PROPAGATION OF ELECTROMAGNETIC WAVES INSIDE A CYLINDRICAL METAL TUBE AND ALONG OTHER TYPES OF GUIDES

Thesis by

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Abstract of Thesis

PROPAGATION OF ELECTROMAGNETIC WAVES INSIDE A CYLINDRICAL METAL TUBE AND ALONG OTHER TYPES OF GUIDES

The prime purpose of this paper is to base the discussion of the properties of propagation of electromagnetic waves inside a metal tube upon the theory of complex functions. The general expressions for the field components for different types of excitation systems are obtained in a rigorous manner starting from that of an electric and a magnetic dipole. The formal mathematical generalization is achieved by means of the transformation formulae of cylindrical functions and the results of the theory of integral equations. The integral equations thus obtained are expanded into series by aid of residual calculus for actual numerical calculation.

The residues at the poles of singularities give rise to different "distinct modes" of propagation and thereby a comprehensive discussion of all the important physical properties is made. At the same time, problems arising in practical applications, say for long distance transmission for television purposes, are analyzed and some interesting conclusions obtained. The unique and rigorous analysis is only made possible by the free use of the results ~~in~~ of the theory of complex functions.

A comparison of the properties of propagation with regard especially to the attenuations and the velocities of propagation

inside a hollow cylindrical metal tube guide and that of a concentric system is made. It is hoped that the conclusions obtained therefrom will throw some light on the merits of both systems and will point out those things which require careful consideration in practical design.

SECTION I.

General Mathematical Solution of Wave Equation In Cylindrical Coordinates

In order to bring out the intrinsic characters of cylindrical functions in the solution of wave equation, a brief sketch of the building-up of wave function following the procedure of R. Weyrich* will first be described. It leads naturally to a generalization, to Sommerfeld's integral expression for all kinds of cylindrical functions. A comprehensive grasp of the procedure and results therefrom paves the way for attacking a vast number of problems in mathematical physics and electrical engineering.

The fundamental partial differential equation, written in cartesian-coordinates, is:

$$\Delta \mathcal{U} = \frac{\partial^2 \mathcal{U}}{\partial x^2} + \frac{\partial^2 \mathcal{U}}{\partial y^2} + \frac{\partial^2 \mathcal{U}}{\partial z^2} = a\frac{\partial^2 \mathcal{U}}{\partial t^2} + b\frac{\partial \mathcal{U}}{\partial t} \tag{1.01}$$

where $\mathcal{U}$ is the function to be determined together with certain boundary and initial conditions. a and b are in general real constants. The independent variables are the cartesian coordinates x, y, and z, and the time t. With different characterizing values given to a and b, Equation (1.01) represents a great number of different natural phenomena, such as propagation of electromagnetic waves, displacement of longitudinal elastic strings, vibration of membranes, diffusion of heat, etc.. In virtue of the validity of

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*R. Weyrich, Die Zylinderfunktionen und ihre Anwendungen.

the application of Fourier series analysis to time variational phenomena, we can always put:

$$\mathcal{U} = u(x, y, z)\, e^{-i\omega t}$$

where $u(x,y,z)$ is a function only dependent on position and independent of time and ω is the angular frequency. Substituting the above relation into Equation (1.01), we get:

$$(1.02) \quad \Delta u + k^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + k^2 u = 0$$

with $k^2 = a\omega^2 + ib\omega$. Usually one simply calls relation (1.02) the "wave equation" and k the "wave number"*. It is to be assumed that both the real part k_{re} and the imaginary part k_{im} of k are positive. Equations (1.01) and (1.02) can also be written in spherical polar coordinates R, φ, θ or cylindrical coordinates r, φ, z , for which the transformation formulae are:

$$x = R\cos\varphi\sin\theta\,, \quad y = R\sin\varphi\sin\theta\,, \quad z = R\cos\theta$$

and $\quad x = r\cos\varphi\,, \quad y = r\sin\varphi\,, \quad z = z$

respectively. In these coordinate systems, the wave equation becomes:

$$(1.03) \quad \Delta u + k^2 u \equiv \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial u}{\partial R}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2} + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right) + k^2 u = 0$$

and

$$(1.04) \quad \Delta u + k^2 u \equiv \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} + k^2 u = 0 .$$

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* The corresponding German names are "Wellengleichung" and "Wellenzahl".

A particular solution of (1.02) can be obtained by means of the classical product substitution:

$$u(x, y, z) = X(x)\, Y(y)\, Z(z)$$

where X, Y, Z are only functions of the arguments in the parentheses, respectively. Substituting the above relation into (1.02) and dividing through by u, we have:

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} + k^2 = 0$$

or simply (1.05) $$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} + k^2 = 0$$

Because all three terms must be independent of the arguments x, y, and z, they must satisfy the following familiar differential equations:

$$\frac{X''}{X} = -c_1^2 k^2 \quad , \quad \frac{Y''}{Y} = -c_2^2 k^2 \quad , \quad \frac{Z''}{Z} = -c_3^2 k^2 \quad ,$$

while between the three arbitrary constants c_1, c_2, and c_3, the following relation holds:

(1.06) $$c_1^2 + c_2^2 + c_3^2 = 1$$

The integral solutions of the above differential equations are:

$$X = A_1 e^{ikc_1x} + B_1 e^{-ikc_1x}$$

$$Y = A_2 e^{ikc_2y} + B_2 e^{-ikc_2y}$$

$$Z = A_3 e^{ikc_3z} + B_3 e^{-ikc_3z}$$

with A_ν, B_ν $(\nu = 1, 2, 3)$ as the constants of integration. The product XYZ can therefore be expressed as a summation of particular solutions of the following type:

$$(1.07) \quad u = A e^{ik(c_1 x + c_2 y + c_3 z)}$$

On account of the relation (1.06), c_1, c_2, and c_3, can be thought of as the direction cosines of a space unit vector (n) from the origin. Then:

$$c_1 x + c_2 y + c_3 z = x \cos\alpha + y \cos\beta + z \cos\gamma = p$$

is the projection of the vector with coordinates x, y, z, on the line n, and one particular solution of (1.01) becomes:

$$(1.08) \quad \mathcal{Re}\,[U] = \mathcal{Re}[A e^{i(kp - \omega t)}] = A \cos(kp - \omega t)$$

This is the equation of propagation of "plane waves" with n as the normal to the wave-front, $\frac{\omega}{k}$ the phase velocity, and $\frac{2\pi}{k}$ the wave-length if k is real. All points in a plane perpendicular to n are "in phase" and constitute a plane "wave front". For detailed discussion of the type of Equation (1.08) and of the building-up therefrom of a general integral solution, the reader is referred to the first original researches of many authors among whom especially may be mentioned Sommerfeld, Whittaker and Bateman.*

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* Messenger of Mathematics, XXXVI, (1907) pp. 98-106.
Math. Ann. VII (1902) pp. 342-345.
Proc. London Math. Soc. (2) I. (1904) pp. 451-458; (2) VII (1909) pp. 20-89.
Bateman's "Electrical and Optical Wave Motion".
Whittaker and Watson, "Modern Analysis", Chap. 18.
Riemann-Weber's "Differentialgleichungen der Phys."

From Equation (1.06) we can also think of c_1, c_2, and c_3, as representing the direction cosines of any point on a unit sphere with spherical polar coordinates φ' and θ', then:

$$(1.09) \qquad c_1 = \cos\varphi' \sin\theta', \quad c_2 = \sin\varphi' \sin\theta', \quad c_3 = \cos\theta'$$

Due to the linearity and homogeneity of the wave Equation (1.01) of $\mathcal{U}$ and its derivatives, any linear combination of solutions like (1.08) is also a solution of (1.01).

$$\mathcal{R}e[\mathcal{U}] = \mathcal{R}e[A_1 \mathcal{U}_1 + A_2 \mathcal{U}_2 + \cdots\cdots + A_n \mathcal{U}_n]$$

The corresponding solution for (1.02) is:

$$u = A_1 u_1 + A_2 u_2 + \cdots\cdots + A_n u_n$$

where $A_1, A_2 \cdots A_n$ are the arbitrary coefficients.

Similarly, in the limit, u can be represented by a definite integral for which we can multiply the right side of (1.07) by an arbitrary function of c_1, c_2, and c_3, and then integrate between any chosen limits so far as the resultant integral exists and differentiation under integral sign is allowable. On account of relation (1.09), c_1, c_2, and c_3, are expressed in terms of φ' and θ' and the definite integral takes the form:

$$(1.10) \qquad u = \int_\Gamma e^{ik[x\cos\varphi'\sin\theta' + y\sin\varphi'\sin\theta' + z\cos\theta']} \cdot F(\varphi',\theta')\,d\varphi'd\theta'$$

$$= \int_\Gamma e^{ikr\cos\Omega} \cdot F(\varphi',\theta')\,d\varphi'd\theta' = \int_\Gamma e^{ikp} F(\varphi',\theta')\,d\varphi'd\theta'$$

where Ω is the angle included between the unit vector $n(1, \varphi', \theta')$ and the line drawn from the origin to the field point (x, y, z) or (r, φ, θ).

One very interesting and, at the same time, very significant generalization of the above expression can be realized: although φ' and θ' are described as the azimuthal and zenithal angles for any point on a unit sphere, we can consider φ' and θ' as complex quantities within the limited range $0 \leqq Re[\varphi'] < 2\pi$ $0 \leqq Re[\theta'] < \pi$ in their complex planes. The latter restrictions assure the "uniqueness" of the integrand of (1.10) and therefore the wave function u. That this powerful, ingenious generalization is always allowable can be easily shown by putting complex quantities for φ' and θ' and substituting (1.09) into (1.06). Now in (1.10), we set $F(\varphi', \theta') = \sin\theta'$, then $\sin\theta' d\theta' d\varphi' = dS =$ elementary area on the unit circle and (1.10) becomes:

$$(1.10)a \quad u = \int_\Gamma e^{ikp} \sin\theta' d\theta' d\varphi' = \int_\Gamma e^{ikp}\, dS$$

For simplicity, we can assume the auxiliary polar-axis passing through the field point (x, y, z) or (r, θ, φ), then $p = R\cos\theta'$. If the integration is extended over the whole surface of the unit sphere, we obtain the effect due to a uniform spherical source:

$$(1.11) \quad u = \int_0^{2\pi}\int_0^{\pi} e^{ikR\cos\theta'} \sin\theta' d\theta' d\varphi' = 2\pi \int_0^{\pi} e^{ikR\cos\theta'} \sin\theta' d\theta'$$

$$= \frac{4\pi}{k} \frac{\sin kR}{R}$$

and

$$\mathcal{R}e\,[U] = \mathcal{R}e\,[u\,e^{-i\omega t}] = \frac{4\pi}{k}\,\frac{\sin kR}{R}\,\cos\omega t$$

This represents a "standing spherical wave", obtained from above particular superposition of "plane waves", if k is real. For complex k, merely a damping factor is introduced.

If, however, we choose a path of integration in the θ'-plane, as shown in Fig. 1.1, $(1.10)_a$ becomes:

$$(1.13)\qquad u = -2\pi \int_{g_1 g_2} e^{ikR\cos\theta'}\, d(\cos\theta') = \frac{2\pi\, e^{ikR}}{ikR}$$

and, therefore, we obtain:

$$(1.14)\quad \mathcal{R}e\,[U] = \mathcal{R}e\left[\frac{2\pi}{ik}\,\frac{e^{ikR}}{R}\,e^{-i\omega t}\right] = \frac{2\pi}{k}\,\frac{\sin(kR-\omega t)}{R}$$

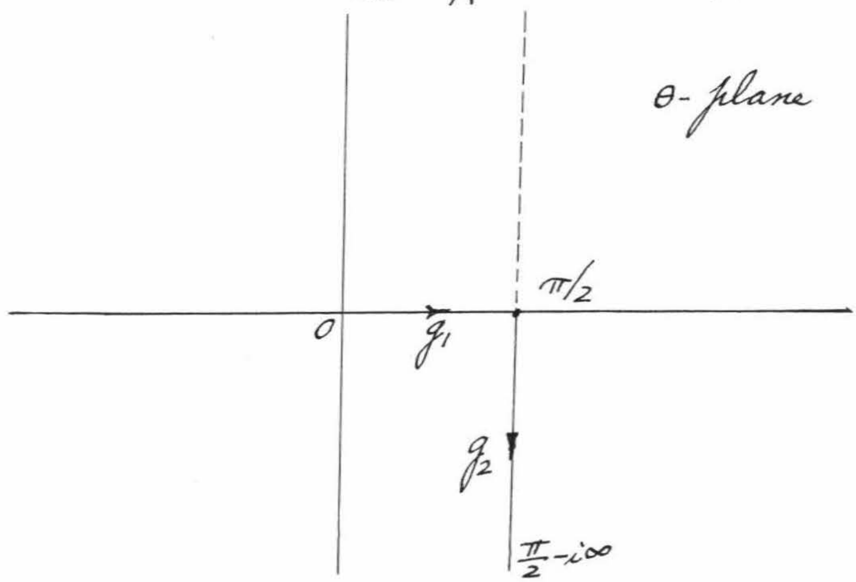

Fig. 1.1.

This always represents a "divergent progressive spherical wave"*

* A. Sommerfeld, (Riemann-Weber) "Differentialgleichungen der Phys." p. 397.

where k may be real or complex. Such a spherical wave function has a singularity at the "single pole" at the origin. "Multiple-pole" spherical wave functions may be obtained by differentiations of (1.11) and (1.13).

In the above paragraphs, starting from plane waves, we succeeded in building up divergent symmetrical spherical waves which, from Equation (1.13) on neglecting the unimportant multiplying factor, can be simply represented as $u = \frac{e^{ikR}}{R}$. Due to the validity of the superposition principle, the field at any point in space due to a continuous and uniform distribution of sources along the polar-axis can be represented by the following definite integral:

$$u^{(1)} = \int_{-\infty}^{+\infty} \frac{e^{ik\sqrt{x^2+y^2+(z-\zeta)^2}}}{\sqrt{x^2+y^2+(z-\zeta)^2}}\, d\zeta = \int_{-\infty}^{+\infty} \frac{e^{ik\sqrt{r^2+(z-\zeta)^2}}}{\sqrt{r^2+(z-\zeta)^2}}\, d\zeta$$

wherein x, y, z and r, φ, z are the cartesian and cylindrical coordinates, respectively, for the field point and $0, 0, \zeta$ is the corresponding coordinates for the source point. Substituting ζ for $(z-\zeta)$ as the new variable, the above expression becomes:

$$(1.15) \quad u^{(1)} = \int_{-\infty}^{+\infty} \frac{\exp.[ik\sqrt{r^2+\zeta^2}]}{\sqrt{r^2+\zeta^2}}\, d\zeta = 2\int_{0}^{+\infty} \frac{\exp.[ik\sqrt{r^2+\zeta^2}]}{\sqrt{r^2+\zeta^2}}\, d\zeta$$

This represents, therefore, a divergent "symmetrical cylindrical wave function" with axis $r=0$ as the "line of singularity". The convergence of the above improper integral is always satisfied for real or complex k when the positive sign of the square root is used, since the imaginary part of k is always taken to be positive. By means of the following substitution:

$$\varsigma = i r \sin\alpha$$

Equation (1.15) reduces to a simpler form:

$$u^{(1)} = i \int_{+i\infty}^{-i\infty} e^{ikr\cos\alpha}\, d\alpha$$

The Hankel cylindrical function of the first kind with degree zero is then defined as: *

$$(1.16) \qquad H_0^{(1)}(kr) = \frac{u^{(1)}}{i\pi} = \frac{1}{\pi}\int_{+i\infty}^{-i\infty} e^{ikr\cos\alpha}\, d\alpha$$

This is immediately a particular form of Sommerfeld's fundamental and important integral expression for cylindrical functions. ǂ

The corresponding "convergent cylindrical wave" function, by the same substitution and simplification is then:

$$u^{(2)} = \int_{-\infty}^{+\infty} \frac{\exp\left[-ik\sqrt{x^2+y^2+(z-\varsigma)^2}\right]}{\sqrt{x^2+y^2+(z-\varsigma)^2}}\, d\varsigma = \int_{-\infty}^{+\infty} \frac{\exp\left[-ik\sqrt{r^2+\varsigma^2}\right]}{\sqrt{r^2+\varsigma^2}}\, d\varsigma$$

$$= i \int_{+i\infty}^{-i\infty} e^{-ikr\cos\alpha}\, d\alpha$$

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* Erste Hankelsche Zylinderfunktion mit dem Index Null.

ǂ A. Sommerfeld, "Über komplexe Integraldarstillingen der Zylinderfunktionen", Arch. d. Math. und Phys. 18, i, 1911.
G. N. Watson, "A Treatise of Bessel Functions", ed. 1927.

and the corresponding Hankel cylindrical function of the second kind with degree zero is defined as:

$$(1.17) \qquad H_0^{(2)}(kr) = \frac{u^{(2)}}{-i\pi} = \frac{-1}{\pi}\int_{+i\infty}^{-i\infty} e^{-ikr\cos\alpha}\,d\alpha$$

$H_0^{(2)}(kr)$ has, therefore, the same path of integration in α-plane with $H_0^{(1)}(kr)$. In order to obtain the same "integrand" for both $H_0^{(1)}(kr)$ and $H_0^{(2)}(kr)$, we make the following substitution for $H_0^{(2)}(kr)$

:

$$\zeta = -ir\sin(\alpha - \pi)$$

then $H_0^{(2)}(kr)$ becomes:

$$(1.18) \qquad H_0^{(2)}(kr) = \frac{1}{\pi}\int_{\pi-i\infty}^{\pi+i\infty} e^{ikr\cos\alpha}\,d\alpha$$

Except for an unimportant multiplying constant, $H_0^{(1)}(kr)$ and $H_0^{(2)}(kr)$ represent divergent and convergent symmetrical cylindrical wave-functions with "source" and "sink" on the axis, respectively. From their definition Equations (1.16) and (1.17), $H_0^{(1)}(kr)$ and $H_0^{(2)}(kr)$ are complex conjugate to each other. These intrinsic close relations of Hankel functions with the cylindrical wave propagation cannot fail to give one an insight to that beautiful branch of knowledge - mathematical physics. Just as "double-pole" and "multiple-pole" spherical wave functions can be obtained by differentiating the "point source" function (1.13), so "double-axis" and "multiple-axis" cylindrical wave functions can be reached by differentiating the symmetrical "axis-source" function $H_0^{(\mu)}(kr)$ $[\mu = 1, 2]$ along any direction r in the following way. Let:

$$D = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} = e^{i\varphi}\left(\frac{\partial}{\partial r} + i\frac{\partial}{r\partial\varphi}\right)$$

be the complex operator, and:

$$D^n = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)^n ,$$

then the "double-axis" or "bi-axis" unsymmetrical cylindrical wave function becomes:

$$(1.19) \quad D H_0^{(\mu)}(kr) = e^{i\varphi}\left(\frac{\partial}{\partial r} + i\frac{\partial}{r\partial\varphi}\right)\frac{1}{\pi}\int_{\mathcal{L}_\mu} e^{ikr\cos\alpha}\,d\alpha$$

$$= (-k)e^{i\varphi}\,\frac{1}{\pi}\int_{\mathcal{L}_\mu} e^{ikr\cos\alpha}\,e^{i(\alpha-\frac{\pi}{2})}\,d\alpha$$

$(\mu = 1, 2)$ ($\mathcal{L}_\mu$ represents suitable path of integration.)

and similarly the "$2n$-multiple axis" unsymmetrical cylindrical wave function becomes:

$$(1.20) \quad D^n H_0^{(\mu)}(kr) = (-k)^n e^{in\varphi}\,\frac{1}{\pi}\int_{\mathcal{L}_\mu} e^{ikr\cos\alpha}\,e^{in(\alpha-\frac{\pi}{2})}\,d\alpha$$

$$(\mu = 1, 2) \quad (n = 0, 1, 2, \ldots\ldots)$$

Presupposing the convergence of these integrals and the feasibility of differentiating under the integral signs, it can easily be shown that (1.19) and (1.20) do satisfy the "wave equation" (1.04) with the function u independent of z.

The general Hankel function of either the first or the second kind of degree n with argument kr is then defined as:

$$(1.21) \qquad H_n^{(\mu)}(kr) = \frac{1}{\pi}\int_{\mathcal{L}_\mu} e^{ikr\cos\alpha} e^{in(\alpha-\frac{\pi}{2})} d\alpha$$

$$(\mu = 1, 2 ; \quad n = 0, 1, 2 \cdots)$$

Putting $z = kr$ in Equation (1.21), we have:

$$(1.21)a \qquad H_n^{(\mu)}(z) = \frac{1}{\pi}\int_{\mathcal{L}_\mu} e^{iz\cos\alpha} e^{in(\alpha-\frac{\pi}{2})} d\alpha$$

$$(\mu = 1, 2 ; \quad n = 0, 1, 2, \cdots\cdots)$$

This is the Somerfeld integral expression for cylindrical functions. The generalization of (1.21) for n to be a complex quantity (say ν) is immediate, as can easily be shown also by a direct substitution in the following way:

$$\int_{\mathcal{L}_\mu} e^{ikr\cos\alpha} f(\alpha)\, d\alpha = \frac{1}{\pi}\int_{\mathcal{L}_\mu} e^{ikr\cos\alpha} e^{i\nu(\alpha-\frac{\pi}{2})} d\alpha$$

This, however, is not required in the present paper. Before giving the corresponding expressions for Bessel- and Newmann- functions from (1.21), its convergence with respect to the different paths of integration $(\mathcal{L}_\mu ; \mu = 1, 2)$ will be carefully considered. It lays the foundation for discussion of certain problems of vital importance in the present paper.

By means of Cauchy's theorem on the integral of a function round a closed contour $(\mathcal{L})$, if $f(\zeta)$ is a function of ζ, analytic at all points inside and on the contour $(\mathcal{L})$, then the following equation always holds.*

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* E. T. Whittaker and G. N. Watson, "A Course of Modern Analysis", Chap. V.

$$\int_{\mathcal{L}} f(\zeta)\, d\zeta = 0$$

We can swing the paths given in Equations (1.17), (1.18), and (1.21)a in such a way so that the function or integrand is analytic throughout the region enclosed between the old and the new paths of integration. The criterion for the convergence of the integral requires that the integrand must vanish identically at the lower and upper limits at infinity; this at the same time assures the closing of the old and the new paths of integration. These characters are similar to that required for the validity of Fourier integral transformation and its application for asymptotic expansion of functions.* We shall now find these new paths $\mathcal{L}_\mu \quad (\mu = 1, 2)$ for $H_n^{(\mu)}(z)$ for all n , real or complex, satisfying the above requirement.

It can be shown that, for the Hankel functions of the first and the second kind, the paths of integration can be deformed in such a way so that we have:

$$(1.22) \qquad H_n^{(1)}(z) = \frac{1}{\pi} \int_{-\eta+i\infty}^{\eta-i\infty} e^{iz\cos\alpha}\, e^{in(\alpha-\frac{\pi}{2})}\, d\alpha$$

and

$$(1.23) \qquad H_n^{(2)}(z) = \frac{1}{\pi} \int_{\eta-i\infty}^{2\pi-\eta+i\infty} e^{iz\cos\alpha}\, e^{in(\alpha-\frac{\pi}{2})}\, d\alpha \quad ,$$

wherein, if z is any complex quantity with phase angle φ or

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* For complete mathematical treatment, the following paper is recommended: A. Haar, "Über asymptotische Entwicklungen von Funktionen", Math. Ann. vol. 96 (1926) pp. 69-107.

$z = |z| e^{i\varphi}$, the following relation must always be observed for the convergence of the integral; i.e.:

$$(1.24) \qquad 0 < (\eta + \varphi) < \pi$$

or

$$-\varphi < \eta < (\pi - \varphi)$$

or

$$-\eta < \varphi < (\pi - \eta)$$

If $z = kr$ is real, then (1.24) reduces to:

$$(1.25) \qquad \varphi = 0 \qquad 0 < \eta < \pi$$

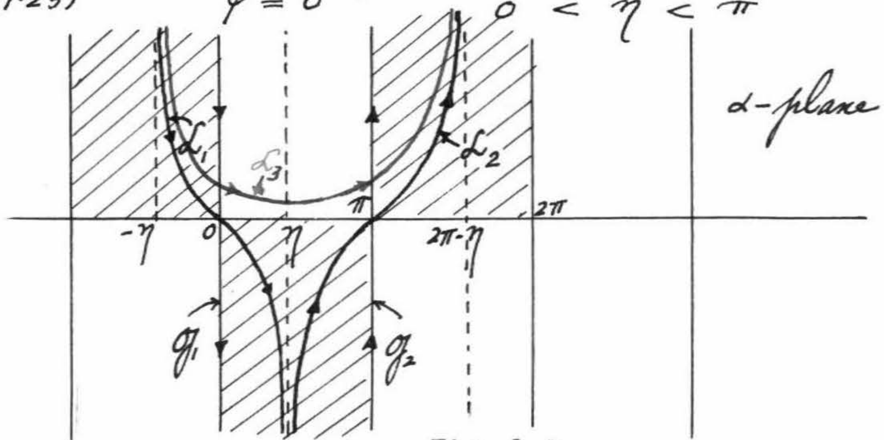

Fig. 1.2

In Fig. 1.2, $\mathcal{G}_1$ and $\mathcal{G}_2$ represent the old paths of integration for Hankel functions of the first and the second kinds, while $\mathcal{L}_1$ and $\mathcal{L}_2$ the corresponding deformed ones. The shaded regions are the limits within which $\mathcal{L}_1$ and $\mathcal{L}_2$ can swing at will; i.e., $0 < \eta < \pi$ for real z. The points $\alpha = 0$ and $\alpha = \pi$ are fixed for $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively.

In case of complex $z = |z|e^{i\varphi}$, the region of swing is changed but the range remains the same.

Then the Sommerfeld definition of Bessel function, with arbitrary complex argument, becomes:

$$(1.26) \quad J_n(z) = \frac{1}{2}\left[H_n^{(1)}(z) + H_n^{(2)}(z) \right]$$

$$= \frac{1}{2\pi}\int_{\alpha_1 \alpha_2} e^{iz\cos\alpha} e^{in(\alpha - \frac{\pi}{2})} d\alpha = \frac{1}{2\pi}\int_{\alpha_3} e^{iz\cos\alpha} e^{in(\alpha - \frac{\pi}{2})} d\alpha$$

and that for Neumann function:

$$(1.27) \quad Y_n(z) \equiv N_n(z) = \frac{1}{2i}\left[H_n^{(1)}(z) - H_n^{(2)}(z) \right]$$

From the above definitions (1.26) and (1.27), we see that $J_n(z)$, also called Bessel function of the first kind, and $Y_n(z) \equiv N_n(z)$, also called Bessel function of the second kind, are real quantities forming in fact the real and the imaginary parts of Hankel's functions, respectively; i.e.:

$$(1.28) \quad \left. \begin{aligned} H_n^{(1)}(z) &\equiv J_n(z) + i\,Y_n(z) \\ H_n^{(2)}(z) &\equiv J_n(z) - i\,Y_n(z) \end{aligned} \right\} (\text{for real } z)$$

With observance of the relation (1.24), a great number of transformation formulae can be obtained from (1.22), (1.23), (1.26), and (1.27). *

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* For excellent treatment of these transformations, the reader is referred to two books: G. N. Watson, "A Treatise on Bessel Functions". R. Weyrich: loc. cit.

It should be noticed here that the Hankel and Neumann functions have a singularity at $z = 0$, but the Bessel function is regular at $z = 0$. This important property serves as a guide for choosing suitable cylindrical functions for the problem at hand. It can be shown easily that for integral-degree n, the Bessel function $J_n(z)$ is a unique and entire function of z and is usually defined with $\eta = \frac{\pi}{2}$ (Fig. 1.2) according to Bessel. This relation (1.24) immediately specifies:

$$-\frac{\pi}{2} < \varphi < \frac{\pi}{2},$$

or z must lie in positive-real half plane. The Hankel functions with integral-degree, however, are not entire functions of z, for which we have eventually:

$$(1.29) \begin{cases} H_n^{(1)}(z e^{m\pi i}) = (-)^{mn}\left[H_n^{(1)}(z) - 2m J_n(z)\right] \\ H_n^{(2)}(z e^{m\pi i}) = (-)^{mn}\left[H_n^{(2)}(z) + 2m J_n(z)\right] \\ H_0^{(1)}(z e^{\pi i}) = H_0^{(1)}(z) - 2J_0(z) = -H_0^{(2)}(z) \\ H_0^{(2)}(z e^{\pi i}) = H_0^{(2)}(z) + 2J_0(z) = H_0^{(1)}(z) + 2H_0^{(2)}(z) \end{cases}$$

These relations will be used in later discussions of the wave potential functions from the point of view of theory of complex functions.

With the help of the above discussions, we can now try to find a general expression for the solution of the wave Equation (1.04) in cylindrical coordinates. By means of the classical product substitution,

$$u = R(r)Z(z)\Phi(\varphi)$$

in (1.04) and dividing through by u, we have:

$$(1.30)\quad \frac{1}{R}\left(\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r}\right) + \frac{1}{\Phi r^2}\frac{\partial^2 \Phi}{\partial \varphi^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} + k^2 = 0$$

On rearranging, there results:

$$(1.31)\quad \frac{1}{R}\left(\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r}\right) + \frac{1}{\Phi r^2}\frac{\partial^2 \Phi}{\partial \varphi^2} + k^2 = \frac{-1}{Z}\frac{\partial^2 Z}{\partial z^2} = \lambda^2$$

where λ^2 is an arbitrary constant, real or complex, independent of the variables r, φ and z. Therefore, (1.31) reduces to the following two equations:

$$(1.32)\quad \begin{cases} \dfrac{d^2 Z}{dz^2} + \lambda^2 Z = 0 \\ \dfrac{r^2}{R}\left(\dfrac{\partial^2 R}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial R}{\partial r}\right) + r^2(k^2 - \lambda^2) = \dfrac{-1}{\Phi}\dfrac{\partial^2 \Phi}{\partial \varphi^2} = \nu^2 \end{cases}$$

where ν^2, being independent of r and φ (also z), may be any arbitrary quantity, real or complex. The second relation again reduces to:

$$(1.33)\quad \frac{d^2 \Phi}{d\varphi^2} + \nu^2 \Phi = 0$$

and

$$(1.34)\quad \frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left[(k^2 - \lambda^2) - \frac{\nu^2}{r^2}\right] R = 0$$

The general integrals for the differential equations (1.32), (1.33), and (1.34) are then:

$$(1.35)\quad \begin{cases} Z = A_1 e^{i\lambda z} + A_2 e^{-i\lambda z} \\ \Phi = B_1 e^{i\nu\varphi} + B_2 e^{-i\nu\varphi} \\ R = C_1 R_\nu^{(1)}(r\sqrt{k^2-\lambda^2}) + C_2 R_\nu^{(2)}(r\sqrt{k^2-\lambda^2}) \end{cases}$$

where $R_\nu^{(1)}$ and $R_\nu^{(2)}$ are any two suitable cylindrical functions defined in (1.22), (1.23), (1.26), and (1.27). Due to the properties of linearity and homogeneity of the wave function, a series summation of the products of these functions is also a solution of (1.04); i.e.:

$$(1.36)\quad u = \sum_{m=1}^{n} Q_m \left(e^{i\lambda z} + A_m e^{-i\lambda z}\right)\left(e^{i\nu\varphi} + B_m e^{-i\nu\varphi}\right)\left(R_\nu^{(1)} + C_m R_\nu^{(2)}\right)$$

where Q_m, A_m, B_m, C_m, being independent of the variables r, φ, and z, may be arbitrary functions of λ and ν. Consequently in the limit for $n \to \infty$, assuming differentiation under integral sign permissible, we have the following definite integral form:

$$(1.37)\quad u = \int_{\mathcal{L}} d\lambda\, d\nu\, e^{i\lambda z} e^{i\nu\varphi}\left[F_1(\lambda,\nu) R_\nu^{(1)}(r\sqrt{k^2-\lambda^2}) + F_2(\lambda,\nu) R_\nu^{(2)}(r\sqrt{k^2-\lambda^2})\right]$$

where $F_1(\lambda,\nu)$ and $F_2(\lambda,\nu)$ are arbitrary functions of λ and ν and $\mathcal{L}$ can be any chosen four-dimensional region of the complex planes of λ and ν. Equation (1.37) is then the most general integral solution of the wave Equation (1.04). In practical cases, there always exist some symmetrical relations and simplifications which will probably bring Equation (1.37) into a manageable form for determination of the characteristics of the phenomena. Although some authors* had tried with certain successes in finding the properties of propagation of electro-magnetic waves under certain boundary conditions starting from Maxwell's field equations without referring to

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* (1) Lord Rayleigh: "On the Passage of Electric Waves through Tubes or the Vibration of Dielectric Cylinders", Phil, Mag. Vol. 43, (1897), pp. 125-132.

