PERTURBATIVE CORRECTIONS TO THE RATIO $\Gamma(\bar{B}\to D\rho^-)/\ \Gamma(\bar{B}\to D\pi^-)$

Thesis by

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In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosphy

California Institute of Technology
Pasadena, California

1993
(Submitted May 25, 1993)

Acknowledgment

I would like to recognize the following for their support: Prof. Mark B. Wise, my advisor whose work and personal advice defined this calculation; Michael Youssefmir, my collaborator with whom I worked very closely to ensure an accurate result; Dr. Murali Sharma, officemate and friend, who was always there to clarify any confusing issues which arose; and finally I wish to thank my parents, Werner and Maria Haas, for their support, emotional and otherwise.

Abstract

QCD interactions involving a heavy quark with energy much smaller than its mass can be understood in the context of an effective field theory in which the heavy quark velocity is held fixed while its mass is taken to infinity. Nonleptonic decays of hadrons containing a heavy quark further simplify when gluons exchanged carry small momenta compared to the heavy quark mass.

Under these assumptions the ratio of rates for $\bar{B}\to D\rho^-$ and $\bar{B}\to D\pi^-$ is investigated. The reliability of these assumptions is tested by calculating first order, one loop QCD corrections, assuming reasonable momentum distributions for the quarks inside the light mesons.

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1. Introduction

The accepted quantum field theory describing the strong interactions is Quantum Chromodynamics, QCD. In contrast with Quantum Electrodynamics, the charge (or color) of QCD is tri-valued. More precisely, QCD is a non-abelian gauge theory. It has been demonstrated that non-abelian theories are "asymptotically" free.[1] That is the coupling constant diverges below a certain scale. In QCD that scale is $\Lambda_{QCD} \sim 300MeV$. The implication of this is that for physical processes with characteristic energies significantly less than Λ_{QCD} ordinary perturbation theory fails. To overcome this shortcoming, symmetries, approximate or otherwise, of the theory are often exploited to make predictions.

One of the original symmetries employed was the $SU(3)$ flavor symmetry of the three light quarks: the up, down and strange. The "three-fold way" was particularly useful in organizing the myriad of hadrons unveiled in the 1950's and 60's. Naively, this symmetry appears to be the result of the closeness of light quark masses. In fact this approximate symmetry appears because the mass differences among the respective quarks is small compared to Λ_{QCD}.

In addition to $SU(3)$ flavor, QCD with only light quarks possesses the approximate $SU(3)_L \times SU(3)_R$ "chiral" symmetry. This symmetry is manifest in the limit $m_q \to 0$ in the QCD Lagrangian; q = u, d and s. The light quarks are taken to be massless with respect to Λ_{QCD}. Corrections to an effective theory with chiral symmetry can be expanded in powers of m_q / Λ_{QCD}. In nature, the $SU(3)_L \times SU(3)_R$ is spontaneously broken to the $SU(3)_V$ vector subgroup.

At the other end of the spectrum, QCD possesses approximate symmetries when one or more quarks possess masses much larger than QCD scale

$$m_Q >> \Lambda_{QCD} \tag{1}$$

Formally, these symmetries are manifest when m_Q is taken to infinity in the QCD Lagrangian. One of these symmetries is the obvious $SU(N)$ flavor symmetry among the heavy quarks, where N is the number of heavy quarks. The second and more powerful symmetry is an internal $SU(2)$ spin symmetry associated with each of the heavy quarks. In contrast with the massless case, corrections to this effective theory are expressed in powers of the dimensionless parameter Λ_{QCD} / m_Q.[2)]

In a theory with only heavy quarks, the usual techniques of perturbation theory could be used to make physical predictions. However, there are light quarks as well. However, the complicated interactions of light quarks in a bound system containing a heavy quark won't affect the motion of the system if $m_Q \to \infty$. This situation is analogous to applying Newtonian laws of motion to the trajectory of a baseball in air. Under suitable conditions, the inextricable influences of the air can be ignored in determining the flight of the ball.[2)]

The most important application of the heavy quark effective field theory is to semileptonic decays, e.g.,

$$\bar{B} \to D e \bar{\nu}_e \tag{2}$$

Predictions regarding decays of the this type are critical in determining elements of the Cabbibo-Kobayashi-Maskawa matrix.[3)] Characterizing transition matrix elements corresponding to the decay (2) is relatively straightforward in the $m_Q \to \infty$ limit.

More challenging are nonleptonic decays

$$\bar{B} \to D \pi^- \tag{3}$$

In this case accounting for the creation of the pion is notoriously difficult. One would like to consider the distinct physical processes

$$\bar{B} \to D \tag{4}$$

$$vacuum \to \pi^- \tag{5}$$

separately. Assuming this sort of "factorization" holds, physical predictions can be made.[4)] Intuitively factorization is a reasonable conjecture if virtual gluon exchanges between the light quarks and heavy quarks are "soft," i.e., their momenta are small compared to that of the heavy quark. Corrections to factorization resulting from "hard" gluon exchanges can be calculated pertubatively in ordinary QCD. Calculation of first-order corrections to factorization for the nonleptonic decay rate ratio

$$\Gamma(\bar{B}\to D\rho^-)\ /\Gamma(\bar{B}\to D\pi^-) \tag{6}$$

is the subject of this thesis.

In Chapter 2 the properties of heavy quark effective field theory are reviewed followed by a discussion of its applications in Chapter 3. A formal justification for factorization is presented in Chapter 4. In Chapter 5 the nonleptonic decay of interest is analyzed, followed by some concluding remarks.

2. Heavy Quark Effective Field Theory

Heavy Quark Effective Field Theory (HQEFT) is an appropriate approximation to the full theory of strong interactions, QCD, for consideration of physical processes in which a heavy quark interacts with light degrees of freedom carrying four momenta much less than the quark mass. The effective theory can be formally constructed by taking the heavy quark mass to infinity while maintaining a fixed four velocity.

Feynman Rules

The Feynman rules for the effective theory are most easily derived by considering the Feynman rules for QCD[5] in the appropriate limit. One first re-expresses the heavy quark four momenta

$$p_Q^\mu = m_Q v_Q^\mu + k^\mu \tag{7}$$

where v^μ is the the quark's four velocity and k^μ is the "residual" momentum which is small compared to the heavy quark mass. The heavy quark propagator in QCD,

$$\frac{i\,(\not{p}_Q + M_Q)}{(p_Q^2 - m_q^2)} \tag{8}$$

thus becomes

$$\frac{i\,(\not{v}+1)}{2v\cdot k} \tag{9}$$

The QCD vertex for heavy quark-gluon interactions is

$$-ig\gamma_{\mu}T^{a} \tag{10}$$

where g is the strong coupling constant and T^a is an $SU(3)$ color generator. Internal factors of $\frac{1+\not{v}}{2}$ associated with incoming/outgoing heavy quark lines will always appear sandwiching the vertex in the effective theory

$$-ig\,\frac{(1+\not{v})}{2}\,\gamma_{\mu}\,\frac{(1+\not{v})}{2}\,T^{a} = -ig\,v_{\mu}T^{a}\left(\frac{1+\not{v}}{2}\right) \tag{11}$$

Further, the action of factors of $(\not{v}+1)$ from the vertex in **(11)** and in the numerator of propagators yield factors of 2 when acting on on-shell spinors. This reduces the Feynman rules for the effective theory to:

$$i/v\cdot k \quad : \quad \textit{heavy quark propagator} \tag{12}$$

$$-igT^{\alpha}v_{\mu} \quad : \quad \textit{heavy quark-gluon vertex} \tag{13}$$

Field Theory

Alternatively, the effective theory can be developed by considering the Lagrangian of the full theory.[6] The quark field portion of the QCD Lagrangian is

$$\bar{Q}(i\not{D}-m_Q)Q \tag{14}$$

where Q is the heavy quark field and $\not{D}$ is the QCD covariant derivative. One begins by re-expressing the heavy quark field with the momentum scaled out:

$$Q = e^{-im_Q v x}\left(h_v^{(Q)} + X_v^{(Q)}\right) \tag{15}$$

The fields $h_v^{(Q)}$ and $X_v^{(Q)}$ must satisfy the constraints

$$\begin{aligned} \not{v}h_v^{(Q)} &= h_v^{(Q)} \\ \not{v}X_v^{(Q)} &= -X_v^{(Q)} \end{aligned} \tag{16}$$

One then uses the $h_v^{(Q)}$ part of Q as a suitable approximation to the fundamental field in the $m_Q \to \infty$ limit . Plugging into the Lagrangian **(14)** yields

$$L_v = \bar{h}_v^{(Q)} \, i \not{D} h_v^{(Q)} \tag{17}$$

By inserting factors of $\left(\frac{1+\not{v}}{2}\right)$ the above can be simplified to

$$L_v = \bar{h}_v^{(Q)} \, i v \cdot D h_v^{(Q)} \tag{18}$$

The propagator obtained in the previous section is now apparent in this effective Lagrangian. The contribution of the $X_v^{(Q)}$ part of Q is suppressed by a factor Λ_{QCD}/m_Q so it may be ignored as $m_Q \to \infty$.

