Hele-Shaw Flow Near Cusp Singularities

Thesis by

Sha Chang

In Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

2013

(Defended September 24, 2012)

Acknowledgements

First, I would like to thank my advisor, Nikolai Makarov, for his introduction and guidance on my study and research during these four years, and for his suggestions and help in this thesis. I always get inspired after talking to him. I would also like to thank Seung-Yeop Lee for generously offering his help whenever I seek for it. He contributed a lot of time and efforts to my research. Without their help, I could not finish this thesis.

Second, I would like to thank Barry Simon and Alexander Kechris for being on my thesis committee.

Third, I would like to thank all my friends for making my life at Caltech colorful.

Last but not least, I would like to thank my parents for their endless love and support. I would also like to thank my husband, for his love and being supportive all the time.

Abstract

This thesis discusses the radial version of the Hele-Shaw problem. Different from the channel version, traveling-wave solutions do not exist in this version. Under algebraic potentials, in the case that the droplets expand, in finite time, cusps will appear on the boundary and classical solutions may not exist afterwards. Physicists have suggested that for $(2p+1, 2)$-cusps, that near cusp singularities of Hele-Shaw flow, after scaling X, Y by some powers of time t respectively, the main part of $Y(X, t)$ is a one-parameter family and does not depend on time t. They have also suggested that the solutions of the Hele-Shaw problem are connected with dispersionless KdV (dKdV) hierarchy. In this study, we rigorously proved that this is the case for (3,2)-cusps when the droplets are simply connected and the external potentials are algebraic. We gave exact solutions and showed that the main parts of the exact solutions are some special solutions of the dispersionless string equation. More over, borrowed from the physical paper [15] with a little more details, we showed the arguments of how these special solutions are related to dKdV hierarchy.

Contents

Chapter 1

Introduction

1.1 Background

In 1898 an English engineer, Henry S. Hele-Shaw, studied liquid flow in a channel [4]. To make the separation interface of laminar flow and turbulent visible, he suggested injecting air into the system. The motion of the interface of the air and the liquid is called Hele-Shaw flow. Hele-Shaw problem is to discover the equations that describe Hele-Shaw flow. In 1958, P. G. Saffman and Sir G. I. Taylor discovered a one-parameter family of exact solutions of Hele-Shaw problem—the so-called Saffman-Taylor fingers [11]. The Saffman-Taylor fingers are traveling-wave solutions, and the shape of the fingers does not depend on time t.

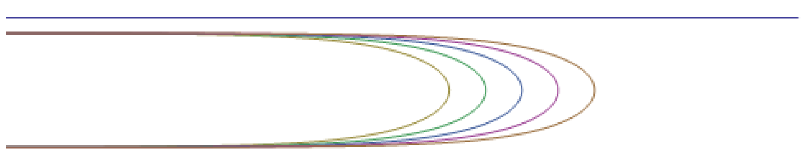

Figure 1.1: Saffman-Taylor fingers.

In this study, we discussed the radial version of Hele-Shaw problem. The radial version is an important version of Hele-Shaw problem and it has attracted a lot of attention in the past few decades [1], [2], [3], [5], [6], [7], [8], [9], [10], [12], [13], [14], [15], [16]. The radial version is to consider a flow between two parallel horizontal planes with a narrow gap in between. It is different from the channel version, since traveling-wave solutions do not exist. Under algebraic potentials, in the case that droplets expand, in finite time, cusps will appear on the boundary, and classical solutions may not exist after that. In [15], R. Teodorescu, P. Wiegmann and A. Zabrodin have suggested that for $(2p + 1, 2)$-cusps, near cusp singularities, after scaling X, Y by some powers of t respectively, the main part of $Y(X, t)$ is a one-parameter family, and has a fingerlike shape that does not depend on time t. In this study, we proved its correctness rigorously for (3,2)-cusps when the droplets are simply connected and the external potential is algebraic.

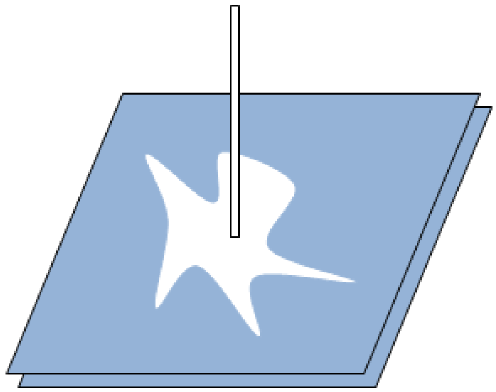

Figure 1.2: Radial version of Hele-Shaw problem.

Physicists have suggested that Hele-Shaw problem is related to integrable systems [7], [8], [15]. Regular Hele-Shaw flow is related to dispersionless 2D Toda hierarchy [8]. In the case that droplets contract, near double points, it is connected to dispersionless AKNS hierarchy [7]. In the case that droplets expand, near cusplike singularities, it is linked to dispersionless Kortweg–de Vries(dKdV) hierarchy [15]. In this study, we showed that the solutions of Hele-Shaw problem are related to a group of special solutions of dispersionless string equation. These special solutions are conjectured as the main parts of Hele-Shaw flow. We added a little more details in the arguments borrowed from [15] and showed how these special solutions are connected with dispersionless Kortweg–de Vries(dKdV) hierarchy. Since after scaling X, Y by some powers of t respectively, these special solutions are one-parameter families, do not depend on time t, and KdV equation describes the motion of solitons, it is not wholly unexpected that these solutions has connection with dKdV hierarchy.

Figure 1.3 is an example of Hele-Shaw flow growing into a deltoid with three (3,2)-cusps.

1.2 Main Result

In this section, we will state the main theorem, while terminologies such as local droplets, algebraic Hele-Shaw potential, etc. will be explained later.

Suppose we have a chain of local droplets S_t under an algebraic Hele-Shaw potential, and $D_t =$

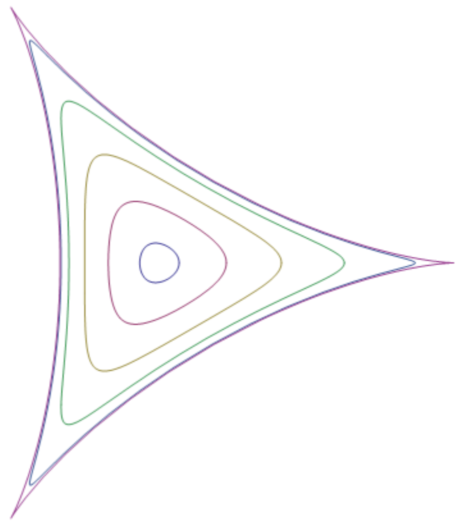

Figure 1.3: Hele-Shaw flow growing into a deltoid.

$\hat{\mathbb{C}} \setminus S_t$ are simply connected. Assume that there is a (3,2)-cusp on ∂D_{t_*} (t_* is normalized to 0). Take the cusp as the origin and the tangent line as X-axis, then we get Cartesian coordinates $X = X(t, \phi)$ and $Y = Y(t, \phi)$, where ϕ are Green's coordinates and t is time.

Theorem 1.2.1. *Suppose the cusp on $\partial D(0)$ is a (3,2)-cusp, without lost of generality, we could assume $X(0, \phi)$, $Y(0, \phi)$ satisfy*

$$\begin{aligned} X(0, \phi) &= \phi^2 + O(\phi^3), \\ Y(0, \phi) &= c\phi^3 + O(\phi^4), \end{aligned} \tag{1.2.1}$$

with $c < 0$ near the cusp, then $\exists \epsilon > 0$, s.t. $\forall t \in (-\epsilon, 0)$, the local boundary of $D(t)$ near the tip is analytic and its Cartesian coordinates have the following form:

$$\begin{aligned} X(t, \phi) &= \sqrt{\tfrac{4}{-3c}}\sqrt{-t} + \phi^2 + o(\sqrt{-t} + \phi^2), \\ Y(t, \phi) &= -\sqrt{-3c}\sqrt{-t}\phi + c\phi^3 + o(|\sqrt{-t}\phi| + |\phi|^3), \end{aligned} \tag{1.2.2}$$

in particular,

$$Y(X,t) = \pm c \left(X - \sqrt{\frac{4}{-3c}}\sqrt{-t} \right)^{1/2} \left(X + \sqrt{\frac{1}{-3c}}\sqrt{-t} \right) + o\left(\left(X - \sqrt{\frac{4}{-3c}}\sqrt{-t} \right)^{3/2} + (-t)^{3/4} \right). \tag{1.2.3}$$

If we divide X, Y by $(-t)^{1/2}$, $(-t)^{3/4}$ respectively, we get

$$\tilde{Y}(\tilde{X}, t) = \pm c \left(\tilde{X} - \sqrt{\frac{4}{-3c}} \right)^{1/2} \left(\tilde{X} + \sqrt{\frac{1}{-3c}} \right) + o\left(\left(\tilde{X} - \sqrt{\frac{4}{-3c}} \right)^{3/2} + 1 \right).$$

Remark 1.2.2. *The main part of $\tilde{Y}(\tilde{X}, t)$ is a one-parameter family, does not depend on time t*

and its shape is a fingerlike curve. Figures 1.4–1.6 show the main part

$$\tilde{Y}(\tilde{X}) = \pm c \left(\tilde{X} - \sqrt{\frac{4}{-3c}} \right)^{1/2} \left(\tilde{X} + \sqrt{\frac{1}{-3c}} \right),$$

when $c = -0.5, -1, -2.$

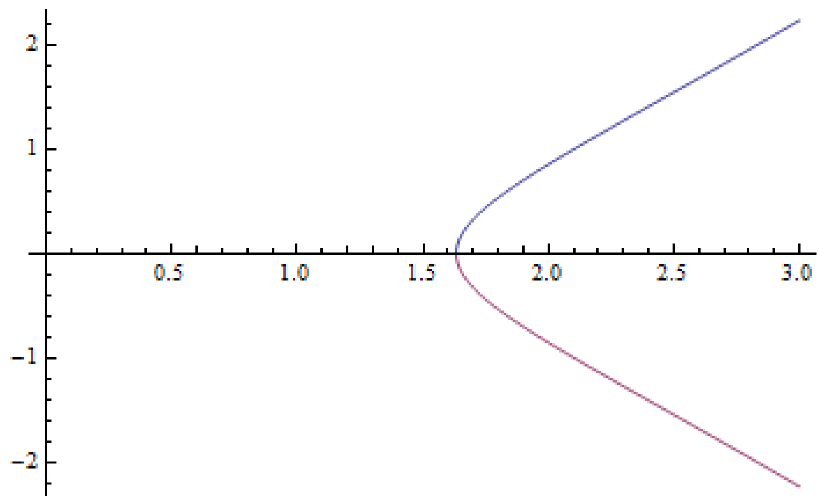

Figure 1.4: $\tilde{Y}(\tilde{X}) = \pm c \left(\tilde{X} - \sqrt{\frac{4}{-3c}} \right)^{1/2} \left(\tilde{X} + \sqrt{\frac{1}{-3c}} \right)$ for $c = -0.5$.

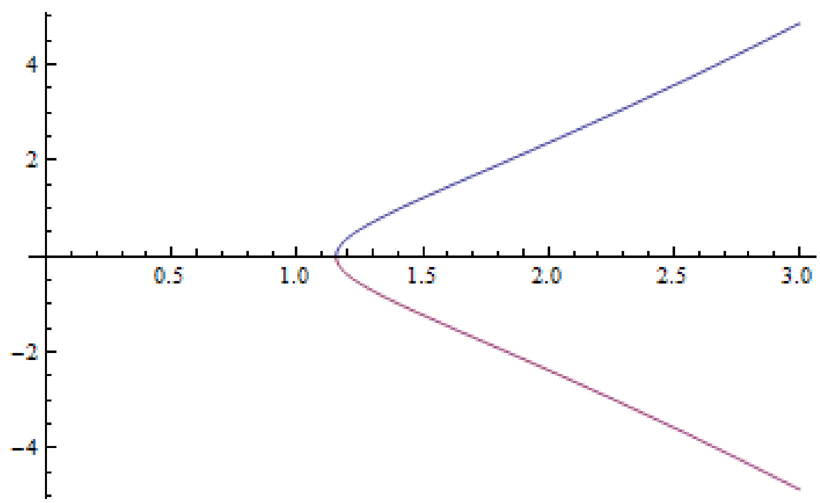

Figure 1.5: $\tilde{Y}(\tilde{X}) = \pm c \left(\tilde{X} - \sqrt{\frac{4}{-3c}} \right)^{1/2} \left(\tilde{X} + \sqrt{\frac{1}{-3c}} \right)$ for $c = -1$.

Hele-Shaw problem is more complicated to solve when $p > 1$. In this study, we solve the easiest case–(3,2)-cusps, but we believe the problems concerning higher-order cusps are still doable. Physicists have given the conjecture concerning $(2p+1, 2)$-cusps, and we will state this conjecture later.

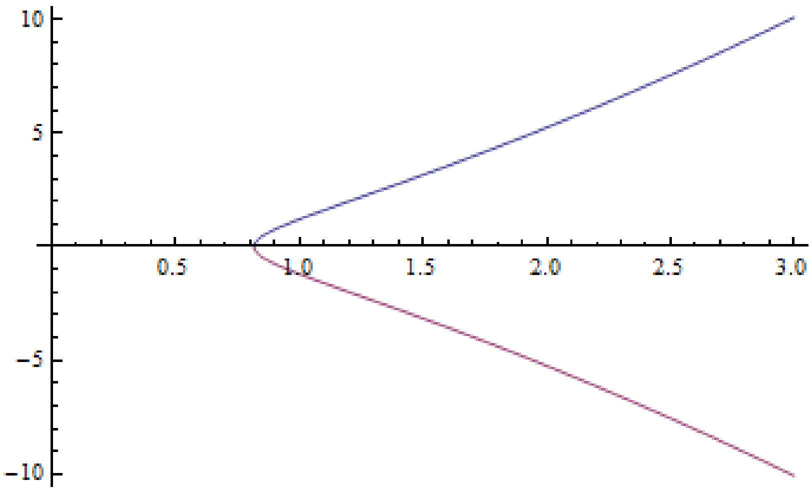

Figure 1.6: $\tilde{Y}(\tilde{X}) = \pm c\left(\tilde{X} - \sqrt{\frac{4}{-3c}}\right)^{1/2}\left(\tilde{X} + \sqrt{\frac{1}{-3c}}\right)$ for $c = -2$.

This thesis is organized as follows: chapters 2–4 review of some known facts. In chapter 2 we reviewed the equilibrium measure and the obstacle problem and stated that local droplets are the support of equlibrium measure in obstacle problems. In chapter 3, we reveiwed some facts about Schwarz function, and talked about Sakai's regularity theorem. Then discussed Laplacian growth problem and (classical and weak) solutions of Hele-Shaw problems. In algebraic and simply connected case, Hele-Shaw flow is a classical solution of Laplacian growth problem, and (local) Schwarz function exists on its boundary. In chapter 4, we talked about Sakai's no turbulence theorem concerning cusps. With this theorem, we know that cusps are laminar-flow points in Hele-Shaw flow. In chapter 5, we introduced Green's coordinates ϕ and derived the dispersionless string equation. In chapter 6, we proposed the main theorem concerning (3,2)-cusps and stated the conjecture concerning higher-order (2p+1,2)-cusps. In chapter 7, we discovered special solutions to dispersionless string equation. These special solutions are main parts of Hele-Shaw flow near cusp singularities. Besides, we showed the relation of these special solutions and dKdV hierarchy. In chapter 8, we described two methods to compute the boundary equations: exterior Faber transform and Löwner equation, and showed the procedure of solving Hele-Shaw problem with three examples using these two methods. In chapter 9, we gave the proof of the main theorem. In chapter 10, we discussed some other possible cases of Hele-Shaw problem.

Chapter 2

Obstacle Problem and Droplets

First of all, we need to formally define (local) droplets, and the motion of its boundary is Hele-Shaw flow. We will define (local) droplets by way of equilibrium measure and obstacle problem. So in this chapter we will review some known facts about equilibrium measure and obstacle problem, and then give the definition of local droplets as the support of equilibrium measure in an obstacle problem.

2.1 Equilibrium Measure

Assume that σ is a finite positive measure on complex plane $\mathbb{C}$ with compact support S_σ. Let

$$U^\sigma(z) = \int \log \frac{1}{|z-\zeta|} d\sigma(\zeta)$$

be the logarithmic potential and

$$I(\sigma) = \int U^\sigma d\sigma$$

be the logarithmic energy.

U^σ is lower semicontinuous, superharmonic on $\mathbb{C}$ and harmonic on $\mathbb{C} \setminus S_\sigma$. Now assume that there is an external field $Q : \mathbb{C} \to (-\infty, +\infty]$ which is lower semicontinuous and

$$\lim_{|z|\to\infty} (Q(z) - \log|z|) = +\infty.$$

Let

$$U_Q^\sigma = \int L_Q(z,\zeta) d\sigma(\zeta)$$

be the Q-potential, where

$$L_Q := \log \frac{1}{|z-\zeta|} + Q(z) + Q(\zeta),$$

and

$$I_Q(\sigma) = \int U_Q^\sigma d\sigma$$

be the Q-energy.

Theorem 2.1.1. *(Frostman) There exists a unique probability measure $\hat{\sigma} \equiv \hat{\sigma}_Q$ such that*

$$I_Q(\hat{\sigma}) = \gamma \equiv \gamma(Q).$$

The equilibrium measure has a compact support, in fact, $\exists M > 0, S_{\hat{\sigma}} \subset \{Q \leq M\}$. Also, $U_Q^{\hat{\sigma}} \leq \gamma$ on $S_{\hat{\sigma}}$; $U_Q^{\hat{\sigma}} \geq \gamma$ q.e.in $\mathbb{C}$.

Therefore, given an external field Q as above, the unique equilibrium measure $\hat{\sigma}$ exists, has compact support $S_{\hat{\sigma}}$, and on the support of $\hat{\sigma}$, the Q-potential $U_Q^{\hat{\sigma}}$ is equal to the Q-energy except on a set of capacity zero.

Now let Q: $\mathbb{C} \to (-\infty, +\infty]$ be a lower semicontinuous function and satisfy

$$\lim_{z \to +\infty} (-Q + t \log |z|) = -\infty, \forall t \tag{2.1.1}$$

in the following context.

