Birkhoff Periodic Orbits, Aubry-Mather Sets, Minimal Geodesics and Lyapunov Exponents

Thesis by

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Abstract

Aubry-Mather theory proved the existence of invariant circles and invariant Cantor set (the ghost circles) for the area-preserving, monotone twist maps of annulus or of cylinders. We are interested in higher dimensional systems. The celebrated KAM theorem established the existence of invariant tori for small perturbations of integrable Hamiltonian systems with nondegenerate Hamiltonian functions, but said nothing about the missing tori. Bernstein-Katok found the Birkhoff periodic orbits, which are viewed as the traces of missing tori, for the system in the KAM theorem but under the stronger condition that the Hamiltonian function is convex. We find the "isolating block", a structure invented by Conley and Zehnder, to demonstrate the existence of Birkhoff periodic orbits for the KAM system.

In the second part, we wanted to establish the existence of minimal closed geodesic which is hyperbolic on the surface of genus greater than one. There is strong evidence that such geodesics exist. We find a curvature condition for the minimal closed geodesic, thus furnishing further evidence.

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Part I: Birkhoff Periodic Orbits for the Monotone Twist Mappings with Non-convex Generating Functions

§1 Introduction: The Motivation

1.1 Poincaré's Last Geometric Theorem and Periodic Orbits of Annulus Mappings. We start with the statement of so called Poincaré's Last Geometric Theorem,taken from Arnold's book [Arn1]:

Suppose that we are given an area-preserving homeomorphic mapping of the planar circular annulus to itself. Assume that the boundary circles of the annulus are turned in different directions under the mapping. Then this mapping has at least two fixed points.

A mapping satisfying the above theorem rises in the following considerations [AA1].

Let $T : (r, \theta) \longmapsto (r, \theta + \lambda(r))$ be an area-preserving "integrable" map, where $(r, \theta + \lambda(r))$, $r \geq 0$, $0 \leq \theta < 2\pi$ is the planar polar coordinate. More often it is called "action-angle" coordinate. All r =constant are the invariant circles of the mapping T. On each invariant circle $r = r_0 > 0$, the restriction map $T\mid_{r=r_0}$ is a circle rotation. Let r_0 satisfy:

(1) $$\lambda(r_0) = 2\pi\frac{m}{n}, \qquad \lambda'(r_0) \neq 0,$$

where m and n are relatively prime positive integers. Then *the composition map*

$(T|_{r=r_0})^n$ of $T|_{r=r_0}$ is an identity mapping and all points on the circle $\Gamma : r = r_0$ are periodic points of $T\ |_{r=r_0}$ of period n.

Note that $\lambda(r_0) = 2\pi\frac{m}{n}$ is the rotation number for the circle map $T\ |_{r=r_0}$. Let us call $r = r_0$ a rational circle for the map T, since its rotation number is a rational multiple of 2π. Consider an **area-preserving small perturbation** T_ϵ of the map T, and consider two T-invariant circle Γ^+ and Γ^- with rotation numbers λ^+ and λ^- respectively, $\lambda^+ > \lambda > \lambda^-$. By the fundamental KAM theorem, one can find T_ϵ-invariant closed curves Γ_ϵ^+ and Γ_ϵ^-, such that $T\ |_{\Gamma_\epsilon^+}$ and $T\ |_{\Gamma_\epsilon^-}$ has rotation number λ_ϵ^+ and λ_ϵ^- respectively, with $|\ \lambda_\epsilon^+ - \lambda^+\ | \ll 1$ and $|\ \lambda_\epsilon^- - \lambda^-\ | \ll 1$. Thus, we obtain the following pictures:

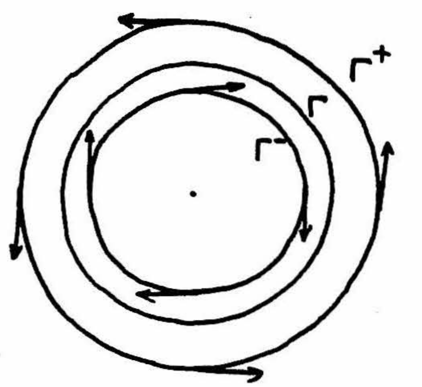

Fig 1.a

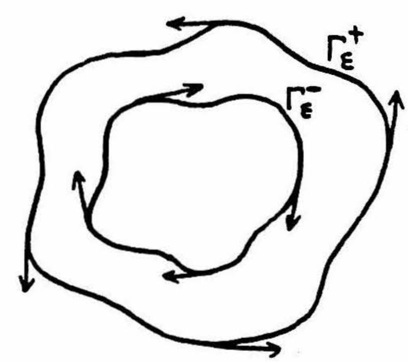

Fig 1.b

The arrows indicate the directions in which the points are moved under the action of T^n and T_ϵ^n respectively. Now the region bounded by Γ_ϵ^+ and Γ_ϵ^- and the mapping T_ϵ^n satisfy the conditions in the theorem. Poincaré proved that, if ϵ is small enough, then T_ϵ has **even** numbers of periodic points of period n, hence T_ϵ has at least **two** periodic orbits of period n. Actually Poincaré proved his Last Geometric Theorem only in this near integrable case. It is G.D.Birkhoff [Bir1] who furnished the first proof of the theorem.

The problem was originated from the study of the three body problem.

The influence of this model problem - the investigation of the small perturbation of an integrable map near an elliptic fixed point - has been far reaching, although it received little attention for quite long. Modern "chaos" theory actually started from here. After proving the existence theorem, Poincaré tried to further analyse the qualitative nature of the perturbation mapping. He was led to the following well known picture which was so complicated that he was "not even attempting to draw" [AA1]:

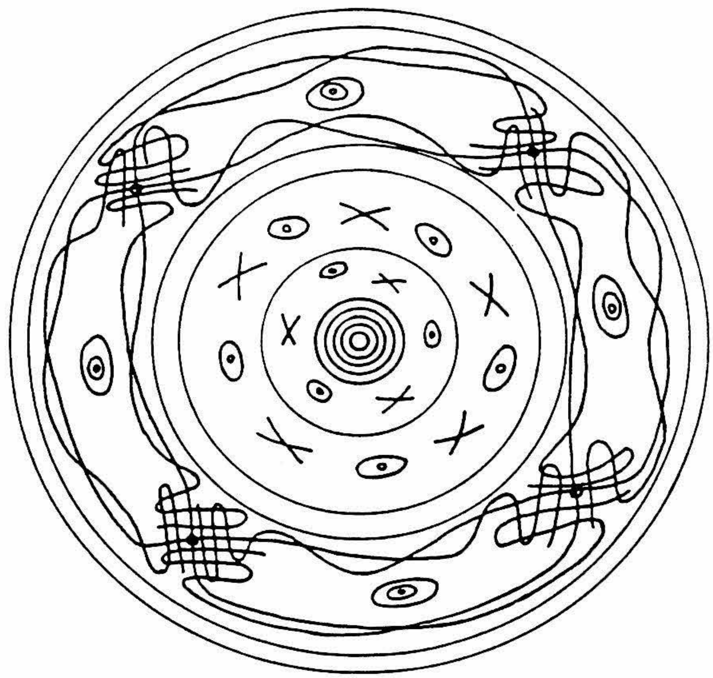

Fig 2

It is not exaggerated to say that many important theorems are detailed analysis of this picture and yet there are still important unsolved problems in this picture.

But it is not in our interest to describe the overall historical development along that line. We will focus on three closely related aspects: the periodic orbits of the annulus mappings, KAM theorem and Aubry-Mather theory.

We start with a look at the developments on the fixed points (periodic points of period one) of the annulus mappings. There are many research papers in this area concerning the generalization of the Poincaré's Last Geometric Theorem. I will only mention one of them. Instead I will pay more attention to another more recent development.

Recall that the condition that the boundary circles are turned in **different** directions under the map is called the **boundary twist.**

Most recent work on the annulus mapping are interested in **monotone twist** mappings [Mat1]. Assume that $f : A = S^1 \times [0,1] \longmapsto A$ is an annulus homeomorphism, let $F : R \times [0,1] \longmapsto R \times [0,1]$ be a lift of f to the universal cover, $F(x,y) = (F_1(x,y), F_2(x,y))$. f is called **monotone twist** if for any $x, F_1(x,y)$ is a strictly monotone function of y (for example, if f is C^1, and $\frac{\partial F_1}{\partial y}$ never vanishes). Geometrically it means that all the points are moved in the same direction under the map, the amounts of the rotation changes strictly monotonically for the points along any segment in the radial direction.

We discuss monotone twist map. The restricted maps to the boundaries are the circle maps, hence we have two rotation numbers, say $\alpha_1(f)$ and $\alpha_2(f)$ with $\alpha_1(f) < \alpha_2(f)$. The following is a part of the landmark paper of J.Mather [Mat2]:

Mather's Theorem. Let $f : A \longmapsto A$ be an orientation preserving, boundary component preserving and **area-preserving monotone twist** homeomorphism, $\frac{p}{q} \in [\alpha_1(f), \alpha_2(f)]$ where p and q are relatively prime integers, then there is a Birkhoff periodic orbit of type (p,q).

An orbit is called Birkhoff if the mapping preserves the order of the orbit. The precise meaning will come later. The order preserving property is one of the most important features of the Aubry-Mather set, but it is a typical low-dimension phenomenon.

The area-preserving condition is weakened by Katok [Kat1]. Katok also observed that there are **two** Birkhoff periodic orbits. Furthermore, his proof is more elementary and more geometric.

Katok's Generalization. Let $f : A \longmapsto A$ be an orientation preserving, boundary component preserving monotone twist homeomorphism, **preserving a measure** positive on open sets, $\frac{p}{q} \in [\alpha_1(f), \alpha_2(f)]$, then there are at least **two** Birkhoff periodic orbits of type (p, q).

The measure preserving condition was further weakened by the **graph intersection property** [Ber1], [Hal1]. We say that f has the graph intersection property if for every continuous $h : S^1 \longmapsto [0, 1], f(\Gamma)$ intersects with Γ, where $\Gamma \in A$ is the graph of h. We can state:

Bernstein-Hall's Generalization. Let $f : A \longmapsto A$ be an orientation preserving, boundary component preserving monotone twist homeomorphism with the **graph intersection** property, $\frac{p}{q} \in [\alpha_1(f), \alpha_2(f)]$, then f has at least a Birkhoff periodic orbit of type (p, q).

Finally we mention the work of Carter [Car1] and Franks [Fra1]. They are somewhat along a quite different line. In proving the following theorem, Franks introduces the notion of chain recurrence which was developped by Charles Conley [Con1], reducing both the geometric "twist" condition and the area-preserving condition (these two are the most essential conditions in recent twist map theorems):

Franks' Generalization of Poincaré-Birkhoff Fixed Point Theorem. Suppose $f : A \longmapsto A$ is a homeomorphism homotopic to the identity, let $pr_1 : R \times [0,1] \longmapsto R$ be the natural projection. If for some $(x, y) \in R \times [0,1]$,

$$\liminf_{n\to\infty} \frac{1}{n}[pr_1 F^n(x,y) - x] \le \frac{p}{q} \le \limsup_{n\to\infty} \frac{1}{n}[pr_1 F^n(x,y) - x] \tag{2}$$

then f has a periodic orbit orbit with rotation number $\frac{p}{q}$.

No condition related to the "twist" condition and to the area-preserving condition appears in the theorem. A chain transitive set is guaranteed by the existence of a point satisfying (2). The final remark is that the periodic orbit in this theorem does not need to be order preserving.

1.2 KAM Theorem and Aubry Mather Theory. By far the most remarkable property of small perturbation of completely integrable Hamiltonian system is the preservation of invariant tori corresponding to irrational frequency vectors which are not too well approximable by rationals. "A simple and novel idea", gave "a solution of a 200 year-old problem" [Arn2].

The "200 year-old problem" is the stability problem of the classical mechanics. Consider the n-body problem in three-dimensional space attracting with each other according to the Newton's law. Let $x_k = (x_k^1, x_k^2, x_k^3)$ describe the position of k-th point with mass m_k, then the motion can be described by the following equations:

$$m_k \frac{d^2 x_k}{dt^2} = \frac{\partial U}{\partial x_k}, k = 1, 2, ..., n$$

and

$$U = \sum_{1 \leq k < l \leq n} \frac{m_k m_l}{|x_k - x_l|}.$$

The equations are defined only for $|x_k - x_l| \neq 0$ for all $k \neq l$. If $r_{kl} = |x_k - x_l|$ approaches to zero we speak of a collision. The stability problem is concerned with the behavior of the solutions of the equations for an **infinite** time interval. Since the solar system can be regarded as a small perturbation of a completely integrable system, the behavior of the solutions of the equations for a **finite** time interval causes no problem. When the time interval becomes infinite, the effect of the small perturbation cannot be ignored. The specific questions are: Are there solutions which do not experience collisions and do not escape? Are there solutions for which

$$\max_{k<l}(r_{kl}, r_{kl}^{-1})$$

is bounded for all time t? If the answer is yes, how many such solutions are there? do they form an open set or at least a set of positive measure in the phase space?

The answers to these questions are yes. And this is the achievement of the KAM theorem. Let me quickly describe the chronicle development of this theorem. The original idea was due to A.N.Kolmogorov, and the first rigorous proof was given by his student V.I.Arnold some eight or nine years later [Arn2]. Arnold needed the Hamiltonian function to be analytic, a smooth condition later played a crucial role in the study of Hamiltonian systems. J.Moser [Mos1] , in an important special case, abandoned the analytic condition and required the Hamiltonian function to be only of class C^{333}. This huge differentiability was brought down to C^4 by

Von Rüssmann [Rüs1] and to C^3 by M.Herman [Her1,2]. A counterexample was provided by M.Herman if the C^2-differentiability is violated.

