ONE. THERMAL DECOMPOSITION OF n-BUTANE

TWO. FLOW IN ENTRANCE SECTION OF PARALLEL PLATES

Thesis by

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ABSTRACT

Part One

The thermal decomposition of n-butane was investigated in a flow reactor at a pressure of 1 atm, in a temperature range of 460° to 560°C, and at low conversion levels, i.e. 0.06 - 0.68% for the 460° runs, 0.5 - 2.3% for the 510° runs, and 3.5 to 8.2% for the 560°C runs. Temperature, velocity, and concentration profiles at the exit end of the reactor were measured to study the effects of energy, momentum, and mass transports on chemical reaction. It was found after analysis of data that the reactor could be treated as an isothermal reactor with plug flow under the prevailing operating conditions.

Two rate expressions were determined for the reaction; one corresponding to a first-order and the other to a second-order rate. They are

$$\text{First-order rate} = 3.34 \times 10^{12}\, e^{\frac{-54,600}{RT}} (C_4H_{10}) \frac{\text{lb}}{\text{ft}^3\ \text{sec}}$$

$$\text{Second-order rate} = 2.55 \times 10^{14}\, e^{\frac{-56,800}{RT}} (C_4H_{10})^2 \frac{\text{lb}}{\text{ft}^3\ \text{sec}}$$

These two expressions equally well represent the experimental data.

On the basis of the products formed and the rates observed, a Rice-type, free-radical mechanism was proposed for the thermal decomposition of n-butane. The mechanism, which is presented in the section on correlation of data, quantitatively describes

the reaction. One major feature of the mechanism is the consideration of secondary reactions at very low conversions.

Part Two

Flow of an incompressible fluid at the entrance section of parallel plates under isothermal, laminar conditions was investigated by solving the two-dimensional Navier-Stokes equations numerically. The Navier-Stokes equations were transformed into finite-difference equations in terms of stream functions Ψ and vorticities ω with a technique developed by de G. Allen. The finite-difference equations were then solved by an iterative procedure on digital computers. From the solution, point velocities and pressure gradients were computed.

Two cases were studied, both with a Reynolds number of 300. Case I had a flat velocity distribution at the entrance to the plates. Case II assumed that potential-flow conditions existed only far upstream from the entrance. For both cases, large velocity and pressure gradients were found near the leading edges of the plates, although they were comparatively smaller in Case II. Also the velocity profiles for small distances from the entrance were found to be slightly concave in the central portion between the plates.

Schlichting and others have solved the boundary layer equation for Case I. Their solutions agree well with the present work at large distances from the entrance but deviate considerably near the leading edges as the boundary-layer equation does not describe the behavior of fluid flow near singular points.

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THERMAL DECOMPOSITION OF n-BUTANE

I. INTRODUCTION

Thermal Decomposition of Paraffinic Hydrocarbons

Extensive studies of the thermal decomposition of paraffinic hydrocarbons, especially lighter ones, have been made, and a comprehensive review of the work was presented by Steacie (1). The results showed that most of the decompositions roughly follow a first-order rate equation even though they are all very complex and involve free radicals. Furthermore, these decompositions are sensitized by free radicals and inhibited by substances such as NO and propylene which readily react with free radicals.

Rice (2) proposed a free-radical theory for the thermal decomposition of organic substances. In his theory the reaction is initiated by a splitting of the C-C bond. Two free radicals are thus produced, whose subsequent reactions decide the reaction products. This is reasonable as the C-C bond in organic substances is known to be considerably weaker than the C-H bond. The bond energy of a C-C bond is approximately 71.0 kcal compared with that of 93.3 kcal of a C-H bond.

To account for the products in the thermal decomposition of a number of saturated hydrocarbons, Rice proposed mechanisms based on his free-radical theory. Later, Rice and Herzfeld (3) showed that, in order to agree with the experimental facts, these mechanisms could be devised to lead to first-order reactions. Moreover, by a suitable choice of the activation energies of the individual reactions, the apparent activation energy of the overall reaction could be made to agree perfectly with the experimental value. Thus, in a general way, the kinetics of the decomposition of hydrocarbons can be explained on the basis of the Rice theory. The explanation involves certain rather definite assumptions about the rates of the individual reactions, and the validity of these assumptions can only be demonstrated by experiment.

The inhibition of hydrocarbon decomposition reactions has been exhaustively investigated by Hinshelwood and his co-workers (4,5). The principal inhibitor used was nitric oxide. As the amount of an inhibitor added was increased, the rate approached a limiting value greater than zero. With different inhibitors, identical limiting rates were obtained for many hydrocarbons. The investigators thus concluded that as the limiting rate was reached, the free-radical reactions were completely suppressed, with the residual rate corresponding to a molecular reaction. Although some evidence suggested that maximum inhibition does not necessarily correspond to complete suppression of free radicals (6,7), it is generally accepted that the thermal decomposition of paraffinic hydrocarbons occurs by simultaneous chain and molecular mechanisms.

Thermal Decomposition of n-Butane

Much work has been done on the thermal decomposition of n-butane, especially in the low-temperature range of 450-650°C. Both flow and static methods have been used. Pease and Durgen (8) studied the reaction in a flow reactor at atmospheric pressure and in a temperature range of 600-650°C. As did many other earlier investigators, they found the reaction to be homogenous and approximately first order, with an activation energy of about 65,000 calories. The main overall reactions were found to be:

$$C_4H_{10} \rightarrow C_3H_6 + CH_4 \tag{1}$$

$$C_4H_{10} \rightarrow C_2H_4 + C_2H_6 \tag{2}$$

The first-order rate constant greatly decreased as the reaction proceeded, and that was attributed to secondary reactions such as rehydrogenation. Within the temperature range studied, the relative amounts of various products were found to remain fairly constant.

Another investigation on this subject was carried out by Steacie and Puddington (9) using a static method at temperatures from 513-572°C and pressures from 5 to 60 cm of Hg. The reaction was studied by measuring the rate of pressure increase and by analyzing the products at various times. Like Pease and Durgen, they found the first-order rate constant in a given run to decrease rapidly as the reaction progressed, and also with diminishing pressure. These two facts suggested that chain processes might be involved. The initial high-pressure rate constant was calculated to be

$$k = 5.1 \times 10^{12}\, e^{\frac{-58,700}{RT}} \ \text{sec.}^{-1} \tag{3}$$

which has been generally accepted.

Later, Echols and Pease (10) investigated the decomposition of n-butane in a Pyrex bulb coated with KCl. The work was done at temperatures from 480° to 535°C and pressures from 100 to 600mm of Hg. They decided, contrary to previous conclusions, that the reaction could not be first-order, and their data corresponded to the equation

$$-\frac{dp}{dt} = k_1 p^{1.5} + k_2 p^{2.0}/(p_o - p) \tag{4}$$

where p is the partial pressure of butane at time t, and p_o that at t = 0. An activation energy of 58 kcal was calculated for the reaction which predominated after the high initial rate. The main course of the reactions were given as

$$C_4H_{10} \rightarrow CH_4 + C_3H_6 \tag{1}$$

$$C_4H_{10} \rightarrow C_2H_4 + C_2H_6 \tag{2}$$

$$C_4H_{10} \rightarrow C_4H_8 + H_2 \tag{5}$$

At 520°C and 25% decomposition, the mole ratio of the products $(CH_4 + C_3H_6)$: $(C_2H_4 + C_2H_6)$: $(C_4H_8 + H_2)$ was 10:4:1.

Frey and Hepp (11) studied the thermal decomposition of several light paraffins in a flow system. For n-butane, they obtained the first-order reaction rate constant

$$k = 3.4 \times 10^{13}\, e^{\frac{-61,400}{RT}} \text{ sec.}^{-1} \tag{6}$$

The investigation, however, was conducted only at two different temperatures. By means of low-temperature fractional distillation, Frey and Hepp were able to analyze the products completely. At 575°C and 11% reaction, the mole ratio of the products $(CH_4 + C_3H_6)$:$(C_2H_4 + C_2H_6)$:$(C_4H_8 + H_2)$ was 6:3:1.

In all cases mentioned above, the extent of decomposition of butane usually ranged between 20-50%. For the purpose of identifying the initial products, the percentages of products formed for finite conversions should be extrapolated to their values at zero decomposition. Since secondary reactions could become significant after, say, 10% decomposition, errors involved in these extrapolations could be very large. Crawford and Steacie (12) carried out the reaction at pressures of 4-13 cm of Hg and at 442°C, with emphasis on low conversion. Due to analytical difficulties, only H_2, CH_4, C_2H_4 and C_2H_6 were determined. The rates of formation of these four gases were found to be constant from 1.5 to 9.0% conversion. As the extent of decomposition exceeded 10%, marked decreases in the rates of formation were noted. This was attributed to the inhibition of the reaction by the products (4). On the other hand, relative amounts of each product remained constant up to 24% conversion, agreeing with Pease and Durgen's results. The initial products obtained by extrapolation were: H_2, 4.8%, CH_4, 58%, C_2H_4, 24.5% and C_2H_6, 12.0%. The ratio of C_2H_4 to C_2H_6 was approximately 2:1, a value much larger than those obtained in other investigations.

Neuhaus and Marek (13) studied the decomposition of n-butane at 600° and 650°C in a flow reactor under atmospheric pressure for the purpose of determining the initial product distribution. The range of conversion was 2.4 to 6.0% at 600°C and 4 to 19% at 650°C. By interpolating to zero conversion, the number of moles of a product formed for each 100 moles of butane decomposed at 600°C was found to be 48.5 for methane and propylene, 34.5 for ethane and ethylene and 16.0 for hydrogen and butene. Similarly, at 650°C, these numbers were 48.0, 37.7 and 12.3 respectively. For higher-temperature work, Egloff and his co-workers (14) pyrolyzed n-butane in noncatalytic metal tubes at 600°-700°C with pressures from 1 to 7 Kg/cm^2. Cambron and Bayley (15) also studied the pyrolysis of butane and other lower paraffins in externally-heated quartz tubes under conditions of streamline and turbulent flow, the latter being obtained in baffled tubes. At high temperatures and very short residence times, the activation energy for the thermal decomposition of n-butane was found to be 43,000 cal/gm-mole. Both investigations, however, were carried out mainly to determine the yield of olefins under various conditions. More recently, Sandler and Y. H. Chung (16) investigated the reaction using both annular and tubular flow reactors made of porcelain or silica. Within the temperature range of 620°-700°C, they found the best value for the first-order reaction rate constant to be:

$$k = 9.55 \times 10^{9}\, e^{-\frac{45,600}{RT}} \text{ sec.}^{-1} \qquad (7)$$

Sandler and Chung pointed out that the discrepancy between values of activation energy reported, 43,000-74,000 cal/gm-mole, was largely due to heat transfer. In many flow-system studies, they claimed reported rates for thermal decomposition were actually heat-transfer rates.

As mentioned previously, the main course of n-butane decomposition can be represented by:

$$C_4H_{10} \rightarrow C_3H_6 + CH_4 \qquad (1)$$

$$C_4H_{10} \rightarrow C_2H_4 + C_2H_6 \qquad (2)$$

In actual cases, however, a preponderance of ethylene over ethane and methane over propylene (12, 15, 16) existed, even at low conversions. Rice (2) explained this result by proposing a mechanism based on his free-radical theory. In his mechanism, the primary steps are:

$$C_4H_{10} \rightarrow CH_3^{\cdot} + C_3H_7^{\cdot} \qquad (8)$$

$$C_4H_{10} \rightarrow 2C_2H_5^{\cdot} \qquad (9)$$

$$C_3H_7^{\cdot} \rightarrow C_2H_4 + CH_3^{\cdot} \qquad (10)$$

followed by chain processes of the usual type:

$$C_4H_{10} + R^{\cdot} \rightarrow RH + C_4H_9^{\cdot} \qquad (11)$$

$$C_4H_9^{\cdot} \rightarrow C_2H_4 + C_2H_5^{\cdot} \qquad (12)$$

$$C_4H_9^{\cdot} \rightarrow 2C_2H_4 + H^{\cdot} \qquad (13)$$

$$C_4H_9^{\cdot} \rightarrow C_3H_6 + CH_3^{\cdot} \qquad (14)$$

where $R = CH_3^{\cdot}$, $C_2H_5^{\cdot}$, or $H^{\cdot}$

The existence of free radicals in the thermal decomposition of n-butane was demonstrated by Rice, Johnston and Evering (17). At 800°-1000°C, they found the free radicals were 70% methyl and 30% ethyl.

The effect of additives on the thermal decomposition of n-butane has also been exhaustively investigated. The reaction was inhibited by NO, and propylene (4,5,6,7,10,18), with NO being 20 to 30 times more effective than the latter (10). Just as for other paraffins, the relative amounts of various products in the case of the fully-inhibited decomposition of n-butane were roughly constant over the pressure range of 2 to 1400 mm at 530°C to 620°C (18). From these results, Peard, Stubbs and Hinshelwood (19) concluded that the pyrolysis of n-butane proceeded by simultaneous chain and molecular mechanisms. A free-radical reaction mechanism due to Rice has been given in the last paragraph. The molecular reactions of n-butane are those given by equations 1, 2 and 4.

The effect of a small amount of oxygen on the decomposition of n-butane was studied by Appleby, Avery, Meerbott and Sartor (20) from 486° to 526°C under both static and flow conditions. The rate of decomposition was increased a hundred-fold with addition of 0.5% oxygen. The initial-reaction products at low oxygen concentrations consisted exclusively of butenes and water. Thus, oxygen probably acted primarily as a dehydrogenating agent under these conditions.

Reasons for the Present Work

A brief review of previous studies on the thermal decomposition of n-butane showed the need for further investigation. Discrepancies among the experimental results often existed. For instance, values of the activation energy from 43,000 to 74,000 cal/gm-mole were reported. Distribution of the initial reaction products also varied considerably. Even the order of the reaction is still an open question. Part of the variability in results probably is attributable to the chain-nature of the reaction. In addition, however, some of the variations appeared to have resulted from lack of care in collection and analysis of the data. One of the possible contributors to the spread in the results was the difficulty in extrapolating product compositions to zero conversion. Most of the butane decomposition studies were made in the range of 20 to 50% decomposition without extrapolation to zero conversion. Because secondary reactions become significant after 10% decomposition, extrapolations for the reported investigations involved large uncertainty. Low-conversion investigations, on the other hand, involved the determination of products existing at very low concentrations.

Another error was involved in the mathematical treatment of the reactors. In both batch and flow reactors, energy transport must be properly considered, and it is not clear in the literature if such transport was under control.

The present work was designed to avoid the sources of error just discussed. A flow reactor in which the temperature, velocity, and composition fields were established was used to investigate the thermal decomposition of n-butane at low conversions in the temperature range of 460° to 560°C. The composition of the reacting mixture was determined accurately by gas chromatography. With low-conversion data, extrapolation to zero conversion was more easily treated. Secondary reactions were not significant under the conditions studied, and the results should give a more satisfactory understanding of the mechanism of decomposition of n-butane.

II. APPARATUS

A schematic diagram of the reactor assembly is shown in Figure 1. The reactor was originally constructed by Kalvinskas (21) for studying the partial oxidation of n-butane. Since that work, however, a large number of modifications have been made to improve the performance of the reactor. The assembly now has the following sections: (1) a feed and flow-regulation section, (2) a preheating section, (3) the reactor, (4) a reactor head, (5) a micromanometer, and (6) a vacuum system. Temperature measurements were made on a Leeds and Northrup type K-3 potentiometer mounted on a central instrument bench in the laboratory. Reaction samples were analyzed by gas chromatography. A description of the chromatograph is given in Appendix A.

Feed and Flow Regulation

Instrumental grade n-butane was used as the feed and was obtained from the Matheson Company in No. 1A cylinders. For temperature control, a hot-water line made of 3/8-inch copper tubing was coiled around the cylinder in use. A Fiber-Glas blanket, which covered both the hot-water line and the cylinder, served as insulation.

From the storage cylinder, butane flowed through a double-gauge pressure regulator into a surge tank. The surge tank was a steel cylinder with a volume of approximately 0.5 cu. ft. The insertion of a surge tank was found necessary in order to reduce fluctuations in the flow.

A needle valve at the exit of the surge tank was used to control the flow of butane into the reactor. From the valve, butane passed through a tee before entering a temperature conditioner. As shown in Figure 1, the other arm of the tee could be connected to either a compressed air line or a nitrogen cylinder. Nitrogen was passed through the reactor between runs. Air, if necessary, could be run through the heated preheating section and reactor to burn out any carbon deposited on the walls from the cracking of butane.

The temperature conditioner consisted of 20 ft. of 1/4-inch, 304-stainless-steel tubing coiled and submerged in a constant-temperature water bath. The bath temperature was controlled to within 0.5°C.

From the temperature conditioner, the gas entered a Fischer and Porter Tri-Flat Rotameter. Use of either a Size 02F-1/8-25-5 tube with a sapphire float or a Size 8F-1/16-16-4 tube with a stainless-steel float allowed a range of flow from 0.004 to 0.075 cu. ft. of butane per minute at 70°F and 760 mm Hg pressure. A six-inch mercury manometer connected to the exit side of the Rotameter measured the gas pressure at that point.

Preheating Section

Butane leaving the Rotameter flowed through the preheating section before entering the reactor. This section consisted of a pre-preheater and a preheater, each with two parallel lines. One of these lines was installed to carry oxygen in partial oxidation studies.

The pre-preheater was made of two 10 ft. lengths of 1/8-inch stainless-steel tubing wound side by side into a helical coil having a diameter of 4 inches and a height of about 9 inches, with both outflow legs passing through the center of the coil. The coil fitted snugly inside a steel casing having a length of 10 inches, an I. D. of 4 inches, and a wall thickness of 1/4 inch. A layer of zirconium oxide cement was coated over the outflow legs, and a 12-ohm heater of No. 22 Nichrome V wire was wrapped on top of the cement. A McDanel porcelain tube with an I. D. of 1 1/2 inches and a length of 12 inches was placed between the outflow legs and the inner surface of the coil. Two lengths of 35 ft. of No. 18 Nichrome V wire with a resistance of 14 ohms each were spiraled into 1/4-inch O. D. coils which in turn were wound around the porcelain tube. The assembly was then placed in a porcelain dome having a height of 13 inches and a diameter of 5 inches. A fourth heater with a resistance of 20 ohms was wrapped vertically on the dome through slits cut 1/2 inch apart around the circumference at each end. Layers of Micro-Quartz and Fiber-Glas over the dome provided the necessary insulation. The pre-preheater was set on a fire-brick base, and the voltage applied to each heater in the pre-preheater, the preheater, and the reactor was regulated by a separate General Radio Variac Type V-5.

The preheater was made of two identical pieces of 3/16-inch stainless-steel tubing, each having a length of 30 inches which could be extended if needed to 40 inches. One end of the preheater was connected to the pre-preheater and the other end to a gas-mixing chamber. A portion of the preheater at the end next to the mixing chamber was housed in a stainless-steel guide tube which had an O. D. of 15/16 inch and a length of 19 inches. A heater ribbon having a resistance of 20 ohms and a width of 1/16-inch was wrapped helically with approximately 1/16-inch spacings between turns on the portion of the preheater tubes inside the guide tube. This ribbon was insulated on both sides with a 1/8-inch layer of asbestos tape.

Two more heaters were wound in series over the portion of the preheater tubes outside the guide tube. Each heater had a resistance of 10 ohms and was 20 ft. of No. 18 Nichrome V wire spiraled into 1/4-inch O. D. coils. They were placed in quartz fiber sleeves before being put on the preheater. A layer of Micro-Quartz insulation was inserted between the preheater tubes and these heaters to provide further electrical insulation.

When the preheater was connected to the reactor, part of the preheater together with the mixing chamber was mounted inside the reactor. The length of this part was varied to obtain different reactor lengths. The guide tube was clamped to a screw jack which guided the movement of the preheater inside the reactor so that the preheater and the reactor would not touch each other. The relative preheater position was measured to within ±0.02 inches with a steel ruler installed next to the screw jack. After the preheater was connected to the reactor, two additional coiled heaters having resistances of 10 ohms each were wrapped on the portion of the guide tube that was not in the reactor. The preheater was then insulated with several layers of Micro-Quartz followed by two layers of Fiber-Glas pipe insulation.

A schematic diagram of the gas-mixing chamber is shown in Figure 2. A function of this chamber was to mix the two inlet streams rapidly by direct impingement upon each other. Two platinum-platinum-10% rhodium thermocouples made of 5-mil wires were mounted in the chamber. The first thermocouple was in the stainless-steel body about 1/32 inch from the surface as shown in Figure 2. The second thermocouple went through the wall of the chamber and was bent 90° at the wall to fix the junction at the center of the outlet side of the chamber. The second thermocouple thus measured the gas-stream temperature at the entrance of the reactor. To insure uniformity in the gas temperature as well as to reduce its volume, the mixing chamber was packed with porcelain pebbles, as in Figure 2.

The thermocouple wires in the mixing chamber were placed in doubleholed porcelain protection tubes to prevent contact with the wall of the chamber. Astroceram cement was used sparingly to mount the porcelain tubes as well as to seal the wires from the gas. The lead wires from the thermocouples, insulated in single-holed porcelain protection tubes broken to 1/2-inch lengths, were led down along the preheater tubes inside the guide tube before branching off.

In order to obtain a flat velocity distribution at the inlet of the reactor, the feed gas was passed through a quartz nozzle before entering the reactor. The nozzle was made of a quartz tube having a length of 1 1/4 inches, an I. D. of 23 mm, and an O. D. of 25 mm. At the upper end, the nozzle was packed with 1/2-inch lengths of quartz tubing, each with an O. D. approximately of 3/64 inch and an I. D. of 1/32 inch. The lower end of the nozzle fitted snugly to the outside wall of the mixing chamber as shown in Figure 2. A thin layer of zirconium oxide cement between the shoulder of the chamber and the nozzle insured their attachment to each other.

Reactor

The reactor itself consisted of two concentric McDanel, high-temperature, porcelain tubes. The inside tube was the reactor tube with an I. D. of 1.0 inch, an O. D. of 1.25 inches, and a length of 30 inches. The outside tube, which was a heater tube, had an I. D. of 1.50 inches, an O. D. of 1.75 inches, and the same length as the reactor tube. These two tubes were held together at both ends by two spacers made of Transite. The reactor was mounted vertically on a steel flange, which in turn was held in place by three bolts on a bracket mounted inside the reactor bay at a height of 37 inches from the floor. The preheater entered the reactor through the center of the flange. A stainless-steel nut at the bottom side of the flange compressed two silicone rubber gaskets to form a gas-tight seal. At the top of the flange, sealing was provided by asbestos string impregnated with a mixture of graphite powder and lubricating oil. The string filled the space between the outer reactor tube and the flange opening and was compressed by a steel ring screwed to the flange. A schematic drawing showing the cross section of the bottom section of the reactor is given in Figure 3.

Five thermocouple holes were drilled six-inches apart along the wall of the heater tube. Through each hole, a chromel-alumel thermocouple housed in a double-holed porcelain protection tube was inserted with its junction touching the outer wall of the inner tube. Astroceram cement was again used to hold the porcelain tubes to the wall of the heater tube and to seal the thermocouple wires from the surrounding atmosphere. A heater made of 30 ft. of No. 20 Nichrome V wire was wrapped around the outer tube between each pair of thermocouples. Asbestos string was used to space the wire to avoid electrical shorting, and a layer of asbestos tape 1/8-inch thick was wrapped tightly over the heaters to hold them in place. Over the tube walls above the first thermocouple and below the fifth one, two 3-ft. Super-Cord heaters made by Glas-Col Company were installed. Each of these heaters had a maximum wattage of 300 and was usable up to 600°C. Insulation consisting of layers of Micro-Quartz, Micro-Glas, and Fiber-Glas completely filled the space between the reactor and a cylindrical steel shield which had a diameter of 14 inches.

Reactor Head

The reactor head was mounted on top of the reactor tubes. The top and side views of the reactor head are shown in Figure 4. A schematic cross-sectional diagram is given in Figure 5. The lower part of the head was a pressure tap which consisted of two concentric rings. The inner ring was made of soapstone and had an I. D. of 1.0 inch, an O. D. of 2.66 inches, and a thickness of 5/8 inch. The outer ring was made of steel and was 7/8 inch in thickness with an O. D. of 4 inches and an I. D. of 2.66 inches for the upper 5/8 inch and of 2 inches for the lower 1/4 inch. The soapstone part was press-fitted in the upper central portion of the steel ring. Two holes, 180° apart, having a diameter of 1/8 inch were drilled through both soapstone and steel for pressure taps. During operation, one tap was closed while the other one was connected to the static side of a micromanometer.

Next to the pressure tap was another set of soapstone and steel rings having the same diameters as the pressure tap rings but only 3/8 inch in thickness. Two grooves, 1/8 inch in diameter, were drilled along the contacting surfaces of these two sets of rings at right angles to the pressure-tap holes. A pitot-tube probe was placed in one of the grooves and a thermocouple probe in the other.

A picture showing both probes can be seen in Figure 6. The probe thermocouple was made of Pt-Pt-10% Rh wires having diameters of 0.003 inch. It was enclosed in a quartz tube which had an O.D. of 1/16 inch and was partitioned into four compartments. The quartz tube was bent 90^{o} at a distance of 2 1/2 inches from the junction. Part of this tube was then welded into a larger quartz tube for protection. The outer tube, having an O.D. of 7/64 inch, slid easily into the 1/8-inch groove. Similarly, the pitot tube was made of a quartz tube with an I.D. of approximately 0.024 inch and an O.D. of 0.06 inch. Part of the pitot tube was also welded to another 7/64-inch O.D. quartz tube, and the effective length of the probe was 2 1/4 inches.

To allow precise positioning, each probe was fastened to a syringe externally mounted. Two 1/8-inch holes were drilled at the center on both ends of each syringe piston to allow the probe to pass through. The outer end of the piston was clamped to a brass frame which in turn was connected to a micrometer head.

To hold each probe to the piston, a silicone rubber gasket was compressed between the end of the piston and the brass frame. When the reactor head was mounted onto the reactor, the probes were placed inside the inner reactor tube. Movement of a micrometer head caused the piston to move inside the syringe barrel and consequently the probe to move across the reactor tube.

The pitot tube was connected both to the kinetic side of the micromanometer and to a sampling line through a tee. By clamping off one of the connecting lines, the pitot tube was used to measure the velocity profile or to take reaction samples across the reactor.

Micromanometer

The micromanometer used in the velocity measurements was similar to a differential pressure gauge. As shown in Figure 7, two identical reservoirs were connected at the bottom by a capillary glass tubing. The area ratio between the reservoirs and the capillary tubing was approximately 1000 to 1. The manometer fluid used was methyl alcohol. Instead of using a liquid drop to observe the movement, an air bubble was inserted in the capillary tubing by means of a syringe through a rubber cap on the side. One reservoir was connected at the top to the pressure tap in the reactor head. The other reservoir could be connected to either the pressure tap or the pitot tube by means of two stopcocks.

Any pressure difference between the pressure tap and the pitot tube was measured in terms of the bubble movement. Silicone grease was used for the stopcocks since it does not readily dissolve in methyl alcohol. To insure a minimum of fluctuations in temperature, the micromanometer was covered with a Lucite hood.

Sampling and Vacuum Systems

Figure 1 shows that the sampling line leaving the reactor led to a vacuum valve. A sampling bulb having a capacity of 250 cc could be placed between the valve and a vacuum line to collect samples from the reactor. A total of 15 sampling bulbs was available. Samples used for chromatographic analysis were drawn from these bulbs by means of a separate manifold system.

The upper part of the reactor head was a stainless-steel nozzle which converged to a minimum diameter of 1/8 inch. A sampling port with a diameter of 0.03 inch was located 2 inches downstream from the nozzle and also connected to the sampling line. Samples were taken both from the sampling port and from the pitot tube.

In operation of the reactor, gases were pulled past the port and into the atmosphere by way of a water-jet exhauster, a stripping column, an air compressor, a heat exchanger and a refrigeration unit.

III. REACTION KINETICS

Behavior of a Tubular Flow Reactor

A flow system, in most cases, is defined if its temperature, pressure, density, composition and velocity are known at every point in the system. For a system of n components flowing in a tubular flow reactor with radial symmetry, therefore, there will be a total of 5 + (n-1) variables. These variables are governed by the same number of equations, namely, the energy balance equation, the equation of continuity, equation of state, two momentum balance equations in longitudinal and radial directions respectively, and material balance equations for n-1 components. These equations are all interrelated and highly non-linear. Although they have been discussed in some detail in many standard textbooks on transfer phenomena, no solution of the complete set of these equations has ever been attempted.

If the fluid is a perfect gas and the flow is under laminar and steady-state conditions, the set of governing equations for the system in question can be written as follows:

(a) Equation of State

$$\rho = PM/RT \tag{15}$$

(b) Equation of Continuity

$$\frac{1}{r}\frac{\partial}{\partial r}(r\ \rho\ u_r) + \frac{\partial}{\partial x}(\rho\ u_x) = 0 \tag{16}$$

(c) Material Balance for Component k

$$\frac{1}{r}\frac{\partial}{\partial r}(r\ \rho_k\ u_r) + \frac{\partial}{\partial x}(\rho_k\ u_x) - \frac{1}{r}\frac{\partial}{\partial r}\left(D_k\ r\ \frac{\partial \rho_k}{\partial r}\right) - \frac{\partial}{\partial x}\left(D_k\ \frac{\partial \rho_k}{\partial x}\right) - R_k = 0 \qquad (17)$$

(d) Energy Balance Equation

$$\rho\ C_p\left(u_r\ \frac{\partial T}{\partial r} + u_x\ \frac{\partial T}{\partial x}\right) - \frac{1}{r}\frac{\partial}{\partial r}\left(k_c\ r\ \frac{\partial T}{\partial r}\right) - \frac{\partial}{\partial x}\left(k_c\ \frac{\partial T}{\partial x}\right) - \lambda R = 0 \qquad (18)$$

(e) Momentum Balance in x-direction

$$\rho\left(u_x\ \frac{\partial u_x}{\partial x} + u_r\ \frac{\partial u_x}{\partial r}\right) = - g\ \frac{\partial P}{\partial x} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\ \frac{\partial u_x}{\partial r}\right) + \frac{\partial^2 u_x}{\partial x^2}\right] \qquad (19)$$

(f) Momentum Balance in r-direction

$$\rho\left(u_x\ \frac{\partial u_r}{\partial x} + u_r\ \frac{\partial u_r}{\partial r}\right) = - g\ \frac{\partial P}{\partial r} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\ \frac{\partial u_r}{\partial r}\right) + \frac{\partial^2 u_r}{\partial x^2}\right] \qquad (20)$$

A number of assumptions have been made in writing down some of these equations. For example, in the momentum balance equations, the viscosity gradients have been neglected. In the component material balance equation, diffusion other than molecular is not considered. In the energy balance equation, only the conductive and convective thermal energy transports plus the heat sink due to chemical reaction are considered important.

Theoretically, equations 15 to 20 can be solved for the 5 + (n-1) variables, provided the boundary conditions are given. In actual cases, so far, such a task has not been achieved. Some simplified approaches are usually adopted. In connection with the present investigation, a project has been initiated to solve this complete set of equations by some numerical method for the thermal decomposition of n-butane in a tubular flow reactor. Only partial success has been accomplished so far, mainly due to instability of the numerical schemes used.

Simplified Tubular Flow Reactor Equation

In the study of chemical kinetics by means of a tubular flow reactor, simplifications are usually made to reduce equations 15 to 20 to only one equation. In the simplified approach, the reactor is considered to be isothermal as well as isobaric, and the composition of the reacting mixture is assumed to be uniform with respect to the radius. Further, longitudinal diffusion is neglected. Under these conditions, only equation 17 has to be considered, which now assumes the form

$$\frac{\partial}{\partial x} (u_x \rho_k) - R_k = 0 \tag{21}$$

Equation 21 is equivalent to the more widely used design equation for flow reactors under steady-state conditions,

$$F \, dX_k = R_k \, dV \tag{22}$$

Equation 22 can be integrated to give

$$\frac{V}{F} = \int_0^{X_k} \frac{dX_k}{R_k} \tag{23}$$

If the relationship between R_k and X_k is known, equation 23 can be used to determine the dependence of X on the ratio V/F. On the other hand, equation 23 can also be used to determine, by trial and error, a proper expression for R_k in terms of X_k if a set of data of X vs V/F is available. In the latter case, several expressions for R_k are usually derived based on theoretical considerations. Each of these expressions is substituted into equation 23. The resultant relation between V/F and X_k is then compared with the experimental data to determine which expression for R_k gives the best fit and hence is probably correct. Presently, the relationship between X and V/F had been obtained experimentally for n-butane, equation 23 was then used to determine the correct rate expression.

Two Possible Rate Expressions for Butane Decomposition

Experimental evidence indicates that the thermal decomposition of n-butane proceeds through a free-radical chain mechanism. A free-radical chain reaction generally consists of four major steps, namely,

(a) Chain-initiating:

$$R_iR_j \xrightarrow{k_a} R_i\cdot + R_j\cdot \tag{24}$$

(b) Chain-carrying:

$$R_iH + R_j\cdot \xrightarrow{k_b} R_jH + R_i\cdot \tag{25}$$

$$R_i\cdot \xrightarrow{k_c} R_j\cdot + M \tag{26}$$

(c) Chain-terminating:

$$R_i\cdot + R_j\cdot \xrightarrow{k_d} R_iR_j \tag{27}$$

In these steps, $R_i\cdot$ and $R_j\cdot$ are free-radicals, R_iR_j and R_iH are stable molecules and M is an unsaturated compound such as an olefin. In a system in which n-butane is the only stable species existing in abundance, the rate of decomposition of n-butane proceeds, therefore, according to equations 24 to 27,

$$-\frac{d(C_4H_{10})}{dt} = (\sum_n k_{a_n})\ (C_4H_{10}) + [\sum_j k_{b_j}(R_j)]\ (C_4H_{10}) \tag{28}$$

assuming that the contribution from the chain-terminating step is negligible. The order of the overall reaction thus depends directly on the relationship between the butane concentration and the free-radical concentrations.

The determination of the concentrations of the free-radicals usually involves the assumption that these concentrations are in the steady state, or

$$\frac{d(R_i\cdot)}{dt} = 0 \tag{29}$$

For j free-radicals, j equations can be set up to be solved for the j concentrations. These equations are in general second-order algebraic equations, however, and are very difficult to solve when n is larger than three. Even if a solution is obtainable for a system with more than three free-radicals, the resulting expressions for the free-radical concentrations would be usually too complicated to be helpful in determining the order of the overall reaction. This happens to be the case for the thermal decomposition of n-butane, which reaction involves five free-radicals from hydrogen to butyl.

On the other hand, it may be surmised that at a given temperature, the dependence of free-radical concentrations on butane concentration varies between zero and first order. The order of the decomposition reaction then lies somewhere between 1 and 2 according to equation 28. To verify this surmise, two reaction rate expressions were tested, one a first-order reaction and the other a second-order reaction.

In the analysis, X_b was taken as the conversion of n-butane at a point. For each mole of n-butane decomposed, $(1 + Z)$ moles of products were formed. A total of $(1 + ZX_b)$ moles for each mole of initial feed were then present at a given point. The mole fraction of butane at that point was thus $(1 - X_b)/(1 + ZX_b)$ and its density

$$\rho_b = \left(\frac{1 - X_b}{1 + ZX_b}\right)\frac{M_b P}{RT} \tag{30}$$

For a first-order reaction, the reaction rate can be written as

$$R_b = k'_{rb}\ \rho_b = k'_{rb}\left(\frac{1 - X_b}{1 + ZX_b}\right)\frac{M_b P}{RT} \tag{31}$$

where k'_{rb} is the first-order reaction rate constant. Substitution into equation 23 gives

$$\frac{V}{F} = \frac{RT}{M_b P k'_{rb}} \int_0^{X_b} \frac{1 + ZX_b}{1 - X_b}\, dX_b \tag{32}$$

When a value of unity was used for Z in accordance with the results shown in Tables VIII to X, equation 32 can be integrated to give

$$k'_{rb} \left(\frac{M_b P}{RT}\right) \left(\frac{V}{F}\right) = - X_b + 2 \ln \frac{1}{1 - X_b} \tag{33}$$

or

$$k'_{rb} = \left(\frac{RT}{M_b P}\right) \left(\frac{F}{V}\right) \left(- X_b + 2 \ln \frac{1}{1 - X_b} \right) \tag{34}$$

Similarly, when the reaction is second-order,

$$R_b = k''_{rb}\, \rho_b^2 = \left(\frac{M_b P}{RT}\right)^2 \left(\frac{1 - X_b}{1 + X_b}\right)^2 k''_{rb} \tag{35}$$

Equation 23 then assumes the form

$$k''_{rb} \left(\frac{M_b P}{RT}\right)^2 \left(\frac{V}{F}\right) = \left[X_b + \frac{4X_b}{1 - X_b} + 4 \ln (1 - X_b)\right] \tag{36}$$

or

$$k''_{rb} = \left(\frac{RT}{M_b P}\right)^2 \left(\frac{F}{V}\right) \left[X_b + \frac{4X_b}{1 - X_b} + 4 \ln (1 - X_b)\right] \tag{37}$$

When the operating conditions and conversion of a run are given, the reaction rate constants k'_{rb} and k''_{rb} can be evaluated for that run using equations 34 and 37 respectively. The reaction rate constant, k_r, is in general a function of temperature only and can be related to temperature by the Arrhenius equation

$$k = Ae^{-E/RT} \tag{38}$$

IV EXPERIMENTAL PROCEDURE

Analysis of Feed Butane

The feed butane was instrument grade (99.5% minimum purity) obtained from the Matheson Company in No. 1A cylinders in 110 pound lots. Three such cylinders were used for the thermal decomposition studies. Four or more samples had been withdrawn from each tank before use and the samples analyzed by gas chromatography. The average compositions of the samples from each tank are listed in Table I. The impurities were mostly trans-2-butene, isobutane, propylene and propane.

Calibration of Equipment

The two Rotameters used were calibrated for butane against a wet test meter over their entire ranges. Calibration was performed at room temperature for several pressures. The flow rates thus obtained were converted to flow rates for standard conditions, which were taken to be 70 °F and 760 mm Hg pressure. The data was then plotted as flow rate of butane at standard conditions against Rotameter readings for given pressures.

The 0.003-inch Pt and Pt-10% Rh wires used in making the probe thermocouple were calibrated by the National Bureau of Standards over the range of 0 - 1100 °C. Calibration was reported to be accurate to within 0.5 °C. The Pt-Pt-10% Rh thermocouples at the mixing chamber had not been calibrated, but were assumed to behave identically as the probe thermocouple. Chromel and alumel thermocouples were also made using a single batch of wire. Seven thermocouples made from these wires were compared directly with the calibrated Pt-Pt-10% Rh thermocouple in a furnace over a temperature range of 380° - 600° C. At a given temperature, the deviation in emf readings among these seven thermocouples was less than 0.1 mV or roughly 2.5° C. Average values of these emf's were plotted against temperature, and the results were used in obtaining reactor temperatures from thermocouple readings.

Inside diameters at both ends of the reactor tube were measured with a micrometer and were 0.981 inch at the upper end and 1.020 inches at the lower end. An average value of 1.00 inch was adopted.

Relationship among the positions of the thermocouple and pitot tube probes and the corresponding micrometer readings were determined with a cathetometer, each with an estimated error of ±0.1 mm. The thermocouple was able to reach within 0.03 inch of the adjacent reactor tube wall and the pitot tube to within 0.15 inch. The bend in the pitot tube was slightly more than 90°, thus preventing it from reaching closer to the wall. When installed, the length of the thermocouple probe inside the reactor tube was 2.25 inches and that of the pitot tube was 2.0 inches.

Surface effects of the reactor tube wall, both in catalyzing the reaction and in terminating the free-radical chains have not been investigated experimentally. The catalytic effects of porcelain were surmised to be small due to consistency of the experimental results.

Operating Procedure

To start the reactor, high-purity nitrogen was allowed to flow through the reactor at a slow rate and exit into the room through a valve in the vacuum line. The variacs which controlled heaters on the reactor as well as the preheating section were turned on. Each variac was set at a voltage which would give approximately the desired temperature of operation. The temperature of the reactor wall, as well as the inlet gas temperature, was recorded continuously on a Micromax recording potentiometer. When steady-state was approached, the variacs were regulated whenever necessary in order to obtain a uniform operating temperature. Final adjustment generally required two to three hours.

An hour or so before starting a run, the flow of nitrogen was stopped and butane flow was started at a predetermined flow rate. The vacuum system was then turned on and the pressure in the reactor adjusted to be slightly above atmospheric (usually a head of 1/4 inch of methyl alcohol or less).

When steady, uniform temperature was reached, the velocity profile was first measured, followed by a complete set of temperature measurements. Samples of the reaction gases were then obtained, followed by a second set of temperature measurements. This completed one run. Each run usually required eight to ten hours. At the end of a run, the butane flow and the vacuum system were turned off. The heaters, however, were left on with nitrogen flowing slowly through the reactor. The heaters were turned off only when no runs were to be made over a period of several days or when repairs were required.

The velocity measurements at any traverse position in the reactor required measurement of the static and velocity heads. One arm of the micromanometer was permanently connected to the pressure tap. The other arm was also connected to the pressure tap when measuring the static head and was connected to the pitot tube for measuring the velocity head. The difference between these two readings gave the relative magnitude of the velocity at that point. Each static-head reading took about three minutes, while a velocity head required fifteen to thirty minutes, depending on the distance over which the air bubble had to travel.

Due to fluctuations in line pressures, flow rates, atmospheric pressures, etc., which all greatly affected the micromanometer readings, several sets of velocity traverse readings were taken for each run.

Thermocouple readings were usually obtained by means of a Type K-3 Universal Potentiometer. When the butane flow was low, however, fluctuations in thermocouple probe readings up to 10 °C were observed. It is suspected that such phenomenom was associated directly with fluctuations in butane flow. For these cases, the traverse readings were recorded continuously on a Speedomax recording potentiometer, with a period of five minutes for each reading. An average temperature was then determined for that position from the recording chart.

For a regular run, six gas samples were collected, four through the pitot tube at different traverse positions, one from the sample port at the reactor head, and one which was a duplicate sample of any of the first five. In runs during which no velocity measurements were made, usually only two samples were taken through the pitot tube, one at the center of the reactor and one 0.15 inch from the reactor wall.

In order to take a gas sample, the vacuum valve between the sampling bulb and the sampling line was closed.

The sampling bulb and attached tubing were evacuated by means of a vacuum pump. The vacuum pump was then cut off and the vacuum valve opened to let the reaction gas flow into the bulb. This procedure was repeated twice for purging purposes before a sample was taken.

In order to avoid leakage of air into the reactor and the sampling line and to obtain a representative sample at a given position, gas was throttled through the vacuum valve at a desired rate observable on a 30-inch mercury manometer attached to the sampling system. Two to three minutes were required to fill a bulb of 250 cc capacity to a pressure of one atmosphere. At that rate, the gas sample was withdrawn from the reactor fast enough to minimize further reaction, and yet slow enough to represent the actual composition at the point of sampling.

Samples thus collected were occasionally stored in the sampling bulb for 24 hours before analysis. Several duplicate samples taken up to 24 hours apart from the same bulbs were analyzed. No noticeable difference in composition within each set of duplicate samples was detected as may be noted in Table II.

V. EXPERIMENTAL PROGRAM

Temperature, reactor length and feed rate of butane were the variables investigated. Pressure was held constant at approximately 745 mm Hg, the atmospheric pressure in the Laboratory.

Temperature

The thermal decomposition of n-butane was investigated in essentially an isothermal reactor with temperatures centering about three values, namely, 460°, 510° and 560°C. In each group, about twenty runs were performed. Despite efforts to attain reaction temperatures exactly at these three values, there were deviations up to 11°C. These deviations, however, were taken into considerations in the determination of the reaction kinetics.

Reactor Length

For each run, the reactor length was either 18 inches or 22 1/4 inches, which was the distance between the reactor inlet and the pitot tube opening. The reactor length was varied to observe end effects and possible variations in the reaction products and velocity profiles for the same retention times. About one-fourth of the runs was made at a reactor length of 22 1/4 inches and the rest at 18 inches. In general, no appreciable differences between the results of the two sets of runs were observed.

Flow Rates

The flow rates of butane were set at values which for the particular temperature levels of operation would allow reasonable amounts of reaction. For each temperature, the feed rates were varied several-fold to give a spread in the degree of decomposition. The range of nominal retention times for the three temperatures, in terms of reactor volume divided by the initial volumetric feed rates, and the amounts of butane decomposed were:

t, °C	Retention Time, sec	Butane Decomposition, %
460	4 - 40	0.05 - 0.65
510	3 - 15	0.5 - 2.3
560	3 - 7.3	3.7 - 7.8

VI. RESULTS

Temperature and Its Control

Because the rate constant of a chemical reaction varies exponentially with the inverse of absolute temperature, it is very important in conducting rate experiments to have precise temperature control. In the present work, attempts were made to maintain the reactor isothermally at a predetermined temperature by adjustment of the variacs controlling all the heaters. The voltage input to these variacs was controlled at 115 ± 0.5 volts by an electromechanical voltage regulator, which thus eliminated possible variations in reactor temperature with time due to fluctuations in line voltages.

The reactor wall temperature at any section could be controlled to within ± 2°C. Because of end effects and the long time required to reach a new steady state after each adjustment of the variac settings, however, the maximum difference in temperatures among different sections of the wall for a run usually amounted to 5°C or higher. Wall temperatures for several typical runs are given in Table III. It can be seen in Table III that the temperature at the No. 1 thermocouple was often several degrees higher than the others. This situation was due to the intentional overheating of the section of the reactor tube above the No. 1 thermocouple to compensate for end effects due to the reactor head. Moreover, the wall temperature recorded by the thermocouple just below the reactor inlet usually varied considerably from the average reactor wall temperature. This temperature was an indication of preheating rather than reaction.

Gas temperature at the inlet of the reactor was controlled by the preheaters and the heaters on the reactor wall below the inlet nozzle. Heating of the feed gas in the preheating section was done gradually and mildly to avoid pyrolysis and carbon formation. Stainless steel begins to catalyze the decomposition of butane into carbon and hydrogen at about 600°-700°C (22). Butane, therefore, was heated to about 400°-500°C in the preheating section before the guide tube and was brought to the reactor temperature in the section enclosed in the guide tube. This latter section had a volume of 0.1 cu.in. compared with a reactor volume of 14 cu.in. with a reactor length of 18 inches. Reaction in the preheater was thus neglegible. Lack of pyrolysis in the preheating section was evidenced by the cleanness of the preheating tubing which was inspected after each series of runs. The gas inlet temperature was controlled to within ± 5°C of the average reactor wall temperature and ocassionally deviated as much as 10°C. In a number of runs, i.e. runs 20, 21, 28, and 34, the inlet temperatures were intentionally set at values up to 38°C lower than the average wall temperatures to study the effect of inlet temperature on the chemical reaction. In all instance, no appreciable effects were noticed.

The thermocouple probe mounted on the reactor head measured the temperature traverses across the reactor at a depth of

2 1/4 inches below the top of the reactor. At low flow rates, the traverse temperatures read low compared with the average wall temperature. The difference was as much as 40^{o}-$50^{o}C$ at the lowest flow rates. At higher flow rates, the traverse temperature approached that of the wall. This is illustrated in Figures 8 - 10. Furthermore, three types of temperature traverses can be observed. In one case, the temperature is high at the center and decreases toward the wall. Another type of temperature traverse has a W-shape. The third type, on the other hand, has an inverse W-shape. These phenomena are due to end effects, which are discussed in detail both in Appendix C and the Discussion section.

Flow Rate

The flow rate was measured with a Rotameter. Errors in flow-rate measurements were mainly due to uncertainties in obtaining the Rotameter reading. The maximum error was estimated to be from ± 1% at high flow rates to ± 3% at low flow rates.

Fluctuations in flow rates were observed for all the runs. One type of fluctuation was a high-frequency oscillation (~ 100 cycles per minute). This was attributed to oscillations in vacuum-line pressure due to vibration. Another type was a long-period drifting mainly due to variations in feed-tank temperature because of changing temperature in the hot water used to control the tank temperature. The drift in flow rate was corrected by occasional adjustment of the flow rate.

Velocity Measurement

Velocity measurement by means of a pitot tube and a micromanometer involved great uncertainties. For instance, Table El in Appendix E shows that the micromanometer reading corresponding to a 1.02 ft/sec linear velocity of butane at 560°C is only 0.77 cm. This was of the same order of magnitude as the noise level in the readings estimated at 0.1 cm. Despite this uncertainty, the taking of velocity measurements over a long period of time (up to eight hours or more) for each run and the making of duplicate runs gave consistent data.

Since the static-head pressure was measured at the pressure tap approximately three inches above the pitot tube, the velocity head was not that usually obtained with a pitot-static tube, but a combination of the velocity head at the pitot tube and the static pressure drop between the pressure tap and the pitot tube. The pressure drop, though unknown, was assumed to be independent of the pitot tube position for each run. The difference between the pitot-tube reading and the static-head reading, therefore, represented a relative measure of the linear gas velocity at the pitot tube.

Chromatographic Analysis

Gas samples withdrawn from the reactor were analyzed by gas chromatography. Description of the chromatographic system, its calibration, method of interpreting data, as well as its sources of error, are given in Appendix A. A gas sample stored in a sample bulb was transferred to a sampler before the sample was analyzed. The volumes of the samplers were calibrated with mercury and were nominally either 1.3 cc. or 0.8 cc. The pressure of the sample in the sampler was usually about 40-45 cm Hg and was measured accurately on a mercury manometer with a cathetometer. The exact amount of each sample was therefore known. After a sample had been analyzed, the area of each chromatogram peak was measured and the absolute amount of each component on the sample calculated. From the amount of each component and the total amount of the sample, the percentage composition of each component, with the exception of n-butane, was computed. The percentage of n-butane was obtained by difference since n-butane was present in more than 85% in all cases. In all these runs, the concentration variations across the reactor were usually very small and independent of radius.

Small amounts of air ($<$ 0.1% by volume) were quite often found in analysis of the samples. Since the ratio of nitrogen to oxygen agreed with that of atmospheric air, this air was believed to be introduced into the samples during transferring operations and, therefore, was not considered as part of the actual reaction mixture.

The accuracy of the chromatographic analysis was indirectly demonstrated by analyzing duplicate samples withdrawn from the same sample bulb. A number of these duplicate analyses were made. Some typical results are shown in Table II. The results also indicated that the storage of a sample in a bulb up to 24 hours did not affect the composition of the sample.

In order to check the steady-state assumption of the reactor operation, duplicate samples were collected from the same location in the reactor at different times during some runs. The chromatographic analysis of some of these samples are given in Table IV.

Summary of Results

A total of 58 isothermal runs were made for the thermal decomposition of n-butane. Of these runs, 19 were made centering around 460°C, 20 around 510°C, and 19 around 560°C. The operating conditions for these runs are listed in Tables V to VII. The given conditions for each run are the arithmetic average of the wall temperatures, the gas-inlet temperature, butane feed rate, length of the reactor, number of samples collected, and finally an indication of whether the velocity profile was measured for the run. The composition of the samples withdrawn from the reactor through the pitot tube are listed in Appendix B. The compositions of several samples withdrawn from the sampling port are also given in Appendix B. Because these samples were taken after the gas passed through a relatively cold region near the exit of the reactor their compositions are listed only for comparison purpose.

VII. CORRELATION OF DATA

Temperature

For a run, the traverse temperatures, wall temperatures and gas-inlet temperature, in general, were not identical. For the purpose of studying the reaction kinetics, the arithmetic average of the wall temperatures was used as the reaction temperature. This determination of the reaction temperature was justified by the following considerations:

(1) The measured temperature traverse was not a good representation of the gas temperature inside the reactor at low flow rates due to end cooling near the exit.

(2) Radial transport of thermal energy in the reactor was very rapid. For one thing, the thermal conductivity of n-butane is quite large at the operating temperatures as shown in Table G2 in Appendix G. Moreover, radial convection probably was even more important than conduction. Radial velocities were developed by expansion of gas due to chemical reaction and by the transformation of velocity profile from flat to parabolic due to viscous effects. Radiation, however, was relatively small at these temperatures. Since the difference between the gas-inlet temperature and the average wall temperature was usually small and the heat of

reaction was not significant at low conversions, the gas temperature at any depth inside the reactor was considered approximately the same as the wall temperature at that depth.

(3) If (2) is true, the actual reaction temperature was then the "reaction-rate average" of wall temperatures, an average based on reaction-rate considerations. If the wall temperatures did not differ from each other considerably, however, the arithmetic average of the wall temperatures approximates the "reaction-rate average" very well. See Appendix D for information on the method to calculate the "reaction-rate average" temperature and a comparison between the "reaction-rate average" and the arithmetic average.

Velocity Profile

The velocity data were collected in the form of kinetic-head and static-head readings at different radial positions. The difference between a kinetic-head and the corresponding static-head reading was a sum of two terms: the actual kinetic head and the difference in static pressure between the pitot tube and the pressure tap. Direct conversion of the manometer readings into velocities was hindered by the fact that the latter term was an unknown, although it was probably a constant in each run independent of radial position. In the determination of velocities, therefore, a velocity profile was first assumed, and the corresponding kinetic-head profile calculated. This kinetic-head profile was then compared with the experimentally-obtained profile to determine whether the assumed velocity profile was the correct one or not. It was found that in all cases the assumption of parabolic velocity profiles agreed well with the experimental results. Further detailed discussion on reduction of the pressure data to obtain velocities is given in Appendix E.

Reaction Products

From the composition of the reaction mixtures, the following information was obtained for each run at two radial positions, $r/r_o = 0.0$ and 0.7: (1) the number of moles of butane decomposed per 100 moles of feed, (2) a carbon-hydrogen balance for the reaction mixture excluding butane, and (3) the number of moles of each product formed per mole of butane decomposed. The method of calculation is shown in Appendix F. In these calculations, the presence of impurities in the feed was neglected. The values of the number of moles of butane decomposed, the carbon-hydrogen mole ratio, and the total number of products formed per mole of butane decomposed for each run are listed in Tables VIII to X. For each set of runs at a given temperature, the number of moles of each product formed per mole of butane decomposed, P_i, was found to be independent of the degree of decomposition for the low-conversion ranges investigated. A plot of several P_i's vs. degree of decomposition of butane is given in Figure 11. The average values for P_i and their standard deviations are listed in Table XI.

Reaction Kinetics

It has been surmised in the Reaction Kinetics section that the order of reaction for the thermal decomposition of n-butane lies between 1 and 2. For the purpose of determining the kinetics, two rate expressions, one a first-order and the other a second-order, had been developed. For each run, the reaction rate constants corresponding to each of these two rate expressions were calculated from equations 34 and 37 respectively for radial positions $r/r_o = 0.0$ and 0.7. The radial position $r/r_o = 0.7$ was as close to the wall as the pitot tube could reach. Difference between these two sets of data thus represents the variation in results across the reactor. A total of four sets of rate constants were thus obtained. A least-squares fit between ln k and 1/T was then carried out for each of the four sets according to equation 38,

$$\ln k = \ln A - E/RT \qquad (38)$$

Values of A and E thus determined, together with their standard deviations, are given in Table XII. Plots of ln k vs 1/T for $r/r_o = 0.7$ for both rate expressions are shown in Figures 12 and 13.

Table XII shows that the experimental data agree with both first- and second-order rate expressions equally well. This is partly because the present work was performed only at one pressure with low conversions. Under such conditions, the concentration of butane varied slightly both within a run and from run to run. The effect of concentration on reaction rate was thus not easily detectable. On the other hand, the agreement does indicate that the order of the reaction lies somewhere between 1 and 2, as predicted.

The values of rate constants obtained by the least-squares fits are also given in Table XII. It can be seen that for a given rate expression, the difference between the rate constants at different radial positions increases with temperature, with those near the wall being the larger. This shows that more reactions occurred near the wall than at the center, even at low conversions probably because of the finite rate of heat transfer, although the difference is very small.

Reaction Mechanism

A reaction mechanism for the thermal decomposition of n-butane is now proposed based on the present experimental results. This mechanism describes the reaction behavior of n-butane for low conversions in the temperature range of 460° to 560° C. The following restrictions have been imposed: (1) In this reaction, the formation of molecules or free radicals with more than four carbon atoms is neglected. Such formation occurs when either two free radicals recombine to form a stable molecule or a free radical is added to a stable molecule to form a larger free radical. The former will produce no molecules with more than four carbon atoms according to (4). The latter case occurs only when the stable molecule is an unsaturated compound (23). At low conversions, the presence of butenes is small. Their reactions with free radicals thus are negligible. Propylene reacts more readily with hydrogen or methyl radicals than with ethyl radicals, thus forming free radicals with no more than four carbon atoms. Reactions of ethylene with free radicals do not produce free radicals with more than four carbon atoms, again according to (4).

(2) Only five free radicals, i.e., $H\cdot$, $CH_3^{\cdot}$, $C_2H_5^{\cdot}$, $C_3H_7^{\cdot}$, $C_4H_9^{\cdot}$, are of importance. Their concentrations are at steady-state. Formation of free radicals such as $CH_2^{\cdot}$ is not likely due to energy considerations.

(3) Free radicals larger than $C_2H_5^{\cdot}$ will decompose into an olefin and a smaller radical rather than react directly with another free radical or a molecule (24).

(4) Chain-carrying reactions involving the splitting of a carbon-carbon bond in a butane molecule such as

$$C_4H_{10} + H\cdot \rightarrow C_3H_7^{\cdot} + CH_4 \qquad (39)$$

are considered not very probable and thus can be omitted (2).

(5) At low conversions, reactions involving stable reaction products, with the possible exception of propylene, are not important and are thus neglected. This is because methane and propylene are the most important reaction products and yet methane is very stable at the temperature range investigated.

With these restrictions in mind, the proposed mechanism may be represented by the following elementary reactions.

Chain Initiating:

$$\text{(I)}\quad C_4H_{10} \xrightarrow{k_1} CH_3\cdot + C_3H_7\cdot \qquad \text{with } k_1 = 10^{13}e^{-\frac{79,000}{RT}} \tag{40}$$

$$\text{(II)}\quad C_4H_{10} \xrightarrow{k_2} 2C_2H_5\cdot \qquad k_2 = 10^{13}e^{-\frac{82,000}{RT}} \tag{41}$$

Chain Carrying:

$$\text{(III)}\quad C_4H_{10}+H\cdot \xrightarrow{k_3} C_4H_9\cdot+H_2 \qquad k_3 = 10^{13}e^{-\frac{9,300}{RT}} \tag{42}$$

$$\text{(IV)}\quad C_4H_{10}+CH_3\cdot \xrightarrow{k_4} C_4H_9\cdot+CH_4 \qquad k_4 = 10^{11}e^{-\frac{8,300}{RT}} \tag{43}$$

$$\text{(V)}\quad C_4H_{10}+C_2H_5\cdot \xrightarrow{k_5} C_4H_9\cdot+C_2H_6 \qquad k_5 = 10^{11}e^{-\frac{7,000}{RT}} \tag{44}$$

$$\text{(VI)}\quad C_2H_5\cdot \xrightarrow{k_6} C_2H_4+H\cdot \qquad k_6 = 10^{14}e^{-\frac{39,500}{RT}} \tag{45}$$

$$\text{(VII)}\quad C_3H_7\cdot \xrightarrow{k_7} C_3H_6+H\cdot \qquad k_7 = 10^{14}e^{-\frac{46,000}{RT}} \tag{46}$$

$$\text{(VIII)}\quad C_3H_7\cdot \xrightarrow{k_8} C_2H_4+CH_3\cdot \qquad k_8 = 10^{14}e^{-\frac{26,000}{RT}} \tag{47}$$

(IX) $C_4H_9\cdot \xrightarrow{k_9} C_4H_8+H\cdot$ $k_9 = 10^{13}e^{-\frac{29,000}{RT}}$ (48)

(X) $C_4H_9\cdot \xrightarrow{k_{10}} C_3H_6+CH_3\cdot$ $k_{10} = 10^{13}e^{-\frac{24,500}{RT}}$ (49)

(XI) $C_4H_9\cdot \xrightarrow{k_{11}} C_2H_4+C_2H_5\cdot$ $k_{11} = 10^{13}e^{-\frac{26,000}{RT}}$ (50)

Chain-Terminating:

(XII) $H\cdot + H\cdot \xrightarrow{k_{12}} H_2$ k_{12} small, 3-body collision (51)

(XIII) $H\cdot + CH_3\cdot \xrightarrow{k_{13}} CH_4$ k_{13} small, 3-body collision (52)

(XIV) $H\cdot + C_2H_5\cdot \xrightarrow{k_{14}} C_2H_6$ $k_{14} = 10^{13}$ (53)

(XV) $CH_3\cdot + CH_3\cdot \xrightarrow{k_{15}} C_2H_6$ $k_{15} = 10^{11}$ (54)

(XVI) $CH_3\cdot + C_2H_5\cdot \xrightarrow{k_{16}} C_3H_8$ $k_{16} = 10^{11}$ (55)

(XVII) $C_2H_5\cdot + C_2H_5\cdot \xrightarrow{k_{17}} C_4H_{10}$ $k_{17} = 10^{11}$ (56)

Secondary Reactions Involving Propylene:

(XVIII) $C_3H_6 + H\cdot \xrightarrow{k_{18}} C_3H_7\cdot$ $k_{18} = 10^{13}e^{-\frac{5,000}{RT}}$ (57)

(XIX) $C_3H_6 + CH_3\cdot \xrightarrow{k_{19}} C_4H_9\cdot$ $k_{19} = 10^{11}e^{-\frac{5,000}{RT}}$ (58)

In the above rate constant expressions, R is the universal gas constant with a value of 1.987 cal/g mole °K, T is the absolute temperature in °K, and the rate constant, k, has units of sec^{-1} for a first-order reaction and cc/(gm mole) (sec) for a second-order reaction. The activation energies and frequency factors are either experimentally-measured values or are estimated from information available for similar reactions. Further information on how these rate constants were obtained is given in Appendix H. Values of the rate constants k_1 through k_{11} together with k_{18} and k_{19} have been evaluated at three temperatures, i.e., 460°, 510°, and 560° C, and are listed in Table XIII.

Based on the above reaction scheme, the rate expressions for the free radicals are, in accordance with the steady-state assumption,

$$\begin{aligned}\frac{d(H\cdot)}{dt} &= k_6(C_2H_5\cdot)+k_7(C_3H_7\cdot)+k_9(C_4H_9\cdot)-k_3(H\cdot)(C_4H_{10}) \\ &\quad -k_{12}(H\cdot)^2-k_{13}(H\cdot)(CH_3\cdot)-k_{14}(H\cdot)(C_2H_5\cdot) \\ &\quad -k_{18}(H\cdot)(C_3H_6) = 0\end{aligned} \tag{59}$$

$$\begin{aligned}\frac{d(CH_3\cdot)}{dt} &= k_1(C_4H_{10})+k_8(C_3H_7\cdot)+k_{10}(C_4H_9\cdot)-k_4(CH_3\cdot)(C_4H_{10}) \\ &\quad -k_{13}(H\cdot)(CH_3\cdot)-k_{15}(CH_3\cdot)^2-k_{16}(CH_3\cdot)(C_2H_5\cdot) \\ &\quad -k_{19}(CH_3\cdot)(C_3H_6) = 0\end{aligned} \tag{60}$$

$$\begin{aligned}\frac{d(C_2H_5\cdot)}{dt} &= 2k_2(C_4H_{10})-k_{11}(C_4H_9\cdot)-k_5(C_2H_5\cdot)(C_4H_{10}) \\ &\quad -k_6(C_2H_5\cdot)-k_{14}(H\cdot)(C_2H_5\cdot)-k_{16}(CH_3\cdot)(C_2H_5\cdot) \\ &\quad -k_{17}(C_2H_5\cdot)^2 = 0\end{aligned} \tag{61}$$

$$\frac{d(C_3H_7\cdot)}{dt} = k_1(C_4H_{10})+k_{18}(H\cdot)(C_3H_6)-k_7(C_3H_7\cdot) -k_8(C_3H_7\cdot) = 0 \tag{62}$$

$$\frac{d(C_4H_9\cdot)}{dt} = k_3(H\cdot)(C_4H_{10})+k_4(CH_3\cdot)(C_4H_{10})+k_5(C_2H_5\cdot)(C_4H_{10}) +k_{19}(CH_3\cdot)(C_3H_6)-k_9(C_4H_9\cdot)-k_{10}(C_4H_9\cdot) -k_{11}(C_4H_9\cdot) = 0 \tag{63}$$

Similarly, the rate expressions for the stable species are, neglecting the chain-initiating and the chain-terminating steps,

$$\frac{d(H_2)}{dt} = k_3(H\cdot)(C_4H_{10}) \tag{64}$$

$$\frac{d(CH_4)}{dt} = k_4(CH_3\cdot)(C_4H_{10}) \tag{65}$$

$$\frac{d(C_2H_6)}{dt} = k_5(C_2H_5\cdot)(C_4H_{10}) \tag{66}$$

$$\frac{d(C_2H_4)}{dt} = k_6(C_2H_5\cdot)+k_8(C_3H_7\cdot)+k_{11}(C_4H_9\cdot) \tag{67}$$

$$\frac{d(C_3H_6)}{dt} = k_7(C_3H_7\cdot)+k_{10}(C_4H_9\cdot)-k_{18}(H\cdot)(C_3H_6) -k_{19}(CH_3\cdot)(C_3H_6) \tag{68}$$

$$\frac{d(C_4H_8)}{dt} = k_9(C_4H_9\cdot) \tag{69}$$

$$-\frac{d(C_4H_{10})}{dt} = [k_3(H\cdot)+k_4(CH_3\cdot)+k_5(C_2H_5\cdot)](C_4H_{10}) \tag{70}$$

It is of interest to note here that equations 64, 65, 66 and 70 can be combined to give

$$-\frac{d(C_4H_{10})}{dt} = \frac{d}{dt}(H_2+CH_4+C_2H_6) \tag{71}$$

or the rate of decomposition of butane is equal to the sum of the rates of formation of hydrogen, methane and ethane.

For a quantitative discussion of the mechanism, the relationship given by equation 71 is first checked. The first-order rate constant, k_r', for butane decomposition has been obtained from experimental data and its values at different temperatures are tabulated in Table XII. Since at any given temperature the number of moles of each product formed for each mole of butane decomposed, P_i, is approximately constant and independent of conversion, the rate of formation of any product i is given by

$$\frac{d(i)}{dt} = P_i\ k_r'\ (C_4H_{10}) \tag{72}$$

For equation 71 to hold,

$$P_{H_2} + P_{CH_4} + P_{C_2H_6} = 1.0 \tag{73}$$

The values of P_i's are given in Table XI. For all three temperatures, ($P_{CH_4} + P_{C_2H_6}$) is approximately 0.95. Values of P_{H_2} are 0.097 at 560° C and not known experimentally for the other two temperatures. At 560° C, therefore, the experimental value for ($P_{H_2} + P_{CH_4} + P_{C_2H_6}$) is 1.05, which compares well with the predicted value of 1.0. The value of 0.097 for P_{H_2} at 560° C is, however, probably a little too high because of some butane cracking at the walls. For the purpose of testing the reaction mechanism, therefore, the values of P_{H_2} were calculated from equation 73 by difference to be 0.056, 0.047 and 0.046 for the three temperatures of 560°, 510°, and 460° C respectively.

From the rates of formation of hydrogen, methane and ethane, the concentrations of $H\cdot$, $CH_3\cdot$, and $C_2H_5\cdot$, can be readily calculated from equations 64 to 66 respectively. For example,

$$(H\cdot) = P_{H_2}\ k_r'/k_3 \qquad (74)$$

The evaluation of ($C_3H_7\cdot$) and ($C_4H_9\cdot$), however, are more difficult. These values can be computed from equations 62 and 63, provided the concentration of propylene is known. Since the concentration of propylene is a function of butane conversion, the concentrations of $C_3H_7\cdot$ and $C_4H_9\cdot$ are also dependent on butane conversion and should be treated as such. This, however, complicates the situation beyond possible solution. As the main interest in the present work is to determine the mechanism of butane decomposition, the effect of secondary reactions due to the presence of propylene may be approximated, at least at low conversions. Two approaches were used for approximation. In the first approach, the presence of propylene is neglected. This is called Case I. In the second approach, which is called Case II from now on, the reaction mixture is considered to be mainly butane with a small but constant percentage of propylene independent of conversion. Case I should apply at very low conversions, such as those for 460° C runs. Case II, on the other hand, should approximate the actual situation reasonably well for the 510° and 560° C runs, with less than 10% conversion. For Case II, the percentage of propylene is chosen to be 0.2% at 460°, 1% at 510°, and 3% at 560° C, approximately half of the average percentages of propylene found in the reaction mixtures at these temperatures. The concentrations of $C_3H_7\cdot$ and $C_4H_9\cdot$ are then calculated from equations 62 and 63 for both cases. The free radical concentrations thus determined are given in Table XIV.

It can now be seen that the assumption which neglects both chain-initiating and chain-terminating steps in the rate of formation or decomposition of stable species is justified. At 560° C, for example, the rate of formation of ethane due to reactions XIV and XV is 2.6×10^{-11} gm mole/(cc)(sec) compared with that of 5.6×10^{-8} gm mole/(cc)(sec) calculated from equation 66.

The rates of formation of the other three products, i.e., ethylene, propylene and butylenes, can now be predicted from equations 67, 68 and 69 respectively. The agreement between the predicted and experimental values is surprisingly well, especially for Case II. These rates are tabulated in Table XV. With the exception of butylenes which existed only in small amounts in the reaction mixtures, the difference between the predicted and experimental rates are only 5% or less. The validity of the proposed mechanism is thus firmly established.

A further test of the mechanism is provided by calculating the rates of formation and disappearance of H·, CH_3·, and C_2H_5·. If the steady-state assumption is to be valid, the rate of formation of a free radical has to be equal to its rate of disappearance. The rates of formation of these three radicals are the sums of all positive terms in equations 59, 60 and 61 respectively. Similarly, their rates of disappearance are the sums of all the negative terms in the same equations. These rates thus calculated are given in Table XVI. The agreement is again better than 5%, with the exception of H· whose rates agree within 30%.

VIII. DISCUSSION

Apparatus

The reactor consisted of a reactor tube and heater tube. Use of a separate heater tube provided a gas jacket between the heater and the reactor tube wall. Due to natural convection, the temperature of the gas in the jacket and consequently the temperature of the reactor tube wall became more uniform than the case in which the reactor was directly heated. The reactor had an I.D. of one inch which was larger than most flow reactors used for laboratory studies of butane decomposition, but was needed to enable traverse measurements.

The thermocouples used in this investigation were made either of platinum and platinum-10% rodium wires or of chromel and alumel wires. These wires tend to disintegrate in butane at high temperatures. For the purpose of extending their useful lives, therefore, these thermocouples were sealed off from the surrounding atmosphere as well as possible. This was done satisfactorily with the probe and inlet thermocouples. The reactor-wall thermocouples, on the other hand, broke down quite regularly at their junctions. For one thing, they were made of chromel-alumel wires which are unstable even at 460° to 560°C. For another, the amount of Astroceram cement that could be put over the junction was limited due to the small size of the thermocouple holes in the wall of the outer tube. Probably there was not enough cement to seal completely the thermocouple junctions from the surrounding butane.

End effects due to the reactor head were observed at low flow rates, even though both the thermocouple and pitot-tube probes were approximately two inches inside the reactor. One way to reduce these end effects would be to make these probes longer so that they could reach farther inside the reactor. Long probes with very small diameters, however, are both difficult to make and structurally very weak. A better way would be to redesign the reactor and mount the probes between two pieces of soapstone which in turn would be sandwiched between two porcelain tubes. The bottom tube would be used as the reactor tube. The top tube would serve no other purposes than to maintain the flow pattern of the gas in the reactor tube as well as to avoid any end cooling of the reactor tube.

Both thermocouple and pitot-tube probes had an O.D. of 0.06 inch which was as small as possible to avoid any possible disturbance of the flow pattern. The probe thermocouple was made of wires having diameters of 0.003 inch to minimize conduction losses along the wires.

For a pitot tube to measure the correct total pressure, its axis has to be aligned properly with the gas flow. Since the flow at the exit of the reactor was Poiseuille, the axis of the tube should have been parallel to the axis of the reactor tube. In the present case, the bend in the pitot was approximately 92^o corresponding to an angle of yaw of 2^o. This, however, did not cause any difficulty in data reduction since, in general, an angle of yaw of 5^o or less produces little or no error in the reading (24). On the other hand, this deviation from a right angle did prevent the pitot tube from reaching within less than 0.15 inch of the wall.

The static pressure was measured at a distance 3 inches downstream from the pitot tube opening. It certainly would be more desirable to measure the static pressure at a point in the reactor tube wall at the same height as the pitot tube opening. The positioning of the static pressure outlet was not critical, however, due to the method used for correlating velocity data.

The micromanometer described in Apparatus was capable of measuring pressure differences as low as 0.0002 cm of methyl alcohol. Although the sensitivity could be improved by increasing the area ratio between the reservoir and the capillary tube, such an increase was not justified because of fluctuations in manometer readings.

Experimental Errors

The operating variables were reactor length, butane feed rate, and temperature. The reactor length was measured to within 0.01 inch and thus introduced practically no errors. The error in measurement of flow rate might have been as high as 3% at the lowest flow rates. This error was due to uncertainties in reading the Rotameter scale. It was also observed that the feed rate of butane was subject to fluctuations. These fluctuations were combinations of high-frequency oscillations and long-period drifting. Drifting in flow rate was quickly noticed by making frequent checks on Rotameter readings and corrected by adjustment of the feed-tank pressure. The high-frequency oscillation was inherent to the system and could not be eliminated. Its effect on conversion and velocity measurement was not known, but was probably insignificant due to the high frequency (~ 100 cycles per minute). On the other hand, this oscillation did induce considerable oscillation in probe thermocouple readings at low flow rates, as noted in the Result section.

The main source of error was the control of temperature. Temperature at different sections of the wall for a run varied as much as 5°C. An end effect caused by the relatively cold reactor head cooled the gas near the exit at low flow rates. Inlet gas temperatures differed in many cases up to 10°C from the average wall temperatures. And finally, chemical reaction had an effect in keeping the gas temperature lower than the wall temperature. These effects are now examined individually.

In consideration of the endothermic nature of the reaction, it should be noted that the heat of reaction for the decomposition of n-butane is only 574 Btu/lb butane at 560°C. At that temperature, butane has a heat capacity of 1.56 Btu/lb °K. If the reaction occurs adiabatically, the temperature drop for a 5% conversion would be 18°C. Because the conversions for 460° and 510°C runs were much smaller than 5% and the reaction was far from adiabatic, the effect due to chemical reaction at these temperatures can be safely neglected. At 560°C, the conversion was of the order of 5%. The effect of heat of reaction then became more significant.

The inlet temperatures for four runs, i.e., runs 20, 21, 28 and 34, were intentionally maintained 15 to 38°C lower than the wall temperatures to study the effect of inlet temperature on conversion. Runs 20 and 21 were operated under the same conditions except that the inlet temperatures were 491° and 483°C respectively. The conversions for these two runs were practically the same as noted in Table IX. Similarly, run 34 had an inlet temperature of 530°C compared with 560°C in run 32. Although the retention time in both cases were the same, run 34 actually had a conversion of 6.6% at $r/r_o = 0.70$ compared with that of 4.9% for run 32, mainly because the average wall temperature of run 34 was 4° higher than that of run 32. This agrees with the fact that within the temperature range studied the rate of decomposition of butane double approximately every 15°C. It was thus concluded that the actual temperature of the reaction was not, within reasonable limits, affected by the gas inlet temperature.

The end effects due to the cold reactor head have been investigated in detail as described in Appendix C. It has been shown that the reactor head cooled the top part of the reactor wall, which in turn cooled the gas next to that portion of the wall, mainly by conduction and convection.

Since the wall temperature three inches below the top of the reactor was known to be approximately the same as the average wall temperature, the cooled portion of the reactor wall could not have been more than three inches in height. The reaction samples, on the other hand, were withdrawn with the pitot tube at a distance of two inches inside the reactor tube. The portion of the reactor length in which the gas temperature could have been lower than the wall temperature was thus less than one inch. Since the reactor length in the present work was at least eighteen inches, the possible error due to neglecting end effects is 5% or less.

The gas temperature in the reactor was closely related to the wall temperature because heat transfer inside the reactor was very fast. If the gas were a perfect conductor, the gas temperature and the wall temperature would then be identical at any height in the reactor. This was the basic assumption made in calculating the average reaction temperature according to Appendix D. In practice, the gas was not a perfect conductor, and the reaction temperature thus calculated would be subject to error. It is estimated that the maximum deviation between the calculated and actual reaction temperature was 2 °C, equivalent to a 10% error in the reaction rate.

Another major source of error, especially at low conversions, was the chromatographic analysis. The minimum detectable amount of a component in a 1-cc sample at atmospheric pressure and room temperature was about 0.3% by volume for hydrogen and 0.005% for all of the other components. Appendix B shows that for runs made at 460° and $510^{\circ}C$, no hydrogen was detected. This probably means that for these runs, hydrogen was present to less than 0.3% by volume rather than totally absent. Another component that often escaped detection was 1-butene. For the chromatographic unit that was used, the retention time of 1-butene was 1129 seconds and that of n-butane 800 seconds. Since the n-butane peak had a pronounced tailing, the peak of 1-butene sometimes was not distinguishable, especially when 1-butene existed in very small quantities in the sample. Other small peaks, such as those of propane, trans-2-butene and cis-2-butene, were probably also missed occasionally. Even when small quantities of a component were detected the percentage error involved in determining the absolute amounts was high. For a component present to the extent of 0.01% by volume, the maximum possible error was estimated to be 30% or more. For this reason, composition data at low conversions are much less dependable than those at high conversions as evidenced by the scattering of rate constants around $460^{\circ}C$ in Figures 12 and 13.

When a feed tank was initially acquired, the composition of its contents was determined by gas chromatography. The results are listed in Table I. Very small amounts of impurities were often found. These impurities consisted mainly of isobutane, trans-2-butene, propylene and propane. These components have larger vapor-liquid equilibrium constants than n-butane. Therefore the per cent by volume of all of them except isobutane, whose equilibrium constant is nearly the same as that of n-butane, decreased rapidly in the discharge from the tank. That was observed experimentally. Since isobutane was not a reaction product, it was treated the same as n-butane in determining the conversions, and the presence of other impurities in the feed was then neglected.

Experimental Results and Their Correlations

Experimental results here refer to the temperature and velocity profiles as well as the composition of the reaction mixture at the exit of the reactor. Samples of the reaction mixtures were also withdrawn through the sampling port in the reactor head. Since these samples were obtained after the stream had passed through a distance of about four inches of "cold" region, they only served to show the distribution pattern of the product in an overall sample. From these samples, it was found that the product distribution was practically independent of radial position, as can be seen in Appendix B.

The measured temperature profiles for several runs at each temperature are shown in Figures 8 to 10. For runs at 460° and 510° C having an average linear velocity greater than 0.2 ft/sec, the temperature is nearly uniform and equal to the wall temperature in the central portion of the reactor, but becomes lower toward the wall. These curves are similar in shape to those shown in Figures F1 and F2 for cases where heat transfer is dominated by conduction and forced convection. The average linear velocities for all of the 560°C runs were greater than 0.2 ft/sec. For these runs, the temperature profiles were lower at both the wall and the center, and higher in between. The conversion for these runs ranged from 3.7 to 8.2%. At these conversion levels, the heat of reaction apparently became significant. This accounts for the lower temperatures at the center. Otherwise, conduction and forced convection still prevailed. When the average linear velocity of a run was much smaller than 0.2 ft/sec., the gas flowed slow enough to allow a considerable density difference to develop between a point close to the reactor head and a lower point in the isothermal region. Furthermore, forced convection decreased because of lower flow rates. Natural convection thus overshadowed all other modes of heat transfer, and the temperature profile changed to a concave shape. For average velocities around 0.2 ft/sec. (Run 53 at 460°C and Run 11 at 510°C), both natural and forced convection were important; the resulting temperature profiles are thus a compromise between the profile for a higher velocity and that for a lower velocity as can be seen in Figures 8 and 9.

Thirty-six runs with velocity measurements were made. Of these, only sixteen velocity profiles were determined. There are several reasons why data for more than half of the runs were bad.

Many of those runs were made at low flow rates, with average linear velocities less than 0.2 ft/sec. The bubble displacement in the manometer corresponding to a velocity of 0.4 ft/sec is only 0.2 cm. Since the noise level in the manometer reading was of the order of 0.1 cm because of the high sensitivity, it was very difficult to obtain meaningful velocity data for these low flow rate runs. As the manometer reading approximately varies with the square of the velocity, the accuracy of measured data would rapidly improve with increasing flow rate. Another factor that worked against reliable velocity data at low flow rates was natural convection. It was reasoned in the last paragraph that due to end cooling, natural convection began to appear at an average linear velocity around 0.2 ft/sec. As natural convection caused circulation in the exit end of the reactor, the flow pattern there was probably distorted. As a matter of fact, all sixteen good runs had average linear velocities greater than 0.2 ft/sec.

Leakage was mainly responsible for bad runs with high flow rates. A one-centimeter bubble displacement is equivalent to a head of less than 0.001 inch of methyl alcohol. If a tiny leakage existed either in the connecting lines between the pitot tube or the pressure tap and the micromanometer or in the stopcocks on the micromanometer, considerable error could have been introduced into the velocity data. Leakage in the stopcocks was most bothersome because it occurred frequently. The silicone grease that was used for these stopcocks did not seal well comparing with many other commonly used stopcock greases such as Kel-F, but was the only one not readily dissolvable in methyl alcohol. Leakage occurred not only during bad runs, but also sometimes during good runs. The only difference was that for a good run, leakage did not occur for a period long enough to allow a sufficient number of data points to be collected.

Although only sixteen out of thirty-six runs had enough velocity data for the determination of velocity profiles, the fact that all these sixteen runs had parabolic velocity distributions at the exit of the reactor is significant. These sixteen runs covered a wide range of conversion and their average linear velocities varied from 0.25 to 0.60 ft/sec. It can thus be concluded that for all the runs in the present work, the velocity profiles were parabolic at the exit in the absence of end cooling.

The conversion and material balance for each run are given in Tables VIII to X. The composition data and hence conversions and material balances were subject to two possible errors: chromatographic analysis and fluctuations in operating conditions with respect to time, as have been discussed.

In general, the composition data and their derivatives, with the exception of conversion, do not seem to vary in any definite pattern with respect to the radial position of the sample. More conversion, however, was observed at points nearer to the wall than to the center of the reactor. This is expected because of the endothermic nature of the reaction as well as the shorter retention time of gas molecules flowing at the center of the reactor. The interesting point is rather that such differences in conversion are smaller than expected. Even with high-conversion runs at 560°C, differences in conversion between $r/r_o = 0.70$ and the center was usually smaller than 5%. Furthermore, the composition of any component quite often varied randomly with respect to radius. It thus seems that (1) heat and diffusional transfer in the reactor were rapid and (2) the transformation of velocity profile from flat to parabolic was probably slower than the case without chemical reaction. In any event, the reactor may be treated as essentially an isothermal, plug-flow reactor based on conversion and composition data.

Material balances were checked by computing for each sample the mole ratio $\Sigma C/\Sigma H$ for all the components other than the butanes. If the presence of impurities is negligible, this ratio has a theoretical value of 0.400. For runs at 460° and 510°C, the calculated values are usually a little higher than 0.4, with a maximum deviation of 2%. For runs at 560°C, the calculated ratios become lower than 0.4 and have a maximum deviation of 3%. Considering the low levels of conversion, such agreements are quite satisfactory. Furthermore, the deviation from the theoretical value of the ratio C/H at lower temperatures can be attributed to the presence of small amounts of hydrogen in the samples which were not detected by the analysis. For runs at 560°C, hydrogen was detected, and the ratio drops below 0.4, possibly because of some carbon formation due to cracking at the walls. For run 60 with a conversion of 4.6 moles of butane per 100 moles of feed with a C/H ratio of 0.392, the calculated carbon deposit is 0.4 mole of carbon per 100 moles of feed or 2.1% of the conversion.

The number of moles of products formed per mole of butane converted also varied with temperature. Its value increased from an average of approximately 1.95 to 460°C to 2.1 at 560°C. If a butane molecule is decomposed into two free radicals, a butyl radical is decomposed into a free radical and an olefin, the presence of propyl radicals is negligible, and free radicals smaller than propyl do not further split into two fractions, the number of moles of products formed for each mole of butane decomposed will be two. If some of the ethyl radicals also decompose into an ethylene molecule and a hydrogen radical, as shown in equation 45, this number will be larger than two. The reason that this number is found to be slightly less than two at lower temperatures is again the lack of detection of hydrogen. Values for P_{H_2} have been estimated to be 0.046 and 0.047 at 460° and 510°C, respectively, based on the proposed reaction mechanism.

When these values are adopted, the number of products formed per mole of butane decomposed becomes two at 460°C and slightly higher than two at 510°C.

The number of moles of a product formed per mole of butane converted, P_i, is also found to be a constant at each temperature for each product, independent of conversion. These average values together with the standard deviations are given in Table XI. Propane in very small quantities was found, mainly at 560°C. Its value of P was not calculated due to the large uncertainties involved. It is of interest to note that P's for methane and ethane remain almost constant with respect to temperature while those for propylene and butenes decrease with increase in temperature. $P_{C_2H_4}$, on the other hand, increases with increase in temperature.

The fact that within the temperature range of 450° to 650°C the relative amounts of the products remain fairly constant has been reported (8, 12). Most investigators (10, 11, 13), however, reported their results as the mole ratio of $(CH_4+C_3H_6)$: $(C_2H_6+C_2H_4)$: $(C_4H_8+H_2)$. Since values of P_i's are different for different species, the use of such a ratio is not only semi-quantitative but also misleading.

Reaction Kinetics

The reaction-rate constant, k_r, for both first-order and second-order rate expressions has been fitted to an

Arrenhius-type expression by the least-squares technique. The values of k_r, used in the fit were computed with equation 18 or 21 for each run. For each rate expression, the rate constant was obtained using the conversion data both at $r/r_o = 0.7$ and 0.0. The activation energies and frequency factors thus determined are listed in Table XII

In using equations 18 and 21, the reactor was assumed to be isothermal, and isobaric, the composition of the reacting mixture uniform with respect to the radius, and longitudinal diffusion negligible. How well did the actual operating conditions agree with these assumptions?

The reactor was essentially isobaric. It was isothermal at low conversions with the possible exception of a very short length at the end. At conversions of 5% or higher, the temperature at the center of the reactor was several degrees lower than the wall temperature because of the heat of reaction. For these runs, data at $r/r_o = 0.7$ should be used as the temperature at that point approximated that of the wall. At lower conversions, the conversion data do not differ appreciably across the reactor.

The composition of the reacting mixture is fairly uniform in the radial direction, as illustrated in Appendix B. Longitudinal diffusion, on the other hand, was found to be negligible.

For the decomposition of n-butane at 560 °C in a reactor two feet in length with an average velocity of 0.1 ft/sec, the conversion was calculated to be 23.66% when neglecting longitudinal diffusion compared with a conversion of 23.65% with diffusion.

With all these considerations, therefore, the validity of equations 18 and 21 has been proved. Furthermore, the rate constants for both first and second-order reactions are, respectively,

$$k_r' = 3.34\text{x}10^{12}\ e^{-\frac{54,600}{RT}} \qquad (75)$$

$$k_r'' = 2.55\text{x}10^{14}\ e^{-\frac{56,800}{RT}} \qquad (76)$$

using the conversion data at $r/r_o = 0.7$. Both constants agree with the experimental data equally well. This is because within the temperature and conversion ranges studied, the concentration of butane varied only to a very small extent. In order to determine the order of the overall reaction accurately, data at several pressures are needed.

Most of the previous investigations reported the rate constant corresponding to a first-order rate expression. Their values for the activation energy ranged from 43 to 74 kcal. Some of these results are given in the Introduction section. The generally accepted value was that of 58,700 kcal by

Steacie and Puddington (9) which agrees reasonably well with the present result. Previous results, however, were based on data at 20 to 50% conversions. At such high conversion levels, secondary reactions, as well as mass and heat transfer in the reactor, could have been of considerable importance in making these results less dependable.

Reaction Mechanism

A free-radical reaction mechanism for the thermal decomposition of n-butane has been proposed in the section of Correlation of Data. Most of the activation energies and some of the frequency factors for the elementary reactions involved in this mechanism were chosen from experimental data collected by Steacie (1) after critical review. The remaining activation energies and frequency factors were estimated from the nature of the reaction, bond-energy considerations, and from values available in Steacie's book for similar reactions. The actual determination of these quantities is described in Appendix H. Because of uncertainties in experimentally measuring or estimating the activation energies and frequency factors, the rate constants for the elementary reactions calculated from these quantities can at best be regarded as semi-quantitative.

One feature of this mechanism is the consideration of secondary reactions due to propylene in the thermal decomposition of n-butane at low conversions. In order to demonstrate the effect of secondary reactions, two cases were investigated.

Case I neglected the presence of propylene. Case II assumed that a constant percentage of propylene existed in the reaction mixture at each temperature independent of conversion. As the amount of propylene in the reaction mixture depends directly on conversion, Case II does not represent any actual situation, but was used to study the effect of propylene as a product on butane decomposition. At low conversions, Case II should still approximate the actual behavior of the reaction mixture reasonably well.

