Solutions of Nonlinear Integral Equations and Their Application to Singular Perturbation Problems

Thesis by

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ABSTRACT

Sufficient conditions for the existence and uniqueness, and estimates, of a continuous vector solution $y = y(t)$ to the integral equation

$$y(t) = u(t) + \epsilon L(t) \int_{t_o}^{t_1} M(s)\ f(s,y(s))\ ds + \epsilon \int_{t_o}^{t} K(t,s)\ f(s,y(s))\ ds,$$

are derived. A successive approximation technique involving a double sequence is used in the proof.

This integral equation result is applied to the second order singular perturbation problem with differential equation

$$\epsilon x'' + p(t,\epsilon)\ x' + q(t,\epsilon)\ x + r(t,\epsilon) + \epsilon h(t,x,x',\epsilon) = 0, \qquad 0 \le t \le 1,$$

and boundary conditions

$$b_1(\epsilon)\ x(o,\epsilon) + b_2(\epsilon)\ x'(o,\epsilon) = \ell_o(\epsilon),$$

$$c_1(\epsilon)\ x(1,\epsilon) + c_2(\epsilon)\ x'(1,\epsilon) = \ell_1(\epsilon).$$

Conditions are established under which a certain sequence generated from this system is the basis for asymptotic expansions of a solution.

The singular perturbation problem with differential equation

$$\epsilon x'' + F(t,x,x',\epsilon) = 0$$

is also studied. Under the assumptions that there exist functions $\xi = \xi(\epsilon)$ and $w = w(t,\epsilon)$ and positive constants A and B such that

$$|\epsilon w'' + F(t,w,w',\epsilon)| \le A\xi \le B\epsilon, \quad 0 \le t \le 1, \quad 0 < \epsilon < \epsilon_o ,$$

and certain relationships including w and w' evaluated at the boundaries hold as $\epsilon \to 0+$, we obtain an asymptotic expansion with leading term $w(t,\epsilon)$ for a solution to this problem.

1. We are concerned with proving the existence and uniqueness of solutions for certain classes of ordinary differential equations which depend in a singular manner on a small positive parameter ϵ. We are further concerned with describing these solutions by obtaining for them asymptotic expansions uniformly valid in the whole interval as $\epsilon \to 0+$. More precisely, let $\rho = \rho(\epsilon)$ be a positive function of ϵ with the property that $\rho(\epsilon) \to 0$ as $\epsilon \to 0+$, and let t belong to the interval I. The formal sum $\sum_{i=1}^{\infty} x_i(t,\epsilon)$ is said to be a uniform asymptotic expansion with scale $\rho(\epsilon)$ for $x(t,\epsilon)$ if there exists a function $\lambda = \lambda(t,\epsilon)$ such that for $\epsilon \to 0+$ and $m = 1,2,\ldots,$

$$x(t,\epsilon) - \sum_{i=1}^{m} x_i(t,\epsilon) = O(\lambda\rho^m) \text{ uniformly for } t \text{ in } I. \tag{1.1}$$

It is the nature of most perturbation problems, where the small parameter multiplies the highest derivative in the differential equation, to exhibit non-uniform convergence, as $\epsilon \to 0+$, in the neighborhood of some point or points in the interval. Such problems are usually called singular perturbation problems. Most of the literature on singular perturbation problems has been concerned with the case when the non-uniformity occurs at one of the end points of the interval. It is the custom in this case to call the region near this end point a boundary layer in analogy with certain hydrodynamic phenomena. Most singular perturbation problems with linear differential equations exhibit boundary layers, and in this case theoretical means exist for determining when a particular end point is part of a boundary layer (see Wasow [1]). Usually, different analytic expressions are developed for the boundary layers and the rest of the interval (see, e.g., Levin and Levinson [2]).

In section 2 some results for integral equations are proven. These are developed in greater generality than needed for the specific applications in the following sections. This is done in the expectation that these results will be found useful in connection with other perturbation problems.

Of fundamental concern in developing the theory for integral equations, which have the same solution as a given boundary value problem involving an ordinary differential equation, is the solution of Volterra integral equations. For such integral equations the convergence technique has been examined in considerable detail for the linear case by Erdelyi [3], [4], and [5], and in some detail for the nonlinear case by Erdelyi [6]. Theorem 2.1 in this thesis is a result for nonlinear Volterra integral equations similar to Erdelyi's result in [6], but differing enough to warrant a separate proof.

Theorem 2.1 is used to get our main integral equation result Theorem 2.2, which states conditions under which a Fredholm equation of the form

$$y(t) = u(t) + \epsilon \int_I K(t,s)\ f(s,y(s))ds, \quad t \in I, \tag{1.2}$$

(more precisely, of the form 2.3) has a unique vector solution $y(t)$ on I. $K(t,s)$ is allowed a discontinuity along $t = s$, and $f(s,y(s))$ is assumed to satisfy a lipschitz condition with respect to $y(s)$.

The nature of the domain $\mathcal{D}$ in which the lipschitz condition on f holds is very important in singular perturbation problems, because it is necessary for $\mathcal{D}$ to include functions $y(t) = y(t,\epsilon)$ with non-uniformities. At the same time one desires to keep $\mathcal{D}$ compact. In the statement of Theorem 2.2 the form of $\mathcal{D}$ is not specified to any great degree. Later, when Theorem 2.2 is applied to singular perturbation problems expected to have a boundary layer at the left endpoint of the interval, $\mathcal{B}$ is specialized to take this into account.

The possibility exists that Theorem 2.2 might be applied to problems having solutions with non-uniformities in the interior of the interval, by choosing the domain $\mathcal{B}$ in an appropriate manner. Little work seems to have been done on such problems. Lagerstrom [7] and his associates are presently engaged in research to produce examples of this nature.

The literature on singular perturbation problems which exhibit boundary layers is quite extensive. Wasow [8] proves the existence of a solution and develops a single uniform asymptotic expansion for that solution in the case of a second order differential equation of the form

$$\epsilon x'' + F(t,x,x',\epsilon) = 0, \quad t_o \leq t \leq t_1, \tag{1.3}$$

with

$$F(t,x,x',\epsilon) = F_1(t,x,\epsilon) + x' F_2(t,x,\epsilon), \tag{1.4}$$

and a set of boundary conditions of the form

$$x(t_o,\epsilon) = \ell_o, \quad x(t_1,\epsilon) = \ell_1 . \tag{1.5}$$

Wasow's theory applies only to problems that have solutions with at worst non-uniformities on the boundary of the interval. Erdelyi [9] proves the existence of a unique solution for a more general problem than 1.3, 1.5, and he shows to some extent the behavior of this solution, as $\epsilon \to 0+$. Using integral equation techniques, Erdelyi is able to replace condition 1.4 by the assumption that $\epsilon^{-1} \dfrac{\partial^2 F}{\partial x'^2}$ is bounded. The boundary conditions for the problem considered by Erdelyi are of the form 1.5, but with ℓ_o and ℓ_1 allowed to depend on ϵ.

A result of Wasow [1] is that most regular second order linear singular perturbation problems with solutions have a single boundary layer. Hence, it is only natural to expect problems involving an equation of the form

$$\epsilon x'' + p(t,\epsilon)x' + q(t,\epsilon)x + r(t,\epsilon) + \epsilon h(t,x,x',\epsilon) = 0, \tag{1.6}$$

which is only "weakly nonlinear", to have under very general conditions a solution with a boundary layer at the same point as the solution of the linear equation

$$\epsilon x'' + p(t,\epsilon)x' + q(t,\epsilon)x + r(t,\epsilon) = 0 . \tag{1.7}$$

In section 5 we treat the problem composed of equation 1.6 and boundary conditions

$$b_1(\epsilon)x(0,\epsilon) + b_2(\epsilon)x'(0,\epsilon) = \ell_o(\epsilon),\ c_1(\epsilon)x(1,\epsilon) + c_2(\epsilon)x'(1,\epsilon) = \ell_1(\epsilon). \tag{1.8}$$

Sufficient conditions to guarantee the existence of a unique solution $x = x(t,\epsilon)$ are stated, and uniform asymptotic expansions exhibiting a boundary layer are given for x and x'. It is characteristic of these asymptotic expansions to consist of parts that may be computed by using only ordinary perturbation methods on problems having non-homogeneous first order linear differential equations and one boundary condition.