We state two versions of this theorem, one is pictorial and rough, and another is analytic and technical which is very suitable for the setting of our work.

KAM Theorem (The invariant tori in a perturbed system) [Arn1]. If an unperturbed system is nondegenerate, then for sufficiently small conservative Hamiltonian perturbations, most non-resonant invariant tori do not vanish, but are only slightly deformed, so that in the phase space of the perturbed system, too, there are invariant tori densely filled with phase curves winding around them conditionally-periodically, with the number of independent frequencies equal to the number of degrees of freedom.

These invariant tori form a majority in the sense that the measure of the complement of their union is small when the perturbation is small.

Before stating the technical one, we discuss some important conditions needed. Let $T^n = R^n/Z^n$ be the n-dimensional torus, let $A^n = T^*(T^n) \cong T^n \times R^n$ be the cotangent bundle of T^n, which has a canonical action angle coordinate (ϕ, r), $\phi = (\phi_1, ..., \phi_n)$, $r = (r_1, ..., r_n)$. A^n also has a natural symplectic form $\Omega = -d\nu$, where $\nu = \sum_{j=1}^n r_j d\phi_j$ is the Liouville 1-form.

A C^1-diffeomorphism F of A^n is called symplectic if $F^*\Omega = \Omega$; it is called exactly symplectic if $F^*\nu - \nu$ is exact. Write

$$DF(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}$$

here $a(x)$, $b(x)$, $c(x)$ and $d(x)$ are all $n \times n$ matrices, then F is symplectic if and only if [Her3]:

$$(3) \qquad [DF(x)]^{-1} = \begin{pmatrix} (d(x))^t & -(b(x))^t \\ -(c(x))^t & (a(x))^t \end{pmatrix}$$

where the superscript t means transpose. F is called **monotone** if $det(b(x)) \neq 0$, for any $x \in A^n$. This is the most important condition we will need. It is a generalization of the monotone twist condition in annulus maps, and it is equivalent to the non-degeneracy condition for the Hamiltonian function in the KAM theorem.

We state another important condition (although we will not need this one). For $x \in T^1 = R^1/Z^1$, let $\| x \|_a = \inf_{p \in Z} |\tilde{x} + p|$, where $\tilde{x}$ is the lift of x to R^1. For $k = (k_1, ..., k_n) \in Z^n$, let $|k| = \sum_{i=1}^n |k_i|$. We say that $\alpha = (\alpha_1, ..., \alpha_n)$ satisfies the Diophantine condition if there exist constants $\gamma > 0, \beta \geq 0$ such that for all $k \in Z^n - \{0\}$

$$(4) \qquad \| \sum_{i=1}^n k_i \alpha_i \|_a \geq \gamma |k|^{-n-\beta}.$$

Theorem of Invariant Torus [Her3]. Let $L(\phi, r) = (\phi + l(r), r)$ be a completely integrable **monotone** (i.e., $det(l(r)) \neq 0$) C^∞-diffeomorphism of A^n. Assume that $r_0 \in R^n$ is such that $l(r_0) = \alpha$ satisfies the Diophantine condition. Then there exists $k_0 \in R^*_+ = \{x \in R : x > 0\}$ and a neighborhood V_L of L in $C^{k_0}(A^n, A^n)$, such that for any **exactly symplectic** $F \in V_L$ of class C^∞, there exist a C^∞-embedding $\Phi : T^n \longmapsto A^n$ and a C^∞-diffeomorphism $H : T^n \longmapsto T^n$ satisfying:

-The image $T = \Phi(T^n)$ is an invariant torus for F, T is homotopic to $r = 0$;

-$H(\mathbf{0}) = \mathbf{0}$.

-Φ and H induce a rotation of the torus T^n:

$$H^{-1} \circ \Phi^{-1} \circ F \circ \Phi \circ H = R_\alpha$$

where $R_\alpha(\phi) = \phi + \alpha \pmod{Z^n}$.

Now we turn to the Aubry-Mather theory. To link it to the KAM theorem, we quote the second unsolved problem raised in Arnold's article [Arn2]:

2°. Large perturbations. Quasi-periodic motions are observed only for very small values of the perturbation parameter ϵ. Do they occur also for large perturbations? For the n-body problem, with any values for the masses, does there exist a set of initial conditions of positive measure giving rise to bounded motions?

The major progress was made about twenty years later when J.Mather [Mat2] discovered the quasi-periodic motion for the **area preservation monotone twist** map of annulus and at the same time two physicists S.Aubry and P.Y.Le Daeron [AL1] developped the theory of minimal energy configurations for the Frenkel-Kontorova model.

Mather's work was also motivated from the work of a physicist I.C.Percival [Per1], who found the quasi-periodic orbit **numerically**. Percival proposed a variational method for finding invariant tori and **Cantori**. This idea is really novel since a traditionally variational principle is employed to find the orbits. Mather successfully realized the idea and rigorously proved the following theorem:

Theorem [Mat2]. Any area preserving monotone twist homeomorphism of the annulus has quasi-periodic orbits of all frequencies belonging the rotation interval.

The precise meaning of "quasi-periodic orbits" will become clear in the next subsection. Let us now describe the work of Aubry and Le Daeron [AL1]. Consider the generalized Frenkel-Kontorova model with the energy

$$\phi(\{u_k\}) = \sum_n L(u_n, u_{n-1}), \tag{5}$$

the model describes a one dimensional chain of atoms which are coupled by springs. Each infinite sequence $\{u_k\}$ describes the state of the chain, where u_k is the abcissa of the k-th atom. The neighboring atoms are coupled by the potential $L(x, y)$ which is a continuous function satisfying

1. There exists a constant B such that

$$L(x, y) \geq B$$

for all x and y;

2. $L(x, y)$ has the period $(2a, 2a)$:

$$L(x + 2a, y + 2a) = L(x, y)$$

for all x and y;

3. $L(x, y)$ is C^2 and there is a constant $C > 0$ such that

$$-\frac{\partial^2 L(x, y)}{\partial x \partial y} > C > 0.$$

Definition. A minimum energy configuration (m.e. configuration, in short) is a sequence $\{u_k\}$ such that any finite change δ_n of a finite set of atoms necessarily increases the energy:

$$\sum_{n=N'}^{N} L(u_n + \delta_n, u_{n-1} + \delta_{n-1}) \geq \sum_{n=N'}^{N} L(u_n, u_{n-1}) \tag{6}$$

for any $N' < N$ and any choice of δ_n with $\delta_n = 0$ for $n < N'$ and $n \geq N$.

We state only a small part of Aubry-Le Daeron's results, which on the one hand is equivalent to Mather's existence theorem, and on the other hand is related to theorems on minimal geodesics on the torus.

Theorem of m.e.configurations. 1. For any value of l, there exists m.e.configurations $\{u_k\}$ of model (5) such that the limit

$$\lim_{|n-m|\to\infty} \frac{u_n - u_m}{n-m} = l \tag{7}$$

exists. Conversely, the above limit is defined for any m.e.configuration of model (5)

2.The set of m.e.configurations is closed for the weak topology, i.e., for any sequence $\{u_n^i\}$ of m.e.configurations such that for each n,

$$\lim_{i\to\infty} u_n^i = u_n,$$

then $\{u_n\}$ is an m.e.configuration.

There are many results on the structure of the m.e.configurations and about the ground state configurations, [AL1], [Au1]. We will not go any further. The relation between Mather's theorem and Aubry-Le Daeron's is as follows. Through the Legendre transformation, the problem of finding the m.e.configuration in the Frenkel-Kontorova model becomes the problem of finding the invariant set for an area-preserving, monotone twist map of an infinite cylinder. Since the boundaries of the cylinder are at the infinity, the rotation interval becomes $(-\infty, \infty)$, this is why one can find m.e.configuration for **any** l, such that (7) holds. Basically, Aubry-Le Daeron's theory is the Lagrangian formulation of Mather's.

Now we come back to Arnold's problem. Consider the set of all area-preserving C^1-diffeomorphisms of the cylinder. The set can be regarded as the collection of all Hamiltonian systems on the cylinder. For the subcollection of all area-preserving monotone twist diffeomorphisms, the Aubry-Mather theory well addresses the existence part of Arnold's question. In addition, it gives an elegant explanation on the missing tori (dead tori or ghost tori, as some people call it) and shows **how** the invariant tori disintegrate when the perturbation becomes larger and larger. Let me elaborate this point by looking at the standard maps [Mat3].

A standard map is an area-preserving monotone twist map, depending on a parameter k, of the infinite cylinder, $f_k : S^1 \times R^1 \longmapsto S^1 \times R^1, f_k(x,y) = (x', y')$ here

$$x' = x + y + \frac{k}{2\pi} \sin 2\pi x \pmod 1 \tag{8}$$
$$y' = y + \frac{k}{2\pi} \sin 2\pi x.$$

Let us observe the change in the phase space as k increases. First let $k = 0$, we have a completely integrable Hamiltonian system. The phase space-the cylinder-is foliated by the invariant tori y =constant. Next, let $k > 0$ and $|k| \ll 1$, then we obtain the small perturbations of the integrable system. The KAM theorem says that most invariant circles are only slightly deformed and that the majority of the phase space is covered by topologically non-trivial invariant closed curves. Between the invariant closed curves are the zones of the instability in which the system may display extremely complicated behavior. Mather [Mat3] showed that for $|k| > \frac{4}{3}$, all invariant closed curves disappear. Numerical studies (see the references in [Mat3]) suggest that there is a number $k_0 \approx o.97$ such that f_k admits a homotopically nontrivial circle when $k \leq k_0$ and that f_k has no such circle when $k > k_0$. There is no rigorous proof of this conjecture. Any way let $k > \frac{4}{3}$, then the phase space is full of holes, and the stability breaks down [Mos2]. But the Aubry-

Mather theory guarantees that there are "many" (not in the sense of measure!) "broken circles"- the Cantori, which are invariant under the perturbed map, the motions on these Cantori are periodic or quasi-periodic. The changes of the phase portraits are symbolically depicted in fig.3.

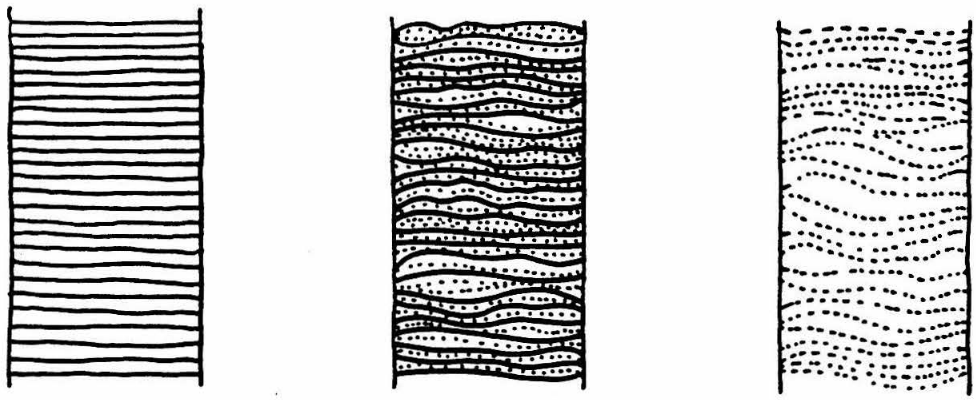

Fig3.a $k=0$ Fig3.b $0<k\ll 1$ Fig3.c $k>\frac{4}{3}$

Finally, we remark that although the Aubry-Mather theory established the existence of "many" invariant Cantori, which are the counterpart of the invariant tori in the KAM theorem, the meaning of "many" here is referred to the fact that for every admissible rotation number, there is at least a corresponding invariant set. It is still an open problem that whether the union of all Aubry-Mather sets has positive Lebesgue measure.

1.3 Aubry-Mather Set as Limit of Periodic Orbits. We discuss the connection between the Birkhoff periodic orbits and the Aubry-Mather sets for an area-preserving monotone twist map of the annulus, thus unify the preceding seemingly irrelevant theorems.

The crucial observation is made by A.Katok [Kat1].

We want to describe this connection in a technical fashion, since we already have good pictures through the previous discussions. As before, $A = S^1 \times [0,1]$ is the annulus, $f : A \longmapsto A$ is an area-preserving monotone twist map, and $F : \tilde{A} \longmapsto \tilde{A}$ is the lift of f where $\tilde{A} = R \times [0,1]$ is the universal cover of A. A closed f-invariant set $E \subset A$ is called an Aubry-Mather set if (1) E intersects every interval $\{\phi\} \times [0,1]$ at most at one point, i.e., E =graph(Φ) where Φ is a continuous function defined on a **closed** subset K of S^1 with values in $[0,1]$; (2) F preserves the order of the covering set $\tilde{E} \subset \tilde{A}$ of E, i.e., for $(x_1, y_1), (x_2, y_2) \in \tilde{E}$, $x_1 < x_2$ implies $x_1' < x_2'$ where $F(x_i, y_i) = (x_i', y_i')$, $i = 1, 2$. If the set E consists of a single orbit, and satisfies this property, then it is called a Birkhoff orbit.

It can be seen that $f|_E$ is topologically conjugate by an order preserving homeomorphism to a restriction of a homeomorphism of the circle to the set K. In particular the rotation number $\rho(E)$ is defined up to an integer.

We shall simply call Aubry-Mather set Mather set. From the definition one sees that every closed subset of a Mather set is again a Mather set, thus we can speak of a minimal Mather set. It follows from the standard Poincaré-Denjoy theory of circle homeomorphism that there are exactly three types of minimal Mather sets. Namely:

1. $\rho(E) = \frac{p}{q}$ is a rational number, and E is a Birkhoff periodic orbit of type (p, q);

2. $\rho(E)$ is irrational, and E is a circle;

3. $\rho(E)$ is irrational, and E is a Cantor set.