Quantitative discussions of both cases are given in Correlation of Data. The predicted results for both cases based on the proposed mechanism are given in Tables XIV to XVI. The agreement between the predicted and experimental results shown in Table XV is surprisingly good for both cases, in view of the uncertainties in the rate constant data used for the elementary reactions. Case II, nevertheless, agrees with the experimental data better than Case I at higher temperatures where more propylene was present in the reaction mixture because of higher conversions. Furthermore, it has been experimentally observed that more methane existed in the reaction mixture than propylene on a molar basis even at the lowest conversion levels. This is not explainable by Case I which predicts equal amounts of methane and propylene according to equations 60, 65 and 68. Case II, on the contrary, does predict a preponderance of methane over propylene because of the secondary reactions XVIII and XIX. The inclusion of these two reactions in the proposed mechanism is thus justified.

In view of the good agreement between experimental and predicted results, the validity of the proposed mechanism, as represented by reactions I through XIX, is definitely established. This mechanism describes the thermal decomposition of n-butane with less than 10% conversion by taking into consideration the secondary reactions of propylene. At conversions higher than 10%, secondary reactions due to ethylene and butenes should also be considered. According to this mechanism, the initial product distribution of butane decomposition is 2.73% H_2, 33.53% CH_4, 12.15% C_2H_6, 15.86% C_2H_4, 33.53% C_3H_6, and 2.20% C_4H_8 at 560°C. These values were calculated from equations 64 to 69 with free radical concentrations given in Table XIV for Case I.

According to the present mechanism, the overall reaction is first-order with respect to butane and first-order with respect to a combination of hydrogen, methyl and ethyl radicals as given by equation 70. The dependence of free radical concentrations on butane concentration are not known, but are surmised to vary between 0 and 1 depending on the radical. The overall reaction then has an order between 1 and 2. This agrees with the observations of Echols and Pease (10).

The first free-radical mechanism proposed for the thermal decomposition of n-butane is that of Rice (2), as described in the Introduction.

Rice, however, did not attempt to discuss his mechanism quantitatively. Most recently, Purnell and Quinn (26) investigated the formation of the products of the reaction in a static system in the temperature range of 420° to 530°C and at initial pressures between 10 and 150 mm Hg. With the help of a very sensitive gas chromatographic unit, these investigators were able to study the reaction at low conversions up to 10%. On the basis of their experimental results, Purnell and Quinn proposed a free-radical mechanism similar to that of Rice, which gives a quantitative account of the elementary reactions occurring. They, however, did not try to quantitatively discuss the overall reaction, as was done in this thesis.

Purnell and Quinn found that the distribution of the major products of the reaction, i.e. CH_4, C_2H_6, C_2H_4, and C_3H_6, was independent of conversion at any temperature and initial pressure. These distributions agree with the present results very well. For example, the mole ratios of CH_4: C_2H_6: C_2H_4: C_3H_6 obtained by Purnell and Quinn at a pressure of 150 mm Hg are 37.7: 12.0: 12.6: 37.7 at 460°C and 36.0: 11.0: 17.0: 36.0 at 518°C. In the present work, these ratios were found to be 37.5: 13.3: 13.3: 35.9 at 460°C and 35.8: 13.5: 16.0: 34.7 at 510°C, both at one atmosphere. Purnell and Quinn, however, found that the yields of methane and propylene were virtually identical, in contrast with the present work.

Because of this fact, they did not incorporate the secondary reactions of propylene in their mechanism, but claimed that the preponderance of ethylene over ethane and the formation of hydrogen were solely due to reaction VI. Purnell and Quinn further claimed that the butenes were not formed by any primary reaction step, such as reaction IX, but were secondary products. They nevertheless failed to show how the butenes were formed.

By varying the initial pressure, Purnell and Quinn were able to show that the overall orders for the formation of the major products lay between 1.4 and 1.6. This would result in an order for the overall decomposition of n-butane somewhere around 1.5, as surmised previously.

IX CONCLUSIONS

Thermal decomposition of n-butane has been studied in a homogeneous flow reactor at atmospheric pressure and in the temperature range of 460° to 560°C. The gas temperature inside the reactor in general remained fairly constant and was equal to the wall temperature for each run. The velocity in the inlet of the reactor was flat across the tube and by measurement was found to be parabolic at the exit. Within the range of conversion of 0.05-8%, concentration gradients across the reactor were found to be insignificant. It was thus concluded that under the operating conditions, the reactor could be treated as essentially an isothermal reactor with plug flow.

The reaction-rate data were then correlated by a least-squares fit to obtain an Arrhenius-type expression for the reaction-rate constant. For a first-order rate expression, the constant is

$$k_r' = 3.34\text{x}10^{12}\, e^{-\frac{54,600}{RT}}$$

and for a second-order rate expression it is

$$k_r'' = 2.55\text{x}10^{14}\, e^{-\frac{56,800}{RT}}$$

Both constants fit the reaction data equally well. The order of the overall reaction thus lies somewhere between 1 and 2.

More likely, the actual overall reaction expression is a combination of first and second-order terms, such as:

$$-\frac{d(C_4H_{10})}{dt} = k_a\ (C_4H_{10}) + k_b(C_4H_{10})^2$$

Based on the reaction-rate data as well as the product distribution, a Rice-type, free-radical mechanism has been proposed for the thermal decomposition of n-butane. Using carefully chosen values of activation energies for the elementary reactions, as shown in Appendix H, the proposed mechanism compares well with the present experimental results. It seems that even at very low conversions (~5%), secondary reactions between propylene and free radicals should be considered, as have been done in the present work. This once again stresses the importance of studying free-radical reactions at very low conversion levels.

X RECOMMENDATIONS FOR FUTURE WORK

1) It is recommended that the present reactor be modified to eliminate the end effects due to the reactor head.

2) It is proposed that an experiment be designed which would enable the measurement of free radical concentrations in the reacting stream. Such a measurement, if possible, testifies in a direct manner the validity of the proposed mechanism.

3) It is proposed that the work on the thermal decomposition of n-butane be extended to temperatures above 600 °C at different pressures and with very low conversions. The use of a microreactor would be ideal for this type of work.

NOMENCLATURE

A	frequency factor, sec^{-1} for a first-order and cc/(gm mole)(sec) or cu.ft./(lb)(sec) for a second-order rate constant
C_p	heat capacity at constant pressure, Btu/(lb)(°K)
D	diffusion coefficient, sq.ft./sec
E	activation energy, cal/gm mole
F	mass flow rate, lb/sec
g	gravitational acceleration, ft/sec^2
k, k_r	reaction-rate constant, same unit as A
k_c	thermal conductivity, Btu/(sec)(sq.ft.)(°K)
k'_r	first-order rate constant, sec^{-1}
k''_r	second-order rate constant, cu.ft./(lb)(sec)
M	molecular weight, lb/lb mole
P	total pressure, atm or lb/sq.ft.
p	partial pressure, atm
P_i	number of moles of component i formed per mole of butane decomposed, dimensionless
R	gas constant, 1.987 Btu/(lb mole)(°R)
R_k	reaction rate for component k, lb/(cu.ft.)(sec)
r	radius, ft or in
r_o	radius of the reactor tube, ft or in
T	temperature, °K or °C
u_r	velocity component in r-direction, ft/sec
u_x	velocity component in x-direction, ft/sec
V	volume of the reactor, cu.ft.

X	moles of butane converted per mole of feed, dimensionless
x	longitudinal distance, ft or in
Z	increase in number of moles for each mole of butane decomposed, dimensionless
λ	heat of reaction, Btu/lb
μ	viscosity, lb/(sec)(ft)
ρ	density, lb/cu.ft.

Subscripts:

b	bulk or butane
k	component k

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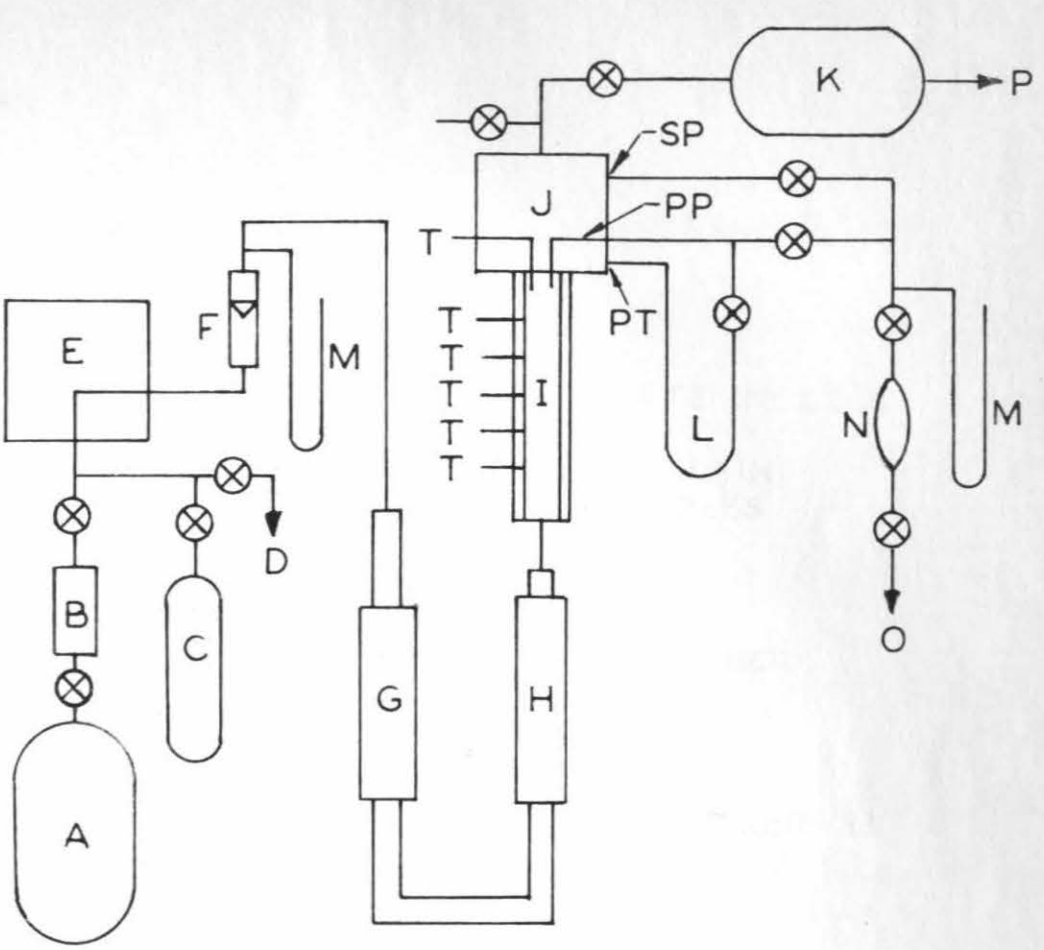


A	BUTANE STORAGE TANK	K	VACUUM SYSTEM
B	SURGE TANK	L	MICROMANOMETER
C	NITROGEN TANK	M	MERCURY MANOMETER
D	COMPRESSED AIR LINE	N	SAMPLING BULB
E	TEMPERATURE CONDITIONER	O	VACUUM LINE
F	ROTAMETER	P	ATMOSPHERE
G	PRE-PREHEATER	PP	PITOT TUBE PROBE
H	PREHEATER	PT	PRESSURE TAP
I	REACTOR	SP	SAMPLING PORT
J	REACTOR HEAD	T	THERMOCOUPLE

Figure 1. Flow sheet of apparatus

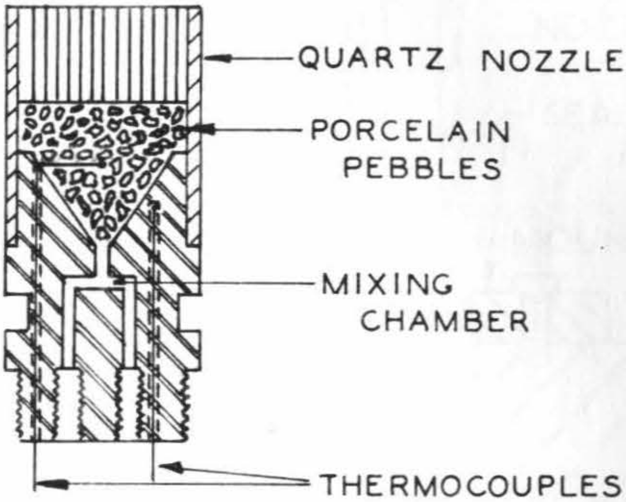


Figure 2. Schematic cross-section of gas-mixing chamber and quartz inlet nozzle

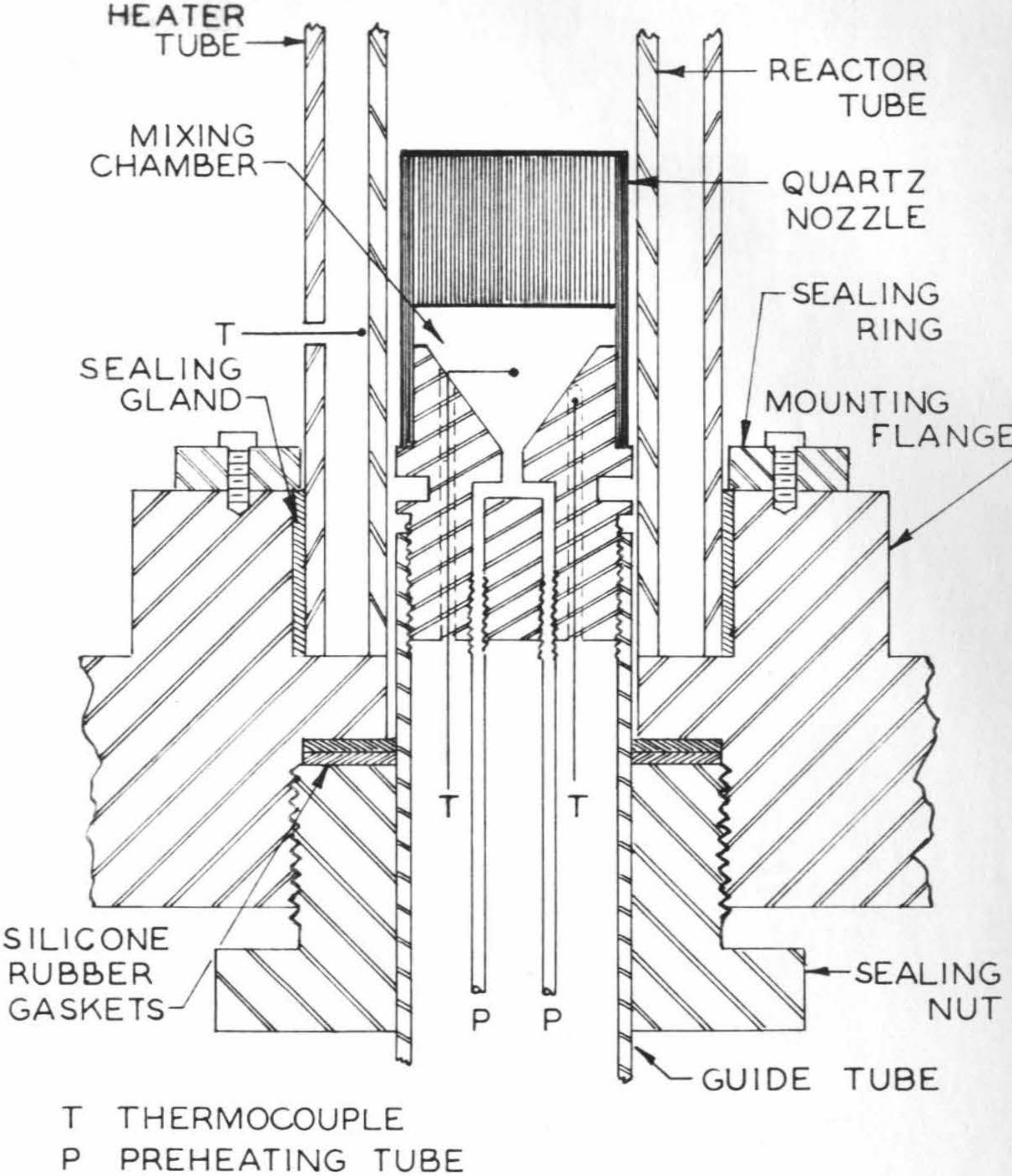
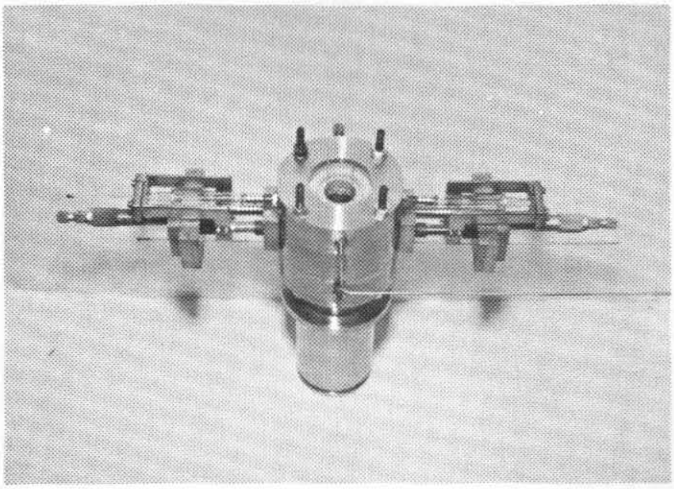
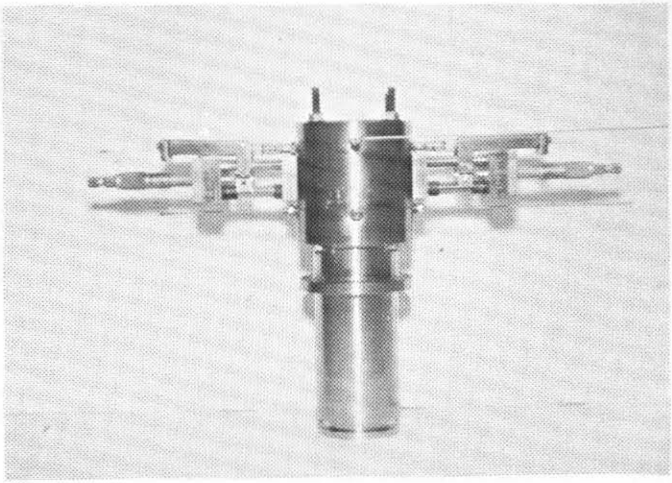


Figure 3. Schematic cross-section of inlet end of reactor



(a) Top View



(b) Side View

Figure 4. Reactor Head

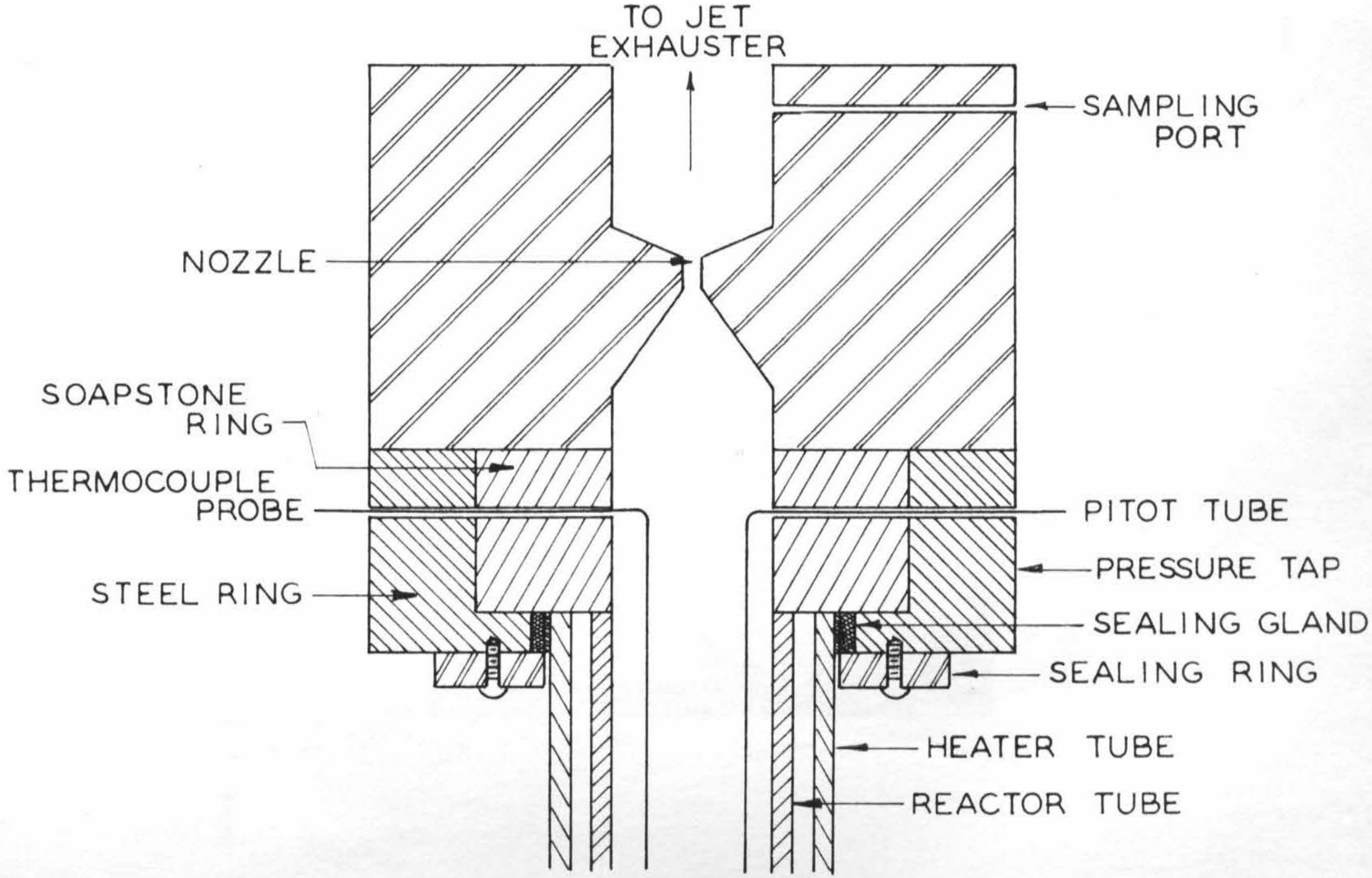


Figure 5. Schematic cross-section of reactor head

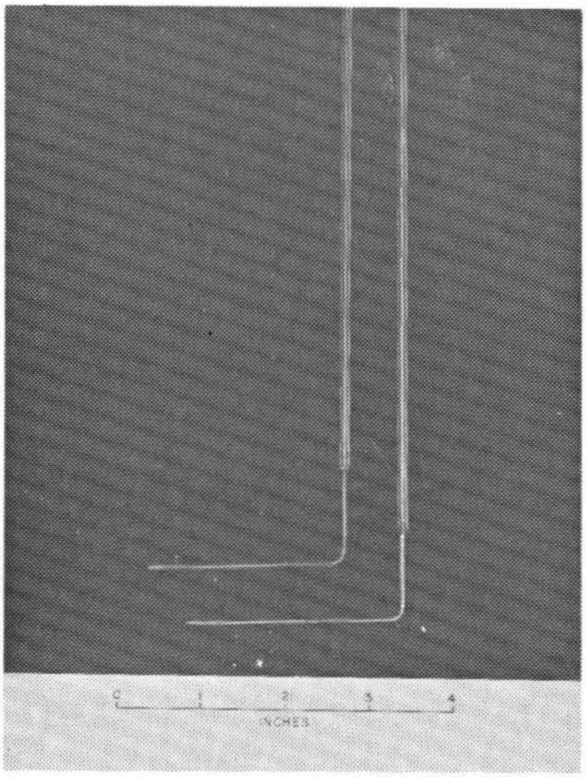


Figure 6. Thermocouple And Pitot-Tube Probes

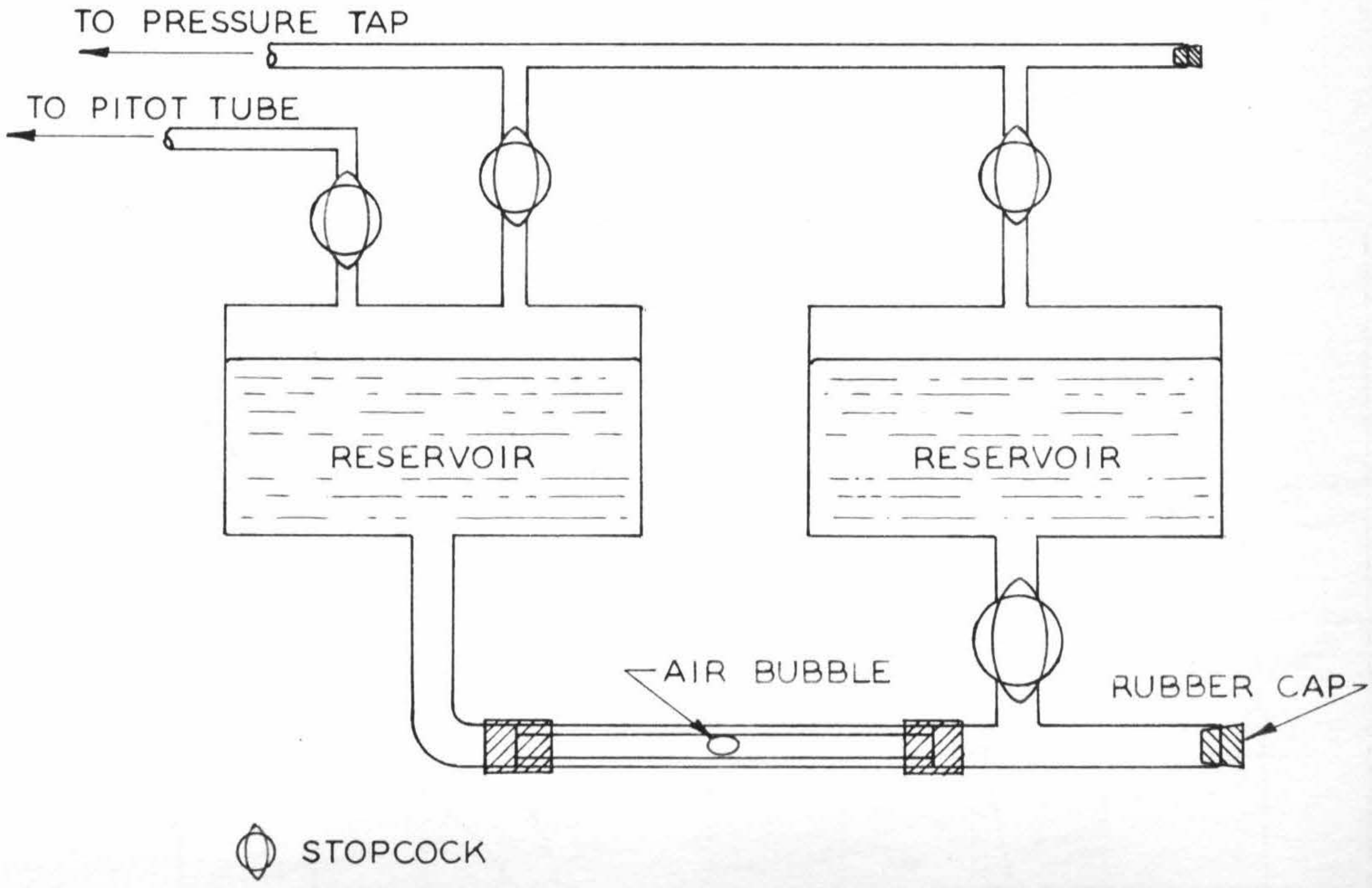


Figure 7. Micromanometer assembly

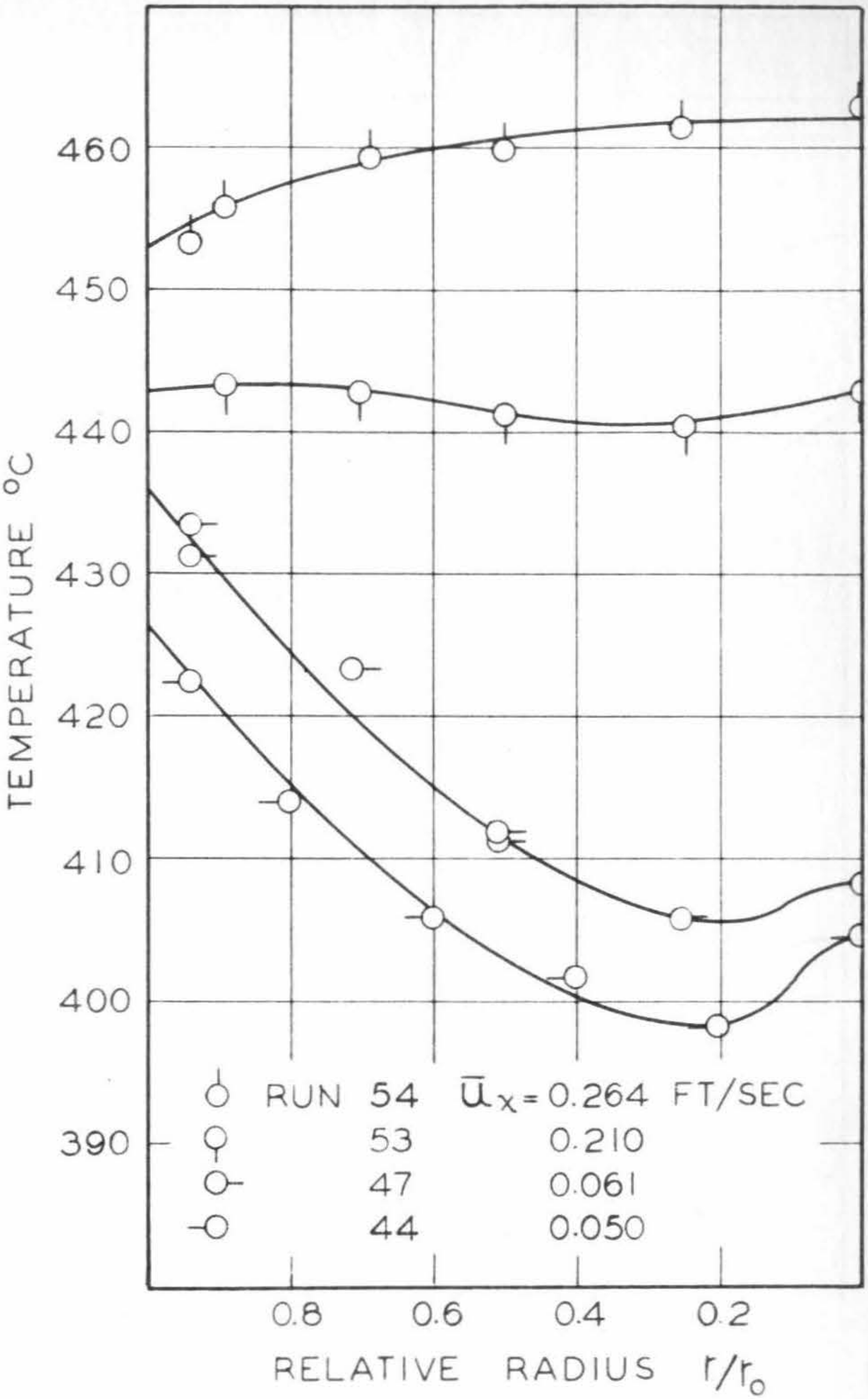


Figure 8. Temperature profiles for several 460° C runs

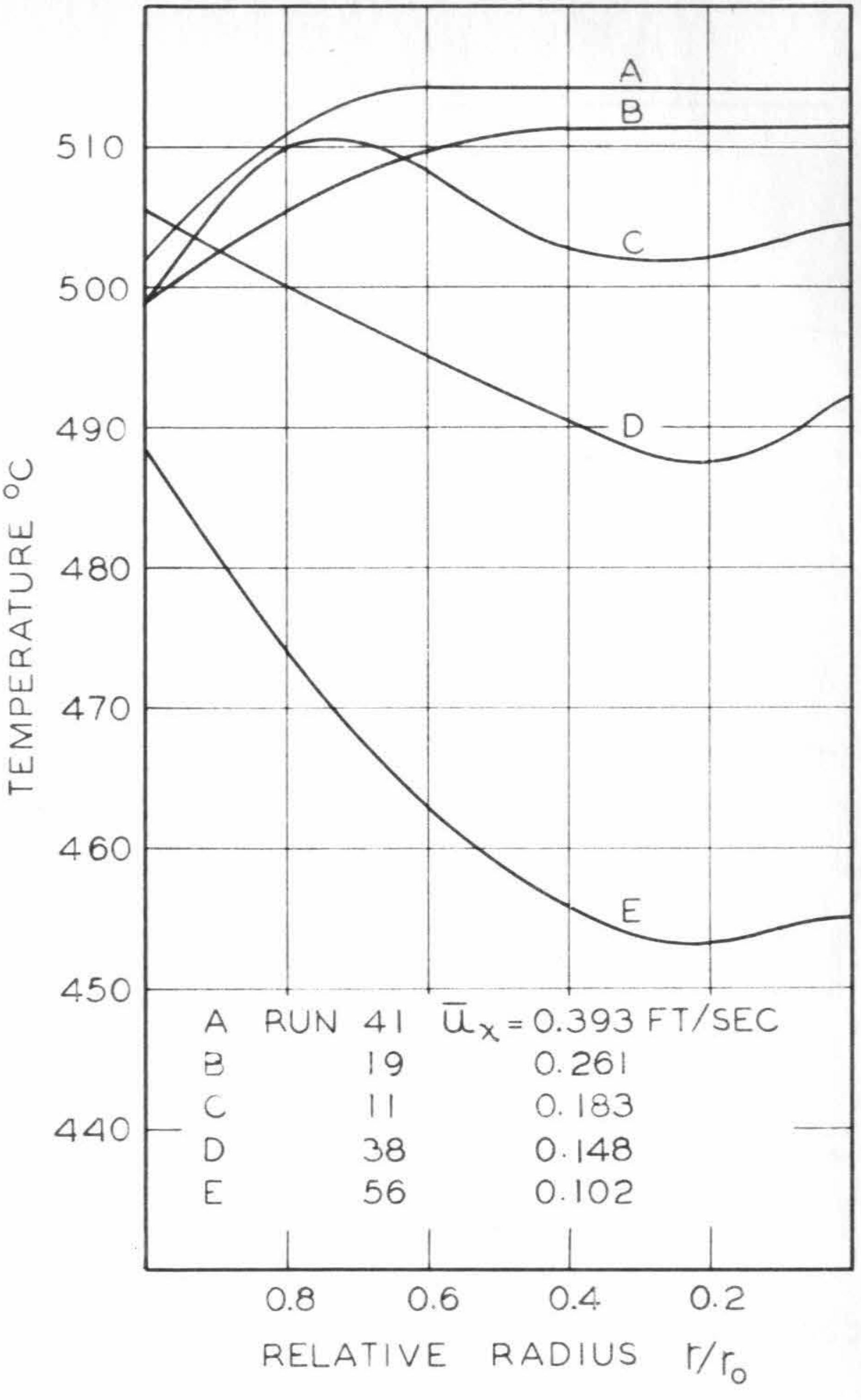


Figure 9. Temperature profiles for several 510°C runs

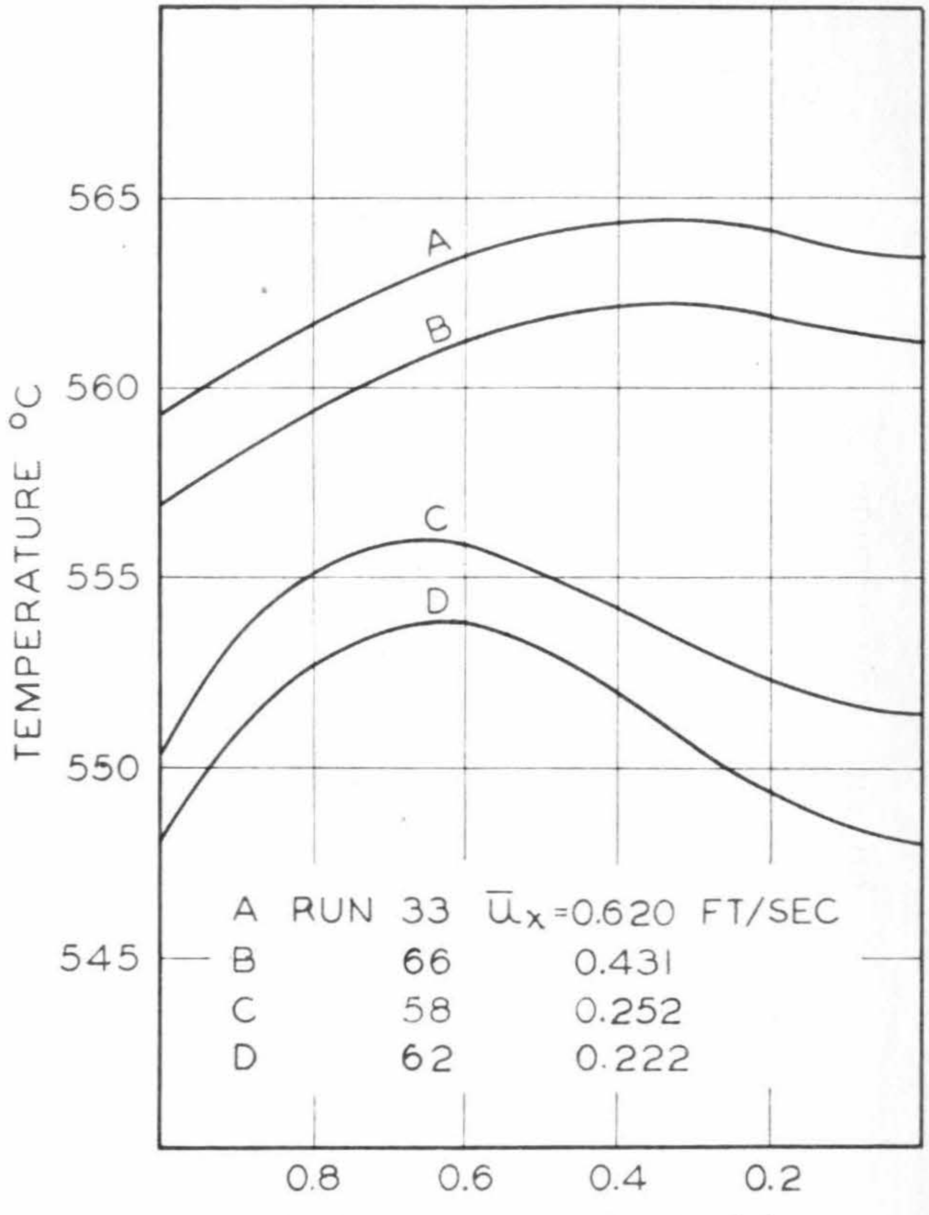


Figure 10. Temperature profiles for several 560°C runs

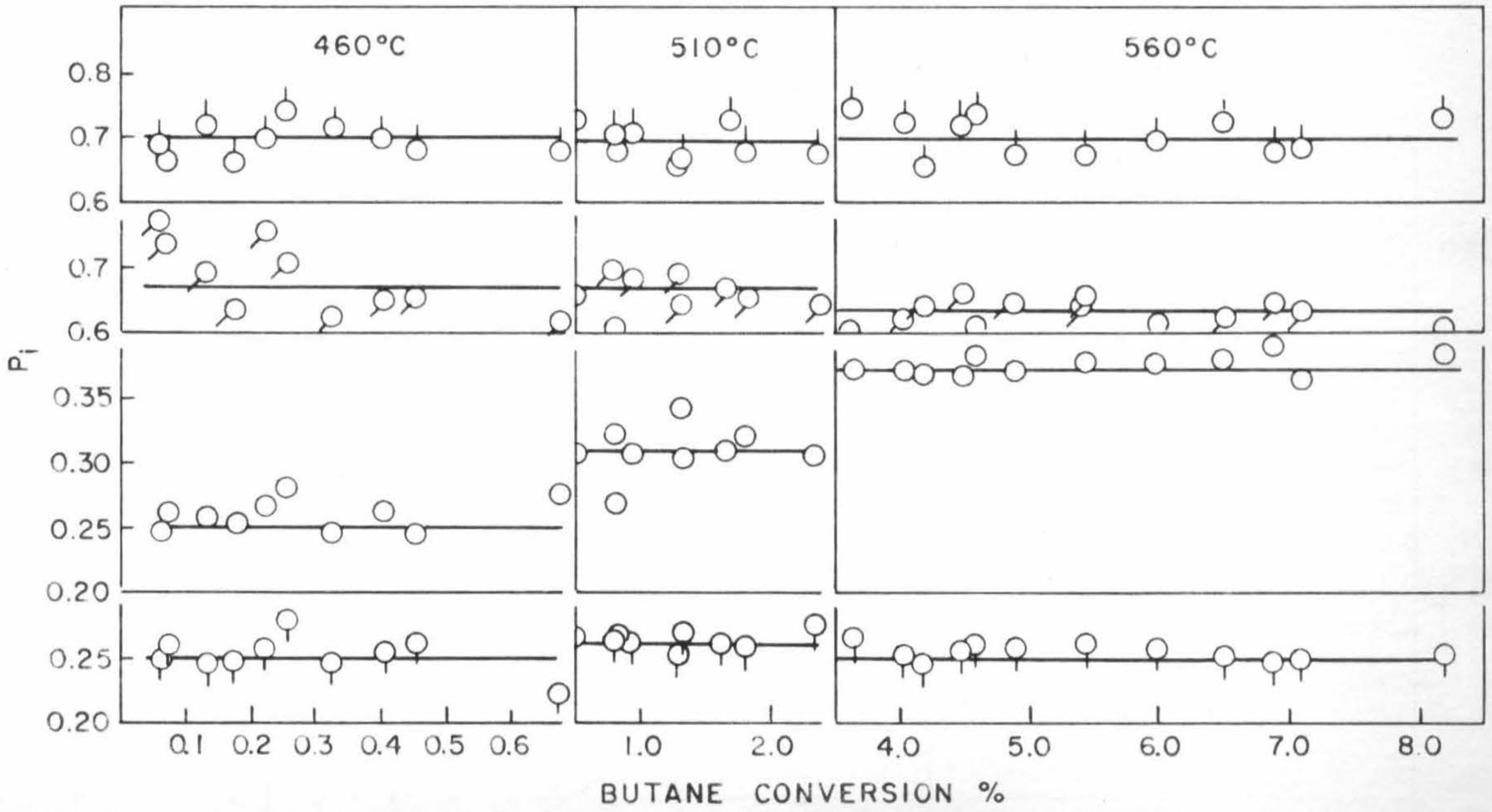


Figure 11. P_i vs butane conversion for methane ⫰, ethane ⫯, ethylene ○ and propylene ⚲

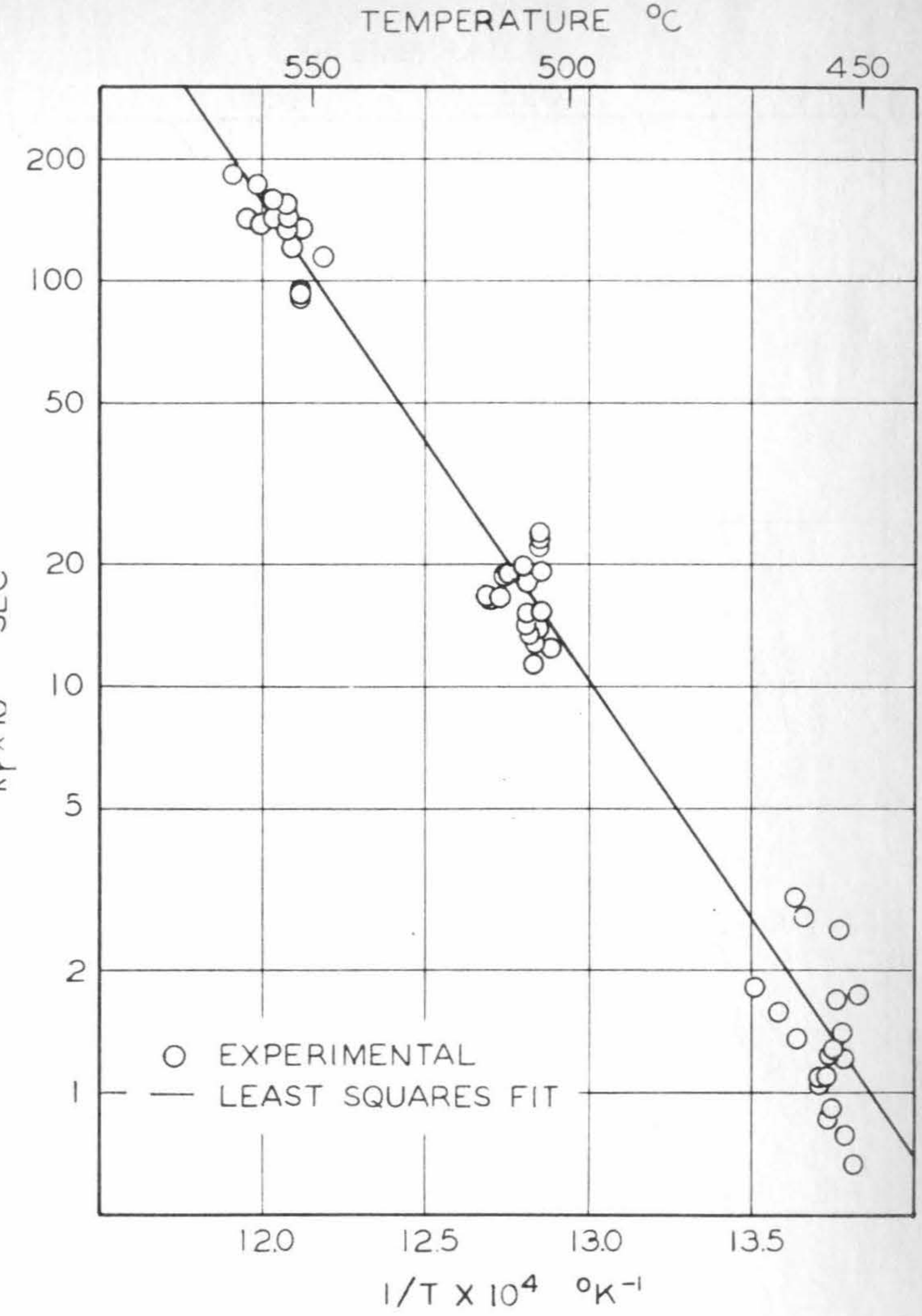


Figure 12. First-order rate constant for n-butane decompostion using data at $r/r_o = 0.70$

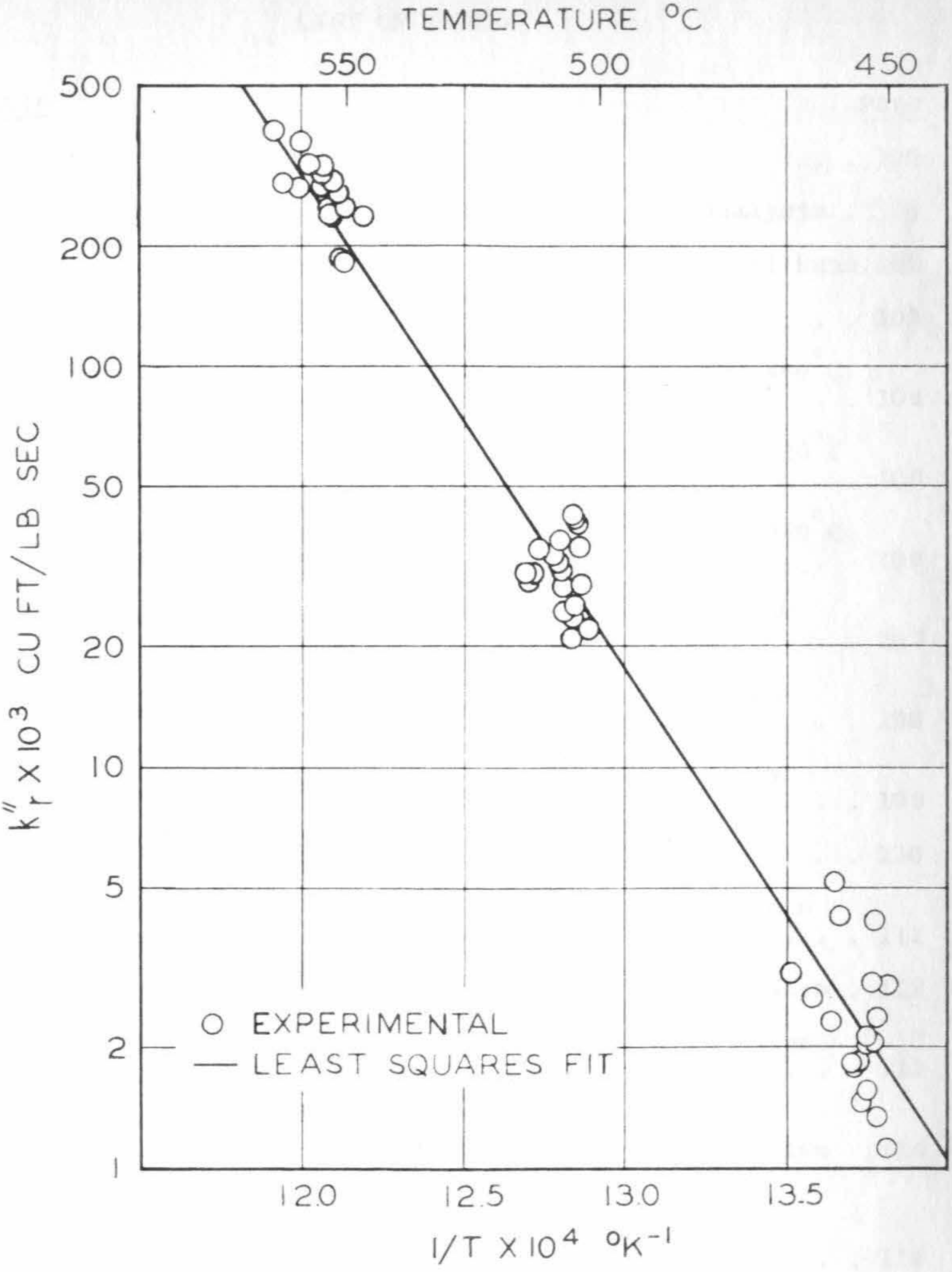


Figure 13. Second-order rate constant for n-butane decompostion using data at $r/r_o = 0.70$

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TABLE I. FEED ANALYSIS

	FEED SOURCE		
	TANK 1	TANK 2	TANK 3
Runs Used In / Component	11-22	24-60	61-70
Propane	--	--	0.071%
Propylene	Trace*	0.022%	0.010%
Isobutane	--	0.148%	0.039%
Trans-2-butene	Trace	0.024%	0.019%
n-Butane**	~ 100%	99.806%	99.861%

---

[*] Less than 0.005%.

[**] Obtained by difference.

TABLE II. COMPARISON OF DUPLICATE CHROMATOGRAPHIC ANALYSIS*

Run No.	13		14		19	
Sample No.	2A	2B	2A	2B	5A	5B
Amount of Sample, cc-cmHg at 25°C	55.242	37.578	51.257	36.591	48.459	39.836
Date Analyzed	5/10/62	5/10/62	5/12/62	5/13/62	6/1/62	6/1/62
Time Analyzed	11:05a.m.	7:30p.m.	1:20p.m.	5:30p.m.	4:20p.m.	11:10p.m.
Composition						
%CH_4	0.214	0.214	0.283	0.266	0.628	0.612
C_2H_6	0.0915	0.0845	0.103	0.107	0.246	0.233
C_2H_4	0.0833	0.0824	0.102	0.087	0.281	0.278
C_3H_6	0.219	0.218	0.261	0.259	0.620	0.591
C_3H_8	0.00	0.00	0.005	0.005	0.000	0.000
1-C_4H_8	0.00	0.00	0.00	0.00	0.0074	--
Trans-2-C_4H_8	--**	0.0170	--	0.014	0.0186	--
Cis-2-C_4C_8	--	0.0130	--	0.023	0.0214	--
Iso-C_4H_{10}	0.00	0.00	0.00	0.00	0.00	0.00
n-C_4H_{10} ***	99.392	99.371	99.246	99.239	98.178	98.286

---

* Samples withdrawn from the same storage bulb.

** Chromatographic peak missed.

*** Obtained by difference.

TABLE III. REACTOR WALL TEMPERATURES FOR SEVERAL TYPICAL RUNS

	WALL TEMPERATURES				
Thermocouple No.	1	2	3	4	5
Distance from Reactor Exit, Inches	1.0	7.0	13.0	19.0	25.0
(a) Runs with 18" Reactor Length					
Run 13	460	464	459	464	--
16	510	507	504	510	--
37	508.5	510	509.5	--	--
45	460	458	457.5	442.5	--
64	562	562.5	558	543.5	--
68	561	559	558	542	--
(b) Runs with 22 1/8" Reactor Length					
Run 24	468	460	468	468.5	465
29	513	509	509.3	506	523.5
32	559	557	565	563	578.5

TABLE IV. COMPARISON OF DUPLICATE SAMPLES

Run No.	11		22		26		33	
Sampling Position, r/r_o	Overall		0.70		Overall		0.70	
Sample No.	1	6	2	6	1	6	2	6
Amount of Sample, cc-cmHg at 25°C	54.173	54.028	49.411	51.885	56.348	56.568	56.767	54.532
Sampling Time	5:25p.m. 5/5	5:33p.m. 5/5	9:29p.m. 6/3	10:37p.m. 6/3	11:10p.m. 6/24	12:03a.m. 6/25	4:08p.m. 7/4	4:50p.m. 7/4
Composition								
%H_2	0.00	0.00	0.00	0.00	0.00	0.00	0.411	0.407
CH_4	1.185	1.275	4.274	4.202	0.271	0.249	2.574	2.648
C_2H_6	0.423	0.444	1.488	1.478	0.104	0.091	0.997	0.991
C_2H_4	0.528	0.559	2.394	2.405	0.099	0.089	1.460	1.422
C_3H_6	1.066	1.081	4.168	3.918	0.244	0.206	2.628	2.487
C_3H_8	0.00	0.00	0.00	0.00	0.00	0.00	0.040	0.047
1-C_4H_8	0.00	0.00	--*	0.045	0.00	0.00	0.047	0.042
Trans-2-C_4H_8	0.021	0.018	0.045	0.043	0.027	0.029	0.067	0.066
Cis-2-C_4H_8	0.016	0.018	0.043	0.040	0.00	0.00	0.032	0.022
iso-C_4H_{10}	0.00	0.00	0.00	0.00	0.146	0.131	0.122	0.109
n-C_4H_{10}**	96.762	96.605	87.588	87.870	99.073	99.205	91.622	91.759

* Chromatographic peak missed.

** Obtained by difference.

TABLE V. SUMMARY OF REACTOR OPERATING CONDITIONS FOR 460°C RUNS

Run No.	Average Wall Temperature, $^{\circ}$C	Inlet Gas Temperature, $^{\circ}$C	Butane Feed Rate, s.c.f.m.	Reactor Length, Inches	Number of Samples Taken	Velocity Measurement
13	461.7	457.0	0.0178	18	6	Yes
14	460.7	459.0	0.0129	18	6	Yes
17	455.2	461.5	0.0129	18	6	Yes
18	452.0	450.0	0.0153	18	6	Yes
24	465.4	458.5	0.0153	22 1/8	6	Yes
25	456.5	451.0	0.0129	22 1/8	6	Yes
26	469.0	465.0	0.0129	22 1/8	6	Yes
27	461.9	455.0	0.0178	22 1/8	6	Yes
44	456.6	462.0	0.0064	18	6	Yes
45	455.6	460.0	0.0049	18	2	No
46	454.3	458.0	0.00627	18	2	No
48	454.8	456.5	0.0070	18	2	No
49	458.3	458.0	0.0064	18	2	No
50	457.2	458.0	0.00747	18	2	No
51	457.1	456.0	0.0091	18	2	No
52	456.2	460.5	0.0117	18	6	Yes
53	452.7	459.5	0.0270	18	6	Yes
54	458.8	462.0	0.0340	18	2	No
55	454.5	458.0	0.0211	18	2	No

TABLE VI. SUMMARY OF REACTOR OPERATING CONDITIONS FOR 510°C RUNS

Run No.	Average Wall Temperature, °C	Inlet Gas Temperature, °C	Butane Feed Rate, s.c.f.m.	Reactor Length, Inches	Number of Samples Taken	Velocity Measurement
11	509.8	506.0	0.0220	18	6	Yes
12	509.0	517.0	0.0315	18	6	Yes
15	506.5	507.0	0.0315	18	6	Yes
16	507.0	509.0	0.0367	18	6	Yes
19	513.7	511.0	0.0315	18	6	Yes
20	506.6	491.0	0.0416	18	6	Yes
21	506.8	483.0	0.0416	18	6	Yes
28	509.3	493.0	0.0315	22 1/8	6	Yes
29	509.3	502.0	0.0367	22 1/8	6	Yes
30	514.8	508.0	0.0315	22 1/8	6	Yes
31	506.5	510.0	0.0416	22 1/8	6	Yes
37	509.5	509.0	0.0120	18	2	No
38	511.6	505.5	0.0178	18	2	No
39	510.0	511.0	0.0242	18	2	No
40	507.5	511.0	0.0154	18	6	No
41	508.1	502.5	0.0474	18	6	Yes
42	516.6	507.5	0.0540	18	2	No
43	515.0	513.5	0.0511	18	6	Yes
56	507.6	510.0	0.0123	18	2	No
57	504.5	506.0	0.0165	18	6	No

TABLE VII. SUMMARY OF REACTOR OPERATING CONDITIONS FOR 560°C RUNS

Run No.	Average Wall Temperature, ^{o}C	Inlet Gas Temperature, ^{o}C	Butane Feed Rate, s.c.f.m.	Reactor Length, Inches	Number of Samples Taken	Velocity Measurement
22	564	554.0	0.0416	18	6	Yes
32	563.4	560.0	0.0549	22 1/8	6	Yes
33	565.9	564.0	0.0675	22 1/8	6	Yes
34	567.6	530.0	0.0549	22 1/8	6	Yes
35	557.3	550.0	0.0416	22 1/8	6	Yes
36	560.0	558.0	0.0549	22 1/8	6	Yes
58	553.5	552.0	0.0271	18	2	No
59	553.7	549.0	0.0340	18	2	No
60	553.5	545.0	0.0324	18	6	Yes
61	548.9	558.0	0.0262	18	2	No
62	556.2	561.0	0.0233	18	2	No
63	554.2	557.0	0.0295	18	2	No
64	557.8	555.5	0.0367	18	6	Yes
65	557.2	559.5	0.0384	18	6	Yes
66	555.8	549.0	0.0469	18	6	Yes
67	553.2	550.0	0.0426	18	6	No
68	555.0	555.5	0.0345	18	6	Yes
69	555.8	546.0	0.0537	18	6	Yes
70	559.4	554.0	0.0510	18	6	No

TABLE VIII. SUMMARY OF CONVERSION AND MATERIAL BALANCE FOR 460°C RUNS

Run No.	Moles C_4H_{10} Converted per 100 Moles Feed		Mole Ratio of C to H in the Product Stream Excluding Butanes		Moles of Products Formed per 1 Mole of Butane Converted	
r/r_o	0	0.70	0	0.70	0	0.70
13	0.3087	0.3316	0.4068	0.4078	1.902	1.900
14	0.4589	0.4079	0.4060	0.4040	1.919	1.862
17	0.3469	0.3796	0.4095	0.4067	1.896	1.889
18	0.2255	0.2212	0.3985	0.4034	2.018	1.973
24	0.2587	0.2426	0.3981	0.3954	2.016	2.079
25	0.2439	0.2285	0.4021	0.4002	1.899	1.974
26	0.3477	0.3311	0.3994	0.3933	1.988	2.018
27	0.1808	0.1823	0.3972	0.4077	1.883	1.890
44	0.3784	0.3816	0.3949	0.4003	2.034	1.853
45	--	0.6780	--	0.4034	--	1.927
46	0.4213	0.3794	0.4012	0.3977	1.931	1.963
48	--	0.3917	--	0.4009	--	1.933
49	0.3293	0.3291	0.3998	0.4022	1.930	1.939
50	0.2758	0.2846	0.4076	0.4089	1.895	1.870
51	0.1897	0.1843	0.3988	0.3953	2.011	2.076
52	0.1352	0.1542	0.3981	0.4037	2.007	2.117
53	--	0.0480	--	0.4103	--	1.833
54	0.0586	0.0609	0.4041	0.4050	2.167	2.087
55	0.0736	--	0.4050	--	1.932	--

TABLE IX. SUMMARY OF CONVERSION AND MATERIAL BALANCE FOR 510°C RUNS

Run No.	Moles C_4H_{10} Converted per 100 Moles Feed		Mole Ratio of C to H in the Product Stream Excluding Butanes		Moles of Products Formed per 1 Mole of Butane Converted	
r/r_o	0	0.70	0	0.70	0	0.70
11	1.6120	1.6300	0.4031	0.4020	1.981	1.996
12	0.975	0.9650	0.4043	0.4020	1.964	1.992
15	1.134	1.310	0.4031	0.4076	1.968	1.941
16	1.144	1.165	0.4057	0.4057	1.952	1.955
19	0.941	1.071	0.4052	0.4048	1.953	1.968
20	0.927	0.969	0.4020	0.4045	1.992	1.970
21	0.816	0.840	0.4071	0.4022	1.941	1.992
28	0.974	0.991	0.3989	0.4021	1.977	1.967
29	0.731	0.805	0.3991	0.3983	1.977	1.995
30	1.132	1.148	0.3964	0.4043	2.037	1.967
31	0.819	0.833	0.4030	0.4034	1.926	1.911
37	2.324	2.353	0.4036	0.4028	1.972	1.931
38	1.854	1.826	0.4029	0.4032	1.980	1.979
39	1.295	1.313	0.4000	0.4044	2.005	1.978
40	1.695	1.591	0.4042	0.4033	1.952	1.975
41	0.452	0.494	0.4000	0.4052	1.978	1.957
42	0.493	0.555	0.4009	0.4038	1.992	1.964
43	0.503	0.562	0.4010	0.4017	1.982	1.979
56	1.665	1.658	0.4031	0.4007	1.978	1.995
57	1.135	1.344	0.3978	0.4067	2.019	1.944

TABLE X. SUMMARY OF CONVERSION AND MATERIAL BALANCE FOR 560°C RUNS

Run No.	Moles C_4H_{10} Converted per 100 Moles Feed		Mole Ratio of C to H in the Product Stream Excluding Butanes		Moles of Products Formed per 1 Mole of Butane Converted	
r/r_o	0	0.70	0	0.70	0	0.70
22	6.328	6.725	0.3972	0.3988	2.102	2.083
32	4.844	4.887	0.3903	0.3931	2.159	2.079
33	3.932	4.187	0.3925	0.3978	2.114	2.054
34	6.240	6.613	0.3996	0.3989	2.071	2.076
35	6.888	6.880	0.3985	0.3987	2.058	2.061
36	5.563	5.705	0.4002	0.3998	2.041	2.052
58	5.148	5.443	0.3947	0.3981	2.099	2.059
59	4.356	4.492	0.3953	0.3957	2.086	2.104
60	4.583	4.642	0.3937	0.3917	2.096	2.108
61	7.378	7.013	0.3916	0.3921	2.101	2.108
62	7.748	8.187	0.3905	0.3930	2.119	2.088
63	6.787	7.070	0.3913	0.4002	2.117	2.046
64	5.935	5.986	0.3914	0.3922	2.123	2.130
65	6.194	6.479	0.3937	0.3934	2.096	2.102
66	4.152	4.040	0.3966	0.3941	2.043	2.058
67	4.375	4.765	0.3968	0.3926	2.055	2.089
68	6.738	6.559	0.3936	0.3917	2.113	2.125
69	3.760	3.684	0.3916	0.3871	2.104	2.139
70	4.171	4.567	0.3901	0.3898	2.124	2.139

TABLE XI. VALUES OF P_i

Temperature, °C	460°		510°		560°	
r/r_o	0.7	0.0	0.7	0.0	0.7	0.0
(a) CH_4						
P	0.704	0.700	0.691	0.705	0.695	0.709
% Std deviation	6.73	5.11	3.65	4.50	5.05	3.72
(b) C_2H_6						
P	0.250	0.258	0.262	0.261	0.249	0.248
% Std deviation	5.02	7.02	3.98	4.06	2.60	2.08
(c) C_2H_4						
P	0.250	0.252	0.309	0.302	0.371	0.371
% Std deviation	5.77	10.20	5.50	5.07	3.04	3.21
(d) C_3H_6						
P	0.671	0.671	0.670	0.660	0.632	0.631
% Std deviation	8.32	7.26	3.73	3.39	3.29	2.98
(e) C_4H_8						
P	0.086	0.087	0.037	0.040	0.026	0.026
% Std deviation	27.4	35.1	22.8	22.0	15.8	16.3
(f) H_2						
P	-	-	-	-	0.097	0.093
% Std deviation	-	-	-	-	15.1	27.7

TABLE XII. FIRST- AND SECOND-ORDER REACTION RATE CONSTANTS CORRELATED BY LEAST SQUARES FIT

$$k_r = Ae^{-\frac{E}{RT}}$$

(a) Values of A and E

	First-Order		Second-Order	
r/r_o	0.7	0.0	0.7	0.0
A*	3.34×10^{12}	2.49×10^{12}	2.55×10^{14}	1.88×10^{14}
% Std deviation of A	2.41	2.50	2.12	2.20
E, cal/gm-mol	54,600	54,200	56,800	56,400
% Std deviation of ΔE	1.96	2.03	1.91	1.98

(b) Values of k_r* Calculated From The Least Squares Fit

	First-Order		Second-Order	
Temperature, °C r/r_o	0.7	0.0	0.7	0.0
460°	1.71×10^{-4}	1.70×10^{-4}	2.92×10^{-3}	2.90×10^{-3}
510°	1.89×10^{-3}	1.83×10^{-3}	3.52×10^{-2}	3.43×10^{-2}
560°	1.54×10^{-2}	1.48×10^{-2}	3.15×10^{-1}	3.02×10^{-1}

---

* A and k_r have the unit of sec^{-1} for first-order and ft^3/lb sec. for second-order rate expressions.

TABLE XIII. NUMERICAL VALUES OF SOME REACTION RATE CONSTANTS COMPUTED FROM EQUATIONS 40 TO 50 AND EQUATIONS 57 AND 58

	k*		
REACTION	460°C	510°C	560°C
I	$2.80x10^{-11}$	$8.93x10^{-10}$	$1.88x10^{-8}$
II	$3.57x10^{-12}$	$1.30x10^{-10}$	$3.07x10^{-9}$
III	$1.69x10^{10}$	$2.69x10^{10}$	$3.63x10^{10}$
IV	$3.35x10^{8}$	$4.82x10^{8}$	$6.64x10^{8}$
V	$8.19x10^{8}$	$1.01x10^{9}$	$1.46x10^{9}$
VI	$1.67x10^{2}$	$0.94x10^{3}$	$0.43x10^{4}$
VII	--	--	85.5
VIII	$1.77x10^{6}$	$5.54x10^{6}$	$1.51x10^{7}$
IX	$2.26x10^{4}$	$8.05x10^{4}$	$2.46x10^{5}$
X	$4.96x10^{5}$	$1.45x10^{6}$	$3.73x10^{6}$
XI	$1.77x10^{5}$	$5.54x10^{5}$	$1.51x10^{6}$
XVIII	$3.23x10^{11}$	$4.02x10^{11}$	$4.88x10^{11}$
XIX	$3.23x10^{9}$	$4.02x10^{9}$	$4.88x10^{9}$

---

*k has the unit of sec^{-1} for a first-order reaction and cc/g-mole sec. for a second-order reaction.

TABLE XIV. FREE RADICAL CONCENTRATIONS CALCULATED FROM THE PROPOSED REACTION MECHANISM

		Relative Concentration, $(R)/(C_4H_{10})_0$*		
R	Case	460°C	510°C	560°C
H	1 and 2	2.81×10^{-11}	2.10×10^{-10}	1.63×10^{-9}
CH_3	1 and 2	2.16×10^{-8}	1.74×10^{-7}	1.10×10^{-6}
C_2H_5	1 and 2	3.15×10^{-9}	3.15×10^{-8}	1.80×10^{-7}
C_3H_7	1	1.58×10^{-17}	1.61×10^{-16}	1.25×10^{-15}
	2	1.71×10^{-13}	2.63×10^{-12}	2.30×10^{-11}
C_4H_9	1	2.46×10^{-10}	9.06×10^{-10}	2.81×10^{-9}
	2	2.49×10^{-10}	9.60×10^{-10}	3.23×10^{-9}

---

*This is the ratio of free radical concentration (R) to that of pure butane $(C_4H_{10})_0$, both in gm-mole/cc at 1 atm and the given temperature.

TABLE XV. COMPARISON OF EXPERIMENTAL RESULTS AND THOSE PREDICTED FROM THE PROPOSED REACTION MECHANISM

	Rate of Formation, $\frac{d(M)}{dt}/(C_4H_{10})_0^*$, sec.$^{-1}$		
	460°C	510°C	560°C
		(a) ethylene	
Experimental	$4.28\text{x}10^{-5}$	$5.84\text{x}10^{-4}$	$5.70\text{x}10^{-3}$
Case I	$4.41\text{x}10^{-5}$	$5.32\text{x}10^{-4}$	$5.01\text{x}10^{-3}$
Case II	$4.49\text{x}10^{-5}$	$5.75\text{x}10^{-4}$	$6.00\text{x}10^{-3}$
		(b) propylene	
Experimental	$1.08\text{x}10^{-4}$	$1.27\text{x}10^{-3}$	$9.73\text{x}10^{-3}$
Case I	$1.22\text{x}10^{-4}$	$1.32\text{x}10^{-3}$	$10.46\text{x}10^{-3}$
Case II	$1.21\text{x}10^{-4}$	$1.27\text{x}10^{-3}$	$9.29\text{x}10^{-3}$
		(c) butenes	
Experimental	$4.28\text{x}10^{-6}$	$7.56\text{x}10^{-5}$	$4.07\text{x}10^{-4}$
Case I	$5.56\text{x}10^{-6}$	$7.29\text{x}10^{-5}$	$6.91\text{x}10^{-4}$
Case II	$5.63\text{x}10^{-6}$	$7.73\text{x}10^{-5}$	$7.95\text{x}10^{-4}$

---

[*] $\frac{d(M)}{dt}$ is the rate of formation of M at zero conversion in $\frac{\text{gm-mole}}{\text{cc sec.}}$, where M is ethylene, propylene, or butenes. $(C_4H_{10})_0$ is the concentration of pure butane in gm-mole/cc. Both at 1 atm and the given temperature.

TABLE XVI. RATES OF FORMATION AND DISAPPEARANCE FOR SOME FREE RADICALS AS CALCULATED FROM THE PROPOSED REACTION MECHANISM

	rate, $\frac{d(R)}{dt}/(C_4H_{10})_o$*, sec.$^{-1}$			
Temperature	Case I		Case II	
	formation	disappearance	formation	disappearance
		(a) $H^{\cdot}$		
460° C	0.61×10^{-5}	0.79×10^{-5}	0.62×10^{-5}	0.82×10^{-5}
510° C	1.02×10^{-4}	0.88×10^{-4}	1.07×10^{-4}	1.01×10^{-4}
560° C	1.47×10^{-3}	0.86×10^{-3}	1.57×10^{-3}	1.21×10^{-3}
		(b) $CH_3\cdot$		
460° C	1.22×10^{-4}	1.20×10^{-4}	1.22×10^{-4}	1.22×10^{-4}
510° C	1.32×10^{-3}	1.31×10^{-3}	1.33×10^{-3}	1.42×10^{-3}
560° C	1.05×10^{-2}	1.07×10^{-2}	1.39×10^{-2}	1.31×10^{-2}
		(c) $C_2H_5\cdot$		
460° C	4.35×10^{-5}	4.33×10^{-5}	4.41×10^{-5}	4.33×10^{-5}
510° C	5.04×10^{-4}	5.25×10^{-4}	5.32×10^{-4}	5.25×10^{-4}
560° C	4.24×10^{-3}	4.60×10^{-3}	4.88×10^{-3}	4.60×10^{-3}

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* $\frac{d(R)}{dt}$ is the rate of formation of R in $\frac{\text{gm-mole}}{(\text{cc})(\text{sec.})}$ where R is $H\cdot$, $CH_3\cdot$, or $C_2H_5\cdot$. $(C_4H_{10})_o$ is the concentration of pure butane in $\frac{\text{gm-mole}}{\text{cc}}$. Both at 1 atm and the given temperature.

PART ONE

APPENDICES

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APPENDIX A: QUANTITATIVE ANALYSIS OF LIGHT HYDROCARBONS BY GAS CHROMATOGRAPHY

ABSTRACT

A three-stage gas chromatograph was developed to allow quantitative resolution of light hydrocarbons through C_4 components along with oxygen, nitrogen, hydrogen, carbon monoxide and carbon dioxide. The system selected, however, did not allow complete resolution of 1-butene and isobutylene. In the analysis, the response of a component in terms of g. moles per unit of chromatogram area relative to that of nitrogen was found useful. The individual response is designated as K and the ratio of responses as N. Within experimental accuracy moderate variations in operating conditions did not change the value of N.

A least-squares analysis of the experimental data was used to determine the value of N for each component. By establishing the value of K for nitrogen each day, it was then possible to compute the amount of each component from its chromtogram area and its value of N. In the calibration of eighteen gases, an average standard deviation of points from the least-square plots was 1.23%

INTRODUCTION

Much work has been done on the analysis of hydrocarbons by gas chromatography. One of the outstanding studies is that by Bloch (A1). By use of a four-stage column he was able to separate more than 70 hydrocarbons from C_1 through C_7 as well as permanent gases. Work reported here is a less extensive study on possible products from the thermal decomposition of butane, but the data was analyzed statistically and constants were obtained for use in the ready computation of the amounts of substances present.

EXPERIMENTAL PROCEDURE

Several partition liquids carried on firebrick were studied qualitatively. These were tetraisobutylene, didecyl phthalate, di-n-butyl maleate, tetraethylene glycol, dimethyl ether, dimethyl sulfolane, and saturated silver nitrate solution in benzyl cyanide. Also, with particular concern for the very low boiling gases, molecular sieves, silica gel, and activated alumina were studied in their roles as absorbent solids.

On the basis of the preliminary data, a three-stage column was developed, in which the first two stages consisted of two sections each. The first section of the first stage was an 8-ft. length of copper tubing with an inside diameter of 3/16 in. It was packed with 30 to 60 mesh firebrick which was coated with 30% by weight, on a dry basis, of di-n-butyl maleate. A second part of that same stage was a 35-ft. length of tubing packed with 30 to 60 mesh firebrick which was coated with 33% by weight of dimethyl sulfolane and 3.5% didecyl phthalate. The first section of the second stage was a 9-in. length of tubing packed with 35 to 48 mesh basic activated alumina. That was followed with an 18-in. length containing 35 to 48 mesh silica gel. The third stage was a 20-ft. length of tubing packed with 28 to 48 mesh molecular sieve, type 13X.

A dual-thermistor thermal conductivity cell was used as a sensing element at the discharge of each stage. The thermistors were matched within 0.1%, and the resistance of each was 8000 ohms. at $25^{\circ}C$. Cells and columns were mounted in an air thermostat in which the temperature was controlled at $30 \pm 0.5^{\circ}C$.

Helium was used as the carrier gas. Its inlet pressure was varied from 18 to 22 psig. and was maintained to within ± 0.2 psig. The gas flowing through the final stage was discharged directly to the atmosphere. Under these conditions a flow rate of 45 to 60 cc./min., as measured at 1 atm. and 70° F., was maintained. For a given rate the variation in flow was not greater than ± 1 cc./min.

The signal from the first stage was recorded on a Brown continuous recorder with a full-scale deflection of 1 mV and a response time of 1 second. The recorder was equipped with a disc integrator. Signals from the second and third stages were each recorded on a Varian G-10 recorder with a full-scale deflection of 10 mV and a response time of 1 second. Before entering the recorder, each signal passed through a seven-step attenuator with a maximum attenuation of 200. A signal was generally attenuated to yield a maximum recordable voltage for the recorder.

The sample cells for the multicomponent mixtures were similar to the design used by Sakaida (A2) with volumes of the order of 1 cc. which were calibrated to within ± 0.001 cc. For quantitative calibration of the chromatograph, standard mixtures were prepared in a 10-liter bottle. In the preparation, the bottle was first evacuated, and each component was then added through a manifold system. The change in total pressure for each gas was observed on a mercury-filled manometer by means of a cathetometer. Finally helium was used to fill the bottle to slightly above atmospheric pressure. A calibration mixture usually consisted of no more than three components besides helium, wherein one of those was always nitrogen which was present as the internal standard. Dalton's law was used in combination with individual compressibility factors in the computation of the composition of the gas in the bottle. The assumption of Dalton's law for the conditions studied was verified experimentally by the fact that each component gave the same response in pressure whether in a mixture or in its pure form.

EXPERIMENTAL RESULTS

Preliminary runs on the three-stage column showed that it was capable of completely separating the components formed in the thermal decomposition of n-butane, except for isobutylene and 1-butene which were only partially resolved. Relative retention times and volumes for the various components are given in Table AI. Hydrogen was the basis for comparison.

Samples of five calibration mixtures were analyzed in a mass spectrometer. The analyses are shown in Table AII and agreed to within 4% or less with the computed compositions.

In the calibration of the chromatographic system, four or more runs were made for each mixture. The area of each chromatogram was measured with a planimeter, and the response, K, defined as g. moles per unit of chromatogram area, was computed. In the use of nitrogen as an internal standard, the ratio of K for a component to that of nitrogen, defined as the normalization factor, N, was not found to vary with modest variations in operating conditions. Average values of N for the first nine components shown in Table AIII were found to be constant with respect to varying amounts of the components. That was not the case for components 10 to 18, and here N was assumed to be a linear function of a normalized chromatogram area. Coefficients in the function were determined by a least-squares fit. The normalized areas used in the correlation were calculated according to the expression:

normalized component area, A_N

$$= \frac{(\text{observed component area})(K_{N_2})}{(K^{o}_{N_2})} \quad ,$$

where $K^o_{N_2}$ is a standard response for nitrogen arbitrarily selected as 0.0300×10^{-8} g. moles/unit chromatogram area. In applying the above equation, the value of K_{N_2} was determined in the same run for which the observed component areas were obtained.

By use of the value of N for a given component, the amount of that component in an unknown sample can be determined by the following equation:

Amount of component A in g. moles

$$= (\text{observed chromatogram area for A})(N_A)(K_{N_2}) \quad .$$

Expected standard deviations for the results are included in Table AIII and were determined from the calibration studies where for most of the components the range of amounts was varied over six to eight-fold and in some cases as high as twenty-fold.

In order to test the invariance of N, the operating conditions were varied over a moderate range. For example, the carrier-gas flow was varied between 45 and 60 std. cc./min., and the thermistor current between 8 and 10 ma. Under these conditions K_{N_2} varied 3-fold but with no changes in N within experimental error.

Although the minimum amount of the calibration samples analyzed corresponded to 0.7% by volume for isobutane in a 1 cc. sample at atmospheric pressure and 25° C., the minimum detectable amount corresponded to approximately 0.005% or

50 ppm. by volume for all the components except hydrogen which had a minimum of about 0.3%. These low concentrations produced signals ranging in maximum from 0.1 to 0.4 mv. in order of decreasing retention times of the components.

DISCUSSION

In the calibration of eighteen gases, the maximum percent standard deviations from the least-squares fit were 7% or less except for 1,3-butadiene. The average maximum percent standard deviation was 4.77%. Relative to an average value for amount of component present, the average percent standard deviation for all eighteen components was 1.23%.

Among the possible sources of error, an important one was the variation in partition efficiency which resulted from variations in the volume of the gas-sample pellet impinging upon the first plate of the column. For the same amount of a component, small pellet volumes gave lower values of N than larger pellet volumes.

There were other errors of less significance than those resulting from feed problem. Sources for the added variability were in the preparation of calibration mixtures, variations in operating conditions during each analysis of a mixture, inaccuracies in area measurements, and finally in the insensitivity of the recorder near the level of zero signal.

It is interesting to note that the results obtained in this study agree very favorably with those of Messner, et al. (A3). A comparison is presented in Table AIV. In their work, the ratios of the relative response of nitrogen to that of other components correspond to the values of N presented here. As shown in the table, however, the relative responses obtained by Messner for components 10 to 18 are constants which agree to within 5% of values of N obtained in this work for zero-values of chromatogram areas. The fact that such values of N obtained from entirely different chromatographic systems agree so well gives added validity to their universal applicability.

Because of changes with time in the character of the component parts of the chromatographic system, daily calibration is required for quantitative studies. The method presented here, wherein a calibration is based upon the daily response of the system to nitrogen in combination with previously determined constants, is very satisfactory in lieu of a calibration for every component, especially if a system with a large number of components is being analyzed each day.

REFERENCE

A1. Bloch, M. G., "Determinations of C_1 Through C_7 Hydrocarbons in a Single Run by a Four-Stage Gas Chromatograph," page 133, Gas Chromatography, Second International Symposium, edited by H. J. Noebels, R. F. Wall, and N. Brenner, Academic Press, New York City, 1961.

A2. Sakaida, R. R., Rinker, R. G., Cuffel, R. F., and Corcoran, W. H., Anal. Chem., 33, 32 (1961).

A3. Messner, A. E., Rosie, D. M., and Argabright, P. A., Anal. Chem., 31, 230 (1959).

TABLE AI. RELATIVE RETENTION TIMES AND RETENTION VOLUMES COMPARED WITH HYDROGEN FOR POSSIBLE PRODUCTS FROM THE THERMAL DECOMPOSITION OF n-BUTANE

Component	Retention Time (Sec.)	Retention Volume (cc.)
	First Stage	
*H_2 + air + CO + methane	0	0
ethane & ethylene	84	77
CO_2	191	175
propane	266	244
propylene	387	355
isobutane	524	481
acetylene	622	570
n-butane	800	734
1-butane + isobutylene	1129	1035
trans-2-butene	1428	1309
cis-2-butene	1671	1532
methyl acetylene	2067	1895
1,3-butadiene	2357	2161
ethyl acetylene	4596	4215
	Second Stage	
H_2 + air + CO + methane	0	0
ethane	137	126
ethylene	179	164

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* Components coming out together or as partially separated peaks.

TABLE AI. (Continued)

Component	Retention Time (Sec.)	Retention Volume (cc.)
	Third Stage	
H_2	0	0
oxygen	43	39
nitrogen	118	108
methane	189	173
CO	261	239

TABLE AII. COMPARISON OF COMPUTED ANALYSES OF CALIBRATION SAMPLES AND MEASUREMENT BY MASS SPECTROMETER[a]

Mixture No.	Component	Mole % Computed	Mole % Reported
1	methane	9.84	9.5
	ethylene	5.87	5.9
	propylene	4.05	4.0
	nitrogen	80.25	80.4
	propane	-	0.2
2	propane	4.85	4.9
	isobutane	5.10	5.0
	nitrogen	90.05	90.1
3	trans-2-butene	4.96	5.0
	1,3-butadiene	2.03	2.0
	nitrogen	93.02	93.0
4	ethane	1.04	1.0
	acetylene	1.09	1.1
	nitrogen	97.87	97.9
5 & 6[b]	methane	2.85	2.7
	propane	3.08	3.0
	nitrogen	94.07	94.1
	ethane	-	0.2

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[a] Courtesy of the California Research Corporation

[b] Duplicate samples, with identical reported analyses

TABLE AIII. SUMMARY OF ANALYSES

Component No.	Component	N	No. of Tests	Amount of Sample, (10^{-8} g.moles)		Standard Deviation, (10^{-8} g.moles)	% Standard Deviation Based On Average Amount	Maximum % Standard Deviation
				Range	Average			
1	Oxygen	1.071	12	9.15-57.60	32.78	0.319	0.973	3.49
2	Methane	1.112	12	7.91-55.12	23.96	0.204	0.851	2.58
3	Carbon Monoxide	1.016	11	7.81-58.79	21.53	0.278	1.291	3.56
4	Ethane	0.761	16	10.61-73.54	40.54	0.313	0.772	2.95
5	Ethylene	0.805	12	18.46-71.44	41.45	0.364	0.878	1.97
6	Carbon Dioxide	0.848	19	5.33-65.30	32.46	0.374	1.152	7.02
7	Propane	0.640	17	10.12-72.89	36.21	0.651	1.798	6.43
8	Propylene	0.663	16	10.39-57.71	27.56	0.279	1.012	2.69
9	Acetylene	0.947	13	8.94-64.49	30.90	0.250	0.809	2.80
10*	Hydrogen	$32.29 + 0.0322A_n$	22	24.30-214.50	88.32	1.54	1.744	6.34
11	Isobutane	$0.523 + 0.995 \times 10^{-5}A_n$	13	3.23-67.35	25.97	0.216	0.832	6.69
12	n-Butane	$0.512 + 1.674 \times 10^{-5}A_n$	15	8.40-59.05	26.86	0.340	1.266	4.05
13	1-Butene	$0.522 + 1.399 \times 10^{-5}A_n$	15	10.77-67.67	33.58	0.425	1.266	3.95
14	Isobutylene	$0.511 + 2.772 \times 10^{-5}A_n$	14	10.61-59.27	37.89	0.358	0.945	3.37
15	Trans-2-Butene	$0.503 + 3.697 \times 10^{-5}A_n$	15	6.94-59.54	26.26	0.360	1.371	5.19
16	Cis-2-Butene	$0.519 + 4.030 \times 10^{-5}A_n$	15	8.45-47.21	26.66	0.513	1.924	6.07
17	Methyl Acetylene	$0.678 + 1.898 \times 10^{-5}A_n$	16	9.53-63.42	32.19	0.425	1.320	4.46
18	1,3-Butadiene	$0.530 + 4.403 \times 10^{-5}A_n$	15	3.98-55.72	26.44	0.486	1.838	12.21

*$K^{o}_{N_2} = 0.0300 \times 10^{-8}$ g.mole/unit chromatogram area for components 10-18.

TABLE AIV. COMPARISON OF NORMALIZATION FACTORS

Component No.	Component	N* (Authors' Values)	N (Computed from Messner (3))
-	N_2	1	1
1	O_2	1.071	1.05
2	Methane	1.112	1.17
3	Carbon Monoxide	1.016	1.00
4	Ethane	0.761	0.82
5	Ethylene	0.805	0.88
6	Carbon Dioxide	0.848	0.88
7	Propane	0.640	0.65
8	Propylene	0.663	0.67
9	Acetylene	0.947	-
10	Hydrogen	32.29	-
11	Isobutane	0.523	0.51
12	n-Butane	0.512	0.49
13	1-Butene	0.522	0.52
14	Isobutylene	0.511	0.51
15	Trans-2-Butene	0.503	0.49
16	Cis-2-Butene	0.519	0.48
17	Methyl Acetylene	0.678	-
18	1,3-Butadiene	0.530	0.53

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[*] For components 10-18, the value of the normalized area, A, was taken as zero to allow comparison with Messner (3).