We consider in section 6 the problem of the "strictly nonlinear" equation 1.3 with boundary conditions 1.8. Instead of considering this problem as merely a "boundary layer problem", as Wasow does in [8] and Erdelyi does in [9], we find conditions sufficient to guarantee the existence of a solution of the form

$$x = w + \epsilon z, \tag{1.9}$$

where $w = w(t,\epsilon)$ is given so that z can be determined as a solution of a weakly nonlinear problem by the theory developed for such problems in section 5. For a very general class of functions F, e.g., $F(t,x,x',\epsilon)$ of class C^1 in t and of class C^2 in x and x', and $\epsilon^{-1} \frac{\partial^2 F}{\partial x'^2} = O(1)$ uniformly (see assumption 6ii for a precise statement of the conditions on F), we find it sufficient that $w(t,\epsilon)$ satisfy

$$\epsilon w''(t,\epsilon) + F(t,w(t,\epsilon), w'(t,\epsilon),\epsilon) = O(\epsilon) \text{ uniformly} \tag{1.10}$$

and a similar weakened version of the boundary conditions. Nothing is said about the non-uniformities of $w(t,\epsilon)$, nor is $w(t,\epsilon)$ determined any further than 1.10 and the other general conditions assumed for it.

One can show that Wasow [8] and Erdelyi [9] have taken

$$w(t,\epsilon) = w_o(t) + \mu w_1(t,\epsilon), \tag{1.11}$$

where $w_o = w_o(t)$ is assumed to satisfy

$$F(t, w_o, w_o', 0) = 0 \tag{1.12}$$

and

$$w_o(1) = \ell_1(0),$$

and where

$$\mu \geq |\ell_o - w_o(0)|. \tag{1.13}$$

For μ sufficiently small, they show the existence of a function $w_1(t,\epsilon)$ so that 1.11 determines w satisfactorily. $w_1(t,\epsilon)$, so determined, can have only a boundary layer type of non-uniformity. Also, the magnitude of μ is limited enough to greatly restrict the generality of the final result for $w(t,\epsilon)$.

It seems certain that the integral equation technique developed in this thesis could be extended to obtain $w(t,\epsilon)$ of the same nature as Wasow's and Erdelyi's result, only for our more general problem. For boundary conditions of the form 1.8, one would choose

$$\mu \geq |\ell_0 - b_1\, w_0(0) - b_2\, w_0'(0)|$$

and then would consider the weakly nonlinear differential equation, where "weakly" now means with respect to the new parameter μ, obtained by substituting 1.11 into 1.10.

The uniform asymptotic expansion we obtain for the general problem defined by 1.6 and 1.8 gives, when $h \equiv 0$, a uniform asymptotic expansion for the general second order linear equation with boundary values of the form 1.8. In this case the expansion is obtained under the assumption that $p(t,\epsilon)$ is from the class C^1 and $q(t,\epsilon)$ is from the class C.

2. In what follows, t and s denote real variables confined to an interval I, which has t_0 as left endpoint and t_1 as right endpoint. If $t_0 = -\infty$, then t_0 is not included in I, and if $t_1 = +\infty$, then t_1 is not included in I. Otherwise, t_0 and t_1 may or may not belong to I. $\mathcal{R}$ is the set $\{(t,s) : t \in I,\ t_0 < s < t\}$; ϵ is a small positive parameter always assumed to be in the interval $0 < \epsilon < \epsilon_0 \leq 1$.

An n-dimensional vector will be thought of as an $n \times 1$ matrix, and by $x^{\dagger}$ will be meant the <u>transpose</u> of the $n \times 1$ matrix x. Whenever we speak of matrices in general, we will mean scalars, column and row vectors, and square matrices compatible with these vectors. All matrices will have real numerical functions for elements. As the norm of the matrix $A = [a_{ij}(t)]$ we take

$$||A|| = \sum_{i,j} |a_{ij}(t)|.$$

If all the $a_{ij}(t)$ are integrable functions of t, by $\int A(t)dt$ is meant the matrix $[\int a_{ij}(t)dt]$. The derivative $A'(t)$ is defined in a like manner. A partial ordering between matrices is defined componentwise by

$$A \leq B \quad \text{if} \quad a_{ij}(t) \leq b_{ij}(t) \quad \text{for all } i, j, \text{ and } t.$$

For later applications we shall need that our results in this section be stated with more precision than the estimates afforded by the norm defined above. Hence, for any matrix A define $|A|$ to be the column vector whose $i^{\underline{th}}$ component is $\sum_j |a_{ij}(t)|$. For scalars this is the usual absolute value, and for a column vector $x = [x_i]$, $|x|$ is the column vector whose $i^{\underline{th}}$ component is $|x_i|$ and $|x^{\dagger}|$ is the scalar $\sum_i |x_i|$.

For any matrices A and B,

$$|AB| \leq |A| \; ||B|| \quad \text{and} \quad ||A|| = ||(|A|)||.$$

Let $\mathcal{D}$ be the set of all vectors $x(t)$ which are continuous on I and satisfy the vector inequality $|x(t)| \leq d(t)$ for some fixed positive vector $d(t)$.

We say that $A(t)$ is <u>integrable over I</u> if $\int_{t_o+}^{t_1-} A(t)dt$ exists (is finite); $A(t)$ is <u>locally integrable over I</u> if for each s belonging to I, $A(t)$ is integrable over (t_o, s); $A(t,s)$ is <u>locally integrable over $\mathcal{R}$</u> if for each t belonging to I, $A(t,s)$ is integrable over the interval (t_o, t) as a function of s; and $A(t,s)$ is <u>integrable over $\mathcal{R}$</u> if $A(t,s)$ is locally integrable over $\mathcal{R}$ and

$$\int_{t_o}^{t_1-} A(t_1-,s)ds = \lim_{t \to t_1-} \int_{t_o}^{t} A(t,s)ds \qquad (2.1)$$

exists.

A function $x = x(t) = x(t,\epsilon)$ is a member of the class $C^k(I)$ if for each ϵ, $o < \epsilon < \epsilon_o$, the first k derivatives of $x(t,\epsilon)$ with respect to t exist on I and these k derivatives and $x(t,\epsilon)$ are continuous on I. This will be denoted in the usual way by $x \in C^k(I)$, $x(t) \in C^k(I)$, or $x(t,\epsilon) \in C^k(I)$.

For functions $\rho(\epsilon)$ and $\tau(\epsilon)$ we say that $\rho(\epsilon) = O(\tau(\epsilon))$ if there exist positive constants α and ϵ^* so that $|\rho(\epsilon)| \leq \alpha \, |\tau(\epsilon)|$ whenever $o < \epsilon < \epsilon^*$, and that $\rho(\epsilon) = o(\tau(\epsilon))$ if for any positive number α there exists another positive number $\epsilon^*(\alpha)$ so that $|\rho(\epsilon)| \leq \alpha \, |\tau(\epsilon)|$ whenever $o < \epsilon < \epsilon^*(\alpha)$. For vectors $x(t,\epsilon)$ and $y(t,\epsilon)$ defined over I, we say that $x(t, \epsilon) = O(y(t, \epsilon))$ uniformly over I (or just "uniformly") if there exists positive constants α and ϵ^* so that $|x(t,\epsilon)| \leq \alpha|y(t,\epsilon)|$ for all t in I and $0 < \epsilon < \epsilon^*$; and that $x(t,\epsilon) = o(y(t,\epsilon))$ uniformly over I (or just "uniformly") if for any positive number α there exists another positive number $\epsilon^*(\alpha)$ so that $|x(t,\epsilon)| \leq \alpha|y(t,\epsilon)|$ for all t in I and $0 < \epsilon < \epsilon^*(\alpha)$. That these last definitions make sense for matrices follows from our previous definition of the absolute value and ordering of matrices.

We will use the following well known result (see [10], p. 37):

Lemma 2.1. Let $f(t)$, $g(t)$, $h(t)$, and $k(t)$ be numerical functions defined over I and such that $k(t)f(t)$, $k(t)g(t)$, and $k(t)h(t)$ are locally integrable over I. If

$$f(t) \leq g(t) + h(t) \int_{t_o}^{t} k(s)f(s)ds ,$$

$$h(t) \geq 0, \quad \text{and} \quad k(t) \geq 0$$

for all t in I, then

$$f(t) \leq g(t) + h(t) \exp(\int_{t_o}^{t} k(s)h(s)ds) \int_{t_o}^{t} k(s)g(s)ds$$

for all t in I.

Our first considerations will be for the unknown vector function $y(t)$ in the nonlinear Volterra integral equation

$$y(t) = u(t) + \epsilon \int_{t_o}^{t} K(t,s)f(s,y(s))ds . \tag{2.2}$$

Sufficient conditions for the existence and uniqueness of a solution $y(t)$ will be stated, and a bound on $y(t) - u(t)$ will be obtained.

The result for equation 2.2 will then be used to find sufficient conditions to guarantee the existence of a unique solution $y(t)$ to the nonlinear integral equation

$$y(t) = u(t) + \epsilon L(t) \int_{t_o}^{t_1-} M(t_1-,s)f(s,y(s))ds + \epsilon \int_{t_o}^{t} K(t,s)\, f(s,y(s))ds. \tag{2.3}$$

K in equations 2.2 and 2.3 and LM in equation 2.3 are matrices compatible with the vectors y, u, and f. The first integral in equation 2.3 is defined in the manner of 2.1.