Considering the space of all closed subsets of A equipped with the Hausdorff topology, Katok proved the following important facts:

Proposition. (a) The set of all Mather sets for a twist homeomorphism (no area-preserving property is needed) is closed in the Hausdorff topology;

(b) The rotation number $\rho(E)$ for the Mather set E is continuous in the Hausdorff topology.

This proposition implies the following important theorem:

Approximation Theorem. If f has Birkhoff periodic orbits of type (p, q) for **every** admissible rational number $\frac{p}{q}$ then f also possesses a minimal Mather set with any irrational rotation number belonging to the twist interval.

Thus, the proof of the existence of the invariant set in the Mather's theorem is reduced to the proof of the existence of Birkhoff periodic orbits of type (p, q) for any rational number $\frac{p}{q}$ belonging to the twist interval $[\alpha_0, \alpha_1]$. It is this observation that enables Katok to prove Mather's theorem in an elementary way.

More importantly, this scheme for approximating Cantorus by a sequence of periodic orbits is suited to numerical experiment; many computational investigations of Cantori depend on approximation by periodic orbits, see, for example [MMP1], [MP1], [Gre1]. Such numerical experiments are very helpful in studying the higher dimensional systems. Based on this scheme, M.Muldoon [Mul1] carried out an ambitious computation of numerical approximation to Birkhoff periodic orbits in search of the Cantori for some four-dimension systems, and got some interesting pictures.

All his examples use "standard-like" perturbations. Let $F_\epsilon : T^2 \times R^2 \longmapsto$

$T^2 \times R^2, F_\epsilon(\mathbf{x}, \mathbf{p}) = (\mathbf{x}', \mathbf{p}')$, here

$$\text{(9)} \qquad \begin{aligned} \mathbf{x}' &= \mathbf{x} + \mathbf{p} - \epsilon \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}) \\ \mathbf{p}' &= \mathbf{p} - \epsilon \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}) \end{aligned}$$

and $\mathbf{x} = (x_0, x_1)$, $\mathbf{p} = (p_0, p_1)$. He chooses three types of functions for $V(\mathbf{x})$:

$$V(\mathbf{x}) = \begin{cases} V_{trig}(\mathbf{x}) = -\frac{1}{M_{trig}}\{\frac{1}{2}(\sin 2\pi x_0 + \sin 2\pi x_1) + \sin 2\pi(x_0 + x_1)\} \\ V_{poly}(\mathbf{x}) = -\frac{1}{M_{poly}}\{[x_0^2(1 - x_0)^2(x_0 - \frac{3}{4})(\frac{1}{4} - x_0)][x_1^2(1 - x_1)^2]\} \\ V_{ff}(\mathbf{x}) = -\frac{1}{2}\{\frac{1}{2}[c(x_0) + c(x_1)] + c(x_0 + x_1)\} \end{cases}$$

with

$$c(x) = \begin{cases} 1 - 24x^2 + 32x^3, & \text{if } x \bmod 1 \le \frac{1}{2} \\ 9 - 48x + 72x^2 - 32x^3, & \text{if } x \bmod 1 \ge \frac{1}{2}, \end{cases}$$

where M_{trig} and M_{poly} are chosen so that $\max_{\mathbf{x} \in T^2} |V(\mathbf{x})| = 1$. V_{ff} is a polynomial approximation to a potential originally introduced as a model of star motion in elliptical galaxies [Fro1].

The map (9) has the generating function

$$\text{(10)} \qquad H_\epsilon(\mathbf{x}, \mathbf{x}') = \frac{1}{2} \| \mathbf{x}' - \mathbf{x} \|^2 - \epsilon V(\mathbf{x})$$

which is a discrete version of Hamiltonian function. The relation between H_ϵ and F_ϵ is: $F_\epsilon(\mathbf{x}, \mathbf{p}) = (\mathbf{x}', \mathbf{p}')$ if and only if

$$\text{(11)} \qquad \mathbf{p}' = \frac{\partial H_\epsilon}{\partial \mathbf{x}'}, \qquad \mathbf{p} = -\frac{\partial H_\epsilon}{\partial \mathbf{x}}.$$

To find a periodic orbit of period q, it is standard to consider an action $L_{\omega,q}$ on the space of periodic states:

$$\text{(12)} \qquad X_{\omega,q} = \{(\mathbf{x}_0, \mathbf{x}_1, ..., \mathbf{x}_{q-1}, \mathbf{x}_q) : \mathbf{x}_q = \mathbf{x}_0 + \omega\}$$

with the action

$$L_{\omega,q}(\mathbf{x}_0, \mathbf{x}_1, ..., \mathbf{x}_{q-1}, \mathbf{x}_q) = \sum_{j=0}^{q-1} H_\epsilon(\mathbf{x}_j, \mathbf{x}_{j+1}), \tag{13}$$

where $\omega \in Z^2$, $\frac{\omega}{q}$ is called the rotation vector, it is obviously a generalization of the rotation number.

In order to find the periodic orbit with rotation vector $\frac{\omega}{q}$, it is equivalent to find the critical point of $L_{\omega,q}$, i.e. , $(\mathbf{x}_0, \mathbf{x}_1, ..., \mathbf{x}_{q-1}, \mathbf{x}_q)$ such that

$$\frac{\partial L_{\omega,q}}{\partial \mathbf{x}_j} = \frac{\partial H_\epsilon}{\partial \mathbf{x}'}(\mathbf{x}_{j-1}, \mathbf{x}_j) + \frac{\partial H_\epsilon}{\partial \mathbf{x}}(\mathbf{x}_j, \mathbf{x}_{j+1}) = 0 \tag{14}$$

for all j. If $(\mathbf{x}_0, \mathbf{x}_1, ..., \mathbf{x}_{q-1}, \mathbf{x}_q)$ is a critical point, then $(\mathbf{x}_1, \mathbf{p}_1)$ where $\mathbf{p}_1 = \frac{\partial H_\epsilon}{\partial \mathbf{x}}(\mathbf{x}_0, \mathbf{x}_1)$, is a peoridic point for F_ϵ with the rotation vector $\frac{\omega}{q}$.

It is interesting to see that, for all these three types of small perturbations, Muldoon's numerical investigations suggest the existence of the Cantori in these four-dimensional systems. It also suggests certain regular behavior of the Cantori which has long been an interesting problem in this field. One thing I want to point out is that in all three types of small perturbations, the unperturbed systems have convex generating functions, thus fall into the class studied by Bernstein-Katok.

1.4 An Outline of Bernstein-Katok's Result. The importance of finding periodic orbits is now clear. The next thing is to show the existence of the periodic orbits with admissible rotation numbers and the regularity of these orbits. These are the work of Bernstein and Katok [BK1]. It is worthwhile to give a brief description of these results. Here instead of summarizing the results and sketching the ideas of proofs, we will pay more attention to the motivation of the work and to their connections with the Poincaré's Geometric Theorem, KAM Theorem and Aubry Mather Theory.

The main results are as follows: for a perturbed Hamiltonian system with n degree of freedom, if the Hamiltonian function is convex (KAM theorem needs only that the Hamiltonian function to be non-degenerate), then there exists n distinct periodic orbits with any admissible rational rotation vector near the corresponding torus of the unperturbed system.

Let us view this result from two different points. Since the KAM theorem is applicable, we know that most invariant tori prevail. The vanishing ones are among those whose rotation vectors violate the Diophantine condition (4). It is clear from (4) that the rational vectors do not satisfy the Diophantine condition most. In other words, the unperturbed tori with rational rotation vectors, in some sense, are most easily broken under the perturbation. Inspired by the Aubry-Mather theory, one can ask the question: where do the missing tori go? the question that Bernstein and Katok ask themselves is: are there special sufficiently simple motions in the perturbed system which are similar to the periodic and quasi-periodic motions on the destroyed tori, and therefore can be viewed as "traces" or "ghosts" of those missing tori?

Next we recall how Poincaré proposed his Geometric Theorem. The picture of the Last Geometric Theorem was born when Poincaré was investigating the behavior of the perturbed system near an unperturbed rational circle. Under the perturbations, the rational circle is very likely to be disrupted. Does there exist any periodic orbit with the same rotation number of the unperturbed circle? This was the question then in Poincaré's mind. Actually this is exactly the set up of Bernstein-Katok's investigation.

We describe Bernstein-Katok's result more technically. Consider the cotangent bundle of an n-torus, $T^*(T^n) \cong T^n \times R^n = \{(\phi_1, ..., \phi_n, r_1, ..., r_n) : \phi_i \in R/Z, r_i \in R, i = 1, 2, ..., n\}$ with the symplectic 2-form:

$$\Omega = \sum_{i=1}^{n} d\phi_i \wedge dr_i.$$

Let $f_0 : T^n \times U \longmapsto T^n \times U$ be an integrable symplectic diffeomorphism, i.e., an Ω-preserving diffeomorphism of the form:

$$f_0(\phi, r) = (\phi + a(r), r), \qquad \phi \in T^n, r \in U,$$

where U is an open set in R^n. Let $F_0 : R^n \times U \longmapsto R^n \times U$ be a lift of f_0 to the universal cover, such that for any $x \in R^n, r \in U$,

$$F_0(x, r) = (x + a(r), r).$$

The symplecticity means that F_0 has a generating function H_0, i.e., $F_0(x, r) = (x', r')$ if and only if

$$r = \frac{\partial}{\partial x} H_0(x, x'), \qquad r' = -\frac{\partial}{\partial x'} H_0(x, x').$$

It is easy to see that H_0 depends only on the difference $x' - x : H_0(x, x') = h(x - x')$. We call the function h the Hamiltonian function of the unperturbed system.

The function h being non-degenerate is a crucial condition in the KAM theorem, it is equivalent to

(R) the mapping $a : U \longmapsto R^n$ is a regular injective map.

Bernstein-Katok required the following condition which is stronger than (R).

(C) h is strictly convex on $a(U)$, i.e., the Hessian of h at every point $\delta \in a(U)$ is a positive definite quadratic form.

Note that every r =constant is an invariant torus for f_0. If for some $r_0, a(r_0) \in R^n$ is a rational vector, then in analogy to the annulus mapping situation, we call

$r = r_0$ a rational torus. In this case $f|_{r=r_0}$ is a periodic map, with period q equals to the least common multiple of all the denominators of the components of $a(r_0)$. If $n = 1$, this is the case Poincaré considered.

Now consider small perturbations H, F, f of H_0, F_0, f_0 respectively. If f has a periodic point with period q, then the lift (x, r) of the periodic point satisfies $F^q(x, r) = (x + \omega, r)$ for some $\omega \in Z^n$. Remember $F_0^q(x, r) = (x + qa(r), r)$, the projection of (x, r) is a periodic orbit of f_0 if and only if $qa(r) = \omega$ is an integer vector . We now state in full the Bernstein-Katok's theorem.

Theorem. Let $a, f_0, F_0, H_0, h, f, F, H$ be described as above, h satisfies (C). Let $\omega = (\omega_1, ..., \omega_n) \in Z^n$, and let q be a positive integer such that $\omega_1, ..., \omega_n, q$ are relatively prime and $\frac{\omega}{q} \in a(U)$. Write $r_{\omega,q} = a^{-1}(\frac{\omega}{q})$. There exists a constant Δ depending only on f_0 but not on ω and q such that for any $\delta < \Delta$, $\| H - H_0 \|_{C^1} = \delta$, the map f has at least $n + 1$ different periodic orbits with rotation vector $\frac{\omega}{q}$ which lie completely inside the $C\delta^{\frac{1}{3}}$ neighborhood of the torus $T^n \times \{r_{\omega,q}\}$ and at least one of those orbits lies inside the $C\delta^{\frac{1}{2}}$ neighborhood of the torus. Here C depends only on the unperturbed map f_0.

When $n = 1$, the condition (C) becomes condition (R) or condition (1), one obtains the result that Poincaré proved.

We will study the case where h satisfies (R).

1.5 The Regularity of Aubry-Mather Sets. The regularity of the invariant circle and of the Aubry-Mather set has always been an important issue in the study of the annulus mappings. The earliest study of the regularity of the invariant closed curves is by Birkhoff [Bir2]. Extensive investigations were carried out by M.Herman [Her1] and A.Fathi [Fat1]. Let us have a taste of the regularity result by looking at a theorem by Birkhoff.

Birkhoff's Theorem. Let $f : S^1 \times R^1 \longmapsto S^1 \times R^1$ be an area preserving, orientation preserving and ends preserving (here ends preserving means that if $f(x,y) = (f_1(x,y), f_2(x,y))$ then $\lim_{y\to\pm\infty} f_2(x,y) = \pm\infty$)monotone twist C^1-diffeomorphism of a cylinder. Let U be an open subset of the cylinder, homeomorphic to $S^1 \times R^1$ and $S^1 \times (-\infty, a] \subset U \subset (-\infty, b]$ for some $a, b \in R$, $a < b$. Then the frontier of U in $S^1 \times R^1$ is the graph of a **Lipschitz** function $\mu : S^1 \longmapsto R^1$, i.e., $\overline{U} - U = \{(x, \mu(x)) : x \in S^1\}$.

The above is actually a version of the Birkhoff's theorem formulated by Mather [Mat3]. A corollary of Birkhoff's theorem is that, if X is an invariant, homotopically non-trivial circle in $S^1 \times R^1$, then X is the graph of a **Lipschitz** function. Two beautiful results about the non-existence of the invariant circle for the standard maps [Mat3] and the non-existence of the caustic for a planar convex billiard [Mat4] are obtained by applying the Birkhoff's theorem. And this regularity theorem also plays a crucial role in the scheme of approximating the Aubry-Mather set by Birkhoff periodic orbits.