Note that the effective theory does not contain pair creation. The heavy quark field, $h_v^{(Q)}$, destroys a heavy quark of velocity v but does not create an anti-quark. Therefore, it is not necessary to include an anti-quark field.

The Lagrangian of the effective theory **(18)** clearly violates Lorentz-invariance as it picks out a particular velocity v. Lorentz invariance can be recovered by taking a superposition of effective Lagrangians[7)] or by viewing the four-velocity as also transforming.

Of course this lack of Lorentz invariance is not surprising since the effective theory was constructed on the premise that the heavy quark four velocity remains fixed while interacting with the light degrees of freedom. However, the effective theory should not be regarded as a nonrelativistic approximation to the full theory. One can consider the situation where two heavy quarks are moving relativistically with respect to one another.[2)]

New Symmetries

The effective theory contains symmetries not apparent in the full theory. These new symmetries provide a great deal of the predictive power of the effective theory.

Because there is no pair creation in the effective theory, there is a global U(1) symmetry associated with heavy quark conservation. The effective Lagrangian **(18)** is left unchanged by the infinitesimal transformation

$$h_v^{(Q)} \rightarrow h_v^{(Q)} + i\varepsilon_0 h_v^{(Q)} \qquad \textbf{(19)}$$

where ε_0 is an arbitrary parameter.

More importantly the effective theory contains an $SU(2)$ symmetry associated with heavy quark spin conservation.[7)] The lack of gamma matrices in the heavy quark-gluon vertex **(13)** makes this symmetry readily apparent. This

$SU(2)$ symmetry can be explicitly demonstrated by first introducing a set of three orthonormal four-vectors $(e_{a\mu})$ that are orthogonal to the heavy quark's four velocity

$$e_{a\mu} e_b^{\mu} = -\delta_{ab} \qquad a,b=1,2,3 \tag{20}$$

$$v_{\mu} e_a^{\mu} = 0 \tag{21}$$

One then constructs the three $SU(2)$ generators by defining the matrices

$$S_a = i \sum_{b,c=1}^{3} \varepsilon_{abc} \left[\not{e}_b , \not{e}_c \right] \tag{22}$$

which satisfy the $SU(2)$ Lie algebra. The effective Lagrangian **(18)** is then left unchanged by the infinitesimal transformation

$$h_v^{(Q)} \to h_v^{(Q)} + i \sum_{a=1}^{3} \varepsilon_a S_a h_v^{(Q)} \tag{23}$$

where ε_a are arbitrary perameters.

Finally, since the heavy quark masses don't appear in the effective theory $(m_Q \to \infty)$, there is a flavor symmetry among heavy quarks moving at the same velocity. An effective Lagrangian containing N heavy quarks

$$L_v = \sum_{i=1}^{N} h_v^{(i)} \, v \cdot D h_v^{(i)} \tag{24}$$

does not distinguish between quarks of different flavors. Thus the $SU(2)$ symmetry associated with spin conservation is more generally an $SU(2N)$ spin-flavor symmetry.

3. Applications of HQEFT

The physical processes of particular interest are the heavy meson non-leptonic decays:

$$\bar{B} \to D\pi^- \tag{25}$$

$$\bar{B} \to D\rho^- \tag{26}$$

More generally, the new symmetry of heavy quark effective theory can be used to establish properties for decays of the type

$$P_Q \to P_Q^* \qquad (e.g., \bar{B} \to \bar{B}^*) \tag{27}$$

and transition elements

$$P_{Q_i} \to P_{Q_j} \qquad (e.g., \bar{B} \to D) \tag{28}$$

Spectroscopic Notation

In order to specify the state of a physical system containing a single heavy quark and light degrees of freedom, it is useful to go to the heavy quark's rest frame.[2)] The total angular momentum of the light degrees of freedom is

$$\vec{S}_l = \vec{S} - \vec{S}_Q \tag{29}$$

where $\vec{S}$ is the total angular momentum. Since $\vec{S}_l$ and $\vec{S}_Q$ are conserved by strong interactions (s_Q, m_Q, s_l, m_q) are a set of good quantum numbers. Thus, for a system containing a single heavy quark there are two distinct total spin states which are degenerate in mass specified by the quantum numbers

$$s_\pm = s_l \pm 1/2 \tag{30}$$

For the case of heavy meson systems where a heavy quark is bound to a single light quark q with zero orbital angular momentum, so that $s_l = \frac{1}{2}$, we have the spin 0 , s_- state denoted by $|P_Q>$ and the spin 1, s_+ state $|P_Q^*>$. These are the B, D and B^*, D^* mesons respectively. The effective theory predicts that the P_Q, P_Q^* states are degenerate in mass.

Transition Matrix Elements

In this section general properties of transitions between various heavy meson states are established. The matrix elements corresponding to these transitions are directly related to the V_{ub} and V_{cb} elements of the Cabibbo-Kobayashi-Maskawa matrix.[2)]

First consider the matrix element between the unexcited heavy meson states $|P_{Q_i}>$ and $|P_{Q_j}>$:

$$\frac{<P_{Q_j}(v')\,|\bar{h}^{(j)}_{v'}\;\gamma_\mu h^{(1)}_v\,|P_{Q_i}(v)>}{\sqrt{m_{P^*_{Q_j}}\;m_{P_{Q_i}}}} = \tilde{f}_+(v+v')_\mu + \tilde{f}_-(v-v')_\mu \tag{31}$$

The form factors $\tilde{f}\pm$ are functions of the relativistic invariant vv'. The factor $\sqrt{m_{P^*_{Q_j}} m_{P_{Q_i}}}$ appears in the denominator of **(31)** in order to have the expression independent of the quark masses. The conventional normalization of these heavy quark meson states in the full theory is

$$<M(p',s')\,|\,M(p,s)> \;=\; 2p^{\circ}\,\delta_{ss'}\delta^{3}(p-p') \qquad (32)$$

The factor $\sqrt{m_{P^*_{Q_j}} m_{P_{Q_i}}}$ in (31) cancels the energy dependence in (32).

By contracting the right-hand side of (31) with $(v-v')^{\mu}$ and using the fact that $\not{v} h_v^{Q_i} = h_{\not{v}}^{Q_i}$ and $\bar{h}_{v'}^{Q_j} \not{v}' = \bar{h}_{v'}^{Q_j}$ we see that

$$\tilde{f}_- = 0 \qquad (33)$$

For the particular case of $v=v'$, i.e., $v{\cdot}v' = 1$, the $\mu = 0$ component of (31) is simply unity. This reflects heavy quark number conservation. Thus,

$$\tilde{f}_+(v{\cdot}v'=1) \;=\; 1 \qquad (34)$$

It is important to note that the light degrees of freedom of the initial and final states carry momenta of order $\Lambda_{QCD}v$ and $\Lambda_{QCD}v'$ respectively. So the invariant momentum (squared) transfer between the light degrees of freedom is order Λ_{QCD}^2 $(v\cdot v'-1)$ which for $|v|\sim|v'|$ is much less than the heavy quark mass.

We next consider the matrix element corresponding to transitions between the meson states $|P^*_{Q_j}>$ and $|P_{Q_i}>$:

$$\frac{<P^*_{Q_j}(v',\varepsilon)|\bar{h}^{(j)}_{v'}\,\gamma_\mu\,h^{(i)}_v\,|P_{Q_i}(v)>}{\sqrt{m_{P^*_{Q_j}}m_{P_{Q_i}}}} = i\tilde{g}(v\cdot v')\varepsilon_{\mu\nu\sigma}\varepsilon^{*}v'^{\lambda}v^{\sigma} \tag{35}$$

where ε specifies the polarization (i.e., spin) of the excited meson.

In addition to (35), there are non-zero transition matrix elements having an insertion of γ_5

$$\frac{<P^*_{Q_j}(v',\varepsilon)|\bar{h}^{(j)}_{v'}\ \gamma_\mu\gamma_5 h^{(i)}_v|P_{Q_i}(v)>}{\sqrt{m_{P^*_{Q_j}}\ m_{P_{Q_i}}}} = \tilde{f}(v{\cdot}v')\varepsilon^*_\mu + \tilde{a}_+(v{\cdot}v')(\varepsilon^*{\cdot}v)(v+v')_\mu + \tilde{a}_-(v{\cdot}v')(\varepsilon^*{\cdot}v)(v-v')_\mu \qquad (36)$$

The form factors, $\tilde{g}$, $\tilde{f}$, $\tilde{a}_+$, and $\tilde{a}_-$ are all functions of $v{\cdot}v'$ and all can be related to the function $\tilde{f}_+$. To see this consider the matrix elements

$$\frac{<P_{Q_j}(v')|\bar{h}^{(j)}_{v'}\ \Gamma h^{(i)}_v|P_{Q_i}(v)>}{\sqrt{m_{P_{Q_i}}\ m_{P_{Q_j}}}} \qquad (37)$$

and

$$\frac{<P^*_{Q_j}(v',\varepsilon)|\bar{h}^{(j)}_{v'}\ \Gamma h^{(i)}_v|P_{Q_i}(v)>}{\sqrt{m_{P^*_{Q_j}}\ m_{P_{Q_i}}}} \qquad (38)$$

where Γ is any collection of gamma matrices.