(2) A. Sommerfeld, "Über die Fortpflangung elektromagnetischer Wellen langs eines Drahtes", Ann. der Phys. Bd. 67, (1899).

(3) Hondros und Dehye, "Elektromagnetische Wellen an dielektrischen Drähten", Ann. der Phys., Bd. 32, (1910), S. 465-476.

(4) Zahn, "Über den Nach-weis elektromagnetischer Wellen an dielektrischen Drähten", Ann. der Phys., Bd. 49, (1916), S. 907-933.

(5) Shriever, "Elektromagnetischen Wellen an dielektrischen Drähten", Ann. der Phys., Bd. 63, (1920) S. 645-673.

(6) J. R. Carson, S. P. Mead, and S. A. Schelkunoff, "Hyper-frequency Wave Guides - Mathematical Theory", G. C. Southworth, "Hyper-frequency Wave Guides - General Considerations and Experimental Results", Bell System Tech. Journal, April, (1936).

(7) W. L. Barrow, "Transmission of Electromagnetic Waves in Hollow Metal Tubes", Proc. I.R.E., Vol. 24, No. 10, Oct. (1936).

(8) L. Brillouin, "Propagation d'ondes Electromagnetiques dans un Tuyan", Revue Generale d'Electricite, Vol. 22, Aug. (1936), pp. 227-239.
L. Brillouin, "Theoretical Study of Dielectric Cables", Electrical Communication, Vol. 16, April (1938), pp. 350-372.

(9) Lan-Jen Chu, "Electromagnetic Waves in Elliptic Hollow Pipes of Metal", Journal of Applied Phys., Vol. 9, No. 9, Sept. (1938).

any exciting system; an attack of some simple exciting system will bring to light certain specific characteristics from the point of view of physical reality in a much more rigid analytical way, and this is the aim of the present paper. The method is not new. R. Weyrich* treated in a formal mathematical way the cases of an electric dipole, a linear antenna and a magnetic dipole placed along the axis of symmetry in a conducting metal tube. Some admirable experimental check of Weyrich's theoretical work had been conducted by L. Bergmann and L. Krugel≠. A very comprehensive formal discussion of all the physical properties, which is lacking in the above-mentioned papers, forms one purpose of this paper. The second purpose is to use the addition theorems in cylindrical functions to achieve an analytic mathematical formulation for certain practical exciting systems; Weyrich's results thus become special cases of some of the more general formulae derived here and serve at the same time as a check. The third purpose of the present paper is to use the standard method developed with regard to the manipulations of the cylindrical functions to the analyses of wave propagation over a plane earth and along concentric transmission lines; some new and interesting phenomena are believed to have been brought out in a rigorous manner.

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* R. Weyrich, "Über einige Randwertprobleme insbesondere der Elektrodynamik", Jour. für reine und angewandte Math., Bd. 172, (1934) S. 133-150.

≠ L. Bergmann und L. Krügel, "Messungen im Strahlungsfeld einer in Innern eines metallischen Hohlzylinders erregten Linear Antenne", Ann. der. Phys. Bd. 21, (1934).

SECTION II.

An Electric Dipole (or an Elementary Current Element) Inside an Infinite Cylindrical Hollow Metal Tube - Integral Solutions

The idea of an electric dipole and that of an infinitesimally small current element can be used alternately for the same phenomenon. The latter leads naturally in its generalization to a linear physical antenna with any possible current distribution along it. In order to describe the field components due to such an exciter in a simple but unique way, we shall introduce here the "general magnetic vector potential" $\mathcal{U}$, whose curl gives the magnetic induction. Before going to the mathematical formulation, a list of the notations to be used in the following analysis will be tabulated: (Gaussian Units are used here.)

$\mathcal{H}$ = Vector magnetic field intensity with components H_z, H_r, and H_φ, in e.m.u. (Gaussian units).

$\mathcal{E}$ = Vector electric field intensity with components E_z, E_r, and E_φ, in e.s.u. (Gaussian units.)

ε = Dielectric constant, dimensionless in Gaussian unit used throughout.

μ = Permeability of medium.

σ = Conductivity of medium.

c = Velocity of light in vacuum space, equals approx. to 3×10^{10} cm./sec.

$\mathcal{U}$ = General magnetic vector potential with components u_z, u_r, and u_φ. (Relation of definition being $\mu \mathcal{H} = \nabla \times \mathcal{U}$)

$\mathcal{U}_m$ = General electric vector potential with components u_{mz}, u_{mr}, and $u_{m\varphi}$. (Relation of definition being $\varepsilon \mathcal{E} = \nabla \times \mathcal{U}_m$).

$\mathcal{N}$ = Poynting vector with components N_z, N_r, and N_φ.

$\mathcal{J}$ = Vector conduction current with components j_z', j_r', and j_φ'.

z, r, φ = Cylindrical coordinates of field point.

ζ, r_0, φ_0 = Cylindrical coordinates of source.

a = Inner radius of the cylindrical hollow metal tube in cm.

ρ = Charge density.

The general Maxwell field equations, in Gaussian units, are:

$$(2.01) \quad \begin{cases} \nabla \times \mathcal{H} = \frac{1}{c}\left(4\pi \mathcal{J} + \varepsilon \frac{\partial \mathcal{E}}{\partial t}\right) = \frac{1}{c}\left(4\pi\sigma \mathcal{E} + \varepsilon \frac{\partial \mathcal{E}}{\partial t}\right) \\ \nabla \times \mathcal{E} = -\frac{\mu}{c}\frac{\partial \mathcal{H}}{\partial t} \\ \nabla \cdot (\mu \mathcal{H}) = 0 \qquad , \qquad \nabla \cdot (\varepsilon \mathcal{E}) = 4\pi\rho \end{cases}$$

The general magnetic vector potential $\mathcal{U}$, called by some authors the Hertzian function, is defined as:

$$(2.02) \qquad \mu \mathcal{H} = \nabla \times \mathcal{U}$$

The following analysis is based upon an electric dipole or an infinitesimally short current element placed at any position inside an infinite cylindrical metal tube with axis of the dipole or the current element parallel to that of the tube. The primary potential function $\mathfrak{U}_0$ has then only a z-component u_z. From Equation (1.13), we have, on suppressing the time factor $e^{-i\omega t}$:

$$(2.03) \qquad u_z = u_0 = \frac{e^{ikR}}{R}$$

where the subscript z is replaced by o , to signify a primary source function.

Since the characteristic constants ε , μ , and σ , are discontinuous at $r=a$, we shall call the dielectric air medium ($r<a$) as medium 1; and the conducting metallic medium ($r>a$) as medium 2. Although any practical hollow metal tube has a finite thickness, we shall consider, however, the outer radius of this tube extending to infinity. This is justified on account of the fact that the electromagnetic waves at very high frequencies (as is necessary here) can rarely penetrate a fractional part of one centimeter of the metal sheath. *

The field components due to a dipole placed at (ζ, r_0, φ_0) inside the cylindrical tube are:

$$\mathfrak{H} = H_\varphi, H_r, \quad H_z = 0 \ ; \quad \mathfrak{E} = E_z, E_r, E_\varphi .$$

To get formal relations between these components, we expand the vector Maxwell field Equations (2.01):

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* Abraham and Becker, "Electricity and Magnetism", p. 190.
Smythe, "Static and Dynamic Electricity", pp. 452-453.

$$
(2.04)\begin{cases}
\frac{1}{r}\left\{\frac{\partial}{\partial r}(rH_\varphi) - \frac{\partial H_r}{\partial \varphi}\right\} = \frac{4\pi\sigma}{c} E_z + \frac{\varepsilon}{c}\frac{\partial E_z}{\partial t} & \text{(a)} \\
-\frac{\partial H_\varphi}{\partial z} = \frac{4\pi\sigma}{c} E_r + \frac{\varepsilon}{c}\frac{\partial E_r}{\partial t} & \text{(b)} \\
\frac{\partial H_r}{\partial z} = \frac{4\pi\sigma}{c} E_\varphi + \frac{\varepsilon}{c}\frac{\partial E_\varphi}{\partial t} & \text{(c)} \\
\frac{1}{r}\frac{\partial E_z}{\partial \varphi} - \frac{\partial E_\varphi}{\partial z} = -\frac{\mu}{c}\frac{\partial H_r}{\partial t} & \text{(d)} \\
\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = -\frac{\mu}{c}\frac{\partial H_\varphi}{\partial t} & \text{(e)}
\end{cases}
$$

From the definition (2.02), we have then:

$$
(2.05)\quad\begin{cases}
\mu H_\varphi = -\frac{\partial u_z}{\partial r} \\
\mu H_r = \frac{1}{r}\frac{\partial u_z}{\partial \varphi}
\end{cases}
$$

Introducing the time factor $e^{-i\omega t}$ into the above equations, there result:

$$
\mathcal{U} = \mathcal{R}e\,[u_z e^{-i\omega t}] = \mathcal{R}e\,[u\, e^{-i\omega t}]
$$

$$
(2.06)\begin{cases}
H_\varphi = \mathcal{R}e\left[-\frac{1}{\mu}\frac{\partial u}{\partial r} e^{-i\omega t}\right] \\
H_r = \mathcal{R}e\left[\frac{1}{\mu r}\frac{\partial u}{\partial \varphi} e^{-i\omega t}\right] \\
H_z = 0 \\
E_z = \mathcal{R}e\left[\frac{i\omega}{ck^2}\left(k^2 u + \frac{\partial^2 u}{\partial z^2}\right) e^{-i\omega t}\right] \\
E_r = \mathcal{R}e\left[\frac{i\omega}{ck^2}\left(\frac{\partial^2 u}{\partial z \partial r}\right) e^{-i\omega t}\right]
\end{cases}
$$

$$
E_\varphi = \mathcal{R}e\left[\frac{i\omega}{ck^2}\frac{1}{r}\frac{\partial^2 u}{\partial z \partial \varphi} e^{-i\omega t}\right]
$$

and

$$(2.07)\qquad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} + k^2 u = 0$$

where

$$(2.08)\qquad k^2 = \frac{\omega^2 \varepsilon \mu + i\, 4\pi\omega\sigma\mu}{c^2}$$

In (2.06) and (2.07), u is the resultant potential function or the sum of the primary source function and the secondary disturbance function due to the presence of the cylindrical metal sheath. All field components (2.06) are the resultant ones obtained from the resultant potential function .

In order to avail ourselves of the integral expression (1.37), we shall first effect a formal mathematical transformation of Equation (2.03) for the primary source function by means of the classical Fourier integral theorem,* which may be written as:

$$(2.09)\qquad u_0(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} u_0(\xi)\, e^{-i\lambda(\xi - x)}\, d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{+i\lambda x} d\lambda \int_{-\infty}^{\infty} u_0(\xi)\, e^{-i\lambda\xi}\, d\xi$$

with certain properties which must be satisfied by the function involved.

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* R. Courant und D. Hilbert, "Methoden der Math. Phys.", S. 65-70.
E. C. Titchmarsh, "Introduction to the Theory of Fourier Integral".

Let now:

$$G(\lambda) = \int_{-\infty}^{\infty} u_0(\xi)\, e^{-i\lambda\xi}\, d\xi$$

where

$$u_0(\xi) = \frac{e^{ikR}}{R} = \frac{\exp.[ik\sqrt{\rho^2+\xi^2}]}{\sqrt{\rho^2+\xi^2}} \quad , \quad (\xi = z - \varsigma)$$

$$(2.10) \quad \therefore \; G(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda\xi} \cdot \frac{\exp.[ik\sqrt{\rho^2+\xi^2}]}{\sqrt{\rho^2+\xi^2}}\, d\xi$$

$$= \int_{-\infty}^{\infty} \cos\lambda\xi \cdot \frac{\exp.[ik\sqrt{\rho^2+\xi^2}]}{\sqrt{\rho^2+\xi^2}}\, d\xi$$

From the following established relation* :

$$\int_{-\infty}^{\infty} \cos\lambda\xi \cdot \frac{\exp.[ik\sqrt{\rho^2+(x-\xi)^2}]}{\sqrt{\rho^2+(x-\xi)^2}}\, d\xi = i\pi H_0^{(1)}(\rho\sqrt{k^2-\lambda^2})\cos\lambda x$$

If we put $x = 0$, we obtain the desired integral result for $G(\lambda)$:

$$(2.10)a \qquad G(\lambda) = i\pi H_0^{(1)}(\rho\sqrt{k^2-\lambda^2})$$

Substituting this into (2.09), the corresponding integral expression of the type (1.37) is obtained:

$$u_0(x) = \frac{i}{2}\int_{-\infty}^{\infty} e^{i\lambda x} H_0^{(1)}(\rho\sqrt{k^2-\lambda^2})\, d\lambda$$

or

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* Riemann - Weber, "Differentialgleichungen der Phys.", S. 541-550.
R. Weyrich, "Uber das strahlungsfeld einer endlichen Antenne zwischen zwei Ebenen", Ann. der Phys., 1929.

$$(2.11)\quad u_0(\rho, z-\zeta) = \frac{e^{ik\sqrt{\rho^2+(z-\zeta)^2}}}{\sqrt{\rho^2+(z-\zeta)^2}} = \frac{i}{2}\int_{-\infty}^{\infty} e^{i\lambda(z-\zeta)} H_0^{(1)}(\rho\sqrt{k^2-\lambda^2})\,d\lambda$$

where $\rho = [r^2 + r_0^2 - 2 r r_0 \cos(\varphi - \varphi_0)]^{1/2}$ and (z, r, φ) and (ζ, r_0, φ_0) being the cylindrical coordinates for the field point and for the source element, respectively.

In order to fit the boundary conditions at $r = a$ on the inner surface of the metal tube, we must expand $H_0^{(1)}(\rho\sqrt{k^2-\lambda^2})$ according to the addition theorem of cylindrical functions. We have, in fact, * :

$$(2.12)\quad H_0^{(1)}(\rho\sqrt{k^2-\lambda^2}) = H_0^{(1)}\left(\sqrt{r^2+r_0^2-2rr_0\cos(\varphi-\varphi_0)}\,\sqrt{k^2-\lambda^2}\right)$$

$$= \begin{cases} \sum_{m=-\infty}^{\infty} H_m^{(1)}(r_0\sqrt{k^2-\lambda^2}) J_m(r\sqrt{k^2-\lambda^2}) e^{im(\varphi-\varphi_0)}, & \text{for } r < r_0 \\ \sum_{m=-\infty}^{\infty} H_m^{(1)}(r\sqrt{k^2-\lambda^2}) J_m(r_0\sqrt{k^2-\lambda^2}) e^{im(\varphi-\varphi_0)} & \text{for } r > r_0 \end{cases}$$

Exactly similar expansions hold true for $H_0^{(2)}(\rho\sqrt{k^2-\lambda^2})$, $J_0(\rho\sqrt{k^2-\lambda^2})$ and $Y_0(\rho\sqrt{k^2-\lambda^2}) \equiv N_0(\rho\sqrt{k^2-\lambda^2})$. Substituting (2.12) into (2.11), we obtain the general primary potential function at (z, r, φ) due to an electric dipole at (ζ, r_0, φ_0):

$$(2.13)\quad u_0(\rho, z-\zeta) = \frac{e^{ik\sqrt{\rho^2+(z-\zeta)^2}}}{\sqrt{\rho^2+(z-\zeta)^2}}$$

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* Riemann - Weber, "Differentialgleichungen der Phys.", Bd. 2, S. 491.
G. H. Watson, "Treatise on Bessel Functions", Chap. XI., — The factor $1/2\pi$ in Watson's book should be unity —; Schelkunoff used similar theorem in finding mutual impedance and radiation resistance; "Modified Sommerfeld Integral", Proc. I.R.E. (1936).

$$= \begin{cases} \frac{i}{2}\sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi_0)} \int_{-\infty}^{\infty} e^{i\lambda(z-s)} H_m^{(1)}(r_0\sqrt{k^2-\lambda^2}) J_m(r\sqrt{k^2-\lambda^2})\, d\lambda & \text{for } r < r_0 \\ \frac{i}{2}\sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi_0)} \int_{-\infty}^{\infty} e^{i\lambda(z-s)} J_m(r_0\sqrt{k^2-\lambda^2}) H_m^{(1)}(r\sqrt{k^2-\lambda^2})\, d\lambda & \text{for } r > r_0 \end{cases}$$

With help of the relations (2.13) and (1.37), we can set up the integral equation for the resultant potential function for medii 1, and 2, respectively:

$$(2.14)\quad \mathcal{U}_1 = \begin{cases} \frac{1}{2}\sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi_0)} \int_{-\infty}^{\infty} e^{i\lambda(z-s)} \left[iH_m^{(1)}(r_0\sqrt{k_1^2-\lambda^2}) + F_1(\lambda) \right] J_m(r\sqrt{k_1^2-\lambda^2})\, d\lambda & \text{for } 0 \leq r < r_0 \\ \frac{1}{2}\sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi_0)} \int_{-\infty}^{\infty} e^{i\lambda(z-s)} \left[iJ_m(r_0\sqrt{k_1^2-\lambda^2}) H_m^{(1)}(r\sqrt{k_1^2-\lambda^2}) + F_1(\lambda) J_m(r\sqrt{k_1^2-\lambda^2}) \right] d\lambda & \text{for } r_0 < r < a \end{cases}$$

$$(2.15)\quad \mathcal{U}_2 = \frac{1}{2}\sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi_0)} \int_{-\infty}^{\infty} e^{i\lambda(z-s)} F_2(\lambda) H_m^{(1)}(r\sqrt{k_2^2-\lambda^2})\, d\lambda \qquad \text{for } r > a$$

Now if we assume finite conductivity for the metal tube, the boundary conditions regarding the tangential components of $\mathcal{H}$ and $\mathcal{E}$ which must be satisfied at $r = a$ yield:

$$(2.16)\quad \begin{cases} \frac{1}{\mu_1}\left(\frac{\partial \mathcal{U}_1}{\partial r}\right)_{r=a} = \frac{1}{\mu_2}\left(\frac{\partial \mathcal{U}_2}{\partial r}\right)_{r=a} \\ \frac{1}{k_1^2}\left(k_1^2 \mathcal{U}_1 + \frac{\partial^2 \mathcal{U}_1}{\partial z^2}\right)_{r=a} = \frac{1}{k_2^2}\left(k_2^2 \mathcal{U}_2 + \frac{\partial^2 \mathcal{U}_2}{\partial z^2}\right)_{r=a} \\ \frac{1}{k_1^2}\left(\frac{\partial^2 \mathcal{U}_1}{\partial z \partial \varphi}\right)_{r=a} = \frac{1}{k_2^2}\left(\frac{\partial^2 \mathcal{U}_2}{\partial z \partial \varphi}\right)_{r=a} \end{cases}$$

The solution of two unknowns $F_1(\lambda)$ and $F_2(\lambda)$ from three equations, with the last two incompatible, is evidently impossible.

This, however, is not merely a mathematical paradox, since the eddy currents produced in the metal sheath give rise to the co-existence of $\mathcal{U}_r$ and $\mathcal{U}_\varphi$ with the primary function $\mathcal{U}_z$ and, therefore, H_z, which is neglected in the beginning. The above reasoning can be put into mathematical form but the labor involved would be prohibitive and is not warranted here. This, however, gives a definite physical reason why H_z and E_z must coexist for unsymmetrical dissipative case. It is simply due to the fact that the eddy current in the metal sheath creates two new components $\mathcal{U}_r$ and $\mathcal{U}_\varphi$ of the general magnetic potential function $\mathcal{U}$.

The above difficulty is overcome if we assume that the conductivity of the metal tube is very high and that we could find the limiting boundary conditions when the tube conductivity (σ_2) approaches infinity. This requires the vanishing of the tangential electric field components for medium 1. at $r=a$; i.e.:

$$E_{z1}\Big|_{r=a} = 0 \quad \& \quad E_{\varphi 1}\Big|_{r=a} = 0$$

or $$\left(k_1^2 + \frac{\partial^2}{\partial z^2}\right)\mathcal{U}_1 = (k_1^2 - \lambda^2)\,\mathcal{U}_1\Big|_{r=a} = 0 \quad \& \quad \left(\frac{\partial^2 \mathcal{U}_1}{\partial z \partial \varphi}\right) = -m\lambda\, \mathcal{U}_1\Big|_{r=a} = 0 .$$

The above two relations give the same result:

$$i J_m\left(r_0\sqrt{k_1^2-\lambda^2}\right) H_m^{(1)}\left(a\sqrt{k_1^2-\lambda^2}\right) + F_1(\lambda) J_m\left(a\sqrt{k_1^2-\lambda^2}\right) = 0$$

or (2.17) $$F_1(\lambda) = -i\,\frac{J_m\left(r_0\sqrt{k_1^2-\lambda^2}\right) H_m^{(1)}\left(a\sqrt{k_1^2-\lambda^2}\right)}{J_m\left(a\sqrt{k_1^2-\lambda^2}\right)}$$

while $\mathcal{U}_2$ needs not be considered.

Substituting (2.17) into (2.14), we obtain:

$$(2.18)\quad u_1 = \begin{cases} \dfrac{i}{2}\sum\limits_{m=-\infty}^{\infty} e^{im(\varphi-\varphi_0)}\displaystyle\int_{-\infty}^{\infty} e^{i\lambda(z-\varsigma)}\left[\dfrac{J_m(a\xi_1)H_m^{(1)}(r_0\xi_1)-J_m(r_0\xi_1)H_m^{(1)}(a\xi_1)}{J_m(a\xi_1)}\right]J_m(r\xi_1)\,d\lambda \\ \qquad = \dfrac{i}{2}\sum\limits_{m=-\infty}^{\infty} e^{im(\varphi-\varphi_0)}\displaystyle\int_{-\infty}^{\infty} e^{i\lambda(z-\varsigma)} W_m(a,r_0) J_m(r\xi_1)\,d\lambda \quad \text{for } r<r_0 \\ \dfrac{i}{2}\sum e^{im(\varphi-\varphi_0)}\displaystyle\int_{-\infty}^{\infty} e^{i\lambda(z-\varsigma)}\left[\dfrac{J_m(a\xi_1)H_m^{(1)}(r\xi_1)-H_m^{(1)}(a\xi_1)J_m(r\xi_1)}{J_m(a\xi_1)}\right]J_m(r_0\xi_1)\,d\lambda \\ \qquad = \dfrac{i}{2}\sum e^{im(\varphi-\varphi_0)}\displaystyle\int_{-\infty}^{\infty} e^{i\lambda(z-\varsigma)} W_m(a,r) J_m(r_0\xi_1)\,d\lambda \quad \text{for } r>r_0 \end{cases}$$

where

$$\xi_1 = \sqrt{k_1^2-\lambda^2}\quad , \quad W_m(a,r) = \frac{J_m(a\xi_1)H_m^{(1)}(r\xi_1)-H_m^{(1)}(a\xi_1)J_m(r\xi_1)}{J_m(a\xi_1)}$$

From the general expression (2.18), it would be easy to obtain integrals for the case with circumferentially arranged dipoles on a circle of radius r_0 , or other irregular setups.

In case the location of the dipole is at $(\varsigma, 0, 0)$, then Equations (2.18) degenerate into one single relation:

$$(2.19)\quad u_1 = \frac{i}{2}\int_{-\infty}^{\infty} e^{i\lambda(z-\varsigma)}\left[\frac{J_0(a\xi_1)H_0^{(1)}(r\xi_1)-J_0(r\xi_1)H_0^{(1)}(a\xi_1)}{J_0(a\xi_1)}\right]d\lambda$$

By aid of the transformation relations (1.29), it can be shown that the integrands of (2.18) and (2.19) are all meromorphous functions of the arguments involved, or, in other words, there is no "branch point" in the whole complex λ-plane. Their evaluation thus reduces to formal expansion by the theory of residues. The symmetrical case (2.19) with dipole along the axis is simply one term of the infinite series for the unsymmetrical case (2.18), which may be called eventually a "cylindrical harmonic expansion". It consistutes consequently a formal analogy to the familiar "circular harmonic expansion". Just as a trigonometric function, cosine or sine, has an infinite number of roots, so does the Bessel function $J_m(x) = 0$. Equations (2.18), after evaluation of the residues at the poles, yield a double infinite series, each term of which represents a "distinct mode" of propagation. The attenuations and velocities of these double infinite "modes" are different from each other and would be independent upon each other if the transmission system is "uniform and homogeneous". The resultant field at any point is thus a superposition of all the modes.

The field components corresponding to the potential function; u_1., (2.18), can be found by substituting (2.18) into (2.06).

The above discussion reveals the fact that a deviation from the symmetrical field configuration by an off-axis location of the exciter causes the total energy emitted to be divided among the different "modes" thus created for different m in (2.18). The energy for each mode is thereby decreased and so do the corresponding field components. It would be of interest to see what form expressions (2.18) assume for a slight off-axis location. According to the theory of complex functions, the integrals are to be expanded into series by evaluating the residues at the poles corresponding to roots of $J_m(a\xi_1) = J_m(a\sqrt{k_1^2-\lambda^2}) = J_m(\chi_m) = 0$,

say $\chi_{m0}, \chi_{m1}, \chi_{m2}, \ldots\ldots\ldots \chi_{mn}, \ldots\ldots$.

$$\therefore \quad u_1 = \frac{1}{2}\sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi_0)} \sum_{n=0}^{\infty} e^{i\lambda_{mn}(z-\zeta)} \cdot \left(\frac{\chi_{mn}}{a^2\lambda_{mn}}\right) \frac{J_m(\frac{r_0}{a}\chi_{mn})\, Y_m(\chi_{mn})}{J_{mn}'(\chi_{mn})} J_m\left(\frac{r}{a}\chi_{mn}\right)$$

When $r_0 \ll a$ and $\frac{r_0}{a}\chi_{mn} \ll 1$ for the first few roots, which and which only need be considered, then u_1 becomes approximately:

$$(2.20) \quad u_1 = \frac{-1}{2}\sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi_0)} \sum_{n=0}^{\infty} \frac{e^{i\lambda_{mn}(z-\zeta)}}{m!} \left(\frac{\chi_{mn}}{a^2\lambda_{mn}}\right) \left(\frac{r_0}{2a}\chi_{mn}\right)^m \frac{Y_m(\chi_{mn})}{J_m'(\chi_{mn})} J_m\left(\frac{r}{a}\chi_{mn}\right)$$

The potential function is proportional to $\left(\frac{r_0}{a}\right)^m$ for each mode of propagation. The corresponding energy is therefore proportional to $\left(\frac{r_0}{a}\right)^{2m}$ for all modes with same m. From this we get a fair picture of the energy distribution among the different modes. Or we may group together the energies for all modes for the same m under a single unit, then the energy unit distribution for different m's has the following shape:

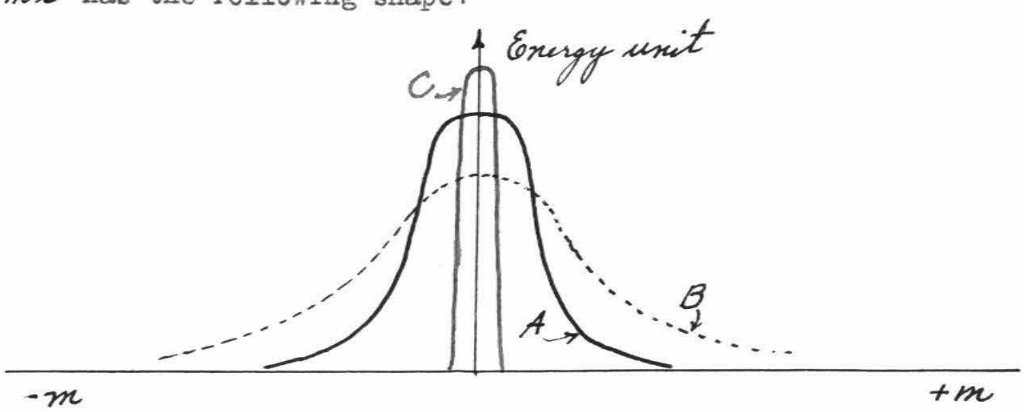

Fig. II-1.

Energy Distribution.

Curves (A) and (B) give a general idea of a "sharp" and a "broad" distribution, respectively. The smaller is the value $\left(\frac{h_0}{a}\right)$, the sharper is the distribution, and the greater is the concentration of energy at $m=0$. (A) degenerates into (C) for the symmetrical case when all energy emitted resides in the single unit $m=0$.

Because in practical application we can not use all the different modes of propagation with different attenuations and velocities, which in fact vitiates the reception, it is then evident that the axially symmetrical operation is the most efficient one for transmission and reception. This is immediately a conclusion of considerable practical importance.

The above development so far has been limited to the condition of perfect conducting metal tube. From this no direct rigorous method can be obtained for accurate calculation of the most important quantity - attenuation constant of propagation. Fortunately rigorous field functions can be derived for metal tube of finite conductivity if the field configuration is symmetrical about the axis. This procedure is at the same time necessary and important, because with air as the dielectric medium, attenuation is primarily due to the finite conductivity of the metal sheath. That, opposite to the off-axis case, this is mathematically possible, at once finds its physical substantiation. With a dipole or linear antenna placed along the axis, the eddy current produced in the metal sheath by the symmetrical field would not give birth to new components of the potential function $\mathcal{U}$ except the z-component as the source possesses. The interpretation given above is believed to have answered in a unique way the question raised by some authors, relating to the field structure for dissipative and non-dissipative cases.

Now we shall consider the symmetrical case with the hollow tube of finite conductivity. Then the primary function u_0 becomes, (refer to (2.11)) :

$$(2.21) \quad u_0(r, z-s) = \frac{i}{2}\int_{-\infty}^{\infty} e^{i\lambda(z-s)} H_0^{(1)}(r\sqrt{k^2-\lambda^2})\,d\lambda$$

$$(2.22) \quad = i\int_0^{\infty} \cos\lambda(z-s)\, H_0^{(1)}(r\sqrt{k^2-\lambda^2})\,d\lambda$$

This is of the form (1.37). Therefore, the resultant general magnetic vector potential or Hertzian function in the dielectric air medium 1 $(r<a)$ is:

$$(2.23) \quad u_1 = \frac{1}{2}\int_{-\infty}^{\infty} e^{i\lambda(z-s)}\left[iH_0^{(1)}(r\sqrt{k_1^2-\lambda^2}) + F_1(\lambda)J_0(r\sqrt{k_1^2-\lambda^2})\right]d\lambda$$

where for the disturbance of the metal sheath $J_0(r\sqrt{k_1^2-\lambda^2})$ is used because it must not be infinite for $r=0$.

It can also be reasoned from the physical side: Since the Bessel function represents a standing wave in radial direction, it is the proper function to be used for the "additional" or "disturbance" solution in medium 1; while $H_0^{(1)}(r\sqrt{k^2-\lambda^2})$, the Hankel function of the first kind represents a divergent symmetrical cylindrical wave, it is the proper function to be employed for the solution in medium 2. This includes the case of perfect conducting hollow metal tube and lossless dielectric medium 1.