Since the Aubry-Mather set, on the one hand is the "trace" of the invariant circle, on the other hand it is a part of a graph of a map from S^1 to R^1, one naturally hopes that the map can be chosen to be Lipschitz, in other words, the ghost circle has the **Lipschitz** property. This is indeed the case. The proof can be found, for example, in [Kat1].

It is very attractive to extend such regularity results to higher dimensional systems. M.Herman [Her3] announced a result for small perturbations of a completely integrable Hamiltonian system. J.Mather [Mat5] claimed a result on the Lipschitz property for the positive definite Lagrangian systems. Bernstein-Katok are able to obtain some interesting regularity result: let $(\phi_i, r_i), i = 0, 1, ..., q-1$ be a periodic orbit which minimizes the action $L_{\omega,q}$, then except perhaps at one point, one has

$$\| r_i - r_j \| \leq C \| \phi_i - \phi_j \|^{\frac{1}{2}},$$

where C is a constant depending on the perturbed function f, but not on ω and q. The numerical results by M.Muldoon seem to indicate that there are certain regularities among the approximating Birkhoff periodic orbits.

All these results require the convexity condition of the generating function or of the Lagrangian functional. The results are quite diverse and partial. In the case when the generating function is non-convex, we expect that the orbits are very irregular for the high-dimensional system, since Herman's example (see section 6) shows that the perturbed periodic orbits can go very far from the unperturbed torus.

1.6 Variational Method for Finding Periodic Orbits. We use the standard variational method, as Bernstein-Katok did, to obtain the periodic orbits. Let me quickly point out the similarities and the differences between our approaches and theirs.

By (12) and (13), the problem of finding a periodic orbit with rotation vector ω/q is equivalent to finding the critical point of the functional $L_{\omega,q}$. If the space $X_{\omega,q}$ is compact (as in the case of annulus maps), then one would immediately obtain at least one critical point. But it is obvious that $X_{\omega,q}$ is not compact.

Conley-Zehnder's well known paper [CZ1,Con1] points out that for a function to have critical points, it is the topological type of the space that matters, whether the space is compact or not is not essential. Since their generating function is convex, Bernstein-Katok immediately have a minimal orbit at hand. The crucial thing they discovered is that the neighborhood of the minimal orbit has the needed topology that guarantees the existence of $n+1$ critical points. Here the convexity condition plays a crucial role.

We make a better application of Conley-Zehnder's ideas. We find two crucial topological structures hiding behind the space $X_{\omega,q}$. One is the topological type of the unperturbed torus and another is the "isolating block". Basically these two are all that are needed to guarantee the existence of $n+1$ critical points of $L_{\omega,q}$. Furthermore, using the size of the "isolating block", we are able to estimate the distance between the perturbed Birkhoff periodic orbits and the unperturbed torus. The estimation turns out to be quite accurate as demonstrated by Herman's example [Her4]. Also, this estimation reveals the nonuniform behavior (with respect to the rotation vectors) which is totally different from that in the convex case.

Finally, let us point out that our proofs are almost along the same line of Conley-Zehnder's proof, although the motivations of the problem are totally different. Actually our situation is simpler. Since they are dealing with the flow, their action space is of infinite dimension. They need the so-called "Liapunov-Schmidt" reduction to reduce this infinite dimension space to a finite dimension space. While our action space is of finite dimension at the very beginning.

§2 Preliminaries and Formulation of Results

Our set up is very much like Bernstein-Katok's. Consider the cotangent bundle of an n-torus, $T^*(T^n) \cong T^n \times R^n = \{(\phi_1, ..., \phi_n, r_1, ..., r_n) : \phi_i \in R/Z, r_i \in R, i = 1, 2, ..., n\}$ with the symplectic 2-form:

$$\Omega = \sum_{i=1}^{n} d\phi_i \wedge dr_i.$$

Let $f_0 : T^n \times U \longmapsto T^n \times U$ be an integrable symplectic diffeomorphism, i.e., an Ω-preserving diffeomorphism of the form:

$$f_0(\phi, r) = (\phi + a(r), r), \qquad \phi \in T^n, r \in U,$$

where U is an open set in R^n. Let $F_0 : R^n \times U \longmapsto R^n \times U$ be a lift of f_0 to the universal cover, such that for any $x \in R^n, r \in U$,

$$F_0(x, r) = (x + a(r), r). \tag{15}$$

Proposition 2.1. f_0 is symplectic if and only if the tangent mapping of a, $da : R^n \longmapsto R^n$ is a symmetric linear mapping.

Proof. Let $f_0(\phi, r) = (\phi', r')$ and $a(r) = (a_1(r_1, ..., r_n), ..., a_n(r_1, ..., r_n))$, then

$$\phi_i' = \phi_i + a_i(r_1, ..., r_n), \qquad r_i' = r_i, \qquad i = 1, 2, ...n,$$

hence $d\phi_i' \wedge dr_i' = d\phi_i \wedge dr_i + \sum_{j=1}^{n} \frac{\partial a^i}{\partial r_j} dr_j \wedge dr_i$, therefore

$$\begin{aligned}\sum_{i=1}^{n} d\phi_i' \wedge dr_i' &= \sum_{i=1}^{n} d\phi_i \wedge dr_i + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial a_i}{\partial r_j} dr_j \wedge dr_i \\ &= \sum_{i=1}^{n} d\phi_i \wedge dr_i + \sum_{1 \le j < i \le n} \left(\frac{\partial a_i}{\partial r_j} - \frac{\partial a_j}{\partial r_i}\right) dr_j \wedge dr_i.\end{aligned}$$

We see that $f_0^*\Omega = \Omega$ if and only if $\frac{\partial a_i}{\partial r_j} = \frac{\partial a_j}{\partial r_i}$ for all i and j, i.e., da is symmetric. Q.E.D.

We discuss the following condition which is equivalent to the nondegeneracy condition of the Hessian of the generating function (generating function is a discrete version of the Hamiltonian function):

(R) $a : U \longmapsto R^n$ is a regular injective map.

Let me explain what the generating function is in our situation and why (R) is the nondegeneracy condition for the Hessian of the generating function. By [Arn1, page 258], we know that F_0 is actually exactly symplectic, i.e., $\sum_{i=1}^n r'_i dx'_i - \sum_{i=1}^n r_i dx_i$ is exact where $(x', r') = F_0(x, r)$. Condition (R) implies that F_0 has a generating function $H_0 = H_0(x, x')$ so that $F_0(x, r) = (x', r')$ if and only if:

$$r = \frac{\partial}{\partial x} H_0(x, x'), \qquad r' = -\frac{\partial}{\partial x'} H_0(x, x'). \tag{16}$$

Proposition 2.2. H_0 depends only on the difference $x' - x$, i.e., there exists a function $h : R^n \longmapsto R$ such that $H_0(x, x') = h(x' - x)$.

Proof. Let $y = x' - x, y' = x' + x$, then $H_0 = H_0(y, y')$. Now

$$\frac{\partial H_0}{\partial y'} = \frac{\partial H_0}{\partial x'}\frac{\partial x'}{\partial y'} + \frac{\partial H_0}{\partial x}\frac{\partial x}{\partial y'} = \frac{1}{2}\left(\frac{\partial H_0}{\partial x'} + \frac{\partial H_0}{\partial x}\right) = \frac{1}{2}(r - r') = 0.$$

Hence H_0 depends only on $x' - x$. Q.E.D.

Now h is our generating function. Since $r = \frac{\partial}{\partial x} H_0(x, x') = -Dh(x' - x) = -Dh(a(r))$, we obtain that $Dh = -a^{-1}$. Therefore we conclude that the regularity of the mapping a is equivalent to that h has non-degenerate Hessian.

Remark 1. Condition (R) is the standard condition for the KAM theorem.

Remark 2. Since

$$DF_0(x) = \begin{pmatrix} I_n & da \\ 0 & I_n \end{pmatrix},$$

condition (R) is the monotonicity condition in the sense of Herman, see (3).

We shall consider the standard perturbation, as M.Muldoon does, i.e., we choose h to be of the form:

$$h(x'-x) = \frac{1}{2}\sum_{i=1}^{s}(x'_i - x_i)^2 - \frac{1}{2}\sum_{i=s+1}^{n}(x'_i - x_i)^2 \tag{17}$$

correspondingly, a will be a linear map, $a(r) = Ar$, where $A = \begin{pmatrix} -I_s & 0 \\ 0 & I_{n-s} \end{pmatrix}$ is a very simple symmetric matrix. This special class of mappings embodies the most important nature of the diffeomorphisms under our consideration, that is, the monotonicity.

When $s = n$, h is a convex function, this is essentially the case considered by Bernstein-Katok.

Let $H = h + P$ be a C^1-small perturbation of h, H induces mappings F and f which is C^0-small perturbation of F_0 and f_0 respectively. H is a generating function of F, that is, $F(x,y) = (x',y')$ if and only if

$$r = \frac{\partial H}{\partial x}(x,x'), \qquad r' = -\frac{\partial H}{\partial x'}(x,x').$$

F is a lift of f. In this setting, f preserves the r-component of the center of masses on each torus $T^n \times \{r_0\}$ for any $r_0 \in U$, or equivalently, for any $m \in Z^n$,

$$H(x+m, x'+m) = H(x,x'). \tag{18}$$

Let $r = r_{\omega,q} \in U$ be a rational torus for the unpertubed map, $s_0 = a(r_{\omega,q}) = \frac{\omega}{q}$ where $\omega \in Z^n$ and $q \in Z$. We want to find the perturbed periodic orbits which are close to the torus $T^n \times \{r_{\omega,q}\}$.

Let $(\phi, r) \in T^n \times R^n$ be a periodic orbit of the map f with period q. Let $(x,r) \in R^n \times R^n$ be a lift of (ϕ,r), then there exists a vector $\omega \in Z^n$ such that

$F^q(x,r) = (x+\omega, r)$. The vector ω/q is called the rotation vector of the point (ϕ, r), it depends on the choice of the lift F but it is uniquely defined modulo Z^n.

We shall prove the following:

Theorem. Let f be a perturbation of an integrable symplectic map f_0, and f has the generating function $H = h + P$ where h is defined by (17) and P satisfies (18). Let $\omega = (\omega_1, \cdots, \omega_n) \in Z^n$, such that $\omega_1, \cdots, \omega_n, q$ are relatively prime and the vector $s_0 = \omega/q \in a(U)$. Denote $r_{\omega,q} = a^{-1}(\omega/q)$.

Conclusion: there exists $\Delta = \Delta(f_0, \omega, q, n)$ such that if $\delta = \| P \|_{C^1} \leq \Delta$, then f has at least $n+1$ different periodic orbits with the rotation vector ω/q which lie completely inside the $C\delta$ neighbourhood of the torus $T^n \times \{r_{\omega,q}\}$, where $C = C(f_0, n, q)$ depends on the unperturbed mapping, the dimension of the manifold and the length of the periodic orbit.

Remark. The main differences between Bernstein-Katok's results and ours are: 1.In our case, in order to guarantee the existence of the periodic orbit, the smallness of the perturbation size depends on the rotation vector of the periodic orbit (this is really bad); 2.in our case ,the deviation of the periodic orbit from the unperturbed torus depends also on the length of the periodic orbit. The longer the length, the farther the orbit wanders away (this is also bad). This is a striking phenomenon, and it is closely related to a question raised by Arnold [Arn2]. We exhibit an example by M.Herman to show that this case does happen.

§3 Space of Periodic Orbits and the Action

Fix ω, q as in the theorem, consider the space $\Psi_{\omega,q}$ of "periodic states", where $\Psi_{\omega,q} \ni x = (x^0, \cdots, x^q), x^i \in R^n$ if and only if it satisfies the following boundary condition:

(19) $$x^q = x^0 + \omega.$$

Define an action on $\Psi_{\omega,q}$:

(20) $$L_{\omega,q}(x) = \sum_{i=0}^{q-1} H(x^i, x^{i+1}).$$

The relation between the critical points and the periodic orbits is as follows: $x \in \Psi_{\omega,q}$ satisfying $x^{i+1} - x^i \in a(U)$, is a critical point if and only if $\{(x^i, r^i) : i = 0, \cdots, q\}$, where $r^i = \frac{\partial H}{\partial x}(x^i, x^{i+1})$, is a periodic orbit for the map F.

We have thus reduced our problem to finding the critical point $x \in \Psi_{\omega,q}$, which satisfies $x^{i+1} - x^i \in a(U)$, of the action $L_{\omega,q}$.

Let G be the group generated by the translations $T_m : (x^0, \cdots, x^q) \to (x^0 + m, \cdots, x^q + m), m \in Z^n$. According to the condition (18), the action $L_{\omega,q}$ is G-invariant. Hence $L_{\omega,q}$ acts on the quotient space $\Phi^*_{\omega,q} = \Psi_{\omega,q}/G$. We still denote this induced action by $L_{\omega,q}$. We will use the coordinate system (v, t) introduced in [BK1]:

$$v = \frac{1}{q}(x^0 + \cdots + x^{q-1}), \qquad t^i = x^i - x^{i-1} - \omega/q, \qquad i = 1, \cdots, q-1.$$

In terms of this coordinate system, we have

$$T_m(v, t) = (v + m, t), t = (t^1, \cdots, t^{q-1}).$$

Hence $\Phi^*_{\omega,q} = T^n \times R^{n(q-1)}$, where v is the torus T^n-coordinate and t is the $R^{n(q-1)}$ coordinate.

§4 Proof of the Theorem in the Simplest Case

We consider the simplest case $n = 2, s = 1$. The general case proceeds along on the same line, and we will indicate the crucial part in the proof of the general situation after the proof of this simple case.

