not been as thoroughly investigated as those in the single-unit demand case. In this dissertation, using the methodologies of both game theory and experimental economics, we explore whether particular properties of the single-unit demand case generalize to the multi-unit demand case, and establish that they often do not. In the next two subsections, we provide a brief survey of relevant previous work.

1.1 Previous Theoretical Literature

1.1.1 Equilibria

In the first major paper on the subject of auctions, Vickrey (1961) introduces the second price sealed bid auction and its multi-unit generalization, the uniform price sealed bid auction. In his paper he assumes an independent private values environment, in which each demander draws a valuation, known only to her, from a distribution which is known to all demanders. A particularly good example of a market in which the independent private values model might be appropriate is a market for a government contract where each contractor knows his own cost for fulfilling the contract and has enough general knowledge to infer the distribution of the other contractors' costs.

The rules of the uniform price sealed bid auction specify that all demanders simultaneously submit sealed bids for the object(s). If there are k objects to be sold, the k highest bids are accepted and the corresponding demanders receive units. Demanders then pay a per-unit price equal to the k+1st highest bid for each unit they receive. In the case of single-unit demands, that is, when there may be multiple units to be sold but demanders purchase at most one unit, this auction form is demand revealing. Each bidder has a dominant strategy to submit a bid equal to the value she has drawn. Recall that a player's dominant strategy maximizes his expected utility no matter what strategies other players follow. The dominant strategy property follows from the fact that the second price sealed bid auction is a Groves mechanism. See Green and Laffont (1977) for a characterization of mechanisms with dominant strategy equilibria.

Forsythe and Isaac (1982) show that the second price auction is the only demand revealing direct mechanism in the single-unit demand environment. The outcome is efficient, that is, it yields the maximum possible total profits to all participants in the market of any feasible allocation, because the item is purchased by the demander who has the highest valuation.

To see the dominant strategy property, consider a bidder named i, who has some valuation v_i for receiving the one object to be sold. Bidder i is a member of a group of n+1 bidders, each of whom draws a valuation from a distribution G, which has strictly positive density on an interval $[0, \overline{v}]$. Let $G_n(x)$ be the probability that the highest valuation drawn by any of the other demanders is less than or equal to x and let $g_n(x) = dG_n(x)$. Let $F_n(x)$ be the probability that the highest bid made by one of bidder i's competitors is less than or equal to x and let $f_n(x) = dF_n(x)$. Assume $f_n(x) > 0$ on the interval $[0, \overline{v}]$. Bidder i chooses a bid b_i to maximize his expected payoff:

$$E\pi = \int_0^{b_i} (v_i - x) f_n(x) dx. \tag{1.1}$$

The first order necessary condition for a maximum to (1.1) is:

$$(v_i - b_i) f_n(b_i) = 0. \tag{1.2}$$

Since $f_n(x)$ has strictly positive density on $[0, \overline{v}]$, it follows that $b_i = v_i$. To see that $b_i = v_i$ is a maximum, note from (1.2) that $\frac{dE\pi}{db_i} > 0$ for $b_i < v_i$ and < 0 for $b_i > v_i$.

Vickrey also provides a discussion of English auctions within the independent private values model. An English auction is a progressive ascending bid auction. There are many variants of English auctions which are strategically equivalent in the independent private values model. The auction form considered in this dissertation is the variant commonly known as the Japanese auction or English clock, used in, among other places, the produce markets of Tokyo and Osaka. This type of auction is modeled by Milgrom and Weber (1982). The price is raised continuously and as it climbs bidders indicate whether they are still taking part in the bidding or quitting. In the single-unit demand case, they communicate whether or not they wish to purchase a unit at the current price. After a buyer drops out he may not resume bidding. As soon as exactly k bidders remain active the auction ends, and the remaining bidders receive units at the current price (the price at which the last bidder dropped out). Each bidder has a dominant strategy to stay in the bidding until the price reaches his valuation and then immediately quit bidding. In the unique dominant strategy equilibrium, the bidders with the k highest valuations receive the objects, resulting in an efficient final allocation.

The uniform price sealed bid auction and the English clock generate the same final allocations and final prices in their respective unique dominant strategy equilibria. Therefore, the two auctions are said to be isomorphic. The dominant strategy equilibria specify that each bidder truthfully reveal his valuation. Since the equilibrium strategies are the same for each player, the equilibria are said to be symmetric.

Another auction mentioned in Vickrey's paper is the first price sealed bid auction. This type of auction is commonly used for government procurement contracts and its multi-unit generalization, the discriminatory sealed bid auction, is used in the treasury bill auction. There is no dominant strategy equilibrium in this type of auction, because a bidder's best response depends on the strategies the other players are using. However, maintaining the same assumptions on F_n, G_n and v_i in our discussion of the second price auction, we can find a symmetric, strictly monotone Bayes-Nash equilibrium to this game. Recall that Nash equilibrium is a set of strategies (of all players) from which no player has an incentive to deviate. The set of dominant strategy equilibria to a game are thus a subset of the Nash equilibria. A Bayes-Nash equilibrium is a Nash equilibrium to a game with a particular informational environment, in which players know the distribution of all players' types but the actual realization only of their own type. Suppose all bidders follow some common strictly monotone bidding rule $B(v)$. The expected profits to bidder i can then be expressed as:

$$E\pi^i(b_i) = \int_0^{B^{-1}(b_i)} (v_i - b_i) g_n(x) dx. \tag{1.3}$$

The first order necessary condition to this problem is:

$$-G_n(B^{-1}(b_i)) + (v_i - b_i) g_n(B^{-1}(b_i)) B^{-1'}(b_i) = 0. \tag{1.4}$$

Using the symmetry and strict monotonicity of the bidding function, it can be shown that if $B(v)$ solves equation (1.4) and $E\pi^i(0) = 0$ for all i, $B(v_i)$ must satisfy:

$$B(v_i) = v_i - \frac{\int_0^{v_i} G_n(x)dx}{G_n(v_i)}. \tag{1.5}$$

Because of the dominant strategy property and the efficient allocations generated by the uniform price sealed bid auction and the English clock in the single unit demand environment, it is natural to consider generalizations of the two auction types for the multi-unit demand environment. There has already been some interesting work on equilibrium models of auctions in multi-unit demand frameworks. Wilson (1979) considers the case when the item to be sold is divisible and demanders have continuous downward sloping demand functions for fractions (shares) of the good. He solves for a market clearing price in an environment with common values when first or second price auctions are employed. In these auctions, bidders report a continuous demand function as their message. He finds the price is positive as is each bidder's resulting share. Engelbrecht-Wiggans and Weber (1979) give an example of a multi-object auction with non-additivity of preferences. This means that bidders' utilities for objects depend on what other objects they receive. They find that conducting multiple simultaneous sales may in general be quite inefficient. Although the example is for the uniform price sealed bid auction, the authors credibly claim that the inefficiency extends to other types of auctions. Palfrey (1980) models the simultaneous but separate sale of heterogeneous objects by first price sealed bid auction when bidders face a constraint on exposure. He finds that symmetric Nash equilibria exist if and only if there are two or less buyers and two or less objects.

If a demander wishes to and is permitted to purchase more than one unit, the demand revealing property no longer necessarily holds for either the uniform price sealed bid auction or the English clock. Demanders may have positive incentive to underreveal their demand functions (see Forsythe and Isaac (1982) for an example) in order to lower the prices they pay. Therefore, the uniform price and English auctions do not necessarily yield efficient outcomes.

Demand revealing mechanisms for multi-unit demand environments have been discovered. Weber (1983) proposes an auction that yields a dominant strategy to bid truthfully on the k most highly valued units to each demander in a market where there are k identical units to be sold. Each bidder simultaneously submits k sealed bids, the highest k bids secure items, and a bidder who receives l items is charged the sum of the l highest rejected bids made by other bidders. This auction is not uniform price as demanders pay different per-unit prices.

In many of the markets where multi-unit auctions are employed, the units are not identical. Examples include the markets for art, real estate, and repossessed motor vehicles. Leonard (1983) generalizes the second price sealed bid auction for an environment where each buyer may purchase at most one of a heterogeneous set of units. Demange, Gale and Satomayer (1986) generalize the English clock for this type of environment. Both of these generalizations are demand revealing mechanisms.

Vickrey suggests an auction form which yields a dominant strategy of truthful revelation in an environment where bidders may purchase more than one unit from a set of (possibly heterogeneous) objects. Each bidder submits a bid for every subset of the objects. The set of objects is distributed among the bidders according to the partition of the set which draws the maximum total bid amount (summing over the high bids on the elements of the partition). Each bidder is charged the difference between the maximum total which could have occurred had his bids not been submitted, and the sum of the high bids placed on the subsets other than the one he receives in the actual maximizing partition. A dominant strategy for each bidder is to bid the amount of his valuations. He also introduces an auction form which is demand revealing in an environment in which buyers have multi-unit demands and multiple identical objects to be sold are supplied elastically. It requires payments to the seller to exceed payments by the buyers, requiring a third party (perhaps a planner) to incur a deficit. All of these demand revealing auctions are variations of Groves mechanisms.

In chapter two we extend the independent private values model to allow demanders to have positive valuations for obtaining up to two units from a fixed supply of identical units. We model the simplest and most obvious generalization of the uniform price sealed bid auction for the two-unit demand environment and provide necessary and sufficient conditions for a bidding rule to be a symmetric monotone Bayes-Nash equilibrium. In any symmetric monotone Bayes-Nash equilibrium, each demander bids an amount equal to her valuation for her more highly valued unit and bids less than her valuation for her lower-valued unit. We also prove that as the number of bidders approaches infinity, the amount of underrevelation on the lower-valued unit approaches zero.

1.1.2 Revenue

Vickrey notes that the three aforementioned auctions generate the same expected revenue (in equilibrium) in the risk neutral independent private values environment. The expected revenue is equal to the expected value of the second highest valuation. This observation is generalizable into an important result known as the revenue equivalence theorem.

Revenue Equivalence Theorem: (Myerson (1981)) Every auction that allocates the goods efficiently and offers zero profit to a zero valuation bidder has the same expected profit for every bidder valuation and the same expected revenue for the seller.

An interesting and simple proof can be found in Milgrom (1989). Let P equal the probability that the bidder wins the item, E denote the expected payment of the winner and X equal the winner's valuation. The bidder's utility is then:

$$U(P, E, X) = P(X - E). \tag{1.6}$$

Let $(P^*(X), E^*(X))$ be the optimal choice for the bidder. Denote the corresponding profits as $U^*(P^*(X), E^*(X); X)$. It can be shown, invoking the envelope theorem, that the first order condition to the bidder's problem is:

$$U^{*'}(X) = U_X(P^*(X), E^*(X); X) = P^*(X). \tag{1.7}$$

Since $U^*(0) = 0$:

$$U^*(X) = \int_0^X P^*(s)ds. \tag{1.8}$$

For any auction in which the allocation is always efficient, $P^*(X)$ equals the probability that all other bidders' valuations are less than X. Since the utility achieved by the bidder is independent of the method of sale, the rents taken by the seller must also be independent of the method of sale. Thus the expected revenue must be the same in any auction mechanism which always generates an efficient allocation in equilibrium. The observation that a description of an auction could be reduced to a mapping $P(X)$ which specified the probability of a bidder's getting the item as a function of his valuation allowed Myerson (1981) to discover the optimal auction, that is, the type of auction which generates the most revenue to the seller. In the optimal mechanism, there is a reserve price greater than or equal to the sellers valuation and the item goes to the bidder with the highest valuation.

Revenue equivalence strongly depends on the assumption, which we keep in chapter two, that demanders are risk neutral. Risk aversion of the bidders results in the first price sealed bid auction yielding higher average revenue (Weber (1983)). The second price sealed bid auction and the English clock have dominant strategy equilibria, which are not affected by the level of risk aversion of the bidders.

The informational structure of the market environment can affect the equilibria and therefore the expected revenue of the auction games. The original independent private values model has been extended by Milgrom and Weber (1982), who develop a general model (called the general symmetric model) of which the independent private values environment is a special case. Their model (they keep the assumption of risk neutrality) allows for private valuations as well as other variables which affect the value of the object to demanders (the variables are allowed to be affiliated which makes the bidders' value estimates statistically dependent). They prove that in the general symmetric model, the English clock generates higher expected revenue than the second price auction, which in turn yields greater expected revenue than the first price auction. An explanation for the higher revenue generated by the English clock offered by Milgrom and Weber is that the English clock allows bidders who are uncertain about their valuations to acquire useful information from the behavior of other bidders. They also show that the English auction yields more revenue than *either* type of sealed bid auction if the seller releases any private information which she has about the value of the object.

In chapter three we show that the uniform price sealed bid auction and the English clock are not isomorphic in the two-unit demand environment. Two examples are provided to illustrate that the auctions can generate different allocations in equilibrium. Either auction may, ex-ante, allocate the commodity more efficiently and either auction may generate higher revenue, invalidating some generalizations of the revenue equivalence theorem to the multi-unit demand case.

1.2 Previous Experimental Work

Much of the experimental work done to date involving auctions of a fixed supply of identical units has been motivated by the theoretical results. We restrict our attention here to the experimental studies of auctions in independent private values environments. In general, in the case of single-unit demands, the predictions of the dominant strategy equilibria of the uniform price sealed bid and English auctions are supported. Coppinger, Smith and Titus (1980) find that observed prices in English and second price auctions are not different from those in the dominant strategy equilibrium and that the two auctions generate highly efficient final allocations. They also observe that the proportion of bidders following their dominant strategies increases as the auction is repeated. Cox, Roberson and Smith (1982) fail to reject the hypothesis that observed prices in the second price auction are equal to those occurring in the dominant strategy equilibrium and observe very efficient allocations. Burns (1985) finds that sequential English auctions converge to the *competitive outcome* whatever the size of the market. McCabe, Rassenti and Smith (1991) consider the behavior of many types of multi-unit auctions in environments with single-unit demands. The English clock and the uniform price sealed bid auction produce slightly different outcomes. There is lower variance in prices and more efficient allocation under the English clock. McCabe et al. also find a very strong tendency for demanders to use their dominant strategy when playing the English clock.

Thus when there is a unique symmetric dominant strategy equilibrium to an auction, it seems to be observed experimentally. However, when there exists a symmetric Bayes-Nash equilibrium but no dominant strategy equilibria to an auction game, the risk neutral Bayes-

Nash equilibrium performs poorly in explaining the data from corresponding experiments. In first price auctions, observed bids are higher than those occuring in the symmetric Bayes-Nash equilibrium under risk-neutrality.

Cox, Roberson and Smith (1982) find that a variation of the Bayes-Nash equilibrium concept, which allows demanders to exhibit varying degrees of risk aversion, better explains their experimental data for first price auctions. Cox, Smith and Walker (1985) reject the multiple-unit (but single-unit demand) generalization of the first price sealed bid symmetric Nash (risk neutral) equilibrium bidding model. They find some support for an alternative model, called the constant relative risk aversion model.

Some experimental work has been concerned with the relative revenue generating properties of different auctions. Coppinger et al. find that the revenue in the second price auction is somewhat below that produced by the English auction. They also find (as do Cox et al. (1982)) that the first price sealed bid auction generates more revenue than the second price sealed bid and the English auction. Smith (1967) compares revenue generated by uniform price and discriminatory sealed bid auctions and concludes that the difference in revenue depends on the amount of excess demand. Miller and Plott (1985) find that the discriminatory auction generates more revenue than the uniform price auction when market demand is inelastic near the competitive equilibrium price and quantity while the uniform price auction takes in more revenue when market demand is relatively elastic. Cox, Smith and Walker (1985) observe that revenue from sealed bid auctions declines as the level of experience of the subjects increases but do not find strong evidence that the discriminatory auction generates either higher or lower revenue than the uniform price auction. Olson and Porter (1992) find that in a market with multiple heterogeneous goods to be sold but in which demanders could only each buy one unit, that a generalization of the English clock generated higher more revenue than a generalization of the second price sealed bid auction.

There are also previous results concerning allocative efficiency of various auctions. Coppinger, Smith, and Titus observe that the final allocations of the English auction are more efficient than those of the second price sealed bid auction which are, in turn, more efficient than those of the first price sealed bid auction. Cox, Roberson, and Smith observe that the second price sealed bid auction generates more efficient allocations than the first price sealed bid auction. Olson and Porter find that the English clock leads to higher efficiencies than the second price sealed bid auction.

In chapter three we report the results from a line of experimentation designed to consider relative revenues and allocative efficiencies of the uniform price auction and the English clock in the two-unit demand case. We also compare the data from a single-unit demand condition to data from a two-unit demand condition. For the parameters in the two-unit demand condition of our experiment, we find that the uniform price auction generates significantly more revenue than the English clock. Also, significantly greater revenue is generated per market period (for both types of market organization) when the markets are conducted only once than when they are repeated and more revenue is generated by the English clock in the single-unit demand environment than in the two-unit demand environment. In all conditions the final allocations are highly efficient.

1.3 This Dissertation

In this dissertation we present a collection of theoretical and experimental results. The focus is on the generalization of the properties of auctions in single-unit demand environments to environments with two-unit demands. The main contributions of the dissertation are the following:

- Necessary and sufficient conditions are derived for a set of bidding strategies to be a symmetric monotone equilibrium to the uniform price auction when demanders wish to purchase two units from a set of identical units.
- A proof of convergence of bidding strategies to truthful revelation as the number of bidders gets large is provided.
- The uniform price sealed bid auction and the English clock are proven not to be isomorphic in the two-unit demand environment. Either type of auction may achieve higher expected efficiency and either may result in higher expected revenue. Both of these results contrast sharply with the results in single-unit demand environments.
- A set of experimental results is reported. The results indicate that the revenue generating properties of the two auctions are different in two-unit demand environments although no significant difference is detected in their allocative efficiency. More revenue is generated by the uniform price sealed bid auction than by the English clock. Also, more revenue is generated per market period if the market is run once than if it is repeated for several periods.

