ON THE SIGNAL SELECTION PROBLEM FOR PHASE COHERENT AND INCOHERENT COMMUNICATION CHANNELS

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ABSTRACT

Landau and Slepian [10] have recently obtained a lower bound for the probability of error for any equienergy signal set in the infinite band Gaussian, additive noise channel. They further claim that the regular simplex signal set achieves equality in their lower bound and thereby proves the optimality of this set.

In the following paper it is proven that the simplex signals achieve equality in the lower bound of Landau and Slepian only when the dimension n is less than or equal to three. There is also shown to be an equivalence between certain optimal signal sets for the phase coherent channel described by Landau and Slepian and certain optimal signal sets for the incoherent case which have been recently discovered by Schaffner and Krieger [11] and [12].

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INTRODUCTION

Ever since Shannon's introduction [14] of the geometric representation of communication systems, there has been much effort by both communication engineers and mathematicians to solve the related geometric problem of finding sets of signal vectors which are optimal in the sense of minimizing the probability of error in the communication channel. Most of the work has been directed at the gaussian additive white noise channel with the signal vectors constrained to have equal energy.

However, to this day almost nothing is known for certain about such globally optimum signal sets, while almost all the successful efforts from the geometric point of view have been concerned either with asymptotic results [15], [1], [18], [7] with showing that certain signal sets achieve local optimums [1], [8], [13], [16] or with simply evaluating the performance of particular signal sets [5].

Renewed interest in this field has recently been generated by Landau and Slepian [10] who obtain a lower bound for the probability of error for any set of signals based upon a difficult generalization of a theorem by L. Fejes Toth [4]. Landau and Slepian further claim that the regular simplex signal set achieves equality in their lower bound and thereby proves "the long conjectured fact that the regular simplex is the code of minimal error probability for transmission over the infinite band Gaussian channel." In this paper it is proven that the simplex signals achieve equality in the lower bound of Landau and Slepian only when the dimension n is less than or equal to three. Toward this end a formulation of the signal selection problem for the phase coherent additive noise channel is developed using the notion of spherical caps.

A similar formulation of the signal selection problem for the phase incoherent additive noise channel is developed using spherical caps. This latter formulation is used to show an equivalence between certain optimal signal sets for the coherent case described by Landau and Slepian and certain optimal signal sets of the incoherent case which have been recently discovered by Schaffner and Krieger [11], [12]. In fact, it is shown that the coherent case in three dimensions is equivalent, in a certain sense, to the incoherent case in four real (or two complex) dimensions.

CHAPTER I. THE PHASE COHERENT CASE

1.1. The Problem.

The coherent, additive noise signal selection problem can be posed in the following way. Let $\{\underline{s}_i\}_{i=1}^{M}$ be a set of M vectors in n dimensional Euclidean space E^n. When the transmitter wishes to inform the receiver that the ith message has occurred he sends the vector $\underline{s}_i$ to the receiver. The receiver observes a vector $\underline{r}$ which is the vector $\underline{s}_i$ corrupted by a noise vector $\underline{n}$ so that $\underline{r} = \underline{s}_i + \underline{n}$. The receiver then makes a decision as to which one of the M messages occurred, based upon the observation $\underline{r}$. The criterion usually used to judge the quality of a transmission scheme is the probability that the receiver makes the correct decision, the probability of being correct P_c, or equivalently the probability that the receiver does not make the correct decision, the probability of error P_e. These quantities are of course related by $P_c + P_e = 1$. An optimal scheme is one that maximizes P_c, or equivalently minimizes P_e, and an optimal set of signals is one that is used in an optimal scheme.

1.2. Decision Rule.

For a fixed set of signals, the receiver must use a decision procedure which maximizes P_c if the scheme is to be optimal. The receivers decision procedure is equivalent to partitioning the E^n space of $\underline{r}$-vectors into M disjoint regions $\mathcal{R}_i$ whose union is E^n. Then if a vector $\underline{r}$ falls in decision region $\mathcal{R}_i$, the receiver decides that the ith message has occurred. Thus if $Pr(\underline{r}/\underline{s}_i)$ represents the probability density on $\underline{r}$ when $\underline{s}_i$ is the vector transmitted, then

$$P_c = \sum_i \int_{\mathcal{R}_i} \Pr(\underline{r}/\underline{s}_i)\ P(\underline{s}_i)\ dV(\underline{r})$$

where $P(\underline{s}_i)$ is the probability with which the ith message occurs and $dV(\underline{r})$ is the n dimensional Euclidean volume element. But

$$P_c = \sum_i \int_{\mathcal{R}_i} \Pr(\underline{r}/\underline{s}_i)\ P(\underline{s}_i)\ dV(\underline{r}) \le \int \max_i \{\Pr(\underline{r}/\underline{s}_i)\ P(\underline{s}_i)\} dV(\underline{r})$$

with equality if the decision regions $\mathcal{R}_i$ are defined such that

$$\underline{r} \in \mathcal{R}_i \Rightarrow \Pr(\underline{r}/\underline{s}_i)\ P(\underline{s}_i) \ge \Pr(\underline{r}/\underline{s}_j)\ P(\underline{s}_j) \quad j = 1, \cdots M$$

for $i = 1, \cdots M$.

1.3. Assumptions.

In the following we shall consider only a special class of the above problem. Namely we shall assume equiprobable messages, equienergy signal vectors, and spherically symmetric, monotone decreasing noise. Equiprobable messages means that $P(\underline{s}_i) = \frac{1}{M}$ $i = 1, \cdots M$. Equienergy signal vectors means that $\|\underline{s}_i\| = E$ $i = 1, \cdots M$ where $\|\cdot\|$ is the Euclidean 2-norm. That is $\|\underline{t}\|^2 = \sum_{j=1}^{n} t_j^{\ 2}$ where t_j are the components of the n dimensional vector $\underline{t}$. Without loss of generality in what follows we may assume that $E = 1$ so that $\|\underline{s}_i\| = 1$ $i = 1, \cdots M$.

Spherically symmetric noise means that the probability density of the noise $Pr(\underline{n})$ is a function only of the norm of $\underline{n}$, so that $Pr(\underline{n}) = f(\|\underline{n}\|^2)$. Monotone decreasing spherically symmetric noise means that f is a monotone decreasing function of $\|\underline{n}\|^2$.

1.4. Description of Decision Regions.

With these assumptions we see that the formula for the density on $\underline{r}$ given that the ith message occurred is

$$Pr(\underline{r}/\underline{s}_i) = f(\|\underline{r}-\underline{s}_i\|^2) .$$

Further the condition that

$$\underline{r} \in \mathcal{R}_i \Rightarrow Pr(\underline{r}/\underline{s}_i)\; P(\underline{s}_i) > Pr(\underline{r}/\underline{s}_j)\; P(\underline{s}_j) \quad i,j = 1, \cdots M$$

is then equivalent to

$$\underline{r} \in \mathcal{R}_i \Rightarrow \|\underline{r}-\underline{s}_i\| \le \|\underline{r}-\underline{s}_j\| \quad i,j = 1, \cdots M$$

But $\|\underline{r}-\underline{s}_i\|^2 = \|\underline{r}\|^2 - 2\langle\underline{r},\underline{s}_i\rangle + 1$ where $\langle\cdot,\cdot\rangle$ is the Euclidian 2-inner product, such that $\langle\underline{t},\underline{w}\rangle = \sum_{j=1}^{n} t_j u_j$ where t_j and u_j are the components of the n dimensional vectors $\underline{t}$ and $\underline{w}$. Thus the condition becomes equivalent to

$$\underline{r} \in \mathcal{R}_i \Rightarrow \langle\underline{r},\underline{s}_i\rangle \ge \langle\underline{r},\underline{s}_j\rangle \quad i,j = 1, \cdots M.$$

If the halfspaces H_{ij} are defined by

$$H_{ij} = \{\underline{r} \mid \langle \underline{r}, \underline{s}_i \rangle > \langle \underline{r}, \underline{s}_j \rangle\} \quad i, j = 1, \cdots M$$

then a sufficient condition that $\mathcal{R}_i$ are chosen to maximize P_c for a fixed signal set is that

$$\mathcal{R}_i \supset \mathcal{R}_i^* = \bigcap_{\substack{j=1 \\ j \neq i}}^{m} H_{ij} \qquad i = 1, \cdots M .$$

We note that this implies that $\mathcal{R}_i$ differs only in a trivial way from $\mathcal{R}_i^*$ since $\bigcup_{i=1}^{M} \mathcal{R}_i^*$ exhausts all of E^n except for portions of regions of the kind $B = \{\underline{r} \mid \langle \underline{r}, \underline{x}_i \rangle = \langle \underline{r}, \underline{x}_j \rangle\}$ which have no n dimensional volume, so that

$$\int_B dV(\underline{r}) = 0 \quad .$$

Thus these left over regions may be arbitrarily assigned to any decision region without affecting P_c.

We note that the regions $\mathcal{R}_i^*$ are convex and radially invariant. Convex means that if $\underline{r}_1$ and $\underline{r}_2$ are elements of $\mathcal{R}_i^*$ then $\lambda \underline{r}_1 + (1-\lambda)\underline{r}_2$ is also in $\mathcal{R}_i^*$ for every $0 \leq \lambda \leq 1$. This is because each halfspace H_{ij} is convex and thus $\mathcal{R}_i^*$, an intersection of halfspaces, must be convex. Radially invariant means that if $\underline{r}$ is an element of $\mathcal{R}_i^*$ then $\alpha \underline{r}$ is an element of $\mathcal{R}_i^*$ for every $\alpha > 0$. Again this follows because each half space H_{ij} is radially invariant and thus $\mathcal{R}_i^*$, an intersection of halfspaces, must be radially invariant.