By Frostman's theorem, the unique equilibrium measure exists for such Q. The Obstacle problem is as follows:
Let

$$Super(t) = \{v : \Delta v \leq 0, \liminf_{z \to \infty} [v + t \log |z|] > -\infty\}.$$

The obstacle problem is to find

$$V = Obs(-Q, t) = \inf\{v \in Super(t) : v \geq -Q\}.$$

If Q is an admissible potential satisfying $\forall A > 0, Q(z) \geq A \log |z|, |z| \to \infty$, then $Obs(-Q, t) = U^{\sigma_t} + c(Q, t)$, where σ_t is the equilibrium measure, $\sigma_t = t\sigma[Q/t]$, and $c(Q, t)$ is a constant. Since $Obs(-Q, t) = -Q$ q.e. on $S_t = supp\ \sigma_t$, $\sigma_t[Q] = -\Delta V_t = \Delta Q$, where $V_t = Obs(-Q, t)$. In fact, if $Q \in W^{2,p}$ for some $p > 1$ in a neighborhood of $S = S_t(Q)$, then $\sigma_t = \Delta Q \cdot \chi_S$.

Therefore, for Q an admissible potential as above, the solution of obstacle problem is the Q-potential with equilibrium measure σ_t, plus constant $c(Q, t)$. And on the support of σ_t, $\sigma_t = \Delta Q$.

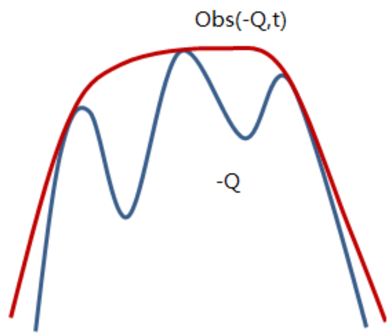

Figure 2.1: The solution $Obs(-Q, t)$ to the obstacle problem given an external field Q.

2.2 Local Droplets

In general, given a lower semicontinuous function Q, Q may not satisfy the growth condition near ∞, even it is as simple as a polynomial. But if we localize the field Q to a closed set $\Sigma \subset \mathbb{C}$ and write $Q_\Sigma = Q \cdot 1_\Sigma + \infty \cdot 1_{\mathbb{C}\setminus\Sigma}$, then Q_Σ is admissible and we could consider the obstacle problem with Q_Σ. Then the equilibrium measure exists and denote it as $\sigma[Q, \Sigma]$, and denote its support as $S[Q, \Sigma]$, i.e., for any Q lower semicontinuous, we could localize Q to some Σ and find the equilibrium measure.

Definition 2.2.1. *(Q-droplet): Let O be an open set in $\mathbb{C}$ and $Q \in \mathbb{C}^2(O)$. Let S be a compact set in O, then S is a Q-droplet if $\exists \Sigma \subset O$, s.t. S is a (Q, Σ) droplet, i.e., $\sigma = \Delta Q \cdot \chi_s$ and S=supp σ, where σ is the equilibrium measure.*

Remark 2.2.2. *S is a Q-droplet is equivalent to S is a (Q, S)-droplet, i.e., $U_S +$ constant $\equiv -Q$ on S. Also, If S is a droplet and $S \subset \Sigma$, then S is a (Q, Σ)-droplet iff $V_s \geq -Q$ on Σ.*

To study Hele-Shaw flow, we consider a family of local droplets, so we need a method to compare two local droplets. Let $t_1 < t_2$ and let S_2 be a Q-droplet, $t_{S_2} = t_2$. Then $S_1 := S_{t_1}(Q, S_2)$ is a Q-droplet. Given two Q-droplets S_{t_1} and S_{t_2}, if $S_{t_1} \subset S_{t_2}$ and $S_{t_1} = S_{t_1}(Q, S_{t_2})$, define this relation as "$\prec$", i.e., $S_{t_1} \prec S_{t_2}$. The symbol "$\prec$" satisfies the transition law: $S_1 \prec S_2 \prec S_3$ implies $S_1 \prec S_3$.

Denote the set $\{V = -Q\}$ as S^* and call it the coincidence set. The set S^* is a compact, nonempty set. V is harmonic in $\mathbb{C} \setminus S^*$ and supp $\sigma \subset S^*$, i.e., local droplets are subsets of S^*.

If $t_1 \leq t_2$, then $V_{t_1} \geq V_{t_2}$ and in particular, $S^*_{t_1} \subset S^*_{t_2}$. Let $S_1 \subset S_2$, then $S_1 \prec S_2$ iff $V_1 \geq V_2$. We say $\{S_t\}$, $0 < t < t_*$, is a chain of Q-droplets if for any $t_1 < t_2$, $S_{t_1} \prec S_{t_2}$, where $t = t_{S_t}$.

Chapter 3

Schwarz Function

Schwarz function is very useful when study local droplets. If there exists a local Schwarz function at a boundary point of a local droplet, Sakai's regularity theorem applies and this boundary point will be in one of the four categories: regular points, cusp points, double points, and degenerate points. This makes it easier to study Hele-Shaw flows. In this chapter we will first review some known facts about (local) Schwarz function, introduce Hele-Shaw flows and Sakai's Regularity theorem, then review Laplacian growth problem, define its classical solutions and weak solutions, and show Hele-Shaw flow is a weak solution of Laplacian growth problem.

3.1 Schwarz Function and Hele-Shaw Flow

Definition 3.1.1. *(local Schwarz function) Let Ω be an open set and $a \in \partial\Omega$, then*

$$F : \bar{\Omega} \cap \Delta \to \mathbb{C};$$

Δ is a disk centered at a, is a local Schwarz function at a if
(1) F is continuous;
(2) F is holomorphic in $\Omega \cap \Delta$;
(3) $F = \bar{z}$ on $\partial\Omega \cap \Delta$.

If S is a Q-droplet, then $U_s \equiv U^{\sigma_S} \in W^{2,p}$ and $\partial U_s(z) = \int_S \frac{d\sigma_s(\zeta)}{\zeta - z} \in W^{1,p} \subset C$, and $\partial U_s + \partial Q = 0$ on ∂S. On the contrary, if S is connected, $S = \textit{clos int } S$, and if $\partial U_S + \partial Q = 0$ on ∂S, then S is a Q-droplet.

In the following part of this chapter, we suppose the external potential is $Q(z) = |z|^2 - H(z)$, where $H(z)$ is harmonic in O, i.e., $\Delta H = 0$ in O, and $h = \partial H$ is analytic in O, where O is some open set containing the droplet. We call this potential Hele-Shaw potential. From the chain of droplets obtained under Hele-Shaw potential, we get Hele-Shaw flow.

If S is a Q-droplet, then $\bar{z} = h(z) - \partial U_s(z)$ on ∂S. So $\Omega = \mathbb{C} \setminus S$ has a local Schwarz function at every boundary point. Also, $\mathbb{C} \setminus S^*$ has a local Schwarz function at every boundary point, and $\sharp(S^* \setminus S) < \infty$.

3.2 Regularity Theorem

Now suppose S is a Q-droplet, $\Omega = \mathbb{C} \setminus S^*$, and a is a point on $\partial\Omega$, then $\exists$ a local Schwarz function $F : \bar{\Omega} \cap \Delta \to \mathbb{C}$, where Δ is a disk centered at a.

(1) If $\partial\Omega \cap \Delta$ is a proper subset of an analytic arc, then a is called a degenerate point.

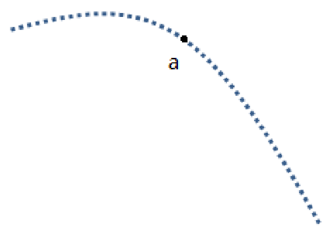

Figure 3.1: The point a is a degenerate point.

(2) If $a \in S^* \setminus S$, then a is called an isolated point.

Figure 3.2: The point a is an isolated point.

(3) If a is not degenerate nor isolated, and $\Delta \cap \Omega$ is a Jordan domain such that the conformal map $\phi : \mathbb{D} \to \Delta \cap \Omega$, $1 \mapsto a$, is analytic at 1 and satisfies $\phi'(1) = 0$, then a is a cusp point.

(4) If $\partial\Omega \cap \Delta$ consists of two analytic arcs, which are tangent to each other at a, then a is called a double point.

M. Sakai gives the following classification of points on a boundary having Schwarz function.

Theorem 3.2.1. *(Sakai's regularity theorem)[12] [13]: Let Ω be an open subset of the unit disk B_1 such that 0 is a nonisolated boundary point of Ω and let $\Gamma = (\partial\Omega) \cap B_1$. If there exists a Schwarz function of $\Omega \cup \Gamma$ in B_1, then, for some small $\delta > 0$, one of the following must occur:*

(1) The point 0 is a regular point.

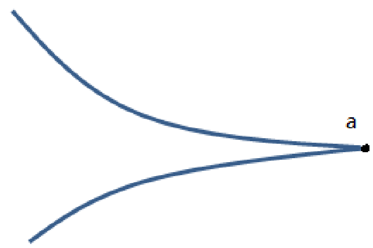

Figure 3.3: The point a is a cusp point.

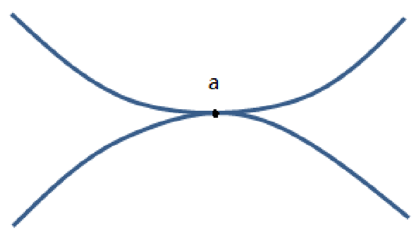

Figure 3.4: The point a is a double point.

(2a) The point 0 is a degenerate point.

(2b) The point 0 is a double point.

(2c) The point 0 is a cusp.

Remark 3.2.2. *For Hele-Shaw flows, if* $0 \in S$*, then* 0 *is not a degenerate point, since the area of an analytic arc is zero so* $0 \notin S$*, contradiction.*

Therefore, for Hele-Shaw flows, if there exists a local Schwarz function near 0, then 0 is a regular point, a double point, or a cusp point.

3.3 Laplacian Growth Problem

Definition 3.3.1. *(a smooth family of curves): Suppose* $\{\Gamma_t\}$ *is a family of simple curves.* $\{\Gamma_t\}$ *is smooth if for any* z *on some curve,* $\exists$ *a local diffeomorphism* $\gamma = \gamma(s,t)$ *from a rectangle in* (s,t)*-plane to a neighborhood of* z *and maps the horizontal segments* $\{(s,t) : s_1 \leq s \leq s_2\}$ *into* Γ_t*,* $\forall t$*.*

Definition 3.3.2. *(normal velocity): Suppose* $\{\Gamma_t\}$ *is a smooth family of curves.* $v_n = v_n(z)$*,* $z \in \Gamma_t$*, is the normal velocity at* z *if*

$$v_n = (\dot{\gamma}, n),$$

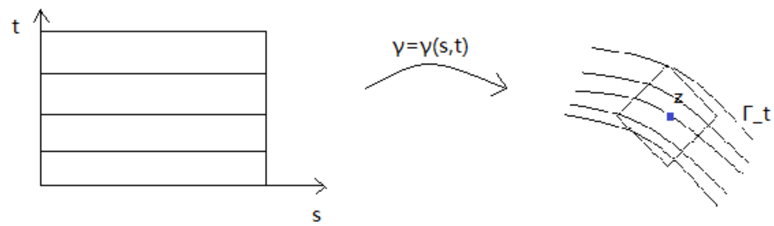

Figure 3.5: A smooth family of curves.

where n *is a unit normal vector at* $z \in \Gamma_t$ *and* $\dot{\gamma} = \frac{\partial}{\partial t}\gamma$.

Remark 3.3.3. $\gamma' = \frac{\partial}{\partial s}\gamma$. *We know that* $n \perp \gamma'$. *(Here* γ' *is taken as a vector because the complex plane* $\mathbb{C}$ *is isomorphic to the vector space* R^2 *with one to one correspondence* $x + iy$ *to* (x, y).*) Suppose* $\gamma = (\gamma_1, \gamma_2)$, $\gamma' = (\gamma_1', \gamma_2')$, *then*

$$n = \frac{1}{|\gamma'|}(-\gamma_2', \gamma_1'),$$

and

$$v_n = (\dot{\gamma}, n) = \frac{1}{|\gamma'|}\Im(\dot{\gamma}\bar{\gamma}').$$

Normal velocity is well-defined, i.e., given different local diffeomorphisms $\gamma, \tilde{\gamma}$ around $z \in \Gamma_t$, $v_n = \tilde{v}_n$. Since $\gamma, \tilde{\gamma}$ are two local diffeomorphisms around $z \in \Gamma_t$, we can find a differentiable function $\sigma = \sigma(s, t)$, s.t.

$$\tilde{\gamma}(\tilde{s}, t) = \gamma(\sigma(s, t), t),$$

then

$$\dot{\tilde{\gamma}} = \gamma'\dot{\sigma} + \dot{\gamma}. \tag{3.3.1}$$

Since $\gamma' \perp n$, $(\dot{\tilde{\gamma}}, n) = (\dot{\gamma}, n)$.

Definition 3.3.4. *(Green function): Suppose* Ω *is an unbounded simply connected domain and* $\Gamma = \partial\Omega$ *is a smooth curve.* $\infty \in \Omega$. $G(z, \infty; \Omega)$ *is a Green function in* Ω *if* G *solves the Dirichlet problem:*

$$\begin{cases} \Delta G = 0 \ \text{in}\ \Omega, \\ G = 0 \ \text{on}\ \Gamma, \\ G \to \log|z| \ \text{as}\ |z| \to +\infty. \end{cases} \tag{3.3.2}$$

Definition 3.3.5. *(Laplacian growth problem): Suppose* $\{D(t)\}$ *is a family of unbounded simply connected domains.* $\{\Gamma(t) = \partial D(t)\}$ *is a smooth family of curves. If* $\exists G(z, \infty; D(t))$ *for each* $D(t)$, *s.t.*

$$v_n = \frac{\partial G}{\partial n} = n \cdot \nabla G$$

on each $\Gamma(t)$, then $\{D(t)\}$ is a solution to this Laplacian growth problem.

3.4 Classical Solutions of Hele-Shaw Problem

Theorem 3.4.1. *Suppose $\{D(t)\}$ is a smooth family of domains and $D(t)$ is simply connected for each t, then by Riemann mapping theorem, $\exists f(\zeta,t) : \hat{\mathbb{C}} \setminus \bar{\mathbb{D}} \to D(t)$, conformal and univalent, and $\{D(t)\}$ is a (classical) solution of Laplacian growth problem if f satisfies*

$$\Re[\dot{f}(\zeta,t)\overline{\zeta f'(\zeta,t)}] = 1$$

on $|\zeta| = 1$.

Proof. Suppose $\{D(t)\}$, a family of unbounded simply connected domains, is a solution to Laplacian growth problem. Then $\exists G(z,t)$, s.t. $v_n = \frac{\partial G}{\partial n} = n \cdot \nabla G$. Since $G(z,t)$ is harmonic in $D(t)$, $\exists$ a holomorphic function $W(z,t)$ in $D(t)$, s.t. $\Re W(z,t) = G(z,t)$. Also, since $D(t)$ is simply connected, $\exists$ conformal and univalent functions $f_t(\zeta) = f(\zeta,t) : \hat{\mathbb{C}} \setminus \bar{\mathbb{D}} \to D(t)$. then $G(f(\zeta,t),t)$ is a solution of Dirichlet problem in $\hat{\mathbb{C}} \setminus \bar{\mathbb{D}}$. So $W(f(\zeta,t),t) = \log\zeta$. Since the normal vector at $\zeta \in \partial\mathbb{D}$ is $n = \zeta\frac{f'}{|f'|}$ and $\frac{\partial W}{\partial z} = \overline{\nabla G}$, we know $v_n = n \cdot \nabla G = \Re(\zeta\frac{\partial W}{\partial z}\frac{f'}{|f'|})$. Also, $\frac{\partial W}{\partial z}f' = \frac{1}{\zeta}$, so $v_n = \frac{1}{|f'|}$. On the other hand, $v_n = \Re(\dot{f}\overline{\zeta\frac{f'}{|f'|}})$, so we have $\Re[\dot{f}(\zeta,t)\overline{\zeta f'(\zeta,t)}] = 1$, which is known as Polubarinova-Galin equation. □

Corollary 3.4.2. *we could derive Löwner-Kufarev equation by using Schwarz-Poisson formula from Polubarinova-Galin equation as follows:*

$$\dot{f}(\zeta,t) = \zeta f'(\zeta,t)\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|f'(e^{i\theta},t)|^2}\frac{e^{i\theta}+\zeta}{e^{i\theta}-\zeta}d\theta,$$

where $\zeta \in \hat{\mathbb{C}} \setminus \mathbb{D}$.

Proof. Since $\Re[\dot{f}(\zeta,t)\overline{\zeta f'(\zeta,t)}] = 1$ on $|\zeta| = 1$,

$$\Re[\frac{\dot{f}(\zeta,t)}{\zeta f'(\zeta,t)}] = \frac{1}{|f'(\zeta,t)|^2},$$

then by Schwarz-Poisson formula, we get

$$\dot{f}(\zeta,t) = \zeta f'(\zeta,t)\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|f'(e^{i\theta},t)|^2}\frac{e^{i\theta}+\zeta}{e^{i\theta}-\zeta}d\theta,$$

where $\zeta \in \hat{\mathbb{C}} \setminus \mathbb{D}$. □

3.5 Weak Solutions of Hele-Shaw Problem

Let C_t be a smooth family of curves such that their interiors K_s increases. then

$$\frac{d}{dt}\int\int_{K_t} f dA = \int_{C_t} f v_n dl.$$

Definition 3.5.1. *(weak solutions) Suppose there are an increasing family of compact sets $K_t = \text{supp } \chi_t$, $\chi_t = \chi_{K_t}$. If for all $f \in C(\mathbb{R}^2)$, the function $t \mapsto \int\int_{K_t} f\Delta Q dA$ is absolutely continuous and for a.e. t we have*

$$\frac{d}{dt}\int\int_{K_t} f\Delta Q dA = \int_{C_t} f d\omega_\infty^{(t)};$$

in short, $\frac{d}{dt}\sigma_t = \omega_\infty^{(t)}$, then we call $\{K_t\}$ a weak solution of Laplacian growth problem.

If $\{S_t\}$ is an increasing family of Q-droplets, then it is a chain. Furthermore, for all t we have $\frac{d}{dt}\chi_{S_t} = \omega_\infty^{(t)}$ in $[L^\infty \cap C]^*$. So the chain of Q-droplets is a weak solution of Laplacian growth problem.

Chapter 4

Cusps

From Sakai's regularity theorem, boundary points of Hele-Shaw flows are regular points, double points, or cusps. In this thesis, we focus on cusps. This chapter introduces maximal/nonmaximal cusps, laminar-flow points and Sakai's theorem on the revolution of cusps. This Sakai's theorem tells that cusps are laminar-flow points, and no tubulents appear near the cusps.

4.1 Maximal and Nonmaximal Cusps

First, a Q-droplet S is maximal if for any $\tilde{S}$, s.t. $S \prec \tilde{S}$, $S = \tilde{S}$. So S is not maximal iff there is a neighborhood U of $\mathcal{P}(S)$ such that $V_S + Q > 0$ in $U \setminus \mathcal{P}(S)$. If S is maximal, there is a cusp on the outer boundary.

