NEUTRINO PROCESSES OF SIGNIFICANCE IN STARS

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ABSTRACT

The transition probabilities and rates of dissipation of energy by the neutrino-antineutrino pairs are calculated for the processes: $e^- + e^+ \rightarrow \nu + \bar{\nu}$ and $e^\pm + \gamma \rightarrow e^\pm + \nu + \bar{\nu}$, in the limit of a nondegenerate electron gas. These quantities are also calculated for the process $\gamma + \gamma \rightarrow \nu + \bar{\nu}$ using a nonlocal weak interaction; the resulting matrix elements are reduced by a symbol manipulation computer program. The energy dissipation rates for these processes, the urca process, and the processes $e^\pm + (z,A) \rightarrow e^\pm + (z,A) + \nu + \bar{\nu}$ and $\gamma + (z,A) \rightarrow (z,A) + \nu + \bar{\nu}$ are compared. The process $e^+ + e^- \rightarrow \nu + \bar{\nu}$ or $e^\pm + \gamma \rightarrow e^\pm + \nu + \bar{\nu}$ is found to dominate in the range of temperatures and densities considered: $10^8\ ^\circ K \leq T \leq 5 \times 10^9\ ^\circ K$, $1\ gm/cm^3 \leq \rho \leq 10^6\ gm/cm^3$.

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I. BACKGROUND AND INTRODUCTION

The existence of the neutrino was first proposed by Pauli in 1933 to account for an experimentally-observed lack of conservation of energy in reactions involving radioactive β-decay. This particle was to react so slightly with matter that, once produced, it could escape from the apparatus unnoticed, carrying with it the energy deficit.

Gamow and Schoenberg (Gamow 1941) proposed that this neutrino, produced in β-decay or in the inverse process of electron capture,

$$(Z, A) \rightarrow (Z+1, A) + e^- + \bar{\nu} \qquad (I.1)$$
$$(Z+1, A) + e^- \rightarrow (Z, A) + \nu$$

could play an important role in the energy transport mechanisms of stars. Once produced, the neutrino, as in the laboratory experiments, would escape from the system, in this case a star, without further interaction and take energy with it. A photon, although much easier to produce, would interact within such a short distance that only those produced near the surface of the star would have much chance of escaping. Gamow and Schoenberg's calculations were based on the Fermi theory of β-decay and on the experimentally-measured parameters of β-decay: maximum electron energy and halflife.

As an example, for the reaction $O^{16} + e^- \rightarrow N^{16} + \nu$, the threshold for electron capture is 6 Mev. This corresponds to the average thermal energy at a temperature of 5×10^{10} °K or 2400 times

the temperature at the center of the sun. Although this is a rather high temperature for stars, this reaction will serve to give an inkling of the possibilities involved. Gamow and Schoenberg considered only the release of gravitational potential energy during the collapse of a star from the solar radius to 1/40th of that radius at a temperature around the threshold. If 1% of the stellar material takes part in the reaction, collapse could occur in about the free-fall time of 1/2 hour. This corresponds to an energy release rate of 1×10^{16} ergs/gm sec for the O^{16} material.

Normally, the energy liberated in nuclear reactions is greater than that dissipated by neutrinos. After the nuclear reactions have run their course, leaving heavy nuclei with little binding energy, the neutrino reactions can take precedence. Assuming a model of the stellar structure, they are able to conclude that in the collapse the neutrino luminosity exceeds the photon luminosity, and hence neutrino processes are of importance in the collapse and in the subsequent nova and supernova explosions. In a less violent way, neutrino processes should play a role in stellar evolution. Since this neutrino process provides a way to take energy right out of the heart of a star, we should investigate any similar processes which might enhance this effect.

In the Fermi theory of β-decay (Fermi 1934), the interaction matrix element is considered to be constant with the density of final states governing the emission spectrum, aside from corrections for the Coulomb interaction between the electron and the nucleus.

But, the more recent theory of weak interactions of Feynman and Gell-Mann (Feynman 1958), Sudarshan and Marshak (Sudarshan 1958), gives a quantitative description of the interaction matrix element. This description not only describes β-decay but predicts other interactions having the same coupling constant. Among these is the vertex: $e^- + e^+ \rightarrow \nu + \bar{\nu}$. From this basic interaction come the processes:

$$e^{\pm} + (Z, A) \longrightarrow e^{\pm} + \nu + \bar{\nu} \qquad \text{(I.2)}$$

$$e^- + e^+ \longrightarrow \nu + \bar{\nu} \qquad \text{(I.3)}$$

$$e^{\pm} + \gamma \longrightarrow e^{\pm} + \nu + \bar{\nu} \qquad \text{(I.4)}$$

$$\gamma + \gamma \longrightarrow \nu + \bar{\nu} \qquad \text{(I.5)}$$

$$\gamma + \gamma \longrightarrow n\gamma + \nu + \bar{\nu} \qquad n = 1, 2, 3, \cdots \qquad \text{(I.6)}$$

$$\gamma + (Z, A) \longrightarrow \nu + \bar{\nu} \qquad \text{(I.7)}$$

Pontecorvo (1959) first suggested the astrophysical importance of (I.2), which is analogous to bremsstrahlung except that in the electron-photon vertex the photon is replaced by a neutrino-antineutrino pair. With the same substitution, (I.3) is analogous to pair annihilation (Chiu 1960), (I.4) is analogous to Compton scattering (Chiu 1961), and (I.5) and (I.6) are analogous to the scattering of light by light through a virtual electron (Chiu 1960). Replacing one of the free photons of (I.5) by the Coulomb field of a nucleus, we obtain (I.7) (Matinyan 1961). In stars we have ions, electrons, photons, and positrons produced in electron-positron pairs by high energy photons. Thus processes (I.2) - (I.7)

can occur in stars and may be of importance.

We will compute the transition probabilities for (I.3), (I.4) and (I.5) and calculate the energy dissipation rates for these processes in the limit that the electron gas is nondegenerate. Reactions (I.2), (I.3), (I.4) and (I.7) have been dealt with in the literature. The results that we obtain and those obtained by others are tabulated in Appendix D.

II. THE INTERACTIONS

Preparatory to calculating the transition probabilities, let us briefly describe the interactions which are involved. A free fermion is described by a four component spinor, ψ, satisfying the first order Dirac equation, $(\not{p} - m)\psi = 0$, or less commonly by a two component spinor, φ, satisfying a second order differential equation, $(p^2 - m^2)\varphi = 0$. The ψ and φ are related by

$$\Psi = (\not{p} + m)\begin{pmatrix} \varphi \\ -\varphi \end{pmatrix}, \quad \varphi = \tfrac{1}{2}(1 - i\gamma_5)\Psi = a\Psi .$$

In Feynman and Gell-Mann's theory of the universal Fermi interaction (Feynman 1958), the fermions interact as two component spinors without gradient couplings. This corresponds, in the Dirac picture, to interaction through a four-component spinor having only two independent components. With ψ as a Dirac spinor, we then use $\psi' = a\psi$ in the interaction. In the representation

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_a = \begin{pmatrix} 0 & -\sigma_a \\ \sigma_a & 0 \end{pmatrix}, \quad \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \tag{II.1}$$

where σ_a are the Pauli spin matrices, we have

$$a = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}; \quad \Psi = \begin{pmatrix} B \\ A \end{pmatrix}; \quad \Psi' = a\Psi = \begin{pmatrix} 0 \\ A \end{pmatrix}$$

so that $(a\psi)$ clearly has only two components. A neutrino then satisfies the massless Dirac equation and has its spin aligned antiparallel to its momentum:

$$\not{p}(a\Psi) = 0, \quad (E + \vec{\sigma}\cdot\vec{p})A = 0 .$$

Experiment decides between parallel and antiparallel alignment.

If we introduce a weak current J_α,

$$J_\alpha = (\bar{e}\,\nu_1) + (\bar{\mu}\,\nu_2) + (\bar{n}\,p)$$
$$= (\bar{\Psi}'_e\,\gamma_\alpha\,\Psi'_{\nu 1}) + (\bar{\Psi}'_\mu\,\gamma_\alpha\,\Psi'_{\nu 2}) + (\bar{\Psi}'_n\,\gamma_\alpha\,\Psi'_p)\,,$$

then for an interaction Lagrangian of

$$\mathcal{L} = \sqrt{8}\,G\,J^*_\mu\,J_\mu \qquad \text{(II.2)}$$

the cross terms describe neutron β-decay ($n \rightarrow p + e^- + \bar{\nu}$), muon β-decay ($\mu \rightarrow e + \nu + \bar{\nu}$), and muon-capture by a proton ($\mu^- + p \rightarrow n + \nu$). These reactions are observed to be well-described by this theory (Klein 1948). The reaction $e^+ + e^- \rightarrow \nu + \bar{\nu}$ is predicted by the cross term $(\bar{e}\nu)(\bar{\nu}e)$ but has not been observed. To account for observed strangeness changing weak interactions, additional terms must be added to the current J_μ. We will have no need for such terms as we deal only with leptonic processes. We will assume that there is no neutral leptonic weak current. It is unlikely that there are such currents, but if there are, it is possible that the neutral currents could cause the cancellation of certain terms in the charged current interaction, e.g. the sum $(\bar{e}\nu)(\bar{\nu}e) + (\bar{e}e)(\bar{\nu}\nu)$ cancels exactly.

From the rate for β-decay (Reiter 1960) the coupling constant has been found to be $GM_p^2 = 1.024 \pm .002 \times 10^{-5}$ where the proton mass, M_p, is introduced to produce a dimensionless quantity.

Alternatively, we might couple this weak current to a vector meson, uxl, which would then mediate weak interactions (Feynman 1958).

It should be remembered that the vector field theory is not renormalizable in the normal sense. However, since no divergences occur in our applications, we will ignore this fact and hope that this does not make these calculations meaningless. Although there has been much discussion of the possible couplings of the uxl with the baryon weak currents, for the leptonic weak current we will simply take a charged uxl and the coupling:

$$\mathcal{L} = \sqrt{4\pi}\, f\, U_\mu^* J_\mu + c.c. \qquad \text{(II. 3)}$$

where U_μ is the uxl field operator. If M is the uxl mass, the two coupling constants are related by $4\pi f^2 = 8GM^2$. One might introduce a neutral uxl, W^o, coupled to a neutral weak current, but this is unnecessary to account for the observed leptonic interactions and introduces the difficulty mentioned above. For the reaction $\gamma+\gamma \rightarrow \nu+\bar{\nu}$, we will need to use an uxl and in this case a neutral current would not affect the calculation. For interactions described by (II. 2), the descriptions of the interaction by (II. 2) and by (II. 3) are the same in the limit that the uxl momentum is much less than the uxl mass. To prevent the K meson from decaying into an uxl, the uxl mass must be at least that of the kaon ($M_K \sim M_p/2$). Recent experiments (Danby 1962) are nicely interpreted with an uxl mass of $M = 0.8\ M_p$. We will use this value when the need arises, but it must be borne in mind that the existence of the uxl is as yet unsubstantiated.

Regarding this interaction, there are two sets of simple mathematical relations which are of interest. First, from the definitions of a and $\bar{a}$, it can be easily verified that the following relations hold:

$$a := \tfrac{1}{2}(1 - i\gamma_5)\,, \quad \bar{a} := \tfrac{1}{2}(1 + i\gamma_5) \Rightarrow a\bar{a} = \bar{a}a = 0,$$
$$aa = a\,, \quad \bar{a}\bar{a} = \bar{a}\,, \quad a\gamma_\mu = -\gamma_\mu \bar{a}\,, \quad \bar{a}\gamma_\mu = -\gamma_\mu a\,. \tag{II.4}$$

Next, let us consider the process A + B → C + D, which is described in the local theory by the matrix: $(\bar{u}_C \gamma_\mu a u_A)(\bar{u}_D \gamma_\mu a u_B)$. This matrix form is subject to exchanges of various particles with corresponding changes in the form of the coupling. Using the representation (II.1) and writing out components, it is seen that

$$\mathcal{M} = \sqrt{8}\, G (\bar{u}_c \gamma_\mu a u_A)(\bar{u}_D \gamma_\mu a u_B) \tag{II.5a}$$

$$= -\sqrt{8}\, G (\bar{u}_c \gamma_\mu a u_B)(\bar{u}_D \gamma_\mu a u_A) \tag{II.5b}$$

$$= 2\sqrt{8}\, G (\bar{u}_{\bar{B}}\, a\, u_A)(\bar{u}_D\, a\, u_{\bar{C}}) \tag{II.5c}$$

$$= 2\sqrt{8}\, G (C_1 D_2 - C_2 D_1)^* (A_1 B_2 - A_2 B_1)$$

where $a u_A{}^T = (0\ 0\ A_1 A_2)$, etc. Exchanging incoming or outgoing particles (i.e. A → B, or C → D) changes the sign of the coupling. Exchanging an ingoing with an outgoing particle while taking the antiparticle in each case (e.g. B → $\overline{C}$, C → $\overline{B}$) changes the coupling constant by a factor of 2 and changes the form of the coupling from (pseudo) vector to (pseudo) scalar. These are the Fierz transformations and they are used when one of the transformed terms is easier to handle than the original.

In addition to the weak interaction we will have use for the electromagnetic interactions of the electron and uxl. Let us introduce the Lagrangian densities for the free electron:

$$\mathcal{L} = \tfrac{i}{2}[\bar{\Psi}\gamma_\mu(\partial_\mu \Psi) - (\partial_\mu \bar{\Psi})\gamma_\mu \Psi] - m\bar{\Psi}\Psi\,,$$

and for the free uxl:

$$\mathcal{L} = -\tfrac{1}{2}\left[\partial_\beta U_\alpha - \partial_\alpha U_\beta\right]^*\left[\partial_\beta U_\alpha - \partial_\alpha U_\beta\right] + M^2 U_\alpha^* U_\alpha \ .$$

We will use the minimal gauge-invariant electromagnetic coupling obtained by the substitution: $\partial_\nu \rightarrow \partial_\nu + iqA_\nu$. If we consider that ψ annihilates electrons or creates positrons and U_μ annihilates positive uxls, W^+, or creates W^-, we use

$$\partial_\nu U_\mu \rightarrow (\partial_\nu + ie A_\nu) U_\mu = \pi_\nu U_\mu$$
$$\partial_\nu \psi \rightarrow (\partial_\nu - ie A_\nu)\psi$$

where A_μ is the photon vector potential. We also introduce an anomalous magnetic moment, γ, for the uxl through the term: $ie\gamma F_{\mu\nu} U_\mu^* U_\nu$, where $F_{\mu\nu} \equiv (\partial_\mu A_\nu - \partial_\nu A_\mu)$ is the electromagnetic field tensor. Then the uxl Lagrangian becomes

$$\mathcal{L} = -\tfrac{1}{2}\left[\pi_\beta U_\alpha - \pi_\alpha U_\beta\right]^*\left[\pi_\beta U_\alpha - \pi_\alpha U_\beta\right] + M^2 U_\alpha^* U_\alpha + ie\gamma F_{\mu\nu} U_\mu^* U_\nu \ .$$

To show the effect of the magnetic moment term, we form the Euler-Lagrange equation of motion for U_μ:

$$\pi_\beta \pi_\beta U_\mu + M^2 U_\mu - \frac{ie(1-\gamma)}{M^2}\pi_\mu \pi_\rho (F_{\rho\nu} U_\nu) + ie(1+\gamma) F_{\mu\nu} U_\nu + ie\,\pi_\mu\left(\frac{\partial^2 A_\rho}{M^2}\right) U_\rho = 0.$$

The last term represents a self-interaction of the uxl with the electromagnetic field source which, in this case, is just the uxl current. If we neglect that term and neglect terms in e^2 as being of second order, write $\partial_\mu \partial_\mu = \partial_t \partial_t - \vec{\nabla} \cdot \vec{\nabla}$, take the root of the $(\partial t)^2$ operator, and go to the nonrelativistic limit with the substitution $U_\mu = \exp(-iMt)\varphi_\mu$, we find that the spatial components of φ_μ satisfy the Schroedinger

type equation

$$i\,\partial_t \vec{\varphi} = -\frac{\nabla^2 \vec{\varphi}}{2M} - \frac{ie}{M}(\vec{\nabla}\cdot\vec{A})\vec{\varphi} - \frac{ie(1+\gamma)}{2M}(\vec{B}\times\vec{\varphi}) + \frac{e(1-\gamma)}{2M^2}\vec{\nabla}(\vec{E}\cdot\vec{\varphi}) - \frac{ie(1+\gamma)}{2M}\varphi_0\vec{E}\,.$$

Thus γ contributes to both the magnetic, $(\vec{B}\times\vec{\varphi})$, and electric, $(\vec{E}\cdot\vec{\varphi})$, interactions.

Subtracting the free Lagrangians from the total Lagrangians leaves the interaction terms:

$$\mathcal{L}_{e\text{-}\gamma} = e A_\mu \bar{\Psi}\gamma_\mu\Psi$$

$$\mathcal{L}_{\pi\text{-}\gamma} = -ie A_\beta(\partial_\beta U_\alpha^*)U_\alpha + ie A_\alpha(\partial_\beta U_\alpha^*)U_\beta + ie A_\beta U_\alpha^*(\partial_\beta U_\alpha) - ie A_\alpha U_\beta^*(\partial_\beta U_\alpha) - e^2 A_\beta A_\beta U_\alpha^* U_\alpha + e^2 A_\alpha A_\beta U_\beta^* U_\alpha\,.$$

For convergent processes the S matrix is a function of the Lagrangian density,

$$S = T\exp\left[i\int\mathcal{L}(x)\,dx\right]$$

where T indicates the time ordered product. We will not go through a full derivation of Feynman's rules(*), but we can obtain the interaction expressions for use in perturbation field theoretic calculations

(*) See for example Bogoliubov and Shirkov (1959).

by taking i times the momentum space representation of the interaction Lagrangian. We use the phase convention of exp(-ikx) for an incoming particle and exp(ikx) for an outgoing particle where k is the particle momentum. In addition we will replace the charge 'e' by '$\sqrt{4\pi}e$' so that in subsequent work $e^2 \simeq 1/137$ in units where $\hbar = c = 1$. The particle propagators can be derived from the field operator commutation rules (see Appendix A). Doing this, the Feynman rules are then as given in Table I.

Using fermion spinors, u, normalized to $\bar{u}u = 2E$, the projection operator, for a particle of mass m and momentum k, is: $\sum_{\text{spins}} \bar{u}u = \not{k} + m$. With this normalization the transition probability, σv, is given by

$$\sigma v = 2\pi \left| \frac{|m|^2}{\prod_{in} 2E \prod_{out} 2E} \right. D \qquad \text{(II.6a)}$$

where D, the density of final states, is

$$D = \frac{1}{2\pi} \prod_{out} \left[2E_i \, 2\pi \delta(p_i^2 - m_i^2) \frac{d^4 p_i}{(2\pi)^4} \right] (2\pi)^4 \delta^4\left(\sum_{in} p - \sum_{out} p\right) \qquad \text{(II.6b)}$$

Here, "in" and "out" indicate all incoming or outgoing particles. With these rules we can calculate the rates for the reactions of interest.

III. THE TRANSITION PROBABILITIES

A. The Process $e^+ + e^- \rightarrow \nu + \bar{\nu}$.

The process $e^+ + e^- \rightarrow \nu + \bar{\nu}$ is described, to lowest order, by the Feynman diagram of Figure 1. Using the rules of Table I, we find for the matrix element:

$$\mathcal{M} = i\sqrt{8}\, G\, (\bar{u}_{\nu 2}\, \gamma_\mu\, a\, u_{e2})(\bar{u}_{e1}\, \gamma_\mu\, a\, u_{\nu 1}) \quad .$$

If we use the second form of the Fierz transformation (II.5b), we obtain

$$\mathcal{M} = -i\sqrt{8}\, G\, (\bar{u}_{\nu 2}\, \gamma_\mu\, a\, u_{\nu 1})(\bar{u}_{e1}\, \gamma_\mu\, a\, u_{e2}) \quad .$$

Taking the absolute square, introducing the traces of the matrices, and using the particle projection operators after summing over outgoing particle spins and averaging over incoming particle spins, we find

$$|\mathcal{M}|^2 = 2G^2\, Sp[k\!\!\!/_2\, \gamma_\mu\, a\, k\!\!\!/_1\, \gamma_\beta\, a]\, Sp[(p\!\!\!/_1 - m)\, \gamma_\mu\, a\, (p\!\!\!/_2 + m)\, \gamma_\beta\, a] \, .$$

We can simplify this expression by using the properties of a so that

$$|\mathcal{M}|^2 = 2G^2\, Sp[k\!\!\!/_2\, \gamma_\mu\, k\!\!\!/_1\, \gamma_\beta\, a]\, Sp[p\!\!\!/_1\, \gamma_\mu\, p\!\!\!/_2\, \gamma_\beta\, a] \, . \qquad \text{(III.1)}$$

Evaluating the density of final states from (II.6) for two outgoing particles, the transition probability is

$$\sigma v = \frac{1}{16\pi^2 E_1 E_2} \int |\mathcal{M}|^2\, \delta(k_1^2)\, \delta(k_2^2)\, \delta^4(k_1 + k_2 - p_1 - p_2)\, d^4k_1\, d^4k_2 \, . \qquad \text{(III.2)}$$

The integral which occurs here has been done by Lenard (1953), who found that

$$I_{\lambda\beta} = \iint d^4k_1\, d^4k_2\, \delta(k_1^2)\,\delta(k_2^2)\,\delta^4(k_1+k_2-P)\, k_{1\lambda}\, k_{2\beta}$$

$$I_{\lambda\beta} = \frac{\pi}{24}\left(2P_\lambda P_\beta + P^2 g_{\lambda\beta}\right). \qquad \text{(III.3)}$$

With this integral and the matrix relations of Appendix B, σv can be reduced to the form:

$$\sigma v = \frac{G^2}{6\pi E_1 E_2}\left(p_1\cdot p_2 + m^2\right)\left(2p_1\cdot p_2 + m^2\right) \qquad \text{(III.4)}$$

where E_1 and E_2 are the energies of the positron and electron and p_1 and p_2 are their 4-momenta. To evaluate this expression in the center of mass system, we let $p_1 = (E,0,0,p)$, $p_2 = (E,0,0,-p)$, $E_T = 2E$. Then we have

$$\sigma v = \frac{G^2}{3\pi}\left(E_T^2 - m^2\right), \qquad \hbar = c = 1.$$

In conventional units with E_T in units of the electron mass

$$\sigma = \left(\frac{c}{v}\right)\frac{(\hbar c)^2}{3\pi}\,\frac{(G M_p^2)^2}{M_p^4}\left(E_T^2 - 1\right) \qquad \text{(III.5)}$$

$$\sigma = 1.4\times 10^{-45}\left(\frac{c}{v}\right)\left(E_T^2 - 1\right) \quad \text{cm}^2,$$

which is the same as the expression obtained by Chiu and Stabler (Chiu 1961) at a later date.

If, instead of the point weak interaction, we use a charged uxl mediated interaction, we have the Feynman diagram of Figure 2. Conservation of momentum gives: $p = p_1 - k_1 = k_2 - p_2$. Feynman's rules then give us the matrix element:

$$M = -4\pi f^2 (\bar{u}_{\nu 2} \gamma_\lambda a u_{e2}) \frac{(g_{\lambda\beta} - p_\lambda p_\beta / M^2)}{(p^2 - M^2)} (\bar{u}_{e1} \gamma_\beta a u_{\nu 1}).$$

The $p_\lambda p_\beta$ term of the uxl propagator prevents the use of a Fierz transformation, and the appearance of k_1 or k_2 in the uxl momentum precludes the use of Lenard's integral. The final states integrals are, however, easily effected in the center of mass system, and the matrix algebra can again be done with the relations of Appendix B. We obtain

$$\sigma v = \frac{G^2}{12\pi} \left[\frac{2(a\beta^2 - \alpha\beta b + \alpha^2 c)}{\beta^2(\alpha^2 - \beta^2)} + \frac{2c}{\beta^2} + \ln\left|\frac{\alpha+\beta}{\alpha-\beta}\right| \frac{(\beta b - 2\alpha c)}{\beta^3} \right]$$

where $a = (4\omega^2 + 4m^4/M^4 + m^4\omega^2/M^4)$

$$b = (2\omega p)(4 - m^4/M^4)$$

$$c = p^2 (4 + m^4/M^4)$$

$$\alpha = M^2 - m^2 + 2\omega^2 \quad , \quad \beta = -2\omega p \ .$$

Here, ω and p are the energy and momentum, respectively, of either one of the particles in the center of mass system. In the limit that the uxl mass is large compared to particle momenta, ($E_T \ll M$, $m \ll M$), this reduces to

$$\sigma v = \frac{G^2}{3\pi} \left[(E_T^2 - m^2) + \frac{1}{M^2} \left(-\frac{E_T^4}{2} + E_T^2 m^2 + m^4 \right) + \cdots \right].$$

The second term becomes significant for $E_T \sim M$, which for $M = 0.8\ M_p$ is 700 Mev. This energy is much greater than the thermal energy in stars so that here the uxl effect is negligible.

B. The Process $e^{\pm} + \gamma \rightarrow e^{\pm} + \nu + \bar{\nu}$.

Next we consider the bremsstrahlung reaction: $e^{\pm} + \gamma \rightarrow e^{\pm} + \nu + \bar{\nu}$. There are two Feynman diagrams in Figure 3 which describe this reaction to lowest order for a local weak interaction. Using the rules of Table I and a Fierz transformation (II.5b), the matrix elements for these two diagrams are

$$\mathcal{M}_A = -i\sqrt{32\pi}\, G e (\bar{u}_{\nu 2} \gamma_\mu a u_{\nu 1})(\bar{u}_{e2} \gamma_\mu a \frac{1}{\not p_1 + \not q - m} \not\epsilon\, u_{e1})$$

$$\mathcal{M}_B = -i\sqrt{32\pi}\, G e (\bar{u}_{\nu 2} \gamma_\mu a u_{\nu 1})(\bar{u}_{e2} \not\epsilon \frac{1}{\not p_2 - \not q - m} \gamma_\mu a u_{e1}) .$$

In this case the transition probability, using (II.6), is

$$\sigma_\gamma = \frac{1}{128\pi^5 E(p_1) E(q)} \int d^4p_2\, \delta(p_2^2 - m^2) \int d^4k_1\, d^4k_2 \times \delta(k_1^2)\delta(k_2^2)\, \delta^4(p_2 + k_1 + k_2 - q - p_1) |\mathcal{M}_A + \mathcal{M}_B|^2 \qquad \text{(III.6)}$$

where $E(p_1)$ is the energy of the initial electron and $E(q)$ is the energy of the initial photon.

