# DIRECTIONAL AND STATIC EQUILIBRIUM
# IN SOCIAL DECISION PROCESSES

Thesis by

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## ABSTRACT

This thesis proposes a model of social decision processes that is applicable to situations in which social change must be incremental. In the limit, only the direction and not the speed of a shift in the status quo can be decided at each point in time. Individual preferences over directions are induced myopically via the inner product of direction (unit) vectors with the gradients of utility functions. Then the direction of shift at each instant is taken to be an equilibrium of a game that has directional outcomes.

Both two-person non-cooperative games in which two candidates adopt directional strategies to maximize their shares of cast votes, and n-person simple games of which absolute majority rule is a special case, are studied. Directional equilibria for the former and directional cores for the latter are characterized. Results include the following: (1) directions "pointing" towards point equilibria are directional equilibria; (2) a mobile candidate will diverge from a rigid, extremist opponent; (3) a status quo x simultaneously approaches each winning coalition's preferred-to-x set if and only if it shifts in an undominated direction; (4) given Euclidean preferences, a status quo that shifts in undominated directions will converge to the point core or to the set of points with empty directional cores; (5) an empty directional core at a point implies local cycling occurs in a neighborhood of the point; (6) stringent

pairwise symmetry conditions must be satisfied by utility gradients at a point that has a nonempty directional core in a majority rule game; and (7) undominated directions exist at boundary points of a global cycling set and "point back into" the cycling set. Results (6) and (7) indicate that for majority games in spaces of dimension greater than three, directional cores are usually empty and global cycling sets are usually the entire space.

The disseration appendix is a self-contained paper in its own right. In a behaviorally-intuitive fashion, it establishes pairwise symmetry conditions for a point contained in the interior or boundary of a convex feasible set to be quasi-undominated in an anonymous simple game.

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# INTRODUCTION

Social decisionmaking is studied by economists predominantly via the concept of equilibrium. Broadly defined, a social system is in equilibrium provided no individual or admissible set of individuals has both the ability and incentive to alter the state of the system. This simple idea is central to the definitions of competitive equilibria, Nash equilibria, and cores for processes that incorporate private goods economies, non-cooperative games, or cooperative games, respectively. Given the basic economic postulate stating that individuals act in rational, maximizing fashions, final social decisions must be equilibria. Herein lies the attractiveness of equilibrium in comparative statics models: the influence upon social decisions of variations in underlying parameters can be predicted without using detailed knowledge of the institutional or dynamical characteristics of the social process.

However, when social change cannot occur quickly, final equilibrium outcomes are of little interest. They will not be achieved for long periods of time, and in fact may not be well-defined because of temporally changing preferences and technologies. When social change is slow, it seems more important to ask what will be the direction of change rather than what will be the final outcome.

In this dissertation a social process in which change is slow is not modeled as some type of game in which a final outcome is chosen. Instead, at each point in time the process is viewed as an instantaneous game whose possible outcomes are directions in which the status quo can shift. This allows the concept of equilibrium to be reapplied to shift directions; at each point in time a directional equilibrium is defined that can be predicted to be the direction in which the status quo shifts. The idea of directional equilibrium contributes not only to our understanding of frozen snapshots of social decision processes, but also provides a behavioral basis for a study of their dynamics. Thus, in situations with fixed preferences and technologies, the convergence properties of a status quo that shifts in equilibrium directions can be studied.

The dissertation is divided into three chapters and an appendix. Plurality games in which two candidates choose directional strategies to maximize plurality are the subject of chapter I. The Nash equilibria of these games in directional strategies are characterized and implications for electoral competition made.

Chapter II deals with simple games, a special case of which is absolute majority rule. The directional cores of these games are characterized and the convergence properties of a status quo that shifts in undominated directions are determined. Furthermore, the existence of a directional core is shown to imply that local cycling in the sense of Schofield [1977] cannot occur. If the game is majority rule, then at the status quo the pairwise symmetries determined in the appendix for constrained static equilibria must hold when the directional core exists.

In chapter III directional cores at special points in the alternative space are investigated for the case of majority rule. Specifically, directional cores are characterized and shown to exist at points contained in boundaries of the top cycle sets studied by Cohen [1977] and McKelvey [1977]. This leads to conclusions about the size of these top cycle sets.

Finally, the appendix, which stands as a self-contained paper, is concerned with conditions for static equilibria in anonymous simple games. Its results, which are used in chapters II and III, generalize the conditions of Plott [1967].

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Chapter 1

# A SIMPLE DIRECTION MODEL OF ELECTORAL COMPETITION

Since the seminal contribution of Downs [1957], spatial models have been used to analyze the electoral process. However, their utility has been severely limited by (at least) four stringent assumptions.[1] First, typical spatial models, henceforth to be called Euclidean models, require that the messages candidates transmit to voters be the points of an Euclidean issue space. A point message indicates a candidate's promised issue outcome. Perfect candidate mobility and a perfect flow of information from candidates to voters are two aspects of this assumption. Secondly, in the basic spatial models all promises are believed -- the issue outcome that a voter believes will occur if a candidate is elected is assumed to be identical to the candidate's point message. Thirdly, every individual's preferences are required to be complete over the entire issue space and often to decline with distance from an ideal point. Finally, candidates are usually assumed to perceive the preferences of all voters over all points in the issue space.

These requirements of Euclidean spatial models have been questioned by political scientists -- Page [1975] is particularly critical. In this paper, a weakening of each of the above assumptions will be shown to lead naturally to a model employing a non-Euclidean outcome space which can be viewed as the set of points on the surface of a hypersphere. Under the primary interpretations to be offered in section 1, this space is composed of the directions in which a status quo point in an Euclidean issue space can shift.

In section 2 the basic model is described as a two-person plurality game in which the candidates adopt shift directions as strategies. Equilibrium directions in this game, however, are shown to be in the core of a corresponding n-person absolute majority rule game. Necessary and sufficient conditions are then easily established for the existence of an equilibrium direction.

In section 3 optimal strategies for a candidate competing against a rigid opponent are investigated. The result is a prediction of candidate divergence, somewhat analogous to that made by Hinich and Ordeshook [1968] within the context of an Euclidean model.

Finally, directional voting is embedded into the framework of Euclidean models in section 4, and the existence of point equilibria is shown to imply the existence of equilibrium directions. Equilibrium direction vectors will be shown to "point" towards equilibrium points, provided the latter exist.

## 1. MOTIVATIONS AND ASSUMPTIONS

Four different conceptualizations of the set of messages that candidates send to voters, the set of possible outcomes that voters perceive, and the relationship between these two sets can serve as foundations to the basic direction model. First, both the messages candidates transmit and the outcomes voters associate with them can be considered as single points in an Euclidean issue space. However, individuals may often map all candidate messages into point outcomes only a marginal distance from the status quo -- the point in the issue space that represents the current state of the world on the relevant issues. The possible causes of this virtual shrinkage of the issue space are twofold: (1) for physical or political reasons, candidate mobility in the message space may be restricted to a neighborhood of the status quo -- truthful and knowledgeable candidates will only choose messages within this neighborhood; (2) based perhaps on past performances, voters may not believe any winning candidate can achieve a large shift of the status quo, regardless of campaign promises (messages). When a candidate's actions can only marginally shift the status quo, only the directions in which he proposes to shift it are important. Strategies can be considered as directions which shall be represented as vectors of unit or zero length.

A second behavioral motivation of the direction model can be based on imperfect communication. Candidates may still attempt to send messages that voters will view as point outcomes. But due to high information costs, voters may not become aware of the exact issue positions that candidates adopt. From Campbell et al. [1960] to Page [1975], empirically-oriented political scientists have been critical of models that assume a perfect flow of information from candidates to voters. However, if candidates are able to at least convey their pro and con opinions and the relative stresses they place upon the issues, they may be able to transmit the directions in which they would shift the status quo.

Thirdly, suppose one of the following is true: (1) as Page [1975] suggests, individual preference orderings are complete or well-defined only in a neighborhood of the familiar status quo; (2) individual indifference surfaces actually take the form of rays emanating from the status quo; or (3) candidates only receive reliable information about preferences near the status quo. Then candidates may have no incentive to adopt more than directions or, equivalently, marginally shifted points as their strategies, since they can know only how voters respond to such strategies.

Two sources of empirical support for directional voting should be mentioned. The first consists of results of spatial experiments conducted by Fiorina and Plott.[2] In their experiments, each voter's payoff function declined with distance from a single point where it achieved its maximum. When a candidate asked: "Who wants me to move into this rectangle?" usually all voters whose optimal points were in the specified rectangle indicated approval of the move. If the voters had utilized subjective estimates of the distances the candidate would move into the specified rectangle, those voters very near the border containing the candidate's current position probably would not have been in favor of such a move. But as it turned out, most who had a utility gradient at the candidate's current point that formed an acute angle with the proposed direction vector favored the move. This behavior suggests direction voting.

The work of Rabinowitz [1977] provides a second source of support for directional voting. Using survey data obtained during the

1968 and 1972 presidential elections, Rabinowitz uses a nonmetric multidimensional scaling procedure to locate voters' ideal points and candidates' campaign positions within two-dimensional issue spaces. He finds few candidates occupying centralist positions, but rather observes candidates adopting peripheral positions surrounding the center of the distribution of voters. He argues that this result can best be explained by what he calls a "dispositional model," in which "it is the direction of a candidate's policy that is critical to developing his support base, not his absolute position."

The above rationalizations for direction strategies have been based upon the concept of an Euclidean issue space. However, if the outcome space into which voters transform candidate messages is cognitive or perceptual in nature, it may not possess the Euclidean structure. In particular, Weisberg [1974] hypothesizes that some political issue spaces can be modeled as closed circles. As an example, Weisberg refers to the Swedish Riksdag, where parties of the so-called left and right sometimes vote together against the moderates. So the fourth conceptualization that can serve as a basis for the direction model, although it would now be inappropriately named, is that the set of perceived outcomes is a non-Euclidean space isomorphic to the surface of a hypersphere.[3]

Assumptions about individual preferences are also required. In the basic model we assume each voter most prefers the status quo to shift in a particular direction. A voter will rank directions negatively with the size of the angle they form with his most preferred direction. Formally, suppose that $v_1$ and $v_2$ are two direction vectors, and s is the direction vector representing some voter's most preferred direction. Then the voter will prefer the direction of $v_1$ to that of $v_2$ if and only if $s'v_1 > s'v_2$. Furthermore, as is usual, we assume a voter's preferences for candidates are identical to his preferences for the directions they adopt.

All of these preference assumptions are analogous to those made in simple Euclidean spatial models -- simply substitute preferred points for preferred directions, and Euclidean distances for angles. But they can be better justified here. Suppose two candidates choose vectors $z_1$ and $z_2$ that are the same distance d from the status quo in the directions of $v_1 = z_1/d$ and $v_2 = z_2/d$. Then the directional preferences described above approximate preferences that can be represented by a differentiable utility function -- $s'(v_1 - v_2)$ is a linear approximation to $[u(z_1) - u(z_2)]/d$ when s is the (normalized) utility gradient at the status quo. The approximation becomes exact if candidates can adopt points only marginally distinct from the status quo.[4]

## 2. THE BASIC MODEL

In the basic direction model, two candidates compete by choosing vectors $v_1$ and $v_2$ of unit or, to allow null shifts, zero length in the set of <u>directions</u> $\mathcal{B} = B \cup \{0\}$, where $B = \{v \in E^n : \| v \| = 1\}$. Each voter i most prefers a vector $s_i \in \mathcal{B}$. An arbitrary probability measure P defined on (Borel) subsets of $\mathcal{B}$

represents the distribution of voters' preferred direction vectors, imposing no limitation on the number of voters. The directional preferences of voter i are represented by the inner product $s_i'v$. Thus the fraction of the electorate who votes for candidate j is $P[s'(v_j - v_k) > 0]$. Geometrically, for the case of $v_j \neq 0$ $(j = 1, 2)$, j's votes are obtained from the fraction of the electorate whose preferred direction vectors lie on the same side as $v_j$ of a hyperplane containing the origin and the mid-vector $v_1 + v_2$. The indifferent voters are those whose ideal direction vectors lie in this dividing hyperplane -- for lack of a more realistic assumption in this setting, they are assumed to abstain. Notice that voters with $s_i = 0$ are assumed to always be indifferent.

Each candidate j is assumed to maximize his plurality:

$$PL_j(v_1,v_2) = P[s'(v_j - v_k) > 0] - P[s'(v_j - v_k) < 0].$$

Because of the symmetry of the two person game played by the candidates, an equilibrium can be defined as a direction that guarantees a nonnegative plurality to any candidate who adopts it.

Definition 1: An equilibrium direction vector $v^*$ is a direction in $B$ for which $PL_1(v^*, v) \geq 0$ for all $v \in B$.

The first task is to show the relationship between equilibrium directions in the two-person plurality game and undominated directions in the n-person absolute majority game. An undominated direction in the latter is one that is not ranked below another by a strict majority of the voters:[5]

Definition 2: A direction vector $v^* \in B$ is undominated provided $P[s'(v^* - v) \geq 0] \geq 1/2$ for all $v \in B$.

It would be disturbing to find equilibrium directions that were not undominated, for then a direction may exist which is preferred by a majority to the direction adopted by the winning candidate. Theorem 1 below shows that this cannot occur. Furthermore, theorem 1 shows that undominated directions are equilibria if $P[s = 0] = 0$, that is, if nobody is indifferent over all directions. This result is not obvious because a positive fraction of the voters may still be indifferent between any two directions $v_1$ and $v_2$, allowing the possibility that $P[s'(v_1 - v_2) \geq 0] \geq 1/2$ even though $PL_1(v_1, v_2) < 0$. (The lengthy proof of theorem 1 is in an Appendix.)

Theorem 1: Equilibrium directions are undominated. Conversely, if $P[s = 0] = 0$, then undominated directions are equilibrium directions.

One use of theorem 1 is to provide necessary conditions for equilibrium directions, since a condition both necessary and sufficient for undominated directions is easily obtained.

Theorem 2: $v^*$ is an undominated direction vector if and only if $P[s'a \geq 0] \geq 1/2$ for all $a \in E^n$ satisfying $a'v^* \geq 0$.

Proof: Suppose $v^*$ is undominated and that $a'v^* > 0$.
We may assume $\|a\| = 1$, and hence, letting $v = v^* - (2a'v^*)a$,

have that $v \in B$. Then $a'v^* > 0$ implies $P[s'a \geq 0] = P[(2a'v^*)s'a \geq 0] = P[s'(v^* - v) \geq 0] \geq 1/2$. If $a'v^* = 0$ and $v^* \neq 0$, there exists a sequence $\{a_1, a_2, \ldots\}$ that converges to a and whose members satisfy $a_n'v^* > 0$. Hence $P[s'a_n \geq 0] \geq 1/2$ for all $a_n$, and $P[s'a \geq 0] \geq \lim_{n\to\infty} P[s'a_n \geq 0] \geq 1/2$ is established by an argument like that used to prove theorem 1. Finally, if $v^* = 0$, then $P[s'a \geq 0] = P[s'(v^* - (-a)) \geq 0] \geq 1/2$ for any $a \in \mathcal{B}$ and hence for any $a \in E^n$.

Conversely, suppose $P[s'a \geq 0] \geq 1/2$ whenever $a'v^* \geq 0$. Since $(v^* - v)'v^* \geq 0$ for any $v \in \mathcal{B}$, $P[s'(v^* - v) \geq 0] \geq 1/2$ and $v^*$ is dominant.

The condition of theorem 2 actually consists of two different parts, namely, that $P[s'a \geq 0] \geq 1/2$ whenever (1) $a'v^* = 0$ and whenever (2) $a'v^* > 0$. Satisfaction of the first part means simply that the individuals whose ideal direction vectors lie upon any hyperplane containing $v^*$ and the origin, or to one side of it, constitute a (weak) majority of all individuals. This property is entirely analogous to the property that Hoyer and Mayer [1974, 1975] define a total median to satisfy for an Euclidean spatial model: every hyperplane containing a total median must bisect the distribution of voters' preferred points.

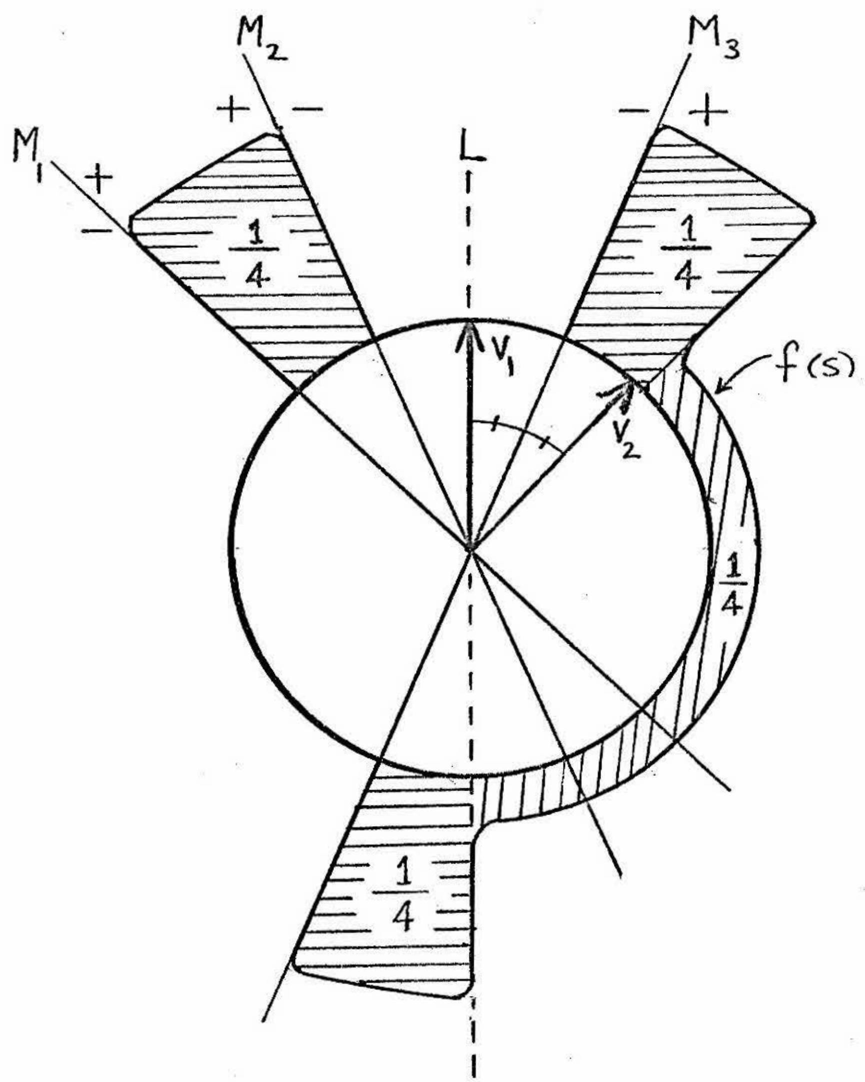
Davis, DeGroot, and Hinich [1972], and later Sloss [1973] and Hoyer and Mayer [1974, 1975], show that in the simple Euclidean model an undominated point exists if and only if it is a total median. But for the direction model, part (2) as well as part (1) of the condition in theorem 2 is needed to obtain existence. Distributions of the electorate exist that satisfy the bisecting property of part (1), but do not allow the existence of undominated directions. A continuous example appears in figure 1, where the distribution of preferred directions is represented by the area between the unit circle B and the curve f(s). Each of the lines $M_1$, $M_2$, and $M_3$ has a greater fraction of the electorate's preferred directions on one side of it than on the other. (The signs "+" and "-" near each line $M_i$ indicate which side of it the greater fraction of voters' preferred directions lie.) No undominated direction can exist, since any direction will lie on the "-" side of some line $M_i$ and so will receive fewer votes than a direction located symmetrically on the opposite side of $M_i$. However, some directions will satisfy part (1) of the condition, such as the vector $v_1$ that lies in the bisecting line L. Since $v_1$ lies on the "-" side of $M_1$, it will receive only 1/4 the votes in a contest against $v_2$.

In a Euclidean model, a candidate who diverges from a fixed opponent will only lose votes. So if the opponent has chosen a total median, the diverging candidate can only decrease his plurality from zero. In the direction model, however, a diverging candidate will gain the votes of voters whose preferred directions directly oppose those of the voters he loses. Even if the opponent has adopted a median-like direction, a diverging candidate may win a strict majority by diverging so as to gain more votes than he loses. The complete

FIGURE 1

A Distribution in Which No Direction Is an Equilibrium

Even Though a Bisecting Direction Vector Exists



condition of theorem 2 eliminates this possibility for an undominated $v^*$ by requiring a majority to have its preferred directions on the same side as $v^*$ of any hyperplane containing the origin.

It can now be shown that the zero direction is an equilibrium or is undominated if and only if the same is true of all directions. Interpreted loosely, this means that a proposal to not shift the status quo is winning if and only if any other proposed shift is also winning. One could say in this case that society is indifferent as to the direction the status quo marginally shifts, just as an individual would be if the status quo were located at an extremum of his utility function.

Corollary 1: The zero direction is an equilibrium (undominated) if and only if all directions are equilibria (undominated).

Proof: By theorem 2, $0 \varepsilon B$ is undominated iff $P[s'a \geq 0] \geq 1/2$ for all $a \varepsilon E^n$, which is true iff all $v \varepsilon B$ are undominated. 0 is an equilibrium provided $PL_1(0,v) \geq 0$ for all $v \varepsilon B$, which is true iff $P[s'v < 0] = P[s'v > 0]$ for all $v \varepsilon B$. But the latter is true iff $PL_1 (v_1, v_2) = 0$ for all $v_1, v_2 \varepsilon B$, or rather, iff every $v \varepsilon B$ is an equilibrium.

## 3. EXPLOITING A FIXED OPPONENT

In this section we show that if one candidate rigidly adopts a direction vector on a particular side of B, to be called $B^-$, then the optimal vector for the opponent to choose lies in $B^+$, the side of B opposite $B^-$. The two vectors will be located symmetrically about the hyperplane that separates $B^+$ from $B^-$. Thus, entirely half the directions will be inferior in the sense that only if both candidates are rigid will they both choose inferior directions. Since $B^+$ shall be defined as the half of B containing the largest fraction of non-indifferent voters, this result may also be interpreted as follows: once an extremist candidate becomes too extreme, the more extreme he becomes the further his opponent should diverge from him. Although this divergence result is similar to that which Hinich and Ordeshook [1968] proved for Euclidean models, it differs fundamentally by not requiring abstention of nonindifferent voters. Furthermore, no symmetry requirements are imposed or equilibriums assumed to exist.

Before formally presenting theorem 3, we need some definitions.

Definition 3: Let $\overline{P} = \sup \{P[s'c > 0] - P[s'c < 0]\}$. Assuming a vector $\overline{c} \in B$ exists such that $\overline{P} = P[s'\overline{c} > 0] - P[s'\overline{c} < 0]$, let $B^- = \{v \in B: v'\overline{c} < 0\}$, and $B^+ = \{v \in B: v'\overline{c} \geq 0\}$.

The direction vector $\overline{c}$ exists if P represents either a continuum of voters or a finite number of voters. Hence it is not restrictive to assume for the remainder of this section that $\overline{c}$ exists. The interesting case is when $\overline{P} > 0$, in which case the following also indicates that any equilibrium direction vector is in $B^+$.

Theorem 3: If $v_2 \in B^-$, then the function $f(v) = PL_1(v, v_2)$ is maximized on B by a vector $\overline{v} = v_2 - (2\overline{c}'\, v_2)\overline{c}$ contained in $B^+$.

Proof: Clearly $\|\overline{v}\| = 1$ and $\overline{v}'\overline{c} = -v_2'\overline{c} > 0$. Hence $\overline{v} \in B^+$. The proof is finished by observing that

$$
\begin{aligned}
PL_1(\bar{v}, v_2) &= P\,[s'(\bar{v} - v_2) > 0] - P\,[s'(\bar{v} - v_2) < 0] \\
&= P\,[(-2\bar{c}'v_2)s'\bar{c} > 0] - P\,[(-2\bar{c}'v_2)s'\bar{c} < 0] \\
&= P\,[s'\bar{c} > 0] - P\,[s'\bar{c} < 0] \\
&= \bar{P}.
\end{aligned}
$$

Theorem 3 is illustrated in figure 2, which also indicates further results obtainable when P exhibits some monotonicity. The half circles $B^+$ and $B^-$ are separated by line M. The optimal vector to choose against a vector in $B^-$ like $v_2$ is a vector in $B^+$ like $v_1$. Notice that if $v^*$ is not perpendicular to M, then $v_1$ is not diverging toward $v^*$ but only away from M and $v_2$ as $v_2$ moves further from M into $B^-$. Hence $v_2$ does have some ability to draw $v_1$ away from $v^*$, but not out of $B^+$. Furthermore, in this situation, if one candidate adopts a vector $s_1$ in $B^+$ that is not $v^*$, then his opponent increasingly receives more votes by choosing vectors increasingly closer to $s_1$, but always between $s_1$ and $v^*$. For example, $s_2$ does better than $s_3$ for candidate 1 when candidate 2 adopts $s_1$. Candidate 1 can insure that the fraction of the electorate voting for him is within any arbitrary amount of the fraction of the electorate whose preferred vectors lie above the line L.

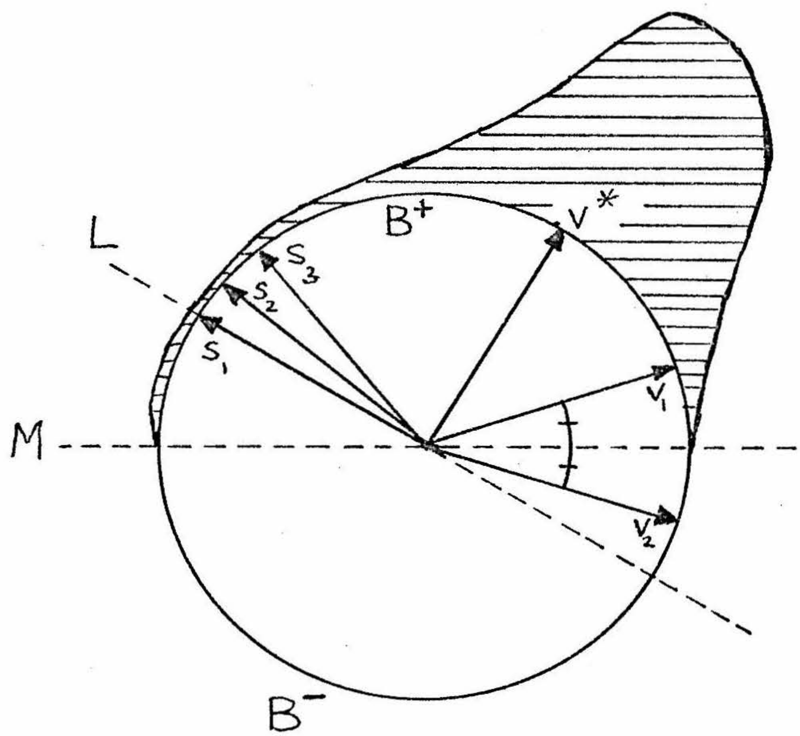
## 4. DIRECTION VOTING IN AN EUCLIDEAN MODEL

We now assume that individuals have well-defined preferences over an Euclidean issue space, but that the outcomes associated with the candidates are restricted to a small neighborhood of the status quo (origin). In fact, we assume the simplest case considered in Euclidean models: each voter i most prefers a point $x_i$ and prefers y to z if and

FIGURE 2

Exploiting a Fixed Opponent When Ideal

Directions Are Distributed Monotonically



only if $\|x_i - y\| < \|x_i - z\|$. In the notation of the basic model, each voter i now most prefers the direction $s_i$ for which $\lambda s_i = x_i$ has a solution $\lambda > 0$. When candidate j chooses a point strategy $z_j$, he is adopting the direction $v_j$ for which $\lambda v_j = z_j$ has a solution $\lambda > 0$. Voting is again assumed to agree with issue preferences, and only indifferent individuals abstain.

The first question concerns the properties that a distribution of voter's preferred points must satisfy for plurality equilibria to exist. We first observe that if $\hat{P}$ is a probability measure representing preferred points in the issue space, it induces a probability measure P on $B$ to represent preferred directions: $P[s \in A] = \hat{P}[x \in C(A)]$ for any (Borel) subset A of B, where $C(A) = \{x \in E^n: \alpha x \in A \text{ for some } \alpha > 0\}$ is the cone spanned by A, and $P[s = 0] = \hat{P}[x = 0]$. Thus the condition of theorem 2 can be considered to apply to $\hat{P}$ as well as P. But we show further that if an equilibrium point exists for a distribution of voters when candidates may choose any points in the issue space, then a corresponding undominated direction exists when outcomes associated with candidates are essentially shift directions. We first need formal definitions.

<u>Definition 4</u>: A point $z \in E^n$ is <u>undominated</u> provided $\hat{P}[\|z - x\| \leq \|y - x\|] \geq 1/2$ for all $y \in E^n$. A point z is an <u>equilibrium</u> in the plurality game provided it satisfies $\hat{P}[\|z - x\| < \|y - x\|] \geq \hat{P}[\|z - x\| > \|y - x\|]$ for all $y \in E^n$.

<u>Definition 5</u>: A point $z \in E^n$ is a <u>total median</u> of $\hat{P}$ provided $\hat{P}[a'(x - z) \geq 0] \geq 1/2$ for all $a \in E^n$.

As previously mentioned, an undominated point is known to be a total median. It is also true that, analogously to theorem 1, undominated points are equilibria.

Lemma 1: In an Euclidean model, $z \in E^n$ is undominated if and only if it is an equilibrium.