APPENDIX B. COMPOSITION OF SAMPLES BY CHROMATOGRAPHIC ANALYSIS

Run No.	Sampling Position	Composition of Samples, Mole %										
	r/r_o	H_2	CH_4	C_2H_6	C_2H_4	C_3H_6	C_3H_8	$1-C_4H_8$	trans-2-C_4H_8	cis-2-C_4H_8	iso-C_4H_{10}	n-C_4H_{10}
11	Overall	-	1.275	0.444	0.559	1.081	-	-	0.018	0.018	-	96.605
	0.70	-	1.133	0.430	0.535	1.064	-	-	0.026	0.014	-	96.798
	0.45	-	1.243	0.424	0.523	1.119	-	-	0.023	0.019	-	96.649
	0.25	-	1.200	0.425	0.529	1.073	-	-	0.030	0.021	-	96.722
	0	-	1.101	0.425	0.507	1.061	-	-	0.027	0.023	-	96.856
12	Overall	-	0.711	0.265	0.306	0.647	-	-	0.023	0.015	-	98.033
	0.70	-	0.686	0.247	0.301	0.639	-	-	0.020	0.011	-	98.096
	0.50	-	0.670	0.241	0.296	0.630	-	-	0.024	0.015	-	98.124
	0.25	-	0.685	0.248	0.297	0.614	-	-	0.016	0.013	-	98.127
	0	-	0.658	0.257	0.296	0.646	-	-	0.023	0.017	-	98.103
13	Overall	-	0.313	0.087	0.091	0.205	-	-	0.025	0.031	-	99.248
	0.70	-	0.214	0.085	0.082	0.218	-	-	0.017	0.013	-	99.371
	0.50	-	0.188	0.076	0.074	0.210	-	-	0.022	0.013	-	99.417
	0.25	-	0.229	0.086	0.083	0.207	-	-	0.016	0.017	-	99.362
	0	-	0.204	0.078	0.068	0.201	-	-	0.019	0.014	-	99.416
14	Overall	-	0.324	0.122	0.123	0.287	-	-	0.014	0.008	-	99.122
	0.70	-	0.283	0.103	0.102	0.261	-	-	0.014	0.023	-	99.244
	0.50	-	0.278	0.111	0.111	0.271	-	-	0.021	0.017	-	99.191
	0.30	-	0.289	0.109	0.104	0.275	-	-	0.016	0.017	-	99.190
	0	-	0.309	0.119	0.111	0.297	-	-	0.021	0.020	-	99.123
15	0.70	-	0.839	0.320	0.385	0.884	-	0.007	0.024	0.022	-	97.519
	0.45	-	0.783	0.304	0.373	0.798	-	0.007	0.024	0.020	-	97.692
	0.25	-	0.844	0.320	0.391	0.770	-	0.007	0.022	0.019	-	97.627
	0	-	0.783	0.283	0.341	0.741	-	0.007	0.019	0.019	-	97.817
16	0.70	-	0.768	0.291	0.353	0.762	-	0.007	0.025	0.022	-	97.773
	0.40	-	0.755	0.293	0.350	0.767	-	0.007	0.025	0.022	-	97.781
	0	-	0.762	0.280	0.334	0.759	-	0.007	0.024	0.020	-	97.814
17	0.70	-	0.254	0.093	0.096	0.256	-	-	0.015	0.013	-	99.286
	0.50	-	0.245	0.079	0.082	0.233	-	-	0.016	0.021	-	99.324
	0.25	-	0.235	0.085	0.083	0.232	-	-	0.015	0.015	-	99.335
	0	-	0.219	0.087	0.090	0.230	-	-	0.015	0.015	-	99.344
18	0.70	-	0.154	0.057	0.059	0.166	-	-	-	-	-	99.564
	0.50	-	0.159	0.056	0.056	0.135	-	-	-	-	-	99.594
	0.25	-	0.148	0.056	0.055	0.156	-	-	-	-	-	99.585
	0.05	-	0.173	0.056	0.061	0.164	-	-	-	-	-	99.546
19	0.70	-	0.729	0.269	0.326	0.728	-	0.007	0.013	0.015	-	97.914
	0.50	-	0.684	0.264	0.313	0.609	-	0.007	0.020	0.016	-	98.087
	0.25	-	0.674	0.253	0.306	0.599	-	0.007	0.019	0.015	-	98.127
	0	-	0.628	0.246	0.281	0.620	-	0.007	0.019	0.021	-	98.178
20	0.70	-	0.663	0.244	0.299	0.652	-	-	0.018	0.016	-	98.108
	0.50	-	0.626	0.226	0.285	0.604	-	-	0.018	0.016	-	98.225
	0.25	-	0.590	0.237	0.301	0.631	-	-	0.022	0.015	-	98.204
	0	-	0.668	0.228	0.289	0.606	-	-	0.024	0.016	-	98.169

Run No.	Sampling Position	Composition of Samples, Mole %										
	r/r_o	H_2	CH_4	C_2H_6	C_2H_4	C_3H_6	C_3H_8	$1\text{-}C_4H_8$	trans-2-C_4H_8	cis-2-C_4H_8	iso-C_4H_{10}	n-C_4H_{10}
21	0.70	-	0.608	0.202	0.267	0.543	-	-	0.020	0.019	-	98.341
	0.50	-	0.577	0.215	0.258	0.556	-	-	0.020	0.019	-	98.355
	0.25	-	0.555	0.209	0.248	0.523	-	-	0.020	0.019	-	98.426
	0	-	0.539	0.200	0.238	0.556	-	-	0.020	0.019	-	98.428
22	Overall	0.600	4.665	1.499	2.420	4.046	-	0.045	0.043	0.040	-	86.642
	0.70	0.600	4.274	1.488	2.394	4.168	-	0.045	0.045	0.043	-	86.943
	0.50	0.444	4.102	1.453	2.346	3.867	-	0.039	0.053	0.040	-	87.656
	0.25	0.656	4.252	1.494	2.413	3.994	-	0.051	0.048	0.033	-	87.059
	0	0.619	4.083	1.425	2.318	3.859	-	0.046	0.047	0.037	-	87.567
24	0.70	-	0.187	0.069	0.062	0.173	-	-	-	-	0.151	99.346
	0.50	-	0.193	0.073	0.073	0.169	-	-	-	-	0.132	99.360
	0.25	-	0.208	0.075	0.074	0.189	-	-	-	-	0.157	99.297
	0	-	0.191	0.073	0.074	0.183	-	-	-	-	0.131	99.349
25	0.70	-	0.170	0.058	0.057	0.145	-	-	0.020	-	0.141	99.409
	0.50	-	0.175	0.062	0.060	0.149	-	-	0.023	-	0.148	99.403
	0.25	-	0.172	0.067	0.059	0.149	-	-	0.020	-	0.148	99.385
	0	-	0.175	0.062	0.052	0.166	-	-	0.018	-	0.153	99.385
26	0.70	-	0.271	0.088	0.077	0.204	-	-	0.027	-	0.130	99.204
	0.50	-	0.241	0.092	0.088	0.223	-	-	0.027	-	0.145	99.184
	0.25	-	0.265	0.085	0.095	0.221	-	-	0.027	-	0.141	99.019
	0	-	0.257	0.092	0.096	0.219	-	-	0.024	-	0.141	99.171
27	0.70	-	0.121	0.044	0.041	0.115	-	-	0.023	-	0.145	99.511
	0.50	-	0.127	0.049	0.042	0.115	-	-	0.023	-	0.147	99.497
	0.25	-	0.125	0.042	0.042	0.114	-	-	0.024	-	0.141	99.511
	0	-	0.119	0.044	0.036	0.116	-	-	0.024	-	0.150	99.511
28	0.70	-	0.668	0.259	0.299	0.675	-	-	0.029	-	0.133	97.937
	0.50	-	0.649	0.259	0.303	0.649	-	-	0.029	-	0.129	97.982
	0	-	0.696	0.243	0.269	0.661	-	-	0.039	-	0.129	97.963
29	0.70	-	0.580	0.211	0.236	0.541	-	-	0.024	-	0.140	98.268
	0.50	-	0.535	0.218	0.238	0.529	-	-	0.030	-	0.142	98.308
	0.25	-	0.557	0.204	0.230	0.507	-	-	0.028	-	0.128	98.346
	0	-	0.519	0.187	0.205	0.496	-	-	0.028	-	0.143	98.422
30	0.70	-	0.763	0.311	0.360	0.766	-	-	0.034	-	0.130	97.636
	0.50	-	0.734	0.299	0.336	0.751	-	-	0.035	-	0.121	97.724
	0.25	-	0.731	0.298	0.332	0.739	-	-	0.035	-	0.146	97.719
	0	-	0.885	0.285	0.342	0.733	-	-	0.035	-	0.121	97.599
31	0.70	-	0.546	0.214	0.217	0.490	0.021	0.020	0.051	0.020	0.117	98.304
	0.50	-	0.555	0.212	0.223	0.463	0.023	0.011	0.045	0.016	0.127	98.325
	0.25	-	0.533	0.215	0.218	0.471	0.022	0.016	0.049	0.016	0.128	98.332
	0	-	0.551	0.209	0.220	0.472	0.021	0.019	0.053	0.017	0.137	98.301

Run No.	Sampling Position	Composition of Samples, Mole %									
	r/r_o	H_2	CH_4	C_2H_6	C_2H_4	C_3H_6	C_3H_8	$1\text{-}C_4H_8$	trans-2-C_4H_8	cis-2-C_4H_8	iso-C_4H_{10}	n-C_4H_{10}
32	Overall	0.751	3.329	1.208	1.753	2.910	0.046	0.034	0.069	0.036	0.119	89.745
	0.70	0.564	3.282	1.164	1.689	2.962	0.041	0.040	0.060	0.043	0.118	90.233
	0.50	0.605	3.202	1.188	1.739	2.945	0.045	0.041	0.060	0.032	0.118	90.025
	0.25	0.757	3.267	1.193	1.744	2.931	0.040	0.039	0.062	0.023	0.116	89.828
	0	0.738	3.221	1.156	1.690	2.915	0.039	0.046	0.067	0.030	0.116	89.982
33	0.70	0.411	2.574	0.997	1.460	2.628	0.040	0.047	0.057	0.032	0.122	91.632
	0.50	0.393	2.688	0.950	1.413	2.457	0.030	0.039	0.058	0.022	0.110	91.840
	0	0.398	2.677	0.958	1.415	2.381	0.030	0.038	0.042	0.023	0.117	91.921
34	0.70	0.610	4.126	1.475	2.377	3.993	0.049	0.063	0.078	0.044	0.119	87.066
	0.50	0.648	4.059	1.416	2.242	3.905	0.045	0.067	0.079	0.033	0.108	87.398
	0.25	0.611	3.989	1.474	2.287	3.847	0.041	0.055	0.071	0.032	0.098	87.495
	0	0.609	3.806	1.419	2.235	3.843	0.045	0.057	0.069	0.028	0.105	87.784
35	0.70	0.509	4.262	1.592	2.546	4.083	0.040	0.071	0.073	0.041	0.097	86.686
	0.50	0.522	4.336	1.589	2.504	4.214	0.042	0.063	0.069	0.043	0.101	86.517
	0.25	0.484	4.138	1.583	2.507	3.998	0.035	0.061	0.074	0.039	0.100	86.981
	0	0.439	4.361	1.590	2.493	4.123	0.042	0.058	0.070	0.037	0.097	86.690
36	0.70	0.511	3.433	1.309	2.049	3.554	0.028	0.046	0.088	0.024	0.098	88.860
	0.50	0.447	3.347	1.284	2.022	3.446	0.033	0.051	0.072	0.030	0.099	89.169
	0.25	0.422	3.348	1.288	2.050	3.593	0.033	0.051	0.063	0.028	0.114	88.980
	0	0.416	3.427	1.275	1.935	3.495	0.031	0.055	0.068	0.030	0.104	89.164
37	0.70	-	1.545	0.643	0.761	1.476	0.036	0.011	0.050	0.019	0.093	95.459
	0	-	1.546	0.601	0.725	1.517	0.024	0.010	0.041	0.016	0.109	95.411
38	0.70	-	1.236	0.475	0.584	1.190	0.012	0.009	0.035	0.010	0.107	96.342
	0	-	1.259	0.488	0.588	1.203	0.010	0.015	0.037	0.007	0.107	96.287
39	0.70	-	0.875	0.351	0.448	0.845	-	0.007	0.037	-	0.102	97.335
	0	-	0.936	0.342	0.399	0.842	-	0.006	0.036	-	0.103	97.335
40	0.70	-	1.076	0.416	0.501	1.037	0.012	0.011	0.030	0.012	0.103	96.802
	0.50	-	1.130	0.408	0.492	1.055	0.012	0.013	0.042	0.018	0.105	96.725
	0.25	-	1.061	0.438	0.527	1.001	0.016	0.017	0.052	0.019	0.109	96.760
	0	-	1.107	0.438	0.510	1.119	0.010	0.012	0.040	0.019	0.109	96.636
41	0.70	-	0.334	0.127	0.149	0.328	-	-	0.025	-	0.089	98.948
	0.50	-	0.321	0.119	0.145	0.283	-	-	0.026	-	0.101	99.005
	0.25	-	0.338	0.112	0.142	0.315	-	-	0.026	-	0.090	98.977
	0	-	0.320	0.130	0.126	0.287	-	-	0.027	-	0.093	99.017
42	0.70	-	0.380	0.146	0.166	0.364	-	-	0.028	-	0.099	98.817
	0	-	0.355	0.130	0.148	0.321	-	-	0.022	-	0.099	98.925
43	0.70	-	0.398	0.149	0.167	0.363	-	-	0.029	-	0.099	98.795
	0.50	-	0.369	0.135	0.167	0.361	-	-	0.029	-	0.104	98.835
	0.25	-	0.391	0.140	0.166	0.367	-	-	0.029	-	0.108	98.799
	0	-	0.365	0.129	0.143	0.325	-	-	0.029	-	0.089	98.920

Run No.	Sampling Position	Composition of Samples, Mole %										
	r/r_o	H_2	CH_4	C_2H_6	C_2H_4	C_3H_6	C_3H_8	$1-C_4H_8$	trans-2-C_4H_8	cis-2-C_4H_8	iso-C_4H_{10}	n-C_4H_{10}
44	0.70	-	0.271	0.088	0.092	0.240	0.030	0.009	0.021	-	0.078	99.218
	0.50	-	0.274	0.095	0.098	0.240	0.031	0.004	0.020	-	0.084	99.154
	0.25	-	0.287	0.102	0.117	0.282	0.035	-	0.020	-	0.089	99.068
	0	-	0.277	0.094	0.103	0.222	0.030	-	0.020	-	0.079	99.154
45	0.70	-	0.458	0.149	0.186	0.416	0.038	0.019	0.032	-	0.080	98.622
46	0.70	-	0.273	0.092	0.095	0.239	0.024	-	0.019	-	0.079	99.179
	0	-	0.282	0.104	0.106	0.273	0.027	-	0.019	-	0.099	99.090
48	0.70	-	0.258	0.100	0.104	0.245	0.028	-	0.019	-	0.080	99.166
49	0.70	-	0.233	0.081	0.081	0.204	0.005	0.005	0.024	0.003	0.084	99.280
	0	-	0.222	0.083	0.080	0.203	0.004	0.005	0.033	0.003	0.072	99.295
50	0.70	-	0.177	0.065	0.065	0.175	0.011	-	0.031	0.004	0.081	99.388
	0	-	0.182	0.064	0.062	0.172	0.003	-	0.031	0.004	0.075	99.404
51	0.70	-	0.149	0.046	0.042	0.137	-	-	-	-	0.076	99.542
	0	-	0.140	0.052	0.051	0.137	-	-	-	-	0.070	99.550
52	0.70	-	0.109	0.038	0.040	0.117	-	-	-	-	0.066	99.608
	0.50	-	0.097	0.040	0.032	0.100	-	-	-	-	0.067	99.664
	0.25	-	0.094	0.043	0.039	0.097	-	-	-	-	0.064	99.663
	0	-	0.097	0.041	0.039	0.094	-	-	-	-	0.060	99.669
53	0.70	-	0.028	0.014	0.014	0.036	-	-	-	-	0.061	99.851
	0.50	-	0.025	0.013	0.013	0.033	-	-	-	-	0.057	99.859
	0.25	-	0.030	0.012	0.012	0.037	-	-	-	-	0.053	99.856
54	0.70	-	0.042	0.015	0.015	0.047	-	-	-	-	0.057	99.816
	0	-	0.041	0.015	0.015	0.045	-	-	-	-	0.056	99.829
55	0	-	0.049	0.020	0.020	0.055	-	-	-	-	0.047	99.810
56	0.70	-	1.182	0.424	0.499	1.087	0.010	-	0.039	0.012	0.040	96.707
	0	-	1.137	0.431	0.515	1.102	0.005	-	0.039	0.012	0.033	96.727
57	0.70	-	0.894	0.319	0.372	0.941	0.003	0.009	0.032	0.008	0.033	97.389
	0.25	-	0.915	0.336	0.393	0.786	0.004	0.008	0.031	0.008	0.051	97.468
	0	-	0.836	0.309	0.355	0.722	0.008	0.009	0.032	0.003	0.025	97.710
58	0.70	0.420	3.433	1.339	1.918	3.357	0.018	0.033	0.042	0.023	-	89.402
	0	0.422	3.506	1.252	1.767	3.164	0.017	0.033	0.044	0.023	-	89.772
59	0.70	0.376	3.073	1.071	1.556	2.830	0.011	0.027	0.040	0.020	-	90.997
	0	0.287	3.056	1.054	1.507	2.692	0.011	0.027	0.040	0.020	-	91.326
60	0.70	0.379	3.285	1.112	1.550	2.848	0.033	0.031	0.048	0.023	-	90.691
	0.50	0.387	3.316	1.165	1.653	2.834	0.030	0.035	0.046	0.028	-	90.506
	0.25	0.380	3.240	1.109	1.579	2.741	0.026	0.031	0.040	0.023	-	90.831
	0	0.380	3.119	1.114	1.594	2.802	0.030	0.035	0.048	0.023	-	90.855

Run No.	Sampling Position	Composition of Samples, Mole %										
	r/r_o	H_2	CH_4	C_2H_6	C_2H_4	C_3H_6	C_3H_8	$1\text{-}C_4H_8$	trans-2-C_4H_8	cis-2-C_4H_8	iso-C_4H_{10}	n-C_4H_{10}
61	0.70	0.578	4.740	1.635	2.464	3.918	0.206	0.058	0.078	0.043	0.039	86.241
	0	0.552	4.951	1.717	2.621	4.101	0.218	0.063	0.073	0.042	0.043	85.619
62	0.70	0.727	5.232	1.849	2.750	4.690	0.227	0.073	0.094	0.055	0.040	84.263
	0	0.749	5.133	1.798	2.691	4.340	0.190	0.076	0.083	0.046	0.042	84.852
63	0.70	0.553	4.458	1.578	2.355	4.119	0.221	0.062	0.080	0.045	0.047	86.482
	0	0.598	4.665	1.535	2.329	3.850	0.211	0.046	0.062	0.057	0.042	86.605
64	0.70	0.642	3.901	1.411	2.095	3.444	0.211	0.050	0.060	0.027	0.046	88.113
	0.50	0.618	3.981	1.412	2.061	3.633	0.220	0.063	0.075	0.040	0.055	87.842
	0.25	0.685	3.881	1.348	2.051	3.478	0.216	0.047	0.067	0.036	0.046	88.245
	0	0.696	3.888	1.376	2.078	3.418	0.217	0.048	0.058	0.033	0.058	88.130
65	0.70	0.525	4.362	1.502	2.273	3.746	0.170	0.044	0.053	0.035	0.041	87.249
	0.50	0.489	4.225	1.388	2.103	3.605	0.169	0.047	0.057	0.034	0.039	87.844
	0.25	0.500	4.122	1.541	2.303	3.616	0.167	0.043	0.054	0.034	0.047	87.573
	0	0.509	4.151	1.434	2.142	3.606	0.173	0.045	0.061	0.034	0.042	87.803
66	0.70	0.285	2.788	0.965	1.423	2.395	0.127	0.027	0.041	0.023	0.045	91.881
	0.50	0.225	3.014	1.056	1.566	2.643	0.139	0.036	0.052	0.027	0.042	91.200
	0.25	0.226	3.017	1.036	1.522	2.573	0.135	0.031	0.047	0.021	0.043	91.123
	0	0.179	2.785	0.971	1.435	2.461	0.126	0.033	0.048	0.028	0.038	91.896
67	0.70	0.393	3.227	1.126	1.621	2.843	0.142	0.041	0.046	0.024	0.046	90.491
	0.50	0.347	3.107	1.087	1.580	2.686	0.137	0.036	0.047	0.031	0.039	90.903
	0.25	0.275	3.066	1.033	1.529	2.689	0.148	0.035	0.048	0.033	0.044	91.100
	0	0.253	2.960	1.001	1.496	2.626	0.139	0.039	0.049	0.032	0.041	91.364
68	0.70	0.695	4.385	1.555	2.270	3.784	0.120	0.062	0.069	0.039	0.040	86.981
	0.50	0.700	4.573	1.485	2.276	3.830	0.120	0.058	0.070	0.043	0.041	86.804
	0.25	0.724	4.573	1.544	2.312	3.764	0.120	0.046	0.070	0.040	0.037	86.770
	0	0.705	4.396	1.547	2.427	3.918	0.132	0.046	0.062	0.036	0.037	86.720
69	0.70	0.372	2.647	0.934	1.297	2.107	0.106	0.031	0.046	0.022	0.035	92.403
	0.50	0.382	2.738	0.953	1.356	2.335	0.119	0.033	0.053	0.027	0.041	91.963
	0.25	0.341	2.832	0.933	1.343	2.353	0.120	0.039	0.059	0.028	0.039	91.913
	0	0.301	2.699	0.891	1.279	2.187	0.115	0.033	0.061	0.030	0.040	92.364
70	0.70	0.498	3.193	1.126	1.641	2.621	0.112	0.034	0.051	0.024	0.032	90.688
	0.50	0.504	3.148	1.041	1.518	2.580	0.115	0.035	0.051	0.037	0.029	90.942
	0.25	0.582	2.933	1.015	1.465	2.530	0.122	0.034	0.053	0.022	0.042	91.202
	0	0.437	2.941	0.989	1.400	2.437	0.123	0.037	0.066	0.031	0.042	91.497

APPENDIX C: END EFFECTS AT THE EXIT OF THE REACTOR

End effects on temperature traverses were observed for many runs. Specifically, the temperature traverses were found to be a strong function of the flow rate. At high flow rates, the temperature traverses were approximately the same as the wall temperatures. As the flow rate was decreased, the temperature traverses read up to 50 °C lower than the average wall temperatures. This was illustrated in Figures 8 to 10.

The reactor head temperature was selected as the cause of this temperature effect. The head was made mainly of steel and was connected to the vacuum system by cast-iron piping. As steel and iron are excellent thermal conductors, the temperature of the reactor head remained considerably lower than the wall temperatures when the reactor was heated. This low reactor head temperature, in turn, affected the gas temperature inside the reactor both directly and indirectly. Directly, the gas temperature was lowered due to conduction, convection and radiation. Indirectly, the reactor wall temperature was lowered by conduction and radiation. Moreover, the traverse thermocouple reading was lowered as much as 5 °C because of conduction loss along the leads. These effects are discussed qualitatively in this appendix. Wherever possible, quantitative estimates are also made.

In these discussions, the dependence of the fluid properties on both temperature and composition will be neglected. Furthermore, the effect of the reactor head will be considered as equivalent to an imaginary boundary four inches above the top thermocouple in the reactor wall. The gas and wall temperatures at this boundary were at a constant temperature, T_h, which was much lower than the average reactor wall temperature for any run. Also this imaginary boundary does not affect the flow pattern of the fluid. Under such circumstances a fairly general energy balance equation for a moving fluid can be written as follows (C1):

$$u \nabla T - \mathcal{K} \nabla^2 T + \frac{s}{C_p \rho} + \frac{4 a \sigma}{C_p \rho} \sum_i F_i \left(T^4 - T_{wi}^4\right) = 0 \qquad \text{(C1)}$$

In equation C1, the first term describes the hydrodynamic transport, the second term thermal transport due to conduction, the third term an energy source term, and the fourth term thermal transport due to radiation. It should be pointed out that in this equation, a is the absorption coefficient over the entire wavelength range. This absorption coefficient will be discussed later in connection with thermal radiation.

Consider now the situation under which conduction and forced convection prevail. For simplicity, u_r is neglected, Equation C1 now assumes the form

$$u_x \frac{\partial T}{\partial x} - \mathcal{K} \left[\frac{\partial^2 T}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r}\right)\right] = 0 \qquad \text{(C2)}$$

or
$$u_x \frac{\partial T}{\partial x} - \mathcal{K} \left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial}{\partial R} \left(2R\frac{\partial T}{\partial R}\right)\right] = 0 \qquad \text{(C2a)}$$

where $R = 0.5\ r^2$ and $\mathcal{K}$ is the thermal diffusivity. If the gas temperature is constant and identical with the average wall temperature, T_{av}, at a distance x_o below the reactor head, the boundary conditions at both ends are then

$$T = T_{av} \quad \text{for} \quad x = 0 \qquad \text{(C3)}$$

$$T = T_h \qquad x = x_o \qquad \text{(C4)}$$

At the side walls, it will be assumed that the temperature varies linearly with distance in the section between the top thermocouple and the reactor head, i.e.,

$$\begin{aligned} T &= T_{av}, \text{ for } r = r_o,\ 0 \le x \le (x_o - 4) = x_1 \\ &= T_{av} - (T_{av} - T_h)(x - x_1)/x_o,\ r = r_o,\ x_1 \le x \le x_o \end{aligned} \qquad \text{(C5)}$$

Finally, the flow is assumed to be Poiseuille, i.e.,

$$u_x = u_{xm} \left[1 - (r/r_o)^2\right] \qquad \text{(C6)}$$

Equation C2a can now be solved for the boundary conditions C3, C4, and C5. Because of the presence of the convective terms, however, an analytical solution is not readily obtainable.

Instead, a numerical method of solution has been devised. This method called for the transformation of equation C2a into a finite-difference equation. The difference equation was then solved by an iterative procedure. In the finite-difference equation, the derivative $\partial T/\partial x$ was replaced by a backward-difference formula while all the other first and second derivatives were substituted with central-difference formulas. It has been found that the numerical solution of equation C2a by an iterative procedure would be stable for all values of u_{xm} only when a backward-difference formula was used for $\partial T/\partial x$. A computer program was then written and the actual computations performed on a Datatron 220 computer at the Computing Center of California Institute of Technology. Solutions were obtained for various values of u_{xm} and T_h. For all these cases, the following quantities were used: $x_o = 6$ inches, $T_{av} = 510°C$, $\mathcal{K} = 0.232 \times 10^{-3}$ ft.^{2}, $\Delta x = 0.2$ inch and $\Delta R = 0.025$ inch^2. The value of thermal diffusivity $\mathcal{K}$ was calculated based on butane properties at 510° C. Two values of T_h were used: 310° C and 360° C. Values of T as function of r for various values of u_{xm} are plotted in Figures C1 and C2 for $x = 2$ inches and $x = 3$ inches. The latter value of x represents approximately the position of the thermocouple probe. The plot at this height therefore demonstrates the effect of wall cooling on the gas temperature at the probe thermocouple. It can be seen that at low flow rates, the gas temperature would drop considerably at the probe.

At x = 2 inches, however, the gas temperature remains essentially the same as T_{av} for even the lowest flow rate.

The above analysis admittedly is not a rigorous one. A number of assumptions have been made. It nevertheless demonstrates the strong dependence of gas cooling on the flow rate due to a cold reactor head. Moreover, the effects of cooling due to radiant and natural convective transfer will be considered. These effects, when superimposed on the results obtained in the above analysis, improve the visualization of the actual situation.

When a gas flowing in a vertical tube under laminar conditions is heated or cooled by the walls, both natural and forced convection occur. Due to changes in viscosity and density, radial flow is developed. In the case of gas being cooled, the gas will flow from the center towards the wall, resulting in a flatter velocity profile (C2). Martinell and Boelter (C3) analytically treated the case of fluid flowing vertically in heated or cooled tubes having uniform wall temperature and obtained the relation

$$Nu = 1.75\ M_1\ \left[\frac{WC_{pb}}{k_b L} + 0.0722\ \frac{D}{L}\ GrPr^{\ 0.75}\ (M_2)\right]^{\frac{1}{3}} \qquad (C7)$$

where the Grashof number Gr, is based on tube diameter and initial Δt, which is positive for both heating and cooling. In equation C7, the first term in the cubic root is the contribution due to forced convection and the second term due to natural convection. For butane flowing in the reactor at an average linear velocity of $\overline{u}_x$ at 510°C and 1 atm with L taken to be 4 inches, the first term has an approximate value of 70 $\overline{u}_x$. On the other hand, if a value of 50°C is used for Δt, the Grashof number is calculated to be 19,000 while the Prandtl number is 0.955. The second term thus has a value of 39.7 based on $M_2 = 1.0$. Since $\overline{u}_x$ ranged from 0.05 to 0.6 ft/sec. in the present work, it is apparent that natural convection is at least of the same order of magnitude as forced convection for these velocities. This analysis, of course, does not necessarily apply to the present situation. For example, the wall temperature in the present work is not uniform. It does, however, point out the importance of natural convection in heat transfer at the exit of the reactor.

What effect will natural convection have on the temperature traverse then? Figure C3 shows a schematic diagram of the portion of the reactor tube next to the reactor head. It is assumed that at a height $x = 0$, the gas and wall temperature are uniform and equal to the average wall temperature. Above $x = 0$, the end effects due to the reactor head begins to become significant and the gas temperature will drop as the flow moves upward. At high flow rates, the gas does not have time to cool much before it exits through the reactor head.

Under such circumstances, natural convection will be suppressed. On the contrary, sufficient cooling of the gas occurs when the flow rate is low enough, and consequently, natural convection takes place. As the gas at x = 0 is hot, the reactor head cold and the reactor wall relatively warm, some circulation of the gas in a pattern shown in Figure C3 will be induced due to natural convection. Consider now the temperature at a height x slightly over the x = 0 line. Cold gas comes down from the top at some radius r between the center and wall, the temperature there is therefore quite low. From this radius, gas flows radially both towards the center and the wall and in doing so is heated by the hot gas at x = 0. The temperature of the gas thus becomes higher the farther the gas moves away from r. As the gas moving towards the wall is also heated by the wall, its temperature rises more sharply than the gas moving towards the center. The resulting temperature profile thus will be similar to the one shown in Figure C4.

Radiant transfer of energy of gases differ from that of solids and liquids. Gases absorb and radiate energy only within certain limited wavelength ranges. Moreover, gases need great thickness to absorb the major part of incident energy, if any absorption occurs. The absorption and radiation of gases are caused by changes in the energy levels of the gas molecules in accordance with Planck's law.

At temperature up to 5000 °F, say, the bulk of radiation is connected with changes in the rotational or vibrational energy of the molecules in the infrared region. An infrared absorption spectrum of n-butane is available (C4). The absorption by a gas of a monochromatic beam with wavelength λ is described by the equation

$$\frac{dI_\lambda}{I_\lambda} = - a_\lambda \, ds \tag{C8}$$

where s is the distance in the gas the beam has travelled, I_λ the intensity of the beam at s and a_λ is the so-called absorption coefficient. The quantity a_λ is roughly proportional to the number of molecules per unit volume and it increases with temperature. Furthermore, a_λ depends strongly on wavelength (C5).

When the temperature and gas concentration are constant, equation C8 can be integrated to result in

$$I_\lambda = I_{\lambda o} \, e^{- a_\lambda s} \tag{C9}$$

where $I_{\lambda o}$ is the intensity of the beam at $s = 0$. By definition, the absorptivity of a gas layer of thickness s is then

$$\alpha_\lambda = 1 - e^{- a_\lambda s} \tag{C10}$$

The emissivity E_λ is equal to the absorptivity according to Kirchhoff's law.

Equations C8 to C10 apply to radiation at some particular wavelength only. In practice, however, only the total radiated heat integrated over all wavelengths is of importance. Theoretically, a_λ for any gas can be evaluated as a function of temperature, pressure and wavelength from the molecular structure of the gas. Such evaluation, even if possible, is usually impractical for other than diatomic molecules (C6). Recently, Bevans and Dunkle (C7) pointed out the applicability of band measurements for engineering calculations. The band absorption of a band is defined as the absorption

$$A_i = \int_{\lambda_1}^{\lambda_2} \alpha_\lambda \, d\lambda \qquad \text{(C11)}$$

associated with a certain transition in the vibrational molecular energy and simultaneous rotational energy changes. Empirical band absorption correlations for CO_2 based on experimental measurements have been given by Edwards (C8). Work (C9) has also been performed to determine the overall emissivity E for a gas at different temperatures as a function of the product ps, where p is the partial pressure of the particular gas and s the thickness of the gas layer. Data are available for CO_2, H_2O, SO_2, CO, NH_3 and mixtures of CO_2 and H_2O at a total pressure of 1 atmosphere. For CO_2 at 510 °C and a ps of 0.25 ft. atm, the value of E at 1 atm total pressure is given to be approximately 0.09. This overall approach, however, applies only to gray surfaces.

The role of radiant thermal transfer in the end effects will now be examined. This transfer is represented by the term

$$\frac{4a\sigma}{C_p\rho} \sum_i F_i \left(T^4 - T_{wi}^4\right)$$

in equation C1. The significance of this term will be determined by comparing its order of magnitude with that of $u_x \frac{\partial T}{\partial x}$ at the thermocouple probe. In order to perform this, however, the values of F_i, a, T_{wi}, T and $\frac{\partial T}{\partial x}$ have first to be obtained. Since the overall emissivity of n-butane is not known, that of CO_2 at 410°C and ps = 0.25 ft. atm will be used as an approximation. The reactor is now divided into two parts. The first part is the reactor head which is 3 inches above the probe thermocouple and has a uniform temperature of 310°C. The second part consists of the rest of the reactor and is at a temperature of 510°C. The value of F_i's can now be obtained

$$F_1 = \frac{(\frac{1}{2})^2 \pi}{4(3)^2 \pi} = \frac{1}{144} = 0.00694$$

$$F_2 = 1 - F_1 = 0.99306$$

The absorption coefficient a can be calculated from E by the relation $E = 1 - e^{-as}$. Since for s = 0.25 ft and p = 1 atm, E = 0.09,

$$a = -\frac{1}{s} \ln (1-E) = 0.372 \text{ ft}^{-1}$$

The value of T and $\partial T/\partial x$ will be obtained from Figure C1 at $x = 3$ inches and $r = 0$ for $u_{xm} = 0.08$ ft/sec,

$$T = 499^\circ C$$

$$\frac{\partial T}{\partial x} = -\ 266^\circ C/ft.$$

At 499° C,

$$\rho = 0.0571\ lb/cu.ft.$$

$$C_p = 1.51\ Btu/lb\ ^\circ C$$

The radiant transfer term becomes

$$\frac{4x0.372x0.173x10^{-8}}{0.0571x1.51x3600}\left(\frac{9}{5}\right)^4\left[(712.1)^4 - 0.00694(583.1)^4 - 0.99306\ (783.1)^4\right] = -0.01644^\circ C/sec.$$

The longitudinal convective term is, on the other hand,

$$0.08x(-266) = -21.3\ ^\circ C/sec.$$

It can thus be seen that even at very low flow rates, the radiant transport is only 0.1% of the longitudinal convective term. Even if the absorption coefficient of n-butane is actually much larger than that of CO_2, the effect of thermal radiation can still be readily neglected.

Finally, possible errors in probe thermocouple readings are to be investigated. In general, the reading of a thermocouple in a flowing fluid is determined by three factors: heat transfer from gas to the thermocouple, heat conduction along the thermocouple leads, and radiant exchange of energy between the thermocouple and the surrounding surfaces. The net gain of energy is zero at steady-state. Mathematically,

$$h_c A_1 (T_g - T_t) + \sigma E A_2 \sum_i F_i (T_{wi}{}^4 - T_t{}^4) + k A_3 \frac{\partial T}{\partial x} = 0 \qquad \text{(C)12}$$

In equation C12, $\frac{\partial T}{\partial x}$ is the temperature gradient in the thermocouple wire. When the thermocouple probe is at the center of the reactor with $u_x = 0.08$ ft/sec. and $T_{av} = 510^{\circ}C$, it has been shown that $T_g = 499^{\circ}C$, $T_{w1} = 310^{\circ}C$, $T_{w2} = 510^{\circ}C$, $F_1 = 0.00694$, and $F_2 = 0.99306$. The heat transfer coefficient h_c can be calculated from the following equation C10:

$$h_c = \frac{0.44\ G^{0.8}\ T^{0.5}\ C_p\ (r_h - 0.081)}{M^{0.1}\ (r_h - 0.027)} \quad \frac{\text{Btu}}{\text{ft}^2\text{hr}^{\circ}\text{F}} \qquad \text{(C13)}$$

whter T is in $^{\circ}R$. For butane, $h_c = 2.61$ under the prevailing conditions. As an approximation A_1 and A_2 are both taken to be equal to the surface area of a sphere having the same diameter as the probe quartz tubing. Thus

$$A_1 = A_2 = 4\pi \left(\frac{1}{32}\right)^2 = \frac{\pi}{256} \text{ sq. in.}$$

A_3 is the cross section of the wires. Its value is $2(\pi \times 0.0015^2) = 4.5 \times 10^{-6}$ sq. in. For quartz, $E = 1$. For Pt at $500^{\circ}C$, k_c is approximately $50^{\text{Btu}/}$(hr) (sq ft) ($^{\circ}F$/ft).

Finally, the temperature gradient in the gas is used for that in the wires, or $\partial T/\partial x = -266\ ^{\circ}C/ft$. Equation C12 can now be written as

$$2.61(\tfrac{9}{5})(772.1 - T_t) + 0.173\times10^{-8}(\tfrac{9}{5})^4[0.00694(583.1)^4 + 0.99306(783.1)^4 - T_t{}^4] - 50(1.152\times10^{-3})(266)(\tfrac{9}{5}) = 0$$

and can be solved for T_t. T_t thus obtained has a value of approximately 767.1°K or 494 °C. The thermocouple therefore reads about 5 °C lower than the actual gas temperature. Since the temperature gradient in the wires is actually smaller than that in the gas, the actual difference is also smaller.

The end effects due to the cold reactor head have been discussed in detail. Because of the cold reactor head, the temperature of the adjacent reactor walls is lowered, which then cause gas flowing past to be cooled by conduction and convection. Radiant transfer, however, is insignificant. Also, errors in thermocouple readings are not excessive.

NOMENCLATURE

A_1	area of gas film around the thermocouple, sq.in.
A_2	surface area of thermocouple junction, sq.in.
A_3	cross-sectional area of thermocouple wires, sq.in.
a	absorption coefficient for thermal radiation, ft.$^{-1}$
C_p	heat capacity at constant pressure, Btu/lb $^\circ$R
D	diameter of reactor tube, ft.
F	view factor
G	mass velocity, lb/ft.^{2}hr
g	gravitational acceleration, 32.2 ft/sec.^{2}
Gr	Grashof number, $L^3\rho^2 g\beta\,\Delta t/\mu^2$, dimensionless
h_c	heat transfer coefficient, Btu/hr ft.^{2} $^\circ$R
I	intensity of radiation beam, Btu/ft.^{2}hr
k	thermal conductivity, Btu/hr ft. $^\circ$R
L	characteristic length, ft.
M	molecular weight
M_1	a unique function of temperatures
M_2	another unique function of temperature
Nu	Nusselt number, hD/k, dimensionless
p	partial pressure, atm
Pr	Prandtl number, $C_p\,\mu/k$, dimensionless
R	$0.5\,r^2$, sq ft or sq. in.
r	radius, ft. or in.
r_h	hydraulic radius of the thermocouple, ft. or in.
r_o	radius of the reactor, ft. or in.

s	distance, ft.
T	temperature °K or °R
t	temperature, °C or °F
$\overline{u}$	average linear velocity, ft/hr
u	velocity ft/hr
u_i	component velocity, ft/hr
u_{xm}	maximum velocity in x-direction, ft/hr
W	mass flow rate, lb/hr
x	longitudinal distance, ft or in.
α	absorption, dimensionless
β	thermal expansion coefficient, °F^{-1}
Δ	difference
E	emissivity, dimensionless
λ	wavelength, microns
$\mathcal{K}$	thermal diffusivity, $k/C_p\rho$, ft.^{2}
μ	viscosity, lb/hr ft.
ρ	density, lb/ft.^{3}
σ	Stefan-Boltzmann constant, 0.171 x 10^{-8} Btu/ft^2 hr °R^4
∇	gradient operator
∇^2	Laplacian operator

Subscripts:

b	bulk
g	gas
t	thermocouple
w	wall
λ	at wavelength λ

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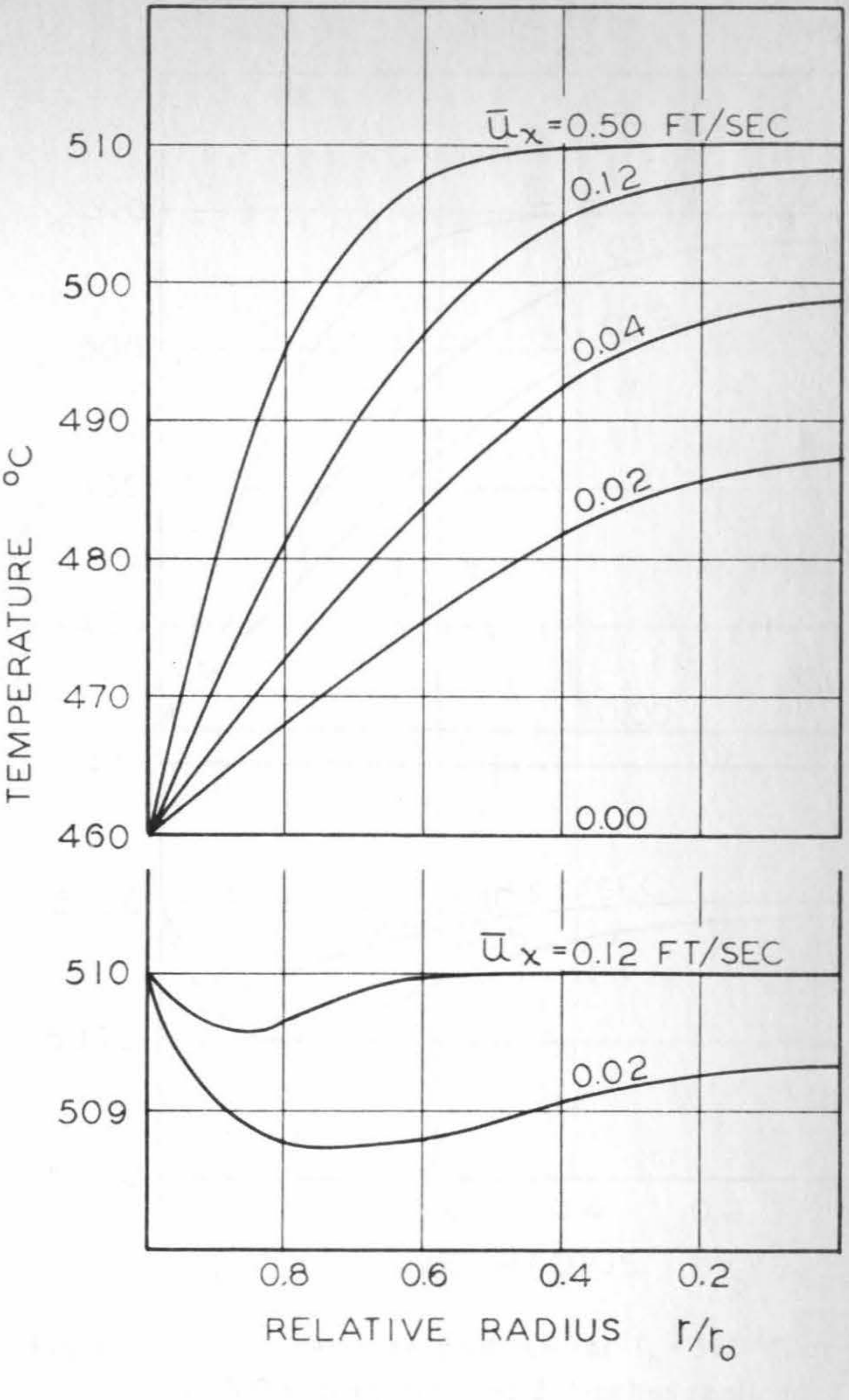

Figure C1. Temperature profiles for T_h = 310°C at x = 3.0 inches (top) and 2.0 inches (bottom)

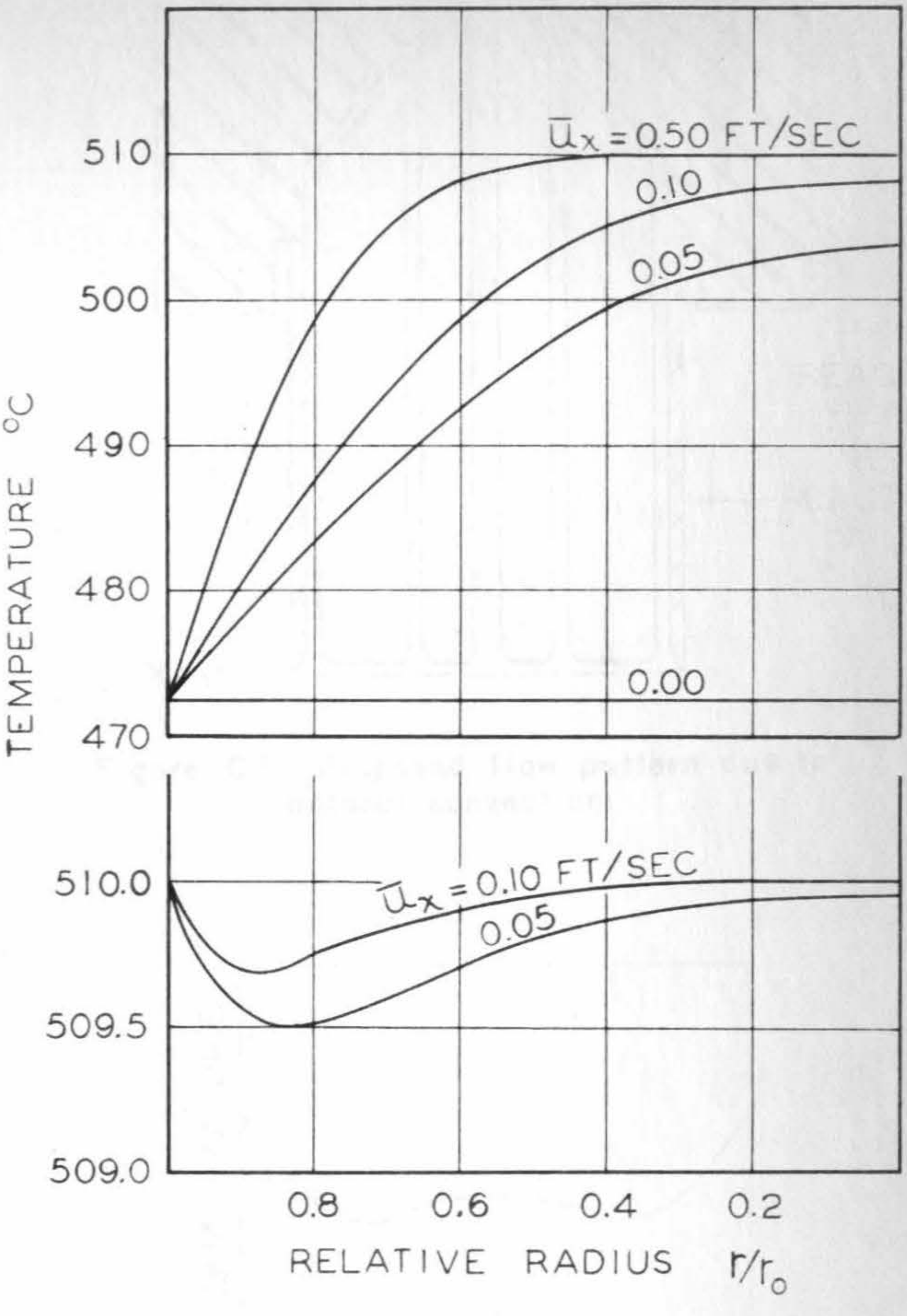

Figure C2. Temperature profiles for T_h = 360°C at x = 3.0 inches (top) and 2.0 inches (bottom)

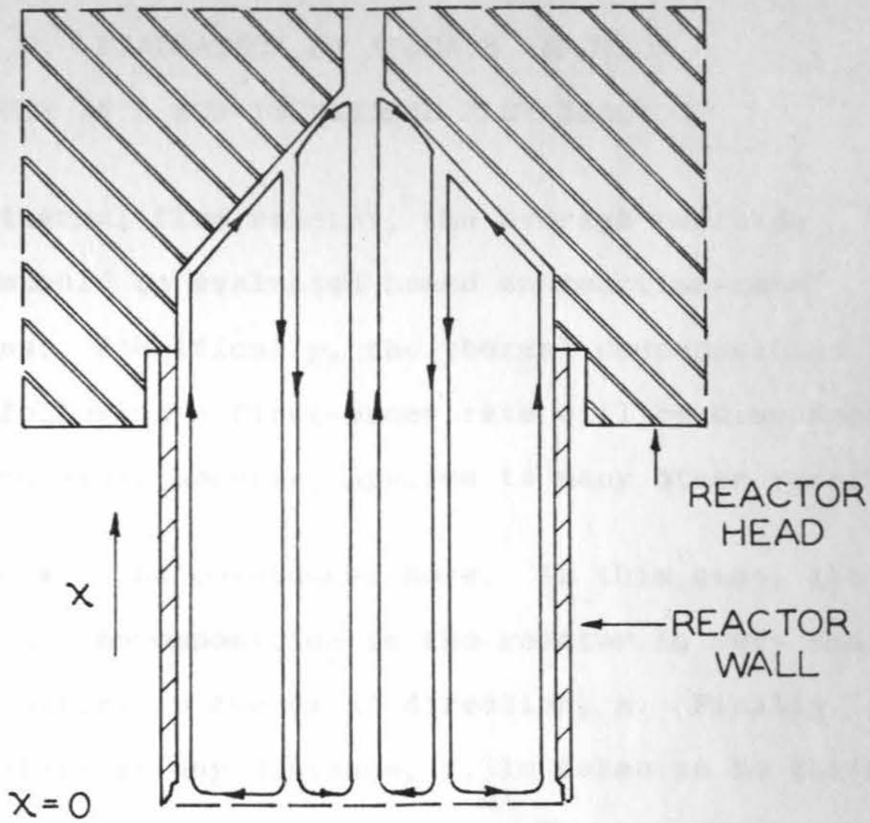

Figure C 3. Proposed flow pattern due to natural convection

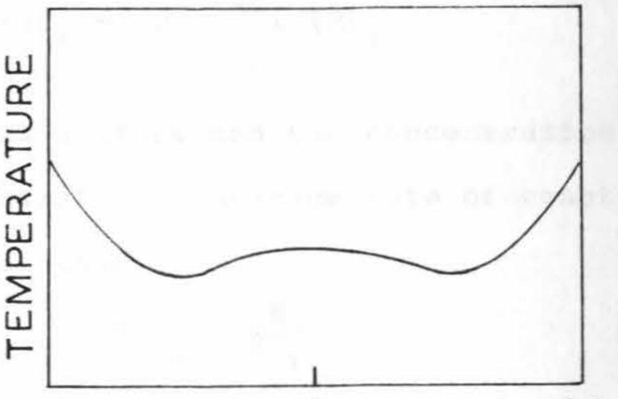

Figure C4. Temperature profile in the reactor where natural convection predominates

APPENDIX D: EVALUATION OF AVERAGE REACTION TEMPERATURE IN A NON-ISOTHERMAL FLOW REACTOR

In a non-isothermal flow reactor, the average reaction temperature should be evaluated based on reaction-rate considerations. Specifically, the thermal decomposition of n-butane following a first-order rate will be discussed. The same discussion, however, applies to many other reactions.

Only one case will be considered here. In this case, the extent of butane decomposition in the reactor is very small. The flow is uniform in the axial direction, x. Finally, the fluid temperature at any distance, x, is taken to be the same as the corresponding wall temperature. The reactor is now divided into m sections of equal length Δx, with each section maintained at a constant temperature T_i. The rate of reaction at any section i is thus:

$$r_i = k_{ri}(B)_i = Ae^{-\frac{E}{RT_i}}(B)_i \tag{D1}$$

Since the flow is uniform and the concentration of butane practically constant, the average rate of reaction for the entire reactor is then

$$r_{av} = \frac{1}{m} A(B) \sum_{i=1}^{m} e^{-\frac{E}{RT_i}} \tag{D2}$$

or the average reaction temperature,

$$T_{rav} = -\frac{R}{E} \ln \frac{1}{m} \sum_{i=1}^{m} e^{-\frac{E}{RT_i}} \qquad \text{(D3)}$$

T_{rav} in equation D3 is called the "reaction-rate average" temperature, as it is calculated from the average reaction rate.

In many cases the T_i's differ from each other only slightly. Under such circumstances,

$$T_{rav} \sim \frac{1}{m} \sum_{i=1}^{m} T_i \qquad \text{(D4)}$$

or the "reaction-rate average" temperature can be approximated by an arithmetic average. The validity of this approximation can probably be demonstrated most clearly with some numerical examples. Let the reaction rate constant be

$$k_r = Ae^{-\frac{29,000}{T}} \qquad \text{(D5)}$$

(I) If $m = 3$, $T_1 = 738.1^{\circ}$, $T_2 = 732.6^{\circ}$ and $T_3 = 730.6^{\circ}$, from equation D3 $Tr_{av} = 733.9^{\circ}C$ compared with the approximation of 733.7°C calculated from equation D4.

(II) Again let $m = 3$ with $T_1 = 782.1^{\circ}$, $T_2 = 777.6^{\circ}$ and $T_3 = 779.1^{\circ}$, the "reaction-rate average" temperature is 779.7° compared with the arithmetic average of 779.6°. Note that the arithmetic average is always lower than the "reaction-rate average".

APPENDIX E: CORRELATION OF VELOCITY DATA

The correlation of velocity data consisted of assuming a velocity profile, calculating a kinetic-head profile corresponding to the velocity profile and comparing the calculated kinetic heads with the measured kinetic heads.

The feed entered in reactor with a practically flat velocity distribution. If no chemical reaction occurred, and since the reactor was essentially isothermal, the velocity would gradually change into a parabolic distribution. Nikuradse (E1) measured the velocity distribution in the inlet portion of a pipe under isothermal and laminar conditions with a flat inlet velocity profile and found that the flow was practically Poiseuille at a distance $\frac{x}{r_o} = 0.16$ Re, where the Reynolds number, Re, was based on the radius of the pipe, r_o. The maximum average linear velocity for all the runs were 0.60 ft./sec. for Run 33 at 566° C. This corresponded to a Reynolds number of 103 or $x/r_o = 16$. As the minimum reactor length was 18 inches, the reactor was long enough for the flow to become parabolic at the exit of the reactor.

Two possible effects on the flow pattern of the gas in the reactor are now considered. The first one is that due to chemical reaction. Because the reaction was endothermic, the gas temperature at the center of the reactor would be lower than that close to the wall. Furthermore, as boundary layers began to develop, the retention time of the gas became longer near the wall. More reaction, therefore, occurred in the boundary layer. The result was a larger increase, percentagewise, in the number of molecules present and consequently, a relatively higher linear velocity in the boundary layer than the case without chemical reaction. The second effect was that due to end cooling. It is shown in Appendix C that when the gas flowing through a tube is cooled by the wall, radial flow in the direction of the wall will develop. This also would result in a relatively higher velocity near the wall. Although both effects probably were insignificant, the discussion here indicates that the velocity profile at the exit of the reactor could not be steeper than a parabolic distribution. As an initial estimate, a parabolic velocity distribution was thus assumed for each run regardless of temperature, flow rate, and other operating conditions.

KINETIC HEAD PROFILE

The pitot tube measures the total pressure, P°, of a moving stream. It was shown experimentally (E2) that

$$P^\circ = P_s + C\ (\frac{1}{2g}\ \rho u^2) \qquad \text{(E1)}$$

where C is a numerical factor dependent on the Reynolds number and the shape of the probe. C is nearly constant and equal to unity for most cases except at very low Reynolds numbers. When the Reynolds number is small, C increases rapidly with decreasing Reynolds number due to viscous effect.

Miss Barker (E3) measured the velocity of a water stream at the center of a 1.1-cm diameter pipe and found that the viscous effect could be neglected when $Re > 30$. The Reynolds number here was based on the inside radius of the pitot tube, r_p. For $Re < 30$, she surmised that the viscous effect showed itself in the form of an additional pressure, comparable with that at the nose of a sphere of radius equal to that of the pitot opening.

$$P \text{ addition} = \frac{3}{2g} \mu u / r_p \tag{E2}$$

Such a supposition was supported by the experimental results.

Later, Hormann (E4) investigated for an incompressible fluid, the effect of viscous force on pitot tube readings both theoretically and experimentally. He obtained the correction factor C as a function of the Reynolds number. The agreement between theoretical and experimental results was good even for very small Reynolds numbers. For a source-shaped tube, which is a tube having a surface contour corresponding to the stagnation stream line of a combination of a point source with a uniform flow, the value of C is given by

$$C = 5.25 - 0.504 \, Re \quad \text{for } Re \leq 6.0 \tag{E3}$$

If the kinetic head, ΔP, is defined as the difference between the total pressure and the static pressure, from equations E1 and E3,

$$\Delta P = P^{\circ} - P_{s} = (5.25 - 0.504\ \mathrm{Re})\ \frac{\rho u^{2}}{2g} \qquad \text{(E4)}$$

The kinetic head for any given velocity can then be readily evaluated.

In equation E4, ΔP has the unit lb/ft.^{2} while the measured kinetic heads were expressed in terms of air bubble displacement, ΔD cm. The relationship between ΔP and ΔD can be obtained as follows: The manometer fluid used was methyl alcohol which has a density of 50 lb/cu. ft. at room temperature. Any pressure difference ΔP between the two sides of the manometer would result in a difference in liquid levels, H_m,

$$H_m = \frac{1}{50}\ \Delta P\ \text{ft.} \qquad \text{(E5)}$$

if the effect of gas density is neglected. As the area ratio between the cross sections of the reservoirs and the bore of the capillary tube was 1000, the resulting air bubble displacement is

$$\Delta D = (\frac{1}{2}\ H_m)(1000)(30.48) = 15240\ H_m \qquad \text{(E6)}$$

Combining equations E5 and E6,

$$\Delta D = 304.8\ \Delta P \qquad \text{(E7)}$$

The velocity profile at the exit of the reactor is represented by the formula

$$u_x = 2\overline{u}_x\ [1 - (\frac{r}{r_o})^2] \qquad \text{(E8)}$$

where the average linear velocity $\overline{u}_x$ is to be calculated by the equation

$$\overline{u}_x = \frac{F}{60A} [1 + (Z-1)X] \frac{TP_s}{T_s P} \qquad (E9)$$

As a numerical example, the kinetic head profile for runs 32, 34 and 36 will be obtained. These three runs were replicate runs at 560°C with a feed rate of 0.0549 cu. ft. per minute measured at 70°F and 760 mm Hg. The conversion correction term (Z - 1)X is calculated from Table X to be 0.056. From equation E9,

$$\overline{u}_x = 0.512 \text{ ft./sec.}$$

By substituting the value of $\overline{u}_x$ into equation E8, point velocities across the reactor can now be calculated. The bubble displacements, ΔD, corresponding to these point velocity are then obtained from equations E4 and E7. These values, together with values of C, are given in Table E1. In evaluating C, the viscosity is calculated to be 1.303×10^{-5} lb/ft. sec. based on a composition of 4.126% CH_4, 1.475% C_2H_6, 2.377% C_2H_4, 3.993% C_3H_6 and 88.029% C_4H_{10}. This composition corresponded to that of run 34 at r/r_o = 0.70. The results of a similar calculation for runs 21 and 31 are also given in Table E1. Since the conversion of these two runs was only 0.8%, the correction due to conversion is neglected.

COMPARING EXPERIMENTAL AND CALCULATED ΔD

Two difficulties arise in reducing the experimental data to obtain ΔD. The first one is that the static-head reading fluctuated. This difficulty was removed by plotting the static head reading as a function of time. For each kinetic head reading, the corresponding static head reading was read from the plot at the same time the kinetic head reading was obtained. Figure E1 shows a typical static head reading - time plot.

The second difficulty lies in the fact that the difference between the kinetic head and static head readings was a combination of two things: the actual kinetic head ΔD plus the difference between the static head at the pressure tap and that at the pitot tube ΔD_s. As the pitot tube could not travel to within 0.15 inch of the wall, ΔD_s could not be obtained by extrapolation without considerable error. To overcome this difficulty, the measured readings $(\Delta D + \Delta D_s)$ were compared with the calculated kinetic head. If both profiles were similar in shape, the value of ΔD_s could then be easily determined by forcing the measured profile to superimpose on the calculated curve. Part of the measured readings for runs 32, 34 and 36 are listed in Table E2. For these runs, the values of ΔD_s were obtained by comparison to be 0.735, 1.90 and 2.10 cm respectively. The resulting values of ΔD are then plotted in Figure E2, together with the calculated profile. A similar plot for runs 21 and 31 is shown in Figure E3.

Figures E2 and E3 show that the experimental data agree well with the calculated kinetic head, which suggests that velocity profiles for these runs were parabolic. Such agreements have been found for sixteen runs out of a total of thirty-six runs in which velocity measurements were made. Of the remaining twenty runs, either the data scattered too much to enable any correlation or the velocity was too low to result in detectable bubble movement. Some typical data for both cases are given in Table E3.

NOMENCLATURE

A	cross sectional area of the reactor, ft^2
C	a correction factor due to viscous force, dimensionless
ΔD	manometer bubble movement due to kinetic head, cm
ΔD_s	manometer bubble movement due to difference in static pressure between the pressure tap and the pitot tube, cm
F	volumeric feed rate measured at T_s and P_s, ft^3/min
g	gravitational acceleration, 32.2 ft/sec^2
H_m	methyl alcohol head, ft
P	pressure, lb/ft^2 or cm Hg
P_s	standard pressure, 76 cm Hg
ΔP	kinetic head measured by the pitot tube, lb/ft^2
r	radius, ft
r_o	radius of the reactor, ft
r_p	radius of the pitot tube opening, ft
Re	Reynolds number, $\rho ur/u$, dimensionless
T	temperature, °K
T_s	standard temperature, 294.2 °K (70° F)
u	point velocity in the direction of the pitot tube axis, ft/sec
u_x	longitudinal velocity, ft/sec
$\overline{u}_x$	average longitudinal velocity, ft/sec
X	conversion of butane, dimensionless
x	distance from the inlet of a tube, ft
z	moles of products formed per mole of butane converted, dimensionless

ρ	density, lb/cu.ft.
μ	viscosity, lb/ft.sec.

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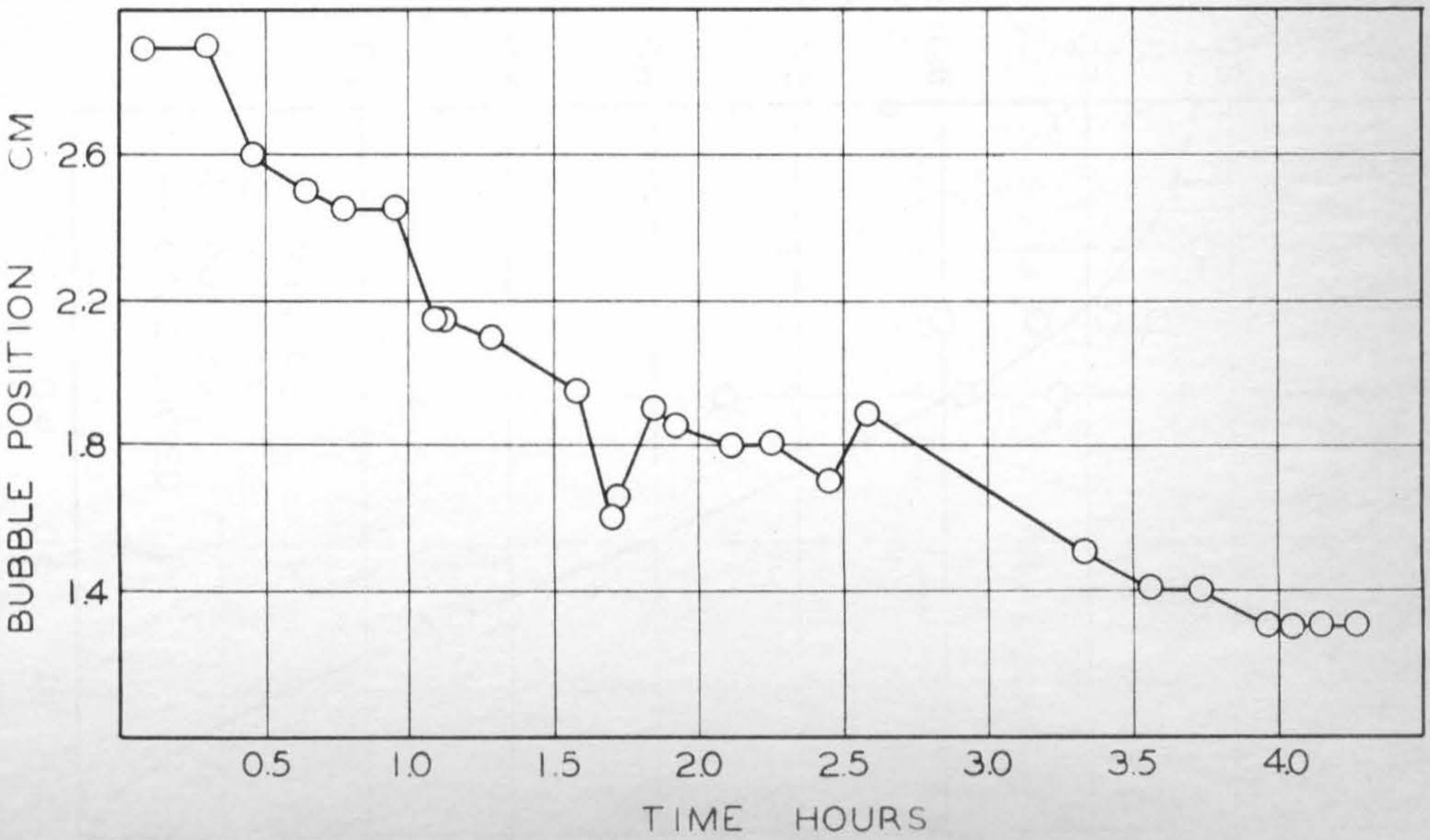

Figure E1. Static head reading as a function of time for run 21

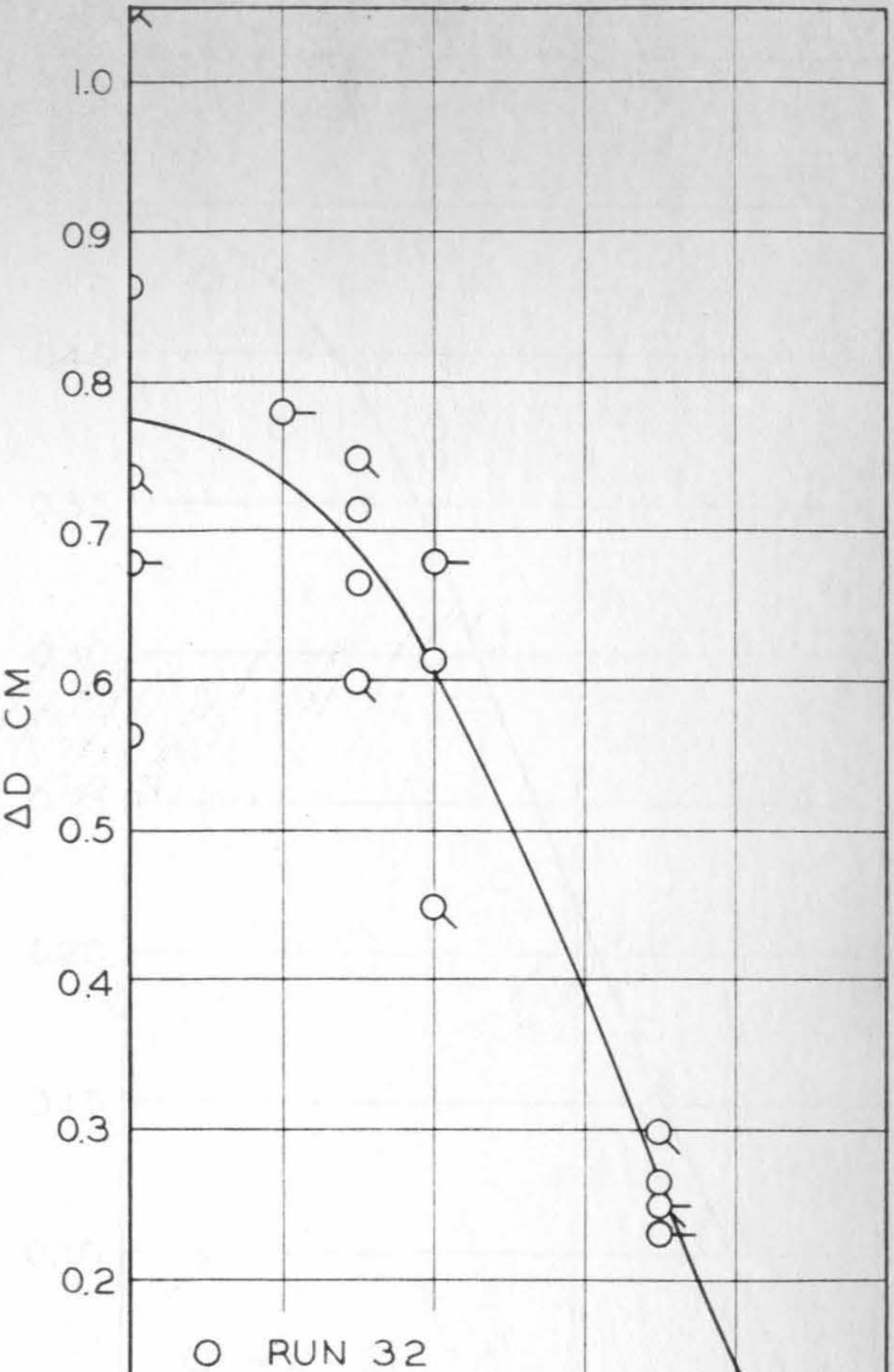

Figure E2. Experimental and calculated kinetic head for runs 32, 34 and 36

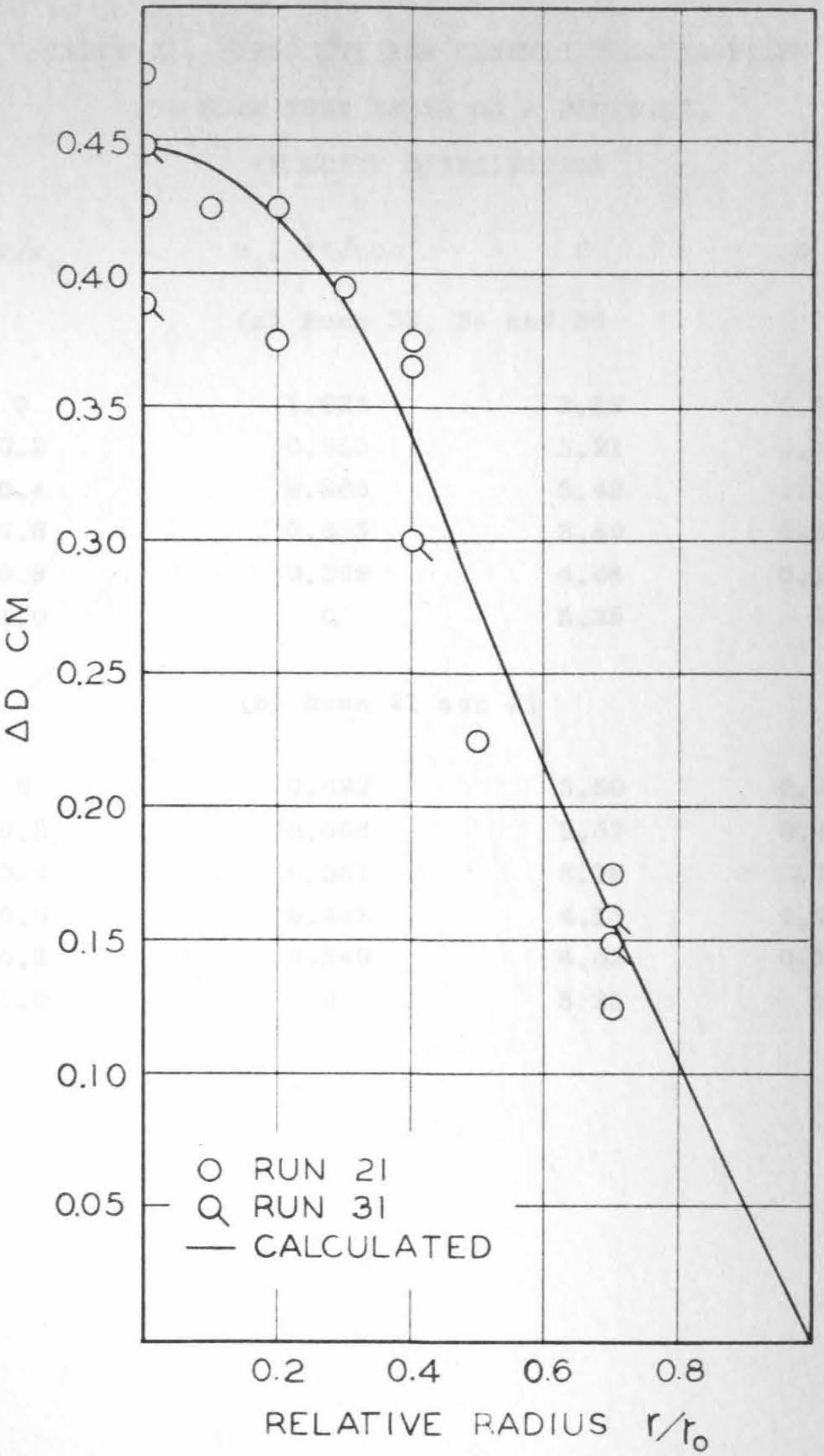

Figure E3. Experimental and calculated kinetic head for runs 21 and 31

TABLE E1. VELOCITY AND KINETIC HEAD PROFILES FOR SOME RUNS BASED ON A PARABOLIC VELOCITY DISTRIBUTION

r/r_o	u_x, ft/sec	C	ΔD cm
(a) Runs 32, 34 and 36			
0	1.024	3.12	0.774
0.2	0.983	3.21	0.735
0.4	0.860	3.46	0.607
0.6	0.655	3.89	0.394
0.8	0.369	4.48	0.145
1.0	0	5.25	0
(b) Runs 21 and 31			
0	0.692	3.50	0.447
0.2	0.664	3.57	0.419
0.4	0.581	3.78	0.340
0.6	0.443	4.13	0.216
0.8	0.249	4.62	0.102
1.0	0	5.25	0

TABLE E2. MEASURED KINETIC HEAD READINGS
FOR RUNS 32, 34 AND 36

Run 32			Run 34			Run 36		
Time	r/r_o	$(\Delta D+\Delta D_s)$ cm	Time	r/r_o	$(\Delta D+\Delta D_s)$ cm	Time	r/r_o	$(\Delta D+\Delta D_s)$ cm
11:03	0.0	1.30	7:15	0.0	2.64	2:26	0.0	2.78
11:15	0.3	1.45	9:33	0.0	2.95	2:59	0.7	2.35
11:28	0.7	1.00	9:55	0.4	2.35	3:33	0.4	2.78
12:00	0.4	1.35	10:26	0.7	2.15	6:53	0.7	2.33
12:51	0.0	1.60	11:15	0.3	2.50	7:31	0.2	2.88
1:21	0.3	1.40	12:55	0.0	3.00			
			1:11	0.3	2.65			
			1:53	0.7	2.20			

TABLE E3. MEASURED KINETIC HEAD READINGS FOR TWO "BAD RUNS"

	Run 65			Run 26	
	($\overline{u}_x$ = 0.359 ft/sec.)			($\overline{u}_x$ = 0.101 ft/sec.)	
r/r_o	Time	$(\Delta D + \Delta D_s)$ cm	r/r_o	Time	$(\Delta D + \Delta D_s)$ cm
0.70	3:34	2.04	0.70	4:01	1.15
	3:59	1.83	0.40	4:13	1.15
	8:20	3.22	0.30	4:35	1.15
0.60	6:02	1.65	0.00	4:23	1.15
0.40	4:21	2.50			
0.30	5:25	1.55			
0.00	4:59	1.98			

APPENDIX F: METHOD FOR CALCULATING CONVERSION AND CHECKING MATERIAL BALANCE

The compositions of samples obtained by chromatographic analysis are listed in Appendix B. From the compositions, the conversions in terms of number of moles of butane converted per 100 moles of feed were calculated for samples at $r/r_0 = 0.0$ and 0.70. Material balances for these samples were also checked by computing the C/H ratio. The results are shown in Tables VIII to X. The method of calculation is given in this appendix.

The basis of calculation is 100 moles of feed with δ moles of components other than butanes. In this treatment, n- and iso-butanes are not distinguished. After the reaction, X moles of butanes are converted into ZX moles of products. The total number of moles now is $100 + (Z-1)X$ with $(100-\delta-X)$ moles of butanes and $(ZX + \delta)$ moles of other components. The total mole percentage of these other components in a sample is thus

$$Y = \frac{ZX + \delta}{100 + (Z-1)X} \, (100) \tag{F1}$$

If the ratio C/H in the δ moles of impurities is R_δ, the overall ratio C/H in the sample excluding the butanes should then be

$$(C/H)_T = \frac{0.4\ ZX + R_\delta\ \delta}{ZX + \delta} \quad . \tag{F2}$$

If the values of Y, Z, δ and R_δ are known or calculable for each sample, equation F1 can be used to calculate X and equation F2 to calculate $(C/H)_T$. $(C/H)_T$ is the theoretical value of the ratio C/H which provides a basis for checking the material balances. The analysis is rigorous and the computations straightforward.

In practice, however, both values of Z and δ were not easily determinable. The amount of impurities in the feed was noticed to decrease rapidly with time due to favorable equilibrium conditions. Since the initial amount of impurities was small, it was assumed that the presence of impurities could be neglected for the present calculations. The value of Z, on the other hand, varied slightly from run to run around an average value of approximately 2.0 (Tables VIII to X). Since X is almost inversely proportional to Z in equation F1, the use of a constant value for Z (say 2.0) to calculate X would introduce undesirable errors. X, therefore, was calculated by another approach.

For 100 moles of a sample, there were Y moles of products and (100 - Y) moles of butanes. The values of ΣC and ΣH were calculated for the Y moles of products. Since the products were formed due to butane decomposition, $\Sigma C/\Sigma H$ has a theoretical value of 0.4. The actual values of the ratio $\Sigma C/\Sigma H$ varied around this theoretical value due to errors in chromatographic analysis and possible carbon deposits on the walls. The number of moles of butanes equivalent to Y moles of products was thus $\Sigma C/4$ and the value of Z was $4Y/\Sigma C$. From equation F1, with $\delta = 0$ and $Z = 4Y/\Sigma C$,

$$X = \frac{100\ (\Sigma C/4)}{(100-Y) + (\Sigma C/4)} \quad . \tag{F3}$$

Equation F3 was used in actual computations to obtain values of X.

The number of moles of any product formed for one mole of butane decomposed, P_i was computed from the formula

$$P_i = Y_i/(\Sigma C/4) \tag{F4}$$

where Y_i is the mole percentage of component i in the product stream.

APPENDIX G: SOME EMPIRICAL FORMULAS FOR CALCULATING PHYSICAL PROPERTIES OF n-BUTANE AND ITS DECOMPOSITION PRODUCTS

Much work has been done in measuring and correlating physical properties of pure hydrocarbons. The correlations usually resulted in empirical formulas with some theoretical basis. Work concerning mixtures has been much less extensive. The properties of hydrocarbon mixtures are invariably obtained from the properties of pure substances by assuming the latter are additive either on a molal or weight basis. After a careful literature survey, the following empirical formulas were chosen to be used in the evaluating of physical properties of n-butane and its main decomposition products, i.e., methane, ethane, ethylene and propylene. The choice was based on the simplicity of the formula without appreciable sacrifice of agreement with experimental data.

I) Heat capacity (G1)

$$C_p = a + bT + cT^2 \tag{G1}$$

II) Viscosity (G2)

$$\mu\xi = (0.291\ T_R - 0.058)^{3/4} \tag{G2}$$

III) Thermal conductivity (G3)

$$k = \mu\ (C_p + \frac{5R}{4M}) \tag{G3}$$

IV) Diffusion coefficient (G4)

$$D_{1-2} = 4.63 \times 10^6 \frac{T^{3/2}}{P(V_{m_1}^{1/3} + V_{m_2}^{1/3})^2} \sqrt{\frac{1}{M_1+M_2}} \tag{G4}$$

V) Heat of reaction (G5)

$$\Delta H_T = \Delta H_{T_s} + \int_{T_s}^{T} \Delta C_p \, dT \qquad \text{(G5)}$$

where ΔH_{T_s} is the heat of reaction at a standard temperature (usually 25°C) and can be calculated from the heat of formation or heat of combustion data.

Values of the coefficients and other qualities needed to evaluate these properties are given in Table G1. Numerical values for n-butane at 460°, 510° and 560°C are shown in Table G2. For a mixture, the heat capacity is additive on a weight basis while the viscosity is additive on a molal basis. Equation G3 applies to mixtures as well as pure substances.

NOMENCLATURE

a,b,c	coefficients used in equation G1
C_p	heat capacity at constant pressure, Btu/lb^{o}K
D_{1-2}	diffusion coefficient between gases 1 and 2, ft^2/sec
H	enthalpy, Btu/lb
k	thermal conductivity, Btu/(sec)(ft^2)(^{o}K/ft)
M	molecular weight, lb/lb mole
P	pressure, atm
R	universal gas constant, 3.5766 Btu/lb mole^{o}K
T	temperature, ^{o}K
T_c	critical temperature, ^{o}K
T_R	reduced temperature, T/T_c, dimensionless
V_m	molecular volume of a gas at its normal boiling point, cc/g-mole
μ	viscosity, lb/ft sec
ξ	a constant for each compound
ΔC_p	change in heat capacity of a reacting system due to chemical reaction at constant pressure, Btu/lb^{o}K
ΔH_T	heat of reaction at temperature T, Btu/lb

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TABLE G1. NUMERICAL VALUES OF COEFFICIENTS AND OTHER PERTINENT QUANTITIES FOR EQUATIONS G1 TO G5

(a) Heat Capacity

Compound	a	$b \times 10^3$	$-c \times 10^6$
Methane	0.3848	2.007	0.4666
Ethane	0.0825	2.503	0.8273
Ethylene	0.1742	1.875	0.3645
Propylene	0.0842	2.131	0.7273
n-Butane	0.0697	2.532	0.8865

(b) Viscosity

Compound	T_c, °K	$\xi \times 10^4$
Methane	191.1	6.905
Ethane	305.5	5.313
Ethylene	282.4	5.298
Propylene	365.1	4.821
n-Butane	425.2	4.777

(c) Diffusion Coefficient and Heat of Reaction

Compound	V_m, cc/g-mole	ΔH_c, heat of combustion at 25°C, Btu/lb
Methane	29.6	21,502
Ethane	51.8	20,416
Ethylene	44.4	20,276
Propylene	66.6	19,683
n-Butane	96.2	19,665

TABLE G2. SOME PROPERTIES OF n-BUTANE

Temperature, ^{o}C	460	510	560
C_p, BTU/lb^{o}K	1.449	1.509	1.564
$\mu \times 10^5$, lb/ft sec.	1.143	1.204	1.269
$k \times 10^5$, BTU/sec. $(ft^2)(\frac{^{o}K}{ft})$	1.744	1.910	2.082
$D \times 10^3$, ft^2/sec.			
n-butane-methane	0.4013	0.4430	0.4862
n-butane-ethane	0.2977	0.3286	0.3607
n-butane-ethylene	0.3206	0.3539	0.3885
n-butane-propylene	0.2490	0.2748	0.3017

APPENDIX H: RATE CONSTANTS FOR ELEMENTARY REACTIONS I THROUGH XIX

The rate constants for elementary reactions I through XIX are given in equations 40 to 58. The frequency factors A and activation energies E in the Arrhenius equation

$$k = A\ e^{-E/RT} \tag{38}$$

for these rate constants were obtained from experimental informations collected by Steacie (1) either directly or indirectly. The choice of these quantities is described here. The page numbers which are referred to in this appendix correspond to those in reference (1).

In a reversible reaction, the activation energies for the forward and reverse reactions, E_1 and E_2, can be related to the heat of reaction at 0^oK, Q_o, by

$$E_1 - E_2 = Q_o \tag{H1}$$

For chain-initiating steps such as reactions I and II, the heats of reaction at 0^oK are equal to the bond dissociation energies, which are given on page 98. The activation energies for the reverse reactions in which two free radicals combine are zero. Therefore, the activation energies for reactions I and II are equal to the corresponding bond dissociation energies, i.e., 79 and 82 kcal respectively.

Reactions III and IV are described on pp. 473 and 561 respectively. The first one has an activation energy around 9.3 kcal and the second reaction 8.3 kcal. These values were adopted.

The heats of reaction for III, IV and V are -10, -7 and -3 kcal respectively. As the difference in E between reactions III and IV is 1 kcal, that between IV and V was surmised to be 1.3 kcal or E for reaction V to be 7.0 kcal.

The activation energy for reaction VI is given on p. 142 to be 39.5 kcal. Those for VII and VIII are found on p. 585 to be 46 and 26 kcal respectively. The activation energies for reactions X and XI are found to be 23 kcal as shown on p. 588. For the proposed mechanism, values of 24.5 and 26.0 kcal were assigned to these two reactions. A value of 29.0 kcal was estimated for reaction IX, which is more endothermic than reaction X and XI.

Reactions XII through XVII are radical-combination processes, the activation energies of which are negligible. For reactions XVIII and XIX, the activation energies are also very small, as discussed on pp. 442 and 557 respectively. They are both around 5 kcal. Other reactions between propylene and free radicals, such as

$$C_3H_6 + H\cdot \rightarrow H_2 + C_3H_5\cdot \qquad (H2)$$

$$C_3H_6 + CH_3\cdot \rightarrow CH_4 + C_3H_5\cdot \qquad (H3)$$

are described on pp. 442 and 556. They have activation energies 2 to 3 kcal higher than those of reactions XVIII and XIX. As the propylene concentration was low, they were neglected.

Experimentally-determined frequency factors are even less dependable than activation energies. A for reaction VI has a value of 3x10^{14} sec^{-1} as shown on p. 142. For the present purpose, a value of 10^{14} sec^{-1} was used. This value was carried over to reactions VII and VIII. For reactions I, II, IX, X and XI which involve a larger moleclue or radical, A was estimated to be 10^{13} sec^{-1}, in agreement with the theoretical value shown on p. 489.

The rest of the reactions are bimolecular reactions with the exception of reactions XII and XIII. A for these reactions has a unit of cc/(gm mole)(sec). For these reactions, a value of 10^{13} for A was used for reactions involving a hydrogen radical and 10^{11} for reactions without hydrogen radicals, in accord with the arguments given on p. 493. Reactions XII and XIII probably occur only at triple collisions as discussed on pp. 137 and 511. Their rates are therefore negligible.

PART TWO

FLOW IN ENTRANCE SECTION OF PARALLEL PLATES

I. INTRODUCTION

The general equations of motion for a Newtonian fluid are the Navier-Stokes equations. Because of the non-linear nature of these equations, very few exact solutions have been derived; and these are generally for the cases in which the non-linear convective terms vanish. A number of approximate solutions have been obtained by neglecting various terms in the equations, such as those obtained by use of boundary layer theory, and the Stokes equation for very slow motion. Although these approximate solutions hold well for some cases, there are still a large number of cases for which no appropriate analytical solutions exist, for instance the flow near the leading edge of a flat plate. Furthermore, few numerical methods for attacking the Navier-Stokes equations are known. The main reason is that there exist no general criteria concerning the stability and convergence of numerical solutions to non-linear partial differential equations, and when conventional numerical methods are applied to the Navier-Stokes equations, the solutions often are unstable. In the present work, a numerical method has been devised for solving the Navier-Stokes equations for the two-dimensional steady-state flow of an incompressible viscous fluid in the inlet length* of two parallel plates under isothermal, laminar conditions.

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*The inlet length is defined here as the distance inside the plates over which the flow pattern changes from potential flow to parabolic velocity distribution.

Two cases have been studied. In the first case, potential flow conditions were assumed to exist at the inlet of the plates. In the second case, these conditions were taken to exist only far upstream from the inlet. Numerical solutions were obtained for both cases with a Reynolds number of 300. Results were computed in terms of stream functions and vorticities. From these, the point velocities and pressure gradients were calculated.

Schlichting (1) obtained an approximate solution for the first case using boundary layer theory. He assumed that the velocity distribution at the inlet was uniform over its width. Owing to viscous effects, boundary layers formed on both walls, and their thickness increased in the downstream direction. Eventually, at a large distance downstream, the velocity profile became parabolic. At small distances downstream from the inlet, the boundary layers grew in the same fashion as those along single plates, with a flat velocity distribution in the center core. In order to satisfy the continuity equation, the velocity in the center had to increase at a rate such as to compensate for the decrease in the rate of flow in the boundary layers. Hence the boundary layers were formed under the influence of an accelerated external flow and their growth was calculated on this basis. This concept was first suggested by Prandtl (2). Farther down the stream, the assumption of a flat velocity distribution in the center core no longer held. Schlichting, however, found it possible to analyze the progressive deviation in the upstream direction of the velocity profile from its asymptotic parabolic distribution. Having obtained both solutions in the form of series expansions, he then joined them together at an intermediate location where both analyses were applicable. In this way, the flow pattern for the whole inlet length was obtained. Schlichting found that the inlet length was approximately (0.04) (2a) (Re), where 2a is the distance between the plates and Re the Reynolds number based on 2a. The acceleration of the flow near the axis in this region gives rise to a pressure drop in addition to that corresponding to Poiseulle's flow, which was evaluated to be - 0.601 $(1/2 \ \rho U_{xo}^2)$ where U_{xo} is the potential flow velocity. Both the influence of friction in the intake and pressure gradients along the width were neglected.* Another approximate solution for this problem based on the momentum equation as well as experimental results has also been reported (3,4).

Work has also been done on the flow characteristics in the inlet length of a cylindrical tube. Again, potential flow conditions at the inlet were always assumed. Boussinesq (5) was the first to conduct a theoretical investigation of this subject. He linearized the Navier-Stokes equation in the direction of bulk-flow and applied it to the whole inlet

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*Recently, Schlichting's solution has been modified and a numerical solution for the problem has been obtained. These are mentioned in the Discussion Section.

length. His calculated velocity profiles were in agreement with the experiments of Nikuradse (6) only at some distance away from the entrance. Based on the momentum equation of the boundary layer, Schiller (7) obtained an approximate solution which agreed with Nikuradse's results for the first one third of the inlet length. It is of interest to note here that Schlichting's approach, as outlined above, was in essence a combination of Boussinesq's and Schiller's techniques. Application of Schlichting's method to flow in the inlet length of a cylindrical tube has been carried out by Atkinson and Goldstein (8).

Still another study of the case of a cylindrical tube is that of Langhaar (9), who linearized the Navier-Stokes equation in the direction of bulk flow, x, by assuming $Du_x/Dt = \nu\beta^2 u_x$; where u_x is the x-direction velocity and β is a function of x alone. D/Dt is the substantial derivative with respect to time. In addition, his basic assumptions, which were common to the theories of Boussinesq, Schiller and Atkinson-Goldstein, were:

(1) at $x = o$, $u_x = \text{const.} = U_{xo}$,

(2) the pressure gradient $- \partial P/\partial x$ is a function of x alone,

(3) the term $\nu\, \partial^2 u_x/\partial x^2$ may be neglected.