Now introduce a four by four matrix containing the heavy meson fields

$$H^{(Q)}(v)=\frac{(\not{v}+1)}{2}[P_Q^{*\mu}\gamma_\mu-P_Q\gamma_5] \tag{39}$$

where the field $P_Q^{*\mu}$ destroys the state $|P_Q^*>$ and the field P_Q destoys the state $|P_Q>$. The heavy vector meson field satisfies the constraint $v^\mu P^*_{Q\,\mu}=0$.

Under a heavy quark spin transformation

$$H^{(Q)}(v)\rightarrow S\ H^{(Q)}(v) \tag{40}$$

where S is an element the $SU(2)$ group of heavy quark spin transformations. Under Lorentz transformations Λ

$$H^{(Q)}\rightarrow D(\Lambda)\ H^{(Q)}\ D(\Lambda)^{-1} \tag{41}$$

where $D(\Lambda)$ is the usual 4 x 4 Dirac representation of the Lorentz group. One can think of $H^{(Q)}(v)$ as representing the bispinor combination of fields

$$H^{(Q)} \sim Q\,\overline{q} \tag{42}$$

It is also convenient to introduce

$$\overline{H}^{(Q)}(v) = \gamma^0 H^{(Q)^+} \gamma^0 \tag{43}$$

which transforms as $\overline{H}^{(Q)}(v) \to \overline{H}^{(Q)}(v) S^{-1}$ under $SU(2)$ spin transformations and as $\overline{H}^{(Q)}(v) \to D(\Lambda)\ \overline{H}^{(Q)}(v)\ D(\Lambda^{-1})$ under Lorentz transformations.

Now the matrix elements of (37) and (38) are calculated from

$$\overline{h}^{(j)}_{v'} \Gamma h^{(i)}_v = -\xi Tr\ [\overline{H}^{(Q_j)}(v') \Gamma H^{(Q_i)}(v)] \tag{44}$$

where ξ is a universal function of $v{\cdot}v'$ The fact that Γ occurs between the two H 's is a consequence of heavy quark spin symmetry. On the outside of the H 's there could occur a factor of

$$(A\not{v} + B\not{v}')$$

But because

$$\not{v} H^{(Q)}(v) = H^{(Q)}(v) \tag{45}$$

$$H^{(Q)}(v) \not{v} = -H^{(Q)}(v) \tag{46}$$

it can be reduced to the form in equation (44)

Performing the trace over the gamma matrices for $\Gamma=\gamma_\mu$ and $\Gamma=\gamma_\mu\gamma_5$ gives[3,8)]

$$\begin{aligned} \tilde{f}_+ &= \xi \ , \tilde{f}_- = 0 \\ \tilde{f} &= (1+v v')\xi \\ (\tilde{a}_+ + \tilde{a}_-) &= -\xi \ , \ \tilde{a}_+ - \tilde{a}_- = 0 \\ \tilde{g} &= \xi \end{aligned} \tag{47}$$

Renormalization

In this section renormalization in the full theory of QCD is reviewed and then compared to renormalization in the effective theory. The QCD Lagrangian is

$$L = -1/4 \ \hat{G}^a_{\mu\nu} \ \hat{G}^{a\mu\nu} + i\sum_{j=1}^{N} \ \bar{\hat{q}}_j \ \gamma^\mu\left(\partial_\mu + i\hat{g}\hat{A}^a_\mu T^a\right)\hat{q}_j \tag{48}$$

where $\hat{G}^a_{\mu\nu}$ is the gluon strength tensor, $\hat{G}^a_{\mu\nu}=\partial_\mu \hat{A}^a_\nu - \partial_\nu \hat{A}^a_\mu - 2gf_{abc}[A^b_\mu, A^c_\nu]$ (f_{abc} are the $SU(3)$ structure constants). The bare fields and coupling, denoted by hats in (48), are related to the renormalized fields and coupling via

$$A^a_\mu = \frac{1}{\sqrt{Z_a}} \hat{A}^a_\mu \quad , \; q_i = \frac{1}{\sqrt{Z_q}} \hat{q}_i \tag{49}$$

$$g = \frac{1}{\sqrt{Z_g}} \mu^{-\varepsilon/2} \hat{g} \tag{50}$$

Perturbative ultraviolet divergences are controlled by dimensional regularization.[9] The dimension of spacetime is taken to be $n = 4-\varepsilon$. The subtraction point μ is introduced so that the renormalized coupling g is dimensionless in n dimensions. Physical quantities are independent of μ.

The renormalized Lagrangian is

$$\begin{aligned} L &= -1/4 \; Z_A \; G^a_{\mu\nu} \; G^{a\mu\nu} + Z_q \sum_{j=1}^{N} i\bar{q}_j \gamma^\mu \left(\partial_u + ig\mu^{\varepsilon/2} A^a_\mu T^a\right) q_j \\ &= -1/4 G^a_{\mu\nu} \; G^{a\mu\nu} + \sum_{j=1}^{N} i\bar{q}_j \gamma^\mu \left(\partial_\mu + ig\mu^{\varepsilon/2} A^a_\mu T^a\right) q_j + \textit{counterterms} \end{aligned} \tag{51}$$

The counterterms are chosen to cancel regulated divergences. Using the method of minimal subtraction (no finite subtractions) the $Z's$ have an expansion in $1/\varepsilon$:

$$Z(g,\varepsilon) = \sum_{p=1}^{\infty} \frac{Z^{(p)}(g)}{\varepsilon^p} \tag{52}$$

The $Z^{(p)}(g)'s$ also have an expansion in terms of the coupling constant g and are determined order by order in perturbation theory.

Although the subtraction point has no physical significance it is interesting to see how the renormalized coupling g depends on μ. In background field gauge

$$Z_g = \frac{1}{\sqrt{Z_A}} \tag{53}$$

gluon wave-function renormalization to one-loop yields

$$Z_g = 1/\sqrt{Z_A} = 1 - (33-2N)g^2/48\pi^2\varepsilon \tag{54}$$

The procedure for calculating the gluon wave function renormalization constant in (54) is well established. One first employs the Feynman trick for combining denominators

$$1/ab = \int_0^1 \frac{dx}{[ax+b(1-x)]^2} \tag{55}$$

After appropriate shifts of momenta, the final integration can be performed using the formula,

$$\int \frac{d^n q (q^2)^\alpha}{(q^2-M^2)^\beta} = i\pi^{n/2}(-1)^{\alpha+\beta}(M^2)^{\alpha-\beta+\frac{n}{2}} \frac{\Gamma(\alpha+\frac{n}{2})\Gamma(\beta-\alpha-\frac{n}{2})}{\Gamma(\frac{n}{2})\Gamma(\beta)} \tag{56}$$

The differential dependence of the renormalized coupling on μ is then given by

$$\begin{aligned} \mu \frac{d}{d\mu} g &= \mu \frac{d}{d\mu}\left(\frac{\mu^{-\varepsilon/2}}{Z_g}\right)\hat{g} \\ &= -\varepsilon g/2 - \mu\left(\frac{d}{d\mu} \ln Z_g\right) g \end{aligned} \tag{57}$$

Taking $\varepsilon \to 0$ and expanding $\ln Z_g$ in powers of g we have

$$u\frac{d}{d\mu}g = \beta(g) = -(33-2N)g^3/48\pi^2 + \textit{higher order} \tag{58}$$

Finally, integrating (56) gives[1)]

$$\alpha_s(\mu) \equiv g^2/4\pi = \frac{12\pi}{(33-2N)\,ln(\mu^2/\Lambda^2_{QCD})} \tag{59}$$

Λ_{QCD} is the characteristic scale of strong interactions and has a value of ~$300MeV$. When doing perturbative calculations, it is convenient to choose a value of μ close to physically relevant energy scale, E. Then logarithms of E^2/μ^2 don't appear in the perturbative expansion in $\alpha_s(E)$.

The result for quark wave function renormalization to one loop is

$$\sqrt{Z_q} = 1-4/3\ g^2/16\pi^2\varepsilon \tag{60}$$

One may also be interested in the renormalization of composite operators, Green's functions of which may not be finite. For example, consider the scalar operator

$$\hat{S}_{jk} = \hat{q}_j\hat{q}_k = Z_q q_j q_k \tag{61}$$

To one loop, the renormalization procedure gives

$$Z_s = 1+8(g^2/16\pi\varepsilon) \tag{62}$$

where

$$S_{jk} = 1/Z_s\hat{S}_{jk} \tag{63}$$

The vector and axial currents in the full theory do not require renormalization. Charges formed from these current are generators for the $SU(N)_L \times SU(N)_R$ chiral symmetry. Matrix elements of these (bare) charges must be μ independent.