Proof: Let $y \in E^n$ and, for $0 \leq \alpha \leq 1$, define $y(\alpha) = \alpha y + (1 - \alpha)z$.

For z undominated, $\hat{P}[\ \| z - x \| \leq \| y(\alpha) - x \|\ ] \geq \hat{P}[\| z - x \|$

$$> \| y(\alpha) - x \|\ ].$$

$$\text{Hence, } \hat{P}\ [\| z - x \| < \| y - x \|] = \lim_{\alpha \to 1-} \hat{P}\ [\| z - x \|$$

$$\leq \| y(\alpha) - x \|] \geq \lim_{\alpha \to 1-} \hat{P}\ [\| z - x \| > \| y(\alpha) - x \|\ ]$$

$$\geq \hat{P}[\ \| z - x \| > \| y - x \|\ ].$$

Therefore z is an equilibrium. The converse is obvious.

Theorem 4: If $z \in E^n$ is an equilibrium point, then any direction $v^*$ satisfying $\lambda v^* = z$ for some $\lambda \geq 0$ is undominated. If $z \neq 0$ or $\hat{P}[x = 0] = 0$, then $v^*$ is also an equilibrium direction.

Proof: Let $v \in B$ be any direction except $v^*$. Then $z'(v^* - v) \geq 0$. Hence

$$P[s'(v^* - v) \geq 0] = \hat{P}[x'(v^* - v) \geq 0]$$

$$\geq \hat{P}[x'(v^* - v) \geq z'(v^* - v)] \geq 1/2$$

since z is undominated and hence a total median. This proves that $v^*$ is undominated. If $z \neq 0$, then $z'(v^* - v) = \lambda(1 - v'v^*) > 0$ and

$$P[s'(v^* - v) > 0] = \hat{P}[x'(v^* - v) > 0]$$
$$\geq \hat{P}[x'(v^* - v) \geq z'(v^* - v)] \geq 1/2.$$

This implies that $v^*$ is an equilibrium if $z \neq 0$. Finally, if $\hat{P}[x = 0] = 0$, then theorem 1 implies that $v^*$ is an equilibrium.

Existence of an undominated direction requires P to satisfy more than the median-like part of the condition in theorem 2, but theorem 4 establishes that no more than a total median condition on $\hat{P}$ is needed. In fact, the converse of theorem 4 is false -- existence of direction equilibria does not guarantee the existence of point equilibria for a corresponding Euclidean model. As a particularly easy example, illustrated in figure 3, suppose there are three voters whose ideal points $P_1$, $P_2$ and $P_3$ are arranged in a triangle to one side of the status quo S. Then no total median exists, but the direction vector $v^*$ that points toward $P_3$ satisfies the condition of theorem 2 and so represents an equilibrium.[6]

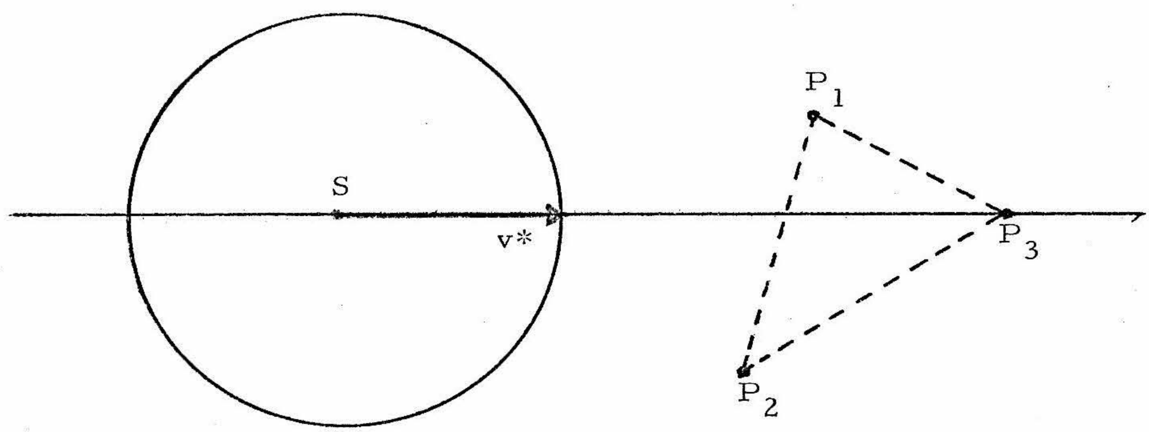
A consequence of theorem 4 and corollary 1 is that the status quo is an equilibrium point if and only if all directions are undominated. Again, the heuristic interpretation is that the status quo is at a social maximum if and only if society is indifferent about the direction the status quo moves.

Theorem 4 also determines a consistency relationship between the two types of equilibria: if point equilibria exist, equilibrium direction vectors will "point" towards them. Suppose we

FIGURE 3

Situation with a Direction Equilibrium

But No Point Equilibrium



now consider the situation in which a candidate may choose either a point or a direction as a strategy. Using another assumption about voter behavior, we can establish another consistency property for each type of strategy: if one candidate has chosen either a direction vector or a point (not the status quo) as his strategy, then his opponent can do no better than to choose the same type of strategy. The additional assumption concerns the voter's decision rule when one candidate chooses a direction and the other a point. We shall suppose the voter believes the candidate who chooses a point can shift the status quo the maintained distance, and that the other candidate would shift the status quo the same amount. Based upon an "equally likely" type of rationale, this assumption implies that voters will always vote as if the two candidates had chosen points on the same hypersphere about the status quo, i. e., voters will direction vote.[7]

The internal consistency property now follows easily. If candidate 2 has chosen a direction, then regardless of the type of strategy candidate 1 chooses, the electorate will behave as if both had chosen directions. But if candidate 2 has chosen a point z, then for any direction v that candidate 1 might choose, he can achieve the same outcome by choosing the point $\|z\| v$.

However, the addition of either infinite or finite information costs might cause direction strategies to dominate point strategies. If the cost of obtaining information about voters' preferences over more than a neighborhood of the status quo is "too" high, or if only information about preferences over directions can be obtained, then the candidates will have no real basis for choosing point strategies. Possessing only uncertain knowledge about voter preferences away from the status quo, the risk-averse candidate may prefer a direction strategy to an exact point. If each candidate is also uncertain as to the amount of information the other candidate possesses about voter preferences away from the status quo, it is even more likely that direction strategies will dominate. This follows because one candidate's choice of a direction strategy essentially forces the opponent to also choose a direction strategy and hence to utilize only information about the distribution of preferred directions, presumably known to both candidates. Formalization of these concepts is left for future work.

## 5. SUMMARY

The direction model of the electoral process allows limits to candidate mobility or voter perception and cognition. It is applicable (1) if only issue outcomes near the status quo are associated with candidates; (2) if only directional information is transmitted to voters; (3) if voter preferences are only well-defined near the status quo or are only defined for directions in which it can shift; or (4) if the outcome space is curved so that it can be modeled as a hypersphere.

Assuming that a voter will vote for the candidate who campaigns for a direction closest to his own preferred direction, plurality equilibria were shown to be undominated. The identity of the two types of solutions was established if nobody was totally indifferent. Then a necessary and sufficient condition for the existence of undominated directions was determined. The first part of the condition, stating that any hyperplane containing the undominated direction vector and the origin bisects the distribution of preferred directions, is analogous to the total median condition in the simple Euclidean models. The remainder of the condition in theorem 2, stating that a majority of the electorate's preferred direction vectors lie on the same side as the undominated direction vector of any hyperplane containing the origin, is not implied by the median-like property in this model because of the "curved" nature of the directional domain space. The second part of the condition is what allows a candidate to diverge from a fixed direction chosen by an extremist opponent, where at least half the feasible directions are defined to be extremist for every distribution of the electorates' preferred directions.

Although the addition of a second part to the characterizing condition for equilibrium seems to further decrease the likelihood of its occurrence, it was shown that in situations where the assumptions of the simple Euclidean model are met, point equilibria exist only if corresponding undominated directions also exist. But the converse of this theorem is false -- some distributions of voter preferences yield direction but not point equilibria. In situations where both types of equilibria exist, contradictory predictions will not occur since equilibrium direction vectors point in the direction of existing equilibrium points.

Finally, it was argued that a candidate has no incentive to adopt a type of strategy different from the type he knows his opponent will choose. This result can be interpreted as an internal stability property for each model. However, it was suggested that when a candidate's uncertainties about voters' preferences away from the status quo and about the extent of his opponent's information is considered, only the direction model may exhibit this internal stability.

APPENDIX

Proof of Theorem 1: Suppose $v^*$ is an equilibrium. Then for any other $v \varepsilon B$, $P[s'(v^* - v) > 0] \geq P[s'(v^* - v) < 0]$. Hence $v^*$ is undominated since $P[s'(v^* - v) \geq 0] \geq P[s'(v^* - v) < 0] = 1 - P[s'(v^* - v) \geq 0]$.

Conversely, suppose $v^*$ is undominated but not an equilibrium, and that $P[s=0] = 0$. For any $a \in E^n$ define the following sets:

$$S_1(a) = \{s \in B: s'a > 0\}$$
$$S_2(a) = \{s \in B: s'a < 0\}$$
$$H(a) = \{s \in B: s'a = 0\}.$$

By assumption, there exists $v \in B$ such that $PL_1(v^*, v) < 0$. Hence, letting $t = v^* - v$, there is an $\epsilon > 0$ such that

$$\text{(i)} \quad P[S_1(t)] < P[S_2(t)] - \epsilon$$

Since $P[\{a\}] > 0$ for only a countable number of $a \in B$, there exists $b \in B$ such that $b'v^* \geq 0$ and $P[H(b)] = 0$. Hence $P[H(b) \cap H(t)] = 0$. Let $H_i = S_i(b) \cap H(t)$ for $i = 1, 2$.

Consider the case $P[H_1] \leq P[H_2]$. For $n > 1$, define $c_n = n^{-1}b + (1 - n^{-1})t$. We now show that $\lim_{n\to\infty} S_i(c_n) = H_i \cup S_i(t)$, or, by definition, that

$$\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} S_i(c_n) = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} S_i(c_n) = H_i \cup S_i(t).$$

First, observe that $s'c_n = n^{-1}s'b + (1 - n^{-1})s't$ monotonically approaches $s't$ as $n \to \infty$. Hence $s \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} S_1(c_n) \Longleftrightarrow s'c_n > 0$ for infinitely many $n \Longleftrightarrow s'c_n \leq 0$ for only finitely many $n \Longleftrightarrow s \in \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} S_1(c_n)$. Also, $s \in H_1 \cup S_1(t) \Longleftrightarrow s't > 0$ or $(s't = 0$ and $s'b > 0) \Longleftrightarrow s'c_n > 0$ for all $n$ sufficiently large $\Longleftrightarrow s \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} S_1(c_n)$. The argument for $i = 2$ is similar.

So by the continuity of a finite measure, and since $H_i$ and $S_i(t)$ are disjoint, there exists an integer $n_0$ such that

$$\text{(ii)} \quad |P[S_i(c)] - P[H_i] - P[S_i(t)]| < \frac{\epsilon}{2}$$

for all $c$ in the arc $A = \{c \in B\colon\ c = \alpha c_{n_0} + \beta t,\ \alpha > 0,\ \beta > 0\}$. For distinct $\hat{\hat{c}}, \hat{c} \in A$, there is no real number $\gamma$ such that $\hat{\hat{c}} = \gamma\hat{c}$. Hence $s \in H(\hat{\hat{c}}) \cap H(\hat{c}) \Longleftrightarrow s'b = 0$ and $s't = 0 \Longleftrightarrow s \in H(b) \cap H(t)$. Thus $H(\hat{\hat{c}}) \cap H(\hat{c}) = H(b) \cap H(t)$ for all distinct $\hat{\hat{c}}, \hat{c} \in A$. So again by a countability argument, there exists $\overline{c} \in A$ such that $P[H(\overline{c})] = P[H(b) \cap H(t)] = 0$. From (i) and (ii), and since we are considering the case $P[H_1] \leq P[H_2]$, we now obtain

$$\begin{aligned}
\text{(iii)} \quad P[S_1(\overline{c})] &< P[H_1] + P[S_1(t)] + \frac{\epsilon}{2} \\
&< P[H_2] + P[S_2(t)] - \epsilon + \frac{\epsilon}{2} \\
&< P[S_2(\overline{c})].
\end{aligned}$$

Now if $v^* = 0$, let $\overline{v} = -\overline{c}$, and otherwise let $\overline{v} = v^* - (2\overline{c}'v^*)\overline{c}$. Clearly, $\overline{v} \in B$. Furthermore, since $b'v^* \geq 0$ and $v'v^* < 1$, when $v^* \neq 0$ we have

$$\overline{c}'v^* = (\alpha n_0^{-1})b'v^* + [\alpha(1 - n_0^{-1}) + \beta](1 - v'v^*) > 0.$$

Thus $P[s'(v^* - \overline{v}) \geq 0] = P[s'\overline{c} \geq 0]$, whether or not $v^* = 0$.

Hence, as $P[H(\overline{c})] = 0$, (iii) implies that

$$\begin{aligned} P[s'(v^* - \overline{v}) \geq 0] &= P[S_1(\overline{c})] + P[H(\overline{c})] \\ &< P[S_2(\overline{c})] \\ &= 1 - P[s'(v^* - \overline{v}) \geq 0]. \end{aligned}$$

Therefore $v^*$ is not undominated, contrary to assumption. The proof is similar for the case $P[H_1] > P[H_2]$.

FOOTNOTES

1. The assumptions of electoral spatial models and many of their predictions are reviewed in Davis, Hinich, and Ordeshook [1970], and Riker and Ordeshook [1973].

2. Personal communication -- but see Fiorina and Plott [1975] for details on similar experiments.

3. In this interpretation the status quo must be on the surface of the hypersphere rather than at its center. The status quo shall be assumed under this interpretation to play no role in the model, just as it plays no role in the usual Euclidean spatial models.

4. See section 2 and appendix A in chapter II for an extensive treatment of directional preferences.

5. Undominated directions to simple games with a finite number of players are the subject of chapter II.

6. However, it is shown in section 7 of chapter II that existence of undominated directions is equivalent to satisfaction of pairwise symmetry conditions similar to those Plott [1967] establishes for his constrained voting equilibria. Their stringency implies that existence is only slightly more "common" for directional than

for point equilibria. See also chapter III.

7. When an individual prefers $z_1$ over $z_2$ if and only if $\| x_i - z_1 \| < \| x_i - z_2 \|$, direction voting exactly agrees with preferences if the outcomes candidates can choose are constrained to lie on the same hypersphere centered at the status quo (origin). This follows trivially for $x_i = 0$. Otherwise, if $\| z_1 \| = \| z_2 \|$, $\| x_i - z_1 \| < \| x_i - z_2 \| \Longleftrightarrow x_i' z_1 > x_i' z_2 \Longleftrightarrow s_i' v_1 > s_i' v_2$, where $s_i = x_i / \| x_i \|$ and $v_j = z_j / \| z_j \|$.

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# Chapter II

## UNDOMINATED DIRECTIONS IN SIMPLE DYNAMIC GAMES

### 1. INTRODUCTION AND SUMMARY

Equilibria to simple games, such as majority rule, in multidimensional spaces require such severe symmetry of preferences as to rarely exist.[1] Consequently, social processes may usually be in disequilibrium. The way they shift the state of the world through time can only be understood when an explicit dynamic mechanism or institution allows sequences of social decisions to be examined.

To date, sequential simple games have been investigated in the context of two "disequilibrium" hypotheses regarding the interconnection between outcomes. Cohen [1977], McKelvey [1976], [1977] and Schofield [1977a] essentially assume that an outcome in one period can be any alternative preferred to the previous outcome by a winning coalition.[2] They show that sequences of outcomes required to satisfy only this dominance property do not satisfy any regularity condition, since such a sequence connects almost any two alternatives in the social choice space. Kramer [1977], in the majority rule context, strengthens their assumption by requiring that an outcome receive a maximal number of votes against the previous outcome. He finds that these "maximally dominating" sequences always enter the minimax set [Simpson, 1969] when each voter has Euclidean preferences, that is, utility that decreases with the Euclidean distance from an ideal point. This convergence is a regularity property that may provide insight into political situations where mobile challengers oppose fixed incumbents.

In this chapter a different hypothesis relating sequential social outcomes is advanced, motivated by the supposition that social change is not instantaneous. More specifically, in any time period only alternatives a small distance from the previous outcome are assumed to be feasible. Taken to its logical and mathematically tractable extreme, this assumption converts the problem into one involving a continuum of social decisions, each of which determines a direction in which to marginally shift the current status quo.

Since social decisions in this setting are directions, the application of cooperative game theory requires that directional preferences be determined from the location of the status quo and the underlying preferences over social states. Directions are represented as vectors of zero or unit length, and in section 2 one direction is said to be preferred to another if it is nearer one's utility gradient evaluated at the status quo. The set of winning coalitions is then used to define a dominance relation on directions, and undominated directions are predicted outcomes to the game. In other words, the status quo is predicted to shift in a direction to which no winning coalition unanimously prefers another direction of shift.

The distinction between the undominated direction hypothesis and the hypothesis that each chosen point dominates the preceeding one should be emphasized. The continuous version of the latter requires the shift direction to dominate only the zero direction that corresponds to a null shift. The undominated hypothesis, on the other hand, requires the shift direction to be undominated in the set of all directions. Neither hypothesis implies the other, as can be seen in the examples of section 2.

In appendix A, which supplements section 2, an alternative directional core is defined via an inducement of directional preferences that is independent of utility gradients. This core is found to be contained in the one defined in section 2. The two are identical if each utility function satisfies a condition we label local symmetry.

For comparative purposes, the local point core so often studied since Plott [1967] is examined in section 3. As it too is defined via utility gradients, a second definition involving small neighborhoods is explored in appendix B. The relationship between the two local point cores is found to be strictly analogous to that between the two directional cores uncovered in appendix A. Furthermore, the first point core and sometimes the second can be defined in terms of the analogous directional cores. Specifically, it is shown that a point is locally undominated if and only if the zero direction corresponding to a null shift is undominated. This result is strengthened in section 3 when its point core is shown to consist of points whose directional cores contain all directions. Finally, section 3 concludes with the demonstration that in the benchmark case of Euclidean preferences, directions that "point" to an existing point core are undominated.

Before dynamics are discussed, an important tangent is pursued in section 4. The directional core is found to be equivalent to the cone whose nonexistence is shown by Schofield [1977a] to imply local cycling. Hence the nonexistence of an undominated direction implies that any two points sufficiently close to the status quo can be connected by a finite sequence of points, each of which dominates the preceeding one.

In section 5 an investigation is begun of the paths generated when the status quo is infinitesimally shifted in undominated directions. A status quo so shifted through a point x is shown, at the time it is at x, to be simultaneously approaching every point in every winning coalition's preferred-to-x set. No path satisfying this "approach" property exists through points with empty directional cores. Thus a point with an empty directional core satisfies a solution-like property in not being able to shift so as to approach simultaneously every point preferred to itself by every winning coalition.

In section 6 it is shown that if preferences are Euclidean, and if the speed of the status quo is bounded below when it follows undominated directions (which it does whenever they exist, by assumption), then the status quo either enters the set of points with empty directional cores or converges to the point core. Thus, for at least this simplest of situations, dynamics based on a local equilibrium concept imply convergence to a set with global solution-like properties.

The last section of the paper contains a discussion of the principal shortcoming of the directional core as a solution-concept: its frequent nonexistence in majority rule situations. Results from the dissertation appendix are used to show that utility gradients at the status quo must satisfy stringent pairwise symmetry conditions akin to Plott's [1967] for an undominated direction to exist. Schofield [1978] has used these conditions to obtain results implying that generically, undominated directions in majority games will not exist at almost all points in spaces of dimension greater than three. Even for majority rule, however, existence of directional cores is more common in some important cases, as is argued in section 7. Furthermore, existence is shown to be more common in games with less decisiveness and anonymity than majority rule. As a polar case, it is observed that undominated directions always exist in a simple game whose winning coalitions form a prefilter.

## 2. THE DIRECTIONAL CORE

The set of possible social states in this paper is simply a Euclidean space $E^m$. The societal status quo can therefore be represented at any time by a point $x \in E^m$. This section describes the static game that is played at each point in time and whose outcomes are shifts in the status quo.

The magnitude of feasible shifts is assumed to be very small (infinitesmal) and independent of their direction. Hence an outcome may be represented by the direction in which x shifts, where a <u>direction</u> is formally defined to be a vector in $E^m$ of unit or zero length. All directions of shift are allowed, so the set of feasible outcomes is $\mathcal{B} = B \cup \{0\}$, where B is the ball consisting of all unit vectors.

The players of the game are represented by an index set

$N = \{1,2,\ldots,n\}$. The preference ordering of each player $i \in N$ over social states is represented by a continuously differentiable utility function $u_i\colon E^m \to R$. From $u_i$, we induce a preference ordering $P_i(x)$ on $\mathcal{B}$ by defining, for any $v_1, v_2 \in \mathcal{B}$,

(2.1) $$v_1 P_i(x) v_2 \Longleftrightarrow v_1 \cdot \nabla u_i(x) > v_2 \cdot \nabla u_i(x).$$

By this ordering on $\mathcal{B}$, the preferred member of a pair of directions is the one closest to the utility gradient. In Appendix A we show that $v_1 P_i(x) v_2$ implies that player i prefers shifting x infinitesimally in direction $v_1$ rather than $v_2$.

Returning to the game, its outcome (direction x shifts) shall be determined by the set $\mathcal{W}$ of winning coalitions that characterize a simple game. Formally, $\mathcal{W}$ is a collection of subsets of N that is

(2.2i) (non-trivial) $\emptyset \notin \mathcal{W}$, $N \in \mathcal{W}$

(2.2ii) (superadditive) $M \in \mathcal{W}$, $M \subset M' \Longrightarrow M' \in \mathcal{W}$

(2.2iii) (proper) $M \in \mathcal{W} \Longrightarrow M^c \notin \mathcal{W}$.

Sometimes we shall assume the game is also strong:

(2.2iv) (strong) $M \in \mathcal{W} \Longleftrightarrow M^c \notin \mathcal{W}$.

Majority rule games, where any coalition containing more than n/2 members is declared winning, are the most common simple games satisfying (2.2i-iii). A majority rule game is strong provided n is odd.

Now the usual solution concept for a cooperative game, the core, can be defined. First define a dominance relation on $\mathcal{B}$ by $v_1 D(x) v_2$ provided there exists a winning coalition $M \in \mathcal{W}$ such that $v_1 P_i(x) v_2$ for all $i \in M$. Then the <u>directional core</u> K(x) is the set of all undominated directions:

(2.3) $$K(x) = \{\bar{v} \in \mathcal{B} \mid \nexists\ v \in \mathcal{B} \ni vD(x)\bar{v}\}.$$

If it is nonempty, the outcome shift is assumed to be in $K(x)$, which is particularly plausible because $K(x)$ is shown in appendix A to contain the core defined there independently of utility gradients.

The nature of the directional core is clarified by the following fundamental characterization. Its statement requires, for any $v \in \mathcal{B}$ and $x \in E^m$, a coalition to be defined by

(2.4) $$M(x,v) = \{i \in N \mid v \cdot \nabla u_i(x) > 0\}.$$

$M(x,v)$ is simply the coalition that prefers the status quo to shift in direction $v$ to remaining at $x$, that is, the set of people that prefers (by $P_i(x)$) $v$ to $0$.

<u>Proposition 2.1:</u> For any $x \in E^m$,

(2.5) $$K(x) = \{\bar{v} \in \mathcal{B} \mid \forall v \in \mathcal{B} \ni v \cdot \bar{v} \leq 0,\ M(x,v) \notin \mathcal{W}\}.$$

<u>Proof</u>: Suppose $\bar{v} \in \mathcal{B}$ and that for all $v \in \mathcal{B}$ satisfying $v \cdot \bar{v} \leq 0$, $M(x,v) \notin \mathcal{W}$. If $\bar{v} \notin K(x)$, then there exists $v' \in \mathcal{B}$ and an $M \in \mathcal{W}$ such that $v' P_i(x) \bar{v}$ for all $i \in M$. Hence for each $i \in M$, $(v' - \bar{v}) \cdot \nabla u_i(x) > 0$.

Since $v' - \bar{v} \neq 0$, we can let $v = \dfrac{v' - \bar{v}}{\|v' - \bar{v}\|}$. Clearly $v \in \mathcal{B}$, and by the Cauchy-Schwarz inequality,

$$v \cdot \bar{v} = \frac{v' \cdot \bar{v} - \bar{v} \cdot \bar{v}}{\|v' - \bar{v}\|} \leq 0.$$

So by the hypothesis, $M(x,v) \notin \mathcal{W}$. But for any $i \in M$, $v \cdot \nabla u_i(x) > 0$ since $(v' - \bar{v}) \cdot \nabla u_i(x) > 0$. Hence $M \subset M(x,v)$, and by superadditivity, we achieve the contradiction $M(x,v) \in \mathcal{W}$. Thus $\bar{v} \in K(x)$.

Conversely, suppose $\bar{v} \in K(x)$. If $\bar{v} = 0$, then for any $v \in \mathcal{B}$,

$vP_i(x)\bar{v}$ for all $i \in M(x,v)$. Because $\bar{v} = 0$ is undominated, $M(x,v) \notin \mathcal{W}$ for all $v \in \mathcal{B}$. Also, $M(x,0) = \emptyset \notin \mathcal{W}$. Hence we need to show $M(x,v) \notin \mathcal{W}$ only for $\bar{v} \neq 0$, $v \neq 0$, and $v \cdot \bar{v} \leq 0$.

For any $i \in N$, $v \cdot \nabla u_i(x)$ can be considered as a continuous function of $v$ on B. So if $v \cdot \nabla u_i(x) > 0$, there is an open neighborhood $U_i(v)$ of $v$ such that $y \cdot \nabla u_i(x) > 0$ for all $y \in U_i(v)$. Hence for any $v \in \mathcal{B}$ and any

$$y \in \bigcap_{i \in M(x,v)} U_i(v) = U(v),$$

$M(x,v) \subset M(x,y)$. By superadditivity, $M(x,y) \notin \mathcal{W}$ implies $M(x,v) \notin \mathcal{W}$. If $v \cdot \bar{v} \leq 0$, furthermore, since $U(v)$ is an open neighborhood of $v$, there is a $y \in U(v)$ such that $y \cdot \bar{v} < 0$. Therefore, to show that $v \cdot \bar{v} \leq 0$ implies $M(x,v) \notin \mathcal{W}$, we need only show that $v \cdot \bar{v} < 0$ implies $M(x,v) \notin \mathcal{W}$.

So suppose $v \cdot \bar{v} < 0$. Let $v' = \bar{v} - 2(v \cdot \bar{v})v$. Then $v' \in B$. If $i \in M(x,v)$, then $v \cdot \nabla u_i(x) > 0$, and so $(v' - \bar{v}) \cdot \nabla u_i(x) = -2(v \cdot \bar{v})(v \cdot \nabla u_i(x)) > 0$. Hence $v'P_i(x)\bar{v}$ for all $i \in M(x,v)$, implying that $M(x,v) \notin \mathcal{W}$ since $\bar{v}$ is undominated.

The content of proposition 2.1 is easily interpretable. Say that a direction $v$ "points away" from another direction $\bar{v}$ provided $v \cdot \bar{v} \leq 0$. Then (2.5) implies a direction $\bar{v}$ is undominated provided no winning coalition prefers a direction pointing away from $\bar{v}$ over the null direction. Stated differently, if the coalition $M(x,v)$ preferring $x$ to shift in direction $v$ is winning, then no direction $\bar{v}$ pointing away from $v$ is undominated.

For future reference, let

(2.6) $\quad I = \{x \in E^m \mid K(x) \neq \emptyset\}$,

and

(2.7) $\quad L = \{x \in E^m \mid K(x) = \emptyset\} = I^c$.

In appendix A, I is shown to be closed, so that L is open. Let $J$ = interior I and $\bar{L}$ = closure L.