Consequently, for medium 2, the general magnetic potential or Hertzian function, except for an unimportant multiplying factor, is:

$$(2.24)\qquad u_2 = \frac{1}{2}\int_{-\infty}^{\infty} e^{i\lambda(z-\varsigma)} F_2(\lambda) H_0^{(1)}\left(r\sqrt{k_2^2-\lambda^2}\right) d\lambda$$

$$= \int_0^{\infty} \cos\lambda(z-\varsigma)\, F_2(\lambda) H_0^{(1)}\left(r\sqrt{k_2^2-\lambda^2}\right) d\lambda$$

The Maxwell theory requires that at the boundary surface of discontinuity of media, the tangential components of the magnetic field intensity $\mathfrak{H}$ and that of the electric field intensity $\mathfrak{E}$ on both sides must be equal. We have then from (2.04) and (2.06), for the present symmetrical case:

$$(2.25)\left\{\begin{aligned} \frac{1}{\mu_1}\left(\frac{\partial u_1}{\partial r}\right)_{r=a} &= \frac{1}{\mu_2}\left(\frac{\partial u_2}{\partial r}\right)_{r=a} \\ \frac{1}{k_1^2}\left(k_1^2 u_1 + \frac{\partial^2 u_1}{\partial z^2}\right)_{r=a} &= \frac{1}{k_2^2}\left(k_2^2 u_2 + \frac{\partial^2 u_2}{\partial z^2}\right)_{r=a} \end{aligned}\right.$$

Substituting u_1 and u_2 from Equations (2.23), (2.24) into the above relations gives:

$$(2.26)\left\{\begin{aligned} \frac{1}{\mu_1}\sqrt{k_1^2-\lambda^2}\left[iH_1^{(1)}\left(a\sqrt{k_1^2-\lambda^2}\right)+F_1(\lambda)J_1\left(a\sqrt{k_1^2-\lambda^2}\right)\right] &= \frac{1}{\mu_2}\sqrt{k_2^2-\lambda^2}\,F_2(\lambda)H_1^{(1)}\left(a\sqrt{k_2^2-\lambda^2}\right) \\ \frac{k_1^2-\lambda^2}{k_1^2}\left[iH_0^{(1)}\left(a\sqrt{k_1^2-\lambda^2}\right)+F_1(\lambda)J_0\left(a\sqrt{k_1^2-\lambda^2}\right)\right] &= \frac{k_2^2-\lambda^2}{k_2^2}F_2(\lambda)H_0^{(1)}\left(a\sqrt{k_2^2-\lambda^2}\right) \end{aligned}\right.$$

Solving (2.26) for $F_1(\lambda)$ and $F_2(\lambda)$, we obtain:

$$(2.27)\begin{cases} F_1(\lambda) = -i\,\dfrac{\mu_1 k_2^2\sqrt{k_1^2-\lambda^2}\,H_0^{(1)}(a\sqrt{k_1^2-\lambda^2})H_1^{(1)}(a\sqrt{k_2^2-\lambda^2}) - \mu_2 k_1^2\sqrt{k_2^2-\lambda^2}\,H_1^{(1)}(a\sqrt{k_1^2-\lambda^2})H_0^{(1)}(a\sqrt{k_2^2-\lambda^2})}{\mu_1 k_2^2\sqrt{k_1^2-\lambda^2}\,J_0(a\sqrt{k_1^2-\lambda^2})H_1^{(1)}(a\sqrt{k_2^2-\lambda^2}) - \mu_2 k_1^2\sqrt{k_2^2-\lambda^2}\,J_1(a\sqrt{k_1^2-\lambda^2})H_0^{(1)}(a\sqrt{k_2^2-\lambda^2})} \\[2em] F_2(\lambda) = \dfrac{i\mu_1 k_2^2(k_1^2-\lambda^2)\left[J_0(a\sqrt{k_1^2-\lambda^2})H_1^{(1)}(a\sqrt{k_1^2-\lambda^2}) - J_1(a\sqrt{k_1^2-\lambda^2})H_0^{(1)}(a\sqrt{k_1^2-\lambda^2})\right]}{\sqrt{k_2^2-\lambda^2}\left[\mu_1 k_2^2\sqrt{k_1^2-\lambda^2}\,J_0(a\sqrt{k_1^2-\lambda^2})H_1^{(1)}(a\sqrt{k_2^2-\lambda^2}) - \mu_2 k_1^2\sqrt{k_2^2-\lambda^2}\,J_1(a\sqrt{k_1^2-\lambda^2})H_0^{(1)}(a\sqrt{k_2^2-\lambda^2})\right]} \end{cases}$$

$$= \frac{2\mu_1 k_2^2\sqrt{k_1^2-\lambda^2}}{\pi a\sqrt{k_2^2-\lambda^2}\left[\mu_1 k_2^2\sqrt{k_1^2-\lambda^2}\,J_0(a\sqrt{k_1^2-\lambda^2})H_1^{(1)}(a\sqrt{k_2^2-\lambda^2}) - \mu_2 k_1^2\sqrt{k_2^2-\lambda^2}\,J_1(a\sqrt{k_1^2-\lambda^2})H_0^{(1)}(a\sqrt{k_2^2-\lambda^2})\right]}$$

The last relation is obtained on account of the Wronskian determinant:*

$$J_0(z)\,H_1^{(1)}(z) - J_1(z)\,H_0^{(1)}(z) = -\frac{2i}{\pi z}$$

Thus substituting the expressions obtained for $F_1(\lambda)$ and $F_2(\lambda)$ into Equations (2.23), (2.24), since all the boundary conditions are satisfied, we obtain the complete solutions of Equation (2.07) $\left[\text{with } \frac{\partial}{\partial\varphi} = 0\right]$ for media 1 and 2. These give the most general analytic expressions for the general magnetic potentials or Hertzian functions for any two media of constants ε_1, μ_1, σ_1, and ε_2, μ_2, σ_2, respectively, with a cylindrical separating surface at $r = a$. These integral expressions must be transformed, by means of the theory of residues, into convenient forms for actual computation. An investi-

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* R. Weyrich, "Zylinderfunktionen und ihre Anwendungen", p. 75. Jahuke and Emde, Functional Tables, p. 144.

gation of the equations for u_1 and u_2 shows that besides the ordinary singularities at the poles of the integrands, there exist also four branch points at $\lambda = \pm k_1$ and $\lambda = \pm k_2$. For the general case that both σ_1 and σ_2 are finite, the actual integration processes are very laborious and do not admit of immediate physical interpretation.

For the specific problem at hand, however, these expressions are susceptible to considerable simplification. The conductivity of the air medium is always negligibly small while that for the metal tube is usually very large. Then:

$$k_1^2 = \frac{\omega^2 \varepsilon_1 \mu_1 + i\,4\pi\omega\sigma_1\mu_1}{c^2} \approx \frac{\omega^2 \varepsilon_1 \mu_1}{c^2} = \frac{\omega^2}{v_0^2}$$

$$k_2^2 = \frac{\omega^2 \varepsilon_2 \mu_2 + i\,4\pi\omega\sigma_2\mu_2}{c^2} \approx \frac{i\,4\pi\omega\sigma_2\mu_2}{c^2} \quad \text{(very large)}$$

and $(k_2^2 - \lambda^2) \approx k_2^2$ (very large).

From the asymptotic expansions of cylindrical functions of large argument, we have: *

$$H_n^{(1)}(z) = \sqrt{\frac{2}{\pi z}}\, e^{i(z - \frac{1}{2}n\pi - \frac{1}{4}\pi)} \left[1 + O\left(\tfrac{1}{z}\right)\right]$$

for $n = 0, 1, 2, \text{------}$,

and $$\lim_{|z|\to\infty} \frac{H_0^{(1)}(z)}{H_1^{(1)}(z)} = e^{i\frac{\pi}{2}} = i$$

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* Jahnke and Emde, 'Functional Tables,' pp. 137-139.

Then $F_1(\lambda)$ and $F_2(\lambda)$ become, after dividing the numerator and the denominator by $H_1^{(1)}(a\sqrt{k_2^2-\lambda^2})$:

$$(2.28) \quad F_1(\lambda) = -i\,\frac{\mu_1 k_2^2\sqrt{k_1^2-\lambda^2}\,H_0^{(1)}(a\sqrt{k_1^2-\lambda^2}) - i\mu_2 k_1^2\sqrt{k_2^2-\lambda^2}\,H_1^{(1)}(a\sqrt{k_1^2-\lambda^2})}{\mu_1 k_2^2\sqrt{k_1^2-\lambda^2}\,J_0(a\sqrt{k_1^2-\lambda^2}) - i\mu_2 k_1^2\sqrt{k_2^2-\lambda^2}\,J_1(a\sqrt{k_1^2-\lambda^2})}$$

$$\approx -i\,\frac{H_0^{(1)}(a\sqrt{k_1^2-\lambda^2})}{J_0(a\sqrt{k_1^2-\lambda^2})}$$

$$(2.29) \quad F_2(\lambda) = \frac{2\mu_1 k_2^2\sqrt{k_1^2-\lambda^2}}{\pi a\sqrt{k_2^2-\lambda^2}\left[\mu_1 k_2^2\sqrt{k_1^2-\lambda^2}\,J_0(a\sqrt{k_1^2-\lambda^2}) - i\mu_2 k_1^2\sqrt{k_2^2-\lambda^2}\,J_1(a\sqrt{k_1^2-\lambda^2})\right]H_1^{(1)}(a\sqrt{k_2^2-\lambda^2})}$$

$$\approx \frac{2}{\pi a\sqrt{k_2^2-\lambda^2}\,J_0(a\sqrt{k_1^2-\lambda^2})\,H_1^{(1)}(a\sqrt{k_2^2-\lambda^2})} \approx \frac{2}{\pi a k_2\,J_0(a\sqrt{k_1^2-\lambda^2})\,H_1^{(1)}(ak_2)}$$

Substituting (2.28) and (2.29) into (2.23) and (2.24), respectively, we have:

$$(2.30) \quad u_1 = \frac{i}{2}\int_{-\infty}^{\infty} e^{i\lambda(z-\zeta)}\cdot\frac{J_0(a\sqrt{k_1^2-\lambda^2})\,H_0^{(1)}(r\sqrt{k_1^2-\lambda^2}) - H_0^{(1)}(a\sqrt{k_1^2-\lambda^2})\,J_0(r\sqrt{k_1^2-\lambda^2})}{J_0(a\sqrt{k_1^2-\lambda^2})}\,d\lambda$$

$$(2.31) \quad u_2 = \frac{1}{\pi a}\int_{-\infty}^{\infty} e^{i\lambda(z-\zeta)}\cdot\frac{H_0^{(1)}(r\sqrt{k_2^2-\lambda^2})}{\sqrt{k_2^2-\lambda^2}\,H_1^{(1)}(a\sqrt{k_2^2-\lambda^2})\,J_0(a\sqrt{k_1^2-\lambda^2})}\,d\lambda$$

$$\approx \frac{H_0^{(1)}(rk_2)}{\pi a k_2\,H_1^{(1)}(ak_2)}\int_{-\infty}^{\infty} e^{i\lambda(z-\zeta)}\cdot\frac{d\lambda}{J_0(a\sqrt{k_1^2-\lambda^2})}$$

The integrands of the above two expressions are then both meromorphous functions of the arguments involved. Their integration reduces to a formal contour integral. But it should be pointed out here that for u_2 (2.31) the closed contour cannot be effected since an integration along an infinite semi-circle yields infinity. Relations (2.30) will be used to calculate the attenuation constant in Section IV. in a logical and quite rigorous manner.

The uniform convergence of (2.30) and (2.31) assures the differentiation under the integral sign and we can thus substitute Equations (2.30) and (2.31) into (2.06) to obtain the field components, remembering, however, for the present symmetrical case $\frac{\partial u}{\partial \varphi} = 0$.

SECTION III.

Linear Antenna Placed Parallel to the Axis Inside an Infinite Cylindrical Metal Tube.

From the expressions obtained above for an ideal electric current element, it is possible to generalize for the practical case of a linearly excited resonanting antenna. From a mathematical point of view, in the theory of integral equations, the expressions for u_1 and u_2 obtained before serve as Green functions and the general solution becomes:

$$(3.01) \qquad U_\nu = \int_{\varsigma_1}^{\varsigma_2} u_\nu(r, z-\varsigma) f(\varsigma)\, d\varsigma \qquad (\nu = 1, 2)$$

where $f(\varsigma)$ is a function of position $(\varsigma, r_0, \varphi_0)$ of the infinitesimal current element. In the above equation, the time factor $e^{-i\omega t}$ is suppressed. This general formulae corresponds to an antenna of finite length extending from ς_1 to ς_2 parallel to the axis with an arbitrary distribution of current along its whole extension. It is known in practice that the antenna is usually excited in its fundamental or harmonic wave length. For an antenna wire of very small dissipating resistance the length of the antenna bears a fractional integer relation to the free-space wave length of oscillation. Therefore, if we assume ℓ to be the length of the antenna, extending from $-\frac{\ell}{2}$ to $+\frac{\ell}{2}$ parallel to the axis and also consider a sinusoidal distribution of current, then we have, for zero current amplitude at both ends:

$$(3.02)\quad f(\varsigma) = \begin{cases} \sin \dfrac{n\pi\varsigma}{\ell} \text{ ---------- for even } n, \\ \cos \dfrac{n\pi\varsigma}{\ell} \text{ ---------- for odd } n. \end{cases}$$

where $n = 1$ gives a half wave length antenna, $n=2$, a full wave length antenna, etc.. The entire space is thus divided into three regions: $z > +\frac{\ell}{2}$; $-\frac{\ell}{2} < z < +\frac{\ell}{2}$; and $z < -\frac{\ell}{2}$. For the first and the third regions, the expressions for the Hertzian function are identical when n is odd and only differ in sign when n is even. For the middle region, the solution is a little more complicated.

The general expressions for the potential functions due to a linear antenna will be derived for the following cases:

(1). Center of Antenna at $(0, r_0, \varphi_0)$:

For this case, we must limit ourselves to the case of perfect conducting cylindrical tube for the reason stated before. From Equations (2.18), (3.01) and (3.02), upon performing the integration along the antenna, we notice that the expressions of the potential function for an antenna differ from that for a dipole only by a factor which, although being a function of the arguments involved, however, does not introduce any additional singularity to the integrand. The latter fact justifies mathematically the legitimacy of the formulation of (3.01). Thus we have:

(a) for $z > \frac{\ell}{2}$ or $z < -\frac{\ell}{2}$, the term $e^{i\lambda(z-\varsigma)}$ in the integrands of (2.18) is simply changed to:

$$(i)\quad M_e(n,\lambda;z) = \frac{4n\pi}{\ell}\,\frac{(-)^n \cos\lambda\frac{\ell}{2}}{\left(\frac{2\pi n}{\ell}\right)^2 - \lambda^2}\,e^{i\lambda z}$$

for even harmonics $(n = 1, 2, \cdots\cdots)$,

and $$(ii)\quad M_o(n,\lambda;z) = \frac{2(2n+1)\pi}{\ell}\,\frac{(-)^n \cos\lambda\frac{\ell}{2}}{\frac{(2n+1)^2\pi^2}{\ell^2} - \lambda^2}\,e^{i\lambda z}$$

for odd harmonics. $(n = 0, 1, 2, \cdots\cdots)$.

(b) for $-\frac{\ell}{2} < z < +\frac{\ell}{2}$, the integration (3.01) must be broken up into two parts, from $-\frac{\ell}{2}$ to z and from z to $+\frac{\ell}{2}$. The term $e^{i\lambda(z-\zeta)}$ in the integrands is then changed to:

$$(i)\quad M_e(n,\lambda;z) = \frac{2}{i}\,\frac{\lambda \sin\frac{2\pi n z}{\ell} - (-)^n\cdot\frac{2\pi n}{\ell}\,e^{i\lambda\frac{\ell}{2}}\sin\lambda z}{\left(\frac{2\pi n}{\ell}\right)^2 - \lambda^2}$$

for even harmonics $(n = 1, 2, \cdots\cdots)$,

and $$(ii)\quad M_o(n,\lambda;z) = \frac{2}{i}\,\frac{\lambda \cos\frac{(2n+1)\pi z}{\ell} + i(-)^n\frac{(2n+1)\pi}{\ell}\,e^{i\lambda\frac{\ell}{2}}\cos\lambda z}{\frac{(2n+1)^2\pi^2}{\ell^2} - \lambda^2}$$

for odd harmonics. $(n = 0, 1, 2, \cdots\cdots)$

(2) Center of Antenna at Origin $(0,0,0)$:

The same changes for the term $e^{i\lambda(z-\zeta)}$ are to be made for this symmetrical case with the potential functions u_1 and u_2 given by (2.23) and (2.24) for the general case with finite conductivity for the metal sheath.

The modifications for linear antenna from the original expressions for current element change both the amplitude and the phase of each "mode" of propagation. One interesting possibility arises if we could make the factors $\left[\left(\frac{2n\pi}{l}\right)^2 - \lambda_\nu^2\right]$ and $\left[\frac{(2n+1)^2\pi^2}{l^2} - \lambda_\nu^2\right]$ in the denominators as small as possible; then the intensity for that special mode (νth. mode) will be greatly augmented, constituting a "real resonance" for the case of a linear antenna. This might be of considerable practical significance in long distance transmission. This will be discussed in the next Section.

SECTION IV.

Characteristics of Propagation of An Electric Dipole or Linear Antenna.

We noticed in the development in the preceding Section that the integral expression for a linear antenna differs from that for an electric dipole only by a factor in the integrand. Since this multiplying factor does not introduce any additional singularity (or pole) in the expansion of the integral expression according to the theory of residues, the general characteristics of propagation, with respect to the fundamental properties of attenuation and phase velocity, are identical for a linear antenna and for an electric dipole. Consequently we need only consider the latter case without losing sight of the properties of a physical antenna. This will be further justified later.

Comparing the expressions (2.18) for a dipole placed off-center at (ζ, s_0, φ_0) with that (2.30) for a dipole at $(\zeta, 0, 0)$, it is evident that (2.30) constitutes merely one term of the infinite series of (2.18); i.e., for $m = 0$. Although the roots of Bessel functions of different order give rise to a superposition of different "modes" of waves, the description for each m is of the same physical character and of similar mathematical procedure. These different "modes" propagate with different attenuations and different velocities and are eventually independent of each other. Consequently, we shall limit the discussion to one mode of the symmetrical case.

The present Section can be divided into two main parts: (A) the first part comprises the formal mathematical transformation of the integral expression (2.30) by the theory of residues; (B) the second part consists of an extensive discussion of the physical properties of propagation.

Part (A). Transformation of the Integral (2.30).

The integral expression (2.30) for the Hertzian function can be transformed into an infinite series according to the theory of residues of complex functions. In order to have a unique definition of (2.30), we must limit ourselves to certain restrictions of the arguments according to the definitions of cylindrical functions and the uniform convergence of the integral.

Firstly, we shall assume:

$$(z-\varsigma) \geqq 0$$

corresponding to measuring the field at one side of the dipole, then we must limit λ to the upper half plane with a positive imaginary component, for otherwise the field intensity will increase with distance, an impossible phenomenon. That is,

$$(4.01) \qquad 0 < \arg(\lambda) < \pi \ .$$

Secondly, we shall assume:

$$(4.02) \qquad -\frac{\pi}{2} < \arg\sqrt{k_1^2-\lambda^2} < \frac{\pi}{2}$$

since the Bessel function $J_n(x)$ is defined with $\eta = \frac{\pi}{2}$.* This restriction can be removed if necessary since the Bessel function is periodic in η .

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* Section I.

Also, because of the relations (1.26) (1.29), the denominator $J_0(a\xi_1)$ and the numerator $\left[J_0(a\xi_1)\,H_0^{(1)}(r\xi_1) - J_0(r\xi_1)\,H_0^{(1)}(a\xi_1)\right]$ are unique and meromorphous functions of $\xi_1 = \sqrt{k_1^2-\lambda^2}$, the integrand of (2.30) consists, therefore, only of the singularities at the poles corresponding to the roots of $J_0(a\sqrt{k_1^2-\lambda^2}) = 0$. Thus by Cauchy's theory we can complete the contour by a semi-circle of infinite radius in the upper half λ-plane and evaluate the integral by finding the residues at the poles. The integrand of (2.30) vanishes identically for $|\lambda| \to \infty$, but not for (2.31).

Now we will take all the roots of $J_0(x) = 0$, $\pm x_1, \pm x_2, \cdots \pm x_\nu, \cdots$ as real,* and on account of (4.02) only the positive real roots $x_1, x_2, \cdots\cdots x_\nu, \cdots$, can be used. Then at any root x_ν, we have:

$$J_0(x_\nu) = J_0(a\sqrt{k_1^2-\lambda_\nu^2}) = 0 \quad ,$$

$$\lambda_\nu^2 = k_1^2 - \frac{x_\nu^2}{a^2} \quad ; \tag{4.03}$$

and the corresponding residue becomes:

$$\lim_{\lambda\to\lambda_\nu} \frac{i}{2} e^{i\lambda_\nu(z-\zeta)} \cdot 2\pi i\left[-J_0(\tfrac{r}{a}x_\nu)\,H_0^{(1)}(x_\nu)\right] \frac{\lambda-\lambda_\nu}{J_0(a\sqrt{k_1^2-\lambda_\nu^2})}$$

$$= \pi e^{i\lambda_\nu(z-\zeta)}\, J_0(\tfrac{r}{a}x_\nu)\, H_0^{(1)}(x_\nu)\, \frac{x_\nu}{a^2\lambda_\nu J_1(x_\nu)}$$

---

* The roots of $J_0(x)=0$ are: $x_1 = 2.4048$, $x_2 = 5.5201$, $x_3 = 8.6537, \cdots$. Jahnke und Emde, Functional Tables, p. 166.

since

$$\lim_{\lambda\to\lambda_\nu} \frac{J_0(a\sqrt{k_1^2-\lambda_\nu^2})}{\lambda-\lambda_\nu} = \left[\frac{\frac{\partial}{\partial\lambda}J_0(a\sqrt{k_1^2-\lambda_\nu^2})}{\frac{\partial}{\partial\lambda}(\lambda-\lambda_\nu)}\right]_{\lambda=\lambda_\nu}$$

$$= \left[-\frac{\partial a\sqrt{k_1^2-\lambda^2}}{\partial\lambda}\cdot J_1(a\sqrt{k_1^2-\lambda^2})\right]_{\lambda=\lambda_\nu} = \frac{a^2\lambda_\nu J_1(x_\nu)}{x_\nu}$$

Therefore, summing over all these residues, we have:

$$(4.04) \quad u_1 = \frac{\pi}{a^2}\sum_{\nu=1}^{\infty} e^{i\lambda_\nu(z-\zeta)}\cdot\frac{x_\nu}{\lambda_\nu}\frac{H_0^{(1)}(x_\nu)}{J_1(x_\nu)}J_0(\tfrac{r}{a}x_\nu)$$

In order that this infinite series actually represents the field, it must be a uniformly convergent series except at $r=0$ and $z=\zeta$, corresponding to the location of the point source. For $r=0$ but $z\neq\zeta$, the series should still be convergent uniformly. From asymptotic expansions of cylindrical functions, the ratio:

$$\lim_{\nu\to\infty}\left|\frac{H_0^{(1)}(x_\nu)}{J_1(x_\nu)}\right| \leq M_\nu = \sqrt{2}$$

remains finite for increasing large argument. (4.04) will, therefore, be uniformly convergent if:

$$\left| e^{i(\lambda_{\nu+1}-\lambda_\nu)(z-\zeta)}\cdot\frac{x_{\nu+1}}{x_\nu}\,\frac{M_{\nu+1}}{M_\nu}\,\frac{\lambda_\nu}{\lambda_{\nu+1}}\right| < 1 \qquad \text{for any } (z-\zeta)$$

This essentially reduces to the criterion that if:

$$\lambda_\nu = \beta_\nu + i\alpha_\nu \;, \quad \lambda_{\nu+1} = \beta_{\nu+1} + i\alpha_{\nu+1}$$

The value of $J_1(x)$ for the first few roots of $J_0(x_\nu)=0$ are :
$J_1(x_1) = +0.5191$, $J_1(x_2) = -0.3403$, $J_1(x_3) = +0.2715$,
$J_1(x_4) = -0.2325$, - - - - - .

then (4.05) $[\alpha_{\nu+1} - \alpha_\nu] > a$ should hold.

It can be shown that this inequality (4.05) is always satisfied by finding the values of λ_ν's from (4.03) for corresponding values of χ_ν's . Now let $\lambda_\nu = \beta_\nu + i\alpha_\nu$, then from (4.03) there result:

$$(\beta_\nu + i\alpha_\nu)^2 = k_1^2 - \frac{\chi_\nu^2}{a^2} = \left(\frac{\omega^2\varepsilon_1\mu_1}{c^2} - \frac{\chi_\nu^2}{a^2}\right) + i\,\frac{4\pi\omega\sigma_1\mu_1}{c^2}$$

or

$$\beta_\nu^2 - \alpha_\nu^2 = \frac{\omega^2\varepsilon_1\mu_1}{c^2} - \frac{\chi_\nu^2}{a^2} \tag{4.06}$$

$$\alpha_\nu\beta_\nu = \frac{2\pi\omega\sigma_1\mu_1}{c^2} \tag{4.07}$$

If $\sigma_1 > 0$, then (4.07) is an equilateral hyperbola with its two branches lying in the first and the third quadrants. σ_1 is usually very small but always positive. While for (4.06) there are two cases: (a) when $\frac{\omega^2\varepsilon_1\mu_1}{c^2} > \frac{\chi_\nu^2}{a^2}$, the hyperbola crosses the real β -axis; (b) when $\frac{\omega^2\varepsilon_1\mu_1}{c^2} < \frac{\chi_\nu^2}{a^2}$, it crosses the imaginary α -axis.

The equilateral hyperbola (4.07) is independent of the roots χ_ν and is, therefore, stationary, while the curve (4.06) travels for different roots χ_ν .

The following graph shows the curves for (4.06) and (4.07). The graph gives two sets of intersections of (4.06) and (4.07) in the first and the third quadrants, respectively. But for the present

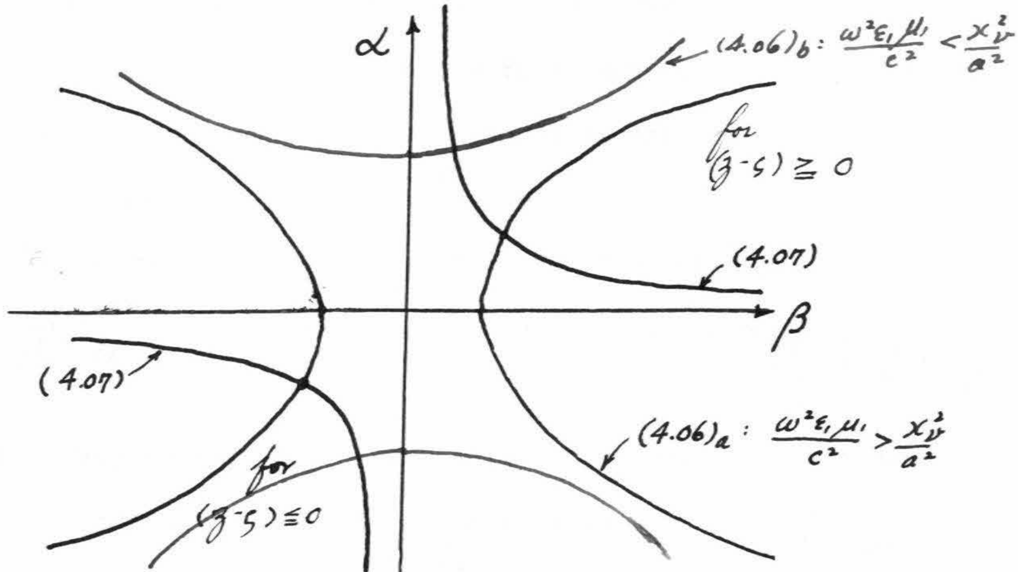

Fig. 4.1.

case, $(z-\varsigma)>0$, λ must have a positive imaginary part, only those values of λ lying in the first quadrant can be used. Generally there are only a few hyperbolas for (4.06)a or even none, while (4.06)b gives values of λ approaching the positive imaginary axis (α) as a limit. It is then evident that the criterion of inequality (4.05) is always satisfied. α is in fact, a monotonically increasing function of x_ν . The series for u_1 (4.04) is, therefore, uniformly convergent and gives the required solution. Although for $\sigma_1 \neq 0$, the resultant Hertzian function u_1 becomes an infinite series; in practical calculation only a few terms are necessary because of its rapid convergence at any considerable distance.

If, instead of (4.01), we measure the field at other side of the source, or:

$$(z-\varsigma) \leqq 0 ,$$

then we have merely to use the roots of λ lying in the third quadrant. Because the roots in the first and the third quadrants of Fig. 4.1 are symmetrical with respect to the origin, we have then in general the same Equation (4.04).

From (4.04), we have, therefore, for the Hertzian function:

$$(4.08)\quad Re\left[u_1 e^{-i\omega t}\right] = Re\left[\frac{\pi}{a^2}\sum_{\nu=1}^{\infty} e^{i[\lambda_\nu(z-\varsigma)-\omega t]} \cdot \frac{x_\nu H_0^{(1)}(x_\nu)}{\lambda_\nu J_1(x_\nu)} J_0\left(\frac{\Lambda}{a}x_\nu\right)\right]$$

The first exponential term represents, in general, the propagation with certain damping; the second ratio term $\frac{x_\nu H_0^{(1)}(x_\nu)}{\lambda_\nu J_1(x_\nu)}$ stands for the amplitude and the "phase" relation between different modes of propagation corresponding to different roots λ_ν's ; and finally the last term $J_0\left(\frac{\Lambda}{a}x_\nu\right)$ depicts the relative intensity distribution of standing waves along the radial direction. Standing waves exist in the radial direction, since $J_0\left(\frac{\Lambda}{a}x_\nu\right)$ is always real. However, before discussing the characteristics of propagation, we shall study one interesting case for $\sigma_1 \to 0$.

If in the limiting case $\sigma_1 \to 0$, then the curve (4.07) becomes $\alpha\beta = 0$ and coincides with the axes. The solutions for λ therefore are the intersections of (4.06) with the real axis and the imaginary axis, respectively. When, say, $\frac{\omega^2\varepsilon_1\mu_1}{c^2} > \frac{x_\nu^2}{a^2}$, the intersections are on the real axis, λ_ν are real. This represents then propagation down the tube without damping. When $\frac{\omega^2\varepsilon_1\mu_1}{c^2} < \frac{x_\nu^2}{a^2}$ $(\nu > n)$, the intersections lie on the imaginary axis. Then the field is damped with increasing distance but no "propagation phenomenon" exists and at any considerable distance it identically vanishes. Consequently the result is greatly simplified and u_1 becomes a finite terminating series:

$$(4.09) \quad u_1 = \frac{\pi}{a^2} \sum_{\nu=1}^{n} e^{i\lambda_\nu (z-\varrho)} \cdot \frac{x_\nu \cdot H_0^{(1)}(x_\nu)}{\lambda_\nu \; J_1(x_\nu)} J_0\left(\frac{r}{a} x_\nu\right) ,$$

For the special case that no root of λ is real; i.e.,:

$$\frac{\omega^2 \varepsilon_1 \mu_1}{c^2} < \frac{x_1^2}{a^2} = \frac{(2.4048)^2}{a^2} = \frac{5.7823}{a^2}$$

or (4.10) $$f < \frac{c \cdot 2.4048}{2\pi a \sqrt{\varepsilon_1 \mu_1}} = \frac{1.15}{a} 10^{10} \text{ cycles/sec.}$$

then no propagation exists. This represents a complete "cut-off", as it may be called. It will be discussed in detail in the second part. One case of great importance occurs if we adjust either the frequency f or the radius a of the tube so that:

$$\frac{\omega^2 \varepsilon_1 \mu_1}{c^2} = \frac{x_\nu^2}{a^2}$$

Then we have $\lambda_\nu = 0$ and u_1 increases without limit for that special limiting "mode" of propagation. Mathematically, u_1 (4.04) is no more a solution, since it loses requirements for uniform convergence. But physically such a phenomenon is of greatest importance; it represents "resonance" between the exciting system and the response of the dielectric medium inside the cylindrical metal tube. Whenever such an ideal resonance happens, the absolute amplitude becomes infinite but no propagation phenomenon exists corresponding to that root of $J_0(x_\nu) = 0$, since then u_1 is independent of the distance $(\mathfrak{z}-\varsigma)$ from the source. However, it must be remembered that this "ideal" case occurs only when $\sigma_2 \to \infty$ and $\sigma_1 \to 0$. In any physical case neither can actually reach the extremal value. Then we have approximately from (4.06) and (4.07):

$$\beta_\nu = \alpha_\nu = \sqrt{\frac{2\pi\omega\sigma_1\mu_1}{c^2}}$$

which indicates that for this special adjustment both the attenuation constant and the phase constant are very small. The velocity for this mode is:

$$v_{p\nu} = \frac{\omega}{\beta_\nu} = c\sqrt{\frac{\omega}{2\pi\sigma_1\mu_1}}$$

which approaches infinity as a limit when $\sigma_1 \to 0$. The group velocity, or the velocity of energy propagation is (refer to(4.29)):

$$v_{g\nu} = \frac{c^2}{\varepsilon_1\mu_1}\frac{1}{v_{p\nu}} = \frac{c}{\varepsilon_1\mu_1}\sqrt{\frac{2\pi\sigma_1\mu_1}{\omega}}$$

which becomes zero when σ_1 becomes zero. The physical picture of this is fascinating. It corresponds to a greater and greater concentration of energy which drifts along at a slower and slower velocity when $\sigma_1 \to 0$. This"mode" will then play a dominating role in reception. The nearer the equilateral hyperbola approaches the axes (Fig. 4.1), the more accurate expression (4.09) represents even the general case, since then terms corresponding to λ_ν for $\nu > n$ are completely negligible at any considerable distance in comparison with those for $\nu \leq n$. In conclusion, we need therefore only compute a few terms for $\mathcal{U}_1$, since in any practical case even with wavelength of the source of a few centimeters and with considerably large tube radius, only the first few smallest roots of $J_0(x_\nu) = 0$ satisfy the relation:

$$\frac{\omega^2 \epsilon_1 \mu_1}{c^2} > \frac{x_\nu^2}{a^2} \tag{4.11}$$

With the above discussions and restrictions, not only the labor in computation is greatly reduced but also the difficulty with the peculiar phenomenon of "resonance" is overcome.