Recall the definition of cusps: If a is not degenerate, and $\Delta \cap \Omega$ is a Jordan domain such that the conformal map $f : \hat{\mathbb{C}} \setminus \mathbb{D} \to \Delta \cap \Omega$, $1 \mapsto a$, is analytic at 1 and satisfies $f'(1) = 0$, then a is a cusp on the boundary.

Definition 4.1.1. *(nonmaximal cusp) A cusp $a \in \partial\mathcal{P}(S)$ is not maximal if there is a neighborhood $N = N(S)$ such that $V > -Q$ in $N \setminus \mathcal{P}(S)$.*

Remark 4.1.2. *The droplet S is maximal iff all cusps on the outer boundary are maximal.*

Suppose $\Phi : \mathbb{C}_+ \to \Omega := \hat{\mathbb{C}} \setminus \mathcal{P}(S)$, $i \to \infty$, $0 \to a$ is the conformal map, and $\Phi(z) = z^2 + a_3 z^3 + ... + (a_\nu + ib_\nu)z^\nu + ...$, $a_j, b_j \in \mathbb{R}$, $\nu \geq 3$ is the first index such that the corresponding coefficient is not real. Let us consider the case when ν is odd, then the power series is univalent in $B(0,\delta) \cap \mathbb{C}_+$ iff $b_\nu > 0$. Let $u = \Re\Phi$ and $v = \Im\Phi$, then on the boundary, $u \sim x^2$ and $v \sim b_\nu x^\nu$, so $v = \pm b_\nu u^{\nu/2} + ...$, $(u \geq 0)$. We call such cusps $(\nu, 2)$-cusps.

It is known that $(3,2)$-cusps are maximal, and $(5,2)$-cusps are not maximal. In general, when $\nu \equiv 1 \mod 4$, $(\nu, 2)$-cusps are not maximal, and when $\nu \equiv 3 \mod 4$, $(\nu, 2)$-cusps are maximal, i.e.,

$(2p+1,2)$-cusps are maximal if p is odd and nonmaximal if p is even.

4.2 Sakai's No Turbulence Theorem

Definition 4.2.1. *(stationary point) Suppose* $z_0 \in \partial\Omega_0$*. If* $\exists t > 0$*, s.t.* $z_0 \in \partial\Omega_t$*, then* $\forall 0 < s < t$*,* $z_0 \in \partial\Omega_s$*. We call such* z_0 *a stationary point.*

Definition 4.2.2. *(laminar-flow point) Suppose* z_0 *is not a stationary point on* $\partial\Omega_0$*, if* $\exists$ *a small disk* B *around* z_0*, s.t.* $\partial\Omega_t \cap B$ *is regular real analytic simple arc for any small* $t > 0$*, we call such* z_0 *a laminar-flow point.*

Theorem 4.2.3. *(Sakai's no turbulence theorem)* [14] *Suppose there exists a Schwarz function around* $a \in \partial\Omega_0$*, if* a *is a regular point or a cusp, then* a *is a laminar-flow point for any injection point* $p_0 \in \Omega_0$*.*

Chapter 5

Green's Coordinates and Dispersionless String Equation

In the main theorem, we will discuss simply connected droplets, in which case conformal maps from the complements of droplets to the complement of unit disk exist, so we can use Green's coordinates around the cusp and derive a PDE from D'Arcy's law. The main arguments are borrowed from physical papers[7], [8], [15], so we will follow physical literature and call it dispersionless string equation. Solutions to dispersionless string equation may correspond to some droplets and so we could solve the Hele-Shaw problem. In this chapter, we will first introduce Green's coordinates and then derive the dispersionless string equation from D'Arcy's law.

5.1 Green's Coordinates

Now, suppose we have a chain of simply connected Q-droplets $\{S_t\}$ and $D(t) = \hat{\mathbb{C}} \setminus S(t)$, and $\{D(t)\}$ is a classical solution of Hele-Shaw problem for $t_* \le t < 0$ and $D(0)$ has a cusp. For each t, there is a conformal map $g_t : D(t) \to \hat{\mathbb{C}} \setminus \bar{D}$ and suppose G_t is the Green function. Suppose the cusp of $D(0)$ is at z_0 and $g_0(z_0) = 1$. Let $z_t \in \Gamma_t$ be the point s.t. $g_t(z_t) = 1$. We call z_t the tip of the droplet. Let U_t be a small disk centered at z_t in $D(t)$, s.t. U_t is simply connected in $D(t)$. Let g, G, U, ϕ denote g_t, G_t, U_t, ϕ_t respectively.

Since $-G$ is harmonic in U, we can choose a branch of $\tilde{G}$, s.t. $\phi = \tilde{G} - iG$ is analytic and single-valued in U. Then

$$\bar{\phi} = \tilde{G} + iG \Rightarrow \phi - \bar{\phi} = -2iG \Rightarrow \partial\phi = \partial(\phi - \bar{\phi}) = -2i\partial G.$$

$G = \log|g|$. Then since

$$\phi = \tilde{G} - i\log|g| = -i(\log|g| + i\tilde{G})$$

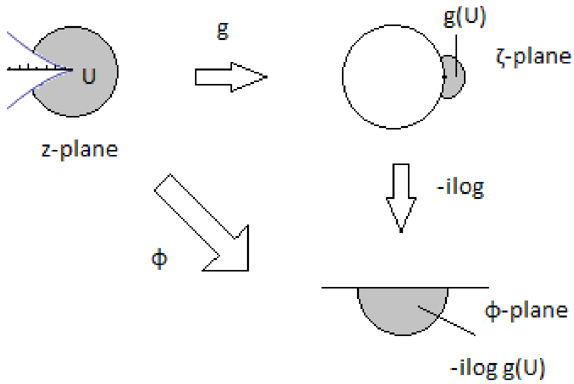

Figure 5.1: Relation of Green's coordinates ϕ and conformal map g.

is analytic, we can choose $\tilde{G}$ =Arg g, so that $\phi = -i \log g$ is single valued in U and $g = e^{i\phi}$.

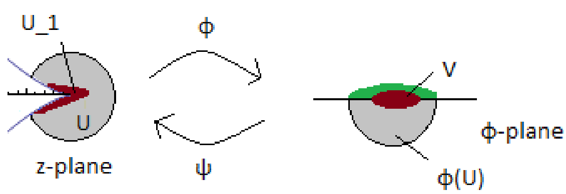

Figure 5.2: Green's coordinates ϕ and its inverse map ψ.

Since $G > 0$ in U, we have $\phi(U) \subset \mathbb{C}_-$ ($\mathbb{C}_-$ is the lower half-plane of $\mathbb{C}$) and $\phi(\Gamma \cap \bar{U}) \subset R$. Define

$$\psi = \phi^{-1} : \phi(U) \to U;$$

then ψ is analytic in $\phi(U)$. Then ψ can be extended analytically into a stripe in $\mathbb{C}_+$. We can now take a symmetric subset V with respect to the real line R, s.t.ψ is analytic in V. Let $U_1 = \phi^{-1}(V \cap \mathbb{C}_-) \subset U$. Now we can define $\mathbb{S}(z) : U_1 \to \mathbb{C}$, s.t.

$$\mathbb{S}(z) = \overline{\psi(\overline{\phi(z)})}.$$

Lemma 5.1.1. $\mathbb{S}(z)$ *is a Schwarz function in* U*.*

Proof. $\mathbb{S}(z)$ is analytic in U and continuous onto $\Gamma \cap \bar{U}$.
If $z \in \Gamma \cap \bar{U}$, then $\mathbb{S}(z) = \overline{\psi(\phi(z))} = \bar{z}$,
so $\mathbb{S}(z)$ is a Schwarz function. $\square$

5.2 Dispersionless String Equation

Lemma 5.2.1. *$\dot{\mathbb{S}}_t = 4\partial G = 2i\partial\phi$ on Γ_t.*

Proof. Consider the case that the boundary Γ_t is real analytic, then by Sakai's regularity theorem, $\mathbb{S}(z)$ can be extended to some stripe inside the droplet $S(t)$. Then for $\triangle t << 1$ and $\forall z \in \Gamma_t \cap \bar{U}_1$, $\overline{\mathbb{S}_{t+\triangle t}(z)}$ is well-defined as the reflection of z about Γ_t. Since Γ_t has normal velocity v_n, we can approximate $\overline{\mathbb{S}_{t+\triangle t}(z) - \mathbb{S}_t(z)} = 2\vec{v}_n \triangle t$, where $\vec{v}_n = v_n \cdot n$. Let $\triangle t \to 0$, we have $\overline{\dot{\mathbb{S}}_t} = 2\vec{v}_n$. then the velocity $\frac{1}{2}\overline{\dot{\mathbb{S}}} = \nabla G \Rightarrow \dot{\mathbb{S}} = 2\overline{\nabla G} = 4\partial G$. □

Since $\phi(z,t)$ is univalent and conformal in U_t for each t, there exists an inverse map $z = z(\phi, t)$ for each t. Now $\mathbb{S}(\phi, t) = \mathbb{S}(z(\phi,t),t)$ on Γ_t, and we could get the following form, which is known as dispersionless String equation.

Lemma 5.2.2. *On boundary Γ_t,*

$$\frac{\partial \mathbb{S}}{\partial \phi}\frac{\partial z}{\partial t} - \frac{\partial \mathbb{S}}{\partial t}\frac{\partial z}{\partial \phi} = -2i.$$

Proof. Since $z = z(\phi, t)$ is the inverse map of $\phi = \phi(z,t)$ and $\mathbb{S}(\phi,t) = \mathbb{S}(z(\phi,t),t)$, for each t

$$\frac{\partial z}{\partial \phi} = \frac{1}{\frac{\partial \phi}{\partial z}}.$$

Now consider z, $\mathbb{S}$ as functions of ϕ and t,

$$\frac{\partial \mathbb{S}}{\partial \phi} = \frac{\partial \mathbb{S}}{\partial z}\frac{\partial z}{\partial \phi},$$

$$\frac{\partial \mathbb{S}}{\partial t} = \frac{\partial \mathbb{S}}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial \mathbb{S}}{\partial t},$$

so

$$\frac{\partial \mathbb{S}}{\partial \phi}\frac{\partial z}{\partial t} - \frac{\partial \mathbb{S}}{\partial t}\frac{\partial z}{\partial \phi} = -\frac{\partial \mathbb{S}}{\partial t}\frac{\partial z}{\partial \phi} = -2i.$$

□

Now let us choose a Cartesian coordinate systerm as follows: Let the cusp point be the origin and the tangent line at the cusp be the x-axis, then on Γ_t, since $\mathbb{S}(z) = \bar{z}$, $X = (z+\mathbb{S})/2$, $Y = (z-\mathbb{S})/(2i)$.

Corollary 5.2.3.

$$\{X, Y\} = \frac{\partial X}{\partial \phi}\frac{\partial Y}{\partial t} - \frac{\partial X}{\partial t}\frac{\partial Y}{\partial \phi} = -1.$$

Proof. Plug $X = (z+\mathbb{S})/2$, $Y = (z-\mathbb{S})/(2i)$ into the equation in previous lemma. □

This equation is also known as dispersionless string equation.

Chapter 6

Main Theorem and Conjecture

In this chapter, we give the main theorem. In the main theorem, we discuss simply connected droplets under Hele-Shaw potentials. Suppose there is a $(3,2)$-cusp on the boundary, we can get the Cartesian coordinates of the Hele-Shaw flows near the cusp using Green's coordinates, and we can see the main parts are one-parameter families and do not depend on time t. Then we state conjectures for higher-order $(2p+1,2)$-cusps and show an example of (5,2)-cusps.

6.1 Main Theorem

Suppose the external potential field is $Q(z) = |z|^2 - H(z)$, where $H(z)$ is harmonic and $h(z) = \partial H(z)$ is a meromorphic function (call it algebraic). From previous chapters, $\exists$ a chain of local droplets S_t. Assume that S_t's are simply connected and there is a (3,2) cusp on ∂S_{t_*}, then from Sakai's no turbulence theorem, for all $t < t_*$, S_t has analytic boundary near the cusp. Also, $\exists$ Schwarz function $\mathbb{S}(z,t)$ in a small neighborhood of the cusp. Write z in the form of Green's coordinate ϕ, we have $\mathbb{S}(\phi,t) = \mathbb{S}(Z(\phi,t),t)$. Since $\mathbb{S}(z,t) = \bar{z}$ on ∂S_t, on the boundary of local droplets, we have the Cartesian coordinates $X(t,\phi), Y(t,\phi)$,

$$X(t,\phi) = \frac{Z(\phi,t) + \mathbb{S}(Z(\phi,t)t)}{2},$$

$$Y(t,\phi) = \frac{Z(\phi,t) - \mathbb{S}(Z(\phi,t),t)}{2i}.$$

For simply connected $D(t)$'s, $\exists$ conformal maps $f_t : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to D(t)$. From the picture below, we see $Z(t,\phi) = f_t(e^{i\phi})$. For $\phi \in \mathbb{R}$, $Z(t,\phi) \in \partial D(t)$, i.e. we could get $X(t,\phi), Y(t,\phi)$ from f_t.

Without lost of generality, let $\tilde{t} = t - t_*$ and denote $\tilde{t}$ as t, then there is a cusp on the boundary of $D(0)$.

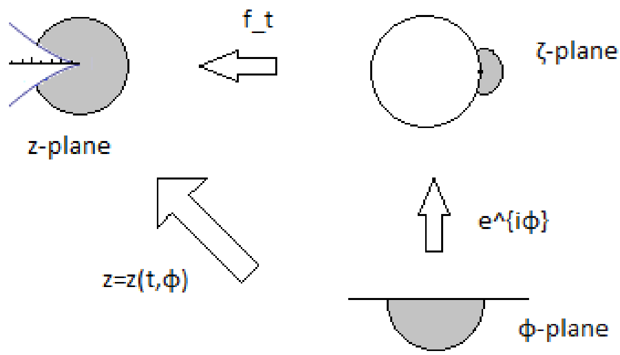

Figure 6.1: Relation of Green's coordinates ϕ and Cartesian coordinates X, Y.

Theorem 6.1.1. *Suppose the cusp on $\partial D(0)$ is a (3,2) cusp, assume $X_0(\phi)$, $Y_0(\phi)$ satisfy*

$$\begin{aligned} X(0,\phi) &= \phi^2 + O(\phi^3), \\ Y(0,\phi) &= c\phi^3 + O(\phi^4), \end{aligned} \tag{6.1.1}$$

with $c < 0$ near the cusp, then $\exists \epsilon > 0$, s.t. $\forall t \in (-\epsilon, 0)$, the local boundary of $D(t)$ near the tip is analytic and its Cartesian coordinates in terms of Green's coordinates have the following form:

$$\begin{aligned} X(t,\phi) &= \sqrt{\tfrac{4}{-3c}}\sqrt{-t} + \phi^2 + o\left(\sqrt{-t} + \phi^2\right), \\ Y(t,\phi) &= -\sqrt{-3c}\sqrt{-t}\phi + c\phi^3 + o\left(|\sqrt{-t}\phi| + |\phi|^3\right), \end{aligned} \tag{6.1.2}$$

in particular,

$$Y(X,t) = \pm c\left(X - \sqrt{\frac{4}{-3c}}\sqrt{-t}\right)^{1/2}\left(X + \sqrt{\frac{1}{-3c}}\sqrt{-t}\right) + o\left(\left(X - \sqrt{\frac{4}{-3c}}\sqrt{-t}\right)^{3/2} + (-t)^{3/4}\right). \tag{6.1.3}$$

If we divide X, Y by $(-t)^{1/2}$, $(-t)^{3/4}$ respectively, we get

$$\tilde{Y}(\tilde{X},t) = \pm c\left(\tilde{X} - \sqrt{\frac{4}{-3c}}\right)^{1/2}\left(\tilde{X} + \sqrt{\frac{1}{-3c}}\right) + o\left(\left(\tilde{X} - \sqrt{\frac{4}{-3c}}\right)^{3/2} + 1\right).$$

Remark 6.1.2. *In general, for*

$$\begin{aligned} X(0,\phi) &= c_2\phi^2 + O(\phi^3), \\ Y(0,\phi) &= c_3\phi^3 + O(\phi^4), \end{aligned} \tag{6.1.4}$$

we could make

$$\tilde{X}(0,\phi) = \phi^2 + O(\phi^3),$$
$$\tilde{Y}(0,\phi) = c_2 c_3 \phi^3 + O(\phi^4), \tag{6.1.5}$$

then $\tilde{X}_t, \tilde{Y}_t$ still satisfies String equation, from the above theorem, we get

$$\tilde{X}(t,\phi) = \sqrt{\tfrac{4}{-3c_2c_3}}\sqrt{-t} + \phi^2 + o\left(\sqrt{-t} + \phi^2\right),$$
$$\tilde{Y}(t,\phi) = -\sqrt{-3c_2c_3}\sqrt{-t}\phi + c_2c_3\phi^3 + o\left(|\sqrt{-t}\phi| + |\phi|^3\right), \tag{6.1.6}$$

then

$$X(t,\phi) = c_2\sqrt{\tfrac{4}{-3c_2c_3}}\sqrt{-t} + c_2\phi^2 + o\left(\sqrt{-t} + \phi^2\right),$$
$$Y(t,\phi) = -\tfrac{1}{c_2}\sqrt{-3c_2c_3}\sqrt{-t}\phi + c_3\phi^3 + o\left(|\sqrt{-t}\phi| + |\phi|^3\right). \tag{6.1.7}$$

Zhukowski's airfoils and deltoid are among the easiest examples of (3,2)-cusps. Here let us look at Zhukowski's airfoil.

The conformal map from $\hat{\mathbb{C}} \setminus \mathbb{D}$ to the following Zhukowski's airfoil is $f(\zeta) = \frac{3}{4}\left(\zeta + \frac{\frac{1}{4}}{\zeta - \frac{1}{2}} - \frac{3}{2}\right)$. The picture is as follows:

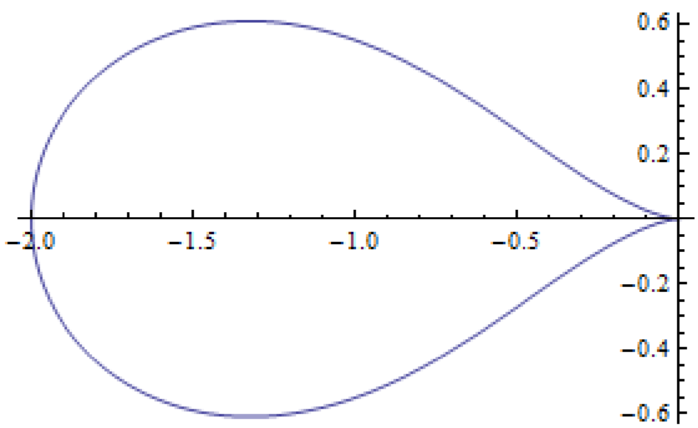

Figure 6.2: Zhukowski's airfoil.