Chapter 2

The Uniform Price Sealed Bid Auction

In this chapter we generalize the independent private values framework to allow demanders to draw positive valuations for obtaining up to two units from a set of identical units of a commodity. We then consider the theoretical properties of a simple generalization of the uniform price auction within the two-unit demand independent private values environment. Necessary and sufficient conditions for a bidding function to be a symmetric monotone Bayes-Nash equilibrium are derived, and an example is provided. It is also shown that as the number of bidders gets large, the only possible symmetric monotone Bayes-Nash equilibrium is truthful bidding.

2.1 The Model

Let there be k (> 1) identical units to be sold and n+1 (> 1) demanders indexed by $i = 1, ..., n+1$. Each demander draws two valuations independently from a fixed and common distribution $\gamma(v)$, where $\gamma(v)$ has strictly positive density on $[0,\overline{v}] \subset R^+$ and $\gamma(v) \in C^2$. Order the two values from higher to lower and index them 1 and 2 respectively; $v_1^i \geq v_2^i \geq 0$ are the valuations of demander i. Define $G(v_1^i, v_2^i) = \text{Prob}(v_1 \leq v_1^i, v_2 \leq v_2^i)$, where v_1 and v_2 are a pair of values independently drawn from $\gamma(v)$. Let $g(v_1, v_2)$ denote the probability density function of G. Since g is a joint density of order statistics drawn from a distribution with positive density on $[0, \overline{v}]$, $g(v_1, v_2) > 0$ for all v_1, v_2 such that $0 \leq v_2 \leq v_1 \leq \overline{v}$. All demanders are risk neutral. Valuations are private information but γ, n, and k are common knowledge.

2.2 The Game

All demanders submit two nonnegative bids. The highest k bids are accepted and the corresponding demanders pay a per-unit price equal to the k + 1st highest bid. A tie for kth highest bid is broken by randomly allocating a unit to one of the tied demanders. A bid which is equal to zero is never accepted.

When buyers are willing to pay for more than one unit, the uniform price sealed bid auction does not have the incentive properties it has in environments with single-unit demands. The following example may clarify this idea. Suppose for simplicity (and only for the example) that there is complete information, so that each buyer knows all of the valuations of the others.

Table 2.1: Buyers' Valuations in Example 1

	buyer 1	buyer 2
Higher Valuation	5	3
Lower Valuation	4	0

Example 1: The number of buyers (n+1) = 2, and the number of units to be sold (k) = 2. Buyers' valuations are given in table 2.1:

Consider buyer 1' s decision on what to bid. He supposes that buyer 2 is going to bid 3 on his highest unit and 0 on his second unit (later we will show this strategy to be the best response of bidder 2). If buyer 1 bids the amount of his valuations, that is 5 on his high unit and 4 on his second, he wins both units. The per-unit price paid is equal to the highest rejected bid, which is 3. Then his profits are (5 - 3) + (4 - 3) = 3. Buyer 2's profits are zero. If instead, buyer 1 bids 5 on his high unit and then 0 on his second unit while buyer 2 bids 3 on his high unit and 0 on his second, each bidder receives one unit, the price is zero, and the profits are 5 and 3 to buyers 1 and 2 respectively. By underbidding on his second unit, buyer 1 was made better off. For buyer 1, bidding his valuation for that unit was not his best response.

2.3 Symmetric Equilibria

2.3.1 Necessary Conditions

Since the environment described in section 2.1 specifies symmetrically informed bidders whose valuations are drawn from a common distribution, it is natural to consider equilibria which consist of symmetric bidding strategies. Therefore, we restrict our attention to symmetric equilibria. In theorem 1 we derive a necessary condition for a bidding strategy to be a symmetric monotone Bayes-Nash equilibrium. Several definitions are required for the statement and proof of the theorem.

Definition 1 *A bidding function* $B(v_1, v_2) = (B_1(v_1, v_2), B_2(v_1, v_2)) : [0, \overline{v}]^2 \rightarrow R^{2+}$ *maps two valuations into two bids.*

Definition 2 *A bidding function,* $B(v_1, v_2)$ *is type M if:*

1) $B(0,0) = (0,0)$,

2) B *is continuous in* v_1 *and* v_2,

3) $B_1(\overline{v}, \overline{v}) \geq B_2(\overline{v}, \overline{v})$,

4) $\exists$ *a function* $\hat{v}_j(v_z)$ *such that* $B_j(v_z, v_j) = 0$; *iff* $v_j \leq \hat{v}_j(v_z)$; $j \neq z; j, z \in 1, 2$

5) $\frac{\partial B_j}{\partial v_j}$ *exists and is* > 0 *if* $B_j > 0$ *and* $\frac{\partial B_j}{\partial v_z}$ *exists and is* $\geq 0; z \neq j$ *if* $B_j > 0$.

Definition 2 describes a general notion of monotonicity. It allows bidding strategies which specify that $B_j(v_1, v_2) = 0$ for all $v_1 \leq \hat{v}_1$ and $v_2 \leq \hat{v}_2$ for any $\hat{v}_1 \in [0, \overline{v}]$ and any $\hat{v}_2 \in [0, \overline{v}]$. It includes, as a special case, bidding functions which are strictly increasing in v_1 and v_2 (where $0 = \hat{v}_2 = \hat{v}_1$). The concept of separability is described in definition 3.

Definition 3 *A bidding function is separable if* $B(v_1, v_2) = (B_1(v_1), B_2(v_2))$. *That is, a demander's bid for his higher (lower) valued unit is independent of his lower (higher) valuation.*

Definition 4

$$H(z_1, z_2, G, m, n, \ell) = \frac{n!}{\ell! m! (n - m - \ell)!} \left(\int_0^{z_1} \int_0^{z_2} g(v_1, v_2) dv_2 dv_1 \right)^{\ell}$$

$$(\int_{z_1}^{\overline{v}}\int_0^{z_2} g(v_1,v_2)dv_2dv_1)^m(\int_{z_1}^{\overline{v}}\int_{z_2}^{\overline{v}} g(v_1,v_2)dv_2dv_1)^{n-m-\ell}. \tag{2.1}$$

The function H(.) is the probability that in a sample of size n drawn from $g(v_1, v_2)$, exactly ℓ observations in the sample have the property that $(v_1 \leq z_1, v_2 \leq z_2)$, exactly m observations have the property that $(v_1 > z_1, v_2 \leq z_2)$ and exactly $n - m - \ell$ have the property that $(v_1 > z_1, v_2 > z_2)$.

Definition 5

$$T(z_1, z_2, z_3, G, n, k) = \frac{\sum_{m,\ell;m+2\ell=2n-k+1} H(z_1, z_2, n, m, \ell)}{\sum_{q=2n-k+2}^{2n}\sum_{m,\ell;m+2\ell=q}(\frac{\partial H}{\partial z_1} + \frac{\partial H}{\partial z_2}\frac{1}{z_3})}. \tag{2.2}$$

The function T(.) is important to the statement and the proofs of theorems 1-3. It is useful in describing the amount by which bidders underbid on their lower valued unit in a Type M equilibrium.

Definition 6 $(F_\nu^{-i}(x)|B)$ = *Prob(at least* ν *bids made by bidders other than i are less than or equal to x if all bidders except for bidder i use B).*

In other words, $(F_\nu^{-i}(x)|B)$ is the cumulative distribution function of the νth order statistic of bids made by n randomly chosen demanders using strategy B.

Theorem 1 *A bidding function* $\beta(v_1, v_2) = (\beta_1(v_1, v_2), \beta_2(v_1, v_2))$ *is a symmetric type M undominated Bayes-Nash equilibrium only if:*

$$\beta_1(v_1) = v_1 \tag{2.3}$$

and:

$$\beta_2(v_2) = \begin{cases} 0 & v_2 \leq v_2^* \\ \overline{\beta}_2(v_2) & v_2 \geq v_2^*, \end{cases} \tag{2.4}$$

where:

$$v_2^* = T(0, v_2^*, \frac{\partial \overline{\beta}_2(v_2^*)}{\partial v_2}, G, n, k) \tag{2.5}$$

and $\overline{\beta}_2$ solves the differential equation:

$$\overline{\beta}_2(v_2) = v_2 - T(\overline{\beta}_2(v_2), v_2, \frac{\partial \overline{\beta}_2(v_2)}{\partial v_2}, G, n, k), \tag{2.6}$$

with the initial conditions:

$$\begin{aligned} v_2^* &= 0; \; \text{if } n > k-1 \\ \overline{\beta}_2(\overline{v}) &= \overline{v}; \; \text{if } n < k-1 \\ \overline{\beta}_2(0) &= 0; \; \text{if } n = k-1. \end{aligned} \tag{2.7}$$

Theorem 1 is proven using lemmas 1-7, which are stated and proven in this subsection. An additional lemma (10), required for technical purposes, is stated and proven in appendix A.

Lemma 1 *A bidding function $\beta(v_1, v_2) = (\beta_1(v_1, v_2), \beta_2(v_1, v_2))$, is a symmetric undominated type M Bayes-Nash equilibrium only if:*

$$\beta_1(v_1) = v_1. \tag{2.8}$$

Proof: Suppose all n+1 bidders are using the same equilibrium bidding function $B^*(v_1, v_2)$. For notational ease, let $F_\nu^{-i*}(x) = F_\nu^{-i}(x)|B^*$. Since B^* is symmetric, γ is fixed and common, and valuations are drawn independently, $F_\nu^{-i*}(x) = F_\nu^*(x); \forall i$. Since B^* is an equilibrium, all bidders are maximizing expected profit. To derive the expected profit to bidder i, as a function of the amount he bids, we must divide all possible outcomes of the auction into four cases.

Case 1: The purchase price is lower than i's highest bid but higher than his second highest, in which case i receives one unit. His profits are the value for his highest unit, v_1^i, minus the price he pays, which is the kth highest or 2n-k+1st lowest of the other players' bids, since there are k others besides i. The profits in case 1 times the probability of the occurrence of case 1 is equal to:

$$\int_{b_2^i}^{b_1^i} (v_1^i - M_{2n-k+1}) f_{2n-k+1}^*(M_{2n-k+1}) dM_{2n-k+1}, \tag{2.9}$$

where M_ν is the νth lowest order statistic of bids made by bidders other than bidder i and b_j^i is the jth highest bid made by bidder i.

Case 2: The price is lower than either of i's bids, implying that i's two bids are among the k highest. Profits are $v_1^i + v_2^i$ minus two times the price he pays. This price is equal to the 2n-k+2nd lowest of the other players' bids. The profits in case 2 times the probability of case 2 equals:

$$\int_0^{b_2^i}(v_1^i + v_2^i - 2M_{2n-k+2})f_{2n-k+2}^*(M_{2n-k+2})dM_{2n-k+2}. \tag{2.10}$$

Case 3: Bidder i's higher bid is accepted and his lower bid sets the price. The higher bid is among the k highest and the lower bid is the k+1st highest, implying that the lower bid is between the 2n-k+1st lowest and the 2n-k+2nd lowest of the other demanders bids. Profits are equal to v_1^i minus i's lower bid. The profits in case 3 times the probability of case 3 is given by:

$$(v_1^i - b_2^i)(F_{2n-k+1}^*(b_2^i) - F_{2n-k+2}^*(b_2^i)). \tag{2.11}$$

Case 4: The price is higher than either of i's bids. Neither of his bids are accepted, he receives no units, and his profits are 0.

By combining the last three equations, we see that the objective function for bidder i is given by:

$$\begin{aligned} E\pi^i = &\int_{b_2^i}^{b_1^i}(v_1^i - M_{2n-k+1})f_{2n-k+1}^*(M_{2n-k+1})dM_{2n-k+1} \\ &+ \int_0^{b_2^i}(v_1^i + v_2^i - 2M_{2n-k+2})f_{2n-k+2}^*(M_{2n-k+2})dM_{2n-k+2} \\ &+(v_1^i - b_2^i)(F_{2n-k+1}^*(b_2^i) - F_{2n-k+2}^*(b_2^i)). \end{aligned} \tag{2.12}$$

In equilibrium, bidder i's two bids, b_1^i and b_2^i, are chosen to maximize the objective function subject to:

$$b_1^i \geq 0, b_2^i \geq 0. \tag{2.13}$$

The first order necessary (Kuhn-Tucker) conditions are given by:

$$\frac{\partial E\pi^i}{\partial b_1^i} = (v_1^i - b_1^i) f_{2n-k+1}^*(b_1^i) = 0; b_1^i > 0,$$
$$\leq 0; b_1^i = 0, \tag{2.14}$$

and:

$$\frac{\partial E\pi^i}{\partial b_2^i} = (v_2^i - b_2^i) f_{2n-k+2}^*(b_2^i) - (F_{2n-k+1}^*(b_2^i) - F_{2n-k+2}^*(b_2^i)) = 0; b_2^i > 0,$$
$$\leq 0; b_2^i = 0. \tag{2.15}$$

If $f_{2n-k+1}^*(b_1^i) > 0$, then B^* is separable and $B_1^*(v_1) = v_1$ for $v_1 \geq 0$. If $f_{2n-k+1}^*(b_1^i) = 0$, then there can be more than one solution but any strategy is weakly dominated by $B_1^*(v_1) = v_1$. Any strategy which involves underbidding results in profits identical to those under truthful revelation if the final price is less than b_1^i, but lower profits than those under truthful revelation if the final price is less than v_1^i but greater than b_1^i. Any strategy which involves overbidding results in profits to bidder i that are identical to those under truthful revelation if the final price is less than v_1^i but are lower than those under truthful revelation if the final price is above v_1^i but less than b_1^i. □

We have shown that in equilibrium, each bidder's higher bid equals his higher valuation. We derive the lower bid in lemmas 2-7.

Lemma 2 *β is a Type M undominated symmetric Bayes-Nash equilibrium only if $\beta_2(v_2) = \overline{\beta}_2(v_2)$; for $v_2 \geq v_2^*$; where $\overline{\beta}_2(v_2)$ solves:*

$$\overline{\beta}_2(v_2) = v_2 - T(\overline{\beta}_2(v_2), v_2, \frac{\partial \overline{\beta}_2(v_2)}{\partial v_2}, G, n, k). \tag{2.16}$$

Proof: The derivation of $B_2^*(v_2)$ requires the derivation of the expression $(F_{2n-k+1}^*(b_2^i) - F_{2n-k+2}^*(b_2^i))/f_{2n-k+2}^*(b_2^i)$ in equation (2.15). By lemma 10, which is in appendix A, $f_{2n-k+2}^*(b_2^i) > 0$. Since B^* is being used by all players and $B_1^*(v_1) = v_1$:

$$Prob(v_1 \leq B_1^{-1*}(b_2^i)) = Prob(v_1 \leq b_2^i). \tag{2.17}$$

Also, because $\frac{\partial B_2^*(v_2)}{\partial v_2} > 0$ for $v_2 > \hat{v}_2$ and $B_2^*(v_2) = 0$ for $v_2 \leq \hat{v}_2$:

$$Prob(B_2^*(v_2) \leq b_2^i) = Prob(v_2 \leq V_2(b_2^i)), \tag{2.18}$$

where the function $V_2(x)$:$[0, B_2^*(\overline{v})] \rightarrow [\hat{v}_2, \overline{v}]$, and $V_2(x) = B_2^{-1*}(x)$. Now consider the probability that a randomly drawn bidder, named y $\neq i$, submits 2 bids that are less than or equal to b_2^i. The probability that two of y's bids are less than b_2^i is the probability of the following event:

$$Prob(b_1^y \leq b_2^i, b_2^y \leq b_2^i) = \int_0^{V_2(b_2^i)} \int_0^{b_2^i} g(v_1, v_2) dv_1 dv_2. \tag{2.19}$$

Similarly, exactly one of y's bids is less than or equal to b_2^i when either of the two following events occurs:

$$Prob(b_1^y > b_2^i, b_2^y \leq b_2^i) = \int_0^{V_2(b_2^i)} \int_{b_2^i}^{\overline{v}} g(v_1, v_2) dv_1 dv_2, \tag{2.20}$$

or

$$Prob(b_1^y \leq b_2^i, b_2^y > b_2^i) = \int_{V_2(b_2^i)}^{\overline{v}} \int_0^{b_2^i} g(v_1, v_2) dv_1 dv_2. \tag{2.21}$$

The last expression equals 0 because it requires that $(v_1^y \leq b_2^i, v_2^y \geq V_2(b_2^i))$, an event that occurs with probability zero.