<u>Assumption</u> <u>2i</u>. $u(t) \in C(I)$; $K(t,s) \in C(\mathcal{R})$; and $f(t,y)$ is defined on $I \times \mathcal{D}$.

<u>Assumption</u> <u>2ii</u>. There exist nonnegative vector functions $m(t)$ and $j(t)$ and a positive scalar function $\kappa(t)$ all continuous on I such that $||f(t,x) - f(t,y)|| \leq m^{\dagger}(t)\ |x - y|$ whenever $(t,x), (t,y) \in I \times \mathcal{D}$ and $|K(t,s)| \leq j(t)\ \kappa(s)$.

The lipschitz condition satisfied by f and the continuity of $m(t)$ imply that $f(t,y)$ as a function of y is continuous on $\mathcal{D}$.

<u>Assumption</u> <u>2iii</u>. There exist locally integrable functions $\delta_o'(t)$ and $\delta_1'(t)$ over I such that $\delta_o'(t) \geq \kappa(t)\ ||f(t,u(t))||$ and $\delta_1'(t) \geq \epsilon\kappa(t)\ m^{\dagger}(t)\ j(t)$ for all t in I.

With

$$\delta_o(t) = \int_{t_o}^{t} \delta_o'(s)ds \quad \text{and} \quad \delta_1(t) = \int_{t_o}^{t} \delta_1'(s)ds ,$$

let

$$\Delta_o(t) = |u(t)| + j(t)\, \epsilon\, \delta_o(t) \exp(\delta_1(t)) . \tag{2.4}$$

Assumption 2iv. $\Delta_o(t) \leq d(t)$, $t \in I$.

Suppose $y(t) \in \mathfrak{D}$ and $K(t,s)f(s,y(s))$ is locally integrable over $\mathcal{R}$; let T be the mapping defined by

$$Ty(t) = u(t) + \epsilon \int_{t_o}^{t} K(t,s)f(s,y(s))ds . \tag{2.5}$$

A fixed point for T belonging to $\mathfrak{D}$ will be constructed by the classical method of successive approximations, setting

$$y_o(t) = u(t) \quad \text{and} \quad y_k(t) = Ty_{k-1}(t), \quad k \geq 1. \tag{2.6}$$

Although Erdelyi [6] works out for considerable generality the technique of taking successive approximations with the operator T, the case covered by our assumptions is not included. We will prove here that the $y_k(t)$ converge to a function $y_*(t)$ satisfying 2.2, and we will determine a bound for $y_*(t) - u(t)$.

Theorem 2.1. Under assumptions 2i to 2iv, $y_k(t)$ as defined by 2.6 exists for all values of k, T as defined by 2.5 has one and only one fixed point $y_*(t)$ for which $y_*(t) \in \mathfrak{D}$ and $\kappa(t)\, m^{\dagger}(t)\, |y_*(t)|$ is locally integrable over I, and

$$|y_*(t) - y_k(t)| \leq j(t)\,\epsilon\,\delta_o(t) \sum_{i=k}^{+\infty} \frac{[\delta_1(t)]^i}{i!} . \tag{2.7}$$

Proof. The main part of the proof will be to show by induction that for $k = 1, 2, \ldots, y_k(t)$ exists and belongs to $\mathcal{B}$, and

$$|y_k(t) - y_{k-1}(t)| \leq j(t)\,\epsilon\,\delta_o(t) \frac{[\delta_1(t)]^{k-1}}{(k-1)!} . \tag{2.8}$$

$u(t) \in \mathcal{B}$ by assumptions 2i and 2iv; hence, $K(t,s)f(s,u(s))$ is defined in $\mathcal{R}$. $K(t,s)f(s,u(s))$ is locally integrable over $\mathcal{R}$, if $|K(t,s)f(s,u(s))|$ is locally integrable over $\mathcal{R}$, and if $K(t,s)f(s,u(s))$ for each $t \in I$ is integrable over all intervals $t^* < s < t$ where $t_o < t^* < t$ (see [11], p. 437). Since all functions involved are continuous on I by assumption and since any interval of the form (t^*,t) where both t^* and t are in I is bounded, the latter requirement is certainly met. $|K(t,s)f(s,u(s))|$ is locally integrable over $\mathcal{R}$, because

$$|K(t,s)f(s,u(s))| \leq j(t)\kappa(s)\,||f(s,u(s))|| \leq j(t)\delta_o'(s) \tag{2.9}$$

by assumptions 2ii and 2iii, and $j(t)\ \delta_o'(s)$ is integrable over $\mathcal{R}$ also by assumption. We conclude that $y_1(t)$ exists, and that

$$|y_1(t) - y_o(t)| \leq j(t)\,\epsilon\,\delta_o(t) \tag{2.10}$$

by equation 2.6, inequality 2.9, and the definition of $\delta_o(t)$. Applying the triangle inequality to inequality 2.10, we obtain

$$|y_1(t)| \leq |u(t)| + j(t)\,\epsilon\,\delta_o(t).$$

Hence, $y_1(t) \in \mathcal{D}$ by assumption 2iv and the continuity of $u(t)$. The induction statement has been verified for $k = 1$.

Next, suppose that the induction statement holds for $k \leq n$ where $n \geq 1$. From this hypothesis, assumptions 2ii and 2iii, and the definitions of δ_o and δ_1, we obtain

$$|K(t,s)f(s,y_n(s))| \leq |K(t,s)| \; ||f(s,y_n(s))|| \leq$$

$$j(t)\kappa(s) \left[||f(s,u(s))|| + \sum_{j=1}^{n} ||f(s,y_j(s)) - f(s,y_{j-1}(s))|| \right] \leq$$

$$j(t)\kappa(s) \left[||f(s,u(s))|| + \sum_{j=1}^{n} m^{\dagger}(s)|y_j(s) - y_{-1}(s)| \right] \leq$$

$$j(t)\kappa(s) \left[||f(s,u(s))|| + \epsilon m^{\dagger}(s)j(s)\,\delta_o(s) \sum_{j=1}^{n} \frac{[\delta_1(s)]^{j-1}}{(j-1)!} \right]$$

$$\leq j(t) \frac{d}{ds} \left[\delta_o(s)\exp(\delta_1(s))\right] = j(t)[\delta_o(s)\exp(\delta_1(s))]'. \qquad (2.11)$$

Inequality 2.11 implies that $|K(t,s)f(s,y_n(s))|$ is locally integrable over $\mathcal{R}$, and it follows as before that $K(t,s)f(s,y_n(s))$ is locally integrable over $\mathcal{R}$. We conclude that $y_{n+1}(t)$ exists. Hence, by equation 2.6, assumptions 2ii and 2iii, and inequality 2.8 for $k = n$,

$$|y_{n+1}(t) - y_n(t)| \leq \epsilon \int_{t_o}^{t} |K(t,s)| \; ||f(s,y_n(s)) - f(s,y_{n-1}(s))||ds$$

$$\leq j(t)\,\epsilon \int_{t_o}^{t} \epsilon\,\kappa(s)\, m^{\dagger}(s)j(s)\,\delta_o(s) \frac{[\delta_1(s)]^{n-1}}{(n-1)!}\, ds$$

$$\leq j(t)\,\epsilon\,\delta_o(t) \frac{[\delta_1(t)]^{n}}{n!} \quad ,$$

which is inequality (2.8) for $k = n + 1$. Hence,

$$|y_{n+1}(t)| \leq |u(t)| + \sum_{k=1}^{n+1} |y_k(t) - y_{k-1}(t)| \leq$$

$$|u(t)| + j(t)\, \epsilon\, \delta_o(t) \exp(\delta_1(t)) = \Delta_o(t) .$$

By assumption 2iv, $|y_{n+1}(t)| \leq d(t)$, and so $y_{n+1}(t) \in \mathcal{D}$. This completely proves the induction statement for $k = n + 1$ given its validity for $k = n$. We conclude that the induction statement holds for all values of $k = 1, 2, \ldots$.

Inequality 2.8 implies that

$$\sum_{k=1}^{+\infty} ||y_k(t)-y_{k-1}(t)|| \leq \epsilon\, ||j(t)||\, \delta_o(t)\exp(\delta_1(t)),$$

and since $j(t)$, $\delta_o(t)$, and $\delta_1(t)$ are all continuous functions on I, the telescoping infinite series $u(t) + \sum_{k=1}^{+\infty} [y_k(t) - y_{k-1}(t)]$ converges absolutely and uniformly on every compact subinterval of I to a continuous function $y_*(t)$. Since $y_*(t) - y_k(t) = \sum_{i=k}^{+\infty} [y_{i+1}(t) - y_i(t)]$, 2.8 implies that

$$|y_*(t) - y_k(t)| \leq \sum_{j=k}^{+\infty} |y_{j+1}(t) - y_j(t)|$$

$$\leq j(t)\, \epsilon\, \delta_o(t) \sum_{j=k}^{+\infty} \frac{[\delta_1(t)]^j}{j!} . \qquad (2.12)$$

Putting $k = 0$ in 2.12 produces

$$|y_*(t)| \leq |y_o(t)| + \epsilon j(t)\, \delta_o(t)\exp(\delta_1(t)) = \Delta_o(t)$$

upon an application of the triangle inequality. It follows by assumption 2iv that $|y_*(t)| \leq d(t)$, and hence $y_*(t) \in \mathcal{D}$.