When $n = 2, s = 1$, our generating function is

$$h(x, x') = \frac{1}{2}(x'_1 - x_1)^2 - \frac{1}{2}(x'_2 - x_2)^2,$$

where $x = (x_1, x_2), x' = (x'_1, x'_2)$.

Let $x^i = (x^i_1, x^i_2), t^i = (t^i_1, t^i_2), v = (v_1, v_2)$, then the action under the new coordinate system (v, t) is

$$L(v,t) = \sum_{i=0}^{q-2} \left\{ \frac{1}{2}(t_1^{i+1} + \omega_1/q)^2 - \frac{1}{2}(t_2^{i+1} + \omega_2/q)^2 + P \right\}$$
$$+ \frac{1}{2}(t^1_1 + \cdots + t_1^{q-1} - \omega_1/q)^2 - \frac{1}{2}(t^1_2 + \cdots + t_2^{q-1} - \omega_2/q)^2 + P.$$

Take the derivative with respect to t^j_i, we have

$$\frac{\partial L}{\partial t^1_1} = 2t^1_1 + t^2_1 + \cdots + t_1^{q-1} + R^1_1(dP)$$
$$\frac{\partial L}{\partial t^2_1} = t^1_1 + 2t^2_1 + \cdots + t_1^{q-1} + R^2_1(dP)$$
$$.. = ...$$
$$\frac{\partial L}{\partial t_1^{q-1}} = t^1_1 + t^2_1 + \cdots + 2t_1^{q-1} + R_1^{q-1}(dP),$$

and

$$\frac{\partial L}{\partial t^1_2} = -2t^1_2 - t^2_2 - \cdots - t_2^{q-1} + R^1_2(dP)$$
$$\frac{\partial L}{\partial t^2_2} = -t^1_2 - 2t^2_2 - \cdots - t_2^{q-1} + R^2_2(dP)$$
$$.. = ...$$
$$\frac{\partial L}{\partial t_2^{q-1}} = -t^1_2 - t^2_2 - \cdots - 2t_2^{q-1} + R_2^{q-1}(dP),$$

where

$$L = L_{\omega,q}.$$

It is easy to estimate that

$$|R_i^j(dP)| \le 2q\delta.$$

Now proceed as in [CZ1], consider the gradient flow $\frac{d}{ds}(v,t) = \nabla L(v,t)$, and let $t_i = (t_i^1, \cdots, t_i^{q-1}) \in R^{q-1}, i = 1,2$. Write down the gradient equation for (t_1,t_2)-component, we obtain

$$\frac{d}{ds}\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} + \begin{pmatrix} R_1(dP) \\ R_2(dP) \end{pmatrix} \tag{21}$$

where

$$A = \begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 2 \end{pmatrix},$$

is a $(q-1)\times(q-1)$ matrix. The crucial thing we need is that A is non-singular, it is easy to verify that this matrix has 1 and q as its eigenvalue, and the eigenvalue 1 has multiplicity $q-2$. Now we arrive at the same situation as the one in [CZ1]. We construct the isolating block in the sense of C.C.Conley. Let $D_1 = \{t_1 \in R^{q-1} : |t_1| \le 3q^2\delta\}, D_2 = \{t_2 \in R^{q-1} : |t_2| \le 3q^2\delta\}$.

Claim. : $B = T^2 \times D_1 \times D_2$ is an isolating block

Proof. for $|t_1| \ge 3q^2\delta$, one has

$$\begin{aligned} \frac{d}{ds}\frac{1}{2}|t_1|^2 &= < t_1, At_1 + R_1(dP) > \\ &\ge |t_1|^2 - 2q^2\delta|t_1| \\ &= |t_1|(|t_1| - 2q^2\delta) \\ &\ge q^2\delta|t_1| > 0. \end{aligned}$$

Similarly, for $|t_2| \ge 3q^2\delta$, $\frac{d}{ds}\frac{1}{2}|t_2|^2 \le -q^2\delta|t_2| < 0$. Hence, B is an isolating block, the claim is proved. Q.E.D.

Now, the Conley-Zehnder argument [CZ1, Thm 4] concludes that there is an invariant set Σ for the gradient flow, lying **inside** the isolating block B. The cup length satisfies:

$$l(\Sigma) \geq 3.$$

To get the 3 geometrically different periodic orbits for our perturbation map, we need to module out the shift map

$$S : \Phi^*_{\omega,q} \to \Phi^*_{\omega,q}, \qquad (x^0, x^1, ..., x^{q-1}, x^q) \to (x^1, x^2, ..., x^q, x^1 + \omega),$$

since L is invariant under S. We use the following argument by Golé [Gol1]. Under the (v, t)-coordinates, it takes the following form:

$$S(v, t^1, t^2, ..., t^{q-1}) = (v + \omega/q, t^2, ..., t^{q-1}, -\sum_{i=1}^{q-1} t^i).$$

Hence $S = S_1 \times S_2$, $S_1 : T^2 \to T^2, v \to v + \omega/q$ is a q-periodic, fixed point free diffeomorphism, and $S_2 : R^{2(q-1)} \to R^{2(q-1)}, (t^1, ..., t^{q-1}) \to (t^2, ...t^{q-1}, -\sum_{i=1}^{q-1} t^i)$ is a linear isomorphism. $\Phi^*_{\omega,q}$ is a q-fold covering of $\Phi_{\omega,q} = \Phi^*_{\omega,q}/\{S^i\}_{i=0}^{q-1}$, the latter is a fiber bundle over T^2, which also has the homotopy type of T^2. Now we have the following covering maps π and deformation retracts k_1, k_2:

$$\begin{array}{ccc} \Phi^*_{\omega,q} = \Psi_{\omega,q}/G & \xrightarrow{k_1} & T^2 \\ \pi \big\downarrow & & \\ \Phi_{\omega,q} = \Phi^*_{\omega,q}/\{S^i\}_{i=0}^{q-1} & \xrightarrow{k_2} & T^2. \end{array}$$

Let $K^* = k_2^* \circ (k_1^*)^{-1} : H^*(\Phi^*_{\omega,q}) \longrightarrow H^*(T^2) \longrightarrow H^*(\Phi_{\omega,q})$. Then K^* is an isomorphism on cohomology. Where H^* is the Alexander cohomology with real coefficient.

L induces an action on $\Phi_{\omega,q}$ which will still be denoted by L. Now the critical points of L on $\Phi_{\omega,q}$ are in one-to-one correspondence with the geometrically different periodic orbits with the rotation vector ω/q of F.

The existence result in the theorem follows from the following lemma.

Lemma. Let $\underline{\Sigma} = \pi(\Sigma)$ be the invariant set for the gradient flow in $\underline{B} = \pi(B)$. Then

$$l(\underline{\Sigma}) \geq l(\Sigma) \geq 3.$$

Proof. We have the following diagram ($\Phi^* = \Phi^*_{\omega,q}, \Phi = \Phi_{\omega,q}$):

$$\begin{array}{ccccc} H^2(\Phi) & \xrightarrow{i_1^*} & H^2(\underline{B}) & \xrightarrow{j_1^*} & H^2(\underline{\Sigma}) \\ K^*\uparrow\downarrow\pi^* & & \downarrow\pi^* & & \downarrow\pi^* \\ H^2(\Phi^*) & \xrightarrow{i_1^*} & H^2(B) & \xrightarrow{j_1^*} & H^2(\Sigma) \end{array}$$

where i_1, i_2, j_1, j_2 are inclusions. We need the following facts:

(1) i_2^* is an isomorphism: B is a deformation retract of Φ^* (see [CZ1]);

(2) $\pi^* \circ K^*$ is the multiplication by $d \neq 0$ in $R \cong H^2(\Phi^*)$;

(3) Forget K^*, the above diagram commutes;

(4) $j_2^*(w^*) \neq 0$, where $w^* = w_1^* \cup w_2^*$, where w_i^*'s generate $H^1(B)$. This is the core of [Thm 4, CZ1].

Take $w = K^* \circ (i_2^*)^{-1}(w^*) = w_1 \cup w_2$, where $w_i = K^* \circ (i_2^*)^{-1} w_i^*$, then w generates $H^n(\Phi)$. Then $j_1^* \circ i_1^*(w) \neq 0$, otherwise $0 = \pi^* \circ j_1^* \circ i_1^*(w) = j_2^* \circ i_2^* \circ \pi^*(w) = j_2^* \circ i_2^* \circ \pi^*(K^* \circ (i_2^*)^{-1}(w^*)) = j_2^* \circ i_2^* \circ (\pi^* \circ K^*) \circ (i_2^*)^{-1}(w^*) = dj_2^* \circ i_2^* \circ (i_2^*)^{-1}(w^*) = dj_2^*(w^*)$, a contradiction. Hence

$$(j_1^* \circ i_1^* w_1) \cup (j_1^* \circ i_1^* w_2) \neq 0$$

and thus, $l(\underline{\Sigma}) \geq l(\Sigma) \geq 3$. Q.E.D.

Now the Conley-Zehnder argument concludes that L has at least 3 critical points **inside** the isolating block. It is left to show that, corresponding to any of those critical points, say $x, x^i - x^{i-1} \in a(U)$, $\forall i \in Z$. We prove this together with the regularity as follows:

Since the critical points found all lie inside B, we get

$$| t^i_j | \leq 3q^2\delta \tag{22}$$

for all i, j. We show that this is the estimation for the distance between the Birkhoff periodic orbit and the unperturbed torus. Let $r = r_{\omega,q} = a^{-1}(\frac{\omega}{q})$ be the unperturbed torus, and $\{(x^i, r^i) : i = 0, 1, ..., q\}$, where $r^i = \frac{\partial H}{\partial x}(x^i, x^{i+1})$, be the Birkhoff periodic of F with the rotation vector $\frac{\omega}{q}$, with $x^i - x^{i-1} \in a(U)$, $\forall i \in Z$. The distance between these periodic points and the unperturbed torus is:

$$r^i - r_{\omega,q} = \frac{\partial H}{\partial x}(x^i, x^{i+1}) - a^{-1}(\frac{\omega}{q}).$$

Remember $H(x, x') = h(x' - x) + P(x, x')$, and $Dh = -a^{-1}$, we have

$$\begin{aligned} r^i - r_{\omega,q} &= -Dh(x^{i+1} - x^i) + \frac{\partial P}{\partial x}(x^i, x^{i+1}) - a^{-1}(\frac{\omega}{q}) \\ &= a^{-1}(x^{i+1} - x^i) - a^{-1}(\frac{\omega}{q}) + \frac{\partial P}{\partial x}(x^i, x^{i+1}) \\ &= a^{-1}(x^{i+1} - x^i - \frac{\omega}{q}) + \frac{\partial P}{\partial x}(x^i, x^{i+1}) \\ &= At^i + \frac{\partial P}{\partial x}(x^i, x^{i+1}), \end{aligned}$$

since we have $a = A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = a^{-1}$. It follows from (22) that

$$|r^i - r_{\omega,q}| \leq 6q^2\delta + 2\delta = 2(3q^2 + 1)\delta. \tag{23}$$

And it is from this inequality, one can find $\Delta = \Delta(f_0, \omega, q)$ such that if $\delta \leq \Delta$, then $x^i - x^{i-1} \in a(U)$, because $\frac{\omega}{q} = a(r_{\omega,q}), r_{\omega,q} \in U$, and $a(U)$ is open.

The proof of the theorem in this simplest case is now complete. Q.E.D.

§5 Proof of the Theorem for the general Case

Now we consider the general case. The proof proceeds along on the same line as in the simplest case.

The generating function is

$$h(x,x') = \frac{1}{2}\sum_{i=1}^{s}(x'_i - x_i)^2 - \frac{1}{2}\sum_{i=s+1}^{n}(x'_i - x_i)^2,$$

where $x = (x_1, x_2, ..., x_n), x' = (x'_1, x'_2, ..., x'_n)$, and s is the signature.