Since the expression $(F^*_{2n-k+1}(b_2^i) - F^*_{2n-k+2}(b_2^i))/f^*_{2n-k+2}(b_2^i) \geq 0$ (because the numerator is a probability and the denominator is a density), it must be the case that $V_2(b_2^i) \geq b_2^i$; but by assumption $v_1^y \geq v_2^y$. The probability that demander y makes 0 bids less than or equal to b_2^i is given by:

$$Prob(b_1^y > b_2^i, b_2^y > b_2^i) = \int_{V_2(b_2^i)}^{\overline{v}} \int_{b_2^i}^{\overline{v}} g(v_1, v_2) dv_1 dv_2. \tag{2.22}$$

Suppose now that each of the n bidders other than bidder i draws one pair of valuations from $G(v_1, v_2)$. Suppose exactly ℓ of the buyers make two bids less than or equal to b_2^i, exactly m buyers make one, and exactly (n-m-ℓ) bidders make zero bids less than or equal to b_2^i. The probability of this event is given by:

Prob(exactly ℓ observations of $B_1^*(v_1^{-i}) \leq b_2^i$, exactly m + ℓ observations of $B_2^*(v_2^{-i}) \leq b_2^i$)

$$= \frac{n!}{\ell!m!(n-m-\ell)!}\left(\int_0^{V_2(b_2^i)}\int_0^{b_2^i} g(v_1,v_2)dv_1dv_2\right)^{\ell}\left(\int_0^{V_2(b_2^i)}\int_{b_2^i}^{\overline{v}} g(v_1,v_2)dv_1dv_2\right)^m \left(\int_{V_2(b_2^i)}^{\overline{v}}\int_{b_2^i}^{\overline{v}} g(v_1,v_2)dv_1dv_2\right)^{n-m-\ell}. \quad (2.23)$$

The previous expression equals $H(b_2^i, V_2(b_2^i), G, n, m, \ell)$ where H is as defined in equation (2.2). It follows that:

$$(F_{2n-k+1}^*(b_2^i) - F_{2n-k+2}^*(b_2^i)) = \sum_{\ell,m;2\ell+m=2n-k+1} H(b_2^i, V_2(b_2^i), G, n, m, \ell), \quad (2.24)$$

$$F_{2n-k+2}^*(b_2^i)) = \sum_{q=2n-k+2}^{2n} \sum_{\ell,m;2\ell+m=q} H(b_2^i, V_2(b_2^i), G, n, m, \ell), \quad (2.25)$$

and

$$f_{2n-k+2}^*(b_2^i)) = \sum_{q=2n-k+2}^{2n} \sum_{\ell,m;2\ell+m=q} \left(\frac{\partial H}{\partial b_2^i} + \frac{\partial H}{\partial V_2}\frac{\partial V_2}{\partial b_2^i}\right). \quad (2.26)$$

The last three equations imply that:

$$(F_{2n-k+1}^*(b_2^i) - F_{2n-k+2}^*(b_2^i))/f_{2n-k+2}^*(b_2^i) = T(b_2^i, V_2(b_2^i), (\frac{\partial V_2}{\partial b_2^i})^{-1}, G, n, k). \quad (2.27)$$

Since all bidders are using the same strategy, b_2^i must equal $B_2^*(v_2^i)$. Therefore:

$$T(b_2^i, V_2(b_2^i), (\frac{\partial V_2}{\partial b_2^i})^{-1}, G, n, k) = T(B_2^*(v_2^i), v_2^i, \frac{\partial B_2^*(v_2)}{\partial v_2}, G, n, k). \quad (2.28)$$

Using equations (2.15),(2.27) and (2.28), we see that equation (2.6) and the second part of equation (2.4) must hold. □

In lemmas 3 and 4 we solve for $\hat{v}_2$. If any demander draws a lower valuation less than equal to $\hat{v}_2$, he submits a lower bid equal to zero. We show that $\hat{v}_2$ is equal to v_2^*, where v_2^* satisfies equation (2.5).

Lemma 3 *$\beta_2(v_2^*) = 0$; where v_2^* satisfies:*

$$v_2^* = T(0, v_2^*, \frac{\partial \overline{\beta}_2(v_2^*)}{\partial v_2}, G, n, k). \tag{2.29}$$

Proof: The first order conditions imply that for $b_2^i = 0$:

$$0 \geq V_2(0) - T(0, V_2(0), (\frac{\partial V_2(0)}{\partial b_2^i})^{-1}, G, n, k). \tag{2.30}$$

Equation (2.30) holds with equality if $V_2(0) = T(0, V_2(0), (\frac{\partial V_2(0)}{\partial b_2^i})^{-1}, G, n, k)$, which implies that $V_2(0) = v_2^*$ and shows that equation (2.5) must hold. □

Lemma 4 *$\beta_2(v_2) = 0$; if $v_2 \leq v_2^*$.*

Proof: Consider $v_2^o < v_2^*$.

$$0 > v_2^o - T(0, V_2(0), (\frac{\partial V_2(0)}{\partial b_2^i})^{-1}, G, n, k). \tag{2.31}$$

By equation (2.15), $B_2^*(v_2^o) = 0$. It is shown that the first part of equation (2.4) must hold in equilibrium. □

In lemmas 5-7 we derive initial conditions for the differential equation in (2.6). The initial conditions depend on the difference between the number of bidders and the number of units to be sold.

Lemma 5 *$v_2^* = 0$; if $n > k-1$.*

Proof: Consider $v_2^* = T(0, v_2^*, \frac{\partial B_2^*(v_2^*)}{\partial v_2}, G, n, k)$. By lemma 10, which is stated and proven in Appendix A, the denominator of the last expression, which is equal to $f^*_{2n-k+2}(0)$, is strictly greater that 0. If the numerator is equal to zero, it would imply that $v_2^* = 0$. The numerator is:

$$\sum_{\ell,m;2\ell+m=2n-k+1} H(0, V_2(0), G, n, m, \ell) = \sum_{\ell,m;2\ell+m=2n-k+1} \frac{n!}{\ell!m!(n-m-\ell)!}$$

$$*(\int_0^0 \int_0^{V_2(0)} g(v_1, v_2)dv_2dv_1)^\ell (\int_0^{\overline{v}} \int_0^{V_2(0)} g(v_1, v_2)dv_2dv_1)^m$$

$$*(\int_0^{\overline{v}} \int_{V_2(0)}^{\overline{v}} g(v_1, v_2)dv_2dv_1)^{n-m-\ell}. \tag{2.32}$$

The last expression equals 0 unless $\ell = 0$. Suppose ℓ equals 0. Then the expression equals:

$$\sum_{m=2n-k+1} \frac{n!}{m!(n-m)!}(\int_0^{\overline{v}} \int_0^{V_2(0)} g(v_1, v_2)dv_2dv_1)^m (\int_0^{\overline{v}} \int_{V_2(0)}^{\overline{v}} g(v_1, v_2)dv_2dv_1)^{n-m}. \tag{2.33}$$

Since $m \geq 0$, the last expression equals 0 if $2n - k + 1 > n$ which occurs if $n + 1 > k$, that is, if the total number of bidders is greater than the number of units. Therefore, $v_2^* = 0$ when $n > k - 1$. □

Lemma 6 *$\overline{\beta}_2(\overline{v}) = \overline{v}$; if $n < k-1$.*

Proof: Consider $T(B_2^*(\overline{v}), \overline{v}, \frac{\partial B_2^*(\overline{v})}{\partial v_2}, G, n, k)$. The denominator of T is positive by lemma 10. If the numerator is equal to zero we know that $B_2^*(\overline{v}) = \overline{v}$. If $v_2^i = \overline{v}$, the numerator of

T equals:

$$\sum_{\ell,m;2\ell+m=2n-k+1} H(B_2^*(\overline{v}),\overline{v},G,n,m,\ell) = \sum_{\ell,m;2\ell+m=2n-k+1} \frac{n!}{\ell!m!(n-m-\ell)!}$$

$$*(\int_0^{B_2^*(\overline{v})}\int_0^{\overline{v}} g(v_1,v_2)dv_2dv_1)^{\ell}(\int_{B_2^*(\overline{v})}^{\overline{v}}\int_0^{\overline{v}} g(v_1,v_2)dv_2dv_1)^m$$

$$*(\int_{B_2^*(\overline{v})}^{\overline{v}}\int_{\overline{v}}^{\overline{v}} g(v_1,v_2)dv_2dv_1)^{n-m-\ell}. \qquad (2.34)$$

The previous equation equals 0 unless n = m + ℓ. It also equals 0 unless $2\ell+m = 2n-k+1$. It can be easily shown that n = m + ℓ and $2\ell + m = 2n - k + 1$ cannot be satisfied simultaneously if $n < k - 1$.

$$(2\ell + m = 2n - k + 1) \Leftrightarrow (2\ell - n - \ell = 2n - k + 1) \Leftrightarrow (\ell = n - k + 1). \qquad (2.35)$$

Noting that $\ell \geq 0$, we see that equation (2.35) is false if n < k-1. We have now shown that if $n + 1 < k$, $T(B_2^*(\overline{v}),\overline{v},\frac{\partial B_2^*(\overline{v})}{\partial v_2},G,n,k) = 0$ and have obtained the initial condition that $B_2^*(\overline{v}) = \overline{v}$ if $n + 1 < k$. The second part of equation (2.7) must hold. □

Lemma 7 $\overline{\beta}_2(0) = 0$; *if* $n = k - 1$.

Proof: Define $\overline{B}_2^*(0)$ to be:

$$\overline{B}_2^*(0) = 0 - T(0,\overline{B}_2^*(0),\frac{\partial \overline{B}_2^*(0)}{\partial v_2},G,n,k). \qquad (2.36)$$

The numerator of T equals:

$$\sum_{\ell,m;2\ell+m=2n-k+1} H(\overline{B}_2^*(0),0,G,n,m,\ell)=\sum_{\ell,m;2\ell+m=2n-k+1}\frac{n!}{\ell!m!(n-m-\ell)!}$$

$$*(\int_0^{\overline{B}_2^*(0)}\int_0^0 g(v_1,v_2)dv_2dv_1)^{\ell}(\int_{\overline{B}_2^*(0)}^{\overline{v}}\int_0^0 g(v_1,v_2)dv_2dv_1)^m$$

$$*(\int_{\overline{B}_2^*(0)}^{\overline{v}}\int_0^{\overline{v}} g(v_1,v_2)dv_2dv_1)^{n-m-\ell}. \quad (2.37)$$

The last expression equals zero unless $\ell = 0$ and m = 0. However, if $\ell = m = 0$, the expression equals zero unless 2n-k+1 equals zero. If 2n - k + 1 = 0, then n $\neq$ k - 1 if $n > 0$. Therefore, for n = k - 1, $T(0,\overline{B}_2^*(0),\frac{\partial \overline{B}_2^*(0)}{\partial v_2},G,n,k)=0$, which implies that $\overline{B}_2^*(0)=0$ for n = k - 1, showing that the third part of equation (2.7) must hold.

Proof of Theorem 1 The proof follows directly from lemmas 1-7. It has now been shown that $B^*(v_1,v_2)$ is a type M undominated symmetric Bayes-Nash equilibrium, only if it equals β. □

2.3.2 Sufficient Conditions

In theorem 1 we provided necessary conditions for a bidding function to be a symmetric undominated Type M Bayes-Nash equilibrium. In theorem 2 sufficient conditions are given for β to be an equilibrium. There are two conditions: A and B. Condition A insures that the appropriate second order conditions are satisfied; if all other demanders use β, the payoff function of bidder i is concave in bidder i's strategy. Condition B insures that β is type M.

Theorem 2 *Suppose that:*

A)

$$\sum_{q=2n-k+2}^{2n} \sum_{m,\ell;2\ell+m=q} \left(\frac{\partial H}{\partial b_2} + \frac{\partial H}{\partial W_2}\frac{\partial^2 W_2}{\partial b_2^2} + \frac{\partial^2 H}{\partial b_2^2} + 2\frac{\partial H}{\partial b_2}\frac{\partial H}{\partial W_2}\frac{\partial W_2}{\partial b_2} + \frac{\partial^2 H}{\partial W_2^2}\left(\frac{\partial W_2}{\partial b_2}\right)^2\right)$$

$$* (W_2(b_2) - b_2) < \sum_{q=2n-k+1}^{2n} \sum_{m,\ell;2\ell+m=q} \left(\frac{\partial H}{\partial b_2} + \frac{\partial H}{\partial W_2}\frac{\partial W_2}{\partial b_2}\right) \tag{2.38}$$

for all b_2 *such that* $0 \leq b_2 \leq \overline{v}$ *where* $W_2(x) : [0, \beta_2(\overline{v})] \rightarrow [v_2^*, \overline{v}]$, $W_2(x) = \beta_2^{-1}(x)$, *where* $H = H(b_2, W_2(b_2), G, n, m, \ell)$ *and:*

B)

$$\frac{1 - \frac{\partial T}{\partial v_2} - \frac{\partial T}{\partial \beta_2}\frac{\partial^2 \beta_2}{\partial v_2^2}}{1 + \frac{\partial T}{\partial \beta_2}} > 0. \tag{2.39}$$

Under conditions A and B, the bidding function $\beta(v_1, v_2) = (\beta_1(v_1, v_2), \beta_2(v_1, v_2))$ *is a symmetric undominated Type M Bayes-Nash equilibrium if it satisfies (2.3) - (2.7)*

The theorem is proven using several lemmas. In lemma 8 we prove that β is Type M. In lemma 9 we prove that β is an equilibrium. Lemmas 10 and 11, required for technical reasons, are proven in appendix A.

Lemma 8 *If condition B holds,* β *is Type M.*

Proof: Clearly $\beta(0,0) = (0,0)$ and β is continuous in v_1 and v_2. $\hat{v}_1(v_2) = 0$, $\hat{v}_2(v_1) = v_2^*$ and $\beta_1(\overline{v}, \overline{v}) = \overline{v} \geq \beta_2(\overline{v}, \overline{v})$. Since $\beta_1(v_1) = v_1, \frac{\partial \beta_1}{\partial v_1} > 0$. Now consider $\beta_2(v_2) = v_2 - T(\beta_2, v_2, \frac{\partial \beta_2}{\partial v_2}, G, n, k)$. The partial derivative of β_2 with respect to v_2 satisfies:

$$\frac{\partial \beta_2}{\partial v_2} = 1 - \left(\frac{\partial T}{\partial \beta_2}\frac{\partial \beta_2}{\partial v_2} + \frac{\partial T}{\partial v_2} + \frac{\partial T}{\partial \beta_2}\frac{\partial^2 \beta_2}{\partial v_2^2}\right). \tag{2.40}$$

Rearranging terms in the last equation yields:

$$\frac{\partial \beta_2}{\partial v_2} = \frac{1 - \frac{\partial T}{\partial v_2} - \frac{\partial T}{\partial \beta_2}\frac{\partial^2 \beta_2}{\partial v_2^2}}{1 + \frac{\partial T}{\partial \beta_2}}, \tag{2.41}$$

which is greater than zero by assumption B. Finally, since β is separable:

$$\frac{\partial \beta_1}{\partial v_2} = \frac{\partial \beta_2}{\partial v_1} = 0. \tag{2.42}$$

We have now shown that β is type M. □

Lemma 9 *If conditions A and B hold, β is a undominated Type M symmetric Bayes-Nash equilibrium.*

Suppose all bidders except for bidder i are using the bidding function $\beta(v_1, v_2)$. The objective function for bidder i is given by:

$$E\pi^i|\beta = \int_{b_2^i}^{b_1^i} (v_1^i - M_{2n-k+1}) f_{2n-k+1}(M_{2n-k+1}) dM_{2n-k+1}$$
$$+ \int_0^{b_2^i} (v_1^i + v_2^i - 2M_{2n-k+2}) f_{2n-k+2}(M_{2n-k+2}) dM_{2n-k+2}$$
$$+(v_1^i - b_2^i)(F_{2n-k+1}(b_2^i) - F_{2n-k+2}(b_2^i)). \tag{2.43}$$

where $F_\nu(x) = F_\nu^{-i}|\beta$. Bidder i chooses b_1^i and b_2^i, to maximize the objective function subject to:

$$b_1^i \geq 0, b_2^i \geq 0. \tag{2.44}$$

By lemma 11 (see Appendix A), the objective function is twice differentiable. The first order necessary conditions are given by:

$$\frac{\partial E\pi^i}{\partial b_1^i} = (v_1^i - b_1^i) f_{2n-k+1}(b_1^i) = 0; b_1^i > 0,$$
$$\leq 0; b_1^i = 0, \tag{2.45}$$

and:

$$\frac{\partial E\pi^i}{\partial b_2^i} = (v_2^i - b_2^i) f_{2n-k+2}(b_2^i) - (F_{2n-k+1}(b_2^i) - F_{2n-k+2}(b_2^i)) = 0; b_2^i > 0,$$
$$\leq 0; b_2^i = 0. \tag{2.46}$$

The second derivatives are (omitting the superscript designating demander for notational ease):

$$\frac{\partial^2 E\pi}{\partial b_1^2} = (v_1 - b_1) f'_{2n-k+1}(b_1) - f_{2n-k+1}(b_1), \tag{2.47}$$

$$\frac{\partial^2 E\pi}{\partial b_1 \partial b_2} = \frac{\partial^2 E\pi}{\partial b_2 \partial b_1} = 0, \tag{2.48}$$

$$\frac{\partial^2 E\pi}{\partial b_2^2} = (v_2 - b_2) f'_{2n-k+2}(b_2) - f_{2n-k+1}(b_2). \tag{2.49}$$

The second order conditions are then:

$$\frac{\partial^2 E\pi}{\partial b_1^2} < 0, \frac{\partial^2 E\pi}{\partial b_2^2} < 0. \tag{2.50}$$

It follows from (2.45) that if $f_{2n-k+1}(b_2^i) > 0$, then $b_1 = v_1$ for $v_1 \geq 0$. If $f_{2n-k+1}(b_2^i) = 0$, then there can be more than one solution to (2.45) but any strategy is weakly dominated by $b_1 = v_1$. Therefore $b_1^i = \beta_1(v_1^i) = v_1^i$. Since β being used by all players besides i and β is type M,

$$Prob(v_1 \leq \beta_1^{-1}(b_2^i)) = Prob(v_1 \leq b_2^i). \tag{2.51}$$

And since $\frac{\partial\beta_2(v_2)}{\partial v_2} > 0$ for $v_2 > v_2^*$:

$$Prob(\beta_2(v_2) \leq b_2^i) = Prob(v_2 \leq W_2(b_2^i)). \tag{2.52}$$

As we did in the proof of theorem 1, we can derive the following equation: Prob(exactly ℓ observations of $\beta_1(v_1^{-i}) \leq b_2^i$, exactly m + ℓ observations of $\beta_2(v_2^{-i}) \leq b_2^i$)

$$= \frac{n!}{\ell!m!(n-m-\ell)!}\left(\int_0^{W_2(b_2^i)}\int_0^{b_2^i} g(v_1,v_2)dv_1dv_2\right)^{\ell}\left(\int_0^{W_2(b_2^i)}\int_{b_2^i}^{\overline{v}} g(v_1,v_2)dv_1dv_2\right)^{m}$$
$$\left(\int_{W_2(b_2^i)}^{\overline{v}}\int_{b_2^i}^{\overline{v}} g(v_1,v_2)dv_1dv_2\right)^{n-m-\ell}. \tag{2.53}$$