In the effective theory, the bare Lagrangian

$$L_v = i\bar{\hat{h}}_v^{(i)}\ v^\mu(\partial_\mu + i\hat{g}T^a\hat{A}_\mu^a)\hat{h}_v^{(i)} \tag{64}$$

becomes

$$L_v = i\bar{h}_v^{(i)} v^\mu(\partial_\mu + ig\mu^{\varepsilon/2}T^a A_\mu^a)h_v^{(i)} + \textit{counterterms} \tag{65}$$

after the usual redefinitions including

$$h_v^{(i)} = 1/\sqrt{Z_Q}\hat{h}_v^{(i)} \tag{66}$$

The procedure for calculating Z_Q differs from wave function renormalization in the full theory because Green's functions contain logrithmic dependence on the quark masses.[2] Since $m_Q \to \infty$ in the effective theory, these contributions are divergent and impose different conditions on the choice of counterterms in (65) than in the ordinary theory. The result for Z_Q to one-loop is

$$\sqrt{Z_Q} = 1+(8/3)g^2/16\pi^2\varepsilon \tag{67}$$

Renormalization of the composite operators

$$\hat{O}_\Gamma = \hat{\bar{q}}_j \ \Gamma \hat{h}_v^{(i)} \tag{68}$$

$$\hat{T}_\Gamma = \hat{\bar{h}}_{v'}^{(j)} \ \Gamma \hat{h}_v^{(i)} \tag{69}$$

where Γ is any collection of gamma matrices, is done in the usual fashion using the dimensional regularization with minimal subtraction scheme yielding

$$Z_O = 1+(12/3)\, g^2/16\pi^2\varepsilon \qquad \textbf{(70)}$$

and

$$Z_T = 1-(16/3)\, g^2/16\pi^2\varepsilon\,[v{\cdot}v'r(v{\cdot}v')-1] \qquad \textbf{(71)}$$

where $\hat{O}_\Gamma = Z_O O_\Gamma$ and $\hat{T}_\Gamma = Z_T T_\Gamma$, and

$$r(v{\cdot}v') = \frac{1}{\sqrt{(v{\cdot}v')^2-1}}\, ln\left(v{\cdot}v'+\sqrt{(v{\cdot}v')^2-1}\right) \qquad (72)$$

Renormalization of $\hat{O}_\Gamma$ and $\hat{T}_\Gamma$ is independent of the Lorentz structure of the gamma matrix because of the heavy quark spin symmetry. While the axial and vector currents don't require renormalization in the full theory, in the effective theory they do. The $v{\cdot}v'$ dependence of Z_Γ is not very surprising since $h_{v'}^{(i)}$, $h_v^{(i)}$ represent different fields. Finally note that at $v{\cdot}v' = 1$, $\hat{T}_\Gamma$ does not require renormalization. This is a manifestation of heavy quark flavor symmetry in the effective theory.

Operators in QCD vs. Operators in HQEFT

In order to compare matrix elements calculated in the effective theory to experimentally measured quantities one must relate operators in the effective theory to those in the full theory QCD.

Consider the vector current in the full theory

$$V_\nu = \hat{\bar{q}}_j \gamma_\nu Q_i \tag{73}$$

versus the renormalized vector current in the effective theory

$$O_{\gamma_\nu} = \bar{q}_j \gamma_\nu h_v^{(i)} + counter\,terms \tag{74}$$

Roughly speaking, virtual loop momenta less than the subtraction point μ are included in the finite part of **(74)** while contributions at momenta greater than μ are subtracted away. If we take $\mu = m_Q$ then the following relationship holds

$$V_\nu = O_{\gamma_\nu} + order\ (\alpha_s(m_{Q_i})) \tag{75}$$

Large logarithms of m_{Q_i}/Λ_{QCD} appearing in matix elements of $O_{\gamma_v}(m_{Q_i})$ can be transferred to a coefficient by scaling the subtraction point down to the QCD scale. In leading logarithm approximation

$$V_v = C_i(\mu) O_{\gamma_v}(\mu) \qquad (76)$$

At $\mu = m_{Q_i}$, equation (75) implies

$$C_i(m_{Q_i}) = 1 + order\ (\alpha_s(m_{Q_i})) \qquad (77)$$

Using the fact that V_v is independent of μ and the renormalization constant Z_O given in (70), differention of (76) with respect to μ yields the renormalization group equation

$$\mu \frac{d}{d\mu} C_i(\mu) - \gamma_o(g) C_i(\mu) = O \qquad (78)$$

where

$$\gamma_o(g) = \mu \frac{d}{d\mu} lnZ_o \\ = -2g^2/8\pi^2 + order\ (g^4) \tag{79}$$

Integrating **(79)** and using the matching condition **(77)** gives

$$C_i(\mu) = \left[\frac{\alpha_s(m_Q)}{\alpha_s(\mu)}\right]^{-6/(33-2N)} \tag{80}$$

The coefficient relating the axial current in the full theory, $\hat{\bar{q}}_j\ \gamma_v \gamma_5 Q_i$, to that in the effective theory, $\bar{q}_j \gamma_v \gamma_5 h_v^{(i)}$ + *counter terms*, is also given by **(80)**.

The matrix elements discussed previously in this chapter involved operators of the form $\bar{h}_{v'}^{(j)} \Gamma h_v^{(i)}$. Relating these operators to bare operators in the full theory is particularly relevant. To leading logarithmic approximation

$$\hat{\bar{Q}}\ \Gamma Q_i = C_{ji}(\mu)\ (\bar{h}_{v'}^{(j)} \Gamma h_v^{(i)} + counter\ terms) \tag{81}$$

A "two-step" application of the renormalization group procedure gives

$$C_{ji}(\mu) = \left[\frac{\alpha_s(m_{Q_i})}{\alpha_s(m_{Q_j})}\right]\left[\frac{\alpha_s(m_{Q_j})}{\alpha_s(\mu)}\right]^{\alpha_L(v\cdot v')} \tag{82}$$

where

$$\alpha_L(v\cdot v') = 8/(33-2N)\,[v\cdot v' r(v\cdot v')-1] \tag{83}$$

4. Factorization

Heavy Quark Effective Field Theory, developed in the last two chapters, is particularly useful in calculating decay rates for semileptonic processes like $\bar{B} \to De\bar{\nu}_e$. In the notation of equations (31) and (82) we have

$$<D(v')\,|\bar{c}\gamma_\mu b\,|B(v)> \; = \; \sqrt{m_b m_c}\; C_{cb}\,\xi(v\cdot v')(v+v')_\mu \tag{84}$$

The leptonic degrees of freedom naturally "factorize." That is knowledge of the weak mixing angles, along with the calculation of (31) give experimentally relevant results for these decays.

However purely hadronic decays such as $\bar{B} \to D\pi^-$ are, in general, very difficult to calculate. Formally, one would like to write[4)]

$$<D\pi^-\,|\,O_1\,|\bar{B}> \; = \; <D|j^\mu\,|B> \; <\pi\,|j_\mu|0> \tag{85}$$

where

$$O_1 \; = \; \bar{b}\gamma^\mu(1-\gamma_5)c\bar{u}\gamma_\mu(1-\gamma_5)d \tag{86}$$

and j^{μ} is the usual $V\!-\!A$ current.

If (85) holds then the purely hadronic decays rates are tractible. Equation (31) gives the heavy meson transition matrix element, while the pion decay constant, f_{π}, defined by

$$f_{\pi} P_{\mu} = \langle \pi^{-}(P) | j_{\mu} | 0 \rangle \qquad (87)$$

is known and has a value of ~132 MeV.

It is possible to justify (85) to all orders of perturbation theory in a particular kinematic limit.[3)] Specifically, in the case where the light quarks are nearly colinear and the heavy quark masses are taken infinity, diagrams with "soft" gluon exchanges do not contribute to the matrix element in question. The contribution of "hard" gluon exchanges can be computed order by order in perturbation theory as corrections to factorization.

The crucial idea is that the large energy transferred to the light quarks is provided, to leading order, by the four-quark operator (i.e., W exchange) rather than by the gluon exchange.

Soft Gluon Exchange

To demonstrate that factorization holds for large energy transfer, we consider the light quark propagator in this limit. One first takes the momenta of the light quarks to be px and $p(1-x)$, $0<x<1$, where p is the pion momentum. The energy transferred to the light quark system is then

$$E = vp \tag{88}$$

It is convenient to re-express **(91)** in terms of a fixed null vector, n, with

$$v{\cdot}n = 1 \tag{89}$$

so that

$$p = En \tag{90}$$

Consider now a gluon, with a small momenta q, exchanged between a light and heavy quark. The light quark propagator, with m_q taken to be zero,

$$i(x\not{p} + \not{q}) / (xp+q)^2 \tag{91}$$

becomes

$$i\not{p}/2p\cdot q = i\not{n}/2n\cdot q \tag{92}$$

under the appropriate assumptions. Namely, the pion is taken to be lightlike

$$p^2 = 0 \tag{93}$$

and gluon exchanges are "soft"

$$|q| << |p| \tag{94}$$

Note that the propagator in **(92)** is independent of the energy transferred as well as the momentum fraction x.