Gauge invariance requires that under the transformation $e_\mu \rightarrow e_\mu + \alpha q_\mu$, where e_μ and q_μ are the photon polarization and momentum respectively, the matrix element be unchanged. If we consider the dependence of the matrix element on the polarization alone, this requires that $\mathcal{M}(\not\epsilon) = \mathcal{M}(\not\epsilon + \alpha \not q)$, and since $\mathcal{M}$ is linear in $\not\epsilon$, $\mathcal{M}(\not q) = 0$. If we direct the z axis along the photon momentum, then $\mathcal{M}(\gamma_0) - \mathcal{M}(\gamma_3) = 0$. Because of this we can average over the photon

polarizations in $|m|^2$ by substituting $(-\frac{1}{2}\gamma_\mu \ldots \gamma_\mu)$ for $(\not{\epsilon} \ldots \not{\epsilon})$. Then the time and longitudinal components cancel:

$$|m|^2 = \tfrac{1}{2}|m(\gamma_3)|^2 + \tfrac{1}{2}|m(\gamma_2)|^2 + \tfrac{1}{2}|m(\gamma_1)|^2 - \tfrac{1}{2}|m(\gamma_0)|^2$$

$$= \tfrac{1}{2}|m(\epsilon_1)|^2 + \tfrac{1}{2}|m(\epsilon_2)|^2$$

where $e_1 = (0,1,0,0)$ and $e_2 = (0,0,1,0)$.

Squaring the matrix element and averaging over the incoming polarizations and summing over the outgoing polarizations,

$$\Sigma |m_A + m_B|^2 = 8\pi G^2 e^2 \, Sp[\not{k}_2 \gamma_\rho \not{k}_1 \gamma_\beta a](A + B + C + D)$$

$$4\alpha^2 A = Sp[(\not{p}_1 + m)\gamma_\eta(\not{p}_1 + \not{q} + m)\gamma_\rho a(\not{p}_2 + m)\bar{a}\gamma_\beta(\not{p}_1 + \not{q} + m)\gamma_\eta]$$

$$-4\alpha\delta B = Sp[(\not{p}_1 + m)\gamma_\eta(\not{p}_1 + \not{q} + m)\gamma_\rho a(\not{p}_2 + m)\gamma_\eta(\not{p}_2 - \not{q} + m)\gamma_\beta a]$$

$$-4\alpha\delta C = Sp[(\not{p}_1 + m)\gamma_\rho a(\not{p}_2 - \not{q} + m)\gamma_\eta(\not{p}_2 + m)\gamma_\beta a(\not{p}_1 + \not{q} + m)\gamma_\eta]$$

$$4\delta^2 D = Sp[(\not{p}_1 + m)\gamma_\rho a(\not{p}_2 - \not{q} + m)\gamma_\eta(\not{p}_2 + m)\gamma_\eta(\not{p}_2 - \not{q} + m)\gamma_\beta a]$$

where the system invariants are

$$\mu = m^2, \quad \alpha = (p_1 \cdot q), \quad \lambda = (p_1 \cdot p_2), \quad \delta = (p_2 \cdot q).$$

The integral over the neutrino momenta in (III.6) can be performed using Lenard's formula (III.3), and with the application of the gamma matrix relations of Appendix B, the transition probability can be reduced to

$$\sigma v = \frac{G^2 e^2}{12\pi^3 E(p_1) E(q)} \int d^4 p_2 \, \delta(p_2^2 - m^2) \, A \, ,$$

$$A = \Big\{ -\mu^3\Big(\frac{1}{\alpha^2}+\frac{1}{\delta^2}\Big)+\mu^2\Big[\lambda\Big(\frac{3}{\alpha^2}+\frac{2}{\alpha\delta}+\frac{3}{\delta^2}\Big)+\Big(\frac{3\delta}{\alpha^2}-\frac{3\alpha}{\delta^2}\Big)+\Big(\frac{3}{\delta}-\frac{3}{\alpha}\Big)\Big]$$

$$-2\mu\lambda^2\Big(\frac{1}{\alpha^2}+\frac{3}{\alpha\delta}+\frac{1}{\delta^2}\Big)+\mu\lambda\Big[\Big(-\frac{4\delta}{\alpha^2}+\frac{4\alpha}{\delta^2}\Big)+\Big(\frac{2}{\delta}-\frac{2}{\alpha}\Big)\Big]$$

$$+\mu\Big[-2+\Big(\frac{\delta}{\alpha}+\frac{\alpha}{\delta}\Big)-\Big(\frac{2\delta^2}{\alpha^2}+\frac{2\alpha^2}{\delta^2}\Big)\Big]+\frac{4\lambda^3}{\alpha\delta}+8\lambda^2\Big(\frac{1}{\alpha}-\frac{1}{\delta}\Big)$$

$$+\lambda\Big[-8+6\Big(\frac{\delta}{\alpha}+\frac{\alpha}{\delta}\Big)\Big]+2(\alpha-\delta)+2\Big(\frac{\delta^2}{\alpha}-\frac{\alpha^2}{\delta}\Big)\Big\} \quad . \tag{III.7}$$

To perform the final integrations we go to the center of mass system with the z axis along the momentum of the incoming electron. The substitution $z = (e_2 + p_2)/m$, where e_2 and p_2 are the final electron energy and spatial momentum, reduces the integrands to $\ln(z)$ and polynomials in z. If we let ξ be the total center of mass energy of the system in units of the electron mass, we finally obtain for the transition probability:

$$\sigma_V = \frac{e^2G^2m^2\xi^2}{36\pi^2(\xi^2-1)(\xi^4-1)}\Big\{\frac{\ln\xi}{(\xi^2-1)}(4\xi^8-4\xi^6-54\xi^4-56\xi^2+50)$$

$$+\frac{1}{12}(-55\xi^6+159\xi^4+510\xi^2-122-147\xi^{-2}+15\xi^{-4})\Big\} \quad . \tag{III.8}$$

In the extreme relativistic limit, $\xi \gg 1$, this reduces to

$$\sigma_V(E.R.) = \frac{e^2G^2m^2\xi^2}{9\pi^2}\Big(\ln\xi-\frac{55}{48}\Big) \quad .$$

Letting ω be the incoming photon energy, we set $\xi = \omega + \sqrt{1+\omega^2}$. In the nonrelativistic limit $|\omega| \ll 1$, $\xi \simeq 1$. We use the expansion

$$\frac{\ln \xi}{(\xi^2-1)} = \frac{1}{2} - \frac{(\xi^2-1)}{4} + \frac{(\xi^2-1)^2}{6} - \frac{(\xi^2-1)^3}{8} + \cdots$$

in (III. 8) and obtain

$$\sigma v\,(N.R.) = \frac{4}{35}\frac{e^2 G^2 \omega^4}{\pi^2 m^2} \qquad \hbar = c = 1$$

or in conventional units

$$\sigma = \frac{4}{35\pi^2}\left(\frac{e^2}{\hbar c}\right)(G M_p^2)^2\left(\frac{\omega}{M_p}\right)^4\left(\frac{\hbar c}{m}\right)\left(\frac{c}{v}\right) \tag{III. 9}$$

$$\sigma = 7.25 \times 10^{-50}\, E_T^4 \left(\frac{c}{v}\right) \quad cm^2$$

where E_T is the total kinetic energy in the center of mass system in units of the electron mass. The relative velocity, v, is c in the N. R. limit but goes to 2c in the extreme relativistic limit. In Figure 4, σv for this process is plotted as a function of the photon energy, and we see that it is smooth through the intermediate region between the N. R. and E. R. limits.

When the energy dissipation rate of this process is computed in the N. R. limit, we shall need to know the average rate, ϵ, at which energy is carried off by the neutrinos in the CM system. When there are only the two neutrinos in the final state, this average is the total energy of the system times the transition probability, but here

$$\epsilon = \int (k_1^\circ + k_2^\circ)\, d(\sigma v) = \int (p_1^\circ + q^\circ - p_2^\circ)\, d(\sigma v)$$

$$= m \cdot \xi \cdot (\sigma v) - \int p_2^\circ \cdot d(\sigma v)$$

where A° or $A_\circ$ denotes the time component of a four-vector. The

second term requires the re-evaluation of the integrals in (III. 7) with an additional factor of p_2^o, the second electron's energy, in the integrand. The integration can be carried out as before. Adding the two terms of ϵ:

$$\epsilon = \frac{e^2 G^2 m^3 \xi^2}{180\pi^2(\xi^2-1)(\xi^9-1)}\left\{\frac{\ln\xi}{(\xi^2-1)}(17\xi^9-5\xi^7-305\xi^5-370\xi^3\right.$$

$$+250\xi+7/\xi)+\frac{1}{120}(-2723\xi^7+7767\xi^5+28172\xi^3$$

$$\left.+1222\xi+12103/\xi+2295\xi^{-3}-270\xi^{-5})\right\} . \qquad \text{(III.10)}$$

In the extreme relativistic limit

$$\epsilon\,(E.R.) = \frac{17\, e^2 G^2 m^3 \xi^3}{180\,\pi^2}\left(\ln\xi - \frac{2723}{2040}\right) .$$

Neglecting the constant term, $(\epsilon/\sigma v) \simeq (17/20)\, m\xi$ so that the neutrinos carry off 85% of the energy in the E. R. limit. For the non-relativistic limit

$$\epsilon\,(N.R.) = \frac{4}{35\pi^2}\quad\frac{e^2 G^2 \omega^5}{m^2} . \qquad \text{(III.11)}$$

Here, $(\epsilon/\sigma v) = \omega$ so that, on the average, the neutrinos get half of the center of mass kinetic energy or, what is the same thing, the photon energy.

The expressions (III. 8) and (III.10) and those derived therefrom differ from the corresponding expressions found by Chiu and Stabler (Chiu 1961) by a factor of 4π. We have used an electron charge of $\sqrt{4\pi}e$ while they use e. However, it would appear that they use $e^2/hc \simeq 1/137$

which, we believe, should be changed to $e^2/hc \simeq \sqrt{4\pi}/137$. Matinyan and Tsilosani (Matinyan 1961) have reported that V. I. Ritus has obtained a much greater value for the neutrino energy loss for this process than that of Chiu and Morrison (Chiu 1960), who used the values of Chiu and Stabler.

In the extreme relativistic limit we will need the average energy carried off by the neutrinos but not in the center of mass (CM) system. Let us find the average rate at which energy and momentum are carried off by the neutrinos in the E. R. limit. From (III.7) we see that $d(\sigma v) = B(p_2)/E(q)E(p_1)$ where $B(p_2)$ is an invariant and $E(q)$ and $E(p_1)$ are independent of p_2. If we then form

$$\bar{P}_{2\mu} = \int d(\sigma v)\, p_{2\mu} \quad , \tag{III.12}$$

we will have a vector quantity. This will be a linear combination of $(q + p_1)$ and $(q - p_1)$ since they are the only vectors we have. In the center of mass system, in the E. R. limit, $(q + p_1)$ has only a time component while $(q - p_1)$ has only a z component. Thus the coefficient of $\bar{p}_2^{\,o}$ in CM is the coefficient of $(q + p_1)$ and the coefficient of $\bar{p}_2^{\,3}$ is the coefficient of $(q - p_1)$. Inspection shows that $\bar{p}_2^{\,2} = \bar{p}_2^{\,1} = 0$. There is only one invariant, $(p_1 \cdot q)$, since we take $m^2 = 0$. We can thus find $\bar{p}_{2\mu}$ in the E. R. limit in any system by using the following center of mass system results obtained from (III.12) using $d(\sigma v)$ from (III.7):

$$\bar{P}_2^{\,o} = \frac{e^2 G^2 m^3 E^3}{60\pi^2}\left(\ln \frac{E}{m} - \frac{3}{40}\right)$$

$$\bar{p}_2^0 = \frac{e^2G^2}{120\pi^2}\frac{(p_1\cdot q)^2}{E(p_1)E(q)}(q+p_1)_0\left[\ln 2(p_1\cdot q) - \frac{3}{20}\right]$$

$$\bar{p}_2^3 = \frac{e^2G^2m^3\xi^3}{60\pi^2}\left(\ln\xi - \frac{63}{40}\right)$$

$$\bar{p}_2^3 = \frac{e^2G^2}{120\pi^2}\frac{(p_1\cdot q)^2}{E(p_1)E(q)}(q-p_1)_3\left[\ln 2(p_1\cdot q) - \frac{63}{20}\right].$$

With these and the relation $\overline{p_1 + q} = \sigma v(p_1 + q)$ we obtain for the E. R. case:

$$\overline{(k_1+k_2)_\mu} = \frac{e^2G^2(p_1\cdot q)^2}{360\pi^2E(p_1)E(q)}\left\{\left[19\ln 2(p_1\cdot q) - \frac{539}{15}\right]q_\mu + \left[20\ln 2(p_1\cdot q) - \frac{329}{6}\right]p_{1\mu}\right\}. \qquad \text{(III.13)}$$

C. The Process $\gamma + \gamma \to \nu + \bar{\nu}$.

Finally, we describe the photoproduction reaction: $\gamma + \gamma \to \nu + \bar{\nu}$. For a local weak interaction we have the single Feynman diagram of Figure 5. This must, of course, be symmetrized with respect to the two incoming photons. Using the rules of Table I and a Fierz transformation (II.5b) we have for the matrix element:

$$M = \frac{-ie^2G\sqrt{2}}{2\pi^3}\left[\bar{u}_{\nu 2}\gamma_\mu a\, u_{\nu 1}\right]\int d^4p\, Sp\left[\gamma_\mu a\frac{1}{\not p+\not q_1-m}\not\epsilon_1\frac{1}{\not p-m}\not\epsilon_2\frac{1}{\not p-\not q_2-m}\right].$$

Gell-Mann (1961) has shown that this matrix element is zero. Since the neutrino-antineutrino pair is created at a point, the particle amplitudes at the origin are nonzero and hence they are in an orbital

angular momentum state of $L = 0$. In the center of mass system the two particles are traveling in opposite directions. The antineutrino spin is along its direction of motion; the neutrino spin is antiparallel to its momentum. Hence, the z component of the spin is 1 so that the spin state is $S = 1$. With $L = 0$, $S = 1$ we have a $J = 1$ state. Angular momentum being conserved in the interaction, the initial photons must be in a $J = 1$ state. But, Yang (1950) has shown that for symmetry reasons two photons can not be in a $J = 1$ state. Hence, the amplitude for $\gamma + \gamma \to \nu + \bar{\nu}$, via a local weak interaction, must be zero.

Let us present the argument more concretely. Consider, first, the vector part of the spur in (III.14):

$$V = \frac{Sp[\gamma_\mu(\not p+\not q_1+m)\not\epsilon_1(\not p+m)\not\epsilon_2(\not p-\not q_2+m)]}{[(p+q_1)^2-m^2][p^2-m^2][(p-q_2)^2-m^2]} + \{SYM\},$$

where SYM denotes the first term with $e_1 \longleftrightarrow e_2$, $q_1 \longleftrightarrow q_2$. The commutation relation of the γ matrices and the trace of any number of γ matrices are invariant under the substitution of the negative transpose for each γ matrix: $\gamma_\mu \to -\gamma_\mu^T$. Making this substitution and taking the transpose of the argument of the spur:

$$V = \frac{-Sp[(-\not p+\not q_2+m)\not\epsilon_2(-\not p+m)\not\epsilon_1(-\not p-\not q_1+m)\gamma_\mu]}{[(p+q_1)^2-m^2][p^2-m^2][(p-q_2)^2-m^2]} + \{SYM\}.$$

Since $\int d^4p$ is even in p, we may substitute -p for p in V. Also, Sp(AB) = Sp(BA) so that exchanging symmetric parts, we find

$$V = \frac{-Sp[\gamma_\mu(\not p+\not q_1+m)\not\epsilon_1(\not p+m)\not\epsilon_2(\not p-\not q_2+m)]}{[(p+q_1)^2-m^2][p^2-m^2][(p-q_2)^2-m^2]} + \{SYM\}.$$

Comparing this with the original form for V, we find that $V = -V = 0$.

This is an example of Furry's Theorem (Furry 1937), which states that because of charge conjugation invariance, the contribution of a diagram having a closed fermion loop with an odd number of γ_μ interactions is cancelled by the contribution of the diagram with the direction of that loop reversed.

Next, let us consider the axial vector part of (III.14):

$$A_\mu = \int d^4p \, \frac{Sp[\gamma_\mu \gamma_5 (\not{p}+\not{q}_1+m) \not{e}_1 (\not{p}+m) \not{e}_2 (\not{p}-\not{q}_2+m)]}{[(p+q_1)^2-m^2][p^2-m^2][(p-q_2)^2-m^2]} + \{SYM\}.$$

A_μ is a function of q_1, q_2, e_1, e_2, and m, bilinear in e_1 and e_2 and symmetric in the two photons. Since A_μ is pseudovector, it contains the permutation symbol $\epsilon_{\alpha\beta\lambda\delta}$, which is antisymmetric in all pairs of indices. In the CM system with transverse photons, we find that $e_1 \cdot q_1 = e_1 \cdot q_2 = e_2 \cdot q_1 = e_2 \cdot q_2 = 0$. Because of these requirements, A_μ must have the form:

$$A_\mu = F_1 (q_1+q_2)_\mu \epsilon(q_1, q_2, e_1, e_2) + F_2 \epsilon(e_1, e_2, q_1-q_2, \mu), \qquad \text{(III.15)}$$

where F_1 and F_2 are functions of the invariants $(q_1 \cdot q_2)$ and m^2, which are symmetric in the two photons. We have used $\epsilon(A,B,C,D,) = \epsilon_{\alpha\beta\lambda\delta} A_\alpha B_\beta C_\lambda D_\delta$. Now, for the axial vector part of the matrix element, we have $\mathcal{M}_A = \text{const}\ (\bar{u}_{\nu 2} \not{A} a u_{\nu 1})$. The Dirac equation for the neutrinos yields $(\bar{u}_{\nu 2} \not{k}_2 a u_{\nu 1}) = (\bar{u}_{\nu 2} \not{k}_1 a u_{\nu 1}) = 0$. Using conservation of momentum, $q_1 + q_2 = k_1 + k_2$, we get zero for the first term of A_μ. In the CM system with transverse photons, working with components, we find that $(e_1, e_2, q_1-q_2, \mu) = (k_1 + k_2)_\mu \, (e_1 \times e_2)_{z \text{ component}}$, so that this term of A_μ is also zero. Thus $\mathcal{M} = 0$. The argument for the second term

hinges on the fact that A is not a function of k_1 or k_2 except in the combination $k_1 + k_2$. This is essentially the same as the local nature of the interaction.

If we now consider a nonlocal weak interaction, the same arguments would apply for a neutral uxl. The appropriate Feynman diagram is given in Figure 6. The matrix element would be

$$M = (\text{const.})\,(\bar{u}_{\nu 2}\gamma_\mu a u_{\nu 1})\int d^4p\, S_p\left[\gamma_\nu a \frac{1}{\not p + \not q_1 - m}\not\epsilon_1 \frac{1}{\not p - m}\not\epsilon_2 \frac{1}{\not p - \not q_2 - m}\right]$$
$$\times \frac{g_{\mu\nu} - (k_1+k_2)_\mu (k_1+k_2)_\nu / M^2}{(k_1+k_2)^2 - M^2} + (S\,Y\,M)\,.$$

Here, the second term of the uxl propagator numerator produces $[u_{\nu 2}(\not k_1 + \not k_2)a u_{\nu 1}] = 0$. Aside from the uxl denominator factor of $[2(q_1 \cdot q_2) - M^2]$, the other piece of the uxl propagator numerator yields the same expression for the matrix element as in the local interaction case (III.14) and hence is zero.

For the charged uxl these arguments do not apply and the matrix element may not vanish. The four Feynman diagrams for this interaction, to lowest order, are given in Figures 7a, b, c, d. Using Feynman's rules from Table I, we find the following matrix elements:

$$M_A = \frac{-e^2 f^2}{\pi^2}\int d^4p\left[(\bar{u}_{\nu 2}\gamma_\nu a \frac{1}{\not p_3 - m}\not\epsilon_2 \frac{1}{\not p_2 - m}\not\epsilon_1 \frac{1}{\not p_1 - m}\gamma_\mu a u_{\nu 1}\right) \qquad \text{(III.16a)}$$
$$\times \left(\frac{g_{\nu\mu} - p_{4\nu}p_{4\mu}/M^2}{p_4^2 - M^2}\right)\Big] + [S\,Y\,M]\,,$$

$$M_B = \frac{e^2 f^2}{\pi^2}\int d^4p\Big\{(\bar u_{\nu_2}\gamma_\nu a\frac{1}{\not p_2 - m}\not e_2\frac{1}{\not p_1 - m}\gamma_\mu a u_{\nu_1})$$

$$\times\left(\frac{g_{\mu\alpha} - p_{4\mu}p_{4\alpha}/M^2}{p_4^2 - M^2}\right)\times\left(\frac{g_{\beta\nu} - p_{3\beta}p_{3\nu}/M^2}{p_3^2 - M^2}\right) \tag{III.16b}$$

$$\times[e_1\cdot(p_3+p_4)g_{\alpha\beta} - e_{1\beta}p_{3\alpha} - e_{1\alpha}p_{4\beta} - \gamma(q_{1\alpha}e_{1\beta} - q_{1\beta}e_{1\alpha})]\Big\} + \{SYM\},$$

$$M_C = \frac{-e^2 f^2}{\pi^2}\int d^4p\Big\{(\bar u_{\nu_2}\gamma_\nu a\frac{1}{\not p_4 - m}\gamma_\mu a u_{\nu_1}) \tag{III.16c}$$

$$\times\left(\frac{g_{\mu\alpha} - p_{1\mu}p_{1\alpha}/M^2}{p_1^2 - M^2}\right)\times\left(\frac{g_{\beta\lambda} - p_{2\beta}p_{2\lambda}/M^2}{p_2^2 - M^2}\right)\times\left(\frac{g_{\rho\nu} - p_{3\rho}p_{3\nu}/M^2}{p_3^2 - M^2}\right)$$

$$\times[e_2\cdot(p_1+p_2)g_{\alpha\beta} - e_{2\beta}p_{2\alpha} - e_{2\alpha}p_{1\beta} + \gamma(q_{2\beta}e_{2\alpha} - q_{2\alpha}e_{2\beta})]$$

$$\times[e_1\cdot(p_2+p_3)g_{\lambda\rho} - e_{1\rho}p_{3\lambda} - e_{1\lambda}p_{2\rho} + \gamma(q_{1\rho}e_{1\lambda} - q_{1\lambda}e_{1\rho})]\Big\} + \{SYM\},$$

$$M_D = \frac{-e^2 f^2}{\pi^2}\int d^4p\Big\{(\bar u_{\nu_2}\gamma_\nu a\frac{1}{\not p_4 - m}\gamma_\mu a u_{\nu_1})$$

$$\times\left(\frac{g_{\mu\alpha} - p_{1\mu}p_{1\alpha}/M^2}{p_1^2 - M^2}\right)\times\left(\frac{g_{\beta\nu} - p_{3\beta}p_{3\nu}/M^2}{p_3^2 - M^2}\right)$$

$$\times[e_{1\alpha}e_{2\beta} - (e_1\cdot e_2)g_{\alpha\beta}]\Big\} + \{SYM\}\quad. \tag{III.16d}$$

In this process, unlike the two previous ones, the uxl momentum is not limited by the external momenta. Here the uxls are in closed loops so we must integrate over all values of uxl momenta.

There is a similar situation in the case of the decay $\mu \rightarrow e + \gamma$. Although we now know that this process is forbidden by the difference between the neutrinos associated with muons and those associated with electrons (Danby 1962), the calculation of the matrix element still may be done as a formal exercise. For a local weak interaction, the process is described by the two diagrams of Figure 8. In both cases the loop integral is $\int d^4p/p$ which, while cubically divergent, is zero since the integrand is odd in p and the interval of integration is even. Introducing a charged uxl we have the three diagrams of Figure 9. We now have two or three particles in the closed loops and the same symmetry argument no longer holds. Indeed, Feinberg (1958) found that for an anomalous magnetic moment of 1, the matrix element is convergent. For no anomalous moment it is divergent, and in either case the first non-vanishing term is $0(G) = 0(m^2/M^2)$.(*) We will return to this process after computing the neutrino pair production matrix element.

Let us gather some tools to aid in our examination of the matrix elements (III.16). We would like to examine these matrix elements to some order in (m^2/M^2), where we consider the particle energies to be of the order of m, since this is the energy range of interest and

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(*) When we use the notation $0(m^n/M^n)$, it is to be understood that the coupling constant is f^2. The only exception will be when G is explicitly displayed.

since such a limitation will ease the calculational difficulties. To do this we will expand the propagator denominators using

$$\frac{1}{A-B} = \frac{1}{A} + \frac{1}{A}B\frac{1}{A} + \frac{1}{A}B\frac{1}{A}B\frac{1}{A} + \cdots$$

to obtain, for example,

$$\frac{1}{(p-k)^2-M^2} = \frac{1}{(p^2-M^2)} + \frac{2(p\cdot k)-k^2}{(p^2-M^2)^2} + \frac{[2(p\cdot k)-k^2]^2}{(p^2-M^2)^3} + \cdots \quad . \qquad \text{(III.17)}$$

Since p is a variable of integration, we must integrate the products of such expansions. The integrals which arise will be of the form:

$$I(s,r,a,b) = \frac{1}{M^{2s}}\int d^4p \, \frac{p^{2r}}{(p^2-m^2)^a(p^2-M^2)^b} \quad .$$

In Appendix C we find that the dominant term, ($m^2 \ll M^2$), in such an integral is:

$$I(s,r,a,b) = 0(m^{2c}/M^{2c})m^{2(r+2-a-b)} \qquad \text{(III.18)}$$

where $c = b + s$ for $a - r - 2 \geq 0$

$c = b + s + a - r - 2$ for $a - r - 2 \leq 0$.

If we consider a product of expansions of the form of (III.17) for uxl and electron propagators all multiplied by some numerator, then for some given terms in the expansions, the product will be characterized by some values for s,r,a,b. Of course, r could be half integral but then the integral, I, vanishes so we take s,r,a,b integral. If we consider further terms in any electron propagator expansion, b and s are fixed. Going out one term in the expansion, $\Delta a = 1$, $\Delta r = 0, \frac{1}{2}$.

Half integral r causes I to vanish. Thus $\Delta(a-r-2) = 1$ and c increases ($c = b + s + a - r - 2$) or remains constant ($c = b + s$). Going out two terms, $\Delta a = 2$, $\Delta r = 0,1$. Here $\Delta(a-r-2) = 2,1$ and c increases or remains constant. In general c increases until $c = b + s$ and then remains constant. In either case $\Delta c \not< 0$ and the exponent of M^2 in the denominator of the resulting integral does not decrease. If we consider further terms in an uxl propagator denominator expansion, a and s are fixed. Going out one term in the expansion, $\Delta b = 1$, $\Delta r = 0$, for integral r. Then (a-r-2) is fixed and $\Delta c = 1$. Going out two terms, $\Delta b = 2$, $\Delta r = 0,1$. If $\Delta r = 0$, then $\Delta c = 2$. For $\Delta r = 1$, if (a-r-2) was ≥ 0 then $\Delta c = \Delta(b+s) = 2$, while if (a-r-2) was <0 then $\Delta c = \Delta(b + s + a - r - 2) = 1$. In all cases further terms in an uxl expansion yield lower exponents for M^2. Thus if for some term in the expansions c is greater than some value c_o, for all further terms $c > c_o$. Also, given any c_o, going out sufficiently far in the uxl expansions we can have $c > c_o$.