His resulting velocity profiles agreed well with Nikuradse's experiments.

In all the analyses mentioned above, certain assumptions were made which are not valid in a region close to the leading edge of the solid boundary, i.e., $\partial^2 u_x/\partial x^2 \ll \partial^2 u_x/\partial y^2$. Carrier and Lin (10) have investigated the nature of the boundary layer near the leading edge of a flat plate. Briefly, their efforts were aimed at modifying Blasius' solution of the boundary layer equation to include the effect of the leading edge. Carrier and Lin obtained solutions applicable to the region of very slow flow near the leading edge. They pointed out, however, that there was another region between those of very slow flow and potential flow in which it was doubtful that any reasonably simple analytic solution could be found. They suggested that some field solution might be possible for that region. In order to obtain a truer picture of what happens in the inlet length of two parallel plates, expecially near the leading edges, numerical solution seems to be the only feasible approach.

Numerical analysis dealing with partial differential equations has progressed rapidly recently. Most of the work in this field, however, considers only linear partial differential equations. As yet, there exist no general criteria concerning the stability and convergence of solutions to nonlinear partial differential equations by numerical methods. Of the comparatively small number of papers on this subject, the one by de G. Allen (11) is of particular interest. In that paper, he described a special technique for solving the Navier-Stokes equations by the relaxation method. It was pointed out that the special treatment was necessary because of the "latent exponentials". Since exponentials cannot be closely represented by polynomials except within a very narrow range, the customary finite difference formulas for derivatives would not work satisfactorily. This difficulty is likely to arise in most problems concerning motions of viscous fluid and to be serious.

Another possible approach to the solution of the Navier-Stokes equations is by means of analog computers. Janssen (12) used this method to investigate flow past a finite flat plate at Reynolds numbers of 0.1 to 10.0, with the exact nonlinear formulation. The solution gave the stream function and the vorticity in the flow field. From these, the pressure distribution along the plate, the local coefficient of friction and the total drag coefficient were calculated numerically. The difference between the computed and experimental drag coefficient for the most part was attributed to truncation error.

Numerical solution of partial differential equations involves the use of finite difference methods. With the finite difference method, a boundary value problem $F(u) = j$ for the continuous operator F is transformed into a matrix equation

$$AU = B \tag{1}$$

Equation 1 can be solved either by a direct or iterative procedure. For non-linear partial differential equations, an iterative method is always used. The number of iterations required for a solution is roughly proportional to the square of the dimension of matrix A. When A is large, acceleration of the rate of convergence is very desirable.

More information on the finite difference method for treating boundary-value problems and methods of accelerating the rate of convergence for iterative procedure is provided in Appendix A.

Review of literature revealed that very few exact solutions of the nonlinear Navier-Stokes equations existed. Existing solutions were obtained either by linearizing these equations or neglecting some of the nonlinear terms in these equations. The purpose of this work was to solve by a numerical method the Navier-Stokes equations for a simple problem without making any assumptions. The problem concerned the flow of an incompressible fluid in the inlet length of two parallel plates under isothermal, steady-state conditions. Due to singularity at the leading edges of the plates, the boundary layer equation does not apply to regions near these singular points.

II. FORMULATION OF THE PROBLEM

For an incompressible, viscous fluid flowing isothermally under steady-state conditions, the problem of flow in the inlet length of parallel plates can be formulated as follows: Let there be an infinite number of flat plates of semi-infinite length, all lying parallel to the x-axis between $x = 0$ and $x = +\infty$. The free-stream* velocity U_{xo} is independent of y and parallel to the x-axis. If the flow is independent of z, the two-dimensional equations of continuity and motion are respectively:

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \tag{2}$$

$$u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = - \frac{\partial P}{\partial x} + \frac{4}{Re} \nabla^2 u_x \tag{3}$$

$$u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} = - \frac{\partial P}{\partial y} + \frac{4}{Re} \nabla^2 u_y \tag{4}$$

Equations 2 to 4 are in dimensionless forms. Re is the Reynolds number based on the hydraulic radius:

$$Re = \frac{4 y_o U_{xo}}{\nu} \tag{5}$$

The definition of symbols and the relationship between a variable and its dimensionless form are listed in the Nomenclature section. Throughout this thesis, dimensionless expressions are used unless otherwise specified.

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*The term "free-stream" is used throughout the text to represent the region in which the effects of the plates on the flow are negligible.

Let the distance between any two adjacent plates be 2 and consider the two plates at y = + 1 and y = - 1. Because of symmetry about the x-axis, it is necessary to be concerned only with the region bounded by y = 0 and y = + 1. For convenience, this is further divided into two regions: The downstream region where $x \geq 0$ and the upstream region where $x \leq 0$. They have, of course, the common boundary line x = 0 at the inlet of the parallel plates. Case I assumes that the velocity at the inlet is uniform in the y-direction. Thus it is necessary to study the flow pattern in the downstream region only. Case II, on the other hand, assumes that free-stream conditions exists only far upstream from the inlet so that the flow conditions in both regions have to be determined. For both cases, the boundary conditions for the downstream region are:

$$\frac{\partial u_x}{\partial y} = 0 \quad , u_y = 0 \text{ for } y = 0 \quad , 0 \leq x \leq + \infty \tag{6a}$$

$$u_x = 0 \quad , u_y = 0 \quad y = 1 \quad , 0 \leq x \leq + \infty \tag{6b}$$

$$u_x = \frac{3}{2} (1-y^2), \ u_y = 0 \quad x = + \infty, \ 0 \leq y \leq 1 \tag{6c}$$

The boundary conditions for Case I at the inlet of the plates are:

$$u_x = 1 \quad , u_y = 0 \text{ for } x = 0 \quad , 0 \leq y < 1 \tag{7}$$

For Case II, the boundary conditions for the upstream region are:

$$\frac{\partial u_x}{\partial y} = 0,\ u_y = 0 \quad \text{for} \quad y = 0 \quad ,\ 0 \geq x \geq -\infty \tag{8a}$$

$$\frac{\partial u_x}{\partial y} = 0,\ u_y = 0 \qquad y = 1 \quad ,\ 0 \geq x \geq -\infty \tag{8b}$$

$$u_x = 1,\ u_y = 0 \qquad x = -\infty,\ 0 \leq y \leq 1 \tag{8c}$$

The stream function ψ is now introduced. It is defined by

$$\begin{aligned} u_x &= \frac{\partial \psi}{\partial y} \\ u_y &= -\frac{\partial \psi}{\partial x} \end{aligned} \tag{9}$$

The continuity equation is thus automatically satisfied. Furthermore, elimination of the pressure terms from equations 3 and 4 gives a single equation with one unknown:

$$\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\,(\nabla^2\psi) - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\,(\nabla^2\psi) = \frac{4}{Re}\nabla^4\psi \tag{10}$$

Equation 10 can also be written as the vorticity transport equation:

$$\frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y} = \frac{4}{Re}\nabla^2\omega \tag{11}$$

with the vorticity ω defined by

$$\omega = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} = -\nabla^2\psi \tag{12}$$

In order to avoid difficulties involved in treating numerically a problem with semi-infinite boundaries, a transformation is used to change the independent variable x to η according to the formula

$$\eta = \eta(x) = 1 - \frac{1}{1+Cx} \tag{13}$$

where C is a constant with a positive sign for the downstream region and a negative sign for the upstream region. This transformation also results in an expansion of scale (in terms of x) near the entrance which is desirable for a diffusional-type problem such as this. In terms of η and y, therefore, both the upstream and downstream regions are transformed into squares bounded by $y = 0$, $y = 1$, $\eta = 0$ and $\eta = 1$. As a result, equations 11 and 12 now assume the forms:

$$\frac{\partial\eta}{\partial x}\left(\frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial\eta} - \frac{\partial\psi}{\partial\eta}\frac{\partial\omega}{\partial y}\right) = \frac{4}{\mathrm{Re}}\left[\frac{\partial^2\eta}{\partial x^2}\frac{\partial\omega}{\partial\eta} + \left(\frac{\partial\eta}{\partial x}\right)^2\frac{\partial^2\omega}{\partial\eta^2} + \frac{\partial^2\omega}{\partial y^2}\right] \qquad (14)$$

$$-\omega = \frac{\partial^2\eta}{\partial x^2}\frac{\partial\psi}{\partial\eta} + \left(\frac{\partial\eta}{\partial x}\right)^2\frac{\partial^2\psi}{\partial\eta^2} + \frac{\partial^2\psi}{\partial y^2} \qquad (15)$$

In terms of ψ and ω, equation 6 for the downstream region can be rewritten as follows:

$$\psi = 0 \quad , \; \omega = 0 \qquad \text{for} \qquad y = 0,\; 0 \le \eta \le 1 \qquad (16a)$$

$$\psi = 1 \quad , \; \frac{\partial\psi}{\partial y} = 0 \qquad y = 1,\; 0 \le \eta \le 1 \qquad (16b)$$

$$\psi = \frac{3}{2}y - \frac{1}{2}y^3, \; \omega = 3y \qquad \eta = 1,\; 0 \le y \le 1 \qquad (16c)$$

For Case I at the inlet of the plates, equation 7 assumes the form

$$\psi = y \quad , \; \omega = 0 \qquad \text{for} \qquad \eta = 0,\; 0 \le y \le 1 \qquad (17)$$

Similarly, for Case II for the upstream region, equation 8 now becomes

$$\psi = 0 \quad , \; \omega = 0 \qquad \text{for} \qquad y = 0,\; 0 \le \eta \le 1 \qquad (18a)$$

$$\psi = 1 \quad , \; \omega = 0 \qquad y = 1,\; 0 \le \eta \le 1 \qquad (18b)$$

$$\psi = y \quad , \; \omega = 0 \qquad \eta = 1,\; 0 \le y \le 1 \qquad (18c)$$

Equations 14 and 15, with boundary conditions 16a through 18c, describe the entire problem.

III. METHOD OF SOLUTION

With the differential equations and the boundary conditions completely defined in the previous section, the next step was to devise a method for solving the differential equations numerically. Preliminary investigation indicated that an iterative procedure would be in order. The procedure required the transformation of differential equations 11, 12, 14, and 15 into difference equations, which are listed below:

$$C'_E\omega_E = C'_A\omega_A + C'_B\omega_B + C'_C\omega_C + C'_D\omega_D \qquad (19)$$

$$2\left[1+\left(\frac{\Delta x}{b}\right)^2\right]\Psi_E = \Delta x^2\omega_E + (\Psi_B+\Psi_D) + \left(\frac{\Delta x}{b}\right)^2(\Psi_A+\Psi_C) \qquad (20)$$

$$C_E\omega_E = C_A\omega_A + C_B\omega_B + C_C\omega_C + C_D\omega_D \qquad (21)$$

$$2\left[C^2(1-\eta_E)^4 \left(\frac{a}{b}\right)^2\right]\Psi_E = a^2\omega_E + \left(\frac{a}{b}\right)^2(\Psi_B+\Psi_D)$$
$$+ C^2(1-\eta_E)^4(\Psi_A+\Psi_C) - aC^2(1-\eta_E)^3(\Psi_A-\Psi_C) \qquad (22)$$

The complete derivations are shown in Appendix B. Equations 19 to 22 are the same as equations B23, B11, B19 and B10 in Appendix B respectively. The coefficients C'_is are defined by equation B20 for equation 21 and B24 for equation 19. Equations 19 and 20 are applicable to a rectangular-meshed x-y net, while equations 21 and 22 apply to a η-y net. The terms a and b are the mesh length of the η-y net while Δx and b are those of the x-y net. Also, a function with a subscript such as Ψ_E represents the value of that function at point E. The relative positions of points in the net are shown in Figure 1.

The boundary conditions for the finite difference equations can be derived from equations 16, 17 and 18. For the downstream region for both cases, they are, referring to Figure 1,

$$\Psi_E = 0 \quad , \omega_E = 0 \qquad \text{for } y_E = 0,\ 0 \leq \eta_E \leq 1 \qquad (23a)$$

$$\Psi_E = 1 \quad , \omega_E = 2(1-\Psi_D)/b^2 \qquad y_E = 1,\ 0 \leq \eta_E \leq 1 \qquad (23b)$$

$$\Psi_E = \frac{3}{2}y_E - \frac{1}{2}y_E^3,\ \omega_E = 3y_E \qquad \eta_E = 1,\ 0 \leq y_E \leq 1 \qquad (23c)$$

The boundary conditions for Case I at the inlet of the plates are:

$$\Psi_E = y_E,\ \omega_E = 0 \quad \text{for } \eta_E = 0,\ 0 \leq y_E < 1 \qquad (24)$$

For Case II, the boundary conditions for the upstream region are:

$$\Psi_E = 0,\ \omega_E = 0 \qquad \text{for} \quad y_E = 0,\ 0 \leq \eta_E \leq 1 \qquad (25a)$$

$$\Psi_E = 1,\ \omega_E = 0 \qquad y_E = 1,\ 0 \leq \eta_E \leq 1 \qquad (25b)$$

$$\Psi_E = y_E, \omega_E = 0 \qquad \eta_E = 1,\ 0 \leq y_E \leq 1 \qquad (25c)$$

Because of differences in boundary conditions, the procedure for solving Case II was more complicated than that for Case I. The two procedures, however, followed the same general scheme. For this reason, only Case II is described in detail here.

In solving Case II, a square-meshed η-y net was chosen for both the upstream and the downstream regions. Figure 2 shows a typical η-y net together with its corresponding x-y net. The square on the right of $\eta=0$ represents the downstream region and that on the left the upstream region. Values of Ψ and ω were assigned to or calculated for the boundary points according to the boundary conditions. Initial estimates of these functions were made for the interior points. Based on equations 19 to 22, new values of Ψ and ω were computed and used to replace the old values of each interior point in a systematic way. This procedure was repeated until the values of Ψ and ω converged to the desired degree of accuracy, and thus to the solution. From the solution, point velocities and pressure gradients were calculated. The actual computations were performed on an IBM 7090 digital computer.

In carrying out the iterative calculations, the values of Ψ were first calculated along the $\eta=0$ line from top down. Equation 20 was used. As shown in Figure 2, the η-y net is symmetrical about the y-axis when transformed into the x-y net. It was thus possible to treat the lines $\eta=0.2$, $\eta=0$ and $\eta=0.2$, say, as part of a rectangular-meshed x-y net and use equation 20 to calculate Ψ at $\eta=0$.

The next step in the iterative process was to compute values of Ψ in the downstream region using equation 22. This was done by temporarily holding the points on the $\eta=0$ line as boundary points. The computations were begun from the upper corner of the column next to the η=0 line, went down to the end of the column and moved to the next column in the direction of increasing η. In the same manner, the values of Ψ were then obtained for the upstream region. Each time a new value of Ψ was calculated, it was substituted for the old value by the process of single-step iteration.

After a new set of Ψ's had been obtained for all the interior points, the same sequence was applied to compute new values of ω based on equation 19 or 21. This then completed one cycle of iteration. At the end of each cycle, convergence was checked to determine whether re-iteration was necessary.

The computation of point velocities and pressure gradients from solutions in terms of Ψ and ω was direct and straightforward. Equations B34, B35, B38, B40, B43, B44 and B45 in Appendix B were derived for this purpose.

In order to accelerate convergence, an overrelaxation method which has been described in Appendix A was employed. Let $\Psi^{(P)}_{m,n}$ be the value of Ψ at point (m,n) after P-th iteration and $\xi^{(P+1)}_{m,n}$ be the value of Ψ calculated for point (m,n) from equation 22 during the (P+1)-th iteration. The new value for Ψ at (m,n) would then be:

$$\Psi^{(P+1)}_{m,n} = \Psi^{(P)}_{m,n} + M^{(P)}_{\Psi} \; (\xi^{(P+1)}_{m,n} - \Psi^{(P)}_{m,n}) \tag{26}$$

$M^{(P)}_{\Psi}$, which is called the overrelaxation factor for Ψ, was chosen to be a constant for each iteration, but could vary from iteration to iteration. Similarly, for ω, an overrelaxation factor $M_{\omega}^{(P)}$ was also employed. $M_{\Psi}^{(P)}$ and $M_{\omega}^{(P)}$ were not necessarily equal for the same iteration, and they had values ranged between 0.5 and 1.5. The choice of values for the overrelaxation factors was entirely empirical, and to some extent, arbitrary.

The actual computational work was performed on digital computers. Case I was solved on a Datatron 220 computer at the Computer Center of Caltech. Case II was solved on an IBM 7090 at the Jet Propulsion Laboratory. The change from a Datatron to an IBM computer for Case II was entirely due to speed considerations, the latter being approximately 100 times faster. A Fortran program for solving Case II is listed in Appendix C.

IV. RESULTS

During the early stages of this study, various numerical methods were explored for obtaining a stable and rapid solution of the problem described in the section on Formulation of the Problem. A detailed description of this phase of the work can be found in Appendix D. In brief, a marching solution was first tried but soon proved to be instable. An iterative method for solving either equation 10 or equations 11 and 12 in the Cartesian coordinates, on the other hand, was found to be stable only for Reynolds numbers less than fifty. For Reynolds numbers above 100, the solutions diverged very rapidly. Substitution of η for x according to equation 13 allowed for use of a coarser net, and consequently, reduced the time required to reach a solution. This substitution, however, did nothing in improving the stability of the solution for any given Reynolds number.

A new approach was then devised. Based on a technique proposed by de G. Allen (11), difference equations 19 to 22 were derived. The method of solution described in the previous section was adapted. For Case I, the downstream region was divided into a 10x10 η-y net. Initially, the values of Ψ were taken to vary linearly with η for any given y. Iterative computations were made with Reynolds numbers of 60, 100, 300 and 1000 on a Datatron 205 computer at Caltech. The value of the constant C in equation 13 was so chosen that the initial values of Ψ did not change markedly during iteration. The value of C, thus, was varied with the Reynolds number. For a Reynolds number of 300, the value of C was chosen to be 1.2. For all four cases, convergence was observed. From 40 to 65 stationary iterations were required to obtain a degree of convergence of 0.001% for Ψ and 0.1% for ω; the number of iterations increased with increasing Reynolds number. The degree of convergence is defined in this work as $\max \dfrac{\xi_{m,n}^{(P+1)} - \Psi_{m,n}^{(P)}}{\Psi_{m,n}^{(P)}}$. Each iteration took about 120 seconds of machine time.

At this time, a Datatron 220 computer was acquired by Caltech's Computer Center to replace the 205. Further computational work, therefore, was transferred to the new computer. Sixty-seven additional iterations were carried out for the case of $Re = 300$ before the Ψ's converged to all the eight significant digits that are available on the 220. ω, being the second derivative of Ψ, was converged only to the sixth place or better. Because of the higher speed of the 220, the time required for each iteration for the 10x10 grid was reduced to 16 seconds. The results in terms of Ψ and ω are given in Table I. The corresponding velocities and pressure gradients are tabulated in Table II.

By means of 4-point Lagragian interpolation, the 10x10 grids were expanded to 20x20 grids for both Ψ and ω. They were used as the starting values for a new series of iterations with the 20x20 grids. After 600 stationary iterations or approximately ten hours on the computer, the solution converged to a degree of 2×10^{-8} for Ψ. The final values of Ψ and ω are listed in Table III. Point velocities and pressure gradients were also calculated and given in Table IV. Figures 3 and 4 show the streamline pattern and constant vorticity lines for Case I. The velocity distributions are plotted for various values of η in Figure 5. In Figures 6 to 8, u_y, $-\frac{\partial P}{\partial x}$ and $-\frac{\partial P}{\partial y}$ are plotted against η as functions of y respectively.

At this point, the possibility of using an overrelaxation method to accelerate convergence was explored. A trial and error procedure was adapted to determine suitable values of the overrelaxation factors. Starting with the result of the 10x10 grid from Datatron 205 for a Reynolds number of 300, it was found that a number of stationary iterations at the beginning would help to orient the values of Ψ and ω. After that, the optimum values for the overrelaxation factors lay somewhere around 1.5. These values should then be decreased slowly as the iterative process progressed and usually the best final values were less than unity. Several runs were made during each of which the overrelaxation factors were held constant. In all these cases, either oscillation or divergence was observed. Although the manner in which these factors should be changed was not fully understood, it was found desirable to decrease M_ω faster than M_Ψ. By manipulating the overrelaxation factors, it was possible to reach a converged solution in fourteen iterations as compared with sixty-seven iterations in the case of stationary iterations. Overrelaxation, therefore, seemed to be useful in accelerating the convergence for this problem.

Case I assumes that potential flow conditions exist at the inlet of the parallel plates. In most actual physical situations, however, this assumption can hardly be valid since some effect of the solid walls on the upstream flow would be expected. In order to explore the effect of the leading edge on the upstream flow, Case II was formulated. A Fortran program for Case II was written for the IBM 7090 computer located at Jet Propulsion Laboratory, Pasadena. This program consists of three parts. Part 1 does the iterative calculations of Ψ and ω, checks and prints out the degrees of convergence for the upstream region, the downstream region, and points on the $\eta=0$ line after each iteration, and decides whether to re-iterate or exit to Part 2. Part 2 of the program calculates and prints out point velocities and pressure gradients from the latest values of Ψ and ω. Part 3 is actually an independent subprogram. Its function is to expand the number of nodes of a network by means of 4-point Lagrangian interpolation. The program is listed in Appendix C.

In solving Case II with the Fortran program, a square-meshed net with mesh length of 0.1 was used for both the upstream and downstream region. The solution for Case I was used for the initial estimate of the downstream region. In the upstream region, potential flow conditions were first assumed. These correspond to $\psi = y$ and $\omega = 0$ at every point. Within 150 iterations, a degree of convergence of 2×10^{-8} was arrived for both ψ and ω. Their values are listed in Table V. The corresponding velocities and pressure gradients are shown in Table VI. The total time required, including the compiling of the Fortran program, is 70 seconds.

The mesh size of the net was then halved to give a 40x20 net. Interpolation of the solution with the 20x10 net again gave the initial estimate. To obtain a degree of convergence of 2×10^{-8} this time, a total of 400 iterations was necessary. Of these, the first 200 were stationary iterations. Over-relaxation factors of 1.3 for ψ and 1.4 for ω were then used. these values were decreased by 0.1 and 0.2 respectively each 60 iterations, thus ending with $M_{\psi} = 1.0$ and $M_{\omega} = 0.8$. Table VII lists the resultant values of ψ and ω for Case II with a 40x20 grid. Streamlines and lines of constant vorticity were constructed from these values and are shown in Figures 9 and 10. In Figures 11 to 18, velocities and pressure gradients are plotted against η for various y's or against y as functions of η. In Tables V to VIII, as well as Figures 9 to 18, negative signs have been added to all the values of η in the upstream region in order to distinguish them from those in the downstream region.

V. DISCUSSION

The simplification of the Navier-Stokes equations to the boundary layer equation by Prandtl proved to be one of the most important contributions in the study of fluid mechanics. It is also well-known, however, that the boundary layer equation, being restricted by certain assumptions, does not apply very well to cases such as flow near the leading edge of a flat plate, where the term $\partial^2 u_x/\partial x^2$ is not negligible with respect to $\frac{\partial^2 u_x}{\partial y^2}$. This is well illustrated in the problem of flow in the inlet length of two parallel plates. Schlichting solved the problem using the boundary layer equation. In this work, the same problem was approached by solving the Navier-Stokes equations numerically. A comparison of the results in terms of u_x is furnished in Table IX. Near the leading edge ($x \ll 1$), the velocity, u_x, for Case I varies more rapidly near the wall than that from Schlichting's solution. This is due to the fact that in this region, $\partial^2 u_x/\partial x^2$ is significant and, therefore, cannot be neglected. When the upstream disturbance is also taken into consideration, (Case II), the velocity gradient, $\partial u_x/\partial y$, then becomes smaller. Plots of u_x vs. η at $y = 0.95$ for all three cases are given in Figure 19. At a distance from the inlet say, ($x \geq 3.0$), the resulting velocity profiles for all three cases agree very well. It is very interesting to note here that different boundary conditions upstream have very little effect on the flow pattern at a distance from the inlet.

In the calculation of pressure drops, Schlichting took into consideration, in addition to the pressure drop due to Poiseuille's flow, a pressure drop resulted from the acceleration of flow near the center. This additional pressure drop was found to be $- 0.601 \, (\frac{1}{2} \rho U_{xo}^2)$ over the inlet length. More recently, Collins and Schowalter (17) improved Schlichting's solution by retaining more terms in the series expansions for both downstream and upstream solutions and found the additional pressure drop to be $- 0.676 \, (\frac{1}{2} \rho U_{xo}^2)$. This last value agrees with the numerical solution of the boundary layer equation for the same problem (18). In all these instances, however, the pressure variations across the width were neglected. Once again, this assumption is rather poor near the leading edges of the plates as witnessed by the large pressure gradients in the y-direction in this region (Figures 8 and 17). For comparison purpose, the pressure differences between the free stream pressure and that at any point η, ΔP_η, are given in Table X for the present work and Schlichting's case. The values ΔP_η, for Cases I and II are obtained by evaluating the integral $\int \frac{\partial P}{\partial \eta} d\eta$ for constant y's using Simpson's rule. They are in general higher than the corresponding values of Schlichting's. Figure 20 plots $(\Delta P_\eta - \Delta P_{s\eta})$ vs. η for both Case I and Schlichting's result, where $\Delta P_{s\eta}$ is the pressure drop due to Poiseuille's flow only.

The streamlines and lines of constant vorticities have been constructed in Figures 3 and 4 for Case I, and in Figures 9 and 10 for Case II. In Case II, deviations from free-stream conditions in the upstream direction are significant up to approximately $x \approx -0.5$. Because of this, the change in the flow pattern for Case II is more gradual as compared with Case I, although they are rather similar in general.

Figures 5 and 11 show the progressive transformation from the uniform velocity profile to the asymptotic parabolic distribution as the fluid moves downstream. It should be noted that in Figure 5 (Case I), the central portions of the velocity profiles for small η's are concave upward. The same phenomenon, although less profound, can be observed in Figure 11. The difference between the maximum velocity and that at the center at the same cross section may be as much as 0.04 as compared with a free-stream velocity of 1.0. This may be attributed to the finite time required for transporting the fluid from near the wall toward the center as the fluid enters the plates. This agrees with the fact that near the entrance, u_y is very large near the wall and then drops slowly towards the center (Figure 6). In Case II, however, part of the transport from near the wall to center is accomplished in the upstream region (Figure 14) and the concavity, therefore, is less profound. Due to the concavity in the center, the velocity profiles thus have points of inflexion. By Rayleigh's theorem (13), this condition implies instability.

It would be interesting to find out whether this phenomenon would become self-exciting with increase in Reynolds number and hence cause the flow to become turbulent. This concave velocity profile was not obtained by Schlichting due to his assumption of a flat velocity distribution in the center core.

The pressure gradient in the direction of flow is zero in the free-stream region and has a value of 0.04 for a Reynolds number of 300 when the flow approaches the asymptotic parabolic velocity distribution. Near the leading edges of the plates, very large pressure gradients exist (Figures 7, 15, and 16). Figure 15 also shows the presence of positive pressure gradients in the direction of flow before the entrance at some distance away from the center. In the upstream region, the viscous effect is relatively small. Positive pressure gradient, therefore, would be expected where deceleration occurs. As soon as the fluid enters the region between the plates, the viscous forces suddenly become large and the pressure gradient changes sign. Such a region of positive pressure gradient can also be found in Case I, except that it is now located near the walls inside the plates. Since part of the effect of deceleration is now counterbalanced by the viscous effect, the magnitude of the positive pressure gradient is much less in Case I than in Case II.

Two kinds of errors arise in solving a differential equation by finite difference methods: the discretization or truncation error and the round-off error. The latter is insignificant in the present work as eight digits for each number were conserved throughout the computations. The discretization error, which results from the replacement of derivatives with finite-difference formulas, depends on the size of the grid and the form of the finite-difference formulas used as well as the continuity and smoothness of the dependent variables in the region of interest. If a proper finite-difference expression has been chosen for the differential equation, the discretization error should tend to zero with the grid size. One empirical method to determine the discretization error semi-quantitatively would be to compare the solution of the finite-difference equation for several grid sizes. Table XI compares the values of Ψ and ω at several positions for two different grid sizes. It can be seen that, except for a small region near the leading edge of the plate, the values of Ψ agree fairly well. The values of ω, being the second derivative of Ψ, differ from each other to a much greater extent, especially when they are small. Fortunately, when the value of ω is small, it has little effect on the values of Ψ, and consequently the velocities and pressure gradients.

The reason that the values of Ψ vary with the grid size in the region near the leading edge is that there exists a singular point ($x=0$, $y=1$). The behaviour of the functions Ψ and ω in this region is, therefore, less smooth as compared with regions farther away from this point. If it is desired to obtain comparable discretization errors throughout the domain, it would be necessary to use smaller grid sizes in this particular region. Although various methods for changing grid sizes in a network are available, none has been tried for the present work.

That the discretization error is larger near the leading edge can also be indirectly verified. For example, the integral $\Delta P_{\eta} = \int_{\eta_1}^{\eta_2} \frac{\partial P}{\partial \eta} d\eta$ where η_1 is in the potential flow region and η_2 far downstream should have the same value for all y's. The values of this integral for several values of y are presented in Table X. For both Cases I and II, ΔP at $y = 0.9$ is lower than those at smaller y's, and the difference is more than can be attributed to round-off errors.

The transformation of the x into η proved to be reasonably successful. With the introduction of η, the original semi-infinite domain became a square. This not only allowed the use of fewer points without losing accuracy but also made it possible to treat the boundary conditions at $x = \infty$ nicely. A number of possible transformations, such as arc tan cx and e^{-ax}, which have similar properties have been studied. Of all these, the present one offered both simplicity and more desirable correspondence to x. For small x, η approximates Cx. For large x, η approaches $1 - \frac{1}{Cx}$. The value of C can consequently be chosen to force the values of Ψ to change as smoothly as possible with respect to η. This has been accomplished to a large extent as can be seen in Figures 3 and 9.

The discretization of equation 11 by the method of de G. Allen introduced exponential terms into the finite-difference equation 30. It has been shown in Appendix B that when the mesh lengths in terms of a and b approach zero, equation 30 can be reduced to a conventional finite-difference equation with forward-difference formulas for the first derivatives and the central-difference formulas for the second derivatives. When the mesh size increases, however, an exponential cannot be closely represented by a polynomial. For this reason, finite-difference equations using conventional formulas for the derivatives were found to result in instability in the present case. On the other hand, use of the conventional finite-difference expressions has the advantage of simplicity and universal applicability, and the possibility should be further explored.

The use of the overrelaxation method in accelerating convergence of the solution of non-linear finite-difference equations was found to be fruitful. It is true that the method used in the present investigation is entirely empirical and most likely not very efficient. It nevertheless did improve the convergence rate by a factor of 2 to 3. The finding that the overrelaxation factors had to be decreased as the iteration proceeded is especially interesting. It is known that the optimum value for the overrelaxation factor for non-linear boundary value problems would change. The fact that it has to be changed, at least in some cases, in order to bring convergence, may shed light on the stability aspect of the problem.

VI. CONCLUSIONS

The two-dimensional flow pattern of an incompressible fluid at the entrance section of two parallel plates has been obtained for two cases, both for a Reynolds number of 300. Using the Navier-Stokes equations, the solutions are presented in terms of the stream function ψ and the vorticity ω. From these, local velocities and pressure gradients have been calculated. In Case I, potential-flow conditions prevail at the entrance. Case II assumes that these conditions exist only far upstream from the entrance. In the latter case, deviation of the upstream flow pattern from free-stream conditions was found to be significant only within a short distance from the plates, i.e., up to $x = -0.5$. For both cases, large velocity and pressure gradients normal to the wall as well as in the direction of flow are observed in the region near the leading edges of the plates, although they are comparatively smaller in Case II. Also, the velocity profiles for small x were found to be slightly concave in the central portion. This is attributed to the finite time required for transporting the fluid towards the center.

Case I has been solved by Schlichting using the boundary layer equation. A comparison of the results shows that the boundary layer equation does not describe the behavior of the system well near the leading edges of the plates. At larger distances from the inlet, the velocity profiles obtained by Schlichting agree very well with those from Case

I, as well as Case II.

Use of finite-difference equations based on the conventional difference formulas for derivatives resulted in the instability of the solution. The finite-difference equations used for the computations of the vorticity were derived based on a technique suggested by de G. Allen to overcome this difficulty. The transformation of x into η helped to reduce the number of nodes and yet retained the accuracy needed. It also made it easier to treat the boundary conditions at infinity.

The use of the overrelaxation method helped to improve the convergence rate of the iterative computations by a factor of 2 to 3. It has been found that the values of overrelaxation factors used had to be reduced gradually during the process to ensure convergence.

VII. RECOMMENDATIONS FOR FUTURE WORK

1) Using the Fortran program listed in Appendix C, work should be continued to determine the flow patterns at the inlet length of two parallel plates over a range of Reynolds numbers.
2) The theory of hydrodynamic stability concerning plane Poiseuille flow in two dimensions has been studied by many authors, often with contradictory conclusions. Possibilities of numerical approach similar to the one described in this thesis should be explored.
3) As described in Appendix C, numerical solution of the present problem using conventional finite-difference expressions converged only for low Reynolds numbers. This information may help in theoretical study of the stability of numerical solutions of the Navier-Stokes equations.

NOMENCLATURE

a	mesh length in η-direction
b	mesh length in y-direction
C	a constant used in equation 13
C_i's	coefficients for ω_i's in equation 20
C_i''s	coefficients for ω_i's in equation 18
M_u	overrelaxation factor for function u, dimensionless
P'	pressure, lb/sq ft
P	dimensionless pressure, $P'/\rho\ U_{xo}^2$
ΔP_η	pressure difference between a point in the free stream and point η, dimensionless
$\Delta P_{s\eta}$	pressure difference between two points if Poiseuilles' flow prevails throughout, dimensionless
Re	Reynolds number, $4y_oU_{xo}/\nu$, dimensionless
u_x'	velocity component in x-direction, ft/sec
u_x	dimensionless velocity component in x-direction, u_x'/U_{xo}
U_{xo}	free-stream velocity, ft/sec
u_y'	velocity component in y-direction, ft/sec
u_y	dimensionless velocity component in y-direction, u_y'/U_{xo}
x'	Cartesian coordinate, ft
x	dimensionless Cartesian coordinate, x'/y_o
y'	Cartesian coordinate, ft
y	dimensionless Cartesian coordinate, y'/y_o
y_o	half width between two parallel plates, ft

∂	partial differential operator
$\frac{D}{D\Theta}$	substantial derivative operator
∇^2	Laplacian operator in Cartesian coordinates, $\partial^2/\partial x^2 + \partial^2/\partial y^2$
∇^4	$\nabla^2\nabla^2 = \partial^4/\partial x^4 + 2\,\partial^4/\partial x^2\partial y^2 + \partial^4/\partial y^4$
λ	an expression defined by equation B22
λ'	an expression defined by equation B26
κ	an expression defined by equation B21
κ'	an expression defined by equation B25
η	a function of x defined by equation 13
ρ	density, lb $\sec^2/\text{ft}^4$
ν	kinematic viscosity, sq ft/sec
Ψ	dimensionless stream function defined by equation 9
ω	dimensionless vorticity, $-\nabla^2\Psi$

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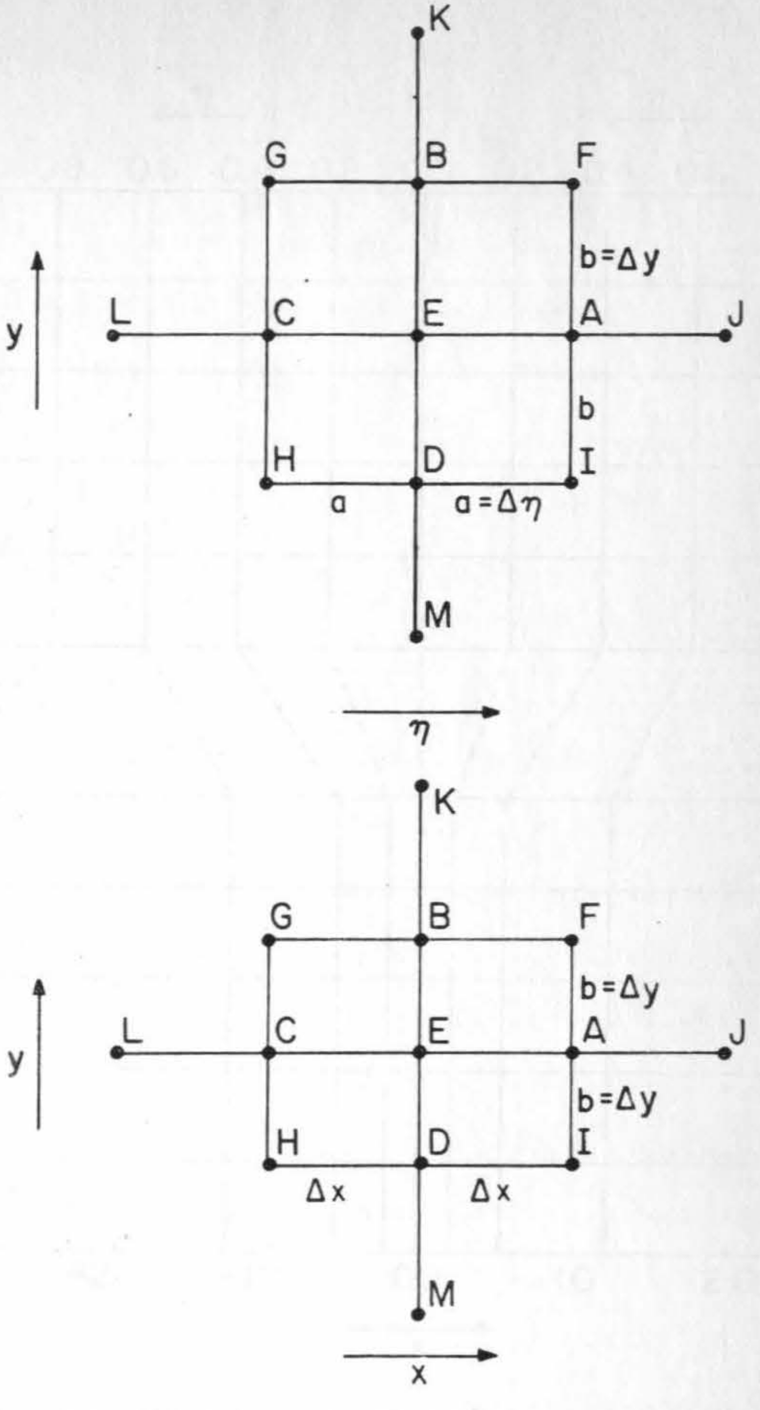

Figure I. Point patterns in $\eta - y$ and $x - y$ nets

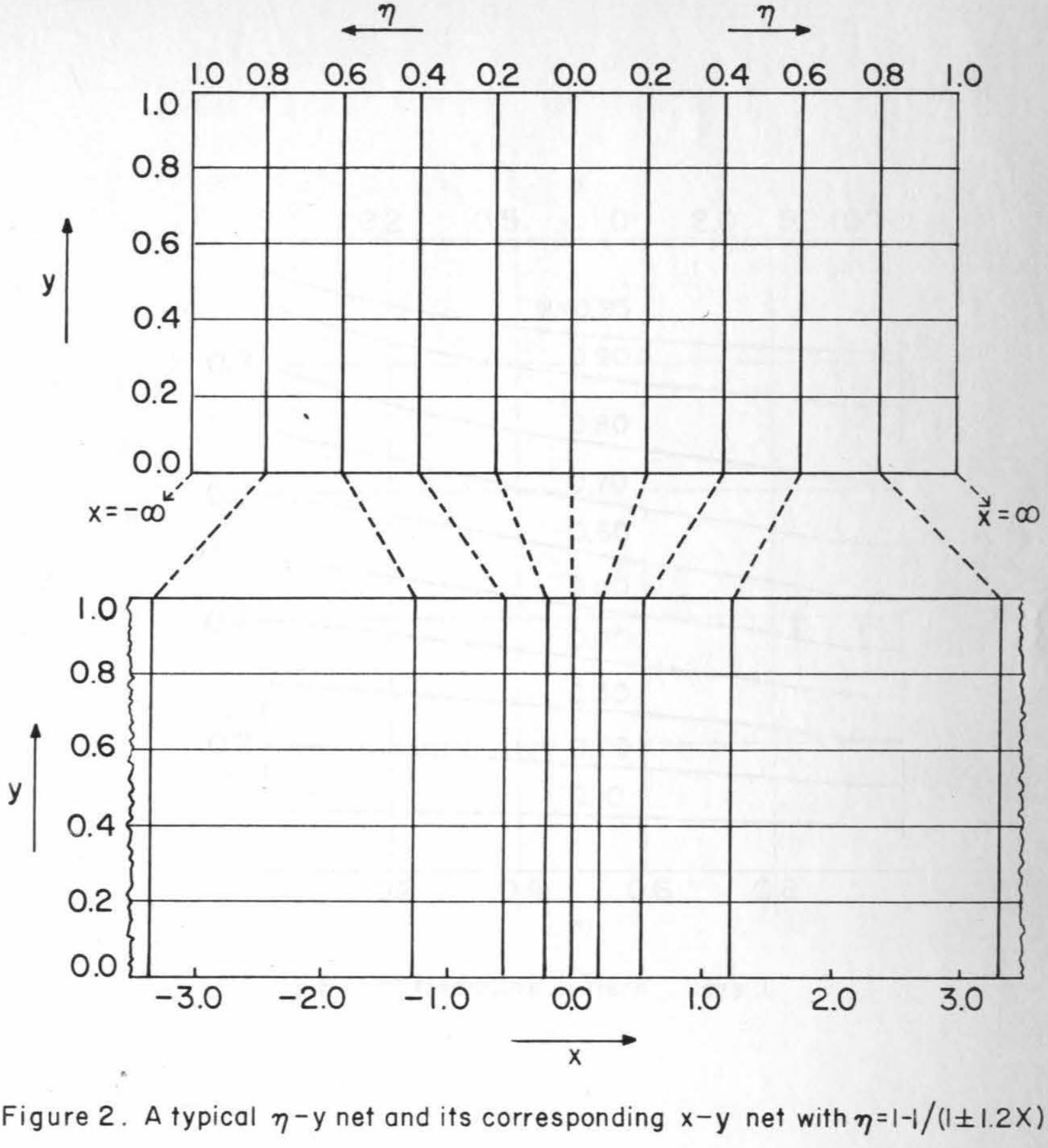

Figure 2. A typical $\eta - y$ net and its corresponding $x - y$ net with $\eta = 1 - 1/(1 \pm 1.2X)$

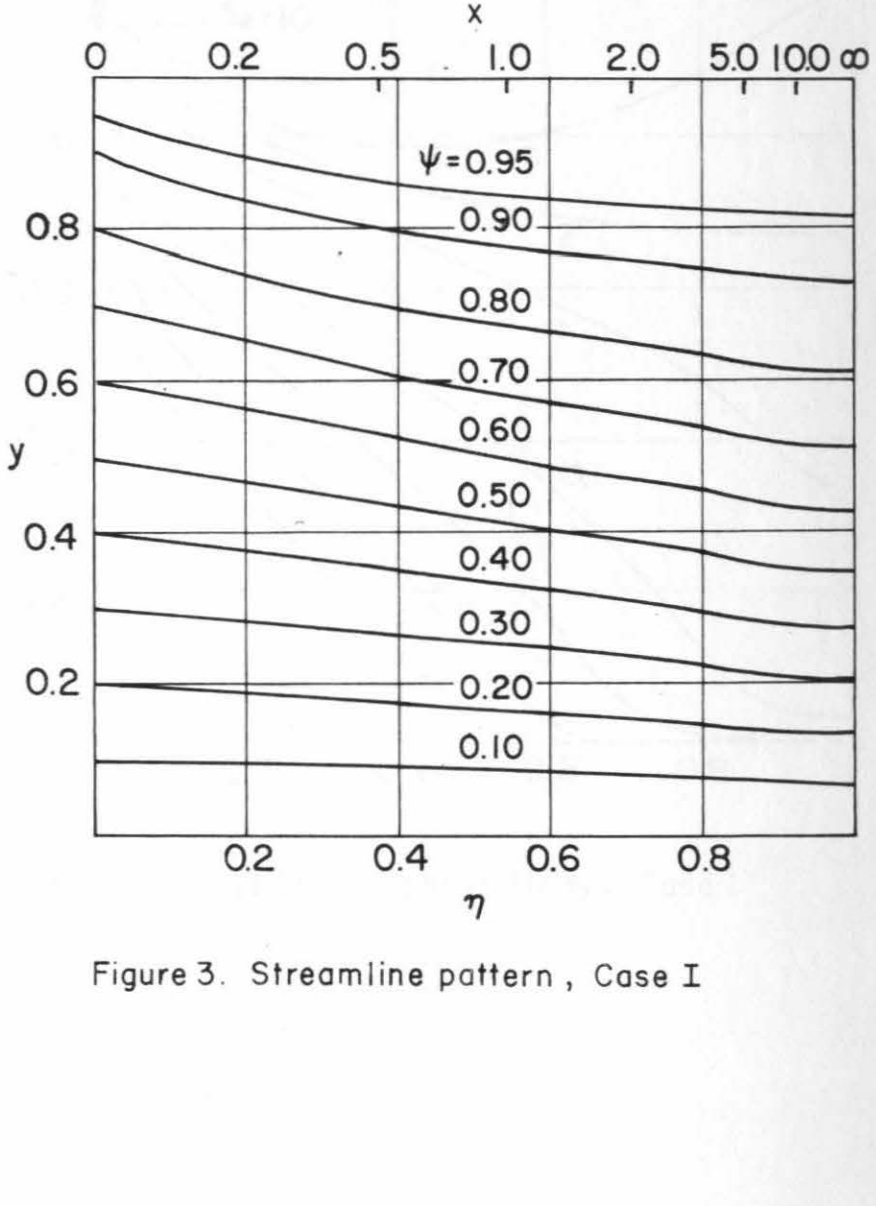

Figure 3. Streamline pattern, Case I

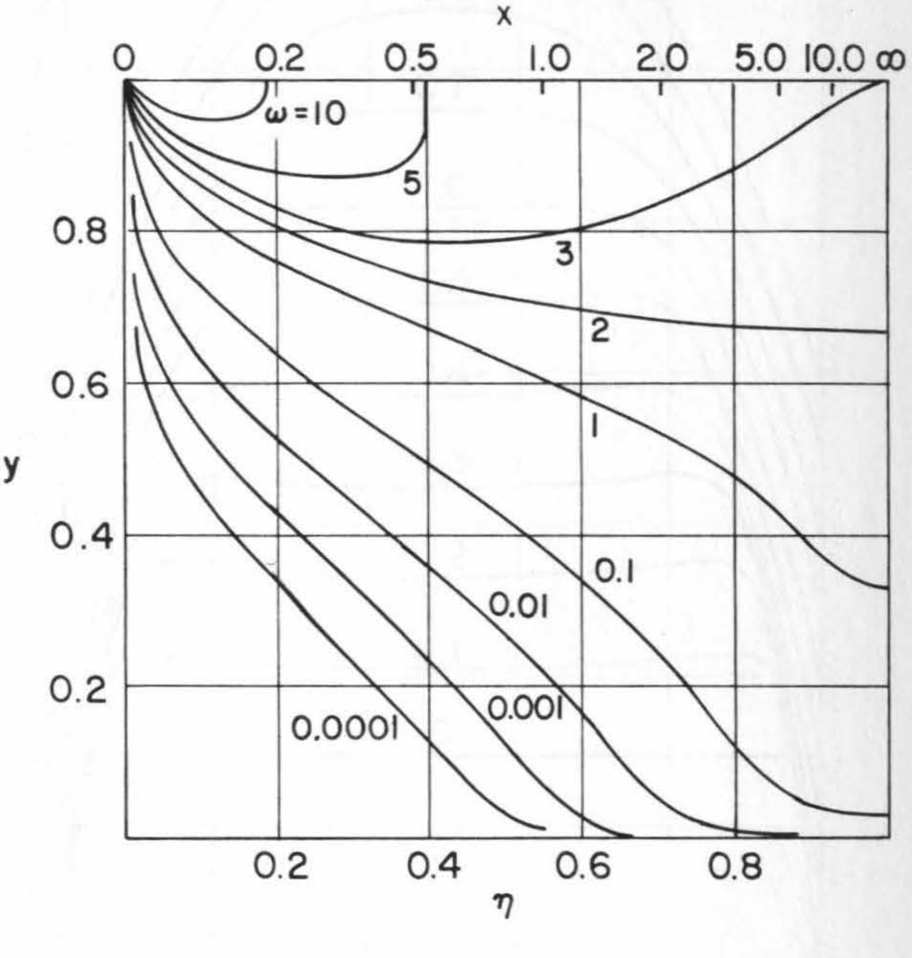

Figure 4. Lines of constant vorticity, Case I

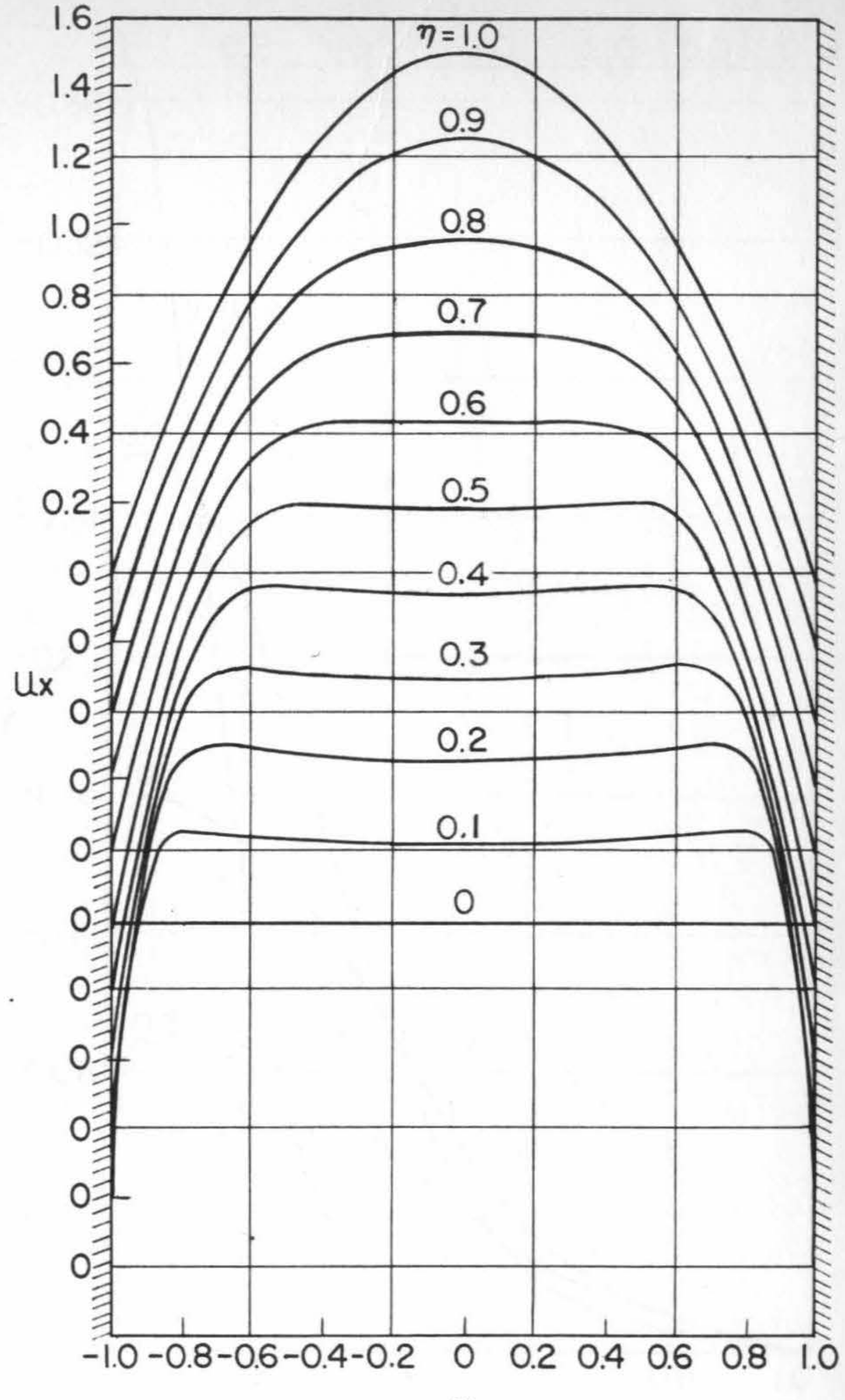

Figure 5 Velocity profiles, Case I

Figure 6. U_y as function of η for several y's, Case I

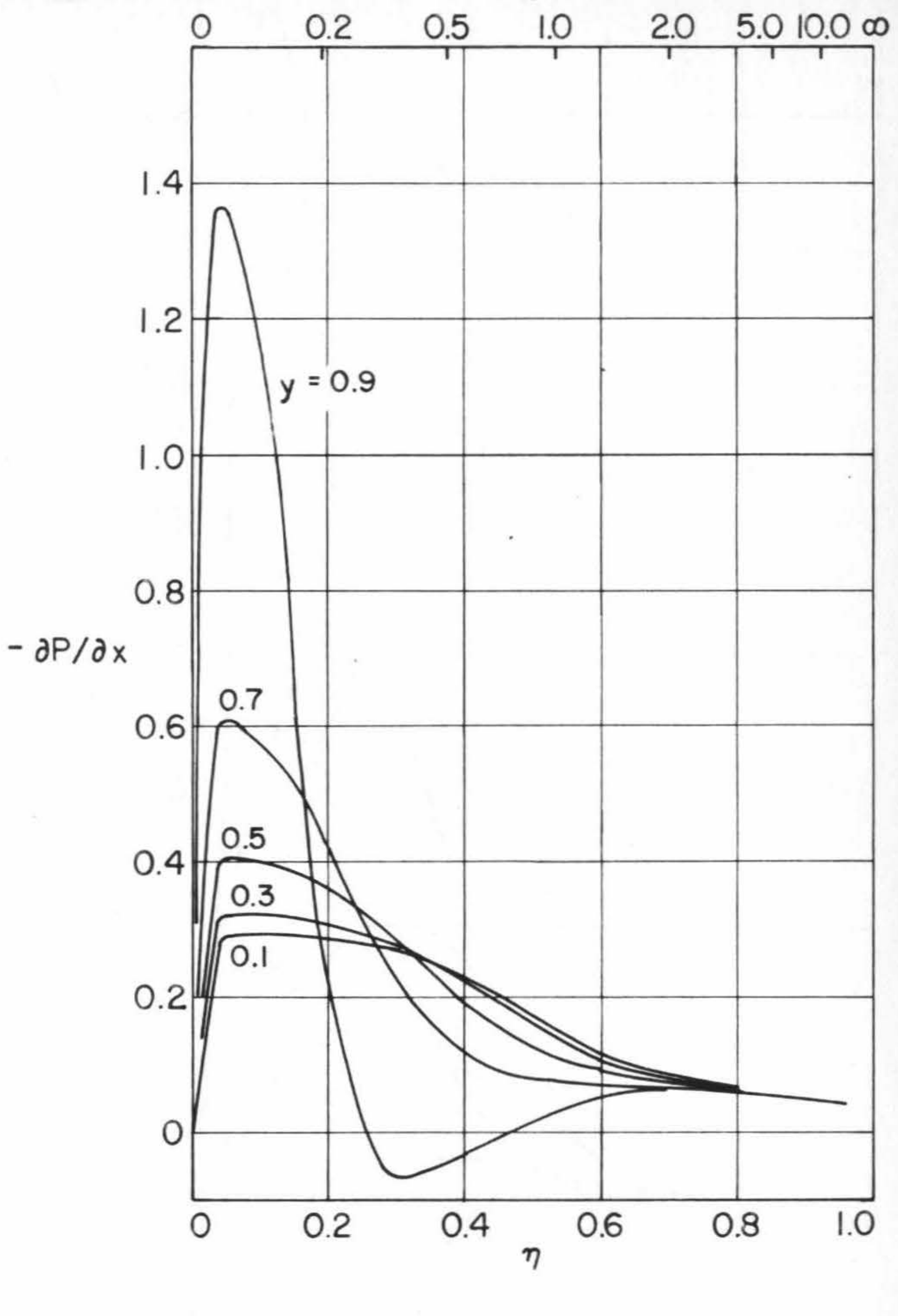

Figure 7. $\partial P/\partial x$ as function of η, Case I

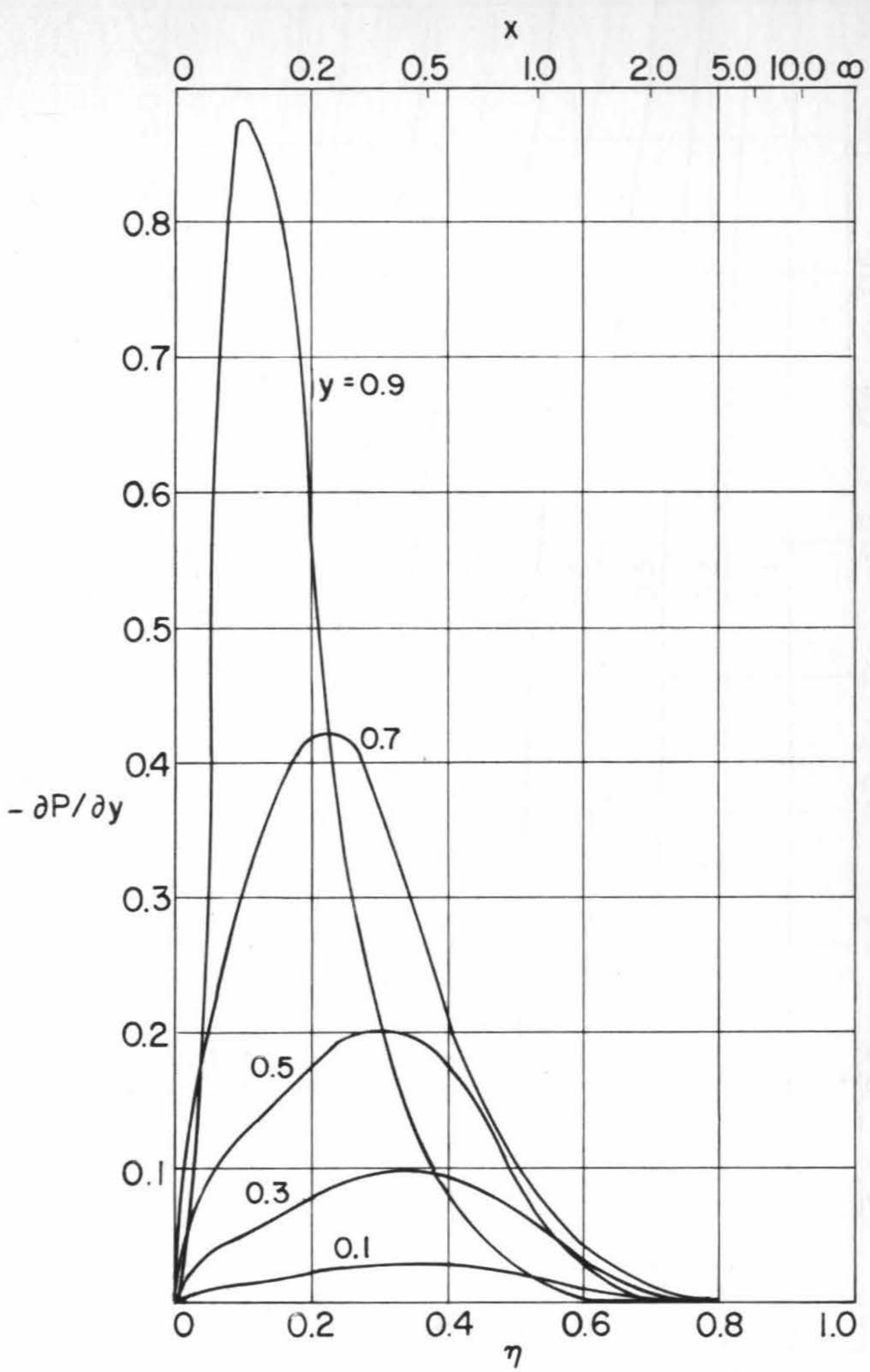

Figure 8. $\partial P/\partial y$ as function of η, Case I

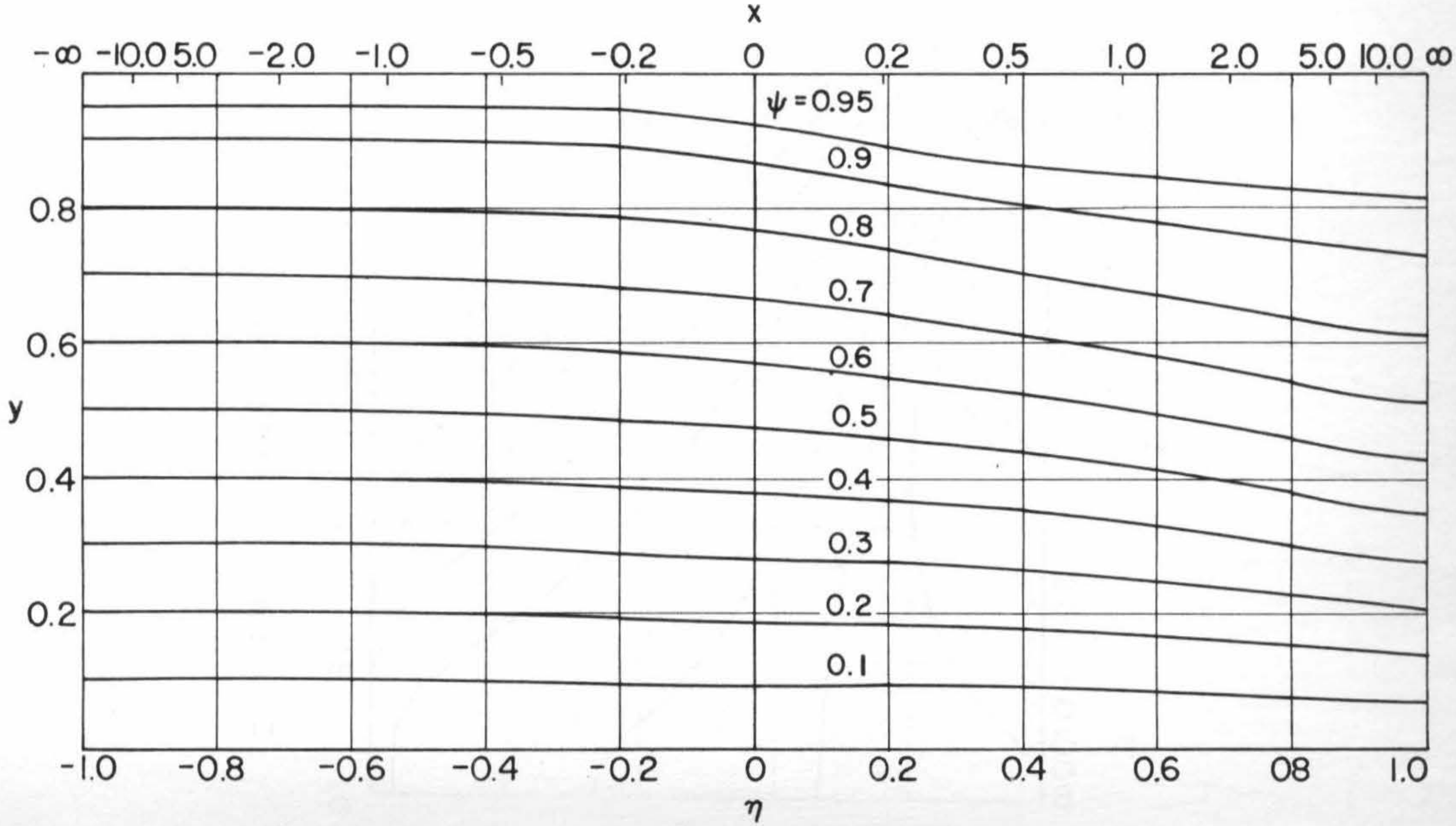

Figure 9. Streamline pattern of Case II

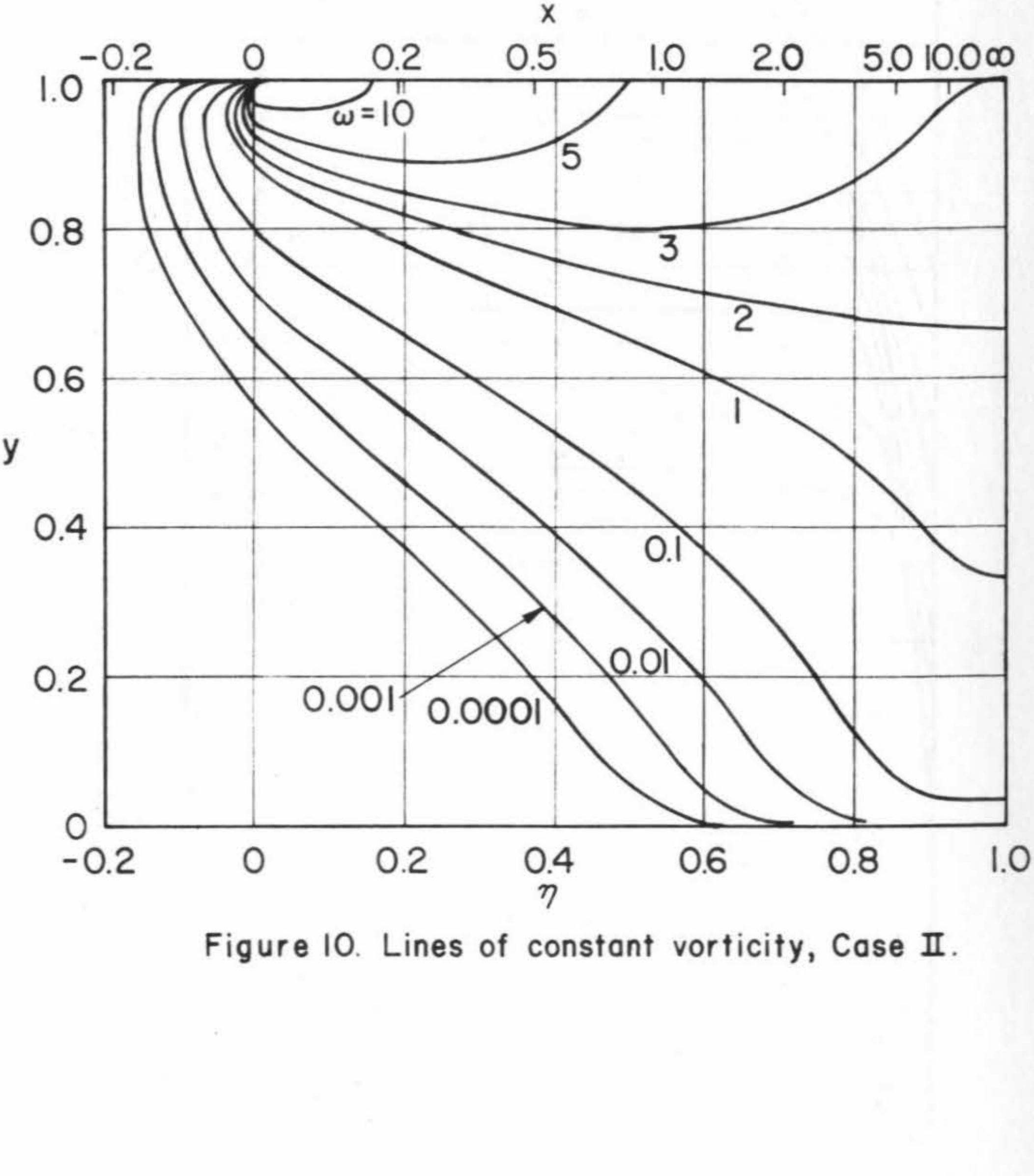

Figure 10. Lines of constant vorticity, Case II.

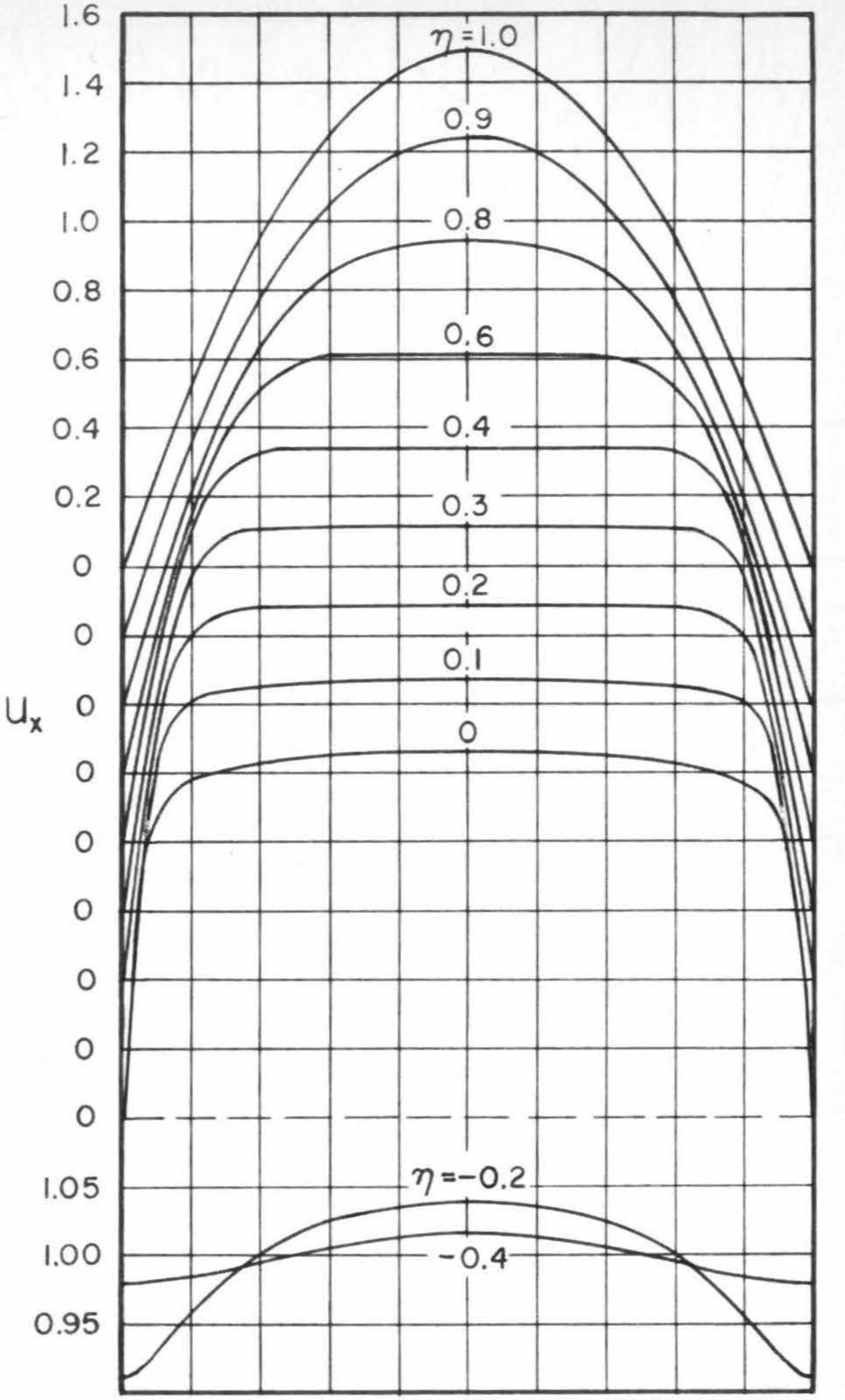

Figure 11. Velocity profiles, Case II.