Then

$$Z(0,\phi) = f(e^{i\phi}) = \frac{3}{4}\left(e^{i\phi} + \frac{\frac{1}{4}}{e^{i\phi} - \frac{1}{2}} - \frac{3}{2}\right),$$

and

$$X(0,\phi) = -\tfrac{3}{2}\phi^2 + O(\phi^3),$$
$$Y(0,\phi) = \tfrac{3}{2}\phi^3 + O(\phi^4). \tag{6.1.8}$$

From the main theorem, we could get

$$\begin{aligned} X(t,\phi) &= -\tfrac{2\sqrt{3}}{3}\sqrt{-t} - \tfrac{3}{2}\phi^2 + o\left(\sqrt{-t} + \phi^2\right), \\ Y(t,\phi) &= \sqrt{3}\sqrt{-t}\phi + \tfrac{3}{2}\phi^3 + o\left(|\sqrt{-t}\phi| + |\phi|^3\right) \end{aligned} \tag{6.1.9}$$

are Cartesian coordinates of $\partial D(t)$ near the tip.

Below is the graph of the main part of Hele-Shaw flow near the cusp in Zhukowski's airfoil near the tip at time $t = -0.0005, -0.0002, 0$ and $-0.8 < \phi < 0.8$.

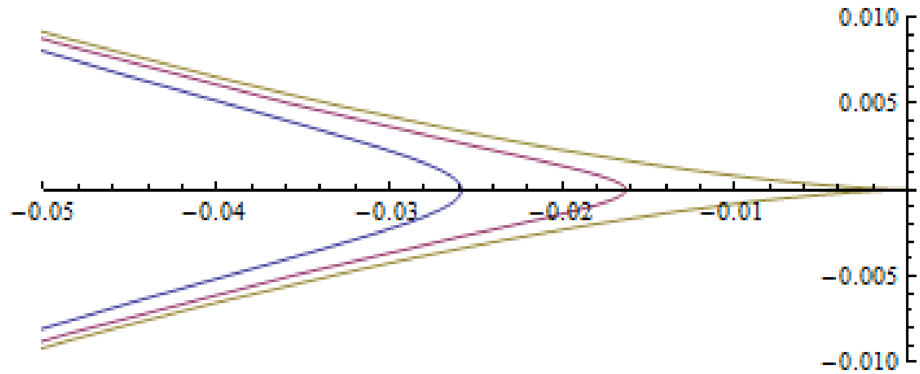

Figure 6.3: Hele-Shaw flow near the (3,2)-cusp in Zhukowski's airfoil.

Make linear transformation to above curves along X-axis s.t. they interect X-axis at the origin, we could get the following picture:

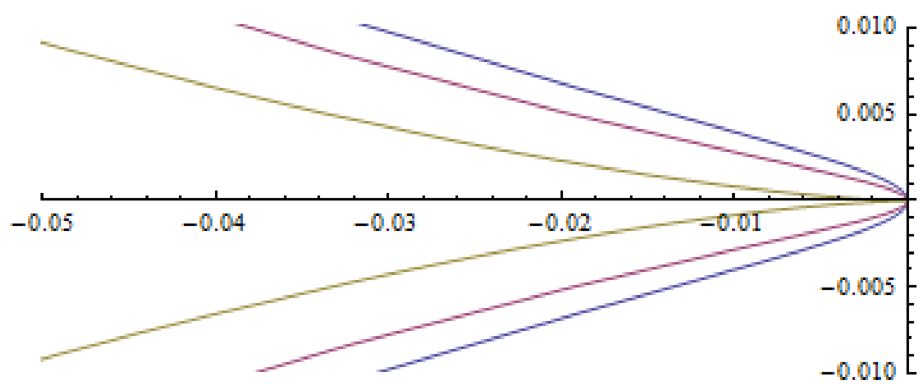

Figure 6.4: Hele-Shaw flow near the (3,2)-cusp in Zhukowski's airfoil after linear transformation along X-axis.

From above equation, we can see that for a (3,2)-cusp, no droplets will exist after time $t = 0$.

6.2 General Conjecture concerning (2p+1,2)-cusps

In [15], Physicists have suggested the following conjecture concerning $(2p+1,2)$-cusps:

Conjecture 6.2.1. *Suppose the cusp on $\partial D(0)$ is a (2p+1,2) cusp, assume $X_0(\phi)$, $Y_0(\phi)$ satisfy*

$$\begin{aligned} X(0,\phi) &= \phi^2 + O(\phi^3), \\ Y(0,\phi) &= c\phi^{2p+1} + O(\phi^{2p+2}), \end{aligned} \tag{6.2.1}$$

with $c < 0$ near the cusp, then $\exists \epsilon > 0$, s.t. $\forall t \in (-\epsilon, 0)$, the local boundary of $D(t)$ around the cusp is analytic and its Cartesian coordinates in terms of Green's coordinates have the following form:

$$\begin{aligned} X(t,\phi) &= a(t) + \phi^2 + o\left(\sqrt[p+1]{-t} + \phi^2\right), \\ Y(t,\phi) &= c\textstyle\sum_{k=0}^{p} \begin{pmatrix} -\frac{1}{2} \\ k \end{pmatrix} (-a(t))^k (a(t)+\phi^2)^{p-k}\phi + o\left(\sum_{k=0}^{p} |(\sqrt[p+1]{-t})^k(\sqrt[p+1]{-t}+\phi^2)^{p-k}\phi|\right), \end{aligned} \tag{6.2.2}$$

where

$$a(t) = 2\left(\frac{(p+1)!t}{2(2p+1)!!c}\right)^{\frac{1}{p+1}},$$

i.e.,

$$Y(X,t) = \pm c\sum_{k=0}^{p} \begin{pmatrix} -\frac{1}{2} \\ k \end{pmatrix} (-a(t))^k X^{p-k}\sqrt{X - a(t)} + o\left((X - a(t))^{(2p+1)/2} + (-t)^{\frac{2p+1}{2(p+1)}}\right).$$

If we divide X, Y by $(-t)^{\frac{1}{p+1}}, (-t)^{\frac{2p+1}{2(p+1)}}$ respectively, we get

$$\tilde{Y}(\tilde{X},t) = \pm c\sum_{k=0}^{p} \begin{pmatrix} -\frac{1}{2} \\ k \end{pmatrix} (-2\left(-\frac{(p+1)!}{2(2p+1)!!c}\right)^{\frac{1}{p+1}})^k \tilde{X}^{p-k}\sqrt{\tilde{X} - 2\left(-\frac{(p+1)!}{2(2p+1)!!c}\right)^{\frac{1}{p+1}}}$$
$$+o\left((\tilde{X} - 2\left(-\frac{(p+1)!}{2(2p+1)!!c}\right)^{\frac{1}{p+1}})^{(2p+1)/2} + 1\right).$$

Remark 6.2.2. *After scaling X, Y, the main part of $\tilde{Y}(\tilde{X},t)$ is a one-parameter family and does not depend on time t.*

It is known that when p is odd, $(2p+1)$-cusps are maximal and when p is even, $(2p+1,2)$-cusps are non maximal. We can see this from $Y(X,t)$ clearly, as when $t > 0$, $a(t)$ does not exists when p is odd and $a(t)$ exists when p is even. Let us take a look at one of the simplest examples of non maximal cusps–(5,2)-cusps.

Conjecture 6.2.3. *Suppose the cusp on $\partial D(0)$ is a (5,2) cusp,*

$$\begin{aligned} X(0,\phi) &= \phi^2 + O(\phi^3), \\ Y(0,\phi) &= c\phi^5 + O(\phi^6), \end{aligned} \tag{6.2.3}$$

with $c<0$ near the cusp, then $\exists \epsilon > 0$, s.t. $\forall t \in (-\epsilon, 0)$, the local boundary of $D(t)$ around the cusp is analytic and its Cartesian coordinates in terms of Green's coordinates have the following form:

$$\begin{aligned} X(t,\phi) &= \sqrt[3]{\tfrac{8t}{5c}} + \phi^2 + o\left(\sqrt[3]{-t} + \phi^2\right), \\ Y(t,\phi) &= \tfrac{15c}{8}\sqrt[3]{\tfrac{64t^2}{25c^2}}\phi + c\tfrac{5}{2}\sqrt[3]{\tfrac{8t}{5c}}\phi^3 + c\phi^5 + o\left(|\sqrt[3]{t^2}\phi| + |\sqrt[3]{-t}\phi^3| + |\phi|^5\right), \end{aligned} \tag{6.2.4}$$

i.e.,

$$Y(X,t) = \pm c\left(X^2 + \frac{1}{2}\sqrt[3]{\frac{8t}{5c}}X + \frac{3}{8}\sqrt[3]{\frac{64t^2}{25c^2}}\right)\sqrt{X - \sqrt[3]{\frac{8t}{5c}}} + o\left(\left(X - \sqrt[3]{\frac{8t}{5c}}\right)^{5/2} + (-t)^{5/6}\right).$$

If we divide X, Y by $(-t)^{\frac{1}{3}}, (-t)^{\frac{5}{6}}$ respectively, we get

$$\tilde{Y}(\tilde{X},t) = \pm c\left(\tilde{X}^2 + \frac{1}{2}\sqrt[3]{-\frac{8}{5c}}\tilde{X} + \frac{3}{8}\sqrt[3]{\frac{64}{25c^2}}\right)\sqrt{\tilde{X} - \sqrt[3]{-\frac{8}{5c}}} + o\left(\left(\tilde{X} - \sqrt[3]{-\frac{8}{5c}}\right)^{5/2} + 1\right).$$

Remark 6.2.4. *From above equation of $Y(X,t)$, different from a $(3,2)$-cusp, for a $(5,2)$-cusp, droplets still exists after time $t=0$. We could see it directly from the following pictures of*

$$\begin{aligned} X(t,\phi) &= \sqrt[3]{\tfrac{8t}{5c}} + \phi^2, \\ Y(t,\phi) &= \tfrac{15c}{8}\sqrt[3]{\tfrac{64t^2}{25c^2}}\phi + c\tfrac{5}{2}\sqrt[3]{\tfrac{8t}{5c}}\phi^3 + c\phi^5, \end{aligned} \tag{6.2.5}$$

when $c=2$, $t=-0.005, 0, 0.005$ and $-0.8<\phi<0.8$.

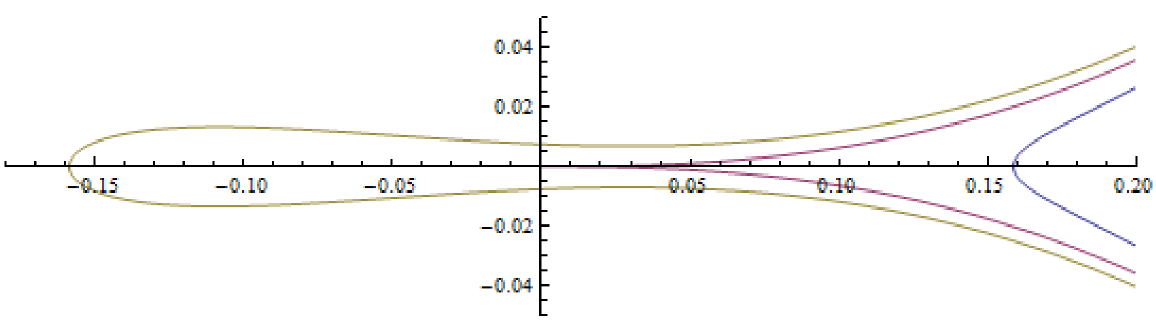

Figure 6.5: Hele-Shaw flow with a (5,2)-cusp will continue to grow after cusp appears.

Chapter 7

Special Solutions and dKdV Hierarchy

Dispersionless string equation is a PDE derived from D'Arcy's law and the solutions of it may correspond to some Hele-Shaw flow. In this chapter, we discover special solutions to the dispersionless string equation. We will see these special solutions are main parts of Hele-Shaw flow equation in the main conjecture. Besides, we will show that these special solutions are related to dispersionless KdV(dKdV) hierarchy. The arguments mainly follow from those in [15], with a little more details.

7.1 Special Solutions of Dispersionless String Equation

First, we will derive an equivalent form of dispersionless String equation.

Lemma 7.1.1. *Suppose* $\begin{cases} X = X(t,\phi) \\ Y = Y(t,\phi) \end{cases}$ *, where* ϕ *is Green's coordinate, and* X, Y *are Cartesian coordinates. Then*

$$\{X, Y\} = \frac{\partial X}{\partial \phi}\frac{\partial Y}{\partial t} - \frac{\partial X}{\partial t}\frac{\partial Y}{\partial \phi} = -1$$

is equivalent to

$$\frac{\partial Y}{\partial t} = -\frac{\partial \phi}{\partial X}.$$

Note: Since $Z(t,\phi)$ is univalent with respect to ϕ for each t, $X(t,\phi)$ is also univalent with respect to ϕ for each t, there exists an inverse function of $\phi = \phi(t, X)$ for each t. then let $\tilde{Y}(t, X) = Y(t, \phi(t, X))$. We write Y instead of $\tilde{Y}$. then we have two new functions $\phi(t, X)$ and $Y(t, X)$.

Proof.

$$\frac{\partial \tilde{Y}}{\partial t} = \frac{\partial Y}{\partial t} + \frac{\partial Y}{\partial \phi}\frac{\partial \phi}{\partial t},$$

and

$$\frac{\partial \phi}{\partial X} = \frac{1}{\frac{\partial X}{\partial \phi}},$$

then

$$-\frac{\partial \phi}{\partial X} = \{X, Y\}\frac{\partial \phi}{\partial X} = \frac{\partial Y}{\partial t} - \frac{\partial \phi}{\partial X}\frac{\partial X}{\partial t}\frac{\partial Y}{\partial \phi}.$$

If we plug $X = X(t, \phi)$ in $\phi = \phi(t, X)$, then we get $\phi = \phi(t, X(t, \phi))$, which is independent of t, so

$$0 = \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial X}\frac{\partial X}{\partial t},$$

so

$$-\frac{\partial \phi}{\partial X} = \frac{\partial \tilde{Y}}{\partial t} = \frac{\partial Y}{\partial t}.$$

□

Now solutions of dispersionless String equation are the same as solutions $\phi(t, X)$, $Y(t, X)$, s.t.

$$\frac{\partial Y}{\partial t} = -\frac{\partial \phi}{\partial X}.$$

We are concerned about some special solutions to this PDE, which are related to dispersionless KdV hierarchy. The form of the special solutions is as follows:

Theorem 7.1.2. $\begin{cases} \phi = \sqrt{X - a(t)} \\ Y = \sum_{j=0}^{p} c_j P_j(X, a(t))\sqrt{X - a(t)} \end{cases}$ *is a special solution of* $\begin{cases} \frac{\partial Y}{\partial t} = -\frac{\partial \phi}{\partial X} \\ Y(0, X) = \sqrt{X}\sum_{j=0}^{p} c_j X^j \end{cases}$, *where* $a(t)$ *satisfies*

$$\sum_{j=0}^{p} c_j 2 \begin{pmatrix} -\frac{1}{2} \\ j+1 \end{pmatrix} (j+1)(-a)^j \dot{a}(t) = -1.$$

Note: $P_j(X, a) = Prin[X^j(1 - \frac{a}{X})^{-1/2}]$, where $Prin(u(X))$ is the principle part of $u(X)$ as $X \to \infty$.

Since

$$(1 - \frac{a}{X})^{-1/2} = \sum_{k=0}^{\infty} \begin{pmatrix} -\frac{1}{2} \\ k \end{pmatrix} (\frac{-a}{X})^k,$$

we have

$$P_j(X, a) = \sum_{k=0}^{j} \begin{pmatrix} -\frac{1}{2} \\ k \end{pmatrix} (-a)^k X^{j-k},$$

e.g.,

$$P_0(X, a) = 1,$$

$$P_1(X, a) = X + \frac{1}{2}a,$$

$$P_2(X,a) = X^2 + \frac{1}{2}aX + \frac{3}{8}a^2.$$

Proof. First, to satisfy the initial conditon, $a(t)$ must go to 0 as $t \to 0$.
Second, we need to find $a(t)$, s.t. $\frac{\partial Y}{\partial t} = -\frac{\partial \phi}{\partial X}$.
We have

$$\begin{aligned} \tfrac{\partial Y}{\partial t} &= \textstyle\sum_{j=0}^{p} c_j \frac{\partial P_j}{\partial a}\sqrt{X-a(t)}\dot{a}(t) + \sum_{j=0}^{p} c_j P_j(X,a(t))\frac{-\dot{a}(t)}{2\sqrt{X-a(t)}} \\ &= \textstyle\sum_{j=0}^{p} c_j[\frac{\partial P_j}{\partial a}2(X-a(t)) - P_j(X,a(t))]\frac{\dot{a}(t)}{2\sqrt{X-a(t)}}. \end{aligned} \tag{7.1.1}$$

Let $Y_j = P_j(X,a(t))\sqrt{X-a(t)}$. Since

$$P_j = \frac{X^{j+1/2}}{\sqrt{X-a}} + O(\frac{1}{X})$$

as $X \to \infty$, so

$$Y_j = X^{j+1/2} + O(\frac{1}{\sqrt{X}}),$$

then

$$\frac{\partial Y}{\partial t} = \sum_{j=0}^{p} c_j \frac{\partial Y_j}{\partial t} = O(\frac{1}{\sqrt{X}}),$$

since

$$\frac{\partial P_j}{\partial a}2(X-a(t)) - P_j(X,a(t))\dot{a}(t)$$

is a polynomial of X, it must be independent of X, otherwise, look into Taylor expansion of

$$\sum_{j=0}^{p} c_j[\frac{\partial P_j}{\partial a}2(X-a(t)) - P_j(X,a(t))]\frac{\dot{a}(t)}{2\sqrt{X-a(t)}},$$

there will be terms of $\sqrt{X}$, contradiction.
Therefore,

$$\frac{\partial P_j}{\partial a}2(X-a(t)) - P_j(X,a(t)) = (-2j-1)\begin{pmatrix} -\frac{1}{2} \\ j \end{pmatrix}(-a)^j = 2\begin{pmatrix} -\frac{1}{2} \\ j+1 \end{pmatrix}(j+1)(-a)^j,$$

then

$$\frac{\partial Y}{\partial t} = \sum_{j=0}^{p} c_j 2\begin{pmatrix} -\frac{1}{2} \\ j+1 \end{pmatrix}(j+1)(-a)^j \frac{\dot{a}(t)}{2\sqrt{X-a(t)}} = -\frac{\partial \phi}{\partial X} = -\frac{1}{2\sqrt{X-a(t)}},$$

so

$$\sum_{j=0}^{p} c_j 2\begin{pmatrix} -\frac{1}{2} \\ j+1 \end{pmatrix}(j+1)(-a)^j \dot{a}(t) = -1.$$