Now we can substitute (4.04) into Equations (2.06) to obtain the field components:

$$H_{\varphi_1} = Re\left\{\frac{\pi}{\mu_1 a^3}\sum_{\nu=1}^{\infty} e^{i[\lambda_\nu(z-s)-\omega t]} \cdot \frac{x_\nu^2 H_0^{(1)}(x_\nu)}{\lambda_\nu J_1(x_\nu)} J_1\left(\frac{r}{a}x_\nu\right)\right\}$$

$$(4.12) \quad \begin{cases} H_{z_1} = H_{r_1} = 0 \\ E_{r_1} = Re\left\{\dfrac{\pi}{a^3}\,\dfrac{\omega}{c k_1^2}\displaystyle\sum_{\nu=1}^{\infty} e^{i[\lambda_\nu(z-s)-\omega t]} \cdot \dfrac{x_\nu^2 H_0^{(1)}(x_\nu)}{J_1(x_\nu)} J_1\left(\dfrac{r}{a}x_\nu\right)\right\} \\ E_{z_1} = Re\left\{\dfrac{\pi}{a^4}\,\dfrac{i\omega}{c k_1^2}\displaystyle\sum_{\nu=1}^{\infty} e^{i[\lambda_\nu(z-s)-\omega t]} \cdot \dfrac{x_\nu^3 H_0^{(1)}(x_\nu)}{\lambda_\nu J_1(x_\nu)} J_0\left(\dfrac{r}{a}x_\nu\right)\right\} \end{cases}$$

$$E_{\varphi_1} = 0$$

As discussed before, it is only necessary to use the first few terms for which (4.11) holds.

Part (B). Characteristics of Propagation.

The formal mathematical development in Part (A) lays down the foundation for the discussions of the physical properties of propagation. Although the rigorous expressions for $F_1(\lambda)$ and $F_2(\lambda)$ (2.22) give rise to branch points at $\lambda = \pm k_1$ and $\lambda = \pm k_2$ which encumber carrying out the integration for u_1 and u_2, fortunately a practical approximation with sufficient accuracy had been attained for the case of very large σ_2 (conductivity of the metal tube). The resultant formula (2.30) is thus free from branch points and its expansion is given in (4.04). The corresponding field components for the air medium are given in (4.12). The number of terms of these expressions equals the number of roots of $J_0(x) = 0$ for which the right side of (4.06) is positive or equal to zero. It should be noted that every such root gives rise to a distinct "mode of propagation" with its attenuation, phase relation, and velocity different from all others. The resultant field at any point along the system is thus a superposition of all these "modes", while each "mode" propagates down the tube guide as if it exists alone. In fact, there is no interaction whatsoever between the different modes, if the transmission system is perfectly uniform. With the above visualization, it suffices to discuss the characteristics of propagation of any of these modes.

(1) Attenuation Constant.

From the researches of Rohde, Schwarz and Handrek* on dielectric

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* Zeits. f. Techn. Phys., Band 16. No. 12, (1935), S. 637.
Band 15. No. 11, (1934), S. 491.
Hochfrequenztech u. Elektroakust, Band 43, No. 5, (1934), S. 156.

loss of various materials at very high frequencies (from 10^6 cycles per sec. up to 500×10^6 cycles per sec.), we are justified in reaching the conclusion that air is probably the only medium which can be used in tube guide. The loss in the air medium is practically negligible and the attenuation is thus completely due to the loss in the metal tube sheath. With this condition we shall assume then $\sigma_1 = 0$ and the curve (4.07) collapses into the real and imaginary axes $[\alpha_\nu \beta_\nu = 0]$. Thus corresponding to root of $J_0(x) = 0$, we have:

$$(4.13) \qquad \lambda_\nu = \beta_\nu + i\,0$$

It is obtained from the intersection of the hyperbola (4.06) on the positive real axes; i.e.:

$$(4.14) \qquad \beta_\nu^2 = \frac{\omega^2 \varepsilon_1 \mu_1}{c^2} - \frac{x_\nu^2}{a^2} = \omega^2\left[\frac{1}{V_0^2} - \frac{x_\nu^2}{a^2\omega^2}\right]$$

$$= \omega^2\left[\frac{1}{V_0^2} - \frac{1}{V_\nu^2}\right] = \left(\frac{2\pi}{V_0}\right)^2\left[f^2 - f_\nu^2\right]$$

or

$$\beta_\nu = \frac{2\pi f}{V_0}\sqrt{1 - \left(\frac{f_\nu}{f}\right)^2}$$

where $V_\nu = \frac{a\omega}{x_\nu}$ may be called the cut-off velocity, and:

$$(4.14)a \qquad f_\nu = \frac{x_\nu V_0}{2\pi a}$$

the corresponding cut-off frequency, where $V_0 = \frac{c}{\sqrt{\varepsilon_1 \mu_1}}$ being the velocity in free air space. The phase velocity for "$\nu th.$ mode" is simply:

$$(4.15) \qquad v_{p\nu} = \frac{\omega}{\beta_\nu} = \frac{v_0}{\sqrt{1-\left(\frac{f_\nu}{f}\right)^2}}$$

Therefore from the function u_1 , only the velocity can be obtained. The attenuation constant must be derived from the relations of energy propagations in the two mediums. With the help of Equations (4.12) it will be possible here to obtain rigorous analytic expression for the attenuation. In measuring the energy propagated or lost, we are interested in the time average values. Thus according to the rules for complex conjugate quantities, we have:

$$Re[g]\,Re[G] = \tfrac{1}{2}(g+g^*)\,\tfrac{1}{2}(G+G^*) = \tfrac{1}{4}(gG+g^*G^*+gG^*+g^*G)$$

In taking the time average:

$$\overline{gG} = 0 \,, \quad \overline{g^*G^*} = 0 \quad \& \quad gG^* = (g^*G)^*$$

Hence

$$(4.16) \qquad \overline{Re[g]\,Re[G]} = \tfrac{1}{2}\,\overline{Re[gG^*]}$$

Thus we shall form the complex Poynting vectors* for the energy propagated in the axial direction inside the tube and for the power lost in the radial direction in the metal sheath. From these the attenuation will be defined in such a way that u_1 , u_2 , and all the field components ((4.04) - (4.12)) are modified by a factor

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* Abraham and Becker, "Classic Theory of Elec. and Magnetism", pp. 193-196.

$e^{-\alpha_\nu(z-\zeta)}$, thus making up together with the phase constant β_ν obtained from (4.06) a complex "propagation constant":

$$(4.17) \qquad \lambda_\nu = \beta_\nu + i\,\alpha_\nu$$

The energy propagated in the axial direction may be divided up into two parts: one residing in the air medium $(r<a)$ constitutes the major part and also the only part which can be picked up by certain receiving device; the other taking place in the metal sheath $(r>a)$ is very small compared with the first and not utilizable. According to (4.16), we form the complex Poynting vector (time average):

$$(4.18) \qquad \overline{\mathcal{H}} = \frac{c}{4\pi}\overline{\{Re[\mathcal{E}]\times Re[\mathcal{H}]\}} = \frac{c}{8\pi}Re[\mathcal{E}\times\mathcal{H}^*]$$

and integrate over a closed surface. In order to make this comply with the definition of Poynting's theorem, the closed surface may be taken as the cylindrical surface at $r=a$ with bases at $\pm z$. Here the Poynting's vector $\mathcal{H}$ has two components, the axial N_z through the two bases at $\pm z$ and the radial N_r through the cylindrical surface. N_z represents the useful part and N_r the loss in metal sheath. We have then:

$$(4.19) \qquad \overline{N_z} = \frac{c}{8\pi}Re[\mathcal{E}\times\mathcal{H}^*]_z = \frac{c}{8\pi}Re.[E_{r_1} H_{\varphi_1}^*]$$

$$= \frac{c}{8\pi}Re\left[\frac{\pi^2}{\mu_1 a^6}\,\frac{\omega}{c k_1^2}\,\frac{x_\nu^4}{\lambda_\nu}\,\frac{H_0^{(1)}(x_\nu)\{H_0^{(1)}(x_\nu)\}^*}{\{J_1(x_\nu)\}^2}\,\{J_1(\tfrac{r}{a}x_\nu)\}^2\right]$$

$$= \frac{\pi}{8\mu_1 a^6}\,\frac{\omega}{k_1^2}\,\frac{x_\nu^4}{\beta_\nu}\left\{\frac{Y_0(x_\nu)}{J_1(x_\nu)}\right\}^2\{J_1(\tfrac{r}{a}x_\nu)\}^2$$

Integrating over the bases at $\pm z$, we get the time averaged value of the utilizable energy propagated in the axial direction:

$$(4.20)\quad \overline{W_\nu} = 2\int_0^{2\pi} d\varphi \int_0^a \overline{N_z}\, r\, dr = \frac{\pi^2}{2\mu_1 a^6}\,\frac{\omega x_\nu^4}{k_1^2 \beta_\nu}\left\{\frac{Y_0(x_\nu)}{J_1(x_\nu)}\right\}^2 \int_0^a \left\{J_1\left(\frac{r}{a}x_\nu\right)\right\}^2 r\,dr$$

$$= \frac{\pi^2}{4\mu_1 a^4}\,\frac{\omega x_\nu^4}{k_1^2\beta_\nu}\left\{Y_0(x_\nu)\right\}^2 = \frac{\pi^4}{\mu_1 v_0}\,\frac{f_\nu^4}{f^2\sqrt{1-\left(\frac{f_\nu}{f}\right)^2}}\left\{Y_0(x_\nu)\right\}^2$$

The last expression is obtained by using the relations (4.14) and (4.14)a. This also shows that only for $f > f_\nu$ is there a real flux of energy leaving the cross-sections at $\pm z$ corresponding to the "$\nu th.$ mode" of propagation. The total energy is simply the superposition of (4.20) for all ν's ; i.e.:

$$(4.20)a\quad \overline{W} = \sum_{\nu=1}^{n} \frac{\pi^2}{4\mu_1 a^4}\,\frac{\omega x_\nu^4}{k_1^2\beta_\nu}\left\{Y_0(x_\nu)\right\}^2$$

$$= \sum_{\nu=1}^{n}\frac{\pi^4}{\mu_1 v_0}\,\frac{f_\nu^4}{f^2\sqrt{1-\left(\frac{f_\nu}{f}\right)^2}}\left\{Y_0(x_\nu)\right\}^2$$

where $f \geq f_n$

The corresponding r-component of $\overline{\mathcal{N}}$ is:

$$(4.21)\quad \overline{N_r} = \frac{c}{8\pi} Re\left[\mathcal{E}\times\mathcal{H}^*\right]_r = \frac{c}{8\pi} Re\left[E_{z1}, H_{\varphi 1}^*\right]_{r=a}$$

$$= \frac{c}{8\pi} Re\left[\frac{\pi^2}{\mu_1 a^7}\,\frac{i\omega}{c k_1^2}\,\frac{x_\nu^5}{\lambda_\nu^2}\,\frac{H_0^{(1)}(x_\nu)\left\{H_0^{(1)}(x_\nu)\right\}^*}{J_1(x_\nu)} J_0(x_\nu)\right]$$

whose real part is zero if the conductivity of the air medium is assumed negligible. Even if we assume a finite conductivity for the air medium, the integration over the cylindrical surface at $r = a$ still gives zero. This, however, is a natural result, since the expression (2.30) for u_1 is obtained for $\sigma_2 \to \infty$.

All energy emitted by the source resides in the air medium within the metal sheath.

Consequently, we are forced to seek another means of finding the loss in the conducting medium 2. This can be achieved since we know from (4.12) all the field components at the inner surface $(r=a)$ of the metallic tube. General formulae taking into account the finite thickness of the metal sheath can be obtained. We shall treat, therefore, in the general sense. (2.31) for u_2 fails to give us a series solution for the different "modes" and their field components, because of the impossibility of forming a closed contour of integration according to Cauchy's theorem. The gist of the present method lies, therefore, in setting up series solutions for the field components in the metal sheath (medium 2) and finding the "corrections" for the roots of $J_0(x)=0$ to take into account of the effect of "finite conductivity" of the metal tube.

Referring to (4.12) and (2.06), we have immediately the desired forms of series solutions for the field components in medium 2, considering, however, only the "νth mode":

$$(4.22)\quad \begin{cases} E_{z2} = Re\left\{e^{i[\lambda_\nu(z-s)-\omega t]}\cdot\left[C_1 H_0^{(1)}(r\sqrt{k_2^2-\lambda_\nu^2})+C_2 J_0(r\sqrt{k_2^2-\lambda_\nu^2})\right]\right\} \\ E_{r2} = Re\left\{e^{i[\lambda_\nu(z-s)-\omega t]}\cdot\frac{-i\lambda_\nu}{\sqrt{k_2^2-\lambda_\nu^2}}\left[C_1 H_1^{(1)}(r\sqrt{k_2^2-\lambda_\nu^2})+C_2 J_1(r\sqrt{k_2^2-\lambda_\nu^2})\right]\right\} \\ H_{\varphi 2} = Re\left\{e^{i[\lambda_\nu(z-s)-\omega t]}\frac{ck_2^2}{i\mu_2\omega\sqrt{k_2^2-\lambda_\nu^2}}\left[C_1 H_1^{(1)}(r\sqrt{k_2^2-\lambda_\nu^2})+C_2 J_1(r\sqrt{k_2^2-\lambda_\nu^2})\right]\right\} \end{cases}$$

The above relations are for the metal sheath of finite thickness. Since, however, we realized that the metal sheath is electrically very thick for high frequencies, we shall put $C_2 \equiv 0$ in order to simplify the mathematical manipulations without impairing the accuracy of calculation. The boundary conditions at $r = a$ require the equality of the tangential components of $\mathcal{E}$ and $\mathcal{H}$ from (4.12) and (4.22). Thus there result:-

$$(4.23)\begin{cases} C_1 H_0^{(1)}\left(a\sqrt{k_2^2-\lambda_\nu'^2}\right) = \dfrac{\pi}{a^4}\dfrac{i\omega}{ck_1^2}\dfrac{x_\nu'^3}{\lambda_\nu'}\dfrac{H_0^{(1)}(x_\nu')}{J_1(x_\nu')}J_0(x_\nu') \\ \dfrac{ck_2^2}{i\mu_2\omega\sqrt{k_2^2-\lambda_\nu'^2}}C_1 H_1^{(1)}\left(a\sqrt{k_2^2-\lambda_\nu'^2}\right) = \dfrac{\pi}{\mu_1 a^3}\dfrac{x_\nu'^2}{\lambda_\nu'}H_0^{(1)}(x_\nu') \end{cases}$$

where x_ν' and λ_ν' with primes indicate the "corrected" values for taking account of the effect of finite conductivity of the metal sheath. From (4.23), we obtain then:

$$(4.24)\quad C_1 = \frac{\pi}{a^4}\frac{i\omega}{ck_1^2}\frac{x_\nu'^3}{\lambda_\nu'}\frac{H_0^{(1)}(x_\nu')}{J_1(x_\nu')}\frac{J_0(x_\nu')}{H_0^{(1)}\left(a\sqrt{k_2^2-\lambda_\nu'^2}\right)} = \frac{\mu_2\pi}{\mu_1 a^3}\frac{x_\nu'^2}{\lambda_\nu'}\frac{i\omega\sqrt{k_2^2-\lambda_\nu'^2}}{ck_2^2}\frac{H_0^{(1)}(x_\nu')}{H_1^{(1)}\left(a\sqrt{k_2^2-\lambda_\nu'^2}\right)}$$

The last two terms give the "relation of compatibility", from which the "corrected root" x_ν' is to be found. Fortunately, we do not need an explicit solution of these x_ν''s from the above complicated transcendental equation. What we need in calculating the attenuation constant is the ratio:

$$(4.35)\quad \frac{x_\nu'}{a}\frac{1}{k_1^2}\frac{1}{H_0^{(1)}\left(a\sqrt{k_2^2-\lambda_\nu'^2}\right)}\frac{J_0(x_\nu')}{J_1(x_\nu')} = \frac{\mu_2}{\mu_1}\frac{\sqrt{k_2^2-\lambda_\nu'^2}}{k_2^2}\frac{1}{H_1^{(1)}\left(a\sqrt{k_2^2-\lambda_\nu^2}\right)}$$

or

$$\frac{J_0(x_\nu')}{J_1(x_\nu')} = \frac{\mu_2 a k_1^2\sqrt{k_1^2-\lambda_\nu^2}}{\mu_1 k_2^2 x_\nu'}\frac{H_0^{(1)}\left(a\sqrt{k_2^2-\lambda_\nu'^2}\right)}{H_1^{(1)}\left(a\sqrt{k_2^2-\lambda_\nu'^2}\right)} \cong \frac{i\mu_2 a k_1^2}{\mu_1 k_2 x_\nu'}$$

from the relation $k_2^2 >> \lambda_\nu'^2$ and the asymptotic expansions for Hankel functions. (Section II.). Substituting the above corrected relations into (4.21) for the energy flux in the radial direction, we get, at $r = a$:

$$(4.21)a \quad \overline{N_r} = \frac{c}{8\pi} Re\left[\frac{\pi^2 i\omega\, x_\nu'^5}{\mu_1 a^7 c k_1^2 \lambda_\nu'^2}\{Y_0(x_\nu')\}^2 \frac{i a \mu_2 k_1^2}{\mu_1 k_2 x_\nu'}\right]$$

$$= Re\left[\frac{-\omega\pi\mu_2\, x_\nu'^4}{8\mu_1 a^6 \mu_1 k_2 \lambda_\nu'^2}\{Y_0(x_\nu')\}^2\right]$$

$$= \frac{-\mu_2 \pi\omega\, x_\nu'^4 c}{16\mu_1^2 a^6 \beta_\nu^2 \sqrt{2\pi\omega\sigma_2\mu_2}}\{Y_0(x_\nu')\}^2$$

Since

$$k_2^2 \simeq \frac{i 4\pi\omega\sigma_2\mu_2}{c^2}$$

$$k_2 = (1+i)\frac{\sqrt{2\pi\omega\sigma_2\mu_2}}{c}$$

$$\frac{1}{k_2} = \frac{c}{\sqrt{2\pi\omega\sigma_2\mu_2}}\,\frac{1-i}{2}$$

From (4.25), we realize that the "corrected" root x_ν' is very nearly equal to x_ν from $J_0(x_\nu) = 0$, where x_ν is a real quantity. Thus putting $x_\nu' = x_\nu$ in (4.21)a, we obtain:

$$(4.21)b \qquad \overline{N_r} = \frac{-\pi\mu_2\omega\, x_\nu^4\, c}{16\mu_1^2 a^6 \beta_\nu^2 \sqrt{2\pi\omega\sigma_2\mu_2}}\{Y_0(x_\nu)\}^2$$

Integrating (4.21)b over a cylindrical surface of unit length, we get the loss in medium 2 per unit length along the axis for the "νth. mode":

$$(4.26) \qquad \overline{q_\nu} = \int_0^{2\pi} \overline{N_r}\, a\, d\varphi = \frac{-\pi^2\omega x_\nu^4 \mu_2 c}{8\mu_1^2 a^5 \beta_\nu^2 \sqrt{2\pi\omega\sigma_2\mu_2}}\{Y_0(x_\nu)\}^2$$

According to the definition of attenuation constant in (4.17), we have:

$$\overline{q_\nu} = \frac{\partial \overline{W_\nu}}{\partial z} = -2\alpha_\nu \overline{W_\nu}$$

Substituting from Equations (4.20) and (4.26), we succeed in finding an explicit expression for the attenuation constant for the "νth. mode".

$$(4.27) \quad \alpha_\nu = \frac{-\overline{q_\nu}}{2\overline{W_\nu}} = \frac{c\mu_2 k_1^2}{2\mu_1 a \beta_\nu \sqrt{2\pi\omega\sigma_2\mu_2}} = \frac{1}{2a}\sqrt{\frac{\mu_2\varepsilon_1}{\mu_1\sigma_2}}\frac{f^{1/2}}{\sqrt{1-(\frac{f_\nu}{f})^2}}$$

For a uniform and homogeneous transmission system, the different "modes" propagate down the axis independent from each other. We must, therefore, calculate the attenuation for each "mode" separately as derived above. (4.27) is obtained in a quite rigorous manner although in a somewhat novel way. It agrees with Kelvin's result derived from the general skin effect without referring to any exciting system and serves as a proof of the latter's validity for the "νth. mode" and for the "νth.mode" only.

The total loss for all "modes" of propagation is then a superposition of (4.26).

$$(4.28) \quad \overline{Q} = \sum_{\nu=1}^{n} \overline{q_\nu} = \sum_{\nu=1}^{n} \frac{-\pi^2\mu_2\omega c x_\nu^4}{8\mu_1^2 a^5 \beta_\nu^2 \sqrt{2\pi\omega\sigma_2\mu_2}} [Y_0(x_\nu)]^2$$

$$= \sum_{\nu=1}^{n} \frac{-\pi^4\varepsilon_1}{2c\mu_1 a}\sqrt{\frac{\mu_2}{\sigma_2}}[Y_0(x_\nu)]^2 \frac{f_\nu^4}{f^{5/2}}\frac{1}{[1-(\frac{f_\nu}{f})^2]}$$

By means of (4.14) and (4.27), we obtain thereby the complete expression of the "propagation constant":

$$(4.17) \qquad \lambda_\nu = \beta_\nu + i\alpha_\nu$$

for the "νth mode". Substituting (4.17) and (4.24) into (4.09), (4.12) and (4.22) gives us complete analytical description of the general potential function and the field components in the two media.

The attenuation (α_ν) is infinite at the cut-off frequency $(f = f_\nu)$, beyond which it firstly decreases and reaches a minimum value.

$$\alpha_{\nu\, min} = \frac{1}{2a}\sqrt{\frac{\mu_2 \varepsilon_1}{\mu_1 \sigma_2}}\left[\frac{3^{3/2}}{2} f_\nu\right]^{\frac{1}{2}} = \frac{1}{4a^{3/2}}\left[\sqrt{\frac{\varepsilon_1}{\mu_1}}\,\frac{\mu_2 x_\nu c}{\mu_1 \sigma_2 \pi}\right]^{1/2}$$

at $f = \sqrt{3}\, f_\nu = \dfrac{\sqrt{3}\, c\, x_\nu}{2\pi a \sqrt{\varepsilon_1 \mu_1}}$, after which it increases monotonically with frequency.

(2) Phase Constant, Phase Velocity, and Group Velocity.

From what has been discussed before, we notice that the phase constant (β_ν) is practically independent of the attenuation effect. The phase constants for the different "modes" of propagation corresponding to the different roots of $J_0(x_\nu) = 0$ are distinct. Rewriting (4.14) and (4.15), we have:

$$(4.14) \qquad \beta_\nu = \sqrt{\frac{\omega^2 \varepsilon_1 \mu_1}{c^2} - \frac{x_\nu^2}{a^2}} = \frac{2\pi f}{v_0}\sqrt{1 - \left(\frac{f_\nu}{f}\right)^2} \quad ,$$

$$(4.15) \qquad V_{p\nu} = \frac{\omega}{\beta_\nu} = v_0 \left[1 - \left(\frac{f_\nu}{f}\right)\right]^{-1/2}$$

where f_ν is the cut-off frequency for the "νth mode". The phase velocity for the "νth mode" is infinite at $f = f_\nu$ and decreases as the frequency increases, approaching the velocity of light in air medium (v_0) at $f=\infty$. There are as many distinct phase velocities as there are "distinct modes" of propagation.

Group velocity, which is of importance when modulated signals are to be transmitted, can be obtained from the Equation (4.14) by differentiating ω against β_ν, keeping in mind that $\frac{x_\nu^2}{a^2}$ is a constant for the "νth mode", we obtain:

$$0 = \frac{\varepsilon_1 \mu_1}{c^2} 2\omega \frac{d\omega}{d\beta_\nu} - 2\beta_\nu$$

The group velocity $v_{g\nu}$ corresponding to the "νth mode" is then: *

$$(4.29) \quad v_{g\nu} = \frac{d\omega}{d\beta_\nu} = \frac{\beta_\nu v_0^2}{\omega} = \frac{v_0^2}{v_{p\nu}} = v_0\sqrt{1-\left(\frac{f_\nu}{f}\right)^2}$$

It is, therefore, zero at the cut-off frequency (f_ν) and approaches the velocity of light as a limit as the frequency continually increases. There are also as many "distinct group velocities" as there are "distinct modes of propagation". Curves for the phase and group velocities are given in the appendix.

(3) Frequency Spectrum.

From the above discussion, we see that corresponding to each mode, both the attenuation and the velocities of propagation change with frequency, a phenomenon very undesirable for wide band modulated signal transmission. For high quality television purposes then, only those parts with very flat characteristics can be used.

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* Max Planck, "Theory of Light."
Försterling, "Lehrbuch der Optik."

The frequency characteristics are distinct for the "distinct modes" of propagation if the transmission system is uniform and homogeneous. When the applied frequency is higher than the first cut-off frequency $f_1 = \frac{2.4048\, v_0}{2\pi a}$ but less than the second cut-off frequency $f_2 = \frac{5.5201\, v_0}{2\pi a}$, then only one single "mode" of propagation exists.

For a modulated television signal with a frequency band covering many million cycles (usually 6 M.C.), if they lie completely within f_1 and f_2, then there is only one single "mode" for the whole band. If, however, they lie within f_2 and f_3, there would be two distinct "modes" with different attenuations and velocities. So with the whole modulated signal within f_n and f_{n+1}, there would be "n distinct modes" with different attenuations and velocities.

For transmission of a modulated signal, we must limit the reception to "one mode" only, since different distinct modes cause interference and distortion of the original composition. Thereby a unique conclusion is reached: That is, we must design the transmission system in such a way sothat the whole modulated band lies within f_1 and f_2. In other words, we must have "single mode" transmission.

Take the case of a cylindrical metal tube guide with a radius of 10 cms., then:

$f_1 = 1.148 \times 10^9$ cycles per second,

$f_2 = 2.636 \times 10^9$ cycles per second,

and the allowed frequency band is approximately:

$$\Delta f = 1.488 \times 10^9 \quad \text{cycles per second,}$$

which is amply sufficient for present day television purposes.

The above conclusion is not only of practical importance, but also of theoretical interest. Since then we need only consider the first term of (4.04) and (4.12).

(4) Phase Displacement and Energy Propagation.

The phase displacement for the field components due to a dipole (electric) is extremely simple. From (4.12), rewriting the expression for E_z, we have (for "νth mode"):

$$E_z = Re\left[\frac{\pi\omega i}{a^4 c k_1^2} e^{-\alpha_\nu(z-\varsigma)} e^{i[\beta_\nu(z-\varsigma)-\omega t]} \cdot \frac{x_\nu^3 H_0^{(1)}(x_\nu)}{\beta_\nu J_1(x_\nu)} J_0\left(\frac{r}{a}x_\nu\right)\right]$$

in which the attenuation constant α_ν, is to be calculated from (4.27). Since:

$$H_0^{(1)}(x_\nu) = J_0(x_\nu) + i\,Y_0(x_\nu) = i\,Y_0(x_\nu)$$

we obtain for real part of E_z:

$$(4.30) \qquad E_z = \frac{-\pi\omega x_\nu^3}{c a^4 k_1^2 \beta_\nu} \frac{Y_0(x_\nu)}{J_1(x_\nu)} J_0\left(\frac{r}{a}x_\nu\right) e^{-\alpha_\nu(z-\varsigma)} \cdot \cos\left(\omega t - \beta_\nu|z-\varsigma|\right)$$

There is, therefore, no phase displacement due to the multiplying factor. The amplitude, at definite r varies according to the factor:

$$\frac{\omega}{a^4 k_1^2 \beta_\nu} = \frac{v_0^2}{a^4 \omega \sqrt{\frac{\omega^2\varepsilon_1\mu_1}{c^2} - \frac{x_\nu^2}{a^2}}} = \frac{v_0^3}{4\pi^2 a^4 f^2 \sqrt{1-\left(\frac{f_\nu}{f}\right)^2}}$$

This shows that the amplitude is infinite at $f = f_v$, a phenomenon called "resonance" by some writers. Such a resonance is ascribed to cause a "localization" of energy without propagation, and can actually be detected by any disturbance "dislodging" this localization of energy. In fact a much more instructive physical depiction can be derived from this phenomenon. On writing out the other field components from (4.12):

$$(4.31) \quad H_\varphi = \frac{\pi x_v^2}{\mu_1 a^3 \beta_v} \frac{Y_0(x_v)}{J_1(x_v)} J_1\left(\frac{r}{a} x_v\right) e^{-\alpha_v (z-s)} \sin[\omega t - \beta_v (z-s)]$$

$$H_z = H_r = 0 \quad ,$$

$$(4.32) \quad E_r = \frac{\pi \omega x_v^2}{a^3 c k_1^2} \frac{Y_0(x_v)}{J_1(x_v)} J_1\left(\frac{r}{a} x_v\right) e^{-\alpha_v (z-s)} \sin[\omega t - \beta_v (z-s)] \quad ,$$

$$E_\varphi = 0 \quad ,$$

one notices that H_φ also contains a factor:

$$\frac{\pi}{a^3 \beta_v} = \frac{v_0}{2a^3 f \sqrt{1 - \left(\frac{f_v}{f}\right)^2}}$$

which becomes infinite at $f = f_v$. While E_r remains finite except at $f = 0$. Thus such a "localization" of energy at $f = f_v$ $(\beta_v = 0)$ means a "readiness" to start propagation as soon as f becomes greater than f_v . The latter can only be accomplished if there is a "stored" amount of energy in space ready for the push. The idea of "localization" of energy is therefore physically and mathematically justified and is susceptible to experimental verification.

(5) Characteristic Impedance and Radiation Resistance.