The previous expression equals $H(b_2^i, W_2(b_2^i), G, n, m, \ell)$, and equations (2.54) and (2.55) follow:

$$(F_{2n-k+1}(b_2^i) - F_{2n-k+2}(b_2^i)) = \sum_{\ell,m;2\ell+m=2n-k+1} H(b_2^i, W_2(b_2^i), G, n, m, \ell), \tag{2.54}$$

$$f_{2n-k+2}(b_2^i)) = \sum_{q=2n-k+1}^{2n} \sum_{\ell,m;2\ell+m=q} \left(\frac{\partial H}{\partial b_2^i} + \frac{\partial H}{\partial W_2}\frac{\partial W_2}{\partial b_2^i}\right). \tag{2.55}$$

The last two equations imply that:

$$(F_{2n-k+1}(b_2^i) - F_{2n-k+2}(b_2^i))/f_{2n-k+2}(b_2^i) = T\left(b_2^i, W_2(b_2^i), \left(\frac{\partial W_2(b_2^i)}{\partial b_2^i}\right)^{-1}, G, n, k\right). \tag{2.56}$$

We can rewrite the first order condition in (2.46) for $b_2^i > 0$ as:

$$b_2^i = v_2^i - T(b_2^i, W_2(b_2^i), (\frac{\partial W_2(b_2^i)}{\partial b_2^i})^{-1}, G, n, k). \tag{2.57}$$

One solution to the last equation is to set $b_2^i = \beta_2(v_2^i)$:

$$\beta_2(v_2^i) = v_2^i - T(\beta_2(v_2^i), W_2(\beta_2(v_2^i), (\frac{\partial W_2}{\partial \beta_2})^{-1}, G, n, k). \tag{2.58}$$

Recalling the fact that $W(x) = \beta_2^{-1}(x)$, we see that:

$$\beta_2(v_2^i) = v_2^i - T(\beta_2(v_2^i), v_2^i, \frac{\partial \beta_2(v_2^i)}{\partial v_2}, G, n, k). \tag{2.59}$$

The first order conditions imply that for $b_2^i = 0$:

$$\begin{aligned} 0 &\geq W_2(0) - T(0, W_2(0), (\frac{\partial W_2(0)}{\partial b_2^i})^{-1}, G, n, k), \\ 0 &\geq W_2(0) - T(\beta_2(v_2^i), v_2^i, \frac{\partial \beta_2}{\partial v_2}, G, n, k). \end{aligned} \tag{2.60}$$

The last inequality holds with equality when $W_2(0) = v_2^*$. Now consider some $v_2^o < v_2^*$.

$$0 > v_2^o - T(0, v_2^*, (\frac{\partial W_2(0)}{\partial b_2^i})^{-1}, G, n, k). \tag{2.61}$$

By the first order condition (2.46), $b_2(v_2^o) = \beta_2(v_2^o) = 0$. We have now shown that the best response to β must satisfy (2.3)-(2.6).

It remains to derive initial conditions for the differential equation in (2.59). Consider $v_2^* = T(0, v_2^*, \frac{\partial \beta_2(v_2^*)}{\partial v_2}, G, n, k) = \frac{F_{2n-k+1}(0) - F_{2n-k+2}(0)}{f_{2n-k+2}(0)}$. By lemma 10 $f_{2n-k+2}(0) > 0$. If $F_{2n-k+1}(0) - F_{2n-k+2}(0) = 0$ it would imply that $v_2^* = 0$.

$$F_{2n-k+1}(0) - F_{2n-k+2}(0) = \sum_{\ell,m;2\ell+m=2n-k+1} \frac{n!}{\ell!m!(n-m-\ell)!}$$

$$\left(\int_0^0 \int_0^{W_2(0)} g(v_1, v_2)dv_2dv_1\right)^\ell \left(\int_0^{\overline{v}} \int_0^{W_2(0)} g(v_1, v_2)dv_2dv_1\right)^m$$

$$\left(\int_0^{\overline{v}} \int_{W_2(0)}^{\overline{v}} g(v_1, v_2)dv_2dv_1\right)^{n-m-\ell}. \tag{2.62}$$

The last expression equals 0 unless $\ell = 0$. If ℓ equals 0, the expression in equation (2.62) equals:

$$\sum_{m;m=2n-k+1} \frac{n!}{m!(n-m)!} \left(\int_0^{\overline{v}} \int_0^{W_2(0)} g(v_1, v_2)dv_2dv_1\right)^m \left(\int_0^{\overline{v}} \int_{W_2(0)}^{\overline{v}} g(v_1, v_2)dv_2dv_1\right)^{n-m}. \tag{2.63}$$

Since $m \geq 0$ the last expression equals 0 if $n + 1 > k$ implying that $v_2^* = 0$ if $n + 1 > k$. Now consider $T(\beta_2(\overline{v}), \overline{v}, \frac{\partial \beta_2(\overline{v})}{\partial v_2}, G, n, k)$. If the numerator is equal to zero, $\beta_2(\overline{v}) = \overline{v}$. The numerator equals:

$$\sum_{\ell,m;2\ell+m=2n-k+1} H(\beta_2(\overline{v}), \overline{v}, G, n, m, \ell) = \sum_{\ell,m;2\ell+m=2n-k+1} \frac{n!}{\ell!m!(n-m-\ell)!}$$

$$*\left(\int_0^{\beta_2(\overline{v})} \int_0^{\overline{v}} g(v_1, v_2)dv_2dv_1\right)^\ell \left(\int_{\beta_2(\overline{v})}^{\overline{v}} \int_0^{\overline{v}} g(v_1, v_2)dv_2dv_1\right)^m$$

$$*\left(\int_{\beta_2(\overline{v})}^{\overline{v}} \int_{\overline{v}}^{\overline{v}} g(v_1, v_2)dv_2dv_1\right)^{n-m-\ell}. \tag{2.64}$$

The previous equation equals 0 unless n = m + ℓ. It is also 0 unless $2\ell + m = 2n - k + 1$. n = m + ℓ and $2\ell + m = 2n - k + 1$ can not both be true if $n < k - 1$. Therefore, if $n + 1 < k, T(\beta_2(\overline{v}), \overline{v}, \frac{\partial \beta_2(\overline{v})}{\partial v_2}, G, n, k) = 0$ and the initial condition that $\beta_2(\overline{v}) = \overline{v}$ if $n + 1 < k$ follows. Now suppose n+1= k and consider $\overline{\beta}_2(0)$ where:

$$\overline{\beta}_2(0) = 0 - T(\overline{\beta}_2(0), 0, \frac{\partial \overline{\beta}_2(0)}{\partial v_2}, G, n, k). \tag{2.65}$$

The numerator of $T(\overline{\beta}_2(0), 0, \frac{\partial \overline{\beta}_2(0)}{\partial v_2}, G, n, k)$ equals:

$$\sum_{\ell, m; 2\ell + m = 2n-k+1} H(\overline{\beta}_2(0), 0, G, n, m, \ell) = \sum_{\ell, m; 2\ell + m = 2n-k+1} \frac{n!}{\ell! m! (n - m - \ell)!}$$

$$* (\int_0^{\overline{\beta}_2(0)} \int_0^0 g(v_1, v_2) dv_2 dv_1)^{\ell} (\int_{\overline{\beta}_2(0)}^{\overline{v}} \int_0^0 g(v_1, v_2) dv_2 dv_1)^m$$

$$(\int_{\overline{\beta}_2(0)}^{\overline{v}} \int_0^{\overline{v}} g(v_1, v_2) dv_2 dv_1)^{n-m-\ell}. \tag{2.66}$$

The last expression equals zero unless $\ell = 0$ and m = 0. However, if $\ell = m = 0$, the expression equals zero unless 2n-k+1 equals zero. If 2n - k + 1 = 0 then n $\neq$ k - 1 if $n > 0$. Therefore, for n = k - 1, $T = 0$ which implies that $\overline{\beta}_2(0) = 0$ when n = k - 1.

We have now shown that $\beta(v_1, v_2)$ is a solution to the first order conditions in (2.45) and (2.46). If the appropriate second order conditions hold, β is a best response. The second order conditions are:

$$\frac{\partial^2 E\pi}{\partial b_1^2} = (v_1 - b_1) f'_{2n-k+1}(b_1) - f_{2n-k+1}(b_1) < 0 \tag{2.67}$$

and

$$\frac{\partial^2 E\pi}{\partial b_2^2} = (v_2 - b_2)f'_{2n-k+2}(b_2) - f_{2n-k+1}(b_2) < 0. \tag{2.68}$$

$\frac{\partial^2 E\pi}{\partial b_1^2} < 0$ is satisfied if $b_1 = v_1$ and $f_{2n-k+1}(b_2)$ is greater than zero. If $f_{2n-k+1}(b_2) = 0$, then many solutions are possible but $b_1 = v_1$ dominates any solution that has $b_1 \neq v_1$. $\frac{\partial^2 E\pi}{\partial b_2^2} < 0$ is insured by condition A. To see this, consider:

$$f_{2n-k+2}(b_2^i) = \sum_{q=2n-k+2}^{2n} \sum_{m,\ell;2\ell+m=q} (\frac{\partial H}{\partial b_2^i} + \frac{\partial H}{\partial W_2}\frac{\partial W_2}{\partial b_2^i}). \tag{2.69}$$

Equation (2.69) implies that $f'_{2n-k+2}(b_2^i)$

$$= \sum_{q=2n-k+2}^{2n} \sum_{2\ell+m=q} (\frac{\partial H}{\partial b_2^i} + \frac{\partial H}{\partial W_2}\frac{\partial^2 W_2}{\partial b_2^{i2}} + \frac{\partial^2 H}{\partial b_1^{i2}} + 2\frac{\partial H}{\partial b_2^i}\frac{\partial H}{\partial W_2}\frac{\partial W_2}{\partial b_2^i} + \frac{\partial^2 H}{\partial W_2^2}(\frac{\partial W_2}{\partial b_2^i})^2) \tag{2.70}$$

and it is now apparent from equations (2.38), (2.69) and (2.70) that assumption A is satisfied if and only if $\frac{\partial^2 E\pi|\beta}{\partial b_2^2} < 0$ for all b_2 such that $0 \leq b_2 \leq \overline{v}$. Assumption A guarantees that the function $E\pi^i$ is strictly concave in b_2^i when all demanders besides bidder i use the strategy β. We have now shown that β is a best response to itself under assumptions A and B. □

There is underrevelation on the lower valued unit for the following reason: in the event that a demander's lower bid is the k+1st highest, he has some incentive to underbid on it in order to lower the price he pays for the unit he receives (the fact that the lower bid is the k+1st highest implies that the higher bid is among the k highest, and therefore the demander receives exactly one unit). There is no incentive to underbid on the higher valued unit, since in the event that the demander's higher bid is the k+1st highest, he wins no units, and his profits are zero. Overbidding is always a dominated strategy.

The two symmetric equilibrium bids are separable, indicating that the extent of under-revelation on the lower unit depends only upon the rank of the unit, the distribution of valuations, the number of bidders and the number of units sold, and is independent of the bidder's higher valuation and his higher bid. The independence results from the fact that the price paid is independent of the amount of the higher bid, and therefore the gains from lowering the final price depend only upon how many bids are accepted in the event that the lower bid is the k+1st highest.

It is important to realize that if a symmetric equilibrium to this game exists, it does not preclude the existence of asymmetric Nash equilibria. Consider a case where there are two units to be sold to two buyers, each of whom values two units. One buyer bids $\overline{v}$ for both of her units and the other bids 0 for both of her units. No single demander has any positive incentive to change her strategy.

2.3.3 A Simple Example

The following is an example of an undominated Type M symmetric equilibrium.

Suppose $\gamma(v)$ is uniform on the interval from 0 to 1, n + 1 = 2 and k = 3. We know that $\beta_1(v_1^i) = v_1^i$ in an undominated symmetric equilibrium. The calculation of $\beta_2(v_2^i)$, which equals $v_2^i - T(.)$, proceeds in the following manner. First note that $\frac{n!}{\ell! m!(n-m-\ell)!} = 1$. Using the fact that $v_1 \geq v_2$, we can derive the following equation which gives the probability that a bidder makes exactly two bids less than or equal to b_2^i:

$$\int_0^{V_2(b_2^i)} \int_0^{b_2^i} g(v_1, v_2) dv_1 dv_2 = \int_0^{b_2^i} \int_0^{b_2^i} g(v_1, v_2) dv_1 dv_2 = (b_2^i)^2. \tag{2.71}$$

The probability that a randomly chosen bidder makes exactly one bid less than or equal to b_2^i equals:

$$\int_0^{V_2(b_2^i)}\int_{b_2^i}^{\overline{v}} g(v_1,v_2)dv_1dv_2 = 2V_2(b_2^i) - (V_2(b_2^i))^2 - (b_2^i)^2. \tag{2.72}$$

and again using $v_1 \geq v_2$, we can derive the probability that a randomly chosen bidder makes exactly zero bids that are less than or equal to b_2^i. The probability is given by:

$$\int_{V_2(b_2^i)}^{\overline{v}}\int_{b_2^i}^{\overline{v}} g(v_1,v_2)dv_1dv_2 = \int_{V_2(b_2^i)}^{\overline{v}}\int_{V_2(b_2^i)}^{\overline{v}} g(v_1,v_2)dv_1dv_2 = (1 - V_2(b_2^i))^2. \tag{2.73}$$

It can be easily verified that:

$$(b_2^i)^2 + 2V_2(b_2^i) - (V_2(b_2^i))^2 - (b_2^i)^2 + (1 - V_2(b_2^i))^2 = 1. \tag{2.74}$$

The numerator of T equals the following expression (note 2n-k+1 = 0):

$$\sum_{(m,\ell;m+2\ell=2n-k+1)} H(b_2^i, V_2(b_2^i), G(v_1,v_2), n, m, \ell) = (1 - V_2(b_2^i))^2. \tag{2.75}$$

Next, we derive the denominator of T. $F_{2n-k+2}(b_2^i)$ equals:

$$(b_2^i)^2 + 2V_2(b_2^i) - (V_2(b_2^i))^2 - (b_2^i)^2. \tag{2.76}$$

The derivative of this expression is:

$$f_{2n-k+2}(b_2^i) = (2 - 2V_2(b_2^i))V_2'(b_2^i). \tag{2.77}$$

Since the equilibrium is symmetric, $V_2(b_2^i) = v_2^i$. The solutions to the first order necessary conditions are given by:

$$\begin{aligned} b_1^i = \beta_1(v_1) &= v_1^i; b_1^i > 0, \\ &\geq v_1^i; b_1^i = 0, \end{aligned} \tag{2.78}$$

$$\begin{aligned} b_2^i = \beta_2(v_2) &= v_2^i - \frac{(1-v_2^i)^2}{(2-2v_2^i)V_2'(b_2^i)}; b_2^i > 0, \\ &\geq v_2^i - \frac{(1-v_2^i)^2}{(2-2v_2^i)V_2'(b_2^i)}; b_2^i = 0, \end{aligned} \tag{2.79}$$

with the initial condition $\beta_2(1) = 1$ because $n < k - 1$. Solving for $V_2'(b_2^i)$ we obtain:

$$V_2'(b_2^i) = \frac{1 - v_2^i}{2(v_2^i - b_2^i)}. \tag{2.80}$$

A solution can be found by setting $b_2^i = (v_2^i)^2$ which implies that $V_2(b_2^i) = (b_2^i)^{\frac{1}{2}}$ and also that $V_2' = \frac{1}{2}(v_2^i)^{-1}$. We obtain:

$$\beta(v_1, v_2) = (v_1, (v_2)^2) \tag{2.81}$$

It can be readily verified that the second order conditions are satisfied.

2.4 Large Numbers of Bidders

If a large number of bidders wants to buy a fixed supply of units, one might expect that each bidder has little incentive to underreveal his demand. As the number of bidders approaches infinity, it seems reasonable to believe that, if demanders are playing symmetric equilibrium strategies, the amount of underrevelation would converge to zero, each bidder would submit two bids equal to his valuations, and the result would be an efficient final allocation at the competitive equilibrium prices. In theorem 1, we provided necessary conditions for a bidding function to be an undominated Type M symmetric Bayes-Nash equilibrium. The equilibrium must involve underbidding by an amount equal to T(.) on demanders' lower valued units. If it could be shown that as the number of bidders gets large, T converges to zero, then the equilibrium bidding strategies must converge to truthful revelation. The following theorem implies that in fact, as the number of bidders becomes large and the supply remains constant, the only possible symmetric undominated type M Bayes-Nash equilibrium is truthful bidding.