We will show next that $y_*(t)$ is a fixed point for T. Since $f(t,y)$ is a continuous function of y, we have

$$f(t,y_*(t)) = \lim_{k \to \infty} f(t,y_k(t)). \tag{2.13}$$

In a manner similar to the derivation of 2.11, we get

$$\kappa(s)||f(s,y_k(s))|| \leq [\delta_o(s)\exp(\delta_1(s))]' \tag{2.14}$$

for all k. The prime in 2.14 denotes a derivative with respect to s. Letting $k \to \infty$ in 2.14, we get by 2.13 that

$$\kappa(s)||f(s,y_*(s))|| \leq [\delta_o(s)\exp(\delta_1(s))]'. \tag{2.15}$$

Inequality 2.15 implies that $|\kappa(t,s)f(s,y_*(s))|$ is locally integrable over $\mathcal{R}$, and it follows as before that $\kappa(t,s)f(s,y_*(s))$ is locally integrable over $\mathcal{R}$. Because of 2.14,

$$\lim_{k \to \infty} \int_{t_o}^{t} K(t,s)f(s,y_k(s))ds = \int_{t_o}^{t} K(t,s)f(s,y_*(s))ds \tag{2.16}$$

by the Lebesgue dominated convergence theorem. We conclude that

$$y_*(t) = \lim_{k \to \infty} Ty_k(t) = Ty_*(t).$$

Let $x(t)$ be any fixed point of T in $\mathcal{D}$ for which $\kappa(t)\ m^{\dagger}(t)|x(t)$ is locally integrable over I. We need this last condition in order to conclude from the assumptions that

$$0 \leq m^{\dagger}(t)|y_*(t) - x(t)| \leq \epsilon m^{\dagger}(t) \int_{t_0}^{t} |K(t,s)[f(s,y_*(s)) - f(s,x(s))]|ds$$

$$\leq \epsilon m^{\dagger}(t)\ j(t) \int_{t_0}^{t} \kappa(s)\ m^{\dagger}(s)|y_*(s) - x(s)|ds < +\infty\ . \qquad (2.17)$$

Lemma 2.1 may be applied to 2.17 for the special case $g(t) \equiv 0$ in order to conclude that

$$m^{\dagger}(t)|y_*(t) - x(t)| = 0, \quad t \in I. \qquad (2.18)$$

Since nothing has been assumed to prevent $m^{\dagger}(t)$ from being zero, we need to make one more computation with the true norms, i.e.,

$$|y_*(t) - x(t)| \leq \epsilon \int_{t_0}^{t} |K(t,s)|\ ||f(s,y_*(s)) - f(s,x(s))||ds$$

$$\leq \epsilon \int_{t_0}^{t} |K(t,s)|\ m^{\dagger}(s)\ |y_*(s) - x(s)|ds \leq 0,\ t \in I,$$

by 2.18. Thus, $y_*(t) = x(t)$ for all t in I. The proof of Theorem 2.1 is now complete.

Equation 2.3 will be considered next. In order to understand the basis for the considerations that follow, let $\mathcal{B}_0$ be the set of vectors $x(t)$ such that $x(t)$ is defined over I, $M(t,s)f(s,x(s))$ is integrable over $\mathcal{R}$, and

$$u^*(t) = u(t) + Ux = u(t) + \epsilon L(t) \int_{t_0}^{t_1-} M(t_1-,s)f(s,x(s))ds \qquad (2.19)$$

satisfies the assumptions of Theorem 2.1 for $u(t)$. Let $\mathcal{D}_1$ be the subset of vectors $y(t)$ of $\mathcal{D}$ for which $\kappa(t)m^{\dagger}(t)|y(t)|$ is locally integrable over I. We consider the mapping S of $\mathcal{D}_o$ into $\mathcal{D}_1$ given by

$$Sx = z, \tag{2.20}$$

where $z = Tz + Ux$. Theorem 2.1 implies that S is single-valued and maps all of $\mathcal{D}_o$ into $\mathcal{D}_1$. We will determine sufficient conditions to imply that S has precisely one fixed point in the intersection of $\mathcal{D}_o$ and $\mathcal{D}_1$.

The proof will be again by successive approximations, setting

$$y_o^o(t) = u(t),\ y_o^k(t) = u(t) + U\, y_*^{k-1}(t), \qquad k = 1, 2, \cdots; \tag{2.21}$$

$$y_*^k(t) = U\, y_*^{k-1}(t) + Ty_*^k(t), \qquad k = 0, 1, \cdots ; \tag{2.22}$$

and putting

$$\delta_o^k(t) \geq \int_{t_o}^{t} \kappa(s) \; ||f(s, y_o^k(s))|| ds . \tag{2.23}$$

<u>Assumption</u> <u>2v</u>. $M(t,s) \in C(\mathcal{R})$, and as $t \to t_1-$, $M(t,s)/\kappa(s)$ converges uniformly for s in I to a function of s which is bounded in some non-degenerate subinterval of I with left endpoint t_o and in some non-degenerate subinterval of I with right endpoint t_1.

<u>Lemma</u> <u>2.2.</u> If assumption 2v holds and $v(s)$ is a vector function such that $||v(s)||$ is integrable over I, then $\frac{M(t,s)}{\kappa(s)}\, v(s)$ is integrable over $\mathcal{R}$.

Proof. In order to show that $\frac{M(t,s)}{\kappa(s)} v(s)$ is integrable over $\mathcal{R}$, we must show the existence of the double limit

$$\lim_{t \to t_1 -} \lim_{\tilde{t} \to t_0 +} \int_{\tilde{t}}^{t} \frac{M(t,s)}{\kappa(s)} v(s)ds. \qquad (2.24)$$

This will follow from the special boundness condition found in assumption 2v and the uniform convergence of $M(t,s)/\kappa(s)$, also part of assumption 2v. For the sake of brevity let the capital letter I with a subscript zero and any superscript denote a non-degenerate subinterval of I with left endpoint t_o, and let I with a subscript one and any superscript denote a non-degenerate subinterval of I with right endpoint t_1.

We will prove the existence of the limits in 2.24 by showing that for any positive number α, there exist intervals I_o^o and I_1^o such that

$$\left|\left| \int_{s_1}^{s_4} \frac{M(s_4,s)}{\kappa(s)} v(s)ds - \int_{s_2}^{s_3} \frac{M(s_3,s)}{\kappa(s)} v(s)ds \right|\right| \leq \alpha, \qquad (2.25)$$

when s_1 and s_2 are in I_o^o and s_3 and s_4 are in I_1^o. Suppose that $s_1 < s_2 < s_3 < s_4$. Then, the left member of inequality 2.25 is bounded by

$$\int_{s_1}^{s_2} \frac{||M(s_4,s)||}{\kappa(s)} ||v(s)||ds + \int_{s_3}^{s_4} \frac{||M(s_4,s)||}{\kappa(s)} ||v(s)||ds +$$

$$\int_{s_2}^{s_3} \frac{||M(s_4,s) - M(s_3,s)||}{\kappa(s)} ||v(s)||ds . \qquad (2.26)$$

Assumption 2v implies that there exists intervals I_o^1 and I_1^1 and a constant c_o such that

$$\frac{||M(t,s)||}{\kappa(s)} \leq c_o \tag{2.27}$$

when (t,s) is in $I_1^1 \times I_o^1$ or $I_1^1 \times I_1^1$. Since $||v(s)||$ is integrable over I, there exist intervals I_o^o and I_1^2 included in I_o^1 and I_1^1 respectively, such that

$$\int_s^t ||v(s)||ds \leq \alpha/3c_o \tag{2.28}$$

when s, t are both in I_o^o or I_1^2. Substitution of inequalities 2.27 and 2.28 into expression 2.26 gives that the sum of the first two integrals is bounded by $\frac{2}{3}\alpha$ when s_1 and s_2 are in I_o^o and s_3 and s_4 are in I_1^2 . Suppose that

$$\int_{t_o}^{t_1} ||v(s)||ds \leq c_1 \ . \tag{2.29}$$

Because $\frac{M(t,s)}{\kappa(s)}$ converges uniformly at $t \to t_1-$, there exists an interval I_1^o included in I_1^2 such that

$$\frac{||M(s_4,s) - M(s_3,s)||}{\kappa(s)} \leq \frac{\alpha}{3c_1} \tag{2.30}$$

when (s_4,s), $(s_3,s) \in I_1^o \times I$. It follows that the third integral in 2.26 is bounded by $\alpha/3$ when s_4 and s_3 are in I_1^o. We conclude that intervals I_o^o and I_1^o exist so that 2.25 holds, and hence, so that $\frac{M(t,s)}{\kappa(s)}$ $v(s)$ is integrable over $\mathcal{R}$.