The conception of characteristic impedance is helpful due to its frequent occurrence in conventional electric circuit theory. It is always defined with respect to some current flowing in the circuit. So we shall follow the same logic procedure in defining the "characteristic impedance" offered by an infinite hollow metal tube guide to an electric dipole as the "ratio of the time averaged surface integration of the complex Poynting's vector, (Equation (4.18)) to the time averaged square of the current flowing in the metal sheath in the axial- or z-direction" : *

$$\overline{\mathcal{H}} = \frac{c}{8\pi} Re[\mathcal{E} \times \mathcal{H}^*]$$

$$(4.33) \qquad \overline{P} = \frac{-c}{8\pi}\int_V \nabla\cdot[\mathcal{E}\times\mathcal{H}^*]\,d\tau = \frac{-c}{8\pi}\int_S [\mathcal{E}\times\mathcal{H}^*]\cdot\overline{n}\,dS$$

$$= \frac{1}{2}\int_V \sigma\, \mathcal{E}\cdot\mathcal{E}^*\,d\tau + 2i\omega\int_V\left[\frac{\mu}{8\pi}\frac{1}{2}\mathcal{H}\cdot\mathcal{H}^* - \frac{\varepsilon}{8\pi}\frac{1}{2}\mathcal{E}\cdot\mathcal{E}^*\right]d\tau$$

$$= \overline{W} + 2i\omega(\overline{U}_{mag.} - \overline{U}_{el.}) = \overline{W} + 2i\omega\overline{U}$$

$\overline{P}$ is therefore a complex quantity whose real part represents the mean Joule heat developed per second and whose imaginary part twice the amount of the difference of the mean magnetic energy and the mean electric energy. Thus, if $\mathcal{J}$ is the total current of the

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* Abraham and Becker, "Classic Theory of Electricity and Magnetism", pp. 196-.

system, we define:

$$(4.34)\quad \begin{cases} \frac{1}{2} R\, \mathcal{J}\cdot\mathcal{J}^* = \overline{W} \\ \frac{1}{2} X\, \mathcal{J}\cdot\mathcal{J}^* = 2\omega\,\overline{\mathcal{U}} \end{cases}$$

The second equation is true because we assumed that the system is dissipative. For a non-dissipative system $\overline{\mathcal{U}} = 0$ and $X = 0$. The current can be easily found from the following relation:

$$\int_0^{2\pi} H_{\varphi 1}\, a\, d\varphi = \frac{4\pi \mathcal{J}}{c}$$

$$(4.35)\quad \mathcal{J} = \frac{ca}{2} Re\left[\frac{\pi}{\mu_1 a^3} \frac{x_\nu^2 H_0^{(1)}(x_\nu)}{\beta_\nu J_1(x_\nu)} J_1(x_\nu) e^{-\alpha_\nu(z-s)} e^{-i[\omega t-\beta_\nu(z-s)]}\right]$$

$$= Re.\left[\frac{\pi c\, x_\nu^2}{2a^2\mu_1\beta_\nu} H_0^{(1)}(x_\nu) e^{-\alpha_\nu(z-s)} e^{i[\omega t-\beta_\nu(z-s)]}\right]$$

$$(4.36)\quad \mathcal{J}\cdot\mathcal{J}^* = \frac{\pi^2 c^2 x_\nu^4}{4a^4\mu_1^2\beta_\nu^2}[Y_0(x_\nu)]^2 e^{-2\alpha_\nu(z-s)} = \frac{\pi^2 c^2 x_\nu^4}{4a^4\mu_1^2\beta_\nu^2}[Y_0(x_\nu)]^2$$

From (4.20) we have, considering only one mode:

$$\overline{W}_\nu = \frac{\pi^2 \omega x_\nu^4}{4a^4\mu_1 k_1^2\beta_\nu}[Y_0(x_\nu)]^2 e^{-2\alpha_\nu(z-s)}$$

and $\overline{\mathcal{U}} \simeq 0$

$$(4.37)\ \therefore\quad R_\nu = \frac{2\overline{W}_\nu}{\mathcal{J}\cdot\mathcal{J}^*} = \frac{2\beta_\nu}{\omega\varepsilon_1} = \frac{2}{c}\sqrt{\frac{\mu_1}{\varepsilon_1}}\sqrt{1-\left(\frac{f_\nu}{f}\right)^2}$$

and $\quad X_\nu = 0$

The reactive component is nearly zero since the attenuation is small as discussed before. The characteristic impedance has practically only a pure resistive component, which may properly be called the "radiation resistance" of the system for "νth. mode" of propagation. There are as many distinct "radiation resistances" as there are "distinct modes" of propagation. If only the first "mode" is considered, then:

$$(4\cdot 38) \qquad R_1 = \frac{2}{c}\sqrt{\frac{\mu_1}{\varepsilon_1}}\sqrt{1-\left(\frac{2\cdot 4048\, c}{2\pi f a \sqrt{\varepsilon_1\mu_1}}\right)^2}$$

At the point of minimum attenuation, $R_{(min\,\alpha)} = \frac{2}{c}\sqrt{\frac{2}{3}}$ for air dielectric medium.

The radiation resistance R_ν is then zero at $f = f_\nu$ and increases as the frequency increases. It approaches the limit

$$R_\nu = \frac{2}{c}\sqrt{\frac{\mu_1}{\varepsilon_1}} = \frac{1}{1\cdot 5\times 10^{10}}\ e.s.u. = 600 \ \text{ohms}$$

when $f \gg f_\nu$. This casts some light on the design of couplings or absorbers for matching purposes. At minimum attenuation the radiation resistance becomes:

$$R_{\nu(min\,\alpha)} = \sqrt{\frac{2}{3}}\,600 = 490 \ \text{ohms}.$$

At $f = f_2$, R_1 becomes:

$$R_1(f=f_2) = \frac{2}{c}\sqrt{\frac{\mu_1}{\varepsilon_1}}\sqrt{1-\left(\frac{f_1}{f_2}\right)^2} = \frac{2}{c}\sqrt{1-\left(\frac{2\cdot 4048}{5\cdot 5201}\right)^2} = 540 \ \text{ohms}$$

The variation of the "characteristic impedance" R_ν is therefore comparatively small for the whole usable range of frequency. For a band of a few megacycles per second, it is practically constant.

(6) Field Structure.

We shall limit the study to the field structure in the dielectric medium. Referring to Equations (4.12), we see that along the axial- or z-direction all field components have a sinusoidal distribution with a gradually but very slowly decreasing amplitude due to the attenuation factor $e^{-\alpha_\nu(z-s)}$. The spatial wave length in the air medium is then:

$$(4.39) \qquad \mathcal{L}_\nu = \frac{2\pi}{\beta_\nu} = \frac{2\pi}{\sqrt{\frac{\omega^2\varepsilon_1\mu_1}{c^2} - \frac{x_\nu^2}{a^2}}} = \frac{v_0}{f\sqrt{1-\left(\frac{f_\nu}{f}\right)^2}}.$$

Corresponding to each mode of propagation, the apparent wave length is thus infinitely long when $f = f_\nu$ and decreasing and approaching that in free space only when $f = \infty$.

The equations for the lines of magnetic intensity $(H\varphi_1)$ are simply concentric circles around the axis and the distribution of intensity in the radial direction (r) is proportional to Bessel's function of the first order; i.e.:

$$H\varphi_1 \sim J_1\left(\frac{r}{a}x_\nu\right)$$

where x_ν is the νth. root of $J_0(x)=0$. Because the zero roots of $J_0(x)=0$ and $J_1(x)=0$ are alternate; i.e., between two roots of $J_0(x)=0$, there is a root for $J_1(x)=0$ and vice versa, $J_1\left(\frac{r}{a}x_\nu\right)$ has, therefore $(\nu-1)$ nodes for the range $0 \leq r < a$. We have, in fact, a standing wave in the radial direction in the air medium; the amplitude of this standing wave changes sinusoidally with time but its nodes remain fixed.

Similar discussion applies to the configuration of E_{r_1}. The structure for E_{z_1} is, however, proportional to $J_0(\frac{r}{a}x_\nu)$; it has then one node at the inner surface of the metal sheath and $(\nu-1)$ nodes for the range $0 \leq r < a$. The equation of the lines of electric intensity can be obtained from solving the following differential equation:

$$(4.40) \qquad \frac{dz}{E_{z_1}} = \frac{dr}{E_{r_1}}$$

or

$$\frac{\pi\omega x_\nu^2}{a^3 c k_1^2}\,\frac{J_1(\frac{r}{a}x_\nu)}{J_1(x_\nu)}\cos\left[\beta_\nu(z-s)-\omega t+\varphi_\nu\right] dz$$

$$= \frac{-\pi\omega x_\nu^3}{a^4 c k_1^2 \beta_\nu}\,\frac{J_0(\frac{r}{a}x_\nu)}{J_1(x_\nu)}\sin\left[\beta_\nu(z-s)-\omega t+\varphi_\nu\right] dr$$

where the phase displacement angle φ_ν is given by:

$$\varphi_\nu = \arctan\left[\frac{Y_0(x_\nu)}{J_0(x_\nu)}\right] = \frac{\pi}{2}$$

The above differential equation then simplifies to: *

$$\frac{J_0(\frac{r}{a}x_\nu)}{J_1(\frac{r}{a}x_\nu)}\, d\left(\frac{r}{a}x_\nu\right) = \frac{\sin\left[\beta_\nu(z-s)-\omega t\right]}{\cos\left[\beta_\nu(z-s)-\omega t\right]}\, d(\beta_\nu z)$$

or

$$d\left\{\log\left[\frac{r}{a}x_\nu J_1(\frac{r}{a}x_\nu)\right]\right\} = -d\left\{\log\cos\left[\beta_\nu(z-s)-\omega t\right]\right\}$$

Integrating gives:

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* Jahnke and Emde, "Functional Tables", p. 146.

$$\log\left[\frac{r}{a}\chi_\nu J_1\left(\frac{r}{a}\chi_\nu\right)\right] = -\log\left\{\cos\left[\beta_\nu(z-\varsigma)-\omega t\right]\right\} + C'$$

or

$$\left[\frac{r}{a}\chi_\nu J_1\left(\frac{r}{a}\chi_\nu\right)\right]\cdot\cos\left[\beta_\nu(z-\varsigma)-\omega t\right] = C \tag{4.41}$$

This gives the equation of the electric lines of force for any time t. The second cosine term indicates that it is propagated along the z-direction with a phase velocity $\frac{\omega}{\beta_\nu}$. The first term $\left[\frac{r}{a}\chi_\nu J_1\left(\frac{r}{a}\chi_\nu\right)\right]$ represents a standing wave distribution in the radial direction for the range $0 \leq r < a$. Equation (4.41) is for the νth mode only. There are as many distinct field structures as there are "distinct modes" of propagation. The integration constant C in (4.41) can be determined from the given exciting strength. Since the field configuration is independent of the azimuthal angle φ (4.41) holds for all planes passing through the axis.

(7) Terminal Device.

It will be shown here that for certain simple terminal devices it is possible to make a rigorous mathematical analysis. Up to now the development has been based upon the case of an infinitely long metal tube. But in any practical set-up the transmitting and the receiving ends must be terminated by some device which, of course, should be so designed as to increase the efficiency of transmission and reception.

For a linear axial antenna the simplest efficient termination at the transmitter end is a closed end made of perfect conducting material. The existence of such an end simply introduces an image situated the same distance behind the end as the exciter is situated in front of it. Then the resultant potential function (Equation (4.04)) becomes (See (8.02)a.)

$$(4.42)\quad \mathcal{U}_1 = \frac{\pi}{a^2}\sum_{\nu=1}^{\infty}\left[e^{i\lambda_\nu(z-s)} + e^{i\lambda_\nu(z+s)}\right]\frac{x_\nu H_0^{(1)}(x_\nu)}{\lambda_\nu J_1(x_\nu)}J_0\left(\frac{r}{a}x_\nu\right)$$

$$= \frac{2\pi}{a^2}\sum_{\nu=1}^{\infty} e^{i\lambda_\nu z}\cos\lambda_\nu z\ \frac{x_\nu H_0^{(1)}(x_\nu)}{\lambda_\nu J_1(x_\nu)}J_0\left(\frac{r}{a}x_\nu\right)$$

which reduces to:

$$\mathcal{U}_1 = \frac{2\pi}{a^2}\sum_{\nu=1}^{\infty} e^{i\lambda_\nu z}\cdot\frac{x_\nu H_0^{(1)}(x_\nu)}{\lambda_\nu J_1(x_\nu)}J_0\left(\frac{r}{a}x_\nu\right)$$

for $s = 0$ when the dipole is placed infinitely close to the perfect reflecting end. The intensity is thus simply doubled everywhere.

The same procedure immediately follows for any physical linear antenna since every element of the antenna has a corresponding image behind the closed end. Such a type of ending is considered as the most simple and at the same time the most efficient termination for transmission purposes.

Other irregular terminal devices disturb the configuration of the field and make any attempt for rigid analysis impossible.

The terminal device to be used at the receiving end is definitely much more difficult to design. Any scheme except that with the same "characteristic impedance" of the transmission system (i.e. complete absorption of the incident energy) causes the formation of a standing wave which may be undesirable and even a nuisance for high quality transmission. A standing wave in the present case eventually represents an interference between the exciting and the receiving devices. The paramount requirements for a satisfactory receiving termination are thus: (a) maximum pick-up of the incident wave to be fed to the detection device, and (b) complete absorption of the incident energy to avoid forming of any standing wave. A detailed analysis of devices achieving complete absorption of incident energy forms a distinct subject by itself and will not be attempted here. The general principle outlined above might be of some value in practical design.

(8) Stability Problem.

The problem of stability arises when the transmission system becomes heterogeneous or deviating from the ideal straight, circular cylindrical tube. For long distance transmission the two main unavoidable deviations from the ideal system are: (a) cross-section not being circular all along the length, some portion may assume oval or elliptic shape, and (b) bending of the tube at certain locations as found necessary in installation. For a linear exciter, a slight deviation from circular cross-section is of no serious consequence; although a small φ-component of electric intensity E_φ may be introduced at such locations but the magnetic intensity remains essentially circular (H_φ), the attenuation constant and velocity of propagation might have undergone a negligible modification. A bending of the tube especially at sharp angle, however, may introduce a strong new field configuration depending on a $e^{i\varphi}$ -factor and many lesser intense field configurations depending on factors $e^{in\varphi}$ $(n = 2, 3, \cdots\cdots)$. (These kinds of field configurations depending on $e^{in\varphi}$ for $n = 1, 2, 3, \cdots\cdots$, are called by some writers E_1 , E_2 , E_3 , ·····-waves.) This means at the same time a great reduction of the original symmetrical field intensity. From the above approximate qualitative argument we may conclude that, for carefully designed transmission system, bendings, even if not completely avoidable, should be performed with as small a curvature as possible and the less frequent the better.

SECTION V.

A Magnetic Dipole Placed at Point (ρ, r_0, φ_0) With Its Axis Parallel to the z-axis Inside an Infinite Cylindrical Hollow Metal Tube

The postulate of the existence of a magnetic dipole with two infinitely large fictitious "magnetic charges" of opposite polarity at an infinitely small distance apart from each other, dated back to the ancient conception of a magnet. Later researches, however, discredited the physical existence of "true magnetism" and unified the old parallel independent theories of electricity and magnetism. But the physical argument of reality does not penetrate into the mathematical analysis, since with proper care the hypothesis of an ideal magnetic dipole gives us a formal mathematical analogy to the case of an ideal electric dipole. And only through such a hypothesis can we obtain the corresponding analytical expressions for the field components due to a current loop of finite dimension in a simple way. A current loop of finite dimension is thus thought of as a magnetic shell whose boundary coincides with the loop. The postulate of a magnetic shell is not new and its properties have been discussed by many authors. What is attempted here is to use the idea of a magnetic shell as a mathematical intermediate bridge to reach an analytical expression in proper coordinates for the field of a current loop with uniform spatial current distribution along the loop, and thenceforth generalize for a non-uniform current distribution by means of a scheme similar to that used in obtaining the "$2n$-multiple axis" unsymmetrical cylindrical waves functions (1.20).

Sommerfeld* first used the idea of a magnetic dipole antenna for a formal mathematical formulation of the corresponding Hertzian function with the dipole situated above a perfect conducting earth.

Analogous to the introduction of a "general magnetic potential" $\mathfrak{U}$ for the case of an electric dipole (2.06), we shall now define a new function, say $\mathfrak{U}_m$, for the case of a "magnetic dipole" so that:

$$(5.01) \qquad \varepsilon \mathcal{E} = \nabla \times \mathfrak{U}_m$$

This function, $\mathfrak{U}_m$, may be proposedly called "general electric potential", since its curl gives the electric field $\mathcal{E}$ multiplied by the dielectric constant ε .

For the present discussion the magnetic dipole is supposed to be placed at (ζ, r_0, φ_0) with its axis parallel to that of the cylindrical guiding tube of radius a. Thus $\mathfrak{U}_m$ has only a z- component, or:

$$(5.02) \qquad \mathfrak{U}_m = u_{mz} \quad , \quad u_{mr} = u_{m\varphi} = 0$$

From the analogous formula derived in Section II., (2.11), we obtain, for the primary "source" general electric potential function, the following expressions:

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* Riemann-Webers, "Differentialgleichungen der Physik", S. 564-565.

$$(5.03)\qquad u_{m0} = \frac{e^{ikR}}{R} = \frac{e^{ik\sqrt{\rho^2+(z-\zeta)^2}}}{\sqrt{\rho^2+(z-\zeta)^2}}$$

$$= \frac{i}{2}\int_{-\infty}^{\infty} e^{i\lambda(z-\zeta)} H_0^{(1)}(\rho\sqrt{k^2-\lambda^2})\,d\lambda$$

where $\rho = \sqrt{r^2+r_0^2-2rr_0\cos(\varphi-\varphi_0)}$

On expanding the integrand according to the addition theorem of cylindrical functions, we have, omitting subscript m:

$$(5.04)\qquad u_0 = \frac{i}{2}\sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi_0)}\int_{-\infty}^{\infty} e^{i\lambda(z-\zeta)} J_m(r\sqrt{k^2-\lambda^2})\,H_m^{(1)}(r_0\sqrt{k^2-\lambda^2})\,d\lambda \qquad \text{for } r < r_0$$

$$(5.05)\qquad u_0 = \frac{i}{2}\sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi_0)}\int_{-\infty}^{\infty} e^{i\lambda(z-\zeta)} J_m(r_0\sqrt{k^2-\lambda^2})\,H_m^{(1)}(r\sqrt{k^2-\lambda^2})\,d\lambda \qquad \text{for } r > r_0$$

For the present case, the field components are:

$$\mathfrak{E} = E_\varphi,\ E_r,\ E_z = 0\ ;\quad \mathfrak{H} = H_z,\ H_r,\ H_\varphi,$$

and the corresponding Maxwell field equations become (in Gaussian units):

$$(5.06)\begin{cases} \text{(a)}\quad \dfrac{\partial H_r}{\partial z} - \dfrac{\partial H_z}{\partial r} = \dfrac{4\pi\sigma}{c}E_\varphi + \dfrac{\varepsilon}{c}\dfrac{\partial E_\varphi}{\partial t} \\ \text{(b)}\quad \dfrac{1}{r}\dfrac{\partial H_z}{\partial \varphi} - \dfrac{\partial H_\varphi}{\partial z} = \dfrac{4\pi\sigma}{c}E_r + \dfrac{\varepsilon}{c}\dfrac{\partial E_r}{\partial t} \\ \text{(c)}\quad \dfrac{1}{r}\left[\dfrac{\partial}{\partial r}(rE_\varphi) - \dfrac{\partial E_r}{\partial \varphi}\right] = -\dfrac{\mu}{c}\dfrac{\partial H_z}{\partial t} \end{cases}$$

(d) $$-\frac{\partial E_\varphi}{\partial z} = -\frac{\mu}{c}\frac{\partial H_r}{\partial t}$$

(e) $$\frac{\partial E_r}{\partial z} = -\frac{\mu}{c}\frac{\partial H_\varphi}{\partial t}$$

Since all quantities must be real, we have:

$$\mathcal{U} = Re[u e^{-i\omega t}]$$

$$(5.07)\quad \begin{cases} E_\varphi = Re[-\frac{1}{\varepsilon}\frac{\partial u}{\partial r} e^{-i\omega t}] \\ E_r = Re[\frac{1}{\varepsilon r}\frac{\partial u}{\partial \varphi} e^{-i\omega t}] \\ E_z = 0 \\ H_z = Re[\frac{c}{\varepsilon\mu i\omega}(k^2 u + \frac{\partial^2 u}{\partial z^2}) e^{-i\omega t}] \\ H_r = Re[\frac{c}{\varepsilon\mu i\omega}\frac{\partial^2 u}{\partial z \partial r} e^{-i\omega t}] \end{cases}$$

$$H_\varphi = Re[\frac{1}{\varepsilon\mu i\omega r}\frac{\partial^2 u}{\partial z\partial\varphi} e^{-i\omega t}]$$

and

$$(5.08)\quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} + k^2 u = 0$$

Therefore, the system of field configuration is unique and Equations (5.04) and (5.05) are the formal solutions of (5.08) for $r < r_0$ and $r > r_0$, respectively.

As discussed before in II., the above field configuration with $E_z = 0$ can only be realized for perfect conducting metal tube and we shall limit ourselves to such a case.

The disturbance function for medium 1 $(r<a)$ must be finite and, therefore, only the Bessel function $J_m(r\sqrt{k^2-\lambda^2})$ can be used. We obtain then for the total "general electric potential" in medium 1:

$$(5.09)\quad \mathcal{U}(\rho, z-\varsigma) = \begin{cases} \frac{1}{2}\sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi_0)} \int_{-\infty}^{\infty} e^{i\lambda(z-\varsigma)} \left[iJ_m(r\xi_1) H_m^{(1)}(r_0\xi_1) + F_1(\lambda) J_m(r\xi_1) \right] d\lambda & \text{for } r<r_0 \quad (a) \\ \frac{1}{2}\sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi_0)} \int_{-\infty}^{\infty} e^{i\lambda(z-\varsigma)} \left[iJ_m(r_0\xi_1) H_m^{(1)}(r\xi_1) + F_1(\lambda) J_m(r\xi_1) \right] d\lambda & \text{for } r>r_0 \quad (b) \end{cases}$$

That for medium 2 $(r>a)$ need not be considered.

The boundary condition at $r=a$ requires that the tangential component of electric field intensity must be zero, thus from (5.07) we have:

$$F_1(\lambda) = \frac{-i J_m(r_0\sqrt{k_1^2-\lambda^2})\, H_m^{(1)\prime}(a\sqrt{k_1^2-\lambda^2})}{J_m'(a\sqrt{k_1^2-\lambda^2})}$$

where the prime indicates differentiation with respect to the argument involved. Equations $(5.09)_a$ and $(5.09)_b$ then become, respectively:

$$(5.10)\quad \mathcal{U}(\rho, z-\varsigma) = \begin{cases} \frac{i}{2}\sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi_0)} \int_{-\infty}^{\infty} e^{i\lambda(z-\varsigma)} \frac{J_m'(a\xi_1) H_m^{(1)}(r_0\xi_1) - J_m(r_0\xi_1) H_m^{(1)\prime}(a\xi_1)}{J_m'(a\xi_1)} J_m(r\xi_1)\, d\lambda & \text{for } r<r_0 \quad (a) \\ \frac{i}{2}\sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi_0)} \int_{-\infty}^{\infty} e^{i\lambda(z-\varsigma)} \frac{J_m(r_0\xi_1) J_m'(a\xi_1) H_m^{(1)}(r\xi_1) - J_m(r_0\xi_1) H_m^{(1)\prime}(a\xi_1) J_m(r\xi_1)}{J_m'(a\xi_1)}\, d\lambda & \text{for } r>r_0 \quad (b) \end{cases}$$

The field components can then be obtained by substituting the above two equations into Equations (5.07).

Now we shall consider the special case of a magnetic dipole placed at origin $(0,0,0)$, then $\varsigma=0,\ r_0=0,\ m=0$ and (5.10)a and (5.10)b reduce to:

$$(5.11) \quad u = \frac{i}{2}\int_{-\infty}^{\infty} e^{i\lambda z}\cdot\frac{J_0'(a\xi_1)H_0^{(1)}(r\xi_1) - H_0^{(1)\prime}(a\xi_1)J_0(r\xi_1)}{J_0'(a\xi_1)}\,d\lambda$$

$$= \frac{i}{2}\int_{-\infty}^{\infty} e^{i\lambda z}\cdot\frac{J_1(a\xi_1)H_0^{(1)}(r\xi_1) - H_1^{(1)}(a\xi_1)J_0(r\xi_1)}{J_1(a\xi_1)}\,d\lambda$$

From the functional relations (1.29), it can easily be shown that (5.10)a, (5.10)b, and (5.11) are all meromorphous functions and their formal integration can be carried out by expanding into an infinite series according to the theory of residues.

SECTION VI.

A Circular Magnetic Shell (Current Loop) Of Radius b with Center at $(\zeta, 0, 0)$ Inside an Infinite Cylindrical Metal Tube

Just as the expression of a current element serves as the Green's function for integrating along the axis of the extension of a linear antenna, so does the expression of a magnetic dipole for integrating over the plane area occupied by a uniform current loop. We obtain, therefore, the potential function as follows:

$$(6.01) \qquad U = \int \mathcal{U}\, f(r_0, \varphi_0)\, dS_0$$

where, for $\mathcal{U}$, expressions (5.10) are to be used for $r < r_0$ and $r > r_0$, respectively. At first glance, it seems impossible to use the idea of a "magnetic shell" to obtain the effect of a current loop with an arbitrary distribution of current along the loop, because the conventional magnetic shell is usually thought of as a uniform one.

Two new methods are described here. Each of them has its own physical realization. The first one is for perfect general arbitrary current distribution and includes eventually the second method as a special case. But due to special significance of the second method, it will be considered as a separate one.

(1) First Method - Arbitrary Current Distribution.

The gist of this method lies in the fact that a linear arbitrary current distribution along the loop corresponds to two types of distribution of the strength of the magnetic shell elements over the "area" bounded by the loop. Referring to Figure VI-1 and using polar coordinates for the plane area of the loop, the two types of distribution are: firstly, the strength of the magnetic shell element dS_0 is constant along radial direction for fixed φ_0 as $d\varphi_0 \to 0$; and, secondly, the strength of the element dS_0 follows the same distribution as the current for different φ_0 at any r_0. Consequently, for a circular loop $f(r_0, \varphi_0)$ in (6.01) is independent upon r_0 and may properly be written as $f(\varphi_0)$, which is just the function for the current distribution along the loop. We have then:

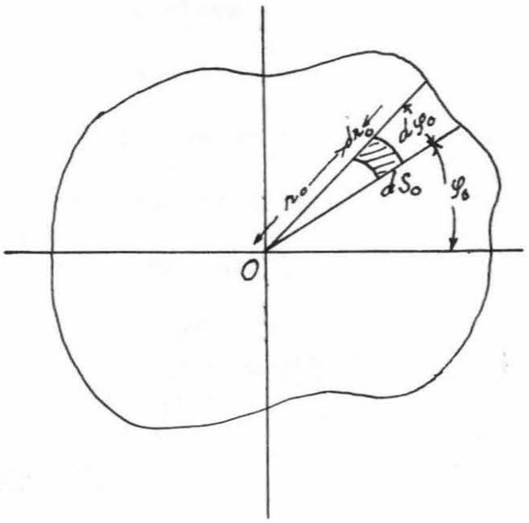

Fig. VI.-1.

$$(6.02) \qquad U = \int u f(\varphi_0)\, dS_0 = \iint u f(\varphi_0)\, r_0\, d\varphi_0\, dr_0$$

where the integration with respect to r_0 can be performed first with given $u = u(z, r, \varphi; \zeta, r_0, \varphi_0)$ leaving the integration with respect to φ_0 depending upon the current distribution. It should be noticed that for a loop of arbitrary shape, $f(r_0, \varphi_0)$ is always a function of both r_0 and φ_0. The above argument also shows that Equation (6.01) is perfectly general for any shaped closed loop with any arbitrary current distribution. This idea can also be used to find rigorous analytic expressions for the "electric potential function" for, say, rectangular or circular loop placed parallel above the earth surface.

From the above general consideration, we shall, however, limit the discussion to a circular loop with its plane orientated perpendicular to the axis and its center at $(\varsigma, 0, 0)$. Then, substituting Equations (5.10) for $\mathcal{U}$ in (6.02), there result:

$$(6.03)\quad \bar{U} = \begin{cases} \dfrac{i}{2}\displaystyle\int_0^{2\pi} \sum_m e^{im(\varphi-\varphi_0)} f(\varphi_0)\,d\varphi_0 \left\{ \int_{-\infty}^{\infty} e^{i\lambda(z-\varsigma)} \left[\int_a^b \frac{J_m'(a\varepsilon_1)H_m^{(1)}(r_0\varepsilon_1) - J_m(r_0\varepsilon_1)H_m^{(1)\prime}(a\varepsilon_1)}{J_m'(a\varepsilon_1)}\, r_0\,dr_0 \right] J_m(r\varepsilon_1)\,d\lambda \right\} & \text{for } r<r_0 \\[2ex] \dfrac{i}{2}\displaystyle\int_0^{2\pi} \sum_m e^{im(\varphi-\varphi_0)} f(\varphi_0)\,d\varphi_0 \left\{ \int_{-\infty}^{\infty} e^{i\lambda(z-\varsigma)} \left[\int_0^b J_m(r_0\varepsilon_1)\, r_0\,dr_0 \right] \frac{J_m'(a\varepsilon_1)H_m^{(1)}(r\varepsilon_1) - J_m(r\varepsilon_1)H_m^{(1)\prime}(a\varepsilon_1)}{J_m'(a\varepsilon_1)}\,d\lambda \right\} & \text{for } r>r_0 \end{cases}$$

Unfortunately, there is no formal simple way of carrying out the indicated integration with respect to r_0 for general integer value of m except for $m=0$. The failure of this general method for $m \neq 0$ compels us to leave it as it stands until some new scheme of integration is to be devised.

For $m=0$, however, we obtain immediately:

$$(6.03)_a\quad U = \begin{cases} \dfrac{ib}{2}\displaystyle\int_0^{2\pi} f(\varphi_0)\,d\varphi_0 \int_{-\infty}^{\infty} e^{i\lambda(z-\varsigma)} \cdot \frac{J_1(a\varepsilon_1)H_1^{(1)}(b\varepsilon_1) - J_1(b\varepsilon_1)H_1^{(1)}(a\varepsilon_1)}{\varepsilon_1 J_1(a\varepsilon_1)} J_0(r\varepsilon_1)\,d\lambda & \text{for } r<r_0 \\[2ex] \dfrac{ib}{2}\displaystyle\int_0^{2\pi} f(\varphi_0)\,d\varphi_0 \int_{-\infty}^{\infty} e^{i\lambda(z-\varsigma)} \cdot \frac{J_1(a\varepsilon_1)H_0^{(1)}(r\varepsilon_1) - J_0(r\varepsilon_1)H_1^{(1)}(a\varepsilon_1)}{\varepsilon_1 J_1(a\varepsilon_1)} J_1(b\varepsilon_1)\,d\lambda & \text{for } r>r_0 \end{cases}$$

Integrating again with given $f(\varphi_0)$ yields therefore only one term of the infinite series, because $f(\varphi_0)$ can always be resolved into Fourier harmonics.

If the current distribution around the circular loop is uniform, then Equation (6.03) reduces to:

$$(6.04)\quad U = \begin{cases} i\pi b \displaystyle\int_{-\infty}^{\infty} e^{i\lambda(z-\zeta)} \frac{J_1(a\varepsilon_1)H_1^{(1)}(b\varepsilon_1) - J_1(b\varepsilon_1)H_1^{(1)}(a\varepsilon_1)}{\varepsilon_1 J_1(a\varepsilon_1)} J_0(r\varepsilon_1)\, d\lambda & \text{for } r < r_0, \\ i\pi b \displaystyle\int_{-\infty}^{\infty} e^{i\lambda(z-\zeta)} \frac{J_1(a\varepsilon_1)H_0^{(1)}(r\varepsilon_1) - J_0(r\varepsilon_1)H_1^{(1)}(a\varepsilon_1)}{\varepsilon_1 J_1(a\varepsilon_1)} J_1(b\varepsilon_1)\, d\lambda & \text{for } r > r_0. \end{cases}$$

Except for an unimportant multiplying factor, (6.04) gives the complete solution for the potential function due to a uniform current loop inside an infinite cylindrical metal tube.