We can also say something about the exponent of k, where k represents any of the external momenta, in the terms of the matrix element. To assure gauge invariance the photons enter through the field tensors $F_{\mu\nu}$. $\mathcal{M}$ is bilinear in the two field tensors and $F_{\mu\nu}$ has one factor of k so that together we have k^2. Examination of the matrix elements (III.16) shows that each term has an odd number of vectors in it. If we again use the expansions for the denominators, we see that all vectors other than k enter in the pairs $(e_1 e_2)$, (p^2), (m^2). Thus for any term we have k^n, where n is odd and greater than or

equal to two so that $n \geq 3$. We can drop terms having k^n, $n < 3$, since they must cancel to preserve gauge invariance.

Another fact that will be of aid is that the $1/M^4$ numerator terms in the expression

(uxl propagator)(uxl-single photon interaction)(uxl propagator)

cancel. These terms are simply

$$A = \frac{1}{M^4}(p-q)_\alpha (p-q)_\beta \left[(p-q+p)\cdot e\, g_{\beta\lambda} - (p-q)_\lambda e_\beta - p_\beta e_\lambda - \gamma(q_\beta e_\lambda - q_\lambda e_\beta)\right] p_\lambda p_\beta \,. \tag{III.19}$$

But, in our coordinates $q^2 = e_1 \cdot q_1 = e_1 \cdot q_2 = e_2 \cdot q_1 = e_2 \cdot q_2 = 0$ so that $A = 0$.

With these tools we can deal effectively with the matrix elements (III.16). First, let us treat the divergent loop integrals. All four matrix elements have primitive divergences. They are linear, cubic, quintic, and cubic in $\mathcal{M}_A$, $\mathcal{M}_B$, $\mathcal{M}_C$, and $\mathcal{M}_D$, respectively. If we neglect $1/M^4$ terms from single photon-uxl interactions, these are reduced to linear, linear, cubic, and cubic, respectively. Expanding the propagator denominators and going out sufficiently far to assure gauge invariance, we see that $\mathcal{M}_A$ and $\mathcal{M}_B$ are convergent and that $\mathcal{M}_C$ and $\mathcal{M}_D$ are logarithmically divergent. These divergent pieces are:

$$\mathcal{M}_D(\text{DIV}) = -\frac{e^2 f^2}{\pi^2 M^4} \int d^4p \left\{ (\bar{u}_{\nu 2}\, \gamma_\nu\, a \frac{1}{\not{p}_4 - m}\, \gamma_\mu\, a\, u_{\nu 1}) \right.$$

$$\left. \times \frac{p_{1\mu} p_{1\alpha} p_{3\rho} p_{3\nu}}{(p_1^2 - M^2)(p_3^2 - M^2)} \left[e_{1\alpha} e_{2\beta} - (e_1 \cdot e_2) g_{\alpha\beta} \right] \right\} + \{\text{SYM}\} \,.$$

$$M_c(\mathrm{DIV}) = -\frac{e^2 f^2}{\pi^2 M^4}\int d^4p\,\{(\bar{u}_{\nu 2}\gamma_\nu a \frac{1}{\not p_4 - m}\gamma_\mu a u_{\nu 1})$$

$$\times \frac{p_{1\mu}\, p_{1\alpha}\, g_{\beta\lambda}\, p_{3\rho}\, p_{3\nu}}{(p_1^2 - M^2)(p_2^2 - M^2)(p_3^2 - M^2)}$$

$$\times[e_{2\cdot}(p_1+p_2)g_{\alpha\beta} - e_{2\beta}p_{2\alpha} - e_{2\alpha}p_{1\beta} - \gamma(q_{2\alpha}e_{2\beta} - q_{2\beta}e_{2\alpha})]$$

$$\times[e_{1\cdot}(p_2+p_3)g_{\lambda\rho} - e_{1\rho}p_{3\lambda} - e_{1\lambda}p_{2\rho} - \gamma(q_{1\lambda}e_{1\rho} - q_{1\rho}e_{1\lambda})]\}+\{\mathrm{SYM}\}\,.$$

In our coordinates

$$e_1\cdot p_1 = e_1\cdot p_2 = e_1\cdot p_3 = e_1\cdot p_5 \qquad p_2\cdot q_1 = p_3\cdot q_1$$

$$e_2\cdot p_1 = e_2\cdot p_2 = e_2\cdot p_3 = e_2\cdot p_5 \qquad p_1\cdot q_2 = p_2\cdot q_2 \quad .$$

Using the Dirac equation

$$[\bar{u}_{\nu 2}\not p_3 a(\not p_4 + m)\not p_1 a u_{\nu 1}] = p^2[\bar{u}_{\nu 2}\not p\, a\, u_{\nu 1}]\,.$$

We create a common denominator of $D = (p_4^2 - m^2)(p_1^2 - M^2)(p_2^2 - M^2)\times(p_3^2 - M^2)(p_5^2 - m^2)$, suspend writing the constant coefficient, and use the notation $(\not A) = (\bar{u}_{\nu 2}\not A a u_{\nu 1})$. Further, let $\xi = 1 - \gamma$ and drop terms which must cancel by gauge invariance or which are convergent. Then, including the symmetric parts, we find:

$$M_D(\mathrm{DIV}) = \int d^4p\ \frac{p^2(\not p)\, p_2^2\, p_5^2[2(e_1\cdot p_3)(e_2\cdot p_1) - 2(e_1\cdot e_2)(p_1\cdot p_3)]}{D},$$

$$\mathcal{M}_C(\mathrm{DIV}) = \int d^4p \; \frac{p^2(p^2)}{D}\{-(e_1\cdot p_3)(e_2\cdot p_1)[2p_2^2p_5^2 + \xi^2(p_5^2+p_2^2)(q_1\cdot q_2)]$$

$$+ (e_1\cdot e_2)[p_2^2p_5^2(p_2^2+p_5^2) + \xi\, p_2^2p_5^2(p_2-p_5)\cdot(q_1-q_2)$$

$$-\xi^2 p_5^2(p_3\cdot q_1)(p_1\cdot q_2) - \xi^2 p_2^2(p_3\cdot q_2)(p_1\cdot q_1)]\} \quad .$$

Since $2(p_1\cdot p_3) = p_2^2 + p_5^2$, all the terms of $\mathcal{M}_D(\mathrm{DIV})$ are cancelled by terms of $\mathcal{M}_C(\mathrm{DIV})$. We can expand the denominator to get

$$\frac{1}{D} = \frac{1}{(p^2-m^2)(p^2-M^2)^4} + \frac{4p\cdot(k_2-k_1)}{(p^2-m^2)(p^2-M^2)^5} + \cdots \quad .$$

All further terms of this expansion lead to convergent integrals. If, in the integral, p enters the denominator only in the form p^2, we may make these substitutions in the numerator (see Appendix C):

$$p_\alpha p_\beta \longrightarrow \frac{p^2}{4} g_{\alpha\beta} \; ,$$

$$p_\alpha p_\beta p_\lambda p_\delta \longrightarrow \frac{p^4}{24}(g_{\alpha\beta}g_{\lambda\delta} + g_{\alpha\lambda}g_{\beta\delta} + g_{\alpha\delta}g_{\beta\lambda}) \; .$$

With these substitutions all remaining divergent terms which satisfy the gauge invariance requirement cancel. Thus $\mathcal{M}_A+\mathcal{M}_B+\mathcal{M}_C+\mathcal{M}_D = \mathcal{M}$ is convergent.

We now turn to the convergent terms. Examining all terms in $\mathcal{M}$ with the aid of the relations (III.18) and the gauge invariance requirements, we find that the leading term is $0(m^2/M^2)$. Such terms yield a matrix element of order $(f^2/M^2) \sim (G)$, which does not vanish if $M^2 \to \infty$. These terms come only from $\mathcal{M}_A$ and there only from

the $g_{\mu\nu}$ term of the uxl propagator. This contribution is proportional to

$$\int d^4p \Big\{ (\bar{u}_{\nu 2} \gamma_\mu a \frac{1}{\not p + \not q_2 - m} \not e_2 \frac{1}{\not p - m} \not e_1 \frac{1}{\not p - \not q_1 - m} \gamma_\nu a u_{\nu 1}) \times \frac{g_{\mu\nu}}{(p+q_2-k_2)^2 - M^2} \Big\} + \{ \text{SYM} \}$$

We expand the uxl denominator,

$$\frac{1}{(p+q_2-k_2)^2-M^2} = \frac{1}{(p^2-M^2)} + \frac{2p\cdot(k_2-q_2)}{(p^2-M^2)^2} + \frac{2(k_2\cdot q_2)}{(p^2-M^2)^2} + \cdots ;$$

and apply (III.18) to see that only the first term in the series contributes to 0(G). The matrix element, to this order, is the same as for the local weak interaction (III.14) but for the factor: $-M^2/(p^2-M^2)$. We may write it as

$$M = \frac{\sqrt{2}\, e^2 G}{2\pi^3} \int d^4p \left(\frac{-M^2}{p^2-M^2} \right) \Big\{ (\bar{u}_{\nu 2} \gamma_\mu a \frac{1}{\not p + \not q_2 - m} \not e_2 \frac{1}{\not p - m} \not e_1 \frac{1}{\not p - \not q_1 - m} \gamma_\mu a u_{\nu 1}) \Big\} + \{ \text{SYM} \}$$

where the effect of the uxl is to introduce a sort of convergence factor. The Fierz transformation and all of the other arguments used in the local interaction case can be used here to show that to 0(G) this matrix element is still zero.

This is not the same as in the case of the muon decay. Here, the leading term in the matrix element behaved the same with and without an uxl. In the μ-decay in the local interaction case, the matrix element vanished, while introducing an uxl produced terms of 0(G). But in the μ-decay case the local interaction matrix element

was cubically divergent. After the introduction of the uxl, the corresponding terms still vanished. That is

$$I = \int \frac{d^4p}{p} \xrightarrow{uxl} \int \frac{d^4p}{p[(p-k)^2 - M^2]} = I'$$

and $I = 0$ but $I' \neq 0$. However, expanding $[(p-k)^2 - M^2]$ we have

$$I' = \int \frac{d^4p}{p[p^2 - M^2]} + \cdots$$

and here the leading term vanishes. The remaining terms still contain divergences and are able to contribute to $0(G)$. On the other hand, the matrix element for $\gamma + \gamma \rightarrow \nu + \bar{\nu}$, for a local weak interaction if gauge invariance requirements are considered, is convergent. In general, if a process is convergent for a local weak interaction, then, since it appears that the zero order effect of introducing an uxl is to include a cutoff factor, $[-M^2/(p^2 - M^2)]$, and since we would expect a convergent integral to be independent of a cutoff, the uxl should have no effect to zero order. However, we certainly have not proved this general conclusion with any rigor.

There are higher order effects. Examination of the matrix elements (III.16) with the relations (III.18) shows that all four Feynman diagrams contribute to $0(m^4/M^4)$. It is a straightforward though lengthy task to evaluate these terms. The actual evaluation was done on a high-speed digital computer with a program devised to do dot products, matrix permutations, substitution of integrals and other work of a symbol-manipulation nature. This program is described in Appendix C. We work in the CM system with transverse photons and let

$$T_1 = k_1\cdot e_1 = -k_2\cdot e_1 \qquad T_2 = k_2\cdot e_2 = -k_1\cdot e_2$$

$$a = -k_1\cdot k_2 = -q_1\cdot q_2 \qquad b = k_1\cdot q_2 = k_2\cdot q_1 \qquad \text{(III.20)}$$

$$f = k_1\cdot q_1 = k_2\cdot q_2 \qquad a+b+f=0$$

$$(\not{A}) = (\bar{u}_{\nu 2}\not{A} u_{\nu 1})$$

The contributions to this order are then found to be:

$$M_A = \frac{-e^2f^2}{iM^4}\int_0^1\int_0^1 z\,dz\,dx\Big\{[(\phi_1)T_2-(\phi_2)T_1]a\Big[4(\ln\Delta)\Big\{4z(1-z)(1-x)+xz\Big\}+\frac{2}{3}\Big\{38z(1-z)(1-x)+17xz\Big\}\Big]$$

$$+[(\phi_1)T_2+(\phi_2)T_1](b-f)xz(1-2z+xz)\Big[-16\ln\Delta-\frac{136}{3}+\frac{8m^2}{\Delta}\Big]$$

$$+(\phi_1)(e_1\cdot e_2)(b-f)xz\Big[4(1+2xz)\ln\Delta+\frac{2}{3}(55xz+14)-\frac{8m^2}{\Delta}(1+xz)\Big]\Big\}$$

where $\Delta = [1 + 2a\,z(1-z)(1-x)/m^2]$,

$$M_B = \frac{e^2f^2}{iM^4}\Big\{[(\phi_1)T_2-(\phi_2)T_1]a\Big(\frac{5}{3}+\frac{4\gamma}{3}\Big)+(\phi_1)(e_1\cdot e_2)(b-f)\Big(\frac{13}{18}+\frac{\gamma}{2}\Big)\Big\},$$

$$M_C = \frac{-e^2f^2}{iM^4}\Big\{[(\phi_1)T_2-(\phi_2)T_1]a\Big(\frac{7}{36}-\frac{\gamma}{3}-\frac{2\gamma^2}{3}\Big)+(\phi_1)(e_1\cdot e_2)(b-f)\Big(-\frac{5}{9}-\frac{7\gamma}{6}-\frac{2\gamma^2}{3}\Big)\Big\},$$

$$M_D = \frac{-e^2f^2}{iM^4}\Big\{[(\phi_1)T_2-(\phi_2)T_1]a\Big(-\frac{5}{36}\Big)\Big\}.$$

The integrals remaining in $\mathcal{M}_A$ were not done by the computer but by hand to yield:

$$\mathcal{M}_A = \frac{-e^2 f^2}{i M^4}\left\{[(\phi_1)T_2-(\phi_2)T_1]a\,\mathrm{Re}\left[-\frac{17}{18}+\frac{13}{3\omega^2}-\left(\frac{5}{12\omega^3}+\frac{1}{3\omega}\right)\sqrt{\omega^2-1}\,\ln(1-\rho)\right.\right.$$

$$\left.-\frac{2\omega^2+1}{4\omega^4}\left[\ln(1-\rho)\right]^2\right]$$

$$+(\phi_1)(e_1\cdot e_2)(b-f)\,\mathrm{Re}\left[-\frac{11}{18}+\frac{29}{3\omega^2}+\left(\frac{-4}{3\omega}+\frac{13}{3\omega^3}\right)\sqrt{\omega^2-1}\,\ln(1-\rho)\right.$$

$$\left.\left.+\frac{2\omega^2+1}{4\omega^4}\left[\ln(1-\rho)\right]^2\right]\right\}$$

where 2ω is the total center of mass energy and $\rho = 2\omega^2(1-\sqrt{1-1/\omega^2})$. $\mathcal{M}$ has the form:

$$-\frac{iM^4}{e^2f^2}\mathcal{M} = F_1(a)[(\phi_1)T_2-(\phi_2)T_1]a + F_2(a)(\phi_1)(e_1\cdot e_2)(b-f)\ .$$

Using (II.6), the transition probability is

$$\sigma v = \frac{1}{16\pi^2\omega^2}\int d^4k_1\, d^4k_2\,\delta(k_1^2)\delta(k_2^2)|\mathcal{M}|^2\delta^4(k_1+k_2-q_1-q_2)\ .$$

We take the incoming photons along the z-axis, each having its polarization in the x or y directions, and average over the four combined polarization states. The neutrino momenta are taken to lie in the x, z-plane. Here the integrations are easily effected. Then

$$\sigma v = \frac{\omega^6 e^4 f^4}{30\,\pi M^8}\left[(F_1-F_2)^2+6F_1^2\right],$$

$$F_1 = \mathrm{Re}\left\{-\tfrac{2}{3}\gamma^2 - \tfrac{5}{3}\gamma - \tfrac{23}{9} + \tfrac{13}{3\omega^2} - \left(\tfrac{5}{3\omega^3} + \tfrac{4}{3\omega}\right)\sqrt{\omega^2-1}\,\ln(1-\rho) - \frac{(2\omega^2+1)}{4\omega^4}\ln^2(1-\rho)\right\},$$

$$(F_1 - F_2) = \mathrm{Re}\left\{-\tfrac{2}{3} - \frac{16}{3\omega^2} - \frac{6}{\omega^3}\sqrt{\omega^2-1}\,\ln(1-\rho) - \frac{(1+2\omega^2)}{2\omega^4}\ln^2(1-\rho)\right\}.$$

We can evaluate these expressions in the nonrelativistic limit, $|\omega| \ll m$,

$$\sigma v(\mathrm{N.R.}) = \frac{e^4 f^4 \omega^6}{30\pi M^8}\left[\left(\frac{53}{18}\right)^2 + 6\left|\frac{2}{3}\gamma^2 + \frac{5}{3}\gamma + \frac{29}{18}\right)^2\right] \qquad \text{(III. 21a)}$$

and in the extreme relativistic limit, $m \ll \omega \ll M$,

$$\sigma v(\mathrm{E.R.}) = \frac{e^4 f^4 \omega^6}{30\pi M^8}\left[\left(\frac{2}{3}\right)^2 + 6\left|\frac{2}{3}\gamma^2 + \frac{5}{3}\gamma + \frac{23}{9} + \frac{8}{3}\ln(2\omega)\right)^2\right]. \qquad \text{(III. 21b)}$$

There is no real value of γ for which these expressions vanish. In the N. R. case the bracketed coefficient has a minimum value of 10.6 for $\gamma = -5/4$ and is equal to 10.9, 24.2, 102. for $\gamma = -1, 0, 1$ respectively. Using $\gamma = 0$,

$$\sigma v(\mathrm{N.R.}) = \frac{1571}{3888\pi^3}\left(\frac{e^2}{\hbar c}\right)^2 (G M_p^2)^2 (\hbar c)^6 c\,\frac{\omega^6}{M_p^4 M^4} \qquad \text{(III. 22)}$$

in conventional units. Using $M = 0.8 M_p$,

$$\sigma(\mathrm{N.R.}) = 2.0 \times 10^{-63}\left(\frac{\omega}{m}\right)^6\left(\frac{c}{v}\right) \quad \mathrm{cm}^2$$

which is much smaller than either of the other two cross sections (III.5) and (III.9) for $(\omega/m) \ll 1$. Although this cross section is small, it does show that perfectly reasonable calculations can be made with uxls.

This result is much smaller than the estimate given by Matinyan and Tsilosani (1961), who did not show that the $0(G)$ term vanishes.

IV. ENERGY DISSIPATION RATES

We will now calculate the energy dissipation rates, in a hot electron, positron, photon atmosphere, due to the three transition probabilities: (III.4), (III.8) and (III.21). The electron and positron densities are described by a Fermi distribution:

$$dn = \frac{2}{(2\pi\hbar c)^3} \frac{p^2\, dp\, d\Omega}{\exp[(\epsilon-\mu)/kT] + 1} \qquad \text{(IV.1a)}$$

where ϵ is the total energy and μ, the chemical potential, serves to normalize the total density. The photon gas is described by a Bose distribution with zero chemical potential:

$$dn = \frac{2}{(2\pi\hbar c)^3} \frac{\epsilon^2 d\epsilon\, d\Omega}{\exp[(\epsilon)/kT] - 1} \,. \qquad \text{(IV.1b)}$$

These distributions are in what we will call the lab system wherein the gas center of mass is at rest.

For the reaction $e^+ + e^- \rightarrow \nu + \bar{\nu}$ we need the positron and electron distributions. The radiation is in equilibrium with the e^+e^- pairs through the reaction

$$e^+ + e^- \longleftrightarrow n\gamma \qquad\qquad n = 2, 3, 4 \ldots$$

Following Landau and Lifschitz (1958), the photon gas chemical potential is zero and so $\mu_+ + \mu_- = 0$ where $\mu_\pm$ are the potentials of the electron and positron gases. In the absence of residual matter, the positron and electron densities are equal: $n_+ = n_-$. Then the densities are given by

$$n_\pm = 2\left(\frac{mkT}{2\pi(\hbar c)^2}\right)^{3/2} \exp(-mc^2/kT) \,.$$

If we approximate the transition probability for the pair production reaction $\gamma + \gamma \rightarrow e^+ + e^-$ by $2\pi r_o^2 c$, multiply this by the square of the number of photons with energy greater than an electron mass, and divide by the equilibrium density of electrons and positrons, we have the reciprocal of a characteristic relaxation time, τ, for the approach to equilibrium between photon and electron-positron densities:

$$\tau = \left(\frac{\pi^3 (\hbar c)^2 \exp(2mc^2/kT)}{4 m kT \alpha^4 c^2} \right)^{1/2}$$

where r_o is the classical radius of the electron and α is the fine structure constant. Evaluating this

$$\tau = 1.6 \times 10^{-16} T_9^{-1/2} \exp(5.9/T_9)$$

where T_9 is the absolute temperature in units of 10^9°K. For $T_9 > .17$, $\tau < 1$ sec, which is small compared to the evolution times in stars. For $T_9 \geq .3$ we have $\tau < 10^{-7}$ sec. These times are upper limits as other processes will enter due to the presence of matter.

Having established that equilibrium conditions exist, we can now calculate the energy dissipation rate, ϵ, using

$$\mathcal{E} = \iint dn_+ dn_- (\sigma v) E(\nu\bar{\nu}) .$$

The distributions are given in (IV.1); the transition probability is given in (III.4). $E(\nu\bar{\nu})$ is the energy taken off by the neutrinos and is here equal to the total energy of the interacting particles. Working in the lab system of coordinates we find

$$\mathcal{E} = \frac{G^2}{96(\hbar c)^6 \pi^7} \iint \frac{(E_1E_2 - \vec{p}_1\cdot\vec{p}_2 + m^2)(2E_1E_2 - 2\vec{p}_1\cdot\vec{p}_2 + m^2)}{E_1E_2\,[\exp((E_1-\mu_-)/kT) + 1]} \times \frac{(E_1+E_2)\,p_1^2\,p_2^2\,dp_1\,dp_2\,d\Omega_1\,d\Omega_2}{[\exp((E_2-\mu_+)/kT) + 1]}.$$

In the nondegenerate limit we neglect the 1 in the distribution denominators. Using $\mu_+ + \mu_- = 0$ we find

$$\mathcal{E} = \frac{G^2}{12\pi^5(\hbar c)^6} \left\{ \frac{(E_1+E_2)}{E_1E_2} \frac{(E_1E_2 - p_1p_2z + m^2)(2E_1E_2 - 2p_1p_2z + m^2)p_1^2p_2^2\,dp_1\,dp_2\,dz}{\exp[(E_1+E_2)/kT]} \right.$$

In this limit we see that $(dn_-\,dn_+)$ is independent of the residual electron density, $n_o = n_- + n_+$, since μ_+ and μ_- enter only through the combination: $(\mu_+ + \mu_-)$. In the N. R. limit, $E = m + p^2/2m$. The integrand has its maximum at $(p^2/2m^2) = (kT/m)$ so that for $(kT) \ll m$ the N. R. approximation is valid. Then the first term in (p/m) is

$$\mathcal{E} = \frac{1}{\pi^4}(GM_p^2)^2\left(\frac{m}{M_p}\right)^4 (mc^2)\frac{(kT)^3}{(\hbar c)^4} c \quad \exp(-2mc^2/kT)$$

$$\mathcal{E} = 0.49 \times 10^{19} \frac{T_9^3}{\rho} \quad \exp(-11.9/T_9) \ \text{erg/gm-sec}$$

where ρ is the density in gm/cm^3. In the E. R. limit we take $E = p \gg m$ and find

$$\mathcal{E} = \frac{128}{\pi^5} \frac{(GM_p^2)^2 (kT)^9 c}{(\hbar c)^9 M_p^4} \qquad \text{(IV.3)}$$

$$\mathcal{E} = 0.46 \times 10^{16} \frac{T_9^9}{\rho} \quad \text{erg/gm-sec.}$$

These results in both the E. R. and N. R. limits are in agreement with those later obtained by Chiu and Stabler (1961). Chiu (1961a) has also calculated the effect of slight degeneracy of the electron gas.

Next we consider the process $e^{\pm}+\gamma \rightarrow e^{\pm}+\nu+\bar{\nu}$. The transition probability (III. 9) and the average $(\nu\bar{\nu})$ pair energy were calculated in the CM system in the N. R. limit while the particle distributions are given in the lab system. The relativistic transformations involve

$$\vec{\beta}_{CM} = \frac{\vec{p}_- + \vec{p}_\gamma}{E_- + E_\gamma} \quad .$$

But in the nonrelativistic limit $\beta_{CM} \sim p/m \sim 0$ so that we can neglect the transformations. Then

$$\mathcal{E} = \int d(\sigma v)\, dn_-\, dn_\gamma\, (k_1^\circ + k_2^\circ) = \int \epsilon\, dn_-\, dn_\gamma$$

where ϵ is as defined in (III. 11). Using this we obtain

$$\mathcal{E} = \frac{4}{35}\frac{e^2G^2}{\pi^4 m^2}\int \frac{\omega^7\, d\omega}{\exp(\omega/kT)-1}\int dn_- \quad .$$

The last factor is simply the electron-positron density. Integrating we obtain

$$\mathcal{E} = \frac{32\pi^4}{525}\frac{e^2G^2}{m^2}(kT)^8\frac{c}{(\hbar c)^2}N_e \qquad \text{(IV. 4)}$$

$$= 1.0\times 10^8 \frac{1}{\mu_e} T_9^8 \quad \text{erg/gm-sec} \quad ,$$

where $1/\mu_e$ is the average number of electrons per nucleon. The value (IV. 4) is in agreement with the work of Chiu and Stabler (1961) except for the factor of 4π discussed after the σv calculation. Our result is 4π times larger than theirs.