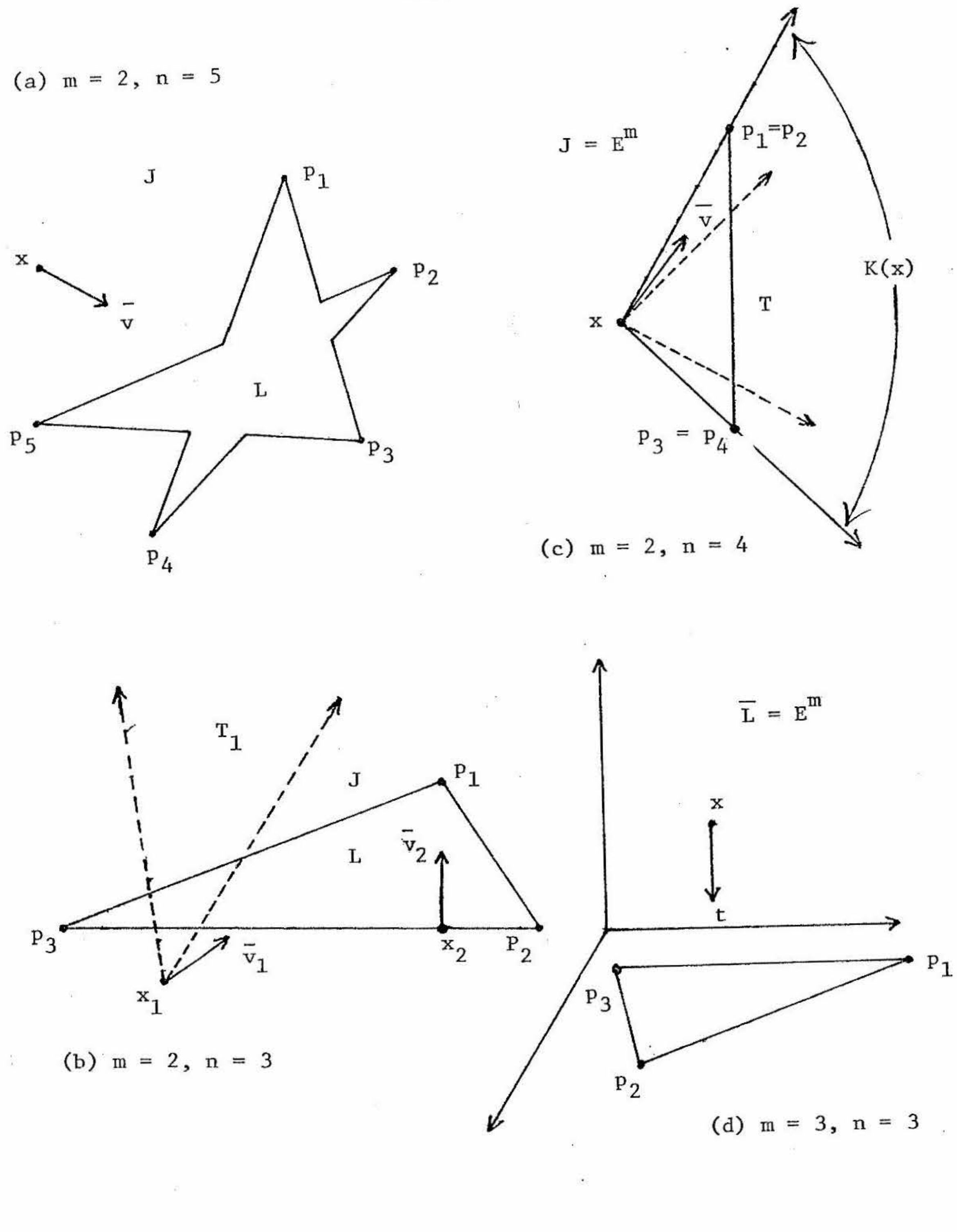
Examples of undominated directions are easily constructed that utilize the benchmark Euclidean preferences so pervasive in the spatial model literature. A person $i \in N$ is said to have Euclidean preferences if there is a point $p_i \in E^m$ such that

(2.8) $\quad u_i(x) = -\| p_i - x \|^2$

represents them. The point $p_i$ is i's ideal point, and his indifference surfaces are spheres centered at $p_i$. At any point x, the gradient $\nabla u_i(x) = 2(p_i - x)$ is a vector "pointing" from x to $p_i$.

When preferences are Euclidean and the game is majority rule, expression (2.5) simply says that $\bar{v} \in K(x)$ if any hyperplane containing x has no more than half the ideal points on any open side of it not containing $\bar{v}$. Thus when n is odd, as in figures 2.1a,b,d, an undominated direction at x is unique and must point towards a $p_i$ satisfying the median-like property that any hyperplane containing $p_i$ and x bisects the whole set of ideal points. In figure 2.1b the cone $T_1$ contains the directions that all three people prefer to a null shift at $x_1$, but the undominated direction $\bar{v}_1 \notin T_1$. At $x_2$ in figure 2.1b and at x in figure 2.1c, no winning coalition prefers the undominated direction shown to the zero

FIGURE 2.1


(a) m = 2, n = 5
J
x
v
p1
p2
p3
p4
p5
L
J = E^m
p1=p2
v
K(x)
T
x
p3 = p4
(c) m = 2, n = 4
T1
J
p1
L
v2
p3
v1
x1
x2
p2
(b) m = 2, n = 3
L = E^m
x
t
p1
p3
p2
(d) m = 3, n = 3

direction, that is, no winning coalition is better off if those status quos shift in the undominated directions indicated. In figure 2.1d, $m = 3$ and $x$ is floating above the two-dimensional triangle $P_1P_2P_3$. Everybody would prefer $x$ to shift in a direction such as $t$, but nevertheless there is no undominated direction.

## 3. THE POINT CORE

In this section, to further clarify the nature of the directional core $K(x)$, it is contrasted to a local point core often considered in the literature. To this end, define $K \subset E^m$ to be the set of points $x$ for which there is no direction $v \in B$ and coalition $M \in \mathcal{W}$ such that $v \cdot \nabla u_i(x) > 0$ for all $i \in M$. In the previous notation, this point core is simply

(3.1) $\quad K = \{x \in E^m \mid \forall\ v \in \mathcal{B},\ M(x,v) \notin \mathcal{W}\}.$

Although, as is shown in appendix B, $K$ is only a linear approximation to the set of locally undominated points, it has been discussed widely under various guises: it is the "local core" to the dynamic game of Schofield [1977b], the set of "Plott equilibriums" in Sloss [1973], and, in the context of majority rule, the set of "equilibriums" in Plott [1967], of "total medians" in Hoyer and Mayer [1975], and of "quasi-undominated" points in the dissertation appendix.

The definition of $K$ can also be written

(3.2) $\quad K = \{x \in E^m \mid 0 \in K(x)\},$

which says that $x$ is in the point core provided <u>no</u> direction in $B$ dominates the zero direction. This is in contrast to the condition implied by (2.5) for the directional core $K(x)$ to be non-empty, namely, that only some closed <u>half</u> of $B$ not dominate the zero direction. In this sense the existence conditions for $K(x)$ are weaker than those for $K$. This is further indicated by the following corollary to proposition 1, which indicates $x \in K$ if and only if <u>every</u> direction is undominated at $x$.

<u>Corollary 3.1</u>: Expression (3.2) can be strengthened to

(3.3) $\quad K = \{x \in E^m \mid K(x) = \mathcal{B}\}$.

<u>Proof</u>: By (3.2) we need only show that $K \subset \{x \in E^m \mid K(x) = \mathcal{B}\}$. Suppose $x \in K$ and $\bar{v} \in \mathcal{B}$. By (3.1), $M(x,v) \notin \mathcal{W}$ for all $v \in \mathcal{B}$. Hence by (2.5), $\bar{v} \in K(x)$. This proves $K(x) = \mathcal{B}$.

There is a closer relationship between the cores $K$ and $K(x)$ in the case of Euclidean preferences. Proposition 3.1 states that in this case any direction pointing from $x$ to $K$ is undominated -- a result clearly having content only when $K \neq \emptyset$.

<u>Proposition 3.1</u>: Let $x \in E^m$. If preferences are Euclidean, then

(3.4) $\quad \{\bar{v} \in \mathcal{B} \mid \exists\, \lambda \geq 0 \ni x + \lambda\bar{v} \in K\} \subset K(x)$.

<u>Proof</u>: It must be shown that if $z \in K$, then $K(x)$ contains the $\bar{v} \in \mathcal{B}$ for which $z = x + \lambda\bar{v}$ for some $\lambda \geq 0$. Suppose $v \in \mathcal{B}$ satisfies $v \cdot \bar{v} \leq 0$. Then $v \cdot (z - x) \leq 0$. Since $\nabla u_i(x) = 2(p_i - x)$, $v \cdot (p_i - x) > 0$ for all $i \in M(x,v)$. So for all $i \in M(x,v)$, $v \cdot (p_i - z) = v \cdot (p_i - x) - v \cdot (z - x) > 0$. This proves that $M(x,v) \subset M(z,v)$. Since $M(z,v) \notin \mathcal{W}$ because $z \in K$, superadditivity

implies $M(x,v) \notin \mathcal{W}$. Thus by proposition 2.1, $\bar{v} \in K(x)$.

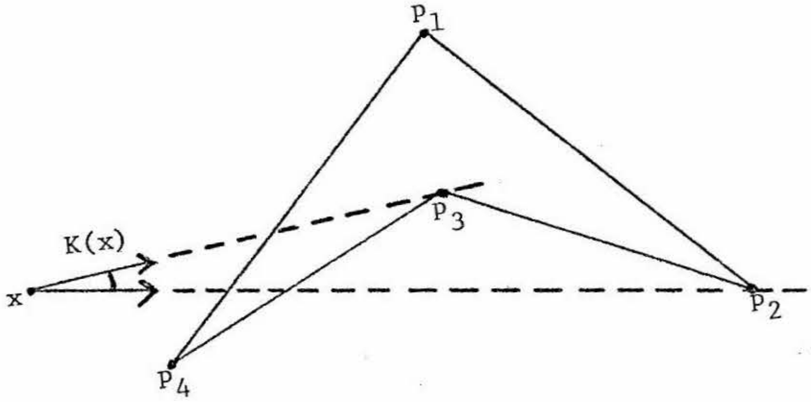
The reverse of inclusion (3.4) is not always true, as figure 3.1 indicates. In this figure, $N = \{1,2,3,4\}$ and three- and four-person coalitions are winning. At the point x, K(x) contains all directions between the directions that point to $p_2$ and $p_3$, but only $p_3$ is contained in K. The reason that all directions in K(x) in this example do not point to K is that the number of players in this majority rule game is even, which means that the game is not strong. The next proposition states that in strong simple games where preferences are Euclidean and $K \neq \emptyset$, K(x) is exactly the set of directions that point to K.

<u>Proposition 3.2</u>: If preferences are Euclidean, the game is strong, and $K \neq \emptyset$, then

(3.5) $\quad K(x) = \{\bar{v} \in \mathcal{B} \mid x + \lambda\bar{v} \in K \text{ for some } \lambda \geq 0\}$.

<u>Proof</u>: In view of proposition 3.1, it is only necessary to show that K(x) is contained in the set on the right of (3.5). So let $\bar{v} \in \mathcal{B}$, and suppose $x + \lambda\bar{v} \notin K$ for all $\lambda \geq 0$. We must show $\bar{v} \notin K(x)$. We can assume $\bar{v} \neq 0$, for $\bar{v} = 0$ and $x \notin K$ imply $\bar{v} \notin K(x)$. In Appendix B it is shown that K is closed, and a simple argument shows it is convex and bounded when preferences are Euclidean. Since $\{x + \lambda\bar{v} \mid \lambda \geq 0\}$ is disjoint from K, a separating hyperplane theorem shows the existence of $v \in B$ such that $v \cdot \bar{v} < 0$ and $v \cdot (z - x) > 0$ for any $z \in K$. Let $M = \{i \in N \mid v \cdot (p_i - x) \leq 0\}$, and let $z \in K$. Then if $i \in M$, $- v \cdot (p_i - z) = v \cdot (z - x) - v \cdot (p_i - x) > 0$. So, as $z \in K$,

FIGURE 3.1



$M \subset M(z,-v) \notin \mathcal{W}$. Superadditivity now implies $M \notin \mathcal{W}$. Since the game is strong and $M(x,v)$ is the complement of $M$, $M(x,v) \in \mathcal{W}$. So by proposition 2.1, $\bar{v} \notin K(x)$.

## 4. LOCAL CYCLES AND DIRECTIONAL CORES

A brief digression is now pursued in order to point out a connection between $K(x)$ and an important cone studied by Schofield [1977a]. A second characterization of $K(x)$ is provided that allows an immediate application of Schofield's Null Dual Theorem to show that $K(x) = \emptyset$ implies the dominance relation over points is cyclic in a neighborhood of $x$. Stated differently, a sufficient condition for $K(x)$ to be nonempty is that local cycling not occur in the vicinity of $x$. More notation is unfortunately necessary.

The <u>(local) Pareto optimal</u> set for a coalition $M \subset N$ is

(4.1) $\mathcal{P}(M) = \{x \in E^m \mid \nexists\, v \in B \ni M \subset M(x,v)\}$ ,

that is, $x$ is (locally) Pareto optimal for $M$ if there is no direction in which everyone in $M$ wants $x$ to shift. Notice that $K = \bigcap_{M \in \mathcal{W}} \mathcal{P}(M)$. The <u>preference co-cone</u> of a coalition $M \subset N$ at a point $x$ is simply the convex cone generated by the utility gradients of those in $M$:

(4.2) $\tilde{D}(x,M) = \{y \in E^m \mid y = \sum_{i \in M} \lambda_i \nabla u_i(x),\ \text{all } \lambda_i \geq 0,\ \text{some } \lambda_i > 0\}$.

As Schofield [1977a] demonstrates, $0 \in \tilde{D}(x,M)$ if and only if $x \in \mathcal{P}(M)$. Define a related cone by

(4.3)
$$D(x,M) = \begin{cases} \mathcal{B} \text{ if } x \in \mathcal{P}(M) \\ \tilde{D}(x,M) \cap \mathcal{B} \text{ if } x \notin \mathcal{P}(M). \end{cases}$$

Thus $0 \in \tilde{D}(x,M)$, $D(x,M) = \mathcal{B}$, and $x \in \mathcal{P}(M)$ are all equivalent statements. The next proposition provides an important characterization of the directional core $K(x)$ in terms of these cones.

Proposition 4.1: At any $x \in E^m$,

(4.4) $K(x) = \bigcap_{M \in \mathcal{W}} D(x,M)$.

Proof: $K(x) \subset \bigcap_{M \in \mathcal{W}} D(x,M)$ is first shown. Suppose $\bar{v} \in K(x)$. Then we must show $\bar{v} \in D(x,M)$ whenever $M$ is winning, which is nontrivial only when $D(x,M) \neq \mathcal{B}$. In this case the closed convex cone $\tilde{D}(x,M)$ does not contain 0. Assume $\bar{v} \notin \tilde{D}(x,M)$. Then $\tilde{D}(x,M)$ and $\bar{v}$ may be strictly separated with a hyperplane containing the origin, that is, there exists $v \in B$ such that $v \cdot \bar{v} \leq 0$ and $v \cdot y > 0$ for all $y \in \tilde{D}(x,M)$. As $\nabla u_i(x) \in \tilde{D}(x,M)$ for all $i \in M$, the latter inequality implies that $M \subset M(x,v)$. Superadditivity then implies $M(x,v) \in \mathcal{W}$, which by proposition 2.1 contradicts $\bar{v} \in K(x)$. Therefore, we know $\bar{v} \in \tilde{D}(x,M) \cap \mathcal{B} = D(x,M)$.

Now suppose $\bar{v} \in \bigcap_{M \in \mathcal{W}} D(x,M)$. We must show $\bar{v} \in K(x)$. Suppose $v \in \mathcal{B}$ satisfies $v \cdot \bar{v} \leq 0$. For any $i \in M(x,v)$, $v \cdot \nabla u_i(x) > 0$, which implies that $v \cdot y > 0$ for all $y \in \tilde{D}(x,M(x,v))$. Hence $0,\bar{v} \notin \tilde{D}(x,M(x,v))$. If $M(x,v) \in \mathcal{W}$, then by hypothesis, $\bar{v} \in D(x,M(x,v)) = \mathcal{B} \cap \tilde{D}(x,M(x,v))$, a contradiction. Hence $M(x,v) \notin \mathcal{W}$. So by proposition 2.1, $\bar{v} \in K(x)$.

Proposition 4.1 allows the immediate conclusion that the emptiness of $K(x)$ implies local cycling, once the latter is properly defined. Say a point $x_1$ is <u>continuously reachable</u> from a point $x_0$ provided there is a continuous path $c: [0,1] \to E^m$, differentiable on the intervals $I_1 = (0,t_1)$, $I_2 = (t_1,t_2), \ldots I_k = (t_{k-1},1)$, such that

(4.5i) $\quad c(0) = x_0$,

(4.5ii) $\quad c(1) = x_1$, and

(4.5iii) $\quad M_j \equiv \bigcap_{t \in I_j} M(c(t), c'(t)) \in \mathcal{W} \quad (j = 1,2,\ldots,k).$

So at each point $t \in I_j$, the winning coalition $M_j$ unanimously prefers the point $c(t)$ to shift along the curve $c$ rather than not shift at all.[3]

For any points $y,z \in E^m$, say that $y$ dominates $z$ if there exists $M \in \mathcal{W}$ such that $u_i(y) > u_i(z)$ for all $i \in M$. Since $c'(t) \cdot \nabla u_i(c(t)) > 0$ at each $t \in I_j$ and $i \in M_j$, it is easy to show that $u_i(c(t_{j+1})) > u_i(c(t_j))$ for every $i \in M_j$. Hence, if $x_1$ is continuously reachable from $x_0$, there is a sequence of points $x_0 = c(0), c(t_1),\ldots,c(t_{k-1}), c(1) = x_1$ such that each point dominates the preceeding one. This dominance relation is cyclic if $x_0 = x_1$. <u>Local cycling</u> is said to <u>occur at x</u> provided there is a neighborhood $U$ of $x$ such that any point in $U$ is continuously reachable from $x$ by a path that stays in $U$.[4]

The Null Dual Theorem of Schofield [1977a] states that local cycling occurs at $x$ if $\bigcap_{M \in \mathcal{W}} D(x,M)$ is empty. Proposition 4.1 therefore immediately implies

<u>Corollary 4.1</u>: Local cycling occurs at $x$ if $K(x) = \emptyset$.

## 5. THE APPROACH PROPERTY

In this section an examination of dynamics is initiated by characterizing points with nonempty directional cores in terms of certain paths containing them. Specifically, the directional core at x is nonempty if and only if there is a path through x that possesses a type of optimality that will soon be defined.

Because the global properties of paths are of interest, utility functions are often subsequently assumed to be <u>pseudo-concave</u>, that is, to satisfy for each $i \in N$

$$(5.1) \qquad (y - x) \cdot \nabla u_i(x) \leq 0 \Longrightarrow u_i(y) \leq u_i(x).$$

The next proposition will also require the <u>preferred-to-x set</u> of a coalition $M \subset N$ to be defined by

$$(5.2) \qquad \mathcal{P}(x,M) = \{y \in E^m \mid u_i(y) > u_i(x) \text{ for all } i \in M\}.$$

The set $\mathcal{P}(x,M)$ is open and, if utility functions are pseudoconcave, also convex.

If A is either a set or point in $E^m$, and $c: [0,\infty] \to E^m$ is a continuous, differentiable (almost everywhere) path, let the function $g_c(\cdot;A) : [0,\infty] \to R^+$ be the distance from c(t) to A :

$$(5.3) \qquad g_c(t;A) = \inf_{y \in A} \|y - c(t)\|.$$

Denote by $g_c'(t;Z)$ the derivative of $g_c$ at t. Say that the path c has the <u>approach property</u> at the point $c(\bar{t})$ provided that for all $M \in \mathcal{W}$ and $y \in \mathcal{P}(c(\bar{t}),M)$,

$$(5.4) \qquad g_c'(\bar{t};y) < 0.$$

The approach property can be interpreted as a pointwise optimality condition on paths, since a path satisfying the approach property at $x = c(\bar{t})$ is moving at time $\bar{t}$ simultaneously towards the preferred-to-x set of every winning coalition. One consequence of the following proposition is that a path satisfying the approach property at a point x exists if and only if $K(x) \neq \emptyset$.

Proposition 5.1: Fix $x \in E^m$. If there is a path c having the approach property at $x = c(\bar{t})$, then

$$\frac{c'(\bar{t})}{\|c'(\bar{t})\|} \in K(x). \tag{5.5}$$

Conversely, if each $u_i$ is pseudoconcave and c is a path satisfying $c(\bar{t}) = x$ and (5.5), then c satisfies the approach property at x.

Proof: Suppose c has the approach property at $x = c(\bar{t})$. Let $\bar{v} = \frac{c'(\bar{t})}{\|c'(\bar{t})\|}$. Suppose $v \in \mathcal{B}$ satisfies $v \cdot \bar{v} \leq 0$. By the continuity of each $u_i$ and the finiteness of $M(x,v)$, there exists $\lambda > 0$ such that $u_i(x + \lambda v) > u_i(x)$ for all $i \in M(x,v)$. Hence, letting $y = x + \lambda v$, we have $y \in \mathcal{P}(x,M(x,v))$. Since

$$g_c'(\bar{t}\,;y) = -\|c'(\bar{t})\| \quad (v \cdot \bar{v}) \geq 0,$$

(5.4) implies that $M(x,v) \notin \mathcal{W}$. Proposition 2.1 now implies $\bar{v} \in K(x)$, or rather, (5.5).

Conversely, suppose c is a path satisfying $c(\bar{t}) = x$ and (5.5), and assume utility functions are pseudoconcave. Let $y \in \mathcal{P}(x,M)$ for some $M \in \mathcal{W}$. Then by pseudoconcavity, $(y - x) \cdot \nabla u_i(x) > 0$ for each $i \in M$. Thus, by (5.5) and proposition 2.1, $M \in \mathcal{W}$ implies $(y - x) \cdot c'(\bar{t}) > 0$. Hence c satisfies the approach property at x:

$$g_c'(\bar{t};y) = \frac{-c'(\bar{t}) \cdot (y - x)}{g_c(\bar{t};y)} < 0.$$

Proposition 5.1 confers the optimal-like approach property to paths that always travel in undominated directions when such exist, as will be explicitly stated in the next section. Furthermore, proposition 5.1 confers a solution property of sorts to the set L, since a point contained in L has an empty directional core and therefore cannot simultaneously approach every point preferred to itself by every winning coalition.[5] Therefore points in either K or L satisfy desirable properties; points $x \varepsilon K$ strongly because the preferred-to-x set of every winning coalition is empty, and points $x \varepsilon L$ weakly because they cannot simultaneously approach all winning coalitions' preferred-to-x sets.

Again, stronger results are obtainable if preferences are Euclidean. This section concludes with the following results that will be important for the convergence theorem of the next section.

Lemma 5.1: Suppose preferences are Euclidean, and assume $\bar{x} \notin P(M)$ for some $M \subset N$. Let $\bar{z} \varepsilon P(M)$ satisfy

$$\|\bar{z} - \bar{x}\| = \inf_{z \varepsilon P(M)} \|z - \bar{x}\|.$$

Then $\bar{z} \varepsilon P(\bar{x},M)$.

Proof: It is well-known that $P(M)$ is the convex hull of $\{p_i \mid i \varepsilon M\}$. Hence $\bar{z}$ exists, since $P(M)$ is closed. As $P(M)$ is also convex, there is a supporting hyperplane at $\bar{z}$ with normal $(\bar{z} - \bar{x})$, that is, (5.6) $\quad (z - \bar{z}) \cdot (\bar{z} - \bar{x}) \geq 0$

for all $z \in P(M)$. Since each $p_i \in P(M)$, let $z = p_i$ in (5.6), subtract $\bar{x} \cdot (\bar{z} - \bar{x})$ from both sides, and rearrange to yield

$$(p_i - \bar{x}) \cdot (\bar{z} - \bar{x}) \geq (\bar{z} - \bar{x}) \cdot (\bar{z} - \bar{x}).$$

Hence

$$\begin{aligned}\|p_i - \bar{x}\|^2 &= (p_i - \bar{x}) \cdot (p_i - \bar{x}) \\ &> (p_i - \bar{x}) \cdot (p_i - \bar{x}) - 2(p_i - \bar{x}) \cdot (\bar{z} - \bar{x}) + (\bar{z}-\bar{x}) \cdot (\bar{z}-\bar{x}) \\ &= \|(p_i - \bar{x}) - (\bar{z} - \bar{x})\|^2 \\ &= \|p_i - \bar{z}\|^2 .\end{aligned}$$

As preferences are Euclidean, this proves $u_i(\bar{z}) > u_i(\bar{x})$ for each $i \in M$, or rather, $\bar{z} \in P(\bar{x},M)$.

Using this lemma, the following corollary proves that any path through $\bar{x}$ approaches each $P(M)$ if and only if its tangent vector at $\bar{x}$ is contained in $K(\bar{x})$. While this property does not by itself have an optimal interpretation like the approach property, it will provide the cornerstone of the next section's convergence result.

<u>Corollary 5.1</u>: Suppose preferences are Euclidean, and fix $\bar{x} \in E^m$. If c is a path differentiable at $c(\bar{t}) = \bar{x}$ such that

(5.7) $$\frac{c'(\bar{t})}{\|c'(\bar{t})\|} \in K(\bar{x}),$$

then for all $M \in \mathcal{W}$ such that $x \notin P(M)$,

(5.8) $\quad g_c'(\bar{t}; P(M)) < 0.$

Proof: Suppose $M \varepsilon \mathcal{W}$ and $\overline{x} \notin \mathcal{P}(M)$. Define $z(t) \varepsilon \mathcal{P}(M)$ by

$$\|z(t) - c(t)\| = \inf_{z\varepsilon\mathcal{P}(M)} \|z - c(t)\| = g_c(t;\mathcal{P}(M)).$$

Since $c(t)$ is continuous and $\mathcal{P}(M)$ convex, $z(t)$ is continuous. We first show $\phi(t) = z(t) \cdot (\overline{z} - \overline{x})$ is differentiable with $\phi'(\overline{t}) = 0$, where we have let $\overline{z} = z(\overline{t})$.

Because $\mathcal{P}(M)$ is convex, there is a supporting hyperplane at $z(t)$ with normal $z(t) - c(t)$:

$$(5.9) \quad (z - z(t)) \cdot (z(t) - c(t)) \geq 0$$

for all $z \varepsilon \mathcal{P}(M)$. Hence

$$\liminf_{t\to\overline{t}+} \left[\frac{(z(t) - \overline{z}) \cdot (\overline{z} - \overline{x})}{t - \overline{t}}\right] \geq 0,$$

and

$$\limsup_{t\to\overline{t}+} \left[\frac{(z(t) - \overline{z}) \cdot (\overline{z} - \overline{x})}{t - \overline{t}}\right]$$

$$= \limsup_{t\to\overline{t}+} \left[\frac{(z(t) - \overline{z}) \cdot (z(t) - c(t))}{t - \overline{t}}\right] \leq 0.$$

Hence the right hand derivative at $\overline{t}$ of $\phi(t)$, equal to

$$\lim_{t\to\overline{t}+} \left[\frac{(z(t) - \overline{z}) \cdot (\overline{z} - \overline{x})}{t - \overline{t}}\right],$$

exists and is 0. A similar argument establishes the same for the left hand derivative, so that $\phi'(\overline{t})$ exists and $\phi'(\overline{t}) = 0$.

Since $\phi'(\overline{t})$ and $c'(\overline{t})$ exist,

$$g_c'(\overline{t};\mathcal{P}(M)) = \frac{d[(z(t)-c(t)) \cdot (\overline{z} - \overline{x})]}{dt}\bigg|_{t=\overline{t}} g_c(\overline{t};\mathcal{P}(M))^{-1}$$

$$= [\phi'(\overline{t}) - c'(\overline{t}) \cdot (\overline{z} - \overline{x})]\; g_c(\overline{t};\; \overline{z})^{-1}$$

also exists. As $\phi'(\bar{t}) = 0$,

$$(5.10) \quad g_c'(\bar{t};P(M)) = -[c'(\bar{t}) \cdot (\bar{z} - \bar{x})]\, g_c(\bar{t};\bar{z})^{-1}$$
$$= g_c'(\bar{t};\bar{z}).$$

But by lemma 5.1, $\bar{z} \varepsilon P(\bar{x},M)$. Hence if (5.7) is true, proposition 5.1 implies $g_c'(\bar{t};\bar{z}) < 0$ since Euclidean utility functions are pseudo-concave. Thus (5.7) implies (5.8).

## 6. THE DYNAMIC PROCESS

Now consider paths that the status quo traces if at each time its direction of shift is contained in the directional core whenever it is nonempty. The requirement that a direction of movement be undominated whenever possible is a behavioral restriction. These paths are generated when the outcome of the simple game is an infintesimal shift of the status quo, after which a new game is played at the new status quo, and the entire process repeated indefinitely. One key assumption here is that players do not respond to realizations that current actions determine the location of future status quos and hence which games will subsequently be played. Whether this "sincere" behavior is a result of myopia, moral injunctions against large-scale gaming, etc., it probably occurs in many situations.

This dynamic process is modeled here as simply as possible. The ultimate goal is to obtain convergence of some sort to the set $K \cup L$ that was argued previously to satisfy solution-like properties. However, for mathematical convenience, convergence to the closed set $K \cup \bar{L}$ is investigated. The simplest assumption sufficient for convergence is merely that the speed of the status quo $x$ is bounded below by $s > 0$ when $x$ is not in $K \cup \overline{L}$, that is, when $x \in J\backslash K$. An upper bound $S$ on the speed is also a convenient assumption. Finally, in order to minimally restrict the direction of motion, it is required to be undominated only when $x \in J\backslash K$ rather than when $x \in I$. Summarizing, the status quo is assumed to follow a path $x: [0,\infty] \to E^m$, differentiable almost everywhere, satisfying

(6.1) $\quad x'(t) \in F(x(t))$,

where $F: E^m \to 2^{E^m}$ is a correspondence defined by

(6.2)
$$F(x) = \begin{cases} \{y \in E^m \mid \ \|y\| \leq S\} & \text{if } x \in K \cup \overline{L} \\ \{y \in E^m \mid \ s \leq \|y\| \leq S \text{ and } \frac{y}{\|y\|} \in K(x)\} & \text{if } x \in J\backslash K. \end{cases}$$

The correspondence $F$ maps points into truncated, convex closed cones, and is shown to be uppersemicontinuous in Appendix C.

It now immediately follows that such a path almost always satisfies the approach property whenever possible.

Corollary 6.1: Provided all preferences are pseudoconcave, a path $x$ satisfying (6.1) and (6.2) has the approach property at all $x(t) \in J$.

Proof: If $x(t) \in J\backslash K$, then from (6.2), $\frac{x'(t)}{\|x'(t)\|} \in K(x)$. Hence proposition 5.1 immediately implies that $x$ satisfies the approach property at $x(t)$. If $x(t) \in K$, then the approach property is vacuously satisfied at $x(t)$ since $P(x(t),M) = \emptyset$ for each $M \in \mathcal{W}$.