Assumption 2vi. $L(t) \in C(I)$, and there exists a locally integrable function $\delta_2'(t)$ over I such that $\delta_2'(t) \geq \kappa(t)\, m^{\dagger}(t)\, |L(t)|$ for all t in I.

We define

$$\delta_2(t) = \int_{t_o}^{t} \delta_2'(s)ds. \qquad (2.31)$$

Assumption (2vii). The limits as $t \to t_1-$ of $\delta_o(t)$, $\delta_1(t)$, and $\delta_2(t)$ all exist.

Assumption (2vii) implies directly that $[\delta_2(t)\exp(\delta_1(t))]'$ and $[\delta_o(t)\exp(\delta_1(t))]'$ are integrable over I. Hence, we obtain from assumption 2v and Lemma 2.2 the existence of two bounded, continuous functions $\bar{\delta}_o(t)$ and $\bar{\delta}_2(t)$ on I such that

$$\bar{\delta}_o(t) \geq \int_{t_o}^{t} \frac{||M(t,s)||}{\kappa(s)} \; [\delta_o(s)\exp(\delta_1(s))]' \, ds, \qquad (2.32)$$

$$\bar{\delta}_2(t) \geq \int_{t_o}^{t} \frac{||M(t,s)||}{\kappa(s)} \; [\delta_2(s)\exp(\delta_1(s))]' \, ds. \qquad (2.33)$$

Let

$$\Delta_1(t) = [\,|L(t)| + j(t)\, \epsilon\, \delta_2(t)\exp(\delta_1(t))]\, \frac{\bar{\delta}_o(t_1)}{1-\epsilon\bar{\delta}_2(t_1)} \,. \qquad (2.34)$$

Assumption (2viii). $\epsilon\, \bar{\delta}_2(t_1) < 1$ and $\Delta_o(t) + \epsilon\, \Delta_1(t) \leq d(t)$ for all t in I.

Theorem 2.2. Under assumptions 2i to 2viii, $y_o^k(t)$ and $y_*^k(t)$ as defined by equations 2.21 and 2.22 exist for all values of k, S as defined by 2.20 has one and only one fixed point y(t) for which $y(t) \in \mathcal{D}$ and $\kappa(t)m^{+}(t)\,|y(t)|$ is integrable over I,

$$|y(t) - y_*^k(t)| \leq [\epsilon\, \bar{\delta}_2(t_1)]^k\, \epsilon\, \Delta_1(t), \qquad (2.35)$$

and

$$|y(t) - u(t)| \leq [\Delta_o(t) - u(t)] + \epsilon\, \Delta_1(t). \quad (2.36)$$

Proof. Once again the main part of the proof will be by induction on k = 1, 2, ⋯ . The induction statement is: solutions $y_*^k(t)$ and $y_o^k(t)$ exist for equations 2.22 and 2.21, $y_*^k(t) \in \mathcal{D}$, $\kappa(t)m^{+}(t)|y_*^k(t)|$ is locally integrable over I,

$$|y_o^k(t) - y_o^{k-1}(t)| \leq [\epsilon\, \bar{\delta}_2(t_1)]^{k-1}\; \bar{\delta}_o(t_1)\, \epsilon\, |L(t)|, \qquad (2.37)$$

$$|y_o^k(t)| \leq |u(t)| + \frac{\epsilon\, \bar{\delta}_o(t_1)}{1 - \epsilon\, \bar{\delta}_2(t_1)}\; |L(t)|, \qquad (2.38)$$

and there exists $\delta_o^k(t)$ satisfying inequality 2.23 and

$$\delta_o^k(t) \leq \delta_o(t) + \frac{\epsilon\, \bar{\delta}_o(t_1)}{1 - \epsilon\, \bar{\delta}_2(t_1)}\; \delta_2(t). \qquad (2.39)$$

(2.38) and (2.39) imply that

$$|y_o^k(t)| + \epsilon\delta_o^k(t)\exp(\delta_1(t))\; j(t) \leq \Delta_o(t) + \epsilon\Delta_1(t) \leq d(t)$$

by assumption 2viii. This means that assumption 2iv of Theorem 2.1 is satisfied by the integral equation 2.22. It follows from Theorem 2.1

that equation 2.22 will have a unique solution $y_*^k(t)$. Thus, if inequalities 2.38 and 2.39 hold for a particular k, then there exists a unique solution $y_*^k(t)$ to equation 2.22, $y_*^k(t) \in \mathcal{D}$, $\kappa(t)\, m^\dagger(t)|y_*^k(t)|$ is locally integrable over I, and in general $y_*^k(t)$ satisfies all the conclusions of Theorem 2.1.

Consider the induction statement for $k = 1$. $y_*^o(t)$ exists and is in S by Theorem 2.1. Hence, $y_o^1(t)$ will exist if $M(t,s)f(s,y_*^o(s))$ is integrable over $\mathcal{R}$. Inequality 2.15 implies that $\kappa(s)||f(s,y_*^o(s))||$ is integrable over I, because $[\delta_o(s)\exp(\delta_1(s))]'$ is integrable over I by assumption 2vii. Lemma 2.2 implies then that $M(t,s)f(s,y_*^o(s))$ $= \frac{M(t,s)}{\kappa(s)}\kappa(s)f(s,y_*^o(s))$ is integrable over $\mathcal{R}$. We obtain from 2.15 the inequality

$$\int_{t_o}^{t_1-} ||M(t_1-,s)||\ ||f(s,y_*^o(s))||ds \leq \int_{t_o}^{t_1-} \frac{||M(t_1-,s)||}{\kappa(s)}[\delta_o(s)\exp(\delta_1(s))]'ds$$

It follows from the definition of $\overline{\delta}_o(t)$ given in (2.32) that

$$\int_{t_o}^{t_1-} ||M(t_1-,s)||\ ||f(s,y_*^o(s))||ds \leq \overline{\delta}_o(t_1). \qquad (2.40)$$

Hence,

$$|y_o^1(t) - y_o^o(t)| \leq \epsilon\ \overline{\delta}_o(t_1)\ |L(t)|, \qquad (2.41)$$

which is 2.37 for $k = 1$. Inequality 2.38 for $k = 1$ follows directly from 2.41 by the triangle inequality. Finally, $\delta_o^1(t)$ may be chosen so that 2.23 and 2.39 hold, because

$$\int_{t_o}^{t} \kappa(s)||f(s,y_o^1(s))||ds \le \int_{t_o}^{t} \kappa(s)\ ||f(s,u(s))||ds \ +$$

$$\int_{t_o}^{t} \kappa(s)||f(s,y_o^1(s)) - f(s,y_o^o(s))||ds \le$$

$$\delta_o(t) + \int_{t_o}^{t} \kappa(s)m^{+}(s)\ |y_o^1(s) - y_o^o(s)|\ ds\ ,$$

and

$$\int_{t_o}^{t} \kappa(s)m^{+}(s)\ |\ y_o^1(s) - y_o^o(s)|\ ds \le \epsilon\bar{\delta}_o(t_1)\ \delta_2(t)$$

by 2.41 and the definition of $\delta_2(t)$. By a previous discussion we can now conclude that $y_*^1(t)$ exists, $y_*^1(t) \in \mathcal{D}$, and $\kappa(t)m^{+}(t)|y_*^1(t)|$ is locally integrable over I. This completes the proof of the induction statement for $k = 1$.

Suppose next that the induction statement holds for $k = 1,2,\cdots,n$ and $n \ge 1$. Then for $k = n$, 2.38 and 2.39 imply the existence of a solution $y_*^n(t)$ for equation 2.22 when $k = n$. At the same time, the induction hypothesis assumes the existence of a solution $y_*^n(t)$. Both these solutions are members of $\mathcal{D}$ and satisfy the integrability condition of Theorem 2.1, which implies uniqueness of solution. Hence, they must be the same function $y_*^n(t)$.