The factor $1/\mu_e$ deserves further consideration. Again following

Landau and Lifschitz, the number of electrons and positrons in the extreme and nonrelativistic limits are

$$n_+ + n_- = \left[n_o^2 + 2\left(\frac{mkT}{\pi(\hbar c)^2}\right)^3 \exp(-2mc^2/kT)\right]^{1/2} \qquad \text{N.R., } T \ll m,$$

$$n_+ + n_- = 0.366\left(\frac{kT}{\hbar c}\right)^3 \qquad \text{E.R., } T \gg m.$$

In the N. R. limit the effect of atomic electrons, n_o, is considered while in the E. R. limit we will see that $n_+ + n_- \gg n_o$ even for high densities. We take n_o to be $\frac{1}{2}$ the nucleon number density and then plot, in Figure 10, that matter density at which the number of pair produced electrons and positrons equals the number of residual electrons. For the N. R. case we have set $n_o = 0$ in the expression for $(n_+ + n_-)$. The residual or pair produced electrons will dominate N_e, depending upon which side of the curve the chosen temperature and density lie. On the same graph we plot, as a function of temperature, that equivalent matter density for which the electron gas becomes degenerate. The temperature used is such that kT is 1/3 of the Fermi energy. For the N. R. limit we have plotted $(n_+ + n_-)/n_o$ as a function of temperature for various densities in Figure 11. At $T_9 = 1$ and $\rho = 2000$ gm/cm^3, N_e is enhanced by a factor of 1.7 over the n_o value. In the extreme relativistic region, Chiu and Stabler consider only the residual electrons but, unless $\rho > 10^9$ gm/cm^3, the pair produced electrons dominate. At such high densities, degeneracy sets in. Using the pair production electron density, we have in the N. R. limit

$$\mathcal{E} = 1.25 \times 10^{14}\ T_9^{9.5}\left(\frac{1}{\rho}\right)\exp(-5.95/T_9) \quad \text{erg/gm-sec.}$$

In the extreme relativistic limit we may use (III.13) for the average momentum carried off by the neutrinos to find:

$$\mathcal{E} = \int \overline{(k_1 + k_2)}_0 \, dn(p) \, dn(q) \, .$$

Using (III.13) and (IV.1), taking the nondegenerate limit, and neglecting the electron mass, we obtain

$$\mathcal{E} = \frac{272}{45\pi^4} e^2 G^2 (kT)^6 N_e \left[\ln(2kT/m) - C + \frac{233}{510} \right]$$

$$\text{where} \quad -C = \int_0^\infty e^{-x} \ln x \, dx = \Psi(0) \, , \quad \Psi(z) = \frac{d}{dz} \ln(z!)$$

$$\mathcal{E} = 0.8 \times 10^8 \frac{T_9^6}{\mu_e} \left[\log_{10} T_9 - 0.524 \right] D_e \qquad \text{(IV.5)}$$

where $D_e \equiv (n_+ + n_-)/n_0$. Using the pair production electron density

$$\mathcal{E} = 0.41 \times 10^{13} T_9^9 \left(\frac{1}{\rho}\right) \left[\log_{10} T_9 - 0.524 \right] \text{ erg/gm-sec.}$$

We must let $T_9 > 20$ so that (III.13) is non-negative. Chiu and Stabler (1961) obtained an answer of the same form as (IV.5) but with an overall coefficient of 0.2×10^8 and the constant in the bracket equal to +0.6. Since the method of calculation is not given, we can not tell where the difference arises. However, since the temperature range in which this expression is valid is rather high for stars, the practical difference, in this case, is small.

If the electron gas is degenerate, there will be a strong decrease in the energy dissipation due to this process. Here, as in the other processes, the integrations over initial particle densities used must be

better approximated, but here, in addition, the degeneracy will affect the final state since it contains an electron. If degeneracy is complete, scattering becomes impossible unless the final electron passes the Fermi surface. The effect of degeneracy has been calculated by Chiu and Stabler.

Finally there is the neutrino energy loss due to the two photon annihilation reaction. Again we have

$$\mathcal{E} = \int (\sigma v)(s_o + t_o)\, dn(s)\, dn(t)$$

where (s_o, s), (t_o, t) are the four-momenta of the two photons. We found in (III.22) that σv is proportional to s_o^6 in the CM system. After summing over photon spins and integrating over the final states, there is but one invariant: $(s \cdot t)$. We can write $\sigma v = (\text{invariant})/s_o t_o$ so that σv is proportional to $(s \cdot t)^4/s_o t_o$. Then

$$\sigma v = \frac{1571}{2^8 3^5 \pi^3} \frac{e^4 G^2}{M^4} \frac{(s \cdot t)^4}{s_o t_o} .$$

Using the distribution (IV.1) we have

$$\mathcal{E} = \frac{1571\, e^4 G^2}{2^9 3^5 \pi^7 M^4} \int \frac{(s_o + t_o)\, s^5 t^5 (1 - z)^4\, ds\, dt\, dz}{[\exp(s/kT) - 1][\exp(t/kT) - 1]} ,$$

where z is the cosine of the angle between the two photon momenta. Integrating we find

$$\mathcal{E} = \frac{1571 \cdot 2^4\, \zeta(7)\, e^4 G^2 (kT)^{13}}{7 \cdot 3^5\, \pi\, M^4} \qquad \text{(IV.6)}$$

$$\mathcal{E} = 0.47 \times 10^{-3}\, T_9^{13} \left(\frac{1}{\rho}\right) \quad \text{erg/gm-sec} .$$

$\zeta(z)$ is the Riemann zeta function $\zeta(7) = 1.008\ldots$. As we would expect, this contribution is very small. Neglecting the $\ln(2\omega)$ term in

the E. R. cross section (III. 21) except in the coefficient, this temperature dependence also holds in the range $m < T < M$:

$$\mathcal{E} \simeq \left[0.76 + 4.6 \log_{10}^{2}(T_9/3.1)\right] \times 10^{-3} T_9^{13} \left(\frac{1}{\rho}\right) \text{ erg/gm-sec} \qquad \text{(IV.7)}$$

for zero anomalous moment.

V. CONCLUSIONS

In conclusion we will compare the energy dissipation rates of the various neutrino processes, and, although a detailed discussion of the astrophysical effects of these processes is beyond the scope of this paper, we will mention some of the implications of these reactions which have been investigated by others.

When comparing the dissipation rates, we will include three reactions for which we have not calculated the rates. We will use the results of S. G. Matinyan and N. N. Tsilosani (1961) for the process (I.7), G. M. Gandel'man and V. S. Pinaev's (1959) results for the bremsstrahlung reaction, and the results of G. Gamow (1941a) for the urca-process for three elements weighted according to estimated relative abundances.

The transition probabilities and energy dissipation rates for all but the urca-process are listed in Appendix D. These energy dissipation rates are plotted as functions of the temperature for various values of the density in the N. R. limit in Figures 12, 13, 14, 15, 16, 17. For these graphs we have taken the average (Z/A) to be $1/2$ and the average (Z^2/A) to be 6, corresponding to Mg^{24} which is of interest for the high density, low luminosity dwarf stars (Gandel'man 1959). Finally, in Figure 18 we indicate the ranges of density and temperature for which the various processes dominate. For the processes involving incoming electrons, we have used the pair produced density where it is important.

We see, approximately, that for $T_9 > .5$, the electron pair annihilation process dominates, while for $T_9 < .5$, the Compton

scattering process is most important.

The rate for the process $\gamma + \gamma \rightarrow \gamma + \nu + \bar{\nu}$ has not yet been calculated and it may be of importance. We can estimate it as $(M/m)^4 e^2 d$ times the $\gamma + \gamma \rightarrow \nu + \bar{\nu}$ rate. $(M/m)^4$ is included since we will have a term of $0(G)$; e^2 accounts for the added photon interaction; $d(\lesssim 1)$ stands for the effect of an additional particle in the final state. Estimating in this way we have $\mathcal{E} \sim 10^{14}$ erg/gm sec at $\rho = 1$, $T_9 = 3$. This is less than that due to electron pair annihilation which, having the same density dependence, thus dominates.

It has been shown by Gandel'man and Pinaev (1959) that for low photon luminosity conditions the neutrino luminosity can exceed the photon luminosity. They consider a stellar model with an isothermal core. If we let T_c and ρ_c be the core temperature and density, then

$$T = T_c \;, \quad \rho = \rho_c \qquad \text{for} \quad r < \xi R = 0.169\, R$$

$$T = T_c \left(\frac{R/r - 1}{1/\xi - 1} \right), \quad \rho = \rho_c \left(\frac{R/r - 1}{1/\xi - 1} \right)^{3.25} \qquad \text{for} \quad \xi R \leq r \leq R .$$

Using their values for the constants of the stellar model and (IV.4) for the energy dissipation rate in the $e^{\pm} \pm \gamma \rightarrow e^{\pm} + \nu + \bar{\nu}$ reaction, it can be found that

$$L_\gamma = 0.8 \times 10^{52}\, T_c^{8}\, \rho_c^{-2.5} \quad \text{erg/sec}$$

$$L_\nu = 0.4 \times 10^{43}\, T_c^{9.5}\, \rho_c^{-0.5} \quad \text{erg/sec}$$

$$L_\nu / L_\gamma = 5 \times 10^{-10}\, T_c^{1.5}\, \rho_c^{2}$$

where L_ν and L_γ are the neutrino and photon luminosities, respectively. ρ_c is in gm/cm^3 and T_c is in T_9 units. At high densities and temperatures, the neutrinos can take the lead. At $T_c = 1$ and $\rho_c = 1 \times 10^6$ the neutrino luminosity is 500 times the photon luminosity so that neutrino processes then control the evolution of the star.

Gamow and Schoenberg (1941a) and Chiu (1961b,c) have performed analyses of the collapse of the stellar atmosphere due to the rapid dissipation of energy by the neutrino processes once the temperature has risen high enough for them to become prominent. Fowler and Hoyle (1962) find that the neutrino processes or a similar energy dissipation mechanism is necessary to provide the proper time scale for the production of the elements around Fe^{54} in the observed proportions. On the other hand, the abundance of red supergiant stars in the region of h and χ Persei is found by Hayashi and Cameron (1962) to exceed that expected on the basis of a decrease in the lifetime of such stars due to the energy loss caused by neutrino emission. This situation has yet to be clarified.

Since all of these reactions, aside from the urca-process, depend upon the unobserved $(\bar{e}\nu)(\bar{\nu}e)$ coupling, a substantiation of its existence would be welcome. Observation of a charged uxl coupled to a leptonic weak current and the absence of cancelling neutral currents would lend weight to the existence of this and other non-cross terms in the $J^*_\mu J_\mu$ coupling.

APPENDIX A. Particle Propagators.

We will sketch the derivation of the momentum space propagators for the electron and uxl to obtain the correct phases. (For a more complete treatment and for the derivation of expressions used in this appendix, see Bogoliubov and Shirkov (1959).)

The propagator for a particle moving between points x and y is the vacuum expectation value of the time-ordered product of the particle's field operators. Thus, for an electron created by $\bar{\psi}$ or annihilated by ψ, we have

$$P = \langle T(\psi(y)\bar{\psi}(x))\rangle_0$$

$$= \begin{cases} \langle \psi(y)\bar{\psi}(x)\rangle_0 & x_0 < y_0 \\ -\langle \bar{\psi}(x)\psi(y)\rangle_0 & x_0 > y_0 \end{cases}$$

where $\langle\ \rangle_0$ indicates the vacuum expectation value. If we break up $\bar{\psi}$ and ψ into creation and annihilation operators, we have

$$\psi = \psi^{(+)} + \psi^{(-)}, \qquad \bar{\psi} = \bar{\psi}^{(+)} + \bar{\psi}^{(-)}$$

where $\psi^{(-)}$ annihilates electrons and $\psi^{(+)}$ creates positrons. Then since ${}_0\langle\psi^{(+)} \equiv \psi^{(-)}\rangle_0 \equiv {}_0\langle\bar{\psi}^{(+)} \equiv \bar{\psi}^{(-)}\rangle_0 \equiv 0$,

$$P = \begin{cases} \langle [\psi^{(-)}(y), \bar{\psi}^{(+)}(x)]_+\rangle_0 & x_0 < y_0 \\ \langle [\bar{\psi}^{(-)}(x), \psi^{(+)}(y)]_+\rangle_0 & x_0 > y_0 \end{cases}$$

where $[\ ,\ ]_+$ is the anticommutator. Since the anticommutators are c-numbers and the vacuum is normalized to 1, we find

$$P = \begin{cases} [\psi^{(-)}(y), \bar{\psi}^{(+)}(x)]_+ & x_0 < y_0 \\ -[\bar{\psi}^{(-)}(x), \psi^{(+)}(y)]_+ & x_0 > y_0 \end{cases}$$

The electron commutator is

$$[\psi^{(-)}(y), \bar{\psi}^{(+)}(x)]_+ = (i\not{\partial}_y + m)\frac{1}{i} D^{(-)}(y-x)$$

$$[\bar{\psi}^{(-)}(x), \psi^{(+)}(y)]_+ = (i\not{\partial}_y + m)\frac{1}{i} D^{(+)}(y-x)$$

where $D^{(\pm)}(x) = \frac{\mp i}{(2\pi)^4} \int e^{ikx}\, \theta(\pm k^0)\, \delta(k^2-m^2)\, d^4k$

$$\theta(x) = 0, 1 \quad \text{as} \quad x<0,\ x>0$$

and the Pauli Jordan function, $D(x) = D^{(+)}(x) + D^{(-)}(x)$, satisfies the Klein-Gordon equation: $(\Box - m^2)D(x) = 0$.

Then

$$P = \frac{1}{i}(i\not{\partial}_y + m)[\theta(y^0) D^{(-)}(y) - \theta(-y^0) D^{(+)}(y)]$$

$$= \frac{1}{i}(i\not{\partial}_y + m) D^{(c)}(y) = \frac{1}{i}(i\not{\partial}_y + m) \int \frac{e^{iky}\, d^4k}{(m^2-k^2-i\epsilon)(2\pi)^4}$$

where $D^c(y)$ is the causal Green's function of the Klein-Gordon equation. This Green's function allows an effect only inside the forward light-cone of a source. Then in the momentum representation

$$P(\text{electron}) = \frac{1}{i} \frac{(\not{k} + m)}{m^2 - k^2 - i\epsilon}$$

for an electron of momentum k or a positron of momentum -k.

Similarly the charged vector uxl commutator is

$$[U^{*(-)}_{\mu}(x),\, U^{(+)}_{\nu}(y)]_{-} = \left(g_{\mu\nu} + \frac{\partial x_\mu \partial x_\nu}{M^2}\right) i\, D^{(-)}(x-y)$$

or $$[U^{(-)}_{\mu}(y),\, U^{*(-)}_{\nu}(x)]_{-} = -\left(g_{\mu\nu} + \frac{\partial x_\mu \partial x_\nu}{M^2}\right) i\, D^{(+)}(x-y)$$

so that

$$P(u\times 1) = \langle T(U_\mu(y)\, U^{*}_{\nu}(x))\rangle_0$$

$$= \begin{cases} \langle U^{(-)}_{\mu}(y)\; U^{*(+)}_{\nu}(x)\rangle_0 & x_0 < y_0 \\ \langle U^{*(-)}_{\mu}(x)\; U^{(+)}_{\nu}(y)\rangle_0 & x_0 > y_0 \end{cases}$$

$$= [U^{(-)}_{\mu}(y),\, U^{*(+)}_{\nu}(x)]_{-}\theta(y_0 - x_0) + [U^{*(-)}_{\mu}(x),\, U^{(+)}_{\nu}(y)]_{-}\theta(x_0 - y_0)$$

$$= i\left(g_{\mu\nu} + \frac{\partial x_\mu \partial x_0}{M^2}\right) D^{c}(x-y)$$

$$= i\left(g_{\mu\nu} + \frac{\partial x_\mu \partial x_0}{M^2}\right) \int \frac{e^{ik(x-y)}\; d^4k}{(m^2 - k^2 - i\epsilon)\,(2\pi)^4}$$

or

$$P(u\times 1) = i\,\frac{(g_{\mu\nu} - k_\mu k_\nu / M^2)}{M^2 - k^2 - i\epsilon}$$

in the momentum representation.

APPENDIX B. Matrix Relations.

The γ matrices are defined by the commutation relation $\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu = 2g_{\mu\nu}$ where $g_{\mu\nu} = 0$ for $\mu\neq\nu$, $g_{\mu\nu} = 1, -1, -1, -1$ for $\mu = \nu = 0, 1, 2, 3$. We define $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$ and let $\epsilon_{\alpha\beta\lambda\delta}$ be the totally antisymmetric fourth-rank tensor. Taking any representation we find $Sp(\gamma_\mu) = 0$, $\mu = 0,1,2,3,5$. We let $\not{A} = A_\mu\gamma_\mu$. Then the following relations can be easily verified:

$$\gamma_\mu\gamma_\mu = 4 \quad , \qquad \gamma_\mu \not{A} \not{B} \gamma_\mu = 4(A\cdot B)$$

$$\gamma_\mu\not{A}\gamma_\mu = -2\not{A} \quad , \qquad \gamma_\mu \not{A} \not{B} \not{C} \gamma_\mu = -2 \not{C} \not{B} \not{A}$$

$$\gamma_\mu \not{A} \not{B} \not{C} \not{D} \gamma_\mu = 2\not{C} \not{B} \not{A} \not{D} + 2 \not{D} \not{A} \not{B} \not{C}$$

$$Sp(1) = 4 \quad , \qquad Sp(\gamma_\mu) = 0 \ , \ \mu = 0,1,2,3,5$$

Sp (odd number of γ matrices) = 0

$$Sp\ (\not{A} \not{B}) = 4(A\cdot B), \qquad Sp\ (\not{A} \not{B} \gamma_5) = 0$$

$$Sp\ (\not{A} \not{B} \not{C} \not{D}) = 4(A\cdot B)(C\cdot D) + 4(A\cdot D)\ (B\cdot C) - 4(A\cdot C)\ (B\cdot D)$$

$$Sp\ (\not{A} \not{B} \not{C} \not{D} \gamma_5) = - A_\alpha B_\beta C_\lambda D_\delta\ \epsilon_{\alpha\beta\lambda\delta}$$

$$Sp\ (\not{A}_1\not{A}_2\cdots\not{A}_n) = \sum_{i=1}^{n-1} (-1)^{i+1}\ (A_n\cdot A_i)\ Sp(\not{A}_1\cdots\not{A}_{i-1}\not{A}_{i+1}\cdots\not{A}_{n-1})$$

$$n \text{ even}, \quad n \geq 2 \ .$$

APPENDIX C. Program for Reduction of Matrix Elements.

Since, in perturbation field theory with a small coupling constant, the matrix elements of higher order processes become less significant, the fact that they quickly become prohibitively lengthy to reduce does not render the theory useless. There are, however, cases in which one desires to reduce fairly complicated expressions such as (III.16). It is the length and tediousness of these calculations that makes them difficult to do without error; the basic methods used are well-known. It may well be that this burden could be lifted by more powerful analytical methods. On the other hand, we can take advantage of the repetitive nature of the work and code the problem for reduction by a high-speed digital computer.(*)

A program has been written to reduce the matrix elements (III.16). The program is specific to this problem, but in general conception and in many particular aspects the methods could be applied to other problems. It should be emphasized that this is not a problem in numerical analysis. The variables are given code numbers since that is the kind of symbol a computer handles, but these numbers need have no ordinal significance. Real numbers enter as coefficients and integral exponents. The coefficients are treated as rational fractions with integral numerator and denominator. This means that no numerical inaccuracies are introduced.

First, we will describe the major steps involved in the calculation and then redescribe them in greater detail. Next, we will cover a

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(*) Mathews (1960) has written a program to do the dot products in some expressions arising in the quantum theory of gravitation. The method used was different from that used in the program under description.

few of the coding techniques which were used. Finally, some possible extensions of the techniques will be mentioned.

A. Description.

To evaluate the matrix elements of expression (III.16) to $0(m^4/M^4)$ one would:

1. Perform the dot products indicated between propagator numerators and the interaction terms.
2. Do the integration over the undetermined momentum.
   a. Introduce a cut-off factor for divergent terms.
   b. Produce a common denominator using Feynman parameterization.
   c. Do the momentum and parametric integrals.
3. Reduce the matrix combinations between the neutrino spinors.
4. Symmetrize the result with respect to the two photons.

The flow chart of Figure 19 describes the process in greater detail. Rather than collecting terms at each stage of the calculation, if an operation produces many terms from one, we usually set up a reservoir of terms and proceed on with just one of them until it is fully reduced. Then we return to the reservoirs to pick up and further process the remaining terms until all have been dealt with. An 'R' on the flow chart indicates that such a reservoir has been set up.

The following written description, which parallels the flow chart but is more detailed, is broken up into three parts, the pieces of which alternate. 'COM' will indicate commentary which describes the

calculation and derives any needed relations. These pieces follow each other and can be read separately. 'PRO' sections describe the work more from the program view. 'EX' labels an example of the work. Each of the diagrams was done separately. Those quantities which form data for a particular diagram are labeled 'INPUT'. Some sections of PRO which will be used more than once are given subroutine names (e.g. SORT) and the occurrence of that name at some other location indicates that that subroutine is also to be applied at that location.

* * *

COM. Let us treat one diagram at a time. For the time being we can ignore the $(p_i^2 - m_i^2)$ factors in all propagator denominators and consider just the propagator and interaction numerators in (III.16). Conservation of momentum allows us to replace q_2 in all expressions by $q_2 = k_1 + k_2 - q_1$.

PRO. Each of the propagator numerators or interaction expressions are INPUT as lists of terms. All of these lists together constitute R1. (A term consists of a numerical coefficient numerator, exponents of scalars (e.g. $1/M^2$, m, uxl anomalous moment), symbolized vectors (e.g. p_α, $q_{1\alpha}$, $k_{1\alpha}$, $k_{2\alpha}$, $e_{1\alpha}$, $e_{2\alpha}$, m_α, γ_α, $g_{\alpha\beta}$), invariants (e.g. p^2, $p\cdot q_1$, $p\cdot k_1$, $p\cdot k_2$, $p\cdot e_1$, $p\cdot e_2$, $k_1\cdot e_1$, $k_1\cdot e_2$, $k_2\cdot e_1$, $k_2\cdot e_2$, $e_1\cdot e_2$, $k_1\cdot k_2$, $k_1\cdot q_1$, $k_2\cdot q_1$), and matrix quantities (e.g. $\not{p}_1$, $\not{q}_1$, $\not{k}_1$, $\not{k}_2$, $\not{e}_1$, $\not{e}_2$, γ_μ). The neutrino spinors are not symbolized since they always appear at the beginning and end of the matrix quantities and we can simply act as if they were there. The matrix (a) is not symbolized since it always occurs as (au_ν), and the effects of the a's can be derived without their

explicit symbolization. For electron propagators, the electron mass is treated as both a vector and scalar at this time so that when doing dot products we may have: $m(m_\mu \gamma_\mu) \equiv m(1)$.)

EX. For M_D we obtain from expression (III.16) the following numerator lists:

$$[g_{\mu\alpha} - \tfrac{1}{M^2} p_\mu p_\alpha - \tfrac{1}{M^2} p_\mu k_{1\alpha} - \tfrac{1}{M^2} k_{1\mu} p_\alpha - \tfrac{1}{M^2} k_{1\mu} k_{1\alpha}]$$

$$[g_{\beta\nu} - \tfrac{1}{M^2} p_\beta p_\nu + \tfrac{1}{M^2} p_\beta k_{2\nu} + \tfrac{1}{M^2} k_{2\beta} p_\nu - \tfrac{1}{M^2} k_{2\beta} k_{2\nu}]$$

$$[e_{1\alpha} e_{1\beta} - (e_1 \cdot e_2) g_{\alpha\beta}]$$

$$[p_\rho + m m_\rho]$$

$$[\gamma_\nu \gamma_\rho \gamma_\mu] .$$

COM. In (III.19) we showed that the $(1/M^4)$ terms from adjacent uxl propagators at an uxl-single photon interaction cancel. We can therefore disregard such terms.

PRO. Decide on one term from each of the numerator lists in R 1. On each successive return to R 1, we take another combination of numerator terms until all are exhausted, and we then proceed to the symmetrization of the reduced terms (see page 77).

EX. Choose terms 2, 2, 1, 1, 1 in the five lists respectively.

PRO. (In the INPUT we list which pairs of numerator lists are uxl propagators from an uxl-photon-uxl interaction.) For each such pair of lists, we test to see if the combination of the terms chosen from these lists have a factor $(1/M^4)$. If there are any pairs which satisfy this condition, we return to R 1; if not, we continue.

EX. There are no such pairs here and so we continue.

COM. The properties of the matrices a and $\bar{a}$ were displayed in (II.4). From these we see that if A represents the product of an odd number of γ matrices then $aAa = A\bar{a}a = 0$; if A is the product of an even number of γ matrices then $aAa = Aa$. In all of the matrix combinations of (III.16) the A's which occur are the product of an even number of γ matrices or electron masses, and all electron masses occur in such an A. Thus the condition $\bar{a}a$ can occur if and only if we have an odd power of the electron mass. We therefore disregard terms with odd powers of the electron mass.

PRO. Take the terms decided upon and combine them. (When we combine terms, we form a new term by multiplying the coefficients, adding the exponents of the two terms, and placing their symbols next to each other in order.) Test the m exponent. If it is odd, return to R 1; if it is even, continue.

EX. The terms chosen yield

$$1(1/M^4)\, p_\mu p_\alpha p_\beta p_\nu e_{1\alpha} e_{2\beta} p_\rho \gamma_\nu \gamma_\rho \gamma_\mu \,.$$

The m exponent is zero, which is even, so we continue.

COM. With the exception of the two cases mentioned (i.e. $1/M^4, m^{2r+1}$), we multiply out all of the terms in the numerators performing the indicated four-vector dot products. We discard those terms in which any of the invariants formed is known to be zero. Since the γ matrices do not commute, we must preserve their order and can not, at this time, carry out the dot product implied by $\gamma_\mu \ldots \gamma_\mu$.

PRO. (SORT subroutine) Take the symbols and group them consecutively so that all matrices are to the right of all other symbols without changing the order of the matrix symbols.

(DOT PRODUCT subroutine) Take the leftmost symbol having a free vector index and find the symbol having the same index. Use the appropriate relation from the following set to reduce the pair:

$A_\alpha \ldots B_\alpha \rightarrow \ldots (A \cdot B)$ for A, B: nonmatrix vectors

$A_\alpha \ldots g_{\alpha\beta}$, $A_\alpha \ldots g_{\beta\alpha} \rightarrow \ldots A_\beta$ for A: nonmatrix vector

$g_{\alpha\beta} \ldots A_\alpha$, $g_{\beta\alpha} \ldots A_\alpha \rightarrow \ldots A_\beta$ for A: vector

$A_\alpha \ldots \gamma_\alpha \rightarrow \ldots \not{A}$ for A: nonmatrix vector

$\ldots g_{\alpha\alpha} \ldots \rightarrow 4 \ldots \ldots$

$m_\alpha \ldots \gamma_\alpha \rightarrow \ldots$

$\gamma_\alpha \ldots \gamma_\alpha$ unaltered.

When the first relation is used, test the resulting invariant. If it is any of $(q_1 \cdot q_1)$, $(k_1 \cdot k_1)$, $(k_2 \cdot k_2)$, $(q_1 \cdot e_1)$, or $(q_1 \cdot e_2)$, return to the preceding reservoir. If not, repeat this process until no further reduction is possible. (SORT).