Two types of convergence will be discussed now. If $c$ is a path in $E^m$ and $A \subset E^m$, $c$ is said to converge to $A$ provided

$\lim_{t\to\infty} g_c(t;A) = 0$. The path c is said to <u>enter</u> A provided that given any $T \geq 0$, there is a time $t \geq T$ such that $c(t) \in A$.

The next proposition is that an x(t) satisfying (6.1) and (6.2) will converge to K if $K \neq \emptyset$ or will enter $\overline{L}$ if $K = \emptyset$, provided that preferences are Euclidean. Hence in this case the path converges to the set $K \cup \overline{L}$ that was argued to have solution properties in the previous section. From corollary 5.1 we see that x(t) will move into one Pareto set $\mathcal{P}(M)$ after another, never leaving any after entering, as long as $x(t) \in J$. So what occurs is that x(t) keeps moving simultaneously towards all winning coalitions' Pareto sets that do not contain it until it has either moved into them all $(x \in K)$ or can no longer approach them all simultaneously $(x \in \overline{L})$.

<u>Proposition 6.1</u>: Suppose all preferences are Euclidean. If $K \neq \emptyset$, then an x(t) satisfying (6.1) and (6.2) converges to K, and does so monotonically if the game is strong. If $K = \emptyset$, then x(t) enters $\overline{L}$.

<u>Proof</u>: Suppose first that $K \neq \emptyset$. Then proposition 3.1 implies $J = E^m$, so that $x(t) \in J$ always. Let $M \in \mathcal{W}$. Corollary 5.1 now implies that $g_x(t;\mathcal{P}(M))$ is strictly decreasing in t when $x(t) \notin \mathcal{P}(M)$. As $g_x(t;\mathcal{P}(M))$ is bounded below by 0,

$$(6.3) \qquad d^* = \lim_{t\to\infty} g_x(t;\mathcal{P}(M))$$

exists. It was shown in proving corollary 5.1 that $g_x(t;\mathcal{P}(M))$ was differentiable when x(t) was differentiable, which is almost everywhere. Hence (6.3) implies

(6.4) $\lim_{t\to\infty} g_x'(t;P(M)) = 0.$

Since x(t) is always approaching the compact set $P(M)$, the range of the path x is contained in a compact set. As the range of $x'$ is also contained in a compact set, there is a sequence $t_v \to \infty$ as $v \to \infty$ such that $\bar{x} = \lim_{v\to\infty} x(t_v)$ and $\bar{x}' = \lim_{v\to\infty} x'(t_v)$ exist. Let $z(t) \varepsilon P(M)$ satisfy

$$\|z(t) - x(t)\| = g_x(t;P(M)),$$

and let $\bar{z} = \lim_{v\to\infty} z(t_v)$. Then

$$\begin{aligned}(6.5) \qquad \bar{x}' \cdot (\bar{z} - \bar{x}) &= -\lim_{v\to\infty} g_x(t_v;z(t_v))\ g_x'(t_v;z(t_v)) && \text{(by 5.10))}\\ &= -\lim_{v\to\infty} g_x(t_v;P(M))\ g_x'(t_v;P(M))\\ &= -d{*}0 = 0 && \text{(by (6.3) and 6.4)).}\end{aligned}$$

Since F is uppersemicontinuous, $\bar{x}' \varepsilon F(\bar{x})$. If $d* \neq 0$, then $\bar{x} \notin P(M)$ and so $\bar{x} \notin K$. Hence by (6.1) and (6.2), $\|\bar{x}'\| \geq s > 0$ and

$$(6.6) \qquad \frac{\bar{x}'}{\|\bar{x}'\|} \ \varepsilon\ K(\bar{x}).$$

Let c(t) be a path such that $c(\bar{t}) = \bar{x}$ and $c'(\bar{t}) = \bar{x}'$. Then, as $\|\bar{z} - \bar{x}\| = g_c(\bar{t};P(M))$, (6.6), corollary 5.1, and $g_c(\bar{t});P(M)) > 0$ imply the contradiction

$$\begin{aligned}\bar{x}' \cdot (\bar{z} - \bar{x}) &= -g_c(\bar{t};\bar{z})\ g_c'(\bar{t};\bar{z}) && \text{(by (5.10))}\\ &= -\ g_c(\bar{t};P(M))\ g_c'(\bar{t};P(M))\\ &> 0.\end{aligned}$$

This proves that $d* = 0$. Hence x(t) converges to $P(M)$ for each $M \varepsilon \mathcal{W}$, which means that x(t) converges to $K = \bigcap_{M\varepsilon\mathcal{W}} P(M)$.

If $K \neq \emptyset$ and the game is strong, proposition 3.2 implies that $x'(t)$ always "points" at K. This can be used to show $g'(t;K) < 0$ when $x(t) \notin K$, so that $x(t)$ monotonically converges to K.

Now suppose $K = \emptyset$. If there was a T such that $x(t) \in J$ for all $t \geq T$, the above argument establishes the existence of a limit point $\bar{x}$ such that $\bar{x} \in \mathcal{P}(M)$ for all $M \in \mathcal{W}$. But then $\bar{x} \in K$, which is impossible. Hence $\bar{L}$ exists and $x(t)$ enters $\bar{L}$ when $K = \emptyset$.

The proof of proposition 6.1 could have used more of the special structure of Euclidean preferences, that is, it could have first been shown via proposition 4.1 that undominated directions "point" towards all winning coalitions' Pareto sets, which indicates that $x(t)$ must converge to them all if $x(t) \in J$ always. However, the above proof used the Euclidean assumption only via the monotinicity property of corollary 5.1. This should allow some elements of the proof to be useful in proving convergence to $K \cup \bar{L}$ under a less restrictive preference assumption.

## 7. THE EXISTENCE PROBLEM IN MAJORITY GAMES

The value of the hypothesis that game outcomes will be undominated is in its use as a predictor. In situations where social change is slow, so that the status quo can never shift far, it can be predicted to shift in undominated directions - provided they exist. Unfortunately, cores "infrequently" exist when power is evenly spread among individuals and the dominance relation is highly decisive, such as occurs in majority rule games defined by

$$\mathcal{W} = \{M \subset N \mid \frac{n}{2} < |M|\}.$$

For a majority game in a space of dimension greater than two, the necessary conditions for a nonempty directional core are similar and only slightly less restrictive than the conditions necessary for a nonempty point core. Expression (2.5) applied to a majority game states that K(x) contains a direction $\bar{v}$ only if each closed halfspace determined by any hyperplane through 0 and $\bar{v}$ contains at least half the utility gradients. Thus (2.5) is a condition of symmetry about a _line_ determined by 0 and $\bar{v}$. Expression (3.1), on the other hand, is a condition of symmetry about the _point_ 0, since it says that the point core K contains x only if each closed halfspace determined by any hyperplane through 0 contains at least half the utility gradients. If the dimension of the space is greater than two, then intuitively symmetry about a line is only slightly less restrictive than symmetry about a point. Each type of symmetry can be shown equivalent to stringent symmetry conditions involving _pairs_ of utility gradients.

Such pairwise symmetries are shown necessary in the dissertation appendix for the existence of various point cores. The point cores investigated there are allowed to be contained in the boundary of a feasible set, which means that only certain directions of shift are feasible. The restrictions on feasible directions allow the results of the appendix to be applied here to show that symmetries involving pairs of gradients are also required for directional cores to exist in majority games. In fact, we will broaden the discussion to generalized majority games, which are defined by a fraction

$.5 \leq \lambda \leq 1$ and

$$\mathcal{W} = \{M \subset N \mid \lambda n \leq |M|\}.$$

Inspection of (2.5) and (3.1) leads to the key observation: $\bar{v} \in K(x)$ whenever x is in the point core K in situations where feasible directions in which x can shift are defined by

$$F = \{v \in \mathcal{B} \mid v \cdot \bar{v} \leq 0\}.$$

The conditions for $x \in K$ when the cone of feasible shift directions is F is one of the special cases considered in the dissertation appendix and, if $\lambda = .5$ and n is odd, in Plott [1967].

To apply results in the appendix, let T be a two dimensional subspace of $E^m$ containing $\bar{v} \in K(x)$, and let $N_T = \{i \in N \mid \nabla u_i(x) \in T\}$. Let $Q \subset N_T$ be a maximal subset of $N_T$ that can be partitioned into pairs $\{i,j\}$ for which neither $\nabla u_i(x)$ nor $\nabla u_j(x)$ is a multiple of $\bar{v}$, but there is an $\alpha_i > 0$ and $\alpha_j > 0$ such that

(7.1) $\quad \alpha_i \nabla u_i(x) + \alpha_j \nabla u_j(x) \in \{0, \bar{v}\}.$

Finally, let $R \subset N_T \setminus Q$ be defined by $R = \{i \in N \mid \nabla u_i(x) = \alpha\bar{v}$ for some $\alpha \geq 0\}$. Then corollary 4 in the appendix implies that a necessary condition for $\bar{v} \in K(x)$ is a bound on $|Q|$:

(7.2) $\quad |N_T| - |R| \geq |Q| > |N_T| - 2|R| - (2\lambda - 1)n.$

To interpret this, suppose the game is majority rule with n odd. Then $\lambda = .5$ and we have

(7.3) $\quad |N_T| - |R| \geq |Q| > |N_T| - 2|R|.$

If $|R| = 0$, then $|N_T| \geq |Q| > |N_T|$, which is impossible. Hence (7.3) implies $|R| \geq 1$. If $|R| = 1$, then (7.3) implies $|Q| = |N_T| - 1$.

In this case, by applying (7.3) to all two dimensional subspaces containing $\overline{v}$, we see that the n-1 people in $N \setminus R$ can be partitioned into pairs whose gradients satisfy (7.1). This is exactly the condition obtained by Plott [1967] for constrained majority rule.

The pairwise symmetry condition (7.3) that applies to majority games is intuitively restrictive, which leads us to believe that $K(x) \neq \phi$ is "uncommon" in a majority game. Subsequent to the original appearance of this pairwise symmetry condition,[6] it was used by Schofield [1978] to show that $K(x) \neq \phi$ is "uncommon" in a formal sense.[7] Specifically, he investigates majority games in which $m > 2$ if n is odd and $m > 3$ if $n \geq 2$ is even. In these cases he shows that if the n-tuple of utility functions is contained in a particular subset of $\prod_{i=1}^{n} C^2$ that is dense with respect to a natural topology,[8] then the set

$$L = \{x \in E^m \mid K(x) = \phi\}$$

is dense in $E^m$. Thus L is _generically_ dense in a majority game if the dimension of the state space is greater than two or three.

If the set L is dense in $E^m$, then convergence to $L \cup K$ of paths that follow existing undominated directions is trivially true. However, despite Schofield's genericity result, there are still two reasons why directional cores in majority games are of interest. First, L is not generically dense in majority games if the outcome space is two dimensional. This is exactly the setting of experiments designed by Fiorina and Plott [1975] and McKelvey, Ordeshook, and Winer [1976] to test various solution concepts. Therefore, in the analysis of these experiments the solution properties of

L must be considered. In fact, Schofield [1977b] has argued that outcomes of these experiments do tend to cluster in L.

Second, the genericity of L being dense in majority games has meaning only in situations where preferences are determined in a "random" or "uncontrollable" fashion. Only then is it "unlikely" that the n-tuple of utility functions will not be in the dense subset of $\pi_{i=1}^{n} C^2$ that implies the denseness of L. However, in some cases the majority game is embedded in a larger model which allows preferences over the alternative space relevant to the majority game to be endogenously controlled. One step in this direction has been taken by Slutsky [1977] who investigates an economy where voting determines an allocation of public goods and a competitive market determines the allocation of private goods. Preferences over public goods are influenced by tax rates, and Slutsky shows that often the point core exists if the tax rates are properly chosen. It seems clear that a similar, dynamic model of an economy can be constructed in which the proper choice of tax rates can insure the existence of directional cores in public goods space.

It must not be forgotten that generic nonexistence of directional cores was obtained for majority games. In simple games with less decisiveness (fewer winning coalitions) and especially with less anonymity, directional cores exist more often. An indication that less decisiveness leads to more existence is that the only sufficient condition obtained in Schofield [1977a] for L to be generically dense in generalized majority games is for $m \geq \max\{2q-1, q+1\}$, where $\mathcal{W}$ consists of coalitions of size $q < n$ or larger.

Although this does not seem the tightest possible bound on m, directional cores will apparently frequently exist in games requiring large majorities operating with many individuals in spaces of low dimension.

Weakening anonymity results in existence to an even greater extent. Define a coalition

$$R = \bigcap_{M \in \mathcal{W}} M. \tag{7.4}$$

If $R \neq \phi$ then the collection of winning coalitions $\mathcal{W}$ that also satisfies (2.2i) and (2.2ii) is a <u>prefilter</u>. Brown [1973] shows that any dominance relation obtained from a prefilter when individual preferences are acyclic is also acyclic. The coalition R is called a <u>collegium</u> and occupies a uniquely powerful position, since (1) the point core K contains the nonempty Pareto set $P(R)$, and (2) the directional core $K(x)$ contains the nonempty cone $D(x,R)$, as can be seen from (4.4).[9] Hence, when a collegium exists, directional cores always exist and, under the conditions of proposition 6.1, the status quo will converge to the point core.

APPENDIX A

This appendix to section 2 first investigates the relationship between $K(x)$ and a directional core $\hat{K}(x)$ defined by a more complete inducement of preferences upon $\mathcal{B}$ than is represented by $P_i(x)$. It concludes with a proof that I is closed.

The best way to induce preferences from $E^m$ to $\mathcal{B}$ that is in keeping with the spirit of the model is to define a preference ordering $\hat{P}_i(x)$ on $\mathcal{B}$ by

(A.1) $v_1\hat{P}_i(x)v_2 \iff \exists\, \bar{\lambda} > 0 \ni u_i(x + \lambda v_1) > u_i(x + \lambda v_2) \quad \forall\, 0 < \lambda < \bar{\lambda}$. Player i will prefer shifting x in direction $v_1$ to shifting it in direction $v_2$, when both shifts are very small and of equal magnitude, if $v_1\hat{P}_i(x)v_2$. As $u_i$ is continuously differentiable, $\sim v_1\hat{P}_i(x)v_2$ and $\sim v_2\hat{P}_i(x)v_1$ imply $u_i(x + \lambda v_1) = u_i(x + \lambda v_2)$ for all $\lambda \geq 0$ less than some $\bar{\lambda} > 0$. Thus an indifference relation defined from $\hat{P}_i(x)$ truly indicates that a player is indifferent between small shifts. This is not true of an indifference relation defined from $P_i(x)$, since there are cases where $v_1\hat{P}_i(x)v_2$ but not $v_1P_i(x)v_2$.

The ordering $P_i(x)$ is a linear approximation to $\hat{P}_i(x)$ and is seen in lemma A1 below to be contained in $\hat{P}_i(x)$. The condition for $P_i(x) = \hat{P}_i(x)$ on $B = \mathcal{B}\setminus\{0\}$ is that $u_i$ be <u>locally symmetric</u> (<u>about its gradient</u>) at x, which is defined to mean that for any $v_1$, $v_2 \in B$, there exists $\bar{\lambda} > 0$ such that

(A.2) $$(v_1 - v_2) \cdot \nabla u_i(x) \leq 0 \Longrightarrow u_i(x + \lambda v_1) \leq u_i(x + \lambda v_2)$$

for all $0 < \lambda < \bar{\lambda}$. The name of this property results from the fact that (A.2) is satisfied provided that whenever $v_1$, $v_2 \in B$ are equidistant from the gradient $\nabla u_i(x)$, $\lambda v_1$ and $\lambda v_2$ must be on the same indifference curve for small $\lambda > 0$. Euclidean functions as defined in (2.8) and linear functions are two examples of functions everywhere locally symmetric.

Lemma A1: $P_i(x) \subset \hat{P}_i(x)$. $P_i(x) = \hat{P}_i(x)$ on B if and only if $u_i$ is locally symmetric at x.

Proof: If $v_1 P_i(x) v_2$, then $(v_1 - v_2) \cdot \nabla u_i(x) > 0$. Let $f(\lambda) = u_i(x + \lambda v_1) - u_i(x + \lambda v_2)$, and observe that

$$\lim_{\lambda \to 0+} \frac{f(\lambda)}{\lambda} = f'(0) = (v_1 - v_2) \cdot \nabla u_i(x) > 0.$$

Hence for small $\lambda > 0$, $f(\lambda) > 0$, that is, $v_1 \hat{P}_i(x) v_2$.

Now suppose that $\hat{P}_i(x) = P_i(x)$ and that $(v_1 - v_2) \cdot \nabla u_i(x) \leq 0$ for a particular $v_1, v_2 \in B$. Then not $v_1 P_i(x) v_2$, and hence not $v_1 \hat{P}_i(x) v_2$. Thus there is no $\bar{\lambda} > 0$ such that $f(\lambda) > 0$ for all $0 < \lambda < \bar{\lambda}$. Since f is continuously differentiable, this implies the existence of $\bar{\lambda} > 0$ such that $f(\lambda) \leq 0$ for all $0 < \lambda < \bar{\lambda}$. This proves that $u_i$ is locally symmetric at x.

Conversely, suppose $u_i$ is locally symmetric at x, and that $v_1 \hat{P}_i(x) v_2$. Then $f(\lambda) > 0$ for all small $\lambda > 0$. If $(v_1 - v_2) \cdot \nabla u_i(x) \leq 0$, then by (A.2), there exists $\bar{\lambda} > 0$ such that $f(\lambda) \leq 0$ for all $0 < \lambda < \bar{\lambda}$. As this is impossible, $(v_1 - v_2) \cdot \nabla u_i(x) > 0$, implying $v_1 P_i(x) v_2$.

If $v_1, v_2 \in \mathcal{B}$, say that $v_1$ (<u>$P(x)$</u>) <u>$\hat{P}(x)$-dominates</u> $v_2$ provided

$M \in \mathcal{W}$ exists such that $v_1\hat{P}_i(x)v_2$ $(v_1P_i(x)v_2)$ for all $i \in M$. Then $K(x)$ is the set of $P(x)$-undominated directions, and similarly define the core $\hat{K}(x)$ to be the set of $\hat{P}(x)$-undominated directions. $K(x)$ is a linear approximation to $\hat{K}(x)$, and the following proposition specifies their relationship. Without further assumptions it implies that the propositions of this paper characterizing directions in $K(x)$ are also true for directions in $\hat{K}(x)$.

Proposition A1: $\hat{K}(x) \subset K(x)$. Conversely, $\hat{K}(x) = K(x)$ provided either (i) every $u_i$ is linear, or (ii) every $u_i$ is locally symmetric at $x$ and, for each $M \in \mathcal{W}$, $x \notin \mathcal{P}(M)$ or $x \in \text{interior}\{y \mid \nexists\, z \ni u_i(z) > u_i(y)\ \forall\, i \in M\} \equiv IP(M)$.[10]

Proof: Lemma A1 implies $\hat{K}(x) \subset K(x)$. If each $u_i$ is linear, then $\hat{P}_i(x) = P_i(x) \Rightarrow \hat{K}(x) = K(x)$. Now suppose each $u_i$ is locally symmetric at $x$, and let $\overline{v} \in K(x)$. Case 1: $\overline{v} \neq 0$. If $\overline{v} \notin \hat{K}(x)$, then $\exists\, M \in \mathcal{W} \ni 0\ \hat{P}_i(x)\overline{v}\ \forall\, i \in M$, as by lemma A1 $\nexists\, v \neq 0$ that $\hat{P}(x)$-dominates $\overline{v}$. We know $x \in \mathcal{P}(M)$, for otherwise $\exists\, v \neq 0 \ni v\,P_i(x)\,0\ \ \forall\, i \in M$, which by lemma A1 and transitivity implies the contradiction $vP_i(x)\overline{v}\ \ \forall\, i \in M$. Also, $0\ \hat{P}_i(x)\ \overline{v}$ implies $\exists\, \overline{\lambda} > 0 \ni \forall\, 0 < \lambda < \overline{\lambda}$, $u_i(x) > u_i(x + \lambda\overline{v})\ \forall\, i \in M$. Hence $x \notin IP(M)$. This final contradiction proves $\overline{v} \in \hat{K}(x)$.
Case 2: $\overline{v} = 0$. Then by (3.1), $x \in \mathcal{P}(M)\ \forall\, M \in \mathcal{W}$, so that $x \in IP(M)\ \forall\, M \in \mathcal{W}$. But then no $v \in \mathcal{B}$ $\hat{P}(x)$-dominates $\overline{v} = 0$, so that $\overline{v} \in \hat{K}(x)$.

Proposition A2: I is closed.

Proof: Let $\{x_t\}$ be a sequence of points in I converging to $\bar{x}$. Let $v_t \in K(x_t)$. Since $\mathcal{B}$ is compact we can choose a subsequence $\{v_k\}$ of $\{v_t\}$ such that $\lim v_k = \bar{v} \in \mathcal{B}$. We show that $\bar{v} \in K(\bar{x})$ and hence $\bar{x} \in I$.

If $\bar{v} = 0$, then as $0 \in \mathcal{B}$ is an isolated point of $\mathcal{B}$, there exist $k_0$ such that $v_k = 0$ for all $k \geq k_0$. Suppose $i \in M(\bar{x},v)$ for some $v \in \mathcal{B}$. Then $v \cdot \nabla u_i(\bar{x}) > 0$. As $\nabla u_i$ is continuous, there exists $k(i)$ such that $v \cdot \nabla u_i(x_k) > 0$ for all $k \geq k(i)$. Hence $M(\bar{x},v) \subset M(x_k,v)$ for all $k \geq \bar{k} = \max\limits_{i \in M(\bar{x},v)} \{k_0,k(i)\}$. Since $v_k = 0 \in K(x_k)$ for $k \geq \bar{k}$, proposition 2.1 implies $M(x_k,v) \notin \mathcal{W}$ for $k \geq \bar{k}$. Superadditivity now implies $M(\bar{x},v) \notin \mathcal{W}$, , and proposition 2.1 now implies $v \in K(\bar{x})$.

So assume $\bar{v} \neq 0$, and suppose $v \cdot \bar{v} \leq 0$ for some $v \in \mathcal{B}$. As in the proof of proposition 2.1, the finiteness of N can be used to show existence of a $y \in \mathcal{B}$ near v such that $y \cdot \bar{v} < 0$ and $M(\bar{x},v) \subset M(\bar{x},y)$. As in the previous paragraph, the continuity of $\nabla u_i$ implies the existence of $\bar{k}$ such that $M(\bar{x},y) \subset M(x_k,y)$ for all $k \geq \bar{k}$. Furthermore, since $v_k \to \bar{v}$, there exists $\hat{k}$ such that $y \cdot v_k < 0$ for all $k \geq \hat{k}$. If $M(\bar{x},v) \in \mathcal{W}$, then $M(x_k,y) \in \mathcal{W}$ for all $k \geq \max\{\bar{k},\hat{k}\}$, which implies $v_k \notin K(x_k)$ for $k \geq \max\{\bar{k},\hat{k}\}$ by proposition 2.1. This contradiction shows $M(\bar{x},v) \notin \mathcal{W}$ for any v such that $v \cdot \bar{v} \leq 0$. Proposition 2.1 now implies $\bar{v} \in K(\bar{x})$.

APPENDIX B

The purpose of this appendix is analagous to that of appendix A, namely, to examine the relationship between the local point core K and a truly local point core $\hat{K}$ defined here. Just as K(x) was viewed as a linear approximation to $\hat{K}(x)$, K will be considered to linearly approximate $\hat{K}$.

Say that $x \in E^m$ is locally undominated provided a neighborhood U of x exists such that for any $z \in U$, $\{i \in N \mid u_i(z) > u_i(x)\} \notin \mathcal{W}$. The local point core $\hat{K}$ is the set of locally undominated points in $E^m$. Say that a function $u_i$ is locally pseudoconcave at x provided there exists a radius $\lambda_i > 0$ such that for any $v \in \mathcal{B}$,

$$v \cdot \nabla u_i(x) \leq 0 \Longrightarrow u_i(x + \lambda v) \leq u_i(x) \tag{B.1}$$

for all $0 < \lambda < \lambda_i$. (Observe that local pseudoconcavity is equivalent to pseudoconcavity (see (5.1)) if $\lambda_i = \infty$.)

The following lemma, stronger than necessary for proposition B1, is of independent interest because it shows when $\hat{K}$ can be defined in terms of $\hat{K}(x)$ just as K is defined in (3.2) in terms of K(x).

Lemma B1:

$$\hat{K} \subset \{x \in E^m \mid 0 \in \hat{K}(x)\}. \tag{B.2}$$

If each $u_i$ is locally pseudoconcave at each x contained in the right hand side of (B.2), then

$$\hat{K} = \{x \in E^m \mid 0 \in \hat{K}(x)\}. \tag{B.3}$$

Proof: Suppose $x \in \hat{K}$. If $0 \notin \hat{K}(x)$, then there exists $M \in \mathcal{W}$ and $v \in \mathcal{B}$ such that for each $i \in M$, $v\hat{P}_i(x)0$. Hence for each $i \in M$, there is a $\overline{\lambda}_i > 0$ such that $u_i(x + \lambda v) > u_i(x)$ for all $0 < \lambda < \overline{\lambda}_i$. As M is finite, $\overline{\lambda} = \min_{i \in M}\{\overline{\lambda}_i\} > 0$ and for all $i \in M$, $u_i(x + \lambda v) > u_i(x)$ if $0 < \lambda < \overline{\lambda}$. But now any neighborhood of x contains a point $x + \lambda v$ that dominates x via M, which contradicts $x \in \hat{K}$. Hence $0 \in \hat{K}(x)$.

Conversely, suppose $0 \in \hat{K}(x)$ and each $u_i$ is locally pseudo-concave at x with a radius of $\lambda_i > 0$. As N is finite, $\overline{\lambda} = \min_{i \in N} \{\lambda_i\} > 0$. If $x \notin \hat{K}$, there exists $v \in \mathcal{B}$ and $\lambda > 0$ such that $\lambda < \overline{\lambda}$ and $u_i(x + \lambda v) > u_i(x)$ for i contained in some $M \in \mathcal{W}$. Hence by (B.1), $v \cdot \nabla u_i(x) > 0$ for all $i \in M$, i.e., $vP_i(x)0$ for all $i \in M$. By lemma A1, $v\hat{P}_i(x)0$ for all $i \in M$, which contradicts $0 \in \hat{K}(x)$. Hence $x \in K$.

Proposition B1: $\hat{K} \subset K$. If each $u_i$ is locally pseudoconcave at each $x \in K$, then $\hat{K} = K$.[11]

Proof: Suppose $x \in \hat{K}$. Then $0 \in \hat{K}(x)$ by lemma B1, and by proposition A1, $0 \in K(x)$. Hence by (3.2), $x \in K$. Conversely, suppose $x \in K$ and each $u_i$ is locally pseudoconcave at x. Let $\overline{\lambda} = \min_{i \in N} \{\lambda_i\} > 0$, and let $U = \{x + \lambda v \mid v \in \mathcal{B}, 0 \leq \lambda < \overline{\lambda}\}$. If for some $x + \lambda v \in U$, $M = \{i \in N \mid u_i(x + \lambda v) > u_i(x)\} \in \mathcal{W}$, then local pseudoconcavity and superadditivity imply $M \subset M(x,v) \in \mathcal{W}$. This contradiction to $x \in K$ shows x is locally undominated in U, so that $x \in \hat{K}$.

Thus propositions true for elements of K are true for elements of $\hat{K}$, and local pseudoconcavity is sufficient for the converse.

Proposition B2: K is closed.

Proof: Let $\{x_t\}$ be a sequence in K converging to $\bar{x}$. Let $v_t = 0 \varepsilon K(x_t)$. Now apply the first half of the proof of proposition A2.

APPENDIX C

In this appendix $F(x)$, defined in (6.2), is shown to be uppersemicontinuous. To do this, suppose $\{x_k\}$ and $\{y_k\}$ are two sequences in $E^m$ such that $x_k \to \bar{x}$, $y_k \in F(x_k)$, and $y_k \to \bar{y}$. Then $\bar{y} \in F(\bar{x})$ must be shown.

If $\bar{x} \in K \cup \bar{L}$, the proof is trivial because each $\|y_k\| \leq S$ implies that $\|\bar{y}\| \leq S$, which shows by (6.2) that $\bar{y} \in F(\bar{x})$. So suppose $\bar{x} \in J\setminus K$.

Because $J\setminus K$ is an open set, $x_k \in J\setminus K$ for large k. Hence $v_k = \dfrac{y_k}{\|y_k\|}$ is contained in $K(x_k)$ for large k, by (6.2). Inspection of the first paragraph of the proof to proposition A2 now reveals that it proves $\bar{v} \in K(\bar{x})$, where $v_k \to \bar{v} = \dfrac{\bar{y}}{\|\bar{y}\|}$. Since $s \leq \|\bar{y}\| \leq S$ because $s \leq \|y_k\| \leq S$ for large k, this shows that $y \in F(\bar{x})$.

FOOTNOTES

1. Median-like symmetry conditions are discussed, for example, in Davis, DeGroot and Hinich [1972], Sloss [1973], Hoyer and Mayer [1975], and Calvert [1977]. The more explicit pairwise symmetry conditions necessary in majority rule are discussed in Plott [1967], McKelvey and Wendell [1976], Slutsky [1978], and the appendix to this dissertation.

2. Schofield actually investigates continuous-time processes that have the property that for small $\varepsilon$, the outcome at time $t + \varepsilon$ is preferred by a majority to the outcome at time $t$.

3. Notice that this does not say that $c'(t) \in K(c(t))$. Hence it is not necessarily true that a status quo moving along $c$ is shifting in undominated directions. This kind of path is discussed in sections 5 and 6.

4. Notice that (4.5iii) implies $\|c'(t)\| > 0$. Hence the local cycling property implies the existence of a nondegenerate path from $x$ to $x$ that stays near $x$, which accounts for the name "local cycling."