Since the existence of $y_*^n(t)$ satisfying 2.22 can be considered a direct result of Theorem 2.1, the inequalities included in the proof of Theorem 2.1 may be used here for $y_*^n(t)$. For example, inequality 2.15 becomes

$$\kappa(s)||f(s,y_*^n(s))|| \leq [\delta_o^n(s)\exp(\delta_1(s))]', \tag{2.42}$$

where the prime denotes a derivative with respect to s. It follows from assumption 2vii and inequality 2.39, which holds for $k = n$ by hypothesis, that $[\delta_o^n(s)\exp(\delta_1(s))]'$ is integrable over I. Hence $\kappa(s)||f(s,y_*^n(s))||$ is integrable over I, and by Lemma 2.2, $M(t,s)f(s,y_*^n(s))$ and $||M(t,s)||\ ||f(s,y_*^n(s))||$ are integrable over $\mathcal{R}$, which proves $y_o^{n+1}(t)$ as defined by equation 2.21 exists. Furthermore, for a proper choice of $\delta^n(s)$, inequalities 2.39 and 2.42 imply that

$$\kappa(s)||f(s,y_*^n(s))|| \leq [\delta_o(s)\exp(\delta_1(s) + \frac{\epsilon\ \overline{\delta}_o(t_1)}{1-\epsilon\ \overline{\delta}_2(t_1)}\ \delta_2(s)\exp(\delta_1(s))]' . \tag{2.43}$$

We obtain now from equations 2.21 and 2.22

$$|y_o^{n+1}(t) - y_o^n(t)| \leq \epsilon|L(t)| \int_{t_o}^{t_1-} ||M(t_1-,s)||m^+(s)|y_*^n(s)-y_*^{n-1}(s)|ds \tag{2.44}$$

and

$$m^+(s)|y_*^n(s) - y_*^{n-1}(s)| \leq m^+(s)|y_o^n(s) - y_o^{n-1}(s)| +$$

$$\epsilon m^+(s)j(s) \int_{t_o}^{s} \kappa(r)m^+(r)|y_*^n(r) - y_*^{n-1}(r)|dr \tag{2.45}$$

by assumption 2ii. The integrals in inequalities 2.44 and 2.45 exist because of the part of the induction hypothesis which says $\kappa(t)m^+(t)|y_*^k(s)|$ is locally integrable over I for $k \leq n$.

Lemma 2.1 may be applied to 2.45 to give

$$m^{+}(s)|y_{*}^{n}(s) - y_{*}^{n-1}(s)| \leq m^{+}(s)|y_{o}^{n}(s) - y_{o}^{n-1}(s)| +$$

$$\epsilon\ m^{+}(s)j(s)\ \exp(\delta_1(s)) \int_{t_o}^{s} \kappa(r)m^{+}(r)|y_{o}^{n}(r) - y_{o}^{n-1}(r)|dr.$$

It follows from inequality 2.37 for $k = n$ that

$$\kappa(s)m^{+}(s)|y_{*}^{n}(s) - y_{*}^{n-1}(s)| \leq$$

$$[\exp(\delta_1(s)) \int_{t_o}^{s} \kappa(r)m^{+}(r)|y_{o}^{n}(r) - y_{o}^{n-1}(r)|dr]' \leq$$

$$\epsilon\ \overline{\delta}_o(t_1)\ [\epsilon\overline{\delta}_2(t_1)]^{n-1}\ [\delta_2(s)\exp(\delta_1(s))]'. \tag{2.46}$$

Hence,

$$\int_{t_o}^{t_1-} ||M(t_1-,s)||m^{+}(s)|y_{*}^{n}(s)-y_{*}^{n-1}(s)|ds \leq \overline{\delta}_o(t_1)[\epsilon\overline{\delta}_2(t_1)]^{n}. \tag{2.47}$$

Inequality 2.37 for $k = n + 1$ now follows from inequalities 2.44 and 2.47. Because $\epsilon\overline{\delta}_2(t_1) < 1$, one can prove the validity of inequality 2.38 for $k = n + 1$ from the identity

$$y_{o}^{n+1} = u(t) + \sum_{k=1}^{n+1} [y_{o}^{k}(t) - y_{o}^{k-1}(t)]$$

and the validity of 2.37 for $k \leq n + 1$.

Next,

$$\kappa(s)||f(s,y_{o}^{n+1}(s))|| \leq \kappa(s)||f(s,u(s)) +$$

$$\kappa(s) \sum_{k=1}^{n+1} ||f(s,y_o^k(s)) - f(s,y_o^{k-1}(s))|| \leq \delta_o'(s) +$$

$$\kappa(s) \sum_{k=1}^{n+1} m^\dagger(s)|y_o^k(s) - y_o^{k-1}(s)| \leq \delta_o'(s) +$$

$$\frac{\epsilon\bar{\delta}_o(t_1)}{1-\epsilon\bar{\delta}_2(t_1)} \delta_2'(s) . \qquad (2.48)$$

Inequality 2.48 implies that there exists a choice for $\delta_o^{n+1}(t)$ so that both inequalities 2.23 and 2.39 are satisfied. As a matter of fact, in all our considerations we could just as well choose

$$\delta_o^n(t) = \delta_o(t) + \frac{\epsilon\bar{\delta}_o(t_1)}{1-\epsilon\bar{\delta}_2(t_1)} \delta_2(t) \quad \text{for all} \quad n.$$

This implies, as we have just shown by induction, that inequality 2.23 holds for all values of k. Inequalities 2.39 and 2.43 hold trivially for such a choice of $\delta_o^n(t)$.

By an earlier discussion in this proof, we conclude that $y_*^{n+1}(t)$ exists satisfying equation 2.22 and the conclusions of Theorem 2.1 hold for $y_*^{n+1}(t)$. This completes the proof of the induction statement.

For $k = 1, 2, \ldots$,

$$\kappa(s)m^\dagger(s) \; |y_*^k(s) - y_*^{k-1}(s)| \leq \epsilon \; \bar{\delta}_o(t_1)[\epsilon\bar{\delta}_2(t_1)]^{k-1}[\delta_2(s)\exp(\delta_1(s))]' \qquad (2.49)$$

follows in exactly the same way 2.46 was established for the particular case $k = n$. Since a solution $y_*^k(t)$ to equation 2.22 exists for

$k = 0,1,\ldots,$ we may use 2.22 and assumption 2ii to obtain

$$|y_*^k(t) - y_*^{k-1}| \le |y_o^k(t)-y_o^{k-1}(t)| + \epsilon j(t) \int_{t_o}^{t} \kappa(s)m^{+}(s)|y_*^k(s)-y_*^{k-1}(s)|ds.$$

Hence, inequalities 2.49 and 2.37 imply that

$$|y_*^k(t) - y_*^{k-1}(t)| \le \epsilon\, \overline{\delta}_o(t_1)[\epsilon\, \overline{\delta}_2(t_1)]^{k-1}[|L(t)|+\epsilon\, \delta_2(t)\exp(\delta_1(t))j(t)]$$

$$\text{for } k = 1,2,\ldots \quad . \tag{2.50}$$

Since $\epsilon\overline{\delta}_2(t_1) < 1$ by assumption (2viii) and since in general $||x|| = ||(|x|)||$, it follows that

$$\sum_{k=1}^{+\infty} ||y_*^k(t)-y_*^{k-1}(t)|| \le \frac{\epsilon\overline{\delta}_o(t_1)}{1-\epsilon\overline{\delta}_2(t_1)} [||L(t)|| + \epsilon\delta_2(t)\exp(\delta_1(t))||j(t)||] . \tag{2.51}$$

The right side of 2.51 is continuous on I, so $\sum_{k=1}^{+\infty} ||y_*^k(t)-y_*^{k-1}(t)||$ is bounded on each compact subinterval of I. Thus, the telescoping series $y_*^o(t) + \sum_{k=1}^{+\infty} [y_*^k(t) - y_*^{k-1}(t)]$ converges absolutely on I to a continuous function $y(t)$.

Inequality 2.50 gives further that

$$|y(t)-y_*^k(t)| \le \sum_{j=k}^{+\infty} |y_*^{j+1}(t) - y_*^j(t)| \le$$

$$\frac{[\epsilon\, \overline{\delta}_2(t_1)]^k\, \epsilon\, \overline{\delta}_o(t_1)}{1 - \epsilon\overline{\delta}_2(t_1)} [|L(t)| + \epsilon j(t)\delta_2(t)\exp(\delta_1(t))]=[\epsilon\overline{\delta}_2(t_1)]^k\epsilon\Delta_1(t),$$

which is precisely inequality 2.35 in the conclusions. Putting $k = 0$ in 2.35 and applying the triangle inequality, we get

$$|y(t)| \leq |y_*^o(t)| + \epsilon\Delta_1(t),$$

or

$$|y(t)| \leq \Delta_o(t) + \epsilon\Delta_1(t)$$

by inequality 2.7 of Theorem 2.1. Assumption (viii) gives now that $y(t) \in \mathcal{D}$. Since $|y(t) - u(t)| \leq |y(t)-y_*^o(t)|+|y_*(t)-u(t)|$ by the triangle inequality, inequality 2.36 of the conclusions follows from 2.7 and 2.35.