1.5. An Alternate Expression for P_c.

Using our knowledge of the decision regions and our assumptions about the noise density we can derive an alternate expression for P_c by considering the integration over E^n to be first an integration over the surface of an n dimensional sphere in E^n and then integrating over all radii for the sphere. That is, if we let $r = \|\underline{r}\|$, then $dV(\underline{r})$ the n dimensional volume element becomes $dS(\underline{r})\cdot dr$ where $dS(\underline{r})$ represents a surface element of the n dimensional sphere of radius r. It is geometrically evident that $dS(\underline{r}) = r^{n-1}$ $dS(\frac{\underline{r}}{r})$ where $dS(\frac{\underline{r}}{r})$ represents a differential surface element on the n dimensional sphere of radius 1. Alternately, this result may be derived in a completely analytic way by the use of n dimensional spherical coordinates as in Appendix I.

In any event we can write

$$P_c = \frac{1}{M} \sum_i \iint_{\mathcal{R}_i} f(\|\underline{r}-\underline{s}_i\|^2)\ dS(\frac{\underline{r}}{r})\ r^{n-1}\ dr$$

$$= \int_0^\infty \left\{ \frac{1}{M} \sum_i \int_{R_i} f(\|\underline{r}-\underline{s}_i\|^2)\ dS(\frac{\underline{r}}{r}) \right\} r^{n-1}\ dr$$

where R_i is the region formed by the intersections of $\mathcal{R}_i$ and the surface of an n dimensional sphere of radius r, radially projected onto the unit sphere. Note that R_i does not depend upon r because of the fact that $\mathcal{R}_i$ is radially invariant.

It will be useful to consider the expression inside the brackets so let us define

$$U(r) = \frac{1}{M} \sum_i \int_{R_i} f(\|\underline{r}-\underline{s}_i\|^2) \, dS(\frac{\underline{r}}{r})$$

and thus

$$P_c = \int_0^\infty U(r) \, r^{n-1} \, dr$$

It is clear that if a set of signals can be found that maximize $U(r)$ for every r, then this set of signals must be optimal.

1.6. Characterization of Optimal Signal Sets by Landau and Slepian.

In what follows it will be convenient to let $\underline{x}$ represent a generic unit length vector in E^n, so that $\|\underline{x}\| = 1$. Then $U(r)$ may be written as

$$U(r) = \frac{1}{M} \sum_i \int_{R_i} f(\|r\underline{x}-\underline{s}_i\|^2) \, dS(\underline{x})$$

Further, let us define the spherical cap of angle θ about a unit vector $\underline{s}$ to be

$$C_{\underline{s}}(\theta) = \{\underline{x} | \langle \underline{x}, \underline{s} \rangle > \cos \theta\}$$

Landau and Slepian [10] have recently shown that among those signal sets satisfying the constraints mentioned above, $U(r)$ will be maximized for each r if there exists a signal set, an angle ϕ, and a largest set K of (i,j) pairs satisfying the following conditions.

1. $C_{\underline{s}_i}(\phi) \supset R_i \qquad i = 1, \cdots M$

2. $C_{\underline{s}_i}(\phi) \cap H_{ji}$ are congruent for $(i,j) \in K$ and

$$R_i = C_{\underline{s}_i}(\phi) - \bigcup_{j \in K_i} H_{ji} \qquad i = 1, \cdots M \text{ where}$$

$$K_i = \{j | (j,i) \in K\} .$$

3. $C_{\underline{s}_i}(\phi) \cap H_{ji} \cap H_{\ell i} = \Phi$ for every $j \neq \ell$ and $(j,i) \in K$ and $(\ell,i) \in K$, where Φ is the empty set.

The proof that Landau and Slepian use is based upon the following two facts which they prove. First, that for a given cap angle $\theta \leq \pi/2$, that $h = \int_{C_{\underline{s}}(\theta) \cap D} f(\|r\underline{x}-\underline{s}_i\|^2)\, dS(\underline{x})$ is minimized over all convex, radially invariant regions D for which

$$w = \int_{C_{\underline{s}}(\theta) \cap D} dS(\underline{x})$$

is fixed, by picking D to be a half space not containing $\underline{s}$.

Secondly, that if D is taken to be a half space not containing $\underline{s}$, then h may be considered as a function of w and $h(w)$ is convex upward.

Let c represent the surface content of a particular cap C about a vector $\underline{s}$ and s_P represent the surface content of the unit sphere so that

$$c = \int_{C_{\underline{s}}(\theta)} dS(\underline{x})$$

$$s_P = \int dS(\underline{x})$$

Then Landau and Slepian show that a general upper bound for $U(r)$ for any set of M signals is given by

$$MU(r) \le M \int_{C_{\underline{s}}(\theta)} f(\|r\underline{x}-\underline{s}\|)\, dS(\underline{x}) - 2kh\left(\frac{Mc-s_P}{2k}\right)$$

where $2k$ represents the total number of hyperplanes necessary to form the boundaries for each of the decision regions R_i. This formula is valid for any cap for which $\theta < \pi/2$ and $Mc - s_P \ge 0$. Furthermore the right-hand side of the equation is monotone increasing in k and equality holds in the equation if and only if the three conditions mentioned above are met. We can relax f to being non-increasing with the result that the three conditions are still sufficient but no longer necessary for equality in the above equation.

Landau and Slepian further claim that when $M = n + 1$, the regular simplex signal set of $n + 1$ vectors in n dimensions satisfies the three conditions necessary for equality in their upper bound, when the k boundaries are taken to be the hyperplanes equidistant from each pair of signal vectors. We will now show that this claim is true only for $n \le 3$.

1.7. The Regular Simplex Signal Set.

The regular simplex signal set $\{\underline{s}_i\}_{i=1}^{n+1}$ of $n + 1$ vectors in n dimensions is uniquely defined by the following equations [17].

$$\|\underline{s}_i\| = 1 \qquad\qquad i = 1, \cdots, n+1$$

$$\langle \underline{s}_i, \underline{s}_j \rangle = -\frac{1}{n} \quad i \neq j \quad i,j = 1, \cdots, n+1$$

Furthermore, it is easy to show that

$$\sum_{i=1}^{n+1} \underline{s}_i = 0$$

but that any subset of n simplex vectors is linearly independent.

We would now like to characterize the spherical decision regions R_i^*, which are the radial projections of the optimal decision regions $\mathcal{R}_i^*$ onto the surface of the unit sphere. Because of the fact that $\mathcal{R}_i^*$ is convex and radially invariant, R_i^* has the property that if $\underline{x}_j \in R_i^*$ $j = 1, \cdots k$ and α_j $j = 1, \cdots k$ are positive constants, then the vector

$$\underline{x} = \frac{\sum \alpha_j \underline{x}_j}{\|\sum \alpha_j \underline{x}_j\|} \in R_i^* .$$

Let us refer to this property as spherical convexity.

Thus it follows that there are certain extreme vectors whose spherical convex combinations generate R_i^*. That is, R_i^* is the spherical convex hull of these extreme vectors, and furthermore no extreme vector is expressible as a spherical convex combination of other extreme vectors.

We may thus characterize the regions R_i^* for the regular simplex signal set by their extreme vectors.

<u>Theorem</u>: The extreme vectors for the region R_i^* for the regular simplex signal set is the set of n vectors $\{-\underline{s}_j\}_{\substack{j=1 \\ j\neq i}}^{n+1}$.

<u>Proof</u>: We first note that if $\underline{x}$ is of the form

$$\underline{x} = \sum_{\substack{j=1 \\ j\neq i}}^{n+1} \alpha_j(-s_j)$$

where $\alpha_j > 0 \quad j = 1, \cdots n+1 \quad j \neq i$

then $\langle \underline{x}, \underline{s}_i \rangle - \langle \underline{x}, \underline{s}_k \rangle = \langle \underline{x}, \underline{s}_i - \underline{s}_k \rangle$

$$= \alpha_k \langle -\underline{s}_k, \underline{s}_i - \underline{s}_k \rangle$$

$$= \alpha_k \left(1 + \frac{1}{n}\right)$$

$$> 0 \qquad \text{for} \quad k \neq i$$

Hence $\underline{x} \in H_{ij}$ for every $j \neq i$

$$\Rightarrow \underline{x} \in \mathcal{R}_i^* = \bigcap_{\substack{j=1 \\ j\neq i}}^{n+1} H_{ij}$$

$$\Rightarrow \underline{x} \in R_i^*$$

Thus we have that the spherical convex hull of the given vectors is a subset of R_i^* .

To show the converse we note that since $\{\underline{s}_i\}_{i=1}^{n}$ are n linearly independent vectors in n dimensional Euclidian space we can represent any $\underline{x}$ in the form

$$\underline{x} = \sum_{j=1}^{n} \beta_j(-\underline{s}_j)$$

for some constants β_j for $j = 1, \cdots n$. If $\beta_1 < 0$ then we can replace $(-\underline{s}_1)$ by

$$(-\underline{s}_1) = - \sum_{j=2}^{n+1} (-\underline{s}_j)$$

to yield

$$\underline{x} = \sum_{j=2}^{n} (\beta_j-\beta_1)(-\underline{s}_j) + (-\beta_1)\, \underline{s}_{n+1}$$

$$= \sum_{j=2}^{n+1} \beta_j'(-\underline{s}_j)$$

where $\beta_j' > \beta_j$ for $j = 2, \cdots n$ and $\beta_{n+1}' > 0$. Similarly if $\beta_2' < 0$, $-\underline{s}_2$ can be replaced by

$$(-\underline{s}_2) = - \sum_{\substack{j=1 \\ j\neq 2}}^{n+1} (-\underline{s}_j)$$

to yield an expression for $\underline{x}$ in terms of $\{-\underline{s}_j\}_{\substack{j=1 \\ j\neq 2}}^{n+1}$ in which all the β's are increased. Proceeding in this manner, all negative β's can be eliminated to yield an expression for $\underline{x}$ of the form

$$\underline{x} = \sum_{\substack{j=1 \\ j\neq i}}^{n+1} \alpha_j(-\underline{s}_j)$$

for some i, where $\alpha_j \geq 0 \quad j = 1, \cdots n+1 \quad j \neq i$. Thus the spherical convex hulls of the given sets of vectors exhaust the surface of the sphere which implies that the open convex hull of the vectors $\{-\underline{s}_j\}_{\substack{j=1 \\ j\neq i}}^{n+1}$ is R_i^*. Finally we note that the set $\{-\underline{s}_j\}_{\substack{j=1 \\ j\neq i}}^{n+1}$ is linearly independent, so that in particular no vector is a spherical convex combination of any of the other vectors in the set.