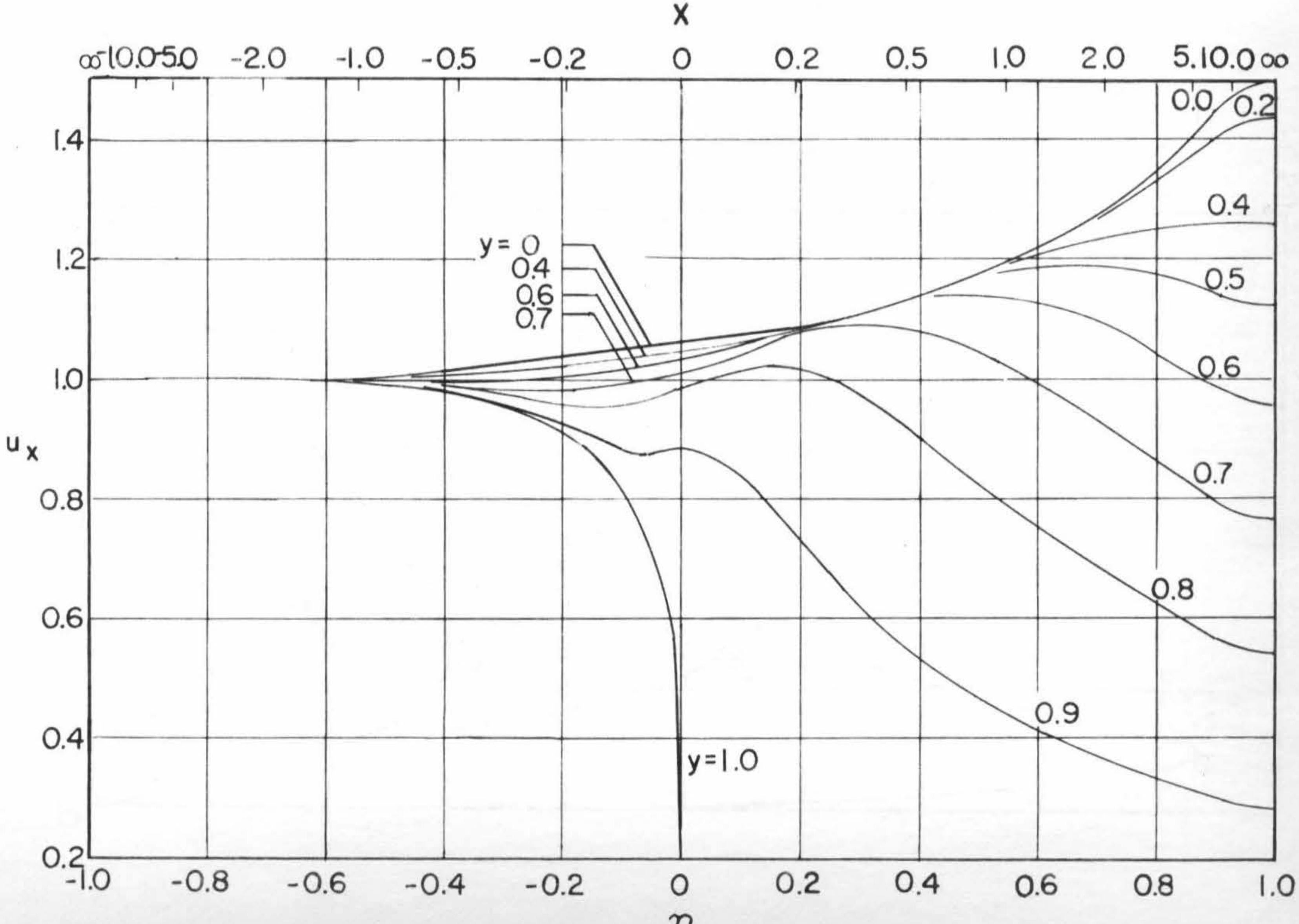

Figure 12. u_x as function of η, Case II

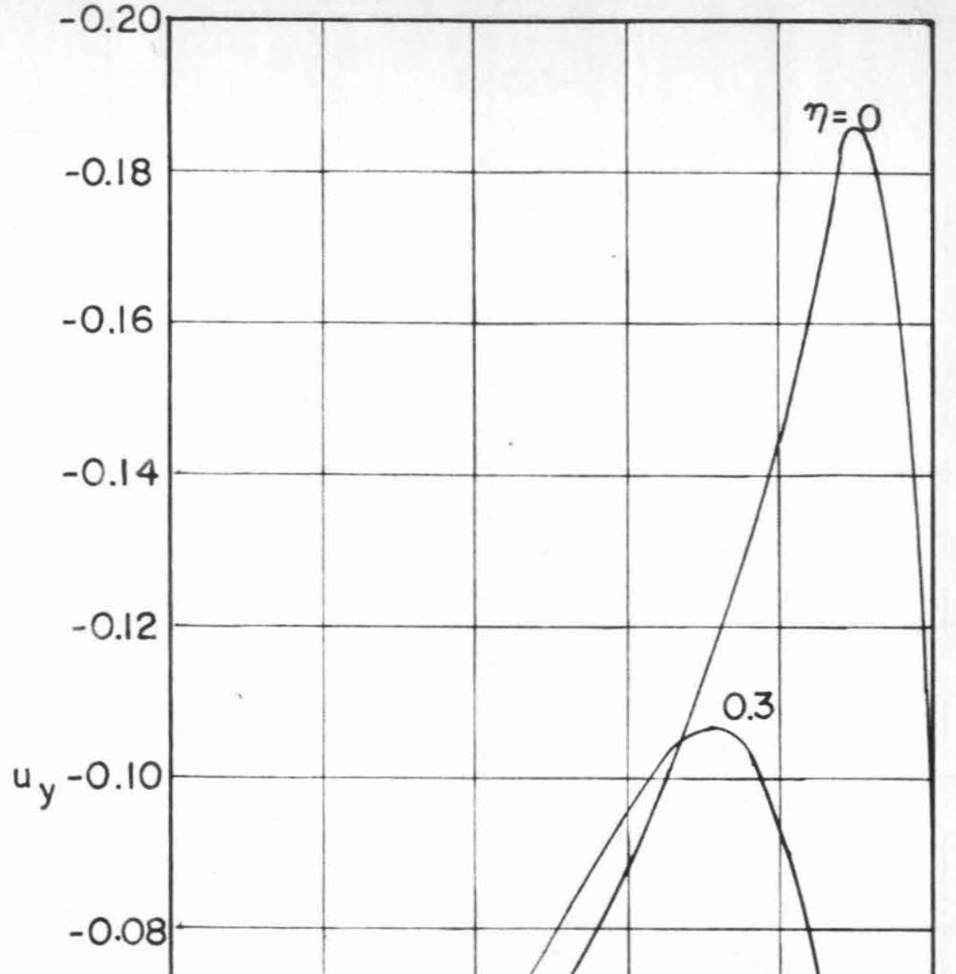

Figure 13. u_y as function of y, Case II

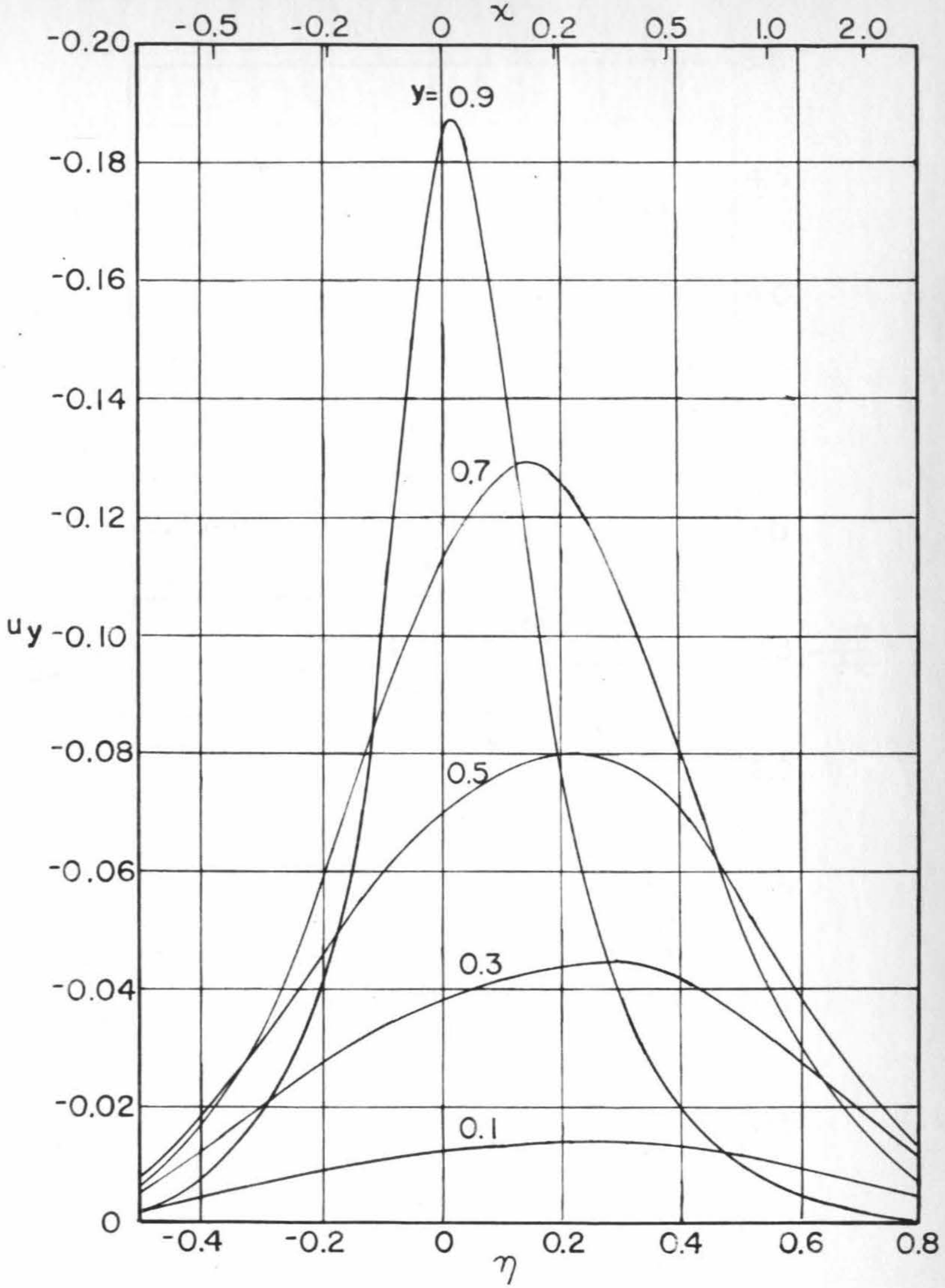

Figure 14. u_y as function of η, Case II

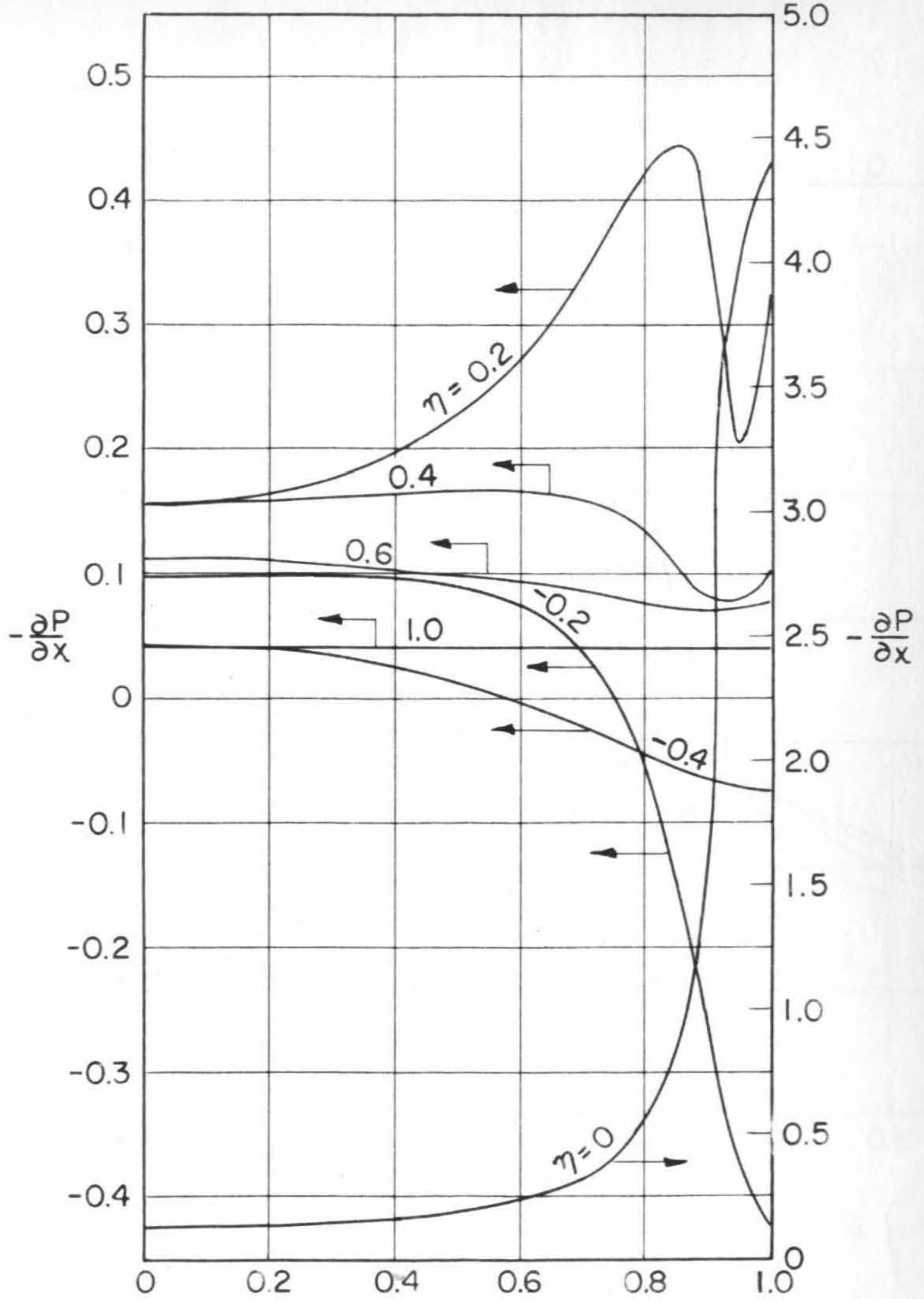

Figure 15. $\partial P/\partial x$ as function of y, Case II

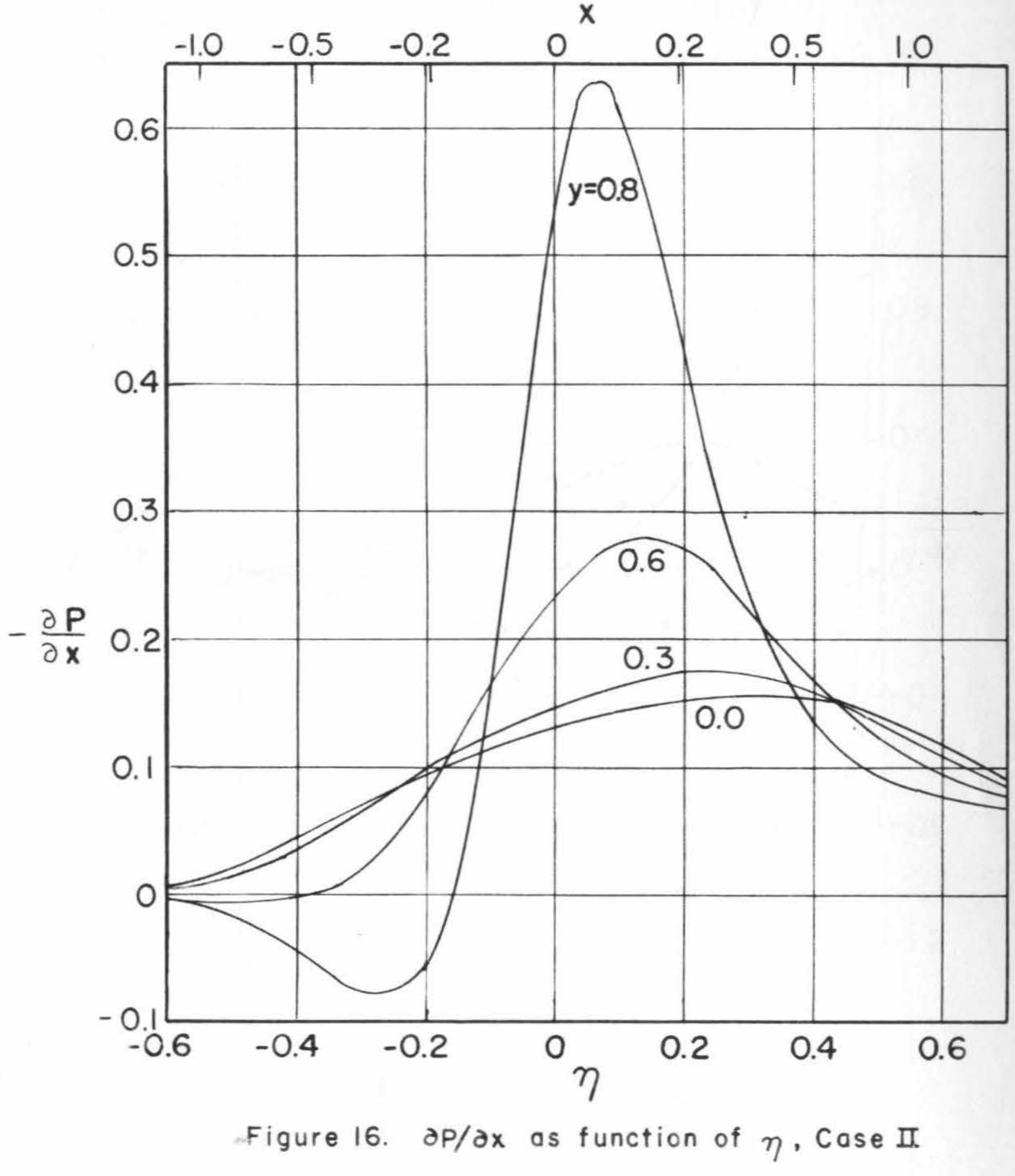

Figure 16. $\partial P/\partial x$ as function of η, Case II

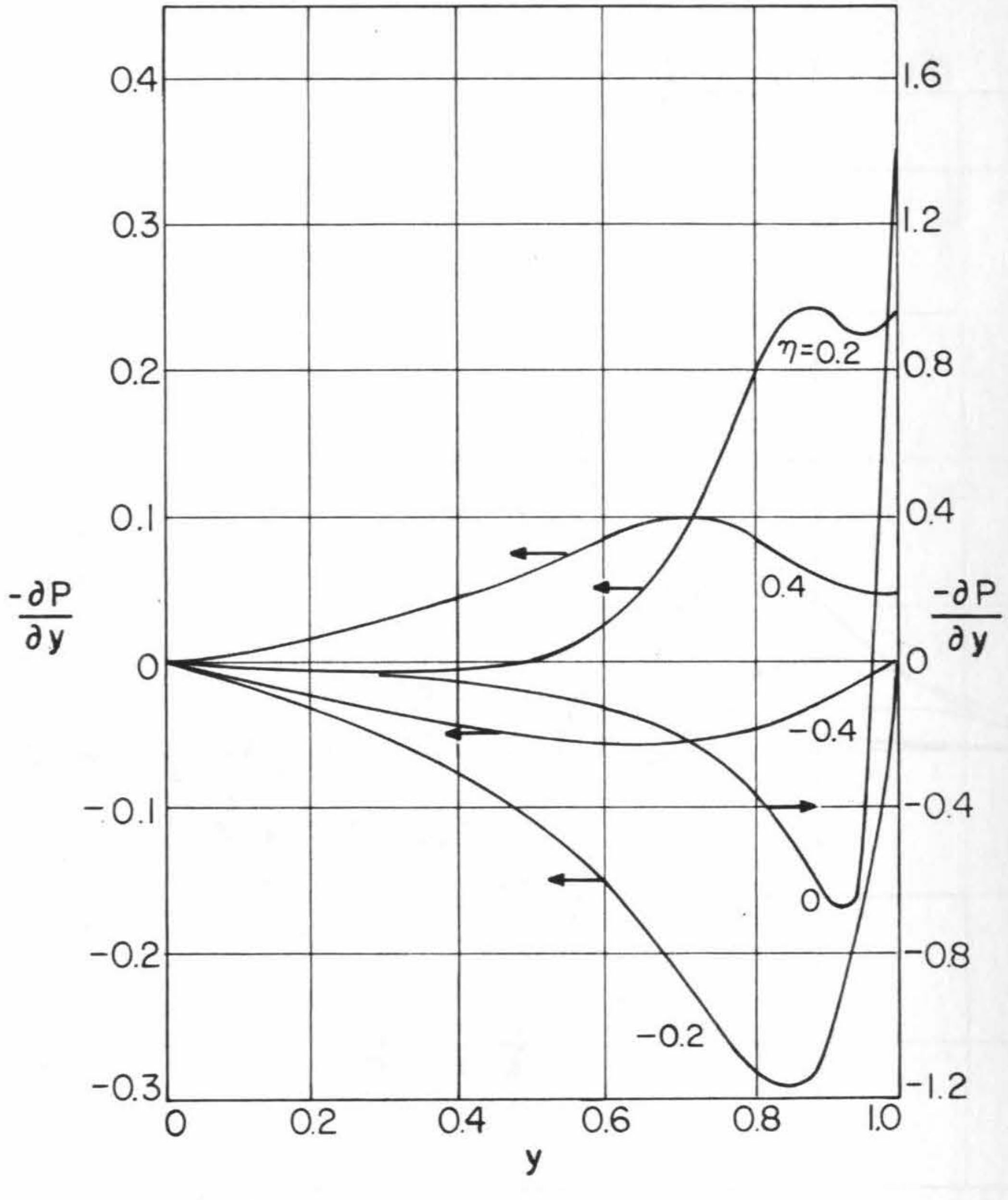

Figure 17. $\partial P/\partial y$ as function of y, Case II

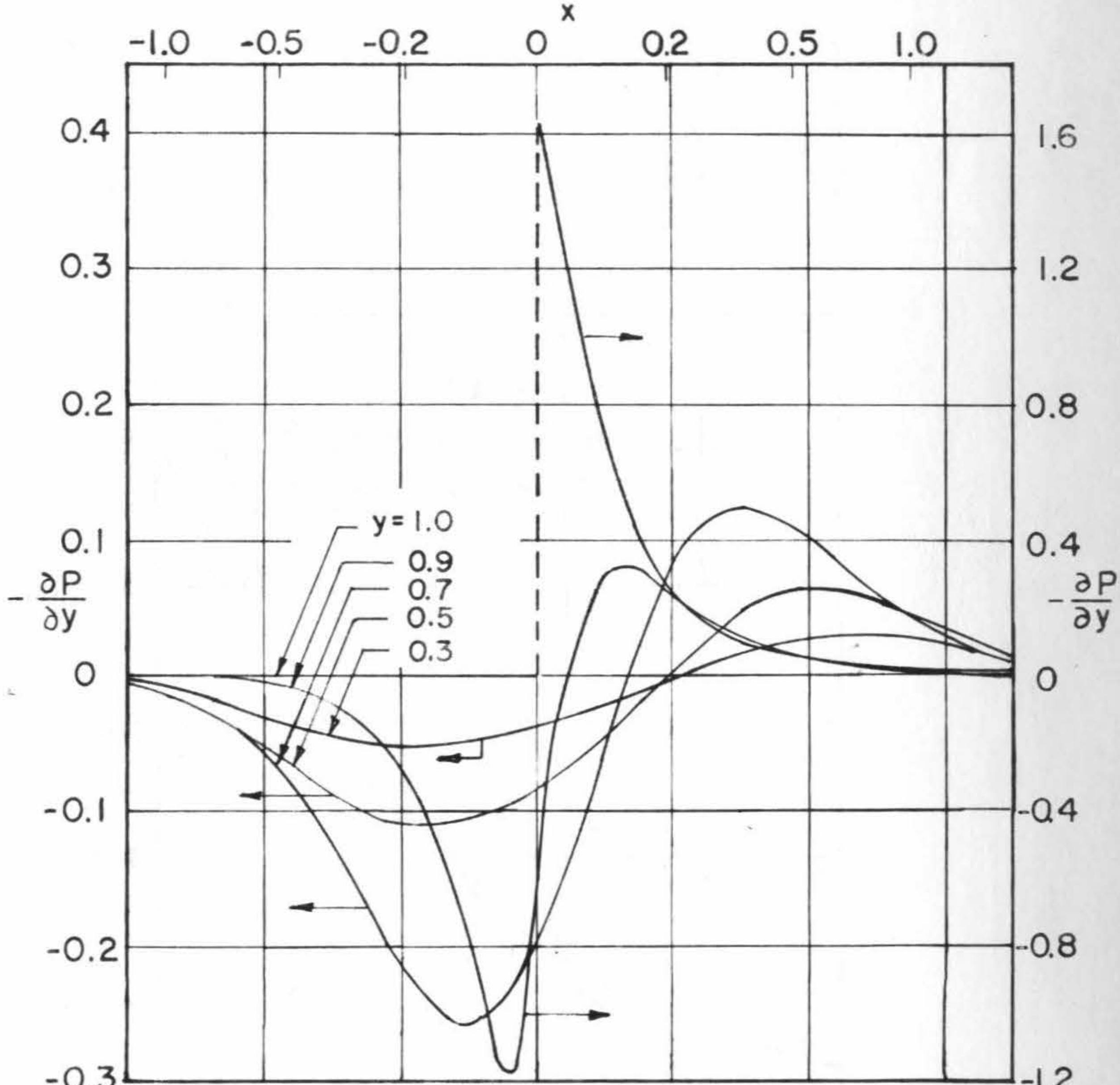

Figure 18. $\partial P/\partial y$ as function of η, Case II

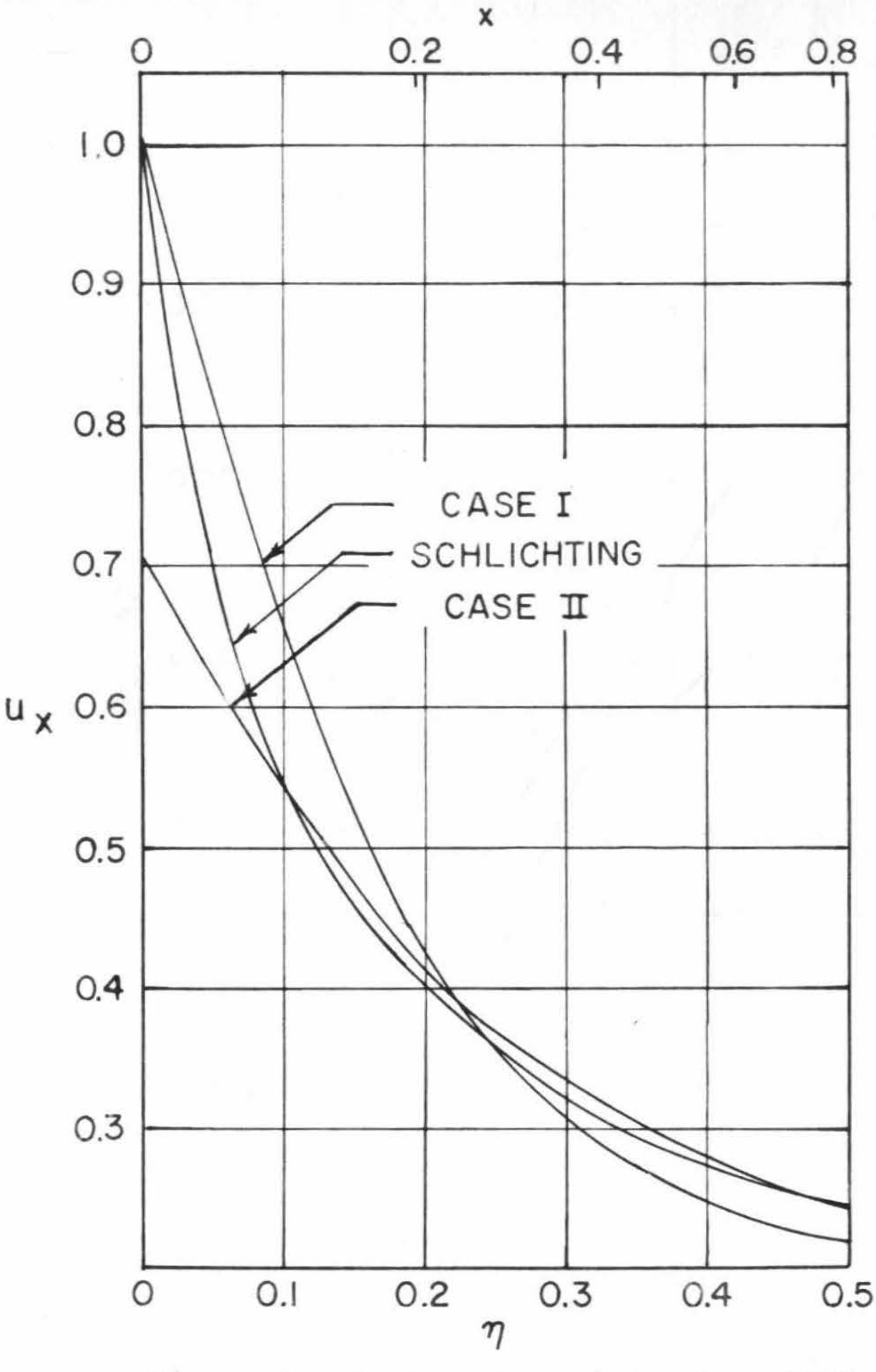

Figure 19. Comparison of u_x at $y = 0.95$

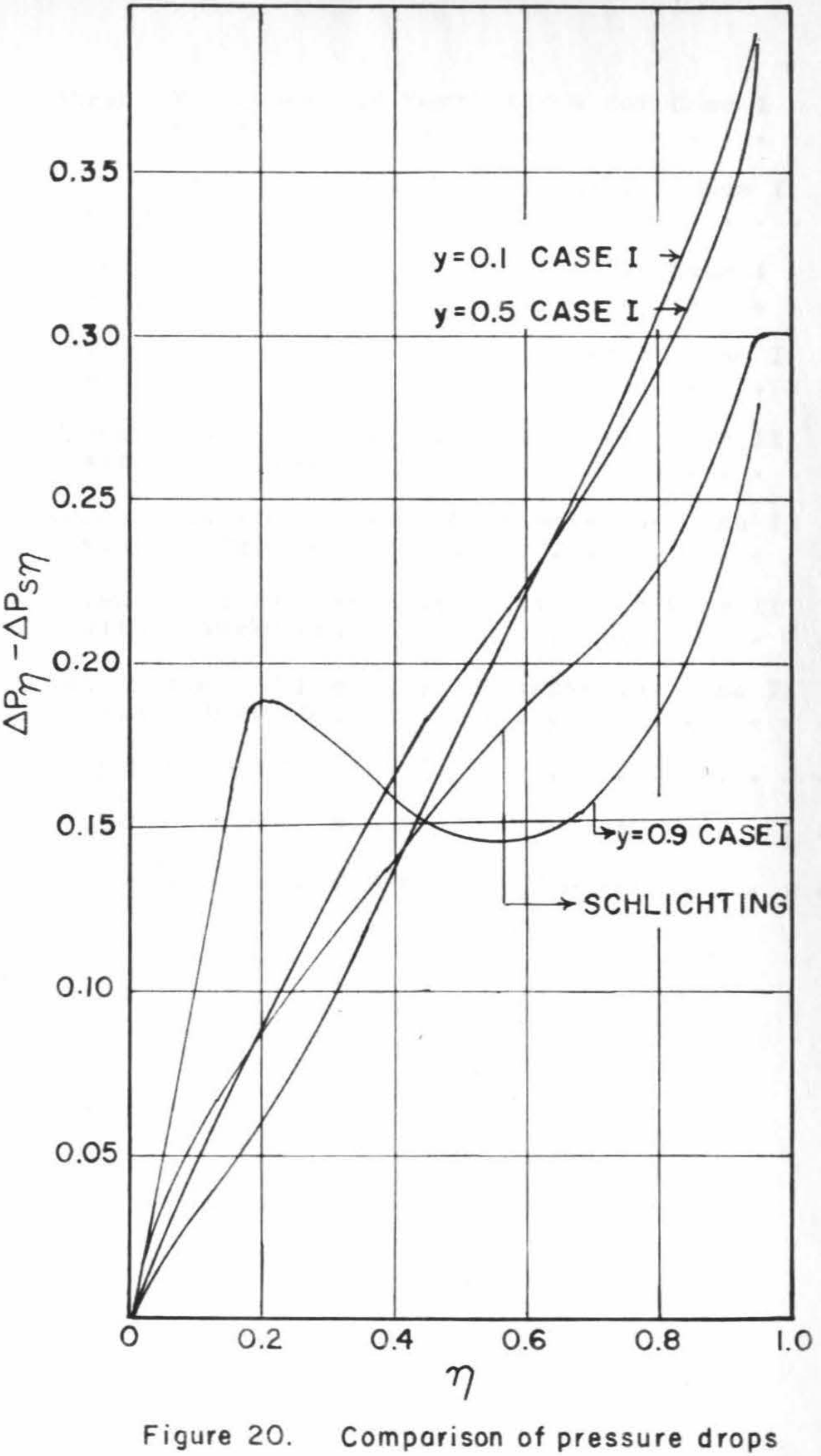

Figure 20. Comparison of pressure drops

LIST OF TABLES

TABLE I. Stream Functions And Vorticities For Case I With A 10 x 10 Grid

```
       ETA*= 0.          X=   0.                       ETA=  0.600000    X=   1.250000
                                                
   Y           PSI*          OMEGA*                       Y           PSI           OMEGA
1.000000      1.00000000    0.20000000E 02**    1.000000      1.00000000    0.40552479E 01
0.900000      0.90000000    0.                  0.900000      0.97972376    0.36134278E 01
0.800000      0.80000000    0.                  0.800000      0.92337727    0.28999195E 01
0.700000      0.70000000    0.                  0.700000      0.83827864    0.19823246E 01
0.600000      0.59999999    0.                  0.600000      0.73376811    0.11610050E 01
0.500000      0.50000000    0.                  0.500000      0.61812450    0.59531698E 00
0.400000      0.40000000    0.                  0.400000      0.49658094    0.27218769E-00
0.300000      0.30000000    0.                  0.300000      0.37349419    0.11237676E-00
0.200000      0.20000000    0.                  0.200000      0.24916802    0.42020328E-01
0.100000      0.09999999    0.                  0.100000      0.12460700    0.13298459E-01
0.            0.            0.                  0.            0.            0.

       ETA=  0.100000    X=   0.092593             ETA=  0.700000    X=   1.944444

   Y           PSI           OMEGA                 Y           PSI           OMEGA
1.000000      1.00000000    0.14672340E 02      1.000000      1.00000000    0.37682880E 01
0.900000      0.92663830    0.35342986E 01      0.900000      0.98115855    0.33046845E 01
0.800000      0.82673205    0.80653650E 00      0.800000      0.92928754    0.27436709E 01
0.700000      0.72264945    0.16930134E-00      0.700000      0.85004759    0.20694099E 01
0.600000      0.61842515    0.32220045E-01      0.600000      0.75023969    0.13962123E 01
0.500000      0.51464365    0.55474820E-02      0.500000      0.63663369    0.84646147E 00
0.400000      0.41127796    0.86522315E-03      0.400000      0.51473310    0.46601263E-00
0.300000      0.30822077    0.12236816E-03      0.300000      0.38832238    0.23494934E-00
0.200000      0.20537391    0.15687124E-04      0.200000      0.25967669    0.10780112E-00
0.100000      0.10265648    0.18019937E-05      0.100000      0.13002624    0.40715417E-01
0.            0.            0.                  0.            0.            0.

       ETA=  0.200000    X=   0.208333             ETA=  0.800000    X=   3.333333

   Y           PSI           OMEGA                 Y           PSI           OMEGA
1.000000      1.00000000    0.10105528E 02      1.000000      1.00000000    0.34637799E 01
0.900000      0.94947235    0.50567573E 01      0.900000      0.98268110    0.30447897E 01
0.800000      0.85553583    0.18110210E 01      0.800000      0.93491928    0.25919473E 01
0.700000      0.74913613    0.53990456E 00      0.700000      0.86125736    0.20956417E 01
0.600000      0.64056772    0.13903631E-00      0.600000      0.76667657    0.15869097E 01
0.500000      0.53240742    0.31427093E-01      0.500000      0.65627988    0.11197987E 01
0.400000      0.42501505    0.62992677E-02      0.400000      0.53474574    0.73650720E 00
0.300000      0.31825805    0.11271117E-02      0.300000      0.40590477    0.45037056E-00
0.200000      0.21194538    0.18062302E-03      0.200000      0.27260785    0.24979272E-00
0.100000      0.10590781    0.25578371E-04      0.100000      0.13684501    0.10961176E-00
0.            0.            0.                  0.            0.            0.

       ETA=  0.300000    X=   0.357143             ETA=  0.900000    X=   7.499999

   Y           PSI           OMEGA                 Y           PSI           OMEGA
1.000000      1.00000000    0.70028779E 01      1.000000      1.00000000    0.31406479E 01
0.900000      0.96458561    0.51894899E 01      0.900000      0.98429675    0.27987906E 01
0.800000      0.88167188    0.25791176E 01      0.800000      0.94060370    0.24496874E 01
0.700000      0.77698138    0.10144486E 01      0.700000      0.87241067    0.20917004E 01
0.600000      0.66560159    0.33664863E-00      0.600000      0.78329805    0.17303813E 01
0.500000      0.55319455    0.96765218E-01      0.500000      0.67688052    0.13774107E 01
0.400000      0.44136123    0.24437340E-01      0.400000      0.55668937    0.10457671E 01
0.300000      0.33030817    0.54713681E-02      0.300000      0.42604215    0.74422557E 00
0.200000      0.21987449    0.10915249E-02      0.200000      0.28795458    0.47266480E-00
0.100000      0.10984138    0.19041782E-03      0.100000      0.14512630    0.22999185E-00
0.            0.            0.                  0.            0.            0.

       ETA=  0.400000    X=   0.555556             ETA=  1.000000    X=   ∞

   Y           PSI           OMEGA                 Y           PSI           OMEGA
1.000000      1.00000000    0.52833839E 01      1.000000      1.00000000    0.29999999E 01
0.900000      0.97358308    0.46696924E 01      0.900000      0.98549999    0.26999999E 01
0.800000      0.90171482    0.29608935E 01      0.800000      0.94399998    0.23999999E 01
0.700000      0.80255619    0.14611612E-00      0.700000      0.87849998    0.20999999E 01
0.600000      0.69119231    0.60156233E 00      0.600000      0.79199998    0.17999999E 01
0.500000      0.57580429    0.21369444E-00      0.500000      0.68749999    0.14999999E 01
0.400000      0.45977595    0.66657608E-01      0.400000      0.56799999    0.12000000E 01
0.300000      0.34415592    0.18450596E-01      0.300000      0.43649999    0.89999999E 00
0.200000      0.22908986    0.45574936E-02      0.200000      0.29600000    0.59999999E 00
0.100000      0.11444001    0.97635676E-03      0.100000      0.14950000    0.30000000E-00
0.            0.            0.                  0.            0.            0.

       ETA=  0.500000    X=   0.833333

   Y           PSI           OMEGA
1.000000      1.00000000    0.44615699E 01
0.900000      0.97769215    0.40641070E 01
0.800000      0.91505247    0.30162832E 01
0.700000      0.82312000    0.17907541E 01
0.600000      0.71444833    0.88931057E 00
0.500000      0.59804489    0.38204153E-00
0.400000      0.47881439    0.14484055E-00
0.300000      0.35891349    0.49056937E-01
0.200000      0.23908675    0.14926474E-01
0.100000      0.11947533    0.39035829E-02
0.            0.            0.
```

*ETA = η, PSI = ψ, OMEGA = ω

** This is a floating -point expression. A floating-point expression 0. YYY E ZZ is equivalent to

$0.\text{YYY} \times 10^{\text{ZZ}}$

TABLE II. Velocities And Pressure Gradients For Case I With A 10 x 10 Grid

ETA = 0. X = 0.

Y	U(X)*	U(Y)*	- ∂P/∂η	- ∂P/∂X	- ∂P/∂Y
1.000000	0.09999999E 01	0.	0.	0.	0.
0.900000	0.09999999E 01	0.	0.	0.	0.
0.800000	0.09999999E 01	0.	0.	0.	0.
0.700000	0.09999999E 01	0.	0.	0.	0.
0.600000	0.09999999E 01	0.	0.	0.	0.
0.500000	0.09999999E 01	0.	0.	0.	0.
0.400000	0.09999999E 01	0.	0.	0.	0.
0.300000	0.09999999E 01	0.	0.	0.	0.
0.200000	0.09999999E 01	0.	0.	0.	0.
0.100000	0.09999999E 01	0.	0.	0.	0.
0.	0.09999999E 01	0.	0.	0.	0.

ETA = 0.100000 X = 0.092593

Y	U(X)	U(Y)	- ∂P/∂η	- ∂P/∂X	- ∂P/∂Y
1.000000	0.	0.	0.15278520E 01	0.14850721E 01	0.59185881E 00
0.900000	0.86633973E 00	-0.24043563E-00	0.98164010E 00	0.95415416E 00	0.23952106E-00
0.800000	0.10199442E 01	-0.26990413E-00	0.49872073E-00	0.48475654E-00	0.26700811E-00
0.700000	0.10415745E 01	-0.23880159E-00	0.48063836E-00	0.46718048E-00	0.19847048E-00
0.600000	0.10400789E 01	-0.19715914E-00	0.44478095E-00	0.43232708E-00	0.14393902E-00
0.500000	0.10357159E 01	-0.15750004E-00	0.39963251E-00	0.38844278E-00	0.10320868E-00
0.400000	0.10321143E 01	-0.12157319E-00	0.36187767E-00	0.35174509E-00	0.72330318E-01
0.300000	0.10295701E 01	-0.88734121E-01	0.33456933E-00	0.32520138E-00	0.48684673E-01
0.200000	0.10278714E 01	-0.58054545E-01	0.31658734E-00	0.30772289E-00	0.29998437E-01
0.100000	0.10268496E 01	-0.28712002E-01	0.30648056E-00	0.29789910E-00	0.14305925E-01
0.	0.10265648E 01	0.	0.30323825E-00	0.29474758E-00	0.

ETA = 0.200000 X = 0.208333

Y	U(X)	U(Y)	- ∂P/∂η	- ∂P/∂X	- ∂P/∂Y
1.000000	0.	0.	0.87652270E 00	0.67316942E 00	0.31771135E-00
0.900000	0.72232086E 00	-0.14725367E-00	0.71672498E 00	0.55044477E 00	0.34719498E-00
0.800000	0.10016811E 01	-0.21096890E-00	0.50557079E 00	0.38827836E-00	0.41995411E-00
0.700000	0.10748405E 01	-0.20863462E-00	0.48820481E-00	0.37494129E-00	0.32693389E-00
0.600000	0.10836435E 01	-0.18115753E-00	0.48749418E-00	0.37439552E-00	0.22677037E-00
0.500000	0.10777633E 01	-0.14803544E-00	0.46443924E-00	0.35668932E-00	0.15603158E-00
0.400000	0.10707468E 01	-0.11551978E-00	0.43551198E-00	0.33447319E-00	0.10737102E-00
0.300000	0.10653483E 01	-0.84815580E-01	0.41101611E-00	0.31566036E-00	0.71937365E-01
0.200000	0.10617511E 01	-0.55682185E-01	0.39373836E-00	0.30239105E-00	0.44328319E-01
0.100000	0.10597768E 01	-0.27589977E-01	0.38372691E-00	0.29470226E-00	0.21158174E-01
0.	0.10590781E 01	0.	0.38047245E-00	0.29220284E-00	0.

ETA = 0.300000 X = 0.357143

Y	U(X)	U(Y)	- ∂P/∂η	- ∂P/∂X	- ∂P/∂Y
1.000000	0.	0.	0.41119909E-00	0.24178506E-00	0.13480832E-00
0.900000	0.59164062E 00	-0.70885536E-01	0.31043108E-00	0.18253347E-00	0.20455660E-00
0.800000	0.94002113E 00	-0.13576622E-00	0.39671474E-00	0.23326826E-00	0.32401112E-00
0.700000	0.10803514E 01	-0.15705499E-00	0.41340135E-00	0.24307999E-00	0.42767376E-00
0.600000	0.11189942E 01	-0.14883630E-00	0.46008066E-00	0.27052742E-00	0.25940152E-00
0.500000	0.11212017E 01	-0.12758678E-00	0.48660612E-00	0.28612439E-00	0.19025703E-00
0.400000	0.11144318E 01	-0.10219704E-00	0.49037593E-00	0.28834105E-00	0.13461396E-00
0.300000	0.11074337E 01	-0.76139741E-01	0.48400564E-00	0.28459532E-00	0.91527694E-01
0.200000	0.11023339E 01	-0.50404772E-01	0.47612880E-00	0.27996373E-00	0.56918299E-01
0.100000	0.10993724E 01	-0.25084640E-01	0.47059061E-00	0.27670728E-00	0.27310854E-01
0.	0.10984138E 01	0.	0.46863697E-00	0.27555854E-00	0.

ETA = 0.400000 X = 0.555556

Y	U(X)	U(Y)	-∂P/∂η	-∂P/∂X	-∂P/∂Y
1.000000	0.	0.	0.18941096E-00	0.81825534E-01	0.47336485E-01
0.900000	0.49142592E-00	-0.27446126E-01	0.68073444E-01	0.29407727E-01	0.85573410E-01
0.800000	0.85513443E 00	-0.72102074E-01	0.26816816E-00	0.11584865E-00	0.16657075E-00
0.700000	0.10526124E 01	-0.99659404E-01	0.74332865E 00	0.32111797E-00	0.13811983E-00
0.600000	0.11337595E 01	-0.10550896E-00	-0.57719892E-02	-0.24934993E-02	0.20654835E-00
0.500000	0.11570817E 01	-0.96876752E-01	0.46365377E-00	0.20029842E-00	0.17161078E-00
0.400000	0.11582418E 01	-0.80898823E-01	0.50812765E 00	0.21951114E-00	0.13113661E-00
0.300000	0.11534304E 01	-0.61787488E-01	0.53294553E 00	0.23023246E-00	0.93450027E-01
0.200000	0.11485795E 01	-0.41498482E-01	0.54504298E 00	0.23545856E-00	0.59789982E-01
0.100000	0.11454493E 01	-0.20809331E-01	0.55026126E 00	0.23771286E-00	0.29138012E-01
0.	0.11444000E 01	0.	0.55155600E 00	0.23827218E-00	0.

ETA = 0.500000 X = 0.833333

Y	U(X)	U(Y)	- ∂P/∂η	- ∂P/∂X	- ∂P/∂Y
1.000000	0.	0.	0.17665018E-00	0.52995054E-01	0.16252880E-01
0.900000	0.42473767E-00	-0.92110185E-02	0.70461104E-01	0.21138331E-01	0.29166768E-01
0.800000	0.77286076E 00	-0.32493676E-01	0.21141163E-00	0.63423487E-01	0.62449200E-01
0.700000	0.10030706E 01	-0.53583662E-01	0.31924910E-00	0.95774730E-01	0.97220629E-01
0.600000	0.11253755E 01	-0.63863697E-01	0.37961054E-00	0.11388316E-00	0.11208452E-00
0.500000	0.11781696E 01	-0.63480320E-01	0.44346708E-00	0.13304012E-00	0.10668900E-00
0.400000	0.11956570E 01	-0.55807470E-01	0.50594276E 00	0.15178283E-00	0.89794100E-01
0.300000	0.11986782E 01	-0.44007404E-01	0.55470403E 00	0.16641121E-00	0.68326776E-01
0.200000	0.11971907E 01	-0.30117239E-01	0.58679916E 00	0.17603975E-00	0.45604748E-01
0.100000	0.11954737E 01	-0.15250500E-01	0.60425980E 00	0.18127794E-00	0.22761148E-01
0.	0.11947533E 01	0.	0.60908791E 00	0.18272637E-00	0.

TABLE II. - Continued

ETA = 0.600000 X = 1.250000

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.000000	0.	0.	0.30681950E-00	0.58909345E-01	0.73461754E-02
0.900000	0.38311966E-00	-0.33277487E-02	0.23299527E-00	0.44735093E-01	0.97432472E-02
0.800000	0.70722561E 00	-0.13665669E-01	0.28532755E-00	0.54782891E-01	0.18034111E-01
0.700000	0.94804578E 00	-0.25850486E-01	0.38766764E-00	0.74432188E-01	0.30469544E-01
0.600000	0.11007706E 01	-0.34359705E-01	0.44868717E-00	0.86147939E-01	0.39922377E-01
0.500000	0.11839859E 01	-0.37045248E-01	0.49518789E-00	0.95076077E-01	0.42506883E-01
0.400000	0.12231516E 01	-0.34481953E-01	0.54718790E 00	0.10506008E-00	0.38915512E-01
0.300000	0.12390646E 01	-0.28232535E-01	0.59815583E 00	0.11484592E-00	0.31391636E-01
0.200000	0.12444958E 01	-0.19766352E-01	0.63796462E 00	0.12248921E-00	0.21758527E-01
0.100000	0.12458400E 01	-0.10128883E-01	0.66148301E 00	0.12700474E-00	0.11092729E-01
0.	0.12460700E 01	0.	0.66659432E 00	0.12798611E-00	0.

ETA = 0.700000 X = 1.944444

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.000000	0.	0.	0.57234998E 00	0.61813799E-01	0.43849152E-02
0.900000	0.35356731E-00	-0.15969618E-02	0.51916923E 00	0.56070278E-01	0.40630631E-02
0.800000	0.65555483E 00	-0.62326880E-02	0.52242777E 00	0.56422201E-01	0.50026900E-02
0.700000	0.89523925E 00	-0.12408510E-01	0.59053167E 00	0.63777421E-01	0.74885029E-02
0.600000	0.10670494E 01	-0.17770565E-01	0.65328162E 00	0.70554417E-01	0.10221914E-01
0.500000	0.11775329E 01	-0.20603903E-01	0.69398379E 00	0.74950252E-01	0.11700635E-01
0.400000	0.12415566E 01	-0.20392992E-01	0.73316403E 00	0.79181717E-01	0.11377387E-01
0.300000	0.12752820E 01	-0.17501713E-01	0.77819087E 00	0.84044615E-01	0.95481395E-02
0.200000	0.12914906E 01	-0.12657507E-01	0.82006102E 00	0.88566592E-01	0.67653418E-02
0.100000	0.12983835E 01	-0.66085248E-02	0.84596851E 00	0.91364601E-01	0.34845205E-02
0.	0.13002624E 01	0.	0.84589714E 00	0.91356893E-01	0.

ETA = 0.800000 X = 3.333333

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.000000	0.	0.	0.11638619E 01	0.55865375E-01	0.20680448E-02
0.900000	0.32540859E-00	-0.75316787E-03	0.11195718E 01	0.53739452E-01	0.16748254E-02
0.800000	0.60711968E 00	-0.27158796E-02	0.11126456E 01	0.53406988E-01	0.16318934E-02
0.700000	0.84121858E 00	-0.53671396E-02	0.11552861E 01	0.55453739E-01	0.20632923E-02
0.600000	0.10248973E 01	-0.79340046E-02	0.12163477E 01	0.58384690E-01	0.27710251E-02
0.500000	0.11596541E 01	-0.96592373E-02	0.12636911E 01	0.60657175E-01	0.33697904E-02
0.400000	0.12518755E 01	-0.10069506E-01	0.12970714E 01	0.62259431E-01	0.35450787E-02
0.300000	0.13106894E 01	-0.90527442E-02	0.13270660E 01	0.63699172E-01	0.31917195E-02
0.200000	0.13452987E 01	-0.67866906E-02	0.13527828E 01	0.64933576E-01	0.23915698E-02
0.100000	0.13630792E 01	-0.36240121E-02	0.13611584E 01	0.65335608E-01	0.12642913E-02
0.	0.13684501E 01	0.	0.13376604E 01	0.64207704E-01	0.

ETA = 0.900000 X = 7.499999

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.000000	0.	0.	0.37984149E 01	0.45580987E-01	0.22503684E-03
0.900000	0.29698148E-00	-0.16913342E-03	0.37704450E 01	0.45245348E-01	0.17142309E-03
0.800000	0.55943042E 00	-0.54484212E-03	0.37884341E 01	0.45461217E-01	0.16872752E-03
0.700000	0.78652829E 00	-0.10345577E-02	0.38386030E 01	0.46063244E-01	0.23432465E-03
0.600000	0.97765077E 00	-0.15194050E-02	0.38921390E 01	0.46705677E-01	0.35023897E-03
0.500000	0.11330433E 01	-0.18732072E-02	0.39122510E 01	0.46947020E-01	0.47163341E-03
0.400000	0.12541918E 01	-0.19952554E-02	0.38783763E 01	0.46540524E-01	0.54622176E-03
0.300000	0.13436739E 01	-0.18357138E-02	0.37955675E 01	0.45546819E-01	0.53546468E-03
0.200000	0.14045792E 01	-0.14035290E-02	0.37027513E 01	0.44433024E-01	0.42462173E-03
0.100000	0.14397728E 01	-0.75929885E-03	0.35528908E 01	0.42634697E-01	0.23670264E-03
0.	0.14512830E 01	0.	0.34737496E 01	0.41685002E-01	0.

ETA = 1.000000 X = ∞

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.000000	0.	0.	∞	0.39999999E-01	0.
0.900000	0.28500003E-00	0.	∞	0.39999999E-01	0.
0.800000	0.54000001E 00	0.	∞	0.39999999E-01	0.
0.700000	0.76500001E 00	0.	∞	0.39999999E-01	0.
0.600000	0.96000001E 00	0.	∞	0.39999999E-01	0.
0.500000	0.11250000E 01	0.	∞	0.39999999E-01	0.
0.400000	0.12599999E 01	0.	∞	0.39999999E-01	0.
0.300000	0.13649999E 01	0.	∞	0.39999999E-01	0.
0.200000	0.14399999E 01	0.	∞	0.39999999E-01	0.
0.100000	0.14850000E 01	0.	∞	0.39999999E-01	0.
0.	0.15000000E 01	0.	∞	0.39999999E-01	0.

---

* $U(X) = u_x$, $U(Y) = u_y$.

TABLE III. Stream Functions And Vorticities For Case I With A 20 x 20 Grid

```
      ETA=   .000000       X=    .000000
  Y            PSI                OMEGA

1.00000     .10000000, 01      .40000000, 02
.950000     .95000000, 00      .00000000, 00
.900000     .90000000, 00      .00000000, 00
.850000     .85000000, 00      .00000000, 00
.800000     .80000000, 00      .00000000, 00
.750000     .75000000, 00      .00000000, 00
.700000     .70000000, 00      .00000000, 00
.650000     .65000000, 00      .00000000, 00
.600000     .60000000, 00      .00000000, 00
.550000     .55000000, 00      .00000000, 00
.500000     .50000000, 00      .00000000, 00
.450000     .45000000, 00      .00000000, 00
.400000     .40000000, 00      .00000000, 00
.350000     .35000000, 00      .00000000, 00
.300000     .30000000, 00      .00000000, 00
.250000     .25000000, 00      .00000000, 00
.200000     .20000000, 00      .00000000, 00
.150000     .15000000, 00      .00000000, 00
.100000     .10000000, 00      .00000000, 00
.050000     .50000000,-01      .00000000, 00
.000000     .00000000, 00      .00000000, 00

      ETA=   .050000       X=    .043859
  Y            PSI                OMEGA

1.00000     .10000000, 01      .27544000, 02
.950000     .96557008, 00      .80517251, 01
.900000     .91719414, 00      .23279126, 01
.850000     .86599006, 00      .65905389, 00
.800000     .81433723, 00      .18140153, 00
.750000     .76275421, 00      .48537867,-01
.700000     .71133519, 00      .12657277,-01
.650000     .66007542, 00      .32259221,-02
.600000     .60895026, 00      .80546338,-03
.550000     .55793456, 00      .19736216,-03
.500000     .50700704, 00      .47512031,-04
.450000     .45615042, 00      .11244870,-04
.400000     .40535091, 00      .26172576,-05
.350000     .35459735, 00      .59908669,-06
.300000     .30388068, 00      .13483572,-06
.250000     .25319335, 00      .29829500,-07
.200000     .20252902, 00      .64835473,-08
.150000     .15188219, 00      .13836860,-08
.100000     .10124799, 00      .28935572,-09
.050000     .50621981,-01      .57157810,-10
.000000     .00000000, 00      .00000000, 00

      ETA=   .100000       X=    .092592
  Y            PSI                OMEGA

1.00000     .10000000, 01      .18155600, 02
.950000     .97730552, 00      .10459326, 02
.900000     .93360654, 00      .45850328, 01
.850000     .88267796, 00      .17520172, 01
.800000     .82979661, 00      .61065830, 00
.750000     .77666896, 00      .19823836, 00
.700000     .72375814, 00      .60731647,-01
.650000     .67114701, 00      .17725270,-01
.600000     .61879471, 00      .49634820,-02
.550000     .56667105, 00      .13405967,-02
.500000     .51472880, 00      .35064140,-03
.450000     .46293292, 00      .89082197,-04
.400000     .41125508, 00      .22032608,-04
.350000     .35967239, 00      .53139768,-05
.300000     .30816615, 00      .12513596,-05
.250000     .25672086, 00      .28795089,-06
.200000     .20532333, 00      .64782089,-07
.150000     .15396217, 00      .14251988,-07
.100000     .10262728, 00      .30614992,-08
.050000     .51309447,-01      .61852846,-09
.000000     .00000000, 00      .00000000, 00

      ETA=   .150000       X=    .147058
  Y            PSI                OMEGA

1.00000     .10000000, 01      .12274400, 02
.950000     .98465707, 00      .10004342, 02
.900000     .94720616, 00      .59058989, 01
.850000     .89860901, 00      .28649233, 01
.800000     .84564009, 00      .12212080, 01
.750000     .79142245, 00      .47249953, 00
.700000     .73714243, 00      .16910072, 00
.650000     .68315981, 00      .56703770,-01
.600000     .62953296, 00      .17985422,-01
.550000     .57622847, 00      .54349555,-02
.500000     .52319421, 00      .15733893,-02
.450000     .47038097, 00      .43822353,-03
.400000     .41774724, 00      .11781956,-03
.350000     .36525884, 00      .30656853,-04
.300000     .31288759, 00      .77357455,-05
.250000     .26060988, 00      .18958946,-05
.200000     .20840564, 00      .45182218,-06
.150000     .15625744, 00      .10478284,-06
.100000     .10414976, 00      .23618912,-07
.050000     .52068401,-01      .49730806,-08
.000000     .00000000, 00      .00000000, 00

      ETA=   .200000       X=    .208333
  Y            PSI                OMEGA

1.00000     .10000000, 01      .89404000, 01
.950000     .98882458, 00      .86373032, 01
.900000     .95739136, 00      .63123103, 01
.850000     .91258725, 00      .37122648, 01
.800000     .86091389, 00      .18773225, 01
.750000     .80643890, 00      .84814485, 00
.700000     .75118214, 00      .35006417, 00
.650000     .69597686, 00      .13396187, 00
.600000     .64109457, 00      .48042089,-01
.550000     .58657970, 00      .16278505,-01
.500000     .53240006, 00      .52446532,-02
.450000     .47850502, 00      .16147035,-02
.400000     .42484532, 00      .47692572,-03
.350000     .37137812, 00      .13556398,-03
.300000     .31806704, 00      .37174654,-04
.250000     .26488121, 00      .98538586,-05
.200000     .21179406, 00      .25286127,-05
.150000     .15878238, 00      .62883041,-06
.100000     .10582537, 00      .15135279,-06
.050000     .52903926,-01      .33748936,-07
.000000     .00000000, 00      .00000000, 00

      ETA=   .250000       X=    .277777
  Y            PSI                OMEGA

1.00000     .10000000, 01      .70935200, 01
.950000     .99113312, 00      .72972557, 01
.900000     .96457676, 00      .61350492, 01
.850000     .92406866, 00      .42015019, 01
.800000     .87480355, 00      .24550939, 01
.750000     .82103709, 00      .12703490, 01
.700000     .76541841, 00      .59613245, 00
.650000     .70931347, 00      .25774887, 00
.600000     .65331629, 00      .10384724, 00
.550000     .59763167, 00      .39324545,-01
.500000     .54229518, 00      .14090396,-01
.450000     .48727954, 00      .48028059,-02
.400000     .43253987, 00      .15639963,-02
.350000     .37803064, 00      .48824139,-03
.300000     .32371063, 00      .14651679,-03
.250000     .26954359, 00      .42359581,-04
.200000     .21549782, 00      .11819141,-04
.150000     .16154515, 00      .31865263,-05
.100000     .10766012, 00      .82866429,-06
.050000     .53819189,-01      .19783032,-06
.000000     .00000000, 00      .00000000, 00

      ETA=   .300000       X=    .357142
  Y            PSI                OMEGA

1.00000     .10000000, 01      .60243200, 01
.950000     .99246969, 00      .62549998, 01
.900000     .96952993, 00      .56969552, 01
.850000     .93308315, 00      .43734554, 01
.800000     .88682977, 00      .28819117, 01
.750000     .83460842, 00      .16778692, 01
.700000     .77932758, 00      .88290476, 00
.650000     .72278719, 00      .42667377, 00
.600000     .66593865, 00      .19155986, 00
.550000     .60921296, 00      .80596378,-01
.500000     .55276688, 00      .31995634,-01
.450000     .49663050, 00      .12050453,-01
.400000     .44078270, 00      .43248818,-02
.350000     .38518588, 00      .14844430,-02
.300000     .32979989, 00      .48869680,-03
.250000     .27458669, 00      .15467915,-03
.200000     .21951173, 00      .47159153,-04
.150000     .16454358, 00      .13867463,-04
.100000     .10965336, 00      .39228789,-05
.050000     .54814095,-01      .10086449,-05
.000000     .00000000, 00      .00000000, 00

      ETA=   .350000       X=    .448717
  Y            PSI                OMEGA

1.00000     .10000000, 01      .53584800, 01
.950000     .99330196, 00      .54993974, 01
.900000     .97295515, 00      .52044364, 01
.850000     .93997467, 00      .43254012, 01
.800000     .89685447, 00      .31394761, 01
.750000     .84673236, 00      .20225007, 01
.700000     .79242547, 00      .11779462, 01
.650000     .73597181, 00      .62956356, 00
.600000     .67862955, 00      .31226245, 00
.550000     .62107803, 00      .14497658, 00
.500000     .56363652, 00      .63431698,-01
.450000     .50642771, 00      .26296924,-01
.400000     .44947850, 00      .10375759,-01
.350000     .39277403, 00      .39105099,-02
.300000     .33628395, 00      .14120461,-02
.250000     .27997397, 00      .48971010,-03
.200000     .22381021, 00      .16344510,-03
.150000     .16776052, 00      .52563787,-04
.100000     .11179460, 00      .16226378,-04
.050000     .55883682,-01      .45031049,-05
.000000     .00000000, 00      .00000000, 00
```

TABLE III. - Continued

```
     ETA=   .400000       X=     .555555
   Y            PSI                OMEGA

1.00000      .10000000, 01      .49172000, 01
.950000      .99385350, 00      .49567356, 01
.900000      .97536144, 00      .47544531, 01
.850000      .94517536, 00      .41559348, 01
.800000      .90498205, 00      .32484572, 01
.750000      .85720480, 00      .22780562, 01
.700000      .80434368, 00      .14503448, 01
.650000      .74846635, 00      .84865854, 00
.600000      .69102831, 00      .46113139, 00
.550000      .63293027, 00      .23459066, 00
.500000      .57466922, 00      .11246944, 00
.450000      .51648736, 00      .51085794,-01
.400000      .45848335, 00      .22080137,-01
.350000      .40068222, 00      .91140522,-02
.300000      .34307485, 00      .36036322,-02
.250000      .28563783, 00      .13683018,-02
.200000      .22834273, 00      .49994561,-03
.150000      .17116010, 00      .17596567,-03
.100000      .11406088, 00      .59327437,-04
.050000      .57016752,-01      .17764007,-04
.000000      .00000000, 00      .00000000, 00

     ETA=   .450000       X=     .681818
   Y            PSI                OMEGA

1.00000      .10000000, 01      .46122400, 01
.950000      .99423472, 00      .45631358, 01
.900000      .97708375, 00      .43778040, 01
.850000      .94908528, 00      .39389542, 01
.800000      .91144818, 00      .32484856, 01
.750000      .86600947, 00      .24404041, 01
.700000      .81486577, 00      .16777188, 01
.650000      .75995439, 00      .10643887, 01
.600000      .70280082, 00      .62848997, 00
.550000      .64446353, 00      .34795580, 00
.500000      .58560269, 00      .18173801, 00
.450000      .52659096, 00      .90005341,-01
.400000      .46761696, 00      .42444996,-01
.350000      .40876255, 00      .19127608,-01
.300000      .35005223, 00      .82617255,-02
.250000      .29148200, 00      .34288894,-02
.200000      .23303472, 00      .13702156,-02
.150000      .17468775, 00      .52757444,-03
.100000      .11641645, 00      .19414134,-03
.050000      .58195624,-01      .62591374,-04
.000000      .00000000, 00      .00000000, 00

     ETA=   .500000       X=     .833333
   Y            PSI                OMEGA

1.00000      .10000000, 01      .43940800, 01
.950000      .99450747, 00      .42734551, 01
.900000      .97834200, 00      .40750911, 01
.850000      .95203674, 00      .37196506, 01
.800000      .91654139, 00      .31816263, 01
.750000      .87327061, 00      .25218479, 01
.700000      .82393298, 00      .18494812, 01
.650000      .77024559, 00      .12600220, 01
.600000      .71369383, 00      .80222087, 00
.550000      .65541411, 00      .48020607, 00
.500000      .59619167, 00      .27173458, 00
.450000      .53652236, 00      .14604463, 00
.400000      .47669364, 00      .74849630,-01
.350000      .41685721, 00      .36706498,-01
.300000      .35708346, 00      .17275006,-01
.250000      .29739771, 00      .78216421,-02
.200000      .23779925, 00      .34136450,-02
.150000      .17827844, 00      .14361629,-02
.100000      .11881800, 00      .57583586,-03
.050000      .59398639,-01      .19925778,-03
.000000      .00000000, 00      .00000000, 00

     ETA=   .550000       X=     1.01851
   Y            PSI                OMEGA

1.00000      .10000000, 01      .42304000, 01
.950000      .99471207, 00      .40556414, 01
.900000      .97928991, 00      .38359295, 01
.850000      .95430062, 00      .35208273, 01
.800000      .92056304, 00      .30827574, 01
.750000      .87921184, 00      .25433710, 01
.700000      .83163392, 00      .19664977, 01
.650000      .77930083, 00      .14249739, 01
.600000      .72358243, 00      .97071559, 00
.550000      .66561758, 00      .62431067, 00
.500000      .60626562, 00      .38075480, 00
.450000      .54612291, 00      .22110817, 00
.400000      .48557271, 00      .12270626, 00
.350000      .42484425, 00      .65287155,-01
.300000      .36406417, 00      .33397006,-01
.250000      .30329569, 00      .16464383,-01
.200000      .24256498, 00      .78355556,-02
.150000      .18187770, 00      .35961627,-02
.100000      .12122865, 00      .15669649,-02
.050000      .60607179,-01      .57956825,-03
.000000      .00000000, 00      .00000000, 00

     ETA=   .600000       X=     1.25000
   Y            PSI                OMEGA

1.00000      .10000000, 01      .40971200, 01
.950000      .99487866, 00      .38851808, 01
.900000      .98004658, 00      .36465982, 01
.850000      .95610808, 00      .33506783, 01
.800000      .92381778, 00      .29761850, 01
.750000      .88413215, 00      .25275685, 01
.700000      .83819457, 00      .20380900, 01
.650000      .78724786, 00      .15560937, 01
.600000      .73251050, 00      .11254749, 01
.550000      .67506514, 00      .77287833, 00
.500000      .61579325, 00      .50546729, 00
.450000      .55535815, 00      .31585705, 00
.400000      .49422549, 00      .18917352, 00
.350000      .43270225, 00      .10890633, 00
.300000      .37097863, 00      .60421171,-01
.250000      .30916534, 00      .32374837,-01
.200000      .24732210, 00      .16773198,-01
.150000      .18547762, 00      .83801540,-02
.100000      .12364268, 00      .39542798,-02
.050000      .61818201,-01      .15556485,-02
.000000      .00000000, 00      .00000000, 00

     ETA=   .650000       X=     1.54761
   Y            PSI                OMEGA

1.00000      .10000000, 01      .39739200, 01
.950000      .99503263, 00      .37413272, 01
.900000      .98071294, 00      .34921272, 01
.850000      .95766721, 00      .32082061, 01
.800000      .92661201, 00      .28757293, 01
.750000      .88838787, 00      .24935945, 01
.700000      .84396148, 00      .20776778, 01
.650000      .79438354, 00      .16567639, 01
.600000      .74071498, 00      .12625466, 01
.550000      .68394675, 00      .91990101, 00
.500000      .62493721, 00      .64187059, 00
.450000      .56437988, 00      .42985531, 00
.400000      .50280104, 00      .27695339, 00
.350000      .44057787, 00      .17207978, 00
.300000      .37796637, 00      .10333260, 00
.250000      .31513219, 00      .60071279,-01
.200000      .25217728, 00      .33816150,-01
.150000      .18916118, 00      .18340134,-01
.100000      .12611662, 00      .93291149,-02
.050000      .63060285,-01      .38820049,-02
.000000      .00000000, 00      .00000000, 00

     ETA=   .700000       X=     1.94444
   Y            PSI                OMEGA

1.00000      .10000000, 01      .38444000, 01
.950000      .99519452, 00      .36063577, 01
.900000      .98137368, 00      .33572649, 01
.850000      .95916181, 00      .30865162, 01
.800000      .92923893, 00      .27860559, 01
.750000      .89236053, 00      .24541685, 01
.700000      .84936186, 00      .20982343, 01
.650000      .80113832, 00      .17341607, 01
.600000      .74860514, 00      .13822699, 01
.550000      .69264558, 00      .10617332, 01
.500000      .63406279, 00      .78616648, 00
.450000      .57354576, 00      .56179434, 00
.400000      .51165404, 00      .38803490, 00
.350000      .44881940, 00      .25949101, 00
.300000      .38535980, 00      .16826682, 00
.250000      .32149976, 00      .10589587, 00
.200000      .25739160, 00      .64597225,-01
.150000      .19313501, 00      .37879045,-01
.100000      .12879334, 00      .20651394,-01
.050000      .64406388,-01      .90330761,-02
.000000      .00000000, 00      .00000000, 00

     ETA=   .750000       X=     2.50000
   Y            PSI                OMEGA

1.00000      .10000000, 01      .36979200, 01
.950000      .99537767, 00      .34668420, 01
.900000      .98208872, 00      .32281657, 01
.850000      .96073081, 00      .29755794, 01
.800000      .93193702, 00      .27046860, 01
.750000      .89638689, 00      .24144733, 01
.700000      .85480872, 00      .21086028, 01
.650000      .80797009, 00      .17955852, 01
.600000      .75665633, 00      .14873840, 01
.550000      .70164019, 00      .11968660, 01
.500000      .64364932, 00      .93511155, 00
.450000      .58333870, 00      .70952649, 00
.400000      .52127183, 00      .52317579, 00
.350000      .45791359, 00      .37519729, 00
.300000      .39363212, 00      .26185693, 00
.250000      .32870880, 00      .17777670, 00
.200000      .26335166, 00      .11700153, 00
.150000      .19771044, 00      .73743022,-01
.100000      .13189110, 00      .42774749,-01
.050000      .65968944,-01      .19532222,-01
.000000      .00000000, 00      .00000000, 00
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TABLE III. - Continued

```
        ETA=    .800000          X=     3.33333
    Y               PSI                 OMEGA

 1.00000        .10000000, 01       .35319200, 01
 .950000        .99558512, 00       .33162577, 01
 .900000        .98288001, 00       .30955960, 01
 .850000        .96243687, 00       .28662005, 01
 .800000        .93483005, 00       .26257260, 01
 .750000        .90066212, 00       .23737390, 01
 .700000        .86056463, 00       .21121982, 01
 .650000        .81519297, 00       .18455956, 01
 .600000        .76521516, 00       .15805298, 01
 .550000        .71129512, 00       .13247232, 01
 .500000        .65407341, 00       .10857464, 01
 .450000        .59414810, 00       .86982724, 00
 .400000        .53205907, 00       .68105315, 00
 .350000        .46827797, 00       .52108759, 00
 .300000        .40320388, 00       .38934348, 00
 .250000        .33716516, 00       .28344921, 00
 .200000        .27042531, 00       .19982204, 00
 .150000        .20319201, 00       .13420109, 00
 .100000        .13562784, 00       .82052293,-01
 .050000        .67861656,-01       .38814580,-01
 .000000        .00000000, 00       .00000000, 00
```

```
        ETA=    .850000          X=     4.72222
    Y               PSI                 OMEGA

 1.00000        .10000000, 01       .33518400, 01
 .950000        .99581020, 00       .31562397, 01
 .900000        .98373003, 00       .29578234, 01
 .850000        .96425588, 00       .27545898, 01
 .800000        .93789632, 00       .25453650, 01
 .750000        .90517502, 00       .23299839, 01
 .700000        .86663125, 00       .21094476, 01
 .650000        .82281701, 00       .18859750, 01
 .600000        .77429179, 00       .16628755, 01
 .550000        .72161409, 00       .14442176, 01
 .500000        .66533114, 00       .12343382, 01
 .450000        .60596799, 00       .10372911, 01
 .400000        .54401746, 00       .85635214, 00
 .350000        .47993196, 00       .69367022, 00
 .300000        .41411781, 00       .55010620, 00
 .250000        .34693353, 00       .42524536, 00
 .200000        .27869067, 00       .31754072, 00
 .150000        .20965777, 00       .22452739, 00
 .100000        .14006651, 00       .14305672, 00
 .050000        .70119632,-01       .69513581,-01
 .000000        .00000000, 00       .00000000, 00
```

```
        ETA=    .900000          X=     7.50000
    Y               PSI                 OMEGA

 1.00000        .10000000, 01       .31717600, 01
 .950000        .99603534, 00       .29981777, 01
 .900000        .98457547, 00       .28234328, 01
 .850000        .96605728, 00       .26467340, 01
 .800000        .94092283, 00       .24676461, 01
 .750000        .90961988, 00       .22861419, 01
 .700000        .87260239, 00       .21026431, 01
 .650000        .83032950, 00       .19180335, 01
 .600000        .78326293, 00       .17336246, 01
 .550000        .73186398, 00       .15510631, 01
 .500000        .67658924, 00       .13721852, 01
 .450000        .61788599, 00       .11988328, 01
 .400000        .55618780, 00       .10326576, 01
 .350000        .49191007, 00       .87494334, 00
 .300000        .42544693, 00       .72646703, 00
 .250000        .35716947, 00       .58741669, 00
 .200000        .28742513, 00       .45736704, 00
 .150000        .21653881, 00       .33530928, 00
 .100000        .14481533, 00       .21972467, 00
 .050000        .72543326,-01       .10868937, 00
 .000000        .00000000, 00       .00000000, 00
```

```
        ETA=    .950000          X=     15.8333
    Y               PSI                 OMEGA

 1.00000        .10000000, 01       .30331200, 01
 .950000        .99620868, 00       .28780387, 01
 .900000        .98522250, 00       .27227709, 01
 .850000        .96742954, 00       .25671686, 01
 .800000        .94321891, 00       .24111608, 01
 .750000        .91298058, 00       .22547542, 01
 .700000        .87710560, 00       .20980366, 01
 .650000        .83598572, 00       .19411757, 01
 .600000        .79001316, 00       .17844128, 01
 .550000        .73957979, 00       .16280479, 01
 .500000        .68507663, 00       .14724188, 01
 .450000        .62689273, 00       .13178721, 01
 .400000        .56541443, 00       .11647301, 01
 .350000        .50102454, 00       .10132584, 01
 .300000        .43410172, 00       .86363523, 00
 .250000        .36501998, 00       .71592864, 00
 .200000        .29414858, 00       .57008226, 00
 .150000        .22185210, 00       .42591347, 00
 .100000        .14849095, 00       .28312143, 00
 .050000        .74422087,-01       .14130711, 00
 .000000        .00000000, 00       .00000000, 00
```

```
        ETA=    1.00000          X=     ∞
    Y               PSI                 OMEGA

 1.00000        .10000000, 01       .30000000, 01
 .950000        .99631250, 00       .28500000, 01
 .900000        .98550000, 00       .27000000, 01
 .850000        .96793750, 00       .25500000, 01
 .800000        .94400000, 00       .24000000, 01
 .750000        .91406250, 00       .22500000, 01
 .700000        .87850000, 00       .21000000, 01
 .650000        .83768750, 00       .19500000, 01
 .600000        .79200000, 00       .18000000, 01
 .550000        .74181250, 00       .16500000, 01
 .500000        .68750000, 00       .15000000, 01
 .450000        .62943750, 00       .13500000, 01
 .400000        .56800000, 00       .12000000, 01
 .350000        .50356250, 00       .10500000, 01
 .300000        .43650000, 00       .90000000, 00
 .250000        .36718750, 00       .75000000, 00
 .200000        .29600000, 00       .60000000, 00
 .150000        .22331250, 00       .45000000, 00
 .100000        .14950000, 00       .30000000, 00
 .050000        .74937500,-01       .15000000, 00
 .000000        .00000000, 00       .00000000, 00
```

* This is again a floating-point expression. A floating-point expression 0. YYY , ZZ is equivalent to $0.YYY \times 10^{ZZ}$

TABLE IV. Velocities And Pressure Gradients For Case I With A 20 × 20 Grid

ETA = .000000 X = .000000

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.00000	.10000000, 01	.00000000, 00	.00000000, 00	.00000000, 00	.00000000, 00
.950000	.10000000, 01	.00000000, 00	.00000000, 00	.00000000, 00	.00000000, 00
.900000	.10000000, 01	.00000000, 00	.00000000, 00	.00000000, 00	.00000000, 00
.850000	.10000000, 01	.00000000, 00	.00000000, 00	.00000000, 00	.00000000, 00
.800000	.10000000, 01	.00000000, 00	.00000000, 00	.00000000, 00	.00000000, 00
.750000	.10000000, 01	.00000000, 00	.00000000, 00	.00000000, 00	.00000000, 00
.700000	.10000000, 01	.00000000, 00	.00000000, 00	.00000000, 00	.00000000, 00
.650000	.10000000, 01	.00000000, 00	.00000000, 00	.00000000, 00	.00000000, 00
.600000	.10000000, 01	.00000000, 00	.00000000, 00	.00000000, 00	.00000000, 00
.550000	.10000000, 01	.00000000, 00	.00000000, 00	.00000000, 00	.00000000, 00
.500000	.10000000, 01	.00000000, 00	.00000000, 00	.00000000, 00	.00000000, 00
.450000	.10000000, 01	.00000000, 00	.00000000, 00	.00000000, 00	.00000000, 00
.400000	.10000000, 01	.00000000, 00	.00000000, 00	.00000000, 00	.00000000, 00
.350000	.10000000, 01	.00000000, 00	.00000000, 00	.00000000, 00	.00000000, 00
.300000	.10000000, 01	.00000000, 00	.00000000, 00	.00000000, 00	.00000000, 00
.250000	.10000000, 01	.00000000, 00	.00000000, 00	.00000000, 00	.00000000, 00
.200000	.10000000, 01	.00000000, 00	.00000000, 00	.00000000, 00	.00000000, 00
.150000	.10000000, 01	.00000000, 00	.00000000, 00	.00000000, 00	.00000000, 00
.100000	.10000000, 01	.00000000, 00	.00000000, 00	.00000000, 00	.00000000, 00
.050000	.10000000, 01	.00000000, 00	.00000000, 00	.00000000, 00	.00000000, 00
.000000	.10000000, 01	.00000000, 00	.00000000, 00	.00000000, 00	.00000000, 00

ETA = .050000 X = .043859

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.00000	.00000000, 00	.00000000, 00	.47995752, 01	.51979400, 01	.27113698, 01
.950000	.82805900, 00	-.29571878, 00	.35399698, 01	.38337874, 01	.27639127, 00
.900000	.99580020, 00	-.36395882, 00	.12545578, 01	.13586861, 01	.32790059, 00
.850000	.10285691, 01	-.35390230, 00	.86145830, 00	.93295934, 00	.32356871, 00
.800000	.10323585, 01	-.32269728, 00	.72963000, 00	.79018930, 00	.30206390, 00
.750000	.10300204, 01	-.28882483, 00	.63720231, 00	.69009011, 00	.26310906, 00
.700000	.10267879, 01	-.25730065, 00	.56119575, 00	.60777500, 00	.21845089, 00
.650000	.10238493, 01	-.22896796, 00	.49912068, 00	.54054770, 00	.17726691, 00
.600000	.10214086, 01	-.20354670, 00	.44903760, 00	.48630773, 00	.14287441, 00
.550000	.10194322, 01	-.18054747, 00	.40876280, 00	.44269012, 00	.11526873, 00
.500000	.10178414, 01	-.15951290, 00	.37633861, 00	.40757472, 00	.93332145,-01
.450000	.10165613, 01	-.14006352, 00	.35017945, 00	.37924435, 00	.75848542,-01
.400000	.10155307, 01	-.12189251, 00	.32905025, 00	.35636143, 00	.61671410,-01
.350000	.10147023, 01	-.10475198, 00	.31200765, 00	.33790429, 00	.49981495,-01
.300000	.10140400, 01	-.88439404,-01	.29833867, 00	.32310078, 00	.40166466,-01
.250000	.10135166, 01	-.72786913,-01	.28750622, 00	.31136924, 00	.31714598,-01
.200000	.10131116, 01	-.57651663,-01	.27911344, 00	.30227986, 00	.24299421,-01
.150000	.10128103, 01	-.42910301,-01	.27285854, 00	.29550580, 00	.17634822,-01
.100000	.10126021, 01	-.28453442,-01	.26852991, 00	.29081790, 00	.11489383,-01
.050000	.10124799, 01	-.14181311,-01	.26598570, 00	.28806252, 00	.56649581,-02
.000000	.10124396, 01	.00000000, 00	.26514719, 00	.28715441, 00	.00000000, 00

ETA = .100000 X = .092592

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.00000	.00000000, 00	.00000000, 00	.21114605, 01	.20523397, 01	.15244070, 01
.950000	.66393500, 00	-.18552554, 00	.17225095, 01	.16742793, 01	.94275714, 00
.900000	.94627560, 00	-.29171683, 00	.11990041, 01	.11654320, 01	.87605570, 00
.850000	.10380993, 01	-.31705619, 00	.89801924, 00	.87287471, 00	.67962404, 00
.800000	.10600900, 01	-.30426379, 00	.76278540, 00	.74142741, 00	.50141202, 00
.750000	.10603847, 01	-.27865529, 00	.67103163, 00	.65224275, 00	.37988723, 00
.700000	.10552695, 01	-.25084637, 00	.59654363, 00	.57984041, 00	.29645331, 00
.650000	.10496343, 01	-.22438027, 00	.53536032, 00	.52037024, 00	.23524246, 00
.600000	.10447096, 01	-.20006384, 00	.48555721, 00	.47196161, 00	.18837464, 00
.550000	.10406591, 01	-.17781680, 00	.44514396, 00	.43267993, 00	.15177987, 00
.500000	.10373813, 01	-.15733929, 00	.41232097, 00	.40077599, 00	.12285735, 00
.450000	.10347372, 01	-.13832094, 00	.38560495, 00	.37480802, 00	.99770850,-01
.400000	.10326053, 01	-.12049232, 00	.36384337, 00	.35365576, 00	.81070252,-01
.350000	.10308893, 01	-.10362968, 00	.34615596, 00	.33646360, 00	.65650043,-01
.300000	.10295153, 01	-.87547165,-01	.33187882, 00	.32258622, 00	.52672947,-01
.250000	.10284282, 01	-.72088671,-01	.32051124, 00	.31153693, 00	.41568324,-01
.200000	.10275869, 01	-.57120746,-01	.31166305, 00	.30293649, 00	.31828361,-01
.150000	.10269605, 01	-.42527430,-01	.30504560, 00	.29650433, 00	.23078282,-01
.100000	.10265273, 01	-.28205204,-01	.30045449, 00	.29204177, 00	.15019954,-01
.050000	.10262728, 01	-.14059202,-01	.29775221, 00	.28941515, 00	.74039083,-02
.000000	.10261889, 01	.00000000, 00	.29686002, 00	.28854794, 00	.00000000, 00

ETA = .150000 X = .147058

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.00000	.00000000, 00	.00000000, 00	.69821084, 00	.60534880, 00	.77082080, 00
.950000	.52793900, 00	-.99870250,-01	.46117197, 00	.39983610, 00	.72309873, 00
.900000	.86048060, 00	-.20621438, 00	.73856943, 00	.64033970, 00	.82262879, 00
.850000	.10156607, 01	-.25931354, 00	.71363040, 00	.61871756, 00	.77289665, 00
.800000	.10718656, 01	-.26978681, 00	.67592387, 00	.58602600, 00	.62505930, 00
.750000	.10849766, 01	-.25810537, 00	.63842726, 00	.55351644, 00	.48131136, 00
.700000	.10826264, 01	-.23776608, 00	.59497437, 00	.51584278, 00	.37165651, 00
.650000	.10760947, 01	-.21531814, 00	.55063836, 00	.47740346, 00	.29186442, 00
.600000	.10693134, 01	-.19333978, 00	.50992404, 00	.44210415, 00	.23265089, 00
.550000	.10633875, 01	-.17260799, 00	.47453768, 00	.41142417, 00	.18739546, 00
.500000	.10584750, 01	-.15320982, 00	.44458141, 00	.38545209, 00	.15192213, 00
.450000	.10544697, 01	-.13501010, 00	.41949895, 00	.36370559, 00	.12359252, 00
.400000	.10512213, 01	-.11782738, 00	.39863930, 00	.34562028, 00	.10057708, 00
.350000	.10485965, 01	-.10148867, 00	.38140544, 00	.33067852, 00	.81547188,-01
.300000	.10464896, 01	-.85840716,-01	.36732284, 00	.31846891, 00	.65520767,-01
.250000	.10448195, 01	-.70750234,-01	.35599342, 00	.30864630, 00	.51734867,-01
.200000	.10435244, 01	-.56101229,-01	.34710170, 00	.30093718, 00	.39627820,-01
.150000	.10425588, 01	-.41791220,-01	.34041079, 00	.29513616, 00	.28744921,-01
.100000	.10418904, 01	-.27727440,-01	.33574904, 00	.29109442, 00	.18716802,-01
.050000	.10414976, 01	-.13324132,-01	.33299764, 00	.28870896, 00	.92277205,-02
.000000	.10413680, 01	.00000000, 00	.33208785, 00	.28792017, 00	.00000000, 00

TABLE IV. - Continued

ETA = .200000 X = .208333

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.00000	.00000000, 00	.00000000, 00	.10524194, 00	.80825813,-01	.37824102, 00
.950000	.42608700, 00	-.49736064,-01	-.11013459, 00	-.84583370,-01	.43565597, 00
.900000	.76237330, 00	-.13340620, 00	.28907942, 00	.22201300, 00	.57664951, 00
.850000	.96477470, 00	-.19553011, 00	.46461223, 00	.35682220, 00	.65323585, 00
.800000	.10614835, 01	-.22397537, 00	.52289895, 00	.40158640, 00	.61331390, 00
.750000	.10973175, 01	-.22744043, 00	.54735016, 00	.42036493, 00	.51651348, 00
.700000	.11046204, 01	-.21715952, 00	.54942673, 00	.42195973, 00	.41665525, 00
.650000	.11008757, 01	-.20086010, 00	.53583618, 00	.41152219, 00	.33355906, 00
.600000	.10939716, 01	-.18265597, 00	.51456238, 00	.39518391, 00	.26849630, 00
.550000	.10869451, 01	-.16437657, 00	.49128098, 00	.37730380, 00	.21775242, 00
.500000	.10807468, 01	-.14669544, 00	.46892309, 00	.36013294, 00	.17758380, 00
.450000	.10755474, 01	-.12978101, 00	.44873514, 00	.34462859, 00	.14523448, 00
.400000	.10712690, 01	-.11360739, 00	.43109279, 00	.33107927, 00	.11870122, 00
.350000	.10677828, 01	-.98087424,-01	.41600877, 00	.31949474, 00	.96589738,-01
.300000	.10649691, 01	-.83120947,-01	.40335653, 00	.30977782, 00	.77820790,-01
.250000	.10627298, 01	-.68610892,-01	.39297277, 00	.30180309, 00	.61601185,-01
.200000	.10609883, 01	-.54467942,-01	.38469789, 00	.29544798, 00	.47270869,-01
.150000	.10596869, 01	-.40609612,-01	.37840509, 00	.29061511, 00	.34334557,-01
.100000	.10587846, 01	-.26959564,-01	.37398553, 00	.28722089, 00	.22375296,-01
.050000	.10582537, 01	-.13446051,-01	.37136248, 00	.28520639, 00	.11036580,-01
.000000	.10580785, 01	.00000000, 00	.37049419, 00	.28453954, 00	.00000000, 00

ETA = .250000 X = .277777

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.00000	.00000000, 00	.00000000, 00	-.80488177,-01	-.54329520,-01	.19245600, 00
.950000	.35423300, 00	-.24604492,-01	-.25255675, 00	-.17047581, 00	.24588649, 00
.900000	.67064460, 00	-.81935347,-01	.64730962,-02	.43693400,-02	.35740953, 00
.850000	.89773210, 00	-.13834732, 00	.24497082, 00	.16535531, 00	.46756942, 00
.800000	.10303157, 01	-.17493219, 00	.36594896, 00	.24701555, 00	.50691138, 00
.750000	.10938514, 01	-.19014426, 00	.43286333, 00	.29218275, 00	.47710489, 00
.700000	.11172362, 01	-.18998172, 00	.47296041, 00	.31924828, 00	.41447589, 00
.650000	.11210212, 01	-.18096972, 00	.49222090, 00	.33224911, 00	.34708041, 00
.600000	.11168180, 01	-.16769754, 00	.49624127, 00	.33496286, 00	.28706382, 00
.550000	.11102111, 01	-.15277450, 00	.49104717, 00	.33145684, 00	.23693476, 00
.500000	.11035213, 01	-.13747603, 00	.48127174, 00	.32485843, 00	.19572161, 00
.450000	.10975531, 01	-.12234699, 00	.46981537, 00	.31712538, 00	.16167904, 00
.400000	.10924890, 01	-.10757731, 00	.45829524, 00	.30934929, 00	.13323878, 00
.350000	.10882924, 01	-.93202380,-01	.44754995, 00	.30209622, 00	.10913852, 00
.300000	.10848705, 01	-.79196737,-01	.43800016, 00	.29565011, 00	.88404236,-01
.250000	.10821281, 01	-.65511990,-01	.42982711, 00	.29013330, 00	.70256623,-01
.200000	.10799844, 01	-.52094272,-01	.42311407, 00	.28560200, 00	.54087021,-01
.150000	.10783770, 01	-.38888100,-01	.41789880, 00	.28208169, 00	.39377890,-01
.100000	.10772597, 01	-.25838932,-01	.41418174, 00	.27957268, 00	.25703131,-01
.050000	.10766012, 01	-.12893640,-01	.41195591, 00	.27807024, 00	.12690599,-01
.000000	.10763837, 01	.00000000, 00	.41121499, 00	.27757012, 00	.00000000, 00

ETA = .300000 X = .357142

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.00000	.00000000, 00	.00000000, 00	-.10461668, 00	-.61514613,-01	.10440370, 00
.950000	.30470100, 00	-.12752779,-01	-.22622528, 00	-.13302047, 00	.13997229, 00
.900000	.59386540, 00	-.49264933,-01	-.11005231, 00	-.64710760,-01	.21207541, 00
.850000	.82700160, 00	-.93527338,-01	.93446207,-01	.54946370,-01	.30510336, 00
.800000	.98474730, 00	-.12965940, 00	.23883743, 00	.14043641, 00	.37207093, 00
.750000	.10750219, 01	-.15108818, 00	.32771690, 00	.19269754, 00	.38999750, 00
.700000	.11182123, 01	-.15880151, 00	.38890663, 00	.22867710, 00	.36852233, 00
.650000	.11338893, 01	-.15675103, 00	.43182149, 00	.25391104, 00	.32773235, 00
.600000	.11357423, 01	-.14884196, 00	.45899290, 00	.26988783, 00	.28243079, 00
.550000	.11317177, 01	-.13786459, 00	.47361506, 00	.27848566, 00	.23981563, 00
.500000	.11258246, 01	-.12548707, 00	.47949918, 00	.28194552, 00	.20217435, 00
.450000	.11198418, 01	-.11259123, 00	.47997615, 00	.28222598, 00	.16962407, 00
.400000	.11144462, 01	-.99599144,-01	.47738765, 00	.28070394, 00	.14150985, 00
.350000	.11098281, 01	-.86691133,-01	.47332914, 00	.27831754, 00	.11706222, 00
.300000	.11059919, 01	-.73931121,-01	.46877680, 00	.27564076, 00	.95586410,-01
.250000	.11028816, 01	-.61330634,-01	.46431403, 00	.27301665, 00	.76452973,-01
.200000	.11004311, 01	-.48876853,-01	.46033017, 00	.27067414, 00	.59147506,-01
.150000	.10985837, 01	-.36546375,-01	.45705600, 00	.26874893, 00	.43224150,-01
.100000	.10972949, 01	-.24310742,-01	.45463472, 00	.26732522, 00	.28287910,-01
.050000	.10965336, 01	-.12139218,-01	.45315307, 00	.26645401, 00	.13989311,-01
.000000	.10962819, 01	.00000000, 00	.45265365, 00	.26616035, 00	.00000000, 00

ETA = .350000 X = .448717

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.00000	.00000000, 00	.00000000, 00	-.74118289,-01	-.37577973,-01	.59661056,-01
.950000	.27044900, 00	-.70159167,-02	-.15628124, 00	-.79234589,-01	.82061920,-01
.900000	.53327290, 00	-.29565755,-01	-.12121287, 00	-.61454930,-01	.12525243, 00
.850000	.76100680, 00	-.61307504,-01	.15174891,-01	.76936700,-02	.18969994, 00
.800000	.93242310, 00	-.92032059,-01	.15273163, 00	.77434940,-01	.25200339, 00
.750000	.10442900, 01	-.11456364, 00	.24896143, 00	.12622345, 00	.28954329, 00
.700000	.11076055, 01	-.12683162, 00	.31757195, 00	.16100898, 00	.29684688, 00
.650000	.11379592, 01	-.13019334, 00	.37133895, 00	.18826885, 00	.28176481, 00
.600000	.11489378, 01	-.12720457, 00	.41329917, 00	.20954268, 00	.25503700, 00
.550000	.11499303, 01	-.12024676, 00	.44406650, 00	.22514172, 00	.22454226, 00
.500000	.11465032, 01	-.11104486, 00	.46510264, 00	.23580704, 00	.19449577, 00
.450000	.11415802, 01	-.10067428, 00	.47856739, 00	.24263367, 00	.16657533, 00
.400000	.11365368, 01	-.89742295,-01	.48663495, 00	.24672392, 00	.14125545, 00
.350000	.11319455, 01	-.78566443,-01	.49107544, 00	.24897525, 00	.11841256, 00
.300000	.11280006, 01	-.67304047,-01	.49316964, 00	.25003701, 00	.97718294,-01
.250000	.11247374, 01	-.56029279,-01	.49385031, 00	.25038211, 00	.78828428,-01
.200000	.11221345, 01	-.44773170,-01	.49376617, 00	.25033945, 00	.61394929,-01
.150000	.11201561, 01	-.33545756,-01	.49334003, 00	.25012340, 00	.45090572,-01
.100000	.11187684, 01	-.22346126,-01	.49285814, 00	.24987908, 00	.29611552,-01
.050000	.11179460, 01	-.11167470,-01	.49250307, 00	.24969906, 00	.14672675,-01
.000000	.11176736, 01	.00000000, 00	.49237262, 00	.24963292, 00	.00000000, 00

TABLE IV. - Continued

ETA = .400000 X = .555555

Y	U(X)	U(Y)	$-\partial P/\partial\eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.00000	.00000000, 00	.00000000, 00	-.24404689,-01	-.10542826,-01	.35131392,-01
.950000	.24638600, 00	-.40295232,-02	-.80065687,-01	-.34588377,-01	.49187400,-01
.900000	.48678140, 00	-.17835552,-01	-.80094976,-01	-.34601030,-01	.74444516,-01
.850000	.70379390, 00	-.39357835,-01	-.23833333,-02	-.10296000,-02	.11490749, 00
.800000	.87970560, 00	-.63044827,-01	.10852993, 00	.46884930,-01	.16128304, 00
.750000	.10063837, 01	-.83277115,-01	.20136041, 00	.86987700,-01	.19905106, 00
.700000	.10873845, 01	-.96942096,-01	.27000518, 00	.11664224, 00	.21922347, 00
.650000	.11331537, 01	-.10360474, 00	.32498615, 00	.14039402, 00	.22159682, 00
.600000	.11553608, 01	-.10441988, 00	.37201553, 00	.16071071, 00	.21115398, 00
.550000	.11635909, 01	-.10102536, 00	.41166976, 00	.17784134, 00	.19354542, 00
.500000	.11644291, 01	-.94893854,-01	.44363537, 00	.19165048, 00	.17296925, 00
.450000	.11618587, 01	-.87105240,-01	.46827523, 00	.20229490, 00	.15184091, 00
.400000	.11580514, 01	-.78358147,-01	.48664016, 00	.21022855, 00	.13132425, 00
.350000	.11540850, 01	-.69070406,-01	.49996789, 00	.21598613, 00	.11184305, 00
.300000	.11504439, 01	-.59478969,-01	.50947500, 00	.22009320, 00	.93496226,-01
.250000	.11473212, 01	-.49714689,-01	.51616328, 00	.22298254, 00	.76213890,-01
.200000	.11447773, 01	-.39849883,-01	.52078569, 00	.22497942, 00	.59849513,-01
.150000	.11428185, 01	-.29925633,-01	.52388928, 00	.22632017, 00	.44231359,-01
.100000	.11414335, 01	-.19966392,-01	.52585817, 00	.22717073, 00	.29175241,-01
.050000	.11406088, 01	-.99875894,-02	.52694472, 00	.22764012, 00	.14493352,-01
.000000	.11403350, 01	.00000000, 00	.52728858, 00	.22778867, 00	.00000000, 00

ETA = .450000 X = .681818

Y	U(X)	U(Y)	$-\partial P/\partial\eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.00000	.00000000, 00	.00000000, 00	.36072873,-01	.13094453,-01	.21117888,-01
.950000	.22916300, 00	-.23739111,-02	-.24478264,-02	-.88856100,-03	.29812026,-01
.900000	.45149440, 00	-.10819432,-01	-.14784730,-01	-.53668570,-02	.44357040,-01
.850000	.65635570, 00	-.24906809,-01	.23852011,-01	.86582800,-02	.68209223,-01
.800000	.83075810, 00	-.41960404,-01	.10329214, 00	.37495050,-01	.98581910,-01
.750000	.96582410, 00	-.58318890,-01	.18470214, 00	.67046880,-01	.12804038, 00
.700000	.10605508, 01	-.71109159,-01	.24957355, 00	.90595200,-01	.14950186, 00
.650000	.11206495, 01	-.79058641,-01	.30115501, 00	.10931927, 00	.15987421, 00
.600000	.11549086, 01	-.82275837,-01	.34616223, 00	.12565689, 00	.16004179, 00
.550000	.11719813, 01	-.81616339,-01	.38713969, 00	.14053171, 00	.15284055, 00
.500000	.11787257, 01	-.78126493,-01	.42374611, 00	.15381984, 00	.14119029, 00
.450000	.11798573, 01	-.72727050,-01	.45515564, 00	.16522150, 00	.12729658, 00
.400000	.11782841, 01	-.66103352,-01	.48104647, 00	.17461987, 00	.11247399, 00
.350000	.11756473, 01	-.58715213,-01	.50177611, 00	.18214473, 00	.97483164,-01
.300000	.11728055, 01	-.50851254,-01	.51799754, 00	.18803311, 00	.82672504,-01
.250000	.11701751, 01	-.42686549,-01	.53044121, 00	.19255016, 00	.68181775,-01
.200000	.11679425, 01	-.34327167,-01	.53976476, 00	.19593461, 00	.54041412,-01
.150000	.11661827, 01	-.25839574,-01	.54650033, 00	.19837962, 00	.40221550,-01
.100000	.11649213, 01	-.17268345,-01	.55104484, 00	.20002928, 00	.26662344,-01
.050000	.11641645, 01	-.86462498,-02	.55366344, 00	.20097983, 00	.13284712,-01
.000000	.11639124, 01	.00000000, 00	.55450752, 00	.20128623, 00	.00000000, 00

ETA = .500000 X = .833333

Y	U(X)	U(Y)	$-\partial P/\partial\eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.00000	.00000000, 00	.00000000, 00	.10722213, 00	.32166640,-01	.13094400,-01
.950000	.21658000, 00	-.14320500,-02	.79819090,-01	.23945727,-01	.18233699,-01
.900000	.42470730, 00	-.66184800,-02	.64578626,-01	.19373588,-01	.26310975,-01
.850000	.61800610, 00	-.15646020,-01	.80703210,-01	.24210963,-01	.39563688,-01
.800000	.78766130, 00	-.27344580,-01	.13262163, 00	.39786490,-01	.57588322,-01
.750000	.92608410, 00	-.39607110,-01	.19810100, 00	.59430300,-01	.77206012,-01
.700000	.10302502, 01	-.50304450,-01	.25641803, 00	.76925410,-01	.94231830,-01
.650000	.11023915, 01	-.58039320,-01	.30318090, 00	.90954270,-01	.10564498, 00
.600000	.11483148, 01	-.62344830,-01	.34299890, 00	.10289967, 00	.11056934, 00
.550000	.11750216, 01	-.63462150,-01	.37995413, 00	.11398624, 00	.10978462, 00
.500000	.11889175, 01	-.61988790,-01	.41509850, 00	.12452955, 00	.10478037, 00
.450000	.11949803, 01	-.58595850,-01	.44768296, 00	.13430489, 00	.97013520,-01
.400000	.11966515, 01	-.53867250,-01	.47671283, 00	.14301385, 00	.87612455,-01
.350000	.11961018, 01	-.48245100,-01	.50159280, 00	.15047784, 00	.77302975,-01
.300000	.11946000, 01	-.42035820,-01	.52222473, 00	.15666742, 00	.66524350,-01
.250000	.11928421, 01	-.35441070,-01	.53882450, 00	.16164735, 00	.55522792,-01
.200000	.11911877, 01	-.28590780,-01	.55175136, 00	.16552541, 00	.44429060,-01
.150000	.11898125, 01	-.21569850,-01	.56138220, 00	.16841466, 00	.33310047,-01
.100000	.11887981, 01	-.14436600,-01	.56802946, 00	.17040884, 00	.22195417,-01
.050000	.11881800, 01	-.72346650,-02	.57191076, 00	.17157323, 00	.11092816,-01
.000000	.11879727, 01	.00000000, 00	.57314940, 00	.17194482, 00	.00000000, 00

ETA = .550000 X = 1.01851

Y	U(X)	U(Y)	$-\partial P/\partial\eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.00000	.00000000, 00	.00000000, 00	.19177898, 00	.46602293,-01	.86365440,-02
.950000	.20710100, 00	-.90199170,-03	.17152547, 00	.41680690,-01	.11396311,-01
.900000	.40411450, 00	-.41421294,-02	.15681425, 00	.38105865,-01	.15529239,-01
.850000	.58726870, 00	-.98933562,-02	.16104526, 00	.39133999,-01	.22292031,-01
.800000	.75088780, 00	-.17681627,-01	.19246481, 00	.46768950,-01	.31929300,-01
.750000	.88929120, 00	-.26393542,-01	.24125600, 00	.58625210,-01	.43335870,-01
.700000	.99911010, 00	-.34655663,-01	.29079518, 00	.70663230,-01	.54494489,-01
.650000	.10805149, 01	-.41315516,-01	.33231139, 00	.80751670,-01	.63427438,-01
.600000	.11368325, 01	-.45724508,-01	.36646502, 00	.89051000,-01	.68959028,-01
.550000	.11731681, 01	-.47752002,-01	.39708882, 00	.96492585,-01	.70909543,-01
.500000	.11949467, 01	-.47631839,-01	.42669094, 00	.10368590, 00	.69753413,-01
.450000	.12069291, 01	-.45770969,-01	.45569773, 00	.11073455, 00	.66239096,-01
.400000	.12127866, 01	-.42602395,-01	.48320065, 00	.11741776, 00	.61072281,-01
.350000	.12150854, 01	-.38503447,-01	.50817065, 00	.12348547, 00	.54807277,-01
.300000	.12154856, 01	-.33765263,-01	.52989378, 00	.12876419, 00	.47820397,-01
.250000	.12149919, 01	-.28596555,-01	.54803711, 00	.13317302, 00	.40358813,-01
.200000	.12141799, 01	-.23140525,-01	.56256761, 00	.13670393, 00	.32581449,-01
.150000	.12133633, 01	-.17494007,-01	.57361008, 00	.13938725, 00	.24593008,-01
.100000	.12127053, 01	-.11723972,-01	.58132053, 00	.14126089, 00	.16465302,-01
.050000	.12122865, 01	-.58795356,-02	.58583452, 00	.14235779, 00	.82529210,-02
.000000	.12121435, 01	.00000000, 00	.58720724, 00	.14269136, 00	.00000000, 00

TABLE IV. - Continued

ETA = .600000 X = 1.25000

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.00000	.00000000, 00	.00000000, 00	.29436000, 00	.56517120,-01	.63078400,-02
.950000	.19953500, 00	-.61547520,-03	.27841775, 00	.53456208,-01	.75036632,-02
.900000	.38770580, 00	-.27322176,-02	.26510365, 00	.50899902,-01	.92820612,-02
.850000	.56228800, 00	-.64638528,-02	.26335742, 00	.50564625,-01	.12268274,-01
.800000	.71975930, 00	-.11614022,-01	.28092603, 00	.53937799,-01	.16760127,-01
.750000	.85623210, 00	-.17617977,-01	.31496031, 00	.60472380,-01	.22488704,-01
.700000	.96884290, 00	-.23668915,-01	.35456916, 00	.68077280,-01	.28642274,-01
.650000	.10568407, 01	-.28958803,-01	.39032026, 00	.74941490,-01	.34198748,-01
.600000	.11218272, 01	-.32894496,-01	.41946750, 00	.80537760,-01	.38327544,-01
.550000	.11671725, 01	-.35192006,-01	.44399789, 00	.85247596,-01	.40605040,-01
.500000	.11970699, 01	-.35849452,-01	.46692669, 00	.89649925,-01	.41036284,-01
.450000	.12156776, 01	-.35053382,-01	.48974932, 00	.94031871,-01	.39871242,-01
.400000	.12265590, 01	-.33078393,-01	.51236926, 00	.98374899,-01	.37454767,-01
.350000	.12324686, 01	-.30208550,-01	.53402807, 00	.10253339, 00	.34116318,-01
.300000	.12353691, 01	-.26692224,-01	.55376505, 00	.10632289, 00	.30121044,-01
.250000	.12365653, 01	-.22726080,-01	.57084531, 00	.10960230, 00	.25659656,-01
.200000	.12368772, 01	-.18455616,-01	.58486921, 00	.11229489, 00	.20866523,-01
.150000	.12367942, 01	-.13984281,-01	.59568140, 00	.11437083, 00	.15837010,-01
.100000	.12365942, 01	-.93849024,-02	.60326328, 00	.11582655, 00	.10643878,-01
.050000	.12364268, 01	-.47099635,-02	.60763145, 00	.11666524, 00	.53473097,-02
.000000	.12363640, 01	.00000000, 00	.60874692, 00	.11687941, 00	.00000000, 00

ETA = .650000 X = 1.54761

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.00000	.00000000, 00	.00000000, 00	.42193704, 00	.62024746,-01	.50771840,-02
.950000	.19287100, 00	-.46431420,-03	.40815721, 00	.59999110,-01	.53437672,-02
.900000	.37365420, 00	-.19508370,-02	.39608292, 00	.58224190,-01	.58385466,-02
.850000	.54100930, 00	-.44889831,-02	.39139261, 00	.57534714,-01	.68421343,-02
.800000	.69279340, 00	-.79690905,-02	.40009286, 00	.58813651,-01	.85488434,-02
.750000	.82650530, 00	-.12095718,-01	.42243272, 00	.62097610,-01	.10943667,-01
.700000	.94004330, 00	-.16415916,-01	.45235190, 00	.66495730,-01	.13768720,-01
.650000	.10324650, 01	-.20418976,-01	.48211333, 00	.70870660,-01	.16594890,-01
.600000	.11043679, 01	-.23659120,-01	.50717551, 00	.74554800,-01	.18971181,-01
.550000	.11577777, 01	-.25843246,-01	.52738780, 00	.77526008,-01	.20576905,-01
.500000	.11956687, 01	-.26856223,-01	.54488014, 00	.80097381,-01	.21267525,-01
.450000	.12213617, 01	-.26735786,-01	.56163735, 00	.82560691,-01	.21066036,-01
.400000	.12380201, 01	-.25619968,-01	.57861845, 00	.85056913,-01	.20099646,-01
.350000	.12483467, 01	-.23692210,-01	.59570067, 00	.87567999,-01	.18530479,-01
.300000	.12544568, 01	-.21140319,-01	.61204816, 00	.89971080,-01	.16507682,-01
.250000	.12578909, 01	-.18131597,-01	.62679617, 00	.92139038,-01	.14154332,-01
.200000	.12597101, 01	-.14802165,-01	.63926314, 00	.93971682,-01	.11563600,-01
.150000	.12606066, 01	-.11256363,-01	.64901832, 00	.95405694,-01	.88046245,-02
.100000	.12610090, 01	-.75714702,-02	.65582114, 00	.96405709,-01	.59302015,-02
.050000	.12611662, 01	-.38046348,-02	.65950609, 00	.96947396,-01	.29827956,-02
.000000	.12612057, 01	.00000000, 00	.65988940, 00	.97003742,-01	.00000000, 00

ETA = .700000 X = 1.94444

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.00000	.00000000, 00	.00000000, 00	.58775875, 00	.63477946,-01	.42186240,-02
.950000	.18626400, 00	-.37264320,-03	.57457525, 00	.62054128,-01	.40440059,-02
.900000	.36032710, 00	-.14858424,-02	.56326353, 00	.60832462,-01	.39685925,-02
.850000	.52134750, 00	-.33086880,-02	.55711711, 00	.60168648,-01	.41148467,-02
.800000	.66801280, 00	-.57510108,-02	.56033712, 00	.60516409,-01	.45809987,-02
.750000	.79877070, 00	-.86389416,-02	.57420015, 00	.62013617,-01	.53962896,-02
.700000	.91222210, 00	-.11715019,-01	.59584470, 00	.64351228,-01	.64912977,-02
.650000	.10075672, 01	-.14673474,-01	.61989389, 00	.66948541,-01	.77066184,-02
.600000	.10849274, 01	-.17216658,-01	.64178291, 00	.69312555,-01	.88427250,-02
.550000	.11454235, 01	-.19108915,-01	.65973724, 00	.71251622,-01	.97148450,-02
.500000	.11909982, 01	-.20209078,-01	.67447866, 00	.72843696,-01	.10202106,-01
.450000	.12240875, 01	-.20475525,-01	.68779646, 00	.74282018,-01	.10263778,-01
.400000	.12472636, 01	-.19948453,-01	.70113899, 00	.75723011,-01	.99212100,-02
.350000	.12629424, 01	-.18722577,-01	.71498922, 00	.77218836,-01	.92354350,-02
.300000	.12731964, 01	-.16919010,-01	.72894620, 00	.78726190,-01	.82803660,-02
.250000	.12796820, 01	-.14662738,-01	.74222937, 00	.80160773,-01	.71269133,-02
.200000	.12836475, 01	-.12068330,-01	.75394093, 00	.81425621,-01	.58336513,-02
.150000	.12859826, 01	-.92332008,-02	.76329803, 00	.82436188,-01	.44451493,-02
.100000	.12872863, 01	-.62364384,-02	.76969021, 00	.83126543,-01	.29942298,-02
.050000	.12879334, 01	-.31413517,-02	.77262782, 00	.83443805,-01	.15058781,-02
.000000	.12881277, 01	.00000000, 00	.77164872, 00	.83338062,-01	.00000000, 00

ETA = .750000 X = 2.50000

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.00000	.00000000, 00	.00000000, 00	.82161066, 00	.61620800,-01	.33200000,-02
.950000	.17911300, 00	-.29295000,-03	.80816978, 00	.60612734,-01	.30300578,-02
.900000	.34646860, 00	-.11297475,-02	.79730766, 00	.59798075,-01	.28023970,-02
.850000	.50151700, 00	-.24562950,-02	.79063725, 00	.59297794,-01	.26967797,-02
.800000	.64343920, 00	-.41933400,-02	.79085304, 00	.59313978,-01	.27577745,-02
.750000	.77128300, 00	-.62261925,-02	.79921762, 00	.59941322,-01	.30019633,-02
.700000	.88416800, 00	-.84020775,-02	.81460153, 00	.61095115,-01	.34080910,-02
.650000	.98152390, 00	-.10540987,-01	.83379968, 00	.62534976,-01	.39161766,-02
.600000	.10632990, 01	-.12457515,-01	.85324070, 00	.63993053,-01	.44408778,-02
.550000	.11300701, 01	-.13987155,-01	.87056616, 00	.65292462,-01	.48910323,-02
.500000	.11830149, 01	-.15007965,-01	.88533885, 00	.66400414,-01	.51910129,-02
.450000	.12237749, 01	-.15451755,-01	.89854722, 00	.67391042,-01	.52953087,-02
.400000	.12542511, 01	-.15303772,-01	.91144194, 00	.68358146,-01	.51902765,-02
.350000	.12763971, 01	-.14593927,-01	.92488361, 00	.69366271,-01	.48912390,-02
.300000	.12920479, 01	-.13383060,-01	.93889654, 00	.70417241,-01	.44283212,-02
.250000	.13028046, 01	-.11749050,-01	.95283298, 00	.71462474,-01	.38381771,-02
.200000	.13099836, 01	-.97752825,-02	.96559114, 00	.72419336,-01	.31557780,-02
.150000	.13146056, 01	-.75427500,-02	.97592504, 00	.73194378,-01	.24107389,-02
.100000	.13174150, 01	-.51258750,-02	.98264209, 00	.73698157,-01	.16259351,-02
.050000	.13189110, 01	-.25914510,-02	.98469636, 00	.73852227,-01	.81816890,-03
.000000	.13193788, 01	.00000000, 00	.98120936, 00	.73590702,-01	.00000000, 00

TABLE IV. - Continued

ETA = .800000 X = 3.33333

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.00000	.00000000, 00	.00000000, 00	.11981238, 01	.57509946,-01	.23050240,-02
.950000	.17120000, 00	-.20761440,-03	.11843678, 01	.56849658,-01	.20608787,-02
.900000	.33148250, 00	-.78782880,-03	.11741885, 01	.56361050,-01	.18592055,-02
.850000	.48049960, 00	-.16920336,-02	.11679502, 01	.56061611,-01	.17337667,-02
.800000	.61774750, 00	-.28604640,-02	.11672441, 01	.56027721,-01	.17059222,-02
.750000	.74265420, 00	-.42183024,-02	.11729311, 01	.56300695,-01	.17824283,-02
.700000	.85469150, 00	-.56748144,-02	.11845814, 01	.56859908,-01	.19535023,-02
.650000	.95349470, 00	-.71265216,-02	.12004154, 01	.57619943,-01	.21934756,-02
.600000	.10389785, 01	-.84650208,-02	.12179977, 01	.58463891,-01	.24634473,-02
.550000	.11114175, 01	-.95874720,-02	.12352147, 01	.59290308,-01	.27188740,-02
.500000	.11714702, 01	-.10407273,-01	.12510519, 01	.60050495,-01	.29174243,-02
.450000	.12201434, 01	-.10862059,-01	.12656111, 01	.60749335,-01	.30262870,-02
.400000	.12587013, 01	-.10917902,-01	.12795930, 01	.61420465,-01	.30259622,-02
.350000	.12885519, 01	-.10568817,-01	.12937806, 01	.62101470,-01	.29110056,-02
.300000	.13111281, 01	-.98331312,-02	.13083509, 01	.62800845,-01	.26878988,-02
.250000	.13277857, 01	-.87478704,-02	.13227818, 01	.63493531,-01	.23709698,-02
.200000	.13397315, 01	-.73627248,-02	.13358745, 01	.64121978,-01	.19784065,-02
.150000	.13479747, 01	-.57347184,-02	.13460080, 01	.64608385,-01	.15288830,-02
.100000	.13533036, 01	-.39241968,-02	.13514440, 01	.64869312,-01	.10396539,-02
.050000	.13562784, 01	-.19923302,-02	.13506077, 01	.64829172,-01	.52571206,-03
.000000	.13572331, 01	.00000000, 00	.13423267, 01	.64431683,-01	.00000000, 00

ETA = .850000 X = 4.72222

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.00000	.00000000, 00	.00000000, 00	.19318548, 01	.52160080,-01	.12965760,-02
.950000	.16270000, 00	-.12155940,-03	.19184803, 01	.51798969,-01	.11446080,-02
.900000	.31554320, 00	-.45777420,-03	.19097788, 01	.51564030,-01	.10161118,-02
.850000	.45833710, 00	-.97751070,-03	.19054113, 01	.51446107,-01	.92860150,-03
.800000	.59080860, 00	-.16450506,-02	.19061248, 01	.51465370,-01	.89260869,-03
.750000	.71265070, 00	-.24185952,-02	.19121768, 01	.51628774,-01	.91147649,-03
.700000	.82358010, 00	-.32501952,-02	.19230441, 01	.51922193,-01	.98135425,-03
.650000	.92339460, 00	-.40868631,-02	.19374114, 01	.52310110,-01	.10914741,-02
.600000	.10120292, 01	-.48728979,-02	.19534569, 01	.52743337,-01	.12256046,-02
.550000	.10896065, 01	-.55535922,-02	.19693617, 01	.53172766,-01	.13640875,-02
.500000	.11564610, 01	-.60792741,-02	.19838109, 01	.53562896,-01	.14862962,-02
.450000	.12131368, 01	-.64092303,-02	.19961511, 01	.53896082,-01	.15733522,-02
.400000	.12603603, 01	-.65147571,-02	.20064455, 01	.54174029,-01	.16105223,-02
.350000	.12989965, 01	-.63806670,-02	.20150906, 01	.54407448,-01	.15883943,-02
.300000	.13299843, 01	-.60056235,-02	.20223100, 01	.54602371,-01	.15034654,-02
.250000	.13542714, 01	-.54011637,-02	.20279737, 01	.54755291,-01	.13574541,-02
.200000	.13727576, 01	-.45899514,-02	.20313864, 01	.54847433,-01	.11562980,-02
.150000	.13862416, 01	-.36036360,-02	.20314370, 01	.54848801,-01	.90889870,-03
.100000	.13953814, 01	-.24806223,-02	.20268337, 01	.54724510,-01	.62593810,-03
.050000	.14006651, 01	-.12640509,-02	.20164466, 01	.54444059,-01	.31899091,-03
.000000	.14023926, 01	.00000000, 00	.19996617, 01	.53990866,-01	.00000000, 00

ETA = .900000 X = 7.50000

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.00000	.00000000, 00	.00000000, 00	.38573844, 01	.46288613,-01	.44364800,-03
.950000	.15424600, 00	-.47817600,-04	.38463100, 01	.46155720,-01	.38658883,-03
.900000	.29978060, 00	-.17909640,-03	.38421633, 01	.46105960,-01	.33783300,-03
.850000	.43652640, 00	-.38083920,-03	.38434358, 01	.46121230,-01	.30354877,-03
.800000	.56437400, 00	-.63871080,-03	.38496487, 01	.46195785,-01	.28745952,-03
.750000	.68320440, 00	-.93666720,-03	.38598778, 01	.46318534,-01	.29089740,-03
.700000	.79290380, 00	-.12569220,-02	.38726891, 01	.46472270,-01	.31288074,-03
.650000	.89339460, 00	-.15802452,-02	.38861775, 01	.46634130,-01	.35034615,-03
.600000	.98465520, 00	-.18865644,-02	.38980825, 01	.46776990,-01	.39840881,-03
.550000	.10667369, 01	-.21558840,-02	.39061781, 01	.46874138,-01	.45088660,-03
.500000	.11397799, 01	-.23694588,-02	.39085902, 01	.46903083,-01	.50089580,-03
.450000	.12040144, 01	-.25109688,-02	.39041015, 01	.46849219,-01	.54157644,-03
.400000	.12597592, 01	-.25676364,-02	.38924216, 01	.46709060,-01	.56676192,-03
.350000	.13074087, 01	-.25311096,-02	.38738905, 01	.46486687,-01	.57154170,-03
.300000	.13474060, 01	-.23980692,-02	.38495500, 01	.46194601,-01	.55275021,-03
.250000	.13802180, 01	-.21703740,-02	.38208167, 01	.45849801,-01	.50913391,-03
.200000	.14063066, 01	-.18549492,-02	.37892153, 01	.45470584,-01	.44139568,-03
.150000	.14260980, 01	-.14633196,-02	.37562708, 01	.45075250,-01	.35203823,-03
.100000	.14399549, 01	-.10109328,-02	.37234260, 01	.44681113,-01	.24508431,-03
.050000	.14481533, 01	-.51629460,-03	.36921084, 01	.44305301,-01	.12573915,-03
.000000	.14508665, 01	.00000000, 00	.36637768, 01	.43965322,-01	.00000000, 00

ETA = .950000 X = 15.8333

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.00000	.00000000, 00	.00000000, 00	.13785004, 02	.41355013,-01	.26496000,-04
.950000	.14777500, 00	-.83148000,-05	.13779317, 02	.41337953,-01	.22613073,-04
.900000	.28779140, 00	-.27735900,-04	.13780620, 02	.41341861,-01	.19544163,-04
.850000	.42003590, 00	-.56406600,-04	.13789163, 02	.41367489,-01	.17848522,-04
.800000	.54448960, 00	-.92315100,-04	.13802034, 02	.41406102,-01	.17859075,-04
.750000	.66113310, 00	-.13327860,-03	.13816505, 02	.41449515,-01	.19688100,-04
.700000	.76994860, 00	-.17692830,-03	.13829714, 02	.41489143,-01	.23238916,-04
.650000	.87092440, 00	-.22074000,-03	.13838720, 02	.41516161,-01	.28220608,-04
.600000	.96405930, 00	-.26211210,-03	.13840638, 02	.41521916,-01	.34181452,-04
.550000	.10453653, 01	-.29845560,-03	.13832954, 02	.41498863,-01	.40539959,-04
.500000	.11268706, 01	-.32732280,-03	.13813699, 02	.41441097,-01	.46639720,-04
.450000	.11966220, 01	-.34654530,-03	.13781835, 02	.41345507,-01	.51803268,-04
.400000	.12586819, 01	-.35436600,-03	.13737487, 02	.41212462,-01	.55389503,-04
.350000	.13131271, 01	-.34957290,-03	.13681890, 02	.41045672,-01	.56859178,-04
.300000	.13600456, 01	-.33159210,-03	.13617576, 02	.40852729,-01	.55821315,-04
.250000	.13995314, 01	-.30054090,-03	.13548022, 02	.40644066,-01	.52073005,-04
.200000	.14316788, 01	-.25724610,-03	.13477487, 02	.40432462,-01	.45622095,-04
.150000	.14565763, 01	-.20321070,-03	.13410487, 02	.40231462,-01	.36690263,-04
.100000	.14743002, 01	-.14054010,-03	.13351558, 02	.40054675,-01	.25698357,-04
.050000	.14849095, 01	-.71825220,-04	.13304731, 02	.39914193,-01	.13233129,-04
.000000	.14884417, 01	.00000000, 00	.13273349, 02	.39820049,-01	.00000000, 00

TABLE IV. - Continued

ETA = 1.00000 X = ∞

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.00000	.00000000, 00	.00000000, 00	∞	.40000000,-01	.00000000, 00
.950000	.14625000, 00	.00000000, 00	∞	.40000000,-01	.00000000, 00
.900000	.28500000, 00	.00000000, 00	∞	.40000000,-01	.00000000, 00
.850000	.41625000, 00	.00000000, 00	∞	.40000000,-01	.00000000, 00
.800000	.54000000, 00	.00000000, 00	∞	.40000000,-01	.00000000, 00
.750000	.65625000, 00	.00000000, 00	∞	.40000000,-01	.00000000, 00
.700000	.76500000, 00	.00000000, 00	∞	.40000000,-01	.00000000, 00
.650000	.86625000, 00	.00000000, 00	∞	.40000000,-01	.00000000, 00
.600000	.96000000, 00	.00000000, 00	∞	.40000000,-01	.00000000, 00
.550000	.10462500, 01	.00000000, 00	∞	.40000000,-01	.00000000, 00
.500000	.11250000, 01	.00000000, 00	∞	.40000000,-01	.00000000, 00
.450000	.11962500, 01	.00000000, 00	∞	.40000000,-01	.00000000, 00
.400000	.12600000, 01	.00000000, 00	∞	.40000000,-01	.00000000, 00
.350000	.13162500, 01	.00000000, 00	∞	.40000000,-01	.00000000, 00
.300000	.13650000, 01	.00000000, 00	∞	.40000000,-01	.00000000, 00
.250000	.14062500, 01	.00000000, 00	∞	.40000000,-01	.00000000, 00
.200000	.14400000, 01	.00000000, 00	∞	.40000000,-01	.00000000, 00
.150000	.14662500, 01	.00000000, 00	∞	.40000000,-01	.00000000, 00
.100000	.14850000, 01	.00000000, 00	∞	.40000000,-01	.00000000, 00
.050000	.14962500, 01	.00000000, 00	∞	.40000000,-01	.00000000, 00
.000000	.15000000, 01	.00000000, 00	∞	.40000000,-01	.00000000, 00

TABLE V. Stream Functions And Vorticities For Case II With A 20 x 10 Grid