It should also be noted that the propagator **(92)** leads to an effective theory which forms a natural addendum to HQEFT.[4)] Following arguments similar to those discussed in Chapter 2, the Feynman rules for the expanded effective theory can be expressed in a more concise fashion. Since vertices will always be sandwiched between factors of $\not{n}$, we have

$$\frac{i}{n\cdot q} : \textit{light quark propagator} \tag{95}$$

$$-iqT^a n^\mu: \textit{ quark-gluon-quark vertex} \tag{96}$$

These rules are analogous to those of HQEFT with the substitution of the timelike vector v by the null vector n. The absence of gamma matrices in (95, 96) further expands the $SU(2N)$ symmetry of HQEFT to include the internal $SU(2)$ spin symmetry of the light quarks.

To see that factorization holds for matrix elements involving the operators

$$O_1 = \bar{b}\gamma^\alpha(1-\gamma_5)c\bar{u}\gamma_\mu(1-\gamma_5)d \tag{97}$$

$$O_8 = \bar{b}T^a\gamma^\mu(1-\gamma_5)c\bar{u}T^a\gamma_\mu(1-\gamma_5)d \tag{98}$$

one need only consider the gauge

$$n{\cdot}A = 0 \tag{99}$$

In this gauge the light quarks decouple and factorization holds trivially. This result can also be shown in ordinary covariant gauges where for O_1 a cancellation occurs between the gluon coupling to the quark and anti-quark propagators.

For the octet operator O_8 the cancellation is not quite so straightforward. The sign difference between the effective propagators will yield a commutator of $SU(3)$ color generators. However, since final states are color singlets, there can be no contribution from these diagrams.

Finally, it should be apparent that once factorization holds for one-loop soft gluon exchanges, it also holds at higher orders by recursively arguing away each gluon exchange.

Hard Gluons

The effects of hard gluon exchange can be evaluated systemically, order by order in perturbation using the full theory, QCD. Formally, these perturbative corrections can be contained in a hard scattering amptitude integrated against the final meson wavefunction. In HQEFT this hard scattering amptitude should be function of m_b, as it does have perturbative expansion in $\alpha_s(m_b)$, as well as the momentum fraction x. The final meson wavefunction, although not known, should, in an intuitive sense, reasonably describe the momentum distribution between the two light quarks.

5. Calculation of $\Gamma(\bar{B}\to D\rho^-)$ / $\Gamma(\bar{B}\to D\pi^-)$

The Heavy Quark Effective Field Theory along with the complimentary idea of factorization developed in the previous chapters allow us to make definite predictions on the nonleptonic decays of heavy mesons. To understand the expected level at which factorzation holds it is important to understand the corrections. Nonperturbative Λ_{QCD}/m_Q corrections are not computable, however, perturbative corrections of order $\alpha_s(m_b)$ can be computed.

Consider the specific nonleptonic decays

$$\bar{B} \to D\,\pi^- \tag{100}$$

$$\bar{B} = D\rho^- \tag{101}$$

We are also interested in the excited decays

$$\bar{B} \to D^*\,\pi^- \tag{102}$$

$$\bar{B} \rightarrow D^* \rho^- \tag{103}$$

Experimentally the ratio of decay rates for the processes **(100)** and **(101)** is [11]

$$\Gamma(B^- \rightarrow D^o \rho^-) / \Gamma(B^- \rightarrow D^o \pi^-) = 2.55 \pm 1.0 \tag{104}$$

Both the excited and non-excited decays are governed by the effective Hamiltonian

$$H_{eff}^{|\Delta c=1|} = \frac{4G_F}{\sqrt{2}} V_{cb} [C_1(m_b) O_1(m_b) + C_8(m_b) O_8(m_b)] \tag{105}$$

where

$$O_1 = \left[(\bar{c}\frac{(1+\gamma_5)}{2}\gamma_\mu b)(\bar{d}\frac{(1+\gamma_5)}{2}\gamma^\mu u) \right] (m_b) \tag{106}$$

$$O_8 = \left[(\bar{c}\frac{(1+\gamma_5)}{2}\gamma_\mu T^a b)\ (\bar{d}\frac{1+\gamma_5)}{2}\gamma^\mu T^a u) \right] (m_b) \tag{107}$$

Spinor induces on the quark fields and $SU(3)$ color operators in the above opertors have been suppressed. V_{cb} is the $b \to c$ element of the Cabbibo-Maskawa matrix. G_F is the fermi constant,

$$G_F/(hc)^3 \cong 1.17 x 10^{-5}\ GeV^{-2} \tag{108}$$

The coefficients C_1 and C_8 are given by[12)]

$$C_1 = \frac{2}{3}\left[\frac{\alpha_s(m_W)}{\alpha_s(m_b)}\right]^{6/23} + \frac{1}{3}\left[\frac{\alpha_s(m_W)}{\alpha_s(m_b)}\right]^{-12/23} \tag{109}$$

$$C_8 = \left[\frac{\alpha_s(m_W)}{\alpha_s(m_b)}\right]^{6/23} - \left[\frac{\alpha_s(m_W)}{\alpha_s(m_b)}\right]^{-12/23} \tag{110}$$

These coefficients as well as the four quark operators O_1 and O_8 are evaluated at the scale m_b. The running of these coefficients sums up leading logarithms of (m_W/m_b) coming from virtual gluon exchanges between the heavy and light quarks. Running to the scale m_c is ignored.

The matrix elements of the operaters for the decays $\bar{B} \rightarrow D\pi^-$ and $\bar{B} \rightarrow D\rho^-$ simplify in the limit in which the relevant quark masses are taken to infinity but their ratio

$$r \equiv m_c / m_b \tag{111}$$

is held fixed. Specifically in this limit the outgoing meson is light-like and the matrix elements with "soft" gluon exhanges from the heavy or spectator quarks to the light quarks will factorize.[13)] What remains is to calculate the perturbative corrections to this coming from "hard" gluons. To first order in Λ_{QCD}/m_b, Λ_{QCD}/m_c, and $\Lambda_{QCD}/(m_b - m_c)$, these corrections can be written as a sum of the product of matrix elements of operators in the effective heavy quark theory[6)] with an integral over the meson "wavefunction" $\phi(x,m_b)$ multiplied by a "hard scattering" amplitude $T(x,r,m_b)$ where x is the fraction of the meson momentum carried by the up quark and (1-x) is the fraction carried by the down quark. This wavefunction can be thought of as the amplitude for the up quark to carry a momentum fraction x of the meson wave function.

The transition matrix elements of interest can thus be written as

$$\begin{aligned}
&<D^{(*)}(v')\pi^-(P)\,|\mathcal{O}_i\,(m_b)\,|\bar{B}^{(*)}(v)> \\
&= \frac{1}{4}<D(v')\,|\bar{h}_c h_b|\bar{B}(v)> m_b f_\pi (1-r)\int_0^1 dx\; T_i^{(s)}(x,r,m_b)\phi_\pi(x,m_b) \\
&+\frac{1}{4}<D^*(v')\,|\bar{h}_c\gamma_5 h_b|\bar{B}^*(v)> m_b f_\pi (1+r)\int_0^1 dx\, T_i^{(p)}(x,r,m_b)\phi_\pi(x,m_b)
\end{aligned} \tag{112}$$

Here $\bar{h}_c$ and h_b are the heavy quark fields in the effective theory. The amplitudes $T_i^{(s)}$, $T_i^{(p)}$ correspond to transitions between unexcited, excited heavy meson states respectively. Equation **(112)** does not presume factorization. Rather, the QCD corrections to this nonleptonic weak decay are incorporated in the hard scattering amplitudes, $T_i^{(s,\,p)}$. The pion decay constant, f_π, is defined by

$$<\pi^-(P)|\bar{d}\gamma_\mu\gamma_5 u|0> \;=\; if_\pi P_\mu \tag{113}$$

All the non-perturbative physics which goes into creating π^- is absorbed in f_π. The factors $m_b(1-r)$ and $m_b(1+r)$ in **(112)** arise by substituting

$$P \;=\; m_b v \;-\; m_c v' \tag{114}$$

For the decay $\bar{B} \to D\rho^-$ the transition matrix element in the heavy quark theory becomes

$$
\begin{aligned}
&<D^{(*)}(v')\rho^-(P)|O_i(m_b)|\bar{B}^{(*)}(v)> \\
&= \frac{1}{4}<D(v')|\bar{h}_c h_b|\bar{B}(v)> m_b f_\rho (1-r)\int_0^1 dx\, T_i^{(s)}(x,r,m_b)\,\phi_\pi(x,m_b) \\
&+ \frac{1}{4}<D^*(v')|\bar{h}_c \gamma_5 h_b|\bar{B}^*(v)> m_b f_\rho (1+r)\int_0^1 dx\, T_i^{(p)}(x,r,m_b)\,\phi_\pi(x,m_b)
\end{aligned}
\tag{115}
$$

The decay constant, f_ρ, is defined by

$$< \rho^-(P,\varepsilon)|\bar{d}_i \gamma_\mu \gamma_5 u_i|0> \; = \; i f_\rho m_\rho \varepsilon_\mu^* \tag{116}$$