Let $x^i = (x^i_1, x^i_2, ..., x^i_n)$, $t^i = (t^i_1, t^i_2, ..., t^i_n)$, $v = (v_1, v_2, ..., v_n)$, then the action under the new coordinate system (v,t) is

$$L(v,t) = \sum_{i=0}^{q-2}\left\{\frac{1}{2}\sum_{j=1}^{s}(t_j^{i+1} + \omega_j/q)^2 - \frac{1}{2}\sum_{j=s+1}^{n}(t_j^{i+1} + \omega_j/q)^2 + P\right\}$$
$$+\frac{1}{2}\sum_{j=1}^{s}(t_j^1 + \cdots + t_j^{q-1} - \omega_j/q)^2 - \frac{1}{2}\sum_{j=s+1}^{n}(t_j^1 + \cdots + t_j^{q-1} - \omega_j/q)^2 + P.$$

Take the derivative with respect to t^j_i, we have

$$\begin{aligned}
\frac{\partial L}{\partial t_1^1} &= 2t_1^1 + t_1^2 + \cdots + t_1^{q-1} + R_1^1(dP)\\
\frac{\partial L}{\partial t_1^2} &= t_1^1 + 2t_1^2 + \cdots + t_1^{q-1} + R_1^2(dP)\\
.. &= ......\\
\frac{\partial L}{\partial t_1^{q-1}} &= t_1^1 + t_1^2 + \cdots + 2t_1^{q-1} + R_1^{q-1}(dP),\\
&\vdots\\
\frac{\partial L}{\partial t_s^1} &= 2t_s^1 + t_s^2 + \cdots + t_s^{q-1} + R_s^1(dP)\\
\frac{\partial L}{\partial t_s^2} &= t_s^1 + 2t_s^2 + \cdots + t_s^{q-1} + R_s^2(dP)\\
.. &= ......\\
\frac{\partial L}{\partial t_s^{q-1}} &= t_s^1 + t_s^2 + \cdots + 2t_s^{q-1} + R_s^{q-1}(dP),
\end{aligned}$$

and

$$
\begin{aligned}
\frac{\partial L}{\partial t_{s+1}^1} &= -2t_{s+1}^1 - t_{s+1}^2 - \cdots - t_{s+1}^{q-1} + R_{s+1}^1(dP) \\
\frac{\partial L}{\partial t_{s+1}^2} &= -t_{s+1}^1 - 2t_{s+1}^2 - \cdots - t_{s+1}^{q-1} + R_{s+1}^2(dP) \\
.. &= ...... \\
\frac{\partial L}{\partial t_{s+1}^{q-1}} &= -t_{s+1}^1 - t_{s+1}^2 - \cdots - 2t_{s+1}^{q-1} + R_{s+1}^{q-1}(dP), \\
&\vdots \\
\frac{\partial L}{\partial t_n^1} &= -2t_n^1 - t_n^2 - \cdots - t_n^{q-1} + R_n^1(dP) \\
\frac{\partial L}{\partial t_n^2} &= -t_n^1 - 2t_n^2 - \cdots - t_n^{q-1} + R_n^2(dP) \\
.. &= ...... \\
\frac{\partial L}{\partial t_n^{q-1}} &= -t_n^1 - t_n^2 - \cdots - 2t_n^{q-1} + R_n^{q-1}(dP),
\end{aligned}
$$

where

$$L = L_{\omega,q}.$$

It is easy to estimate that

$$|R_i^j(dP)| \leq 2nq\delta.$$

Now proceed as in [CZ1], consider the gradient flow $\frac{d}{ds}(v,t) = \nabla L(v,t)$, and let $t_i = (t_i^1, \cdots, t_i^{q-1}) \in R^{q-1}$, $i = 1, 2, ..., n$. Write down the gradient equation for $(t_1, t_2, ..., t_n)$-component, we obtain

$$
\frac{d}{ds}\begin{pmatrix} t_1 \\ \vdots \\ t_s \\ t_{s+1} \\ \vdots \\ t_n \end{pmatrix} = \begin{pmatrix} A & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & A & 0 & 0 & 0 \\ 0 & 0 & 0 & -A & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & -A \end{pmatrix} \begin{pmatrix} t_1 \\ \vdots \\ t_s \\ t_{s+1} \\ \vdots \\ t_n \end{pmatrix} + \begin{pmatrix} R_1(dP) \\ \vdots \\ R_s(dP) \\ R_{s+1}(dP) \\ \vdots \\ R_n(dP) \end{pmatrix},
$$

the number of A's is in this big matrix is the signature of the Hessian, where

$$A = \begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 2 \end{pmatrix},$$

as before. The crucial thing we need is that the big matrix is non-singular since it is in block diagonal form and A is non-singular.

Now again we arrive at the same situation as the one in [CZ1]. We construct the isolating block in the sense of C.C.Conley. Let $(t_1, t_2, ..., t_n) = (T_1, T_2)$, where $T_1 = (t_1, ..., t_s)$, and $T_2 = (t_{s+1}, ..., t_n)$, let $|T_1|^2 = \sum_{i=1}^{s} |t_i|^2$, and $|T_2|^2 = \sum_{i=s+1}^{n} |t_i|^2$. Finally let $D_1 = \{T_1 \in R^{s(q-1)} : |T_1| \leq 3q^2n^2\delta\}, D_2 = \{T_2 \in R^{(n-s)(q-1)} : |T_2| \leq 3q^2n^2\delta\}$.

Claim. : $B = T^n \times D_1 \times D_2$ is an isolating block

Proof. for $|T_1| \geq 3q^2n^2\delta$, one has

$$\begin{aligned} \frac{d}{ds}\frac{1}{2}|T_1|^2 &= \sum_{i=1}^{s} < t_i, At_i + R_i(dP) > \\ &\geq \sum_{i=1}^{s}(|t_i|^2 - 2q^2n\delta|t_i|) \\ &= |T_1|^2 - 2nq^2\delta\sum_{i=1}^{s}|t_i| \\ &\geq |T_1|^2 - 2q^2n^2\delta|T_1| \\ &\geq q^2n^2\delta|T_1| > 0. \end{aligned}$$

Similarly, for $|T_2| \geq 3q^2n^2\delta$, $\frac{d}{ds}\frac{1}{2}|T_2|^2 \leq -q^2n^2\delta|T_2| < 0$. Hence, B is an isolating block, the claim is proved. Q.E.D.

Now the Conley-Zehnder argument [CZ1, Thm 4] concludes that there is an invariant set Σ for the gradient flow, lying **inside** the isolating block B. The cup length satisfies:

$$l(\Sigma) \geq n+1.$$

To get the $n+1$ geometrically different periodic orbits for our perturbation map, we still use Golé's argument. Module out the shift map

$$S : \Phi^*_{\omega,q} \to \Phi^*_{\omega,q}, \qquad (x^0, x^1, ..., x^{q-1}, x^q) \to (x^1, x^2, ..., x^q, x^1 + \omega),$$

since L is invariant under S. Under the (v,t)-coordinates, it takes the following form:

$$S(v, t^1, t^2, ..., t^{q-1}) = (v + \omega/q, t^2, ..., t^{q-1}, -\sum_{i=1}^{q-1} t^i).$$

Hence $S = S_1 \times S_2$, $S_1 : T^n \to T^n, v \to v + \omega/q$ is a q-periodic, fixed point free diffeomorphism, and $S_2 : R^{n(q-1)} \to R^{n(q-1)}, (t^1, ..., t^{q-1}) \to (t^2, ...t^{q-1}, -\sum_{i=1}^{q-1} t^i)$ is a linear isomorphism. $\Phi^*_{\omega,q}$ is a q-fold covering of $\Phi_{\omega,q} = \Phi^*_{\omega,q}/\{S^i\}_{i=0}^{q-1}$, the latter is a fiber bundle over T^n, which also has the homotopy type of T^n. Now we have the following covering maps π and deformation retracts k_1, k_2:

$$\begin{array}{ccc} \Phi^*_{\omega,q} = \Psi_{\omega,q}/G & \xrightarrow{k_1} & T^n \\ \pi \downarrow & & \\ \Phi_{\omega,q} = \Phi^*_{\omega,q}/\{S^i\}_{i=0}^{q-1} & \xrightarrow{k_2} & T^n. \end{array}$$

Let $K^* = k_2^* \circ (k_1^*)^{-1} : H^*(\Phi^*_{\omega,q}) \longrightarrow H^*(T^n) \longrightarrow H^*(\Phi_{\omega,q})$. Then K^* is an isomorphism on cohomology. Where H^* is the Alexander cohomology with real coefficient.

L induces an action on $\Phi_{\omega,q}$ which will still be denoted by L. Now the critical points of L on $\Phi_{\omega,q}$ are in one-to-one correspondence with the geometrically different periodic orbits with the rotation vector ω/q of F.

The existence result in the theorem follows from the following lemma.

Lemma. Let $\underline{\Sigma} = \pi(\Sigma)$ be the invariant set for the gradient flow in $\underline{B} = \pi(B)$. Then

$$l(\underline{\Sigma}) \geq l(\Sigma) \geq n + 1.$$

Proof. We have the following diagram ($\Phi^* = \Phi^*_{\omega,q}, \Phi = \Phi_{\omega,q}$):

$$\begin{array}{ccccc}
H^n(\Phi) & \xrightarrow{i_1^*} & H^n(\underline{B}) & \xrightarrow{j_1^*} & H^n(\underline{\Sigma}) \\
K^* \uparrow \downarrow \pi^* & & \downarrow \pi^* & & \downarrow \pi^* \\
H^n(\Phi^*) & \xrightarrow{i_1^*} & H^n(B) & \xrightarrow{j_1^*} & H^n(\Sigma)
\end{array}$$

where i_1, i_2, j_1, j_2 are inclusions. We need the following facts:

(1) i_2^* is an isomorphism: B is a deformation retract of Φ^* (see [CZ1]);

(2) $\pi^* \circ K^*$ is the multiplication by $d \neq 0$ in $R \cong H^n(\Phi^*)$;

(3) Forget K^*, the above diagram commutes;

(4) $j_2^*(w^*) \neq 0$, where $w^* = w_1^* \cup \ldots \cup w_n^*$, where w_i^*'s generate $H^1(B)$. This is the core of [Thm 4, CZ1].

Take $w = K^* \circ (i_2^*)^{-1}(w^*) = w_1 \cup \ldots \cup w_n$, where $w_i = K^* \circ (i_2^*)^{-1} w_i^*$, then w generates $H^n(\Phi)$. Then $j_1^* \circ i_1^*(w) \neq 0$, otherwise $0 = \pi^* \circ j_1^* \circ i_1^*(w) = j_2^* \circ i_2^* \circ \pi^*(w) = j_2^* \circ i_2^* \circ \pi^*(K^* \circ (i_2^*)^{-1}(w^*)) = j_2^* \circ i_2^* \circ (\pi^* \circ K^*) \circ (i_2^*)^{-1}(w^*) = dj_2^* \circ i_2^* \circ (i_2^*)^{-1}(w^*) = dj_2^*(w^*)$, a contradiction. Hence

$$(j_1^* \circ i_1^* w_1) \cup \ldots \cup (j_1^* \circ i_1^* w_n) \neq 0$$

and thus, $l(\underline{\Sigma}) \geq l(\Sigma) \geq n + 1$. Q.E.D.

Now the Conley-Zehnder argument concludes that L has at least $n + 1$ critical points **inside** the isolating block. It is left to show that, corresponding to any of those critical points, say $x, x^i - x^{i-1} \in a(U)$, $\forall i \in Z$. We prove this together with the regularity as follows:

Since the critical points found all lie inside B, we get

$$| t_j^i | \leq 3q^2n^2\delta$$

for all i, j. We show that this is the estimation for the distance between the Birkhoff periodic orbit and the unperturbed torus. Let $r = r_{\omega,q} = a^{-1}(\frac{\omega}{q})$ be the unperturbed torus, and $\{(x^i, r^i) : i = 0, 1, ..., q\}$, where $r^i = \frac{\partial H}{\partial x}(x^i, x^{i+1})$, be the Birkhoff periodic of F with the rotation vector $\frac{\omega}{q}$, with $x^i - x^{i-1} \in a(U)$, $\forall i \in Z$. The distance between these periodic points and the unperturbed torus is:

$$r^i - r_{\omega,q} = \frac{\partial H}{\partial x}(x^i, x^{i+1}) - a^{-1}(\frac{\omega}{q}).$$

Remember $H(x, x') = h(x' - x) + P(x, x')$, and $Dh = -a^{-1}$, we have

$$\begin{aligned} r^i - r_{\omega,q} &= -Dh(x^{i+1} - x^i) + \frac{\partial P}{\partial x}(x^i, x^{i+1}) - a^{-1}(\frac{\omega}{q}) \\ &= a^{-1}(x^{i+1} - x^i) - a^{-1}(\frac{\omega}{q}) + \frac{\partial P}{\partial x}(x^i, x^{i+1}) \\ &= a^{-1}(x^{i+1} - x^i - \frac{\omega}{q}) + \frac{\partial P}{\partial x}(x^i, x^{i+1}) \\ &= At^i + \frac{\partial P}{\partial x}(x^i, x^{i+1}), \end{aligned}$$

since we have $a = A = \begin{pmatrix} -I_s & \mathbf{0} \\ \mathbf{0} & I_{n-s} \end{pmatrix} = a^{-1}$. It follows from above estimation that

$$|r^i - r_{\omega,q}| \leq 3q^2n^3\delta + 2\delta.$$

And it is from this inequality, one can find $\Delta = \Delta(f_0, \omega, q, n)$ such that if $\delta \leq \Delta$, then $x^i - x^{i-1} \in a(U)$, because $\frac{\omega}{q} = a(r_{\omega,q}), r_{\omega,q} \in U$, and $a(U)$ is open.

The proof of the theorem in the general case is now complete. Q.E.D.

§6 A Question of Arnold and Herman's Example

In the introduction, when relating the KAM theorem and the Aubry-Mather Theory, we quoted a question raised by Arnold in his well known paper. Now we quote another question raised in the same paper, to indicate the importance of our estimation (22). Here is the question [Arn2]:

1°. Zones of the Instability. How do the trajectories that begin in the "gaps" of 1.3 behave? Can they, for $n > 2$, depart very far from the torus p =const.? In particular, are the equilibrium configurations and the periodic solutions of general elliptic type stable when the number of the degree of freedom exceeds 2? The simplest problem is the canonical mapping of four-dimensional space.

When the generating function is convex, the $n+1$ Birkhoff periodic orbits found in [BK1] cannot depart very far from the unperturbed torus. Our estimation (22) seems to indicate that trajectories may be very far away from the unperturbed torus. We carry out the calculation for Herman's example which says that this can actually happen. For a fixed small perturbation, the larger the period, the farther the trajectory departs. Therefore, for two very close unperturbed tori, their "traces" can be a long distance away. The example happens to be a canonical mapping of the four-dimensional space (Arnold call the symplectic map the canonical map).

Consider the integrable map $L_B : T^*(T^2) \to T^*(T^2)$

$$L_B(\theta, r) = (\theta + rB, r)$$

where

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\theta = (\theta_1, \theta_2)$, $r = (r_1, r_2)$. Note that B has eigenvalue 1, -1.

To construct the small perturbation, let $\phi(\theta_1, \theta_2) = \frac{\delta}{2\pi} sin(2\pi\theta_1)$ be a function on T^2, and $G_\phi(\theta, r) = (\theta, r + \frac{\partial \phi}{\partial \theta}) = (\theta_1, \theta_2, r_1 + \phi'(\theta_1), r_2)$. Finally, let $F = G_\phi \circ L_B$.