5. See Schofield [1977b] for a very different argument for considering $L$ as a solution set.

6. In the initial version of this chapter, Matthews [1977].

7. Schofield [1978] actually shows that local acyclicity is "uncommon" in a formal sense, but his method is to show that K(x) is "commonly" empty in majority games.

8. $C^2$ is the space of real, continuously twice differentiable functions on $E^m$, and a set A is dense in a topological space X provided the closure of A is X.

9. Schofield [1977 a,c] also observes that $K(x) \neq \phi$ if $\mathcal{W}$ is a prefilter.

10. The local Pareto set $\mathcal{P}(M)$ is defined in expression (4.1) as $\{x \mid \nexists\, v \in B \ni v \cdot \nabla u_i(x) > 0 \ \forall\ i \in M\}$. IP(M) is the interior of the global Pareto set of M. If each $u_i$ is pseudoconcave, as defined in (5.1), then IP(M) = interior$\mathcal{P}(M)$ and condition (ii) requires that $x \notin$ boundary$\mathcal{P}(M)$ $\forall\ M \in \mathcal{W}$. Figure 2.1(b) provides a counterexample when this condition is not required, for there each $u_i$ is locally symmetric at $x_2$, but $K(x) = \{\overline{v}_2\}$, although $\hat{K}(x) = \emptyset$. $(0\ \hat{P}_i(x_2)\overline{v}_2$ for $i = 2,3.)$

11. When N is not finite, neither $\hat{K} \subset K$ or $K \subset \hat{K}$ is true in general, even assuming local pseudoconcavity. Pseudoconcavity, however, implies both $K \subset \hat{K}$ and $\hat{K} = \{$globally undominated points$\}$. See Calvert [1977] and Sloss [1973] for further discussion of K and $\hat{K}$ when N is arbitrarily large.

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# Chapter III

## CONSTRAINED PLOTT EQUILIBRIA, DIRECTIONAL EQUILIBRIA, AND GLOBAL CYCLING SETS

Although the possibility of majority rule intransitivities has been recognized since at least the time of Condorcet, their ubiquity has only recently been revealed. Previously, democratic theorists had hoped that the smallest set of alternatives that collectively dominate all other alternatives -- the top cycle set -- would be small enough to uphold faith in the unbiased selectivity of majority rule (see Schwartz [1970]). McKelvey [1976] demonstrated that this hope was unfounded, at least for the case of multidimensional alternative spaces. He showed that in a special case any alternative can be reached from any other alternative by a finite sequence of majority rule decisions, which implies that the top cycle set is the entire alternative space. Another implication is that the final outcome of a majority rule procedure is determined completely by the agenda, or rather, by the person or institution that constructs the agenda.

Cohen [1977] shows that more generally the top cycle set is a member of the class of sets P(x), each of which is defined to be the set of all alternatives that can be reached from an alternative x via a sequence of majority decisions. Both Cohen [1977] and McKelvey [1977] deduce conditions necessary for a set P(x) not to include all alternatives. If P(x) is a proper subset of the alternative space, then it must have a boundary. McKelvey argues that the conditions necessarily satisfied at boundary points are severe enough to imply that the boundary rarely exists.

In this chapter we strengthen Cohen's [1977] and McKelvey's [1977] results by showing that extremely strong conditions must be satisfied at boundary points of P(x) when preferences are representable by differentiable utility functions. The characteristic properties of boundary points are local, and to exploit this fact we apply the results of Schofield [1977a,b] and of chapter II that concern continuous, local intransitivities and a continuous, dynamic majority rule process, respectively.

Specifically, in section 2 we first define global cycling sets and Schofield's continuous local cycling. We then observe that the latter cannot occur at boundary points of the former. This allows the results of chapter II to be applied in section 3 to conclude that undominated directions exist at the boundary points of any P(x). It is also shown in section 3 that, if some assumptions are satisfied, the undominated directions point back into the cycling set from its boundary points. This implies that in a dynamic setting such as that of chapter II, the top cycle set possesses dynamic as well as static stability properties.

However, the existence of undominated directions at boundary points of a cycling set implies, as is shown in chapter II, a severe pairwise symmetry condition on utility gradients. This condition is derived by first observing that the existence of undominated directions at a boundary point implies that the point must be a constrained Plott equilibrium (see Plott [1967]). The pairwise symmetry condition, which implies the "weak symmetry" condition McKelvey [1977] shows to hold at boundary points, is discussed in section 4. It further strengthens the conclusion that "usually" the set of global intransitivities is the entire alternative space.

## 1. LOCAL AND GLOBAL CYCLING

Let $N = \{1, 2, \ldots, n\}$ be the set of voters, with n odd. Each voter i has preferences over the open alternative space $X \subset R^m$ that are representable by a differentiable utility function $u_i$ whose indifference surfaces have no interiors (i.e., are "thin"). Under majority rule the set $W \subset 2^N$ of winning coalitions consists of all subsets of N with at least $\frac{n+1}{2}$ members, and the (absolute) majority rule relation P is defined by $xPy \iff \{i \in N \mid u_i(x) > u_i(y)\} \in W$. Define another relation Q by $xQy \iff yPx$.

Suppose the status quo is a point x in X. Any point y that can be achieved by a finite sequence of majority rule decisions starting at x is the outcome of <u>some</u> social process based on majority rule. That is, there exists an agenda that guarantees y as the outcome whenever y can be <u>reached via P</u> from x. This formally means that a finite sequence $x = x_0, x_1, \ldots, x_k = y$ exists such that $x_j P x_{j-1}$ for $j = 1, 2, \ldots, k$. Let $P(x)$ be the set of all points that can be reached via P from x, and let $Q(x)$ be the set of points that can be reached via Q from x. The sets $P(x)$ and $Q(x)$ are easily shown to be open by Cohen [1977] and McKelvey [1977], and the latter further shows $Q(x)$ to be the complement of the closure $\overline{P(x)}$ when indifference surfaces are thin.

If $\overline{P(x)} = X$, then arbitrarily close to any point in X is another point that can be reached from x. For this not to happen, $\overline{P(x)}$ must be a proper subset of X and have a nonempty boundary $\partial\overline{P(x)} = \overline{P(x)} \cap \overline{Q(x)}$.[1] Both Cohen [1977] and McKelvey [1977] establish properties that any $y \in \partial\overline{P(x)}$ must satisfy. For convenience and in accordance with McKelvey [1977], define $P^1(y) = \{z \in X \mid zPy\}$ and $Q^1(y) = \{z \in X \mid zQy\}$. Then properties necessary for $y \in \partial\overline{P(x)}$ are that P(x) contain and essentially be the set of points that defeat x:[2]

(1) $P^1(y) \subset P(x)$

(2) $\overline{P^1(y)} = \overline{P(x)}$

(3) $\overline{Q^1(y)} = \overline{Q(x)}$

Loosely speaking, (1) is true because any point that beats y will beat some point in P(x) by the continuity of utility functions and hence must itself be in P(x). Property (2) says that every open set in P(x) contains a point that beats y. If (2) is false, then the thin indifference curve assumption implies that y beats some point in P(x) -- a contradiction to $y \notin P(x)$. Property (3) is an immediate corollary of (2).

When preferences are strictly convex, or rather, when utility functions are strictly quasiconcave[3], Cohen [1977] shows that (2) and (3) can be strengthened to

(4) $P^1(y) = P(x)$

(5) $Q^1(y) = Q(x)$

She also shows for this case that each $P(x)$ is convex when $X$ is convex, and that a particular $P(\bar{x})$ exists, which we shall denote as $V$, that is a top cycle set in the following sense: any point in $V$ is reachable from any other (not necessarily different) point in $V$, and any point in $V$ beats any point not in $V$. Formally, $V$ satisfies $P(x) = V$ for all $x \in V$. This top cycle set $V$ thus can be considered a solution set. However, if $V$ is the whole space, then majority rule alone tells us nothing about what the outcome of a democratic process might be, regardless of the location of the status quo $x$.

Considering again an arbitrary $P(x)$, one obvious implication of Cohen's [1977] and McKelvey's [1977] propositions about any $y$ contained in the boundary $\partial\overline{P(x)}$ is that there are points in every neighborhood of $y$ that cannot be reached from $y$. More precisely, we have:

Proposition 1: Let $y \in \partial\overline{P(x)}$. Then, given any neighborhood $N(y)$ of $y$, there exist points in $N(y)$ which cannot be reached via $P$ from $y$.

Proposition 1 will allow us to conclude that any neighborhood of $y \in \partial\overline{P(x)}$ contains points that cannot be continuously reached from $y$, where continuous reachability is a concept explored by Schofield [1977a,b] and now to be defined. Define first a direction to be any vector in $E^m$ of zero or unit length, and denote the set of directions by $\mathcal{B}$. Let $B = \mathcal{B} \setminus \{0\}$. For any $v \in B$ and $z \in X$, define a coalition $M(z,v)$ by

$$M(z,v) = \{i \in N \mid v \cdot \nabla u_i(z) > 0\}.$$

$M(z,v)$ can be interpreted as the set of voters who prefer the point z to shift in direction v rather than not shift at all. (More will be said about directional preferences in the next section.)

Say that a point $x_1$ is continuously reachable from a point $x_0$ provided there is a continuous path $c: [0,1] \to E^m$, differentiable on the intervals $I_1 = (0,t_1)$, $I_2 = (t_1,t_2)$, ..., $I_k = (t_{k-1}, 1)$, such that

$$c(0) = x_0,$$

$$c(1) = x_1, \text{ and}$$

$$M_j \equiv \bigcap_{t \in I_j} M(c(t),c'(t)) \in W \qquad (j = 1,2,\ldots k).$$

So at each point $t \in I_j$, the winning coalition $M_j$ prefers the point $c(t)$ to shift along the curve c rather than not shift at all. Since $c'(t) \cdot \nabla u_i(c(t)) > 0$ at each $t \in I_j$ and $i \in M_j$, it is easy to show that $u_i(c(t_{j+1})) > u_i(c(t_j))$ for every $i \in M_j$. Therefore, if $x_1$ is continuously reachable from $x_0$, there is a sequence of points $x_0 = c(0)$, $c(t_1)$, ..., $c(t_{k-1})$, $c(1) = x_1$ by which x can be reached from $x_0$. Thus continuous reachability implies reachability, although the converse is false.[4] Proposition 1 now immediately implies:

Proposition 2: Let $y \in \partial\overline{P(x)}$. Then in any neighborhood of y there are points that cannot be continuously reached from y.

## 2. DIRECTIONAL CORES

We show in this section that the boundary properties of $\overline{P(x)}$ imply the existence at boundary points of directional cores, which are the subject of chapters I and II. The <u>directional core</u> $K(y)$ is a subset of the set of directions $\mathcal{B}$ defined by

$$\bar{v} \in K(y) \iff \{i \in N \mid (v - \bar{v}) \cdot \nabla u_i(y) > 0\} \notin W$$

for all $v \in \mathcal{B}$. The directions in $K(y)$ are said to be <u>undominated at y</u>.

The interpretation of $K(y)$ is simple. Suppose y is the status quo and all feasible alternatives are very close to y. Then the choice to be made is essentially a direction in which to shift y. If $(v - \bar{v}) \cdot \nabla u_i(y) > 0$, then voter i prefers a shift in direction v to a shift in direction $\bar{v}$ when both shifts are sufficiently small and of equal magnitude.[5] Therefore the directional core contains any direction which cannot be beaten by absolute majority rule when every voter votes in accordance with the above inducement of his directional preferences.

An undominated direction can be usefully characterized in terms of directions $v \neq 0$ that are preferred by a majority to the zero direction. A characterization obtained in chapter II is simply that $\bar{v}$ is undominated provided all directions that beat the zero direction are on the same open side as $\bar{v}$ of a hyperplane normal to $\bar{v}$. Formally, $\bar{v} \in \mathcal{B}$ is undominated at y if and only if

(6) $$M(y,v) \notin W$$

for all $v \in B$ such that $v \cdot \overline{v} \leq 0$.

A result of Schofield [1977a] is shown in chapter II to imply the existence of a neighborhood about y whose every point can be continuously reached from y whenever $K(y) = \emptyset$. Since no such neighborhood can occur at a boundary point of $\overline{P(x)}$, as was discussed in section I, undominated directions exist at boundary points of $\overline{P(x)}$.

<u>Theorem 1</u>: $K(y) \neq \emptyset$ for all $y \in \partial\overline{P(x)}$.

The remaining task of this section is to determine which directions are undominated at any $y \in \partial\overline{P(x)}$. We shall show that directions that are in some sense "perpendicular" to $\partial\overline{P(x)}$ and that "point towards" $P(x)$ are undominated. More definitions are needed to make these terms precise.

For any $z \in \overline{P(x)}$, a nonzero vector $v \in B$ is <u>tangent to P(x) at z</u> if there is a sequence $\{z_k\} \subset P(x)$ such that $z_k \to z$ and

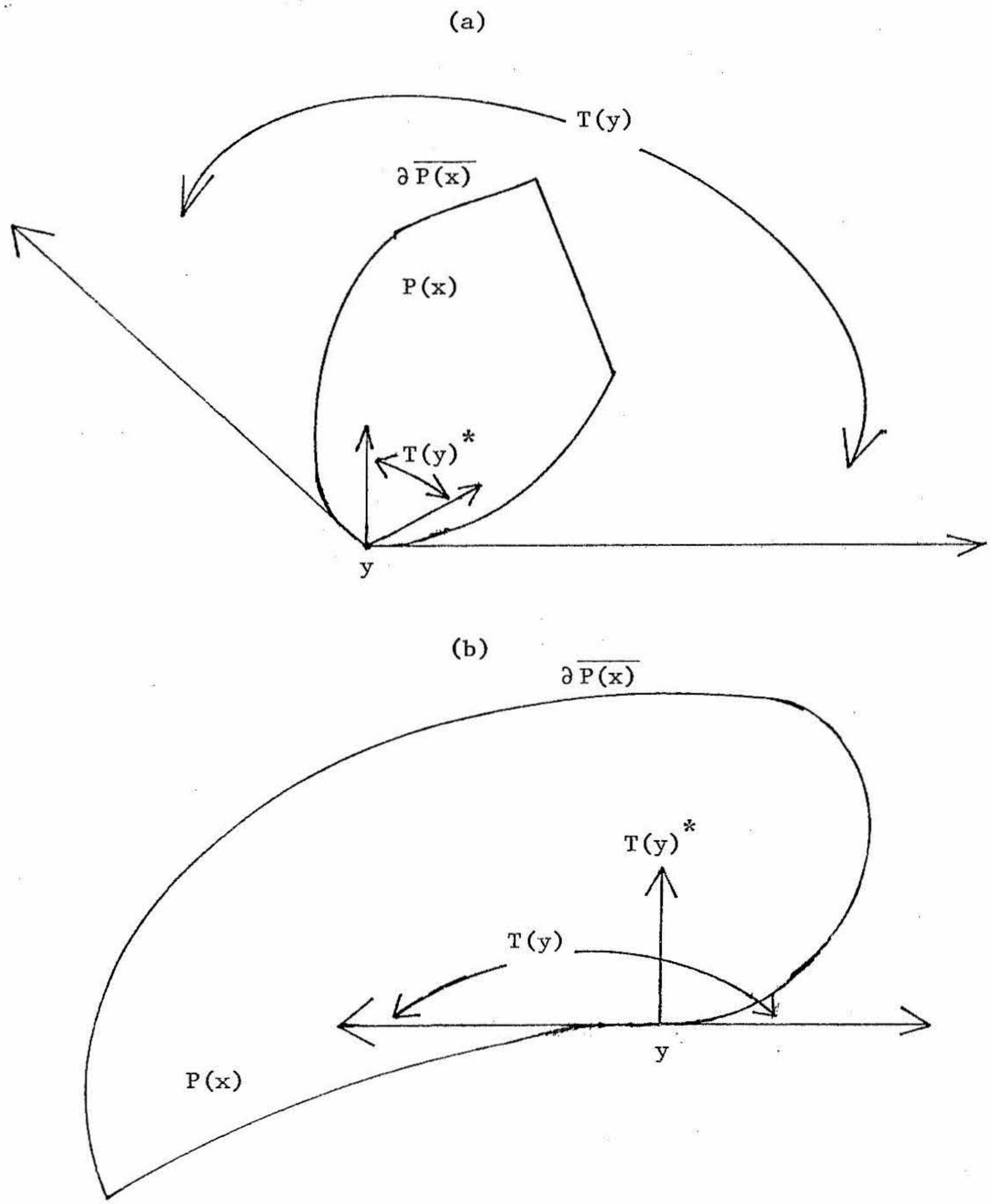
$$\lim_{k\to\infty} \frac{z_k - z}{\|z_k - z\|} = v.$$

Denote the set of directions $v \in B$ that are tangent to $P(x)$ at z by $T(z)$, the <u>tangent cone of P(x) at z</u>. Then the <u>(inner) normal cone of P(x) at z</u> is defined as the nonnegative dual of $T(z)$:

$$T(z)^* = \{\overline{v} \in B \mid v \cdot \overline{v} \geq 0 \text{ for all } v \in T(z)\}.$$

For most sets $P(x)$, the normal cone $T(y)^*$ is easily visualized for $y \in \partial\overline{P(x)}$. Examples are depicted in figure 1. If $P(x)$ is convex in a neighborhood of y, as in figure 1a, then

FIGURE 1



$T(y)^*$ consists of the inward normals of hyperplanes that support $P(x)$ at y. If the boundary of $P(x)$ is smooth at y, as in figure 1b, then the tangent cone $T(y)$ can be regarded as a closed halfspace tangent to the boundary at y, and $T(y)^*$ is a single vector normal to that hyperplane and pointing straight into $P(x)$.

We now show that the inner normal cone $T(y)^*$ is almost the directional core $K(y)$. The significance of this will be discussed after the theorem is presented. The proofs of the following two lemmas are in the chapter appendix.

Lemma 1: If $y \in \overline{P(x)}$, $v \in B$, and $M(y, v) \in W$, then v is contained in the interior of $T(y)$.

The essential equivalence of $T(y)^*$ and $K(y)$ requires one more assumption on utility functions, which is

$$\text{(A)} \qquad \nabla u_i(y) = 0 \implies u_i(y) = \max_{z \in X} u_i(z)$$

for all $y \in X$ and $i \in N$. Assumption (A) allows a proof of a near converse to lemma 1 when $y \in \partial\overline{P(x)}$:

Lemma 2: If $y \in \partial\overline{P(x)}$, $v \in$ interior $T(y)$, and assumption (A) holds, then there exists a direction $v'$ in any neighborhood of v such that $M(y,v') \in W$.

Theorem 2: Let $y \in \partial\overline{P(x)}$. Then

$T(y)^* \subset K(y)$.

Furthermore, if (A) is true, then

$$K(y) \setminus \{0\} \subset (\text{interior } T(y))^*.$$

<u>Proof</u>: Let $\bar{v} \in T(y)^*$, which implies $\bar{v} \neq 0$. Suppose that $M(y,v) \in W$ for some $v \in B$. Then by lemma 1, $v \in$ interior $T(y)$. An elementary argument establishes now, from the definition of $T(y)^*$, that $v \cdot \bar{v} > 0$. Therefore, by the characterization (6) of undominated directions, $\bar{v}$ is undominated. Hence, $T(y)^* \subset K(y)$.

Now assume (A) is true, and let $\bar{v} \in K(y) \setminus \{0\}$. If $\bar{v} \notin$ (interior $T(y))^*$, then, as $\bar{v} \neq 0$, a direction $v \in$ interior $T(y)$ exists such that $v' \cdot \bar{v} < 0$. Let U be a neighborhood of v such that $v' \cdot \bar{v} < 0$ for all $v' \in U$. Then by lemma 2, there exists $v' \in U$ such that $M(y,v') \in W$, contrary to the characterization (6) of all $\bar{v} \in K(y)$. Therefore $K(y) \setminus \{0\} \subset (\text{interior } T(y))^*$.

Two important applications of theorem 2 are to the more specific situations examined in Cohen [1977] and McKelvey [1977], respectively. Cohen assumes utility functions are strictly quasi-concave, but if this is strengthened to strict pseudoconcavity, a strong result is obtained:[6]

<u>Corollary 1</u>: If $y \in \partial\overline{P(x)}$ and each $u_i$ is strictly pseudoconcave, then

$$K(y) = T(y)^*.$$

<u>Proof</u>: By theorem 2, $T(y)^* \subset K(y)$. Since the pseudoconcavity of utility functions implies (A), theorem 2 also implies $K(y) \subset$ (interior $T(y))^*$. As strict quasiconcavity of utility functions follows from

their strict pseudoconcavity, $P(x)$ is convex. But then $T(y)$ is convex, which implies $T(y)^* = (\text{interior } T(y))^*$ (see e.g., Bazaraa and Shetty [1976]). $P(x)$ open and convex also implies that $T(y)$ has an interior, so that $0 \notin K(y)$ by lemma 2. Therefore $K(y) = T(y)^*$.

McKelvey [1977], on the other hand, does not assume quasi-concave utility functions. Instead, he lets $I^1{}_i(y) = \{z \in X \mid u_i(z) = u_i(y)\}$ be an indifference curve through y and then assumes a condition on indifference surfaces called <u>diversity of preferences</u>:

(DP) For all open $S \subset X$ and $y \in \partial\overline{S}$, $I^1{}_i(y) \cap I^1{}_j(y)$ has no interior in the relative topology in $\partial\overline{S}$.

As McKelvey puts it, "this condition guarantees that individual indifference contours never exactly coincide." He shows that (DP) implies the existence of $j \in N$ for which $u_j(y) = u_j(y')$ for all $y, y' \in \partial\overline{P(x)}$. This and the differentiability of $u_j$ immediately imply:

<u>Corollary 2:</u> If conditions (A) and (DP) are satisfied and $y \in \partial\overline{P(x)}$, then $T(y)^*$ is a singleton and

$$K(y) = T(y).^*$$

<u>Proof</u>: Because $\partial\overline{P(x)}$ is contained in a "thin" indifference surface of a differentiable utility function, $T(y)$ is a halfspace and $T(y)^*$ is a single vector collinear with $\nabla u_j(y)$, as in figure 1b. The convexity of $T(y)$ implies $T(y)^* = (\text{interior } T(y))^*$, and the existence of the interior of $T(y)$ implies $0 \notin K(y)$ by lemma 2. Hence $K(y) = T(y)^*$.

The significance of $K(y)$ being the inner normal cone to $P(x)$ is clearest when (DP) is satisfied, utility functions are pseudo-concave, and $P(x) = V$, the top cycle set. For then the undominated direction at a boundary point of V is perpendicular to the boundary and points straight back into V. If the status quo is continually shifting infinitesimal amounts in undominated directions when they exist, as seems likely under a sequential decision process when feasible sets are small (see chapter 2), then once the status quo enters V it cannot escape. Thus the results of this section imply a type of dynamic as well as static stability to the top cycle set.

## 3. CONSTRAINED PLOTT EQUILIBRIA AND PAIR SYMMETRY

McKelvey [1977], assuming (DP), shows that for any $P(x)$ there is some $j \varepsilon N$ such that voters' utility gradients at any $y \varepsilon \overline{\partial P(x)}$ satisfy a symmetry condition about $\nabla u_j(y)$. Specifically, he shows that the set of utility gradients at y is <u>weakly symmetric with respect to j</u>, which means that for any $i \neq j$ there exists a third individual k such that $\nabla u_i(y)$, $\nabla u_j(y)$ and $\nabla u_k(y)$ are independent. In this section we observe that, with respect to some $j \varepsilon N$, a somewhat stronger symmetry condition is satisfied at most boundary points if (DP) holds. Furthermore, similar symmetry is shown to be satisfied at boundary points even if (DP) does not hold.

First, observe that the characterization of $K(y)$ in (6) implies that if $\overline{v} \varepsilon K(y)$, then y is a constrained Plott equilibrium relative to the set $\{v \in B \mid v \cdot \overline{v} \leq 0\}$ (see chapter II). By this we mean that if the only feasible directions in which y can shift "point" away from $\overline{v}$, then the zero direction is undominated -- y is a constrained equilibrium as first defined in Plott [1967]. This result is easily visualized for a point $y \in \partial\overline{P(x)}$ when P(x) is convex. Then, if v is any direction such that $v \cdot \overline{v} \leq 0$, $\overline{v}$ being in the inner normal cone and the openness of P(x) imply that points in the direction v from y lie outside P(x). Since no majority will prefer points outside the set P(x) to y, no majority will vote to move in the direction v.

Stringent symmetry conditions on the voters' gradients must hold at y if y is a constrained equilibrium. The conditions are collectively termed "pair symmetry". The simplest case, investigated by Plott [1967], exists when no more than one voter's gradient points in the direction $\overline{v} \in K(y)$. In this case, pair symmetry requires that:

(a) for some $j \in N$, $\nabla u_j(y) = \alpha\overline{v}$ for some $\alpha \geq 0$.

(b) all other voters in N can be partitioned into distinct pairs so that for each pair (i,k) there are positive numbers $\alpha_i$ and $\alpha_k$ such that $\alpha_i \nabla u_i(y) + \alpha_k \nabla u_k(y) \in \{0,\overline{v}\}$.

Condition (a) says that somebody's utility gradient must be a nonnegative multiple of $\overline{v}$. Condition (b), necessary when no more than one gradient is a nonnegative multiple of $\overline{v}$, requires that either j and k's gradients point in opposite directions or $\overline{v}$ lies in the convex cone generated by them. Figure 2 demonstrates this condition in the case where $|N| = 5$ and $X = R^2$.

Pair symmetry, in its (a) and (b) version, differs from weak symmetry by requiring that $N \setminus \{j\}$ be partitioned into pairs $\{i,k\}$ such that the vectors $\nabla u_i(y)$, $-\nabla u_j(y)$, and $\nabla u_k(y)$ are positively dependent. Thus the (a) and (b) version of pair symmetry implies weak symmetry. If (DP) holds, then on a subset of $\partial\overline{P(x)}$ that is dense in $\partial\overline{P(x)}$, no two utility gradients can be collinear. In this case, by theorem 1 and the above remarks, (a) and (b) must hold on a dense subset of $\partial\overline{P(x)}$. Furthermore, by theorem 2, for any $y \in \partial\overline{P(x)}$ there is a $\overline{v} \in T(y)^* \subset K(y)$ that is collinear with $\nabla u_j(y)$, where j is the individual indifferent on $\partial\overline{P(x)}$. Thus we have McKelvey's weak symmetry result strengthened on a dense subset of $\partial\overline{P(x)}$:

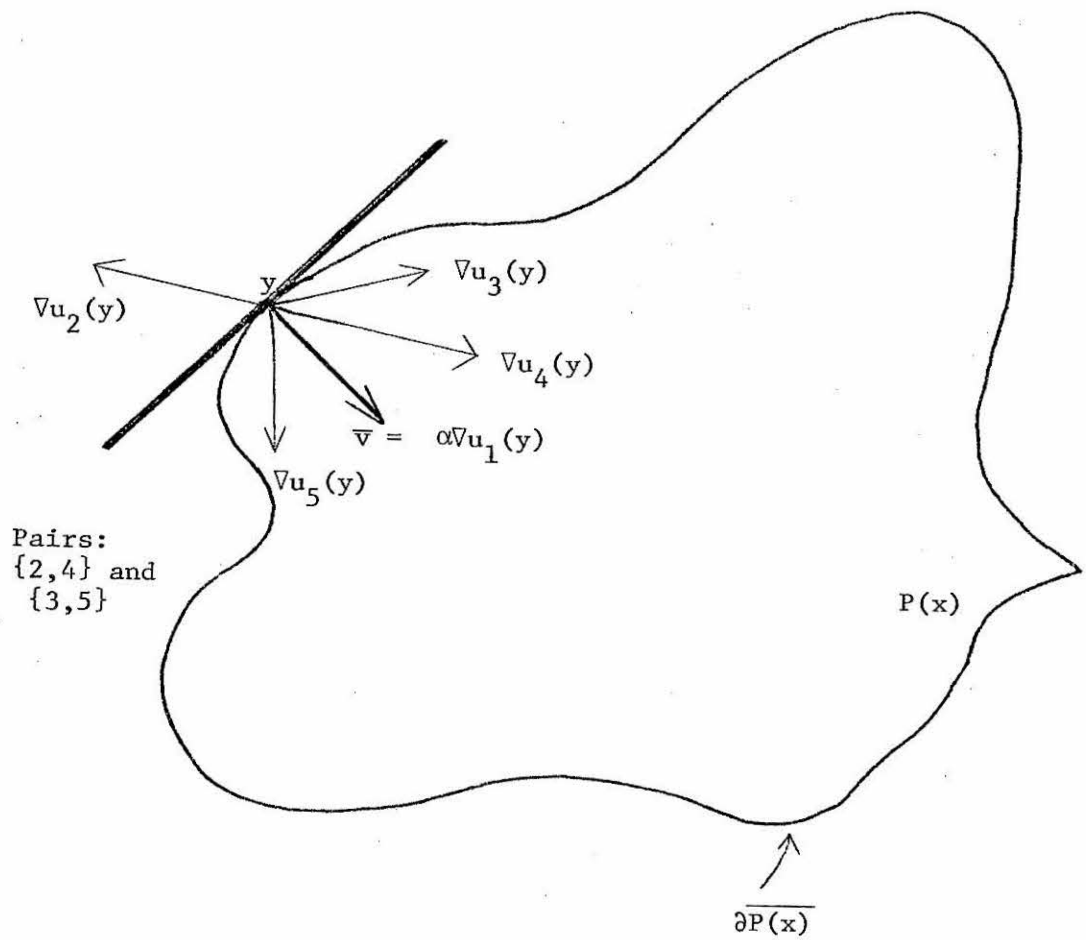
Corollary 3: If (DP) holds, then there exists $j \in N$ such that (a) and (b) are satisfied, with (a) referring to j, at every y in a dense subset of $\partial\overline{P(x)}$.