EX. Doing the dot products and sorting we find the term:

$$1\ (1/M^4)\ (p \cdot e_1)(p \cdot e_2)\ \not{p}\not{p}\not{p} \ .$$

COM. Some matrix operations can be performed without permuting matrices. The Dirac equation for the neutrinos gives $\not{k}_1 u_{\nu 1} = 0$ and

$\bar{u}_{\nu 2} \not{k}_2 = 0$. The commutation relation for the γ matrices gives: $\not{A}\not{A} = A^2$, $\gamma_\mu \gamma_\mu = 4$. We use these relations where possible and discard terms which yield a zero.

PRO. (MATRIX ZEROES AND PAIRS subroutine) Test the first and last matrix symbols. If we have $\not{k}_2 \ldots$ or $\ldots \not{k}_1$, return to the preceding reservoir. Starting from the left, test each neighboring matrix pair. If we have $\ldots \not{A}\not{A} \ldots$, test to see if A is k_1, k_2, or q_1. If so, return to the preceding reservoir; if not, use $\ldots \not{A}\not{A} \ldots = \ldots A^2 \ldots$ or $\ldots \gamma_\mu \gamma_\mu \ldots = 4 \ldots \ldots$. (SORT). Repeat testing for pairs until none are found and then continue.

EX. We perform $\not{p}\not{p} = p^2$ to find the term:

$$1\ (1/M^4)\ (p \cdot e_1)(p \cdot e_2)(p^2)\ \not{p}\ .$$

COM. We now have to do the integral over the loop momentum, p, by creating a single denominator through Feynman parameterization, completing the square of that denominator, integrating over the momentum variable of that denominator, and finally integrating over the Feynman parameters. In case the p integral is primitively divergent for any term, we will multiply that term by a simple convergence factor of $-\Lambda^2/(p^2-\Lambda^2)$. This is justified since we have shown that all primitive divergences cancel. Since we desire to know $\mathcal{M}$ only to $0(m^4/M^4)$, we will discard terms which do not contribute to this order. Also, we will apply the gauge invariance condition described in the text.

First, let us derive some useful relations. The basic parameterization relation is:

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[(1-x)a + xb]^2} \quad .$$

Differentiating with respect to b yields:

$$\frac{1}{ab^n} = n \int_0^1 \frac{x^{n-1}\, dx}{[(1-x)a + xb]^{n+1}} \quad .$$

Combining these for n = 2 we have:

$$\frac{1}{abc} = 2 \int_0^1\int\int \frac{z\, dz\, dx}{[(1-z)a + z(1-x)b + xzc]^3} \quad .$$

In general, for n denominators each having an exponent β_i we find:

$$\frac{1}{a_1^{\beta_1} a_2^{\beta_2} \cdots a_n^{\beta_n}}$$

$$= \int_0^1 \cdots \int_0^1 \frac{(\beta-1)! \left\{ \prod_{i=1}^{n} [(1-x_i) \prod_{j=1}^{i-1} x_j]^{(\beta_i - 1)} \right\} \left\{ \prod_{i=1}^{n-1} (x_i^{n-i-1}\, dx_i) \right\}}{\left\{ \prod_{i=1}^{n} (\beta_i - 1)! \right\} L_0^{\beta}}$$

$$\beta = \Sigma \beta_i \quad ; \qquad \begin{aligned} L_0 &= (1-x_1)a_1 + x_1 L_1 \\ &\vdots \\ L_{i-1} &= (1-x_i)a_i + x_i L_i \\ &\vdots \\ L_{n-1} &= a_n \end{aligned} \quad . \qquad \text{(C.1)}$$

Integrals over the momentum in a closed loop are of the form:

$$I_n^r = \int \frac{p^{2r}\, d^4p}{(p^2 - L)^n} \quad .$$

The simplest convergent loop integral is I_3^o, which can be done as a contour integral, remembering that L has a small imaginary part (Feynman 1949). Thus

$$I_3^o = \int \frac{d^4p}{(p^2 - L + i\epsilon)^3} = \frac{\pi^2}{2iL} \quad .$$

There are some simple recursion relations which allow us to find all I_n^r from I_3^o. Differentiating with respect to L we obtain:

$$I_{n+1}^r = \frac{1}{n} \frac{d}{dL} I_n^r \quad .$$

Forming a factor of (p^2-L) in the numerator we find:

$$I^{r+1}_n = L\, I^r_n + I^r_{n-1} \ .$$

Combining these we find:

$$I^r_n = \frac{2}{(n-r-1)!}\left(1+\frac{L}{n}\frac{d}{dL}\right)\cdots\left(1+\frac{L}{n-r+1}\frac{d}{dL}\right)\left(\frac{d}{dL}\right)^{n-r-3} I^0_3 \ . \qquad (C.2)$$

Since the denominator (p^2-L) contains only invariants, if the numerator contained $p_\alpha p_\beta$, we might replace this by $(g_{\alpha\beta}/4)p^2$ since $g_{\alpha\beta}$ is the only tensor at our disposal. The 1/4 is a normalization factor: $g_{\alpha\beta}g_{\alpha\beta}/4 = 1$. If we let $G_r(\alpha_1 \ldots \alpha_{2r}) = G(r)$ represent the sum of all nonredundant terms of products of r factors of $g_{\alpha\beta}$ using 2r indices each once, we may write $G(r)p^{2r}/K(r)$ for $p_\alpha \ldots p_{\alpha_{2r}}$ where K(r), the normalization factor, is determined by:

$$(g_{\alpha_1\alpha_2}\cdots g_{\alpha_{2r-1}\alpha_{2r}})\, G(\alpha_1\cdots\alpha_{2r}) = K(r) \ . \qquad (C.3)$$

For given r, G(r) has $(2r)!/(2^r r!) = (2r-1)!!$ terms and $K(r) = 2(r-1)K(r-1) = 2^r(r-1)!$. Using relations (C.2) and (C.3) we finally have:

$$\int \frac{p^{2m}\, p_{\alpha_1}\cdots p_{\alpha_{2r}}\, d^4p}{(p^2-L)^n}$$

$$= \frac{\pi^2}{i}\, \frac{G(r)\,(-1)^{n-m-r-3}}{2^r\,(r+1)!}\, \frac{(n-m-r-3)!\,(m+r+1)!}{(n-1)!\,L^{n-m-r-2}} \ . \qquad (C.4)$$

In order to apply these relations, the uxl propagators will be expanded according to

$$\frac{1}{A-B} = \frac{1}{A} + \frac{1}{A}B\frac{1}{A} + \frac{1}{A}B\frac{1}{A}B\frac{1}{A} + \cdots$$

where $A \equiv (p^2-M^2)$. These expansions are then multiplied together to form a power series in $(p^2-M^2)^{-1}$.

PRO. The uxl denominator expansion is INPUT as a list of lists which then constitute R 2. (Each list is characterized by some value of the number of electron propagators for the diagram, A, some value for the $1/(p^2-M^2)$ exponent, B, and some value of the power of p in the numerator for each term of the list, RD. Neither the electron propagator denominator nor the $1/(p^2-M^2)$ factor are symbolized.)

EX. The denominator for M_D is

$$\frac{1}{(p^2-m^2)(p^2+2p\cdot k_1-M^2)(p^2-2p\cdot k_2-M^2)},$$

which can be broken down into the lists:

$$= \frac{1}{(p^2-m^2)}\Big\{ (p^2-M^2)^{-2}[1]$$
$$+ (p^2-M^2)^{-3}[-2(p\cdot k_1)+2(p\cdot k_2)]$$
$$+ (p^2-M^2)^{-4}[+4(p\cdot k_1)(p\cdot k_1)-4(p\cdot k_1)(p\cdot k_2)+4(p\cdot k_2)(p\cdot k_2)]$$
$$+ (p^2-M^2)^{-5}[-8(p\cdot k_1)(p\cdot k_1)(p\cdot k_1)+8(p\cdot k_1)(p\cdot k_1)(p\cdot k_2)$$
$$-8(p\cdot k_1)(p\cdot k_2)(p\cdot k_2)+8(p\cdot k_2)(p\cdot k_2)(p\cdot k_2)]$$
$$+ \cdots \Big\}.$$

For these lists we have $(A, B, RD) = (1,2,0), (1,3,1), (1,4,2), (1,5,3) \ldots$.

COM. Let us now consider a particular uxl denominator expansion term and a particular term of the numerator. The p integral is of the form

$$\int \frac{d^4p_1 \; p_1^{2r}}{\prod_{i=1}^{a}(p_1^2-2p_1\cdot k_i-2k_i^2-m^2)\prod_{j=1}^{b}(p_j^2-2p_j\cdot k_j-2k_j^2-M^2)}$$

and is convergent or divergent as $(2a+2b-2-2r)$ is >0 or ≤ 0. If the integral is divergent, we multiply by the convergence factor: $-\Lambda^2/(p^2-\Lambda^2)$.

If we have a convergent integral, we can tell if it has terms of $0(m^4/M^4)$. Consider the integral

$$I = \frac{1}{M^{2S}} \int \frac{p^{2R}\, d^4p}{(p^2-m^2)^A\,(p^2-M^2)^B} = 0\left[\left(\frac{m^2}{M^2}\right)^c\right] m^{2(R-A-B-S+2)} .$$

Using the parameterization formulas (C.1) we can write

$$I = \frac{K}{M^{2S}} \int_0^1 z^{A-1}(1-z)^{B-1}\, dz \int \frac{p^{2R}\, d^4p}{(p^2-L_1)^{A+B}}$$

where $L_1 = (1-z)M^2 - zm^2$, K denotes constant factors. Doing the p integral using expression (C.4) we find

$$I = \frac{K}{M^{2S}} \int \frac{z^{A-1}\,(1-z)^{B-1}\, dz}{L_1^{A+B-R-2}} .$$

If we let $L = (1-z) - z\Delta$, $\Delta = m^2/M^2$, then

$$I = \frac{K}{M^{2(S+A+B-R-2)}} \int_0^1 \frac{z^{A-1}\,(1-z)^{B-1}\, dz}{L^{A+B-R-2}} .$$

Substituting $x = L(z)$ and expanding z and (1-z) by the Binomial Theorem we have

$$I = \sum_i \sum_j \frac{K_{ij}}{M^{2(S+A+B-R-2)}} \frac{\Delta^j}{(1-\Delta)^{A+B-1}} \int_\Delta^1 dx\; x^{R-A+1+i-j}$$

where $0 \leq i \leq A-1$, $0 \leq j \leq B-1$. If we consider that $\ln(\Delta) = 0(1)$, then

$$I \simeq \sum_i \sum_j \frac{K_{ij}}{M^{2(S+A+B-R-2)}} \frac{[\Delta^j + K\,\Delta^{R-A+2+i}]}{(1-\Delta)^{A+B-1}} .$$

Since we want the dominant term as $\Delta \to 0$, we let $j = i = 0$ and see that

$$C = B + S \qquad \text{for} \quad A - R - 2 \geq 0$$
$$C = B + S + A - R - 2 \qquad A - R - 2 \leq 0 \ . \tag{C.5}$$

With this we can test convergent terms and discard those for which $C > 2$.

PRO. Find RN, the exponent of p in the numerator term. Let S equal the exponent of $1/M^2$. Consider the next one of the denominator lists. If none remain, then return to R 1. (For this list we have A, B, RD.) Compute the degree of divergence: DD = RN + RD + 4 - 2A - 2B. Compare DD with zero. If $DD \geq 0$, set the C/D indicator to DIV (divergent); if not, set C/D indicator to CONV (convergent).

If CONV, compute X = 2A-RN-RD-4. If X is odd, increase it by 1. (Odd X means that we have p^{2r+1} in the numerator so we effectively take the next term in the electron propagator denominator expansion by increasing the p exponent by 1 and the $1/(p^2-m^2)$ exponent by 1. If there is actually no next term in that expansion, we will have lost nothing.) If $X \geq 0$, compute $C = B + S$; if not, compute $C = B + S + X/2$. If $C > 2$, return to R 2; if not, continue.

If C/D is DIV, change the sign of the term coefficient. (Neither Λ^2 nor $(p^2-\Lambda^2)^{-1}$ are symbolized but are implied by the DIV setting.)

EX. We find the p power to be RN = 5, and the $(1/M^2)$ exponent to be S = 2. Suppose that the next list is the fifth denominator list in R 2, the list of lists. Then A = 1, B = 5, RD = 3, and we compute $DD = 0 \geq 0$ so that C/D is set to DIV. Since the integral is primitively divergent,

we do not test its relevance but instead change the sign so that we have

$$-1\ (1/M^4)\ (p\cdot e_1)(p\cdot e_2)(p^2)\ \not{p}\ .$$

COM. Since the electron propagator denominator parameterization need be done only once for each diagram, we do that manually. We combine the A electron propagators with A-1 parameters, (z,x), combine these with the $1/(p^2-M^2)$ factor using another parameter, u, and combine all this with the convergence factor denominator using the parameter w. Starting with the parameterizing formula (C.1) we find

$$\frac{1}{(p^2-\Lambda^2)^{(1,0)}(p^2-M^2)^B(p_1^2-m^2)\cdots}$$

$$= \int_0^1 \frac{[A+B+(0,1)]!}{(B-1)!}\ \frac{w^{B-1}\ (1-u)^{B-1}\ d\tau}{L_0^{A+B+(1,0)}}$$

where $d\tau = w\ dw\ du$, $w^2u\ dw\ du\ dz$, $w^3u^2z\ dw\ du\ dz\ dx$ for $A = 1, 2, 3$, respectively. Further:

$$L_0(A=3) = (1-w)(p^2-\Lambda^2) + w(1-u)(p^2-M^2) + wu(1-z)(p_1^2-m^2)$$

$$+ wuz(1-x)(p_2^2-m^2) + wuzx(p_3^2-m^2)$$

$$= p^2 - 2wu(p\cdot k) - L'.$$

For $A = 2$, let $x = 0$, and for $A = 1$, let $x = z = 0$ in L_0. If the integral is convergent, we simply set $w = 1$. L' and k_μ are functions of the system momenta and invariants. k_μ depends on x and z while L' depends on w, u, z and x. We will be able to write

$$L' = (1-w)\Lambda^2 + w(1-u)M^2 + wuL(x,z)$$

so that the w and u dependence of the parameterization is known for all of the diagrams. Only the electron parameterization varies. We further define:

$$M^2\epsilon = L(x,z) \quad , \quad \Lambda^2\epsilon' = (1-u)M^2 + uL \quad , \quad \Delta = \delta + \epsilon \, ,$$

$$\Lambda^2\delta'/u^2 = M^2\delta = k^2 \quad , \quad \Delta' = \delta' + \epsilon' \, . \tag{C.6}$$

EX. We do, here, the parameterization for M_D. There is only one electron propagator. We find

$$L_o = (1-w)(p^2-\Lambda^2)+w(1-u)(p^2-M^2)+wu(p^2-m^2)$$

$$= p^2-(1-w)\Lambda^2-w(1-u)M^2-wum^2$$

so that $k = 0$; $L(x,z) = m^2$; $\delta' = \delta = 0$; $\epsilon = \Delta = m^2/M^2$; $\epsilon' = \Delta' = (1-u)M^2/\Lambda^2 - um^2/\Lambda^2$.

PRO. Set CD = 1, 0 as C/D is DIV or CONV. Compute the integer $(A+B+CD-1)!/(B-1)!$ and multiply the coefficient of the term by it. Multiply the term by $w^{B-1}(1-u)^{B-1}$. Multiply the term by w, w^2u, w^3u^2z as $A = 1, 2, 3$. Set up a denominator exponent: $DN = A+B+CD$. (Only the exponents of $w, u, z, x, 1-u$ will be carried.)

EX. For our term we set CD = 1 and multiply the term by $30w^5(1-u)^4$ to give

$$-30\,(1/M^4)\,w^5(1-u)^4\,(p\cdot e_1)(p\cdot e_2)(p^2)\,\not{p} \, .$$

For the denominator exponent we have DN = 7.

PRO. Set up a reservoir, R 3, of the terms of the denominator list chosen from the list of lists. Take a term from R 3, unless all have been used; in which case, return to R 2. Combine that denominator term from R 3 with the numerator term.

EX. Take the second term of the list to give

$$-240\ (1/M^4)\ w^5(1-u)^4(p\cdot e_1)(p\cdot e_2)(p^2)\not{p}\ (p\cdot k_1)(p\cdot k_1)(p\cdot k_2)\ .$$

COM. The denominator now has the form $(p^2-2uwp\cdot k-L')$. To use the formula (C.4) for loop integrals we want the denominator in the form (p'^2-L''). Substituting $p_\mu = p_{1\mu} + uwk_\mu$ accomplishes this. Then $L'' = L'+w^2u^2k^2$. This substitution must also be made in the numerator.

PRO. (To facilitate further manipulation we want p_α to enter the expression in only one way.) Test each symbol in turn starting from the left. If it contains a $(p\cdot A)$, use the relation

$$\ldots(p\cdot A)\ldots \rightarrow \ldots A_\alpha\ldots p_\alpha\ ,$$

where A_α is any vector. Repeat this until p enters only as p_α.

EX. The term becomes

$$-240\ (1/M^4)w^5(1-u)^4e_{1\alpha}e_{2\beta}p_\lambda\gamma_\delta k_{1\mu}k_{1\gamma}k_{2\rho}p_\alpha p_\beta p_\lambda p_\delta p_\mu p_\nu p_\rho\ .$$

PRO. Starting from the left, test each symbol to see if it is a p_α. If it is, produce two terms using $p_\alpha \rightarrow p_{1\alpha} + uwk_\alpha$. Place the first in R 4 and repeat the search procedure until the term no longer

contains p_α. On subsequent returns to R 4, take another term from R 4 and repeat the procedure. When R 4 is empty, return to R 3.

EX. Since $k = 0$ in the example, we get simply

$$-240(1/M^4)w^5(1-u)^4 e_{1\alpha}e_{2\beta}p_1 \gamma_\delta k_{1\mu}k_{1\nu}k_{2\rho}p_{1\alpha}p_{1\lambda}p_{1\beta}p_{1\delta}p_{1\mu}p_{1\nu}p_{1\rho}.$$

COM. We can now apply formula (C. 4) to effect the integral over p_1.

PRO. Starting from the left, test each symbol to see if it is a p_1. If it is, find the index mate and test for $p_1 \ldots p_1$. If this is the case, use $p_{1\alpha}\cdots p_{1\alpha} = p_1^2$; if not, delete the p_1 but save the index. (This index is for later use in producing G(r).) Continue until there are no more $p_{1\alpha}$ symbols. Count the number of p_1^2's, RPS, and the number of $p_{1\alpha}$'s, RPA. If RPA is odd, return to R 4; if not, set RP = RPA/2. Compute Z = DN-RPS-RP-3, N = (RPS+RP+1)!/(RP+1)!, and DR = 2^{RP}(DN-1)!/Z!. Multiply the term coefficient by N, set up a term coefficient denominator of DR, and change the sign of the numerator if Z is odd. Set up a new denominator exponent DE = DN-RPS-RP-2.

EX. We find RPS = 1, RPA = 6(even), RP = 3, Z = 0(even), N = 5, DR = 5760, DE = 1. The indices saved are $(\alpha, \beta, \delta, \mu, \nu, \rho)$. We have the term:

$$-(1200/5760)(1/M^4)w^5(1-u)^4 e_{1\alpha}e_{2\beta}\gamma_\delta k_{1\mu}k_{1\nu}k_{2\rho}.$$

COM. The integration over the parameters w, u, z, x remains to be done. The w and u integrals will be of the form:

F(s,n,t,m,r) ≡ the 0(r) terms of I(s,n,t,m) where

$$I(s,n,t,m) \equiv \frac{1}{K^{2s}} \int_0^1 \frac{\xi^n (1-\xi)^t \, d\xi}{[(1-\xi)K^2 + \xi E + \xi^2 D]^m} \qquad m \geq 0$$

$$I(s,n,t,-1) \equiv \frac{1}{K^{2s}} \int_0^1 \xi^n (1-\xi)^t \, d\xi \, \ln[(1-\xi) + \xi E/K^2 + \xi^2 D/K^2].$$

The order is with respect to the parameters $\epsilon \equiv E/K^2$ or $\delta \equiv D/K^2$, which are much less than 1. For $m \equiv -1$ we will use the logarithm form. If we let $z = [(1-\xi) + \epsilon\xi + \delta\xi^2]$ and expand the integrand of I in a series, we find:

$$I(s,n,t,m) = \frac{1}{K^{2(s+m)}} \int_\Delta^1 \frac{(1-z)^n z^t}{z^m} \left\{ -t\epsilon \left(\frac{1-z}{z}\right) - t\delta \frac{(1-z)^2}{z} \right.$$

$$\left. + 1 + (n+1)\epsilon + (n+2)(1-z)\delta + \cdots \right\} \qquad m \geq 0$$

$$\text{or} \quad I(s,n,t,-1) = \frac{1}{K^{2s}} \int_\Delta^1 (1-z)^n z^t \ln z \, \{ \cdots \} \qquad \text{(C.7)}$$

where $\Delta \equiv \epsilon + \delta$. These terms are sufficient for our purposes.

This then requires the integrals:

$$J(a,b,c) = \int_\Delta^1 \frac{(1-z)^a z^b}{z^c} \qquad c \geq 0 \; ; \; a,b \geq 0$$

$$J(a,b,-1) = \int_\Delta^1 (1-z)^a z^b \ln z \qquad a,b \geq 0$$

where $0 \ll \Delta \ll 1$. J(a,b,c,r) is the 0(r) term of J(a,b,c). We find for $c \geq 0$, $r \equiv 0$:

$$J(a,b,c,0) = \sum_{i=0}^{a} \frac{a!\,(-1)^i}{(a-i)!\,i!\,(b-c+i+1)} \quad ; \quad b-c+i+1 \neq 0$$

$$- \frac{(-1)^{c-b-1}\, a!\, \ln \Delta}{(a+b-c+1)!\,(c-b-1)!} \qquad \text{if} \quad 0 \leq c-b-1 \leq a$$

for $c \geq 0$, $r \neq 0$, $b-c+1 < r < a+b-c+1$:

$$J(a,b,c,r) = \frac{-(-1)^{r+c-b-1}\, a!\, \Delta^r}{(a+b-r-c+1)!\,(r+c-b-1)!}$$

for $c \geq 0$, and all other r, $J(a,b,c,r) = 0$.

for $c = -1$, $r = 0$:

$$J(a,b,-1,0) = -\sum_{i=0}^{a} \frac{(-1)^i\, a!}{(a-i)!\,i!\,(b+i+1)^2}$$

for $c = -1$, $r \neq 0$, $b+1 \leq r \leq a+b+1$:

$$J(a,b,-1,r) = \frac{a!\, \Delta^r\, (-1)^{r-b-1}}{r^2\,(a+b-r+1)!\,(r-b-1)!}$$

$$- \frac{a!\, \Delta^r \ln \Delta\, (-1)^{r-b-1}}{r\,(a+b-r+1)!\,(r-b-1)!}$$

for $c = -1$, all other r: $J(a,b,-1,r) = 0$.

In the w integral, $K^2 = \Lambda^2$. Then $s = -1$, since Λ^2 only arises from the cutoff factor. Since we want the cutoff-independent term, let $r = 0$. For ϵ, δ, Δ we use $\epsilon', \delta', \Delta'$ as defined in (C.6). Since no (1-w) factors appear, $t = 0$, and since the p_1 integral always leaves a denominator, we have $m > 0$. The resultant terms of $F(-1,n,0,m,0)$ will contain Δ' in the denominator or else a factor $\ln \Delta'$. From its definition we have $\Lambda^2\Delta'/M^2 = (1-u)+u\epsilon+u^2\delta$. If it occurs, the $\ln(\Lambda^2/M^2)$ term is dropped since we know that M is

cutoff independent. If no w integral is performed, the denominator is already of the form: $M^2\Delta'$. In either case we again need the I(s,n,t,m) integrals.

For the u integral $K^2 = M^2$ and ϵ, δ, Δ are as in (C.6). We set $r = 2$ since we only want the $0(m^4/M^4)$ terms. The resulting terms will have coefficients, numerator factors of ϵ, δ, Δ and perhaps a factor of $\ln\Delta$ or a power of Δ in the denominator.

PRO. (Because it would be too involved, and because there is no symbol manipulation involved, we will not give a detailed description of the evaluation of these integrals. We determine the parameters for F(s,n,1,m,r). Then we use each term of the series (C.7) in turn and calculate the J(a,b,c,r) needed. These terms are collected. There will be at most two terms in each case. These terms constitute R 5 for the w integral and R 6 for the u integral.)

EX. For our term, for the w integral we need F(-1,5,0,1,0), which has the two terms: $-\ln\Delta - 137/60$. We take the $\ln\Delta$ piece which, together with the $(1-u)^4/M^4$, specifies $F(2,0,4,-1,2) = -1/25$ for the u integral. Our term yields

$$-(1200/144000)(1/M^4)\, e_{1\alpha} e_{2\beta} \gamma_\delta\, k_{1\mu} k_{1\nu} k_{2\rho}\,.$$

COM. We can now apply the criterion based on the requirement of gauge invariance: the number of momentum factors in the numerator must be 3 or greater. If the denominator contains momenta, we will keep the term since there may be gauge invariant terms in the expansion of the denominator. This allows us to drop many terms and is

essential to the convergence of the matrix element.

PRO. (As INPUT we have the maximum number of momentum factors in the terms of $\epsilon, \delta, \Delta, k$. The vectors k_1, k_2, and q_1 have one power of momentum. As INPUT we have a momentum power of 3 or 0 for the $1/\Delta$ and $\ln\Delta$ factors, depending on whether or not they contain any momenta.) Take the numerator exponents of ϵ, δ, Δ obtained from the parametric integrals and multiply each by its respective momentum power. Sum these numbers. Test each symbol and add its momentum power to the sum. If there is a $1/\Delta$ or $\ln\Delta$, add the appropriate momentum power to the sum. If this sum is less than 3, return to R 6 for the next term; if not, we continue.

EX. The momentum values of $\epsilon, \delta, \Delta, 1/\Delta$ and $\ln\Delta$ are all zero. None of these factors occur but the vectors yield a momentum power of 3 so that we may continue.

COM. In doing the x and z integrals we must remember that $\epsilon, \delta, \Delta, k$ may depend on these parameters. After substituting for $\epsilon, \delta, \Delta, k$ we find that for $A = 3$ the integrals are complicated by the appearance of $1/\Delta$ and $\ln\Delta$ with $\Delta = [1-2z(1-z)(1-x)(k_1 \cdot k_2)/M^2]$. For $A = 2$ there are only the integrals:

$$\int_0^1 x^n \, dx = \frac{1}{n+1}, \qquad n \neq -1 .$$

For $A = 1$ there is no x or z integral. Accordingly, for $A = 3$ we will postpone the x, z integrations until after we have collected all of the final terms, but we will now do the integrals required for $A = 2, 1$.