If we assume that the ρ is light-like so that its polarization is dominated by the longitudinal component, we have

$$< \rho(P,\varepsilon)|\bar{d}_i \gamma_\mu \gamma_5 u_i|0> \; = \; i f_\rho m_\rho \varepsilon_\mu^* \simeq i f_\rho P_\mu \tag{117}$$

Zeroeth Order

To zeroeth order in $\alpha_s(m_b)$ the ratio of decay rates simplifies.[15,16] To this order hard gluon exchanges are ignored and factorization holds exactly. Explicitly

$$<D^{(*)}(v')X|O_1\ |\bar{B}^{(*)}(v)=<D^{(*)}(v')|\bar{c}\gamma_\mu\frac{(1-\gamma_5)}{2}b|B^{(*)}(v)> \tag{118}$$

$$x\ <X|\bar{d}\gamma_\mu\left(\frac{1-\gamma_5}{2}\right)u|0>$$

and

$$<D^{(*)}(v')X|O_8\ |\bar{B}^{(*)}(v)> = O \tag{119}$$

where X is either π^- or ρ^-. In terms of the hard scattering amplitudes, T_i, equations **(118)** and **(119)** imply

$$T_1^{(s,\,p)} = 1+O\ (\alpha_s(m_b)) \tag{120}$$

and

$$T_8^{(s,\,p)} = O\ (\alpha_s(m_b)) \tag{121}$$

To zeroeth order in $\alpha_s(m_b)$ the ratio of decay rates, R, is

$$R \cong \left(\frac{f_\rho}{f_\pi}\right)^2 \left| \frac{<D(v')|\bar{h}_c h_b|\bar{B}(v)>_{\rho^-}}{<D(v')|\bar{h}_c h_b|\bar{B}(v)>_{\pi^-}} \right|^2 x \text{ (phase space)} \qquad \textbf{(122)}$$

The heavy meson matrix elements in **(122)** do not cancel exactly. In general, the $v{\cdot}v'$ dependence of these transition matrix elements is given by the universal function $\xi(v{\cdot}v')$. Since $m_\pi \neq m_\rho$, the argument of the universal functon must differ in the two transitions we are considering. Treating the pion as massless, the following kinematic relations hold[10)]

$$(v{\cdot}v')_{\bar{B}\to D\rho^-} = \frac{m_b^2+m_c^2-m_\rho^2}{2m_b m_c} \qquad \textbf{(123)}$$

$$(v{\cdot}v')_{\bar{B}\to D\pi^-} = \frac{m_b^2+m_c^2}{2m_b m_c} \qquad \textbf{(124)}$$

The ratio of the transition matrix elements in **(122)** thus differs from unity by a term of order $m_\rho^2/m_c m_b$ which can be ignored in the heavy quark limit so that

$$R = \left(\frac{f_\rho}{f_\pi}\right)^2 \tag{125}$$

to zeroeth order in $\alpha_s(m_b)$. Here we are ignoring the difference in phase space factors for the two decays. To leading order we have

$$phase\ space \simeq 1 - \frac{m_\rho^2}{m_B^2 + m_D^2} \tag{126}$$

This factor differs from unity by roughly 5%, which is well within the experimental uncertainty. For notational simplicity we shall ignore it in the following discussion.

Taking $f_\rho \simeq 190\ MeV$ and $f_\pi = 132\ MeV$ we have $R \simeq 2$. Empirically the ratio of decay rates is

$$R = \frac{\Gamma(B^- \to D^o \rho^-)}{\Gamma(B^- \to D^o \pi^-)} = 2.55 \pm 1.0 \tag{127}$$

The large uncertainty in the experimental value of R makes it difficult to ascertain the reliability of the zeroeth order calculation. By calculating first order corrections to factorization, we can at least judge whether the prediction **(125)** is "perturbatively" sound

$\alpha_s(m_b)$ Corrections

In terms of the relevant hard scattering amplitudes, $T_i^{(s)}(x,m_b,r)$, and the π^- and ρ^- wave functions, the ratio of rates is

$$R = \left(\frac{f_\rho}{f_\pi}\right)^2 \left| \frac{\int_0^1 dx T_1^{(s)} \phi_\rho + \left(\frac{C_8}{C_1}\right) \int_0^1 dx T_8^{(s)} \phi_\rho}{\int_0^1 dx T_1^{(s)} \phi_\pi + \left(\frac{C_8}{C_1}\right) \int_0^1 dx T_8^{(s)} \phi_\pi} \right|^2 \qquad \textbf{(128)}$$

Expanding **(128)** to first order in $\alpha_s(m_b)$ yields

$$\simeq \left(\frac{f_\rho}{f_\pi}\right)^2 [1 + 2(C_8/C_1) Re \int_0^1 dx (\phi_\rho - \phi_\pi) T_8^{(s)} + 2 Re \int_0^1 dx (\phi_\rho - \phi_\pi) T_1^{(s)}] \qquad \textbf{(129)}$$

The first-order perturbative corrections to $T_1^{(s)}$ come from Feynman graphs with gluon running between either the heavy quark or the light quarks. In the first case, the corrections are independent of the momentum fraction carried by the light quarks and so drop out of the ratio. In the second case, they are absorbed into the definition of f_π , f_ρ and the respective wave functions. Thus we are left with

$$R \cong \left(\frac{f_\rho}{f_\pi}\right)^2 \left[1+2\left(\frac{C_8}{C_1}\right) Re\int_0^1 dx(\phi_\rho-\phi_\pi)\, T_8^{(s)}\right] \tag{130}$$

to order $\alpha_s(m_b)$.

Perturbative corrections to $T_8^{(s)}$ come from the Feynman diagrams in Fig. 1. These one-loop graphs have gluons running between the heavy quarks and light quarks. Taking the momentum fraction carried by the up quark to be x while the down quark carries a fraction (1-x), the hard scattering amplitude is

$$\begin{aligned} T_8^{(s)}(x) = \frac{2\alpha_s(m_b)}{9\pi}&[a\, I_{1,1}(a) - \frac{1}{2} b\, I_{1,1}(b) + c\, I_{1,1}(c) - \frac{1}{2} d I_{1,1}(d) \\ &+ I_{2,0}(a) - \frac{1}{2}(1+r)\, I_{2,0}(b) + I_{2,0}(c) - \frac{1}{2}(1+\frac{1}{r})\, I_{2,0}(d) \\ &+ \frac{1}{2} J(a) - \frac{1}{8} J(b) + \frac{1}{2} J(c) - \frac{1}{8} J(d)], \end{aligned} \tag{131}$$

where

$$
\begin{aligned}
a &= x(1-r^2) \\
b &= (1-x)(1-r^2) \\
c &= -\frac{b}{r^2} \\
d &= -\frac{a}{r^2}
\end{aligned}
\tag{132}
$$

$$I_{1,1}(a) = -\frac{1}{2}(1-a)\left(a+\frac{1}{1-a}ln|a|\right) \tag{133}$$

$$I_{2,0}(a) = \frac{1}{2(1-a)}\left(1+\frac{a}{1-a}ln|a|\right) \tag{134}$$

$$J(a) = \frac{-a ln|a|}{2(1-a)} \tag{135}$$

All terms that do not show an x-dependence have been dropped since they do not contribute to the ratios.

In calculating the Feynman diagrams leading to **(131)**, there is a certain dependence on the scheme for the renormalization of O_8. To know what scheme to use one must match, at one loop, the theory in which the W is integrated out with the theory in which it is not and then scale down to m_b to two-loop order. However, in mass independent subtraction any such scheme dependence is x independent and so drops out of the ratio of rates along with other x independent terms. We have also ignored any imaginary parts coming from terms in **(133, 134, 135)** that fix the branch of logarithms; these terms do not contribute to the ratio of rates in the order we are working.

The expressions for the wave functions may now be substituted into the expression for the ratio. To get an idea of the magnitude of the ratio we substitute reasonable guesses for the "wavefunction" ϕ_π and ϕ_ρ and then integrate them against the hard scattering amplitude $T_8^{(s)}$.

The wavefunction

$$\phi_\pi(x) = \delta\left(x - \frac{1}{2}\right) \qquad \textbf{(136)}$$

is a reasonable guess in accordance with the phenomenological quark model. The wave function

$$\phi_\rho(x) = 6x(1-x) \tag{137}$$

is analogous to the wave function of ref. (8) when the scale of their wave function is taken to infinity. In addition, the "constant" wavefunctions: ϕ_π (or ϕ_ρ) = 1 were considered to get an idea of the spread of

$$\int_0^1 dx(\phi_\rho - \phi_\pi)T_8^{(s)} \tag{138}$$

The result of these computations, which were evaluated using *Mathematica*, is a 1% contribution to the ratio of rates $\Gamma(\bar{B} \to D\rho^-)/\Gamma(\bar{B} \to D\pi^-) \cong (f_\rho/f_\pi)^2$ at order $\alpha_s(m_b)$. For the various choices of wavefunctions, the results, at fixed r, varied by less than a factor of 2. These correctons are in fact smaller than those discussed in the previous section coming from the kinemetric differences in the two decays. The quantity

$$\xi(v \cdot v')_{\bar{B} \to Dp^-} / \xi(v \cdot v')_{\bar{B} \to D\pi^-} \tag{139}$$

gives rise to a 5% correction. The exact value of **(139)**, to leading order in $m_\rho^2 / m_b m_c$, depends on precise knowledge of $\xi(v \cdot v')_{\bar{B} \to D\pi^-}$.