One implication of conditions (a) and (b) concerns the projections $\nabla u_i^P(y)$ of the utility gradients $\nabla u_i$ onto the hyperplane normal to the undominated direction $\overline{v}$. Conditions (a) and (b) imply the following:

(a') for some $j \in N$, $\nabla u_j^P(y) = 0$

(b') all other voters in N can be partitioned into pairs

FIGURE 2


y
$\nabla u_3(y)$
$\nabla u_2(y)$
$\nabla u_4(y)$
$\overline{v} = \alpha\nabla u_1(y)$
$\nabla u_5(y)$
Pairs:
{2,4} and
{3,5}
P(x)
$\overline{\partial P(x)}$

(i,k) such that there exist positive real numbers

$\alpha_i$, $\alpha_k$ satisfying $\alpha_i \ \Delta u_i^P(y) + \alpha_k \ \Delta u_k^P(y) = 0$.

Conditions (a') and (b') are identical to Plott's conditions for an unconstrained voting equilibrium (see Plott [1967]), implying that any $y \in \partial\overline{P(x)}$ would be a voting equilibrium if only the points in $\partial\overline{P(x)}$ were feasible. Figure 3 shows the projections from three of the voter gradients in figure 2.

More general pair symmetry conditions necessary for $\bar{v} \in \mathcal{B}$ to be undominated at y, or rather, for y to be a constrained voting equilibrium, are derived in the dissertation appendix. One formulation is reproduced here.[7] Let $N_T$ be the voters whose utility gradients are contained in a two dimensional subspace T that also contains $\bar{v}$. Define the following subsets of N:

$R^+ = \{i \in N \mid \nabla u_i(y) = \alpha\bar{v} \text{ for } \alpha \geq 0\}$

$R^- = \{i \in N \mid \nabla u_i(y) = \alpha v \text{ for } \alpha < 0\}$

$Q$ = maximal subset of $N_T \setminus (R^+ \cup R^-)$ that can be partitioned into pairs (j,k) for which there exist positive numbers $\alpha_j, \alpha_k$ satisfying $\alpha_j \ \nabla u_j(y) + \alpha_j \ \nabla(y) \in \{\bar{v},0\}$.

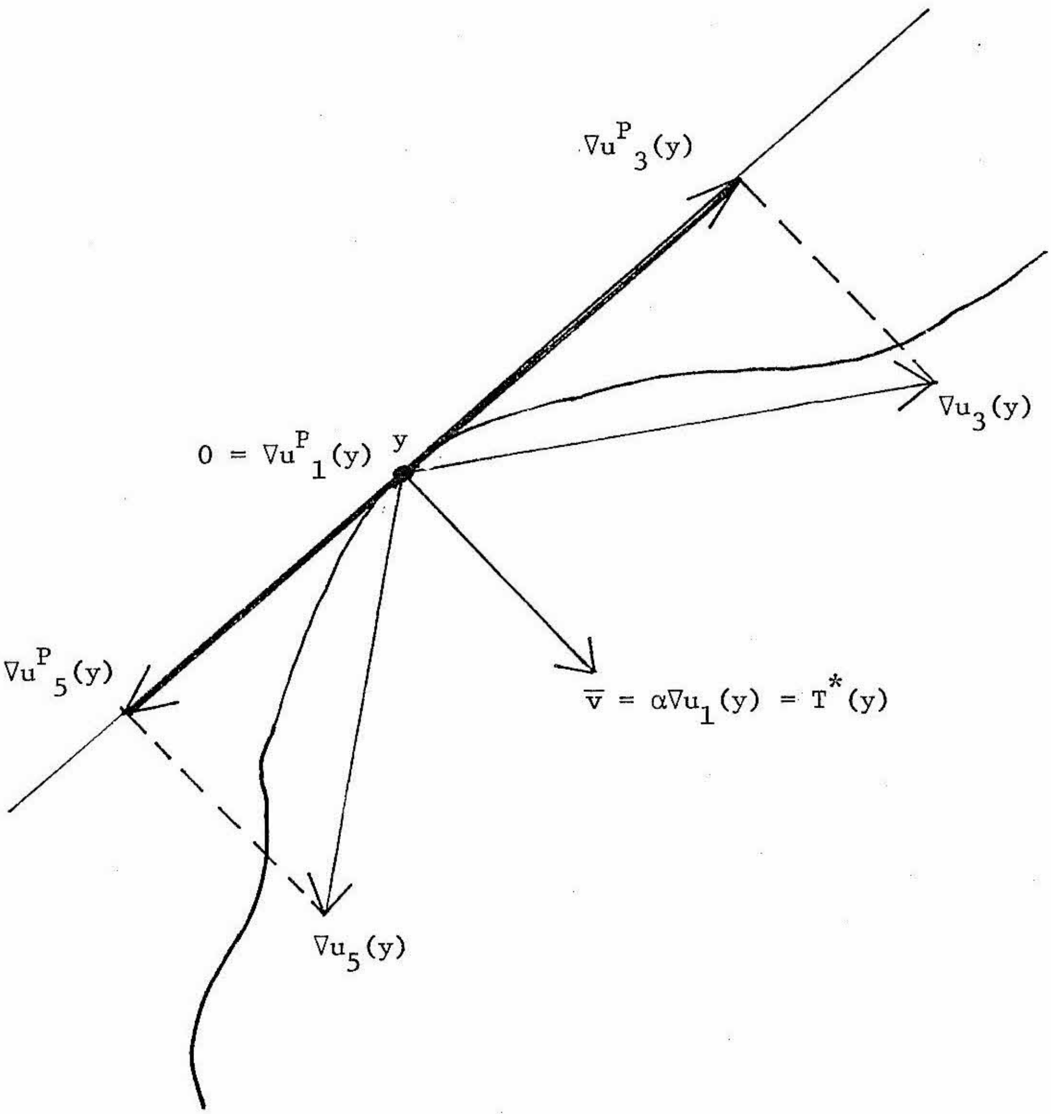
Then for $\bar{v} \in K(y)$ it is necessary by corollary 4 of the appendix that

(a'') $|R^+| > |R^-|$

(b'') $1/2 \ |Q| + |R^+| \geq \dfrac{|N_T|+1}{2}$

Observe that condition (b'') says that the coalition formed by all the

FIGURE 3


$\nabla u^P{}_3(y)$
$\nabla u_3(y)$
$0 = \nabla u^P{}_1(y)$
y
$\nabla u^P{}_5(y)$
$\overline{v} = \alpha\nabla u_1(y) = T^*(y)$
$\nabla u_5(y)$

people in $R^+$ and half the people in Q is a majority in $N_T$. If $|R^+| = 1$ and $|R^-| = 0$, then summing (b") over all two dimensional subspaces containing $\bar{v}$ results in conditions (a) and (b).

The pair symmetry conditions, as well as the results of previous sections, are collected as Theorem 3.

<u>Theorem 3</u>: Let $x \in X$, $P(x)$ be the set of points in X which can be reached from x, and let y be contained in the boundary $\partial\overline{P(x)}$. Then

(i) the directional core $K(y)$ is nonempty;

(ii) the inner normal cone $T(y)^* \subset K(y)$, and, given assumption (A), $K(y)\setminus\{0\} \subset (\text{interior } T(y))^*$;

(iii) letting $\bar{v} \in K(y)$, y is a constrained Plott equilibrium in a situation where its feasible set of shift directions is $\{v \in \mathcal{B} | v \cdot \bar{v} \leq 0\}$;

(iv) the voters' utility gradients satisfy the pair symmetry conditions (a") and (b") with respect to any $\bar{v} \in K(y)$.

## 4. DISCUSSION

The usefulness of the top cycle set V as a solution concept requires that it be "small". We have shown that if it is small in the sense of its closure having a boundary, then at every point of the boundary an undominated direction exists that, in some sense, "points back into V." The restrictiveness of this condition is perhaps best seen via one of its implications, namely, that the multitude of points on the boundary all satisfy the same pairwise symmetry condition as constrained voting equilibria. Since this condition closely resembles the symmetry condition satisfied at unconstrained equilibria, it can be heuristically said that the closure of the top cycle set is "usually" the entire space for the same reason that voting equilibria "rarely" exist.

The set $L = \{y|K(y) = \emptyset\}$ has also been proposed as a solution set (Schofield, 1977a,b; chapter II). One reason is that Schofield's results imply that if L is connected, then any point in L can be continuously reached from any other point in L, so that L is a continuous cycling set analogous to the discrete cycling set V.[8] Secondly, as is shown in chapter II, in some cases a status quo that continuously shifts in undominated directions eventually enters L. Furthermore, in two dimensions L empirically seems to be a reasonable solution set because, as Schofield [1977b] notes, the outcomes of experimental games (Fiorina and Plott, 1975) are often in L, a relatively small set in two dimensions.

The size of L in general is thus an important question. In the two dimensional case with Euclidean preferences, L is bounded. Clearly then, since McKelvey [1976] shows for the same case that V is the entire space, sufficient conditions for V to be small must be more stringent than conditions for L to be small. However, the necessary condtiions (i), (iii), and (iv) of theorem 3 all apply to points not in L as well as to points in the boundary of P(x), since they were derived by showing $\partial\overline{P(x)} \cap L = \emptyset$. Hence some of the reasons for believing V is usually large also serve as reasons to believe L is large. An unanswered question is whether condition (ii) of theorem 3 is strong enough to characterize the difference between V and L, so that it can be viewed as the reason "why" V is large but L small in two dimensions.

When the dimension of the space is greater than two, the conditions (i), (iii) and (iv) of theorem 3 appear more restrictive. This is because they are conditions for a constrained voting equilibrium, which resemble the conditions for an unconstrained equilibrium in a space of one dimension less (see figure 3). Hence undominated directions exist at more points in two dimensions for the same reason voting equilibria often exist in one dimension. We conjecture that L and V will usually be large, i.e., dense in the space, when the dimension of the space is greater than two.

Schofield [1977a] has formalized this intuition in cases where the dimension of the space is larger than the number of voters. Given this dimensional assumption, he shows that L is generically connected and dense in the space, where generically (= "usually") refers to a property that is satisfied whenever the n-tuple of utility functions belongs to some dense set in an appropriate function space. But the conditions characterizing points not in L, such as pair symmetry, appear so restrictive that we conjecture that L is generically dense whenever the alternative space has dimension greater than two.[9]

## APPENDIX

The proofs of lemmas 1 and 2 of section 2 are outlined here. Throughout the appendix, if S is a subset of a topological space (either X or B), let $S^\circ$ be the interior of S.

Lemma 1: If $y \varepsilon \overline{P(x)}$, $v \varepsilon B$, and $M(y,v) \varepsilon W$, then $v \varepsilon T(y)^\circ$.

Proof: If $v \notin T(y)^\circ$, then there exists $v' \varepsilon B$ near enough to v so that $M(y,v') \varepsilon W$, but such that $v' \notin T(y)$. Since $M(y,v')$ is a finite set, $M(y,v') \varepsilon W$ and the definition of a gradient can be used to show the existence of $\overline{\lambda} > 0$ such that $(y + \lambda v')Py$ for all $0 < \lambda \leq \overline{\lambda}$. Hence, $y + \lambda v' \varepsilon P(x)$ for all $0 < \lambda \leq \overline{\lambda}$, since $P^1(y) \subset P(x)$. But $v' \notin T(y)$ implies the existence of $0 < \lambda' < \overline{\lambda}$ such that $y + \lambda' v' \notin P(x)$, a contradiction. Therefore $v \varepsilon T(y)^\circ$.

Lemma 2: Suppose (A) is true, i.e., that for all $i \varepsilon N$, $y \varepsilon X$,

$$\nabla u_i(y) = 0 \iff u_i(y) = \max_{z \varepsilon X} u_i(z) \text{ for all } i \varepsilon N,\ y \varepsilon X.$$

If $y \varepsilon \partial\overline{P(x)}$, $v \varepsilon T(y)^\circ$, and $U \subset B$ is any neighborhood of v, then there exists $v' \varepsilon U$ such that $M(y,v') \varepsilon W$.

Proof: For any $v' \varepsilon B$, partition N into four sets defined by

$$N^+(v') = \{i \varepsilon N | v' \cdot \nabla u_i(y) > 0\} = M(y,v')$$

$$N^-(v') = \{i \varepsilon N | v' \cdot \nabla u_i(y) < 0\}$$

$$N_1^0(v') = \{i \varepsilon N | v' \cdot \nabla u_i(y) = 0,\ \nabla u_i(y) \neq 0\}$$

$$N_2^0(v') = \{i \varepsilon N | \nabla u_i(y) = 0\}$$

If the lemma is false, then $M(y,v') \notin W$ for all $v' \varepsilon U \cap T(y)^\circ \neq \emptyset$.

Let $a \in U \cap T(y)^\circ$ satisfy

$$|M(y,a)| = \max_{v' \in U \cap T(y)^\circ} |M(y,v')| < \frac{n+1}{2} .$$

Then there is a neighborhood $A \subset U \cap T(y)^\circ$ of a such that for all $v' \in A$, $N^+(a) = N^+(v')$ and $N^-(a) \subset N^-(v')$. Now, $\{v' \in B | v \cdot \nabla u_i(y) = 0\}$ is of dimension m-1 and hence nowhere dense in B whenever $\nabla u_i(y) \neq 0$. Since A is open and thus of dimension m, it cannot be covered by a finite number of these nowhere dense sets of dimension m-1. Therefore there exists $b \in A$ such that $N^-(b) = N^-(a) + N_1^0(a)$. We now obtain

$$|N^-(b) + N_2^0(b)| = |N^-(a) + N_1^0(a') + N_2^0(a)|$$

$$= n - |M(y,a)| \geq \frac{n+1}{2} .$$

Since $b \in T(y)$, there exists a sequence $\{y_k\} \subset P(x)$ such that

$$y_k \to y \text{ and } \frac{y_k - y}{\|y_k - y\|} \to b.$$

Because $N^-(b)$ is finite and

$$b \cdot \nabla u_i(y) = \lim_{k\to\infty} \frac{u_i(y_k) - u_i(y)}{\|y_k - y\|}$$

(see, e.g., Hestenes [1975]), there is a $K > 0$ such that $u_i(y_K) < u_i(y)$ for $i \in N^-(b)$. The continuity of each $u_i$ and the openness of P(x) now imply the existence of a neighborhood $V \subset P(x)$ of $y_K$ such that $u_i(z) < u_i(y)$ for all $z \in V$, $i \in N^-(b)$. Furthermore, assumption (A) and the assumption that indifference surfaces have no interiors imply the existence of $\bar{z} \in V$ such that $u_i(\bar{z}) < u_i(y)$ for all $i \in N_2^0(b)$. Therefore,

$$|\{i \in N \mid u_i(\overline{z}) < u_i(y)\}| \geq |N^-(b) + N_2^0(b)| \geq \frac{n+1}{2} .$$

This implies $yP\overline{z}$, which, since $\overline{z} \in P(x)$ but $y \notin P(x)$, is a contradiction of the definition of $P(x)$. The lemma is proved.

FOOTNOTES

1. $\partial\overline{P(x)}$ is equal to what McKelvey [1977] calls the frontier of $P(x.)$

2. Actually, McKelvey [1977] only shows $P^1(y) \subset \overline{P(x)}$, and Cohen [1977] assumes convex preferences to show $P^1(y) = P(x)$. However, lemma 4 in Cohen [1977] uses only the continuity of each $u_i$ to show that any two boundary points of $P(x)$ are not comparable by $P$, so that McKelvey's result can be strengthened to $P^1(y) \subset P(x)$. Expression (2) is proved rigorously in McKelvey [1977].

3. A real-valued function f is strictly quasiconcave if $f(x) \geq f(y)$ implies $f(z) > f(y)$ for all $z = \lambda x + (1 - \lambda)y$, $0 < \lambda < 1$, $x \neq y$. The preferences represented by a strictly quasiconcave utility function are strictly convex in the sense used in Cohen [1977].

4. McKelvey shows that when there is no core point, $X = R^m$, and each person's preferences decrease exactly with Euclidean distance from a bliss point, then any point in the space can be reached from any other point. This theorem is not true if reachability is replaced with continuous reachability, as examples in Schofield [1977a] indicate.

5. Notice that $(v - \overline{v}) \cdot \nabla u_i(y) > 0$ means that direction v is closer

than $\bar{v}$ to the utility gradient $\nabla u_i(y)$. Hence, loosely speaking, utility increases faster if y shifts in direction v rather than $\bar{v}$. [See also appendix A of chapter II.]

6. A differentiable function f is strictly pseudoconcave provided that whenever $z \neq y$, $(z - y) \cdot \nabla f(y) \leq 0 \Rightarrow f(z) < f(y)$. Strict pseudoconcavity clearly implies strict quasiconcavity and assumption (a).

7. A somewhat more general formulation is presented and applied to undominated directions in chapter II.

8. However, it is not true that any point in L can be continuously reached from any point not in L. (For example, any core point is not in L and no point can be continuously reached from a core point.) Hence L is not the exact analog of the discrete <u>top</u> cycle set V.

9. Schofield [1978] formally proved this conjecture after its initial appearance in Cohen and Matthews [1977], as is discussed in section 7 of chapter II.

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# CONCLUSION

The observant reader will have noticed a changing tenor as he progressed through the text. In chapter I the idea of directional strategies was offered as a positive contribution to our understanding of political competition. Its assumptions captured commonly recognized "folk" phenomena like the incrementality of social change at national levels and the directional nature of candidates' campaign platforms. There was optimism that a viable explanatory model had been achieved when frequently observed facts, such as a candidate's divergence from a rigid, extremist opponent, were predicted. The model satisfied necessary consistency properties in that a candidate whose opponent uses directional strategies was shown to have no incentive to adopt another type of strategy himself. Finally, the model was shown to be compatible with standard Euclidean models by the demonstration that directional equilibria "point" at equilibrium points.

Some of the normative results of chapter II heightened the optimism. Specifically, the dynamics generated by the adoption of undominated directions implied that the status quo x shifts, whenever possible, so as to (instantaneously) approach each winning coalition's preferred-to-x set. Furthermore, with Euclidean preferences we obtained the desirable stability property of convergence to either the point core or to the set of points that cannot contain a path satisfying the approach property.

However, it was also shown in chapter II that local cycling occurs at a point if its directional core is empty. This connection provides a solid behavioral explanation of the local cycling phenomena. But it also leads to pessimism regarding the existence of undominated directions in majority games, since local cycling is generic for such games in spaces of high dimension. The pessimism was increased when results obtained in the appendix were applied to show that a stringent, pairwise symmetry condition must be satisfied by utility gradients at points with nonempty directional cores. In fact, this symmetry condition has recently been used by Schofield to show that the emptiness of directional cores (and hence the existence of local cycling) is generic for majority rule games in spaces of dimension greater than three.

The turnaround was completed in chapter III, the main result of which was based on the fact that directional cores are empty in majority games. It was shown there that undominated directions exist at boundary points of a top cycle set and that they "point back into" the top cycle set. Although this implies dynamic stability properties, the major conclusion was that because undominated directions do not usually exist, neither do boundaries of top cycle sets. Hence top cycle sets must usually be the entire space of alternatives.

Because of the intuitive attractiveness of directional strategies and outcomes, it is important to have established the properties of directional models. Some of these properties do not rely upon the existence of equilibrium. Furthermore, the strength of equilibrium as a solution concept makes necessary the investigation of its consequences in any model. Even in majority rule situations, directional equilibria are theoretically valuable for understanding why local cycling is so pervasive and top cycling sets so large. So how can the theory of directional models be developed further? First, the frequent nonexistence of directional equilibria in majority games, or indeed, in all anonymous simple games with very decisive dominance relations, calls for an approach to the study of political processes that considers institutional factors and the formation of expectations and tastes in addition to considering equilibrium phenomena. Nonexistence of equilibria in pristine environments suggests that the additional factors are important for understanding underlying regularities of political processes. Secondly, there is promise that future work can extend the concept of directional outcomes to more general cooperative games that allow equilibria by restricting the power of coalitions and/or the preferences of individuals. Thus, directional cores may provide a behaviorally-based adjustment mechanism for a private goods economy, for example.

# APPENDIX

## PAIRWISE SYMMETRY CONDITIONS FOR VOTING EQUILIBRIA

It is common knowledge that characterizations of majority rule equilibria in multidimensional spaces take the form of pairwise symmetry conditions on utility gradients.[1] Plott [1967], the initial investigator of these conditions, shows that if exactly one utility gradient at an interior point is zero and the number of people is odd, then the point is an equilibrium if and only if the set of nonzero gradients can be partitioned into pairs of exactly opposing vectors. This degree of symmetry seems unlikely to occur. Hence it must be concluded that this type of equilibrium does not usually exist.

However, the condition that all nonzero gradients must be paired is necessary only for equilibria at which only one gradient is zero. One object of this paper is to derive necessary conditions that do not *a priori* restrict the number of zero gradients. These more general conditions are determined also for the more general case of $\lambda$-majority rule, in which a coalition is winning only if it constitutes more than a fraction $\lambda$ of the voters.[2] The amount of pairwise symmetry required for equilibrium is still restrictive, however, unless many gradients are zero or $\lambda$ is near one.

Conditions necessary for equilibrium may be less restrictive for equilibria contained in the boundary of a feasible set. Since often the feasible set is a proper subset of the space, such equilibria are certainly worthy of investigation. Plott [1967] makes an initial step in this direction by investigating situations in which the feasible set is a half-space and the equilibrium is contained in the defining hyperplane. His conditions are generalized here by allowing the equilibrium to be contained in the boundary of any convex feasible set, as well as by allowing more than one gradient to "point out" of the feasible set and by considering $\lambda$-majority rule. We find that the type of pairwise symmetry required at boundary equilibria is of a lesser degree than that required at interior equilibria. But the symmetry still appears restrictive unless (1) the boundary is highly pointed at the equilibrium, (2) many gradients are zero or "point out" of the feasible set, or (3) $\lambda$ is near one.

A fundamental characteristic of majority rule is that if two people with diametrically opposed preferences are removed from the set of voters, then any equilibrium remains an equilibrium. The votes of the two individuals merely "cancel each other out." This basic fact is what causes pairwise symmetry conditions to be necessary for equilibrium, as the subsequent proofs are designed to show. All the symmetry conditions are derived as corollaries to theorems stating that various sets of individuals that "disagree" in some sense can be deleted without upsetting equilibrium. This intuitive approach results in relatively concise proofs.

Sufficient conditions involving pairwise symmetries on gradients are important because properties of pairs are relatively easy to verify. The ones derived in section 3 generalize those of

Plott [1967], McKelvey and Wendell [1976], and Slutsky [1978] by allowing the point to be on the boundary of a convex feasible set, by allowing more than one gradient at the point to be zero or to "point out" of the feasible set, and by allowing for $\lambda$-majority rule.

## 1. PRELIMINARIES

The set of feasible alternatives is a convex subset V of a Euclidean space W. Denote by x a particular point of V, not necessarily in the interior. Let the set of voters be denoted by $N = \{1, 2, \ldots, n\}$. Each voter has a differentiable utility function defined on W. The gradient of the utility function of voter i evaluated at x is denoted by $u_i \in W$.[3] We are to investigate pairwise symmetries in the set $\{u_1 \ldots, u_n\}$ of gradients associated with x being a voting equilibrium.

The cone of feasible directions in which x can shift is

$$F = \{v \in W \mid \exists\, \alpha > 0 \ni x + \alpha v \in V\}.$$

Observe that F is a convex cone that includes the origin. If $x \in \text{interior}(V)$, then $F = W$, whereas $x \in \text{boundary}(V)$ implies that F is contained in a halfspace.

Much of the subsequent discussion concerns the dual of F,

$$F^* = \{y \in W \mid v \cdot y \leq 0 \quad \forall\, v \in F\} \equiv D.$$

Notice that D is a closed convex cone containing the origin, and that $D = \{0\}$ if and only if $F = W$. If $u_i \in D$ then $v \cdot u_i \leq 0$ for all $v \in F$, so that voter i is "happy" with x in the sense of not marginally benefiting by any feasible shift of x.

Define also a cone

$$E = \{y \in W \mid y \notin D, -y \in D\}.$$

E is a convex cone without the origin that may be empty. In particular, $E = \emptyset$ whenever $D = \{0\}$ or D is a subspace of positive dimension. If $u_i \in E$, then i is "unhappy" with x in the sense that $v \cdot u_i \geq 0$ for any $v \in F$, and there exists $\bar{v} \in F$ such that $\bar{v} \cdot u_i > 0$.

Examples of possible cones F, D, and E are illustrated in figure 1. In the figure and hereafter a cone generated by vectors $y_1, \ldots, y_\ell$ is defined by

$$C(y_1, \ldots, y_\ell) = \{y \in W \mid y = \alpha_1 y_1 + \ldots + \alpha_\ell y_\ell,$$
$$\alpha_i \geq 0, \Sigma\alpha_i > 0\}.$$

Also, if $M = \{i_1, \ldots, i_\ell\} \subset N$, the notation $C(M) = C(u_{i_1}, \ldots, u_{i_\ell})$ will be used for convenience.

Define for any cone C the following derived cones:

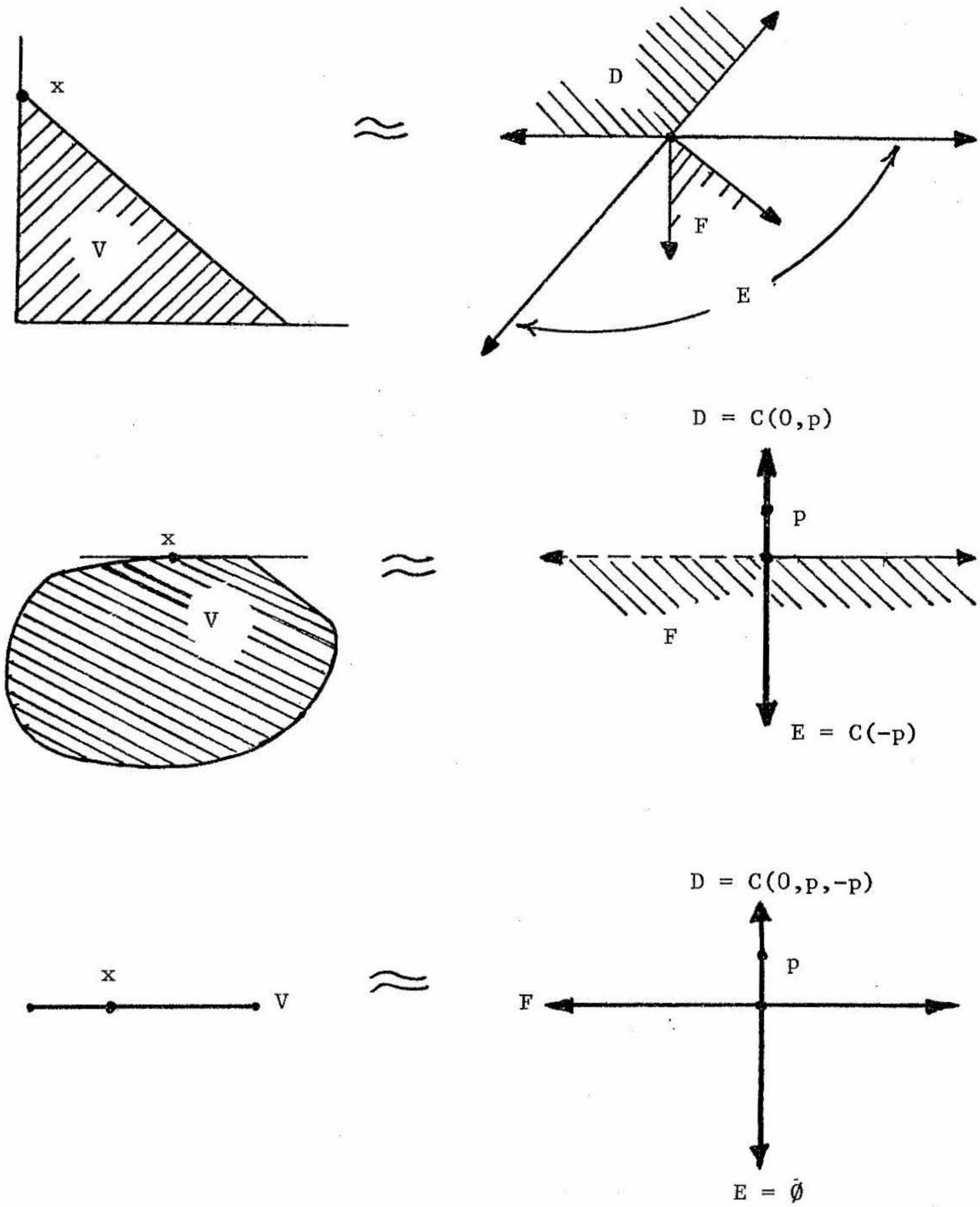
$$C^+ = \{y \in W \mid y \cdot c > 0 \quad \forall \quad c \in C\}$$

$$C^- = \{y \in W \mid y \cdot c < 0 \quad \forall \quad c \in C\}$$

$$C^0 = \{y \in W \mid y \cdot c = 0 \quad \forall \quad c \in C\}$$

Without fear of ambiguity, for any $v \in W$ let $v^+$, $v^-$, and $v^0$ denote $C(v)^+$, $C(v)^-$, and $C(v)^0$. Then $v^+$ and $v^-$ are halfspaces and $v^0$ is a subspace. Observe that $u_i \in v^+$ implies that $v \cdot u_i > 0$, so that voter i benefits if x shifts in direction v. For any subsets $M \subset N$ and $C \subset W$, let

FIGURE 1



$$S_M(C) = |\{i \varepsilon M \mid u_i \varepsilon C\}|,$$

and let $S(C) = S_N(C)$. Hence $S_M(v^+)$ is the number of voters in M who benefit by a shift in direction v. For convenience we also adopt the convention that if an upper case letter denotes a subset of voters, then the corresponding lower case letter denotes their number, e.g., $n = |N|$ and $M \subset N$ implies $m = |M|$.