Theorem 3 *Let* $z_1(n) : N \to [0,\overline{v}], z_2(n) : N \to [0,\overline{v}]$ *and* $z_3(n) : N \to R^+$. *and let* $z_1(n) \leq z_2(n)$ *for all n. Then for k fixed, as* $n \to \infty, T(z_1(n), z_2(n), z_3(n), G, k, n) \to 0$

Proof: Recall that:

$$T(z_1, z_2, z_3, G, n, k) = \frac{\sum_{m,\ell;m+2\ell=2n-k+1} H(z_1, z_2, n, m, \ell)}{\sum_{q=2n-k+2}^{2n} \sum_{m,\ell;m+2\ell=q} (\frac{\partial H}{\partial z_1} + \frac{\partial H}{\partial z_2}\frac{1}{z_3})} \tag{2.82}$$

$$= \frac{\sum_{m,\ell;m+2\ell=2n-k+1} \frac{n!}{\ell!m!(n-m-\ell)!} T_{num}}{\sum_{q=2n-k+2}^{2n} \sum_{m,\ell;m+2\ell=q} \frac{n!}{\ell!m!(n-m-\ell)!} T_{denom}} \tag{2.83}$$

where:

$$T_{num}(z_1(n), z_2(n), z_3(n), G, k, n) = (\int_0^{z_2} \int_0^{z_1} g(v_1, v_2) dv_1 dv_2)^{\ell}$$

$$* (\int_0^{z_2} \int_{z_1}^{\overline{v}} g(v_1, v_2) dv_1 dv_2)^m (\int_{z_2}^{\overline{v}} \int_{z_1}^{\overline{v}} g(v_1, v_2) dv_1 dv_2)^{n-m-\ell}. \tag{2.84}$$

Let n increase to n+1 for k fixed. Then it must be the case that ℓ also increases to $\ell + 1$ since the summation in the last equation is over all combinations of m and ℓ such that: $m + 2\ell = 2n - k + 1$. Therefore, $n \to \infty \Rightarrow \ell \to \infty$. Let $\alpha = n - \ell$. α stays constant as n and ℓ increase. Since the integrand in the next equation is less than or equal to one:

$$\lim_{n\to\infty} (\int_0^{z_2(n)} \int_0^{z_1(n)} g(v_1, v_2) dv_1 dv_2)^{n-\alpha} = 0; z_1 < \overline{v},$$

$$= 1; z_1 = \overline{v}. \tag{2.85}$$

Also:

$$(\int_0^{z_2(n)} \int_{z_1(n)}^{\overline{v}} g(v_1, v_2) dv_1 dv_2)^m (\int_{z_2(n)}^{\overline{v}} \int_{z_1(n)}^{\overline{v}} g(v_1, v_2) dv_1 dv_2)^{\alpha - m} = A(n, z_1, z_2, G), \tag{2.86}$$

where $0 \leq A(.) < 1$ and equals 0 if $z_1 = \overline{v}$. It follows that:

$$\lim_{n\to\infty} T_{num}(z_1(n), z_2(n), z_3(n), G, n, k) = 0. \tag{2.87}$$

Now we consider $T_{denom}(z_1(n), z_2(n), z_3(n), n, k, G)$, which equals (using the product rule):

$$(\int_0^{z_2} \int_0^{z_1} g(v_1, v_2) dv_1 dv_2)^{n-\alpha} \frac{\partial A(.)}{\partial z_1}$$

$$+(n-\alpha)(\int_0^{z_2} g(z_1, v_2) dv_2)^{(n-\alpha-1)} A(.) + (\int_0^{z_2} \int_0^{z_1} g(v_1, v_2) dv_1 dv_2)^{n-\alpha} \frac{\partial A(.)}{\partial z_2} \frac{1}{z_3}$$

$$+ (n-\alpha)(\int_0^{z_2} g(v_1, z_2) dv_1)^{(n-\alpha-1)} * A(n) * \frac{1}{z_3(n)}. \tag{2.88}$$

We know from equation (2.85) that:

$$\lim_{n\to\infty}\left(\int_0^{z_2(n)}\int_0^{z_1(n)} g(v_1,v_2)dv_1dv_2\right)^{n-\alpha} = 0; z_1 < \overline{v},$$
$$= 1; z_1 = \overline{v}. \tag{2.89}$$

Let:

$$\frac{\partial A(n,z_1(n),z_2(n),.)}{\partial z_1} = X(n). \tag{2.90}$$

$-\infty < X(n) < \infty$. The derivative exists and is finite because $g(v_1,v_2) \in C^1$ on $[0,\overline{v}]^2$. By assumption, $z_1(n) \leq z_2(n)$ and (by the definition of $g(v_1,v_2)$), $v_1 \geq v_2$. Therefore, $\int_0^{z_2(n)} g(z_1(n),v_2)dv_2 = Prob(v_2 \leq z_2(n)|v_1 = z_1(n)) = 1$. Taking the limit as n gets large, we find:

$$\lim_{n\to\infty}(n-\alpha)\left(\int_0^{z_2(n)} g(z_1(n),v_2)dv_2\right)^{(n-\alpha-1)} = \lim_{n\to\infty}(n-\alpha) = \infty. \tag{2.91}$$

Let $Z(n) = \frac{\partial A(n,.)}{\partial z_2}$. $-\infty < Z(n) < \infty$. The derivative exists and is finite because $g(v_1,v_2) \in C^1$ on $[0,\overline{v}]^2$. $\frac{1}{z_3}$ is > 0 because by assumption $z_3(n) > 0$. The remaining term of T_{denom} is:

$$(n-\alpha)\left(\int_0^{z_1(n)} g(v_1,z_2(n))dv_1\right)^{(n-\alpha-1)} * A(.) * \frac{1}{z_3(n)}. \tag{2.92}$$

Note that $(\int_0^{z_1} g(v_1,v_2^i)dv_2)$ is equal to $Prob(v_1 \leq z_1(n)|v_2 = z_2(n))$. In the expression of the last equation, $z_1(n) < z_2(n)$ while $v_1 \geq v_2$. Therefore, $(\int_0^{z_1(n)} g(v_1,z_2)dv_2) = 0$ for all n. $\frac{1}{z_3}$ is > 0 by assumption. Now, notice that:

$$\frac{\sum_{m,\alpha;2(n-\alpha)+m=2n-k+1}\left(\frac{n!}{(n-\alpha)!m!(\alpha-m)!}T_{num}\right)}{\sum_{q=2n-k+2}^{2n}\sum_{m,\alpha;2(n-\alpha)+m=q}\left(\frac{n!}{(n-\alpha)!m!(\alpha-m)!}T_{denom}\right)} \tag{2.93}$$

$$\leq \frac{\sum_{m,\alpha;2(n-\alpha)+m=2n-k+1}(\frac{n!}{(n-\alpha)!m!(\alpha-m)!}T_{num})}{\sum_{m,\alpha;2(n-\alpha)+m=2n-k+2}(\frac{n!}{(n-\alpha)!m!(\alpha-m)!}T_{denom})} \tag{2.94}$$

$$\leq \sum_{m,\alpha;2(n-\alpha)+m=2n-k+1} (\frac{\frac{n!}{(n-\alpha)!m!(\alpha-m)!}}{\frac{n!}{(n-\alpha+1)!(m-1)!(\alpha-m)!}} \frac{T_{num}}{T_{denom}}) \tag{2.95}$$

Therefore:

$$\lim_{n\to\infty} T(.) \leq \sum_{2(n-\alpha)+m=2n-k+1} (\lim_{n\to\infty} \frac{n-\alpha+1}{m} * \lim_{n\to\infty} \frac{T_{num}}{T_{denom}}) \tag{2.96}$$

$$\leq \lim_{n\to\infty} \sum_{m,\alpha;2(n-\alpha)+m=2n-k+1} \frac{(n-\alpha+1)*0}{(n-\alpha)*m*A(.)} \tag{2.97}$$

Since $m \geq 1$ (because $2\ell + m = 2n - k + 1, k \geq 2$ and $\ell \leq n$), the limit in the last expression is clearly 0 if $A(n) > 0$.

Now recall that in any type M equilibrium, $\overline{\beta}_2(v_2) \leq v_2$ and $\frac{\partial \overline{\beta}_2}{\partial v_2} > 0$ for $v_2 \geq v_2^*$. Also, in any type M equilibrium, $A(n, \beta_2(v_2), v_2, G) > 0$ for $v_2 < \overline{v}$. Therefore we can let $z_1(n) = \overline{\beta}_2(v_2, n, .), z_2(n) = v_2$, and $z_3(n) = \frac{\partial \overline{\beta}_2(v_2,n)}{\partial v_2}$. It must be the case that:

$$\lim_{n\to\infty} T(\overline{\beta}_2(v_2), v_2, \frac{\partial \overline{\beta}_2}{\partial v_2}, G, n, k) = 0. \tag{2.98}$$

□

2.4.1 Digression: A Note on Replication

It might seem reasonable to compare the behavior of an economy with n+1 demanders and k units to be sold with that of an economy with 2(n+1) demanders and 2k units to be sold. One might conjecture that the expected revenue of the uniform price sealed bid auction in the latter economy would be twice the expected revenue in the former. It can be shown that the conjecture is false in the multi-unit demand case but it is easy to show that it is also false even in the single-unit demand case. The following example clarifies the idea.

Example 3: n + 1 = 2, k = 1, and demanders draw one valuation from G(v) which we assume for the example is uniform on [0,1]. In the dominant strategy equilibrium, $b^i = v^i$. The expected revenue is equal to the expected value of the lower of 2 order statistics in a sample of 2 drawn from G.

$$Rev1 = k\int_{-\infty}^{+\infty} v\frac{(n+1)!}{(k-1)!(n+1-k)!}G(v)^{k-1}(1-G(v))^{n+1-k}g(v)dv \tag{2.99}$$

$$= \int_{-\infty}^{+\infty} v(2-2v)dv. \tag{2.100}$$

Now suppose n + 1 = 4, and k = 2. The expected revenue from the auction is equal to the expected value of the second lowest order statistic from a sample of 4 drawn from G. The expected revenue equals:

$$Rev2 = 2\int_{-\infty}^{+\infty} 6v^2(1-v)^2dv \neq 2 * Rev1. \tag{2.101}$$

Chapter 3

Uniform Price Auction vs. English Clock

As we saw in chapter one, the uniform price sealed bid auction and the English clock are isomorphic in the independent private values environment when demands are single-unit. The focus of this chapter is the difference in the bidding strategies, revenue and final allocations between the two types of auctions under multi-unit demands. As is illustrated in section 3.1, the type of auction which achieves higher expected efficiency and induces higher bids, leading to higher expected revenue, depends on the distribution of valuations of demanders.

In sections 3.2-3.4 we describe the design of an experiment, which we use to consider some empirical properties of the two types of auctions. In section 3.5 we give some theoretical predictions of experimental outcomes. In section 3.6 we analyze the data in detail and describe the results. We compare the allocative efficiency and market revenue across three different dimensions:

1) whether the uniform price sealed bid auction or the English clock is employed,

2) whether individual demands are single-unit or two-unit,

3) whether the auction is run once or is repeated.

We also test statistically the predictions of the dominant strategy equilibria of both types of auctions in the single-unit demand environment, the symmetric monotone Bayes-Nash equilibrium of the uniform price auction in the two-unit demand environment, as well as revenue equivalence of the two auctions.

3.1 Theoretical Non-Equivalence

The purpose of this section is to show that the uniform price sealed bid auction and the English clock do not necessarily generate identical expected revenues and allocative efficiencies in their respective symmetric equilibria in multi-unit demand environments.

Assume the following independent private values framework. Let there be m demanders indexed by i so that i=1,...,m. There are k identical units of a commodity to be sold. Each demander i draws w_i valuations $v_1^i, ..., v_{w_i}^i$ independently from a distribution $\Gamma^i(v)$. Demander i knows $m, k, v_1^i, ..., v_{w_i}^i, \Gamma^1(v), ..., \Gamma^m(v)$ and $w_1, ..., w_m$. He does not know the actual valuations of the other buyers.

Theorem 4 *The English clock and the uniform price sealed bid auction may generate inefficient allocations in their respective symmetric equilibria. Either the English clock or the uniform price sealed bid auction may generate greater expected revenue or allocative efficiency.*

Proof: Consider examples 4 and 5.

Example 4: $m = 3, k = 2, w_1 = 2, w_2 = w_3 = 1$.

$$\Gamma^2(v) = \Gamma^3(v) = \begin{cases} 0 & v < 1, \\ .01 & 1 \leq v < 8, \\ 1 & v \geq 8. \end{cases} \tag{3.1}$$

An approximation of $d\Gamma^2(v)$ is illustrated in figure 3.1. Let $\Gamma^1(v) = 0$ for $v < 10$ and $= 1$ for $v \geq 10$. Also, let $v_1^1 = 10, v_2^1 = 10, v_1^2 = 8, v_1^3 = 1$ and the allocation mechanism used be the uniform price sealed bid auction. Assume that all ties are broken in buyer 1's favor. We know that $b_1^i = v_1^i$ if $w_i = 1$ (a single-unit demander bids equal to his valuation). Therefore buyer 1, supposing that $v_1^2 = b_1^2$ and that $v_1^3 = b_1^3$, faces the following expected payoff. $E\pi^1$:

$$= \begin{cases} 10 + 10 - 2((P_1 + P_2)(8) + P_3(1)) & b_1^1 \geq 8, b_2^1 \geq 8, \\ P_1(10 - 8) + P_2(10 - b_2^1) + P_3(10 + 10 - 2(1)) & b_1^1 \geq 8, 8 > b_2^1 \geq 1 \\ P_3(10 + 10 - 2(1)) + P_2(10 - b_2^1) & 8 > b_1^1 \geq 1, 8 > b_2^1 \geq 1 \\ P_1(10 - 8) + (P_2 + P_3)(10 - 1) & b_1^1 \geq 8, b_2^1 < 1 \\ (P_2 + P_3)(10 - 1) & 8 > b_1^1 \geq 1, b_2^1 < 1 \\ 0 & b_1^1 < 1, b_2^1 < 1. \end{cases} \tag{3.2}$$

where $P_1 = .99^2, P_2 = 1 - .99^2 - .01^2$, and $P_3 = .01^2$. The expected profits for each of the six types of strategies in (3.2) are $4.0014, 2.16 - .0198b_2^1, .1998 - .0198b_2^1, 2.1393, .1791$ and 0 respectively for rows 1 - 6. Bidder 1 maximizes expected profit by submitting $b_1^1, b_2^1 \geq 8$, that is, by choosing a strategy from the first row of (3.2). Each of the other two bidders, for whom $w_i = 1$, submits a bid equal to his valuation. Bidder one wins both units and pays a per-unit price of 8. The allocation is efficient because the units are obtained by the demander(s) to whom they have the greatest value.

Suppose now that the mechanism utilized is the English clock. If $\underline{x} = 0$, then $q^1(0) = 2, q^2(0) = q^3(0) = 1$. At price 1 bidder 3 drops his demand from one unit to zero units because he does not want to win the unit at a price greater than its value to him. This leaves total demand at 3 units. At price $1 + \epsilon$ bidder 1 can update his beliefs about the other two bidders. Since $\text{Prob}(v_1^2 = 8 | v_1^2 > 1) = 1$, bidder 1 knows that d_1^2 will equal 8. Bidder 1 reasons that his profits, if $d_2^1 > 8$, would be (10 - 8) + (10 - 8) = 4. The final price would be 8 because bidder 2 is a single-unit demander with a dominant strategy of truthful revelation. If, however, $d_2^1 = 1 + \epsilon$, (bidder 1 reduces his quantity demanded to one unit immediately after bidder 3 drops out), his profits are $(10 - 1 - \epsilon) = (9 - \epsilon) > 4$, and so he drops his demand to one unit at price $1 + \epsilon$. The final allocation is the following: buyers one and two each receive a unit and the market price is $1 + \epsilon$. The allocation is not efficient.

In example 4, the two mechanisms yield different ex-post revenue. Notice that they also do not yield the same expected revenue. There are three other possible configurations of demand in the example. They are:

1) $v_1^1 = 10, v_2^1 = 10, v_1^2 = 1, v_1^3 = 8$,

2) $v_1^1 = 10, v_2^1 = 10, v_1^2 = 1, v_1^3 = 1$,

3) $v_1^1 = 10, v_2^1 = 10, v_1^2 = 8, v_1^3 = 8$.

Case one obtains the same outcome as in example 4 which we have just discussed, but with demanders 2 and 3 reversed. In case two, if either type of auction is used then buyer

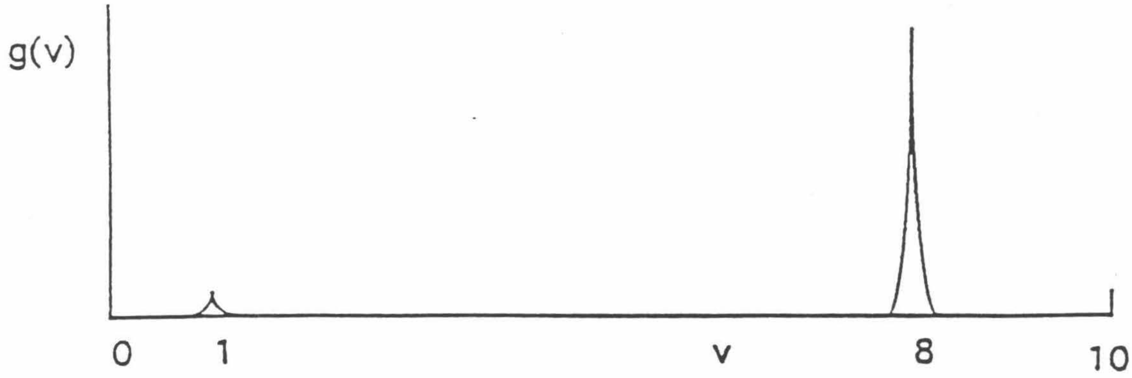

Figure 3.1: Probability Density Function for Example 4

one obtains both units and pays a per-unit price of 1. In case three, if either type of auction is used, buyer one obtains both units and pays a per-unit price of 8. In all realizations the revenue generated by the uniform price sealed bid auction is greater than or equal (and in some realizations strictly greater) to that generated by the English clock. Therefore, the expected revenue of the uniform price sealed bid auction is greater than that of the English clock for the parameters of example 4.