Excited Decays

The formalism applied to the nonleptonic decays $\bar{B} \to D\pi^-$ and $\bar{B} \to D\rho^-$ can be used for the excited decays

$$\bar{B} \to D^* \pi^- \tag{140}$$

$$\bar{B} \to D^* \rho^- \tag{141}$$

Indeed, the symmetries of the heavy quark effective theory naturally relate spin 0 and spin 1 heavy quark mesons of the same flavor as discussed in the second chapter. The heavy meson transition matrix elements are again fully described by the universal function ξ. In terms of the hard scattering amplitude, $T_8^{(p)}$ we have

$$\Gamma(\bar{B}\to D^*\pi^-)/\Gamma(\bar{B}\to D^*\pi) \cong \left(\frac{f_\rho}{f_\pi}\right)^2\left[1+2\frac{C_8}{C_1}Re\int_0^1 dx(\phi_\rho-\phi_\pi)T_8^{(p)}\right] \qquad (142)$$

Calculation of $T_8^{(p)}$ is performed in the same fashion as in the scalar case. Explicit factors of $1+r$ in **(131)** are replaced by $1-r$. For reasonable wavefunctions the order $\alpha_s(m_b)$ correction to the zeroeth order prediction

$$\Gamma(\bar{B}^*\to D^*\rho)/\Gamma(\bar{B}^*\to D^*\pi) \cong (f_\rho/f_\pi)^2 \qquad (143)$$

are again about 1%. The experimental data shows $\Gamma(\bar{B}^{o*}\to D^{+*}\rho^-)/\Gamma(\bar{B}^{o*}\to D^{+*}\pi^-) \cong 2.7\pm1.0$.[11]

6. Conclusion

The Heavy Quark Effective Field Theory along with the complimentary notion of factorization predicts that to zeroeth order in the coupling $\alpha_s(m_b)$ the ratio of decay rates for the nonleptonic processes $\bar{B}\to D\rho^-$ and $\bar{B}\to D\pi^-$ is

$$\Gamma(\bar{B}\to D\rho^-)\Gamma(\bar{B}\to D\pi^-) \cong (f_\rho/f_\pi)^2 \cong 2 \qquad (144)$$

This result also applies to the excited heavy decays $\bar{B}\to D^*\rho^-$ and $\bar{B}\to D^*\pi^-$.

First-order corrections to this prediction arise from the one-loop Feynman diagrams with hard gluons being exchanged between the light and heavy quarks. The contribution of these corrections is small. We find for a reasonable spread of π^- and ρ^- wave functions corrections to be of order 1%.

There are additional corrections to **(144)** due to kinematical differences between the two decays. These corrections are of order 5%.

Experimentally the ratio of decays is

$$\Gamma(B^-\to D^o\rho^-)/\Gamma(B^-\to D^0\pi^-) = 2.55\pm1.0 \qquad \textbf{(145)}$$

The zeroeth order prediction falls within the range of experimental uncertainty. It is reassuring to discover that this prediction withstands the one loop $\alpha_s(m_b)$ corrections calculated in this thesis although Λ_{QCD}/m_b corrections may disrupt this picture.. It would appear that the strong interactions, in particular the nonleptonic decays discussed herein are well approximated by an effective field theory in which the heavy quark masses are taken to infinity (keeping their ratios fixed) and matrix elements between hadronic final states factorize.

Nonetheless, until improvements in the experimental uncertainty for the decay rates are made, the reliability of this approach cannot be conclusively ascertained.

Appendix

The expression for the hard scattering amplitude $T_8^{(s)}$ in equation **(131)** is computed from the Feyman diagrams in **Figure 1**. The details of this calculation are presented in this section.

First consider the amplitude associated with the Feyman diagram where a gluon, with momentum q, is exchanged between the outgoing anti-up quark and the incoming bottom quark:

$$\int \frac{d^n q}{(2\pi)^n} [\bar{u}(p_c)\gamma_\nu(1-\gamma_5)T^b \frac{i[(\not{p}_b+\not{q})+m_b]}{[(p_b+q)^2-m_b^2]} igT^a\gamma^\mu u(p_b)]$$
$$x[\bar{u}(p_d)\gamma_\nu(1-\gamma^5)T^b \frac{i(\not{p}_u-\not{q})}{(p_u+q)^2} igT^a\gamma_\mu v(p_u)](\frac{-i}{q^2}) \qquad \textbf{(A1)}$$

Here p_b, p_u, p_d, and p_c are the respective quark momenta. The incomoing/outgoing spinors, the quark/ gluon propagators, and quark-gluon-quark vertex factors follow from the Feynman rules for ordinary QCD. Note that we are ignoring the up quark mass.

T^a and T^b are $SU(3)$ color generators satisfying

$$T^a T^b = \frac{\delta^{ab}}{6} \tag{A2}$$

$$Trace[T^a T^b] = \frac{\delta^{ab}}{2} \tag{A3}$$

thus yielding an overall color factor of $\frac{2}{9}$ for this and all other diagrams. Factors of $\gamma_\nu(1-\gamma_5)$ appearing in **(A1)** are due to the chiral structure of the effective operator O_8 given in equation **(110)**.

In massaging **(A1)** one first employs the Feynman trick for combining denominators:

$$\frac{1}{q^2[(p_b+q)^2-m_b^2](q+p_u)^2} = 2\int_0^1 dz \int_0^{1-z} dy \frac{1}{[(q+p_b z+p_u y)^2-(p_b z+p_u y)^2]} \tag{A4}$$

After shifting the loop momenta integration

$$q \rightarrow q - p_b z - p_u y$$

the expression for the amplitude becomes

$$\left(\tfrac{2}{9}\right) 2ig^2\int dz\, dy\int \frac{d^nq}{(2\pi)^n}\{\bar{u}(p_c)[\gamma_\nu(1-\gamma_5)[\not{q}-\not{p}_b(1-z)-y\not{p}_u+m_b]\gamma^\mu]u(p_b)\}$$
$$x\ \{\bar{u}(p_d)[\gamma_\nu(1-\gamma_5)[\not{q}-\not{p}_b z+\not{p}_u(1-y)]\gamma_\mu]v(p_u)\}\frac{1}{(q^2-M^2)^3} \qquad \text{(A5)}$$

where $M^2=(p_b z+p_u y)^2$. At this point it is useful to have an expression with a chiral structure that most closely resembles that of the transition matix element. We would like an expression in which only factors of $\gamma_\nu(1-\gamma_5)$ appear between the heavy quark and light quark spinors . To achieve this the various ("slashed") momenta terms in **(A5)** must be commuted through γ_μ thus allowing them to act on the spinors. Using the anti -commutator relation

$$\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu} \qquad \textbf{(A6)}$$

and the action of p_b, p_u on spinors

$$\not{p}_b u(p_b) = m_b$$

(A7)

$$\not{p}_u v(p_u) = 0$$

The result is

$$\left(\tfrac{2}{9}\right)\ 4ig^2 \int dz\ dy \int \frac{d^n q}{(2\pi)^n} \frac{1}{(q^2-M^2)^3}$$

$$x\ [\bar{u}(p_c)\gamma_\nu(1-\gamma_5)(zm_b\not{p}_u - 2zp_u{\cdot}p_b)u(p_b)][\bar{u}(p_d)\gamma^\nu(1-\gamma_5)v(p_u)]$$ (A8)

$$x\ [\bar{u}(p_c)\gamma_\nu(1-\gamma_5)u(p_b)][\bar{u}(p_d)\gamma^\nu(1-\gamma_5)(-2yp_u{\cdot}p_b - zm_b^2)v(p_u)]$$

$$x\ [\bar{u}(p_c)\gamma_\nu(1-\gamma_5)\not{q}u(p_b)][\bar{u}(p_d)\gamma^\nu(1-\gamma_5)\not{q}v(p_u)]$$

We are now in a position to work on the integrals, or at least re- express them in a more manageable form. The integration over q ,ignoring the $\not{q}\ \not{q}$ piece for the moment, can be handled using

(A9)

$$\int \frac{d^n q}{(2\pi)^n} \frac{1}{(q^2-M^2)^3} = \frac{-i(M^2)^{-1}}{2(16\pi^2)}$$

We also define the "generalized" integral form

$$I_{\alpha,\beta}(a)=\int_0^1 dz\int_0^{1-z} dy\frac{z^{\alpha}y^{\beta}}{(z^2+ayz)} \tag{A10}$$

The dimensionless variable a contains all the relevant kinematic information, i.e., x-dependence, and equals $\frac{2p_u p_b}{m_b^2}$.