```
         ETA= -1.000000       X= -  ∞

     Y              PSI               OMEGA
1.000000        1.00000000      0.
0.900000        0.89999998      0.
0.800000        0.80000000      0.
0.700000        0.70000000      0.
0.600000        0.59999999      0.
0.500000        0.49999999      0.
0.400000        0.40000000      0.
0.300000        0.30000000      0.
0.200000        0.20000000      0.
0.100000        0.09999999      0.
0.              0.              0.

         ETA= -0.900000       X=  -7.499999

     Y              PSI               OMEGA
1.000000        1.00000000      0.
0.900000        0.89999998      0.
0.800000        0.79999999      0.
0.700000        0.69999999      0.
0.600000        0.59999999      0.
0.500000        0.49999999      0.
0.400000        0.39999998      0.
0.300000        0.29999999      0.
0.200000        0.20000000      0.
0.100000        0.09999999      0.
0.              0.              0.

         ETA= -0.800000       X=  -3.333333

     Y              PSI               OMEGA
1.000000        1.00000000      0.
0.900000        0.90000093      0.
0.800000        0.80000176      0.
0.700000        0.70000242      0.
0.600000        0.60000285      0.
0.500000        0.50000299      0.
0.400000        0.40000284      0.
0.300000        0.30000240      0.
0.200000        0.20000175      0.
0.100000        0.10000092      0.
0.              0.              0.

         ETA= -0.700000       X=  -1.944444

     Y              PSI               OMEGA
1.000000        1.00000000      0.
0.900000        0.90002799      0.
0.800000        0.80005310      0.
0.700000        0.70007299      0.
0.600000        0.60008565      0.
0.500000        0.50008988      0.
0.400000        0.40008531      0.
0.300000        0.30007244      0.
0.200000        0.20005256      0.
0.100000        0.10002761      0.
0.              0.              0.

         ETA= -0.600000       X=  -1.250000

     Y              PSI               OMEGA
1.000000        1.00000000      0.
0.900000        0.90022106      0.
0.800000        0.80041900      0.
0.700000        0.70057399      0.
0.600000        0.60066988      0.
0.500000        0.50069941      0.
0.400000        0.40066066      0.
0.300000        0.30055866      0.
0.200000        0.20040403      0.
0.100000        0.10021178      0.
0.              0.              0.

         ETA= -0.500000       X=  -0.833333

     Y              PSI               OMEGA
1.000000        1.00000000      0.
0.900000        0.90083916      0.10252753E-29
0.800000        0.80157924      0.88065375E-30
0.700000        0.70213985      0.46319564E-30
0.600000        0.60246950      0.19163717E-30
0.500000        0.50254725      0.68656199E-31
0.400000        0.40237921      0.22358907E-31
0.300000        0.30199277      0.68043745E-32
0.200000        0.20143083      0.19609239E-32
0.100000        0.10074671      0.51001661E-33
0.              0.              0.

         ETA= -0.400000       X=  -0.555556

     Y              PSI               OMEGA
1.000000        1.00000000      0.
0.900000        0.90216868      0.61900789E-19
0.800000        0.80401476      0.41005242E-19
0.700000        0.70532279      0.16782335E-19
0.600000        0.60600153      0.54591769E-20
0.500000        0.50605505      0.15485550E-20
0.400000        0.40554716      0.40055192E-21
0.300000        0.30457403      0.96823536E-22
0.200000        0.20324688      0.22162610E-22
0.100000        0.10168272      0.46650954E-23
0.              0.              0.

         ETA= -0.300000       X=  -0.357143

     Y              PSI               OMEGA
1.000000        1.00000000      0.
0.900000        0.90460014      0.15583476E-11
0.800000        0.80822561      0.77991103E-12
0.700000        0.71048675      0.24895769E-12
0.600000        0.61135613      0.64460059E-13
0.500000        0.51113917      0.14707135E-13
0.400000        0.40954686      0.30710784E-14
0.300000        0.30804370      0.59889296E-15
0.200000        0.20563272      0.11031529E-15
0.100000        0.10289589      0.18859117E-16
0.              0.              0.

         ETA= -0.200000       X=  -0.208333

     Y              PSI               OMEGA
1.000000        1.00000000      0.
0.900000        0.90889052      0.32946760E-06
0.800000        0.81483657      0.12212911E-06
0.700000        0.71778075      0.30528057E-07
0.600000        0.61839394      0.63553734E-08
0.500000        0.51731942      0.11790463E-08
0.400000        0.41504563      0.20089284E-09
0.300000        0.31193116      0.31949265E-10
0.200000        0.20824677      0.47876962E-11
0.100000        0.10420830      0.66900650E-12
0.              0.              0.

         ETA= -0.100000       X=  -0.092593

     Y              PSI               OMEGA
1.000000        1.00000000      0.
0.900000        0.91666494      0.28813207E-02
0.800000        0.82450245      0.76541896E-03
0.700000        0.72696678      0.14919141E-03
0.600000        0.62638012      0.25004745E-04
0.500000        0.52393366      0.37854358E-05
0.400000        0.42027833      0.52943381E-06
0.300000        0.31581365      0.69267405E-07
0.200000        0.21081353      0.85411441E-08
0.100000        0.10548501      0.98611201E-09
0.              0.              0.

         ETA=  0.                 X=   0.

     Y              PSI               OMEGA
1.000000        1.00000000      0.13746776E 02
0.900000        0.93126611      0.26435994E 01
0.800000        0.83734640      0.43835387E-00
0.700000        0.73722056      0.64876184E-01
0.600000        0.63454057      0.87660169E-02
0.500000        0.53037667      0.10997055E-02
0.400000        0.42523547      0.12950481E-03
0.300000        0.31942976      0.14416540E-04
0.200000        0.21317977      0.15235861E-05
0.100000        0.10665548      0.15194982E-06
0.              0.              0.

         ETA=  0.100000       X=   0.092593

     Y              PSI               OMEGA
1.000000        1.00000000      0.11040855E 02
0.900000        0.94479553      0.41864666E 01
0.800000        0.85175294      0.10810953E 01
0.700000        0.74910790      0.22429542E-00
0.600000        0.64389807      0.39790753E-01
0.500000        0.53764816      0.62596583E-02
0.400000        0.43076117      0.89388782E-03
0.300000        0.32342669      0.11762390E-03
0.200000        0.21578114      0.14397942E-04
0.100000        0.10793842      0.16318689E-05
0.              0.              0.

         ETA=  0.200000       X=   0.208333

     Y              PSI               OMEGA
1.000000        1.00000000      0.86174531E 01
0.900000        0.95651273      0.48832553E 01
0.800000        0.86832219      0.17885015E 01
0.700000        0.76435035      0.50473469E 00
0.600000        0.65635968      0.11827458E-00
0.500000        0.54742163      0.23991484E-01
0.400000        0.43819059      0.43300483E-02
0.300000        0.32879183      0.70842411E-03
0.200000        0.21926752      0.10635768E-03
0.100000        0.10965609      0.14551819E-04
0.              0.              0.

         ETA=  0.300000       X=   0.357143

     Y              PSI               OMEGA
1.000000        1.00000000      0.67917674E 01
0.900000        0.96604116      0.49122541E 01
0.800000        0.88477433      0.23772044E 01
0.700000        0.78155457      0.87490235E 00
0.600000        0.67192578      0.26465096E-00
0.500000        0.56009039      0.68645569E-01
0.400000        0.44798385      0.15692203E-01
0.300000        0.33591941      0.32224760E-02
0.200000        0.22391678      0.60214970E-03
0.100000        0.11195076      0.10105981E-03
0.              0.              0.
```

TABLE V. - Continued

```
        ETA=  0.400000       X=   0.555556
     Y               PSI                OMEGA
1.000000          1.00000000      0.55820941E 01
0.900000          0.97208953      0.45680230E 01
0.800000          0.89922122      0.27454748E 01
0.700000          0.80023418      0.12690274E 01
0.600000          0.69584505      0.48264877E-00
0.500000          0.57558084      0.15741839E-00
0.400000          0.46037006      0.45197792E-01
0.300000          0.34510554      0.11632405E-01
0.200000          0.23997198      0.27153654E-02
0.100000          0.11495548      0.56134817E-03
0.                0.              0.

        ETA=  0.500000       X=   0.833333
     Y               PSI                OMEGA
1.000000          1.00000000      0.48175140E 01
0.900000          0.97591243      0.41195094E 01
0.800000          0.91085637      0.28861513E 01
0.700000          0.81764205      0.16171505E 01
0.600000          0.70881455      0.75490654E 00
0.500000          0.59323210      0.30430561E-00
0.400000          0.47515021      0.10854062E-00
0.300000          0.35637604      0.34835161E-01
0.200000          0.23752204      0.10155377E-01
0.100000          0.11873373      0.25817136E-02
0.                0.              0.

        ETA=  0.600000       X=   1.250000
     Y               PSI                OMEGA
1.000000          1.00000000      0.43000593E 01
0.900000          0.97849970      0.37131269E 01
0.800000          0.91993110      0.28604969E 01
0.700000          0.83297132      0.18714982E 01
0.600000          0.72762828      0.10486230E 01
0.500000          0.61215907      0.51500078E 00
0.400000          0.49186043      0.22622366E-00
0.300000          0.36955237      0.90189508E-01
0.200000          0.24652426      0.32789584E-01
0.100000          0.12328468      0.10179666E-01
0.                0.              0.

        ETA=  0.700000       X=   1.944444
     Y               PSI                OMEGA
1.000000          1.00000000      0.38930922E 01
0.900000          0.98053454      0.33762594E 01
0.800000          0.92732348      0.27485757E 01
0.700000          0.84668934      0.20206252E 01
0.600000          0.74596138      0.13264581E 01
0.500000          0.63210983      0.78324500E 00
0.400000          0.51056745      0.42107762E-00
0.300000          0.38493620      0.20804124E-00
0.200000          0.25731701      0.93943179E-01
0.100000          0.12881911      0.35104311E-01
0.                0.              0.
```

```
        ETA=  0.800000       X=   3.333333
     Y               PSI                OMEGA
1.000000          1.00000000      0.35164683E 01
0.900000          0.98241766      0.30791166E 01
0.800000          0.93404856      0.26027779E 01
0.700000          0.85966769      0.20822117E 01
0.600000          0.76449542      0.15568321E 01
0.500000          0.65375804      0.10844286E 01
0.400000          0.53230567      0.70485355E 00
0.300000          0.40381280      0.42677472E-00
0.200000          0.27109270      0.23494624E-00
0.100000          0.13605198      0.10261630E-00
0.                0.              0.

        ETA=  0.900000       X=   7.499999
     Y               PSI                OMEGA
1.000000          1.00000000      0.31565710E 01
0.900000          0.98421714      0.28094103E 01
0.800000          0.94034071      0.24544445E 01
0.700000          0.87152164      0.20903302E 01
0.600000          0.78260281      0.17236982E 01
0.500000          0.67605221      0.13673478E 01
0.400000          0.55583455      0.10347246E 01
0.300000          0.43527644      0.73433769E 00
0.200000          0.28738113      0.46701828E-00
0.100000          0.14482033      0.22620650E-00
0.                0.              0.

        ETA=  1.000000       X=     ∞
     Y               PSI                OMEGA
1.000000          1.00000000      0.29999999E 01
0.900000          0.98549999      0.26999999E 01
0.800000          0.94399998      0.23999999E 01
0.700000          0.87849998      0.20999999E 01
0.600000          0.79199998      0.17999999E 01
0.500000          0.68749999      0.14999999E 01
0.400000          0.56799999      0.12000000E 01
0.300000          0.43649999      0.89999999E 00
0.200000          0.29600000      0.59999999E 00
0.100000          0.14950000      0.30000000E-00
0.                0.              0.
```

TABLE VI. Velocities And Pressure Gradients For Case II With A 20 × 10 Grid

ETA = -1.000000 X = $-\infty$

Y	U(X)	U(Y)	$-\partial P/\partial\eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.000000	0.09999999E 01	0.	0.	0.	0.
0.900000	0.09999999E 01	0.	0.	0.	0.
0.800000	0.09999999E 01	0.	0.	0.	0.
0.700000	0.09999999E 01	0.	0.	0.	0.
0.600000	0.09999999E 01	0.	0.	0.	0.
0.500000	0.09999999E 01	0.	0.	0.	0.
0.400000	0.09999999E 01	0.	0.	0.	0.
0.300000	0.09999999E 01	0.	0.	0.	0.
0.200000	0.09999999E 01	0.	0.	0.	0.
0.100000	0.09999999E 01	0.	0.	0.	0.
0.	0.09999999E 01	0.	0.	0.	0.

ETA = -0.900000 X = -7.499999

Y	U(X)	U(Y)	$-\partial P/\partial\eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.000000	0.10000001E 01	0.	0.47311193E-04	-0.56773442E-06	0.
0.900000	0.10000000E 01	-0.56773432E-07	0.44144701E-04	-0.52973650E-06	-0.27251282E-07
0.800000	0.99999998E 00	-0.10594727E-06	0.37066630E-04	-0.44479963E-06	-0.51069319E-07
0.700000	0.09999999E 01	-0.14573338E-06	0.27380884E-04	-0.32857066E-06	-0.70166654E-07
0.600000	0.09999999E 01	-0.17166140E-06	0.14342378E-04	-0.17210857E-06	-0.82397512E-07
0.500000	0.09999999E 01	-0.17993154E-06	-0.55879354E-06	0.67055236E-08	-0.86367152E-07
0.400000	0.09999999E 01	-0.17032029E-06	-0.14714891E-04	0.17657872E-06	-0.82075588E-07
0.300000	0.99999994E 00	-0.14461579E-06	-0.27241174E-04	0.32689414E-06	-0.69630112E-07
0.200000	0.99999995E 00	-0.10494144E-06	-0.37252907E-04	0.44703496E-06	-0.50371850E-07
0.100000	0.09999999E 01	-0.55264691E-07	-0.43725593E-04	0.52470721E-06	-0.26527025E-07
0.	0.99999999E 00	0.	-0.46053900E-04	0.55264689E-06	0.

ETA = -0.800000 X = -3.333333

Y	U(X)	U(Y)	$-\partial P/\partial\eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.000000	0.99999070E 00	0.	0.13980884E-02	-0.67108246E-04	0.
0.900000	0.99999719E 00	-0.67108870E-05	0.13278667E-02	-0.63737605E-04	-0.92280035E-05
0.800000	0.99999753E 00	-0.12747645E-04	0.11259560E-02	-0.54045889E-04	-0.17540058E-04
0.700000	0.99999458E 00	-0.17520189E-04	0.81359027E-03	-0.39052334E-04	-0.24106960E-04
0.600000	0.99999718E 00	-0.20558237E-04	0.42224833E-03	-0.20267920E-04	-0.28285839E-04
0.500000	0.10000000E 01	-0.21573007E-04	-0.84880950E-05	0.40742857E-06	-0.29683248E-04
0.400000	0.10000028E 01	-0.20476878E-04	-0.43596511E-03	0.20926326E-04	-0.28173147E-04
0.300000	0.10000054E 01	-0.17387867E-04	-0.81901775E-03	0.39312852E-04	-0.23923893E-04
0.200000	0.10000074E 01	-0.12614429E-04	-0.11209991E-02	0.53807959E-04	-0.17358277E-04
0.100000	0.10000087E 01	-0.66266208E-05	-0.13140157E-02	0.63072755E-04	-0.91175628E-05
0.	0.10000092E 01	0.	-0.13805587E-02	0.66266818E-04	0.

ETA = -0.700000 X = -1.944444

Y	U(X)	U(Y)	$-\partial P/\partial\eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.000000	0.99972054E 00	0.	0.11003294E-01	-0.11883558E-02	0.
0.900000	0.99973448E 00	-0.11886880E-03	0.10427736E-01	-0.11261955E-02	-0.27937941E-03
0.800000	0.99977478E 00	-0.22530556E-03	0.87727755E-02	-0.94745978E-03	-0.52920388E-03
0.700000	0.99983729E 00	-0.30842721E-03	0.62418103E-02	-0.67411552E-03	-0.72374790E-03
0.600000	0.99991556E 00	-0.36019519E-03	0.31284042E-02	-0.33786765E-03	-0.84426080E-03
0.500000	0.10000017E 01	-0.37606716E-03	-0.23300787E-03	0.25164851E-04	-0.88037168E-03
0.400000	0.10000871E 01	-0.35522841E-03	-0.35073229E-02	0.37879089E-03	-0.83060868E-03
0.300000	0.10001637E 01	-0.30037455E-03	-0.63917290E-02	0.69030675E-03	-0.70161632E-03
0.200000	0.10002341E 01	-0.21723177E-03	-0.86376891E-02	0.93287043E-03	-0.50700646E-03
0.100000	0.10002627E 01	-0.11386430E-03	-0.10059950E-01	0.10864746E-02	-0.26561326E-03
0.	0.10002761E 01	0.	-0.10545901E-01	0.11389573E-02	0.

ETA = -0.600000 X = -1.250000

Y	U(X)	U(Y)	$-\partial P/\partial\eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.000000	0.99778943E 00	0.	0.40470925E-01	-0.77704178E-02	0.
0.900000	0.99790502E 00	-0.77876329E-03	0.38064186E-01	-0.73083238E-02	-0.23151469E-02
0.800000	0.99823736E 00	-0.14650941E-02	0.31302935E-01	-0.60101636E-02	-0.43360032E-02
0.700000	0.99874560E 00	-0.19841909E-02	0.21355622E-01	-0.41002796E-02	-0.58341661E-02
0.600000	0.99937089E 00	-0.22884965E-02	0.96769702E-02	-0.18579783E-02	-0.66774623E-02
0.500000	0.10000460E 01	-0.23590779E-02	-0.23327570E-02	0.44788935E-03	-0.68289124E-02
0.400000	0.10007037E 01	-0.22021436E-02	-0.13507869E-01	0.25935109E-02	-0.63270529E-02
0.300000	0.10012832E 01	-0.18435251E-02	-0.22970404E-01	0.44103177E-02	-0.52627894E-02
0.200000	0.10017344E 01	-0.13231479E-02	-0.30108973E-01	0.57809229E-02	-0.37586843E-02
0.100000	0.10020701E 01	-0.69033623E-03	-0.34533609E-01	0.66304530E-02	-0.19551493E-02
0.	0.10021178E 01	0.	-0.36031157E-01	0.69179822E-02	0.

ETA = -0.500000 X = -0.833333

Y	U(X)	U(Y)	$-\partial P/\partial\eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.000000	0.99160843E 00	0.	0.96563771E-01	-0.28969131E-01	0.
0.900000	0.99210778E 00	-0.29214285E-02	0.89087756E-01	-0.26726326E-01	-0.99090282E-02
0.800000	0.99349651E 00	-0.53936429E-02	0.69261560E-01	-0.20778468E-01	-0.17946339E-01
0.700000	0.99554874E 00	-0.71238129E-02	0.42655765E-01	-0.12796729E-01	-0.23088548E-01
0.600000	0.99796303E 00	-0.79974866E-02	0.14458437E-01	-0.43375313E-02	-0.25173797E-01
0.500000	0.10004514E 01	-0.80334617E-02	-0.11792430E-01	0.35377289E-02	-0.24564064E-01
0.400000	0.10027723E 01	-0.73297339E-02	-0.34133208E-01	0.10239962E-01	-0.21827240E-01
0.300000	0.10047418E 01	-0.60230567E-02	-0.51685358E-01	0.15505607E-01	-0.17542736E-01
0.200000	0.10062303E 01	-0.42642903E-02	-0.64180563E-01	0.19254169E-01	-0.12215085E-01
0.100000	0.10071541E 01	-0.22064242E-02	-0.71625993E-01	0.21487798E-01	-0.62552328E-02
0.	0.10074671E 01	0.	-0.74096660E-01	0.22228997E-01	0.

ETA = -0.400000 X = -0.555556

Y	U(X)	U(Y)	$-\partial P/\partial\eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.000000	0.97831324E 00	0.	0.18397111E-00	-0.79475522E-01	0.
0.900000	0.97992421E 00	-0.81237296E-02	0.16221704E-00	-0.70077760E-01	-0.32198653E-01
0.800000	0.98422941E 00	-0.14356147E-01	0.11105159E-00	-0.47974287E-01	-0.53667434E-01
0.700000	0.99006612E 00	-0.18029288E-01	0.53814171E-01	-0.23247721E-01	-0.62751461E-01
0.600000	0.99633873E 00	-0.19281536E-01	0.33126841E-02	-0.14310795E-02	-0.62347411E-01
0.500000	0.10022718E 01	-0.18558547E-01	-0.36463622E-01	0.15752284E-01	-0.55997370E-01
0.400000	0.10074051E 01	-0.16346127E-01	-0.65755736E-01	0.28406477E-01	-0.46421821E-01
0.300000	0.10115013E 01	-0.13069951E-01	-0.86183099E-01	0.37231098E-01	-0.35332933E-01
0.200000	0.10144565E 01	-0.50760600E-02	-0.99451663E-01	0.42963117E-01	-0.23663148E-01
0.100000	0.10162344E 01	-0.46422277E-02	-0.10687978E-00	0.46172066E-01	-0.11838751E-01
0.	0.10166272E 01	0.	-0.10926722E-00	0.47203439E-01	0.

TABLE VI. - Continued

ETA = -0.300000 X = -0.357143

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.000000	0.95395857E 00	0.	0.32063125E-00	-0.18853117E-00	0.
0.900000	0.95887195E 00	-0.19762207E-01	0.25614250E-00	-0.15061179E-00	-0.96606018E-01
0.800000	0.97056698E 00	-0.31816126E-01	0.13179978E-00	-0.77498271E-01	-0.13510048E-00
0.700000	0.98414738E 00	-0.36626377E-01	0.30222269E-01	-0.17770693E-01	-0.13388009E-00
0.600000	0.99673789E 00	-0.36433660E-01	-0.36969180E-01	0.21737877E-01	-0.11561851E-00
0.500000	0.10072463E 01	-0.33117244E-01	-0.78140606E-01	0.45946676E-01	-0.92803939E-01
0.400000	0.10154773E 01	-0.27925500E-01	-0.10256874E-00	0.60310418E-01	-0.70580871E-01
0.300000	0.10215706E 01	-0.21629967E-01	-0.11675873E-00	0.68654136E-01	-0.50448149E-01
0.200000	0.10257790E 01	-0.14699646E-01	-0.12471263E-00	0.73331030E-01	-0.32380766E-01
0.100000	0.10281635E 01	-0.74251780E-02	-0.12871821E-00	0.75686309E-01	-0.15808038E-01
0.	0.10289889E 01	0.	-0.12993542E-00	0.76402029E-01	0.

ETA = -0.200000 X = -0.208333

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.000000	0.91105484E 00	0.	0.54960889E 00	-0.42209961E-00	0.
0.900000	0.92581715E 00	-0.46328828E-01	0.35897228E-00	-0.27569070E-00	-0.28708469E-00
0.800000	0.95554885E 00	-0.62503050E-01	0.81043612E-01	-0.62241492E-01	-0.29214682E-00
0.700000	0.98221717E 00	-0.63283307E-01	-0.50953681E-01	0.39132426E-01	-0.22738344E-00
0.600000	0.10023066E 01	-0.57538517E-01	-0.10499516E-00	0.80636282E-01	-0.16508909E-00
0.500000	0.10167415E 01	-0.49130858E-01	-0.12593228E-00	0.96715994E-01	-0.11754841E-00
0.400000	0.10269412E 01	-0.39672845E-01	-0.13334031E-00	0.10240535E-00	-0.82509406E-01
0.300000	0.10339943E 01	-0.29836635E-01	-0.13535792E-00	0.10395488E-00	-0.55980496E-01
0.200000	0.10386142E 01	-0.19894315E-01	-0.13543947E-00	0.10401751E-00	-0.34796082E-01
0.100000	0.10412738E 01	-0.99422225E-02	-0.13508075E-00	0.10374202E-00	-0.16694693E-01
0.	0.10420830E 01	0.	-0.13490392E-00	0.10360621E-00	0.

ETA = -0.100000 X = -0.092593

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.000000	0.83335057E 00	0.	0.93273112E 00	-0.90661464E 00	0.
0.900000	0.87748777E 00	-0.10874540E-00	0.39543816E-00	-0.38436588E-00	-0.83193090E 00
0.800000	0.94849084E 00	-0.10939776E-00	-0.13037907E-00	0.12672845E-00	-0.50121567E 00
0.700000	0.99061161E 00	-0.94477469E-01	-0.18732624E-00	0.18208110E-00	-0.28749833E-00
0.600000	0.10151655E 01	-0.78472620E-01	-0.17701577E-00	0.17205932E-00	-0.17661678E-00
0.500000	0.10305090E 01	-0.63458217E-01	-0.16135824E-00	0.15684021E-00	-0.11539404E-00
0.400000	0.10406000E 01	-0.49522633E-01	-0.14873195E-00	0.14456746E-00	-0.77530818E-01
0.300000	0.10473739E 01	-0.36443229E-01	-0.13964808E-00	0.13573793E-00	-0.51428335E-01
0.200000	0.10516432E 01	-0.23974377E-01	-0.13361747E-00	0.12987617E-00	-0.31592835E-01
0.100000	0.10540676E 01	-0.11893292E-01	-0.13018435E-00	0.12653919E-00	-0.15072665E-01
0.	0.10548801E 01	0.	-0.12907037E-00	0.12545640E-00	0.

ETA = 0. X = 0.

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.000000	0.	0.	0.12336863E 01	0.14804235E 01	0.38964686E-00
0.900000	0.81326801E 00	-0.15190515E-00	0.10538635E 01	0.12646362E 01	-0.23227257E-00
0.800000	0.97022779E 00	-0.14715268E-00	0.45188198E-00	0.54225838E 00	-0.24559270E-00
0.700000	0.10140791E 01	-0.11956205E-00	0.28900034E-00	0.34680041E-00	-0.18475034E-00
0.600000	0.10342194E 01	-0.94596511E-01	0.21401937E-00	0.25682323E-00	-0.12734823E-00
0.500000	0.10465755E 01	-0.74058258E-01	0.17253742E-00	0.20704491E-00	-0.87806862E-01
0.400000	0.10547744E 01	-0.56607377E-01	0.14803938E-00	0.17764726E-00	-0.60732333E-01
0.300000	0.10602784E 01	-0.41110378E-01	0.13310764E-00	0.15972917E-00	-0.40988678E-01
0.200000	0.10638714E 01	-0.26825140E-01	0.12412081E-00	0.14894497E-00	-0.25443409E-01
0.100000	0.10658988E 01	-0.13248464E-01	0.11928201E-00	0.14313842E-00	-0.12207551E-01
0.	0.10665847E 01	0.	0.11775179E-00	0.14130215E-00	0.

ETA = 0.100000 X = 0.092593

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.000000	0.	0.	0.94025081E 00	0.91392378E 00	0.31407805E-00
0.900000	0.74123828E 00	-0.12464253E-00	0.85145168E 00	0.82761102E 00	0.11433577E-00
0.800000	0.97843815E 00	-0.15054233E-00	0.53841885E 00	0.52334312E 00	0.21132277E-01
0.700000	0.10392743E 01	-0.13185078E-00	0.39022869E-00	0.37930228E-00	-0.40458361E-01
0.600000	0.10572986E 01	-0.10604089E-00	0.30322365E-00	0.29473338E-00	-0.52087488E-01
0.500000	0.10656844E 01	-0.82838511E-01	0.24618454E-00	0.23929136E-00	-0.45680553E-01
0.400000	0.10711073E 01	-0.62961871E-01	0.20936506E-00	0.20350283E-00	-0.35670564E-01
0.300000	0.10749001E 01	-0.45499612E-01	0.18610589E-00	0.18089493E-00	-0.25792138E-01
0.200000	0.10774413E 01	-0.29586465E-01	0.17196656E-00	0.16715150E-00	-0.16670168E-01
0.100000	0.10789057E 01	-0.14582966E-01	0.16434806E-00	0.15974630E-00	-0.81711894E-02
0.	0.10793842E 01	0.	0.16194081E-00	0.15740646E-00	0.

ETA = 0.200000 X = 0.208333

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.000000	0.	0.	0.64829826E 00	0.49789305E-00	0.18695021E-00
0.900000	0.65838907E 00	-0.81583210E-01	0.58814798E 00	0.45169763E-00	0.16449983E-00
0.800000	0.96281189E 00	-0.12680210E-00	0.51158941E 00	0.39320786E-00	0.15252821E-00
0.700000	0.10598125E 01	-0.12613274E-00	0.42118860E-00	0.32347284E-00	0.83188342E-01
0.600000	0.10846435E 01	-0.10762639E-00	0.36250860E-00	0.27840660E-00	0.32036474E-01
0.500000	0.10908454E 01	-0.86178184E-01	0.31432162E-00	0.24139900E-00	0.75139558E-02
0.400000	0.10931490E 01	-0.66135090E-01	0.27676532E-00	0.21255577E-00	-0.21358503E-02
0.300000	0.10946153E 01	-0.47972043E-01	0.25008357E-00	0.19206417E-00	-0.48555639E-02
0.200000	0.10956787E 01	-0.31240817E-01	0.23275318E-00	0.17875443E-00	-0.43951790E-02
0.100000	0.10963775E 01	-0.15407383E-01	0.22309045E-00	0.17133345E-00	-0.24827962E-02
0.	0.10965609E 01	0.	0.21999127E-00	0.16895328E-00	0.

ETA = 0.300000 X = 0.357143

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.000000	0.	0.	0.42619350E-00	0.25060178E-00	0.94838385E-01
0.900000	0.57612836E 00	-0.44619799E-01	0.34014023E-00	0.20000245E-00	0.11126052E-00
0.800000	0.92043094E 00	-0.90843154E-01	0.43133275E-00	0.25362366E-00	0.14776223E-00
0.700000	0.10642427E 01	-0.10549846E-00	0.40120400E-00	0.23590795E-00	0.12889132E-00
0.600000	0.11093728E 01	-0.98446980E-01	0.38284674E-00	0.22511388E-00	0.87305600E-01
0.500000	0.11197095E 01	-0.82788093E-01	0.36474609E-00	0.21447070E-00	0.53296334E-01
0.400000	0.11208848E 01	-0.65207671E-01	0.34346817E-00	0.20195928E-00	0.31345811E-01
0.300000	0.11203753E 01	-0.47962304E-01	0.32372016E-00	0.19034745E-00	0.17466458E-01
0.200000	0.11198431E 01	-0.31471110E-01	0.30875593E-00	0.18154848E-00	0.97024744E-02
0.100000	0.11195838E 01	-0.15580204E-01	0.29968675E-00	0.17621581E-00	0.42358176E-02
0.	0.11195076E 01	0.	0.29665827E-00	0.17443506E-00	0.

TABLE VI. - Continued

ETA = C.400000 X = 0.555556

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.00CCCC	C.	C.	0.31258491E-00	0.13520948E-00	0.44039814E-01
C.90CCCC	C.50389*5CE CC	-C.21321552E-01	C.20631585E-00	0.89128447E-01	0.56227710E-01
0.80CCCC	C.85927*76E CC	-C.56337219E-01	0.35070504E-00	0.15150457E-00	0.91049282E-01
C.7CCCCC	C.1C46E*08E C1	-C.7686EC51E-01	0.37574331E-00	0.16232111E-00	0.10583328E-00
0.60CCCC	C.11232*66E C1	-C.79679742E-01	0.38285314E-00	0.16539255E-00	0.93288837E-01
C.5CCCCC	C.11473749E C1	-C.71586076E-01	0.39494664E-00	0.17061695E-00	0.71063154E-01
C.40CCCC	C.11523765E C1	-C.58679318E-01	0.40081573E-00	0.17315239E-00	0.50280061E-01
C.3CCCCC	C.115199C4E C1	-C.44186313E-01	0.39957620E-00	0.17261691E-00	0.33622450E-01
C.2CCCCC	C.11507*02E C1	-0.29387363E-C1	0.39533022E-00	0.17078265E-00	0.20505212E-01
0.10CCCC	C.11458*99E C1	-C.14651153E-C1	C.39156146E-00	0.16915455E-00	0.96963452E-02
C.	C.11495*48E C1	C.	0.39004246E-00	0.16849834E-00	0.

ETA = C.500000 X = 0.833333

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.CCCCCC	C.	0.	0.31C22427E-00	0.93067281E-01	0.20698187E-01
C.90CCCC	C.44571*13E-C0	-C.96152498E-02	0.21C76243E-00	0.63228730E-01	0.24861351E-01
C.8CCCCC	C.79185792E CC	-C.31064819E-01	0.31959519E-00	0.95878556E-01	0.42658168E-01
0.70CCCC	C.1C102*91E C1	-C.49105714E-01	0.38510072E-00	0.11553022E-00	0.60146527E-01
C.60CCCC	C.11215497E C1	-0.56674840E-01	0.40502119E-00	0.12150636E-00	0.64467368E-01
0.50CCCC	C.11683717E C1	-C.54867344E-01	0.42978168E-00	0.12893450E-00	0.57670863E-01
C.4CCCCC	C.11E42*C2E C1	-C.47235554E-01	0.45708095E-00	0.13712428E-00	0.46146704E-01
C.30CCC0	C.11E814C8E C1	-C.3667C261E-01	0.47744758E-00	0.14323428E-00	0.33797868E-01
C.2CCCCC	C.11E82715E C1	-C.2482E420E-01	0.48920428E-00	0.14676128E-00	0.21968396E-01
C.1CCCCC	C.11876702E C1	-C.124938C0E-01	0.49458018E-00	0.14837405E-00	0.10793401E-01
C.	C.11873*73E C1	C.	0.49562594E-00	0.14868778E-00	0.

ETA = C.600000 X = 1.250000

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.00C0C0	C.	C.	0.40759199E-00	0.78257664E-01	0.10418358E-01
0.90CCCC	C.4CC34451E-00	-0.44372249E-02	0.32844592E-00	0.63061618E-01	0.10793725E-01
0.80CCCC	C.72764788E C0	-0.15808425E-01	0.37972569E-00	0.72907333E-01	0.16559631E-01
C.70CCCC	C.96151412E CC	-C.279814C5E-01	0.45761981E-00	0.87863004E-01	0.25298740E-01
0.60CCCC	C.11C4C612E C1	-C.3566C956E-01	0.49094352E-00	0.94261157E-01	0.30975007E-01
C.50CCCC	C.11788*91E 01	-0.37322623E-01	0.51542228E 00	0.98961079E-01	0.31231278E-01
C.40CCCC	C.1213C*34E C1	-0.3400C550E-01	0.54901273E 00	0.10541044E-00	0.27360567E-01
0.30CCCC	C.12266*C9E C1	-C.27417755E-01	0.58383922E 00	0.11209713E-00	0.21344184E-01
0.20CCCC	C.12913*85E C1	-C.19003168E-01	0.61067162E 00	0.11724895E-00	0.14455435E-01
C.10CCCC	C.12926712E C1	-0.96819654E-02	0.62592120E 00	0.12017687E-00	0.72698268E-02
0.	C.1232E468E C1	C.	0.62875566E 00	0.12072109E-00	0.

ETA = C.700000 X = 1.944444

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.00CCCC	C.	C.	0.638C6528E 00	0.68911051E-01	0.54233842E-02
0.90CCC0	C.36938759E-CC	-C.21156996E-02	0.57592122E 00	0.62199493E-01	0.48160607E-02
C.80CCCC	C.66922*98E CC	-C.762343C9E-02	0.58739269E 00	0.63438412E-01	0.58248174E-02
0.70CCCC	C.90681*54E C0	-0.14416039E-01	0.65117907E 00	0.70327340E-01	0.84223060E-02
0.60CCCC	C.10728975E C1	-C.19908261E-01	0.69811418E 00	0.75396333E-01	0.11006797E-01
0.50CCCC	C.11769696E C1	-C.22485040E-01	0.72547035E 00	0.78350799E-01	0.12114757E-01
C.40CCCC	C.12358681E C1	-0.2184C423E-01	0.75778046E 00	0.81840290E-01	0.11413349E-01
0.30CCCC	C.12662*22E C1	-C.1850C628E-01	0.79966220E 00	0.86363519E-01	0.93516776E-02
C.20CCCC	C.12805*54E C1	-C.13266557E-01	0.83975046E 00	0.90693051E-01	0.65163270E-02
0.10C0CC	C.12865*5CE C1	-C.68943415E-02	0.86492570E 00	0.93411977E-01	0.33228719E-02
0.	C.12881911E C1	C.	0.86567468E 00	0.93492867E-01	0.

ETA = C.800000 X = 3.333333

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.00CCCC	C.	C.	0.12148658E 01	0.58313560E-01	0.23033426E-02
0.90CCCC	C.32975718E-C0	-0.88382363E-03	0.11642403E 01	0.55883539E-01	0.18585636E-02
0.80CCCC	C.61374985E CC	-C.31241333E-02	0.11618107E 01	0.55766917E-01	0.18501352E-02
C.7CCCCC	C.E4776569E CC	-0.605575C2E-02	0.12076292E 01	0.57966205E-01	0.23821115E-02
0.60CCCC	C.1C293482E C1	-0.87939441E-02	0.12623393E 01	0.60592290E-01	0.31807785E-02
0.50CCCC	C.11609487E C1	-C.1C546170E-01	0.12990617E 01	0.62354963E-01	0.38056816E-02
C.40C0C0	C.12499762E C1	-0.10864105E-01	0.13253812E 01	0.63618297E-01	0.39381809E-02
C.30CCCC	C.13C60648E C1	-0.96816579E-02	0.13541070E 01	0.64997142E-01	0.34992789E-02
0.20CCCC	C.13388041E C1	-C.72153890E-02	0.13823701E 01	0.66353765E-01	0.25871935E-02
0.10CCCC	C.13554634E C1	-C.38402925E-02	0.13944442E 01	0.66933325E-01	0.13666206E-02
0.	C.13605798E C1	C.	0.13735440E 01	0.65930114E-01	0.

ETA = C.900000 X = 7.499999

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.CCCCOC	C.	C.	0.38573404E 01	0.46288092E-01	0.25051383E-03
C.9CCCOC	C.25825647E-CC	-C.18493968E-03	6.38264253E 01	0.45917112E-01	0.18996381E-03
0.80CCCC	C.5614775CE CC	-C.59708530E-03	0.38451005E 01	0.46141213E-01	0.18603746E-03
C.70CCCC	C.78868548E CC	-C.11299379E-02	0.38964088E 01	0.46756913E-01	0.25691768E-03
C.6CCCCC	C.97934715E C0	-C.16502736E-02	0.39466012E 01	0.47359222E-01	0.38127790E-03
0.5CCCCC	C.11338412E C1	-C.2C221177E-02	0.39576039E 01	0.47491255E-01	0.50949004E-03
0.40CCCC	C.12538788E C1	-C.21416597E-02	0.39121702E 01	0.46946050E-01	0.58587992E-03
C.3CCCCC	C.13422871E C1	-C.19612320E-02	0.38195246E 01	0.45834303E-01	0.57101344E-03
C.20CCCC	C.14C22*C5E C1	-C.14944381E-02	0.37027855E 01	0.44433434E-01	0.45367420E-03
C.10CCCC	C.14369*56E C1	-C.80688124E-03	C.35856149E 01	0.43027387E-01	0.25081922E-03
C.	C.14482*33E C1	C.	0.34871783E 01	0.41846146E-01	0.

ETA = 1.CCCCC0 X = ∞

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.CCCCCC	C.	C.	∞	0.39999999E-01	0.
C.9CCCCC	C.285CC*C3E-CC	C.	∞	0.39999999E-01	0.
C.8CCCCC	C.54COC*C1E CC	C.	∞	0.39999999E-01	0.
C.7CCCCC	C.765OC*C1E CC	C.	∞	0.39999999E-01	0.
C.6CCCCC	C.96COC*C1E CC	C.	∞	0.39999999E-01	0.
C.5CCCCC	C.1125C*CCE C1	C.	∞	0.39999999E-01	0.
C.4CCCCC	C.12599999E C1	C.	∞	0.39999999E-01	0.
C.3CCCCC	C.13649999E C1	C.	∞	0.39999999E-01	0.
C.2CCCC0	C.14399999E C1	C.	∞	0.39999999E-01	0.
C.1CCCCC	C.1485C*CCE C1	C.	∞	0.39999999E-01	0.
C.	C.15COC*CCE C1	C.	∞	0.39999999E-01	0.

TABLE VII. Stream Functions And Vorticities For Case II With A 40 x 20 Grid

```
        ETA= -1.000000        X=  -  ∞

      Y                PSI               ØMEGA
 1.000000          1.00000000     0.
 0.950000          0.95000000     0.
 0.900000          0.89999998     0.
 0.850000          0.84999999     0.
 0.800000          0.80000000     0.
 0.750000          0.74999999     0.
 0.700000          0.70000000     0.
 0.650000          0.64999998     0.
 0.600000          0.59999999     0.
 0.550000          0.55000000     0.
 0.500000          0.49999999     0.
 0.450000          0.45000000     0.
 0.400000          0.40000000     0.
 0.350000          0.34999999     0.
 0.300000          0.30000000     0.
 0.250000          0.24999999     0.
 0.200000          0.20000000     0.
 0.150000          0.15000000     0.
 0.100000          0.09999999     0.
 0.050000          0.05000000     0.
 0.                0.             0.
```

```
        ETA= -0.700000        X=  -1.944444

      Y                PSI               ØMEGA
 1.000000          1.00000000     0.
 0.950000          0.95001166     0.
 0.900000          0.90002304     0.
 0.850000          0.85003386     0.
 0.800000          0.80004384     0.
 0.750000          0.75005271     0.
 0.700000          0.70006028     0.
 0.650000          0.65006635     0.
 0.600000          0.60007080     0.
 0.550000          0.55007350     0.
 0.500000          0.50007441     0.
 0.450000          0.45007347     0.
 0.400000          0.40007073     0.
 0.350000          0.35006624     0.
 0.300000          0.30006012     0.
 0.250000          0.25005253     0.
 0.200000          0.20004366     0.
 0.150000          0.15003372     0.
 0.100000          0.10002295     0.
 0.050000          0.05001161     0.
 0.                0.             0.
```

```
        ETA= -0.650000        X=  -1.547619

      Y                PSI               ØMEGA
 1.000000          1.00000000     0.
 0.950000          0.95003982     0.
 0.900000          0.90007865     0.
 0.850000          0.85011552     0.
 0.800000          0.80014948     0.
 0.750000          0.75017969     0.
 0.700000          0.70020542     0.
 0.650000          0.65022602     0.
 0.600000          0.60024101     0.
 0.550000          0.55025004     0.
 0.500000          0.50025289     0.
 0.450000          0.45024951     0.
 0.400000          0.40024000     0.
 0.350000          0.35022464     0.
 0.300000          0.30020378     0.
 0.250000          0.25017796     0.
 0.200000          0.20014784     0.
 0.150000          0.15011412     0.
 0.100000          0.10007764     0.
 0.050000          0.05003930     0.
 0.                0.             0.
```

```
        ETA= -0.600000        X=  -1.250000

      Y                PSI               ØMEGA
 1.000000          1.00000000     0.
 0.950000          0.95010158     0.
 0.900000          0.90020058     0.
 0.850000          0.85029440     0.
 0.800000          0.80038068     0.
 0.750000          0.75045720     0.
 0.700000          0.70052209     0.
 0.650000          0.65057375     0.
 0.600000          0.60061102     0.
 0.550000          0.55063301     0.
 0.500000          0.50063933     0.
 0.450000          0.45062992     0.
 0.400000          0.40060513     0.
 0.350000          0.35056565     0.
 0.300000          0.30051254     0.
 0.250000          0.25044715     0.
 0.200000          0.20037112     0.
 0.150000          0.15028629     0.
 0.100000          0.10019469     0.
 0.050000          0.05009851     0.
 0.                0.             0.
```

```
        ETA= -0.550000        X=  -1.018519

      Y                PSI               ØMEGA
 1.000000          1.00000000     0.
 0.950000          0.95021302     0.26258718E-34
 0.900000          0.90042032     0.44232248E-34
 0.850000          0.85061630     0.50455405E-34
 0.800000          0.80079579     0.46789384E-34
 0.750000          0.75095417     0.37673295E-34
 0.700000          0.70108747     0.27271678E-34
 0.650000          0.65119255     0.18149822E-34
 0.600000          0.60126710     0.11279999E-34
 0.550000          0.55130965     0.66235085E-35
 0.500000          0.50131958     0.37077540E-35
 0.450000          0.45129713     0.19926144E-35
 0.400000          0.40124331     0.10332227E-35
 0.350000          0.35115979     0.52068108E-36
 0.300000          0.30104893     0.25588498E-36
 0.250000          0.25091361     0.12119023E-36
 0.200000          0.20075721     0.56515649E-37
 0.150000          0.15058347     0.25284977E-37
 0.100000          0.10039647     0.10872085E-37
 0.050000          0.05020049     0.31911933E-38
 0.                0.             0.
```

```
        ETA= -0.500000        X=  -0.833333

      Y                PSI               ØMEGA
 1.000000          1.00000000     0.
 0.950000          0.95039009     0.13450667E-27
 0.900000          0.90076888     0.21765863E-27
 0.850000          0.85112551     0.23411486E-27
 0.800000          0.80145007     0.20213974E-27
 0.750000          0.75173400     0.15027393E-27
 0.700000          0.70197022     0.99898455E-28
 0.650000          0.65215342     0.60839317E-28
 0.600000          0.60228004     0.34522779E-28
 0.550000          0.55234820     0.18481476E-28
 0.500000          0.50235762     0.94234017E-29
 0.450000          0.45230956     0.46105197E-29
 0.400000          0.40220652     0.21774041E-29
 0.350000          0.35205209     0.99738088E-30
 0.300000          0.30185091     0.44486089E-30
 0.250000          0.25160832     0.19383154E-30
 0.200000          0.20133033     0.82696964E-31
 0.150000          0.15102348     0.34545256E-31
 0.100000          0.10069466     0.13942090E-31
 0.050000          0.05035105     0.48895035E-32
 0.                0.             0.
```

```
        ETA= -0.450000        X=  -0.681818

      Y                PSI               ØMEGA
 1.000000          1.00000000     0.
 0.950000          0.95064943     0.36270661E-22
 0.900000          0.90127800     0.55871664E-22
 0.850000          0.85186607     0.56104781E-22
 0.800000          0.80239616     0.44698509E-22
 0.750000          0.75285388     0.30453616E-22
 0.700000          0.70322815     0.18482316E-22
 0.650000          0.65351140     0.10254496E-22
 0.600000          0.60369945     0.52954122E-23
 0.550000          0.55379114     0.25785344E-23
 0.500000          0.50378793     0.11956201E-23
 0.450000          0.45369354     0.53192277E-24
 0.400000          0.40351348     0.22841740E-24
 0.350000          0.35325473     0.95128669E-25
 0.300000          0.30292533     0.38573005E-25
 0.250000          0.25253423     0.15276664E-25
 0.200000          0.20209093     0.59241667E-26
 0.150000          0.15160546     0.22512493E-26
 0.100000          0.10108811     0.83045968E-27
 0.050000          0.05054941     0.27093450E-27
 0.                0.             0.
```

```
        ETA= -0.400000        X=  -0.555556

      Y                PSI               ØMEGA
 1.000000          1.00000000     0.
 0.950000          0.95101073     0.10993011E-17
 0.900000          0.90198427     0.15927746E-17
 0.850000          0.85288639     0.14757872E-17
 0.800000          0.80368826     0.10743511E-17
 0.750000          0.75436766     0.66576637E-18
 0.700000          0.70490953     0.36681934E-18
 0.650000          0.65530547     0.18467776E-18
 0.600000          0.60555283     0.86554505E-19
 0.550000          0.55565380     0.38270015E-19
 0.500000          0.50561424     0.16121545E-19
 0.450000          0.45544285     0.65192562E-20
 0.400000          0.40515023     0.25455134E-20
 0.350000          0.35474835     0.96417945E-21
 0.300000          0.30425008     0.35561186E-21
 0.250000          0.25366884     0.12810449E-21
 0.200000          0.20301826     0.45185053E-22
 0.150000          0.15231223     0.15624999E-22
 0.100000          0.10156459     0.52629420E-23
 0.050000          0.05078921     0.15923447E-23
 0.                0.             0.
```

TABLE VII. - Continued

```
          ETA= -0.350000        X=  -0.448718

     Y                PSI              ØMEGA
 1.000000          1.00000000    0.
 0.950000          0.95150056    0.62742257E-14
 0.900000          0.90293527    0.84184471E-14
 0.850000          0.85424592    0.71001118E-14
 0.800000          0.80538730    0.46739778E-14
 0.750000          0.75632896    0.26145697E-14
 0.700000          0.70705432    0.13008860E-14
 0.650000          0.65755846    0.59217964E-15
 0.600000          0.60784530    0.25131939E-15
 0.550000          0.55792502    0.10076561E-15
 0.500000          0.50781192    0.38540404E-16
 0.450000          0.45752276    0.14164502E-16
 0.400000          0.40707560    0.50304693E-17
 0.350000          0.35648902    0.17340512E-17
 0.300000          0.30578162    0.58224726E-18
 0.250000          0.25497176    0.19098856E-18
 0.200000          0.20407739    0.61346638E-19
 0.150000          0.15311605    0.19324555E-19
 0.100000          0.10210488    0.59442157E-20
 0.050000          0.05106067    0.16647164E-20
 0.                0.            0.

          ETA= -0.300000        X=  -0.357143

     Y                PSI              ØMEGA
 1.000000          1.00000000    0.
 0.950000          0.95215853    0.96909539E-11
 0.900000          0.90419858    0.11803395E-10
 0.850000          0.85602278    0.89228562E-11
 0.800000          0.80756529    0.52538323E-11
 0.750000          0.75879154    0.26332369E-11
 0.700000          0.70969199    0.11773234E-11
 0.650000          0.66027468    0.48305823E-12
 0.600000          0.61055855    0.18529411E-12
 0.550000          0.56056841    0.67306826E-13
 0.500000          0.51033171    0.23367556E-13
 0.450000          0.45987637    0.78076752E-14
 0.400000          0.40922963    0.25239535E-14
 0.350000          0.35841750    0.79267433E-15
 0.300000          0.30746455    0.24266443E-15
 0.250000          0.25639386    0.72609495E-16
 0.200000          0.20522720    0.21282587E-16
 0.150000          0.15398513    0.61204348E-17
 0.100000          0.10268734    0.17221177E-17
 0.050000          0.05135280    0.44631887E-18
 0.                0.            0.

          ETA= -0.250000        X=  -0.277778

     Y                PSI              ØMEGA
 1.000000          1.00000000    0.
 0.950000          0.95304870    0.52579227E-08
 0.900000          0.90587623    0.56693345E-08
 0.850000          0.85832321    0.37759461E-08
 0.800000          0.81030679    0.19664267E-08
 0.750000          0.76180533    0.87677296E-09
 0.700000          0.71283583    0.35069937E-09
 0.650000          0.66343552    0.12936043E-09
 0.600000          0.61364973    0.44790608E-10
 0.550000          0.56352506    0.14734846E-10
 0.500000          0.51310585    0.46454784E-11
 0.450000          0.46243290    0.14126088E-11
 0.400000          0.41154296    0.41633565E-12
 0.350000          0.36046910    0.11938763E-12
 0.300000          0.30924099    0.33411912E-13
 0.250000          0.25788552    0.91486443E-14
 0.200000          0.20642722    0.24559678E-14
 0.150000          0.15488876    0.64739946E-15
 0.100000          0.10329138    0.16728256E-15
 0.050000          0.05165529    0.40211875E-16
 0.                0.            0.

          ETA= -0.200000        X=  -0.208333

     Y                PSI              ØMEGA
 1.000000          1.00000000    0.
 0.950000          0.95428278    0.12120318E-05
 0.900000          0.90812735    0.11215299E-05
 0.850000          0.86128964    0.64580563E-06
 0.800000          0.81370372    0.29412241E-06
 0.750000          0.76540551    0.11588755E-06
 0.700000          0.71647320    0.41313712E-07
 0.650000          0.66699447    0.13675317E-07
 0.600000          0.61705191    0.42724416E-08
 0.550000          0.56671821    0.12738392E-08
 0.500000          0.51605546    0.36531387E-09
 0.450000          0.46511636    0.10135698E-09
 0.400000          0.41394565    0.27327678E-10
 0.350000          0.36258163    0.71849193E-11
 0.300000          0.31105748    0.18472250E-11
 0.250000          0.25940233    0.46546082E-12
 0.200000          0.20764222    0.11516698E-12
 0.150000          0.15580071    0.28021251E-13
 0.100000          0.10389961    0.66970947E-14
 0.050000          0.05195946    0.15018307E-14
 0.                0.            0.
```

```
          ETA= -0.150000        X=  -0.147059

     Y                PSI              ØMEGA
 1.000000          1.00000000    0.
 0.950000          0.95607109    0.13639501E-03
 0.900000          0.91120236    0.10419635E-03
 0.850000          0.86510108    0.50805543E-04
 0.800000          0.81783766    0.20007344E-04
 0.750000          0.76959403    0.69243948E-05
 0.700000          0.72055466    0.21941425E-05
 0.650000          0.67087489    0.65147810E-06
 0.600000          0.62067807    0.18390054E-06
 0.550000          0.57006077    0.49836247E-07
 0.500000          0.51909908    0.13055231E-07
 0.450000          0.46785375    0.33228716E-08
 0.400000          0.41637418    0.82493281E-09
 0.350000          0.36470120    0.20036346E-09
 0.300000          0.31286932    0.47727346E-10
 0.250000          0.26090827    0.11171769E-10
 0.200000          0.20884416    0.25738823E-11
 0.150000          0.15670045    0.58441322E-12
 0.100000          0.10449859    0.13066958E-12
 0.050000          0.05225870    0.27612403E-13
                   0.            0.

          ETA= -0.100000        X=  -0.092593

     Y                PSI              ØMEGA
 1.000000          1.00000000    0.
 0.950000          0.95884109    0.82810203E-02
 0.900000          0.91548409    0.49623283E-02
 0.850000          0.86994907    0.20002148E-02
 0.800000          0.82274383    0.67324327E-03
 0.750000          0.77431694    0.20368095E-03
 0.700000          0.72498731    0.57339092E-04
 0.650000          0.67497391    0.15312787E-04
 0.600000          0.62442966    0.39260907E-05
 0.550000          0.57346467    0.97418182E-06
 0.500000          0.52216090    0.23525331E-06
 0.450000          0.47058138    0.55518158E-07
 0.400000          0.41877589    0.12843572E-07
 0.350000          0.36678492    0.29196391E-08
 0.300000          0.31464215    0.65340865E-09
 0.250000          0.26237635    0.14418115E-09
 0.200000          0.21001261    0.31407283E-10
 0.150000          0.15757328    0.67601404E-11
 0.100000          0.10507882    0.14366028E-11
 0.050000          0.05254832    0.29017024E-12
                   0.            0.

          ETA= -0.050000        X=  -0.043860

     Y                PSI              ØMEGA
 1.000000          1.00000000    0.
 0.950000          0.96350293    0.29085883E-00
 0.900000          0.92148674    0.12606420E-00
 0.850000          0.87595651    0.40714610E-01
 0.800000          0.82835694    0.11585104E-01
 0.750000          0.77944750    0.30633684E-02
 0.700000          0.72963889    0.77132518E-03
 0.650000          0.67917375    0.18741576E-03
 0.600000          0.62820828    0.44301939E-04
 0.550000          0.57685000    0.10241769E-04
 0.500000          0.52517702    0.23239382E-05
 0.450000          0.47324850    0.51889249E-06
 0.400000          0.42111084    0.11421957E-06
 0.350000          0.36880153    0.24820922E-07
 0.300000          0.31635179    0.53305446E-08
 0.250000          0.26378816    0.11323003E-08
 0.200000          0.21113385    0.23805215E-09
 0.150000          0.15840954    0.49557102E-10
 0.100000          0.10563412    0.10204981E-10
 0.050000          0.05282532    0.20043205E-11
 0.                0.            0.
          ETA=  0.                  X=   0.

     Y                PSI              ØMEGA
 1.000000          1.00000000    0.22736007E 02
 0.950000          0.97157998    0.62024548E 01
 0.900000          0.92948017    0.15998890E 01
 0.850000          0.88295694    0.39805771E-00
 0.800000          0.83444630    0.96420543E-01
 0.750000          0.78478716    0.22879688E-01
 0.700000          0.73435555    0.53383476E-02
 0.650000          0.68335858    0.12273230E-02
 0.600000          0.63192717    0.27838559E-03
 0.550000          0.58015174    0.62348542E-04
 0.500000          0.52809855    0.13796434E-04
 0.450000          0.47581834    0.30178276E-05
 0.400000          0.42335133    0.65284421E-06
 0.350000          0.37073029    0.13973123E-06
 0.300000          0.31798273    0.29601236E-07
 0.250000          0.26513229    0.62087570E-08
 0.200000          0.21219969    0.12897498E-08
 0.150000          0.15920354    0.26539902E-09
 0.100000          0.10616098    0.54034530E-10
 0.050000          0.05308800    0.10501622E-10
 0.                0.            0.
```

TABLE VII. - Continued

```
      ETA=  0.050000      X=   0.043860

    Y             PSI             ØMEGA
1.000000       1.00000000     0.17398572E 02
0.950000       0.97825179     0.85688835E 01
0.900000       0.93779965     0.31769177E 01
0.850000       0.89072090     0.10096575E 01
0.800000       0.84123395     0.29074664E-00
0.750000       0.79067720     0.78165720E-01
0.700000       0.73949701     0.19986854E-01
0.650000       0.68787537     0.49198911E-02
0.600000       0.63591027     0.11753178E-02
0.550000       0.58366710     0.27398986E-03
0.500000       0.53119478     0.62569453E-04
0.450000       0.47853198     0.14035664E-04
0.400000       0.42571034     0.30989908E-05
0.350000       0.37275641     0.67449216E-06
0.300000       0.31969288     0.14487676E-06
0.250000       0.26653963     0.30737456E-07
0.200000       0.21331441     0.64458536E-08
0.150000       0.16003329     0.13367282E-08
0.100000       0.10671122     0.27385440E-09
0.050000       0.05336226     0.53471288E-10
0.             0.             0.
```

```
      ETA=  0.100000      X=   0.092593

    Y             PSI             ØMEGA
1.000000       1.00000000     0.13425100E 02
0.950000       0.98321863     0.90178491E 01
0.900000       0.94585957     0.44177175E 01
0.850000       0.89924327     0.17628285E 01
0.800000       0.84908446     0.61283632E 00
0.750000       0.79760697     0.19283506E-00
0.700000       0.74556211     0.56279393E-01
0.650000       0.69319209     0.15492298E-01
0.600000       0.64058360     0.40703618E-02
0.550000       0.58777874     0.10294613E-02
0.500000       0.53480636     0.25221387E-03
0.450000       0.48169008     0.60137288E-04
0.400000       0.42845058     0.14004938E-04
0.350000       0.37510630     0.31942899E-05
0.300000       0.32167380     0.71508034E-06
0.250000       0.26816817     0.15738259E-06
0.200000       0.21460333     0.34100433E-07
0.150000       0.16099212     0.72811081E-08
0.100000       0.10734680     0.15309817E-08
0.050000       0.05367897     0.30553753E-09
0.             0.             0.
```

```
      ETA=  0.150000      X=   0.147059

    Y             PSI             ØMEGA
1.000000       1.00000000     0.10747606E 02
0.950000       0.98656549     0.85507371E 01
0.900000       0.95286500     0.51573114E 01
0.850000       0.90779938     0.24954388E 01
0.800000       0.85761945     0.10314052E 01
0.750000       0.80544762     0.37901831E-00
0.700000       0.75254755     0.12713630E-00
0.650000       0.69935825     0.39655834E-01
0.600000       0.64601544     0.11659093E-01
0.550000       0.59255905     0.32639159E-02
0.500000       0.53900342     0.87670931E-03
0.450000       0.48535768     0.22728272E-03
0.400000       0.43163063     0.57128944E-04
0.350000       0.37783152     0.13972992E-04
0.300000       0.32396974     0.33351309E-05
0.250000       0.27005474     0.77862598E-06
0.200000       0.21609581     0.17813385E-06
0.150000       0.16210204     0.39993390E-07
0.100000       0.10808235     0.88068394E-08
0.050000       0.05404545     0.18290112E-08
0.             0.             0.
```

```
      ETA=  0.200000      X=   0.208333

    Y             PSI             ØMEGA
1.000000       1.00000000     0.89915394E 01
0.950000       0.98876058     0.77961667E 01
0.900000       0.95858616     0.54580179E 01
0.850000       0.91583586     0.30951778E 01
0.800000       0.86639220     0.14899530E 01
0.750000       0.81395847     0.63205492E 00
0.700000       0.76036578     0.24252047E-00
0.650000       0.70637210     0.85761932E-01
0.600000       0.65224566     0.28345942E-01
0.550000       0.59806569     0.88513374E-02
0.500000       0.54384942     0.26331288E-02
0.450000       0.48959786     0.75113724E-03
0.400000       0.43530992     0.20653643E-03
0.350000       0.38098592     0.54966682E-04
0.300000       0.32662793     0.14206143E-04
0.250000       0.27223925     0.35752302E-05
0.200000       0.21782412     0.87809770E-06
0.150000       0.16338737     0.21083075E-06
0.100000       0.10893415     0.49451751E-07
0.050000       0.05446985     0.10853233E-07
0.             0.             0.
```

```
      ETA=  0.250000      X=   0.277778

    Y             PSI             ØMEGA
1.000000       1.00000000     0.78048586E 01
0.950000       0.99024393     0.70478869E 01
0.900000       0.96313672     0.54593616E 01
0.850000       0.92305496     0.35157407E 01
0.800000       0.87500580     0.19304159E 01
0.750000       0.82284252     0.93204951E 00
0.700000       0.76885518     0.40538137E-00
0.650000       0.71417361     0.16171565E-00
0.600000       0.65927526     0.59990481E-01
0.550000       0.60433225     0.20918407E-01
0.500000       0.54939359     0.69150353E-02
0.450000       0.49446586     0.21819177E-02
0.400000       0.43954407     0.66075445E-03
0.350000       0.38462225     0.19289751E-03
0.300000       0.32969598     0.54484002E-04
0.250000       0.27476285     0.14933279E-04
0.200000       0.21982202     0.39814547E-05
0.150000       0.16487385     0.10345068E-05
0.100000       0.10991956     0.26165903E-06
0.050000       0.05496091     0.61364727E-07
0.             0.             0.
```

```
      ETA=  0.300000      X=   0.357143

    Y             PSI             ØMEGA
1.000000       1.00000000     0.69538413E 01
0.950000       0.99130770     0.63988394E 01
0.900000       0.96675280     0.52920896E 01
0.850000       0.92937253     0.37614208E 01
0.800000       0.88317598     0.23089854E 01
0.750000       0.83180580     0.12505648E 01
0.700000       0.77780683     0.61003341E 00
0.650000       0.72264877     0.27248918E-00
0.600000       0.66706149     0.11291285E-00
0.550000       0.61136083     0.43856692E-01
0.500000       0.55566358     0.16101170E-01
0.450000       0.50000247     0.56254912E-02
0.400000       0.44437942     0.18808672E-02
0.350000       0.38878737     0.60454904E-03
0.300000       0.33321819     0.18750545E-03
0.250000       0.27766500     0.56293276E-04
0.200000       0.22212256     0.16401128E-04
0.150000       0.16658713     0.46460906E-05
0.100000       0.11105604     0.12771793E-05
0.050000       0.05552746     0.32222097E-06
0.             0.             0.
```

```
      ETA=  0.350000      X=   0.448718

    Y             PSI             ØMEGA
1.000000       1.00000000     0.63069521E 01
0.950000       0.99211631     0.58560017E 01
0.900000       0.96966422     0.50476556E 01
0.850000       0.93483292     0.38637643E 01
0.800000       0.89073794     0.26018436E 01
0.750000       0.84059822     0.15583604E 01
0.700000       0.78699563     0.84328289E 00
0.650000       0.73164298     0.41822582E-00
0.600000       0.67551953     0.19236611E-00
0.550000       0.61911868     0.82857292E-01
0.500000       0.56266031     0.33687586E-01
0.450000       0.50622857     0.13013668E-01
0.400000       0.44984750     0.48027371E-02
0.350000       0.39351743     0.17010175E-02
0.300000       0.33723108     0.58036949E-03
0.250000       0.28097969     0.19135980E-03
0.200000       0.22475512     0.61134326E-04
0.150000       0.16855041     0.18958171E-04
0.100000       0.11235961     0.56884918E-05
0.050000       0.05617767     0.15486351E-05
0.             0.             0.
```

```
      ETA=  0.400000      X=   0.555556

    Y             PSI             ØMEGA
1.000000       1.00000000     0.57936131E 01
0.950000       0.99275798     0.54024324E 01
0.900000       0.97204991     0.47794174E 01
0.850000       0.93953554     0.38623186E 01
0.800000       0.89762234     0.28036106E 01
0.750000       0.84903107     0.18314552E 01
0.700000       0.79620665     0.10874566E 01
0.650000       0.74097839     0.59359409E 00
0.600000       0.68452965     0.30095012E-00
0.550000       0.62753905     0.14296500E-00
0.500000       0.57035603     0.64104978E-01
0.450000       0.51314194     0.27300333E-01
0.400000       0.45596135     0.11100002E-01
0.350000       0.39883373     0.43278141E-02
0.300000       0.34175943     0.16241241E-02
0.250000       0.28473178     0.58848852E-03
0.200000       0.22774220     0.20642041E-03
0.150000       0.17078201     0.70202617E-04
0.100000       0.11384317     0.23034222E-04
0.050000       0.05691818     0.67694832E-05
0.             0.             0.
```

TABLE VII. - Continued

```
        ETA=  0.450000      X=   0.681818

    Y                 PSI             ØMEGA
1.000000          1.00000000      0.53740267E 01
0.950000          0.99328247      0.50199487E 01
0.900000          0.97403798      0.45146157E 01
0.850000          0.94359133      0.37931719E 01
0.800000          0.90382398      0.29221030E 01
0.750000          0.85697766      0.20544758E 01
0.700000          0.80525230      0.13241518E 01
0.650000          0.75047093      0.78854538E 00
0.600000          0.69394761      0.43740151E-00
0.550000          0.63652597      0.22785429E-00
0.500000          0.57869605      0.11213946E-00
0.450000          0.52071749      0.52449442E-01
0.400000          0.46271510      0.23427135E-01
0.350000          0.40474176      0.10034715E-01
0.300000          0.34681471      0.41366538E-02
0.250000          0.28893515      0.16461643E-02
0.200000          0.23109746      0.63395594E-03
0.150000          0.17329368      0.23654290E-03
0.100000          0.11551521      0.84876403E-04
0.050000          0.05775345      0.26889722E-04
0.                0.              0.

        ETA=  0.500000      X=   0.833333

    Y                 PSI             ØMEGA
1.000000          1.00000000      0.50234734E 01
0.950000          0.99372066      0.46939935E 01
0.900000          0.97571950      0.42656132E 01
0.850000          0.94710393      0.36844108E 01
0.800000          0.90937702      0.29723050E 01
0.750000          0.86436705      0.22213260E 01
0.700000          0.81398360      0.15376718E 01
0.650000          0.75994682      0.99022076E 00
0.600000          0.70361818      0.59678666E 00
0.550000          0.64596318      0.33868038E-00
0.500000          0.58760472      0.18201672E-00
0.450000          0.52891234      0.93106103E-01
0.400000          0.47008855      0.45531929E-01
0.350000          0.41123530      0.21369957E-01
0.300000          0.35239860      0.96582615E-02
0.250000          0.29359516      0.42155523E-02
0.200000          0.23482750      0.17809942E-02
0.150000          0.17609151      0.72864290E-03
0.100000          0.11738031      0.28559309E-03
0.050000          0.05868593      0.97272182E-04
0.                0.              0.

        ETA=  0.550000      X=   1.018519

    Y                 PSI             ØMEGA
1.000000          1.00000000      0.47248601E 01
0.950000          0.99409392      0.44132000E 01
0.900000          0.97716229      0.40370416E 01
0.850000          0.95016666      0.35558316E 01
0.800000          0.91434069      0.29713713E 01
0.750000          0.87117819      0.23338243E 01
0.700000          0.82229943      0.17174017E 01
0.650000          0.76926035      0.11854813E 01
0.600000          0.71339334      0.77036886E 00
0.550000          0.65572926      0.47340751E-00
0.500000          0.59699794      0.27637056E-00
0.450000          0.53767726      0.15393487E-00
0.400000          0.47805870      0.82123298E-01
0.350000          0.41830833      0.42110813E-01
0.300000          0.35851429      0.20818809E-01
0.250000          0.29871912      0.99495590E-02
0.200000          0.23894040      0.46054857E-02
0.150000          0.17918260      0.20633554E-02
0.100000          0.11944360      0.88147257E-03
0.050000          0.05971826      0.32156681E-03
0.                0.              0.

        ETA=  0.600000      X=   1.250000

    Y                 PSI             ØMEGA
1.000000          1.00000000      0.44649362E 01
0.950000          0.99441883      0.41680862E 01
0.900000          0.97842155      0.38294350E 01
0.850000          0.95286661      0.34203401E 01
0.800000          0.91879462      0.29353423E 01
0.750000          0.87743049      0.23992170E 01
0.700000          0.83015731      0.18587358E 01
0.650000          0.77831557      0.13633779E 01
0.600000          0.72315792      0.94815169E 00
0.550000          0.66572186      0.62691949E 00
0.500000          0.60680490      0.39542937E-00
0.450000          0.54697721      0.23874614E-00
0.400000          0.48662081      0.13843190E-00
0.350000          0.42597675      0.77318203E-01
0.300000          0.36518861      0.41711395E-01
0.250000          0.30433704      0.21784893E-01
0.200000          0.24346413      0.11029719E-01
0.150000          0.18258964      0.54007143E-02
0.100000          0.12172084      0.25065459E-02
0.050000          0.06085848      0.97487085E-03
0.                0.              0.