We will prove next that $y(t)$ is a fixed point for the mapping S defined by equation 2.20, or what is the same, that $y(t)$ is a solution to integral equation 2.3. In order to do this, let $k \to +\infty$ in equation 2.22. We will show that

$$\lim_{k\to+\infty} \int_{t_o}^{t_1-} M(t_1-,s)f(s,y_*^k(s))ds = \lim_{t\to t_1-} \lim_{k\to+\infty} \int_{t_o}^{t} M(t,s)f(s,y_*^k(s))ds$$

$$= \int_{t_o}^{t_1-} M(t_1-,s)f(s,y(s))ds \;, \qquad (2.52)$$

and that

$$\lim_{k\to+\infty} \int_{t_o}^{t} K(t,s)f(s,y_*^k(s))ds = \int_{t_o}^{t} K(t,s)f(s,y(s))ds. \quad (2.53)$$

Since $y_*^n(t)$ for $n = 0,1,\ldots$ is a unique solution by Theorem 2.1, inequality 2.15 found in the proof of Theorem 2.1 will be given as inequality 2.43 when $n = 0, 1, \ldots$. Letting $n \to \infty$ in 2.43, one obtains

$$\kappa(s)\ ||f(s,y(s))|| \leq [\delta_0(s)\exp(\delta_1(s)) + \frac{\epsilon\bar{\delta}_0(t_1)}{1-\epsilon\bar{\delta}_2(t_1)}\ \delta_2(s)\exp(\delta_1(s))]' , \tag{2.54}$$

because f is a continuous function of y and $\lim_{n\to\infty} y_*^n(s) = y(s)$. The right side of inequality 2.54 is integrable over I by assumption 2vii. Hence, $K(t,s)f(s,y(s))$ is locally integrable over $\mathcal{R}$. Furthermore, the interchange of limit and integral in 2.53 is valid by the Lebesgue dominated convergence theorem.

Writing

$$||M(t,s)f(s,y_*^n(s))|| \leq \frac{||M(t,s)||}{\kappa(s)}\ \kappa(s)||f(s,y_*^n(s))|| ,$$

we see by inequality 2.43 and the definitions 2.32 and 2.33 of $\bar{\delta}_0$ and $\bar{\delta}_2$ that $||M(t,s)f(s,y_*^n(s))||$ is bounded for all n by an integrable function over $\mathcal{R}$. Hence, the interchange of limits necessary to get the second equality in 2.52 is valid by the Lebesgue dominated convergence theorem.

The only dependence of the first integral in 2.52 on k is through $f(s,y_*^k(s))$. In the convergence of the integral as $t \to t_1-, \kappa(s)||f(s,y_*^k(s))||$ is uniformly bounded with respect to k by a function integrable over I. This implies that $\lim_{t\to t_1-}$ is uniform with respect to k. The proof would proceed in a manner similar to the proof of Lemma 2.2. It follows that $\lim_{t \to t_1-}$ and $\lim_{k \to +\infty}$ may be interchanged to produce the first equality of 2.52. The validity of 2.52 and 2.53 implies that $y(t)$ is a solution of equation 2.3.

In the proof of Theorem 2.2 there remains only to prove the uniqueness part of the conclusions. Suppose that $z(t)$ is any solution of equation 2.3, or, in other words, that $z(t)$ is any fixed point of the mapping S, such that $\kappa(s)m^{+}(s)\,|z(s)|$ is integrable over I. Then,

$$m^{+}(t)|y(t)-z(t)| \leq \epsilon m^{+}(t)|L(t)| \int_{t_o}^{t_1-} ||M(t_1-,s)||\;||f(s,y(s))-f(s,z(s))||ds$$

$$+ \epsilon\, m^{+}(t) \int_{t_o}^{t} |K(t,s)|\;||f(s,y(s)) - f(s,z(s))||ds \leq$$

$$\epsilon\, m^{+}(t)|L(t)| \int_{t_o}^{t_1-} ||M(t_1-,s)||m^{+}(s)|y(s)-z(s)|ds +$$

$$\epsilon\, m^{+}(t)j(t) \int_{t_o}^{t} \kappa(s)m^{+}(s)|y(s)-z(s)|ds \; . \qquad (2.55)$$

The integral in 2.55 exist by Lemma 2.2 and the integrability of $\kappa(s)m^{+}(s)\;|y(s)|$ and $\kappa(s)m^{+}(s)\;|z(s)|$. Applying Lemma 2.1 to inequality 2.55, we obtain

$$m^{+}(t)|y(t)-z(t)| \leq [\epsilon m^{+}(t)|L(t)| + \epsilon^2 m^{+}(t)j(t)\;\delta_2(t)\exp(\delta_1(t))] \cdot$$

$$\int_{t_o}^{t_1-} ||M(t_1-,s)||m^{+}(s)|y(s)-z(s)|ds \; ,$$

which yields with another application of Lemma 2.1

$$m^{+}(t)|y(t)-z(t)| = 0 \; . \qquad (2.56)$$

Inequality 2.55 is valid as a vector inequaltiy without the factor $m^{\dagger}(t)$. Hence, it follows from 2.56 that $y(t) = z(t)$ for all t in I.

For the applications of Theorem 2.2 to boundary value problems with second order ordinary differential equations, it will be sufficient to use a two-dimensional form of equation 2.3. We need to further specify the "lipschitz function" $m(t)$ and the domain $\mathcal{D}$ by specifying $d(t)$.

<u>Assumption</u> <u>2ix.</u> $u^{\dagger}(t) = u^{\dagger}(t,\epsilon) = (u_1(t,\epsilon), u_2(t,\epsilon)) = O(1)$ uniformly.

Assumption 2ix is essentially a restriction on the transformations between the boundary value problem and its equivalent integral equation formulations. Only transformations yielding uniformly bounded $u(t,\epsilon)$ can be used.

<u>Assumption</u> <u>2x.</u> $\sigma(t,\epsilon)$ is a nonnegative numerical function defined for t in I and $o < \epsilon < \epsilon_o$ such that $\sigma(t,\epsilon) \in C(I)$, $\sigma(t,\epsilon) = O(1)$ uniformly, and $\sigma(t,\epsilon) \geq |u_2(t,\epsilon)|$.

For two new positive parameters μ and ν independent of t, define

$$d^{\dagger}(t) = (\mu, \nu + \sigma(t,\epsilon)). \tag{2.57}$$

μ and ν can be considered measures of the size of the domain $\mathcal{D}$. The presence of $\sigma(t,\epsilon)$ permits non-uniformities to occur in the second components of vectors in $\mathcal{D}$.

We define next

$$m^{\dagger}(t) = (\nu + \sigma(t,\epsilon),\ 1)\ \epsilon^{-1}\ \omega\ , \tag{2.58}$$

where $\omega = \omega(\mu)$ is a nondecreasing numerical function of μ. With Δ_o and Δ_1 defined in 2.4 and 2.34 respectively, let

$$\Delta = \epsilon^{-1}(\Delta_o - |u|) + \Delta_1. \tag{2.59}$$

For m given by 2.58, we actually have that $\Delta = \Delta(t,\epsilon;\, \mu,\nu)$ and $\overline{\delta}_2 = \overline{\delta}_2(t,\epsilon;\, \mu,\nu)$ are functions also of μ and ν.

<u>Assumption</u> <u>2xi.</u> There exist positive constants μ^*, ν^*, and ϵ_o^* such that $\delta_2(t_1,\epsilon;\mu^*,\nu^*)$ and $\Delta(t,\epsilon;\mu^*,\nu^*)$ exist for $o < \epsilon \leq \epsilon_o^*$ and t in I,

$$\epsilon\overline{\delta}_2(t_1,\epsilon;\mu^*,\nu^*) = o(1), \tag{2.60}$$

and

$$\Delta(t,\epsilon;\mu^*,\nu^*) = O(1) \text{ uniformly.} \tag{2.61}$$

Assumptions 2ix to 2xi are sufficient to imply that $\Delta_o + \epsilon\Delta_1 \leq d$, when μ is sufficiently large, $\nu = \epsilon\pi$, and ϵ is sufficiently small. More specifically, let

$$\mu > \tilde{\mu} = \sup_{t\in I}\ \limsup_{\epsilon \to 0+} \max\,(|u_1(t,\epsilon)|,\sigma(t,\epsilon)). \tag{2.62}$$

Then, for μ so-chosen, let

$$\pi > \sup_{t\in I}\ \limsup_{\epsilon \to 0+} ||\Delta(t,\epsilon;\mu,\nu^*)|| \,. \tag{2.63}$$

With μ and π fixed and satisfying 2.62 and 2.63, we require that ϵ_o be sufficiently small so that

$$0 < \epsilon_o \leq \min\,\left(\frac{\nu^*}{\pi}\,,\ \frac{\mu-\tilde{\mu}}{\pi},\ \epsilon_o^*\right), \tag{2.64}$$

$$\epsilon\, \bar{\delta}_2(t_1,\epsilon;\, \mu,\nu^*) \le \text{constant} \;<1 \quad \text{and} \tag{2.65}$$

$$||\Delta(t,\epsilon;\mu,\nu^*)|| \le \pi \quad \text{for} \quad o < \epsilon < \epsilon_o. \tag{2.66}$$