Example 5: Suppose the distribution from which bidders 2 and 3 draw their valuation(s) is the one whose density function is illustrated in figure 2 in which,

$$\Gamma^2(v) = \Gamma^3(v) = \begin{cases} 0 & v < 2, \\ .5 & 2 \leq v < 8, \\ 1 & v \geq 8. \end{cases} \tag{3.3}$$

Bidder 1 draws two valuations from a distribution in which $\text{Prob}(v_1^1 = v_2^1 = 10) = 1$. Suppose $m = 3, k = 2, w_1 = 2$ and $w_2 = w_3 = 1$, the allocation mechanism employed is the uniform price sealed bid auction and $v_1^1 = 10, v_2^1 = 10, v_1^2 = 8, v_1^3 = 8$. Recall that

$b_1^i = v_1^i = 8$ for $i \in 2, 3$. Consider bidder 1's bids. His expected payoff, $E\pi^1$, equals:

$$\begin{cases} (P_1+P_2)(10+10-2(8))+P_3(10+10-2(2)) & b_1^1 \geq 8, b_2^1 \geq 8, \\ P_1(10-8)+P_2(10-b_2^1)+P_3(10+10-2(2)) & b_1^1 \geq 8, 8 > b_2^1 \geq 2, \\ P_3(10+10-2(2))+P_2(10-b_2^1) & 8 > b_1^1 \geq 2, 8 > b_2^1 \geq 2, \\ P_1(10-8)+(P_2+P_3)(10-2) & b_1^1 \geq 8, b_2^1 < 2, \\ (P_1+P_2)(10-8) & 8 > b_1^1 \geq 2, b_2^1 < 2, \\ 0 & b_1^1 < 2, b_2^1 < 2. \end{cases} \tag{3.4}$$

where $P_1 = .25, P_2 = .5$ and $P_3 = .25$. The expected profits of the strategies in the six rows of (3.4) are $7, 9.5 - .5b_2^1, 9 - .5b_2^1, 6.5, 1.5$ and 0 respectively. Bidder 1's best response is to submit $b_1^1 \geq 8$ and $b_2^1 = 2$ (a strategy from the second row of (3.4)). In the final allocation, bidder one receives one unit and either bidder two or three receives the other. The allocation is not efficient.

Suppose the English clock mechanism is employed. If $\underline{x} = 0$, then $q^1(0) = 2, q^2(0) = q^3(0) = 1$. At price $2 + \epsilon$, where ϵ equals the minimum price increment, bidder i notices that $q^2(0) = q^3(0) = 1$. This allows bidder 1 to update his beliefs about 2 and 3's valuations. He conjectures that $v_1^2 = v_1^3 = 8$ and that 8 will be the final price. At this point his best response is to reduce his quantity demanded to one unit at a price greater than 8. Since $d_1^2 = d_1^3 = 8$, bidder one receives two units and the final allocation is efficient.

The English clock yields higher expected revenue ex-ante for the parameters in example 5. There are three possible configurations of market demand other than the one we discussed. They are:

1) $v_1^1 = 10, v_2^1 = 10, v_2^1 = 8, v_3^1 = 2,$

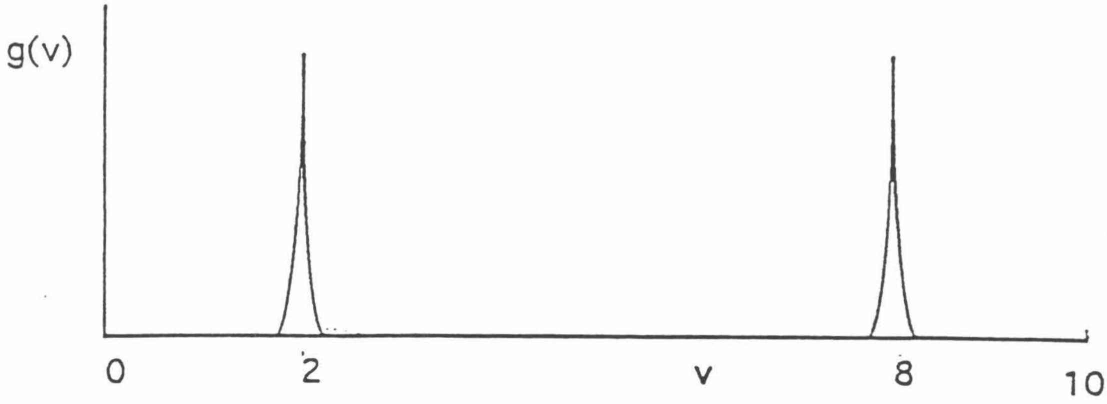

Figure 3.2: Probability Density Function for Example 5

2) $v_1^1 = 10, v_2^1 = 10, v_2^1 = 2, v_3^1 = 8,$

3) $v_1^1 = 10, v_2^1 = 10, v_2^1 = 2, v_3^1 = 2.$

Case one obtains the following outcome for both types of auctions: buyers one and two each receive a unit and pay a per-unit price of two. In case two the outcome is the same as in case one but with demanders two and three reversed. In case three under both types of auctions, bidder one receives both units and pays a per-unit price of two. Under all four possible configurations of demand, revenue from the English clock is greater than or equal to that from the sealed bid auction. In one realization, the revenue from the English clock is strictly greater. Therefore the expected revenue of the English clock is strictly greater than the uniform price sealed bid auction for the parameters of example 5.

For example 4, the final allocation from the English clock may not be efficient and the expected revenue generated is lower when the English clock is used than when the uniform price auction is employed. For example 5, the final allocation from the uniform price sealed bid auction may not be efficient and the expected revenue is higher under the English clock than under the uniform price sealed bid auction. □

3.1.1 Discussion

In this section we compared generalizations of two common auction forms and incorporated the notion of multi-unit demands. We saw that the theoretical strategic isomorphism and revenue equivalence results, which hold when demands are single-unit in the independent private values environment, do not necessarily hold when demands are multi-unit. The difference between the two mechanisms is that the English clock allows demanders to update their beliefs about other demanders' valuations while the price is increasing. The extra information may lead bidders to behave in ways that either raise or lower revenues and efficiency of final allocations when compared to the uniform price sealed bid auction.

These results certainly do not refute the revenue equivalence theorem. The revenue equivalence theorem considers only auctions in which the equilibrium final allocations are efficient. With multi-unit demands, since inefficient allocations can occur in equilibrium, the revenue equivalence theorem does not apply. However, we have shown that it does not necessarily extend to multi-unit demand generalizations of single-unit auctions to which it does apply.

3.2 Experimental Design: Motivation

This section describes the design of our laboratory experiment in which we investigate the revenue and efficiency properties of the two auctions in a variety of conditions. The experimental design allows us to contrast the data along three dimensions:

1) Sealed Bid vs. English Clock: We have seen in the previous subsection that when demanders are playing equilibrium strategies, the two auctions may generate different prices from each other. This suggests that there should also be differences empirically. The nature and extent of these differences is of obvious interest to a seller or buyers trying to choose an auction form to maximize or minimize revenue or to a planner choosing a mechanism to maximize efficiency.

2) One Shot vs. Repeated Games: In some markets, items are generally sold only once, while in other markets similar units of a good are auctioned repeatedly or periodically. A seller's or planner's preferred type of auction may differ in the two circumstances. It also may be possible for a seller or a planner to choose between a simultaneous and a sequential auction to sell multiple identical units. Since a one-shot game and a repeated game in which players do not know when the game ends have fundamentally different equilibrium properties, the outcome is likely to be different in the two conditions. For these reasons we contrast the outcome in one-shot versus repeated games.

3) Single-Unit vs. Two-Unit Demands: We have seen that when demands are multi-unit, demanders, even when behaving non-cooperatively, may have an incentive to underreveal their demands. In the experiments we compare outcomes from environments in which market demand is identical but in which individual demands are varied. In the single-unit demand condition, all demanders have positive valuations for one unit. In the two-unit demand condition, all demanders wish to buy two units. In a way, the two-unit demanders can be thought of as two one-unit demanders colluding to maximize the sum of their individual profits and so the experiments can give insight as to the impact of these "cartels" on the market as a whole.

3.3 Experimental Design: Environment and Parameters

3.3.1 Supply and Demand Conditions

There were 4 units inelastically supplied every period. The distribution of valuations $\gamma(v)$ was uniform from 0 to 1000.[1] In the single-unit demand experiments, 6 demanders each drew one valuation from $\gamma(v)$ independently every market period. In the two-unit demand condition, 3 demanders drew two valuations from $\gamma(v)$ every period.

3.3.2 Information Conditions

Demanders knew their own two valuations, the distribution of valuations, the supply available and the number of buyers. They did not know the actual valuations of the other buyers.

3.4 Procedures

3.4.1 Procedures Common to All Conditions

The one-shot games consisted of one trading period while the repeated games were divided into a series of trading periods. Each subject was given a redemption value sheet at the beginning of each period (see appendix B for samples of the materials distributed to subjects) listing the monetary value for which they could redeem from the experimenter any units of a commodity acquired during the trading period. To obtain the units, subjects purchased

[1] Valuations were randomly computer generated and then rounded upward to the nearest integer. Participants were told that they had an equal chance of drawing any integer value between 1 and 1000 and that the valuations had been generated by a computer before the experiments were run. Valuations were drawn independently each period but the same valuations were used for several experimental sessions, enabling the experimenter to control for demand draws when making comparisons across the various conditions.

Table 3.1: Number of Experiments Conducted for Each Treatment and the Number of Market Periods per Experiment

Treatment	Num. Experiments	Num. Periods per Expt.
SB/OS/2U	15	1
SB/R/2U	4	20
E/OS/2U	15	1
E/R/2U	4	20
SB/OS/1U	6	1
SB/R/1U	1	10
E/R/1U	1	10

them in an auction with cash obtained through an unlimited loan from the experimenter for the duration of the experimental session. The profits to the demanders were the sum of the differences of the redemption values for the units bought and the prices paid for them.

All of the experiments were conducted at the Laboratory of Experimental Economics and Political Science at the California Institute of Technology and all of the subjects were undergraduates at the California Institute of Technology. The length of the experiments for the repeated play conditions averaged 40 minutes for the uniform price auction and 90 minutes for the English clock. The sessions of one-shot games averaged 15 minutes for the sealed bid and 20 minutes for the English clock. Table 3.1 contains some information concerning the number of experiments conducted and the number of market periods in each experiment.

The following subsections outline some of the procedures which were specific to certain experimental conditions.

3.4.2 Specific Procedures

Uniform Price Auction

Repeated Play: Subjects could enter bids on their computer terminal. The program then indicated to them the per-unit price and how many of their bids were accepted. The 4 highest bids were accepted and the winners of the units paid a per-unit price equal to the 5th highest bid.

One Shot: Each experimental session, subjects filled out by hand their bids on a form distributed to them. They were told that they were to be grouped in a market with bidders from different experimental sessions. The experimenter played them against these opponents at a later date. This procedure was to insure that they could not learn anything about their opponents from the practice periods or by any other means.[2] The form can be found in appendix B.3.

English Clock

This[3] experiment was not computerized. While the experimenter transcribed the information on the blackboard, each subject indicated the number of units she demanded at the current price. At the beginning of each period, the current price was zero francs.[4] The

[2]Payment and verification were handled in the following way. Subjects were enrolled in an introductory economics course. Students in the course could participate in experiments and the money they earned would be given to them upon completion of the course. This procedure had been in effect for several years and so subjects knew they would actually be paid.

[3]The single-unit demand treatment was only run as a repeated game and not as a one-shot game. The one-shot games would have involved many additional experiments and in light of the very strong tendency for demanders to truthfully bid in the repeated games (even in the first period), and previous experimental research, our priors were very strong that we would observe the dominant strategy equilibrium outcomes in the one-shot games.

[4]"Francs" is a common name for the experimental currency in terms of which units of commodities are valued. Francs are convertible to U.S. dollars at the end of the experimental session at an exchange rate known to all participants at the beginning of the experiment.

experimenter then allowed the price to rise in small increments until a bidder wished to drop a unit from the bidding. The price at which each demander reduced her quantity demanded was posted on the board for the remainder of the market period. The clock price continued to rise until there were exactly 4 units demanded.[5] The remaining bidders received the units and paid a per-unit price equal to the price at which the clock stopped. The procedure was the same whether the game was played once or repeatedly.

3.5 Predictions

3.5.1 Point Predictions

For the different conditions of this study the current state of knowledge of game theory can give us predictions of varying degrees of precision and strength. These are reviewed below for the treatments to which they apply.

Single-Unit Demand

As we saw in chapter one, in the single-unit demand environment, there always exists a symmetric dominant strategy equilibrium with truthful bidding for the uniform price sealed bid auction. As we have also seen in chapter one, the dominant strategy equilibrium of the uniform price auction has been widely observed in the previous experimental literature.

For the English clock game there is also a dominant strategy equilibrium in the single-unit demand environment. In the equilibrium, each demander stays in the bidding until the price reaches her valuation and then immediately exits from the bidding. This equilibrium

[5] If more than one demander drops out at the same increment, leaving less than k units demanded, then the experimenter backtracks, i.e., lowers the price until one of the demanders gets back in.

Table 3.2: Predicted Average Ex-Post Per-Unit Revenue for the Sealed Bid Auction and the English Clock Under Single-Unit Demands (in francs)

Auction	Revenue	
	One-Shot	Repeated
Sealed Bid	287	281
English	-	281

has been observed consistently in previous experimental studies.

The predicted average ex-post market revenue is given in table 3.2. The revenues differ between the one-shot and the repeated games because the actual realizations of market demand differ (although the distribution from which demand is drawn is identical) in the two conditions. The demand draws in the one-shot games are a subset of those in the repeated games. The demand draws for all conditions of our study are displayed in appendix C. The predicted revenue is identical across the two auctions, however, because the realizations of demand are the same across the two auctions and the revenue occuring in the *two respective* dominant strategy equilibria of the two auctions is identical. Since the units are allocated to the demanders with the highest valuations, the allocative efficiency is 100 percent in all market periods, where we define the percentage efficiency as the percentage of the maximum possible gains from trade realized during a market period.

Two-Unit Demand

In chapter two we characterized symmetric monotone undominated Bayes-Nash equilibria to the uniform price sealed bid auction in the two-unit demand environment. A strictly monotone symmetric equilibrium for the experimental parameters is derived in appendix

Table 3.3: Predicted Average Ex-Post Per-Unit Revenue and Efficiency for the Sealed Bid Auction Under Two-Unit Demands

Auction	Revenue	
	One-Shot	Repeated
Revenue (in francs)	84	81
Efficiency (in percent)	93	93

C. If all demanders were to play the symmetric equilibrium strategy the average ex-post revenue and market efficiency would be as given in table 3.3.

3.5.2 Predictions of Differences Across Treatments

Single-Unit Demands vs. Two Unit Demands - We predict that both revenue and allocative efficiency will be lower in the two-unit demand environment. The prediction is based on the fact that the unilateral incentive for strategic underrevelation exists only in the two-unit demand environment. The underrevelation is likely to lead to revenue and efficiency less than in the single-unit demand environment. This can be seen for the sealed bid auction by consulting tables 3.2 and 3.3.

Sealed Bid vs. English Clock - We predict revenue equivalence and equal allocative efficiency under single-unit demands across the two auctions. However, under two-unit demands there seems to be a strong possibility that both metrics would differ. Since we do not have an equilibrium model of the English clock in the two-unit demand environment we can not predict higher revenue or allocative efficiency for either mechanism.

One-Shot vs. Repeated - In our repeated games, subjects were not aware which period was to be the last. Since the periods were short, at most four minutes, the probability that any given period was the last must have been perceived to be very small. Similarly, due to the shortness of the market periods, subjects' discount rates must have been close to 1. Thus, our repeated game condition may be thought of as an infinitely repeated as well as a finitely repeated game. By the Folk theorem, the equilibrium outcomes of the one-shot games are a subset of the possible equilibrium outcomes in any period of an infinitely repeated game. Also, a set of strategies consisting of playing the Nash equilibrium strategy in every period of a finitely repeated game constitute a Nash equilibrium of the finitely repeated game.

In previous single-unit demand experimental studies of both types of auctions the dominant strategy equilibrium to the one-shot game has been observed in repeated games. In the two-unit demand case, however, it would not surprise us to see different outcomes. In particular, it is possible that cooperative behavior among demanders, resulting in prices lower than in the non-cooperative symmetric equilibrium outcome of the one-shot game, can be sustained in the repeated game. We would therefore predict no differences in revenue and ex-post efficiency in the single-unit demand condition but likely differences in the two-unit demand condition, with lower per period revenue in the repeated games.

3.6 Results

In this section, we report the results of our study. When making some of the inferences below, we implicitly assume, by using each period as a data point, that the outcomes in each period are independent. This is a plausible assumption only in the one-shot games. Nevertheless, we report the same statistics for the repeated games as the one-shot games, because the statistics are useful in making comparisons across treatments.

Results 1 - 6 are discussed in section 3.6.3. In result 1, we consider the accuracy of the point predictions of the game theoretic equilibria discussed in subsection 3.5.1. The interesting contrasts across treatments are reported in results 2 - 6. In section 3.7, some observations concerning individual behavior are presented.

3.6.1 Equilibria

Our principal observations concerning the validity of the theoretical equilibrium point predictions for the conditions of our experiment are summarized in result 1.

Result 1 *In the single-unit demand condition, prices are not different than those predicted in section 3.5.1. However, in the two-unit demand uniform price sealed bid auction experiments, prices differ from those occuring in the symmetric monotone Bayes-Nash equilibrium.*

Support: We reject, using a sign test[6] the hypothesis that the observed prices in the two-unit demand uniform price auction are less than or equal to those occuring in the strictly

[6]The sign test is a procedure for testing hypotheses about the median, call it μ, of a continuous distribution. If X denotes the random variable whose distribution is under investigation, then $P(X \leq \mu) = P(X \geq \mu) = .5$. The general null hypothesis has the form $H_0 : \mu = \mu_0$.