<u>(2) Second Method - Trapezoidal Current Distribution.</u>

The generalization from (6.04) to cases with trapezoidal current distribution can be accomplished by the same scheme used to obtain the general cylindrical waves functions, (1.19) and (1.20). Thus for a "bi-axis" current loop with current in the two halves of the circular loop "opposite in phase" but of "same magnitude", (trapezoidal in shape) the potential function at the field point becomes:

$$(6.05)\quad U_{(2)} = D_0 U = \left(\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial y_0}\right)U = e^{i\varphi_0}\left(\frac{\partial}{\partial b} + i\frac{\partial}{b\,\partial\varphi_0}\right)U$$

$$= -DU = -\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)U = -e^{i\varphi}\left(\frac{\partial}{\partial r} + i\frac{\partial}{r\,\partial\varphi}\right)U$$

where U is to be substituted from (6.04), and D_0 and D refers to operations upon the "source" and upon the "field point", respectively.

$$(6.06)\ U_{(2)} = \begin{cases} i\pi b \displaystyle\int_{-\infty}^{\infty} e^{i\varphi} e^{i\lambda(z-\zeta)} \cdot \frac{J_1(a\xi_1)H_1^{(1)}(b\xi_1) - J_1(b\xi_1)H_1^{(1)}(a\xi_1)}{J_1(a\xi_1)} J_1(r\xi_1)\,d\lambda & \text{for } r<r_0, \\ i\pi b \displaystyle\int_{-\infty}^{\infty} e^{i\varphi} e^{i\lambda(z-\zeta)} \cdot \frac{J_1(a\xi_1)H_1^{(1)}(r\xi_1) - J_1(r\xi_1)H_1^{(1)}(a\xi_1)}{J_1(a\xi_1)} J_1(b\xi_1)\,d\lambda & \text{for } r>r_0. \end{cases}$$

or

$$(6.06)_a\ U_{(2)} = \begin{cases} i\pi b\cos\varphi \displaystyle\int_{-\infty}^{\infty} e^{i\lambda(z-\zeta)} \cdot \frac{J_1(a\xi_1)H_1^{(1)}(b\xi_1) - J_1(b\xi_1)H_1^{(1)}(a\xi_1)}{J_1(a\xi_1)} J_1(r\xi_1)\,d\lambda & \text{for } r<r_0, \\ i\pi b\cos\varphi \displaystyle\int_{-\infty}^{\infty} e^{i\lambda(z-\zeta)} \cdot \frac{J_1(a\xi_1)H_1^{(1)}(r\xi_1) - J_1(r\xi_1)H_1^{(1)}(a\xi_1)}{J_1(a\xi_1)} J_1(b\xi_1)\,d\lambda & \text{for } r>r_0. \end{cases}$$

The factor $\cos\varphi$ is used in the last form for a trapezoidal current distribution which is positive (or negative) for $-\frac{\pi}{2}<\varphi<\frac{\pi}{2}$ and negative (or positive) for $\frac{\pi}{2}<\varphi<\frac{3\pi}{2}$. Equations (6.06) are formal solutions of the differential Equation (5.08). Similarly we can obtain the potential functions for "quadruple-axis" current loop, "sextuple-axis" current loop, etc.. The corresponding expressions for "$2n$-multiple axis" current loop is:

$$(6.07)\quad U_{(2n)} = (-D)^n U = \begin{cases} i\pi b\, e^{in\varphi} \displaystyle\int_{-\infty}^{\infty} e^{i\lambda(z-s)} (\xi_1)^{n-1} \cdot \frac{J_1(a\xi_1) H_1^{(1)}(b\xi_1) - J_1(b\xi_1) H_1^{(1)}(a\xi_1)}{J_1(a\xi_1)} J_n(r\xi_1)\, d\lambda & \text{for } r < r_0, \\ i\pi b\, e^{in\varphi} \displaystyle\int_{-\infty}^{\infty} e^{i\lambda(z-s)} (\xi_1)^{n-1} \cdot \frac{J_1(a\xi_1) H_n^{(1)}(r\xi_1) - J_n(r\xi_1) H_1^{(1)}(a\xi_1)}{J_1(a\xi_1)} J_1(b\xi_1)\, d\lambda & \text{for } r > r_0. \end{cases}$$

The integration of (6.04), (6.06), and (6.07) can be carried out by a formal expansion at the poles of the integrand according to Cauchy's theory since they are all meromorphous functions of the arguments involved.

SECTION VII.

Characteristics of Propagation of a Current Loop Inside a Cylindrical Metal Tube

In this section we shall limit our discussion to the case of a uniform current loop. Equations (6.04) give the potential functions for $r<b$ and $r>b$, respectively. The characteristics with trapezoidal current distribution can be obtained from that for the symmetrical case with only slight modifications. The procedure in the present section follows along parallel lines as that in Section IV..

Part (A) Transformation of the Integral Expressions (6.04).

Since relations (6.04) are unique and meromorphous functions of λ and the argument of the cylindrical functions, a formal solution can be obtained by aid of the calculus of residues. According to Cauchy's theory, the closed contour is to be achieved by means of an infinite semi-circle above the real axis in the λ-plane. This is permissible since integration along this semi-circle yields nothing. Relations (4.01), (4.02) and some of the discussions there hold true word for word for the present case.

The poles of (6.04) are the roots of

$$a\sqrt{k_1^2-\lambda^2}\,J_1(a\sqrt{k_1^2-\lambda^2}\,) = y\,J_1(y) = 0 \quad ,$$

which will be denoted by $y = y_0,\ y_1,\ y_2,\ \cdots,\ y_\nu,\ \cdots$ *.

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* The first few roots of $J_1(y)=0$ are: $y_0=0$, $y_1=3.8317$, $y_2=7.0156$, $y_3=10.1735$, ... Jahnke and Emde, "Functional Tables", p. 166.

Corresponding to the νth. root, y_ν , we have:

$$\lambda_\nu^2 = k_1^2 - \frac{y_\nu^2}{a^2} = \left(\frac{\omega^2\epsilon_1\mu_1}{c^2} - \frac{y_\nu^2}{a^2}\right) + i\,\frac{4\pi\omega\sigma_1\mu_1}{c^2}$$

Let $\lambda_\nu = \beta_\nu + i\alpha_\nu$, then we have:

$$\beta_\nu^2 - \alpha_\nu^2 = \frac{\omega^2\epsilon_1\mu_1}{c^2} - \frac{y_\nu^2}{a^2}\,, \tag{7.01}$$

$$\alpha_\nu\beta_\nu = \frac{2\pi\omega\sigma_1\mu_1}{c^2} \approx 0\,. \tag{7.02}$$

The two branches of the hyperbola (7.01) cross the real $(\beta_\nu-)$ or the imaginary $(\alpha_\nu-)$ axis according to the right side being positive or negative. Those of the hyperbola (7.02) lie in the first and third quandrants, and practically coincide with the axes for the case $\sigma_1 \to 0$.

The residue at $y_0 = 0$ is a little different from that at others and is to be evaluated by means of the expansions of the cylindrical function at very small argument.

$$\lim_{\epsilon_1\to 0} J_0(r\epsilon_1) \approx 1$$

$$\lim_{\epsilon_1\to 0} J_1(r\epsilon_1) \approx \frac{r\epsilon_1}{2}$$

$$\lim_{\epsilon_1\to 0} H_0^{(1)}(r\epsilon_1) \approx \frac{2i}{\pi}\log r\epsilon_1\gamma$$

$$\lim_{\epsilon_1\to 0} H_1^{(1)}(r\epsilon_1) \approx \frac{2i}{\pi}\left[\frac{r\epsilon_1}{2}\log r\epsilon_1\gamma - \frac{1}{r\epsilon_1}\right]$$

$$\lim_{\epsilon_1\to 0} \frac{J_1(a\epsilon_1)H_1^{(1)}(b\epsilon_1) - J_1(b\epsilon_1)H_1^{(1)}(a\epsilon_1)}{\epsilon_1\, J_1(a\epsilon_1)} J_0(r\epsilon_1)$$

$$\approx \frac{4i}{\pi a\epsilon_1^2}\left[\frac{ab\epsilon_1^2}{4}\log\frac{b}{a} - \frac{a}{2b} + \frac{b}{2a}\right]$$

$$\approx \frac{-4i\,(a^2-b^2)}{2\pi a^2 b\,\varepsilon_1^2} = \frac{-i(a^2-b^2)}{\pi a^2 b k_1}\left[\frac{1}{k_1-\lambda}+\frac{1}{k_1+\lambda}\right]$$

Residue for the first equation of (6.04) then becomes at $\lambda_1 = k_1$:

$$Res.[\lambda = k_1] = \frac{-2i\pi(a^2-b^2)}{a^2 k_1} e^{ik_1(z-\zeta)}$$

The residues at the other poles for the first equation of (6.04) are all of the following form:

$$Res.[\lambda=\lambda_\nu] = (i\pi ab)(2\pi i)\left[\frac{-J_1(\frac{b}{a}y_\nu)H_1^{(1)}(y_\nu)J_0(\frac{r}{a}y_\nu)}{y_\nu}\right]\lim_{\lambda\to\lambda_\nu}\left[\frac{\lambda-\lambda_\nu}{J_1(a\sqrt{k_1^2-\lambda^2})}\right]$$

$$= -\frac{2\pi^2 b}{a\lambda_\nu}\frac{J_1(\frac{b}{a}y_\nu)H_1^{(1)}(y_\nu)J_0(\frac{r}{a}y_\nu)}{J_1'(y_\nu)}.$$

Since

$$\lim_{\lambda=\lambda_\nu}\frac{\lambda-\lambda_\nu}{J_1(a\sqrt{k_1^2-\lambda_\nu^2})} = \frac{1}{\frac{\partial}{\partial\lambda}J_1(a\sqrt{k_1^2-\lambda^2})}\Bigg|_{\lambda=\lambda_\nu} = \frac{-y_\nu}{a^2\lambda_\nu J_1'(y_\nu)}$$

For the second equation of (6.04), we have:

$$\lim_{\varepsilon_1\to 0}\frac{J_1(a\varepsilon_1)H_0^{(1)}(r\varepsilon_1)-J_0(r\varepsilon_1)H_1^{(1)}(a\varepsilon_1)}{\varepsilon_1 J_1(a\varepsilon_1)}J_1(b\varepsilon_1)$$

$$\approx \frac{i2b}{\pi a\varepsilon_1}\left[\frac{a\varepsilon_1}{2}\log r\varepsilon_1\gamma - \frac{a\varepsilon_1}{2}\log a\varepsilon_1\gamma + \frac{1}{a\varepsilon_1}\right]$$

$$\approx \frac{ib}{\pi a^2 k_1}\left[\frac{1}{k_1-\lambda}+\frac{1}{k_1+\lambda}\right]$$

$$Res.[\lambda=k_1] = \frac{i\pi b^2}{a^2 k_1}e^{ik_1(z-\zeta)}$$

$$Res.[\lambda=\lambda_\nu] = (i\pi ab)(2\pi i)\left[\frac{-J_1(\frac{b}{a}y_\nu)H_1^{(1)}(y_\nu)J_0(\frac{r}{a}y_\nu)}{y_\nu}\right]\lim_{\lambda\to\lambda_\nu}\left[\frac{\lambda-\lambda_\nu}{J_1(a\sqrt{k_1^2-\lambda^2})}\right]$$

$$= -\frac{2\pi^2 b}{a}\frac{J_1(\frac{b}{a}y_\nu)H_1^{(1)}(y_\nu)J_0(\frac{r}{a}y_\nu)}{\lambda_\nu J_1'(y_\nu)}$$

Therefore, the series expansions for (6.04) become:

$$(7.03)\quad \mathcal{U} = \begin{cases} \dfrac{-2i\pi(a^2-b^2)}{a^2 k_1} e^{ik_1(z-\zeta)} - \displaystyle\sum_{\nu=1}^{\infty}\frac{2\pi^2 b}{a}\frac{J_1(\frac{b}{a}y_\nu)H_1^{(1)}(y_\nu)}{\lambda_\nu J_1'(y_\nu)}J_0(\tfrac{r}{a}y_\nu)e^{i\lambda_\nu(z-\zeta)} & \text{for } r<b \\ \dfrac{i\pi b^2}{a^2 k_1} e^{ik_1(z-\zeta)} - \displaystyle\sum_{\nu=1}^{\infty}\frac{2\pi^2 b}{a}\frac{J_1(\frac{b}{a}y_\nu)H_1^{(1)}(y_\nu)}{\lambda_\nu J_1'(y_\nu)}J_0(\tfrac{r}{a}y_\nu)e^{i\lambda_\nu(z-\zeta)} & \text{for } r>b. \end{cases}$$

The two expressions, therefore, only differ in the first term corresponding to the zero root $y_0 = 0$. All the discussions for Equation (4.04) in Section IV. can be applied here with slight modifications. In practical computation, only the first few terms of the summation are necessary for which λ_ν's are real and the following relation holds:

$$(7.04)\qquad \frac{\omega^2\epsilon_1\mu_1}{c^2} > \frac{y_\nu^2}{a^2}$$

The expressions for the field components for the air medium can be obtained by substituting (7.03) for $\mathcal{U}$ in (5.07), remembering $\frac{\partial}{\partial\varphi} = 0$ for the present symmetrical case. They are:

$$E_{\varphi_1} = Re\left[\frac{-1}{\epsilon_1}\sum_{\nu=1}^{\infty}\frac{2\pi^2 b}{a}\frac{J_1(\frac{b}{a}y_\nu)H_1^{(1)}(y_\nu)}{\lambda_\nu J_1'(y_\nu)}\cdot\frac{y_\nu}{a}J_1(\tfrac{r}{a}y_\nu)e^{i[\lambda_\nu(z-\zeta)-\omega t]}\right]$$

$$(7.05)\quad \begin{cases} H_{z_1} = Re\left[\dfrac{-c}{\varepsilon_1\mu_1 i\omega}\sum_{\nu=1}^{\infty}(k_1^2-\lambda_\nu^2)\dfrac{2\pi^2 b}{a}\dfrac{J_1(\frac{b}{a}y_\nu)H_1^{(1)}(y_\nu)}{\lambda_\nu J_1'(y_\nu)}J_0(\frac{r}{a}y_\nu)e^{i[\lambda_\nu(z-\zeta)-\omega t]}\right] \\ H_{r_1} = Re\left[\dfrac{c}{\varepsilon_1\mu_1\omega}\sum_{\nu=1}^{\infty}\dfrac{2\pi^2 b}{a^2}y_\nu\dfrac{J_1(\frac{b}{a}y_\nu)H_1^{(1)}(y_\nu)}{J_1'(y_\nu)}J_1(\frac{r}{a}y_\nu)e^{i[\lambda_\nu(z-\zeta)-\omega t]}\right] \end{cases}$$

In the above expression for H_{z_1}, the first term before the summation sign is neglected because they are lacking in the expressions for E_{φ_1} and H_{r_1} and consequently plays no physical role.

Part (B) Characteristics of Propagation.

The formal mathematical development in the previous section and the transformation formulae obtained in Part (A) lay the foundation for a rigorous discussion of the physical properties of propagation due to a circular loop antenna inside a conducting cylindrical metal tube. Similar to the case of a linear antenna, each root, say y_ν, of $J_1(y)=0$ gives rise to a "distinct mode" of propagation. The "attenuations" and "velocities" for different modes are different from each other and would be independent upon each other if the transmission system is uniform and homogeneous. In fact, each "distinct mode" propagates down the tube guide as if it exists alone. We shall, therefore, limit the discussion to the characteristics of one mode. As pointed out before, for satisfactory operation only one mode could be allowed.

(1) Attenuation Constant.

With loss in the dielectric air medium negligible, attenuation in the system is completely due to the finite conductivity of the metal sheath. We have, in fact, $\sigma_1 = 0$ but $\sigma_2 \neq \infty$. This can be taken into account by the same scheme developed in Section IV. Consider, say, the $\nu th.$ mode, its phase constant is computed from:

$$\beta_\nu^2 = \frac{\omega^2 \varepsilon_1 \mu_1}{c^2} - \frac{y_\nu^2}{a^2}$$

or (7.06) $$\beta_\nu = \frac{2\pi f}{v_0}\sqrt{1-\left(\frac{f_\nu}{f}\right)^2}$$

where $f_\nu = \dfrac{y_\nu v_0}{2\pi a}$ may be called the "cut-off" frequency of the "$\nu th.$ mode". Its attenuation constant α_ν can then be found by setting up the field expressions in the second medium and obtaining therefrom the "corrected roots y_ν'". Thus the z-component of the Poynting vector represents the useful part of energy transmitted and the r-component of the same stands for the loss in the metal sheath. We have then:

$$(7.07)\quad \overline{N_z} = \frac{c}{8\pi} Re[\mathcal{E} \times \mathcal{H}^*]_z = \frac{-c}{8\pi} Re[E_{\varphi 1} H_{r1}^*]$$

$$= \frac{c}{8\pi} Re\left[\frac{c}{\varepsilon_1^2 \mu_1 \omega \lambda_\nu}\left(\frac{2\pi^2 b y_\nu}{a^2}\right)^2 H_1^{(1)}(y_\nu)\{H_1^{(1)}(y_\nu)\}^* \left\{\frac{J_1(\frac{b}{a}y_\nu)J_1(\frac{r}{a}y_\nu)}{J_1'(y_\nu)}\right\}^2\right]$$

$$= \frac{c^2}{8\pi\varepsilon_1^2\mu_1\omega}\left(\frac{2\pi^2 b y_\nu}{a^2}\right)^2 \frac{1}{\beta_\nu}\left[\frac{J_1(\frac{b}{a}y_\nu)J_1(\frac{r}{a}y_\nu)Y_1(y_\nu)}{J_1'(y_\nu)}\right]^2$$

Integrating over the bases at $\pm z$, the time averaged value for energy propagation becomes:

$$(7.08)\quad \overline{W_\nu} = 2\int_0^{2\pi}\int_0^a \overline{N_z}\, r\,dr\,d\varphi = \frac{c^2}{2\varepsilon_1^2\mu_1\omega}\left(\frac{2\pi^2 b y_\nu}{a^2}\right)^2\frac{1}{\beta_\nu}\left[\frac{J_1(\frac{b}{a}y_\nu)Y_1(y_\nu)}{J_1'(y_\nu)}\right]^2\int_0^a\left[J_1(\tfrac{r}{a}y_\nu)\right]^2 r\,dr$$

$$= \frac{c^2}{4\varepsilon_1^2\mu_1\omega}\left(\frac{2\pi^2 b y_\nu}{a}\right)^2\frac{1}{\beta_\nu}\left[\frac{J_1(\frac{b}{a}y_\nu)Y_1(y_\nu)J_0(y_\nu)}{J_1'(y_\nu)}\right]^2$$

for the νth-mode. The total energy propagated is simply a superposition of those for all "modes".

$$(7.09)\quad \overline{W} = \sum_{\nu=1}^{n}\overline{W_\nu} = \sum_{\nu=1}^{n}\frac{\pi^4 b^2 y_\nu^2 c^2}{\varepsilon_1^2\mu_1\omega a^2\beta_\nu}\left[\frac{J_1(\frac{b}{a}y_\nu)Y_1(y_\nu)J_0(y_\nu)}{J_1'(y_\nu)}\right]^2$$

where (7.04) holds for $\nu = n$.

The corresponding r-component of the complex Poynting vector is:

$$(7.10)\quad \overline{N_r} = \frac{c}{8\pi} Re[\mathcal{E}\times\mathcal{H}^*]_r = \frac{-c}{8\pi} Re[E_{\varphi 1} H_{z1}^*]_{r=a}$$

$$= \frac{-c}{8\pi} Re\left[\frac{c}{\varepsilon_1^2\mu_1 i\omega}\left(\frac{2\pi^2 b}{a\lambda_\nu}\right)^2\left(-\frac{y_\nu}{a}\right)^3\left\{\frac{J_1(\frac{b}{a}y_\nu)Y_1(y_\nu)}{J_1'(y_\nu)}\right\}^2 J_0(y_\nu)J_1(y_\nu)\right]$$

This is zero if y_ν is the root of $J_1(y) = 0$, which results from the assumption that $\sigma_1 \to 0$ & $\sigma_2 \to \infty$. In order to find the "corrected root y_ν'" for σ_2 large but not infinite, we formulate the series expressions for the field components for the second medium, which is assumed electrically thick. Referring to (5.07), we obtain:

$$(7.11)\begin{cases} E_{\varphi 2} = Re\left\{\left[A_1 H_1^{(1)}(r\sqrt{k_2^2-\lambda_\nu^2}) + A_2 J_1(r\sqrt{k_2^2-\lambda_\nu^2})\right]e^{i[\lambda_\nu(z-s)-\omega t]}\right\} \\ H_{z2} = Re\left\{\frac{c\sqrt{k_2^2-\lambda_\nu^2}}{\mu_2 i\omega}\left[A_1 H_0^{(1)}(r\sqrt{k_2^2-\lambda_\nu^2}) + A_2 J_0(r\sqrt{k_2^2-\lambda_\nu^2})\right]e^{i[\lambda_\nu(z-s)-\omega t]}\right\} \\ H_{r2} = Re\left\{\frac{-c\lambda_\nu}{\mu_2\omega}\left[A_1 H_1^{(1)}(r\sqrt{k_2^2-\lambda_\nu^2}) + A_2 J_1(r\sqrt{k_2^2-\lambda^2})\right]e^{i[\lambda_\nu(z-s)-\omega t]}\right\} \end{cases}$$

Here only the term for the νth mode is written. A_2 will be taken identically equal to zero, $A_2 \equiv 0$, for the high frequencies involved.

The boundary condition at $r = a$ requires the continuity of the tangential components of the electric and magnetic field. This gives:

$$(7.12)\begin{cases} A_1 H_1^{(1)}(a\sqrt{k_2^2-\lambda_\nu^2}) = \frac{-1}{\varepsilon_1}\left(\frac{2\pi^2 b\, y_\nu'}{a^2\lambda_\nu'}\right)\frac{H_1^{(1)}(y_\nu')J_1(\frac{b}{a}y_\nu')J_1(y_\nu')}{J_1'(y_\nu)} \\ \frac{c\sqrt{k_2^2-\lambda_\nu^2}}{\mu_2 i\omega} A_1 H_0^{(1)}(a\sqrt{k_2^2-\lambda_\nu^2}) = \frac{-c}{\varepsilon_1\mu_1 i\omega}\left(\frac{2\pi^2 b\, y_\nu'^2}{a^3\cdot\lambda_\nu'}\right)\frac{H_1^{(1)}(y_\nu')J_1(\frac{b}{a}y_\nu')J_0(y_\nu')}{J_1'(y_\nu)} \end{cases}$$

where y_ν' and λ_ν' with primes are the "corrected values" taking into account the effect of the finite conductivity of the metal sheath:

$$(7.13)\quad \therefore\ A_1 = \frac{-2\pi^2 b\, y_\nu'}{\varepsilon_1 a^2 \beta_\nu'} \frac{H_1^{(1)}(y_\nu')J_1(\frac{b}{a}y_\nu')J_1(y_\nu')}{J_1'(y_\nu')\, H_1^{(1)}(a\sqrt{k_2^2-\lambda_\nu'^2})}$$

$$= \frac{-\mu_2}{\varepsilon_1\mu_1} \frac{2\pi^2 b\, y_\nu'^2}{a^3\beta_\nu'\sqrt{k_2^2-\lambda_\nu'^2}} \frac{H_1^{(1)}(y_\nu')J_1(\frac{b}{a}y_\nu')J_0(y_\nu')}{J_1'(y_\nu')\, H_0^{(1)}(a\sqrt{k_2^2-\lambda_\nu'^2})}$$

The transcendental equation to be solved for the "corrected roots", y_ν', is then:

$$(7.14)\qquad J_1(y_\nu') = \frac{\mu_2\, y_\nu'}{\mu_1 a\sqrt{k_2^2-\lambda_\nu'^2}} \frac{H_1^{(1)}(a\sqrt{k_2^2-\lambda_\nu'^2})}{H_0^{(1)}(a\sqrt{k_2^2-\lambda_\nu'^2})} J_0(y_\nu')$$

$$\approx \frac{\mu_2\, y_\nu'}{i\mu_1 a k_2} J_0(y_\nu') \approx \frac{\mu_2\, y_\nu}{i\mu_1 a k_2} J_0(y_\nu)$$

Substituting $J_1(y_\nu')$ from above into Equation (7.10), since y_ν' is very nearly equal to y_ν, we have:

$$(7.15)\quad \overline{N_r} = \frac{-c}{8\pi} Re\left[\frac{c}{\varepsilon_1^2\mu_1 i\omega}\left(\frac{2\pi^2 b}{a\lambda_\nu}\right)^2\left(\frac{y_\nu}{a}\right)^3\left\{\frac{J_1(\frac{b}{a}y_\nu)J_0(y_\nu)Y_1(y_\nu)}{J_1'(y_\nu)}\right\}^2 \frac{\mu_2 y_\nu}{i\mu_1 a k_2}\right]$$

$$= \frac{-c^2\mu_2\pi^3 b^2 y_\nu^4}{4\,\varepsilon_1^2\mu_1^2\omega\sqrt{\frac{|k_2|}{2}}\,\beta_\nu^2 a^6}\left[\frac{J_1(\frac{b}{a}y_\nu)J_0(y_\nu)Y_1(y_\nu)}{J_1'(y_\nu)}\right]^2$$

To obtain the loss in medium 2 per unit length along the axis, we integrate the above equation over a cylindrical surface at $r = a$ of unit length:

$$(7.16) \qquad \bar{q}_\nu = \int_0^{2\pi} \overline{N_n}\, a\, d\varphi = \frac{-c^3\pi^4\mu_2 b^2 y_\nu^4}{2\epsilon_1^2\mu_1^2\omega\beta_\nu^2 a^5\sqrt{2\pi\omega\sigma_2\mu_2}}\left[\frac{Y_1(y_0)J_1(\frac{b}{a}y_\nu)J_0(y_\nu)}{J_1'(y_\nu)}\right]^2$$

The attenuation constant per unit length of the metal sheath for the $\nu th.$ mode is then:

$$(7.17). \qquad \alpha_\nu = \frac{-\bar{q}_\nu}{2\,\overline{W_\nu}} = \frac{c\,\mu_2\, y_\nu^2}{4\mu_1 a^3\beta_\nu\sqrt{2\pi\omega\sigma_2\mu_2}}$$

$$= \frac{1}{4a}\sqrt{\frac{\mu_2\epsilon_1}{\mu_1\sigma_2}}\,\frac{f_\nu^2}{f^{3/2}\sqrt{1-(\frac{f_\nu}{f})^2}}$$

For a uniform and homogeneous system of transmission, the attenuation constants for the different "distinct modes" of propagation are different from and independent of each other. Formula (7.17), thus obtained in a formal manner, is of great importance in the discussion of propagation phenomenon especially for long distances. The attenuation for a current loop antenna (7.17) follows a different law as compared with that for a linear antenna (4.27). It is infinite at the cut-off frequency $f=f_\nu$ and is monotonically decreasing as the frequency increases.

As pointed out before, for practical satisfactory operation without interference or distortion, probably only "one mode" of propagation can be utilized. Taking the example of a metal tube with an inner radius of 10 cms., the first and the second cut-off frequencies are:

$$f_1 = \frac{y_1 \, 3 \times 10^{10}}{2\pi \, 10} = 1.83 \times 10^9 \quad \text{cycles per sec.}$$

and
$$f_2 = \frac{y_2 \, 3 \times 10^{10}}{2\pi \, 10} = 3.35 \times 10^9 \quad \text{cycles per sec.}$$

respectively. The frequency band available is thus:

$$\Delta f = f_2 - f_1 = 1.52 \times 10^9 \quad \text{cycles per sec.}$$

For $f > f_1$, the variation of attenuation is fairly slow except at the first cut-off value $(f = f_1)$. Because the frequency must be kept below f_2 for single (first) "mode" operation, the monotonically decreasing character of the attenuation constant (α) is not so fascinating as it might appear at first glance. The slightly lower attenuation of (7.17) compared with (4.27) for the available range is, however, an advantage for long distance transmission.

(2) Phase Constant, Phase Velocity, and Group Velocity.

The phase constant is computed from (7.06). The phase velocity and group velocity for the "νth mode" are given as:

$$(7.18) \qquad V_{p\nu} = \frac{\omega}{\beta_\nu} = \frac{v_o}{\sqrt{1 - \left(\frac{f_{c\nu}}{f}\right)^2}}$$

and
$$(7.19) \qquad V_{g\nu} = \frac{v_o^2}{V_{p\nu}} = v_o\sqrt{1 - \left(\frac{f_{c\nu}}{f}\right)^2}$$

respectively. The cut-off frequency is calculated by means of Equation (7.06)a.

There are as many distinct phase constants as there are "distinct modes" of propagation.

(3) Frequency Spectrum.

The discussion in Section IV. can be used here word for word.

(4) Phase Displacement and Energy Propagation.

Slightly modified development from that given in Section IV. can be applied here.

(5) Characteristic Impedance and Radiation Resistance.

These will not be discussed here, since the definition is artificial and its physical significance not evident.

(6) Field Structure.

It is completely described by the Equations (7.05) for the field components. The spatial wave length in the air medium inside the tube guide is:

$$(7.20) \qquad \mathcal{L}_\nu = \frac{2\pi}{\beta_\nu} = \frac{v_0}{f\sqrt{1-\left(\frac{f_\nu}{f}\right)^2}}$$

where $v_0 = \frac{c}{\sqrt{\epsilon_1 \mu_1}}$ and f_ν, the cut-off frequency. The electric lines form concentric circles. The differential equation for the magnetic lines $\mathcal{L}_z$, for the $\nu th.$ mode, is:

$$(7.21) \qquad \frac{dz}{H_z} = \frac{dr}{H_r}$$

or

$$\frac{J_0(\frac{r}{a}y_\nu)}{J_1(\frac{r}{a}y_\nu)} d\left(\frac{r}{a}y_\nu\right) = \frac{\sin[\lambda_\nu(z-\varsigma)-\omega t]}{\cos[\lambda_\nu(z-\varsigma)-\omega t]} d(\lambda_\nu z)$$

Integrating gives:

$$\log\left[\frac{r}{a}y_\nu J_1\left(\frac{r}{a}y_\nu\right)\right] = -\log\cos[\lambda_\nu(z-\varsigma)-\omega t] + C'$$

$$(7.22) \qquad \therefore \left[\frac{r}{a}y_\nu J_1\left(\frac{r}{a}y_\nu\right)\right]\cos[\lambda_\nu(z-\varsigma)-\omega t] = C$$

It is, therefore, the same as (4.39).

(7) Terminal Device.

Just opposite to the case discussed in Section IV., a perfect conducting disc end weakens the resultant potential function when the loop antenna is placed near to it. Referring to (7.03) and (8.16), we have, for the resultant potential function:

$$(7.23) \quad U = -2i \sum_{\nu=1}^{\infty} \frac{2\pi^2 b}{a} \frac{H_1^{(1)}(y_\nu) J_1\left(\frac{b}{a} y_\nu\right)}{\lambda_\nu J_1'(y_\nu)} J_0\left(\frac{r}{a} y_\nu\right) e^{i\lambda_\nu z} \sin \lambda_\nu \zeta$$

which becomes zero when $\zeta = 0$. The first unimportant term in (7.03) is neglected here. It is therefore desirable to adjust the distance, ζ , of the loop before the end plate so that:

$$\lambda_\nu \zeta = \beta_\nu \zeta = \frac{\zeta\, 2\pi f}{v_0} \sqrt{1 - \left(\frac{f_\nu}{f}\right)^2} = (2n+1)\frac{\pi}{2}$$

$$(n = 0, 1, 2, \text{------})$$

then we have $\sin \lambda_\nu \zeta = (-)^n$.

(8) Stability Problem.

Remarks in Section IV. hold true here with slight modifications and with the interchange of the roles of the electric and the magnetic fields. But the effect of disturbance due to any non-homogeneity of the transmission system is slightly greater for the present case of a loop antenna than that for the case of a linear antenna.

SECTION VIII.

Propagation over Plane Earth Surface

Only a brief formulation by means of the standard procedure developed above will be attempted. This problem has been subject to close theoretical study and experimental investigation by a great number of scientists during the past two decades. The pioneer work of Sommerfeld was followed by Poincare, Nicholson, Watson, Epstein, Reyrich, Van der Pol, and many others. The complete references can be found in the various papers by these authors.

Here two cases will be considered: an electric dipole (vertical antenna) and a magnetic dipole (current loop).

Part (A) Electric Dipole Placed at (ζ, r_0, φ_0) Above an Infinite Plane Earth Surface $(z=0)$.