PRO. Take the exponents of ϵ, k, Δ in the numerator and delete them after placing the appropriate number of symbols for each of them.

(Since we will substitute for these factors, they must be symbolized. As INPUT we have ϵ, δ, Δ, and k in terms of coefficients, invariants, vectors, and the parameters.) Starting from the left, test each symbol of the term. If it is one of $\epsilon, \delta, \Delta, k$, substitute for it thereby producing many terms. All terms produced excepting one are put into R 7. Continue with this term until it has no further substitutable factors. On subsequent returns to R 7, take a term and repeat the substitution procedure on it. If there are no more terms, return to R 6.

EX. In this case, ϵ, δ, Δ, k do not appear.

PRO. If $A = 1, 3$, ignore the x, z exponents; if $A = 2$, delete the z exponent, n, after multiplying the coefficient denominator by $(n+1)$.

EX. Here, $A = 1$ so that nothing need be done.

PRO. Take the indices saved from the p_1's and create a reservoir, R 8, each term of which is the combination of the original term with one of the terms of G(r). (These terms can be generated using the recursion relation

$$G(\alpha_1 \cdots \alpha_{2r}) = \sum_{i=1}^{2r-1} g_{\alpha_{2r}\alpha_i} G(\alpha_1 \cdots \alpha_{i-1}\alpha_{i+1} \cdots \alpha_{2r-1})$$

$$G(\alpha\beta) = g_{\alpha\beta} \qquad .)$$

EX. We have the indices $(\alpha, \beta, \delta, \mu, \nu, \rho)$. There are 15 terms to G(3). We take one term to give the term:

$$-(1200/144000)(1/M^4)e_{1\alpha}e_{2\beta}\gamma_\delta k_{1\mu}k_{1\nu}k_{2\rho}g_{\alpha\delta}g_{\beta\mu}g_{\nu\rho} \cdot$$

PRO. Do (DOT PRODUCTS) and (SORT). If there is a zero dot product, return to R 8.

EX. Our term, with all integrals done, becomes

$$-(1200/144000)(1/M^4)(e_2\cdot k_1)(k_1\cdot k_2)\,\not{e}_2 \ .$$

COM. With the integrations complete (except for x, z if $A = 3$), we turn to simplifying the matrix expressions. We might have any combination of an odd number of matrices between the neutrino spinors. By permuting these matrix quantities using the γ matrix commutation relation and using the neutrino Dirac equation, we can reduce any term to a combination of the forms: $(\not{q}_1)$, $(\not{e}_1)$, $(\not{e}_2)$ and $(\not{e}_1\not{e}_2\not{q}_1)$. Using spur techniques, one can verify that

$$|-(\not{e}_1\not{e}_2\not{q}_1)+(k_1\cdot e_1)(\not{e}_2)+(k_2\cdot e_2)(\not{e}_1)+(e_1\cdot e_2)(\not{q}_1)|^2=0 \ .$$

We can use this to eliminate $(\not{e}_1\not{e}_2\not{q}_1)$. If we consider the polarizations in combination with the matrix quantities, we have the basic forms: $T_1T_2(\not{q}_1)$, $(e_1\cdot e_2)(\not{q}_1)$, $T_1(\not{e}_2)$ and $T_2(\not{e}_1)$. T_1 and T_2 are defined in (III.20). These forms will be multiplied by combinations of invariants and perhaps a factor of $1/\Delta$ or $\ln\Delta$. If we collect all these terms, we have the reduced but non-symmetrized matrix element.

PRO. Set up a reservoir, R 9, and place this term in it. Take a term from R 9, unless there are none; in which case, return to R 8. Apply (MATRIX ZEROES AND PAIRS). If a zero is produced, return to R 9; if not, continue. Do the 7 tests listed below, in sequence. If any of the conditions is satisfied, then follow the procedure given and skip further tests:

1. If there is a γ_μ, move the leftmost one one place to the right, producing 2 terms. For a neighbor of γ_ν, use:

$$(\cdots \gamma_\mu \gamma_\nu \cdots \gamma_\mu \cdots) \rightarrow -(\cdots \gamma_\nu \gamma_\mu \cdots \gamma_\mu \cdots) + 2(\cdots \quad \cdots \gamma_\nu \cdots) .$$

For any other neighbor, use

$$(\cdots \gamma_\mu \not{A} \cdots \gamma_\mu \cdots) \rightarrow -(\cdots \not{A} \gamma_\mu \cdots \gamma_\mu \cdots) + 2(\cdots \quad \cdots \not{A} \cdots)$$

where $A \equiv k_2$, k_1, q_1, e_1, or e_2. (To do this, the mate must be located. To prevent the situation $\gamma_\mu \gamma_\nu \rightarrow \gamma_\nu \gamma_\mu \rightarrow \gamma_\mu \gamma_\nu \ldots$ (ad infinitum), we must look separately for a γ_μ with each index and deal with $\gamma_{\alpha 2}$ only after $\gamma_{\alpha 1}$ is no longer in the expression.)

2. If there is a $\not{k}_2$, move the rightmost one one place to the left, producing two terms:

$$(\ldots \not{A} \not{k}_2 \ldots) \rightarrow -(\ldots \not{k}_2 \not{A} \ldots) + 2(\ldots (k_2 \cdot A) \ldots)$$

where $A \equiv k_1$, q_1, e_1, or e_2.

3. If there is a $\not{k}_1$, move the leftmost one one place to the right, producing two terms:

$$(\ldots \not{k}_1 \not{A} \ldots) \rightarrow -(\ldots \not{A} \not{k}_1 \ldots) + 2(\ldots (k_1 \cdot A) \ldots)$$

where $A \equiv q_1$, e_1, or e_2.

4. If there is a $\not{q}_1$, which is not at the extreme right, move the leftmost one one place to the right, producing one term:

$$(\ldots \not{q}_1 \not{A} \ldots) \rightarrow -(\ldots \not{A} \not{q}_1 \ldots) \qquad A \equiv e_1, \text{ or } e_2 .$$

5. If there is an $\not{e}_2$ which is not either in the combination $\not{e}_2 \not{q}_1$ or at the extreme right, move it one place to the right, producing 2 terms:

$$(\ldots \not{e}_2 \not{A} \ldots) \rightarrow -(\ldots \not{A} \not{e}_2 \ldots) + 2(\ldots (e_2 \cdot A) \ldots), \quad A = e_1$$

6. If we have the combination $\not{e}_1\not{e}_2\not{q}_1$, produce three terms:

$$(\ldots\not{e}_1\not{e}_2\not{q}_1\ldots) \rightarrow (\ldots(e_1\cdot k_1)\not{e}_2\ldots)+(\ldots(e_2\cdot k_2)\not{e}_1\ldots)$$
$$+(\ldots(e_1\cdot e_2)\not{q}_1\ldots) .$$

7. If the invariant $(k_1\cdot k_2)$ is present, take the leftmost one and produce two terms:

$$(\ldots(k_1\cdot k_2)\ldots)\rightarrow(\ldots(k_1\cdot q_1)\ldots)+(\ldots(k_2\cdot q_1)\ldots) .$$

If any of these operations were carried out, take the terms produced, (SORT) each, and place them in R 9, and then return to R 9; if not, continue.

EX. We make the substitution $(k_1\cdot k_2) = (k_1\cdot q_1)+(k_2\cdot q_1)$ and take, here, the first term to give

$$-(1200/144000)(1/M^4)(e_2\cdot k_1)(k_1\cdot q_1)\not{e}_1 .$$

PRO. Delete the matrix factor symbol but keep an 'exponent' to denote which it was (i.e. $\not{e}_1$, $\not{e}_2$, $\not{q}_1$). Delete the polarization symbols but keep an 'exponent' to denote which they were (i.e. $T_1, T_2, T_1T_2, e_1\cdot e_2$), and change the sign of the coefficient once for each factor of $(k_1\cdot e_2)$ or $(k_2\cdot e_1)$. Delete the $(k_1\cdot q_1)$ and $(k_2\cdot q_1)$ symbols but keep an exponent for each. Reduce the coefficient to lowest terms, and change the signs of the numerator and denominator if the latter was negative.

EX. Reducing the coefficient we have

$$(1/120)(1/M^2)^2\ (T_1)^1\ (f)^1\ (\not{e}_1)$$

PRO. (R 10 stores the fully reduced terms. We keep the cumulative coefficient and the various exponents for each possible type of term. There is a common denominator, LCD, for R 10, and each time a term is to be added to R 10, this common denominator and the coefficient of the term as well as all coefficients in R 10 are adjusted, if necessary, to create a new LCD for the term and the contents of R 10. Then the coefficient numerator of the term is added to the accumulated numerator already in R 10 for that type of term.) Let the term coefficient be N/D. Reduce D/LCD to lowest terms: D'/LCD'. Unless $D' = 1$, multiply LCD and all coefficients in R 10 by D'. Multiply N by LCD' and add the product to the appropriate sum in R 10. Return to R 9.

EX. Assume that there were no previous terms so that LCD = 1. We then set LCD = 120 and add 1 to the appropriate total for terms of the form: $(1/M^2)^2\ T_1\ f\ \not{e}_1$.

COM. To symmetrize the expression we create a second expression with the substitutions $e_1 \longleftrightarrow e_2$, $q_1 \longleftrightarrow q_2$ and add the two together. We then have a fully reduced and symmetrized matrix element for one diagram.

PRO. (We get here after exhausting all terms in R 1.) Create a reservoir R 11 and copy all nonzero terms of R 10 into R 11. Take each term from R 11 and change exponents according to the scheme:

$$b \longleftrightarrow f,\quad T_1 \longleftrightarrow T_2,\quad \not{e}_1 \longleftrightarrow \not{e}_2\ .$$

If the total number of $\not{e}_1$'s, T_1's and T_2's is odd, change the sign of the coefficient. Add the term to R 10 and return to R 11 until all terms in R 11 have been treated.

EX. If the only term in R10 were the one we were using as an example, we would add to it: $-(1/M^2)^2\ T_2\ b\ (\not{e}_2)$.

PRO. (The English symbols for the variables are built into the program.) Take each term from R10. Reduce the accumulated coefficient and LCD to lowest terms. Print the coefficient as a fraction and the letter equivalents for the variables. When all terms in R10 have been printed, we are through.

EX. If LCD = 36 and the term were

$$-24\ R^2\ f\ T_1\ (\not{e}_1), \qquad \text{(R is the uxl anomalous moment)}$$

we would get a printout of

```
-   2/   3    R**2     F     T1      (E2)
```

B. Coding

Since this work was done on an IBM 7090, the symbols were coded as 36 bit binary or 12 digit octal words. The symbol numbers were chosen so that the code for an invariant is the algebraic absolute sum of the codes for the constituent vectors. For example, if we denote "code for p" by C(p), we set:

$C(p) = 100\ 000\ 000\ 000$ (octal)

$C(k_1) = 010\ 010\ 000\ 000$

$C(p \cdot k_1) = 110\ 010\ 000\ 000 = C(p) + C(k_1)$.

A vector index is a number between 1 and 15 (dec) in the first 4

bits of the word. E.g.

$$C(p_\alpha) = 100\ 000\ 000\ 003 \qquad \alpha = 3$$

$$C(k_{1\beta}) = 010\ 010\ 000\ 006 \qquad \beta = 6\ .$$

A kronecker delta, or more precisely the metric tensor $g_{\alpha\beta}$, is coded in two words, each having two indices. The first index, in the first 4 bits, has one of the indices of the $g_{\alpha\beta}$. The second index, in the next 4 bits, is a number between 1 and 15, which is the same for the two halves of a $g_{\alpha\beta}$ and serves to identify them as belonging to the same $g_{\alpha\beta}$. E. g.

$$C(g_{\alpha\beta}) = \begin{matrix} 000\ 000\ 000\ 023 \\ 000\ 000\ 000\ 026 \end{matrix} \qquad \alpha = 3,\ \beta = 6$$

where 1 is the second index. (The 1 appears as a 2 in the second digit since the octal numbers specify 3 bits, while each index is alloted 4 bits.)

A gamma matrix without indices is $C(\gamma) = 000\ 000\ 010\ 000$. It has two index locations in the first and third 4 bits. The first index is used in dot products, while the second is used to identify $\gamma_\mu \ldots \gamma_\mu$ pairs in matrix reductions. The second index is always present. We have

$$C(\gamma_\alpha) = 000\ 000\ 011\ 002 \qquad \alpha = 2$$

$$\text{or} = 000\ 000\ 011\ 000\ .$$

Slashed quantities are coded as $\gamma \cdot A = \not{A}$ with the addition rule for dot products holding. Thus

$C(\not{p}) = 100\ 000\ 010\ 000 = C(p) + C(\gamma)$.

The coefficient is kept as a signed integral numerator and signed integral denominator. Exponents are kept several to a word sometimes signed and sometimes unsigned.

Thus a sample term might be

$$(1/2)(1/M^4)\ (1-u)^5 (p\cdot k_1)\, g_{\alpha\beta}\, p_\alpha \not{e}_1 \not{e}_2 \gamma_\beta$$

which would be coded:

-000 000 000 001	coefficient numerator
000 050 000 002	exponent word: (1-u) exp. and $(1/M^2)$ exp
110 010 000 000	$(p\cdot k_1)$
000 000 000 021	half of $g_{\alpha\beta}$ (α=1)
100 000 000 001	p_α
000 000 000 023	half of $g_{\alpha\beta}$ (β = 3)
001 001 010 000	$\not{e}_1$
001 002 010 000	$\not{e}_2$
000 000 011 403	γ_β
⋮	
000 000 000 002	coefficient denominator.

A table of symbols is kept and any symbol which may occur can be identified by comparison with the elements of the table.

The program was written in FAP (Fortran Assembly Program), involved about 4000 commands, used 20,000 words of storage, and executed for about 8 minutes per diagram. The input data for each diagram

was about 150 words.

Numerous internal checks were included so that faulty data was soon rejected. When terms of an expansion were used, it was required either that the last term be of higher order than that needed or that the last term be of no lower order than that needed. Since, prior to the writing of the program, part of the calculation had been done by hand, that part served as a check on the program. Aside from the possibility of using incorrect formulas, the mistakes introduced by an incorrect program should be quite different from those introduced by random human error so that the check is fairly good.

C. Generalization

One would like to be able to do a wider range of similar problems by computer techniques. The difficulty lies not in the impossibility of writing appropriate programs but in the time required to write them. FORTRAN or any of the other programming programs provides an easy way to make computer programs for arithmetic problems. Provided that the appropriate subroutines are available, one need only write out an arithmetic expression and FORTRAN codes a program to evaluate it. It is not hard to conceive of a similar method for use with symbols and real-number fractions. However, the writing of FORTRAN was a huge task and one would guess that writing a symbolic programming program (*)

(*) One such program is called LISP (Woodward 1961). In LISP information is stored in lists of elements, an element being a single symbol or another list. The LISP language consists of a set of basic operations which can be performed on a list and which can be compounded to accomplish any sort of operation on the lists. The author has not thoroughly investigated the feasibility of using LISP type languages in the type of problem discussed here.

would also be a large task. There is also the problem that generalized methods accomplish things less economically in terms of machine time.

Another approach is the development of a library of subroutines which could be grouped together to do part or all of a problem. Many of the parts of the program we have described are of, or could be generalized to be of, this nature. The method of indexing used for four-vector dot products would serve in general, although the rule $C(A)+C(B)=C(A\cdot B)$ might be replaced by a tabulated function. This would ease the symbol coding restriction imposed by the addition rule.

All of the operations in the matrix permutation procedure could be embodied in the rule: "If there is a ___ (some matrix quantity), move it one place to the ___ (right or left) unless it would pass ___ (some matrix quantity)." The programmer would have to figure out which are the simplest basic forms and in what order to move the matrix quantities. The flexibility might be increased by symbolizing a, $\bar{a}$, and the spinors. It would also certainly be desirable to be able to reduce spurs of matrices in addition to matrix quantities between spinors. The matrix relations $\not{A}\not{A} = A^2$, $\gamma_\mu\gamma_\mu = 4$ are common to all such problems.

A major difficulty is presented by the integrations. The momentum integrals are of a common form but this is not true of the parametric integrals. Without the relation $\Lambda^2 >> M^2 >> m^2$, the entire method of doing these integrals would be different. The only recourse may be to build up a collection of subroutines for exact rational functions and use those needed after working out the general integral formulas, for the particular problem, by hand.

Finally, we might write a general substitution subroutine for replacing one symbol by others, and we might try to devise a general scheme for collecting terms in a way that would be economical with machine storage space.

An easy method of coding these types of problems for machine reduction would certainly find use in computing the matrix elements for higher-order processes.

APPENDIX D. Tabulation of the Cross Sections and Energy Dissipation Rates.

For each process we list the transition probability and the rate at which energy is carried away by the neutrinos in the N. R. and non-degenerate electron gas limits. Each is first in units with $\hbar = c = k = 1$. Masses are in energy units. Energies may be in units of m_e.

1. $e^+ + e^- \rightarrow \nu + \bar{\nu}$ (III.4), (IV.2)

$$\sigma v = \frac{G^2}{3\pi}(E_T^2 - m^2) \; ; \quad \sigma = 1.4 \times 10^{-45}(E_T^2 - 1)\left(\frac{c}{v}\right) \text{ cm}^2$$

E_T = total CM energy

$$\mathcal{E} = \frac{G^2 m^5}{\pi^4} T^3 e^{-\frac{2m}{T}}$$

$$= 0.5 \times 10^{19} \frac{T_9^3}{\rho} \exp(-11.9/T_9) \text{ erg/gm-sec.}$$

2. $e^{\pm} + \gamma \rightarrow e^{\pm} + \nu + \bar{\nu}$ (III.8) (IV.4)

$$\sigma v = \frac{4}{35\pi^2} \frac{e^2 G^2 \omega^4}{m^2} \; ; \quad \sigma = 7.2 \times 10^{-50} E_T^4 \left(\frac{c}{v}\right) \text{ cm}^2$$

$E_T = \omega$ = photon energy in CM system

$$\mathcal{E} = \frac{32\pi^4}{525} e^2 G^2 \frac{T^8}{m^2} Ne$$

$$= 1.0 \times 10^8 \frac{1}{\mu_e} T_9^8 \, De \text{ erg/gm-sec.}$$

3. $\gamma + \gamma \rightarrow \nu + \bar{\nu}$ (III.21) (IV.6)

$$\sigma v = \frac{1571}{3888\pi^3}\frac{e^4 G^2 \omega^6}{M^4}\,; \quad \sigma = 2.0\times10^{-63}\left(\frac{\omega}{m}\right)^6\left(\frac{c}{v}\right)\ cm^2$$

ω = energy of one photon in CM system,

$$\mathcal{E} = \frac{64\cdot1571}{1701\,\pi}\,\rho(7)\,\frac{e^4G^2}{M^2}\,T^{13}$$

$$= 0.5\times10^{-3}\,T_9^{13}\,\rho \quad erg/gm\text{-}sec.$$

4. $e^{\pm} + (z,A) \rightarrow e^{\pm} + \nu + \bar{\nu}$ (Gandel'man 1959)

$$\sigma v = \frac{8}{525\pi^3}\frac{z^2e^4G^2E^3}{m}\,; \quad \sigma = 3.5\times10^{-52}\,z^2E^3\left(\frac{c}{v}\right)\ cm^2$$

E = electron kinetic energy

$$\mathcal{E} = \frac{4096\sqrt{2\pi}}{1575\,\pi^4}\,\frac{z^2G^2e^4}{m^{3/2}}\,T^{9/2}\,N_{nuclei}\,N_e$$

$$= 0.14\,\frac{\rho^2}{\nu\mu_e}\,T_9^{9/2}\,D_e \quad erg/cm^3\text{-}sec.$$

5. $\gamma + (z,A) \rightarrow \nu + \bar{\nu}$ (Matinyan 1961)

$$\sigma v = \frac{7}{576\pi^2}\,z^2e^2G^2\omega^2\,; \quad \sigma = 1.7\times10^{-52}\,z^2\left(\frac{\omega}{m}\right)^2\left(\frac{c}{v}\right)\ cm^2$$

ω = photon energy

$$\mathcal{E} = \frac{1}{648\pi}\,z^2G^2e^4\,T^6\,N_{nuclei}$$

$$= 1.4\times10^{9}\,\frac{\rho}{\nu}\,T_9^6 \quad erg/cm^3\text{-}sec.$$

In these expressions $GM_p^2 = 1.02\text{x}10^{-5}$, $(e^2/hc) = 1/137$, Z = nuclear charge, $T_9 = 10^{-9}$°K, ρ = matter density in gm/cm^3, m = electron mass, M = uxl mass, M_p = proton mass. Further $(1/\mu_e) = \Sigma(C_iZ_i/A_i)$, $(1/\nu) = \Sigma(C_iZ_i^2/A_i)$ where C_i is the concentration of the nuclear species (Z_i, A_i) and $\Sigma C_i = 1$. To account for the electron-positron density due

to pair production, we have introduced the factor D_e:

$$D_e = \left[1 + \frac{2\mu_e^2}{N_A^2 \rho^2}\left(\frac{mT}{\pi}\right)^3 \exp\left(-2m/T\right)\right]^{1/2}$$

$$= \left[1 + \frac{1.04 \times 10^{12}}{\rho^2} T_9^3 \exp\left(-11.9/T_9\right)\right]^{1/2}$$

where N_A is Avogadro's number. We have used $\mu_e = 2$, $1/\nu = 6$. The energy dissipation rates are plotted in Figures 13, 14, 15, 16, and 17.

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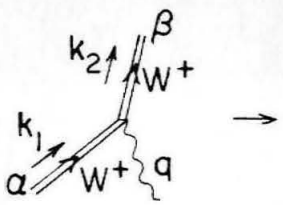
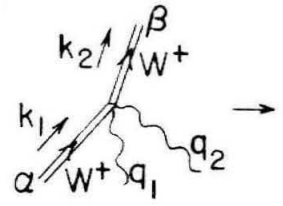
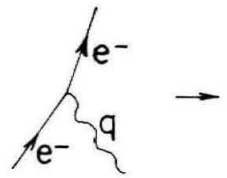


$$-i\sqrt{4\pi}\, e[(k_1+k_2)\cdot e\, g_{\alpha\beta} - k_{1\beta} e_\alpha - k_{2\alpha} e_\beta$$

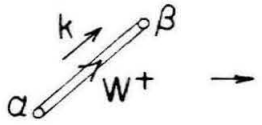
$$-\gamma(q_\alpha e_\beta - q_\beta e_\alpha$$



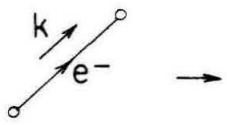
$$+i4\pi e^2 [e_{1\alpha} e_{2\beta} + e_{1\beta} e_{2\alpha} - 2\, g_{\alpha\beta}(e_1\cdot e_2)]$$



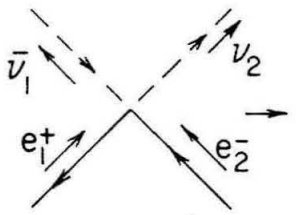
$$i\sqrt{4\pi}\; e \not{\epsilon}$$



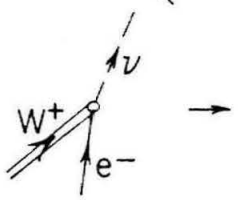
$$-i\,\frac{g_{\alpha\beta} - k_\alpha k_\beta/M^2}{k^2 - M^2 + i\epsilon}$$



$$+i\,\frac{\not{k}+m}{k^2 - m^2 + i\epsilon}$$



$$i\sqrt{8}\; G\,(\bar{u}_{\nu 2}\,\gamma_\mu a\, u_{e2})(\bar{u}_{e1}\gamma_\mu a\, u_{\nu 1})$$



$$i\sqrt{4\pi}\, f\;\; (\bar{u}_e\,\gamma_\mu a\, u_\nu)$$

$\int d^4p/(2\pi)^4$ over undetermined momenta.

Momentum is conserved at each vertex.

TABLE I. Feynman Rules used in forming the matrix elements. (see page 11)

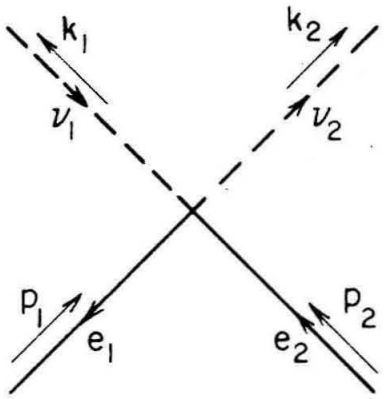


Figure 1. Feynman Diagram for:

$e^- + e^+ \rightarrow \nu + \bar{\nu}$

(see page 12)

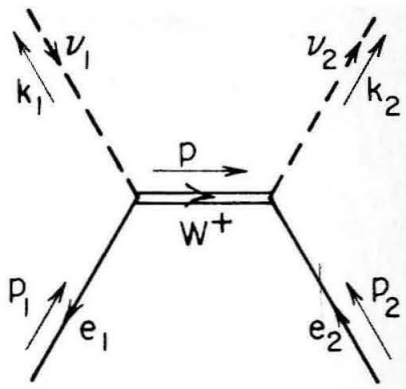


Figure 2. Feynman Diagram for:

$e^- + e^+ \rightarrow \nu + \bar{\nu}$

(see page 13)

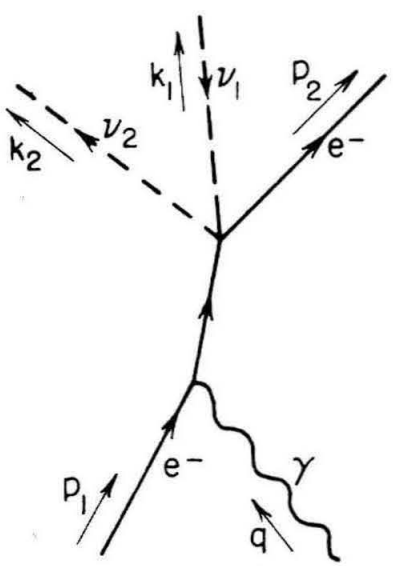


Fig. 3a.

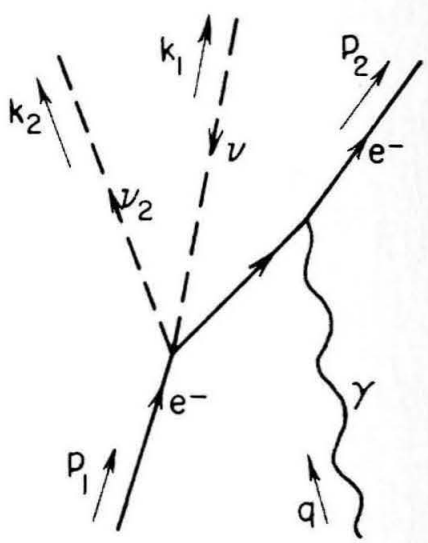


Fig. 3b.