Write down the lift map for F (still denoted by F) and its iterates explicitly:

(24) $$F(x, r) = (x_1 + r_2, x_2 + r_1, r_1 + \delta cos(2\pi(x_1 + r_2)), r_2).$$

For $j \geq 2$, the j-th iterate of F is

$$F^j(x, r) = (x_1^j, x_2^j, r_1^j, r_2^j)$$

$$= (x_1 + jr_2, x_2 + jr_1 + \delta \sum_{k=1}^{j-1} (j-k) cos(2\pi(x_1 + kr_2)),$$

$$r_1 + \delta \sum_{k=1}^{j} cos(2\pi(x_1 + kr_2)), r_2).$$

Take any integer $q \geq 2$, and an integer vector $\omega = (1, 1)$. Then the unperturbed torus with rotation vector ω/q is $r = r_{\omega,q} = (1/q, 1/q)$. Through an elementary calculation, we find that $\{F^i(-1/(2q), x_2, 1/q, 1/q),\ i \in Z\}$ is a Birkhoff periodic orbit with rotation vector ω/q, where x_2 can be arbitary. The closeness of the Birkhoff periodic orbit to the unperturbed torus is measured by $x^{j+1} - x^j - \omega/q$, in this example $x^{j+1} - x^j - \omega/q = (0, r_1^j - r_1)$. For $1 \leq j \leq q$,

$$r_1^j - r_1 = \delta \sum_{k=1}^{j} cos(2\pi(k - 1/2)r_2)$$

$$= \delta sin(2\pi j r_2)/(2 sin(\pi r_2))$$

where $r_2 = 1/q$. Take $j = [q/4]$ the integer part of $q/4$, for large q, we have

$$| r_1^j - r_1 | \approx \frac{\delta q}{2\pi}.$$

This estimation verifies the dependence of the distance on the period q and shows that our estimation (22) is optimal. It also reveals a major distinction between the convex generating function case and the non-convex generating function case and, to a large extent, answers the question of Arnold.

Part II: Minimal Geodesics and Lyapunov Exponents

§7 Introduction

In this part we are interested in a very old object, the geodesic flow on a compact Riemannian surface of genus greater than one. To be more precise, we are interested in finding minimal closed geodesic that is hyperbolic. A geodesic is called minimal if its lift to the universal cover minimizes the distance between any pair of points on the lift. The earlier systematic study of the minimal geodesic (they were called class geodesics) as a whole by M.Morse [Mor1] and by Hedlund [Hed1] dated back to the twenties. By comparing the surface to the model space, i.e., the Poincaré disc and the Euclidean plane, respectively, they showed, for each homotopy type of an infinite curve on the surface, the existence of the minimal geodesics with the given type for an **arbitrary** compact surface of genus greater than one and of genus equal to one, respectively. They gave a complete description of the structure of all minimal geodesics and proved many nice properties of such geodesics.

The intense study in the past decade of Aubry-Mather sets causes a revival of this old object [Mos2], [Ban1], [Ban2]. It is now recognized that the Aubry-Mather set corresponding to a rotation number is very similar to the set of all minimal geodesics having the same homotopy type. Simply looking at the definition of the Aubry-Le Daeron's minimum energy configurations (see (6)) and at the definition of the minimal geodesics, one can see the similarity. In Aubry-Le Daeron's setting, each configuration is a bi-infinite sequence $\{u_k\}_{k \in Z}$ of real numbers with the energy $\phi(\{u_k\}) = \sum_n L(u_n, u_{n-1})$. Remember a configuration is called a minimum energy configuration if

$$\sum_{n=N'}^{N} L(u_n, u_{n-1}) \leq \sum_{n=N'}^{N} L(u_n + \delta_n, u_{n-1} + \delta_{n-1}),$$

for any $N' < N$ and any choice of δ_n with $\delta_n = 0$ for $n < N'$ and $n \geq N$.

Next let $\gamma : R \mapsto M$ be a curve in a simply connected Riemannian surface M with a metric σ, let l_σ be the induced length function. γ is called minimal if for any $a, b \in R, a < b$,

$$l_\sigma(\gamma|_{[a,b]}) \leq l_\sigma(\gamma'|_{[a,b]}),$$

for any γ' with $\gamma'(a) = \gamma(a), \gamma'(b) = \gamma(b)$.

The similarity of these two objects is readily seen. Basically, one is dealing with the discrete system while another the continuous system.

Actually Bangert [Ban1] provided a rigorous treatment for the torus. He proved the following: let g be a Riemannian metric on the torus $T^2 = R^2/Z^2$, g induces a metric $\tilde{d}$ on the universal cover R^2. Choose a new coordinate on T^n so that the coordinate line $s \longmapsto (i, s)$, $i \in Z$, are minimal geodesics. Then define the potential

$$H(\xi, \eta) = \tilde{d}((0, \xi), (1, \eta)).$$

The function H, in Aubry-Le Daeron's notation, is L. Bangert showed that 1. a minimum energy configuration $\{x_k\}_{k \in Z}$ determines a minimal geodesic through the points (i, x_i) for all $i \in Z$; 2.conversely, the intersection of a minimal geodesic $c(s) = (\xi(s), \eta(s))$ with lines $\{i\} \times R$, $i \in Z$, determines a minimal energy configuration $x = \{x_k\}_{k \in Z}$.

The other similarities between the Aubry-Mather set and the set of equivalent geodesics are the properties they possess. There are too many such properties. For details see the survey paper [Ban1].

We are interested in a problem of a different nature. Generally, a closed geodesic can be elliptic or parabolic or hyperbolic. Our problem is, for the surface of genus greater than one, is there a minimal closed geodesic that is hyperbolic?

M.Morse showed, that there are many minimal closed geodesics on the surface of genus greater than one.

Geodesic flow on a compact surface of genus greater than one may serve as the simplest nice example with **non-uniformly** hyperbolic behavior. Applying smooth ergodic theory and combining the classical regularization theorem, Katok [Kat2] showed: 1. (in contrast to Morse' geometric approach) minimal closed geodesics have exponential growth; 2. hyperbolic closed geodesics has the exponential growth; 3. the topological entropy for the geodesic flow restricted to the set of all unit tangent vectors which project into minimal geodesics is positive. The question is: is there even one minimal closed geodesic which is hyperbolic?

I followed the idea in a theorem of E.Hopf, and showed the following rigidity result: the integration of the Gaussian curvature, that is, the average Gaussian curvature, along any closed minimal geodesic is always non-positive. If it is zero, then the Gaussian curvature **vanishes everywhere** along that geodesic. Thus for a closed minimal geodesic, three cases can happen:

I. the Gaussian curvature is zero everywhere along this geodesic, hence the Lyapunov exponents are zero;

II. the integration of the Gaussian curvature over this geodesic is negative, the Lyapunov exponents are zero;

III. the integration of the Gaussian curvature over this geodesic is negative, the Lyapunov exponents are non-zero.

Furthermore, combining Katok's results, variational principle of entropy and Ruelle's inequality, one can show that the majority of the closed minimal geodesics falls into case II and case III.

We believe that the majority of the minimal closed geodesics belong to case III, since negative curvature is the main cause for the hyperbolicity [Ano1]. In the absence of focal point along a single minimal closed geodesic the negativeness of the **average Gaussian curvature** implies existence of the non-zero Lyapunov exponents. This is proved in [Pes1]. Unfortunately, we are unable to prove this conjecture at the present stage.

§8 Riemannian Surface of Genus Greater Than One

Unless specifically pointed out, from now on (M, σ) will be a $C^{2+\delta}(\delta > 0)$ compact Riemannian surface of genus greater than one, σ an arbitrary fixed Riemannian metric. The facts stated in this section are taken from [Mor1]. First we introduce some notations:

$(D, \overline{\sigma}_0)$ - the Poincaré disc, $\overline{\sigma}_0$ the hyperbolic metric with constant curvature -1, a geodesic in $(D, \overline{\sigma}_0)$ is called a hyperbolic line;

Γ - the standard discrete group such that $M_0 = D/\Gamma$ is of the genus of M. Γ is the fundamental group of M, which will be regarded as the deck transformation group of D preserving the metric $\overline{\sigma}$, where

$\overline{\sigma}$ - the lift metric of σ to the universal cover D of M; and

$\overline{d} = \overline{d}_{\overline{\sigma}}$ - the distance on D induced by the metric $\overline{\sigma}$;

$d = d_\sigma$ - the distance on M induced by the metric σ;

D_σ-the distance on the unit tangent bundle $SM = S^\sigma M$ induced by the metric σ.

A curve $\gamma : [a, b] \to D$ is called a minimal geodesic segment if it is the shortest curve in metric σ connecting $\gamma(a)$ and $\gamma_(b)$ in metric σ.

An infinite curve $\gamma : (-\infty, \infty) \to D$ is called minimal geodesic if for any $-\infty < a < b < \infty, \gamma|_{[a,b]}$ is a minimal geodesic segment.

Let γ_1, γ_2 be two infinite curves in D, they are said to be of the same type if

$$\sup_{t \in (-\infty,\infty)} \overline{d}(\gamma_1(t), \gamma_2) + \sup_{t \in (-\infty,\infty)} \overline{d}(\gamma_1, \gamma_2(t)) < \infty.$$

We have the following facts:

Fact 1. For any hyperbolic line γ_0 in $(D, \overline{\sigma}_0)$, there exists at least one minimal geodesic γ which is of the type of γ_0; on the other hand for any minimal geodesic γ, γ is of the same type of some hyperbolic line γ_0 in $(D, \overline{\sigma}_0)$.

A curve γ is called periodic if there is a deck transformation $\phi \in \Gamma$ such that $\phi\gamma = \gamma$.

Fact 2. For any periodic hyperbolic line γ_0 in $(D, \overline{\sigma}_0)$, there is at least one periodic minimal geodesic γ which is of the type of γ_0.

Fact 3. Any minimal periodic geodesic projects into a closed geodesic which is shortest among its free homotopy class, and this is a one to one correspondence.

A proof of Fact 3 is virtually given in [FHS1]. By Fact 3, we can speak of minimal geodesic or shortest curve in a free homotopy class alternatively. Due to the above facts and also due to a result on the word growth rate of the fundamental group by J.Milnor [Mil1], we know that there are many minimal closed geodesics.

§9 Statement of Results

For a Riemannian surface (M, σ) let $K = K_\sigma$ be the Gaussian curvature, $\phi_t = \phi_t^\sigma : SM \to SM$ the geodesic flow on the unit tangent bundle with the unit speed. Our first result is:

Theorem 9.1. Let (M, σ) be any Riemannian surface (with no assumption on the genus), $\gamma : [0, T] \rightarrow M$ be a shortest closed geodesic in its free homotopy class, with the length T, then

$$\int_0^T K(\gamma(t))dt \leq 0, \tag{25}$$

in case the equality holds, $K(\gamma(t)) \equiv 0$.

We now introduce the notion of Lyapunov exponents. Let $(V, <,>)$ be a compact C^1 Riemannian manifold. $<,>$ induces a norm $\| \bullet \|$. Let $\phi_t : V \longmapsto V$ be a C^1-flow on V. For $v \in V, \xi \in T_v(V)$, one defines

$$\chi^+(v, \xi) = \limsup_{t \to \infty} \frac{1}{t} \log \| (d\phi_t)_v(\xi) \| . \tag{26}$$

Oseledec's theorem [Osel] asserts that $\chi^+(v, \bullet)$ assumes only finitely many values on $T_v(V)$:

$$\chi_1(v) < \chi_2(v) < \ldots < \chi_{s(v)}(v),$$

where $s(v) \leq dimV$ for any $v \in V$. These numbers are called the Lyapunov exponents of ϕ_t at v, and they play the fundamental role in the modern smooth dynamical systems. In our case, $V = SM$, ϕ_t is the geodesic flow on SM. V is three-dimensional, hence there are at most three exponents for any $v \in SM$. By a lemma in [FK1], if $\chi(v)$ is an exponent, so is $-\chi(v)$. Note that $\chi = 0$ is always an exponent corresponding to the flow direction, hence ϕ_t has non-zero Lyapunov exponents if and only if the largest exponent is positive. Denote the largest exponent $\chi^+(v)$, then $\chi^+(v) \geq 0$.

We want to describe some other ways to determine the Lyapunov exponent $\chi^+(v)$. Since our surface is compact, the Gaussian curvature is bounded from below. By a result of Eberlein [Ebe1], [Pes2], one can find the Lyapunov exponent for any $v \in SM$ by computing the exponential growth rate of the Jacobi field along the geodesic $\gamma(t) = \pi\phi_t(v)$, where $\pi : SM \longmapsto M$ is the canonical projection.

Definition 9.1. Let $\gamma(t)$, $t \in R$, be a geodesic in M. A function $y : R \longmapsto R$ is called a Jacobi field along γ if y satisfies the equation

$$y''(t) + K(\gamma(t))y(t) = 0. \tag{27}$$

Proposition 9.1. For any $v \in SM$ such that $\pi\phi_t(v)$ is a minimal geodesic, let $y(t)$ be the solution of the (27) ($\gamma(t) = \pi\phi_t(v)$) with $y(0) = 1$ and $\lim_{t \longrightarrow -\infty} y(t) = 0$, then the largest Lyapunov exponent

$$\chi^+(v) = \limsup_{t \longmapsto \infty} \frac{1}{t} \log |y(t)|, \tag{28}$$

and if $\phi_t(v)$ is a closed orbit, then v is a regular point, and (28) becomes

$$\chi^+(v) = \lim_{t \longmapsto \infty} \frac{1}{t} \log |y(t)|. \tag{29}$$

For a geodesic γ in M, we speak of the Lyapunov exponents of γ as the Lyapunov exponents of v, where v is any unit tangent vector to γ.