From (2.13), we obtain the expressions for the primary general magnetic vector potential (or Hertzian function) at any point (z, r, φ); for $z > \zeta$, we have:

$$(8.01)\quad U_0 = \begin{cases} \dfrac{i}{2}\displaystyle\sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi_0)} \int_{-\infty}^{\infty} e^{i\lambda(z-\zeta)} J_m\left(r\sqrt{k^2-\lambda^2}\right) H_m^{(1)}\left(r_0\sqrt{k^2-\lambda^2}\right) d\lambda & \text{for } r<r_0 \\ \dfrac{i}{2}\displaystyle\sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi_0)} \int_{-\infty}^{\infty} e^{i\lambda(z-\zeta)} J_m\left(r_0\sqrt{k^2-\lambda^2}\right) H_m^{(1)}\left(r\sqrt{k^2-\lambda^2}\right) d\lambda & \text{for } r>r_0. \end{cases}$$

for $z < \varsigma$, we have:

$$(8.01)_a \quad \mathcal{U}_0 = \begin{cases} \frac{i}{2}\sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi_0)}\int_{-\infty}^{\infty} e^{-i\lambda(z-\varsigma)} J_m(r\sqrt{k^2-\lambda^2})H_m^{(1)}(r_0\sqrt{k^2-\lambda^2})d\lambda & \text{for } r<r_0 \\ \frac{i}{2}\sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi_0)}\int_{-\infty}^{\infty} e^{-i\lambda(z-\varsigma)} J_m(r_0\sqrt{k^2-\lambda^2})H_m^{(1)}(r\sqrt{k^2-\lambda^2})d\lambda & \text{for } r>r_0 \end{cases}$$

We shall denote air medium $(z>0)$ by 1 and the earth medium by 2. For the general case of plane earth of finite conductivity, the disturbance due to its presence can be taken into account by the following expressions: (consider $r>r_0$ only) for $z>\varsigma$:

$$(8.02) \quad \mathcal{U} = \frac{i}{2}\sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi_0)}\int_{-\infty}^{\infty}\left[e^{i\lambda(z-\varsigma)} + F_1(\lambda)e^{i\lambda z}\right]J_m(r_0\sqrt{k_1^2-\lambda^2})H_m^{(1)}(r\sqrt{k_1^2-\lambda^2})d\lambda ,$$

for $0 < z < \varsigma$:

$$(8.03) \quad \mathcal{U}_1 = \frac{i}{2}\sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi_0)}\int_{-\infty}^{\infty}\left[e^{-i\lambda(z-\varsigma)} + F_1(\lambda)e^{i\lambda z}\right]J_m(r_0\sqrt{k_1^2-\lambda^2})H_m^{(1)}(r\sqrt{k_1^2-\lambda^2})\,d\lambda ,$$

and for $z < 0$:

$$(8.04) \quad \mathcal{U}_2 = \frac{i}{2}\sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi_0)}\int_{-\infty}^{\infty}\left[e^{-i\lambda(z-\varsigma)} + F_2(\lambda)e^{-i\lambda z}\right]J_m(r_0\sqrt{k_2^2-\lambda^2})H_m^{(1)}(r\sqrt{k_2^2-\lambda^2})d\lambda .$$

The boundary conditions at $z=0$ require that the tangential components of the magnetic and electric field intensities must be equal for the two mediums; i.e.:

(a) $H_{\varphi 1} = H_{\varphi 2}$ (b) $H_{r1} = H_{r2}$

(c) $E_{r1} = E_{r2}$ (d) $E_{\varphi 1} = E_{\varphi 2}$

On referring to (2.06), we have:

$$
(8.05)\begin{cases}
\text{(a)}\ \frac{\varepsilon_1}{\mu_1}\left[e^{i\lambda\varsigma}+F_1(\lambda)\right]J_m(r_0\xi_1)H_m^{(1)\prime}(r\xi_1)=\frac{\varepsilon_2}{\mu_2}\left[e^{i\lambda\varsigma}+F_2(\lambda)\right]J_m(r_0\xi_2)H_m^{(1)\prime}(r\xi_2)\\
\text{(b)}\ \frac{1}{\mu_1}\left[e^{i\lambda\varsigma}+F_1(\lambda)\right]J_m(r_0\xi_1)H_m^{(1)}(r\xi_1)=\frac{1}{\mu_2}\left[e^{i\lambda\varsigma}+F_2(\lambda)\right]J_m(r_0\xi_2)H_m^{(1)}(r\xi_2)\\
\text{(c)}\ \frac{\varepsilon_1}{k_1^2}\left[-e^{i\lambda\varsigma}+F_1(\lambda)\right]J_m(r_0\xi_1)H_m^{(1)\prime}(r\xi_1)=\frac{\varepsilon_2}{k_2^2}\left[-e^{i\lambda\varsigma}-F_2(\lambda)\right]J_m(r_0\xi_2)H_m^{(1)\prime}(r\xi_2)\\
\text{(d)}\ \frac{1}{k_1^2}\left[-e^{i\lambda\varsigma}+F_1(\lambda)\right]J_m(r_0\xi_1)H_m^{(1)}(r\xi_1)=\frac{1}{k_2^2}\left[-e^{i\lambda\varsigma}-F_2(\lambda)\right]J_m(r_0\xi_2)H_m^{(1)}(r\xi_2)
\end{cases}
$$

Since it is impossible to solve for two unknowns from the four simultaneous equations, we conclude that the assumption of $H_z = 0$ is not legitimate due to the existence of eddy current in the earth medium.

However, if we assume the limiting condition of a perfect conducting earth, then the boundary conditions for the vanishing of the tangential components of electric intensity at $z=0$ give rise to a single relation for any r :

$$\left[-e^{i\lambda\varsigma}+F_1(\lambda)\right]=0$$

or

$$F_1(\lambda)=e^{i\lambda\varsigma}$$

Substituting this into (8.02) and (8.03), we have; for $r>r_0$:

$$(8.02)_a\quad U=\frac{i}{2}\sum_{m=-\infty}^{\infty}e^{im(\varphi-\varphi_0)}\int_{-\infty}^{\infty}e^{i\lambda z}\left[e^{-i\lambda\varsigma}+e^{+i\lambda\varsigma}\right]J_m\left(r_0\sqrt{k_1^2-\lambda^2}\right)H_m^{(1)}\left(r\sqrt{k_1^2-\lambda^2}\right)d\lambda$$

for $z>\varsigma$

$$(8.03)a \quad \mathcal{U}_1 = \frac{i}{2}\sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi_0)}\int_{-\infty}^{\infty}\left[e^{-i\lambda(z-\varsigma)} + e^{i\lambda(z+\varsigma)}\right]J_m(r_0\sqrt{k_1^2-\lambda^2})H_m^{(1)}(r\sqrt{k_1^2-\lambda^2})\,d\lambda$$
$$\text{for } 0<z<\varsigma$$

Following the same method used in (III.), we can obtain the expressions for a linear antenna with arbitrary current distribution along it. Although this constitutes merely a formal integration using (8.02)a, (8.03)a, as the Green's functions, yet the practical evaluation is rather prohibitive.

For the simple case with the dipole situated at $(\varsigma, 0, 0)$, (8.02)a and (8.03)a become:

$$(8.06) \quad \mathcal{U} = i\int_{-\infty}^{\infty} e^{i\lambda z}\cos\lambda\varsigma\, H_0^{(1)}(r\sqrt{k_1^2-\lambda^2})\,d\lambda \qquad \text{for } z>\varsigma$$

$$(8.07) \quad \mathcal{U}_1 = i\int_{-\infty}^{\infty} e^{i\lambda\varsigma}\cos\lambda z\, H_0^{(1)}(r\sqrt{k_1^2-\lambda^2})\,d\lambda \qquad \text{for } z<\varsigma .$$

Further, if $\varsigma = 0$, with dipole at origin, (8.06) and (8.07) reduce into a single expression:

$$(8.08) \quad \mathcal{U} = i\int_{-\infty}^{\infty} e^{i\lambda z} H_0^{(1)}(r\sqrt{k_1^2-\lambda^2})\,d\lambda = 2i\int_0^{\infty}\cos\lambda z\, H_0^{(1)}(r\sqrt{k_1^2-\lambda^2})\,d\lambda .$$

Comparing with (2.11), we see that this simply means doubling of the field everywhere due to the presence of the perfect conducting earth.

From the exponential forms of (8.02)a, (8.03)a, (8.06) and (8.07), the disturbance of the earth simply constitutes an image radiating dipole situated at same distance below the earth surface.

The characteristic properties of propagation over plane earth surface of finite conductivity can be taken into account for the symmetrical case with dipole at $(\varsigma, 0, 0)$. For this purpose, we must transform the original source function for a dipole with respect to the argument of integration so as to facilitate satisfying the boundary conditions.

$$(8.09)\quad u_0 = \frac{e^{ikR}}{R} = \frac{i}{2}\int_{-\infty}^{\infty} e^{i\lambda(z-\varsigma)} H_0^{(1)}(r\sqrt{k_1^2-\lambda^2})\,d\lambda$$

$$= \begin{cases} \frac{1}{2}\int_{-\infty}^{\infty} \frac{\lambda\,d\lambda}{\sqrt{\lambda^2-k_1^2}} H_0^{(1)}(r\lambda)\, e^{-\sqrt{\lambda^2-k_1^2}\,(z-\varsigma)} & \text{for } z>\varsigma \\ \frac{1}{2}\int_{-\infty}^{\infty} \frac{\lambda\,d\lambda}{\sqrt{\lambda^2-k_1^2}} H_0^{(1)}(r\lambda)\, e^{+\sqrt{\lambda^2-k_1^2}\,(z-\varsigma)} & \text{for } 0<z<\varsigma \end{cases}$$

With the help of the above formula, we can now formulate the expressions taking account of the presence of the earth.

$$u_1 = \begin{cases} (8.10)\quad \frac{1}{2}\int_{-\infty}^{\infty} H_0^{(1)}(r\lambda)\, e^{-\sqrt{\lambda^2-k_1^2}\,z}\left[\frac{\lambda e^{\sqrt{\lambda^2-k_1^2}\,\varsigma}}{\sqrt{\lambda^2-k_1^2}} + F_1(\lambda)\right]d\lambda & z>\varsigma \\ (8.11)\quad \frac{1}{2}\int_{-\infty}^{\infty} H_0^{(1)}(r\lambda)\, e^{+\sqrt{\lambda^2-k_1^2}\,z}\left[\frac{\lambda e^{-\sqrt{\lambda^2-k_1^2}\,\varsigma}}{\sqrt{\lambda^2-k_1^2}} + F_1(\lambda)\right]d\lambda & 0<z<\varsigma \end{cases}$$

$$(8.12)\quad u_2 = \frac{1}{2}\int_{-\infty}^{\infty} H_0^{(1)}(r\lambda)\, e^{+\sqrt{\lambda^2-k_2^2}\,z}\left[\frac{\lambda e^{-\sqrt{\lambda^2-k_2^2}\,\varsigma}}{\sqrt{\lambda^2-k_2^2}} + F_2(\lambda)\right]d\lambda \qquad z<0$$

The boundary conditions at $z=0$ require,

$$(8.13)\begin{cases} \text{(a)}\quad H_{\varphi 1} = H_{\varphi 2} \quad \text{or} \quad \frac{1}{\mu_1}\frac{\partial u_1}{\partial r} = \frac{1}{\mu_2}\frac{\partial u_2}{\partial r} \\ \text{(b)}\quad E_{r1} = E_{r2} \quad \text{or} \quad \frac{1}{k_1^2}\frac{\partial^2 u_1}{\partial r\,\partial z} = \frac{1}{k_2^2}\frac{\partial^2 u_2}{\partial r\,\partial z} \end{cases}$$

From (8.11) and (8.12) we have then:

$$\text{(a)} \quad \frac{1}{\mu_1}\left[\frac{\lambda e^{-\sqrt{\lambda^2-k_1^2}\,\zeta}}{\sqrt{\lambda^2-k_1^2}} + F_1(\lambda)\right] = \frac{1}{\mu_2}\left[\frac{\lambda e^{-\sqrt{\lambda^2-k_2^2}\,\zeta}}{\sqrt{\lambda^2-k_2^2}} + F_2(\lambda)\right]$$

$$\text{(b)} \quad \frac{\sqrt{\lambda^2-k_1^2}}{k_1^2}\left[0 + F_1(\lambda)\right] = \frac{\sqrt{\lambda^2-k_2^2}}{k_2^2}\left[0 + F_2(\lambda)\right]$$

Since differentiation of the primary function $\mathcal{U}_0$ against z is zero; i.e.:

$$\frac{\partial}{\partial z}\left[\frac{e^{ikR}}{R}\right]_{z=0} = \left[\frac{z}{R}\frac{\partial}{\partial R}\left(\frac{e^{ikR}}{R}\right)\right]_{z=0} = 0$$

Solving (a) and (b) for $F_1(\lambda)$ and $F_2(\lambda)$, we obtain:

$$(8.14) \qquad F_1(\lambda) = \frac{\lambda k_1^2}{\sqrt{\lambda^2-k_1^2}} \; \frac{\mu_1\sqrt{\lambda^2-k_1^2}\, e^{-\sqrt{\lambda^2-k_2^2}\,\zeta} - \mu_2\sqrt{\lambda^2-k_2^2}\, e^{-\sqrt{\lambda^2-k_1^2}\,\zeta}}{\mu_2 k_1^2\sqrt{\lambda^2-k_2^2} - \mu_1 k_2^2\sqrt{\lambda^2-k_1^2}}$$

$$(8.14)a \qquad F_2(\lambda) = \frac{\lambda k_2^2}{\sqrt{\lambda^2-k_2^2}} \; \frac{\mu_1\sqrt{\lambda^2-k_1^2}\, e^{-\sqrt{\lambda^2-k_2^2}\,\zeta} - \mu_2\sqrt{\lambda^2-k_2^2}\, e^{-\sqrt{\lambda^2-k_1^2}\,\zeta}}{\mu_2 k_1^2\sqrt{\lambda^2-k_2^2} - \mu_1 k_2^2\sqrt{\lambda^2-k_1^2}}$$

Substituting these into (8.10), (8.11), and (8.12), one gets the complete solutions, whose integrations are, however, complicated by the presence of branch points at $\lambda = \pm k_1$ and $\lambda = \pm k_2$.

If in the above discussed case $\zeta = 0$ for a dipole at origin, (8.14) and (8.14)a reduce to:

$$(8.15) \qquad F_1(\lambda) = \frac{\lambda k_1^2}{\sqrt{\lambda^2-k_1^2}} \; \frac{\mu_1\sqrt{\lambda^2-k_1^2} - \mu_2\sqrt{\lambda^2-k_2^2}}{\mu_2 k_1^2\sqrt{\lambda^2-k_2^2} - \mu_1 k_2^2\sqrt{\lambda^2-k_1^2}}$$

$$(8.15)a \qquad F_2(\lambda) = \frac{\lambda k_2^2}{\sqrt{\lambda^2-k_2^2}} \; \frac{\mu_1\sqrt{\lambda^2-k_1^2} - \mu_2\sqrt{\lambda^2-k_2^2}}{\mu_2 k_1^2\sqrt{\lambda^2-k_2^2} - \mu_1 k_2^2\sqrt{\lambda^2-k_1^2}}$$

This corresponds to the case discussed by Sommerfeld.*

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* Loc. cit.

Before concluding, it might be of use to mention that from (8.02)a, (8.03)a, it is possible to derive a quite general expression for many dipoles arranged in a certain way to obtain directional effect.

Part (B) Magnetic Dipole at $(\varsigma, r_0, \varphi_0)$ Above an Infinite Plane Earth Surface $(z=0)$.

The primary functions are given in (5.04) and (5.05) and are identical to (8.01) and (8.01)a. The secondary disturbance can be taken into account by exactly the same formulae (8.02), (8.03), and (8.04). The vanishing of E_φ and E_r (refer to (5.07)) at the surface $z=0$ of a perfect conducting earth gives rise, however, to:

$$e^{i\lambda\varsigma} + F_1(\lambda) = 0$$

Thus:

$$(8.16) \quad u = \begin{cases} \frac{i}{2}\sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi_0)} \int_{-\infty}^{\infty} \left[e^{i\lambda(z-\varsigma)} - e^{i\lambda(z+\varsigma)}\right] J_m\left(r_0\sqrt{k_1^2-\lambda^2}\right) H_m^{(1)}\left(r\sqrt{k_1^2-\lambda^2}\right) d\lambda, & z>\varsigma \\ \frac{i}{2}\sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi_0)} \int_{-\infty}^{\infty} \left[e^{i\lambda(\varsigma-z)} - e^{i\lambda(\varsigma+z)}\right] J_m\left(r_0\sqrt{k_1^2-\lambda^2}\right) H_m^{(1)}\left(r\sqrt{k_1^2-\lambda^2}\right) d\lambda, & 0<z<\varsigma \end{cases}$$

The above expressions indicate that neither the transmitter nor the detector should be located too close to the earth surface, since the integrands contain factors $\sin\lambda\varsigma$ and $\sin\lambda z$, respectively. Corresponding to each m, there is a "distinct harmonic mode" of propagation. The resultant field at any point is just a superposition of all these harmonics. For a current loop of radius b with center at $(\varsigma, 0, 0)$, we obtain immediately:

$$(8.17) \quad u = \begin{cases} 2\pi \int_{-\infty}^{\infty} e^{i\lambda z} \sin\lambda\varsigma \, J_0\left(b\sqrt{k_1^2-\lambda^2}\right) H_0^{(1)}\left(r\sqrt{k_1^2-\lambda^2}\right) d\lambda & z>\varsigma \\ 2\pi \int_{-\infty}^{\infty} e^{i\lambda\varsigma} \sin\lambda z \, J_0\left(b\sqrt{k_1^2-\lambda^2}\right) H_0^{(1)}\left(r\sqrt{k_1^2-\lambda^2}\right) d\lambda & 0<z<\varsigma \end{cases}$$

The integrands are not unique and meromorphous functions of the arguments of the cylindrical functions. Besides completing the closed contour by means of a semi-circle above the real axis, we must draw a branch cut through $\lambda = +k_1$. The writer intends at some later time to make a detailed analysis of wave propagation over plane and spherical earth surface by the present method for a physical antenna.

SECTION IX.

Propagation Characteristics Of Concentric Transmission Lines

The prime purpose of this section is to obtain a simple, explicit expression for the propagation constant of a concentric transmission system so that a clear view of the relative merits of the concentric system compared with a hollow tube guide can be grasped. By means of the asymptotic properties of the cylindrical functions, the desired result can be obtained in a very straightforward manner.

Because of the explicit relations of the current and voltage between the two conductors, the method of attack follows in a general sense that used for "conventional transmission line circuits", but at the same time guided by Maxwell's field equations, so that we will be aware of the approximations which can be made without impairing the accuracy desired. The circuit diagram is shown in Fig. IX-1. The current in the central conductor is

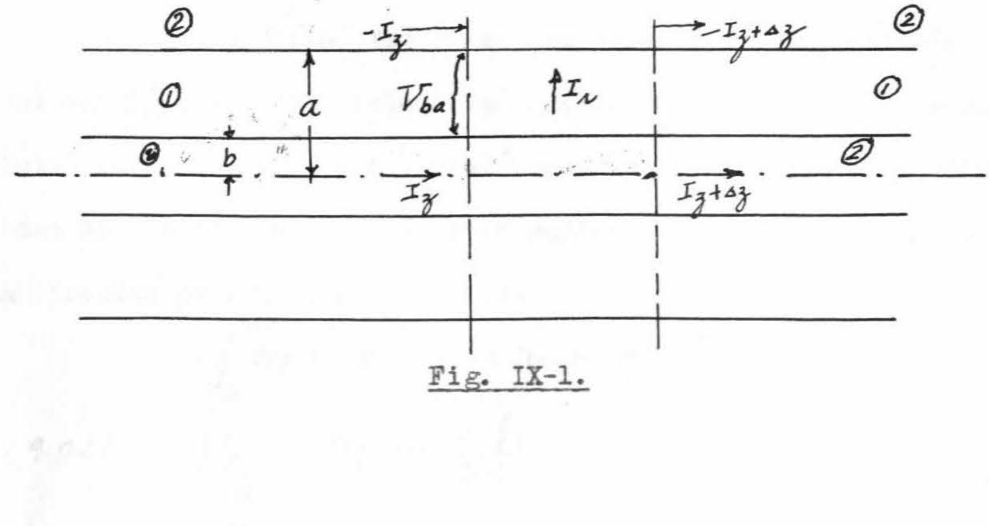

Fig. IX-1.

assumed in the positive z-direction and the return path is formed by the outside hollow tube. For the present symmetrical field configuration, Maxwell equations reduce to:

$$
(9.01)\quad \begin{cases}
\dfrac{1}{r}\dfrac{\partial(rH_\varphi)}{\partial r} = \dfrac{4\pi\sigma}{c}E_z + \dfrac{\varepsilon}{c}\dfrac{\partial E_z}{\partial t} = \dfrac{ck^2}{i\omega\mu}E_z \\[2ex]
-\dfrac{\partial H_\varphi}{\partial z} = \dfrac{4\pi\sigma}{c}E_r + \dfrac{\varepsilon}{c}\dfrac{\partial E_r}{\partial t} = \dfrac{ck^2}{i\omega\mu}E_r \\[2ex]
\dfrac{\partial E_r}{\partial z} - \dfrac{\partial E_z}{\partial r} = -\dfrac{\mu}{c}\dfrac{\partial H_\varphi}{\partial t} = \dfrac{i\omega\mu}{c}H_\varphi
\end{cases}
$$

where $\quad \mathcal{E} = E_z,\ E_r,\ E_\varphi = 0\ ;\ \mathcal{H} = H_\varphi,\ H_z = H_r = 0\ .$

The assumptions to be made in the present analysis can be summarized as follows:

(a) $\quad k_1^2 = \dfrac{\omega^2\varepsilon_1\mu_1}{c^2} + i\dfrac{4\pi\omega\sigma_1\mu_1}{c^2} \approx \dfrac{\omega^2\varepsilon_1\mu_1}{c^2}\ ,\quad \sigma_1 \to 0$

(b) $\quad k_2^2 = \dfrac{\omega^2\varepsilon_1\mu_2}{c^2} + i\dfrac{4\pi\omega\sigma_2\mu_2}{c^2} \approx i\dfrac{4\pi\omega\sigma_2\mu_2}{c^2}\ ,\quad \sigma_2 \to \text{large}$

(c) Outside conductor electrically thick.

These coincide with those used for the hollow tube guide in Sections II. - IV. and are nearly realized in practice. The case that σ_1 is not equal to zero can be taken into account very easily.

Now we shall find the relations between the current I_z, the voltage V_{ba}, and the field intensities. J_z is used to represent the total current within a circular cross-section of radius r, and I_z that in the central conductor of radius b. Integrating H_φ around a circular path at radius r gives:

$$\int_0^{2\pi} H_\varphi\, r\, d\varphi = 2\pi r H_\varphi = \frac{4\pi}{c} J_z$$

$$(9.02)\qquad \therefore\quad H_\varphi = \frac{2 J_z}{c r}$$

Substituting (9.02) into (9.01) yields, with assumed time factor $e^{-i\omega t}$:

$$E_z = \frac{2i\omega\mu}{c^2k^2}\,\frac{1}{r}\frac{\partial \mathcal{J}_z}{\partial r} \tag{9.03}$$

$$E_r = \frac{-2i\omega\mu}{c^2k^2}\,\frac{1}{r}\frac{\partial \mathcal{J}_z}{\partial z} \tag{9.04}$$

$$\frac{\partial E_r}{\partial z} = \frac{2i\omega\mu}{c^2}\frac{1}{r}\mathcal{J}_z + \frac{\partial E_z}{\partial r} \tag{9.05}$$

The voltage difference V_{ba} between the two conductors is then simply, since $\mathcal{J}_z = I_z$ remains constant for $b \leq r \leq a$:

$$V_{ba} = \int_b^a E_r\,dr = \frac{-2i\omega\mu_1}{c^2k_1^2}\log\frac{a}{b}\,\frac{\partial I_z}{\partial z}$$

or (9.06) $$\frac{\partial I_z}{\partial z} = \frac{i c^2 k_1^2}{2\omega\mu_1 \log\frac{a}{b}}\,V_{ba} = -Y^* V_{ba}$$

where $$Y^* = \frac{-i c_1^2 k_1^2}{2\omega\mu_1 \log\frac{a}{b}} = \frac{2\pi\sigma_1 - i\frac{1}{2}\omega\varepsilon_1}{\log\frac{a}{b}}$$

or $$Y = G + i\omega C = \frac{2\pi\sigma_1}{\log\frac{a}{b}} + i\omega\frac{\varepsilon}{2\log\frac{a}{b}}$$

This is immediately the definition of the shunt admittance for the concentric transmission system. The shunt conductance and the shunt capacitance per unit length are:

$$G = \frac{2\pi\sigma_1}{\log\frac{a}{b}} \quad \text{and} \quad C = \frac{\varepsilon_1}{2\log\frac{a}{b}} \tag{9.07}$$

respectively.

Similarly, by integrating (9.05), we obtain:

---

* The star signs are used here to indicate the complex conjugate values of Y and Z. These are necessary because time factor $e^{-i\omega t}$ is used here.

$$\frac{\partial}{\partial z}\int_b^a E_r\,dr = \frac{2i\omega\mu_1}{c^2}\log\frac{a}{b}\,I_z + \int_b^a \frac{\partial E_z}{\partial r}\,dr$$

or (9.08) $$\frac{\partial V_{ba}}{\partial z} = \frac{2i\omega\mu_1}{c^2}\log\frac{a}{b}\,I_z + [E_z(a) - E_z(b)]$$

Equations (9.06) and (9.08) are formally analogous to the partial differential equations of the conventional transmission lines. Now the question is how $E_z(a)$ and $E_z(b)$ can be evaluated. This has to resort to Maxwell's field Equations (9.01). Referring to Equations (4.12), we see that the solutions of the field components for the different rigions can be written as:

(a) for $0 \le r \le b$ inside the central conductor

$$(9.09)\begin{cases} E_z = Re\left[\sum_{\nu=1}^{\infty} e^{-i\omega t} e^{i\lambda_\nu z} \cdot A_\nu J_0(r\sqrt{k_2^2-\lambda_\nu^2})\right] \\ E_r = Re\left[\sum_{\nu=1}^{\infty} e^{-i\omega t} e^{i\lambda_\nu z} \frac{-i\lambda_\nu}{\sqrt{k_2^2-\lambda_\nu^2}} A_\nu J_1(r\sqrt{k_2^2-\lambda_\nu^2})\right] \\ H_\varphi = Re\left[\sum_{\nu=1}^{\infty} e^{-i\omega t} e^{i\lambda_\nu z} \frac{c k_2^2}{i\mu_2\omega\sqrt{k_2^2-\lambda_\nu^2}} A_\nu J_1(r\sqrt{k_2^2-\lambda_\nu^2})\right] \end{cases}$$

(b) for $b < r < a$ in the dielectric air medium

$$(9.10)\begin{cases} E_z = Re\left[\sum_{\nu=1}^{\infty} e^{-i\omega t} e^{i\lambda_\nu z}\left\{B_{1\nu} J_0(r\sqrt{k_1^2-\lambda_\nu^2}) + B_{2\nu} H_0^{(1)}(r\sqrt{k_1^2-\lambda_\nu^2})\right\}\right] \\ E_r = Re\left[\sum_{\nu=1}^{\infty} e^{-i\omega t} e^{i\lambda_\nu z} \frac{-i\lambda_\nu}{\sqrt{k_1^2-\lambda_\nu^2}}\left\{B_{1\nu} J_1(r\sqrt{k_1^2-\lambda_\nu^2}) + B_{2\nu} H_1^{(1)}(r\sqrt{k_1^2-\lambda_\nu^2})\right\}\right] \\ H_\varphi = Re\left[\sum_{\nu=1}^{\infty} e^{-i\omega t} e^{i\lambda_\nu z} \frac{c k_1^2}{i\mu_1\omega\sqrt{k_1^2-\lambda_\nu^2}}\left\{B_{1\nu} J_1(r\sqrt{k_1^2-\lambda_\nu^2}) + B_{2\nu} H_1^{(1)}(r\sqrt{k_1^2-\lambda_\nu^2})\right\}\right] \end{cases}$$

(c) for $a < r < \infty$ inside outer conductor

$$(9.11)\begin{cases} E_z = Re\left[\sum_{\nu=1}^{\infty} e^{-i\omega t} e^{i\lambda_\nu z} D_\nu H_0^{(1)}(r\sqrt{k_2^2-\lambda_\nu^2})\right] \\ E_r = Re\left[\sum_{\nu=1}^{\infty} e^{-i\omega t} e^{i\lambda_\nu z} \frac{-i\lambda_\nu}{\sqrt{k_2^2-\lambda_\nu^2}} D_\nu H_1^{(1)}(r\sqrt{k_2^2-\lambda_\nu^2})\right] \end{cases}$$

$$H_\varphi = Re\left[\sum_{\nu=1}^{\infty} e^{-i\omega t} e^{i\lambda_\nu z} \frac{c k_2^2}{i\mu_2\omega\sqrt{k_2^2-\lambda_\nu^2}} D_\nu H_1^{(1)}(r\sqrt{k_2^2-\lambda_\nu^2})\right]$$

where λ_ν's are the eigenvalues corresponding to different modes of propagation and are to be determined from the boundary conditions at $r=a$ and $r=b$. Integrating H_φ in (9.09) around a circle at $r=b$, we obtain:

$$(9.12)\quad \int_0^{2\pi} H_\varphi b\, d\varphi = 2\pi b\left[\sum_{\nu=1}^{\infty} e^{-i\omega t} e^{i\lambda_\nu z} \frac{c k_2^2}{i\mu_2\omega\sqrt{k_2^2-\lambda_\nu^2}} A_\nu J_1(r\sqrt{k_2^2-\lambda_\nu^2})\right] = \begin{cases} \dfrac{4\pi I_z}{+c} \\ \displaystyle\sum_{\nu=1}^{\infty} \dfrac{4\pi I_{z\nu}}{+c} \end{cases}$$

This shows that I_z must also be comprised of the same number of different "modes".

This determines the A_ν's. If we substitute the above relation in E_z, there results:

$$(9.12)a\quad E_z(b) = \sum_{\nu=1}^{\infty} \frac{i\omega\mu_2\sqrt{k_2^2-\lambda_\nu^2}}{c k_2^2} \frac{J_0(b\sqrt{k_2^2-\lambda_\nu^2})}{J_1(b\sqrt{k_2^2-\lambda_\nu^2})} \frac{2 I_{z\nu}}{bc}$$

Similar integration of H_φ in (9.11) around a circle at $r=a$ gives:

$$(9.13)\quad \int_0^{2\pi} H_\varphi a\, d\varphi = 2\pi a\left[\sum_{\nu=1}^{\infty} e^{-i\omega t} e^{i\lambda_\nu z} \frac{c k_2^2}{i\mu_2\omega\sqrt{k_2^2-\lambda_\nu^2}} D_\nu H_1^{(1)}(a\sqrt{k_2^2-\lambda_\nu^2})\right] = \frac{4\pi I_z}{-c} = \sum_{\nu=1}^{\infty} \frac{4\pi I_{z\nu}}{-c}$$

This then determines the D_ν's. Substituting into E_z, we have:

$$(9.13)a\quad E_z(a) = \sum_{\nu=1}^{\infty} \frac{i\omega\mu_2\sqrt{k_2^2-\lambda_\nu^2}}{c k_2^2} \frac{H_0^{(1)}(a\sqrt{k_2^2-\lambda_\nu^2})}{H_1^{(1)}(a\sqrt{k_2^2-\lambda_\nu^2})} \frac{-2 I_{z\nu}}{ac}$$

One criticism might be raised here about the deduction of the Equations (9.12)a, and (9.13)a. On the left-hand sides of (9.12) and (9.13), there is the factor $e^{i\lambda_\nu z}$ assumed for the field components; while on the right-hand sides of both, the z-dependence for I_z is still unknown. We shall suppose for the present, that I_z also has such an exponential factor $e^{i\lambda_\nu z}$. Although λ_ν is involved in the expressions for $E_z(a)$ and $E_z(b)$, and can only be rigorously determined from the boundary conditions at $r = a$ and $r = b$; fortunately for most practical cases the conductivity of the conductors is so high that the relation $(k_2^2 \gg \lambda_\nu^2)$ holds. Then $E_z(a)$ and $E_z(b)$ assume the following simple forms:

$$(9.14) \quad E_z(b) = \frac{2i\omega\mu_2}{bc^2 k_2} \frac{J_0(bk_2)}{J_1(bk_2)} \sum_{\nu=1}^{\infty} I_{z\nu} \approx \frac{-2i^2\omega\mu_2}{bc^2 k_2} I_z$$

$$= \frac{\omega\mu_2 (1-i)}{bc\sqrt{2\pi\omega\sigma_2\mu_2}} I_z$$

and

$$(9.15) \quad E_z(a) = \frac{-2i\omega\mu_2}{ac^2 k_2} \frac{H_0^{(1)}(ak_2)}{H_1^{(1)}(ak_2)} \sum_{\nu=1}^{\infty} I_{z\nu} \approx \frac{2\omega\mu_2}{ac^2 k_2} I_z$$

$$= \frac{\omega\mu_2(1-i)}{ac\sqrt{2\pi\omega\sigma_2\mu_2}} I_z$$

since *:

$$\lim_{\sigma_2\to\infty} \frac{J_0(bk_2)}{J_1(bk_2)} \approx -i$$

and
$$\lim_{\sigma_2\to\infty} \frac{H_0^{(1)}(ak_2)}{H_1^{(1)}(ak_2)} \approx i$$

from the asymptotic properties of cylindrical functions. Now we can substitute (9.14) and (9.15) into (9.08) and obtain:

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* Jahnke and Emde, "Functional Tables", pp. 264-266.

$$(9.08)a \quad \frac{\partial V_{ba}}{\partial z} = \left[\frac{i 2\omega\mu_1}{c^2} \log\frac{a}{b} + \frac{\omega\mu_2(1-i)}{a c\sqrt{2\pi\omega\sigma_2\mu_2}} - \frac{\omega\mu_2(1-i)}{b c\sqrt{2\pi\omega\sigma_2}\mu_2} \right] I_z$$

$$= -\left[\frac{\omega\mu_2}{c\sqrt{2\pi\omega\sigma_2\mu_2}}\left(\frac{1}{b}-\frac{1}{a}\right) - i\left(\frac{2\omega\mu_1}{c^2}\log\frac{a}{b} + \frac{\omega\mu_2}{c\sqrt{2\pi\omega\sigma_2\mu_2}}\left(\frac{1}{b}-\frac{1}{a}\right)\right)\right] I_z$$

$$= -Z^* I_z$$

where the series impedance of the transmission system is defined as:

$$Z = R + i\omega L = \frac{\omega\mu_2}{c\sqrt{2\pi\omega\mu_2\sigma_2}}\left(\frac{1}{b}-\frac{1}{a}\right) + i\left[\frac{2\omega\mu_1}{c^2}\log\frac{a}{b} + \frac{\omega\mu_2}{c\sqrt{2\pi\omega\sigma_2\mu_2}}\left(\frac{1}{b}-\frac{1}{a}\right)\right]$$

The series resistance and the series inductance per unit length are then:

$$(9.16) \qquad R = \frac{\omega\mu_2}{c\sqrt{2\pi\omega\sigma_2\mu_2}}\left(\frac{1}{b}-\frac{1}{a}\right)$$

and

$$L = \frac{2\mu_1}{c^2}\log\frac{a}{b} + \frac{\mu_2}{c\sqrt{2\pi\omega\sigma_2\mu_2}}\left(\frac{1}{b}-\frac{1}{a}\right)$$

respectively.