Figure 3. Feynman Diagrams for:

$e^- + \gamma \rightarrow e^- + \nu + \bar{\nu}$

(see page 15)

Figure 4. The transition probability for the process $e^{\pm} + \gamma \rightarrow e^{\pm} + \nu + \bar{\nu}$. This is given as a function of the center of mass photon energy in the N.R., E.R., and intermediate ranges. (See page 18).

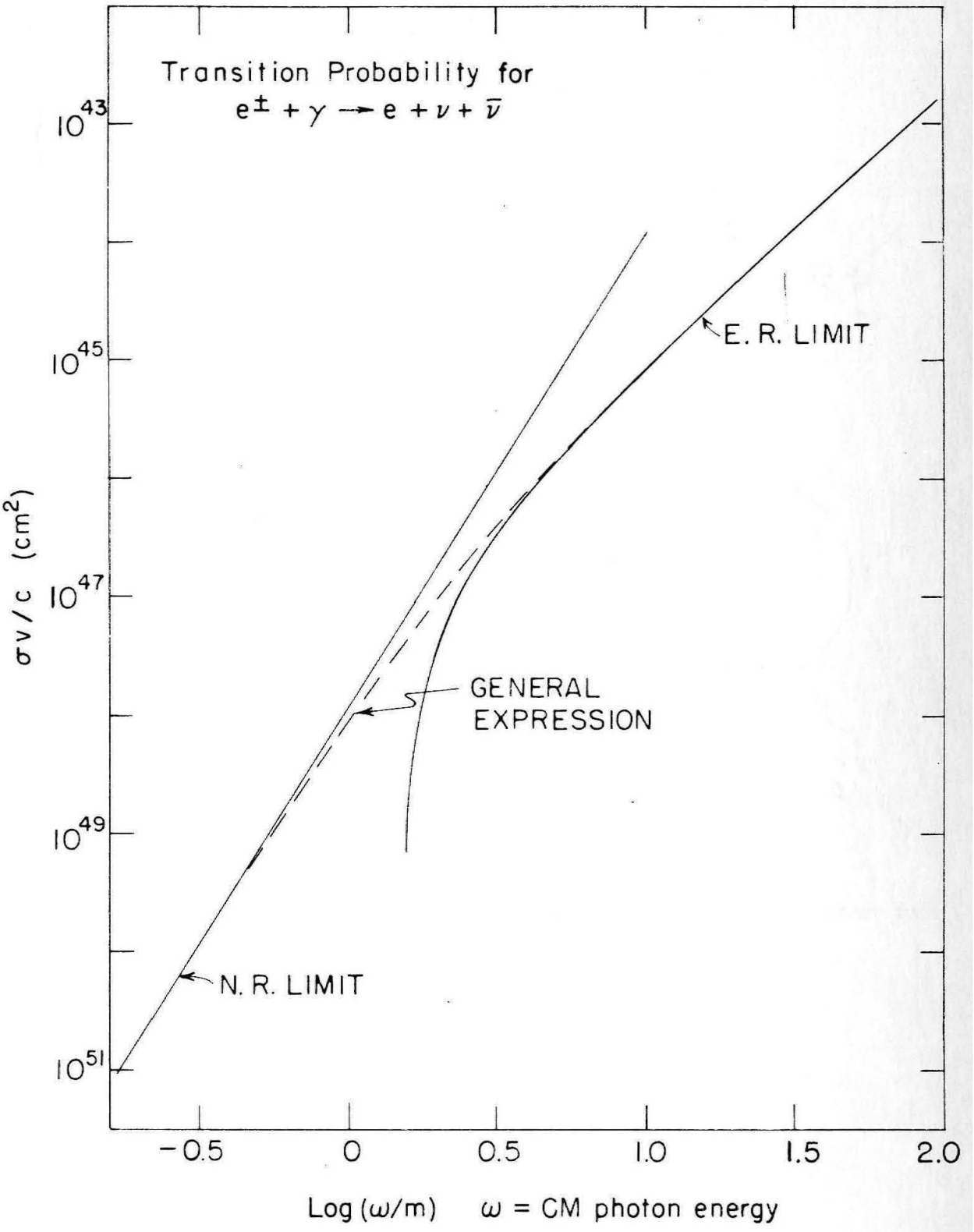

Transition Probability for
$e^{\pm} + \gamma \rightarrow e + \nu + \bar{\nu}$
10^{43}
10^{45}
10^{47}
10^{49}
10^{51}
$\sigma v / c$ (cm^2)
E. R. LIMIT
GENERAL EXPRESSION
N. R. LIMIT
−0.5
0
0.5
1.0
1.5
2.0
Log (ω/m) ω = CM photon energy

Figure 4

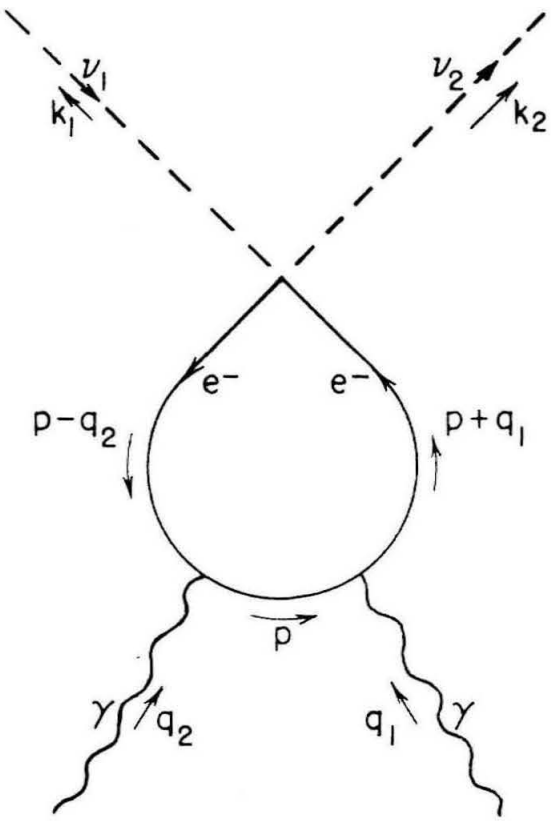


Figure 5. Feynman Diagram for:

$\gamma + \gamma \rightarrow \nu + \bar{\nu}$

with local weak interaction
(see page 21)

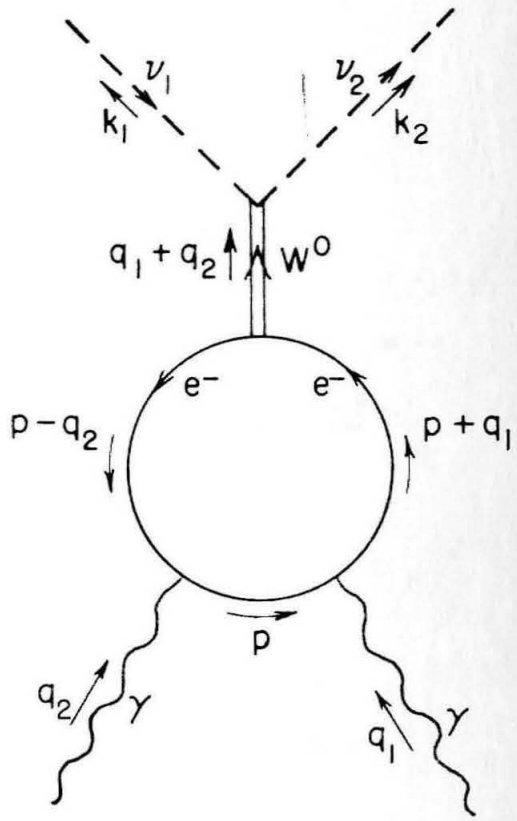
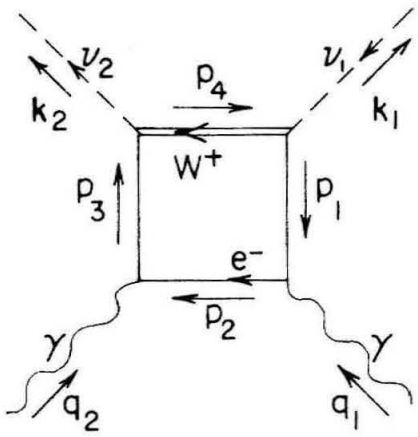


Figure 6. Feynman Diagram for:

$\gamma + \gamma \rightarrow \nu + \bar{\nu}$

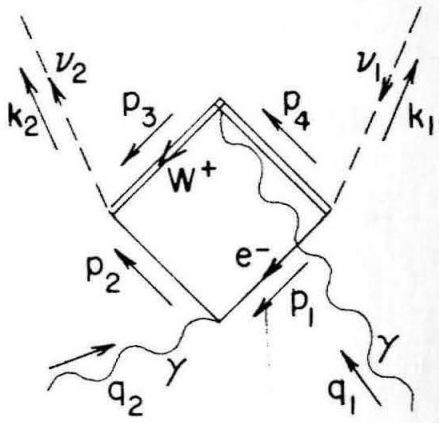
with neutral uxl
(see page 24)



$p_1 = p-k_1$ $\quad p_3 = p+k_2$

$p_2 = p-k_1+q_1$ $\quad p_4 = p$

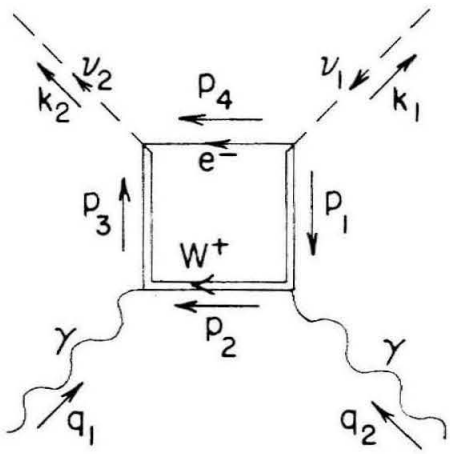
Figure 7a.



$p_1 = p-k_1$ $\quad p_3 = -p+q_1$

$p_2 = p+k_2-q_1$ $\quad p_4 = -p$

Figure 8b.

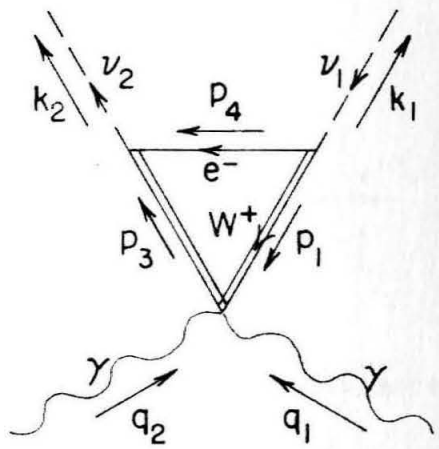


$p_1 = -p-k_1$ $\quad p_4 = p$

$p_2 = -p+k_2-q_1$ $\quad p_5 = -p+k_2-q_2$

$p_3 = -p+k_2$

Figure 7c.



$p_1 = -p-k_1$ $\quad p_4 = p$

$p_2 = -p+k_2-q_1$ $\quad p_5 = -p+k_2-q_2$

$p_3 = -p+k_2$

Figure 7d.

Figure 7. Feynman Diagrams for: $\gamma + \gamma \rightarrow \nu + \bar{\nu}$ with charged uxl
(see page 24)

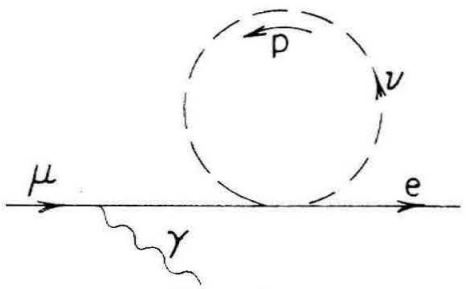


Fig. 8a

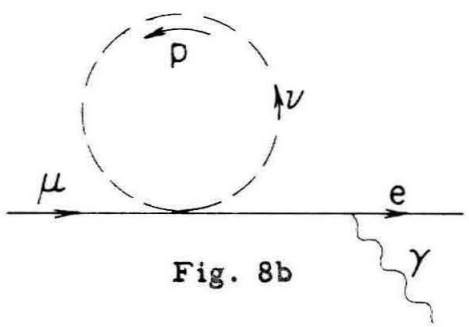


Fig. 8b

Figure 8. Feynman Diagrams for:

$\mu \rightarrow e + \gamma$

with local weak interaction
(see page 26)

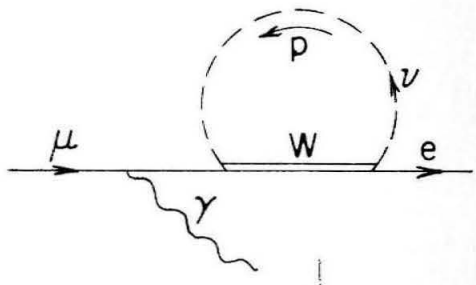


Fig. 9a

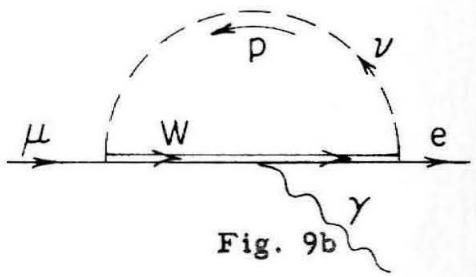


Fig. 9b

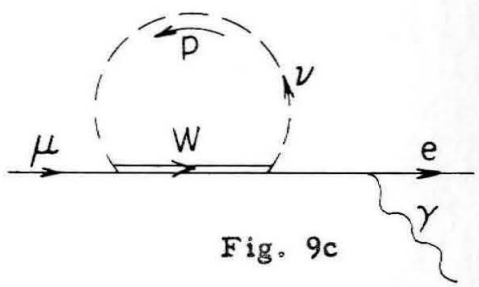


Fig. 9c

Figure 9. Feynman Diagrams for:

$\mu \rightarrow e + \gamma$

with charged uxl
(see page 26)

Figure 10. Comparison of the electron densities due to pair production and due to residual electrons from matter. As a function of temperature we plot that matter density at which the atomic electron density equals the electron-positron density caused by pair production. This is given in the N.R. and E.R. ranges with a smooth curve joining the two regions. Off of this curve, one or the other source of electrons dominates.

The regions of degeneracy and nondegeneracy for the electron gas are given considering that all electrons are of atomic origin ($Z/A = \frac{1}{2}$) and that degeneracy sets in when kT = (Fermi Energy)/3. (See page 42.)

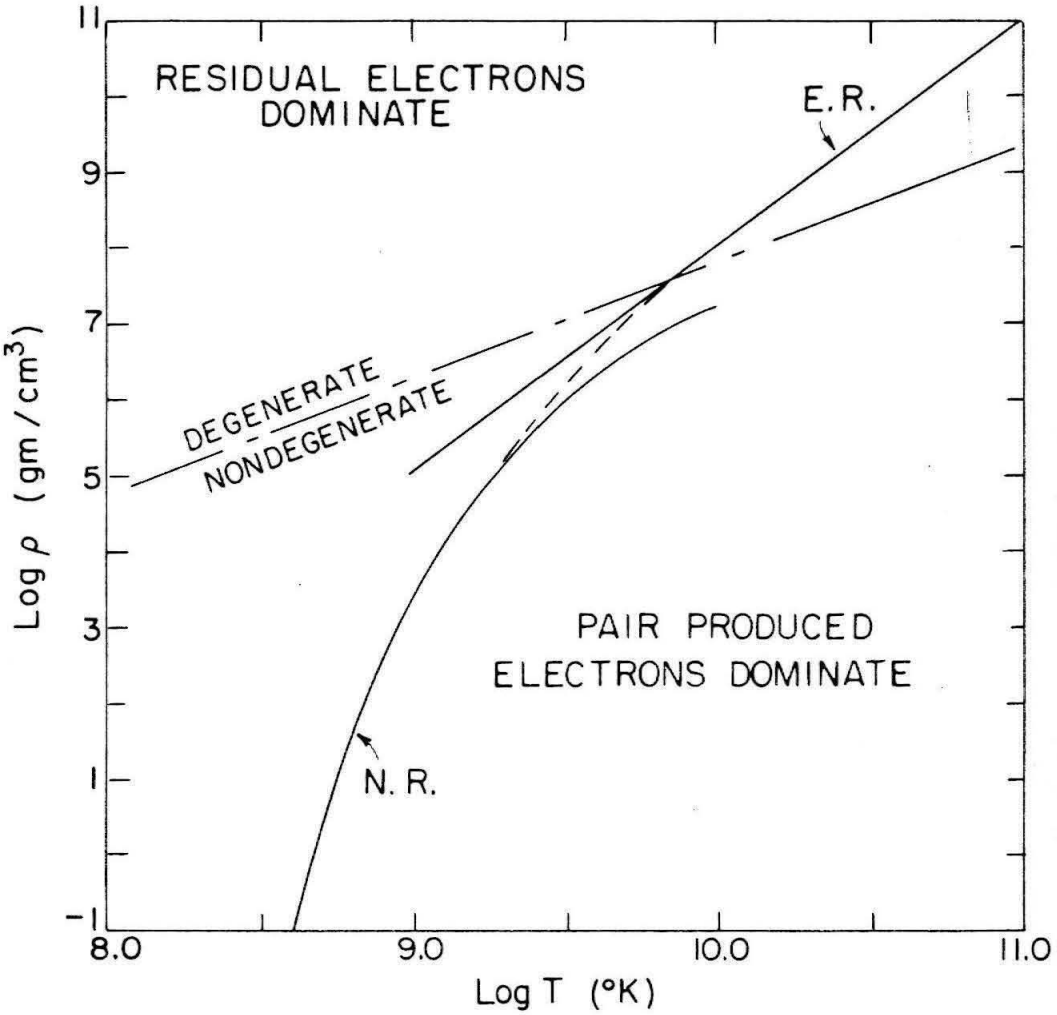

RESIDUAL ELECTRONS
DOMINATE
E.R.
DEGENERATE
NONDEGENERATE
PAIR PRODUCED
ELECTRONS DOMINATE
N.R.
Log ρ (gm/cm³)
Log T (°K)
11
9
7
5
3
1
-1
8.0
9.0
10.0
11.0

Figure 10

Figure 11. The enhancement of the electron density due to pair production. D_e = (electron density due to pair production)/(electron density due to matter) is given as a function of temperature for various matter densities. (See page 42.)

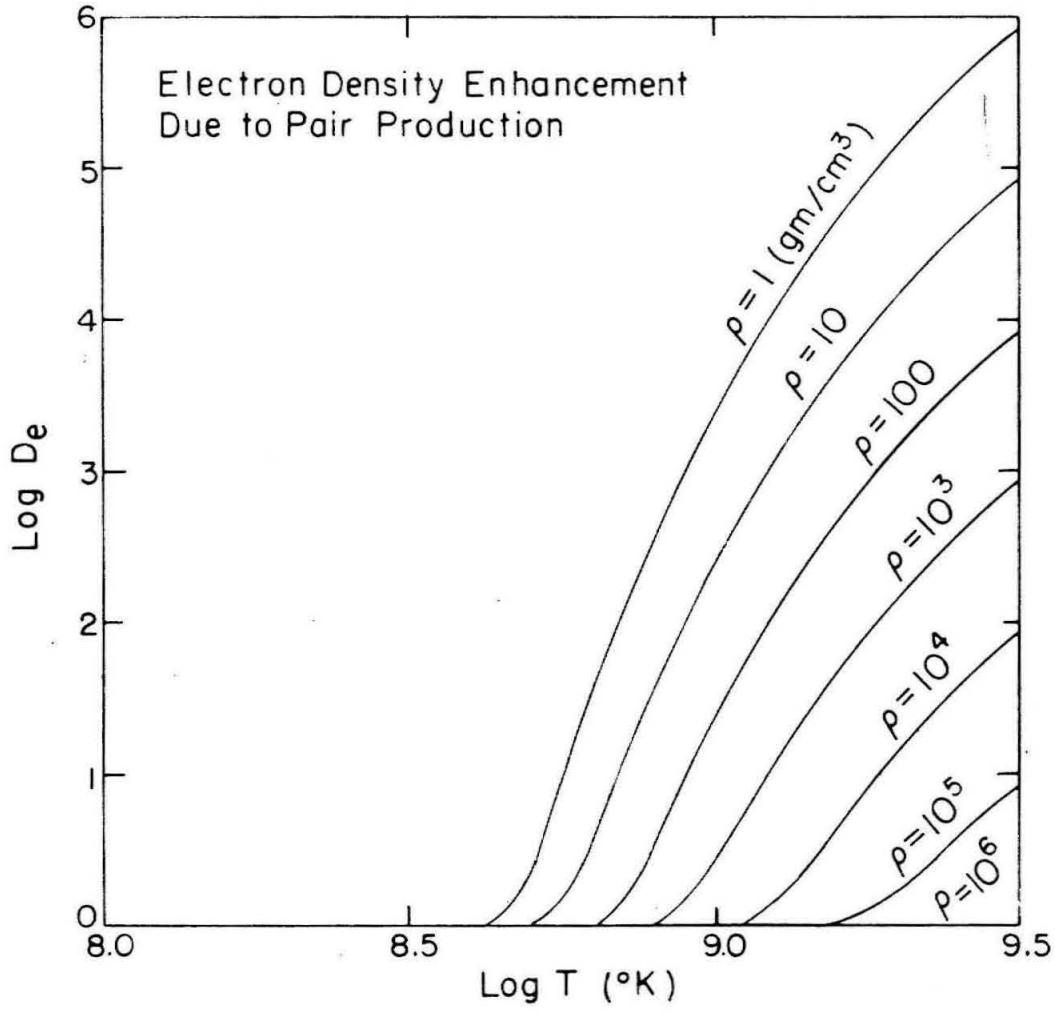

Electron Density Enhancement
Due to Pair Production
Log D_e
Log T (°K)
6
5
4
3
2
1
0
8.0
8.5
9.0
9.5
$\rho = 1$ (gm/cm^3)
$\rho = 10$
$\rho = 100$
$\rho = 10^3$
$\rho = 10^4$
$\rho = 10^5$
$\rho = 10^6$

Figure 11

Figure 12. Rate of energy emission through $\nu\bar{\nu}$ pairs for the urca processes. (See page 46.)

Figure 13. Rate of energy emission through $\nu\bar{\nu}$ pairs for the process $e^+ + e^- \rightarrow \nu + \bar{\nu}$ in the nondegenerate limit. For $T < 2 \times 10^9\ ^oK$ we use the expression obtained for the N.R. limit. For higher temperatures a smooth curve joining the N.R and E.R. expressions is used. (See pages 40, 46.)

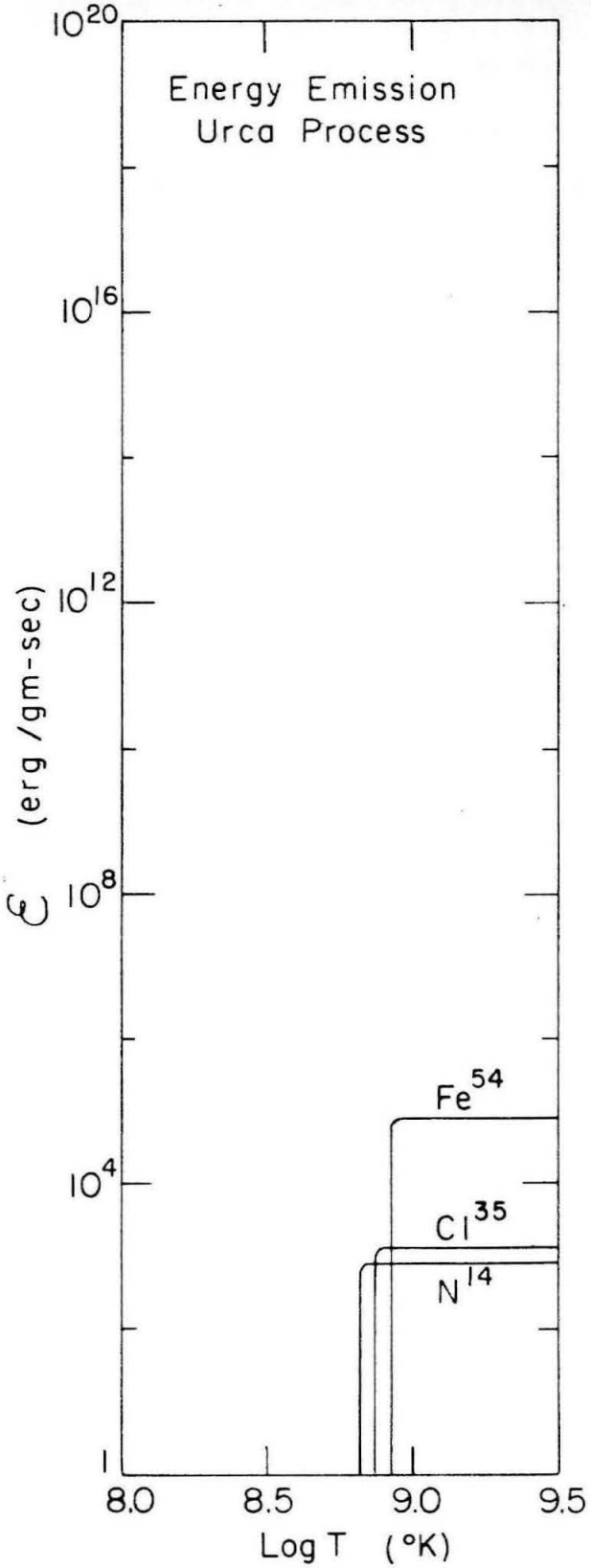


Figure 12

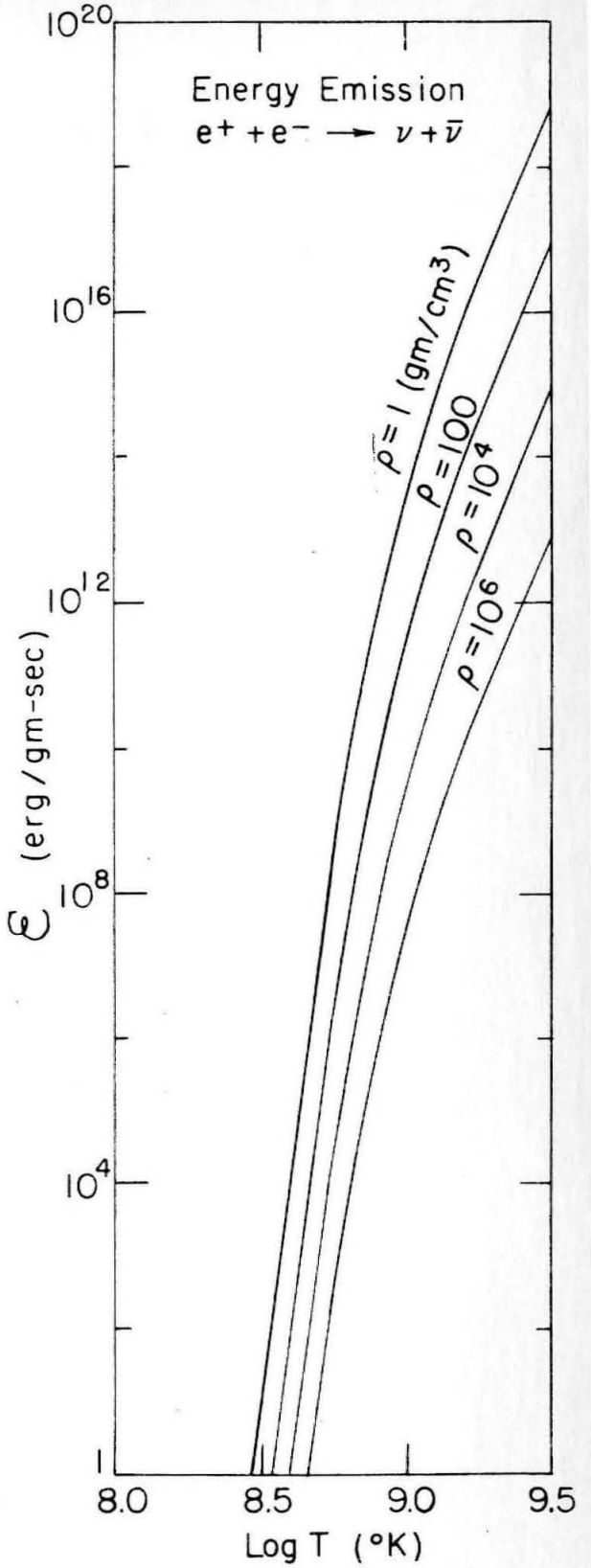


Figure 13

Figure 14. Rate of energy emission through $\nu\bar{\nu}$ pairs for the process $e^{\pm} + \gamma \rightarrow e^{\pm} + \nu + \bar{\nu}$ in the nondegenerate limit. For $T < 1 \times 10^9\ ^oK$ we use the expression obtained for the N.R. limit. For higher temperatures a smooth curve joining the N.R. and E.R. expressions is used. (See pages 41, 42, 43 and 46.)