Therefore,

if $a(t)$ satisfies $\sum_{j=0}^{p} c_j 2 \begin{pmatrix} -\frac{1}{2} \\ j+1 \end{pmatrix} (-a)^{j+1} = t$ and $a(0) = 0$,

$\begin{cases} \phi = \sqrt{X - a(t)} \\ Y = \sum_{j=0}^{p} c_j P_j(X, a(t)) \sqrt{X - a(t)} \end{cases}$ is a special solution of $\phi(t, X)$, $Y(t, X)$, s.t. $\frac{\partial Y}{\partial t} = -\frac{\partial \phi}{\partial X}$,
with initial condition $Y(0, X) = \sqrt{X} \sum_{j=0}^{p} c_j X^j$. □

Example:

1) Suppose $Y(0, X) = c_1 X^{3/2}$, then $\begin{cases} \phi = \sqrt{X - a(t)} \\ Y = c_1 P_1(X, a(t)) \sqrt{X - a(t)} \end{cases}$ is a solution of $\frac{\partial Y}{\partial t} = -\frac{\partial \phi}{\partial X}$,
where $P_1(X, a(t)) = X + \frac{1}{2} a(t)$, and $a(t)$ satisfies $c_1 2 \begin{pmatrix} -\frac{1}{2} \\ 2 \end{pmatrix} (2)(-a)\dot{a}(t) = -1$, i.e., $a(t) = \sqrt{\frac{4t}{3c_1}}$.
So

$$Y(t, X) = c_1 \left(X + \sqrt{\frac{t}{3c_1}} \right) \sqrt{X - \sqrt{\frac{4t}{3c_1}}}.$$

2)Suppose $Y(0, X) = c_2 X^{5/2}$, then $\begin{cases} \phi = \sqrt{X - a(t)} \\ Y = c_2 P_2(X, a(t)) \sqrt{X - a(t)} \end{cases}$ is a solution of $\frac{\partial Y}{\partial t} = -\frac{\partial \phi}{\partial X}$,
where $P_2(X, a(t)) = X^2 + \frac{1}{2} a(t) X + \frac{3}{8} a(t)^2$, and $a(t)$ satisfies $c_2 2 \begin{pmatrix} -\frac{1}{2} \\ 3 \end{pmatrix} (3)(-a)^2 \dot{a}(t) = -1$, i.e.,
$a(t) = \sqrt[3]{\frac{8t}{5c_2}}$. So

$$Y(t, X) = c_2 \left(X^2 + \frac{1}{2} \sqrt[3]{\frac{8t}{5c_2}} X + \frac{3}{8} \sqrt[3]{\frac{64t^2}{25c_2^2}} \right) \sqrt{X - \sqrt[3]{\frac{8t}{5c_2}}}.$$

From these two examples, we can see that these special solutions are main parts of the exact solutions of Hele-Shaw problem.

7.2 Dispersionless KdV Hierarchy

Now we will discuss the relation between the special solutions and dispersionless KdV hierarchy.

Integrable hierarchy is a set of differential equations that commute. Suppose a function $u = u(t; t_1, t_2, ...)$ has a set of differential equations:

$$\frac{\partial u}{\partial t_j} = F_j(u, \dot{u}, \ddot{u}, ...),$$

where $\dot{u} = \frac{\partial u}{\partial t}$.

If

$$[\frac{\partial}{\partial t_j}, \frac{\partial}{\partial t_k}]u = \frac{\partial}{\partial t_j}\frac{\partial u}{\partial t_k} - \frac{\partial}{\partial t_k}\frac{\partial u}{\partial t_j} = 0,$$

then

$$\{\frac{\partial u}{\partial t_j} = F_j(u, \dot{u}, \ddot{u}, ...)\}$$

is called integrable hierarchy.

If

$$F_j(u, \dot{u}, \ddot{u}, ...) = -\frac{(2j+1)!!}{j!2^j}u^j\dot{u},$$

then it is dispersionless KdV hierarchy.

From previous section, the special solutions of dispersionless string equation with given initial condition $Y(0, X) = \sqrt{X}\sum_{n=0}^{p} c_n X^n$ are

$$\begin{cases} \phi(X,t) = \sqrt{X - a(t)} \\ Y(X,t) = \sum_{n=0}^{p} c_n P_n(X, a(t))\sqrt{X - a(t)} \end{cases},$$

where $P_n(X, a(t)) = Prin[X^n(1 - \frac{a}{X})^{-1/2}]$ and $\sum_{n=0}^{p} c_n 2 \begin{pmatrix} -\frac{1}{2} \\ n+1 \end{pmatrix} (-a)^{n+1} = t.$

First, define KdV times t_{2n+1}'s:

$$t_{2n+1} = \frac{2}{2n+1}c_{n-1},$$

for $n = 1, 2, 3, ...$ and $t_1 = -t$.

Give different initial values of c_n's, we could get different $a(t)$'s, so $a(t)$ can be seen as a function of t, t_{2n+1}'s. We will show that

$$\{\frac{\partial a(t; t_1, t_3, ...)}{\partial t_{2n+1}} = -\frac{(2n+1)!!}{n!2^n}a^n\dot{a}\},$$

so it is dKdV hierarchy.

Let

$$\omega_n(X,t) = P_n(X, a(t))\sqrt{X - a(t)},$$

for $n = 0, 1, 2, ...,$

then $Y(X,t)$ can be rewritten as

$$Y(X,t)=\sum_{n=1}^{p+1}\frac{2n+1}{2}t_{2n+1}\omega_{n-1}.$$

Lemma 7.2.1.

$$\frac{\partial Y}{\partial t_{2n+1}}=\frac{\partial \omega_n}{\partial X}.$$

Proof.

$$\omega_n(X,t)=P_n(X,a(t))\sqrt{X-a(t)}=Prin[X^n(1-\frac{a}{X})^{-\frac{1}{2}}]\sqrt{X-a(t)}$$

$$=(X^n(1-\frac{a}{X})^{-\frac{1}{2}}-[X^n(1-\frac{a}{X})^{-\frac{1}{2}}]_-)\sqrt{X-a(t)}=X^{n+\frac{1}{2}}-[X^n(1-\frac{a}{X})^{-\frac{1}{2}}]_-\sqrt{X-a(t)},$$

where $[\cdot]_-$ is the sum of negative powers,

and

$$[X^n(1-\frac{a}{X})^{-\frac{1}{2}}]_-\sqrt{X-a(t)}=O(\frac{1}{\sqrt{X}}).$$

If we substitute X by $\phi^2+a(t)$, we get

$$\tilde{\omega}_n(\phi,t)=\omega_n(\phi^2+a,t)=(\phi^2+a(t))^{n+\frac{1}{2}}-[X^n(1-\frac{a}{X})^{-\frac{1}{2}}]_-\sqrt{X-a(t)}|_{X=\phi^2+a}$$

and

$$[X^n(1-\frac{a}{X})^{-\frac{1}{2}}]_-\sqrt{X-a(t)}|_{X=\phi^2+a}=O(\frac{1}{\phi}).$$

Because $\tilde{\omega}_n(\phi,t)$ is a polynomial of ϕ, so $\tilde{\omega}_n(\phi,t)$ is the polynomial part of $(\phi^2+a(t))^{n+\frac{1}{2}}$ in ϕ.

It is clear that $\frac{\partial Y}{\partial t_{2n+1}}=\frac{2n+1}{2}\omega_{n-1}+\sum_{j=1}^{p+1}\frac{2j+1}{2}t_{2j+1}\frac{\partial \omega_{j-1}}{\partial t_{2n+1}}$, and $\frac{\partial \omega_{j-1}}{\partial t_{2n+1}}=O(\frac{1}{\sqrt{X}})$.

Since $\frac{\partial \omega_n}{\partial X}=(n+\frac{1}{2})X^{n-\frac{1}{2}}+O(\frac{1}{X^3/2})$, if we substitute X by $\phi^2+a(t)$, we know $\frac{\partial \omega_n}{\partial X}=$ poly$(\phi)+O(\frac{1}{\phi})$, while $O(\frac{1}{X^3/2})=O(\frac{1}{\phi^3})$, so $\frac{\partial \omega_n}{\partial X}=$poly$(X)\sqrt{X-a(t)}+O(\frac{1}{\sqrt{X}})$.

Since $\frac{\partial Y}{\partial t_{2n+1}}$ and $\frac{\partial \omega_n}{\partial X}$ has the same principle part,

$$\frac{\partial Y}{\partial t_{2n+1}}=\frac{\partial \omega_n}{\partial X}.$$

□

Lemma 7.2.2.

$$\frac{\partial Y}{\partial t_{2n+1}}=\{Y,\omega_n\}.$$

Proof. Since $\{X, Y\} = -1$,

$$-\frac{\partial \omega_n}{\partial X} = \{X, Y\}\frac{\partial \omega_n}{\partial X} = (\frac{\partial X}{\partial \phi}\frac{\partial Y}{\partial t} - \frac{\partial X}{\partial t}\frac{\partial Y}{\partial \phi})\frac{\partial \omega_n}{\partial X},$$

and from $\tilde{\omega}_n(\phi, t) = \omega_n(X(\phi, t), a(t))$,

$$\frac{\partial \tilde{\omega}_n}{\partial \phi} = \frac{\partial \omega_n}{\partial X}\frac{\partial X}{\partial \phi}.$$

Moreover, as $\tilde{\omega}_n(\phi, t)$ is the polynomial part of $(\phi^2 + a(t))^{n+\frac{1}{2}}$ in ϕ,

$$\frac{\partial \tilde{\omega}_n}{\partial t} = \frac{\partial \omega_n}{\partial X}\frac{\partial X}{\partial t}.$$

So

$$-\frac{\partial \omega_n}{\partial X} = \frac{\partial \tilde{\omega}_n}{\partial \phi}\frac{\partial Y}{\partial t} - \frac{\partial \tilde{\omega}_n}{\partial t}\frac{\partial Y}{\partial \phi} = \{\tilde{\omega}_n, Y\},$$

and

$$\frac{\partial Y}{\partial t_{2n+1}} = \frac{\partial \omega_n}{\partial X} = \{Y, \tilde{\omega}_n\}.$$

Denote $\tilde{\omega}_n$ as ω_n. Then

$$\frac{\partial Y}{\partial t_{2n+1}} = \{Y, \omega_n\}.$$

□

Lemma 7.2.3.

$$\frac{\partial X}{\partial t_{2n+1}} = \{X, \omega_n\}.$$

Proof. Since $Y(t, \phi)$ is a univalent function near the cusp, there exists an inverse function $\phi = \phi(t, Y)$. And $\frac{\partial \phi}{\partial Y} = \frac{1}{\frac{\partial Y}{\partial \phi}}$. Let $X = \tilde{X}(t, Y) = X(t, \phi(t, Y))$. Now

$$\frac{\partial X}{\partial t_{2n+1}} = \frac{\partial X}{\partial \phi}\frac{\partial \phi}{\partial Y}\frac{\partial Y}{\partial t_{2n+1}} = \frac{\partial X}{\partial \phi}\frac{\partial \phi}{\partial Y}\{Y, \omega_n\}.$$

Since $\tilde{X}(t, Y) = X(t, \phi(t, Y))$ and $Y = Y(t, \tilde{X}(t, Y))$, $\frac{\partial X}{\partial Y} = \frac{\partial X}{\partial \phi}\frac{\partial \phi}{\partial Y}$ and $1 = \frac{\partial Y}{\partial X}\frac{\partial X}{\partial Y}$, also

$$\frac{\partial Y}{\partial t} = \sum_{n=1}^{p+1}\frac{2n+1}{2}t_{2n+1}\frac{\partial \omega_{n-1}}{\partial t} = \sum_{n=1}^{p+1}\frac{2n+1}{2}t_{2n+1}\frac{\partial \omega_{n-1}}{\partial X}\frac{\partial X}{\partial t} = \frac{\partial Y}{\partial X}\frac{\partial X}{\partial t},$$

$$\frac{\partial X}{\partial t_{2n+1}} = \frac{\partial X}{\partial \phi}\frac{\partial \phi}{\partial Y}(\frac{\partial Y}{\partial \phi}\frac{\partial \omega_n}{\partial t} - \frac{\partial Y}{\partial t}\frac{\partial \omega_n}{\partial \phi}) = \frac{\partial X}{\partial \phi}\frac{\partial \omega_n}{\partial t} - \frac{\partial X}{\partial t}\frac{\partial \omega_n}{\partial \phi} = \{X, \omega_n\}.$$

□

Theorem 7.2.4.

$$\frac{\partial a}{\partial t_{2n+1}} = -\frac{(2n+1)!!}{n!2^n} a^n \frac{\partial a}{\partial t}.$$

Proof. Since $X(t, \phi) = \phi^2 + a(t)$ and $\omega_n(t, \phi) = P_n(\phi^2 + a(t), a(t))\phi = \sum_{k=0}^{n} \begin{pmatrix} -\frac{1}{2} \\ k \end{pmatrix} (-a)^k(\phi^2 + a)^{n-k}\phi$,

$$\frac{\partial a}{\partial t_{2n+1}} = \{X, \omega_n\}$$

$$= 2\phi \sum_{k=0}^{n} \begin{pmatrix} -\frac{1}{2} \\ k \end{pmatrix} (-k(-a)^{k-1}\dot{a}(\phi^2 + a)^{n-k}\phi + (-a)^k(n-k)(\phi^2 + a)^{n-k-1}\dot{a}\phi)$$

$$-\dot{a} \sum_{k=0}^{n} \begin{pmatrix} -\frac{1}{2} \\ k \end{pmatrix} (-a)^k((\phi^2 + a)^{n-k} + (n-k)(\phi^2 + a)^{n-k-1}2\phi^2)$$

$$= 2\phi \sum_{k=0}^{n} \begin{pmatrix} -\frac{1}{2} \\ k \end{pmatrix} (-k)(-a)^{k-1}\dot{a}(\phi^2 + a)^{n-k}\phi - \dot{a} \sum_{k=0}^{n} \begin{pmatrix} -\frac{1}{2} \\ k \end{pmatrix} (-a)^k(\phi^2 + a)^{n-k}$$

$$= -\frac{(2n+1)!!}{n!2^n} a^n \frac{\partial a}{\partial t}.$$

□

Remark 7.2.5. *Since $t_1 = -t$,*

$$\frac{\partial a}{\partial t_1} = -\frac{\partial a}{\partial t},$$

so

$$\frac{\partial a}{\partial t_{2n+1}} = \frac{(2n+1)!!}{n!2^n} a^n \frac{\partial a}{\partial t_1}.$$

And the first equation is Hopf-Burgers equation

$$\frac{\partial a}{\partial t_3} = \frac{3a}{2} \frac{\partial a}{\partial t_1}.$$

Chapter 8

Algebraic Quadrature Domains

In this chapter, we will describe two methods to compute Hele-Shaw flows: one is exterior Faber transform and the other is Löwner equation. We will show three examples using these two methods. In chapter 9, we will prove the main theorem using Löwner equation.

8.1 Exterior Faber Transform

The main theorem is about simply connected droplets under algebraic Hele-Shaw potentials $Q(z) = |z|^2 - H(z)$, where $\partial H(z)$ is a meromorphic function, so there exist conformal maps from $\hat{\mathbb{C}} \setminus \mathbb{D}$ to the complements of the droplets. It is known that the conformal and univalent map from $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ to Ω has the following form:

$$f(\zeta) = r\zeta + c_0 + c_{-1}\zeta^{-1} + c_{-2}\zeta^{-2} + ...,$$

where $r > 0$.
Let $\Phi = \Phi[\cdot, f] : P_n \to P_n$ be defind by $p \to \Phi p = [p \circ f^{-1}]_\infty$, i.e. $p \circ f^{-1} = \Phi p + o(1)$.

Examples:

$$\Phi[1] = 1, \Phi[\zeta] = \frac{\omega - c_0}{r}, \Phi[\zeta^2] = \frac{(\omega - c_0)^2}{r^2} - \frac{2c_{-1}}{r},$$

and for $\lambda \in \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$,

$$\Phi[\frac{1}{\lambda - \zeta}] = \frac{f'(\lambda)}{f(\lambda) - \omega}.$$

Remark 8.1.1. *If $F \in \mathbf{A}(\mathbb{D})$, then define $F_*(z) = \overline{F(\frac{1}{\bar{z}})}$ and $F_*(z) \in \mathbf{A}(\hat{\mathbb{C}} \setminus \overline{\mathbb{D}})$.*
If $F \in \mathbf{A}(\hat{\mathbb{C}} \setminus \overline{\mathbb{D}})$, then define $F_*(z) = \overline{F(\frac{1}{\bar{z}})}$ and $F_*(z) \in \mathbf{A}(\mathbb{D})$.
On $\mathbb{T}$, $F_{**} = F$.

Theorem 8.1.2. *(Faber transform) [3] Let $Q(z) = |z|^2 - H(z)$ be a Hele-Shaw potential in O and* *let $h = \partial H$. Suppose $K \subset O$ has no holes and $f(\zeta) = r\zeta + g(\zeta)$, for $\zeta \in \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, is the conformal map onto K^c. Then K is a droplet iff*

$$\Phi[g_*; r\zeta + g] = h.$$

Remark 8.1.3. *Functions f and h has the same form. Using exterior Faber transform, given $Q(z) = z^2 - H(z)$, s.t. $h(z) = \partial H(z)$ is a meromorphic function, then $f(t, \zeta)$ will be meromorphic, we call this case algebraic case and the main theorem is about simply connected, algebraic case.*

8.2 Löwner Equation in Algebraic Case

Suppose the external potential is $Q(z) = |z|^2 - H(z)$, where $H(z)$ is harmonic and $h(z) = \partial H(z)$ is a meromorphic function. Also, suppose $D(t)$'s are simply connected unbounded domains. Then by exterior Faber transform, there exists a conformal map $f_t(\zeta) : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to D(t)$, and $f_t(\zeta)$ is a meromorphic function. Assume that $f_t'(\infty) > 0$, then $f_t'(\zeta) = c(t) \prod_j \frac{\zeta - a_j(t)}{\zeta - b_j(t)}$, where $c(t) > 0$. In this section, we will derive Löwner equation in algebraic case and then derive a system of ordinary differential equations involving all the coefficients $c(t), a_j(t)$'s and $b_j(t)$'s.