Thus we have a characterization for the R_i^* in terms of the set of extreme vectors $\{-\underline{s}_j\}_{j=1}^{n+1}$. This characterization in fact suggests where the regular simplex signal set derives its name from. The convex hull of any $n+1$ non-degenerate points in n space determine a convex n dimensional polytope called a simplex [3]. In particular, the set $\{-\underline{s}_j\}_{j=1}^{n+1}$ determines a regular polytope called a regular simplex centered at the origin. The regular simplex signal set is then the set of $n+1$ vectors which pass through the center of the $n+1$ faces of the polytope. A face is an $n-1$ dimensional polytope formed by the convex hull of a subset containing n of the points.

1.8. Landau and Slepian's Conditions for an Optimal Code as Applied to the Regular Simplex Signal Set.

We are now ready to investigate the applications of Landau and Slepian's conditions for an optimal code to the regular simplex signal set. We first note that the second of the three conditions mentioned above is trivially satisfied by the total symmetry of the simplex set. We further note that if the angle ϕ in the first condition is taken equal to θ^* defined by $\cos\theta^* = \frac{1}{n}$, then the first condition will be met also. This follows since each of the extreme vectors $\{-\underline{s}_j\}_{\substack{j=1\\ j\neq i}}^{n+1}$ of R_i^* makes an angle θ^* with $\underline{s}_i$:

$$\langle \underline{s}_i, (-\underline{s}_j)\rangle = \frac{1}{n} = \cos\theta^*$$

Thus the cap of angle θ^* about $\underline{s}_i$, which is spherically convex, contains the extreme vector of R_i^* and hence must contain R_i^*.

We next show that ϕ must be taken at least as large as θ^* to satisfy condition 1.

Theorem: If $\phi = \theta^* - \delta$ for $0 < \delta < \theta^* < \frac{\pi}{2}$ then $\{\bigcup_{j=1}^{n+1} C_{\underline{s}_j}(\phi)\} \cap C_{-\underline{s}_1}(\delta) = \Phi$.

Proof: We first note that $C_{\underline{s}_1}(\phi) \cap C_{-\underline{s}_1}(\delta) = \Phi$

since $\underline{x} \in C_{\underline{s}_1}(\phi) \Rightarrow \langle \underline{x}, \underline{s}_1\rangle > \cos\phi > 0$

and $\underline{x} \in C_{\underline{s}_1}(\delta) \Rightarrow \langle \underline{x}, -\underline{s}_1\rangle > \cos\delta \Rightarrow \langle \underline{x}, \underline{s}_1\rangle < -\cos\delta < 0.$

We next show the impossibility of having

$\underline{x} \in C_{\underline{s}_j}(\phi) \cap C_{-\underline{s}_1}(\delta)$ for any $j \neq 1$.

$\underline{x} \in C_{\underline{s}_j}(\phi) \Rightarrow \langle \underline{x}, \underline{s}_j\rangle > \cos\phi$

$\underline{x} \in C_{\underline{s}_1}(\delta) \Rightarrow \langle \underline{x}, -\underline{s}_1\rangle > \cos\delta$

Adding $\sin\delta$ times the first equation to $\sin\phi$ times the second yields

$$\sin\theta^* = \sin(\phi+\delta) = \sin\delta\cos\phi + \sin\phi\cos\delta <$$

$$\langle \underline{x}, \sin\delta\,\underline{s}_j + \sin\phi(-\underline{s}_1)\rangle$$

but by the Schwartz inequality we have

$$\langle \underline{x}, \sin\delta\,\underline{s}_j - \sin\phi\,\underline{s}_1\rangle \le \|\sin\delta\,\underline{s}_j + \sin\phi(-\underline{s}_1)\|$$

$$= \sqrt{\sin^2\delta + \sin^2\phi + 2\sin\delta\sin\phi\cos\theta^*}$$

$$= \sqrt{\sin^2\delta + \sin^2(\theta^*-\delta) + 2\sin\delta\sin(\theta^*-\delta)\cos\theta^*}$$

$$= \sqrt{\sin^2\delta + \{\sin^2\theta^*\cos^2\delta + \cos^2\theta^*\sin^2\delta - 2\sin\theta^*\cos\theta^*\sin\delta\cos\delta\} + 2\{\sin\theta^*\cos\theta^*\sin\delta\cos\delta - \cos^2\theta^*\sin^2\delta\}}$$

$$= \sqrt{\sin^2\delta + \sin^2\theta^*\cos^2\delta - \cos^2\theta^*\sin^2\delta}$$

$$= \sqrt{\sin^2\theta^*\sin^2\delta + \sin^2\theta^*\cos^2\delta}$$

$$= \sin\theta^*$$

Hence any $\underline{x} \in C_{\underline{s}_j}(\phi) \cap C_{-\underline{s}_1}(\delta)$ for $j \neq 1$ must satisfy

$$\sin\theta^* < \langle \underline{x}, \sin\delta\,\underline{s}_j + \sin\phi(-\underline{s}_1)\rangle \le \sin\theta^*$$

which is of course impossible. Hence we have

$$C_{\underline{s}_j}(\phi) \cap C_{-\underline{s}_1}(\delta) = \Phi \quad \text{for } j = 1, \cdots n+1$$

$$\Rightarrow \{\bigcup_{j=1}^{n+1} C_{\underline{s}_j}(\phi)\} \cap C_{-\underline{s}_1}(\delta) = \Phi$$

Thus if ϕ is taken less than θ^* we find that $C_{\underline{s}_j}(\phi) \supset R_j^*$ for $j = 1, \cdots n+1$ is impossible since the union of the R_j^* covers the entire surface of the sphere except for a region of zero content.

We now investigate the possibility of simultaneously satisfying the third condition for an optimal signal set. Let us consider the particular vector $\hat{\underline{s}}$ given by

$$\hat{\underline{s}} = \frac{\underline{s}_1 + \underline{s}_2 + \underline{s}_3}{||\underline{s}_1 + \underline{s}_2 + \underline{s}_3||} \quad \text{for} \quad n \geq 3 .$$

The angle $\hat{\theta}$ that $\hat{\underline{s}}$ makes with $\underline{s}_1$ is given by

$$\langle \hat{\underline{s}}, \underline{s}_1 \rangle = \cos \hat{\theta} = \sqrt{\frac{1}{3} \frac{n-2}{n}} .$$

It will be helpful to first prove the following result.

Theorem: If $\phi = \hat{\theta} + \delta$ for $\delta > 0$, then $C_{\hat{\underline{s}}}(\delta) \subset C_{\underline{s}_1}(\phi)$.

Proof: We will show that the complement of $C_{\underline{s}_1}(\phi)$ intersected with $C_{\hat{\underline{s}}}(\delta)$ is void. Again we consider an $\underline{x}$ in the intersection

$$\underline{x} \in \text{complement } C_{\underline{s}_1}(\phi) \Rightarrow \langle \underline{x}, \underline{s}_1 \rangle \leq \cos \phi$$

$$\underline{x} \in C_{\hat{\underline{s}}}(\delta) \qquad \Rightarrow \langle \underline{x}, \hat{\underline{s}} \rangle > \cos \delta$$

Thus adding $-\sin \delta$ times the first equation to $\sin \phi$ times the second yields

$$\sin \hat{\theta} = \sin(\phi - \delta) = \sin \phi \cos \delta - \sin \delta \cos \phi$$

$$< \langle \underline{x}, \sin \phi \, \hat{\underline{s}} - \sin \delta \, \underline{s}_1 \rangle$$

Again using the Schwartz inequality we find

$$\langle \underline{x}, \sin \phi\, \hat{\underline{s}} - \sin \delta\, \underline{s}_1 \rangle \le \|\sin \phi\, \hat{\underline{s}} - \sin \delta\, \underline{s}_1\|$$

$$= \sqrt{\sin^2\delta + \sin^2\phi - 2 \sin \phi \sin \delta \cos \hat{\theta}}$$

$$= \sqrt{\sin^2\delta + \sin^2(\hat{\theta}+\delta) - 2 \sin(\hat{\theta}+\delta) \sin \delta \cos \hat{\theta}}$$

$$= \sqrt{\sin^2\delta + \{\sin^2\hat{\theta}\cos^2\delta + \cos^2\hat{\theta}\sin^2\delta + 2\sin\hat{\theta}\cos\hat{\theta}\sin\delta\cos\delta\} - 2\{\sin\hat{\theta}\cos\hat{\theta}\sin\delta\cos\delta + \cos^2\hat{\theta}\sin^2\delta\}}$$

$$= \sqrt{\sin^2\delta + \sin^2\hat{\theta} \cos^2\delta - \cos^2\hat{\theta} \sin^2\delta}$$

$$= \sqrt{\sin^2\hat{\theta} \sin^2\delta + \sin^2\hat{\theta} \cos^2\delta}$$

$$= \sin \hat{\theta}$$

Thus we again have an impossible condition that

$$\sin \hat{\theta} < \langle \underline{x}, \sin \phi\, \hat{\underline{s}} - \sin \delta\, \underline{s}_1 \rangle \le \sin \hat{\theta}$$

and hence $C_{\hat{\underline{s}}}(\phi) \subset C_{\underline{s}_1}(\delta)$.

It is further evident by the symmetry of the vectors $\underline{s}_1$, $\underline{s}_2$, $\underline{s}_3$ and $\hat{\underline{s}}$ that under the conditions of the above theorem $C_{\hat{\underline{s}}}(\delta) \subset C_{\underline{s}_2}(\phi)$ and $C_{\hat{\underline{s}}}(\delta) \subset C_{\underline{s}_3}(\phi)$. If we are given an angle δ such that $0 < \delta < \pi$, let us determine what part of the cap $C_{\hat{\underline{s}}}(\delta)$ lies in the intersection of the two half spaces H_{21} and H_{31}.