        ETA=  0.650000      X=   1.547619

    Y                 PSI             ØMEGA
1.000000          1.00000000      0.42321383E 01
0.950000          0.99470983      0.39498676E 01
0.900000          0.97954717      0.36407223E 01
0.850000          0.95529128      0.32354398E 01
0.800000          0.92284016      0.28773417E 01
0.750000          0.88322596      0.24273901E 01
0.700000          0.83758540      0.19625419E 01
0.650000          0.78709041      0.15168677E 01
0.600000          0.73286133      0.11204690E 01
0.550000          0.67589135      0.79195916E 00
0.500000          0.61700032      0.53672963E 00
0.450000          0.55682220      0.34964275E-00
0.400000          0.49581902      0.21949059E-00
0.350000          0.43430960      0.13310695E-00
0.300000          0.37250408      0.78154944E-01
0.250000          0.31053305      0.44509588E-01
0.200000          0.24847510      0.24595349E-01
0.150000          0.18637465      0.13124300E-01
0.100000          0.12425536      0.65880245E-02
0.050000          0.06212880      0.27167621E-02
0.                0.              0.

        ETA=  0.700000      X=   1.944444

    Y                 PSI             ØMEGA
1.000000          1.00000000      0.40156900E 01
0.950000          0.99498039      0.37497865E 01
0.900000          0.98058744      0.34667794E 01
0.850000          0.95753199      0.31542554E 01
0.800000          0.92660054      0.28066755E 01
0.750000          0.88866851      0.24282333E 01
0.700000          0.84468874      0.20334408E 01
0.650000          0.79565457      0.16432551E 01
0.600000          0.74254619      0.12797891E 01
0.550000          0.68627489      0.96056346E 00
0.500000          0.62763929      0.69544949E 00
0.450000          0.56730076      0.48640474E-00
0.400000          0.50577915      0.32921670E-00
0.350000          0.44346382      0.21602242E-00
0.300000          0.38063364      0.13764138E-00
0.250000          0.31748044      0.85240271E-01
0.200000          0.25413121      0.51250079E-01
0.150000          0.19066717      0.29677111E-01
0.100000          0.12713864      0.16017374E-01
0.050000          0.06357644      0.69592880E-02
0.                0.              0.

        ETA=  0.750000      X=   2.500000

    Y                 PSI             ØMEGA
1.000000          1.00000000      0.38061439E 01
0.950000          0.99524231      0.35594565E 01
0.900000          0.98158661      0.33021133E 01
0.850000          0.95967792      0.30264381E 01
0.800000          0.93020809      0.27285662E 01
0.750000          0.89392517      0.24098583E 01
0.700000          0.85162964      0.20774551E 01
0.650000          0.80415617      0.17432590E 01
0.600000          0.75234327      0.14214422E 01
0.550000          0.69699768      0.11253837E 01
0.500000          0.63386064      0.86514327E 00
0.450000          0.57858278      0.64617430E 00
0.400000          0.51671050      0.46934040E-00
0.350000          0.45368423      0.33184858E-00
0.300000          0.38984545      0.22857029E-00
0.250000          0.32544588      0.15331943E-00
0.200000          0.26068307      0.99825267E-01
0.150000          0.19567616      0.62341329E-01
0.100000          0.13052039      0.35901560E-01
0.050000          0.06527945      0.16317382E-01
0.                0.              0.

        ETA=  0.800000      X=   3.333333

    Y                 PSI             ØMEGA
1.000000          1.00000000      0.35970569E 01
0.950000          0.99550368      0.33724373E 01
0.900000          0.98257656      0.31415225E 01
0.850000          0.96179678      0.28998382E 01
0.800000          0.93376986      0.26449972E 01
0.750000          0.89913458      0.23772933E 01
0.700000          0.85856210      0.21000417E 01
0.650000          0.81274746      0.18193588E 01
0.600000          0.76239417      0.15432812E 01
0.550000          0.70819385      0.12804143E 01
0.500000          0.65080473      0.10385054E 01
0.450000          0.55083201      0.82332849E 00
0.400000          0.52981359      0.63810942E 00
0.350000          0.46521197      0.48350502E-00
0.300000          0.40041272      0.35799291E-00
0.250000          0.33472832      0.25847115E-00
0.200000          0.26840615      0.18088239E-00
0.150000          0.20163853      0.12073708E-00
0.100000          0.13457417      0.73474564E-01
0.050000          0.06732956      0.34654003E-01
0.                0.              0.
```

TABLE VII . - Continued

ETA= 0.850000 X= 4.722222

Y	PSI	ØMEGA
1.000000	1.00000000	0.33874691E 01
0.950000	0.99576566	0.31869707E 01
0.900000	0.98356405	0.29831909E 01
0.850000	0.96390498	0.27738237E 01
0.800000	0.93731242	0.25576083E 01
0.750000	0.90432768	0.23345442E 01
0.700000	0.86550923	0.21060378E 01
0.650000	0.82142925	0.18748832E 01
0.600000	0.77266645	0.16450126E 01
0.550000	0.71979634	0.14210287E 01
0.500000	0.66337945	0.12076095E 01
0.450000	0.60394979	0.10089105E 01
0.400000	0.54200429	0.82808612E 00
0.350000	0.47799496	0.66700350E 00
0.300000	0.41232423	0.52616061E 00
0.250000	0.34534372	0.40477492E-00
0.200000	0.27735623	0.30098188E-00
0.150000	0.20862040	0.21208051E-00
0.100000	0.13935757	0.13477530E-00
0.050000	0.06975996	0.65383977E-01
0.	0.	0.

ETA= 0.900000 X= 7.499999

Y	PSI	ØMEGA
1.000000	1.00000000	0.31869768E 01
0.950000	0.99601628	0.30112433E 01
0.900000	0.98450448	0.28342246E 01
0.850000	0.96590730	0.26550341E 01
0.800000	0.94067284	0.24731997E 01
0.750000	0.90925594	0.22887183E 01
0.700000	0.87211803	0.21021002E 01
0.650000	0.82972597	0.19143716E 01
0.600000	0.78254931	0.17270218E 01
0.550000	0.73105673	0.15418828E 01
0.500000	0.67571127	0.13609549E 01
0.450000	0.61696544	0.11861996E 01
0.400000	0.55525623	0.10193299E 01
0.350000	0.49100085	0.86163227E 00
0.300000	0.42459349	0.71383592E 00
0.250000	0.35640351	0.57604551E 00
0.200000	0.28677521	0.44773287E-00
0.150000	0.21602909	0.32778081E-00
0.100000	0.14446471	0.21456613E-00
0.050000	0.07236473	0.10606932E-00
0.	0.	0.

ETA= 0.950000 X= 15.833332

Y	PSI	ØMEGA
1.000000	1.00000000	0.30359446E 01
0.950000	0.99620507	0.28804749E 01
0.900000	0.98520894	0.27247799E 01
0.850000	0.96740089	0.25687111E 01
0.800000	0.94317108	0.24121968E 01
0.750000	0.91291083	0.22552490E 01
0.700000	0.87701250	0.20979666E 01
0.650000	0.83586933	0.19405354E 01
0.600000	0.78987493	0.17832202E 01
0.550000	0.73942263	0.16263486E 01
0.500000	0.68490460	0.14702869E 01
0.450000	0.62671100	0.13154075E 01
0.400000	0.56522906	0.11620536E 01
0.350000	0.50084215	0.10105047E 01
0.300000	0.43392918	0.86094492E 00
0.250000	0.36486401	0.71343917E 00
0.200000	0.29401539	0.56792077E 00
0.150000	0.22174712	0.42418954E-00
0.100000	0.14341846	0.28192201E-00
0.050000	0.07438508	0.14069206E-00
0.	0.	0.

ETA= 1.000000 X= ∞

Y	PSI	ØMEGA
1.000000	1.00000000	0.29999999E 01
0.950000	0.99631249	0.28499999E 01
0.900000	0.98549999	0.26999999E 01
0.850000	0.96793749	0.25500000E 01
0.800000	0.94399998	0.23999999E 01
0.750000	0.91406249	0.22499999E 01
0.700000	0.87849998	0.20999999E 01
0.650000	0.83768748	0.19499999E 01
0.600000	0.79199998	0.17999999E 01
0.550000	0.74181249	0.16499999E 01
0.500000	0.68749999	0.14999999E 01
0.450000	0.62943748	0.13499999E 01
0.400000	0.56799999	0.12000000E 01
0.350000	0.50356250	0.10500000E 01
0.300000	0.43649999	0.89999999E 00
0.250000	0.36718749	0.74999999E 00
0.200000	0.29600000	0.59999999E 00
0.150000	0.22331250	0.45000000E-00
0.100000	0.14950000	0.30000000E-00
0.050000	0.07493749	0.15000000E-00
0.	0.	0.

TABLE VIII. Velocities And Pressure Gradients For Case II With A 40 x 20 Grid

ETA = -1.000000 X = $-\infty$

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.000000	0.99999999E 01	0.	0.	0.	0.
0.950000	0.99999999E 01	0.	0.	0.	0.
0.900000	0.99999999E 01	0.	0.	0.	0.
0.850000	0.99999999E 01	0.	0.	0.	0.
0.800000	0.99999999E 01	0.	0.	0.	0.
0.750000	0.99999999E 01	0.	0.	0.	0.
0.700000	0.99999999E 01	0.	0.	0.	0.
0.650000	0.99999999E 01	0.	0.	0.	0.
0.600000	0.99999999E 01	0.	0.	0.	0.
0.550000	0.99999999E 01	0.	0.	0.	0.
0.500000	0.99999999E 01	0.	0.	0.	0.
0.450000	0.99999999E 01	0.	0.	0.	0.
0.400000	0.99999999E 01	0.	0.	0.	0.
0.350000	0.99999999E 01	0.	0.	0.	0.
0.300000	0.99999999E 01	0.	0.	0.	0.
0.250000	0.99999999E 01	0.	0.	0.	0.
0.200000	0.99999999E 01	0.	0.	0.	0.
0.150000	0.99999999E 01	0.	0.	0.	0.
0.100000	0.99999999E 01	0.	0.	0.	0.
0.050000	0.99999999E 01	0.	0.	0.	0.
0.	0.99999999E 01	0.	0.	0.	0.

ETA = -0.700000 X = -1.944444

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.000000	0.99976489E 00	0.	0.75382298E-02	-0.81412884E-03	0.
0.950000	0.99976957E 00	-0.40715933E-04	0.74436089E-02	-0.80390979E-03	-0.11622500E-03
0.900000	0.99977799E 00	-0.80409545E-04	0.71619718E-02	-0.77349297E-03	-0.22938458E-03
0.850000	0.99979307E 00	-0.11808425E-03	0.67007835E-02	-0.72368464E-03	-0.33688265E-03
0.800000	0.99981152E 00	-0.15279740E-03	0.60704503E-02	-0.65560865E-03	-0.43571204E-03
0.750000	0.99983558E 00	-0.18366426E-03	0.52903952E-02	-0.57136270E-03	-0.52370558E-03
0.700000	0.99986352E 00	-0.20995259E-03	0.43799238E-02	-0.47303178E-03	-0.59849361E-03
0.650000	0.99989481E 00	-0.23098648E-03	0.33599523E-02	-0.36287487E-03	-0.65833814E-03
0.600000	0.99992849E 00	-0.24625897E-03	0.22647211E-02	-0.24458988E-03	-0.70172748E-03
0.550000	0.99996388E 00	-0.25546432E-03	0.11165857E-02	-0.12059126E-03	-0.72760762E-03
0.500000	0.10000003E 01	-0.25833696E-03	-0.58757760E-04	0.63458382E-05	-0.73540566E-03
0.450000	0.10000367E 01	-0.25484875E-03	-0.12258773E-02	0.13239475E-03	-0.72518884E-03
0.400000	0.10000723E 01	-0.24511635E-03	-0.23591202E-02	0.25478499E-03	-0.69716324E-03
0.350000	0.10001060E 01	-0.22938922E-03	-0.34308698E-02	0.37053395E-03	-0.65223894E-03
0.300000	0.10001370E 01	-0.20808175E-03	-0.44177037E-02	0.47711202E-03	-0.59132291E-03
0.250000	0.10001646E 01	-0.18169686E-03	-0.52925821E-02	0.57159888E-03	-0.51618550E-03
0.200000	0.10001881E 01	-0.15094064E-03	-0.60374445E-02	0.65204403E-03	-0.42871671E-03
0.150000	0.10002071E 01	-0.11651114E-03	-0.66387101E-02	0.71698071E-03	-0.33076605E-03
0.100000	0.10002210E 01	-0.79261289E-04	-0.70751229E-02	0.76411328E-03	-0.22497239E-03
0.050000	0.10002294E 01	-0.40118974E-04	-0.73407345E-02	0.79279934E-03	-0.11388255E-03
0.	0.10002322E 01	0.	-0.74311655E-02	0.80256590E-03	0.

ETA = -0.650000 X = -1.547619

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.000000	0.99920362E 00	0.	0.17971378E-01	-0.26417926E-02	0.
0.950000	0.99921353E 00	-0.13219491E-03	0.17738926E-01	-0.26076223E-02	-0.40151624E-03
0.900000	0.99924304E 00	-0.26097269E-03	0.17047522E-01	-0.25059859E-02	-0.79255909E-03
0.850000	0.99929169E 00	-0.38300381E-03	0.15916518E-01	-0.23397283E-02	-0.11622149E-02
0.800000	0.99935830E 00	-0.49515592E-03	0.14380225E-01	-0.21138931E-02	-0.15012815E-02
0.750000	0.99944063E 00	-0.59460329E-03	0.12482623E-01	-0.18349457E-02	-0.18006862E-02
0.700000	0.99953466E 00	-0.67885973E-03	0.10276505E-01	-0.15106462E-02	-0.20529955E-02
0.650000	0.99964410E 00	-0.74587719E-03	0.78267984E-02	-0.11505394E-02	-0.22521758E-02
0.600000	0.99975981E 00	-0.79412231E-03	0.51960757E-02	-0.76382317E-03	-0.23941065E-02
0.550000	0.99988118E 00	-0.82246701E-03	0.24557341E-02	-0.36099293E-03	-0.24752032E-02
0.500000	0.10000052E 01	-0.83042937E-03	-0.31917767E-03	0.46919119E-04	-0.24950650E-02
0.450000	0.10001388E 01	-0.81798201E-03	-0.30657659E-02	0.45066760E-03	-0.24535737E-02
0.400000	0.10002486E 01	-0.78556852E-03	-0.57177605E-02	0.84051083E-03	-0.23525538E-02
0.350000	0.10003423E 01	-0.73413627E-03	-0.82129234E-02	0.12072998E-02	-0.21950250E-02
0.300000	0.10004467E 01	-0.66504335E-03	-0.10493112E-01	0.15424876E-02	-0.19856715E-02
0.250000	0.10005593E 01	-0.58009142E-03	-0.12507834E-01	0.18386516E-02	-0.17299673E-02
0.200000	0.10006383E 01	-0.48138059E-03	-0.14218877E-01	0.20901750E-02	-0.14338140E-02
0.150000	0.10007019E 01	-0.37127657E-03	-0.15586642E-01	0.22912366E-02	-0.11050270E-02
0.100000	0.10007481E 01	-0.25245859E-03	-0.16581435E-01	0.24374711E-02	-0.75103095E-03
0.050000	0.10007764E 01	-0.12773114E-03	-0.17187721E-01	0.25265951E-02	-0.37976241E-03
0.	0.10007860E 01	0.	-0.17392045E-01	0.25566307E-02	0.

ETA = -0.600000 X = -1.250000

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.000000	0.99796832E 00	0.	0.34568863E-01	-0.66372217E-02	0.
0.950000	0.99799424E 00	-0.33253669E-03	0.34096545E-01	-0.65465367E-02	-0.10516561E-02
0.900000	0.99807180E 00	-0.65600394E-03	0.32688490E-01	-0.62761902E-02	-0.20721797E-02
0.850000	0.99819899E 00	-0.96150358E-03	0.30393934E-01	-0.58356354E-02	-0.30319232E-02
0.800000	0.99837706E 00	-0.12409114E-02	0.27299938E-01	-0.52415881E-02	-0.39033071E-02
0.750000	0.99858589E 00	-0.14870024E-02	0.23505015E-01	-0.45129630E-02	-0.46636810E-02
0.700000	0.99883445E 00	-0.16935395E-02	0.19136316E-01	-0.36741728E-02	-0.52932195E-02
0.650000	0.99911071E 00	-0.18557452E-02	0.14335718E-01	-0.27524579E-02	-0.57785548E-02
0.600000	0.99940747E 00	-0.19701003E-02	0.92390613E-02	-0.17738998E-02	-0.61095072E-02
0.550000	0.99971489E 00	-0.20344448E-02	0.39922619E-02	-0.76651429E-03	-0.62832450E-02
0.500000	0.10000308E 01	-0.20480490E-02	-0.12648262E-02	0.24284665E-03	-0.62983089E-02
0.450000	0.10003419E 01	-0.20114493E-02	-0.64059674E-02	0.12299457E-02	-0.61597819E-02
0.400000	0.10006426E 01	-0.19263339E-02	-0.11313932E-01	0.21722749E-02	-0.58753311E-02
0.350000	0.10009358E 01	-0.17954921E-02	-0.15880619E-01	0.30490789E-02	-0.54557674E-02
0.300000	0.10011849E 01	-0.16226864E-02	-0.20015156E-01	0.38429102E-02	-0.49139293E-02
0.250000	0.10014141E 01	-0.14124513E-02	-0.23642591E-01	0.45393775E-02	-0.42641647E-02
0.200000	0.10016087E 01	-0.11699903E-02	-0.26694552E-01	0.51253540E-02	-0.35233071E-02
0.150000	0.10017643E 01	-0.90115070E-03	-0.29117953E-01	0.55906472E-02	-0.27081975E-02
0.100000	0.10018778E 01	-0.61215340E-03	-0.30878936E-01	0.59287559E-02	-0.18368740E-02
0.050000	0.10019469E 01	-0.30945830E-03	-0.31946563E-01	0.61337402E-02	-0.92783783E-03
0.	0.10019701E 01	0.	-0.32302924E-01	0.62021615E-02	0.

TABLE VIII. - Continued

ETA = -0.550000 X = -1.018519

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.000000	0.99573970E 00	0.	0.57454455E-01	-0.13961432E-01	0.
0.950000	0.99575484E 00	-0.70105834E-03	0.56584597E-01	-0.13750057E-01	-0.23075069E-02
0.900000	0.99596716E 00	-0.13809701E-02	0.54015793E-01	-0.13125838E-01	-0.45340601E-02
0.850000	0.99624829E 00	-0.20195846E-02	0.49866639E-01	-0.12117593E-01	-0.66052479E-02
0.800000	0.99662133E 00	-0.25986340E-02	0.44328497E-01	-0.10771824E-01	-0.84554005E-02
0.750000	0.99708719E 00	-0.31026204E-02	0.37634984E-01	-0.91453011E-02	-0.10031193E-01
0.700000	0.99761413E 00	-0.35189609E-02	0.30051716E-01	-0.73025668E-02	-0.11295655E-01
0.650000	0.99820768E 00	-0.38386031E-02	0.21856085E-01	-0.53110287E-02	-0.12223916E-01
0.600000	0.99882908E 00	-0.40557172E-02	0.13322356E-01	-0.32373326E-02	-0.12807257E-01
0.550000	0.99947920E 00	-0.41679133E-02	0.47014901E-02	-0.11424620E-02	-0.13047703E-01
0.500000	0.10001750E 01	-0.41754631E-02	-0.37772481E-02	0.91787130E-03	-0.12958413E-01
0.450000	0.10007427E 01	-0.40815438E-02	-0.11910515E-01	0.28942550E-02	-0.12559810E-01
0.400000	0.10013735E 01	-0.38913789E-02	-0.19536797E-01	0.47474416E-02	-0.11877852E-01
0.350000	0.10019437E 01	-0.36120655E-02	-0.26515745E-01	0.64433260E-02	-0.10941837E-01
0.300000	0.10024618E 01	-0.32522394E-02	-0.32739495E-01	0.79556972E-02	-0.97839507E-02
0.250000	0.10029171E 01	-0.28216232E-02	-0.38124526E-01	0.92642599E-02	-0.84373759E-02
0.200000	0.10033014E 01	-0.23308805E-02	-0.42602726E-01	0.10352463E-01	-0.69342931E-02
0.150000	0.10036074E 01	-0.17913549E-02	-0.46127808E-01	0.11209057E-01	-0.53077051E-02
0.100000	0.10038757E 01	-0.12149504E-02	-0.48668787E-01	0.11826515E-01	-0.35890045E-02
0.050000	0.10039446E 01	-0.61368971E-03	-0.50200755E-01	0.12198783E-01	-0.18097278E-02
0.	0.10040099E 01	0.	-0.50711980E-01	0.12323011E-01	0.

ETA = -0.500000 X = -0.833333

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.000000	0.99219829E 00	0.	0.86601111E-01	-0.25980333E-01	0.
0.950000	0.99231124E 00	-0.13092308E-02	0.85088933E-01	-0.25526680E-01	-0.45316078E-02
0.900000	0.99264577E 00	-0.25730430E-02	0.80661571E-01	-0.24198471E-01	-0.88653799E-02
0.850000	0.99318802E 00	-0.37493034E-02	0.73603392E-01	-0.22081017E-01	-0.12623846E-01
0.800000	0.99391513E 00	-0.48011317E-02	0.64338734E-01	-0.19301620E-01	-0.16260865E-01
0.750000	0.99475854E 00	-0.56991353E-02	0.53386629E-01	-0.16015988E-01	-0.19073961E-01
0.700000	0.99580571E 00	-0.64220352E-02	0.41283463E-01	-0.12385038E-01	-0.21204483E-01
0.650000	0.99690184E 00	-0.69565325E-02	0.28552230E-01	-0.85656691E-02	-0.22634193E-01
0.600000	0.99805728E 00	-0.72970613E-02	0.15664256E-01	-0.46992768E-02	-0.23379849E-01
0.550000	0.99922411E 00	-0.74444709E-02	0.30136019E-02	-0.90408057E-03	-0.23481134E-01
0.500000	0.10003863E 01	-0.74050425E-02	-0.90797569E-02	0.27239271E-02	-0.22997329E-01
0.450000	0.10015110E 01	-0.71892029E-02	-0.20373764E-01	0.61121292E-02	-0.21993056E-01
0.400000	0.10025746E 01	-0.68105420E-02	-0.30690276E-01	0.92070829E-02	-0.20539271E-01
0.350000	0.10035560E 01	-0.62848179E-02	-0.39908990E-01	0.11972696E-01	-0.18705752E-01
0.300000	0.10044777E 01	-0.56292302E-02	-0.47953103E-01	0.14385930E-01	-0.16556645E-01
0.250000	0.10052057E 01	-0.48618614E-02	-0.54779897E-01	0.16433964E-01	-0.14151942E-01
0.200000	0.10058483E 01	-0.40011908E-02	-0.60367114E-01	0.18110134E-01	-0.11546028E-01
0.150000	0.10063566E 01	-0.30659716E-02	-0.64706971E-01	0.19412091E-01	-0.87861898E-02
0.100000	0.10067242E 01	-0.20745205E-02	-0.67800761E-01	0.20340228E-01	-0.59164397E-02
0.050000	0.10069436E 01	-0.10467543E-02	-0.69654866E-01	0.20896459E-01	-0.29758702E-02
0.	0.10070211E 01	0.	-0.70273586E-01	0.21082076E-01	0.

ETA = -0.450000 X = -0.681818

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.000000	0.98701149E 00	0.	0.12251592E-00	-0.44473279E-01	0.
0.950000	0.98722003E 00	-0.22529261E-02	0.11993406E-00	-0.43536066E-01	-0.83407012E-02
0.900000	0.98783759E 00	-0.44118688E-02	0.11244026E-00	-0.40815815E-01	-0.16200293E-01
0.850000	0.98881533E 00	-0.63920129E-02	0.10072734E-00	-0.36564026E-01	-0.23161514E-01
0.800000	0.99012189E 00	-0.81246086E-02	0.85767430E-01	-0.31133577E-01	-0.28932890E-01
0.750000	0.99168018E 00	-0.95601894E-02	0.68650535E-01	-0.24920144E-01	-0.33346424E-01
0.700000	0.99342480E 00	-0.10669708E-01	0.50427210E-01	-0.18305077E-01	-0.36365051E-01
0.650000	0.99528453E 00	-0.11441914E-01	0.31990632E-01	-0.11612599E-01	-0.38047965E-01
0.600000	0.99720762E 00	-0.11880243E-01	0.14051051E-01	-0.51005317E-02	-0.38513229E-01
0.550000	0.99911526E 00	-0.11999324E-01	-0.28707709E-02	0.10420898E-02	-0.37920962E-01
0.500000	0.10009759E 01	-0.11821526E-01	-0.18436276E-01	0.66923685E-02	-0.36438978E-01
0.450000	0.10027444E 01	-0.11373813E-01	-0.32450401E-01	0.11779495E-01	-0.34234444E-01
0.400000	0.10043980E 01	-0.10685665E-01	-0.44821416E-01	0.16270174E-01	-0.31455117E-01
0.350000	0.10058914E 01	-0.97873955E-02	-0.55534967E-01	0.20159192E-01	-0.28229475E-01
0.300000	0.10072049E 01	-0.87090235E-02	-0.64623893E-01	0.23458473E-01	-0.24666632E-01
0.250000	0.10083439E 01	-0.74746750E-02	-0.72148642E-01	0.26189957E-01	-0.20854209E-01
0.200000	0.10092977E 01	-0.61271823E-02	-0.78179421E-01	0.28379130E-01	-0.16860851E-01
0.150000	0.10100783E 01	-0.46761260E-02	-0.82785456E-01	0.30051120E-01	-0.12741966E-01
0.100000	0.10105404E 01	-0.31578112E-02	-0.86030065E-01	0.31228913E-01	-0.85379138E-02
0.050000	0.10108911E 01	-0.15905386E-02	-0.87957405E-01	0.31928538E-01	-0.42804533E-02
0.	0.10109883E 01	0.	-0.88595920E-01	0.32160319E-01	0.

ETA = -0.400000 X = -0.555556

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.000000	0.97978547E 00	0.	0.16678534E-00	-0.72051269E-01	0.
0.950000	0.98015733E 00	-0.36768900E-02	0.16231194E-00	-0.70118759E-01	-0.14857229E-01
0.900000	0.98124733E 00	-0.71594057E-02	0.14953098E-00	-0.64597382E-01	-0.28515255E-01
0.850000	0.98296005E 00	-0.10280950E-01	0.13015892E-00	-0.56228654E-01	-0.40034032E-01
0.800000	0.98518737E 00	-0.12921652E-01	0.10643522E-00	-0.45980018E-01	-0.48871157E-01
0.750000	0.98778725E 00	-0.15012334E-01	0.80573183E-01	-0.34807615E-01	-0.54894432E-01
0.700000	0.99062189E 00	-0.16529091E-01	0.54428834E-01	-0.23513256E-01	-0.58255594E-01
0.650000	0.99356703E 00	-0.17483324E-01	0.29355951E-01	-0.12681770E-01	-0.59293722E-01
0.600000	0.99651472E 00	-0.17910053E-01	0.62234281E-02	-0.26885208E-02	-0.58431714E-01
0.550000	0.99938886E 00	-0.17858361E-01	-0.14501473E-01	0.62646385E-02	-0.56086350E-01
0.500000	0.10021094E 01	-0.17363642E-01	-0.32652496E-01	0.14105878E-01	-0.52636755E-01
0.450000	0.10046401E 01	-0.16542223E-01	-0.48259171E-01	0.20847961E-01	-0.48394721E-01
0.400000	0.10065445E 01	-0.15388370E-01	-0.61462782E-01	0.26551921E-01	-0.43613466E-01
0.350000	0.10090014E 01	-0.13972130E-01	-0.72465360E-01	0.31305034E-01	-0.38482851E-01
0.300000	0.10107551E 01	-0.12435132E-01	-0.81484400E-01	0.35201260E-01	-0.33138100E-01
0.250000	0.10123181E 01	-0.10530122E-01	-0.88729482E-01	0.38331135E-01	-0.27673696E-01
0.200000	0.10135460E 01	-0.85815022E-02	-0.94391090E-01	0.40776950E-01	-0.22153721E-01
0.150000	0.10145768E 01	-0.65257902E-02	-0.98625810E-01	0.42608077E-01	-0.16611232E-01
0.100000	0.10152700E 01	-0.43924484E-02	-0.10156914E-00	0.43877868E-01	-0.11067732E-01
0.050000	0.10156459E 01	-0.22082460E-02	-0.10329620E-00	0.44623958E-01	-0.55323762E-02
0.	0.10157044E 01	0.	-0.10386518E-00	0.44869758E-01	0.

TABLE VIII. - Continued

ETA = -C.35CCCC X = -0.448718

Y	U(Y)	U(Y)	-∂P/∂η	-∂P/∂x	-∂P/∂Y
1.CCCCCC	C.56558685E CC	C.	C.22267048E-00	-0.11289393E-00	0.
C.95CCCC	C.57064734E CC	-C.58193417E-02	C.21462933E-00	-0.10881706E-00	-0.26245376E-01
C.9CCCCC	C.57254641E CC	-C.11226564E-01	C.19230003E-00	-0.97496112E-01	-0.49394859E-01
C.85CCCC	C.57547971E CC	-C.15901454E-01	C.16007174E-00	-0.81156371E-01	-0.67396730E-01
C.80CCCC	C.57516961E CC	-C.19656572E-01	0.12297055E-00	-0.62346069E-01	-0.79529316E-01
C.75CCCC	C.58332971E CC	-C.22429065E-01	0.85204963E-01	-0.43198915E-01	-0.86118107E-01
C.7CCCCC	C.58770492E CC	-C.24247036E-01	0.49630456E-01	-0.25162640E-01	-0.88084398E-01
C.65CCCC	C.59209026E CC	-C.25193850E-01	C.17830504E-01	-C.90400653E-02	-0.86527888E-01
0.60CCCC	C.59633448E CC	-C.25378986E-01	-0.95861405E-02	0.48601731E-02	-0.82481589E-01
0.55CCCC	C.10003737E C1	-C.24917118E-01	-C.32624848E-01	0.16540798E-01	-0.76799585E-01
C.5CCCCC	C.10040925E C1	-C.23417568E-01	-0.51625733E-01	0.26174246E-01	-0.70131412E-01
C.45CCCC	C.10073441E C1	-C.22477546E-01	-0.67078524E-01	0.34008811E-01	-0.62946031E-01
C.40CCCC	C.10103773E C1	-C.20682564E-01	-0.79501446E-01	0.40307232E-01	-0.55504111E-01
C.35CCCC	C.10125757E C1	-C.18602628E-01	-C.89385457E-01	0.45318425E-01	-0.48141491E-01
C.3CCCCC	C.10151725E C1	-C.16297367E-01	-0.97162546E-01	0.49261411E-01	-0.40833723E-01
C.25CCCC	C.10170422E C1	-C.13815907E-01	-C.10318865E-00	0.52316646E-01	-0.33679765E-01
C.2CCCCC	C.10185770E C1	-C.11195281E-01	-C.10775726E-00	0.54632928E-01	-0.26693513E-01
C.15CCCC	C.10197751E C1	-C.84816088E-02	-0.11109421E-00	0.56324763E-01	-0.19865593E-01
C.1CCCCC	C.10205835E C1	-C.56923353E-02	-0.11336015E-00	0.57473597E-01	-0.13167372E-01
C.05CCCC	C.10210487E C1	-C.28573721E-02	-C.11467530E-00	0.58140377E-01	-0.65601324E-02
C.	C.10212134E C1	C.	-C.11510796E-00	0.58359735E-01	0.

ETA = -C.3CCCCC X = -0.357143

Y	U(Y)	U(Y)	-∂P/∂η	-∂P/∂x	-∂P/∂Y
1.CCCCCC	C.55682949E CC	C.	C.29626144E-00	-0.17420172E-00	0.
0.95CCCC	C.55801421E CC	-C.51030703E-02	C.28101528E-00	-0.16523698E-00	-0.46990305E-01
C.9CCCCC	C.56135750E CC	-C.17292889E-01	C.24060257E-00	-0.14147431E-00	-0.85584255E-01
C.85CCCC	C.56633785E CC	-C.23974482E-01	C.18659736E-00	-0.10971925E-00	-0.11168154E-00
C.8CCCCC	C.57231735E CC	-C.28926610E-01	C.12981034E-00	-0.76328480E-01	-0.12540314E-00
C.75CCCC	C.57873308E CC	-C.32201058E-01	0.77231129E-01	-0.45411903E-01	-0.12918909E-00
C.7CCCCC	C.58516955E CC	-C.33955228E-01	C.32125689E-01	-0.18889905E-01	-0.12603076E-00
C.65CCCC	C.59133439E CC	-C.34557084E-01	-0.47513208E-02	0.27937766E-02	-0.11855757E-00
C.6CCCCC	C.59706762E CC	-C.34130073E-01	-C.33981609E-01	0.19981185E-01	-0.10872858E-00
C.55CCCC	C.10022483E C1	-C.32428250E-01	-C.56685632E-01	C.33333503E-01	-0.97675772E-01
C.5CCCCC	C.10065704E C1	-C.31128296E-01	-C.74097382E-01	0.43569260E-01	-0.86612787E-01
C.45CCCC	C.10110708E C1	-C.28871629E-01	-0.87326849E-01	0.51348187E-01	-0.76007819E-01
C.4CCCCC	C.10145985E C1	-C.26268080E-01	-C.97318055E-01	0.57223016E-01	-0.65689966E-01
C.35CCCC	C.10176507E C1	-C.23402879E-01	-0.10482009E-00	0.61634211E-01	-0.55448961E-01
C.3CCCCC	C.10202363E C1	-C.20341104E-01	-C.11041905E-00	0.64926403E-01	-0.46780285E-01
C.25CCCC	C.10223735E C1	-C.17132519E-01	-0.11455626E-00	0.67359080E-01	-0.38141895E-01
C.2CCCCC	C.10240974E C1	-C.13816541E-01	-C.11756342E-00	0.69127292E-01	-0.29956710E-01
C.15CCCC	C.10253985E C1	-C.10423475E-01	-0.11968182E-00	0.70372912E-01	-0.22143830E-01
C.1CCCCC	C.10263732E C1	-C.69766162E-02	-C.12108324E-00	0.71196949E-01	-0.14609074E-01
C.05CCCC	C.10268734E C1	-C.34963772E-02	-0.12188192E-00	0.71666566E-01	-0.72582479E-02
C.	C.10270561E C1	C.	-C.12214202E-00	0.71819511E-01	0.

ETA = -C.25CCCC X = -0.277778

Y	U(Y)	U(Y)	-∂P/∂η	-∂P/∂x	-∂P/∂Y
1.CCCCCC	C.53902603E CC	C.	C.39894659E-00	-0.26928894E-00	0.
C.95CCCC	C.54123766E CC	-C.14338737E-01	C.36791099E-00	-0.24833991E-00	-0.87090344E-01
C.9CCCCC	C.54725490E CC	-C.26519167E-01	C.29170404E-00	-0.19690023E-00	-0.14984202E-00
C.85CCCC	C.55569447E CC	-C.35551304E-01	C.20141301E-00	-0.13595378E-00	-0.18245912E-00
C.8CCCCC	C.56517983E CC	-C.41434344E-01	C.11811064E-00	-0.79724684E-01	-0.19104595E-00
C.75CCCC	C.57470962E CC	-C.44644295E-01	C.50271136E-01	-0.33933016E-01	-0.16443133E-00
C.7CCCCC	C.58365907E CC	-C.45773187E-01	-C.12760373E-02	0.86132520E-03	-0.16985579E-00
C.65CCCC	C.59186943E CC	-C.45358583E-01	-C.38912360E-01	0.26265843E-01	-0.15206405E-00
C.6CCCCC	C.59910460E CC	-C.43830177E-01	-C.65751013E-01	0.44381934E-01	-0.13376398E-00
C.55CCCC	C.10054788E C1	-C.41511089E-01	-C.84625447E-01	0.57122175E-01	-0.11630713E-00
C.5CCCCC	C.10109715E C1	-C.38635275E-01	-0.97783106E-01	0.66003596E-01	-0.10026015E-00
C.45CCCC	C.10156788E C1	-C.35365529E-01	-C.10689553E-00	0.72154482E-01	-0.85749854E-01
C.4CCCCC	C.10196780E C1	-C.31833135E-01	-C.11316904E-00	0.76399106E-01	-0.72718543E-01
C.35CCCC	C.10230196E C1	-C.28107823E-01	-0.11746465E-00	0.79288636E-01	-0.60989568E-01
0.3CCCCC	C.10258757E C1	-C.24252206E-01	-C.12038130E-00	0.81257382E-01	-0.50379170E-01
C.25CCCC	C.10281777E C1	-C.20307171E-01	-C.12234172E-00	0.82580665E-01	-0.40682779E-01
C.2CCCCC	C.10295476E C1	-C.16301385E-01	-0.12363814E-00	0.83455743E-01	-0.31721731E-01
C.15CCCC	C.10313593E C1	-C.12255163E-01	-C.12447452E-00	0.84020300E-01	-0.23320396E-01
C.1CCCCC	C.10323346E C1	-C.81828223E-02	-C.12498844E-00	0.84367201E-01	-0.15325928E-01
C.05CCCC	C.10325138E C1	-C.40945724E-02	-C.12526365E-00	0.84552960E-01	-0.75959764E-02
C.	C.10331058E C1	C.	-C.12534933E-00	0.84610799E-01	0.

ETA = -C.2CCCCC X = -0.208333

Y	U(Y)	U(Y)	-∂P/∂η	-∂P/∂x	-∂P/∂Y
1.CCCCCC	C.51434434E CC	C.	C.55270102E 00	-0.42447437E-00	0.
C.95CCCC	C.51872655E CC	-C.23711936E-01	C.48402826E-00	-0.37173370E-00	-0.17056940E-00
C.9CCCCC	C.52593148E CC	-C.40904672E-01	C.33469839E-00	-0.25704836E-00	-0.26559594E-00
C.85CCCC	C.54423829E CC	-C.52054022E-01	0.18789418E-00	-0.14430273E-00	-0.29143137E-00
C.8CCCCC	C.55884125E CC	-C.57837064E-01	C.75466266E-01	-0.57958041E-01	-0.27769268E-00
C.75CCCC	C.57230516E CC	-C.59817236E-01	-0.14794572E-02	0.11362231E-02	-0.24749504E-00
C.7CCCCC	C.58411045E CC	-C.59280622E-01	-0.51248816E-01	0.39359089E-01	-0.21352771E-00
C.65CCCC	C.59421393E CC	-C.57134397E-01	-0.82452530E-01	0.63323542E-01	-0.18144052E-00
C.6CCCCC	C.10027825E C1	-C.53977601E-01	-C.10161752E-00	0.78038411E-01	-0.15316828E-00
C.55CCCC	C.10099444E C1	-C.50194247E-01	-0.11314382E-00	0.86894459E-01	-0.12894545E-00
C.5CCCCC	C.10160184E C1	-C.46028020E-01	-0.11989561E-00	0.92079835E-01	-0.10836383E-00
C.45CCCC	C.10210080E C1	-C.41632207E-01	-C.12367519E-00	0.94982544E-01	-0.90899074E-01
C.4CCCCC	C.10253473E C1	-C.37103747E-01	-C.12562510E-00	0.96480081E-01	-0.75908066E-01
C.35CCCC	C.10288170E C1	-C.32502507E-01	-C.12647379E-00	0.97131871E-01	-0.62912731E-01
C.3CCCCC	C.10317072E C1	-C.27865561E-01	-C.12668079E-00	0.97290846E-01	-0.51483013E-01
C.25CCCC	C.10341525E C1	-C.23214711E-01	-0.12654530E-00	0.97186789E-01	-0.41277447E-01
C.2CCCCC	C.10360163E C1	-C.18562101E-01	-0.12625447E-00	0.96963434E-01	-0.32003497E-01
C.15CCCC	C.10374761E C1	-C.13912783E-01	-0.12593240E-00	0.96716087E-01	-0.23432709E-01
C.1CCCCC	C.10384124E C1	-C.92713734E-02	-C.12565803E-00	0.96505366E-01	-0.15358318E-01
C.05CCCC	C.10389061E C1	-C.46341810E-02	-C.12547530E-00	0.96365031E-01	-0.76039569E-02
C.	C.10391953E C1	C.	-C.12541124E-00	C.96315833E-01	0.

TABLE VIII. - Continued

ETA = -C.15CCCC   X = -0.147059

Y	U(Y)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial x$	$-\partial P/\partial Y$
1.CCCCCC	C.87857°28E CC	C.	0.8010C835E 00	-0.69447423E 00	0.
0.95CCCC	C.88797A36E CC	-C.3952C545C-01	C.63616814F 00	-0.55155777E 00	-0.35893674E-00
C.90CCCC	C.9C97C°1CE CC	-C.63783CC3E-01	C.33679835E-00	-0.29200416E-00	-0.47113584E-00
C.85CCCC	C.933647C8E CC	-C.75077255E-01	0.11689648E-00	-0.10134925E-00	-0.44494864E-00
C.80CCCC	C.95507^48E CC	-C.78377751E-01	-0.11385564E-01	0.98712841E-02	-0.37618242E-00
C.75CCCC	C.97282958C CC	-C.77262168F-01	-C.7954C130E-01	0.68961293E-01	-0.30612373E-00
C.70CCCC	C.98719135E CC	-C.73817299F-01	-0.11381721E-00	0.98679525E-01	-0.24693610E-00
0.65CCCC	C.99876°9CE CC	-C.69181723E-01	-0.12999923E-00	0.11270933E-00	-0.19978287E-00
0.60CCCC	C.1CC81412E C1	-C.639651C1E-01	-0.13670995E-00	0.11852752E-00	-0.16273432F-00
0.55CCCC	C.1C157°98E C1	-C.58491837F-01	-C.13853318E-00	0.12010827E-00	-0.13349014E-00
C.50CCCC	C.1C22C?C1E C1	-C.529342C9E-01	-0.13789928E-00	0.11955867C-00	-0.11008160E-00
C.45CCCC	C.1C272489E C1	-C.47381718E-01	-0.13611542E-00	0.11801206E-00	-0.91033664E-01
0.40CCCC	C.10315756E C1	-C.41878254C-01	-C.13388730F-00	0.11608028E-00	-0.75229406E-01
0.35CCCC	C.1035C485E C1	-C.36442521E-C1	-C.13159407F-00	0.11409206E-00	-0.61857510E-01
C.30CCCC	C.10379?92E C1	-C.31079168E-01	-0.12944181E-00	0.11222604E-00	-0.50322784E-01
C.25CCCC	C.10402°16E C1	-C.25784739E-01	-C.12754272E-00	0.11057954F-00	-0.40167547E-01
C.20CCCC	C.1042C781E C1	-C.20551273E-01	-0.12595509E-00	0.10920306E-00	-0.31055105E-01
C.15CCCC	C.10434°57E C1	-C.15368177E-C1	-C.1247C657E-00	0.10812060E-00	-0.22683638E-01
C.10CCCC	C.10444174E C1	-C.10223749E-01	-0.12380870E-00	0.10734214E-00	-0.14845417E-01
C.05CCCC	C.10449°59E C1	-C.51053666E-02	-C.12327009E-00	0.10687516E-00	-0.73406183E-02
C.	C.1045174CE C1	C.	-0.123C9104E-00	0.10671993E-00	0.

ETA = -C.10CCCC   X = -0.092593

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial x$	$-\partial P/\partial Y$
1.CCCCCC	C.82317°14E CC	C.	C.12258179E 01	-0.11914949E 01	0.
0.95CCCC	C.84515°C7E CC	-C.722375C7E-01	0.8C612157E 00	-0.78355016F 00	-0.81044006E 00
C.90CCCC	C.88892^28E CC	-C.99964C9CE-01	C.21382045E-00	-0.20783348F-00	-0.80457725E 00
C.85CCCC	C.9274C?68E CC	-C.10551476E-00	-C.5155C543E-01	0.50107127E-01	-0.62064307E 00
C.80CCCC	C.95632121E CC	-C.10224741E-C0	-C.14758693E-00	0.14345449F-00	-0.45691234E-00
0.75CCCC	C.97756°2CE C0	-C.957757CCE-01	-0.17599535E-00	0.17106748E-00	-0.33952098E-00
C.70CCCC	C.99343^39E C0	-C.88298734F-01	-C.17932548E-00	0.17430436F-00	-0.25833898E-00
C.65CCCC	C.10055764E C1	-0.806649C7E-01	-C.17390177E-00	0.16903251E-00	-0.20134698E-00
C.60CCCC	C.1C15C923E C1	-C.73193683E-01	-0.16591752E-00	0.16127183E-00	-0.16011465E-00
0.55CCCC	C.1C226°75E C1	-0.65991312E-01	-0.15772329E-00	0.15330703E-00	-0.12928123F-00
C.50CCCC	C.1C288?29E C1	-0.59077553E-01	-0.15017413E-00	0.14596925E-00	-0.10550699E-00
C.45CCCC	C.10338°CCE C1	-C.52436853E-01	-0.14354370E-00	0.13952447E-00	-0.86622628E-01
C.40CCCC	C.10379^45E C1	-C.4604C352E-01	-0.13786836E-00	0.13400804E-00	-0.71235039E-01
0.35CCCC	C.10413°73E C1	-C.39855316E-01	-0.13309214E-00	0.12936556E-00	-0.58379818E-01
0.30CCCC	C.1044C°55E C1	-C.338497C1E-01	-0.12913854E-00	0.12552267E-00	-0.47386291E-01
C.25CCCC	C.10462955E C1	-C.27992617E-01	-0.12592932E-00	0.12240329E-00	-0.37764996E-01
0.20CCCC	C.1048C?C7E C1	-C.22255787E-01	-0.12339657E-00	0.11994146E-00	-0.29150221E-01
0.15CCCC	C.10493?78E C1	-C.16612359E-01	-0.12148619E-00	0.11808458E-00	-0.21287550E-01
0.10CCCC	C.1C502495E C1	-C.11037450E-01	-0.12015132E-00	0.11678708E-00	-0.13922603E-01
C.05CCCC	C.1C507°82E 01	-C.55075725E-02	-0.11936162E-00	0.11601949E-00	-0.68823179E-02
C.	C.1C509°64E C1	C.	-0.1191C029E-00	0.11576547E-00	0.

ETA = -C.C50000   X = -0.043860

Y	U(Y)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial x$	$-\partial P/\partial Y$
1.CCCCCC	C.72994143E CC	C.	0.19313469E 01	-0.20916486E 01	0.
C.95CCCC	C.78513?65E CC	-C.13796218E-00	0.88926540E 00	-0.96307441E 00	-0.17956883E 01
0.90CCCC	C.87546423E CC	-C.15157745E-00	-0.21375983E-00	0.23150190E-00	-0.11596213E 01
C.85CCCC	C.93129?99E CC	-C.14087524E-00	-0.34228904E-00	0.37069903E-00	-0.70917774E 00
C.80CCCC	C.96509^1CE CC	-C.12673775E-00	-0.31339455E-00	0.33940630E-00	-0.46282919E-00
0.75CCCC	C.98718^47E CC	-C.11339238E-C0	-0.27018671E-00	0.29261221E-00	-0.32422747E-00
0.700CCC	C.10027?74E C1	-0.101458C4E-00	-0.23429107E-00	0.25373723E-00	-0.23986119E-00
0.650CCC	C.1C143^6CE C1	-C.9C806C25E-01	-0.20667358E-00	0.22382749E-00	-0.18450433E-00
C.60CCCC	C.1C232?75E C1	-C.81198049E-01	-0.18552116E-00	0.20091942E-00	-0.14584241E-00
C.55CCCC	C.10303126E C1	-C.7242C938E-01	-0.16914121E-00	0.18317993E-00	-0.11748873E-00
C.50CCCC	C.1036C149E C1	-0.64304676E-01	-0.15630360E-00	0.16927680E-00	-0.95823503E-01
0.45CCCC	C.10406^17E C1	-C.56716366E-01	-0.14614163E-00	0.15827138E-00	-0.78715276E-01
C.4CCCCC	C.10444^96E C1	-C.49551919E-01	-0.13804505E-00	0.14950278E-00	-0.64788407E-01
C.35CCCC	C.10475904E C1	-C.42728348E-01	-0.13157825E-00	0.14249924E-00	-0.53156846E-01
0.30CCCC	C.10501?36E 01	-C.36178476E-01	-0.12642859E-00	0.13692216E-00	-0.43188085E-01
C.25CCCC	C.1C521?94E C1	-Q.29846748E-01	-0.12236851C-00	0.13252509E-00	-0.34459371E-01
C.20CCCC	C.1C537°62E C1	-C.23686083E-01	-0.11923415E-00	0.12913059E-00	-0.26632784E-01
0.15CCCC	C.1C549°71E C1	-C.17655712E-01	-0.11690199E-00	0.12660485E-00	-0.19438514E-01
0.10CCCC	C.1C558421E 01	-C.11719758E-01	-0.11529190E-00	0.12486113E-00	-0.12731165E-01
0.05CCCC	C.1C563412E 01	-C.58447998E-02	-0.11434874E-00	0.12383968E-00	-0.62953536E-02
C.	C.1C565^64E C1	C.	-0.11403636E-00	0.12350138E-00	0.

ETA = C.   X = 0.

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial x$	$-\partial P/\partial Y$
1.CCCCCC	C.	C.	C.36741226E 01	0.44089471E 01	0.16225801E 01
C.95CCCC	C.70519°35E CC	-C.16813695E-00	C.33479514E 01	0.40175416E 01	-0.51692142E 00
C.9CCCCC	C.88623^47E CC	-C.18596719E-00	C.12956875E 01	0.15548250E 01	-0.62995769E 00
C.85CCCC	C.95C33°69E CC	-C.168314C5E-CC	C.68877749E 00	0.82653299E 00	-0.49720012E-00
C.80CCCC	C.98169781E CC	-C.14679790E-C0	0.45287930E-00	0.54345516C 00	-0.35627800E-00
C.75CCCC	C.1CCC9^74E C1	-C.12801859E-00	C.33636098E-00	0.40363318E-00	-0.25911898E-00
C.70CCCC	C.1C142°57E C1	-C.11238252E-00	0.26867602E-00	0.32241122E-00	-0.19604822E-00
C.65CCCC	C.1C242°37E C1	-C.99198409E-01	C.225C8546E-00	0.27010255E-00	-0.15349664E-00
C.60CCCC	C.1032C^84E C1	-C.87802632E-01	C.19504945E-00	0.23405934E-00	-0.12315840E-00
0.55CCCC	C.1C382°62E C1	-C.77714965E-01	C.17337064E-00	0.20804477E-00	-0.10044441E-00
C.5CCCCC	C.10433?39E C1	-C.68607455E-01	C.1572C919E-00	0.18865103E-00	-0.82772990E-01
C.45CCCC	C.10474?21E C1	-C.6C231698E-01	0.14488314E-00	0.17385977E-00	-0.68556542E-01
C.40CCCC	C.1C508°C6E C1	-C.52434389E-01	0.13533289E-00	0.16239947F-00	-0.56809319E-01
C.35CCCC	C.1C536°58E C1	-C.450855C2C-01	C.12786996F-00	0.15344395E-00	-0.46865259E-01
C.30CCCC	C.1C559°CCE C1	-C.38088339E-01	0.12202911E-00	0.14643494E-00	-0.38252752E-01
C.25CCCC	C.1C578?C5E C1	-C.31366753E-C1	C.11748439E-00	0.14098127E-00	-0.30618133E-01
C.20CCCC	C.1C592°75E C1	-C.24858378E-C1	C.11401070E-00	0.13681284E-00	-0.23719211E-01
0.15CCCC	C.1C603°7CE C1	-C.1851C852E-C1	C.11144627E-00	0.13373552E-00	-0.17383970E-01
C.10CCCC	C.1C611°53E C1	-C.12278830E-C1	0.10968637E-00	0.13162365E-00	-0.11381426E-01
C.05CCCC	C.1C616^98E C1	-C.61211165E-C2	C.10865840E-00	0.13039008E-00	-0.56354588E-02
C.	C.1C617^C1E C1	C.	0.10831929E-00	0.12998315E-00	0.

TABLE VIII. - Continued

ETA = C.C5CCCC X = 0.043860

Y	U(Y)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.CCCCCC	C.	C.	C.21741308E 01	0.23545835E 01	0.11475386E 01
C.95CCCC	C.6220C753E CC	-C.12604648E-C0	0.2026C792E 01	0.21942437E 01	0.32303695E-00
C.9CCCCC	C.8753C°89E CC	-C.17738851E-CC	C.1355C053E 01	0.14674707E 01	0.12976304E-01
C.85CCCC	C.965657C1E CC	-C.1763EC92E-C0	C.85867037E 00	0.92993999E 00	-0.13950942E-00
0.80CCCC	C.1CC04769E C1	-C.1585313CE-C0	C.58645986E 00	0.63513602E 00	-0.17075958E-00
C.75CCCC	C.1C173493E C1	-C.13883855E-00	0.43343342E-00	0.46940839E-00	-0.15573520E-00
C.7CCCCC	C.1C28C182E C1	-C.12136712E-C0	0.34111056E-00	0.36942273E-00	-0.13223519E-00
C.65CCCC	C.10358473E C1	-C.10649656E-0C	0.28138046E-00	0.30473503E-00	-0.11087556E-00
C.60CCCC	C.1042C926E C1	-C.93749104E-01	C.24C48692E-0C	0.26044733E-00	-0.93146227E-01
C.55CCCC	C.10471548E C1	-C.8260C439E-C1	0.21126349E-00	0.22879836E-00	-0.78515255E-01
C.5CCCCC	C.1C513511E C1	-C.72645579E-C1	C.1897C335E-00	0.20544872E-00	-0.66302427E-01
C.45CCCC	C.1C548443E C1	-C.6359C853E-C1	0.17342062E-00	0.18781453E-00	-0.55957549E-01
C.4CCCCC	C.1C577°56E C1	-C.55224855E-C1	0.16C91228E-00	0.17426800E-00	-0.47050163E-01
C.35CCCC	C.1C60174EE C1	-C.47392177E-01	0.15121110E-00	0.16376162E-00	-0.39256976E-01
C.3CCCCC	C.1C621°77E C1	-C.39974204E-C1	C.14366804E-00	0.15559249E-00	-0.32325117E-01
C.25CCCC	C.1C637°47E C1	-C.32878666E-01	0.13782826E-00	0.14926800E-00	-0.26063996E-01
C.2CCCCC	C.1C65C433E C1	-C.26031412E-C1	C.13338239E-00	0.14445313E-00	-0.20315728E-01
C.15CCCC	C.1C66C°18E C1	-C.1937C43CE-01	C.13011657E-00	0.14091624E-00	-0.14916854E-01
C.1CCCCC	C.1C6671C3E C1	-C.12842424E-01	0.12787892E-00	0.13849287E-00	-0.98083951E-02
C.05CCCC	C.10671122E C1	-C.64001927E-02	0.12657170E-00	0.13707715E-00	-0.48611663E-02
C.	C.1C672452E C1	C.	C.12614175E-0C	0.13661151E-00	0.

ETA = C.1CCCCC X = 0.092593

Y	U(Y)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.CCCCCC	C.	C.	0.12091222E 01	0.11752668E 01	0.69400644E 00
C.95CCCC	C.5414C434E CC	-C.80809166E-C1	0.10777845E 01	0.10476065E 01	0.42869773E-00
C.9CCCCC	C.83975°6CE CC	-C.14643519E-0C	0.10838675E 01	0.10535192E 01	0.28380413E-00
C.85CCCC	C.96775°C7E CC	-C.1660C283E-00	C.84260909E 00	0.81901602E 00	0.12346790E-00
C.8CCCCC	C.1C163°29E C1	-C.15926708E-00	0.63651009E 00	0.61868780E 00	0.13977483E-01
C.75CCCC	C.1C352234E C1	-C.14356853E-CC	0.49399044E-00	0.48015870E-00	-0.37049899E-01
C.7CCCCC	C.10441487E C1	-C.12685125E-00	C.39770690E-00	0.38657111E-00	-0.54894045E-01
C.65CCCC	C.10497°51E C1	-C.11161358E-0C	C.33134420E-00	0.32206656E-00	-0.58212571E-01
C.6CCCCC	C.1C541°35E C1	-C.98222253E-01	0.28418942E-00	0.27623211E-00	-0.55753034E-01
C.55CCCC	C.1C577°23E C1	-C.86429728E-C1	C.24971338E-00	0.24272140E-00	-0.51109891E-01
C.5CCCCC	C.1060E°65E C1	-C.75895585E-C1	0.22391307E-00	0.21764349E-00	-0.45725498E-01
C.45CCCC	C.1C635°77E C1	-C.66345806E-01	C.20425398E-00	0.19853486E-00	-0.40235534E-01
C.4CCCCC	C.10658°77E C1	-C.57545226E-C1	C.18907824E-00	0.18378405E-00	-0.34899461E-01
C.35CCCC	C.1C677°77E C1	-C.4933C087E-C1	0.17727444E-00	0.17231075E-00	-0.29803693E-01
C.3CCCCC	C.1C693°12E C1	-C.41571047E-C1	C.16807617E-00	0.16337004E-00	-0.24983082E-01
C.25CCCC	C.10707°47E C1	-C.34166791E-C1	C.16095208E-00	0.15644541E-00	-0.20419496E-01
C.2CCCCC	C.10717°04E C1	-C.27035211E-01	0.15553093E-00	0.15117606E-00	-0.16060823E-01
C.15CCCC	C.1C725°52E C1	-C.20108164E-01	C.15154319E-00	0.14729998E-00	-0.11896928E-01
C.1CCCCC	C.10731°15E C1	-C.13327377E-C1	0.14881094E-00	0.14464423E-00	-0.78560711E-02
C.05CCCC	C.10734°8CE C1	-C.66406277E-C2	C.14721683E-00	0.14309476E-00	-0.39072122E-02
C.	C.10735°95E C1	C.	0.14669222E-00	0.14258483E-00	0.

ETA = C.15CCCC X = 0.147059

Y	U(Y)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.CCCCCC	C.	C.	C.67569975E 00	0.58583167E 00	0.40600255E-00
C.95CCCC	C.47135°C3E-CC	-C.4804E7C9E-C1	0.52508347E 00	0.45524736E-00	0.33087420E-00
C.9CCCCC	C.7676°108E CC	-C.11033960E-C0	C.75187072E 00	0.65187190E 00	0.30621091E-00
C.85CCCC	C.95245°48E CC	-C.14385783E-00	0.72182859E 0C	0.62582538E 00	0.23252980E-00
C.80CCCC	C.1C235175E C1	-C.15005807E-CC	C.61287674E 00	0.53136413E 00	0.13726961E-00
C.75CCCC	C.1C507189E C1	-C.14176749E-C0	0.51139686E 00	0.44338106E-00	0.64420753E-01
C.7CCCCC	C.10608°37E C1	-C.12834775E-C0	C.431C5990E-00	0.37372893E-00	0.20738564E-01
C.65CCCC	C.10653°1CE C1	-C.11427068E-00	0.36968160E-00	0.32051394E-00	-0.27290939E-02
C.60CCCC	C.1C679°19E C1	-C.1C111CC4E-C0	C.32288575E-00	0.27994194E-00	-0.14583591E-01
C.55CCCC	C.10701°01E C1	-C.89187850E-01	0.28692921E-00	0.24876761E-00	-0.20043885E-01
0.50CCCC	C.1072C136E C1	-C.78403395E-01	C.25904407E-00	0.22459120E-00	-0.21946035E-01
C.45CCCC	C.10737°78E C1	-C.6856C433E-01	0.23724431E-00	0.20569081E-00	-0.21818463E-01
C.40CCCC	C.1C752°15E C1	-C.5947C474E-C1	0.2201C896E-00	0.19083447E-00	-0.20526599E-01
C.35CCCC	C.10766°9CE C1	-C.50976385E-C1	C.2066C367E-00	0.17912538E-00	-0.18562705E-01
C.30CCCC	C.1C777°78E C1	-C.42952301E-C1	C.19597834E-00	0.16991321E-00	-0.16222285E-01
C.25CCCC	C.10787°92E C1	-C.35296191E-01	C.18769842E-00	0.16273452E-00	-0.13662358E-01
C.20CCCC	C.10795°69E C1	-C.27924240E-C1	0.18135951E-00	0.15723869E-00	-0.10994466E-01
C.15CCCC	C.1C801°45E C1	-C.20766735E-01	C.17668179E-00	0.15318311E-00	-0.82707280E-02
C.1CCCCC	C.10805°58E C1	-C.13762323E-01	C.17347059E-00	0.15039900E-00	-0.55181142E-02
C.C5CCCC	C.10808°35E C1	-C.68569542E-02	0.17159159E-00	0.14876992E-00	-0.27611168E-02
C.	C.1C809°91E C1	C.	0.17097449E-00	0.14823488E-00	0.

ETA = C.2000CC X = 0.208333

Y	U(Y)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.0CCCCC	C.	C.	0.41505998E-00	0.31876605E-00	0.24303222E-00
C.95CCCC	C.41413°36E-CC	-C.2825C426E-01	0.26827346E-00	0.20603400E-00	0.22296727E-00
C.9CCCCC	C.72924711E CC	-C.78886811E-C1	0.48430160E-00	0.37194362E-00	0.24144343E-00
C.85CCCC	C.92193969E CC	-C.11716283E-CC	0.57294924E 00	0.44002500E-00	0.23503731E-00
0.80CCCC	C.10187739E C1	-C.1335272CE-C0	0.54736684E 00	0.42037772E-00	0.18729008E-00
C.75CCCC	C.10602°41E C1	-C.13359277E-C0	0.49289457E-00	0.37854302E-00	0.12822763E-00
C.7CCCCC	C.10758°35E C1	-C.12524259E-C0	0.43893385E-00	0.33710119E-00	0.79804091E-01
C.65CCCC	C.10812°11E C1	-C.11378196E-C0	0.391828C6E-00	0.30092394E-00	0.46360088E-01
C.60CCCC	C.1083C°41E C1	-C.10183541E-C0	C.35226023E-00	0.27053585E-00	0.24813722E-01
0.55CCCC	C.10839°22E C1	-C.90418164E-01	0.31953690E-00	0.24540433E-00	0.11295156E-01
C.5CCCCC	C.10846°82E C1	-C.79796558E-C1	0.29269522E-00	0.22478992E-00	0.29726448E-02
C.45CCCC	C.10853°5CE C1	-C.6995C8C6E-01	0.27C8C246E-00	0.20797628E-00	-0.19838168E-02
0.40CCCC	C.10861°93E C1	-C.6077522CE-C1	0.25303261E-00	0.19432904E-00	-0.47259260E-02
C.35CCCC	C.10868°98E C1	-C.52152812E-C1	C.23868562E-00	0.18331055E-00	-0.60099927E-02
C.3CCCCC	C.10874°67E C1	-C.43977525E-01	0.22719529E-00	0.17448598E-00	-0.63070308E-02
C.25CCCC	C.1088C°81E C1	-C.36158264E-C1	C.21811269E-0C	0.16751054E-00	-0.59563061E-02
C.20CCCC	C.10885°87E C1	-C.28617295E-01	0.21109179E-00	0.16211849E-00	-0.51710490E-02
C.15CCCC	C.10888°97E C1	-C.21287569E-01	0.20587495E-00	0.15811196E-00	-0.40849947E-02
C.1CCCCC	C.10891°5CE C1	-C.14105849E-01	0.20227110E-00	0.15534420E-00	-0.28179789E-02
C.C5CCCC	C.10893°14E C1	-C.70307526E-02	0.20015641E-00	0.15372012E-00	-0.14343084E-02
C.	C.10893°71E C1	C.	0.19946048E-00	0.15318564E-00	0.

TABLE VIII. - Continued

ETA = 0.250000 X = 0.277778

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.000000	0.	0.	0.29905055E-00	0.20185912E-00	0.15318311E-00
0.950000	0.36863782E-00	-0.17193076E-01	0.17478351E-00	0.11797886E-00	0.14694792E-00
0.900000	0.67188971E 00	-0.55124775E-01	0.31302265E-00	0.21129028E-00	0.17031383E-00
0.850000	0.88130914E 00	-0.91372515E-01	0.43982083E-00	0.29687905E-00	0.19237915E-00
0.800000	0.10021743E 01	-0.11329052E-00	0.47052173E-00	0.31760217E-00	0.18423740E-00
0.750000	0.10615062E 01	-0.12046952E-00	0.45486478E-00	0.30703372E-00	0.15217586E-00
0.700000	0.10866890E 01	-0.11772708E-00	0.42718202E-00	0.28834786E-00	0.11437503E-00
0.650000	0.10957991E 01	-0.10986748E-00	0.39804830E-00	0.26868260E-00	0.81700338E-01
0.600000	0.10984135E 01	-0.10000685E-00	0.37032951E-00	0.24997241E-00	0.56862257E-01
0.550000	0.10988165E 01	-0.89742169E-01	0.34503312E-00	0.23289736E-00	0.38950160E-01
0.500000	0.10986638E 01	-0.79745542E-01	0.32258817E-00	0.21774701E-00	0.26279420E-01
0.450000	0.10984951E 01	-0.70231110E-01	0.30312053E-00	0.20460635E-00	0.17403071E-01
0.400000	0.10984760E 01	-0.61219139E-01	0.28654423E-00	0.19341736E-00	0.11228810E-01
0.350000	0.10984810E 01	-0.52659716E-01	0.27266376E-00	0.18404803E-00	0.69971669E-02
0.300000	0.10985940E 01	-0.44484243E-01	0.26122759E-00	0.17632861E-00	0.41441218E-02
0.250000	0.10987795E 01	-0.36623795E-01	0.25199369E-00	0.17009573E-00	0.22867759E-02
0.200000	0.10988899E 01	-0.29014476E-01	0.24474189E-00	0.16520078E-00	0.11377032E-02
0.150000	0.10990744E 01	-0.21598379E-01	0.23928677E-00	0.16151857E-00	0.47581687E-03
0.100000	0.10991793E 01	-0.14322826E-01	0.23548797E-00	0.15895437E-00	0.14788191E-03
0.050000	0.10991956E 01	-0.71388259E-02	0.23324803E-00	0.15744242E-00	0.25352817E-04
0.	0.10992184E 01	0.	0.23250754E-00	0.15694258E-00	0.

ETA = 0.300000 X = 0.357143

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.000000	0.	0.	0.25170155E-00	0.14800051E-00	0.10143223E-00
0.950000	0.33247703E-00	-0.11009612E-01	0.15348369E-00	0.90248410E-01	0.98772997E-01
0.900000	0.61935164E 00	-0.38381733E-01	0.22329435E-00	0.13129708E-00	0.11626548E-00
0.850000	0.83576821E 00	-0.69254402E-01	0.34023869E-00	0.20006035E-00	0.14278813E-00
0.800000	0.97566731E 00	-0.92504978E-01	0.40104029E-00	0.23581169E-00	0.15525869E-00
0.750000	0.10536914E 01	-0.10440353E-00	0.41295188E-00	0.24281570E-00	0.14666840E-00
0.700000	0.10915703E 01	-0.10666589E-00	0.40584987E-00	0.23863972E-00	0.12502832E-00
0.650000	0.11074533E 01	-0.10271593E-00	0.39306860E-00	0.23112433E-00	0.10021693E-00
0.600000	0.11128794E 01	-0.95516304E-01	0.37831581E-00	0.22244969E-00	0.77793416E-01
0.550000	0.11139790E 01	-0.86944249E-01	0.36280982E-00	0.21333217E-00	0.59468792E-01
0.500000	0.11135835E 01	-0.78008267E-01	0.34738658E-00	0.20426331E-00	0.45128404E-01
0.450000	0.11128415E 01	-0.69164768E-01	0.33272748E-00	0.19564375E-00	0.34085363E-01
0.400000	0.11121509E 01	-0.60584103E-01	0.31933442E-00	0.18776864E-00	0.25633312E-01
0.350000	0.11116122E 01	-0.52303642E-01	0.30749473E-00	0.18080690E-00	0.19168705E-01
0.300000	0.11112736E 01	-0.44306386E-01	0.29733011E-00	0.17483010E-00	0.14222131E-01
0.250000	0.11109562E 01	-0.36555034E-01	0.28886285E-00	0.16985136E-00	0.10414132E-01
0.200000	0.11107786E 01	-0.29006628E-01	0.28205005E-00	0.16584543E-00	0.74508991E-02
0.150000	0.11106651E 01	-0.21618116E-01	0.27683569E-00	0.16277938E-00	0.51023156E-02
0.100000	0.11105966E 01	-0.14347495E-01	0.27316183E-00	0.16061915E-00	0.31801698E-02
0.050000	0.11105604E 01	-0.71544851E-02	0.27097730E-00	0.15933465E-00	0.15253785E-02
0.	0.11105492E 01	0.	0.27025213E-00	0.15890825E-00	0.

ETA = 0.350000 X = 0.448718

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.000000	0.	0.	0.23718627E-00	0.12025344E-00	0.69403429E-01
0.950000	0.30335777E-00	-0.73529085E-02	0.16100718E-00	0.81630641E-01	0.68034735E-01
0.900000	0.57283794E 00	-0.26856344E-01	0.18836761E-00	0.95502378E-01	0.79367291E-01
0.850000	0.78926780E 00	-0.51526416E-01	0.27716108E-00	0.14052066E-00	0.10122953E-00
0.800000	0.94234698E 00	-0.73243057E-01	0.34752057E-00	0.17619292E-00	0.11981872E-00
0.750000	0.10374730E 01	-0.87332093E-01	0.37729455E-00	0.19128833E-00	0.12526309E-00
0.700000	0.10895523E 01	-0.93287082E-01	0.38458757E-00	0.19498589E-00	0.11783738E-00
0.650000	0.11147610E 01	-0.92931172E-01	0.38393834E-00	0.19465673E-00	0.10318245E-00
0.600000	0.11252429E 01	-0.88563614E-01	0.38026178E-00	0.19279271E-00	0.86567377E-01
0.550000	0.11285921E 01	-0.82023574E-01	0.37458636E-00	0.18991528E-00	0.70949448E-01
0.500000	0.11289010E 01	-0.74490746E-01	0.36734066E-00	0.18624171E-00	0.57413574E-01
0.450000	0.11281780E 01	-0.66617117E-01	0.35911885E-00	0.18207325E-00	0.46094932E-01
0.400000	0.11271114E 01	-0.58720370E-01	0.35058348E-00	0.17774582E-00	0.36766920E-01
0.350000	0.11261641E 01	-0.50935058E-01	0.34230569E-00	0.17354897E-00	0.29098752E-01
0.300000	0.11253773E 01	-0.43304092E-01	0.33471017E-00	0.16969805E-00	0.22780353E-01
0.250000	0.11247595E 01	-0.35828586E-01	0.32805647E-00	0.16632462E-00	0.17535303E-01
0.200000	0.11242928E 01	-0.28491571E-01	0.32250696E-00	0.16351102E-00	0.13117974E-01
0.150000	0.11239550E 01	-0.21268087E-01	0.31814991E-00	0.16130200E-00	0.93288405E-02
0.100000	0.11237773E 01	-0.14130748E-01	0.31502030E-00	0.15971529E-00	0.59818762E-02
0.050000	0.11235961E 01	-0.70509608E-02	0.31313920E-00	0.15876157E-00	0.29211339E-02
0.	0.11235833E 01	0.	0.31251091E-00	0.15844303E-00	0.

ETA = 0.400000 X = 0.555556

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial X$	$-\partial P/\partial Y$
1.000000	0.	0.	0.24146961E-00	0.10431486E-00	0.48336409E-01
0.950000	0.27950093E-00	-0.50377599E-02	0.18203695E-00	0.78639963E-01	0.47582683E-01
0.900000	0.53222440E 00	-0.18894606E-01	0.18705010E-00	0.80805641E-01	0.54586267E-01
0.850000	0.74427567E 00	-0.37836333E-01	0.24695133E-00	0.10668297E-00	0.70214352E-01
0.800000	0.90504667E 00	-0.56531674E-01	0.31334458E-00	0.13536485E-00	0.87541405E-01
0.750000	0.10141569E 01	-0.70759171E-01	0.35305966E-00	0.15252177E-00	0.98651327E-01
0.700000	0.10805767E 01	-0.78868788E-01	0.37023458E-00	0.15994133E-00	0.10044482E-00
0.650000	0.11167658E 01	-0.81336720E-01	0.37762589E-00	0.16313438E-00	0.94641215E-01
0.600000	0.11343934E 01	-0.79609334E-01	0.38174611E-00	0.16491432E-00	0.84664366E-01
0.550000	0.11417761E 01	-0.75199461E-01	0.38406117E-00	0.16591442E-00	0.73340264E-01
0.500000	0.11439710E 01	-0.69274408E-01	0.38452592E-00	0.16611519E-00	0.62296770E-01
0.450000	0.11439468E 01	-0.62592103E-01	0.38319457E-00	0.16554005E-00	0.52228897E-01
0.400000	0.11430920E 01	-0.55588078E-01	0.38044232E-00	0.16435108E-00	0.43315535E-01
0.350000	0.11420191E 01	-0.48485102E-01	0.37682228E-00	0.16278722E-00	0.35516970E-01
0.300000	0.11410195E 01	-0.41401327E-01	0.37286516E-00	0.16107775E-00	0.28699391E-01
0.250000	0.11401723E 01	-0.34367577E-01	0.36899196E-00	0.15940452E-00	0.22706987E-01
0.200000	0.11394975E 01	-0.27398509E-01	0.36551797E-00	0.15790376E-00	0.17392378E-01
0.150000	0.11385901E 01	-0.20490928E-01	0.36264687E-00	0.15666344E-00	0.12598383E-01
0.100000	0.11386783E 01	-0.13632149E-01	0.36051624E-00	0.15574301E-00	0.81902289E-02
0.050000	0.11384717E 01	-0.68073960E-02	0.35920952E-00	0.15517851E-00	0.40329459E-02
0.	0.11383636E 01	0.	0.35876772E-00	0.15498765E-00	0.

TABLE VIII. - Continued

ETA = 0.450000 X = 0.681818

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial x$	$-\partial P/\partial y$
1.000000	0.	0.	0.26011204E-00	0.94420671E-01	0.33933516E-01
0.950000	0.25962025E-00	-0.34945358E-02	0.21307997E-00	0.77348032E-01	0.33465470E-01
0.900000	0.49691141E-00	-0.13320611E-01	0.20741134E-00	0.75290316E-01	0.37675393E-01
0.850000	0.70213096E 00	-0.27473252E-01	0.24447077E-00	0.88742890E-01	0.48030375E-01
0.800000	0.86613470E 00	-0.42669475E-01	0.29996866E-00	0.10888862E-00	0.61503249E-01
0.750000	0.98571480E 00	-0.55669555E-01	0.34304799E-00	0.12452642E-00	0.73062456E-01
0.700000	0.10650472E 01	-0.64530358E-01	0.36693600E-00	0.13319776E-00	0.79191502E-01
0.650000	0.11130468E 01	-0.68855384E-01	0.37965138E-00	0.13781345E-00	0.79340377E-01
0.600000	0.11394496E 01	-0.69291332E-01	0.38850172E-00	0.14102612E-00	0.74997702E-01
0.550000	0.11525156E 01	-0.66879554E-01	0.39613496E-00	0.14379699E-00	0.68130441E-01
0.500000	0.11580847E 01	-0.62612744E-01	0.40258603E-00	0.14613873E-00	0.60270909E-01
0.450000	0.11598054E 01	-0.57246571E-01	0.40738461E-00	0.14788061E-00	0.52333868E-01
0.400000	0.11597572E 01	-0.51281743E-01	0.41038235E-00	0.14896879E-00	0.44757803E-01
0.350000	0.11590038E 01	-0.45017724E-01	0.41180541E-00	0.14948536E-00	0.37706060E-01
0.300000	0.11580460E 01	-0.38620180E-01	0.41205955E-00	0.14957762E-00	0.31197909E-01
0.250000	0.11571725E 01	-0.32174063E-01	0.41158842E-00	0.14940659E-00	0.25195320E-01
0.200000	0.11564146E 01	-0.25719642E-01	0.41075832E-00	0.14910527E-00	0.19623667E-01
0.150000	0.11558724E 01	-0.19273484E-01	0.40985885E-00	0.14877876E-00	0.14405963E-01
0.100000	0.11554022E 01	-0.12835817E-01	0.40908954E-00	0.14849950E-00	0.94555140E-02
0.050000	0.11551520E 01	-0.64165286E-02	0.40857735E-00	0.14831357E-00	0.46829555E-02
0.	0.11550490E 01	0.	0.40839419E-00	0.14824709E-00	0.

ETA = 0.500000 X = 0.833333

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial x$	$-\partial P/\partial y$
1.000000	0.	0.	0.29287100E-00	0.87861300E-01	0.23889060E-01
0.950000	0.24280503E-00	-0.24343505E-02	0.25505765E-00	0.76517295E-01	0.23512162E-01
0.900000	0.46616133E-00	-0.43725569E-02	0.24474032E-00	0.73422097E-01	0.25914501E-01
0.850000	0.66342480E 00	-0.19725953E-01	0.26583497E-00	0.79750491E-01	0.32360008E-01
0.800000	0.82736880E 00	-0.31550131E-01	0.30847320E-00	0.92541959E-01	0.41736074E-01
0.750000	0.95393412E 00	-0.42601591E-01	0.34980241E-00	0.10494072E-00	0.51278824E-01
0.700000	0.10442022E 01	-0.51141387E-01	0.37763642E-00	0.11329092E-00	0.58281977E-01
0.650000	0.11036542E 01	-0.56366253E-01	0.39415132E-00	0.11824539E-00	0.61426932E-01
0.600000	0.11358763E 01	-0.58337173E-01	0.40571265E-00	0.12171379E-00	0.60890783E-01
0.550000	0.11601744E 01	-0.57605870E-01	0.41616263E-00	0.12484878E-00	0.57665806E-01
0.500000	0.11705083E 01	-0.54905666E-01	0.42630835E-00	0.12789250E-00	0.52850460E-01
0.450000	0.11751417E 01	-0.50875313E-01	0.43559445E-00	0.13067833E-00	0.47282082E-01
0.400000	0.11767703E 01	-0.46030774E-01	0.44339605E-00	0.13301881E-00	0.41480763E-01
0.350000	0.11768954E 01	-0.40699738E-01	0.44944502E-00	0.13483351E-00	0.35717823E-01
0.300000	0.11764015E 01	-0.35098705E-01	0.45383902E-00	0.13615170E-00	0.30115841E-01
0.250000	0.11757109E 01	-0.29351938E-01	0.45686463E-00	0.13705938E-00	0.24711198E-01
0.200000	0.11750764E 01	-0.23528845E-01	0.45885415E-00	0.13765624E-00	0.19504286E-01
0.150000	0.11744718E 01	-0.17666755E-01	0.46010439E-00	0.13803131E-00	0.14468317E-01
0.100000	0.11740558E 01	-0.11785198E-01	0.46084177E-00	0.13825253E-00	0.95671876E-02
0.050000	0.11738031E 01	-0.58944294E-02	0.46121778E-00	0.13836533E-00	0.47600387E-02
0.	0.11737186E 01	0.	0.46131326E-00	0.13839398E-00	0.

ETA = 0.550000 X = 1.018519

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial x$	$-\partial P/\partial y$
1.000000	0.	0.	0.34201384E-00	0.83109362E-01	0.16843070E-01
0.950000	0.22837706E-00	-0.16965568E-02	0.31109607E-00	0.75596345E-01	0.16451219E-01
0.900000	0.43527760E-00	-0.65659812E-02	0.29909737E-00	0.72680661E-01	0.17664208E-01
0.850000	0.62821404E 00	-0.14003333E-01	0.30974093E-00	0.75267047E-01	0.21378512E-01
0.800000	0.78988470E 00	-0.22884770E-01	0.34044567E-00	0.82728299E-01	0.27302931E-01
0.750000	0.92041761E 00	-0.31763617E-01	0.37667857E-00	0.91532892E-01	0.34097356E-01
0.700000	0.10191783E 01	-0.39302103E-01	0.40571923E-00	0.98589772E-01	0.40066577E-01
0.650000	0.10890408E 01	-0.44636082E-01	0.42492609E-00	0.10325704E-00	0.43966032E-01
0.600000	0.11353108E 01	-0.47481576E-01	0.43819991E-00	0.10648257E-00	0.45378540E-01
0.550000	0.11639535E 01	-0.48013589E-01	0.44975632E-00	0.10929079E-00	0.44577546E-01
0.500000	0.11805199E 01	-0.46656427E-01	0.46144798E-00	0.11213186E-00	0.42153855E-01
0.450000	0.11893924E 01	-0.43697636E-01	0.47320327E-00	0.11498839E-00	0.38707660E-01
0.400000	0.11936892E 01	-0.40173411E-01	0.48423548E-00	0.11766922E-00	0.34696313E-01
0.350000	0.11954441E 01	-0.35821706E-01	0.49385116E-00	0.12000582E-00	0.30413555E-01
0.300000	0.11958920E 01	-0.31075731E-01	0.50172685E 00	0.12191962E-00	0.26028790E-01
0.250000	0.11957388E 01	-0.26102772E-01	0.50785752E 00	0.12340938E-00	0.21626227E-01
0.200000	0.11953453E 01	-0.20987009E-01	0.51242667E 00	0.12451968E-00	0.17243188E-01
0.150000	0.11949479E 01	-0.15790433E-01	0.51567176E 00	0.12530824E-00	0.12892895E-01
0.100000	0.11946433E 01	-0.10547464E-01	0.51781129E 00	0.12582814E-00	0.85751109E-02
0.050000	0.11944760E 01	-0.52792991E-02	0.51900029E 00	0.12611707E-00	0.42809162E-02
0.	0.11943452E 01	0.	0.51931680E 00	0.12619398E-00	0.

ETA = 0.600000 X = 1.250000

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial x$	$-\partial P/\partial y$
1.000000	0.	0.	0.41229161E-00	0.79159991E-01	0.11919250E-01
0.950000	0.21578453E-00	-0.11825466E-02	0.38649113E-00	0.74206299E-01	0.11471531E-01
0.900000	0.41552716E-00	-0.45789671E-02	0.37417740E-00	0.71842063E-01	0.11911002E-01
0.850000	0.59626929E 00	-0.98392724E-02	0.37818007E-00	0.72610576E-01	0.13804409E-01
0.800000	0.75428121E 00	-0.16318974E-01	0.39905302E-00	0.76618181E-01	0.17159400E-01
0.750000	0.88637707E 00	-0.23131713E-01	0.42861525E-00	0.82294130E-01	0.21406766E-01
0.700000	0.99122915E 00	-0.29349074E-01	0.45636074E-00	0.87621264E-01	0.25636669E-01
0.650000	0.10699038E 01	-0.34233712E-01	0.47701664E-00	0.91587197E-01	0.28981195E-01
0.600000	0.11259771E 01	-0.37378535E-01	0.49155433E-00	0.94378432E-01	0.30921864E-01
0.550000	0.11635701E 01	-0.38711228E-01	0.50340917E 00	0.96654562E-01	0.31357744E-01
0.500000	0.11874463E 01	-0.38404569E-01	0.51516343E 00	0.98911380E-01	0.30492927E-01
0.450000	0.12018408E 01	-0.36758280E-01	0.52756412E 00	0.10129231E-00	0.28656088E-01
0.400000	0.12100046E 01	-0.34095810E-01	0.54012251E 00	0.10370352E-00	0.26169232E-01
0.350000	0.12143720E 01	-0.30722823E-01	0.55196493E 00	0.10597727E-00	0.23281820E-01
0.300000	0.12163971E 01	-0.26860399E-01	0.56237582E 00	0.10797615E-00	0.20159789E-01
0.250000	0.12172448E 01	-0.22682726E-01	0.57097888E 00	0.10962794E-00	0.16906525E-01
0.200000	0.12174740E 01	-0.18306606E-01	0.57769850E 00	0.11091811E-00	0.13578981E-01
0.150000	0.12174728E 01	-0.13808738E-01	0.58264069E 00	0.11186701E-00	0.10210267E-01
0.100000	0.12173115E 01	-0.92385596E-02	0.58595514E 00	0.11250339E-00	0.68171801E-02
0.050000	0.12172084E 01	-0.46282250E-02	0.58778024E 00	0.11285380E-00	0.34116216E-02
0.	0.12171457E 01	0.	0.58815974E 00	0.11292667E-00	0.

TABLE VIII. - Continued

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                                  ETA =   C.650uCU      X =   1.547619

    Y              U(X)              U(Y)             - ∂P/∂η            - ∂P/∂X            - ∂P/∂Y
1.00CCCC      C.                 C.                0.51205588E 00     0.75272218E-01     0.84847739E-02
0.95CCCC      C.20452°27E-CC    -C.82548586E-03    0.48993316E-00     0.72020177E-01     0.79948444E-02
0.90CCCC      C.39418ᶠ48E-CC    -C.318386C2E-02    0.47782107E-00     0.70239701E-01     0.79715561E-02
0.85CCCC      C.56707ʳ17E CC    -0.68581011E-02    0.47760934E-00     0.70208576E-01     0.87422744E-02
0.80CCCC      C.72065⁷24E CC    -C.11474704E-01    0.49088863E-00     0.72160632E-01     0.10385249E-01
0.75CCCC      C.85254751E CC    -C.16508121E-01    0.51356157E 00     0.75493553E-01     0.12696700E-01
C.70CCCC      C.96135ᶠ49E CC    -C.21361197E-01    0.53814039E 00     0.79106640E-01     0.15251118E-01
0.65CCCC      C.10472407E C1    -0.25488317E-01    0.55883186E 00     0.82148286E-01     0.17554476E-01
0.60CCCC      C.11119ᶜC4E C1    -0.2850C751E-01    0.57425626E 00     0.84415673E-01     0.19208089E-01
C.55CCCC      C.11586ᵀCCE C1    -C.30212966E-01    0.58632211E 00     0.86189353E-01     0.20009164E-01
C.50CCCC      C.11906ᶜ15E C1    -C.30626549E-01    0.59758964E 00     0.87845681E-01     0.19948656E-01
0.45CCCC      C.12118129E C1    -C.29875612E-01    0.60956457E 00     0.89605996E-01     0.19149659E-01
C.40CCCC      C.12251⁷39E C1    -C.28162757E-01    0.62238591E 00     0.91490732E-01     0.17786862E-01
C.35CCCC      C.12331493E C1    -C.25705989E-01    0.63535250E 00     0.93396822E-01     0.16029906E-01
C.30CCCC      C.12377ᴬ76E C1    -C.22704191E-01    0.64754394E 00     0.95188963E-01     0.14014173E-01
0.25CCCC      C.12402°97E C1    -0.19320795E-01    0.65820709E 00     0.96756446E-01     0.11834350E-01
C.20CCCC      C.12415ᴾ39E C1    -0.1568C619E-01    0.66690908E 00     0.98035639E-01     0.95530523E-02
C.15CCCC      C.12421ⁿ74E C1    -C.11873969E-01    0.67348307E 00     0.99002016E-01     0.72080658E-02
0.1CCCCC      C.12424ᶠ84E C1    -0.79641656E-02    0.67791872E 00     0.99654055E-01     0.48243119E-02
0.050CCC      C.12425ᶠ35E C1    -C.39954057E-02    0.68023764E 00     0.99994937E-01     0.24174463E-02
0.            C.12425759E C1     0.                0.68038345E 00     0.10001637E-00     0.

                                  ETA =   C.7000C0      X =   1.944444

    Y              U(X)              U(Y)             - ∂P/∂η            - ∂P/∂X            - ∂P/∂Y
1.00CCCC      C.                 C.                0.65655185E 00     0.70907601E-01     0.60349274E-02
0.95CCCC      C.19412ᶠ62E-CC    -0.57508439E-03    0.63692590E 00     0.68787998E-01     0.55570406E-02
0.90CCCC      C.37448⁷99E-CC    -C.22025871E-02    0.62511737E 00     0.67512677E-01     0.53163861E-02
0.85CCCC      C.53986ᶜCCE CO    -0.47375724E-02    0.62219201E 00     0.67196739E-01     0.54887503E-02
0.80CCCC      C.68863481E CO    -C.79573658E-02    0.62983526E 00     0.68022210E-01     0.61434846E-02
C.75CCCC      C.81911ᴾC2E CO    -0.11555149E-01    0.64625745E 00     0.69795806E-01     0.72227080E-02
0.70000C      C.93013ᶜ42E CC    -0.15167779E-01    0.66668255E 00     0.72001716E-01     0.85521968E-02
0.65CCCC      0.1C214⁷54E 01    -0.18431C24E-01    0.68611863E 00     0.74100813E-01     0.98902594E-02
C.60CCCC      C.1C937ⁿ66E 01    -C.2104C497E-01    0.70197059E 00     0.75812826E-01     0.10999636E-01
0.55CCCC      C.1149Cᴬ89E 01    -0.22794831E-01    0.71452694E 00     0.77168912E-01     0.11707760E-01
0.50CCCC      C.11897413E 01    -0.23609149E-01    0.72566812E 00     0.78372158E-01     0.11937158E-01
C.45CCCC      C.12186ʳ12E 01    -C.23501430E-01    0.73720111E 00     0.79617722E-01     0.11696479E-01
C.4CCCCC      C.12383ᶠ93E 01    -C.22562798E-01    0.74994239E 00     0.80993780E-01     0.11052100E-01
C.35CCCC      C.12514ᶠ51E C1    -C.20924377E-01    0.76367038E 00     0.82476402E-01     0.10093836E-01
C.30000C      C.12598⁷37E C1    -0.18728689E-01    0.77752107E 00     0.83972277E-01     0.89098940E-02
0.25CCCC      C.1265C742E C1    -0.1611C188E-01    0.79047039E 00     0.85370804E-01     0.75736281E-02
C.20CCCC      C.12681326E C1    -C.13184606E-01    0.80163198E 00     0.86576255E-01     0.61391946E-02
C.150000      C.12699258E C1    -C.10045630E-01    0.81037477E 00     0.87520477E-01     0.46441629E-02
0.10CCCC      C.12709ʳ71E C1    -0.67662317E-02    0.81628728E 00     0.88159028E-01     0.31125727E-02
0.05CCCC      C.12713ᴾ64E 01    -0.34027119E-02    0.81907262E 00     0.88459844E-01     0.15608230E-02
0.            C.12715789E C1     C.                0.81841429E 00     0.88388745E-01     0.

                                  ETA =   C.750000      X =   2.500000

    Y              U(X)              U(Y)             - ∂P/∂η            - ∂P/∂X            - ∂P/∂Y
1.00CCCC      C.                 C.                0.87711108E 00     0.65783334E-01     0.41817421E-02
0.95CCCC      C.18413⁷95E-CC    -0.39246865E-03    0.85910968E 00     0.64433229E-01     0.37781207E-02
0.90CCCC      C.35564⁷93E-CO    -0.14918391E-02    0.84759829E 00     0.63569874E-01     0.34956680E-02
0.85CCCC      C.51378ᶠ18E CO    -C.31985902E-02    0.84301407E 00     0.63226058E-01     0.34250624E-02
C.80CCCC      C.65752⁷52E CC    -C.53769910E-02    0.84672945E 00     0.63504712E-01     0.36103354E-02
0.75CCCC      C.78578442E CC    -0.78495520E-02    0.85810392E 00     0.64357796E-01     0.40409987E-02
C.70CCCC      C.89768998E 00    -0.10405020E-01    0.87433717E 00     0.65575290E-01     0.46524985E-02
C.65CCCC      C.99286⁷77E CO    -C.12819669E-01    0.89174806E 00     0.66881107E-01     0.53401221E-02
0.60CCCC      C.10715ᴾ48E C1    -0.14885987E-01    0.90754990E 00     0.68066245E-01     0.59833986E-02
0.55CCCC      C.11348⁷62E C1    -0.16435220E-01    0.92092098E 00     0.69069076E-01     0.64750292E-02
0.500CCC      C.11841489E C1    -0.17374082E-01    0.93275285E 00     0.69956467E-01     0.67411765E-02
C.45CCCC      C.12215ʳ14E 01    -C.17648438E-01    0.94463380E 00     0.70847538E-01     0.67505484E-02
C.40CCCC      C.12489ᴾ55E C1    -0.17275824E-01    0.95777365E 00     0.71833026E-01     0.65091446E-02
0.350CCC      C.12686ᶠ04E C1    -0.16311119E-01    0.97253126E 00     0.72939847E-01     0.60492349E-02
0.30CCCC      C.12823⁴33E 01    -0.14834312E-01    0.98835043E 00     0.74126285E-01     0.54150338E-02
C.25CCCC      C.12916⁷39E C1    -0.12935917E-01    0.10041030E 01     0.75307731E-01     0.46518559E-02
0.20CCCC      C.12977⁷72E C1    -0.10706199E-01    0.10184389E 01     0.76382929E-01     0.37992527E-02
0.150CCC      C.13C16⁷67E C1    -0.822852C2E-02    0.10300427E 01     0.77253211E-01     0.28883216E-02
0.10CCCC      C.13C39ᴬ69E C1    -C.55766550E-02    0.10377772E 01     0.77833297E-01     0.19417901E-02
0.C5CCCC      C.13052ʳ38E 01    -0.28148303E-02    0.10407005E 01     0.78052543E-01     0.97529806E-03
C.            C.13C55ᴾ91E C1     C.                0.1038C205E 01     0.77851542E-01     0.

                                  ETA =   C.80CCC0      X =   3.333333

    Y              U(X)              U(Y)             - ∂P/∂η            - ∂P/∂X            - ∂P/∂Y
1.00CCCC      C.                 C.                0.12478866E 01     0.59898559E-01     0.26827240E-02
0.950CCC      C.17423443E-CC    -0.25120497E-03    0.1231C815E 01     0.59091915E-01     0.23924991E-02
0.9CCCCC      C.33706903E-CC    -C.94917297E-03    0.12201320E 01     0.58566339E-01     0.21641026E-02
0.85CCCC      C.48806ᴬ97E-CC    -0.20285874E-02    0.12150001E 01     0.58320005E-01     0.20431816E-02
0.80CCCC      C.62662199E CC    -C.34100819E-02    0.12166082E 01     0.58397196E-01     0.20542660E-02
C.75CCCC      C.75207⁷62E CC    -0.49932039E-02    0.12247391E 01     0.58787481E-01     0.21986394E-02
0.7CCCCC      C.86387ᵀ2CE CC    -0.66622030E-02    0.12377942E 01     0.59414127E-01     0.24543831E-02
0.65CCCC      C.96167ᶜ29E CC    -0.8291C763E-02    0.12532671E 01     0.60156821E-01     0.27798511E-02
0.60CCCC      C.10455⁷6CE C1    -C.97551297E-02    0.12687807E 01     0.60901475E-01     0.31214673E-02
0.55CCCC      C.11158ᶜ43E C1    -C.10943355E-01    0.12829863E 01     0.61583347E-01     0.34234767E-02
C.5CCCCC      C.11736ᵀ84E C1    -C.11769029E-01    0.12958983E 01     0.62203119E-01     0.36381999E-02
0.45CCCC      C.12199ᵀ14E C1    -C.12176166E-01    0.13084603E 01     0.62806098E-01     0.37326150E-02
0.400CCC      C.12562ʳ03E C1    -C.12141019E-01    0.13218638E 01     0.63449463E-01     0.36914548E-02
C.35CCCC      C.1284Cʳ86E C1    -C.11669150E-01    0.13368251E 01     0.64167609E-01     0.35154981E-02
0.30CCCC      C.13048⁷64E C1    -0.10789815E-01    0.13532100E 01     0.64954083E-01     0.32175848E-02
C.25CCCC      C.1320Cᴬ56E C1    -C.9549C377E-02    0.13700303E 01     0.65761460E-01     0.28174882E-02
0.20CCCC      C.13308ᴾ8CE C1    -C.80031180E-02    0.13856755E 01     0.66512429E-01     0.23373525E-02
C.15CCCC      C.13383ᵀ96E C1    -0.62132353E-02    0.13982093E 01     0.67114049E-01     0.17983954E-02
0.10CCCC      C.1343C°97E C1    -C.42418465E-02    0.14056972E 01     0.67473465E-01     0.12192333E-02
C.C5CCCC      C.13457417E C1    -0.21506435E-02    0.14C64482E 01     0.67509514E-01     0.61542901E-03
0.            C.1346591CE C1     C.                0.13992044E 01     0.67161815E-01     0.
```

TABLE VIII. - Continued

ETA = 0.850000 X = 4.722222

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial x$	$-\partial P/\partial Y$
1.000000	0.	0.	0.19802310E 01	0.53466240E-01	0.14435439E-02
0.950000	0.16435951E-00	-0.13840199E-03	0.19648797E 01	0.53051757E-01	0.12732495E-02
0.900000	0.31860679E-00	-0.52054036E-03	0.19555163E 01	0.52798945E-01	0.11315785E-02
0.850000	0.46251625E-00	-0.11098411E-02	0.19517218E 01	0.52696493E-01	0.10394520E-02
0.800000	0.59577301E 00	-0.18638041E-02	0.19538507E 01	0.52753972E-01	0.10088848E-02
0.750000	0.71803190E 00	-0.27327674E-02	0.19615783E 01	0.52962618E-01	0.10427062E-02
0.700000	0.82898431E 00	-0.36601030E-02	0.19737069E 01	0.53290091E-01	0.11346747E-02
0.650000	0.92842780E 00	-0.45841978E-02	0.19883471E 01	0.53685375E-01	0.12704477E-02
0.600000	0.10163390E 01	-0.54416898E-02	0.20034178E 01	0.54092285E-01	0.14295555E-02
0.550000	0.10928699E 01	-0.61729763E-02	0.20172764E 01	0.54466467E-01	0.15883318E-02
0.500000	0.11584654E 01	-0.67247677E-02	0.20291131E 01	0.54786057E-01	0.17231330E-02
0.450000	0.12137516E 01	-0.70560252E-02	0.20390070E 01	0.55053195E-01	0.18135387E-02
0.400000	0.12595483E 01	-0.71395130E-02	0.20476088E 01	0.55285443E-01	0.18445176E-02
0.350000	0.12968004E 01	-0.69629962E-02	0.20556401E 01	0.55502289E-01	0.18075667E-02
0.300000	0.13265124E 01	-0.65288082E-02	0.20634151E 01	0.55712213E-01	0.17006744E-02
0.250000	0.13496801E 01	-0.58522990E-02	0.20705704E 01	0.55905404E-01	0.15273255E-02
0.200000	0.13672331E 01	-0.49596464E-02	0.20760793E 01	0.56054146E-01	0.12951560E-02
0.150000	0.13795866E 01	-0.38854513E-02	0.20783991E 01	0.56116779E-01	0.10144232E-02
0.100000	0.13886043E 01	-0.26704433E-02	0.20758452E 01	0.56047824E-01	0.69681642E-03
0.050000	0.13935757E 01	-0.13594973E-02	0.20669367E 01	0.55807295E-01	0.35455987E-03
0.	0.13951092E 01	0.	0.20507821E 01	0.55371121E-01	0.