Theorem 8.2.1. *Suppose $f_t : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to D(t)$ has a rational derivative, i.e.,*

$$f_t'(\zeta) = c(t) \prod_j \frac{\zeta - a_j(t)}{\zeta - b_j(t)}, \tag{8.2.1}$$

where $c(t) > 0$.
Let $h = |f_t'|^{-2}$ and H be its harmonic extension. Then

$$H + i\tilde{H} = \frac{1}{|c|^2} \sum_k B_k \frac{\zeta + a_k}{\zeta - a_k} + \frac{1}{|c|^2} \prod \frac{b_k}{a_k}, \tag{8.2.2}$$

where

$$B_k = \frac{(1 - a_k\bar{b}_k)(a_k - b_k)}{(1 - |a_k|^2)a_k} \prod_{j \neq k} \frac{(1 - \bar{b}_j a_k)(a_k - b_j)}{(1 - \bar{a}_j a_k)(a_k - a_j)}. \tag{8.2.3}$$

Proof. On $\partial\mathbb{D}$,

$$|c|^2 h = \prod_j \frac{\zeta - b_j}{\zeta - a_j} \frac{\bar{\zeta} - \bar{b}_j}{\bar{\zeta} - \bar{a}_j} = \prod_j \frac{\zeta - b_j}{\zeta - a_j} \frac{1 - \bar{b}_j\zeta}{1 - \bar{a}_j\zeta} = \sum_j C_j \frac{1}{\zeta - a_j} + \sum_j D_j \frac{1}{1 - \bar{a}_j\zeta} + M. \tag{8.2.4}$$

Compare residues on both sides at a_k's and $\bar{a}_k^{-1}$'s,

$$C_k = \frac{(a_k - b_k)(1 - \bar{b}_k a_k)}{1 - |a_k|^2} \prod_{j \neq k} \frac{(a_k - b_j)(1 - \bar{b}_j a_k)}{(a_k - a_j)(1 - \bar{a}_j a_k)}, \tag{8.2.5}$$

$$D_k = \frac{(\bar{a}_k^{-1} - b_k)(1 - \bar{b}_k \bar{a}_k^{-1})}{\bar{a}_k^{-1} - a_k} \prod_{j \neq k} \frac{(\bar{a}_k^{-1} - b_j)(1 - \bar{b}_j \bar{a}_k^{-1})}{(\bar{a}_k^{-1} - a_j)(1 - \bar{a}_j \bar{a}_k^{-1})}. \tag{8.2.6}$$

Let

$$B_k = \frac{(a_k - b_k)(1 - \bar{b}_k a_k)}{(1 - |a_k|^2) a_k} \prod_{j \neq k} \frac{(a_k - b_j)(1 - \bar{b}_j a_k)}{(a_k - a_j)(1 - \bar{a}_j a_k)}, \tag{8.2.7}$$

then $C_k = B_k a_k$ and $D_k = \bar{B}_k$, and

$$\begin{aligned} |c|^2 h &= \sum B_k \tfrac{a_k}{\zeta - a_k} + \sum \bar{B}_k \tfrac{\bar{\zeta}}{\bar{\zeta} - \bar{a}_k} + M \\ &= \tfrac{1}{2} \sum B_k \tfrac{\zeta + a_k}{\zeta - a_k} + \tfrac{1}{2} \sum \bar{B}_k \tfrac{\bar{\zeta} + \bar{a}_k}{\bar{\zeta} - \bar{a}_k} - i\Im \sum B_k + M \\ &= \tfrac{1}{2} \sum B_k \tfrac{\zeta + a_k}{\zeta - a_k} + \tfrac{1}{2} \sum \bar{B}_k \tfrac{1 + \bar{a}_k \zeta}{1 - \bar{a}_k \zeta} - i\Im \sum B_k + M, \end{aligned} \tag{8.2.8}$$

as $\zeta \to 0$ and $\zeta \to \infty$, we can get $M - i\Im \sum B_k = \Re(\prod \frac{b_k}{a_k})$. □

Corollary 8.2.2. *Löewner equation in algebraic case is in the form of*

$$\dot{f}_t(\zeta, t) = \zeta A(\zeta, t) f_t'(\zeta, t), \tag{8.2.9}$$

where

$$A(\zeta, t) = \frac{1}{|c|^2} \sum_k B_k \frac{\zeta + a_k}{\zeta - a_k} + \frac{1}{|c|^2} \prod \frac{b_k}{a_k}. \tag{8.2.10}$$

Proof. We can see that $H + i\tilde{H} = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|f'(e^{i\theta}, t)|^2} \frac{e^{i\theta} + \zeta}{e^{i\theta} - \zeta} d\theta$. □

Now we will derive a system of ordinary differential equations involving coefficients $a_j(t), b_j(t), c(t)$:

Theorem 8.2.3. *Löwner equation is equivalent to the following* $(2n+1)$ *equations:*

$$\begin{cases} -\dot{a}_j = \gamma a_j + \frac{1}{c^2} \sum 2B_k a_k + \frac{1}{c^2} \sum_{k \neq j} \frac{2B_k a_k^2}{a_j - a_k} + \frac{1}{c^2} 2B_j a_j^2 (\sum_{k \neq j} \frac{1}{a_j - a_k} - \sum \frac{1}{a_j - b_k}), \\ \dot{b}_j = -\gamma b_j - \frac{1}{c^2} \sum 2B_k a_k - \frac{1}{c^2} \sum \frac{2B_k a_k^2}{b_j - a_k}, \\ \frac{\dot{c}}{c} = \gamma. \end{cases} \tag{8.2.11}$$

Proof. Since

$$\frac{d}{dt}(\log f_t') = (\zeta A)' + \zeta A (\log f_t')', \tag{8.2.12}$$

we have

$$\begin{aligned} &\tfrac{\dot{c}}{c} - \sum \tfrac{\dot{a}_k}{\zeta - a_k} + \sum \tfrac{\dot{b}_k}{\zeta - b_k} \\ &= \tfrac{1}{c^2} (\sum_k B_k + \prod \tfrac{b_k}{a_k} - \sum \tfrac{2a_k^2 B_k}{(\zeta - a_k)^2}) + \tfrac{1}{c^2} \zeta (\sum B_k \tfrac{\zeta + a_k}{\zeta - a_k} + \prod \tfrac{b_k}{a_k})(\sum \tfrac{1}{\zeta - a_k} - \sum \tfrac{1}{\zeta - b_k}) \\ &= \tfrac{1}{c^2} (\sum_k B_k + \prod \tfrac{b_k}{a_k} - \sum \tfrac{2a_k^2 B_k}{(\zeta - a_k)^2}) + \tfrac{1}{c^2} (\zeta (\sum B_k + \prod \tfrac{b_k}{a_k}) + \sum 2B_k a_k + \sum \tfrac{2B_k a_k^2}{\zeta - a_k})(\sum \tfrac{1}{\zeta - a_k} - \sum \tfrac{1}{\zeta - b_k}) \\ &= (\gamma - \tfrac{1}{c^2} \sum \tfrac{2a_k^2 B_k}{(\zeta - a_k)^2}) + (\zeta\gamma + \tfrac{1}{c^2} \sum 2B_k a_k + \tfrac{1}{c^2} \sum \tfrac{2B_k a_k^2}{\zeta - a_k})(\sum \tfrac{1}{\zeta - a_k} - \sum \tfrac{1}{\zeta - b_k}), \end{aligned} \tag{8.2.13}$$

where $\gamma = \frac{1}{c^2}(\sum_k B_k + \prod \frac{b_k}{a_k})$.

Compare residues, we get

$$\begin{aligned} -\dot{a}_j &= \gamma a_j + \tfrac{1}{c^2}\sum 2B_k a_k + \tfrac{1}{c^2}\sum_{k\neq j}\tfrac{2B_k a_k^2}{a_j - a_k} + \tfrac{1}{c^2}2B_j a_j^2(\sum_{k\neq j}\tfrac{1}{a_j-a_k} - \sum \tfrac{1}{a_j - b_k}), \\ \dot{b}_j &= -\gamma b_j - \tfrac{1}{c^2}\sum 2B_k a_k - \tfrac{1}{c^2}\sum \tfrac{2B_k a_k^2}{b_j - a_k}. \end{aligned} \tag{8.2.14}$$

Let $\zeta \to \infty$, we have

$$\frac{\dot{c}}{c} = \gamma. \tag{8.2.15}$$

Now we solve these $2n+1$ equations, we can get a_j's, b_j's and c.

□

8.3 Examples

Deltoid and Zhukowski's airfoils are among the easiest examples of Hele-Shaw flows, so here we will solve them and show how to use these two methods.

8.3.1 Deltoid

Let the potential be $Q(z) = |z|^2 - 2R[bz^3]$, $b \in \mathbb{C}$. Take $b = \frac{1}{6}$. Then

$$h(z) = \frac{z^2}{2}, \tag{8.3.1}$$

and

$$\partial Q(z) = \bar{z} - \frac{z^2}{2} \tag{8.3.2}$$

gives that the only local minimum of $Q(z)$ is zero.

By exterior Faber transform, we have

$$f(\zeta) = r\zeta + g(\zeta) = r\zeta + \bar{c}_0 + \bar{c}_1\zeta^{-1} + \bar{c}_2\zeta^{-2}, \tag{8.3.3}$$

$$g_*(\zeta) = c_0 + c_1\zeta + c_2\zeta^2, \tag{8.3.4}$$

$$\frac{z^2}{2} = c_0 + c_1\frac{z - \bar{c}_0}{r} + c_2(\frac{(z-\bar{c}_0)^2}{r^2} - \frac{2\bar{c}_1}{r}), \tag{8.3.5}$$

then we have

$$c_0 - \frac{c_1\bar{c}_0}{r} + \frac{c_2\bar{c}_0^2}{r^2} - \frac{2\bar{c}_1 c_2}{r} = 0, \tag{8.3.6}$$

$$\frac{c_1}{r} - \frac{2c_2\bar{c}_0}{r^2} = 0, \tag{8.3.7}$$

$$\frac{c_2}{r^2} = \frac{1}{2}, \tag{8.3.8}$$

$\Rightarrow$

$$\begin{cases} c_2 = \frac{r^2}{2}, \\ c_1 = r\bar{c}_0, \\ c_0 = \frac{\bar{c}_0^2}{2} + r^2 c_0. \end{cases} \tag{8.3.9}$$

If $c_0 \not\equiv 0$, then we can prove that f is not univalent.

So we need $c_0 \equiv 0$. Then $c_1 \equiv 0$ and $c_2 = \frac{r^2}{2}$.

$$\begin{aligned} Z(t,\phi) &= f(e^{i\phi}) && (8.3.10)\\ &= re^{i\phi} + \frac{r^2}{2}e^{-2i\phi} && (8.3.11)\\ &= r(\sum_{n=0}^{\infty}\frac{(i\phi)^n}{n!}) + \frac{r^2}{2}(\sum_{n=0}^{\infty}\frac{(-2i\phi)^n}{n!})) && (8.3.12)\\ &= (r+\frac{r^2}{2}) + (r\phi - r^2\phi)i + (-\frac{r\phi^2}{2} - r^2\phi^2) + (-\frac{r\phi^3}{6} + \frac{2r^2\phi^3}{3})i + ... && (8.3.13) \end{aligned}$$

Truncate $Z(t,\phi)$ and keep terms until ϕ^3, the Cartesian coordinates will be

$$\begin{cases} X(\phi,t) = (r+\frac{r^2}{2}) + (-\frac{r}{2} - r^2)\phi^2, \\ Y(\phi,t) = (r - r^2)\phi + (-\frac{r}{6} + \frac{2r^2}{3})\phi^3. \end{cases} \tag{8.3.14}$$

Since $q = -2\pi$, By area theorem, we have

$$-2t = r^2 - \frac{r^4}{2}, \tag{8.3.15}$$

with $0 < r \le 1$ and $-\frac{1}{4} \le t < 0$.

Then

$$r = \sqrt{1 - \sqrt{1+4t}}. \tag{8.3.16}$$

Let $\tilde{t} = -(t + \frac{1}{4})$. Then the cusp apprears at $\tilde{t} = 0$, and

$$r = \sqrt{1 - \sqrt{-4\tilde{t}}}, \tag{8.3.17}$$

so

$$\begin{aligned} r + \tfrac{r^2}{2} &= \sqrt{1-\sqrt{-4\tilde{t}}} + \tfrac{1-\sqrt{-4\tilde{t}}}{2} = \tfrac{3}{2} - \sqrt{-4\tilde{t}} + o(\sqrt{-\tilde{t}}), \\ -\tfrac{r}{2} - r^2 &= -\tfrac{\sqrt{1-\sqrt{-4\tilde{t}}}}{2} - (1-\sqrt{-4\tilde{t}}) = -\tfrac{3}{2} + O(\sqrt{-\tilde{t}}), \\ r - r^2 &= \sqrt{1-\sqrt{-4\tilde{t}}} - (1-\sqrt{-4\tilde{t}}) = \tfrac{1}{2}\sqrt{-4\tilde{t}} + o(\sqrt{-\tilde{t}}), \\ -\tfrac{r}{6} + \tfrac{2r^2}{3} &= -\tfrac{\sqrt{1-\sqrt{-4\tilde{t}}}}{6} + \tfrac{2(1-\sqrt{-4\tilde{t}})}{3} = \tfrac{1}{2} + O(\sqrt{-\tilde{t}}), \end{aligned} \tag{8.3.18}$$

then

$$
\begin{aligned}
X(\tilde{t},\phi) &= \tfrac{3}{2} - 2\sqrt{-\tilde{t}} - \tfrac{3}{2}\phi^2 + o(\sqrt{-\tilde{t}} + \phi^2), \\
Y(\tilde{t},\phi) &= \sqrt{-\tilde{t}}\phi + \tfrac{1}{2}\phi^3 + o(|\sqrt{-\tilde{t}}\phi| + |\phi|^3).
\end{aligned}
\tag{8.3.19}
$$

Move the Hele-Shaw flow along X-axis by linear transformation s.t. the cusp point is at the origin, we get

$$
\begin{aligned}
X(\tilde{t},\phi) &= -2\sqrt{-\tilde{t}} - \tfrac{3}{2}\phi^2 + o(\sqrt{-\tilde{t}} + \phi^2), \\
Y(\tilde{t},\phi) &= \sqrt{-\tilde{t}}\phi + \tfrac{1}{2}\phi^3 + o(|\sqrt{-\tilde{t}}\phi| + |\phi|^3),
\end{aligned}
\tag{8.3.20}
$$

and

$$
\{X, Y\} = \frac{\partial X}{\partial \phi}\frac{\partial Y}{\partial \tilde{t}} - \frac{\partial Y}{\partial \phi}\frac{\partial X}{\partial \tilde{t}} = -1. \tag{8.3.21}
$$

The Cartesian coordinates are as stated in the main theorem.

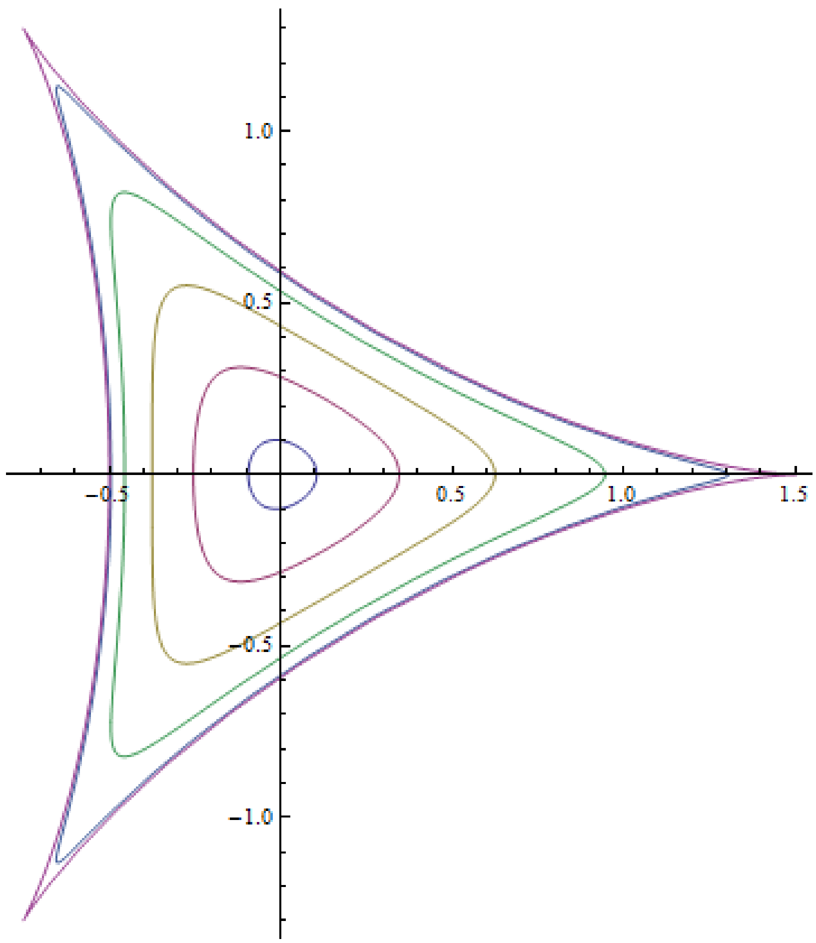

Figure 8.1: Hele-Shaw flow growing into a deltoid.