<u>Theorem</u>:

$$\int_{C_{\hat{\underline{s}}}(\delta) \cap H_{21} \cap H_{31}} dS(\underline{x}) = \frac{1}{3} \int_{C_{\hat{\underline{s}}}(\delta)} dS(\underline{x})$$

Proof: We shall use the invariance of the surface integrals under rotation to obtain a convenient parameterization. $C_{\hat{\underline{s}}}(\delta)$ is defined by the vector $\hat{\underline{s}}$ and H_{21} and H_{31} are defined by the vectors $\underline{h}_{21} = \frac{\underline{s}_2 - \underline{s}_1}{\|\underline{s}_2-\underline{s}_1\|}$ and $\underline{h}_{31} = \frac{\underline{s}_3 - \underline{s}_1}{\|\underline{s}_3-\underline{s}_1\|}$ respectively. We note that $\hat{\underline{s}}$ is perpendicular to both $\underline{h}_{21}$ and $\underline{h}_{31}$. Without loss of generality, then, we may let

$$\hat{\underline{s}} = (1,0, \cdots , 0,0,0)$$

$$\underline{h}_{21} = (0,0, \cdots , 0,\cos\theta,\sin\theta)$$

$$\underline{h}_{31} = (0,0, \cdots , 0,\cos\theta,-\sin\theta)$$

where θ is determined by the inner product of $\underline{h}_{21}$ and $\underline{h}_{31}$ as

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \langle \underline{h}_{21},\underline{h}_{31}\rangle = \frac{1}{2}$$

or $2\theta = \frac{\pi}{3}$ radians. Letting $\underline{x}$ be defined as

$$\underline{x} = (x_1,x_2, \cdots , x_{n-2},x_{n-1},x_n)$$

and using the n dimensional spherical coordinates of Appendix I, we find

$$x_1 = \cos \theta_1$$

$$x_{n-1} = \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-2} \cos \theta_{n-1}$$

$$x_n = \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-2} \sin \theta_{n-1}$$

where $0 < \theta_i < \pi$ for $i = 1, 2, \cdots n-2$

and $-\pi < \theta_{n-1} < \pi$

and $dS(\underline{x}) = \sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \cdots \sin\theta_{n-2}\, d\theta_1 d\theta_2 \cdots d\theta_{n-1}$

The cap $C_{\hat{s}}(\delta)$ is thus defined parametrically as

$$C_{\hat{s}}(\delta) = \{\underline{x} | \theta_1 < \delta\}$$

and the regions H_{21} and H_{31} intersected with the unit sphere by

$$H_{21} = \{\underline{x} | \cos\theta \cos\theta_{n-1} + \sin\theta \sin\theta_{n-1} > 0\}$$

$$H_{31} = \{\underline{x} | \cos\theta \cos\theta_{n-1} - \sin\theta \sin\theta_{n-1} > 0\}$$

or $$H_{21} = \{\underline{x} | \cos(\theta_{n-1}-\theta) > 0\}$$

$$H_{31} = \{\underline{x} | \cos(\theta_{n-1}+\theta) > 0\}$$

Hence the region of intersection is given by

$$H_{31} \cap H_{21} = \{\underline{x} | \cos(\theta_{n-1}-\theta) > 0 \text{ and } \cos(\theta_{n-1}+\theta) > 0\}$$

or $$H_{21} \cap H_{31} = \{\underline{x} | -\frac{\pi}{2} - \theta < \theta_{n-1} < \frac{\pi}{2} - \theta\}$$

but since $\theta = \frac{\pi}{6}$ this becomes

$$H_{21} \cap H_{31} = \{\underline{x} | -\frac{\pi}{3} < \theta_{n-1} < \frac{\pi}{3}\}$$

Hence we calculate

$$\int_{C_{\underline{\hat{s}}}(\delta)\cap H_{21}\cap H_{31}} dS(\underline{x})$$

$$= \int_{-\pi/3}^{\pi/3} \int_0^{\pi} \cdots \int_0^{\pi} \int_0^{\delta} \sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \cdots \sin\theta_{n-2} d\theta_1 d\theta_2 \cdots d\theta_{n-2} d\theta_{n-1}$$

$$= \frac{1}{3} \int_{-\pi}^{\pi} \int_0^{\pi} \cdots \int_0^{\pi} \int_0^{\delta} \sin^{n-2}\theta_1 \sin^{n-2}\theta_2 \cdots \sin\theta_{n-2} d\theta_1 d\theta_2 \cdots d\theta_{n-2} d\theta_{n-1}$$

$$= \frac{1}{3} \int_{C_{\underline{\hat{s}}}(\delta)} dS(\underline{x})$$

Finally we note the relationship between θ^* and $\hat{\theta}$ in the following way:

$$\cos \theta^* = \frac{1}{n}$$

$$\cos \hat{\theta} = \sqrt{\frac{1}{3} \frac{n-2}{n}}$$

Hence θ^* is monotone increasing in n while $\hat{\theta}$ is monotone decreasing in n. Furthermore when $n = 3$ we find

$$\theta^* = \hat{\theta} = \cos^{-1}(\tfrac{1}{3}) \quad \text{for} \quad n = 3 .$$

Thus we have the following facts: The angle ϕ must be taken at least as large as θ^* to satisfy condition 1. If n is greater than 3, then $\hat{\theta}$ is strictly less than θ^* so that if ϕ is chosen to satisfy condition 1, then there exists a δ such that

$C_{\hat{\underline{s}}}(\delta) \subset C_{\underline{s}_1}(\phi)$. Under the above conditions we have therefore

$$C_{\underline{s}_1}(\phi) \cap H_{21} \cap H_{31} \supset C_{\hat{\underline{s}}}(\delta) \cap H_{21} \cap H_{31}$$

so that

$$\int_{C_{\underline{s}_1}(\phi)\cap H_{21}\cap H_{31}} dS(\underline{x}) \geq \frac{1}{3} \int_{C_{\underline{s}_1}(\delta)} dS(\underline{x}) > 0 \quad .$$

Thus if n is greater than 3 and ϕ is chosen to satisfy condition 1, then condition 3 cannot be satisfied.

Hence we have proved the result.

Theorem: The sufficient conditions of Landau and Slepian for the existence of an optimal code are not met by the regular simplex set of $n + 1$ vectors in n dimensions if n is greater than 3.

1.9. Some Conjectures.

There are thus three successively stronger conjectures concerning $n + 1$ signals in n dimensions which remain unresolved.

Conjecture 1: The simplex signals are optimal for the gaussian white noise additive channel, when the signals are constrained to be equiprobable and equienergy.

Conjecture 2: The simplex signals are optimal for the additive noise channel with any spherically symmetric, monotone decreasing noise density, when the signals are constrained to be equiprobable and equienergy.

Conjecture 3: The function $U(r)$ defined above is maximized by the simplex signals for the function f monotone decreasing, when the signals are constrained to be equiprobable and equienergy.

Let us define any noise density of the form

$$Pr(\underline{n}) = \begin{cases} k & \|\underline{n}\| < r_o \\ 0 & \|\underline{n}\| \geq r_o \end{cases}$$

as a spherical ball density. Then since any spherical ball density can be uniformly approximated by a sequence of monotone decreasing densities and since any monotone decreasing density can be uniformly approximated by a sum of spherical ball densities, we have that conjecture 2 will be true if and only if the following conjecture is true.

<u>Conjecture 2'</u>: The simplex signals are optimal for the additive noise channel with any spherical ball noise density, when the signals are constrained to be equiprobable and equienergy.

Furthermore, if the noise has a spherical ball density, then the function f in conjecture 3 will be of the form

$$f(\|r\underline{x}-\underline{s}_i\|^2) = \begin{cases} k & \langle \underline{x}, \underline{s}_i \rangle > \cos\,\theta(r,r_o) \\ 0 & \langle \underline{x}, \underline{s}_i \rangle \leq \cos\,\theta(r,r_o) \end{cases}$$

for some angle $\theta(r,r_o)$ which can be determined graphically from the following diagram:

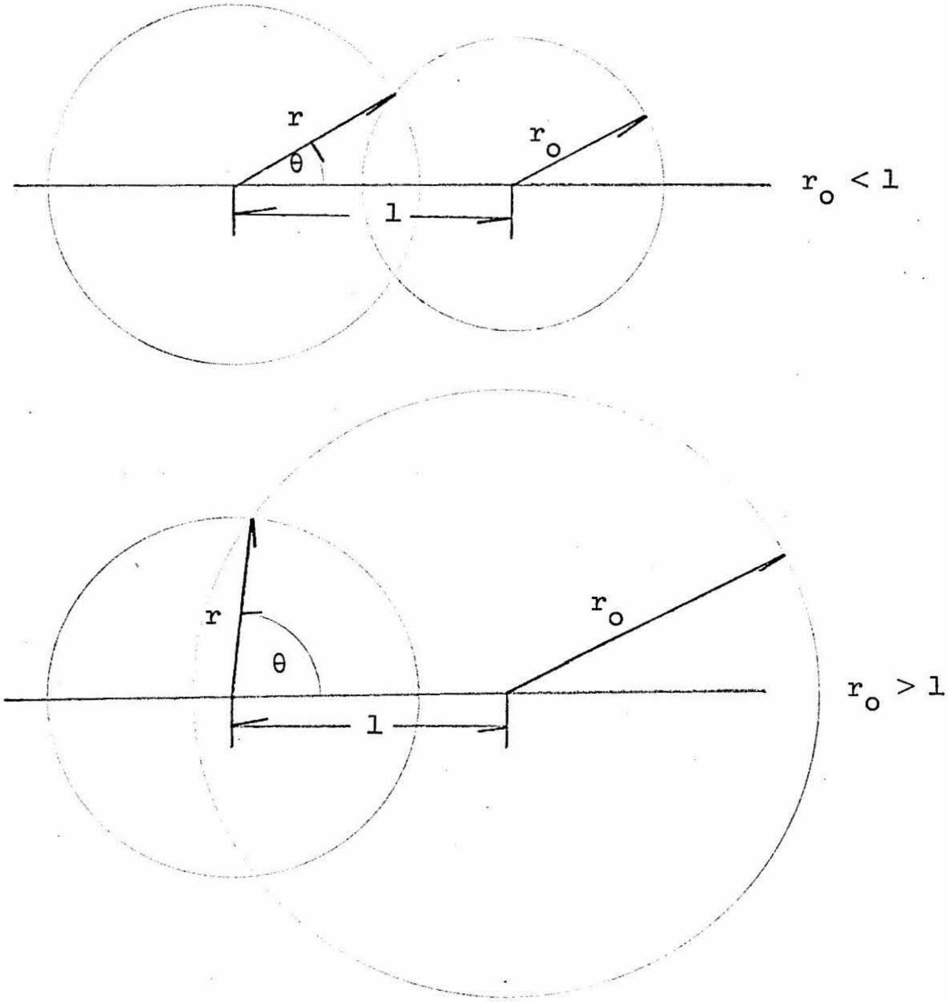

Figure 1. Graphical Determination of $\theta(r,r_o)$ for $r_o < 1$ and $r_o > 1$.