ETA = 0.900000 X = 7.499999

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial x$	$-\partial P/\partial Y$
1.000000	0.	0.	0.39051884E 01	0.46862268E-01	0.48330315E-03
0.950000	0.15495516E-00	-0.52729555E-04	0.38929316E 01	0.46715187E-01	0.42086796E-03
0.900000	0.30108981E-00	-0.19738737E-03	0.38883410E 01	0.46660101E-01	0.36785473E-03
0.850000	0.43831646E-00	-0.41950917E-03	0.38901274E 01	0.46681537E-01	0.33108675E-03
0.800000	0.56651761E 00	-0.70303838E-03	0.38974476E 01	0.46769379E-01	0.31463838E-03
0.750000	0.68554804E 00	-0.10299782E-02	0.39090358E 01	0.46908438E-01	0.31987140E-03
0.700000	0.79529071E 00	-0.13803919E-02	0.39230285E 01	0.47076350E-01	0.34553592E-03
0.650000	0.89568719E 00	-0.17328101E-02	0.39370962E 01	0.47245163E-01	0.38800123E-03
0.600000	0.98669738E 00	-0.20650178E-02	0.39486916E 01	0.47384308E-01	0.44165951E-03
0.550000	0.10683803E 01	-0.23551551E-02	0.39554851E 01	0.47465830E-01	0.49952111E-03
0.500000	0.11409128E 01	-0.25830177E-02	0.39557258E 01	0.47468717E-01	0.55395155E-03
0.450000	0.12045504E 01	-0.27313448E-02	0.39485238E 01	0.47382294E-01	0.59745947E-03
0.400000	0.12596458E 01	-0.27869729E-02	0.39339191E 01	0.47207037E-01	0.62345775E-03
0.350000	0.13066773E 01	-0.27416637E-02	0.39127094E 01	0.46952522E-01	0.62686931E-03
0.300000	0.13459734E 01	-0.25925932E-02	0.38862089E 01	0.46634515E-01	0.60452976E-03
0.250000	0.13781828E 01	-0.23424351E-02	0.38559330E 01	0.46271203E-01	0.55537567E-03
0.200000	0.14037441E 01	-0.19990994E-02	0.38233728E 01	0.45880482E-01	0.48039853E-03
0.150000	0.14231049E 01	-0.15752066E-02	0.37898789E 01	0.45478554E-01	0.38244377E-03
0.100000	0.14366435E 01	-0.10873069E-02	0.37566711E 01	0.45080061E-01	0.26589507E-03
0.050000	0.14446470E 01	-0.55501372E-03	0.37249165E 01	0.44699006E-01	0.13630409E-03
0.	0.14472946E 01	0.	0.36958759E 01	0.44350519E-01	0.

ETA = 0.950000 X = 15.833332

Y	U(X)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial x$	$-\partial P/\partial Y$
1.000000	0.	0.	0.13819533E 02	0.41458607E-01	0.28755765E-04
0.950000	0.14791057E-00	-0.88863833E-05	0.13813341E 02	0.41440029E-01	0.24578348E-04
0.900000	0.28804183E-00	-0.29865061E-04	0.13816723E 02	0.41450178E-01	0.21269476E-04
0.850000	0.42037867E-00	-0.60905823E-04	0.13826912E 02	0.41480743E-01	0.19447050E-04
0.800000	0.54490060E 00	-0.99814401E-04	0.13841354E 02	0.41524070E-01	0.19475690E-04
0.750000	0.66158578E 00	-0.14419647E-03	0.13857209E 02	0.41571636E-01	0.21476913E-04
0.700000	0.77041499E 00	-0.19145857E-03	0.13871482E 02	0.41614455E-01	0.25341698E-04
0.650000	0.87137565E 00	-0.23884539E-03	0.13880980E 02	0.41642948E-01	0.30751325E-04
0.600000	0.96446700E 00	-0.28352004E-03	0.13882596E 02	0.41647796E-01	0.37208798E-04
0.550000	0.10497033E 01	-0.32267292E-03	0.13873624E 02	0.41620880E-01	0.44080973E-04
0.500000	0.11271162E 01	-0.35366160E-03	0.13852023E 02	0.41556079E-01	0.50653531E-04
0.450000	0.11967553E 01	-0.37416150E-03	0.13816817E 02	0.41450460E-01	0.56192806E-04
0.400000	0.12586883E 01	-0.38231297E-03	0.13768259E 02	0.41304783E-01	0.60011996E-04
0.350000	0.13129088E 01	-0.37684942E-03	0.13707870E 02	0.41123616E-01	0.61534093E-04
0.300000	0.13597814E 01	-0.35719519E-03	0.13638423E 02	0.40915275E-01	0.60346832E-04
0.250000	0.13991378E 01	-0.32351972E-03	0.13563760E 02	0.40691287E-01	0.56240950E-04
0.200000	0.14311688E 01	-0.27674366E-03	0.13488343E 02	0.40465035E-01	0.49233086E-04
0.150000	0.14559653E 01	-0.21850228E-03	0.13416978E 02	0.40250941E-01	0.39568258E-04
0.100000	0.14736704E 01	-0.15105868E-03	0.13354363E 02	0.40063096E-01	0.27700642E-04
0.050000	0.14841846E 01	-0.77183035E-04	0.13304721E 02	0.39914169E-01	0.14259941E-04
0.	0.14877015E 01	0.	0.13271460E 02	0.39814388E-01	0.

ETA = 1.000000 X = ∞

Y	U(Y)	U(Y)	$-\partial P/\partial \eta$	$-\partial P/\partial x$	$-\partial P/\partial Y$
1.000000	0.	0.	∞	0.39999999E-01	0.
0.950000	0.14625002E-00	0.	∞	0.39999999E-01	0.
0.900000	0.28500003E-00	0.	∞	0.39999999E-01	0.
0.850000	0.41625001E-00	0.	∞	0.39999999E-01	0.
0.800000	0.54000001E 00	0.	∞	0.39999999E-01	0.
0.750000	0.65625002E 00	0.	∞	0.39999999E-01	0.
0.700000	0.76500001E 00	0.	∞	0.39999999E-01	0.
0.650000	0.86625001E 00	0.	∞	0.39999999E-01	0.
0.600000	0.96000001E 00	0.	∞	0.39999999E-01	0.
0.550000	0.10462499E 01	0.	∞	0.39999999E-01	0.
0.500000	0.11250000E 01	0.	∞	0.39999999E-01	0.
0.450000	0.11962499E 01	0.	∞	0.39999999E-01	0.
0.400000	0.12599999E 01	0.	∞	0.39999999E-01	0.
0.350000	0.13162500E 01	0.	∞	0.39999999E-01	0.
0.300000	0.13649999E 01	0.	∞	0.39999999E-01	0.
0.250000	0.14062499E 01	0.	∞	0.39999999E-01	0.
0.200000	0.14399999E 01	0.	∞	0.39999999E-01	0.
0.150000	0.14662500E 01	0.	∞	0.39999999E-01	0.
0.100000	0.14850000E 01	0.	∞	0.39999999E-01	0.
0.050000	0.14962500E 01	0.	∞	0.39999999E-01	0.
0.	0.15000000E 01	0.	∞	0.39999999E-01	0.

TABLE IX. COMPARISON OF RESULTS: u_x

	$\eta = 0 \quad x = 0$			$\eta = 0.05 \quad x = 0.043859$		
y	Schlichting's	Case I	Case II	Schlichting's	Case I	Case II
0.95	1.0	1.0	0.7052	0.705	0.8281	0.6220
0.9	1.0	1.0	0.8862	1.007	0.9958	0.8753
0.8	1.0	1.0	0.9817	1.040	1.0324	1.0004
0.7	1.0	1.0	1.0143	1.040	1.0268	1.0280
0.6	1.0	1.0	1.0321	1.040	1.0214	1.0421
0.5	1.0	1.0	1.0433	1.040	1.0178	1.0514
0.4	1.0	1.0	1.0509	1.040	1.0155	1.0578
0.3	1.0	1.0	1.0560	1.040	1.0140	1.0622
0.2	1.0	1.0	1.0593	1.040	1.0131	1.0651
0.1	1.0	1.0	1.0612	1.040	1.0126	1.0667
0.0	1.0	1.0	1.0618	1.040	1.0124	1.0672

	$\eta = 0.20 \quad x = 0.208333$			$\eta = 0.50 \quad x = 0.83333$		
y	Schlichting's	Case I	Case II	Schlichting's	Case I	Case II
0.95	0.399	0.4261	0.4141	0.245	0.2166	0.2428
0.9	0.720	0.7624	0.7292	0.467	0.4247	0.4662
0.8	1.046	1.0615	1.0188	0.806	0.7877	0.8274
0.7	1.088	1.1046	1.0759	1.082	1.0303	1.0442
0.6	1.088	1.0940	1.0831	1.130	1.1483	1.1398
0.5	1.088	1.0807	1.0847	1.182	1.1889	1.1705
0.4	1.088	1.0713	1.0861	1.192	1.1967	1.1768
0.3	1.088	1.0650	1.0875	1.196	1.1946	1.1764
0.2	1.088	1.0610	1.0885	1.196	1.1912	1.1750
0.1	1.088	1.0588	1.0892	1.196	1.1888	1.1741
0.0	1.088	1.0581	1.0894	1.196	1.1880	1.1737

	$\eta = 0.80 \quad x = 3.33333$			$\eta = 0.90 \quad x = 7.50000$		
y	Schlichting's	Case I	Case II	Schlichting's	Case I	Case II
0.95	0.172	0.1712	0.1742	0.156	0.1542	0.1550
0.9	0.333	0.3315	0.3371	0.302	0.2998	0.3011
0.8	0.619	0.6177	0.6266	0.568	0.5644	0.5665
0.7	0.858	0.8547	0.8639	0.797	0.7929	0.7953
0.6	1.041	1.0390	1.0455	0.989	0.9847	0.9867
0.5	1.175	1.1715	1.1736	1.143	1.1398	1.1409
0.4	1.265	1.2587	1.2562	1.261	1.2598	1.2596
0.3	1.317	1.3111	1.3048	1.347	1.3474	1.3460
0.2	1.338	1.3397	1.3309	1.404	1.4063	1.4037
0.1	1.346	1.3533	1.3431	1.436	1.4400	1.4366
0.0	1.347	1.3572	1.3466	1.446	1.4509	1.4473

TABLE X. PRESSURE DROPS, ΔP_η, AS FUNCTION OF η

		Case I			Case II*		
η	Schlichting	y=0.9	y=0.5	y=0.1	y=0.9	y=0.5	y=0.1
0	0	0	0	0	-0.0563	0.0463	0.0628
0.05	0.035	0.0506	0.0277	0.0207	0.0119	0.0550	0.0688
0.15	0.078	0.1638	0.0719	0.0508	0.1193	0.0774	0.0837
0.25	0.111	0.1955	0.1186	0.0882	0.1693	0.1066	0.1040
0.35	0.145	0.1863	0.1663	0.1336	0.1925	0.1413	0.1314
0.45	0.180	0.1787	0.2107	0.1861	0.2116	0.1797	0.1675
0.55	0.217	0.1854	0.2525	0.2428	0.2364	0.2225	0.2137
0.65	0.258	0.2123	0.2998	0.3036	0.2743	0.2746	0.2727
0.75	0.317	0.2697	0.3686	0.3822	0.3380	0.3485	0.3557
0.85	0.437	0.3931	0.4998	0.5225	0.4661	0.4842	0.5013
0.95	0.934	0.9108	1.0237	1.0270	0.9882	1.0126	1.0089
1.00	∞	∞	∞	∞	∞	∞	∞

* Relative to free-stream pressure at $-\infty$.

TABLE XI. VARIATION OF ψ AND ω WITH GRID SIZES

ψ

		Case I		Case II	
	y	10×10 Grid	20×20 Grid	20×10 Grid	40×20 Grid
$\eta = 0.10$	0.90	0.92664	0.93361	0.94480	0.94586
$x = 0.0926$	0.50	0.51464	0.51473	0.53765	0.53481
	0.10	0.10266	0.10263	0.10794	0.10735
$\eta = 0.50$	0.90	0.97769	0.97834	0.97591	0.97572
$x = 0.8333$	0.50	0.59804	0.59619	0.59323	0.58760
	0.10	0.11948	0.11882	0.11873	0.11738
$\eta = 0.90$	0.90	0.98430	0.98458	0.98422	0.98450
$x = 7.500$	0.50	0.67688	0.67659	0.67605	0.67571
	0.10	0.14513	0.14482	0.14482	0.14446

TABLE XI. - CONTINUED

ω

		Case I		Case II	
	y	10×10 Grid	20×20 Grid	20×10 Grid	40×20 Grid
$\eta = 0.10$	1.00	14.673	18.156	11.041	13.425
$x = 0.0926$	0.90	3.534	4.585	4.186	4.418
	0.50	5.547×10^{-3}	3.506×10^{-4}	6.260×10^{-3}	2.522×10^{-4}
	0.10	1.802×10^{-6}	3.061×10^{-9}	1.632×10^{-6}	1.531×10^{-9}
$\eta = 0.50$	1.00	4.462	4.394	4.818	5.023
$x = 0.8333$	0.90	4.064	4.075	4.120	4.266
	0.50	0.3820	0.2717	0.3043	0.1820
	0.10	3.904×10^{-3}	5.758×10^{-4}	2.582×10^{-3}	2.856×10^{-4}
$\eta = 0.90$	1.00	3.1406	3.1718	3.1566	3.1870
$x = 7.500$	0.90	2.7988	2.8234	2.8094	2.8342
	0.50	1.3774	1.3722	1.3673	1.3610
	0.10	0.2300	0.2197	0.2262	0.2146

PART TWO

APPENDICES

LIST OF APPENDICES

APPENDIX A: FINITE-DIFFERENCE METHOD FOR SOLVING BOUNDARY-VALUE PROBLEMS

A boundary-value problem, $F(u) = j$, for the continuous operator F, when discretized, takes the form

$$AU = B \tag{1}$$

The term discretization refers to the transformation of differential equations into finite-difference equations. In equation 1, B is a vector of constants determined by j and by the boundary conditions, U the vector of unknown u, and A a matrix determined by F and by the boundary conditions. For linear problems, A is constant. For non-linear problems, A is also a function of U. Methods for solving a matrix equation like equation 1 are ordinarily divided into direct and iterative. The direct methods are confined almost exclusively to linear systems. They can render exact solutions in a finite number of steps if there are no round-off errors. A typical direct method is Gaussian elimination. Iterative methods, on the other hand, consist of the repeated application of a simple algorithm and would result in the exact answer only as a limit of a sequence. When A is large, iterative methods are preferred because they usually take full advantage of the numerous zeros in the matrix (14). Moreover, the round-off errors of an iterative method tend to be self-correcting and are minimized, in contrast to the direct method. Finally, for non-linear systems, iterative methods are invariably used.

In applying the iterative method, equation 1 can be written as a system of N algebraic equations with N unknowns, u_i's. A one-to-one correspondence between the N unknowns and the N equations can thus be obtained with each equation to be solved for the corresponding unknown. For partial difference equations, it is natural to use the equation of a node to solve for the unknown at that node. Let $U^{(P)}$ be the vector U obtained from equation 1 after P-th iteration. The method of simultaneous displacements, or total step iteration, is one which determines $U^{(P+1)}$ based on the values of $U^{(P)}$. On the other hand, with the method of successive displacements, or single-step iteration, a certain order of the systems of equations has to be fixed; i.e., the order in which the components, u_i's, will be computed. In solving equation 1 for $u_i^{(P+1)}$, the latest computed values of u_i's are used; i.e., $u_j^{(P+1)}$'s for $j < i$ and $u_j^{(P)}$'s for $j \geq i$.

The number of iterations required for the solution of a system of linear equations to converge to a given degree of accuracy is roughly proportional to the square of number of equations. For non-linear systems, this number would presumably be more dependent on the number of equations. When the number of nodes involved in a discreted partial differential equation of boundary-value type is large, the time of computation required may become too large, even when handled by a high-speed digital computer. Acceleration of the rate of convergence, therefore, is often a necessity.

Of the various iterative methods available, the one that converges most rapidly is the method of successive overrelaxation, or simply overrelaxation. In using overrelaxation, the value of $u_i^{(P+1)}$ is computed by the formula

$$u_i^{(P+1)} = u_i^{(P)} + M_u \left(v_i^{(P+1)} - u_i^{(P)}\right) \tag{A1}$$

In equation A1, $u_i^{(P)}$ is the value of u_i after the P-th iteration; $v_i^{(P+1)}$ is the value of the function u_i computed by the governing difference equation for point i; and finally M_u is the overrelaxation factor which is ordinarily larger than 1. With M_u equal to unity, the iteration is called stationary and when M_u is smaller than unity, the process is sometimes called underrelaxation.

Overrelaxation is not very useful when coupled with the method of simultaneous displacements. With the method of successive displacements, however, Young (15) found that overrelaxation was very helpful in dealing with a class of matrices with, what he called, property A. Further investigations by others revealed that overrelaxation also improved the rate of convergence for matrices which do not satisfy property A, even though it was theoretically unjustifiable at the time. It was found that, for certain types of matrices, the method of overrelaxation gives convergence for all values of M_u between 0 and 2, and that the most rapid convergence occurs for a value of $M_{u_{opt}}$ between 1 and 2. The determination of $M_{u_{opt}}$ for a system of linear equations involves the determination of the dominant eigenvalue of the matrix A in equation 1, and is a tedious job. For non-linear boundary-value problems, matrix A is a function of U. Under such circumstance, the value of $M_{u_{opt}}$ requires frequent determination without prior knowledge (16). It is, therefore, probably more efficient to determine the value of M_u empirically by trial-and-error. The rate of convergence using successive overrelaxation, with $M_u \sim M_{u_{opt}}$, is roughly P times the rate of successive displacements, wherein P is the dimension of matrix A. When P is large, the use of the overrelaxation method thus results in a substantial saving of time.

APPENDIX B: DERIVATION OF FINITE-DIFFERENCE EQUATIONS

The derivation of finite-difference equations from differential equations involves the replacement of derivatives by difference formulas. Some of the more commonly used formulas for first derivatives are the central-difference, the forward-difference and the backward-difference formulas. Referring to Figure 1, which shows the point pattern of a rectangular mesh net with mesh lengths of a and b, the central-difference formulas can be written as:

$$\begin{aligned} \left(\frac{\partial \Psi}{\partial \eta}\right)_E &= \frac{1}{2a}\left(\Psi_A - \Psi_C\right) \\ \left(\frac{\partial \Psi}{\partial y}\right)_E &= \frac{1}{2b}\left(\Psi_B - \Psi_D\right) \end{aligned} \qquad \text{(B1)}$$

The forward-difference formulas have the form:

$$\begin{aligned} \left(\frac{\partial \Psi}{\partial \eta}\right)_E &= \frac{1}{a}\left(\Psi_A - \Psi_E\right) \\ \left(\frac{\partial \Psi}{\partial y}\right)_E &= \frac{1}{b}\left(\Psi_B - \Psi_E\right) \end{aligned} \qquad \text{(B2)}$$

Similarly, the backward-difference formulas are:

$$\begin{aligned} \left(\frac{\partial \Psi}{\partial \eta}\right)_E &= \frac{1}{a}\left(\Psi_E - \Psi_C\right) \\ \left(\frac{\partial \Psi}{\partial y}\right)_E &= \frac{1}{b}\left(\Psi_E - \Psi_D\right) \end{aligned} \qquad \text{(B3)}$$

For the second derivatives, the difference formulas most frequently used are based on central-differences as given by:

$$\left(\frac{\partial^2 \Psi}{\partial \eta^2}\right)_E = \frac{1}{a^2}\,(\Psi_A+\Psi_C-2\Psi_E) \tag{B4}$$

$$\left(\frac{\partial^2 \Psi}{\partial y^2}\right)_E = \frac{1}{b^2}\,(\Psi_B+\Psi_D-2\Psi_E) \tag{B5}$$

$$\left(\frac{\partial^2 \Psi}{\partial \eta \partial y}\right)_E = \frac{1}{4ab}\,(\Psi_F-\Psi_G+\Psi_H-\Psi_I) \tag{B6}$$

The errors which result in using these formulas are discussed in nearly all standard numerical analysis books. More elaborate formulas than the ones given above are available and are used mostly for large mesh sizes. In this work, central-difference formulas were used in most instances.

Consider now equations 11, 12, 14 and 15. Equations 12 and 15, being linear, present no particular problem. Equation 15 can be rewritten as:

$$-\,\omega = \frac{\partial^2 \Psi}{\partial y^2} + \left(\frac{\partial \eta}{\partial x}\right)^2 \frac{\partial^2 \Psi}{\partial \eta^2} + \frac{\partial^2 \eta}{\partial x^2}\,\frac{\partial \Psi}{\partial \eta} \tag{B7}$$

Since $$\eta = 1 - \frac{1}{1+Cx} \tag{13}$$

then $$\frac{\partial \eta}{\partial x} = C\,(1-\eta)^2 \tag{B8}$$

and $$\frac{\partial^2 \eta}{\partial x^2} = -\,2C^2\,(1-\eta)^3 \tag{B9}$$

Substituting equations B1, B4, B5, B8 and B9 into B7 and solving for Ψ at point E results in

$$\begin{aligned} 2\left[C^2(1-\eta_E)^4 + \left(\frac{a}{b}\right)^2\right] \Psi_E = a^2\omega_E + \left(\frac{a}{b}\right)^2 (\Psi_B+\Psi_D) \\ + C^2(1-\eta_E)^4 (\Psi_A+\Psi_C) - aC^2(1-\eta_E)^3 (\Psi_A-\Psi_C) \end{aligned} \tag{B10}$$

Similarly, the difference expression for equation 12 at point E in the x-y net is:

$$2\left[1+\left(\frac{\Delta x}{b}\right)^2\right] \Psi_E = \Delta x^2\omega_E + (\Psi_B+\Psi_D) + \left(\frac{\Delta x}{b}\right)^2 (\Psi_A+\Psi_C) \tag{B11}$$

The approach used in obtaining equations B10 and B11, however, was not applied to equations 11 and 14. Preliminary investigations indicated that a special device, due to de G. Allen (11), had to be used in order to induce stability. Equation 14, when rearranged, becomes

$$\begin{aligned} \left[\frac{\partial^2\omega}{\partial y^2} + \left(\frac{Re}{4}\frac{\partial\eta}{\partial x}\frac{\partial\Psi}{\partial\eta}\right)\frac{\partial\omega}{\partial y}\right] \\ + \left(\frac{\partial\eta}{\partial x}\right)^2\left[\frac{\partial^2\omega}{\partial\eta^2} + \left(\frac{\frac{\partial^2\eta}{\partial x^2} - \frac{Re}{4}\frac{\partial\eta}{\partial x}\frac{\partial\Psi}{\partial y}}{\left(\frac{\partial\eta}{\partial x}\right)^2}\right)\frac{\partial\omega}{\partial\eta}\right] = 0 \end{aligned} \tag{B12}$$

or symbolically

$$A + \left(\frac{\partial\eta}{\partial x}\right)^2 B = 0 \tag{B12a}$$

where

$$A \equiv \frac{\partial^2\omega}{\partial y^2} + \left(\frac{Re}{4}\frac{\partial\eta}{\partial x}\frac{\partial\Psi}{\partial\eta}\right)\frac{\partial\omega}{\partial y} \equiv \frac{\partial^2\omega}{\partial y^2} + \kappa\frac{\partial\omega}{\partial y} \tag{B13}$$

and

$$B \equiv \frac{\partial^2\omega}{\partial\eta^2} + \left(\frac{\frac{\partial^2\eta}{\partial x^2} - \frac{Re}{4}\frac{\partial\eta}{\partial x}\frac{\partial\Psi}{\partial y}}{\left(\frac{\partial\eta}{\partial x}\right)^2}\right)\frac{\partial\omega}{\partial\eta} \equiv \frac{\partial^2\omega}{\partial\eta^2} + \lambda\frac{\partial\omega}{\partial\eta} \tag{B14}$$

When κ and A are held invariant, equation B13 can easily be solved to give

$$\kappa\omega = Ay + P + Qe^{-\kappa y} \tag{B15}$$

P and Q are constants of integration. Referring to Figure 1 again, the following equations can be written:

$$\begin{aligned} \kappa\omega_B &= Ay_B + P + Qe^{-\kappa y_B} \\ \kappa\omega_E &= Ay_E + P + Qe^{-\kappa y_E} \\ \kappa\omega_D &= Ay_D + P + Qe^{-\kappa y_D} \end{aligned} \tag{B16}$$

Elimination of P and Q from B16 results in an expression for A at point E:

$$A_E = \frac{\kappa_E[e^{\kappa_E b}(\omega_B-\omega_E) - (\omega_E-\omega_D)]}{b\ (e^{\kappa_E b} - 1)} \tag{B17}$$

Similarly, by holding λ and B invariant, solution of equation B14 gives

$$B_E = \frac{\lambda_E[e^{\lambda_E a}(\omega_A-\omega_E) - (\omega_E-\omega_C)]}{a\ (e^{\lambda_E a} - 1)} \tag{B18}$$

Substituting equations B17 and B18 into B12a and solving for ω_E results in:

$$C_E\omega_E = C_A\omega_A + C_B\omega_B + C_C\omega_C + C_D\omega_D \tag{B19}$$

where

$$\begin{aligned} C_A &= \left(\frac{\partial\eta}{\partial x}\right)^2_E \lambda_E \, e^{\lambda_E a}/a(e^{\lambda_E a} - 1) \\ C_B &= \kappa_E \, e^{\kappa_E b}/b(e^{\kappa_E b} - 1) \\ C_C &= \left(\frac{\partial\eta}{\partial x}\right)^2_E \lambda_E/a(e^{\lambda_E a} - 1) \\ C_D &= \kappa_E/b(e^{\kappa_E b} - 1) \end{aligned} \qquad \text{(B20)}$$

and

$$C_E = C_A + C_B + C_C + C_D$$

To evaluate κ and λ, the central-difference formulas are applied to the partial derivatives of Ψ. Thus:

$$\kappa_E = \frac{Re}{4}\left(\frac{\partial\eta}{\partial x}\frac{\partial\Psi}{\partial\eta}\right)_E = \frac{Re}{8a} C(1-\eta_E)^2 \; (\Psi_A - \Psi_C) \qquad \text{(B21)}$$

$$\begin{aligned} \lambda_E &= \left(\frac{\partial^2\eta}{\partial x^2} - \frac{Re}{4}\frac{\partial\eta}{\partial x}\frac{\partial\Psi}{\partial y}\right)\Big/\left(\frac{\partial\eta}{\partial x}\right)^2_E \\ &= -\frac{2}{1-\eta_E} - \frac{Re}{8b}\frac{1}{C(1-\eta_E)^2}(\Psi_B - \Psi_D) \end{aligned} \qquad \text{(B22)}$$

In a like manner, equation 11 can be transformed into the following form:

$$C'_E\omega_E = C'_A\omega_A + C'_B\omega_B + C'_C\omega_C + C'_D\omega_D \qquad \text{(B23)}$$

with

$$\begin{aligned} C'_A &= \lambda'_E \, e^{\lambda'_E \Delta x} / \Delta x (e^{\lambda'_E \Delta x} - 1) \\ C'_B &= \kappa'_E \, e^{\kappa'_E b} / b (e^{\kappa'_E b} - 1) \\ C'_C &= \lambda'_E / \Delta x (e^{\lambda'_E \Delta x} - 1) \\ C'_D &= \kappa'_E / b (e^{\kappa'_E b} - 1) \end{aligned} \tag{B24}$$

and

$$C'_E = C'_A + C'_B + C'_C + C'_D$$

In equation B24,

$$\kappa'_E = \frac{Re}{4} \left(\frac{\partial \Psi}{\partial x}\right)_E = \frac{Re}{8\Delta x} (\Psi_A - \Psi_C) \tag{B25}$$

and

$$\lambda'_E = - \frac{Re}{4} \left(\frac{\partial \Psi}{\partial y}\right)_E = \frac{Re}{8b} (\Psi_D - \Psi_B) \tag{B26}$$

It is now time to take a closer look at the above transformation. Equation B17 can be rewritten as

$$A_E = \frac{\kappa_E \, (e^{\kappa_E b} - 1) \, (\omega_B - \omega_E) + (\omega_B + \omega_D - 2\omega_E)}{b(e^{\kappa_E b} - 1)}$$

or

$$A_E = \kappa_E \, \frac{\omega_B - \omega_E}{b} + \frac{\kappa_E}{b} \, \frac{(\omega_B + \omega_D - 2\omega_E)}{(e^{\kappa_E b} - 1)} \tag{B27}$$

as by definition,

$$A_E = \kappa_E \left(\frac{\partial \omega}{\partial y}\right)_E + \left(\frac{\partial^2 \omega}{\partial y^2}\right)_E \tag{B13}$$

it follows that

$$\left(\frac{\partial\omega}{\partial y}\right)_E = \frac{\omega_B-\omega_E}{b} \tag{B28}$$

and

$$\left(\frac{\partial^2\omega}{\partial y^2}\right)_E = \frac{\kappa_E}{b}\,\frac{(\omega_B+\omega_D-2\omega_E)}{(e^{\kappa_E b}-1)} \tag{B29}$$

By expanding $e^{\kappa_E b}$ in terms of powers of $\kappa_E b$; equation B29 becomes:

$$\begin{aligned}\left(\frac{\partial^2\omega}{\partial y^2}\right)_E &= \frac{\kappa_E}{b}\,\frac{(\omega_B+\omega_D-2\omega_E)}{\kappa_E b\ (1+\kappa_E b/2!+(\kappa_E b)^2/3!+\ldots)} \\ &= \frac{1}{b^2}\,\frac{(\omega_B+\omega_D-2\omega_E)}{1+\kappa_E b/2!+(\kappa_E b)^2/3!+\ldots}\end{aligned} \tag{B30}$$

When b approaches zero, equation B30 gives:

$$\lim_{b\to 0}\left(\frac{\partial^2\omega}{\partial y^2}\right)_E = \frac{(\omega_B+\omega_D-2\omega_E)}{b^2} \tag{B31}$$

which is the conventional central difference formula for a second derivative. Following the same reasoning, we can obtain from equations B14 and B18 these relationships:

$$\left(\frac{\partial\omega}{\partial\eta}\right)_E = \frac{(\omega_A-\omega_E)}{a} \tag{B32}$$

and

$$\lim_{a\to 0}\left(\frac{\partial^2\omega}{\partial\eta^2}\right)_E = \frac{\omega_A+\omega_C-2\omega_E}{a^2} \tag{B33}$$

It is thus shown that for small mesh lengths equation B19 is equivalent to a conventional finite-difference representation of equation B12. The same argument holds for equation B23.

The point velocities and pressure gradients can be calculated directly from the values of stream function and vorticity once the latter have been determined. By definition:

$$(u_x)_E = \left(\frac{\partial \Psi}{\partial y}\right)_E = \frac{1}{2b}\ (\Psi_B - \Psi_D) \tag{B34}$$

$$\begin{aligned}(u_y)_E &= -\left(\frac{\partial \Psi}{\partial x}\right)_E = \frac{1}{2\Delta x}\ (\Psi_C - \Psi_A)\\ &= -\left(\frac{\partial \eta}{\partial x}\right)_E \left(\frac{\partial \Psi}{\partial \eta}\right)_E = \frac{C(1-\eta_E)^2}{2a}\ (\Psi_D - \Psi_B)\end{aligned} \tag{B35}$$

Rearranging equation 6 gives

$$-\left(\frac{\partial P}{\partial x}\right) = u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} - \frac{4}{Re}\ \nabla^2\ u_x \tag{B36}$$

In terms of Ψ and ω, this becomes

$$-\left(\frac{\partial P}{\partial x}\right) = \frac{\partial \Psi}{\partial y}\frac{\partial^2 \Psi}{\partial x \partial y} - \frac{\partial \Psi}{\partial x}\frac{\partial^2 \Psi}{\partial y^2} + \frac{4}{Re}\frac{\partial \omega}{\partial y} \tag{B37}$$

the finite difference form of which is

$$\begin{aligned}-\left(\frac{\partial P}{\partial x}\right)_E = \frac{1}{8\Delta x b^2}\ \big[&(\Psi_B - \Psi_D)(\Psi_F - \Psi_G + \Psi_H - \Psi_I) - 4(\Psi_A - \Psi_C)\\ &(\Psi_B + \Psi_D - 2\Psi_E)\big] + \frac{4}{Reb}\ (\omega_B - \omega_E)\end{aligned} \tag{B38}$$

Similarly, since

$$\begin{aligned}-\left(\frac{\partial P}{\partial y}\right) &= u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} - \frac{4}{Re}\ \nabla^2\ u_y\\ &= \frac{\partial \Psi}{\partial x}\frac{\partial^2 \Psi}{\partial x \partial y} - \frac{\partial \Psi}{\partial y}\frac{\partial^2 \Psi}{\partial x^2} - \frac{4}{Re}\frac{\partial \omega}{\partial x}\end{aligned} \tag{B39}$$

then

$$-\left(\frac{\partial P}{\partial y}\right)_E = \frac{1}{8\Delta x^2 b}\left[(\Psi_A-\Psi_C)\ (\Psi_F-\Psi_G+\Psi_H-\Psi_I) - 4(\Psi_B-\Psi_D)(\Psi_A+\Psi_C-2\Psi_E)\right] - \frac{4}{Re\Delta x}\ (\omega_A-\omega_E) \tag{B40}$$

Note that the forward-difference representations for $\partial\omega/\partial x$ and $\partial\omega/\partial y$ are used, which is in agreement with equations B28 and B32.

When η replaces x as the independent variable, equations B37 and B39 assume the following form:

$$-\left(\frac{\partial P}{\partial x}\right) = \frac{\partial \eta}{\partial x}\ \left(\frac{\partial \Psi}{\partial y}\ \frac{\partial^2 \Psi}{\partial \eta \partial y} - \frac{\partial \Psi}{\partial \eta}\ \frac{\partial^2 \Psi}{\partial y^2}\right) + \frac{4}{Re}\ \frac{\partial \omega}{\partial y} \tag{B41}$$

$$-\left(\frac{\partial P}{\partial y}\right) = \left(\frac{\partial \eta}{\partial x}\right)^2\ \left(\frac{\partial \Psi}{\partial \eta}\ \frac{\partial^2 \Psi}{\partial \eta \partial y} - \frac{\partial \Psi}{\partial y}\ \frac{\partial^2 \Psi}{\partial \eta^2}\right) - \frac{\partial^2 \eta}{\partial x^2}\ \frac{\partial \Psi}{\partial y}\ \frac{\partial \Psi}{\partial \eta} - \frac{4}{Re}\ \frac{\partial \eta}{\partial x}\ \frac{\partial \omega}{\partial \eta} \tag{B42}$$

Based on these equations, the point pressure gradients can be calculated as follows:

$$-\left(\frac{\partial P}{\partial x}\right)_E = \frac{C(1-\eta_E)^2}{8ab^2}\left[(\Psi_B-\Psi_D)\ (\Psi_F-\Psi_G+\Psi_H-\Psi_I) - 4(\Psi_A-\Psi_C)\ (\Psi_B+\Psi_D-2\Psi_E)\right] + \frac{4}{Reb}\ (\omega_B-\omega_E) \tag{B43}$$

$$-\left(\frac{\partial P}{\partial y}\right)_E = \frac{C^2(1-\eta_E)^4}{8a^2 b}\left[(\Psi_A-\Psi_C)(\Psi_F-\Psi_G+\Psi_H-\Psi_I) - 4(\Psi_B-\Psi_D)(\Psi_A+\Psi_C-2\Psi_E)\right] + \frac{C^2(1-\eta_E)^3}{2ab}\ (\Psi_B-\Psi_D)(\Psi_A-\Psi_C) - \frac{4C(1-\eta_E)^2}{Rea}\ (\omega_A-\omega_E) \tag{B44}$$

The values of $-(\partial P/\partial \eta)$ can easily computed from $-(\partial P/\partial x)$ by the relation:

$$-\left(\frac{\partial P}{\partial \eta}\right)_E = -\left(\frac{\partial P}{\partial x}\right)_E \Big/ \left(\frac{\partial \eta}{\partial x}\right)_E \qquad \text{(B45)}$$

APPENDIX C. A FORTRAN PROGRAM FOR SOLVING THE TWO-DIMENSIONAL NAVIER-STOKES EQUATIONS

PAGE 1

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* DATE 7/24/62
*     XEQ
* THIS IS A FORTRAN PROGRAM FOR SOLVING THE TWO-DIMENSIONAL NAVIER-
*  STOKES EQUATIONS BY ITERATION METHOD . THE PROGRAM CONCERNS THE
*  STEADY LAMINAR FLOW OF AN INCOMPRESSIBLE FLUID AT THE ENTRANCE
*  SECTION OF TWO PARALLEL PLATES UNDER ISOTHERMAL CONDITIONS.
* THE DEPENDENT VARIABLES USED ARE THE STREAM FUNCTION AND THE
*  VORTICITY.  THE INDEPENDENT VARIAB ES ARE ETA AND Y.  THE PROGRAM
*  ALSO CALCULATES THE POINT VELOCITIES AS WELL AS PRESSURE GRADIENTS
*  WHEN THE SOLUTION CONVERGES.
* WRITTEN BY YUI LOONG WANG, CHEMICAL ENGINEERING, CALIFORNIA
*  INSTITUTE OF TECHNOLOGY,   JANUARY 1962.
* TWO DIMENSIONAL NAVIER-STOKES EQUATIONS
      DIMENSION PSI(21,21),OMEGA(21,21),PSIB(21,21),OMEGAB(21,21),
     1CO1(19),CO2(19),CO3(19),CO4(19) CO5(19),CO6(19),CO7(19),
     2ORP(19),ORPB(19),ORO(19),OROB(19)
      DIMENSION  ETA(21),X(21),Y(21)
* DEFINITION OF SYMBOLS
*      A      MESH LENGTH IN ETA DIRECTION
*      B      MESH LENGTH IN Y-DIRECTI N
*      C      A CONSTANT USED IN RELAT ING ETA TO X
*      DOR...      DECREASE IN OVERRELAXATION FACTORS PER IA ITERATIONS
*      ETA     A FUNCTION OF X DEFINED BY EQUATION 13
*      IA      NUMBER OF ITERATIONS PE FORMED BEFORE CHANGING THE
*              OVERRELAXATION FACTORS
*      IALLOW      MAXIMUM NUMBER OF ITERATIONS ALLOWED
*      IPERP      NUMBER OF ITERATIONS PERFORMED BEFORE EACH PRINTOUT
*      ITER     NUMBER OF ITERATIONS PERFORMED
*      M2      NUMBER OF COLUMNS IN THE Y-DIRECTION
*      N2      NUMBER OF ROWS IN THE E A-DIRECTION IN EITHER
*              UPSTREAM OR DOWNSTREAM REGION
*      OMEGA,OMEGAB      DIMENSIONLESS VORTICITY AT DOWNSTREAM
*                        AND UPSTREAM REGIONS RESPECTIVELY
*      OR...     OVERRELAXATION FACTORS
*      PPETA,PPETAB        PRESSURE GRADIENTS IN THE ETA-DIRECTION IN THE
*                          DOWNSTREAM AND UPSTREAM REGIONS RESPECTIVELY
*      PPX,PPXB      PRESSURE GRADIENTS IN THE X-DIRECTION IN THE
*                   DOWNSTREAM AND UPSTREAM REGIONS RESPECTIVELY
*      PPY,PPYB      PRESSURE GRADIENT IN THE Y-DIRECTION IN THE
*                    DOWNSTREAM AND UPSTREAM REGIONS RESPECTIVELY
*      PSI,PSIB      DIMENSIONLESS STREAM FUNCTION AT DOWNSTREAM
*                    AND UPSTREAM REGI NS RESPECTIVELY
*      RENO     REYNOLDS NUMBER
*      UX,UXB      VELOCITY IN THE X-DIRECTION IN THE DOWNSTREAM
*                  AND UPSTREAM REGIONS RESPECTIVELY
*      UY,UYB      VELOCITY IN THE Y-DIRECTION IN THE DOWNSTREAM
*                  AND UPSTREAM REGIONS RESPECTIVELY
*      SUBSCRIPTS,
*      I      USED IN THE ETA-DIRECTION
*      J      USED IN THE Y-DIRECTION
* INPUT DATA HERE FOR GRID SIZE
      M2=11
      N2=11
      A=0.1
      B=0.1
      C=1.2
      RENO=300.0
      R=A/B
      R2=R**2
```

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```
      M=M2-2
      N=N2-2
      K=N+1
* INPUT DATA FOR TOLERANCES ALLOWED HERE
      TOLP=2.0E-8
      TOLO=2.0E-8
*THESE ARE THE OVERRELAXATION FACTORS
      DO 1 I=1,N
      ORP(I)=1.3
      ORPB(I)=1.3
      ORO(I)=1.2
    1 OROB(I)=1.2
      DORP=0.1
      DORPB=0.1
      DORO=0.2
      DOROB=0.2
* INPUT DATA HERE FOR ITERATION INFORMATION
* IALLOW IS THE MAXIMUM NUMBER OF ITERATIONS WHICH WILL BE EXECUTED
      IALLOW=300
      IPERP=60
      IA=60
* SENSE LIGHT 2 ON MEANS RESULTING VALUES OF PSI AND OMEGA WILL BE
*          BE PUNCHED ON CARDS
      SENSE LIGHT 2
* READ IN ESTIMATED VALUES FOR PSI AND OMEGA
      READ INPUT TAPE 5,150,((PSI(J,I),J=1,M2),I=1,N2),((OMEGA(J,I),
     1   J=1,M2),I=1,N2),((PSIB(J,I),J=1,M2),I=1,N2),((OMEGAB(J,I),
     2   J=1,M2),I=1,M2)
      DO 2 I=1,K
      ETA(I)=A*FLOATF(I-1)
    2 X(I)=ETA(I)/(C*(1.0-ETA(I)))
      ETA(M2)=1.0
      DO 3 J=1,M2
    3 Y(J)=B*FLOATF(M2-J)
      DELXO = A/(C*(1.0-A))
      ITER=1
* PART I BEGINS HERE . IT CARRYS OUT THE ITERATIVE CALCULATIONS OF
*  PSI AND OMEGA BY DE G ALLEN,S METHOD
* CALCULATING COEFFICIENTS CONSTANT FOR EACH COLUMN
      DO 100 I=1,N
      ALPHA=C*(1.0-ETA(I+1))**2
      BETA=(-C)*ALPHA*(1.0-ETA(I+1))*2.0
      ALPHA2=ALPHA**2
      CO1(I)= ALPHA2+R2)*2.0
      BETA = (-C)*ALPHA*(1.0-ETA)*2.0
      CO2(I)= ALPHA2+(A*BETA/2.0)
      CO3(I)= ALPHA2-(A*BETA/2.0)
      CO4(I) = ALPHA2/A
      CO5(I)= (ALPHA*RENO)/(8.0*A)
      CO6(I)= BETA/ALPHA2
  100 CO7(I)= (-RENO)/(8.0*B*ALPHA)
      COO = DELXO**2/B**2
      COO1 =2.0*(1.0+COO)
  142 IPC=0
```

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```
      WRITE OUTPUT TAPE 6,151
* ITERATING PSI
  136 IDPO=0
      IDP=0
      IDPB=0
      SUMDPO=0
      SUMDP=0
      SUMDPB=0
* CALCULATING PSI FOR ETA EQUALS ZERO
      DO 110 J=1,M
      DUMMY =(DELXO**2*OMEGA(J+1,1)+PSI(J+1,2)+PSIB(J+1,2)
    1         +COO*(PSI(J+2,1)+PSI(J,1 ))/COO1
      DELTA=DUMMY-PSI(J+1,1)
      SUMDPO = SUMDPO+ABSF(DELTA)
      IF(ABSF(DELTA)/DUMMY-TOLP)111,111,112
  111 IDPO=IDPO+1
  112 PSI(J+1,1) =PSI(J+1,1) +DELTA*0.5*(ORP(1)+ORPB(1))
  110 PSIB(J+1,1)=PSI(J+1,1)
      OMEGA(1,1)= 2.0*(1.0-PSI(2,1))/B**2
      OMEGAB(1,1) =OMEGA(1,1)
* CALCULATING PSI FOR DOWNSTREAM REGION
      DO 113 I=1,N
      DO 114 J=1,M
      DUMMY = (A**2*OMEGA(J+1,I+1) + CO2(I)*PSI(J+1,I+2)+ CO3(I)*
    1  PSI(J+1,I) +R2*(PSI(J,I+1)+PSI(J+2,I+1)))/CO1(I)
      DELTA =DUMMY-PSI(J+1,I+1)
      SUMDP  =SUMDP  +ABSF(DELTA)
      IF(ABSF(DELTA)/DUMMY-TOLP) 115,115,114
  115 IDP=IDP+1
  114 PSI(J+1,I+1) =PSI(J+1,I+1) +DELTA*ORP(I)
  113 OMEGA(1,I+1) = 2.0*(1.0-PSI(2,I+1))/B**2
* CALCULATING PSI FOR UPSTREAM REGION
      DO 116 I=1,N
      DO 116 J=1,M
      DUMMY =(A**2*OMEGAB(J+1,I+1) + CO2(I)*PSIB(J+1,I+2) + CO3(I)*
    1  PSIB(J+1,I) +R2*(PSIB(J,I+1) +PSIB(J+2,I+1)))/CO1(I)
      DELTA =DUMMY-PSIB(J+1,I+1)
      SUMDPB =SUMDPB +ABSF(DELTA)
      IF (ABSF(DELTA)/DUMMY-TOLP) 117,117,116
  117 IDPB=IDPB+1
  116 PSIB(J+1,I+1) =PSIB(J+1,I+1) +DELTA*ORPB(I)
* ITERATING OMEGA
      IDOO =0
      IDO=0
      IDOB=0
      SUMDOO=0
      SUMDO=0
      SUMDOB=0
* CALCULATING OMEGA FOR ETA EQUALS ZERO
      DO 120 J=1,M
      XKAPPA=(RENO*(PSI(J+1,2)-PSIB(J+1,2)))/(8.0*DELXO)
      XLAMBD=((-RENO)*(PSI(J,1)-PSI(J+2,1)))/(8.0*B)
      C3=XLAMBD/(DELXO*(EXPF(XLAMBD*DELXO)-1.0))
      C1=C3*EXPF(XLAMBD*DELXO)
      C4=XKAPPA/(B*(EXPF(XKAPPA*B)-1.0))
      C2=C4*EXPF(XKAPPA*B)
      DUMMY=(C1*OMEGA(J+1,2)+C2*OMEGA(J,1)+C3*OMEGAB(J+1,2)
    1        +C4*OMEGA(J+2,1))/(C1+C2+C3+C4)
      DELTA=DUMMY-OMEGA(J+1,1)
      SUMDOO=SUMDOO+ABSF(DELTA)
      IF(ABSF(DELTA)/DUMMY-TOLO) 121,121,122
  121 IDOO=IDOO+1
```

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```
  122 OMEGA(J+1,1)=OMEGA(J+1,1)+DELTA*0.5*(ORO(1)+OROB(1))
  120 OMEGAB(J+1,1)=OMEGA(J+1,1)
* CALCULATING OMEGA FOR DOWNSTREAM REGION
      DO 123 I=1,N
      DO 123 J=1,M
      XKAPPA=CC5(I)*(PSI(J+1,I+2)-PSI(J+1,I))
      XLAMBD=CC6(I)+CO7(I)*(PSI(J,I+1)-PSI(J+2,I+1))
      C3=XLAMBD*CC4(I)/(EXPF(XLAMBD*A)-1.0)
      C1=C3*EXPF(XLAMBD*A)
      C4=XKAPPA/(B*(EXPF(XKAPPA*B)-1.0))
      C2=C4*EXPF(XKAPPA*B)
      DUMMY=(C1*OMEGA(J+1,I+2)+C2*OMEGA(J,I+1)+C3*OMEGA(J+1,I)
    1        +C4*OMEGA(J+2,I+1))/(C1+C2+C3+C4)
      DELTA=DUMMY-OMEGA(J+1,I+1)
      SUMDO=SUMDO+ABSF(DELTA)
      IF(ABSF(DELTA)/DUMMY-TOLO) 124,124,123
  124 IDO=IDO+1
  123 OMEGA(J+1,I+1)=OMEGA(J+1,I+1)+DELTA*ORO(I)
* CALCULATING OMEGA FOR UPSTREAM REGION
      DO 125 I=1,N
      DO 125 J=1,M
      XKAPPA=CC5(I)*(PSIB(J+1,I)-PSIB(J+1,I+2))
      XLAMBD=CC6(I)-CO7(I)*(PSIB(J,I+1)-PSIB(J+2,I+1))
      C3=XLAMBD*CC4(I)/(EXPF(XLAMBD*A)-1.0)
      C1=C3*EXPF(XLAMBD*A)
      C4=XKAPPA/(B*(EXPF(XKAPPA*B)-1.0))
      C2=C4*EXPF(XKAPPA*B)
      DUMMY=(C1*OMEGAB(J+1,I+2)+C2*OMEGAB(J,I+1)+C3*OMEGAB(J+1,I)
    1        +C4*OMEGAB(J+2,I+1))/(C1+C2+C3+C4)
      DELTA=DUMMY-OMEGAB(J+1,I+1)
      SUMDOB=SUMDOB+ABSF(DELTA)
      IF(ABSF(DELTA)/DUMMY-TOLO) 126,126,125
  125 IDOB=IDOB+1
  126 OMEGAB(J+1,I+1)=OMEGAB(J+1,I+1)+DELTA*OROB(I)
* AFTER EACH ITERATION  PRINT OUT THE NUMBER OF CORRECTIONS WITHIN
*         TOLERANCE AS WELL AS THE SUMS OF CORRECTIONS
      WRITE OUTPUT TAPE 6,152,(ITER,IDPO,SUMDPO,IDP,SUMDP,IDPB,
    1    SUMDPB,IDOO,SUMDOO,IDO,SUMDO,IDOB,SUMDOB)
      IP = IDPO +IDP+IDPB+IDOO+IDO+IDOB
* CHECK RESULTS    IF CONVERGED OR NUMBER OF ITERATION REACHS IALLOW
*         GO TO PRINT OUT
      IF(IP-2*M*(2*N+1)) 130,131,131
  130 IF(ITER-IALLOW) 132,131,131
  132 ITER=ITER+1
      IPC =IPC+1
      IF (ITER-IA) 133,134,134
  134 DO 135 I=1,N
      ORP(I)=ORP(I)-DORP
      ORPB(I)=ORPB(I)-DORPB
      ORO(I)=ORO(I)-DORO
  135 OROB(I) OROB(I)-DOROB
      IA=IA+IOA
  133 IF(IPC-IPERP) 136,137,137
  137 SENSE LIGHT 1
* PRINT OUT RESULTS IN TABULAR FORM
  131 WRITE OUTPUT TAPE 6,153
      DO 138  =1,N2
      WRITE OUTPUT TAPE 6,154,(ETA(I),X(I))
      WRITE OUTPUT TAPE 6,155
  138 WRITE OUTPUT TAPE 6 ,156,(Y(J),PSI(J,I),OMEGA(J,I),J=1,M2)
      DO 139 I=1,N2
      E=-ETA(I)
```

```
      XX=-X(I)
      WRITE CUTPUT TAPE 6,154,(E,XX)
      WRITE CUTPUT TAPE 6,155
  139 WRITE CUTPUT TAPE 6 ,156,(Y(J),PSIB(J,I),OMEGAB(J,I),J=1,M2)
      IF (SENSE LIGHT 2) 140,141
* PUNCH RESULTS ON CARDS
  140 WRITE CUTPUT TAPE 15,150,((PSI(J,I),J=1,M2),I=1,N2),((OMEGA(J,I),
     1     J=1,M2),I=1,N2),((PSIB(J,I ,J=1,72),I=1,N2),((OMEGAB(J,I),
     2     J=1,M2),I=1,N2)
  141 IF(SENSE LIGHT 1) 142,143
  150 FORMAT(4E18.8)
  151 FORMAT(120H ITER  IDPO      SUMDPO      IDP      SUMDP      IDPB    SU
     1MDPB      IDCO      SUMDOO      IDO      SUMDO      IDOB      SUMDOB     )
  152 FORMAT(I6,6(I6,E13.6))
  153 FORMAT(24X,72HTABLE      STREAM FUNCTIONS AND VORTICITIES FOR CASE I
     1I USING A 40X20 GRID,24X)
  154 FORMAT(/10X,4HETA=,F10.6,5X,4H    = ,F10.6)
  155 FORMAT(/7X,1HY,12X,3HPSI,14X,5HOMEGA/)
  156 FORMAT(F10.6,F18.8,E18.8)
* PART II BEGINS HERE.  IT CALCULATES THE POINT VELOCITIES AND
*  PRESSURE GRADIENTS FROM THE CONVERGED VALUES OF PSI AND OMEGA.
      DIMENSION UX(21,21),UY(21,21),PPX(21,21),PPY(21,21),PPETA(21,21),
     1 UXB(21,21),UYB(21,21),PPXB(21,21),PPYB(21,21),PPETAB(21,21)
* THESE ARE THE VALUES OF PRESSURE GRADIENTS AND VELOCITIES AT THE
*         BOUNDARY
  143 DO 200 I=1,N2
      UX(1,I)=0.0
      UXB(1,I)=0.0
      UY(1,I)=0.0
      UYB(1,I)=0.0
      UY(M2,I)=0.0
      UYB(M2,I)=0.0
      PPY(M2,I)=0.0
  200 PPYB(M2,I)=0.0
      DO 201 J=1,M2
      UXB(J,N2)=1.0
      UYB(J,N2)=0.0
      UX(J,N2)=1.5*(1.0-Y(J)**2)
      UY(J,N2)=0.0
      PPXB(J,N2)=0.0
      PPYB(J,N2)=0.0
      PPETAB(J,N2)=0.0
      PPX(J,N2)=0.04
      PPY(J,N2)=0.0
  201 PPETA(J,N2)=0.0
* CALCULATING PRESSURE GRADIENTS AND VELOCITIES FOR ETA EQUALS ZERO
      DO 202 J=1;M
      UX(J+1,1)=(PSI(J,1)-PSI(J+2,1))/(2.0*B)
      UY(J+1,1)=(PSIB(J+1,2)-PSI(J+1,2))/(2.0*DELXO)
      PPSIX2=(PSI(J+1,2)+PSIB(J+1,2)-2.0*PSI(J+1,1))/DELXO**2
      PPSIY2=(PSI(J,1)+PSI(J+2,1)-2.0*PSI(J+1,1))/B**2
      PPSIXY=(PSI(J,2)-PSI(J+2,2)-PSIB(J,2)+PSIB(J+2,2))/(4.0*B*DELXO)
      PPX(J+1,1)=UX(J+1,1)*PPSIXY+UY(J+1,1)*PPSIY2
     1           +(4.0*(OMEGA(J,1)-OMEGA(J+1,1)))/(RENO*B)
  202 PPY(J+1,1)=PPSIXY*(-UY(J+1,1))-UX(J+1,1)*PPSIX2
     1           -(4.0*(OMEGA(J+1,2)-OMEGA(J+1,1)))/(RENO*DELXO)
      UX(M2,1)=PSI(M+1,1)/B
      PPSIXY=(PSI(M+1,2)-PSIB(M+1,2))/(2.0*B*DELXO)
      PPX(M2,1)=UX(M2,1)*PPSIXY+(4.0*OMEGA(M+1,1))/(B*RENO)
      PPX(1,1)=(4.0*(OMEGA(1,1)-OMEGA(2,1)))/(RENO*B)
      PPY(1,1)=(4.0*(OMEGA(1,1)-OMEGA(1,2)))/(RENO*DELXO)
      DO 203 J=1,M2
      PPETA(J,1)=PPX(J,1)/C
```

```
      UXB(J,1)=UX(J,1)
      UYB(J,1)=UY(J,1)
      PPXB(J,1)=PPX(J,1)
      PPYB(J,1)=PPY(J,1)
  203 PPETAB(J,1)=PPETA(J,1)
* CALCULATING PRESSURE GRADIENTS AND VELOCITIES FOR DOWNSTREAM REGION
      DO 204 I=2,K
      ALPHA=C*(1.0-ETA(I))**2
      BETA=(-C)*ALPHA*(1.0-ETA(I))*2.0
      DO 205 J=1,M
      UX(J+1, )=(PSI(J,I)-PSI(J+2,I))/(2.0*B)
      UY(J+1,I)=( ALPHA*(PSI(J+1,I-1)-PSI(J+1,I+1)))/(2.0*A)
      PPSIY2 =(PSI(J,I)+PSI(J+2,I)-2.0*PSI(J+1,I))/B**2
      PPSIX2 =(PSI(J+1,I+1)+PSI(J+1,I-1)-2.0*PSI(J+1,I))*ALPHA**2/A**2
     1           +BETA*(PSI(J+1,I+1)-P I(J+1,I-1))/(2.0*A)
      PPSIXY =ALPHA*(PSI(J,I+1)-PSI(J+2,I+1)-PSI(J,I-1)+PSI(J+2,I-1))/
     1                 (4.0*A*B)
      PPX(J+1,I)=UX(J+1,I)*PPSIXY+UY(J+1,I)*PPSIY2+
     1            (4.0*(OMEGA(J,I)-OMEGA(J+1,I)))/(RENO*B)
      PPETA(J+1,I )=PPX(J+1,I)/ALPHA
  205 PPY(J+1,I)=(-UY(J+1,I))*PPSIXY-UX(J+1,I)*PPSIX2
     1           -(4.0*(OMEGA(J+1,I+1)-OMEGA(J+1,I))*ALPHA)/(RENO*A)
      UX(M2,I)=PSI(M+1,I)/B
      PPSIXY =ALPHA*(PSI(M+1,I+1)-PSI(M+1,I-1))/(2.0*A*B)
      PPX(M2,I)=UX(M2,I)*PPSIXY+4.0*OMEGA(M+1,I)/(RENO*B)
       PPETA(M2,I )=PPX(M2,I)/ALPHA
      PPX(1,I =4.0*(OMEGA(1,I)-OMEGA(2,I))/(RENO*B)
      PPETA(1,I)=PPX(1,I)/ALPHA
  204 PPY(1,I)=4.0*(OMEGA(1,I)-OMEGA(1,I+1))*ALPHA/(RENO*A)
* CALCULATING PRESSURE GRADIENTS AND VELOCITIES FOR UPSTREAM REGION
      DO 206 I=2,K
      ALPHA=(-C)*(1.0-ETA(I))**2
      BETA=C*ALPHA*(1.0-ETA(I))*2.0
      DO 207 J=1,M
      UXB(J+1,I)=(PSIB(J,I)-PSIB(J+2,I))/(2.0*B)
      UYB(J+1,I)=(ALPHA*(PSIB(J+1,I-1)-PSIB(J+1,I+1)))/(2.0*A)
      PPSIY2 =(PSIB(J,I)+PSIB(J+2,I)-2.0*PSIB(J+1,I))/B**2
      PPSIX2 =(PSIB(J+1,I+1)+PSIB(J+1,I-1)-2.0*PSIB(J+1,I))*ALPHA**2
     1           /A**2 +BETA*(PSIB(J+1,I+1)-PSIB(J+1,I-1))/(2.0*A)
      PPSIXY =ALPHA*(PSIB(J,I+1)-PSIB(J+2,I+1)
     1           -PSIB(J,I-1)+PSIB(J+2,I-1))/(4.0*A*B)
      PPXB(J+1,I)=UXB(J+1,I)*PPSIXY+UYB(J+1,I)*PPSIY2
     1             +(4.0*(OMEGAB(J,I)-OMEGAB(J+1,I)))/(RENO*B)
      PPETAB(J+1,I)=PPXB(J+1,I)/ALPHA
  207 PPYB(J+1,I)=(-UYB(J+1,I))*PPSIXY-UXB(J+1,I)*PPSIX2
     1           -(4.0*(OMEGAB(J+1,I+1)-OMEGAB(J+1,I)))*ALPHA/(RENO*A)
      UXB(M2,I)=PSIB(M+1,I)/B
      UXB(1,I)=(PSIB(1,I)-PSIB(2,I))/B
      PPSIXY=ALPHA*(PSIB(M+1,I+1)-PSIB(M+1,I-1))/(2.0*A*B)
      PPXB(M2,I)=UXB(M2,I)*PPSIXY+4.0*OMEGAB(M+1,I)/(RENC*B)
      PPETAB(M2,I)=PPXB(M2,I)/ALPHA
      PPXB(1,I)=4.0*(OMEGAB(1,I)-OMEGAB(2,I))/(RENO*B)
     1           +UXB(1,I)*ALPHA*(PSIB(2,I-1)-PSIB(2,I+1))/(A*B*2.0)
      PPETAB(1,I)=PPXB(1,I)/ALPHA
  206 PPYB(1,I)=0.0
      DO 208 I=1,N2
      WRITE OUTPUT TAPE 6,251,(ETA(I),X(I))
      WRITE OUTPUT TAPE 6,252
```

```
  208 WRITE CUTPUT TAPE 6,253,(Y(J),UX(J,I),UY(J,I),PPETA(J,I),
     1                        PPX(J,I),PPY(J,I),J=1,M2)
      DO 209 I=1,N2
      E=-ETA(I)
      XX=-X(I)
      WRITE CUTPUT TAPE 6,251,(E,XX)
      WRITE CUTPUT TAPE 6,252
  209 WRITE CUTPUT TAPE 6,253,(Y(J),UXB(J,I),UYB(J,I),PPETAB(J,I),
     1                         PPXB(J,I),PPYB(J,I),J=1,M2)
  251 FORMAT(/32X,6HETA = ,F10.6,5X,5H X = ,F10.6)
  252 FORMAT(/7X,1HY,11X,4HU(X),14X,4HU(Y),14X,5H P/  ,13X,
     1          5H P/ X,13X,5H P/ Y/)
  253 FORMAT(F10.6,5E18.8)
      CALL EXIT
* PART 3 BEGINS HERE.   IT IS A SUBPROGRAM FOR REDUCING THE MESH SIZES
*         OF A GIVEN RECTANGULAR NET.   BY MEANS OF 4 POINT LAGRAGIAN
*         INTERPOLATION, THE MESH LENGTH IN BOTH DIRECTIONS ARE REDUCED
*         BY A FACTOR OF TWO.   IN THIS PARTICULAR  PROGRAM, ONLY
*         THE DOWNSTREAM REGION WILL BE TREATED.
* THE INITIAL NET HAS TO BE FURNISHED IN PUNCHED CARDS AS DATA
* THE DIMENSIONS OF THE EXPANDED NET SHOULD BE GIEN BELOW
      M=19
      N=19
      A=0.05
      B=0.05
      M2=M+2
      N2=N+2
      READ INPUT TAPE 5,350,((PSI(J,I),J=1,M2,2),I=1,N2,2),
     1   ((OMEGA(J,I),J=1,M2,2),I=1,N2,2)
      DO 300 J=1,M2
      Y=B*FLOATF(M2-J)
      PSI(J,N2)=Y*(1.5-0.5*Y**2)
  300 OMEGA(J,N2)=3.0*Y
      DO 301 I=1,N2
      PSI(1,I)=1.0
      PSI(M2,I)=0.0
  301 OMEGA(M2,I)=0.0
      DO 302 I=1,N,2
      PSI(2,I)=0.0625*(5.0*PSI(1,I)+15.0*PSI(3,I)-5.0*PSI(5,I)+PSI(7,I))
      OMEGA(2,I)=0.0625*(5.0*OMEGA(1,I)+15.0*OMEGA(3,I)-5.0*OMEGA(5,I)
     1                   +OMEGA(7,I))
      PSI(M+1,I)=0.0625*(15.0*PSI(M,I)-5.0*PSI(M-2,I)+PSI(M-4,I))
      OMEGA(M+1,I)=0.0625*(15.0*OMEGA(M,I)-5.0*OMEGA(M-2,I)
     1                    +OMEGA(M-4,I))
      DO 302 J=4,M,2
      PSI(J,I)=0.0625*(9.0*PSI(J+1,I)+9.0*PSI(J-1,I)-PSI(J+3,I)
     1                -PSI(J-3,I))
  302 OMEGA(J,I)=0.0625*(9.0*OMEGA(J+1,I)+9.0*OMEGA(J-1,I)
     1                    -OMEGA(J-3,I)-OMEGA(J+3,I))
      DO 303 J=1,M
      PSI(J+1,2)=0.0625*(5.0*PSI(J+1,1)+15.0*PSI(J+1,3)-5.0*PSI(J+1,5)
     1                   +PSI(J+1,7))
      OMEGA(J+1,2)=0.0625*(5.0*OMEGA(J+1,1)+15.0*OMEGA(J+1,3)
     1                     -5.0*OMEGA(J+1,5)+OMEGA(J+1,7))
      PSI(J+1,N+1)=0.0625*(5.0*PSI(J+1,N2)+15.0*PSI(J+1,N)-5.0*
     1                 PSI(J+1,N-2)+PSI(J+1,N-4))
      OMEGA(J+1,N+1)=0.0625*(5.0*OMEGA(J+1,N2)+15.0*OMEGA(J+1,N)
     1                       -5.0*OMEGA(J+1,N-2)+OMEGA(J+1,N-4))
      DO 303 I=4,N,2
      PSI(J+1,I)=0.0625*(9.0*PSI(J+1,I+1)+9.0*PSI(J+1,I-1)-PSI(J+1,I+3)
     1                   -PSI(J+1,I-3))
  303 OMEGA(J+1,I)=0.0625*(9.0*OMEGA( .+1,I+1)+9.0*OMEGA(J+1,I-1)
     1                    -OMEGA(J+1,I+3)-OMEGA(J+1,I-3))

      DIMENSION PSI(21,21),OMEGA(21,21),PSIB(21,21),OMEGAB(21,21),

      WRITE C TPUT TAPE 6,350,((PSI(J,I),J=1,M2),I=1,N2),
     1   ((OMEGA(J,I),J=1,M2),I=1,N2)
  350 FORMAT(4E18.8)
      CALL EXIT
      END(1,0,0,0,0,0,0,0,0,1,0,0,0,0,0)
```

APPENDIX D: DETAILED DESCRIPTION OF PRELIMINARY INVESTIGATION

The purpose of the preliminary investigations for the problem was to devise a method of solution which would be both stable and rapid. Only Case I was studied for this purpose. To start with, equation 10 was derived in the Cartesian coordinates.

$$\frac{\partial \Psi}{\partial y} \frac{\partial}{\partial x} (\nabla^2 \Psi) - \frac{\partial \Psi}{\partial x} \frac{\partial}{\partial y} (\nabla^2 \Psi) = \frac{\Psi}{Re} \nabla^4 \Psi \tag{10}$$

For equation 10, difference equations were obtained based on backward, central, and forward-difference formulas for the first and third-order derivatives. The difference formulas used for the second and fourth-order derivatives were always based on central differences. These formulas are listed in Appendix B.

A marching solution was first tried with a Reynolds number of 16 for two grid sizes. One was square-meshed with a mesh length of 0.1 and the other a rectangular meshed grid with $x = 0.025$ and $y = 0.1$. In both instances, divergence was observed almost instantly. The instability associated with applying a marching solution technique to a boundary-value problem such as this, however, was not unexpected.

An iterative approach was studied next. An inlet length of 0.08 Re, as given by Schlichting was assumed, although theoretically, it should be infinity. Over this length, a coarse grid of 8x2 was initially set up for hand computation.

Equations 11 and 12 were derived and transformed into two sets of difference equations, using central-difference and backward-difference formulas for the first derivatives respectively. With each set, several iterations were carried out both by the total-step and by the single-step iteration methods for a Reynolds number of 96. The results thus obtained indicated that the backward-difference formulas were stable while the central-difference formulas were not. Also solution by the single-step iteration appeared to converge more rapidly than that by the total-step iteration. Similar conclusions were drawn with equation 10.

Based on these findings, a computer program was written for solving equation 10 for Ψ using the backward-difference formulas and single-step iteration. The scheme of solution followed the same pattern as that described in Method of Solution. Equation 10 was used because it contains only one dependent variable Ψ. The Datatron 205 digital computer then located in the Computer Center of the California Institute of Technology was used. For a Reynolds number of 300, the inlet length was assumed to be 0.08 Re or 24. Over this, a x-y net with a grid of 150×10 was laid. Since the values of Ψ at $x = 0$ as well as $x = 24$ were known, linear interpolation of these values for each value of y gave an initial estimate of Ψ at every interior point. Fourteen iterations were carried out, each requiring approximately 400 seconds.

Very slow rate of convergence was observed.

Hoping that a better initial estimate for the values of Ψ would greatly reduce the number of iterations required to reach a solution, the tabulated values of u_x, as calculated by Schlichting (1), were integrated to give values of Ψ as functions of position. These values were used as a new estimate to start a new series of iteration. This time, the result after ten iterations started to diverge. Investigations revealed that for this Reynolds number and grid size, upstream errors tended to be self exciting, thus causing the solution to diverge. This instability, however, disappeared when the Reynolds number was lowered. A run consisting of 53 iterations was then made on the 205 computer with Re = 30. The solution converged smoothly, though very slowly. The latter was attributed to the large number of nodes in the net.

In order to reduce the number of nodes as well as to avoid the necessity of assuming an inlet length, the transformation

$$\eta = 1 - \frac{1}{1+Cx} \tag{13}$$

was introduced. Equation 10 now assumes the form

$$C(1-\eta)^2 \frac{\partial\Psi}{\partial y}\frac{\partial}{\partial\eta}\left(\nabla^2_{\eta-y}\Psi\right) - \frac{\partial\Psi}{\partial\eta}\frac{\partial}{\partial y}\left(\nabla^2_{\eta-y}\Psi\right) = \frac{4}{Re}\nabla^4_{\eta-y}\Psi \tag{D-1}$$

where $\nabla^2_{\eta-y}\Psi = \frac{\partial^2\Psi}{\partial y^2} + C^2(1-\eta)^4\frac{\partial^2\Psi}{\partial\eta^2} - 2C^2(1-\eta)^3\frac{\partial\Psi}{\partial\eta}$

and $\nabla^{4}_{\eta-y}\Psi = \frac{\partial^4\Psi}{\partial y^4} + 2\,\frac{\partial^2\Psi}{\partial y^2}\;C^2(1-\eta)^4\,\frac{\partial^2\Psi}{\partial\eta^2} - 2\,C^2(1-\eta)^3\,\frac{\partial\Psi}{\partial\eta}$

$$+ C^4(1-\eta)^8\,\frac{\partial^4\Psi}{\partial\eta^4} - 12\,C^4(1-\eta)^7\,\frac{\partial^3\Psi}{\partial\eta^3} + 36\,C^4(1-\eta)^6\,\frac{\partial^2\Psi}{\partial\eta^2}$$

$$- 24\,C^4(1-\eta)^5\,\frac{\partial\Psi}{\partial\eta}$$

Equation (D-1) was again transformed into a finite-difference equation using backward-difference formulas for the first and third derivatives. Iterative computations were then performed on the Datatron 205 for Reynolds numbers of 60, 100, 200 and 300 with a 10 × 10 η-y grid. Except for the case with Reynolds number of 60, the solutions were unstable. Thus, the introduction of the new variable η had no apparent effect on the stability of the solution, even though it is possible to cut down the size of the grid greatly without losing much accurracy.

Finally, the discretion of equations 11 and 12 based on de G. Allen's method was performed. Detailed description from this stage on is given in the Results section.

PROPOSITIONS

PROPOSITION I

For a quantitative discussion of the mechanism of a complex reaction involving free radicals, such as the thermal decomposition of hydrocarbons, the concentrations of the free radicals have to be determined. The usual approach is to assume that these free radicals are at steady-state. The rate equations for these free radicals can then be set up and solved for the free-radical concentrations. From these concentrations, the rates of formation or disappearance of the stable species can then be predicted for comparison with the experimental results. Some typical examples have been worked out by Steacie (1).

When the number of species of free radicals is large, however, the solution of the rate equations of the free radicals becomes very complicated and is sometimes unattainable. It is proposed that under such circumstances some of the free-radical concentrations can be obtained from the rate equations of some of the stable products and the experimentally-determined reaction rates for these stable products. The remaining free-radical concentrations can be computed from their respective rate expressions. The validity of the mechanism can then be checked by comparing the rates of formation or disappearance of the rest of the stable species both determined experimentally and predicted from the mechanism.

This method has been applied to the quantitative discussion of the mechanism for n-butane decomposition, as described on pp. 54-56 in Part One of this thesis under correlation of data.

PROPOSITION II

Finite-difference methods are most useful in solving partial differential equations. In applying the method, the domain of interest is treated as a network of nodes. When the domain is not closed in one direction, say the x-direction, the net will contain infinite number of nodes, which becomes impractical.

When the domain is semi-infinite in the x-direction, i.e., $0 \leq x \leq \infty$, some simple transformation of the x-coordinate such as e^{-ax} may be performed to obtain a new domain of finite dimensions. When $-\infty < x \leq \infty$, however, mere transformation of the coordinates is not sufficient. For such cases, it is proposed that the domain be divided into two regions; one with $x \geq 0$ and the other with $x \leq 0$. Each region may then be treated separately with their solutions to be matched at the common boundary line $x = 0$. This technique was used successfully in solving the two-dimensional Navier-Stokes equations. For detailed information, refer to pp. 202-203 in Part Two of this thesis in the section of method of solution. It was found that with the transformation $\eta = 1 - 1/(1+Cx)$, the use of a 10x10 η-y net results on a solution having the same accuracy as that obtained with a 150x10 x-y net for small x and having better accuracy for large x.

PROPOSITION III

The importance of axial mixing in chemical flow reactor designs has been stressed (2). With axial mixing, the reactor design equation for a tubular flow reactor operated under isothermal, steady-state conditions may be written as

$$\frac{d}{dx}\left(D \frac{dC}{dx} \right) - u_x \frac{dC}{dx} - kC^n = 0 \qquad (1)$$

When the order of the reaction n is other than zero or one, equation 1 is non-linear, and analytical solutions for it are not available. Numerical solutions, however, have been worked out by Fan and Bailie (3) with n = 2 and 3 for constant D and u_x.

It is proposed that when n is an integer, equation 1 can be readily solved analytically by the perturbation method with the resulting solution in the form of, say,

$$C = C_0 + C_1 D + C_2 D^2 + \cdots \qquad (2)$$

Substitution of equation 2 into equation 1 results in a power series of D, the coefficients in which series should each be equal to zero. For the D^o term, the coefficient is

$$u_x \frac{dC_0}{dx} - kC_0{}^n = 0 \qquad (3)$$

if D is taken to be a constant. Equation 3 can be readily solved for C_0 which is also the solution of equation 1 with negligible axial diffusion.

The next coefficient in the series is

$$\frac{d^2C_0}{dx^2} - u_x \frac{dC_1}{dx} - nC_0{}^{n-1}C_1 = 0 \qquad (4)$$

which is a first-order linear differential equation with respect to C_1. This equation can thus be solved for C_1 using the solution of C_0 obtained from equation 3. Similarly, the solutions of C_2, C_3, etc. can be determined. C_1, C_2, C_3, etc. may be regarded as corrections to C_0 due to the existence of axial diffusion.

When D is small, equation 2 converges very rapidly. Equation 1 can then be solved much faster with the perturbation method than with any numerical method. Furthermore, for reactor design purpose, analytical solutions such as equation 2 are always preferred over numerical solutions. A major advantage in having an analytical solution is that the solution can be differentiated with respect to any parameter.

PROPOSITION IV

The rigorous solution of a multicomponent distillation problem at steady-state requires the determination of temperature, compositions, and flow rates at every plate in the column, provided that the pressure in the column is known. For a column with theoretical plates, the conditions at each plate are defined when the temperature as well as the composition and flow rate of, say, the vapor stream are known. For a n-component system, there will therefore be (n+1) variables for each plate governed by the energy balance equation, the dew-point relationship equation and the (n-1) material balance equations for the components. Since the conditions of different plates are interrelated, there will be a total of m(n+1) equations with m(n+1) unknowns for a m-plate column including the condenser and the reboiler, or

$$Mx = b \tag{5}$$

where x is the vector of unknowns, b is a vector determined by the boundary conditions and M a matrix based on the governing equations. x and b both have a dimension of m(n+1). M is a square matrix also with a dimension of m(n+1).

Equation 5 is a non-linear algebraic matrix equation and can be solved only by some iterative procedures. A number of methods of solution were proposed (4,5). In these methods, the matrix equation 5 is invariably solved by the method of block iteration. With block iteration, equation 5 is divided into p sets of equations corresponding to p sets of unknowns. These sets of equations are then solved one at a time in a predetermined sequence for their corresponding unknowns. This would be repeated until all the variables do not change with respect to iteration. The division of equation 5 can be done either by plates (4) or by variables such as temperature, flow rate and composition (5).

The methods mentioned above work well with certain types of distillation problems but not all of them, especially problems with multi-feeds and multi-withdrawals. It is proposed that a more general method should be used for stubborn cases, i.e., to solve equation 5 in a single step each time. By taking advantage of the numerous zeros in the matrix M, equation 5 can be solved on a high-speed digital computer reasonably fast. For example, for a 10-component system with a 30-plate column, each solution of equation 5 is estimated to require approximately one minute on an IBM 7090 digital computer.

As the number of elementary operations required to solve a matrix equation is approximately proportional to the cube of the dimension of the matrix, the use of block iteration greatly reduces the number of operations required in solving equation 5 as compared with the proposed one-step solution . The nature of the block iteration, however, is such that for each iteration it tends to overcorrect some sets of the variables while undercorrect the others. This often leads to instability in the solution. With the proposed one-step solution, such non-uniformity in correction should be reduced or even eliminated. It should therefore be used wherever the method of block iteration gives divergent solutions.

PROPOSITION V

The use of least-squares technique in correlating experimental data is well accepted. If a set of P observations, F_p, can be represented by the expression

$$G = G(X_j , A_k) \tag{6}$$

where X_j's are the independent variables and A_k's are constants, the least-squares fit then calls for the determination of the constants A_k's based on the experimental observations such as to minimize the function

$$T = \sum_{p=1}^{P} [F_p(X_{j_p}) - G_p(X_{j_p}, A_k)]^2 \tag{7}$$

This is equivalent to solving the set of M equations

$$\begin{aligned} U_i &= \frac{\partial T}{\partial A_i} = 0 \\ &= \sum_{p=1}^{P} 2 [F_p - G_p] \frac{\partial G}{\partial A_i} \qquad i=1,2,\cdots, M \end{aligned} \tag{8}$$

for the M constants A_1 through A_M.

When G is linear with respect to A_k's, equation 8 can be solved directly. When G is not linear with respect to A_k's, however, equation 8 is usually solved by the iterative Newton's method. With the Newton's method, a set of values for A_i's is initially assigned. This set of values is then adjusted according to the formula

$$(A_i)_{new} = (A_i)_{old} + \Delta A_i \tag{9}$$

where the adjustments ΔA_i's are computed by solving the set of equations

$$\sum_{i=1}^{M} \frac{\partial U_n}{\partial A_i} \Delta A_i = - U_n \quad n = 1, 2, \cdots, M \tag{10}$$

The values of A_i's are to be repeatedly adjusted by the above procedure until they all converge to the solution.

With the Newton's method, however, convergence is often unattainable. It was found that in many textbooks, the formula for $\partial U_n / \partial A_i$ is given to be (6)

$$\frac{\partial U_n}{\partial A_i} = -2 \sum_{p=1}^{P} \left(\frac{\partial G_p}{\partial A_n} \right) \left(\frac{\partial G_p}{\partial A_i} \right) \tag{11}$$

while the correct expression should be

$$\frac{\partial U_n}{\partial A_i} = 2 \sum_{p=1}^{P} \left[(F_p - G_p) \frac{\partial^2 G_p}{\partial A_n \partial A_i} - \left(\frac{\partial G_p}{\partial A_n}\right) \left(\frac{\partial G_p}{\partial A_i}\right) \right] \tag{12}$$

When equation 12 is used instead of equation 11, the solutions to many cases which previously diverged have been obtained.

NOMENCLATURE

A_k's	constants in G
C	concentration, lb/cu. ft.
C_0, C_1, $\cdots$	pertubated concentrations, lb/cu.ft.
D	diffusion coefficient, sq.ft/sec
F	experimentally-observed value for G
G	any function of the independent variables X_j's
k	reaction-rate constant, $(\text{lb/cu.ft.})^{-(n-1)}\text{sec}^{-1}$
n	order of a reaction, dimensionless
T	a function defined by equation 7
U_i	$\partial T/\partial A_i$
u_x	longitudinal velocity, ft/sec
X_j's	some independent variables
x	Cartesian coordinate, ft.
∇^2	the Laplacian operator, $\partial^2/\partial x^2 + \partial^2/\partial y^2$

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