After the determination of these circuit constants, we can proceed to solve the propagation characteristics in the manner used for conventional transmission circuits. From relations (9.06) and (9.08)a, we have immediately:

$$(9.17) \qquad \frac{\partial^2 I_z}{\partial z^2} = Z^* Y^* I_z = -\lambda^2 I_z$$

and

$$\frac{\partial^2 V_{ba}}{\partial z^2} = Z^* Y^* V_{ba} = -\lambda^2 V_{ba}$$

The solutions of (9.17), for an infinitely long concentric transmission system, are:

$$(9.18)\begin{cases} I_z = A_1 e^{-i\lambda z} + A_2 e^{+i\lambda z} \\ V_{ba} = \sqrt{\frac{Z^*}{Y^*}} A_1 e^{-i\lambda z} - \sqrt{\frac{Z^*}{Y^*}} A_2 e^{i\lambda z} \end{cases}$$

where $i\lambda = \sqrt{Z^* Y^*}$ and A_1 & A_2 are to be determined from the exciting and the receiving conditions. λ is the explicit propagation constant and comprises a real and an imaginary component:

$$\lambda = \beta + i\alpha$$

we have then:

$$\begin{cases} \alpha^2 - \beta^2 = RG - \omega^2 LC \\ 2\alpha\beta = R\omega C + G\omega L \end{cases}$$

Solving the above simultaneous equations for α and β, we obtain in general:

$$(9.19)\begin{cases} \alpha^2 = \frac{1}{2}\left[-(\omega^2 LC - RG) + \sqrt{(\omega^2 LC - RG)^2 + (R\omega C + G\omega L)^2}\right] \\ \beta^2 = \frac{1}{2}\left[-(RG - \omega^2 LC) - \sqrt{(RG - \omega^2 LC)^2 + (R\omega C + G\omega L)^2}\right] \end{cases}$$

The characteristic impedance of the concentric transmission system is simply:

$$Z_0 = \sqrt{\frac{Z^*}{Y^*}}$$

Now if we use the assumptions (a) and (b) stated at the beginning, then $\begin{cases} \sigma_1 = 0 \\ \sigma_2 \to \text{large} \end{cases}$ and $\begin{cases} G = 0 \\ R \to \text{small} \end{cases}$.

The attenuation constant and the phase constant become:

$$(9.20)\qquad \alpha = \sqrt{\frac{C}{L}}\frac{R}{2} \quad \text{and} \quad \beta = \omega\sqrt{LC}$$

respectively.

Substituting into (9.20) the values obtained before for R, C, and L, neglecting the second term in (9.16) for L, then we have:

$$(9.21)\qquad \alpha = \frac{1}{4}\sqrt{\frac{\mu_2\varepsilon_1}{\mu_1\sigma_2}}\left(\frac{1}{b}-\frac{1}{a}\right)\frac{f^{\frac{1}{2}}}{\log\frac{a}{b}}$$

and $(9.22)\quad \beta = \omega\frac{\sqrt{\varepsilon_1\mu_1}}{c} = \frac{\omega}{v_0}$ approximately.

The phase velocity of propagation is thus essentially equal to that in free space:

$$(9.23)\qquad V_p = \frac{\omega}{\beta} = v_0 = \frac{c}{\sqrt{\varepsilon_1\mu_1}}\ .$$

independent upon the frequency of excitation. Comparing (9.23) with (4.15) for the νth mode, we see that the concentric system has a decided advantage over the hollow tube guide which has a phase velocity varying widely when the frequency is near to the cut-off value.

When the frequency is very high $(f >> f_\nu)$, however, (9.23) and (4.15) are practically equal, since:

$$\frac{(9.23)}{(4.15)} = \frac{V_p}{V_{p\nu}} = \sqrt{1-\left(\frac{f_\nu}{f}\right)^2} \sim 1 \quad \text{for } f >> f_\nu$$

The attenuation constants for the two cases for the same inner radius, a, of the metal tube have the following ratio:

$$(9.24)\qquad \frac{(9.21)}{(4.27)} = \frac{\alpha}{\alpha_\nu} = \frac{a-b}{2b\log\frac{a}{b}}\sqrt{1-\left(\frac{f_\nu}{f}\right)^2}$$

In general near the cut-off frequency f_ν, the hollow tube guide has a higher attenuation than the concentric system. When $f \gg f_\nu$:

$$(9.25) \qquad \lim_{f \gg f_\nu} \frac{\alpha}{\alpha_\nu} \approx \frac{\frac{a}{b} - 1}{2 \log \frac{a}{b}}.$$

which varies between 0.6 to 2.0 for the probable range of $\frac{a}{b}$ from 1.5 to 10. It is expected that the attenuation constant (9.21) for a concentric system will be increased somewhat due to the insertion of regular separating insulators used to keep the central conductor in position. Consequently in order to keep down this unavoidable increase of attenuation, it is strongly recommended that only high quality material should be used for the insulators. If this can be realized, a concentric transmission system compares favorably with a hollow tube guide so far as attenuation is concerned.

Besides, a concentric system possesses some decided advantages for usual transmission purposes over a hollow guide. They are:

(1) Ease of matching the transmitter and the receiver to the transmission system.

(2) Nearly hundred per cent efficiency of reception which can never be realized for a hollow guide.

(3) Stability of operation and of field configuration when the physical construction deviates from the ideal case of a straight cylindrical system with uniform circular cross-section. (Compare with the discussion at the end of Section IV.).

Let us, however, not attempt to discredit a hollow tube guide too much, since the economy of engineering application always plays a very important role and further it might find some other fields of use due to its specific physical character.

The following is a rather incomplete list of references for this section:-

(1) S. A. Schekunoff, "The Electromagnetic Theory of Co-axial Transmission Lines and Cylindrical Shields", Bell System Tech. Jour., Oct., 1934.

(2) H. Kruse und O. Zinke, "Currents in Layered Cylindrical Conductors", Hochfrequenztech. u. Elektroakustik, 44, S. 195-203, Dec., 1934.

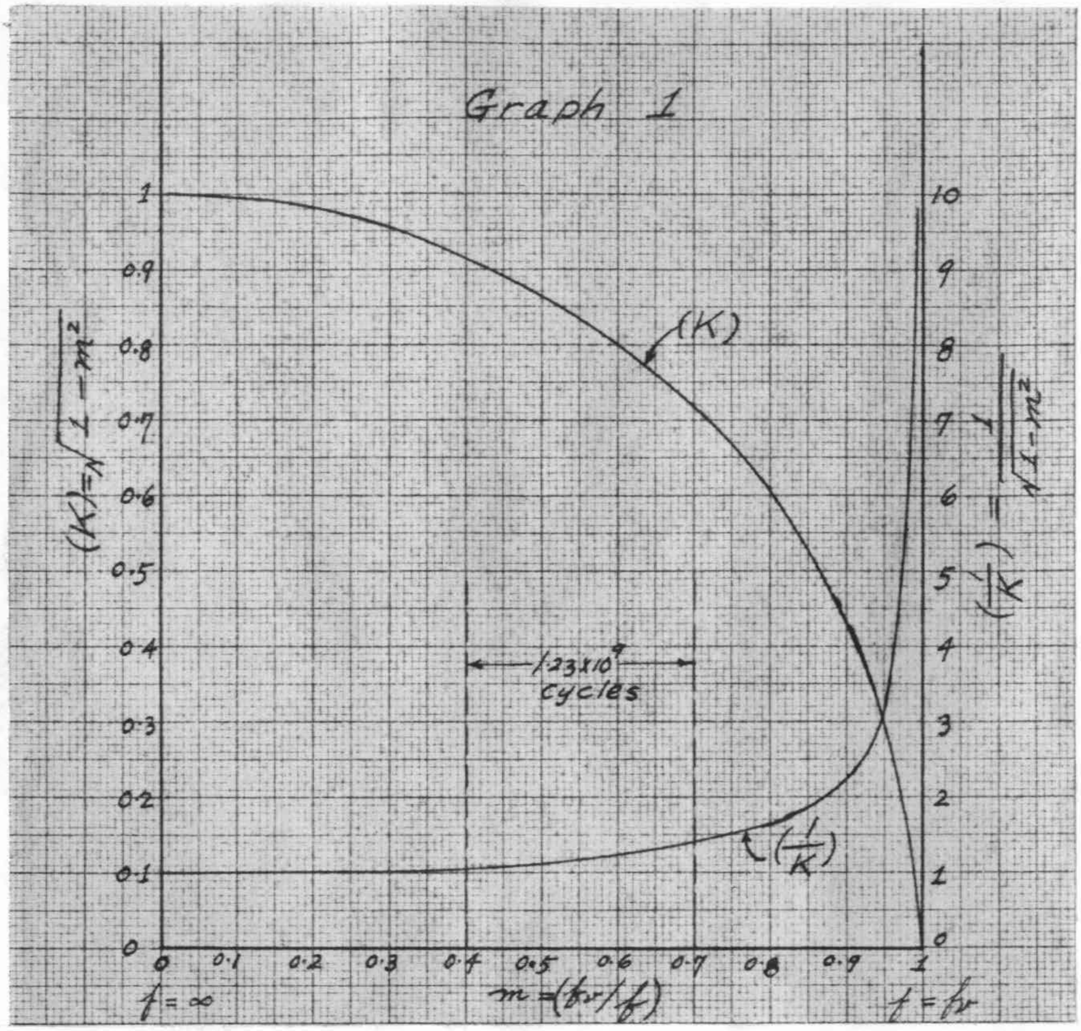
(3) H. Kaden, "Television Cables", Arch. f. Elektrat. 30, S. 691-712, Nov., 1936.

(4) R. Redus, "Co-axial Cables, Their Employment at H. F. for Television", Onde Elec. 17, pp. 325-337 July; pp. 399-426, August, 1938.

(5) J. R. Carson and Gilbert, "Transmission Characteristics of Submarine Cables", Jour. Franklin Inst., Dec., 1921.

(6) J. R. Carson and Gilbert, "Transmission Characteristics of Submarine Cables", B. S. T. J., July, 1922.

APPENDIX

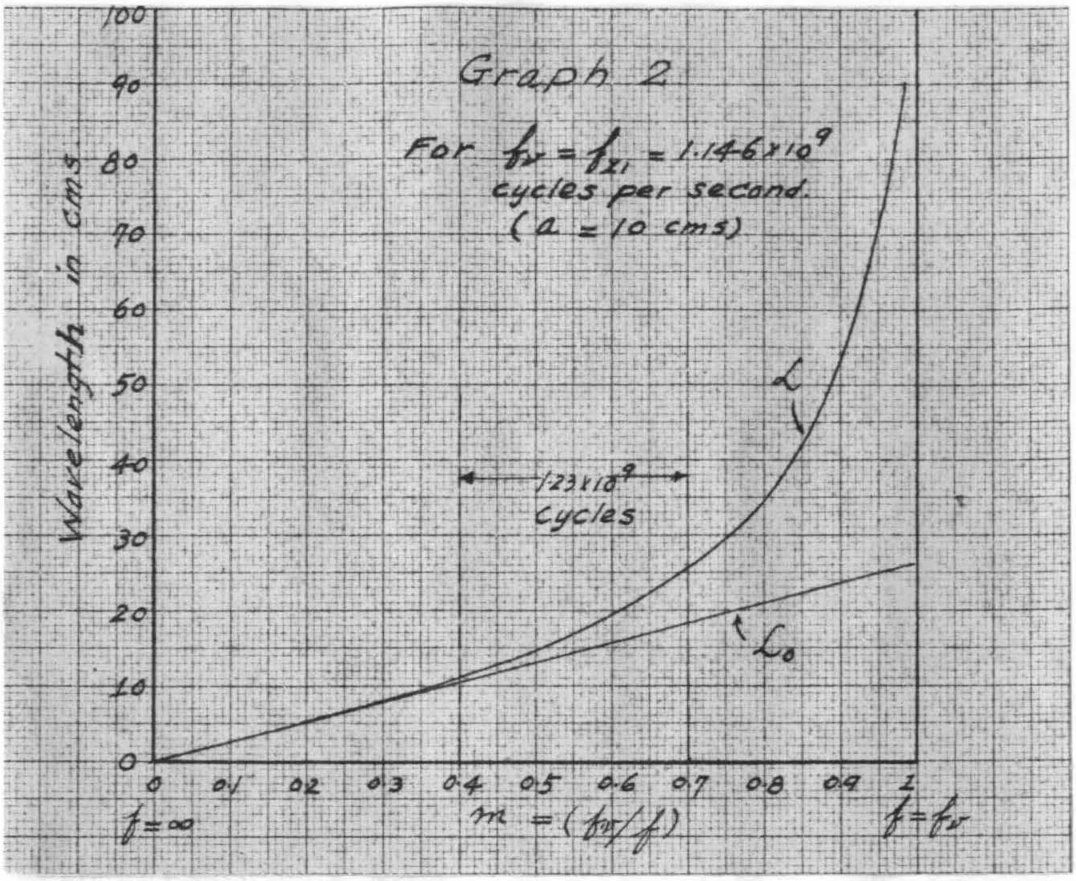


Graph 1.

Phase Velocity $$V_{p\nu} = \frac{c}{\sqrt{\varepsilon_1 \mu_1}} \frac{1}{\sqrt{1-(f_\nu/f)^2}} = \frac{c}{\sqrt{\varepsilon_1 \mu_1}} \frac{1}{K} \qquad \begin{matrix}(4.15)\\(7.18)\end{matrix}$$

Group Velocity $$V_{g\nu} = \frac{c}{\sqrt{\varepsilon_1 \mu_1}} \sqrt{1-\left(\frac{f_\nu}{f}\right)^2} = \frac{c}{\sqrt{\varepsilon_1 \mu_1}} K \qquad \begin{matrix}(4.29)\\(7.19)\end{matrix}$$

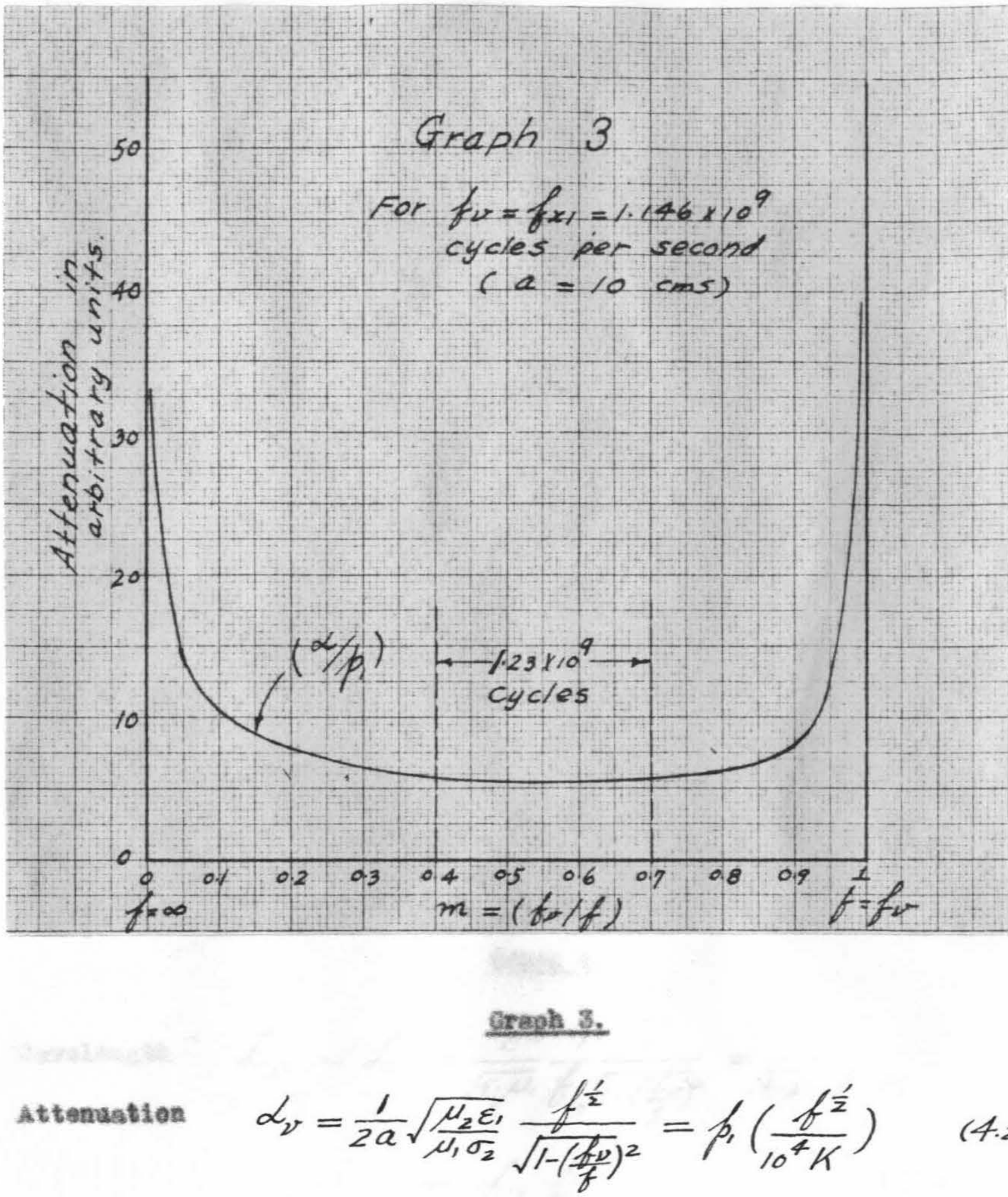
Radiation Resistance $$R_\nu = \frac{2}{c}\sqrt{\frac{\mu_1}{\varepsilon_1}} \sqrt{1-\left(\frac{f_\nu}{f}\right)^2} = \frac{2}{c}\sqrt{\frac{\mu_1}{\varepsilon_1}} K \qquad (4.37)$$



Graph 2.

Wavelength $\quad \mathcal{L}_v = \mathcal{L} = \dfrac{c}{\sqrt{\varepsilon_1 \mu_1}} \dfrac{1}{f} \dfrac{1}{\sqrt{1-(\frac{f_v}{f})^2}} = \dfrac{c}{\sqrt{\varepsilon_1 \mu_1}} \dfrac{1}{f} \dfrac{1}{K}$

$$= \mathcal{L}_0 \frac{1}{K} \qquad (4.39)$$

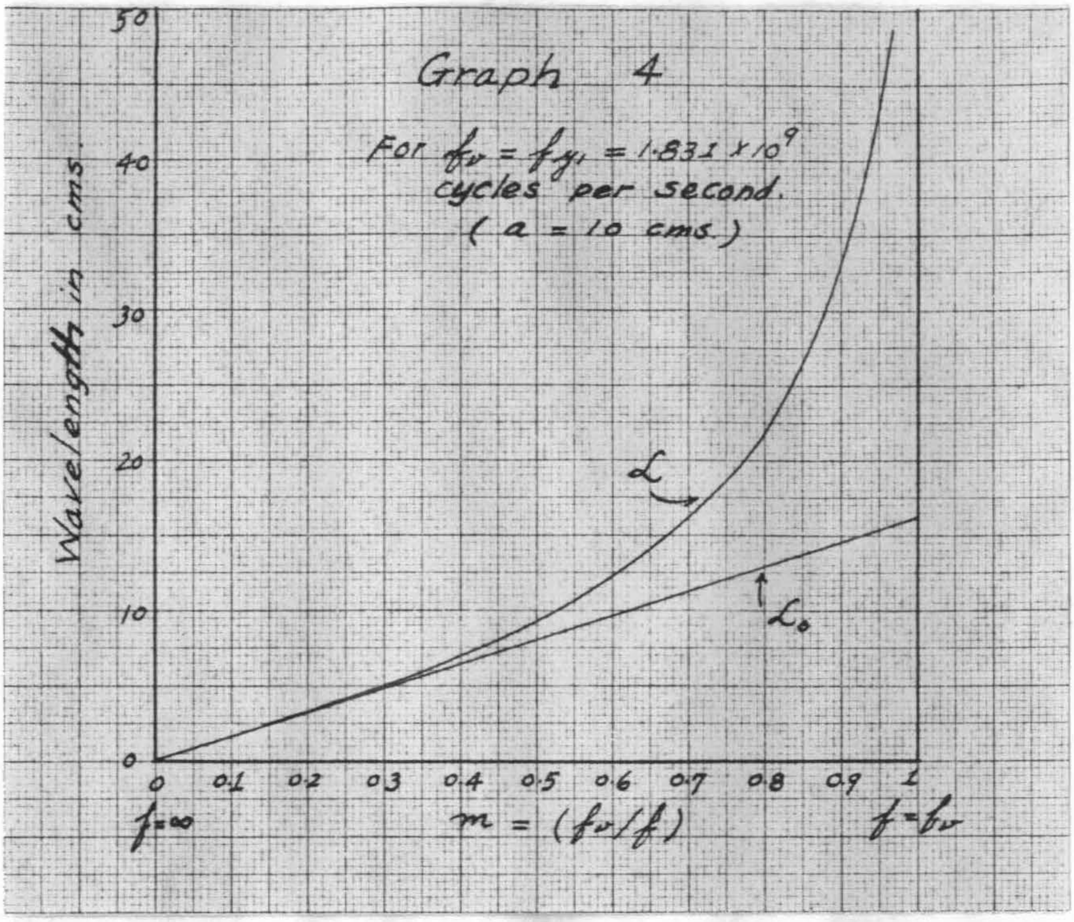


Graph 3.

Attenuation

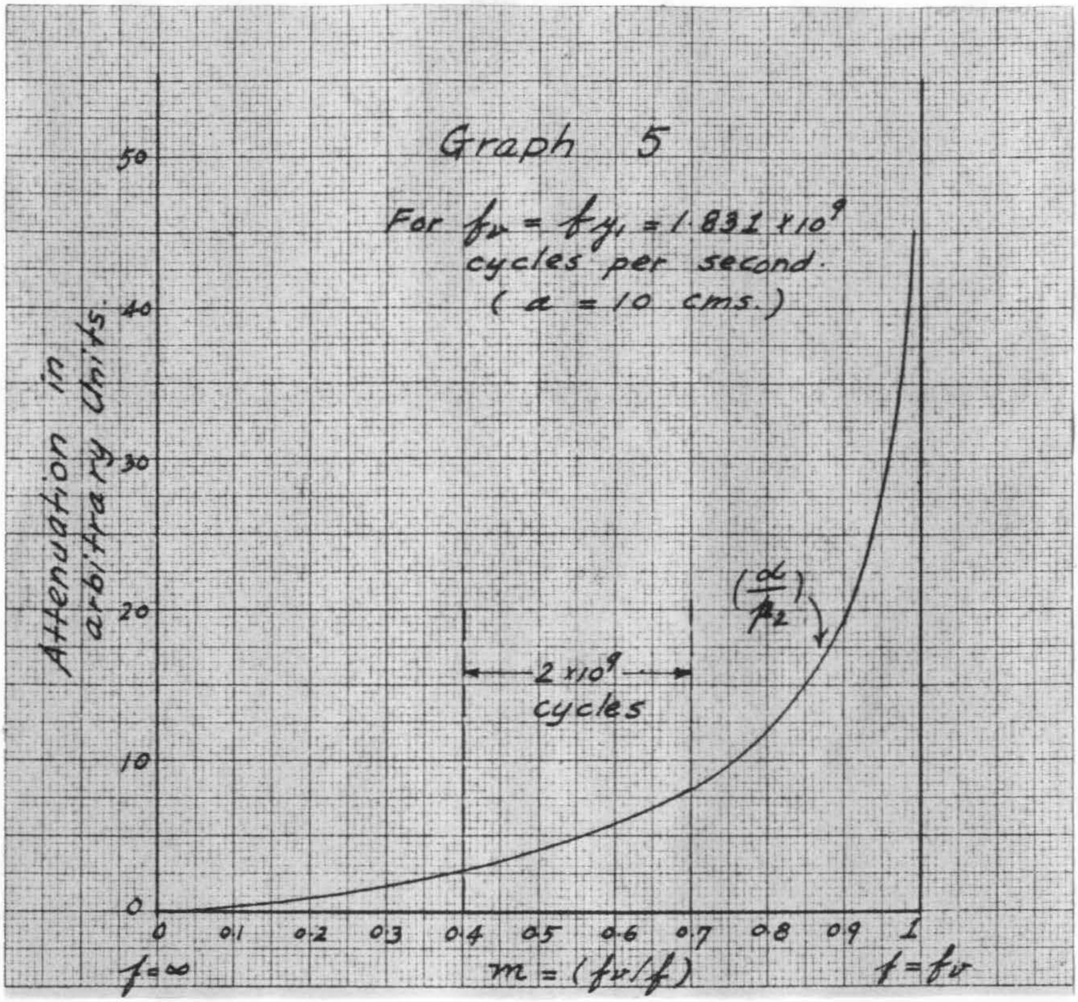
$$\alpha_v = \frac{1}{2a}\sqrt{\frac{\mu_2 \epsilon_1}{\mu_1 \sigma_2}}\ \frac{f^{\frac{1}{2}}}{\sqrt{1-(\frac{f_v}{f})^2}} = p_1\left(\frac{f^{\frac{1}{2}}}{10^4 K}\right) \qquad (4.27)$$

$$p_1 = \frac{10^4}{2a}\sqrt{\frac{\mu_2 \epsilon_1}{\mu_1 \sigma_2}}$$



Graph 4.

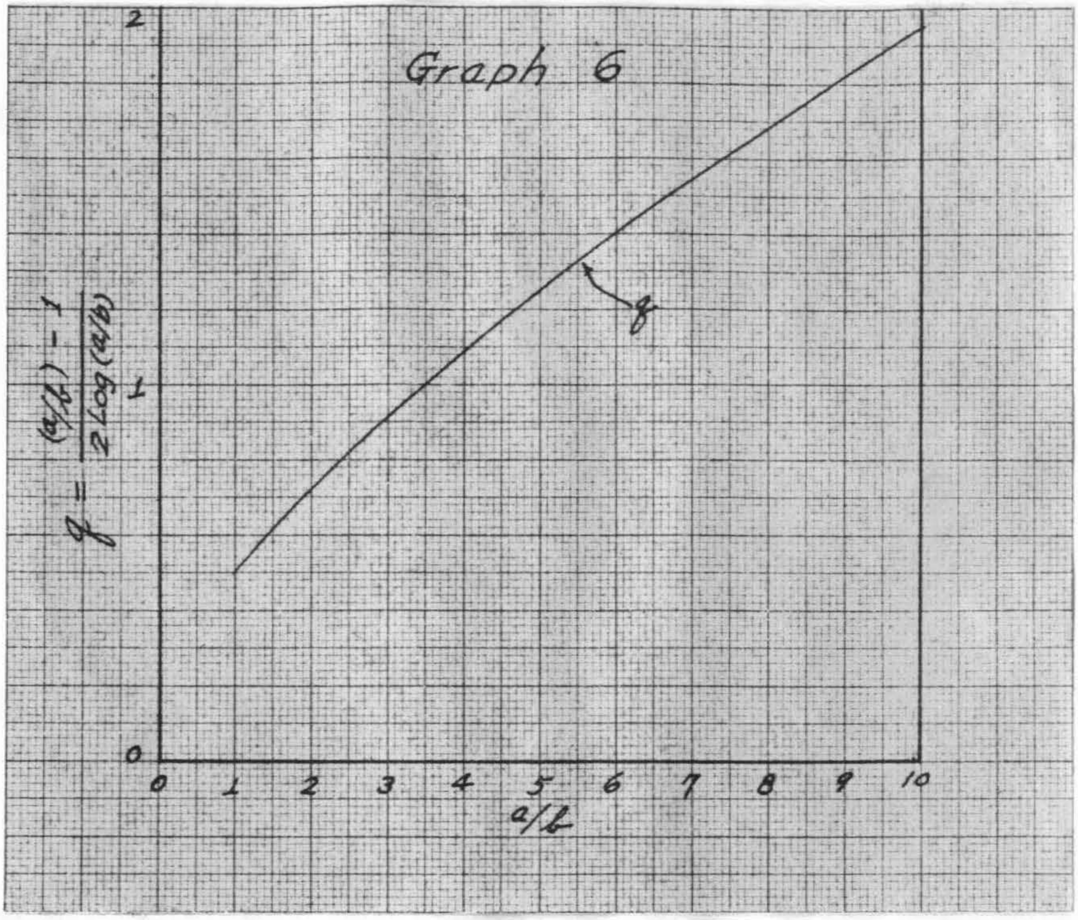
Wavelength
$$\mathcal{L}_v = \mathcal{L} = \frac{c}{\sqrt{\varepsilon_1\mu_1}}\frac{1}{f}\frac{1}{\sqrt{1-\left(\frac{f_v}{f}\right)^2}} = \frac{c}{\sqrt{\varepsilon_1\mu_1}}\frac{1}{f}\frac{1}{K}$$
$$= \mathcal{L}_0 \frac{1}{K} \qquad (7.20)$$



Graph 5.

Attenuation $$\alpha_v = \frac{1}{4a}\sqrt{\frac{\mu_2 \epsilon_1}{\mu_1 \sigma_2}}\,\frac{f^{\frac{1}{2}}(f_v/f)^2}{\sqrt{1-(f_v/f)^2}} = p_2\,\frac{10 f^{\frac{1}{2}}\left(\frac{f_v}{f}\right)^2}{K} \qquad (7.17)$$

$$p_2 = \frac{1}{40\,a}\sqrt{\frac{\mu_2 \epsilon_1}{\mu_1 \sigma_2}}$$



Graph 6.

$$q = \lim_{f \gg f_r} \frac{(9.21)}{(4.27)} = \lim_{f \gg f_r} \frac{\alpha}{\alpha_r} = \frac{\frac{a}{b} - 1}{2 \log \frac{a}{b}} \qquad (9.25)$$