With these definitions in hand, an equilibrium concept can be defined. Let $\lambda$ be a fixed fraction $0 \leq \lambda < 1$. Then we want x to be an equilibrium provided no coalition of size greater than $\lambda n$ can marginally benefit by a feasible shift of x. So define x to be quasi-undominated (q.u.d.) provided

$$v \varepsilon F \Rightarrow S(v^+) \leq \lambda n,$$

and define x to be strictly quasi-undominated (s.q.u.d.) provided

$$v \varepsilon F \Rightarrow S(v^+) < \lambda n.$$

Notice that x is q.u.d. if x is s.q.u.d. Conversely, x is s.q.u.d. if x is q.u.d. and $\lambda n$ is nonintegral, which is the case when n is odd and $\lambda = 1/2$, the majority rule case studied by Plott [1967].

Two alternative concepts of equilibrum for x are local undominance, which requires the existence of a neighborhood U of x such that no point in $U \cap V$ is unanimously preferred to x by a coalition of size greater than $\lambda n$, and global undominance, which requires x to be locally undominated in every neighborhood $U \subset W$. When there is a finite number of voters, each with a differentiable utility function, global undominance implies local undominance.

implies quasi-undominance. The reverse implications require utility functions to first be locally pseudoconcave (see appendix B of chapter II) and then pseudoconcave (Kats and Nitzan [1976]). The reader is referred to the cited references for these results, and to Sloss [1973], McKelvey and Wendell [1976], and Slutsky [1978] for further discussions of the relationship between quasi-undominance and other equilibrium concepts. Hence attention here can be focused solely upon quasi-undominance.

It will be convenient for the determination of quasi-undominance to test only directions contained in the relative interior of F. Lemma 2 below justifies this procedure. It also allows us to assume henceforth that F is a closed convex cone, so that $D^* = F^{**} = F$.[4]

<u>Lemma 1</u>: Let $M \subset N$ and $\overline{v} \in W$. Then there exists a neighborhood U of $\overline{v}$ such that $S_M(v^+) \geq S_M(\overline{v}^+)$ for all $v \in U$.

<u>Proof</u>: Follows from the continuity of an inner product and the finiteness of M.

<u>Lemma 2</u>: Let $M \subset N$ and $\beta > 0$. If $S_M(v^+) \leq \beta$ for all v contained in the relative interior of F, then $S_M(v^+) \leq \beta$ for all $v \in$ closure(F).

<u>Proof</u>: Since F is convex, every neighborhood of any $v \in$ closure(F) contains points in the relative interior of F. Hence the result follows from lemma 1.

Henceforth, without loss of generality, we assume F is closed.

The basic feature of majority rule we wish to exploit is that if the number of people who prefer alternative $a_1$ to $a_2$ is not a majority, and $Q \subset N$ is a set that can be partitioned into pairs with strictly opposite preferences on $\{a_1, a_2\}$, then when Q is deleted, the number of voters preferring $a_1$ to $a_2$ is still not a majority. More generally, if the number of people preferring $a_1$ is less than $\lambda n$, then when Q is deleted, the number of people who prefer $a_1$ is less than $\lambda n - 1/2q$. Now, our general method will be to show that the deletion of coalitions analogous to Q will leave x quasi-undominated, in some sense, in the remaining set of voters. But if $K = N - Q$, the above reasoning indicates that only $S_K(v^+) \leq \lambda n - 1/2(n-k)$ can be guaranteed by $S(v^+) \leq \lambda n$. Hence we shall say that x is q.u.d. in $K \subset N$ provided

$$v \in F \implies S_K(v^+) \leq \lambda n - 1/2(n-k) = \lambda_k k,$$

where $\lambda_k$ is defined by

$$\lambda_k = \lambda + (\lambda - 1/2)(n/k - 1).$$

Similarly, x is s.q.u.d. in K provided

$$v \in F \implies S_K(v^+) < \lambda_k k.$$

We now prove a simple proposition to illustrate the meaning of quasi-undominance in subsets of N. Say that a pair $\{i,j\} \in N$ strongly disagree provided $u_i \notin D$, $u_j \notin D$, and

$$v \cdot u_i > 0 \iff v \cdot u_j < 0$$

for all $v \in W$. Observe that i and j strongly disagree if and only if there is a ray $r \subset W$ not intersecting D such that $u_i \in r$ and $u_j \in -r$. Thus, if D contains no line, i and j strongly disagree exactly when $u_i$ and $u_j$ are a pair of gradients exactly opposing each other in the sense of Plott [1967]. We show that removing or adding pairs of strongly disagreeing voters preserves quasi-undominance. The following lemma is useful.

---

Lemma 3: Let $T \subset W$ be a subspace, and let $\bar{v} \in T$, $\bar{v} \neq 0$. Suppose $Q \subset N$ and $u_i \notin T^0$ for each $i \in Q$. If U is a neighborhood of $\bar{v}$, then there exists $v \in U \cap T$ such that $v \cdot u_i \neq 0$ for all $i \in Q$.

Proof: $U' = U \cap T$ is an open set of T. If $u_i \notin T^0$, then $T \not\subset u_i^0$, so that $\dim(T \cap u_i^0) < \dim(T)$. Hence for each $i \in Q$, $T \cap u_i^0$ is a nowhere dense subset of T. Since a countable union of nowhere dense sets cannot contain an open set (Baire's theorem),

$$U \cap T = U' \not\subset \bigcup_{i \in Q} (T \cap u_i^0).$$

Therefore there exists $v \in U \cap T$ such that $v \cdot u_i \neq 0$ for each $i \in Q$.

Proposition 1: Let Q be a subset of N that can be partitioned into strongly disagreeing pairs, and let $K = N - Q$. Then x is (s.)q.u.d. in K iff x is (s.)q.u.d.

Proof: Suppose x is (s.)q.u.d. Let $\bar{v}$ be contained in the relative interior of F. Let T be the smallest subspace containing F. Hence there is a neighborhood $U'$ of $\bar{v}$ such that $U' \cap T \subset F$. By lemma 1 there exists a neighborhood $U \subset U'$ such that $S_K(v^+) \geq S_K(\bar{v}^+)$ for any $v \in U$. Since $u_i \notin D$ for each $i \in Q$,

$u_i \notin T^0$ for each $i \in Q$. Hence lemma 3 implies the existence of $v \in U \cap T \subset F$ such that $v \cdot u_i \neq 0$ for each $i \in Q$. But Q can be partitioned into pairs of strongly disagreeing individuals, so that $S_Q(v^+) = q/2$. Therefore

$$S_K(\bar{v}^+) \leq S_K(v^+) = S(v^+) - q/2$$
$$\leq \lambda n - q/2 = \lambda_k k,$$

with the second inequality strict if x is s.q.u.d. By lemma 2, this proves x is (s.)q.u.d. in K. Now assume x is (s.)q.u.d. in K. Let $v \in F$. Then $S_Q(v^+) \leq q/2 \Rightarrow S(v^+) \leq S_K(v^+) + q/2 \leq \lambda_k k + q/2 = \lambda n$ (second inequality strict if x is s.q.u.d. in K). So x is (s.)q.u.d.

Proposition 1 actually does not lead to strong pairwise symmetry conditions, even for the case of an interior x. In the next section, symmetry conditions for an interior x are obtained easily by a different route. But a result analogous to proposition 2 regarding the deletion of pairs that disagree in a weaker sense is very useful for the case of a boundary x. Hence define a pair $\{i,j\} \subset N$ to <u>weakly disagree</u> provided $u_i \notin D$, $u_j \notin D$, and for any $v \in F$,

$$v \cdot u_i > 0 \Rightarrow v \cdot u_j < 0$$

$$\text{and} \quad v \cdot u_j > 0 \Rightarrow v \cdot u_i < 0 .$$

Let $\mathcal{D}$ be the symmetric binary relation on N denoting weak disagreement, so that $i\mathcal{D}j$ means i and j weakly disagree. If x is an interior point of V, then $D = \{0\}$ and weak disagreement implies strong disagreement. Otherwise it is possible that $i\mathcal{D}j$ even though $v \cdot u_i < 0$ and $v \cdot u_j < 0$ for some $v \in F$. The next proposition characterizes weakly disagreeing pairs.

Proposition 2: If $u_i \notin D$ and $u_j \notin D$, then $i\mathcal{D}j$ iff $C(u_i,u_j) \cap D \neq \emptyset$.

Proof: $D$ and $C(u_i,u_j) \cup \{0\}$ are closed convex cones. Hence if $C(u_i,u_j) \cap D = \emptyset$, by a separation theorem there exists $v \neq 0$ such that $v \cdot y > 0$ for all $y \in C(u_i,u_j)$ and $v \in D^* = F$. Hence, since $v \cdot u_i > 0$ and $v \cdot u_j > 0$, $i\mathcal{D}j$ is false. Conversely, suppose there exists $y = \alpha_i u_i + \alpha_j u_j \in C(u_i,u_j) \cap D$. Then $\alpha_i > 0$ and $\alpha_j > 0$. Hence, because $v \cdot y \leq 0$ for all $v \in F$, $i\mathcal{D}j$.

Finally, basic necessary conditions are derived via the deletion of individuals who are malcontent in a different way. For any subspace $T \subset W$, say that voter $i \subset N$ is content with T provided $u_i \in T^0$. Let $\mathcal{C}(T) \subset N$ be the subset of N content with T. To interpret $\mathcal{C}(T)$, suppose a subset of public goods is associated with T. Then any $i \in \mathcal{C}(T)$ is content with the allocation of those particular goods at x in the sense of being indifferent to any proposal to change their amounts. Letting $M(T) = N - \mathcal{C}(T)$, each $i \in M(T)$ is discontented with T at x in the sense of preferring a change in allocation of the goods associated with T.

Define a free subspace to be a subspace $T \subset W$ for which $T \subset F$. It is easy to show

Lemma 4: A subspace T is free iff $D \subset T^0$.

A major result of the next section is that quasi-undominance is preserved when M(T) is removed and T is free. Intuitively, if the amounts of the goods associated with T can be increased or decreased freely at x, then the votes of those discontented with the amounts of these goods must "cancel out" for x to be in equilibrium.

## 2. NECESSARY CONDITIONS

Theorem 1: x is (s.)q.u.d. iff x is (s.)q.u.d. in $C(T)$ for every free subspace T.

Remark 1: This theorem actually only provides a necessary condition for x to be (s.)q.u.d., since $T = \{0\}$ is always a free subspace and $C(\{0\}) = N$. Subsequently an example will be presented indicating that a true sufficient condition cannot be obtained by requiring T to be nondegenerate.

Remark 2: The freeness of T is necessary for theorem 1. Consider a case with $W = R^2$, $n = 3$, $\lambda = 1/2$, and with $D = C(0,p)$ with $p = (0,1)$. Let $u_1 = u_2 = p$, and $u_3 = (1,0)$. If T is taken as the line $C(p, -p)$, which is not free, then $C(T) = \{3\}$. But x is clearly not s.q.u.d. in $\{3\}$, even though x is s.q.u.d. in $\{1,2,3\}$.

Lemma 5: Suppose x is q.u.d. If $v \in F$, $a \in v^0$, and $v + a \in F$, then

$$S(v^+) + S(v^0 \cap a^+) \leq \lambda n,$$

with the inequality strict if x is s.q.u.d.

Proof: By the continuity of the inner product, there exists a neighborhood U of v such that $y \cdot u_i > 0$ for all $y \in U$, $u_i \in v^+$.

As F is convex, there exists $0 < \delta \leq 1$ such that $b = v + \delta a \in F \cap U$. Since $b \cdot u_i > 0$ for any $u_i \in v^0 \cap a^+$, and since x is q.u.d., we have

$$S(v^+) + S(v^0 \cap a^+) \leq S(b^+) \leq \lambda n,$$

with the last inequality strict if x is s.q.u.d.

Proof of Theorem 1: (Figure 2 may be helpful.) Suppose x is q.u.d. and $T \neq \{0\}$ is a free subspace. Let $M = C(T)$. Since $i \in N - M \iff u_i \notin T^0$, lemma 3 implies the existence of $v \in T$ such that $v \cdot u_i \neq 0 \iff i \in N - M$. Hence $n = S(v^+) + S(v^-) + m$. We can assume $S(v^-) \leq S(v^+)$, switching v with $-v$ if necessary, so that

$$S(v^+) \geq 1/2(n - m).$$

Let $\overline{v} \in F$. $\overline{v}$ can be expressed as $\overline{v} = a + b$, where $a \in T^0$, $b \in T$. For any $p \in D$, $p \cdot a = p \cdot (\overline{v} - b) = p \cdot \overline{v} \leq 0$, since the freeness of T implies $p \in T^0$. Hence $a \in D^* = F$. T being free also implies $v \in F$, so that $v + a \in F$ by the convexity of F. Applying lemma 5, we now have

$$S(v^+) + S_M(a^+) \leq \lambda n$$

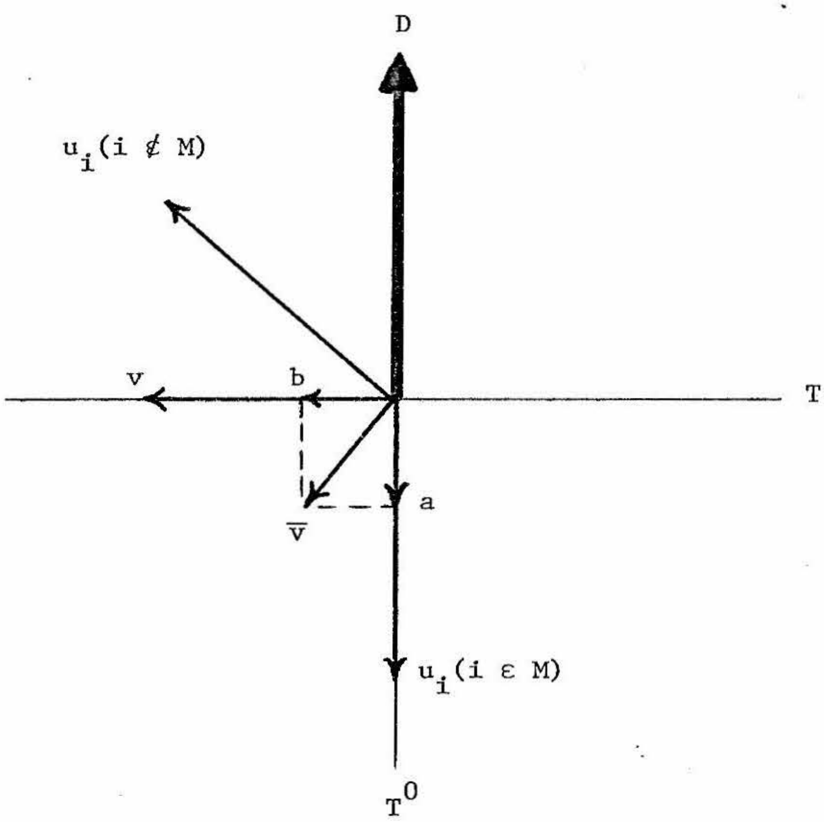
because our choice of v implies $S(v^0 \cap a^+) = S_M(a^+)$. (This inequality is strict if x is s.q.u.d.) Finally, since $i \in M \Rightarrow u_i \in T^0 \Rightarrow \overline{v} \cdot u_i = a \cdot u_i$, we obtain

$$S_M(\overline{v}^+) = S_M(a^+).$$

Putting the pieces together leads to

$$S_M(\overline{v}^+) \leq \lambda n - S(v^+) \leq \lambda n - 1/2(n - m) = \lambda_m m,$$

FIGURE 2



with the first inequality strict if x is s.q.u.d. The theorem is proved.

<u>Corollary 1</u>: Let T be a free subspace and $M = \mathcal{C}(T)$. If x is q.u.d. and $v \in F$, then

$$S_M(v^+) - S_M(v^-) \leq S_M(v^0) + (2\lambda - 1)n,$$

with the inequality strict if x is s.q.u.d.

<u>Proof</u>: Theorem 1 implies $S_M(v^+) \leq \lambda n - 1/2(n - m)$, so the inequality follows from substituting $S_M(v^+) + S_M(v^-) + S_M(v^-)$ for m.

<u>Corollary 2 (Generalized Plott Theorem 1)</u>:

Suppose x is an interior point of V and r is a ray without the origin. If x is q.u.d. then

(i) $|S(r) - S(-r)| \leq S(0) + (2\lambda - 1)n$

(ii) $S(0) \geq (1 - 2\lambda)n$,

with both inequalities strict if x is s.q.u.d. If Q is a maximal subset of N that can be partitioned into disagreeing pairs, then $n = q + S(0)$ whenever either one of the following holds:

(iii) x is q.u.d. and $S(0) < 1 - (2\lambda - 1)n$

(iv) x is s.q.u.d. and $S(0) \leq 1 - (2\lambda - 1)n$.

<u>Proof</u>: $T = r^0$ is a free subspace, since $F = W$. Letting $M = \mathcal{C}(T)$, $i \in M \iff u_i \in -r \cup \{0\} \cup r$. Hence for any $v \in r$, $S_M(v^+) = S(r)$, $S_M(v^-) = S(-r)$, and $S_M(v^0) = S(0)$. Applying corollary 1 first to v and then to -v now results in (i). Expression (i) implies (ii)

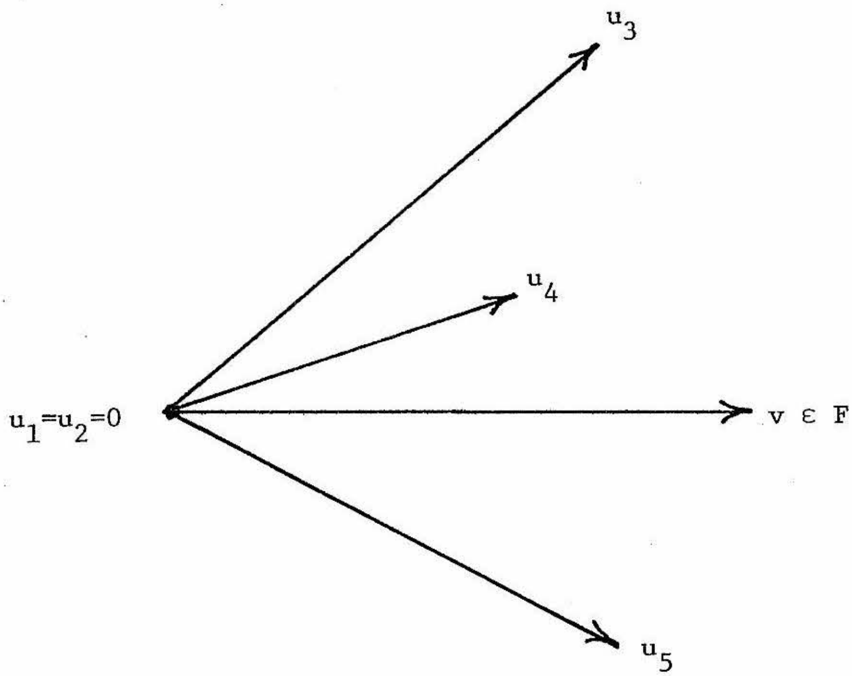
when r is chosen so that no gradients are contained in r or -r. If either (iii) or (iv) hold, then (i) implies $|S(r) - S(-r)| = 0$ for all rays r. This implies $n = q + S(0)$, since $n - q - S(0) = \sum_{i \varepsilon I} |S(r_i) - S(-r_i)|$, where I indexes the lines $\ell_i = -r_i \cup \{0\} \cup r_i$ that contain nonzero gradients.

Remark 3: Corollary 2 states the complete pairwise symmetry required of the set of utility gradients at interior equilibria. The simple example of figure 3, which has $W = R^2$, $n = 5$ and $\lambda = 1/2$, indicates that (i) and (ii) are only necessary conditions, since $S(v^+) = 3$. The example also serves to show that x being s.q.u.d. in $C(T)$ for every free, nondegenerate T does not imply that x is q.u.d., as x is s.q.u.d. in all the subsets content with nondegenerate subspaces: {1,2}, {1,2,3}, {1,2,4}, {1,2,5}.

Remark 4: A converse of corollary 2 is true. Specifically, if $Q \subset N$ can be partitioned into weakly disagreeing pairs and $n = q + S(0)$, then x is q.u.d. if $S(0) \geq (1 - 2\lambda)n$ and x is s.q.u.d. if $S(0) > (1 - 2\lambda)n$. This follows easily from the observation that $S(v^+) = S_Q(v^+) \leq q/2$ for any feasible direction $v \varepsilon F$. This converse is true of any D and is generalized in section 3.

Theorem 1 is only the first step in proving symmetry conditions hold at boundary equilibria. However, it does imply necessary lower bounds on $S(D) - S(E)$ in important cases. This is

FIGURE 3



not unexpected, since the vote of an individual in D "cancels" the vote of an individual in E for any feasible direction, just as the votes of individuals whose gradients are contained in opposing rays cancel. Hence one expects an analog of (i) in corollary 2 to bound $S(D) - S(E)$. But an example will be presented subsequently showing this is not always true. First, the following corollary provides a sufficient condition for $S(D) - S(E)$ to be bounded below.

<u>Corollary 3</u>: Suppose T is a free subspace such that $C(T) = \{i \in N \mid u_i \in D \cup E\}$. If x is q.u.d., then

$$S(D) - S(E) \geq (1 - 2\lambda)n,$$

with the inequality strict if x is s.q.u.d.

<u>Proof</u>: Let $M = C(T)$. Let $\overline{v} \in$ relative interior(F), which exists because F is convex. For each $i \in M$ satisfying $u_i \in E$, there exists some $v \in F$ such that $v \cdot u_i > 0$. Hence lemma 3 can be applied to show the existence near $\overline{v}$ of $v \in F$ such that $v \cdot u_i > 0$ for all $u_i \in E$. Therefore $S(E) = S_M(v^+)$ and $S(D) = S_M(v^-) + S_M(v^0)$, implying $S(D) - S(E) \geq (1 - 2\lambda)n$ by corollary 1.

<u>Remark 5</u>: If $D \cup E$ is a subspace, then the hypothesis of corollary 3 is satisfied for $T = (D \cup E)^0$. One case is $D = \{0\}$, $E = \emptyset$, for which the result is merely (ii) of corollary 2. Another case is $D = C(0,p)$, $E = C(-p)$, which occurs when V is uniquely supported by a hyperplane at x. If $D \cup E$ is not a subspace, the hypothesis may not be satisfied, and the bound

on $S(D) - S(E)$ can be violated if $\dim(W) > 2$. An example with $\dim(W) = 3$, $n = 9$, and $\lambda = 1/2$ is shown in figure 4. There, none of $\{u_1,\ldots,u_6\}$ are in $E \cup D$, $\{u_7,u_8\} \subset E$, and $u_9 \in D$. x is s.q.u.d., since directions v in the corners of F get $S(v^+) = 4 < 9/2$ votes and directions in the middle of F get only 2 votes. But $S(D) - S(E) = -1 \not\geq 0$.

Pairwise symmetries at boundary equilibria will be implied by the following theorem.

Theorem 2: Let T be a two dimensional subspace and $M = \{i \in N \mid u_i \in T\}$. Let Q be a maximal subset of M that can be partitioned into weakly disagreeing pairs, and let $K = M - Q$. Then x is (s.)q.u.d. in K if x is (s.)q.u.d. in T.

Remark 6: This theorem differs from the analogous proposition 1 concerning strongly disagreeing pairs by referring to only a two-dimensional subspace and by requiring Q to be maximal. Neither additional hypothesis can be eliminated. Figure 5(a) depicts a situation with $n = 5$, $\lambda = 1/2$, $W = R^2$, $D = C(0, p)$, and x s.q.u.d. By proposition 2, $2\mathcal{D}5$, $3\mathcal{D}5$, and $2\mathcal{D}4$. If $Q = \{3,5\} \cup \{2,4\}$ is deleted, x is s.q.u.d. in $\{1\}$, but $Q = \{2,5\}$ cannot be deleted because x is not s.q.u.d. in $\{1,3,4\}$. This shows Q must be taken maximal. In figure 5(b), $n = 7$, $\lambda = 1/2$, $W = R^3$, and $D = C(0,p)$. All gradients except $u_4$ and $u_5$ are in the plane of the figure, with $u_5$ receding behind and $u_4$ coming up off the page. The gradients $u_3$, $u_4$ and $u_5$ are all slightly lower than the plane $p^0$ seen in

FIGURE 4

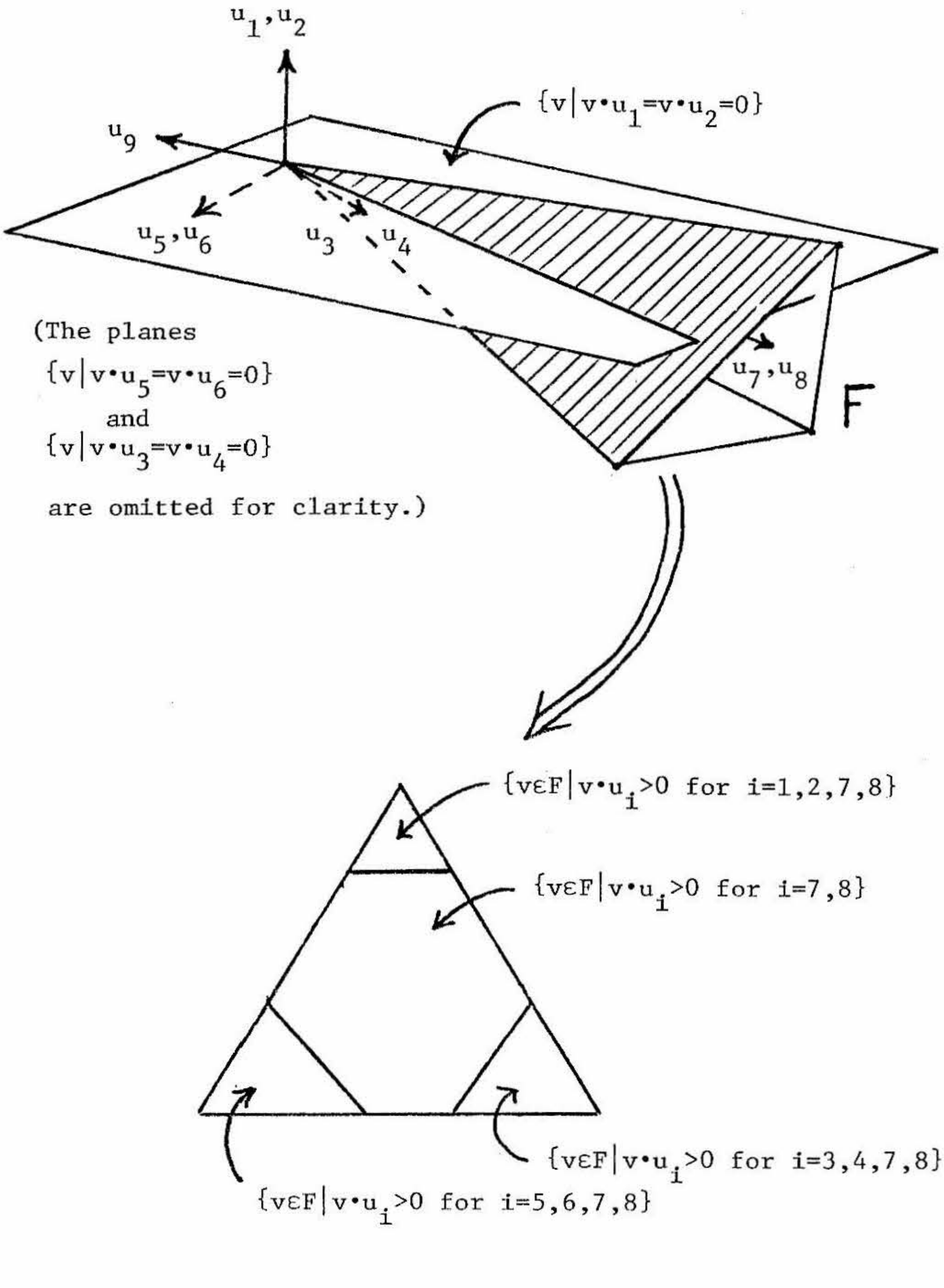
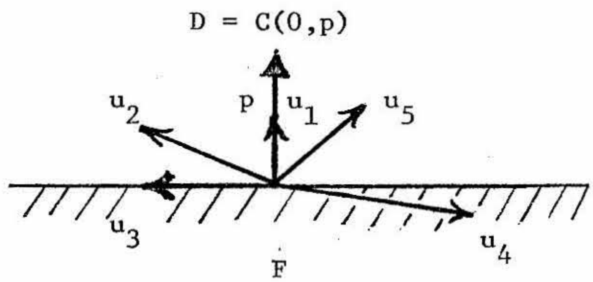
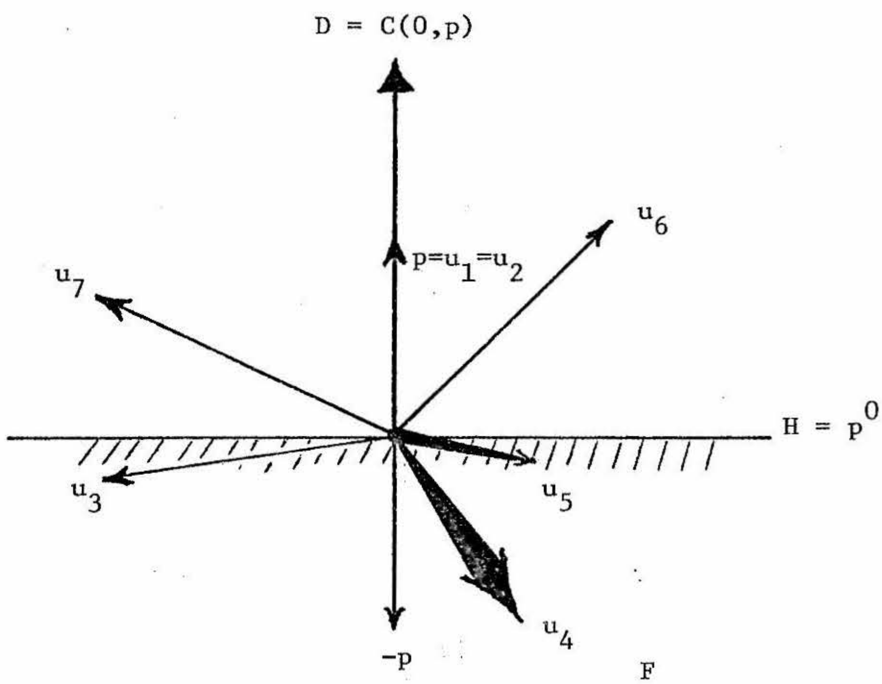

$u_1, u_2$
$\{v \mid v \cdot u_1 = v \cdot u_2 = 0\}$
$u_9$
$u_5, u_6$
$u_3$
$u_4$
$u_7, u_8$
F
(The planes
$\{v \mid v \cdot u_5 = v \cdot u_6 = 0\}$
and
$\{v \mid v \cdot u_3 = v \cdot u_4 = 0\}$
are omitted for clarity.)
$\{v \varepsilon F \mid v \cdot u_i > 0 \text{ for } i=1,2,7,8\}$
$\{v \varepsilon F \mid v \cdot u_i > 0 \text{ for } i=7,8\}$
$\{v \varepsilon F \mid v \cdot u_i > 0 \text{ for } i=3,4,7,8\}$
$\{v \varepsilon F \mid v \cdot u_i > 0 \text{ for } i=5,6,7,8\}$

FIGURE 5




(a)




(b)

cross-section as H. Hence $C = u_3^+ \cap u_4^+ \cap u_5^+$ is a narrow cone containing $-p$. The only disagreeing pair is $\{6,7\}$. If $\{6,7\}$ is deleted, then $S_{\{1,\ldots,5\}}(-p^+) = 3$ and x is not q.u.d. in $\{1,\ldots,5\}$. But, as $u_6^+ \cap C = u_7^+ \cap C = \emptyset$, x is s.q.u.d. in $\{1,\ldots 7\}$. Hence, figure 5(b) shows T must be assumed two dimensional in theorem 2.