8.3.2 Zhukowski's Airfoils via Faber Transform

Hele-Shaw potential: $Q(z) = |z|^2 - H(z) = |z|^2 - B\log|z-2|^2$, $-1 < B < 0$. Then $h(z) = \partial H(z) = \frac{B}{z-2}$.
From $\partial Q(z) = \bar{z} - \frac{B}{z-2}$, we get $z = 1 \pm \sqrt{1+B}$ and $z = 1-\sqrt{1+B}$ is the only local minimum.
By exterior Faber transform, we get
$f(\zeta) = r\zeta + g(\zeta)$ and $\Phi[g_*; r\zeta + g] = h = \frac{B}{z-2}$.
Then $g_*(\zeta) = \frac{\beta}{\zeta-\lambda}$ and $\beta f'(\lambda) = B$, $f(\lambda) = 2$.
Since $f(\zeta) = r\zeta + \frac{\bar{\beta}\zeta}{1-\bar{\lambda}\zeta}$, $f'(\zeta) = r + \frac{\bar{\beta}}{(1-\bar{\lambda}\zeta)^2}$,
$\beta r + \frac{|\beta|^2}{(1-|\lambda|^2)^2} = B$, $r\lambda + \frac{\bar{\beta}\lambda}{1-|\lambda|^2} = 2$.
Suppose $r \in R$, then $\beta \in R$ and $\lambda \in R$.
By area theorem (suppose $q = -2\pi$),
$-2\pi t = \pi r^2 - \frac{\beta^2\pi}{\lambda^2}\sum \frac{n}{\lambda^{2n}} = \pi r^2 - \frac{\pi\beta^2}{(\lambda^2-1)^2}$.
$\Rightarrow$

$$\begin{aligned} -2t &= \frac{-\beta^2}{(\lambda^2-1)^2} + r^2, \\ \beta r + \frac{\beta^2}{(\lambda^2-1)^2} &= B, \\ r\lambda + \frac{\beta\lambda}{1-\lambda^2} &= 2, \end{aligned} \tag{8.3.22}$$

$\Rightarrow$

$$\begin{aligned} r &= \frac{1}{\lambda} - \frac{\lambda t}{2}, \\ \beta &= \frac{(1-\lambda^2)(2+t\lambda^2)}{2\lambda}, \end{aligned} \tag{8.3.23}$$

$r\beta = B - \frac{\beta^2}{(1-\lambda^2)^2} < 0$, so f is univalent iff $|\lambda| \geq 1 + \sqrt{\frac{-\beta}{r}}$.
Also

$$\begin{aligned} \beta_\pm &= -\frac{(\lambda^2-1)^2}{\lambda^3} \pm \frac{\lambda^2-1}{\lambda}\sqrt{\frac{(\lambda^2-1)^2}{\lambda^4} + B}, \\ r_\pm &= \frac{1}{\lambda}(1 + \frac{1}{\lambda^2} \pm \sqrt{\frac{(\lambda^2-1)^2}{\lambda^4} + B}), \\ t_\pm &= -\frac{2}{\lambda^2}(\frac{1}{\lambda^2} \pm \sqrt{\frac{(\lambda^2-1)^2}{\lambda^4} + B}). \end{aligned} \tag{8.3.24}$$

To make f univalent, we need $|\lambda| \geq 1 + \sqrt{\frac{-\beta}{r}}$, i.e. $\lambda^2 - |\lambda| \geq \frac{\lambda^2-1}{\lambda r}$.
Let $\lambda > 1$, then $r > 0$, $(B+1)\lambda^3 - 3\lambda + 2 \geq 0$ and $\lambda \geq \lambda_* =$ maximal root of $(B+1)\lambda^3 - 3\lambda + 2 = 0$.
As $\lambda \to \lambda_*$, $f'(\zeta) \to 0$ at $\zeta = 1$, so we get a cusp.

$$\begin{aligned} Z(t,\phi) &= f(e^{i\phi}) = re^{i\phi} + \frac{\beta e^{i\phi}}{1-\lambda e^{i\phi}} \\ &= (r - \frac{\beta}{\lambda-1}) + (r + \frac{\beta}{(\lambda-1)^2})i\phi + (-\frac{r}{2} + \frac{\beta(\lambda+1)}{2(\lambda-1)^3})\phi^2 + (-\frac{r}{6} - \frac{\beta(\lambda^2+4\lambda+1)}{6(\lambda-1)^4})i\phi^3 + \cdots \end{aligned} \tag{8.3.25}$$

Take a specific $B = -0.5$, then $\lambda_* = 2, r_* = 0.75, \beta_* = -0.75, t_* = -0.25$ then

$$Z(t_*,\phi) = 0.75(e^{i\phi} - \frac{e^{i\phi}}{1-2e^{i\phi}}),$$

$$\begin{aligned}\beta_+ &= -\tfrac{(\lambda^2-1)^2}{\lambda^3} + \tfrac{\lambda^2-1}{\lambda}\sqrt{\tfrac{0.5\lambda^4-2\lambda^2+1}{\lambda^4}},\\ r_+ &= \tfrac{1}{\lambda}(1+\tfrac{1}{\lambda^2}+\sqrt{\tfrac{0.5\lambda^4-2\lambda^2+1}{\lambda^4}}),\\ t_+ &= -\tfrac{2}{\lambda^2}(\tfrac{1}{\lambda^2}+\sqrt{\tfrac{0.5\lambda^4-2\lambda^2+1}{\lambda^4}}).\end{aligned} \tag{8.3.26}$$

Let $l = \frac{1}{\lambda^2}$, then l increases to $\frac{1}{4}$ as $\lambda \to \lambda_*$.
then $t = t_+ = -2l(l + \sqrt{\frac{1}{2} - 2l + l^2})$,

$$\sqrt{\frac{1}{2} - 2l + l^2} = (\frac{1}{16} + \frac{3}{2}(\frac{1}{4} - l) + (\frac{1}{4} - l)^2)^{1/2} = 1 - 3l + o(1 - 4l),$$

$\Rightarrow t = -2l(1-2l) + o(1-4l)$ and $l = \frac{1-\sqrt{1+4t}}{4} + o(\sqrt{1+4t})$.
Let $\tilde{t} = -\frac{1+4t}{4} \leq 0$, then $l = \frac{1-\sqrt{-4\tilde{t}}}{4} + o(\sqrt{-\tilde{t}})$.

$$\begin{aligned} r = r+ &= \tfrac{3}{4} - \tfrac{\sqrt{-\tilde{t}}}{4} + o(\sqrt{-\tilde{t}}),\\ r - \tfrac{\beta}{\lambda-1} &= \tfrac{3}{2} - 2\sqrt{-\tilde{t}} + o(\sqrt{-\tilde{t}}),\\ r + \tfrac{\beta}{(\lambda-1)^2} &= 3\sqrt{-\tilde{t}} + o(\sqrt{-\tilde{t}}),\\ -\tfrac{r}{2} + \tfrac{\beta(\lambda+1)}{2(\lambda-1)^3} &= -\tfrac{3}{2} + O(\sqrt{-\tilde{t}}),\\ -\tfrac{r}{6} - \tfrac{\beta(\lambda^2+4\lambda+1)}{6(\lambda-1)^4} &= \tfrac{3}{2} + O(\sqrt{-\tilde{t}}),\end{aligned} \tag{8.3.27}$$

$\Rightarrow$

$$\begin{aligned} X(\tilde{t},\phi) &= \tfrac{3}{2} - 2\sqrt{-\tilde{t}} - \tfrac{3}{2}\phi^2 + o(\sqrt{-\tilde{t}} + \phi^2),\\ Y(\tilde{t},\phi) &= 3\sqrt{-\tilde{t}}\phi + \tfrac{3}{2}\phi^3 + o(|\sqrt{-\tilde{t}}\phi| + |\phi|^3),\end{aligned} \tag{8.3.28}$$

let $t = 3\tilde{t}$, then

$$\begin{aligned} X(t,\phi) &= \tfrac{3}{2} - \tfrac{2\sqrt{3}}{3}\sqrt{-t} - \tfrac{3}{2}\phi^2 + o(\sqrt{-t} + \phi^2),\\ Y(t,\phi) &= \sqrt{3}\sqrt{-t}\phi + \tfrac{3}{2}\phi^3 + o(|\sqrt{-t}\phi| + |\phi|^3).\end{aligned} \tag{8.3.29}$$

If we move the Hele-Shaw flow by linear transformation s.t. the cusp is at the origin, we will get the same result as using the Löwner equation(details are in the next subsection), and as stated in the main theorem.

8.3.3 Zhukowski's Airfoils via Löwner Equation

We used exterior Faber transform to get the equation of boundaries of Zhukowski's airfoils. Now we have another method: Löwner equation.

Let $f_t(\zeta)$: $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to D(t)$.

Suppose $f_0(\zeta) = \frac{3}{4}(\zeta + \frac{\frac{1}{4}}{\zeta - \frac{1}{2}} - \frac{3}{2})$. Then by exterior Faber transform, $f_t(\zeta)$ is

$$f_t(\zeta) = a(t)\zeta + b(t) + \frac{c(t)}{\zeta - \lambda(t)},$$

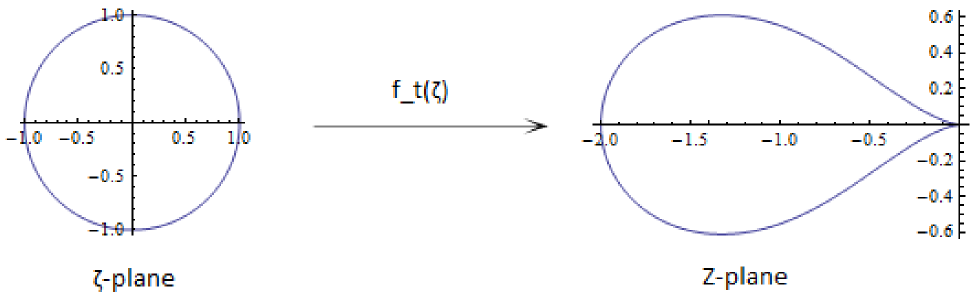

Figure 8.2: Conformal map $f_t(\zeta)$ from $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ to a Zhukowski's airfoil with a (3,2)-cusp.

and

$$f_t'(\zeta) = a(t) - \frac{c(t)}{(\zeta - \lambda(t))^2},$$

also $f_0(1) = 0$, $f_0'(1) = 0$.
$\Rightarrow \lambda(0) = 1 - \sqrt{\frac{c(0)}{a(0)}}$ since $|\lambda(0)| < 1$.
Let $a_1(t) = \lambda(t) - \sqrt{\frac{c(t)}{a(t)}}$, $a_2(t) = \lambda(t) + \sqrt{\frac{c(t)}{a(t)}}$,
$b_1(t) = b_2(t) = \lambda(t)$, $a(0) = \frac{3}{4}$, $b(0) = -\frac{9}{8}$, $c(0) = \frac{3}{16}$, $\lambda(0) = \frac{1}{2}$, $a_1(0) = 0$ and $a_2(0) = 1$. By similar process as in the proof of the main theorem, we get

$$\begin{aligned} a(t) &= \tfrac{3}{4} - \tfrac{\sqrt{3}}{12}\sqrt{-t} + o(\sqrt{-t}), \\ b(t) &= -\tfrac{9}{8} - \tfrac{\sqrt{3}}{6}\sqrt{-t} + o(\sqrt{-t}), \\ c(t) &= \tfrac{3}{16} - \tfrac{7\sqrt{3}}{48}\sqrt{-t} + o(\sqrt{-t}), \\ \lambda(t) &= \tfrac{1}{2} - \tfrac{\sqrt{3}}{6}\sqrt{-t} + o(\sqrt{-t}), \end{aligned} \tag{8.3.30}$$

then

$$Z(t,\phi) = f_t(e^{i\phi}) = a(t)e^{i\phi} + b(t) + \frac{c(t)}{e^{i\phi} - \lambda(t)} = a\sum_{n=0}^{\infty}\frac{(i\phi)^n}{n!} + b - \frac{c}{\lambda} + \frac{c}{\lambda}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{\lambda^k(-ik\phi)^n}{n!}$$

$$= (a + b - \frac{c}{\lambda} + \frac{c}{\lambda}\sum_{k=0}^{\infty}\lambda^k) + (a - \frac{c}{\lambda}\sum_{k=1}^{\infty}k\lambda^k)i\phi + (-\frac{a}{2} - \frac{c}{\lambda}\sum_{k=1}^{\infty}\frac{k^2\lambda^k}{2})\phi^2 + (-\frac{a}{6} + \frac{c}{\lambda}\sum_{k=1}^{\infty}\frac{k^3\lambda^k}{6})i\phi^3 + ...$$

$$= (a + b - \frac{c}{\lambda - 1}) + (a - \frac{c}{(\lambda-1)^2})i\phi + (-\frac{a}{2} + \frac{c(\lambda+1)}{2(\lambda-1)^3})\phi^2 + (-\frac{a}{6} + \frac{c(\lambda^2 + 4\lambda + 1)}{6(\lambda-1)^4})i\phi^3 + ...$$

Since

$$a + b - \frac{c}{\lambda - 1} = -\frac{2\sqrt{3}}{3}\sqrt{-t} + o(\sqrt{-t}),$$

$$a - \frac{c}{(\lambda-1)^2} = \sqrt{3}\sqrt{-t} + o(\sqrt{-t}),$$

$$-\frac{a}{2}+\frac{c(\lambda+1)}{2(\lambda-1)^3}=-\frac{3}{2}+O(\sqrt{-t}),$$

$$-\frac{a}{6}+\frac{c(\lambda^2+4\lambda+1)}{6(\lambda-1)^4}=\frac{3}{2}+O(\sqrt{-t}).$$

Truncate $Z(t,\phi)$ till ϕ^3, we have the Cartesian coordinates as follows:

$$\begin{aligned} X(t,\phi) &= -\tfrac{2\sqrt{3}}{3}\sqrt{-t}-\tfrac{3}{2}\phi^2+o(\sqrt{-t}+\phi^2), \\ Y(t,\phi) &= \sqrt{3}\sqrt{-t}\phi+\tfrac{3}{2}\phi^3+o(|\sqrt{-t}\phi|+|\phi|^3). \end{aligned} \tag{8.3.31}$$

$X(t,\phi)$, $Y(t,\phi)$ satisfy dispersionless string equation,

$$\frac{\partial X}{\partial \phi}\frac{\partial Y}{\partial t}-\frac{\partial Y}{\partial \phi}\frac{\partial X}{\partial t}=-1.$$

Chapter 9

Proof of the Main Theorem

In this chapter, we give the proof of the main theorem: Suppose the external potential is $Q(z) = |z|^2 - H(z)$, where $H(z)$ is harmonic and $h(z) = \partial H(z)$ is a meromorphic function. Also, suppose $D(t)$'s are simply connected unbounded domains. Then by exterior Faber transform, there exists a conformal map $f_t(\zeta) : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to D(t)$, and $f_t(\zeta)$ is a meromorphic function. Assume that $f_t'(\infty) > 0$, then $f_t'(\zeta) = c(t) \prod_j \frac{\zeta - a_j(t)}{\zeta - b_j(t)}$, where $c(t) > 0$. In chapter 8, we derived Löwner equation in algebraic case and a system of ordinary differential equations involving all coefficients $c(t), a_j(t)$'s and $b_j(t)$'s. Now, we will prove the main theorem by solving all the coefficients.

In simply connected and algebraic case,

$$f_t'(\zeta) = c(t) \prod_j \frac{\zeta - a_j(t)}{\zeta - b_j(t)},$$

$c(t) > 0$, is a conformal map from $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ to $D(t)$. So $a_j(t), b_j(t) \in \mathbb{D}$. Then $Z(t, \phi) = f_t(e^{i\phi}) = C_0(t) + C_1(t)i\phi + C_2(t)\phi^2 + C_3(t)i\phi^3 + ...$

Suppose at $t = 0$, there is a (3,2)-cusp at $\phi = 0$. then $Z(0, \phi) = C_2(0)\phi^2 + C_3(0)i\phi^3 + O(\phi^4)$, and one of the a_j's must be 1 at time $t = 0$. Without lost of generality, we can suppose $a_1(0) = 1$.

Let $c(0) = c > 0, a_j(0) = \alpha_j (j \neq 1)$ and $b_j(0) = \beta_j$, where $|\alpha_j| < 1 (j \neq 1)$, $|\beta_j| < 1$, and α_j, β_j are different.

So

$$\begin{aligned} C_0(t) &= Z(t, 0) = f_t(1), \\ C_1(t) &= \tfrac{1}{i} \tfrac{d}{d\phi} Z(t, 0) = f_t'(1), \\ C_2(t) &= \tfrac{1}{2} \tfrac{d^2}{d\phi^2} Z(t, 0) = -\tfrac{1}{2} f_t'(1) - \tfrac{1}{2} f_t''(1), \\ C_3(t) &= \tfrac{1}{6i} \tfrac{d^3}{d\phi^3} Z(t, 0) = -\tfrac{1}{6} f_t'(1) - \tfrac{1}{2} f_t''(1) - \tfrac{1}{6} f_t'''(1). \end{aligned} \tag{9.0.1}$$

Truncate $Z(t, \phi)$ and only keep terms until ϕ^3, to compute the first four terms of $Z(t, \phi)$, we only need to compute $f_t(1), f_t'(1), f_t''(1)$ and $f_t'''(1)$.

Lemma 9.0.1.

$$B_1 = \frac{|1-\beta_1|^2}{1-|a_1(t)|^2}\left(\prod_{k\neq 1}\frac{|1-\beta_k|^2}{|1-\alpha_k|^2}+o(1)\right) = \frac{r+o(1)}{1-|a_1(t)|^2},$$

$$B_j = O(1), \forall j \neq 1,$$

where $r = \frac{\prod_k |1-\beta_k|^2}{\prod_{k\neq 1}|1-\alpha_k|^2} > 0.$

Proof. Since $a_1(t) = 1+o(1), a_j(t) = \alpha_j + o(1)$,for $j \neq 1$, and $b_j(t) = \beta_j + o(1)$,

$$B_1 = \frac{|1-\beta_1|^2+o(1)}{1-|a_1(t)|^2}\prod_{k\neq 1}\frac{|1-\beta_k|^2+o(1)}{|1-\alpha_k|^2+o(1)} = \frac{|1-\beta_1|^2}{1-|a_1(t)|^2}\left(\prod_{k\neq 1}\frac{|1-\beta_k|^2}{|1-\alpha_k|^2}+o(1)\right),$$

$$B_j = \frac{(1-\alpha_j\bar{\beta}_j)(\alpha_j-\beta_j)+o(1)}{(1-|\alpha_j|^2)\alpha_j+o(1)}\prod_{k\neq j}\frac{(1-\bar{\beta}_k\alpha_j)(\alpha_j-\beta_k)+o(1)}{(1-\bar{\alpha}_k\alpha_j)(\alpha_j-\alpha_k)+o(1)} = O(1), \forall j \neq 1.$$

□

Lemma 9.0.2.

$$-\dot{a}_1 = \frac{r}{c^2}(3+2(\sum_{k\neq 1}\frac{1}{1-\alpha_k}-\sum_k\frac{1}{1-\beta_k})+o(1))\frac{1}{1-|a_1(t)|^2}+O(1) = \frac{s+o(1)}{1-|a_1(t)|^2}+O(1),$$

For $j \neq 1$,

$$-\dot{a}_j = \frac{\mu_j+o(1)}{1-|a_1(t)|^2}+O(1),$$

For all j,

$$\dot{b}_j = \frac{\nu_j+o(1)}{1-|a_1(t)|^2}+O(1),$$

and

$$\dot{c}(t) = \frac{r+o(1)}{c}\frac{1}{1-|a_1(t)|^2}+O(1),$$

where

$$s = \frac{r}{c^2}(3+2(\sum_{k\neq 1}\frac{1}{1-\alpha_k}-\sum_k\frac{1}{1-\beta_k})),$$

$$\mu_j = \frac{r}{c^2}(\alpha_j+\frac{2}{\alpha_j-1}+2), \forall j \neq 1,$$

$$\nu_j = -\frac{r}{c^2}(\beta_j+\frac{2}{\beta_j-1}+2), \forall j.$$

Proof.