When $\mu = 0$, any X_i is equally likely to be positive or negative. If, however, the true value of μ is much greater than 0, we would expect most of the observed X_i's to be positive. Define the test statistic Y = the number of X_i's such that $X_i > 0$. For testing H_0, versus $H_a : \mu > 0$, the sign test rejects H_0 when the test-statistic $Y \geq c$. If we regard each X_i as a trial, and the data consist of a set of n identical trials, and if we define a positive X_i to be a success and a non-positive X_i as a failure, then we have p = P(success) = $P(X_i > 0) = P(X_i > \mu) = .5$. Then, when H_0 is true, the statistic Y has a binomial distribution with parameters n and p (p = .5). Therefore, if the null hypothesis is $H_0 : \mu = 0$ and the alternative hypothesis is $H_a : \mu \neq 0$, then we reject H_0 if either $Y \geq c$ or $Y \leq (n - c)$. When p = .5 and $n \geq 10$ the binomial distribution can be approximated by a normal distribution. For our data, in order to compare observed revenue to predicted revenue, we let each X_i equal the observed revenue minus the predicted revenue in a market period.

To compare relative revenue generated by the two auctions, we let X_i equal the observed revenue in the uniform price sealed bid auction minus that observed in the English clock. Since we used the same valuations for the two mechanisms, the difference should be equal to zero if the mechanisms generate the same revenue given the same valuations. Other comparisons across treatments were performed similarly.

Table 3.4: Observed Average Ex-Post Per-Unit Revenue: Single-Unit Demands

Auction	Revenue	
	One-Shot	Repeated
Sealed Bid	302 (6 Periods)	285 (10 Periods)
English	-	271 (10 Periods)

Table 3.5: Observed Average Ex-Post Per-Unit Revenue: Two-Unit Demands

Auction	Revenue	
	One-Shot	Repeated
Sealed Bid	310 (15 Periods)	236 (80 Periods)
English	223 (15 Periods)	164 (80 Periods)

monotone Bayes-Nash equilibrium in the two-unit demand one-shot games ($p < .05$) as well as in the repeated games ($p < .01$). We fail to reject the hypotheses that observed prices are equal to those prevailing in the dominant strategy equilibrium for the English clock and for the sealed bid auction in the single-unit demand condition (for the pooled data from the one-shot and the repeated games) at the 10 percent level.

3.6.2 Revenue

In tables 3.4 and 3.5, we summarize the average per-unit franc revenue achieved across treatments. Principal observations concerning revenue are summarized in results 2 - 4.

Result 2 *The uniform price sealed bid auction generated more revenue than the English clock in the two-unit demand condition. The two auctions generate the same revenue in the single-unit demand condition.*

Support: A sign test rejects the hypothesis that the revenue generated by the English clock is greater than or equal to that generated by the sealed bid auction. The test yields z = 1.803 ($p < .05$) in the one shot, z = 3.16 ($p < .001$) in the repeated version. The revenues generated by the two auctions in the single-unit demand environment are not different from each other ($p > .1$) for the repeated games.

Result 3 *Less revenue was generated per period when the games were repeated than when they were played once.*

Support: We reject the hypothesis that revenue from the repeated games is greater than or equal to the one-shot games at the ($p < .1$) level for both auctions in the two-unit demand condition.

Result 4 *More revenue was generated by the English clock under single-unit demand than under two-unit demand.*

Support: We reject the hypothesis that revenue under single-unit demands is less than or equal to that under two-unit demands at the ($p < .01$) level for the repeated game condition of the English clock. We cannot reject the same hypothesis for the sealed bid auction at the 10 percent level.

3.6.3 Efficiency

We can also consider levels of efficiency observed in the experiment. When considering the levels of efficiency attained it is difficult to describe them as high or low because there is no universally agreed upon threshhold, above which allocative efficiency is said to be high and below which efficiency is considered to be low. We can, however, compare the observed efficiencies with those which would result if units were distributed randomly among the three demanders in the two-unit demand condition, with each demander receiving at most two of the four units sold during each market period.

Result 5 *Both types of market organization generate efficiencies higher than random allocations in the two-unit demand condition.*

Support: We reject the hypothesis that the efficiency of the observed allocations are less than or equal to the expected efficiency resulting if the 4 units were distributed randomly among the demanders in such a way that no demander receives more than two (in the two-unit demand condition). The level of significance with which we reject the hypotheses are: $p < .0001$ for the one-shot uniform price auction, $p < .005$ for the repeated uniform price auction, $p < .0005$ for the one-shot English clock and $p < .05$ for the repeated play version of the English clock.

In result 6 we consider differences in allocative efficiencies across treatments.

Result 6 *The levels of efficiency attained by the two auction forms are not different from each other. There is no difference in the level of efficiency between the one-shot and the repeated games. The efficiencies in the single-unit demand experiments are not different from those in the two-unit demand experiments.*

Support: The majority of the observed efficiencies are 100 percent in all of the seven treatments. A sign test of the differences in efficiencies achieved fails to reject the hypothesis that the levels of efficiency of the two types of auction in the two-unit demand conditions are equal. $z = 0.024$,(p-value $> .1$) for the one-shot game, and $z = .47$,(p-value $> .1$)

in the repeated games. A sign test of the difference in the efficiency fails to reject the hypothesis that the efficiency is the same in the one-shot vs. the repeated games for the sealed bid auction and for the English clock in the two-unit demand condition ($p > .1$ for both auctions). We also fail to reject the hypothesis that the efficiencies are the same in the one-unit vs. the two-unit demand conditions ($p > .1$) for all treatments.

Discussion

In line with previous experimental research into auctions, we fail to reject the point predictions of the dominant strategy equilibrium for either type of auction in the single-unit demand environment. However, prices are higher than in the symmetric monotone equilibrium of the uniform price sealed bid auction in the two-unit demand environment. This is consistent with the previous literature in that symmetric Bayes-Nash (under risk-neutrality) equilibria tend to perform poorly in explaining experimental data. It is possible that the high bids are a reflection of risk aversion on the part of demanders or that some demanders are having difficulty in determining their best response and therefore bid close to their valuations, behaving like price-takers. It may also be the case that the behavior is consistent with an asymmetric equilibrium. We shall examine individual behavior in detail in the next subsection.

In the two-unit demand treatments, the uniform price auction generated significantly more revenue than the English clock, suggesting that a seller with several identical units to sell simultaneously might prefer to use the uniform price auction to the English clock if he has reason to believe that the distribution of demanders' valuations is uniform. It seems likely that the difference in revenue is attributable to the iterative property of the

English clock mechanism in which many times more messages are sent between buyers. Since demanders observe market quantity demanded as the clock price increases, they can more easily update their beliefs about other buyers. In this way, the English clock facilitates strategic behavior. In the uniform price sealed bid auction, strategic behavior was probably more difficult for subjects. We base this assertion on the fact that the rents obtained by the seller in the uniform price sealed bid auction exceeded those which would occur in the (non-cooperative) symmetric monotone Bayes-Nash equilibrium. It may be possible that the difference in revenue is due to the level of excess demand as suggested by Smith (1967) or due to the elasticity of demand as suggested by Miller and Plott (1985). It is beyond the scope of this project to compare these theories.

Another clear result which emerges from the data is that the repeated play version generates lower revenue per period. This is indicative of some type of learning or experience effect in which buyers in later periods improve their extraction of surplus from the seller. Game theory suggests that more cooperative behavior (buyers submitting lower bids) can be a non-cooperative equilibrium in a repeated game when the final period is unknown. Indeed, more cooperative behavior on the part of players seems to be observed than in the one-shot game. Our empirical evidence suggests that a seller may be better off if he can lead buyers to believe that they are playing a one-shot game. Two open questions are 1) whether, in a multi-unit demand setting, he should bundle all of the units he has and sell them all at one time and 2) whether he should tell demanders exactly how many periods will be played.

We found that the English clock generated less revenue in the two-unit demand condition than in the single-unit demand condition, but that the uniform price auction did not. At first glance this seems somewhat surprising. A complete explanation is not possible since we have no reason to suppose that the revenues of the two auctions should be equal in the two-unit demand environment and there is no satisfactory equilibrium model of the English clock in multi-unit demand environments. It seems likely that the difference between the one-unit and two-unit revenues in the English clock was due to strategic behavior. We believe that the difference was not observed for the sealed bid auction because strategic behavior was more difficult.

Both of the mechanisms generate allocations which are more efficient than algorithms which allocate the units randomly even in the two-unit demand conditions. Both mechanisms realize most of the gains from trade despite the possibility of inefficiencies resulting from strategic behavior.

The equal efficiencies generated by the two auctions under single-unit demands was expected considering the results of previous work and the very strong theoretical predictions. The equality of efficiency under two-unit demands is somewhat surprising, especially in light of the different revenues in the one-unit and the two-unit demand conditions and across the two auctions. It seems that even though subjects behaved strategically and manipulated market prices in their favor, they still preserved for the most part the efficient allocations which would result in a competitive equilibrium. The ordinal ranking of bids remained the same as the ordinal ranking of valuations. The evidence suggests that the multi-unit demands do not have a strong tendency to lower efficiency levels, even in a very thin market. However, we concede that replication is needed to confirm our results, because we used only one set of parameters and there is considerable previous evidence (Miller and Plott (1985), and Smith (1967), for example) which suggests that relative revenue generating properties of different auctions depend on the particular parameters of the market.

In the next subsection we further analyze the data at the individual level.

3.6.4 Individual Behavior

Uniform Price Auction

We can assign all bidding strategies in the two-unit demand condition to one of four exhaustive and mutually exclusive classes. The first two classes are very restrictive[7] while the last two are very large.

Strategy 1: Bidders use the strictly monotone Bayes-Nash equilibrium. Recall that the equilibrium strategy involves each bidder submitting a bid equal to his valuation on his more highly-valued unit, and underbidding on his lower-valued unit.

Strategy 2: Bidders behave as price takers. They bid their valuations for two units. If all bidders follow this strategy, the competitive outcome results.

Strategy 3: Undominated: These are all strategies other than strategies 1 and 2 that are not dominated by another strategy. This group includes all strategies which involve both a higher bid equal to the higher valuation and a lower bid less than or equal to the lower valuation. This is the also the set of rationalizable strategies.

Strategy 4: Dominated: This group encompasses all other strategies. Any strategy that involves underbidding on the higher-valued unit is dominated by a strategy in which the demander bids his valuation on the higher unit. Any strategy which includes a bid greater than a demander's valuation for either unit is dominated by a strategy in which

[7]We will say that a bidding strategy is in one of the first two classes if the bid is less than 1 cent away from the exact bidding function specified by the strategy.

Table 3.6: Demanders' Strategies in the Two-Unit Demand Uniform Price Auction

	Strat. 1	Strat. 2	Strat. 3	Strat 4
One Shot	3.6	29.6	12.3	54.5
Repeated Periods 1-10	6.6	20.0	12.3	61.1
Repeated Periods > 10	7.2	30.0	19.5	43.3

demander instead bids his valuation. In table 3.6 we report the percentage of the time that the four types bidding strategies were observed.

Very few subjects used the symmetric monotone Bayes-Nash equilibrium strategy. This result is not unanticipated as the equilibrium bidding function is difficult to calculate and is only a best response if it is being used by all of the other bidders. Over one quarter of the bidders submitted bids equal to their two valuations. It is a strategy which is easy to calculate and is also a natural focal point. Roughly 15 percent of the time, bidders used other strategies which would be undominated in a one-shot game.

More than half of the time, bidders followed strategies which are dominated in one-shot games. However, bidders were less likely to follow dominated strategies as they played the game longer in the repeated games, suggesting that they were "learning" which strategies were dominated. The fact that high efficiencies were nonetheless achieved in the auction suggests that use of dominated strategies did not affect the ordinal ranking of the bids in the market. The persistence of dominated strategies in conjunction with the high observed efficiencies hints that the players using dominated strategies did not lose much by using the strategies, i.e., they were not often punished for their actions.

Let us consider more precisely the nature of the strategies which demanders were using in the experiment. In tables 3.7 and 3.8 we give the number of times each of nine possible

Table 3.7: Demanders' Strategies in the Uniform Price Auction (One-Shot Games, Number of Observations)

	$b_2 < v_2$	$b_2 = v_2$	$b_2 > v_2$
$b_1 < v_1$	13	2	1
$b_1 = v_1$	7	13	1
$b_1 > v_1$	1	0	7

Table 3.8: Demanders' Strategies in the Uniform Price Auction (Repeated Games, Number of Observations)

	$b_2 < v_2$	$b_2 = v_2$	$b_2 > v_2$
$b_1 < v_1$	51	4	10
$b_1 = v_1$	53	60	9
$b_1 > v_1$	2	8	43

subfamilies of strategies were used. n = 45 and n = 240 for tables 3.7 and 3.8 respectively. The most striking feature in the two tables is the concentration of the data along the diagonal from upper left to the lower right. By far the two most frequently observed types of dominated strategies were underbidding on both units, and overbidding on both units.

The relatively high number of players overbidding on both units indicates that final prices were not high enough to cause the overbidders to lose money. In fact, in none of the one-shot games did any player actually receive negative profits during a market period. The high incidence of the strategies $(b_1 > v_1, b_2 > v_2)$ and $(b_1 < v_1, b_2 < v_2)$ hints at a type of asymmetric "equilibrium" behavior in that the players who are underbidding have little incentive to increase their bids if they are playing against overbidders, because the increase in bid does not increase appreciably the probability of receiving units. Similarily, if other players are underbidding, the overbidders receive units at lower prices than those

Table 3.9: Demanders' Strategies in the English Clock with Two-Unit Demands

	At Value	Less than Value	Greater than Value
One Shot	42.3	53.8	3.9
Repeated Periods 1-10	27.3	70.3	2.4
Repeated Periods > 10	17.1	82.9	0

Table 3.10: Demanders' Strategies in the English Clock with Single-Unit Demands

	At Value	Less than Value	Greater than Value
Repeated Periods 1-5	93.3	6.7	0
Repeated Periods > 5	96.7	3.3	0

that would prevail if the underbidders followed undominated strategies. The overbidders, then, have little incentive to change their strategies.

English Clock

When analyzing the data from the English Clock, it is not possible to identify the strategies which demanders are following because the dropout points which are higher than the final price are not observable. However, we can examine the dropout points observed[8] and check whether demanders tended to drop out at prices less than, equal to, or greater than their valuations. In table 3.9, we list the percentage of the time each strategy was followed on a demander's lower-valued unit in the two-unit demand condition. For comparison, the strategies followed in the single-unit demand condition are given in table 3.10.

Note the sharp contrast between the two conditions. There is a strong tendency for

[8]The table includes only the observations where it could be ascertained which of the three strategies were being used.

demanders to bid truthfully in the single-unit demand condition, even in the early periods. In the two-unit demand condition however, well over half of the subjects underrevealed on their lower-valued unit. The tendency toward underrevelation became stronger the more periods that subjects had played, suggesting that subjects were learning to drop their lower-valued units in order to lower the price they paid on their higher-valued units. This underrevelation was not reflected in low efficiencies but was reflected in the low revenues generated by the clock.

Chapter 4

Summary and Concluding Remarks

In chapter two we generalized some important theoretical properties of the uniform price sealed bid auction to an independent private values environment with two-unit demands. We considered a class of bidding functions called type M, essentially a general type of monotonicity. A necessary condition for a bidding function to be a type M symmetric undominated Bayes-Nash equilibrium was derived. The dominant strategy equilibrium of the single-unit demand environment clearly obeys the necessary conditions. In any equilibrium, there is underbidding for each demander's lower valued unit, as demanders, while still behaving non-cooperatively, underreveal demand in an attempt to shift the market price in their favor. An interesting property of type M equilibria, separability, is also obtained.

A sufficient condition for a solution to the necessary conditions to be an equilibrium is also deduced and an example of an equilibrium is provided. As the number of bidders gets large, the underrevelation in equilibrium converges to zero. It is perhaps not surprising that bidders exhibit an increasing tendency to behave like price takers as the market becomes thicker and each bidder's ability to affect the market price declines.

In chapter three we compared generalizations of two common auction forms and incorporated the notion of multi-unit demands. We saw that the theoretical strategic and revenue equivalence results, which hold when demands are single-unit in the independent private values environment, do not necessarily hold when they are multi-unit. We then examined outcomes from a set of laboratory experiments. In the experiments, where there was a uniform distribution of valuations, we observed that the uniform price auction generated more revenue than the English clock in the two-unit demand condition, suggesting that a revenue maximizing seller may want to use the uniform price auction rather than the English clock in environments similar to the two-unit demand experimental environment. We also found that less revenue was generated when the auctions were played repeatedly than when they were played only once. We conclude, citing both theoretical and experimental evidence, that in multi-unit demand environments the two mechanisms are fundamentally different from each other. This result contrasts sharply with much of the previous theoretical and experimental literature concerning single-unit demand independent private values environments.

Appendix A

Additional Lemmas

Lemma 10 *If all n players besides bidder i use a bidding function that is type M and undominated, then $f_\nu(x) > 0$ for $0 \leq x \leq B_2(\overline{v}, \overline{v})$.*

Proof: Recall that F_ν equals Prob(At least ℓ bidders make 2 bids that are less than or equal to x, at least m + ℓ bidders make at least 1 bid that is less than or equal to x)

Consider the probability that a randomly chosen bidder makes 2 bids that are less than or equal to x which is given by:

$$\int_0^x \int_0^{\overline{v}} g(v_1, v_2) dv_2 dv_1. \tag{A.1}$$

The term in the last equation follows from the fact that if B is undominated, that $b_1^y = v_1^y \geq v_2^y \geq B_2(v_1^y, v_2^y)$ (bidding an amount higher than one's valuation is dominated). The term is clearly strictly increasing in x if B is type M and undominated for x such that $0 \leq x \leq \overline{v}$. Now consider the probability that a randomly chosen bidder makes at least one bid that is less than or equal to x, which equals:

$$1 - \int_{B_2^{-1}(x|v_1=x)}^{\overline{v}} \int_{B_2^{-1}(x|v_1=x)}^{\overline{v}} g(v_1, v_2) dv_2 dv_1. \tag{A.2}$$

The last term results from the fact that underbidding on the higher valued unit is dominated by bidding an amount equal to the higher valuation and from the fact that $v_1^y \geq v_2^y$. This last equation is also strictly increasing in x for all x such that $0 \leq x \leq \overline{v}$. It follows that $F_\nu(x)$ is strictly increasing in x and that $f_\nu(x) > 0; 0 \leq x \leq B_2(\overline{v}, \overline{v})$. □

Lemma 11 *$E\pi|\beta$ is twice differentiable.*

Proof: By assumption $\gamma \in C^2$.