That is, f is of the form

$$f(\|r\underline{x}-\underline{s}_i\|^2) = \begin{cases} k & \underline{x} \in C_{\underline{s}_i}(\theta(r,r_o)) \\ 0 & \text{otherwise} \end{cases}$$

where $\theta(r,r_o)$ can clearly take on any value between 0 and π. Hence it follows that conjecture 3 will be true if and only if the following equivalent conjecture is true.

Conjecture 3': The simplex signals maximize

$$v_C(\theta) = \int_{C(\theta)} dS(\underline{x}) \quad \text{for every } \theta \text{ between } 0 \text{ and } \pi$$

where $C(\theta) = \bigcup_{j=1}^{n+1} C_{\underline{s}_j}(\theta)$ over all signal sets such that $\|s_j\| = 1$ for $j = 1, \cdots n+1$.

Conjecture 3' has the interesting property that the dependence upon decision regions has been supressed, although it is not clear that this makes the problem any simpler to solve.

The current state of knowledge of conjecture 3' can be summarized in the following diagram.

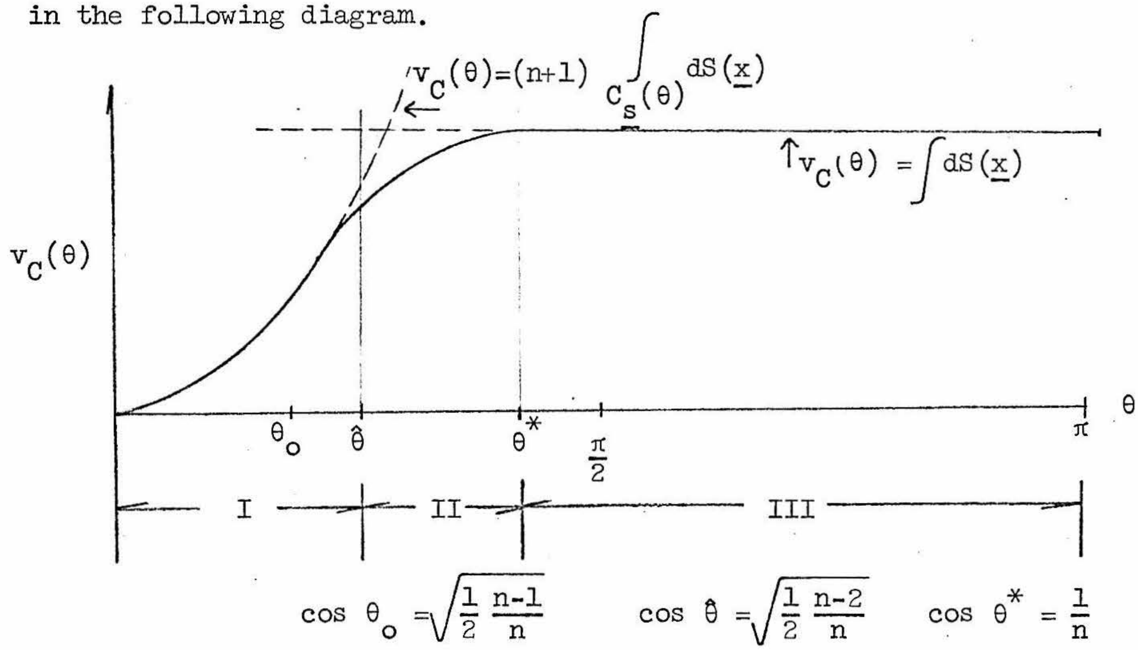

Figure 2. Graph of $v_C(\theta)$ for the Coherent Channel.

Regions I and III are where conjecture 3' is known to be true and region II is where the conjecture is as yet undecided. For θ between 0 and θ_o, $v_c(\theta)$ satisfies the bound

$$v(\theta) = (n+1) \int_{C_{\underline{s}_1}(\theta)} dS(\underline{x})$$

that is, the caps do not intersect. Between θ_o and $\hat{\theta}$, Landau and Slepian's bound can be used to show the optimality of the simplex signals, since the parts of a cap cut off by the hyperplane boundary regions are non-intersecting. For θ greater than θ^*, we again have that the simplex signals are optimal since $v_c(\theta)$ satisfies the bound

$$v_c(\theta) = \int dS(\underline{x})$$

For θ between θ_o and $\hat{\theta}$, the caps intersect each other at most two at a time, while as θ is increased from $\hat{\theta}$ to θ^*, the caps will intersect first three at a time, then four at a time, etc., until finally just before θ^* they intersect n at a time.

For $n = 3$, $\hat{\theta} = \theta^*$ as we have already noted, so that region II is void and the conjecture is true. However, as n increases $\hat{\theta}$ monotonely decreases until $\cos \hat{\theta} = \frac{1}{3}$ or $\hat{\theta} \cong 55^\circ$, while θ^* monotonely increases until $\cos \theta^* = 0$ or $\theta^* = 90^\circ$.

CHAPTER II. THE PHASE INCOHERENT CASE

2.1. The Incoherent Additive Noise Channel.

The incoherent additive noise channel can be modeled analogously to the coherent additive noise channel except that now the signal, noise and received vector are vectors in C^n, the n dimensional complex space. Furthermore the channel in addition to adding the noise $\underline{n}$ to the transmitted signal $\underline{s}$ performs the following operation

$$\underline{r} = \underline{s}\, e^{i\theta} + \underline{n}$$

where θ is a random variable uniformly distributed between $-\pi$ and π [17], [9].

We shall again consider a set of M equiprobable signals $\{\underline{s}_i\}_{i=1}^{M}$ and without loss of generality in what follows, we shall again restrict the signals to have unit energy. That is, $\|\underline{s}_i\| = 1$ for $i = 1, \cdots M$ where $\|\underline{s}_i\|^2 = \sum_{i=1}^{n} |s_{ij}|^2$ and s_{ij} are the complex components of $\underline{s}_i$.

We wish to consider only certain noise densities. As before, we shall require that the noise density $Pr(\underline{n})$ be spherically symmetric. Thus

$$Pr(\underline{n}) = g(\|\underline{n}\|^2)$$

for some function g. Hence we can write the conditional density on $\underline{r}$ when $\underline{s}$ is given as

$$\Pr(\underline{r}/\underline{s}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Pr(\underline{r}/\underline{s},\theta)\, d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\|\underline{r}-\underline{s}e^{i\theta}\|^2)\, d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\|\underline{r}\|^2 + 1 - 2\langle\underline{r},\underline{s}\rangle \cos\theta)\, d\theta$$

where $\langle\underline{r},\underline{s}\rangle = \sum_{j=1}^{n} r_j s_j^*$ and r_j and s_j are the components of $\underline{r}$ and $\underline{s}$ respectively, and s_j^* denotes the complex conjugate of s_j. $\Pr(\underline{r}|\underline{s})$ is therefore a function of only $r = \|\underline{r}\|$ and $|\langle\frac{1}{r}\,\underline{r},\underline{s}\rangle|$, say $h(r,|\langle\frac{1}{r}\,\underline{r},\underline{s}\rangle|)$. The other restriction we wish to place upon the noise is that for each r, $h(r,\gamma)$ will be monotone increasing in γ for $0 \le \gamma \le 1$. This will be true if g is required to be convex upward since

$$h(r,\gamma) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(r^2 + 1 - 2r\gamma\cos\theta)\, d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi/2} [g(r^2 + 1 - 2r\gamma\cos\theta) + g(r^2 + 1 + 2r\gamma\cos\theta)]d\theta$$

and thus

$$\frac{d}{d\gamma}h(r,\gamma) = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} -2r\cos\theta[g'(r^2+1-2r\gamma\cos\theta) - g'(r^2+1+2r\gamma\cos\theta)]\, d\theta$$

and the right-hand side is positive for $0 \le \gamma \le 1$ because $\cos\theta > 0$

in the interval $-\pi/2$ to $\pi/2$ and $g'(x_1) < g'(x_2)$ for $x_1 < x_2$ by the convex upward assumption or g. In particular the complex gaussian white noise process satisfies these properties [6].

When h has the monotone property, it follows in a manner analogous to the coherent case that the optimal decision region $\mathcal{R}_i$ for deciding that $\underline{s}_i$ was sent will be defined by

$$\underline{r} \in \mathcal{R}_i \Rightarrow |\langle \underline{r}, \underline{s}_i \rangle| > |\langle \underline{r}, \underline{s}_j \rangle| \qquad i,j=1, \cdots M$$

Or if the halfspaces H_{ij} are defined by

$$H_{ij} = \{\underline{r} \| \langle \underline{r}, \underline{s}_i \rangle| > |\langle \underline{r}, \underline{s}_j \rangle|\} \qquad i,j=1, \cdots M$$

then the optimal decision regions may be defined to within trivial differences by requiring

$$\mathcal{R}_i \supset \mathcal{R}_i^* = \bigcup_{\substack{j=1 \\ j \neq i}}^{M} H_{ij} \qquad i = 1, \cdots M$$

The regions $\mathcal{R}_i^*$ are radially invariant but not convex.