$$\gamma = \frac{1}{c^2+o(1)}(B_1+O(1)+\prod\frac{\beta_k+o(1)}{\alpha_k+o(1)}) = \frac{1}{c^2+o(1)}B_1+O(1).$$

$$-\dot{a}_1 = \gamma\,(1+o(1))+\frac{1}{c^2+o(1)}\left(2B_1(1+o(1))+O(1)+2B_1(1+o(1))(\sum_{k\neq 1}\frac{1}{1-\alpha_k}-\sum\frac{1}{1-\beta_k}+o(1))\right)$$

$$= \frac{1}{c^2+o(1)}\left(3+o(1)+2(\sum_{k\neq 1}\frac{1}{1-\alpha_k}-\sum\frac{1}{1-\beta_k}+o(1))\right)B_1+O(1)=\frac{s+o(1)}{1-|a_1(t)|^2}+O(1).$$

For $j\neq 1$,

$$-\dot{a}_j=\gamma(\alpha_j+o(1))+\frac{1}{c^2+o(1)}\left(2B_1(\frac{1}{\alpha_j-1}+o(1))+2B_1(1+o(1))\right)+O(1)$$

$$=\frac{1}{c^2+o(1)}\left(\alpha_j+\frac{2}{\alpha_j-1}+2+o(1)\right)B_1+O(1)=\frac{\mu_j+o(1)}{1-|a_1(t)|^2}+O(1).$$

For all j,

$$\dot{b}_j=-\gamma(\beta_j+o(1))-\frac{1}{c^2+o(1)}\left(2B_1(\frac{1}{\beta_j-1}+o(1))+2B_1(1+o(1))\right)+O(1)$$

$$=-\frac{1}{c^2+o(1)}\left(\beta_j+\frac{2}{\beta_j-1}+2+o(1)\right)B_1+O(1)=\frac{\nu_j+o(1)}{1-|a_1(t)|^2}+O(1).$$

And

$$\dot{c}(t)=\frac{1}{c+o(1)}\left(B_1+O(1)\right)=\frac{r+o(1)}{c}\frac{1}{1-|a_1(t)|^2}+O(1).$$

□

Lemma 9.0.3. *Suppose $s=s_1+is_2$, $s_1,s_2\in\mathbb{R}$ and $c_0=\frac{s_2}{s_1}$, then*

$$\begin{aligned}
&a_1(t)=1-\sqrt{s_1t}-ic_0\sqrt{s_1t}+o(\sqrt{-t}),\\
&a_j(t)=\alpha_j+O(\sqrt{-t}),\forall j\neq 1,\\
&b_j(t)=\beta_j+O(\sqrt{-t}),\forall j,\\
&c(t)=c+O(\sqrt{-t}).
\end{aligned}\tag{9.0.2}$$

Proof. Let $a_1(t)=q_1(t)+iq_2(t)$, $q_1(t),q_2(t)\in\mathbb{R}$, $q_1(0)=1,q_2(0)=0$.
Then

$$-\dot{q}_1-i\dot{q}_2=\frac{s_1+is_2+o(1)}{1-q_1^2-q_2^2}+O(1),$$

since $\frac{\dot{q}_1}{\dot{q}_2}=\frac{s_1}{s_2}+o(1)$, $q_2\approx c_0(q_1-1)$, where $c_0=\frac{s_2}{s_1}$.
Solve $-\dot{q}_1=\frac{s_1}{1-q_1^2-c_0^2(q_1-1)^2}$, we get $\frac{1}{3}(q_1-1)^2(q_1+2+c_0^2(q_1-1))=s_1t$.
Since $q_1(t)=1+o(1)$,$(q_1-1)^2\approx s_1t$, so $q_1(t)\approx 1-\sqrt{s_1t}$ and $q_2(t)\approx -c_0\sqrt{s_1t}$.
Check $q_1(t)=1-\sqrt{s_1t}+o(\sqrt{-t})$ and $q_2(t)=-c_0\sqrt{s_1t}+o(\sqrt{-t})$, we know they are solutions of the previous PDE.
Then $a_1(t)=1-\sqrt{s_1t}-ic_0\sqrt{s_1t}+o(\sqrt{-t})$.
Then $-\dot{a}_j(t)=\frac{\mu_j+o(1)}{1-|a_1(t)|^2}+O(1)=\frac{\mu_j+o(1)}{2\sqrt{s_1t}-s_1t-c_0^2s_1t}+O(1)$,
so $a_j(t)=\alpha_j+O(\sqrt{-t})$ for $j\neq 1$.

Similarly, $b_j(t) = \beta_j + O(\sqrt{-t})$ for all j.

$c(t) = c + O(\sqrt{-t})$. □

Theorem 9.0.4.

$$\begin{aligned} C_0(t) &= -\sqrt{\tfrac{4rt}{c^2 s_1}} + o(\sqrt{-t}),\\ C_1(t) &= -s\sqrt{\tfrac{c^2 t}{s_1 r}} + o(\sqrt{-t}),\\ C_2(t) &= -\tfrac{c}{2\sqrt{r}} + O(\sqrt{-t}),\\ C_3(t) &= -\tfrac{c}{2\sqrt{r}}\tfrac{sc^2}{3r} + O(\sqrt{-t}). \end{aligned} \tag{9.0.3}$$

Proof. Now

$$C_1(t) = c(t)\prod_j \frac{1-a_j(t)}{1-b_j(t)} = c\frac{\prod_{j\neq 1}(1-\alpha_j)}{\prod_j(1-\beta_j)}\frac{s}{s_1}\sqrt{s_1 t} + o(\sqrt{-t}).$$

By Löwner equation,

$$\dot{f}_t = \zeta A(\zeta, t) f_t',$$

then

$$\dot{f}_t(1) = \frac{1}{c^2 + o(1)}(B_1\frac{1+a_1(t)}{1-a_1(t)} + O(1))c(t)\prod_j \frac{1-a_j(t)}{1-b_j(t)}$$

$$= \left(\frac{2r}{c}\frac{\prod_{j\neq 1}(1-\alpha_j)}{\prod_j(1-\beta_j)} + o(1)\right)\frac{1}{1-|a_1(t)|^2} + O(1) = (\frac{2r}{cs}\frac{\prod_{j\neq 1}(1-\alpha_j)}{\prod_j(1-\beta_j)} + o(1))(-\dot{a}_1) + O(1),$$

so

$$C_0(t) = f_t(1) = \frac{2r}{c}\frac{\prod_{j\neq 1}(1-\alpha_j)}{\prod_j(1-\beta_j)}\frac{-(a_1(t)-1)}{s} + o(\sqrt{-t}) = \frac{2r}{c}\frac{\prod_{j\neq 1}(1-\alpha_j)}{\prod_j(1-\beta_j)}\frac{\sqrt{s_1 t}}{s_1} + o(\sqrt{-t}).$$

Since

$$f_t''(\zeta) = c(t)(\prod_j \frac{\zeta - a_j(t)}{\zeta - b_j(t)})'$$

$$= c(t)\frac{(\sum_j \prod_{k\neq j}(\zeta - a_k(t)))\prod_j(\zeta - b_j(t)) - \prod_j(\zeta - a_j(t))(\sum_j \prod_{k\neq j}(\zeta - b_k(t)))}{\prod_j(\zeta - b_j(t))^2},$$

we have

$$C_2(t) = -\frac{1}{2}f_t'(1) - \frac{1}{2}f_t''(1) = -\frac{c}{2}\frac{\Pi_{k\neq 1}(1-\alpha_k)}{\Pi_j(1-\beta_j)} + O(\sqrt{-t}).$$

Since

$$f_t'''(\zeta) = c(t)(\frac{(\sum_j \Pi_{k\neq j}(\zeta - a_k(t)))\Pi_j(\zeta - b_j(t)) - \Pi_j(\zeta - a_j(t))(\sum_j \Pi_{k\neq j}(\zeta - b_k(t)))}{\Pi_j(\zeta - b_j(t))^2})'$$

$$= c(t)[\frac{(\sum_j\sum_{k\neq j}\Pi_{l\neq k,j}(\zeta - a_l(t)))\Pi_j(\zeta - b_j(t)) + (\sum_j \Pi_{k\neq j}(\zeta - a_k(t)))(\sum_j \Pi_{k\neq j}(\zeta - b_k(t))}{\Pi_j(\zeta - b_j(t))^2}$$

$$-\frac{(\sum_j \Pi_{k\neq j}(\zeta - a_k(t)))(\sum_j \Pi_{k\neq j}(\zeta - b_k(t))) + \Pi_j(\zeta - a_j(t))(\sum_j\sum_{k\neq j}\Pi_{l\neq k,j}(\zeta - b_l(t)))}{\Pi_j(\zeta - b_j(t))^2}$$

$$-\frac{(\sum_j \Pi_{k\neq j}(\zeta-a_k(t))\Pi_j(\zeta-b_j(t))-\Pi_j(\zeta-a_j(t))\sum_j \Pi_{k\neq j}(\zeta-b_k(t)))(\sum_j 2(\zeta-b_j(t))\Pi_{k\neq j}(\zeta-b_k(t))^2)}{\Pi_j(\zeta-b_j(t))^4}],$$

we have

$$f_t'''(1)=c[\frac{(\sum_j\sum_{k\neq j}\Pi_{l\neq k,j}(1-\alpha_l))\Pi_j(1-\beta_j)}{\Pi_j(1-\beta_j)^2}-\frac{(\Pi_{k\neq 1}(1-\alpha_k)\Pi_j(1-\beta_j))(\sum_j 2(1-\beta_j)\Pi_{k\neq j}(1-\beta_k)^2)}{\Pi_j(1-\beta_j)^4}]$$

$$+O(\sqrt{-t})$$

$$=c[\frac{(2\sum_{k\neq 1}\Pi_{l\neq k,1}(1-\alpha_l))\Pi_j(1-\beta_j)}{\Pi_j(1-\beta_j)^2}-\frac{(\Pi_{k\neq 1}(1-\alpha_k)\Pi_j(1-\beta_j))(\sum_j 2(1-\beta_j)\Pi_{k\neq j}(1-\beta_k)^2)}{\Pi_j(1-\beta_j)^4}]$$

$$+O(\sqrt{-t})$$

$$=\frac{c\Pi_{k\neq 1}(1-\alpha_k)}{\Pi_k(1-\beta_k)}(2\sum_{k\neq 1}\frac{1}{1-\alpha_k}-2\sum_k\frac{1}{1-\beta_k})+O(\sqrt{-t}),$$

then

$$C_3(t)=-\frac{1}{6}f_t'(1)-\frac{1}{2}f_t''(1)-\frac{1}{6}f_t'''(1)=-\frac{c}{2}\frac{\Pi_{k\neq 1}(1-\alpha_k)}{\Pi_j(1-\beta_j)}\left(1+\frac{2}{3}(\sum_{k\neq 1}\frac{1}{1-\alpha_k}-\sum_k\frac{1}{1-\beta_k})\right)+O(\sqrt{-t}).$$

Since we have a $(3,2)$-cusp at $t=0$, $C_2(0)\in R$. i.e. $\frac{\Pi_{k\neq 1}(1-\alpha_k)}{\Pi_j(1-\beta_j)}\in R$ and $\frac{\Pi_{k\neq 1}(1-\alpha_k)}{\Pi_j(1-\beta_j)}=\frac{1}{\sqrt{r}}$, $s_1<0$. Therefore,

$$C_0(t)=-\sqrt{\frac{4rt}{c^2s_1}}+o(\sqrt{-t}),$$

$$C_1(t)=-s\sqrt{\frac{c^2t}{s_1r}}+o(\sqrt{-t}),$$

$$C_2(t)=-\frac{c}{2\sqrt{r}}+O(\sqrt{-t}),$$

$$C_3(t)=-\frac{c}{2\sqrt{r}}\frac{sc^2}{3r}+O(\sqrt{-t}).$$

□

Now we have the main theorem.

Theorem 9.0.5. *Let $X(t,\phi),Y(t,\phi)$ be the Cartesian coordinates, then*

$$X(t,\phi)=-\sqrt{\frac{4r}{s_1c^2}t}-\frac{c}{2\sqrt{r}}\phi^2+o(\sqrt{-t}+\phi^2),$$

$$Y(t,\phi) = \sqrt{\frac{c^2 s_1 t}{r}}\phi - \frac{c}{2\sqrt{r}}\frac{s_1 c^2}{3r}\phi^3 + o(|\sqrt{-t}\phi| + |\phi|^3).$$

Proof. Since $s_2\sqrt{\frac{c^2 t}{s_1 r}}\phi = o(\sqrt{-t} + \phi^2)$, we get the result from the previous theorem. □

Remark 9.0.6. *If we use the notation in the main theorem,* $c_2 = -\frac{c}{2\sqrt{r}} < 0$, $c_3 = -\frac{c}{2\sqrt{r}}\frac{s_1 c^2}{3r} > 0$, *we have*

$$X(t,\phi) = -\sqrt{\frac{4c_2}{3c_3}t} + c_2\phi^2 + o(\sqrt{-t} + \phi^2),$$

$$Y(t,\phi) = \sqrt{\frac{3c_3}{c_2}t\phi} + c_3\phi^3 + o(|\sqrt{-t}\phi| + |\phi|^3),$$

as stated in the main theorem and Remark 6.1.2.

Chapter 10

Next Steps

In the main theorem, we proved that if the external potential is algebraic and droplets are simply connected, near a (3,2)-cusp, Hele-Shaw flow is a one-parameter family after scaling the Cartesian coordinates. How about other cases? For example, if droplets are non-simply-connected droplets, or the cusp is a higher-order cusp, or the external potential is not algebraic, will Hele-Shaw flow still be a one-parameter family after scaling the Cartesian coordinates? And how about other types of singularities such as double points, etc.? We do not have proof for these cases, but would like to take a guess:

1) For non-simply-connected droplets, no conformal maps from the complement of the droplets to the complement of the unit disk exist, so we cannot use either exterior Faber transform or Löwner equation to compute the boundary equation. But we can try to use canonical maps from the complement of the droplets to torus, in this case Hele-Shaw problem is more complicated.

Figure 10.1: Non-simply-connected droplets.

2) For higher-order $(2p+1,2)$-cusps, we have mentioned the conjecture in chapter 6 given by physicists in [15]. Though the computation is more complicated than (3,2)-cusps, we believe that it is doable and the conjecture is correct.

3) For another type of cusps (type II cusps), these cusps in Hele-Shaw flow are still laminar-flow points [14], so we may try to expand the conjecture. This is an interesting problem and its connec-

tion with integrable systems has not been discovered yet.

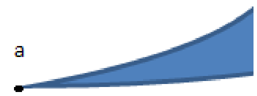

Figure 10.2: A type II cusp.

4) For other types of singularities, such as double points, results have been given by Seung-Yeop Lee, etc. in [7]. In the process of droplets contracting, they discovered that the double point singularities are self similar after scaling, and it is related to disperionless AKNS hierarchy.

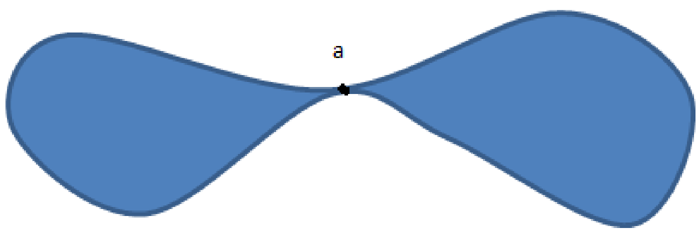

Figure 10.3: A type I double point.

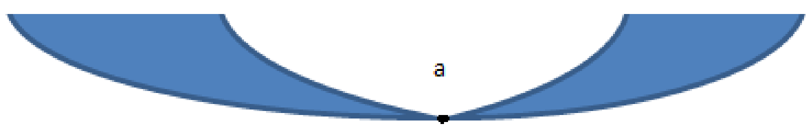

Figure 10.4: A type II double point.

5) For nonalgebraic potentials, our methods are completely inapplicable. It is not obvious to see if the conjecture is still reasonable or not.

Bibliography

[1] B. Gustafsson, *On a differential equation arising in a Hele Shaw flow moving boundary problem*, Ark. Mat., 22 (1984), 251–268.

[2] B. Gustafsson, A. Vasil'ev, *Conformal and Potential Analysis in Hele-Shaw Cells*, Birkhäuser, 2006.

[3] H. Hedenmalm, N. Makarov, *Coulomb gas ensembles and Laplacian growth*, ArXiv:1106.2971v3, 15 Jun 2011.

[4] H. S. Hele-Shaw, *The flow of water*, Nature, 58 (1898), No. 1489, 34–36, 520.

[5] S. D. Howison, *Bubble growth in porous media and Hele-Shaw cells*, Proc. Roy. Soc. Edinburgh, Section A Mathematics, 102, 141–148.

[6] S. D. Howison, *Cusp development in Hele-Shaw flow with a free surface*, SIAM J. Appl. Math., Vol. 46 (1986), No.1, 20–26.

[7] S. Y. Lee, E. Bettelheim, P. Wiegmann, *Bubble break-off in Hele-Shaw flows—Singularities and integrable structures*, ArXiv:nlin/0604054v2, 25 Apr 2006.

[8] M. Mineev-Weinstein, P. B. Wiegmann, A. Zabrodin, *Integrable structure of interface dynamics*, Phys. Rev. Lett. 84, 5106 (2000).

[9] M. Reissig, L. V. Wolfersdorf, *A simplified proof for a moving boundary problem for Hele-Shaw flows in the plane*, Ark. Mat., 31 (1993), 101–116.

[10] S. Richardson, *Hele Shaw flows with a free boundary produced by the injection of fluid into a narrow channel*, J. Fluid Mech. (1972), Vol. 56, part 4, 609–618.

[11] P. G. Saffman, G. I. Taylor, *The penetration of a fluid into a porous medium or Hele-Shaw cell containing a more viscous liquid*, Proc. Royal Soc. London, Ser. A, 245 (1958), No. 281, 312–329.

[12] M. Sakai, *Regularity of a boundary having a Schwarz function*, Acta Math., 166 (1991), 263–297.

[13] M. Sakai, *Regularity of free boundaries in two dimensions*, Ann. Scuola Norm. Sup. Pisa, Cl. Sci. (4) 20 (1993), 323–339.

[14] M. Sakai, *Small Modifications of Quadrature Domains*, Memoirs of the American Mathematical Society, Vol. 206, No. 969, 2010.

[15] R. Teodorescu, P. Wiegmann, A. Zabrodin, *Unstable fingering patterns of Hele-Shaw flows as a dispersionless limit of the Kortweg–de Vries hierarchy*, Phys. Rev. Lett. 95 044502 (2005).

[16] A. N. Varchenko, P. I. Etingof, *Why the Boundary of a Round Drop Becomes a Curve of Order Four*, University Lecture Series, Vol. 3, AMS, 1992.