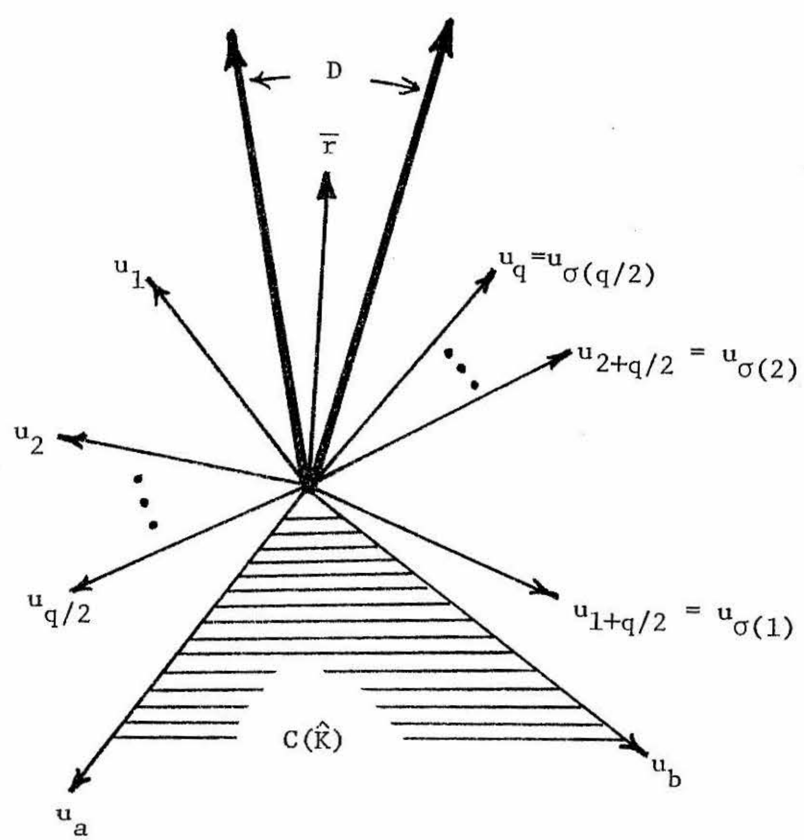
Lemma 6: Let T, M, Q and K be defined as in theorem 2. Suppose $T \cap D \neq \{0\}$ and $T \cap D$ contains no line. Then there exists $\hat{Q} \subset Q$ such that $\hat{q} = q/2$ and $C(\hat{K} \cup \hat{Q}) \cap D = \emptyset$, where $\hat{K} = \{i \in K \mid u_i \notin D\}$.

Proof: Let $\bar{r} \in T \cap D$ be a nondegenerate ray containing the origin. For any nonzero $v \in T$ let $\alpha(v)$ be the angle measured counterclockwise from $\bar{r}$ to v, with the convention $0 \leq \alpha(v) < 2\pi$. Number the members of Q as $1,2,\ldots,q$ so that $i < j$ implies $\alpha(u_i) \leq \alpha(u_j)$, as in figure 6. Because Q can be partitioned into weakly disagreeing pairs, a tedious but straightforward argument that we omit establishes that $i\mathcal{D}(i + q/2)$ for each $1 \leq i \leq q/2$. Let $\sigma(\cdot)$ be defined by $\sigma(i) = i + q/2$, so that $i\mathcal{D}\sigma(i)$ for $i \leq i \leq q/2$.

Let $A \subset Q \cup \hat{K}$. Because $T \cap D$ contains no line and $u_i \notin D$ for any $i \in Q \cup \hat{K}$, it can be shown that $\dim(T) = 2$ implies $C(A) \cap D = \emptyset \iff C(A) \cap \bar{r} = \emptyset$. Thus we need only establish the existence of $\hat{Q} \subset Q$ such that $\hat{q} = q/2$ and $C(\hat{K} \cup \hat{Q}) \cap \bar{r} = \emptyset$.

Now consider $C(\hat{K})$. Let $a \in \hat{K}$ satisfy $\alpha(u_a) \leq \alpha(u_i)$ for all $i \in \hat{K}$ and let $b \in \hat{K}$ satisfy $\alpha(u_b) \geq \alpha(u_i)$ for all $i \in \hat{K}$. Then $C(\hat{K}) \cap \bar{r} \neq \emptyset \iff \alpha(u_b) - \alpha(u_a) \geq \pi \iff C(u_a,u_b) \cap \bar{r} \neq \emptyset$. But then $C(\hat{K}) \cap \bar{r} \neq \emptyset$ implies $a\mathcal{D}b$, contrary to the maximality of Q. Hence

FIGURE 6


D
$\bar{r}$
$u_1$
$u_q = u_{\sigma(q/2)}$
$u_{2+q/2} = u_{\sigma(2)}$
$u_2$
$u_{q/2}$
$u_{1+q/2} = u_{\sigma(1)}$
$C(\hat{K})$
$u_a$
$u_b$

$C(\hat{K}) \cap \overline{r} = \emptyset$ and $C(\hat{K}) = C(u_a, u_b)$.

Suppose $C(\hat{K} \cup \{1\}) \cap \overline{r} = \emptyset$. Then, since $\alpha(u_b) - \alpha(u_1) < \pi$, $\alpha(u_{q/2}) - \alpha(u_1) < \pi$, and $\alpha(u_b) - \alpha(u_a) < \pi$, we have

$$\max\{\alpha(u_b), \alpha(u_{q/2})\} - \min\{\alpha(u_a), \alpha(u_1)\} < \pi.$$

Therefore $C(\hat{K} \cup \{1,\ldots,q/2\}) \cap \overline{r} = \emptyset$, and the lemma is proved. Similarly, the lemma is proved if $C(\hat{K} \cup \{q\}) \cap \overline{r} = \emptyset$. Now for $1 < i \leq q/2$, suppose

$$C(\hat{K} \cup \{i, \sigma(i-1)\}) \cap \overline{r} = \emptyset.$$

Then $u_j \in C(\{i,\sigma(i-1)\})$ for each $i \leq j \leq \sigma(i-1)$. Hence, letting

$$Q_i = \{i, i+1, \ldots, \sigma(i-1)\},$$

we have

$$C(\hat{K} \cup \hat{Q}_i) \cap \overline{r} = C(\hat{K} \cup \{i, \sigma(i-1)\}) \cap \overline{r} = \emptyset,$$

and the lemma is proved.

As the final step, assume the lemma false. Then by the previous paragraph, $C(\hat{K} \cup \{1\}) \cap \overline{r} \neq \emptyset$, implying $1\mathcal{D}b$. Since $C(\hat{K} \cup \{q\}) \cap \overline{r} \neq \emptyset$, $a\mathcal{D}q$. For $1 < i \leq q/2$, $C(\hat{K} \cup Q_i) \cap \overline{r} \neq \emptyset$ implies $i\mathcal{D}b$ or $a\mathcal{D}(\sigma(i-1))$ or $i\mathcal{D}(\sigma(i-1))$. Let $i_0$ be the maximal $1 \leq i \leq q/2$ such that $i\mathcal{D}b$. Let $j_0$ be the minimal $i_0 < j \leq q/2 + 1$ such that $a\mathcal{D}(\sigma(j-1))$. Then substitution of

$$\{i_0,b\} \cup \{i_0+1, \sigma(i_0)\} \cup \ldots \cup \{j_0-1, \sigma(j_0-2)\} \cup \{a, \sigma(j_0-1)\}$$

for $\{i_0, \sigma(i_0)\} \cup \{i_0+1, \sigma(i_0+1)\} \cup \ldots \cup \{j_0-1, \sigma(j_0-1)\}$ in the partition $Q = \{1, \sigma(1)\} \cup \ldots \cup \{q/2, \sigma(q/2)\}$ yields a partition of $Q \cup \{a,b\}$ into weakly disagreeing pairs. This contradiction of Q maximal finishes the proof.

Proof of Theorem 2: Case 1: $T \cap D = \{0\}$. In this case each weakly disagreeing pair in Q is strongly disagreeing and the theorem follows by proposition 1. Case 2: $T \cap D = T$. Then $Q = \emptyset$ and the theorem is trivial. Case 3: $T \cap D \neq T$ contains a line $\ell$. Because $\dim(T) = 2$, there exists nonzero $v \in T$ such that $\ell = v^0$. Since D is convex, $T \cap D = v^0$ or $T \cap D = v^0 \cup v^+$ (switching v and $-v$ if necessary). If $T \cap D = v^0 \cup v^+$ and $u_i, u_j \notin D$ for some $i,j \in M$, then $C(u_i,u_j) \cap D = \emptyset$. Hence $Q = \emptyset$ and the theorem is trivial if $T \cap D = v^0 \cup v^+$. If $T \cap D = v^0$, then for any $i,j \in M$, $i\mathcal{D}j \iff C(u_i,u_j) \cap v^0 \neq \emptyset$. Hence all of $\{u_i \mid i \in K\}$ and half of $\{u_i \mid i \in Q\}$ are contained in one halfspace ($v^+$ or $v^-$). Therefore there exists $\hat{Q} \subset Q$ such that $\hat{q} = q/2$ and $C(\hat{K} \cup \hat{Q}) \cap D = \emptyset$. By lemma 6, such a $\hat{Q}$ also exists for the remaining Case 4: $T \cap D \neq \emptyset$ and $T \cap D$ contains no line. Therefore we must prove the theorem for cases 3 and 4 assuming such a $\hat{Q}$ exists. But then $C(K \cup \hat{Q})$ is a closed, convex and pointed cone not intersecting the convex closed cone D, so a separation theorem implies the existence of $\overline{v} \in D^* = F$ such that $C(\hat{K} \cup \hat{Q}) \subset \overline{v}^+$. Hence, since x is q.u.d. in M,

$$\hat{k} + \hat{q} \leq \lambda_m m = \lambda n - 1/2(n-q-k).$$

This implies, as $\hat{q} = q/2$, that $\hat{k} \leq \lambda n - 1/2(n-k) = \lambda_k k$, with the inequality strict if x is s.q.u.d. in M. Since $S_K(v^+) \leq \hat{k}$ for all $v \in F$, x is (s.)q.u.d. in K.

Corollary 4 (Generalized Plott Theorem 2): Suppose $D = C(0,p_1,p_2)$, with $p_1$ and $p_2$ nonzero but not necessarily distinct. Let T be a two dimensional subspace containing D, $M = \{i \in N \mid u_i \in T\}$, Q a maximal subset of M that can be partitioned into weakly disagreeing pairs, and $\hat{K} = \{i \in M - Q \mid u_i \notin D\}$. Then if x is q.u.d.,

(i) $\hat{k} \leq S(D) + (2\lambda - 1)n$

(ii) $m - S(D) - S(E) \geq q \geq m - 2S(D) - (2\lambda - 1)n$,

with the inequality in (i) and the second inequality in (ii) strict if x is s.q.u.d. Furthermore, if $\overline{Q}$ is the maximal subset of N that can be partitioned into weakly disagreeing pairs, then

$n = \overline{q} + S(D) + S(E)$ if

(iii) x is q.u.d. and $S(D) - S(E) < 1 - (2\lambda-1)n$

or

(iv) x is s.q.u.d. and $S(D) - S(E) \leq 1 - (2\lambda-1)n$.

Proof: Since T contains D, $M = C(T^0)$ and $T^0$ is a free subspace. By theorem 1, x is (s.)q.u.d. in M. Hence by theorem 2, x is (s.)q.u.d. in M - Q. Also, for $D = C(0,p_1,p_2)$, cases 3 or 4 of the proof of theorem 2 apply, so that $\hat{k} \leq \lambda n - 1/2(n - k)$, where $k = \hat{k} + S(D)$. Hence (i) follows. The second inequality in (ii) follows from (i) by substituting $m - q - S(D)$ for $\hat{k}$ in (i). The first inequality in (ii) holds because $E \cup D \subset T$ and no $i \in M$ with $u_i \in E \cup D$ can weakly disagree with anybody. By (ii), $q = m - S(D) - S(E)$ if either (iii) or (iv) hold, so that summing over all two dimensional subspaces containing gradients not in $E \cup D$ yields $n = \overline{q} + S(D) + S(E)$.

Remark 7: Observe the analogy between corollaries 2 and 4. Expression (i) in corollary 2 puts a bound on the minimal set of people whose gradients are in a one dimensional subspace containing $D = \{0\}$ that does not contain a disagreeing pair. Expression (i) in corollary 4 puts a bound on the minimal set of people, whose gradients are in a two dimensional subspace containing a $D \neq \{0\}$, that does not contain a weakly disagreeing pair. Expressions (iii) and (iv) in the two corollaries are obviously similar.

Remark 8: Corollary 4(ii) indicates the pairwise symmetry that must hold at boundary equilibria if D is two dimensional, since $i\mathcal{D}j$ iff $u_i$ and $u_j$ occupy symmetrical positions about D. Observe that D is two dimensional if V is uniquely supported at x by a hyperplane, or if F can be defined as the intersection of only two halfspaces with boundaries containing x. Clearly, less symmetry is required if V is more "pointed" than this at x; it seems that corollaries 2 and 4 indicate the only situations in which required symmetries involve pairs of gradients.

Remark 9: Notice that because D is two dimensional, (ii) of corollary 4 implies the validity of $S(D) - S(E) \geq (1-2\lambda)n$ without requiring the condition that $D \cup E$ be contained in a subspace containing only gradients in $D \cup E$, which was needed in corollary 3.

Remark 10: A converse of corollary 4 is also true: If $\overline{Q} \subset N$ can be partitioned into weakly disagreeing pairs and $n = \overline{q} + S(D) + S(E)$, then x is q.u.d. if $S(D) - S(E) \geq (1 - 2\lambda)n$ and x is s.q.u.d. if $S(D) - S(E) > (1 - 2\lambda)n$. This follows easily from the observation that $S(v^+) \leq S_{\overline{Q}}(v^+) + S(E) \leq \overline{q}/2 + S(E)$ for any feasible $v \in F$. This converse is true for any D and is generalized in section 3.

## 3. SUFFICIENT CONDITIONS

Most conditions sufficient for quasi-undomination are not as general as the necessary ones and, unfortunately, require more notation for their derivation. However, there is one general result providing a necessary as well as a sufficient condition, although it is not often useful if F is "large".

Theorem 3: Let $\{T_\alpha\}$ be a collection of subspaces such that $F \subset \bigcup_\alpha T_\alpha$. Then x is (s.)q.u.d. if and only if for every subspace $T_\alpha$ that intersects F, x is (s.)q.u.d. when every person's gradient is projected onto $T_\alpha$.

Proof: Given a subspace T, write $u_i = a_0^i + a_1^i$, where $a_0^i \in T^0$, $a_1^i \in T$. The set $\{a_1^i\}$ is the set of gradients projected onto T, and the result follows from the fact that $v \cdot u_i > 0$ if and only if $v \cdot a_1^i > 0$ when $v \in F \cap T$.

The usefulness of the criterion provided by theorem 3 is severely limited by the tradeoff between checking many subspaces of low dimension and checking fewer subspaces of higher dimension. To obtain more tractable conditions, we introduce new notation. Let $\overline{M} = \{i \in N \mid u_i \in E \cup D\}$. For any $\overline{M} \subset M \subset N$ and for any $v \in F$, define

$$n_M(v) = S_{M-\overline{M}}(v^+) - S_{M-\overline{M}}(v^- \cup v^0)$$

and

$$n_M = \max_{v \in F} n_M(v).$$

Now we have what will prove to be a very useful result.

<u>Theorem 4</u>: Let $M_1, \ldots, M_h$ be a collection of subsets of $N$ satisfying $N = M_1 \cup \ldots \cup M_h$ and $M_i \cap M_j = \overline{M}$ for $i \neq j$. Then x is q.u.d. if

$$\sum_{i=1}^{h} n_{M_i} \leq S(D) - S(E) + (2\lambda - 1)n,$$

and x is s.q.u.d. if the inequality is strict.

<u>Proof</u>: Let $v \in F$. Then

$$\begin{aligned} S(v^+) &\leq S(E) + S_{N-\overline{M}}(v^+) \\ &= S(E) + \sum_{i=1}^{h} S_{M_i-\overline{M}}(v^+) \\ &\leq S(E) + \sum_{i=1}^{h} n_{M_i} + \sum_{i=1}^{h} S_{M_i-\overline{M}}(v^- \cup v^0) \\ &\leq S(D) + \sum_{i=1}^{h} S_{M_i-\overline{M}}(v^- \cup v^0) + (2\lambda-1)n \\ &\leq S(v^- \cup v^0) + (2\lambda-1)n. \end{aligned}$$

Now $S(V^+) \leq \lambda n$ follows by substituting $n - S(v^+)$ for $S(v^- \cup v^0)$.

The proof that x is s.q.u.d. if strict inequality holds is identical.

Corollary 5: Suppose $x \in$ interior(V). Let Q be a maximal subset of N that can be partitioned into disagreeing pairs. Then x is q.u.d. if $n - q \leq 2S(0) + (2\lambda-1)n$, and x is s.q.u.d. if $n - q < 2S(0) + (2\lambda-1)n$.

Remark 11: Observe that

$$n - q - S(0) = \sum_{i \in I} | S(r_i) - S(-r_i)|,$$

where I indexes the lines $\ell_i = -r_i \cup \{0\} \cup r_i$ that contain nonzero gradients. Hence the sufficient condition for x to be q.u.d. is that

$$\sum_{i \in I} | S(r_i) - S(-r_i) | \leq S(0) + (2\lambda-1)n.$$

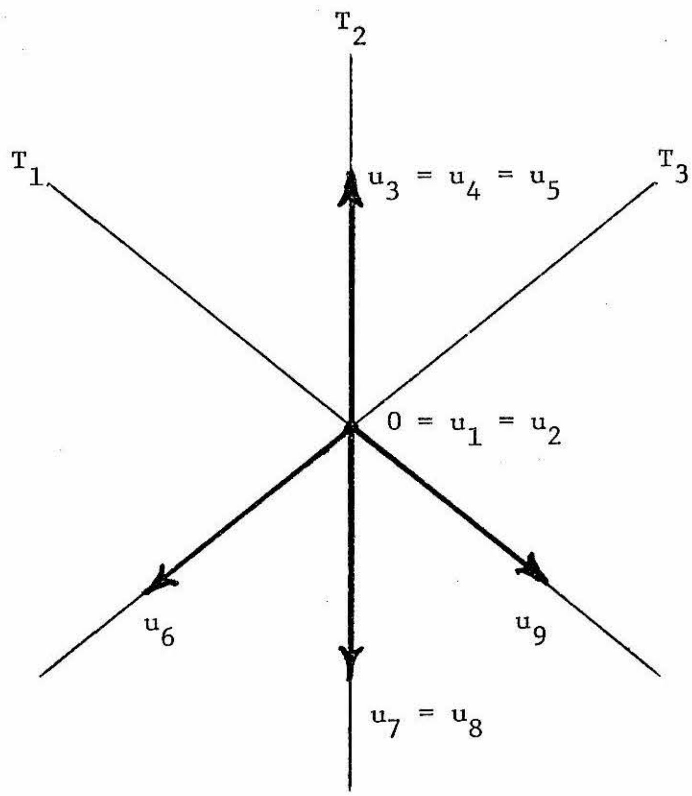
Notice the relationship to (i) in corollary 2.

Proof of Corollary 5: In theorem 4, take $M_i = \{i \in N | u_i \in \ell_i)$ for each $i \in I$. Since $D = \{0\}$, these $M_i$ satisfy the hypothesis of theorem 4. Also, $n_{M_i} = | S(r_i) - S(-r_i) |$. Hence, by remark 11, $n - q \leq 2S(0) + (2\lambda-1)n$ implies $\sum_{i \in I} n_{M_i} \leq S(0) + (2\lambda-1)n =$ $S(D) - S(E) + (2\lambda-1)n$. Therefore the result follows from theorem 4.

Remark 12: The condition of corollary 5 is not necessary for x to be q.u.d., as figure 7 illustrates. There, $n = 9$, $\lambda = 1/2$, $D = \{0\}$, x is s.q.u.d. since max $S(v^+) = 4$, but

$$\begin{aligned} \Sigma n_{M_i} &= \sum_{i=1}^{3} |S(r_i) - S(-r_i)| \\ &= 3 \not\leq 2 = S(0). \end{aligned}$$

FIGURE 7



Remark 13: The simple sufficient condition mentioned in remark 4 is a special case of corollary 5.

Corollary 6: Suppose $x \varepsilon$ boundary(V) with $D = C(0,p)$ $(p \neq 0)$. Let Q be a maximal subset of N that can be partitioned into weakly disagreeing pairs. Then x is q.u.d. if $n - q \leq 2S(D) + (2\lambda-1)n$, and x is s.q.u.d. if $n - q < 2S(D) + (2\lambda-1)n$.

Remark 14: Notice the relationship of this inequality to the second inequality in (ii) of corollary 4.

Proof of corollary 6: Let $T_1,\ldots,T_h$ be a set of two dimensional subspaces that collectively contain all nonzero gradients and that satisfy $D \subset T_i$. Let $M_i = \{i \varepsilon N \mid u_i \varepsilon T_i\}$, and notice $M_1,\ldots M_h$ satisfy the hypothesis of theorem 4. Let $Q_i$ be a maximal subset of $M_i$ that can be partitioned into weakly disagreeing pairs. Then $q = \sum_{i=1}^{h} q_i$. Let $\hat{K}_i = \{j \varepsilon M_i - Q_i \mid u_j \not\in D\}$. Then as in cases 3 and 4 of the proof of theorem 2, there exists $v_i \varepsilon F$ satisfying

$$S_{M_i}(v_i^+) = \hat{k}_i + q_i/2 = S_{M_i-\overline{M}}(v_i^+) + S(E)$$

and

$$S_{M_i-\overline{M}}(v_i^- \cup v_i^0) = q_i/2.$$

This $v_i$ yields the greatest $n_{M_i}(v^+)$, so that $n_{M_i} = \hat{k}_i - S(E)_i$.

Noticing that $n - q = \sum_{i=1}^{h} (\hat{k}_i - S(E)) + S(D) + S(E)$, we have

$$\sum_{i=1}^{h} n_{M_i} = n - q - S(D) - S(E)$$

$$\leq S(D) - S(E) + (2\lambda-1)n.$$

Hence theorem 4 implies corollary 6.

We conclude with a useful theorem that can be easily applied if $D = \{0\}$ or $D = C(0,p)$.

<u>Theorem 5</u> (Partial converse to theorem 1):

Let $T_1,\ldots,T_h$ be any collection of free subspaces such that $C(T_1) \cup \ldots \cup C(T_h) = N$ and $C(T_i) \cap C(T_j) = \overline{M}$ for $i \neq j$. Then x is q.u.d. if

(i) $S(D) - S(E) < 1-(2\lambda-1)n$ and x is q.u.d. in each $C(T_i)$,

and x is s.q.u.d. if

(ii) $S(D) - S(E) \leq 1-(2\lambda-1)n$ and x is s.q.u.d. in each $C(T_i)$.

<u>Lemma 7</u>: For any $M \subset N$ that contains $\overline{M}$, x is q.u.d. in M iff

$$n_M \leq S(D) - S(E) + (2\lambda-1)n,$$

and x is s.q.u.d. in M iff the inequality is strict.

<u>Proof</u>: By lemma 2, there exists $\overline{v}$ ε relative interior (F) such that $S_M(\overline{v}^+) \geq S_M(v^+)$ for all v ε F. By suitable applications of lemmas 1 and 3, $\overline{v}$ ε relative interior (F) can be shown to imply that $\overline{v} \cdot u_i > 0$ for each $u_i$ ε E. Hence, since $\overline{M} \subset M$ and $\overline{v} \cdot u_i \leq 0$ for

all $u_i \in D$,

$$S_M(\overline{v}^+) = S_{M-\overline{M}}(\overline{v}^+) + S(E).$$

Similarly, there exists $\hat{v} \in$ relative interior (F) such that $S_{M-\overline{M}}(\hat{v}^+) \geq S_{M-\overline{M}}(v^+)$ for any $v \in F$ and

$$S_M(\hat{v}^+) = S_{M-\overline{M}}(\hat{v}^+) + S(E).$$

Hence $S_{M-\overline{M}}(\overline{v}^+) = S_M(\overline{v}^+) - S(E) \geq S_M(\hat{v}^+) - S(E) = S_{M-\overline{M}}(\hat{v}^+)$ implies $S_{M-\overline{M}}(v^+)$ is maximized on F at $\overline{v}$.

Therefore, if x is q.u.d. in M then

$$\begin{aligned}
n_{M_i} &= \max_{v\in F}\left\{S_{M-\overline{M}}(v^+) - S_{M-\overline{M}}(v^- \cup v^0)\right\} \\
&= \max_{v\in F}\left\{S_{M-\overline{M}}(v^+) - [m - S_{M-\overline{M}}(v^+) - S(D) - S(E)]\right\} \\
&= S(D) + S(E) - m + 2 \max_{v\in F} S_{M-\overline{M}}(v^+), \\
&= S(D) - S(E) - m + 2S_M(\overline{v}^+) \\
&\leq S(D) - S(E) - m + 2\lambda_m m \\
&= S(D) - S(E) + (2\lambda-1)n,
\end{aligned}$$

with the inequality strict if x is s.q.u.d. in M. The other direction of proof is straightforward and very similar to the proof used in theorem 4.

Proof of Theorem 5: Let $M_i = C(T_i)$ and observe that $M_1,\ldots,M_h$ satisfy the hypothesis of theorem 4. Suppose (i) holds. Then by lemma 7,

$$n_{M_i} \leq S(D) - S(E) + (2\lambda-1)n < 1.$$

Hence, as each $n_{M_i}$ is nonpositive, $\sum_{i=1}^{h} n_{M_i} \leq n_{M_h} \leq S(D) - S(E) + (2\lambda-1)n$.

Therefore x is q.u.d. by theorem 4. If (ii) holds, then by lemma 7,

$$n_{M_i} < S(D) - S(E) + (2\lambda-1)n \leq 1.$$

Therefore $\sum_{i=1}^{h} n_{M_i} \leq n_{M_h} < S(D) - S(E) + (2\lambda-1)n$ and x is s.q.u.d. by theorem 4.

FOOTNOTES

1. Although to my knowledge symmetry conditions for pairs of utility gradients have only been studied previously in three papers: Plott [1967], McKelvey and Wendell [1976], and Slutsky [1978].

2. For interior equilibria, Slutsky [1978] has independently derived pairwise symmetry conditions for $\lambda$-majority rule equilibria. His conditions are similar to some of those derived here.

3. A simple generalization would be to allow W to be a differentiable manifold, F a convex cone in the tangent space $TW_x$ of W at x, and $u_i$ an element of the dual of $TW_x$.

4. For this and other results mentioned below concerning convex cones, refer to any standard source such as Fenchel [1953] or Rockafellar [1970].

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