If we define the sets

$$S_i = \{\underline{x} | \underline{x} = \underline{s}_i e^{i\theta} \; -\pi < \theta \leq \pi\} \qquad i = 1, \cdots M$$

then the optimal decision rule has the following intuitive explanation. Decide $\underline{s}_i$ was transmitted if the distance from $\underline{r}$ to S_i is less than the distance from $\underline{r}$ to S_j for all $j \neq i$.

2.2. An Alternate Expression for P_c for the Incoherent Case.

Because of the radial invariance of the regions $\mathcal{R}_i^*$ we can change the volume integral expression for the probability of being correct P_c into first an integration over the surface of the unit sphere and then a radial integration. If we let $\underline{x}$ again represent a generic unit length vector, then we have

$$P_c = \frac{1}{M} \sum_i \int_{\mathcal{R}_i} h(r, \langle \underline{x}, \underline{s}_i \rangle)\, dV(\underline{x})$$

$$= \int_0^\infty \frac{1}{M} \sum_{i=1}^{M} \int_{R_i} h(r, |\langle \underline{x}, \underline{s}_i \rangle|)\, dS(\underline{x}) \quad r^{2n-1}\, dr$$

where $dV(\underline{x})$ and $dS(\underline{x})$ are analytically defined in Appendix IB and R_i is the radial projection of $\mathcal{R}_i$ onto the unit sphere in C^n.

Thus defining

$$U(r) = \frac{1}{M} \sum_{i=1}^{M} \int_{R_i} h(r, |\langle \underline{x}, \underline{s}_i \rangle|)\, dS(\underline{x})$$

P_c for the incoherent case may be written as

$$P_c = \int_0^\infty U(r)\, r^{2n-1}\, dr\ .$$

2.3. Conjectures for the Incoherent Case.

Our interest is in the long standing [13] conjecture that when $M = n$, the orthogonal signals defined by

$$\underline{s}_1 = (1, 0, \cdots 0, 0)$$

$$\underline{s}_2 = (0,1, \cdots 0,0)$$

$$\vdots$$

$$\underline{s}_n = (0,0, \cdots 0,1)$$

are optimal in the following sense:

<u>Conjecture 4</u>. The orthogonal signals are optimal for the gaussian white noise additive incoherent channel, when the signals are constrained to be equiprobable and equienergy.

From the above we see that conjecture 4 will be true if the following is true.

<u>Conjecture 5</u>. The orthogonal signals are optimal for the incoherent additive noise channel with any spherically symmetric, convex upward noise density when the signals are constrained to be equiprobable and equienergy.

Furthermore, conjecture 5 will be true if the following is true.

<u>Conjecture 6</u>. The function $U(r)$ above is maximized by the orthogonal signals for the function $h(r,\gamma)$ monotone increasing in γ for $0 \leq \gamma \leq 1$ for each r, when the signals are constrained to be equiprobable and equienergy.

If we define a cap of angle θ about a vector $\underline{s}$ for the incoherent case by

$$C_{\underline{s}}(\theta) = \{\underline{x} \,|\, |\langle \underline{x}, \underline{s} \rangle| > \cos\theta\}$$

then by the monotone property of h we immediately get that conjecture

6 is equivalent to the following.

Conjecture 6'. The orthogonal signals maximize $V_I(\theta) = \int_{C(\theta)} dS(\underline{x})$

for every θ between 0 and $\pi/2$ where $C(\theta) = \bigcup_{j=1}^{n} C_{\underline{s}_j}(\theta)$ over all signal sets such that $\|s_j\| = 1$ for $j = 1, \cdots M$.

We can similarly define the state of knowledge of this conjecture by a diagram.

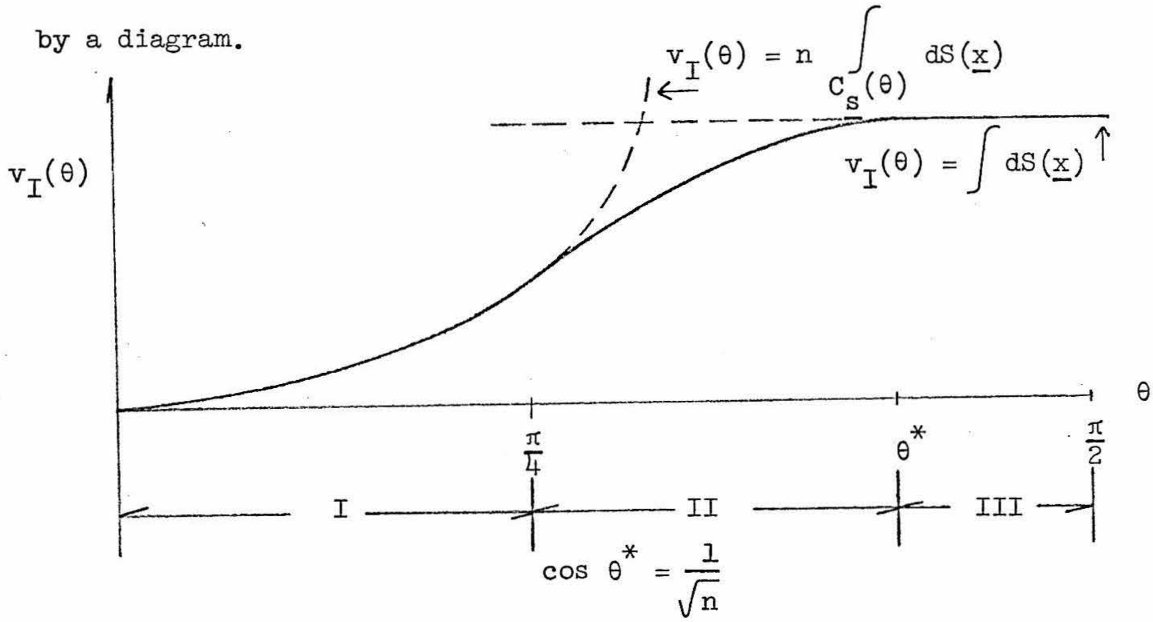

Figure 3. Graph of $v_I(\theta)$ for Incoherent Case.

Regions I and III are where conjecture 6' is known to be true and Region II is where the conjecture is as yet undecided. For θ between 0 and $\pi/4$ $V_I(\theta)$ satisfies the bound

$$V_I(\theta) = n \int_{C_{\underline{s}_i}(\theta)} dS(\underline{x}) .$$

That is, the caps do not intersect as can be easily verified. Between θ^* defined by $\cos\theta^* = \frac{1}{\sqrt{n}}$ and $\pi/2$, $V_I(\theta)$ satisfies the bound

$$V_I(\theta) = \int dS(\underline{x}) \ .$$

That is, the union of the caps covers the entire surface of the sphere as can again be easily verified.

However, as θ increases from $\pi/4$ to θ^*, the caps intersect each other first two at a time, then three at a time, etc., until just before θ^* they intersect each other $n-1$ at a time and no general results are known in these regions. This behavior, in fact, is completely analogous to the behavior of $V_c(\theta)$ in the coherent case.

Furthermore, even if the analogue of the two facts which Landau and Slepian [10] use in their paper could be proven for the incoherent case, it would only show the optimality of $V_I(\theta)$ for $\theta \leq \hat{\theta}$ defined by $\cos \hat{\theta} = \frac{1}{\sqrt{3}}$ where intersections occur at most two at a time. This would, of course, not be sufficient to resolve conjecture 6'. We note, however, that if $n = 3$, then $\hat{\theta} = \theta^*$ so that it would establish the conjecture for $n \leq 3$.

We further note that if $n = 2$, then $\theta^* = \pi/4$ and thus the conjecture is trivially true, which establishes the optimality of 2 orthogonal signals for the incoherent case. Surprisingly enough a proof of this result seems to have been first published by Schaffner and Krieger [11] as late as 1968, although work by Helstrom [8] in 1955 strongly implied the result for the gaussian case.

CHAPTER III. THE RELATION BETWEEN THE COHERENT AND INCOHERENT CASES

3.1. The Relationship Between the Incoherent Case with $n = 2$ and the Coherent Case with $n = 3$.

The recent work by Schaffner and Krieger [11] proves the optimality of certain signal sets with $M = 2, 3, 4, 6$ and 12 for the incoherent case with $n = 2$ by showing they maximize $U(r)$. These are, in fact, the same values of M for which Landau and Slepian were able to find optimal signal sets for the coherent case with $n = 3$.

We will show that this is more than mere coincidence by demonstrating a direct relationship between the incoherent case with $n = 2$ and the coherent case with $n = 3$ in terms of conjecture 6 and conjecture 3.

Theorem: If $\underline{x} = (\cos\theta\, e^{i\alpha}, \sin\theta\, e^{i\beta})$ is a generic unit vector in C^2 and $\underline{x}' = (\cos\theta', \sin\theta'\cos\alpha', \sin\theta'\sin\alpha')$ is a generic unit vector in E^3, then the transformation

$$\begin{aligned} 2\,\theta &\to \theta' \\ \alpha-\beta &\to \alpha' \\ \beta &\to \beta' \\ 2\,\phi &\to \phi' \end{aligned}$$

maps $C_{\underline{s}}(\phi)$ into $C_{\underline{s}'}(\phi')$ and

$dS(\underline{x})$ into $\frac{1}{4}\, dS(\underline{x}')\, d\beta'$

Proof: If $\underline{x} = (\cos\theta\, e^{i\alpha}, \sin\theta\, e^{i\beta})$ and $\underline{s} = (\cos\theta_1 e^{i\alpha_1}, \sin\theta_1 e^{i\beta_1})$

then $|\langle \underline{x}, \underline{s} \rangle| > \cos \phi$ becomes

$$\frac{1}{2}\{1 + \cos 2\theta \cos 2\theta_1 + \sin 2\theta \sin 2\theta_1 [\cos(\alpha-\beta)\cos(\alpha_1-\beta_1) + \sin(\alpha-\beta)\sin(\alpha_1-\beta_1)]\} > \cos^2\varphi$$

or $\langle \underline{x}', \underline{s}' \rangle > \cos \phi'$ where

$$\underline{x}' = (\cos \theta', \sin \theta' \cos \alpha', \sin \theta' \sin \alpha')$$

$$\underline{s}' = (\cos \theta_1', \sin \theta_1' \cos \alpha_1', \sin \theta_1' \sin \alpha_1')$$

and from Appendix I

$$dS(\underline{x}) = \sin \theta \cos \theta \, d\theta \, d\alpha \, d\beta$$

$$= \frac{1}{4} \sin \theta' \, d\theta' \, d\alpha' \, d\beta'$$

$$= \frac{1}{4} dS(\underline{x}') \, d\beta'$$

Thus by observing the form of conjecture 6 and conjecture 3 we have the following theorem.