It is easy to see that if $K(\pi\phi_t(v)) \equiv 0$, then $y(t)$ is a linear function of t, therefore by proposition 9.1, $\chi^+(v) = 0$. Based on this observation and theorem 9.1, we can divide $\{v \in SM : \pi\phi_t(v) \text{ is a minimal closed geodesic in } M\}$ into three classes:

I. $K(\pi\phi_t(v)) \equiv 0$, hence $\chi^+(v) = 0$;

II. $\int_0^T K(\pi\phi_t(v))dt < 0$, and $\chi^+(v) = 0$, where T is the length of $\pi\phi_t(v)$;

III. $\int_0^T K(\pi\phi_t(v))dt < 0$, and$\chi^+(v) > 0$, T as in II.

Correspondingly, one divides the set of minimal closed geodesics into three classes.

Our other results involve estimating the exponential growth rate of the numbers of geodesics from classes I, II, III. Again we need some notations:

$S_I = \{v \in SM : \pi\phi_t(v)$ is a minimal closed geodesic with $K(\pi\phi_t(v)) \equiv 0\}$;

$\overline{S_I}$ - the closure of S_I in SM;

l_σ - the length function induced by the Riemannian metric σ;

$P^s_\sigma(T) :=$ the number of minimal closed geodesic γ with $l_\sigma(\gamma) \leq T$;

$P^{s,I}_\sigma(T) :=$ the number of minimal closed geodesic γ in class I with $l_\sigma(\gamma) \leq T$;

$P^{s,II,III}_\sigma(T) :=$ the number of minimal closed geodesic γ in class II or in class III with $l_\sigma(\gamma) \leq T$.

Since $\overline{S_I}$ is a compact metric space, it is ϕ_t-invariant, we can define the topological entropy for the geodesic flow restricted to $\overline{S_I}$ [DGS1]. We have the following:

Theorem 9.2. The topological entropy for the geodesic flow restricted to $\overline{S_I}$ is zero:

(30) $$h\left(\phi^\sigma|_{\overline{S_I}}\right) = 0.$$

The following theorem is a consequence of Theorem 9.2.

Theorem 9.3. The growth rate of the number of the minimal closed geodesics from the class I is less exponential:

$$\limsup_{T\longmapsto\infty}\frac{1}{T}\log P_\sigma^{s,I}(T)=0. \tag{31}$$

Katok [Kat2] established the following inequality:

$$P_\sigma^s:=\liminf_{T\longmapsto\infty}\frac{1}{T}\log P_\sigma^s(T)\geq\left(\frac{-2\pi E}{v_\sigma}\right)^{\frac{1}{2}}, \tag{32}$$

where E is the Euler characteristic and v_σ is the total surface area under the metric σ. Actually the meaning of our $P_\sigma^s(T)$ is slightly different from Katok's, but his inequality implies the same as our (32).

(31) and (32) immediately lead to the following estimation:

Theorem 9.4. The minimal closed geodesics from both class II and class III have exponential growth:

$$\liminf_{T\longmapsto\infty}\frac{1}{T}\log P_\sigma^{s,II,III}(T)\geq\left(\frac{-2\pi E}{v_\sigma}\right)^{\frac{1}{2}}. \tag{33}$$

§10 Proof of Theorem 9.1

The proof of the theorem 1 relies on the following main lemma.

Main lemma. Let $\{\pi\phi_t(v)\}_{t\in[0,T]}$ be as in the theorem 1, for simplicity write $K(t)=K(\pi\phi_t(v))$, then the Jacobi equation

$$y''(t)+K(t)y(t)=0 \tag{34}$$

has a solution $y(t)$ satisfying:

a) $y(t) \neq 0$, for all $t \in (-\infty, \infty)$;

b) $u(t) := \frac{y'(t)}{y(t)}$ is periodic: $u(T) = u(0)$.

Proof. Let $y(t), -\infty < t < \infty$ be the solution of the Jacobi equation. Consider the Poincaré map $P : (y(0), y'(0)) \longmapsto (y(T), y'(T))$, since (34) is a linear equation, P is a two-dimensional linear map.

Claim A. Zero is not the eigenvalue of P. Otherwise take $0 \neq (y(0), y'(0))$ the eigenvector corresponding to zero, we have $(y(T), y'(T)) = 0$. By the uniqueness theorem of the linear ODE, $y(t) \equiv 0$. Contradicts to $(y(0), y'(0)) \neq 0$.

Claim B. P can only have a real eigenvalue. Suppose P has a pair of conjugate complex numbers as eigenvalues, then P is a rotation followed by a scalar multiplication, if we iterate P many times, we will find a solution $y(t)$ and $t_1, t_2, -\infty < t_1 < t_2 < \infty$, such that $y(t_1) = y(t_2) = 0$. Lift the $\pi\phi_t(v)$ and the Jacobi equation to the universal cover D, by Fact 3 we know that the lift of $\pi\phi_t(v)$ is a minimal geodesic, $y(t_1) = y(t_2) = 0$ would mean that there is a pair of conjugate points along the lift of $\pi\phi_t(v)$ which is impossible, see for example, [CE1].

Take an eigenvalue λ, and its corresponding non-zero eigenvector $(y(0), y'(0))$, we show that the solution with this eigenvector as initial values will satisfy a) and b). First observe $y(0) \neq 0$, otherwise $y(T) = \lambda y(0) = 0$, again we get a pair of conjugate points along the minimal geodesic. We show $y(t) \neq 0$, for all $t \in [0, T]$. First observe that $\lambda > 0$. Otherwise $y(0), y(T) = \lambda y(0), y(2T) = \lambda^2 y(0), \ldots$ will have alternating signs which again contradicts to the minimality of the lift geodesics. Now suppose that $t_0, 0 < t_0 < T$ is the first $t > 0$ such that $y(t_0) = 0$, then by uniqueness $y'(t_0) \neq 0$ hence for $t > t_0, t$ close to $t_0, y(t)$ will have an opposite sign to that of $y(0)$ and $y(T)$, there must be at least one $t_1 \in (t_0, T)$ such that $y(t_1) = 0$, again producing two conjugate points along a minimal geodesic.

Therefore, $y(t)$ satisfies a). We can safely define $u(t) := \frac{y'(t)}{y(t)}$. It is obvious that $u(t)$ is periodic: since $(y(T), y'(T)) = \lambda(y(0), y'(0))$, we have $u(T) = \frac{y'(T)}{y(T)} = \frac{\lambda y'(0)}{\lambda y(0)} = \frac{y'(0)}{y(0)} = u(0)$. Q.E.D.

We can now give another description of the largest Lyapunov exponent $\chi^+(v)$, for v such that $\pi\phi_t(v)$ is a minimal closed geodesic in M.

Proposition 10.1. Let λ be the bigger eigenvalue of the Poincaré mapping P, then

$$\chi^+(v) = \log \lambda. \tag{35}$$

Proof. The Jacobi equation can be transformed into a standard system of first order linear differential equations

$$\begin{cases} y' &= & y_1 \\ y_1' &= & -K(t)y \end{cases}$$

or

$$\begin{pmatrix} y' \\ y_1' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -K(t) & 0 \end{pmatrix} \begin{pmatrix} y \\ y_1 \end{pmatrix},$$

since $\operatorname{trace}\begin{pmatrix} 0 & 1 \\ -K(t) & 0 \end{pmatrix} = 0$, the Poincaré mapping is area preserving (of course this explains why there are two Lyapunov exponents with the same absolute value and with opposite sign). Let λ_1, λ_2 be the eigenvalue of P, then $\lambda_1 > 0, \lambda_2 > 0$. Area preserving means that $\lambda_1\lambda_2 = 1$.Now let y be the solution in Proposition 9.1, then $y(nT) = ay_1(nT) + by_2(nT) = a\lambda_1^n y_1(0) + b\lambda_2^n y_2(0)$ where y_1, y_2 are the solutions of the Jacobi equation whose initial values are eigenvectors of P corresponding to the eigenvalues λ_1, λ_2 respectively, and a, b are constants. Now (35) follows from (29).

It is now clear that v has only zero exponents if and only if one is the only eigenvalue of P and if and only if there is a **periodic** Jacobi field along the geodesic $\pi\phi_t(v)$. In this case there are focal points along the geodesic.

Now we prove theorem 9.1. Take $y(t), u(t)$ obtained in the lemma, $u(t)$ will satisfy the Riccati equation:

$$u'(t) + u^2(t) + K(t) = 0.$$

Integrate the equation over the interval $[0, T]$, we have

$$\int_0^T K(t)dt = -\int_0^T u^2(t)dt \leq 0.$$

If $\int_0^T K(t)dt = 0$, then $\int_0^T u^2(t)dt = 0$, then $u(t) \equiv 0$ on $[0, T]$, or $y'(t) \equiv 0$ on $[0, T]$, since $y(t)$ satisfies the Jacobi equation $y''(t) + K(t)y(t) = 0$, and since $y(t)$ never vanishes by the lemma, we must have $K(t) \equiv 0$, for $t \in [0, T]$. Q.E.D.

§11 Proofs of Theorems 9.2, 9.3 and 9.4

The argumemts in this section are basically suggested by A.Katok.

First of all, the set $\overline{S_I}$ enjoys some nice properties:

Lemma 11.1. $\overline{S_I}$ is closed, ϕ_t-invariant, and $K|_{\overline{S_I}} \equiv 0$. For any $v \in \overline{S_I}$, $\chi^+(v) = 0$ and $\pi\phi_t(v)$ is a minimal geodesic in M.

Proof. The last statement is proved in [Mor1], $\chi^+(v) = 0$ follows from the remark following the proposition 9.1, and others are trivial.

We prove theorem 9.2. Here we do not even need the definition of the entropy.

Proof of theorem 9.2. Suppose $h\left(\phi^\sigma|_{\overline{S_I}}\right) > 0$, where $\phi^\sigma = \{\phi_t^\sigma\}_{t\in R}$ is the geodesic flow on SM. Then by the variational principle of entropy and by Ruelle's inequality [Ruel], there is a ϕ_t-invariant measureμ supported on $\overline{S_I}$, so that

$$\int_{\overline{S_I}} \chi^+(v)d\mu(v) \geq \frac{1}{2}h\left(\phi^\sigma|_{\overline{S_I}}\right) > 0.$$

But by lemma 11.1, we have that $\int_{\overline{S_I}} \chi^+(v)d\mu(v) = 0$, a contradiction. Q.E.D.

Now we recall the definition of entropy. Let $T > 0$, $\delta > 0$. The vectors $v, w \in SM$ are said to be (T,δ)-separated if

$$D_\sigma^T(v,w) := \max_{0\leq t\leq T} D_\sigma(\phi_t v, \phi_t w) \geq \delta, \tag{36}$$

where D_σ is the metric on SM induced by σ. Note that D_σ^T is also a metric on SM.

For a closed ϕ_t-invariant set A in SM, let $S_\sigma(T,\delta,A)$ be the maximal number of (T,δ)-separated orbits of ϕ^σ belonging to the set A, and let $N_\sigma(T,\delta,A)$ be the minimal number of sets of diameter $\leq 2\delta$ in the metric D_σ^T which covers A, clearly

$$N_\sigma(T,\delta,A) \geq S_\sigma(T,2\delta,A). \tag{37}$$

Also, it follows from the proof of proposition 4.5 in [Kat2] that

$$h\left(\phi^\sigma|_A\right) = \lim_{\delta\to 0}\lim_{T\to\infty}\frac{1}{T}\log N_\sigma(T,\delta,A). \tag{38}$$

Now we apply these general results to our case $A = \overline{S_I}$.

Proof of theorem 9.3. First we point out a fact: there exist positive numbers T_σ and δ_σ depending only on the metric σ such that, for any $T \geq T_\sigma$ and $\delta \leq \delta_\sigma$, two non-homotopic closed geodesics are always (T, δ)-separated, see the proof of the Proposition 4.5 in [Kat2].

This fact together with (37) implies that, for $T \geq T_\sigma$ and $\delta \leq \delta_\sigma$,

$$P_\sigma^{s,I}(T) \leq S_\sigma(T, \delta, \overline{S_I}) \leq N_\sigma(T, \frac{\delta}{2}, \overline{S_I}),$$

hence

$$\limsup_{T\to\infty} \frac{1}{T} P_\sigma^{s,I}(T) \leq \lim_{\delta\to 0} \lim_{T\to\infty} \frac{1}{T} \log N_\sigma(T, \frac{\delta}{2}, \overline{S_I}) \leq h\left(\phi^\sigma|_{\overline{S_I}}\right) = 0.$$

Theorem is proved. Q.E.D.

Proof of theorem 9.4. We have an obvious inequality

$$P_\sigma^{s,I}(T) + P_\sigma^{s,II,III}(T) = P_\sigma^{s}(T),$$

for any $T > 0$. The inequality (33) now follows from Katok's inequality (32) and theorem 9.3.

By the remark following the proof of proposition 10.1, we know that a minimal closed geodesic has only zero Lyapunov exponents if and only if there is a non-trivial periodic Jacobi field along the geodesic. From this point of view the geodesics in class I can be viewed as special case of geodesics from class II, since when the Gaussian curvature vanishes identically along a minimal closed geodesic, any non-zero constant Jacobi field is a non-trivial periodic Jacobi field.

We conclude this part with a conjecture:

conjecture. The growth rate of the number of minimal closed geodesics from class II is also less exponential:

$$\limsup_{T\to\infty} \frac{1}{T} \log P_{\sigma}^{s,II}(T) = 0.$$

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