Figure 15. Rate of energy emission through $\nu\bar{\nu}$ pairs for the process $\gamma + \gamma \rightarrow \nu + \bar{\nu}$ in the N.R. limit. (See page 46.)

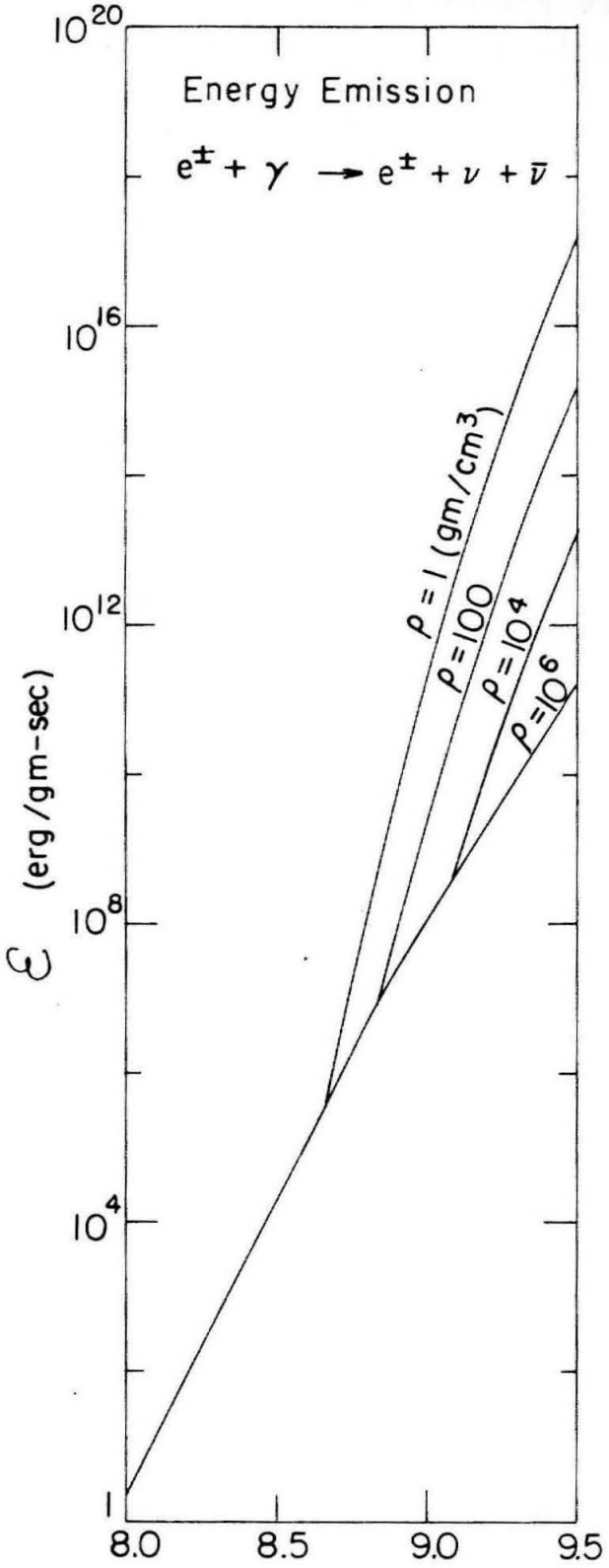


Figure 14

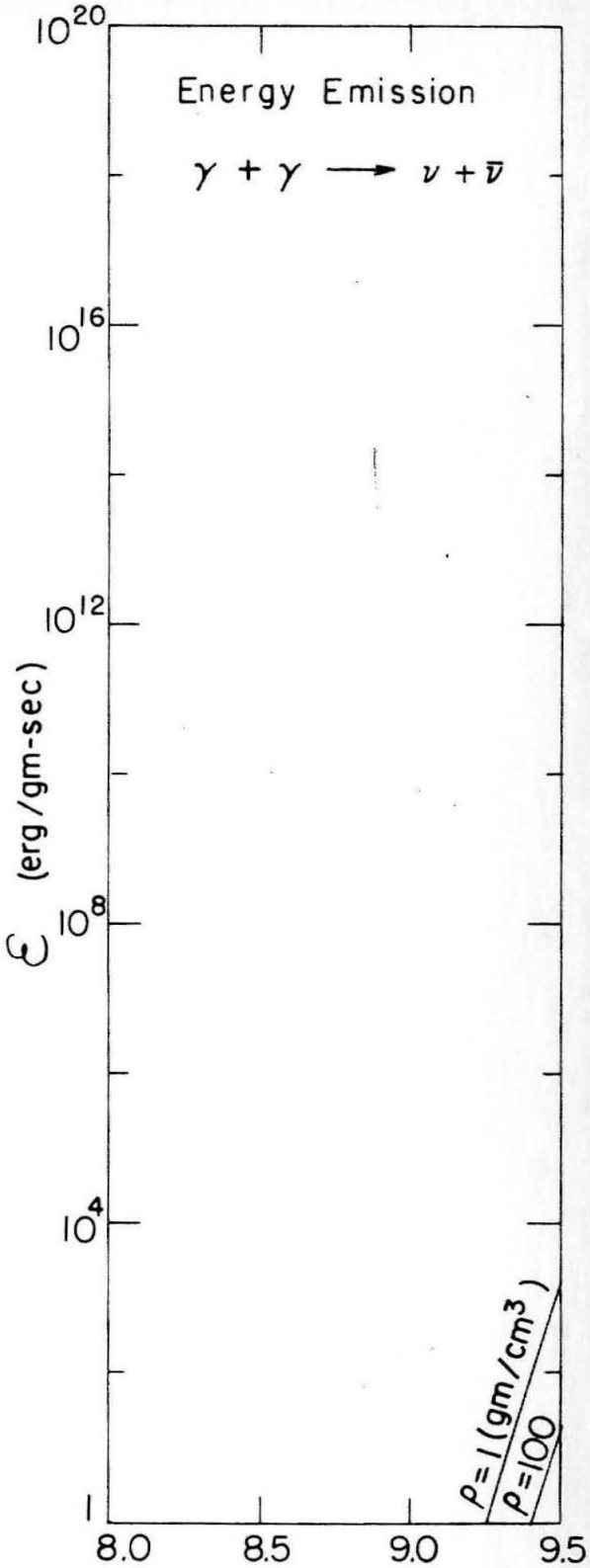


Figure 15

Figure 16. Rate of energy emission through $\nu\bar{\nu}$ pairs for the process $e^{\pm} + (Z,A) \rightarrow e^{\pm} + \nu + \bar{\nu}$ in the N.R. and nondegenerate limits. (See page 46.)

Figure 17. Rate of energy emission through $\nu\bar{\nu}$ pairs for the process $\gamma + (Z,A) \rightarrow \gamma + \nu + \bar{\nu}$ in the N.R. limit. (See page 46.)

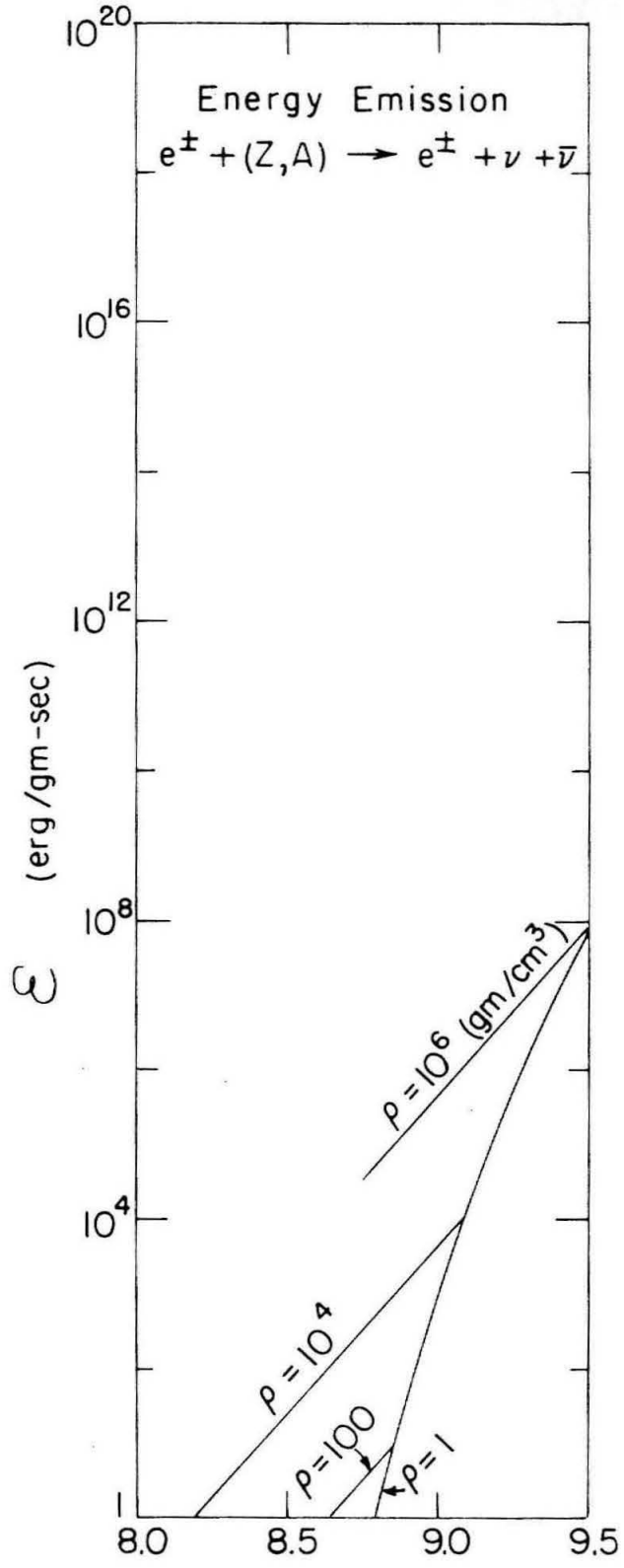


Figure 16

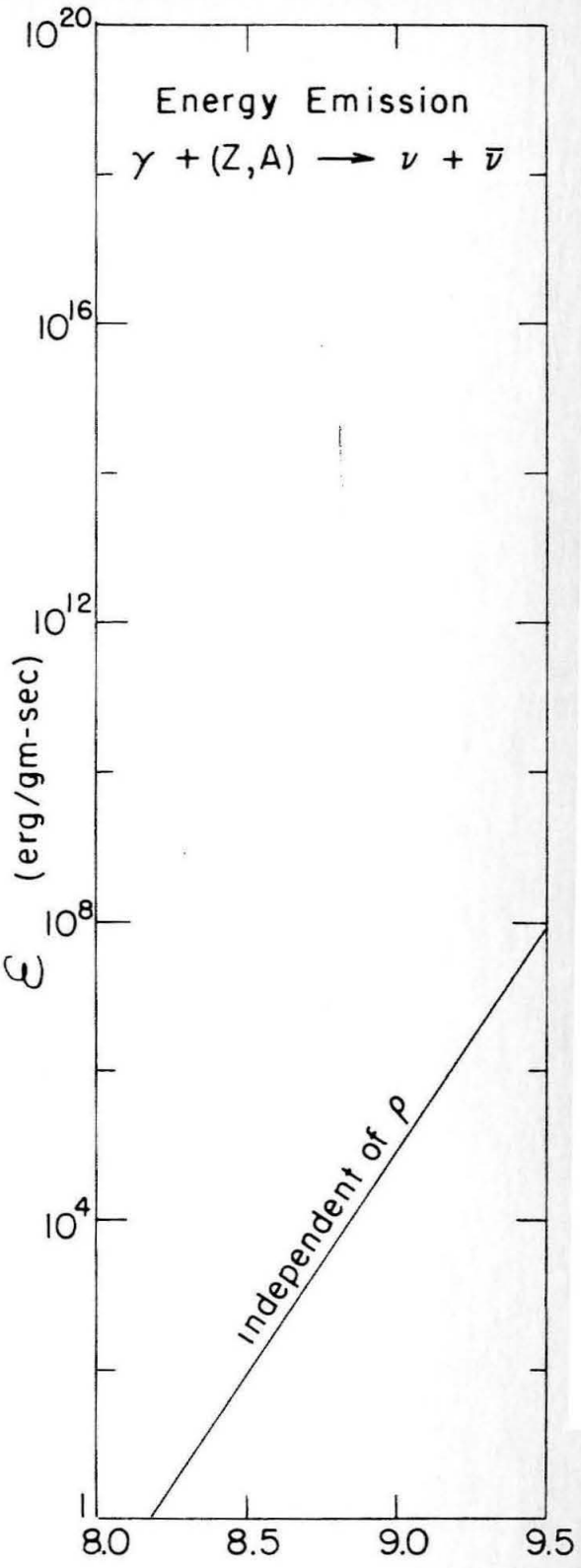


Figure 17

Figure 18. Regions of dominance for energy emission through neutrinos for the various neutrino processes.

(See page 46.)

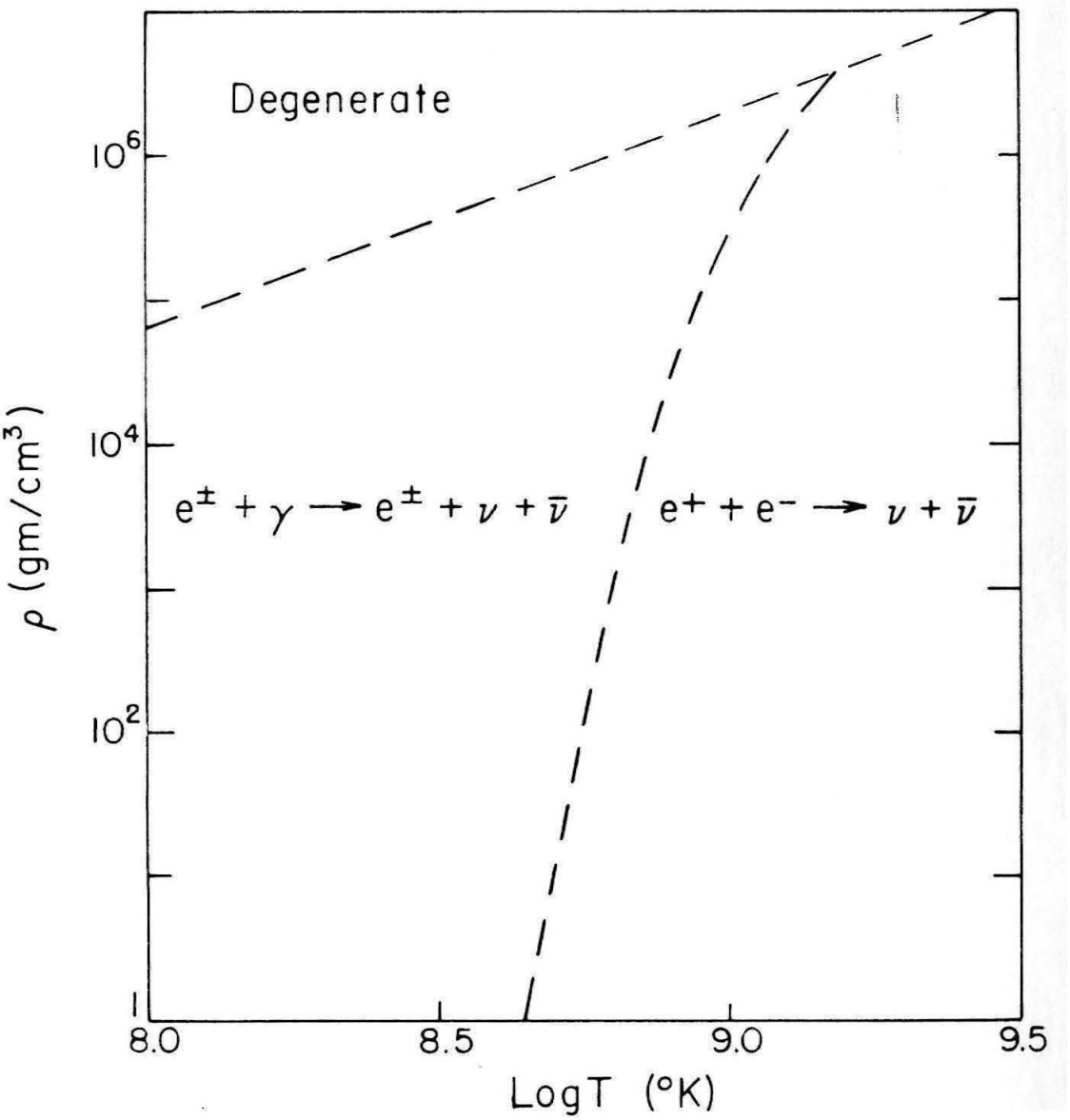
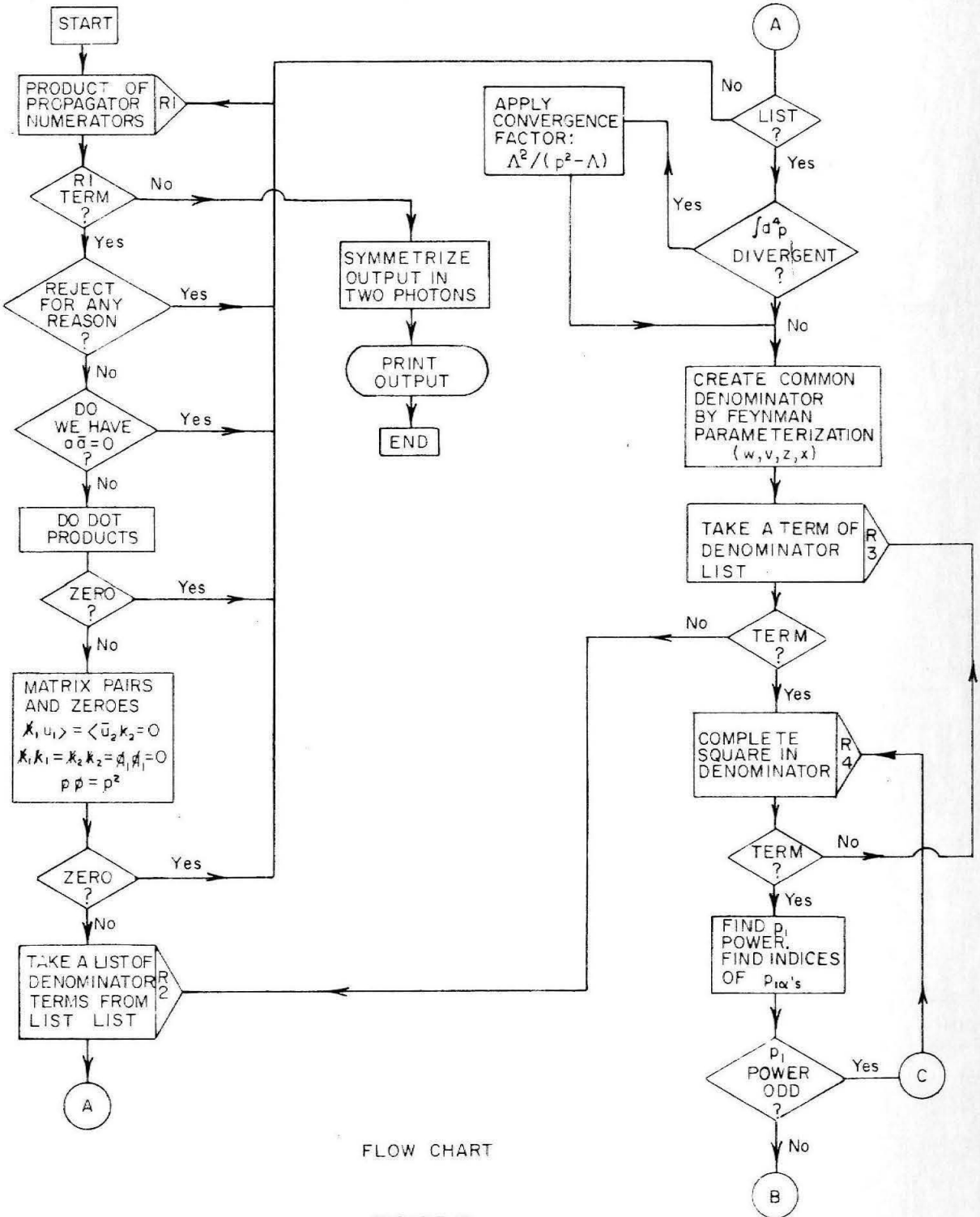

Degenerate
10^6
10^4
10^2
1
ρ (gm/cm^3)
$e^{\pm} + \gamma \rightarrow e^{\pm} + \nu + \bar{\nu}$
$e^+ + e^- \rightarrow \nu + \bar{\nu}$
8.0
8.5
9.0
9.5
LogT (°K)

Figure 18

Figure 19. Flow chart for the computer program used in the reduction of the matrix elements for the process $\gamma + \gamma \rightarrow \nu + \bar{\nu}$. (See page 54.)



FLOW CHART

FIGURE 19

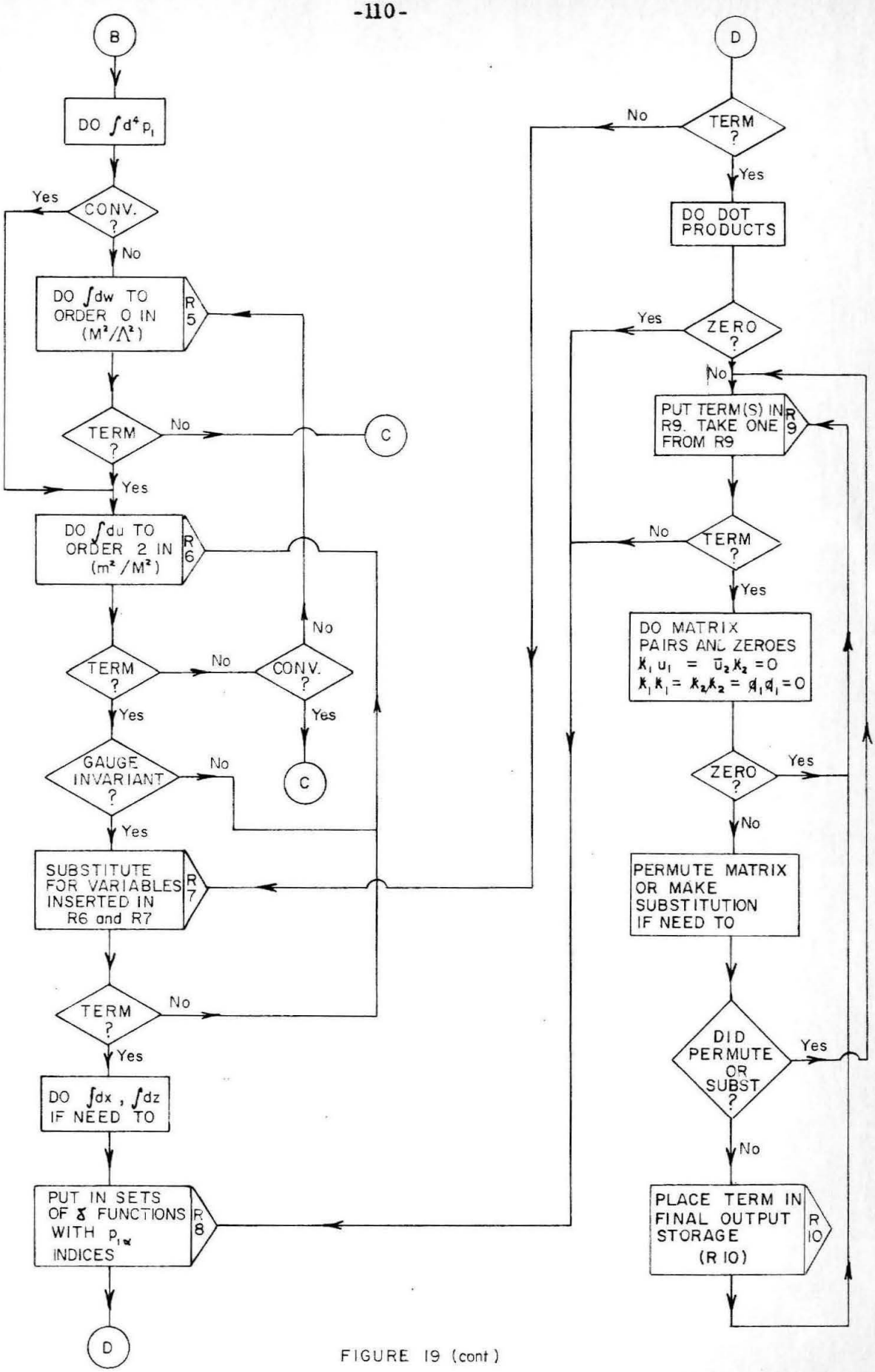


FIGURE 19 (cont)