Theorem: $\{\underline{s}_i'\}_{i=1}^M$ is an optimal signal set in the sense of maximizing $U(r)$ for the coherent case in E^3 if and only if the signal set $\{\underline{s}_i\}_{i=1}^M$ is an optimal signal set in the sense of maximizing $U(r)$ for the incoherent case in C^2 where $\underline{s}_i$ and $\underline{s}_i'$ are related by

$$\underline{s}_i = (\cos \theta_i e^{i\alpha_i}, \sin \theta_i e^{i\beta_i})$$

$$\underline{s}_i' = (\cos 2\theta_i,\ \sin 2\theta_i\ \cos(\alpha_i-\beta_i), \sin 2\theta_i \sin(\alpha_i-\beta_i))$$

and the signals are constrained to be equiprobable and equienergy.

Proof: Obvious.

Thus for each value of M the above transformation must map the optimal signal sets found by Landau and Slepian into the optimal signal sets found by Schaffner and Krieger. This can, in fact, be directly verified. In particular, consider the case of $M = 2$. Then the optimal signals for the coherent case are given by

$$\underline{s}_1' = (1,0,0) = (\cos 0,\ \sin 0 \cos \alpha_1',\ \sin 0 \sin \alpha_1')$$

$$\underline{s}_2' = (-1,0,0) = (\cos \pi,\ \sin \pi \cos \alpha_2',\ \sin \pi \sin \alpha_2')$$

where α_1' and α_2' are arbitrary and are transformed into

$$\underline{s}_1 = (e^{i\alpha_1},\ 0) = (\cos \tfrac{0}{2}\, e^{i\alpha_1},\ \sin \tfrac{0}{2}\, e^{i\beta_1})$$

$$\underline{s}_2 = (0,\ e^{i\beta_2}) = (\cos \tfrac{\pi}{2}\, e^{i\alpha_2},\ \sin \tfrac{\pi}{2}\, e^{i\beta_2})$$

which are the orthogonal signals with α_1 and β_2 arbitrary.

It is another unresolved conjecture as to whether this close of a relationship exists between the coherent and incoherent cases in higher dimensions.

3.2. A Simple Proof of the Theorem in Landau and Slepian's Appendix C for the case of $n = 3$.

The transformation in the previous section section can be used not only to map Schaffner and Krieger's optimal signal sets in C^2 into Landau and Slepian's optimal signal sets in E^3, it can also be used to map Schaffner and Krieger's proof of optimality in C^2 into a proof of the optimality of the transformed signals in E^3. In particular, this will yield an alternate proof of the very difficult general theorem in Landau and Slepian's Appendix C for the special case of $n = 3$. This will hopefully enable us to gain insight into the methods of both pairs of authors. The theorem of interest is the one which describes the particular convex, radially invariant region D which minimizes

$$h = \int_{D \cap C_{\underline{s}}(\theta)} f(\|r\underline{x}-\underline{s}\|^2)\, dS(\underline{x})$$

when

$$w = \int_{D \cap C_{\underline{s}_i}(\theta)} dS(\underline{x})$$

is held fixed. In Section 1.6 it was mentioned that the optimal D is a half space. Hence let us first investigate what happens to h and w when D is in fact a halfspace. The only fact that we will need about f is that $f(\|r\underline{x}-\underline{s}\|^2)$ is increasing in $\langle \underline{x}, \underline{s} \rangle$ for each fixed r. Hence let us put $f(\|r\underline{x}-\underline{s}\|^2) = h(r, \langle \underline{x}, \underline{s} \rangle)$ where h is increasing in $\langle \underline{x}, \underline{s} \rangle$.

If we parameterize $\underline{x}$ by

$$\underline{x} = (\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi)$$

and a halfspace H_i by

$$H_i = \{\underline{r} \mid \langle \underline{r}, \underline{h}_i \rangle > 0\}$$

where

$$\underline{h}_i = (-\sin\theta_i, \cos\theta_i\cos\phi_i, \cos\theta_i\sin\phi_i)$$

then the region $H_i \cap C_{\underline{s}}(\Theta)$ for $0 \le \Theta < \pi/2$ is given parametrically by

$$H_i \cap C_{\underline{s}}(\Theta) = \{\theta,\phi \mid -\sin\theta_i\cos\theta + \cos\theta_i\sin\theta\cos(\phi-\phi_i) > 0 \text{ and } 0 \le \theta < \theta\}$$

the range for ϕ as a function of θ is thus

$$-\cos^{-1}(\mathrm{TAN}\theta_i\mathrm{CTN}\theta) \le \phi - \phi_1 < \cos^{-1}(\mathrm{TAN}\theta_i\mathrm{CTN}\theta)$$

or

$$B_{iL}(\theta) \le \phi < B_{iU}(\theta) .$$

Without loss of generality we can pick $s_1 = (1,0,0)$ so that when $D = H_1$ we can represent h and w by

$$h = \int_{0}^{\Theta} \int_{B_{1L}(\theta)}^{B_{1U}(\theta)} h(r, \cos\theta)\, d\phi\, d\gamma(\theta)$$

and

$$w = \int_{0}^{\Theta} \int_{B_{1L}(\theta)}^{B_{1U}(\theta)} d\phi\, d\gamma(\theta) \quad \text{where} \quad \gamma(\theta) = -\cos\theta$$

In general, when D is formed by the intersection of several half spaces, say H_i for $i = 1, \cdots p$ then h and w are given by

$$h = \int_{0}^{\Theta} \int_{B_{L}(\theta)}^{B_{U}(\theta)} h(r, \cos\theta)\, d\phi\, d\gamma(\theta)$$

and

$$w = \int_{0}^{\Theta} \int_{B_{L}(\theta)}^{B_{U}(\theta)} d\phi\, d\gamma(\theta)$$

where there exists a partition $[\theta^{(1)}, \theta^{(2)}, \cdots \theta^{(p)}]$ of the interval $(0, \Theta)$ such that

$$B_L(\theta) = B_{i_k L}(\theta)$$

and $\quad B_U(\theta) = B_{j_k L}(\theta) \quad$ for $\quad \theta^{(i-1)} \leq \theta \leq \theta^{(i)} \quad i = 1, \cdots p$

for some i_k and j_k.

Next we note that

$$\frac{d}{d\theta_i} B_{1U}(\theta) = - \frac{d}{d\theta_i} B_{1L}(\theta) = \frac{d}{d\theta_i} \cos^{-1}(\mathrm{TAN}\theta_i \mathrm{CTN}\theta) < 0$$

for $-\Theta < \theta_i < \Theta$. Hence it follows that if a single halfspace, say H_o, is to also cut a region of content w off from the cap, then we must have $\theta_o > \theta_i$ for $i = 1, \cdots p$. This is because the content of an intersection of halfspaces can be no bigger than the content of the smallest intersection from any one of the halfspaces and we have just shown that the intersection with H_i is decreasing as θ_i increases.

Hence let us compare

$$h = \int_0^{\Theta} h(r, \cos\theta) \{B_U(\theta) - B_L(\theta)\} \, d\gamma(\theta)$$

with

$$\int_0^{\Theta} \int_{B_{OL}(\theta)}^{B_{OU}(\theta)} h(r, \cos\theta) \, d\phi \, d\gamma(\theta)$$

when θ_o is chosen such that

$$w = \int_0^{\Theta} \{B_U(\theta) - B_L(\theta)\} \quad d\gamma(\theta) = \int_0^{\Theta} \{B_{OU}(\theta) - B_{OL}(\theta)\} \, d\gamma(\theta)$$

We readily see that

$$\frac{dB_{iU}(\theta)}{d\gamma(\theta)} = - \frac{dB_{iL}(\theta)}{d\gamma(\theta)} = \frac{d}{d\gamma(\theta)} \cos^{-1}(\mathrm{TAN}\ \theta_i\ \mathrm{CTN}\ \theta)$$

for $\theta \leq \theta_o < \theta$

and the right-hand side

$$\frac{d}{d\gamma(\theta)} \cos^{-1}(\text{TAN } \theta_i \text{CTN } \theta) = \frac{\csc^3 \theta \text{ TAN } \theta_i}{\sqrt{1 - \text{TAN}^2\theta_i \text{CTN}^2\theta}}$$

is increasing as θ_i increases. But remembering that $\theta_o > \theta_i$ for $i = 1, \cdots p$, we see that

$$\frac{d}{d\gamma(\theta)} \{B_U(\theta)-B_L(\theta)\} < \frac{d}{d\gamma(\theta)}\{B_{OU}(\theta)-B_{OL}(\theta)\}$$

for $\theta > \theta_o$. Hence it follows that there exists a $\hat{\theta}$ such that $0 < \hat{\theta} < \Theta$ and that

$$B_U(\theta) - B_L(\theta) > B_{OU}(\theta) - B_{OL}(\theta) \quad \text{for} \quad 0 \leq \theta < \hat{\theta}$$

$$B_U(\theta) - B_L(\theta) < B_{OU}(\theta) - B_{OL}(\theta) \quad \text{for} \quad \hat{\theta} < \theta \leq \Theta .$$

Thus from a lemma which appears in Appendix A of Landau and Slepian's paper we see that

$$\int_0^{\Theta} h(r,\cos\theta) \ \{B_U(\theta)-B_L(\theta)\} - \{B_{OU}(\theta)-B_{OL}(\theta)\} \ d\gamma(\theta)$$

$$> h(r,\cos\hat{\theta}) \int_0^{\Theta} \ \{B_U(\theta)-B_L(\theta)\} - \{B_{OU}(\theta)-B_{OL}(\theta)\} \ d\gamma(\theta)$$

$$> 0$$

Hence we have proved that h is always smaller when D is formed by the intersection of a finite number of halfspaces and the regions have the same content. The result for any convex, radially invariant D follows since any such D can be uniformly approximated by a sequence of sets, each of which is the intersection of a finite number of halfspaces.