The $\not q \not q$ contribution to the amplitude gives rise to the divergent integral

$$\int\frac{d^n q}{(2\pi)^n}\frac{q^2}{(q^2-M^2)^3}=\frac{i}{2}\left(\frac{n}{2}\right)M^{-4}\left(\frac{M^2}{4\pi}\right)^{\frac{n}{2}}\Gamma\left(2-\frac{n}{2}\right) \tag{A11}$$

Taking $n=4-\epsilon$

$$\int\frac{d^n q\; q^2}{(2\pi)^n}=\frac{2}{\varepsilon}-\frac{1}{2}-\gamma+\ln(4\pi)-\ln(M^2)+order(\epsilon) \tag{A12}$$

where γ is an Euler constant. Fortunately, the infinite part of **(A12)** is independent of x, the momentum fraction carried by the up quark, so it does not contribute to the ratio of rates (recall that the hard scattering amplitude is integrated against the meson wave functions which are normalized to unity). Similar;y, the finite terms in **(A12)** can be ignored since all the x-dependence is contained in M^2 . Defining

$$J(a)=\int_0^1 dy\int_0^{1-y} dz\ \ln(z^2+ayz) \qquad \textbf{(A13)}$$

the amplitude for hard gluon exchange between the up and bottom quarks reduces to

$$\frac{8}{9}\frac{g^2}{16\pi^2}[\bar{u}(p_c)(1+\gamma_5)\gamma_\mu[I_{2,0}(a)+aI_{1,1}a)+J(a)]u(p_b)] \qquad \textbf{(A14)}$$

$$x\ [\bar{u}(p_d)(1+\gamma_5)\gamma^\mu v(p_u)]$$

The amplitudes corresponding to the remaining three diagrams in **Figure 1** are calculated in a similar fashion. For gluon exchange between the down and bottom quark we obtain

$$\frac{8}{9}\frac{g^2}{16\pi^2}[\bar{u}(p_c)(1+\gamma_5)[\frac{\not{p}_b}{m_b}I_{2,0}(b)+\frac{\not{p}_d}{m_b}I_{1,1}(b)+\frac{1}{4}\ J(b)]u(p_b)]$$

$$x\ [\bar{u}(p_d)(1+\gamma_5)\frac{\not{p}_b}{m_b}v(p_u)] \qquad \text{(A15)}$$

where $b=\frac{2p_b p_d}{m_b^2}$.

Although we computed the amplitudes for each diagram separately, gluon exchange between the light quarks and the charm quark is completely analogous to the case of the bottom quark if we let

$$m_b \rightarrow m_c$$

$$p_b \rightarrow -p_c$$

Explicitly, substituting the kinematic variables

$$c=\frac{-2p_c \cdot p_d}{m_c^2}$$

$$d=\frac{-2p_c \cdot p_u}{m_c^2} \qquad \text{(A16)}$$

for a, b in **(A14, A15)** yields the full amplitude. The kinematic variables

a, b, c, and d can be straightforwardly related to the parameters x and r where

$$p_u = x p_\pi$$
$$r = m_c / m_b \tag{A17}$$

These relationships are given in equation (132).

What remains is to relate these amplitudes computed in the full theory QCD to the expession for the heavy meson transition in terms of the hard scattering amplitude $T_8^{(s)}$ (equation (112)). One first makes the identification:

$$\bar{u}(p_d)(1+\gamma_5)\gamma^\mu v(p_u) \rightarrow f_\pi \, p_\pi^\mu \, \phi_\pi \tag{A18}$$

We now use the approximations of the heavy quark effective theory and re-express the pion (or pho) momentum in terms of the heavy quark momenta:

$$p_\pi^\mu = m_b v^\mu - m_c v'^\mu \tag{A19}$$

Further, we identify the heavy quark spinors with the heavy quark fields of the effective theory:

$$u(p_b) \rightarrow h_b^v$$

$$\bar{u}(p_c) \rightarrow \overline{h_c^{v'}} \qquad \textbf{(A20)}$$

Factors of $\gamma_\mu v^\mu$ and $\gamma_\mu v'^\mu$ act on the heavy quark fields according to the constraint equations:

$$\not{v} h_b^v = h_b^v$$

$$\overline{h_c^{v'}} \not{v}' = \overline{h_c^{v'}} \qquad \textbf{(A21)}$$

The contribution to $T_8^{(s)}$ from the amplitude **(A14)** is now easily deduced. Ignoring the γ_5 piece (this contributes to $T_8^{(p)}$, the hard scaterring amplitude for the excited heavy meson decays) we have

$$\frac{2\alpha_s(m_b)}{9\pi}[aI_{1,1}(a)+I_{2,0}(a)+\tfrac{1}{2}J(a)] \tag{A22}$$

as the contribution from up- bottom gluon exhange, where $\alpha_s(m_b)=\frac{g^2}{4\pi}$.

For down-bottom gluon exchange, massaging the amplitude **(A15)** into a form which corresponds to equation **(112)** is only slightly more invloved. After performing the kinematic manipulations, we extract

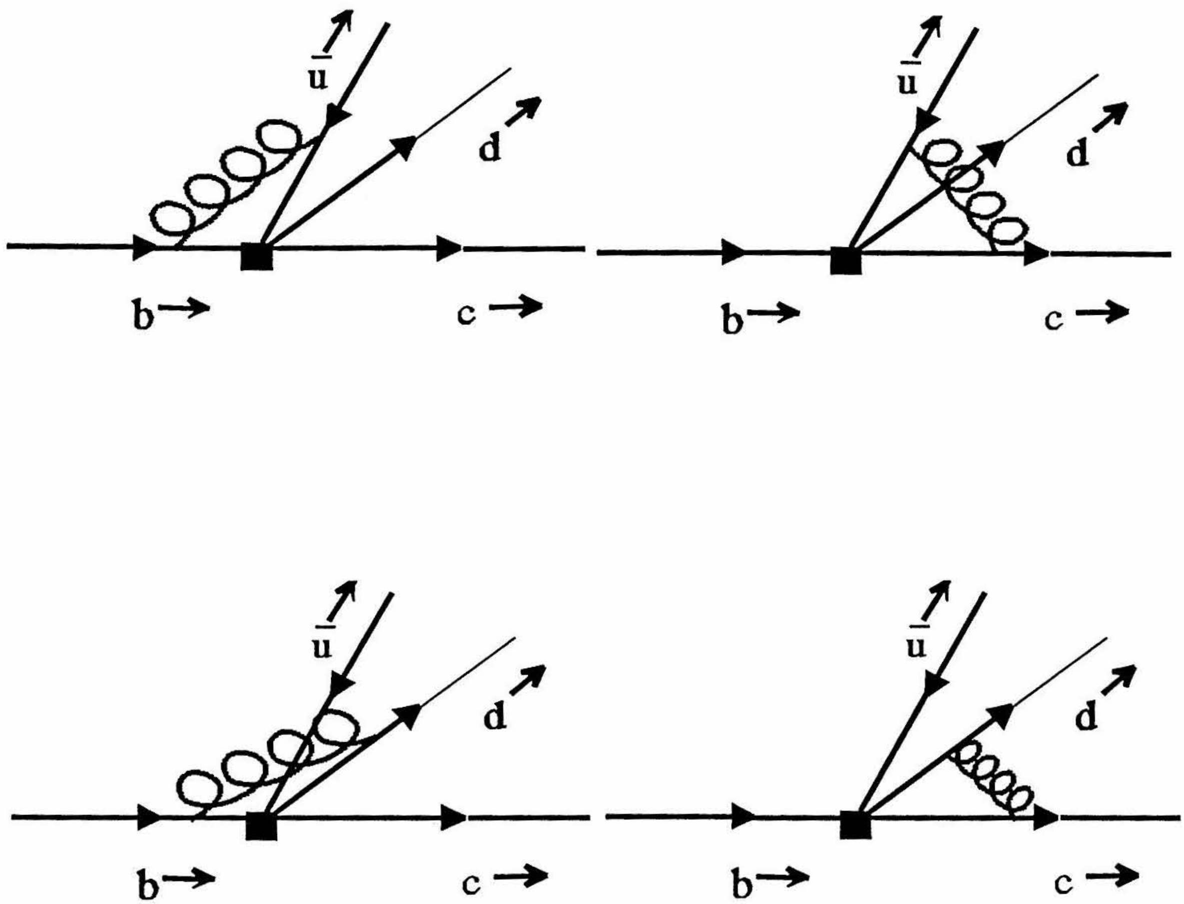
$$\frac{2\alpha_s(m_b)}{9\pi}[-\tfrac{1}{2}bI_{1,1}(b)-\tfrac{1}{2}(1+r)I_{2,0}(b)+\tfrac{1}{8}J(b)] \tag{A23}$$

as the contibution to $T_8^{(s)}$. Finally, the full amplitude is obtained by substituting c, d

for a, b in **(A22,A23)** and adding the four contributions together. The result

is equation **(131)**.

Figure 1



Feyman diagrams contributing to $T_8^{(s)}$ and $T_8^{(p)}$ at order $\alpha_s(m_b)$.

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