It follows that $g(v_1, v_2) \in C^1$ since $g(v_1, v_2) = 2\gamma(v_2) d\gamma(v_2)(\gamma(v_1) - \gamma(v_2)) d\gamma(v_1)$.

Since $g(v_1, v_2) \in C^1$, $\int_0^x \int_0^y g(v_1, v_2) dv_2 dv_1$ is twice differentiable with respect to x and y (for a reference see Marsden pp. 285-286) and thus continuous in x and y (Marsden pp. 161-162).

It follows that $H(x, y, G, n, m, \ell)$, the product of twice differentiable functions, is twice differentiable with respect to x and y (see Marsden pg. 171) and thus continuous in x and y.

Now consider $F_\nu(x) = \sum_{q=\nu}^{2n} \sum_{\ell,m;2\ell+m=q} H(x, \beta^{-1}(x), G, n, m, \ell)$. $F_\nu(x)$ is twice differentiable in x since the sum of differentiable functions is differentiable, (Marsden pg. 199). This implies that $f_\nu(x)$ is differentiable and continuous in x.

The expected profit is given in equation (2.43). $E\pi^i$ is differentiable in b_1^i and b_2^i because an anti-derivative of a continuous function is differentiable (Marsden pg. 286).

The first derivatives equal:

$$\frac{\partial E\pi^i}{\partial b_1^i} = (v_1^i - b_1^i) f_{2n-k+1}(b_1^i) \tag{A.3}$$

and

$$\frac{\partial E\pi^i}{\partial b_2^i} = (v_2^i - b_2^i) f_{2n-k+2}(b_2^i) - F_{2n-k+1}(b_2^i) + F_{2n-k+2}(b_2^i). \tag{A.4}$$

Clearly, both of the expressions are differentiable in b_1^i and b_2^i. We have now shown that $E\pi^i(b_1^i, b_2^i)$ is twice differentiable. □

Appendix B

Experimental Instructions and Forms

In this appendix we include sample instructions for both the English clock and the uniform price sealed bid auction experiments, a sample redemption value sheet, and a sample bid form for the one-shot uniform price auction. In section B.1 are the instructions common to both the the English clock and the uniform price sealed bid auction experiments. The instructions for the two experiments were identical except for the part entitled "The Auction Process," the text of which is given in section B.2 for the English clock, and in section B.3 for the uniform price sealed bid auction. Figures B.1 and B.2, which are in section B.4, contain the sample redemption value sheet and the sample bid form respectively.

B.1 Instructions Common to All Experiments

This is an experiment in the economics of market decision-making. The instructions are simple, and if you follow them carefully and make good decisions, you might earn a considerable amount of money which will be paid to you in cash.

The experiment will be broken up into a series of trading periods in which you will make decisions. At the beginning of each period, you will be given a Record Sheet. The Record Sheet describes the value to you of a fictitious commodity which you can purchase in the market. Your Record Sheet is your own private information. You are not to reveal its contents to anyone.

The currency used in this market is francs. All trading will be in terms of francs. Your final payoff will be in terms of dollars. The conversion rate is ..... francs to one dollar. You will be paid at the end of the experiment.

Each buyer has .... value(s) on his Redemption Value Sheet. Each value is equally likely to be any integer between 1 and 1000. If you have more than one value, your values are ordered on the Redemption Value Sheet from highest to lowest.

You can obtain units of the commodity by participating in the market process which is described below. There are ... units to be sold in each market period.

The Auction Process

Earnings

Please refer to your record sheet to determine your earnings. Your earnings for the period are the redemption values of the units you receive minus the total of the prices you paid for them. For the first unit that you buy in the trading period marked at the top of the page, you will receive the amount listed in row (1) marked 1st Unit Redemption Value. If you buy a second unit during the same trading period, you will receive the additional amount listed in line (4) marked Second Unit Redemption Value, etc... The profits from each purchase, which are yours to keep, are computed by taking the difference between the redemption value and purchase price of the unit bought. That is,

YOUR EARNINGS = REDEMPTION VALUE - PURCHASE PRICE

Suppose for example, that you buy two units and that your redemption value for the first unit is 200 and for the second unit is 180. If you pay a per-unit price of 150, your earnings are:

Earnings From First = 200 - 150 = 50

Earnings From Second = 180 - 150 = 30

Total Earnings = 50 + 30 = 80

The blanks on the table will help you record your profits. The purchase price of the first unit you buy should be recorded on row (2). You should then record the profit on the purchase as directed on row (3). At the end of the period, record the total profit on the last row of the page. Subsequent periods should be recorded similarly.

There will be a practice period at the beginning of the experiment. If you have any question concerning the experiment, please raise your hand.

B.2 Specific Instructions for the English Clock

The Auction Process

Each period, you will be grouped with .... other participants. There will be .... of these groups. Each of the groups will be in a separate market, that is, they will be bidding on a different set of units. You will be in the same group for several periods. The experimenter will inform you if there is a change in the members of the group. There will be .... units available to each group each period.

You can obtain these units by participating in an ascending price auction. The auction proceeds as follows. At the beginning of a period, a "low" price will be posted. You will then be asked to submit a request specifying how many units you would like at that price. If the total number of units requested is more than the number of units available, the price is increased.

Suppose, for example, that a per-unit price of 50 is posted on the board and there are 4 units available to be sold to 3 buyers. Suppose that each of the three buyers requests 2 units. Notice that at the price of 50 there is an overdemand for units. That is, at this price total requests are more than the 4 units available. Since the total amount requested is greater the number of units available, the price will be increased and new requests must be submitted.

The price is now increased to 60 francs and new requests are submitted. The only restriction placed on a participant's new request after a price increase is that the number of units requested must be less than or equal to his number of units requested at the previous price.

The price will continue to increase as long as total orders are greater than the number of units available. The price will stop increasing if total orders equal the number of units available. If a price increase results in total requests of less than the number of units available, then the price will be reduced until a price where total requests equal that number is found. This final price will be the per-unit price charged.

B.3 Specific Instructions for the Uniform Price Sealed Bid Auction

The Auction Process

Each period, you will be grouped with .... other participants. There will be .... of these groups. Each of the groups will be in a separate market, that is, they will be bidding on a different set of units. You will be in the same group for several periods. *The experimenter* will inform you if there is a change in the members of the group. There will be .... units available to each group each period.

During each period you may submit bids for units of the commodity by filling out a bid form which will appear on your computer screen at the beginning of the period. The bid form has slots for ten bids but you will not be allowed to fill in more than .... bid(s). To fill out the bid form, just put the amount of your bid in the column labeled bid per unit and enter a 1 in the column labeled number of units.

Once you have filled out your form, you must submit it to the market. Once all participants have submitted their bids, all of the bids will be ranked from highest to lowest. The highest .... bids in each group will be accepted and receive the .... units awarded to the group. The per unit price for all of the accepted bids is equal to the highest rejected bid, that is, the ...th highest bid in the group. If there is a tie for the lowest accepted bid, the unit(s) is randomly assigned to the tied buyers. Notice that if you have a bid accepted, the price you pay for that unit will never be more than what you have bid.

B.4 Forms

On the next page is a sample redemption value sheet. On the following page is the form used in the one period uniform price auction experiments.

Figure B.1: Redemption Value Sheet

Record of Purchases and Earnings, Buyer No.____

Unit Pur-chased		Trading Period Number
1	1	1st unit redemption value
	2	Purchase price
	3	Profit (row 1 - row 2)
2	4	2nd unit redemption value
	5	Purchase price
	6	Profit (row 4 - row 5)
3	7	3rd unit redemption value
	8	Purchase price
	9	Profit (row 7 - row 8)
4	10	4th unit redemption value
	11	Purchase price
	12	Profit (row 10 - row 11)
5	13	5th unit redemption value
	14	Purchase price
	15	Profit (row 13 - row 14)
6	16	6th unit redemption value
	17	Purchase price
	18	Profit (row 16 - row 17)
7	19	7th unit redemption value
	20	Purchase price
	21	Profit (row 19 - row 20)
8	22	8th unit redemption value
	23	Purchase price
	24	Profit (row 22 - row 23)
9	25	9th unit redemption value
	26	Purchase price
	27	Profit (row 25 - row 26)
10	28	10th unit redemption value
	29	Purchase price
	30	Profit (row 28 - row 29)
	31	Total per period

Figure B.2: Bid Form For One Period Uniform Price Sealed Bid Auction

Unit	REDEMPTION VALUES
1	
2	

Unit	BID
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

Appendix C

Experimental Parameters

In this appendix we solve for a symmetric monotone Bayes-Nash equilibrium for our experimental parameters. The appendix also includes two tables, in which are shown the exact valuations which demanders had in the experiments, as well as some details concerning our procedures.

C.1 Bayes-Nash Equilibrium for the Experimental Parameters

For our experiment n + 1 = 3, k = 4, and $\gamma(v)$ is uniform. From theorem 1, we know that any equilibrium requires $\beta_1(v_1) = v_1$ and $\beta_2(\overline{v}) = \overline{v}$. We also know that $\beta_2(v_2)$ must satisfy equation (2.6) We can calculate $F_{2n-k+1}(b_2) - F_{2n-k+2}(b_2)$, which for the example is $F_1(b_2) - F_2(b_2)$, the probability that two randomly drawn bidders make a total of exactly one bid less than or equal to b_2. It is equal to:

$$2(1 - V_2(b_2))^2(2V_2(b_2) - V_2(b_2)^2 - (b_2)^2) \tag{C.1}$$

Also,

$$F_{2n-k+2}(b_2) = 1 - (1 - V_2(b_2))^4 - 2(1 - V_2(b_2))^2(2V_2(b_2) - V_2(b_2)^2 - (b_2)^2) \tag{C.2}$$

and therefore

$$f_{2n-k+2}(b_2) = 4V_2'(1 - V_2(b_2))^3 - 2(1 - V_2(b_2))^2(2V_2' - 2V_2(b_2)V_2' - 2b_2)$$

$$+ 4V_2'(1 - V_2(b_2))(2V_2(b_2) - V_2(b_2)^2 - (b_2)^2) \tag{C.3}$$

Therefore:

$$b_2 = V_2(b_2) - \frac{2(1 - V_2(b_2))^2(2V_2(b_2) - V_2(b_2)^2 - b_2)}{4b_2(1 - V_2(b_2))^2 - 4(1 - V_2(b_2))V_2'(2V_2(b_2) - V_2(b_2)^2 - (b_2)^2)} \tag{C.4}$$

Solving for V_2', we obtain:

$$V_2' = \frac{1 - V_2(b_2)}{2(V_2(b_2) - b_2)} - \frac{b_2(1 - V_2(b_2))}{2V_2(b_2) - (V_2(b_2))^2 - b_2^2} \tag{C.5}$$

There are many solutions that obey the initial conditions of theorem 1, i.e. that $\beta_2(1) = 1$. Only one solution is strictly monotone in the (0,1) interval and it is depicted in figure C.1.

The second order condition is the following:

$$(V_2 - b_2)(-4(1-V_2)^2 - 8b_2V_2'(1-V_2) + 4V_2'(2V_2' - 2V_2V_2' - 2b_2)$$

$$+4V_2''(2V_2 - V_2^2 - b_2^2)) + (V_2' - 1)(-4b_2(1-V_2)^2 + 4V_2'(2V_2 - V_2^2 - (b_2)^2))$$

$$-2(1-V_2)^2(2V_2' - 2V_2V_2' - 2b_2) + 4(1-V_2)V_2'(2V_2 - V_2^2 - (b_2)^2) < 0 \tag{C.6}$$

We have verified that the last equation is satisfied by the solution depicted in the graph.

Table C.1: Demanders' Valuations in the Two-Unit Demand Condition

Demand Draw	Demander					
	1		2		3	
	v_1^1	v_2^1	v_1^2	v_2^2	v_1^3	v_2^3
1	886	866	696	118	936	222
2	818	377	997	108	299	128
3	486	300	598	121	210	15
4	723	204	738	697	825	694
5	694	557	496	333	987	726
6	571	424	882	583	587	448
7	793	777	804	51	803	508
8	422	188	985	127	635	440
9	877	343	369	364	722	75
10	977	37	788	652	145	118
11	813	741	439	431	953	665
12	897	792	69	49	877	569
13	676	502	472	288	842	99
14	695	358	870	463	132	11
15	978	586	927	864	786	157
16	395	6	914	591	253	180
17	102	13	850	155	772	473
18	915	105	632	195	838	719
19	727	628	830	650	241	110
20	667	241	331	24	812	183

C.2 Valuations

Tables C.1 and C.2 indicate the value to subjects, in terms of "francs", the experimental currency, of obtaining the commodity trading in our experimental market.

In each of the eight two-unit demand repeated play experiments, all twenty demand draws were used. In four of the experiments, i.e. two English clock and two sealed bid auction experiments, the demand draws 1,...,20 were used in periods 1,..20 respectively. In the other four two-unit demand repeated play experiments, draws 11-20 were used in

Figure C.1: Symmetric Equilibrium Lower Bid for the Experimental Parameters

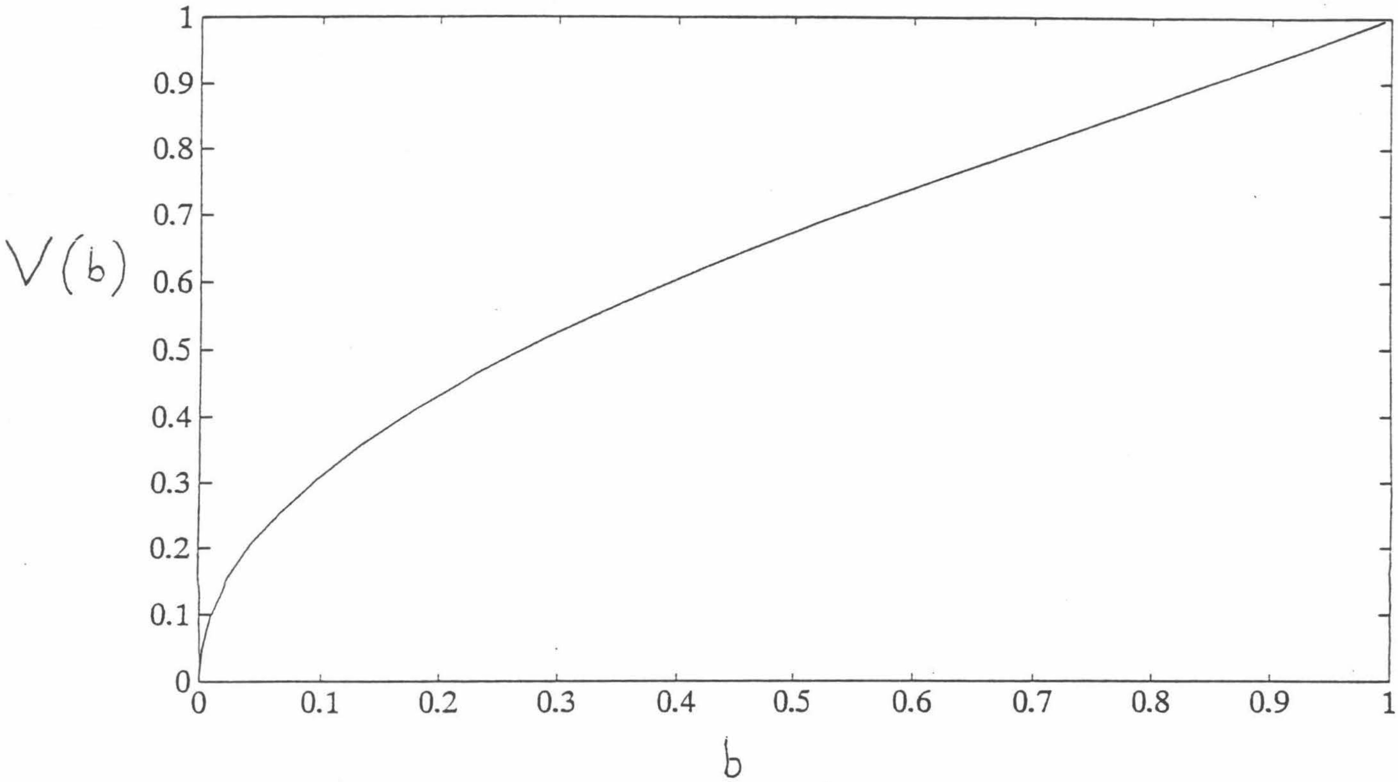

Table C.2: Demanders' Valuations in the Single-Unit Demand Condition

Demand Draw	Demander					
	1	2	3	4	5	6
	v^1	v^2	v^3	v^4	v^5	v^6
1	866	886	118	696	936	222
2	377	818	997	108	299	128
3	486	300	121	598	15	210
4	204	723	697	738	825	694
5	557	694	333	496	987	726
6	571	424	583	882	448	587
7	777	793	804	51	803	508
8	422	188	985	127	440	635
9	343	877	364	369	75	722
10	977	37	652	788	118	145

market periods 1 - 10, and draws 1 - 10 were used in market periods 11-20. There were 30 one-shot games with two-unit demands, 15 using each type of auction. Each of demand draws 1 - 15 was used in one game for each type of auction.

In each of the two repeated play single-unit demand experiments demand draws 1,...,10 were used in markets periods 1,...,10 respectively. In the six one-shot single-unit demand experiments demand draws 1 - 6 were used.

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