As we can see, the method of proof for $n = 3$ depends only upon the ability to compare the derivatives of the boundaries $B_{iU}(\theta)$ and $B_{iL}(\theta)$ for $i = 1, \cdots p$. It is not known, however, if this method can be extended to higher dimensions in the incoherent case, or for that matter, for the coherent case in any method different from Landau and Slepian's.

3.3. Another Expression for P_c in Terms of $V(\theta)$.

We have already noted that conjecture 3 being true implies conjecture 2 is true and that conjecture 6 being true implies conjecture 5 is true. We are interested now in the converse statements.

If we define

$$F(\gamma) = \int_0^\infty f(r^2 + 1 - 2r\gamma)\, r^{n-1}\, dr$$

where f is the monotone function of conjecture 3, then we may express the probability of being correct P_c for the coherent case as

$$P_c = \frac{1}{M} \sum_{i=1}^{M} \int_{R_i} F(\langle \underline{x}, \underline{s}_i \rangle)\, dS(\underline{x})$$

Note that F is monotone in γ since $f(r^2 + 1 - 2r\gamma)$ is monotone in γ for each r. Let us define $P_{c/i}$ by

$$P_{c/i} = \frac{1}{M} \int_{R_i} F(\langle \underline{x}, \underline{s}_i \rangle)\, dS(\underline{x})$$

and without loss of generality let $\underline{s}_1 = (1, 0, \cdots 0)$. Then using the spherical coordinates of Appendix I

$$P_{c/1} = \frac{1}{M} \int_0^\pi F(\cos\theta)\, dv_1(\theta)$$

where $v_i(\theta)$ is given by

$$v_i(\theta) = \int_{R_i \cap C_{\underline{s}_i}(\theta)} dS(\underline{x})$$

Substituting into the original expression for P_c then gives

$$P_c = \frac{1}{M} \int_0^{\pi} F(\cos\theta)\, dv_c(\theta)$$

where $$v_C(\theta) = \sum_{i=1}^{M} \int_{R_i \cap C_{\underline{s}_i}(\theta)} dS(\underline{x})$$

$$= \int_{\bigcup_{i=1}^{M} \{R_i \cap C_{\underline{s}_i}(\theta)\}} dS(\underline{x})$$

$$= \int_{\bigcup_{i=1}^{M} C_{\underline{s}_i}(\theta)} dS(\underline{x})$$

That is, $v_c(\theta)$ is the same function as in conjecture 3' and which is plotted in Figure 3.

Similarly for the incoherent case let us define $H(\gamma)$ by

$$H(\gamma) = \int_0^{\infty} h(r,\gamma)\, r^{2n-1}\, dr$$

where $h(r,\gamma)$ is the monotone function in conjecture 6. Then we have

$$P_c = \int_0^{\pi/2} H(\cos\theta)\, dv_I(\theta)$$

where H is monotone increasing and $v_I(\theta)$ is given by

$$v_I(\theta) = \int_{\bigcup_{i=1}^{M} C_{\underline{s}_i}(\theta)} dS(\underline{x}) \ .$$

Thus $v_I(\theta)$ is the same function as in conjecture 6 and which is plotted in Figure 3. Note that neither F nor H depends on the signal set.

Integrating by parts, we get for the coherent case

$$\begin{aligned} P_c &= \frac{1}{M} \int_0^{\pi} F(\cos\theta)\, dv_C(\theta) \\ &= \frac{1}{M} F(\cos\theta)\, v_C(\theta) \Big|_0^{\pi} + \int_0^{\pi} F'(\cos\theta) \sin\theta\, v_c(\theta)\, d\theta \\ &= \frac{1}{M} F(1)\, s_P + \frac{1}{M} \int_0^{\pi} F'(\cos\theta) \sin\theta\, v_c(\theta)\, d\theta \end{aligned}$$

where s_P is the surface content of the unit sphere in E^n. Similarly for the incoherent case we get

$$P_c = \frac{1}{M} H(1)\, s_{PI} + \int_0^{\pi/2} H'(\cos\theta) \sin\theta\, v_I(\theta)\, d\theta$$

where s_{PI} is the surface content of the unit sphere in C^n. Expressions for s_p and s_{PI} as a function of n are given in Appendix II.

Hence if we consider two signal sets, the first denoted by $^{(1)}$ and the second by $^{(2)}$ we have for the coherent case

$$P_c^{(1)} - P_c^{(2)} = \frac{1}{M}\int_0^{\pi} F'(\cos\theta)\sin\theta\{v_C^{(1)}(\theta) - v_C^{(2)}(\theta)\}\,d\theta$$

and for the incoherent case

$$P_c^{(1)} - P_c^{(2)} = \frac{1}{M}\int_0^{\pi/2} H'(\cos\theta)\sin\theta\{v_I^{(1)}(\theta) - v_I^{(2)}(\theta)\}\,d\theta$$

where F' and H' are both positive. Hence we see that for a particular F or H it is not necessary that $v^{(1)}(\theta) > v^{(2)}(\theta)$ for all θ in order that $P_c^{(1)} > P_c^{(2)}$. However, the inability to describe the class of possible F's and H's makes it difficult to say much more.

APPENDIX I.

GENERALIZED n DIMENSIONAL SPHERICAL COORDINATES

Although there are many ways of generalizing spherical coordinates to higher dimensions, the two that follow are sufficient for our purposes.

A. Spherical Coordinates in E^n.

Consider $\underline{r} = (r_1, r_2, \cdots, r_n)$ with $\|r\| = r$. Then the transfunction

$$r_1 = r \cos \theta_1$$

$$r_2 = r \sin \theta_1 \cos \theta_2$$

$$\vdots$$

$$r_j = r \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{j-1} \cos \theta_j$$

$$\vdots$$

$$r_{n-1} = r \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-2} \cos \theta_{n-1}$$

$$r_n = r \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-2} \sin \theta_{n-1}$$

$$\theta \le \theta_i < \pi \qquad i = 1, 2, \cdots n-2$$

$$-\pi \le \theta_{n-1} < \pi$$

changes $dV(\underline{r})$ into

$$dV(\underline{r}) = \{\sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \cdots \sin_{n-2} d\theta_1 d\theta_2 \cdots d\theta_{n-2}\} r^{n-1} dr$$

$$= dS\ r^{n-1}\ dr$$

where dS can be defined as the expression in brackets. This formula can be verified by inductions by first noticing that for $n = 2$ it yields the circular coordinates

$$r_1 = r \cos \theta_1$$

$$r_2 = r \sin \theta_1 \qquad -\pi \le \theta_1 < \pi$$

$$dV = d\theta_1\ rdr$$

If it is true for $n = k$, then apply it to the last k coordinates for $n = k+1$ with $r^2 = \sum_{i=2}^{k+1} r_i^{\ 2}$. Then applying formula for $n = 2$ to r_1 and r yields the desired result.

B. Spherical Coordinates in C^n.

Consider $\underline{r} = (r_1 e^{j\varphi_1}, r_2 e^{j\varphi_2}, \cdots r_n e^{j\varphi_n})$ with $r_i > 0$ and $r = \|r\|$. Then the transfunctions

$$r_1 = r \cos \theta_1$$

$$r_2 = r \sin \theta_1 \cos \theta_2$$

$$\vdots$$

$$r_j = r \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{j-1} \cos \theta_j$$

$$\vdots$$

$$r_{n-1} = r \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-2} \cos \theta_{n-1}$$

$$r_n = r \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-2} \sin \theta_{n-1}$$

$$\theta \le \theta_i < \pi/2 \qquad i = 1, 2, \cdots n-1$$

yields

$$dV(\underline{r}) = d\varphi_1 d\varphi_2 \cdots d\varphi_n \, r_1 dr_1 \, r_2 dr_2 \cdots r_n dr_n$$

$$= \{d\varphi_1 d\varphi_2 \cdots d\varphi_n \sin^{2n-3} \theta_1 \sin^{2n-5} \theta_2 \cdots \sin \theta_{n-1} \cos \theta_1 \cos \theta_2 \cdots \cos \theta_{n-1} \, d\theta_1 d\theta_2 \cdots d\theta_{n-1}\} r^{2n-1} dr$$

$$= dS \, r^{2n-1} \, dr$$

where dS can be defined as the expression inside the brackets. This formula may also be verified by induction.

APPENDIX II.

SURFACE CONTENT OF n DIMENSIONAL SPHERES

A formula for the surface content for an n dimensional sphere can be readily found by a trick due to Courant [2] which also appears in Coxeter [3].

A. Surface Content in E^n.

Let $\underline{r} = (r_1, r_2, \cdots r_n)$. Let us integrate the function $e^{-\|\underline{r}\|^2}$ over all E^n. Thus

$$s_P \cdot \int e^{-\|r\|^2} r^{n-1}\, dr = \prod_{i=1}^{n} \left[\int_{-\infty}^{\infty} e^{-r_i^2}\, dr_i \right]$$

or the surface content s_P is given by

$$s_P = 2 \frac{\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)}$$

B. Surface Content in C^n.

Using the same trick yields that the surface content s_{PI} is given by

$$s_{PI} = 2 \frac{\pi^n}{\Gamma(n)}$$

or the same value as s_P in E^{2n}.

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