From the nature of Δ_o and Δ_1, one can show that Δ exists for all μ and ν provided it is assumed that it exists for one set of values μ^*, ν^*. This, of course, depends on ϵ being sufficiently small to make $\epsilon\bar{\delta}_2(t_1,\epsilon;\, \mu,\nu) < 1$, which is the reason for 2.65. The other property of Δ important here is that $\Delta(t,\epsilon;\mu,\nu)$ is nonincreasing as ν decreases, given that t, ϵ, and μ are fixed. For $o < \epsilon < \epsilon_o$ we have from 2.64 that $\nu = \epsilon\pi < \nu^*$, and hence, $\Delta(t,\epsilon;\mu,\nu) \le \Delta(t,\epsilon;\mu,\nu^*)$. It follows from requirement 2.66 that $\pi \ge ||\Delta(t,\epsilon;\mu,\nu)||$ for $o < \epsilon < \epsilon_o$. Hence,

$$\Delta_o + \epsilon\Delta_1 \le |u| + \epsilon\Delta \le (\,|u_1| + \nu,\; |u_2| + \nu)^{\dagger}\,.$$

2.64 implies that $|u_1| + \nu \le |u_1| + \mu\text{-}\tilde{\mu}$, and from our choice of μ in 2.62, it follows that $|u_1| + \nu \le \mu$. Since $\sigma \ge |u_2|$, we have $|u_2| + \nu \le \sigma + \nu$, and so

$$\Delta_o + \epsilon\Delta_1 \le (\mu,\sigma + \nu)^{\dagger} = d. \tag{2.67}$$

Corollary. If assumptions 2iv and 2viii are replaced by assumptions 2ix to 2xi and μ satisfies inequality 2.62, then Theorem 2.2 is true for d and m given by 2.57 and 2.58 respectively and ϵ sufficiently small.

The restriction on μ is due to the way the assumptions have been formulated and is not an important requirement. The domain in which the lipschitz condition on f holds has been constructed about the zero vector rather than the vector u.

3. In this section we will be concerned with relating the second order boundary value problem given by

$$\epsilon y' + A(t,\epsilon)y + \epsilon g(t,\epsilon) + \epsilon^2 f(t,y,\epsilon) = 0, \tag{3.1}$$

$$b^{+}(\epsilon)y(0,\epsilon) = \ell_0^*(\epsilon), \quad c^{+}(\epsilon)y(1,\epsilon) = \ell_1^*(\epsilon), \tag{3.2}$$

to an integral equation problem included in the theory of the previous section. All vectors, such as y, g, and f are two-dimensional, and

$$A = \begin{pmatrix} \epsilon a_1(t,\epsilon) & a_2(t,\epsilon) \\ \epsilon a_3(t,\epsilon) & a_4(t,\epsilon) \end{pmatrix} . \tag{3.3}$$

The interval I is taken to be the closed interval $0 \leq t \leq 1$.

<u>Assumption 3i.</u> $a_2, a_4 \in C^1(I)$; $a_1, a_3 \in C(I)$; $a_1, a_2, a_2', a_3, a_4, a_4'$ are $O(1)$ uniformly; and $|a_2(t,\epsilon)|$, $a_4(t,\epsilon)$ are positive and bounded away from zero for t in I and $0 < \epsilon < \epsilon_0$.

(If a_4 is negative, the substitution $t^* = 1 - t$ will change the problem to one of the type that we are considering.)

The form of equation 3.1 is not unique, i.e., higher order terms in ϵ that are linear in y may be considered part of Ay or part of $\epsilon^2 f$. Under assumption 3i and the assumption that f satisfies a lipschitz condition of the type described in assumption 2ii with m given by 2.58, one can always determine $A^*(t,\epsilon)$ and $f^*(t,y,\epsilon)$ so that

$$Ay + \epsilon^2 f \equiv A^*y + \epsilon^2 f^* ,$$

A^* and f^* satisfy conditions of the same form as A and f respectively, and $\epsilon y' + A^* y = 0$ has a special kind of fundamental solution. We will assume that equation 3.1 is already written in the desired form.

Assumption 3ii. Let

$$\epsilon y' + A(t,\epsilon)y = 0$$

have the fundamental solution

$$H(t,\epsilon) = \begin{pmatrix} \exp \frac{\psi(t,\epsilon)}{\epsilon} & \exp \zeta(t,\epsilon) \\ \beta(t,\epsilon)\exp\frac{\psi(t,\epsilon)}{\epsilon} & \alpha(t,\epsilon)\exp \zeta(t,\epsilon) \end{pmatrix},$$

where

$$\beta = a_4/a_2,$$

$$\psi' = -a_2\beta - \epsilon a_1 = -a_4 - \epsilon a_1 ,$$

$$\alpha = -\epsilon^{-1} \int_0^t a_3(s,\epsilon) \exp\left(\int_s^t a_1(r,\epsilon)dr - \epsilon^{-1} \int_s^t a_4(r,\epsilon)dr\right) ds ,$$

$$\zeta' = -a_2\alpha - a_1 .$$

By assumption, a_4 is positive and bounded away from zero and a_1 is uniformly bounded, so,

$$\psi'(t,\epsilon) < o , \quad t \in I, \quad o < \epsilon < \epsilon_o ,$$

if ϵ_o is chosen sufficiently small. For brevity we write

$$\Psi(t,s,\epsilon) = \exp \frac{\psi(t,\epsilon)-\psi(s,\epsilon)}{\epsilon} .$$

Then,

$$\psi(1,o,\epsilon) = O(\epsilon^k) \quad \text{for all} \quad k. \tag{3.4}$$

Assumption 3iii. $g(t,\epsilon) \in C(I)$ and

$$||g(t,\epsilon)|| \leq \tilde{g}(\epsilon)\ [1 + \epsilon^{-1}\ \psi^2(t,o,\epsilon)]\ .$$

The problem defined by differential equation 3.1 and boundary conditions 3.2 is equivalent to the problem of solving an integral equation of the type 2.3, where

$$L(t,\epsilon) = -H(t,\epsilon) \begin{pmatrix} c^{\dagger} H(1,\epsilon) \\ b^{\dagger} H(o,\epsilon) \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{3.5}$$

$$K(t,s,\epsilon) = H(t,\epsilon)H^{-1}(s,\epsilon), \tag{3.6}$$

$$M(t,s,\epsilon) = c^{\dagger} K(t,s,\epsilon), \tag{3.7}$$

$$u(t,\epsilon) = H(t,\epsilon)\begin{pmatrix} c^{\dagger} H(1,\epsilon) \\ b^{\dagger} H(o,\epsilon) \end{pmatrix} \begin{pmatrix} \ell_1^* - \int_o^1 M(1,s,\epsilon)g(s,\epsilon)ds \\ \ell_o^* \end{pmatrix}$$

$$+ \int_o^t K(t,s,\epsilon)g(s,\epsilon)ds. \tag{3.8}$$

Suppose that

$$T_o(\epsilon) = |b_1 + b_2\ \beta(o,\epsilon)|^{-1} \tag{3.9}$$

and

$$T_1(\epsilon) = |c_1 + \epsilon c_3\ \alpha(1,\epsilon)|^{-1}\ . \tag{3.10}$$

If

$$T_o T_1\ \psi(1,o,\epsilon) = o(1), \tag{3.11}$$

then

$$\det \begin{pmatrix} c\ H(1,\epsilon) \\ b\ H(o,\epsilon) \end{pmatrix} = \delta \exp\ (-\epsilon^{-1}\ \ \psi(o,\epsilon)\ -\ \zeta(o,\epsilon)), \tag{3.12}$$

where

$$\delta^{-1} = O(T_o T_1). \tag{3.13}$$

3.11 is hardly a restriction in the light of equation 3.4.

It follows from equations 3.5, 3.12, and 3.13 and $\alpha(o,\epsilon) = 0$ that

$$L(t,\epsilon) = \tilde{u}(t,\epsilon)\ O[T_1(1+|b_1|\ T_o)] \text{ uniformly,} \tag{3.14}$$

where

$$\tilde{u}(t,\epsilon) = \begin{pmatrix} 1 \\ \epsilon + \psi(t,o,\epsilon) \end{pmatrix} . \tag{3.15}$$

We obtain by using $|K(t,s,\epsilon)| = O(\tilde{u}(t,\epsilon))$ uniformly and assumption 3iii that

$$\int_o^t |K(t,s,\epsilon)g(s,\epsilon)|ds \leq \int_o^t |K(t,s,\epsilon)|\ ||g(s,\epsilon)||ds$$

$$= O(\widetilde{\widetilde{gu}}) \quad \text{uniformly} \tag{3.16}$$

and

$$\int_o^1 |M(1,s,\epsilon)g(s,\epsilon)|ds \leq |c|^{\dagger} \int_o^1 |K(1,s,\epsilon)g(s,\epsilon)|ds = O(T_2\tilde{g}) ,$$

where

$$T_2 = T_2(\epsilon) = |c_1| + \epsilon\ |c_2| . \tag{3.17}$$

In order to simplify the estimate for $|u(t,\epsilon)|$, we assume that

$$T_o T_1 |c_1 + c_2\beta(1,\epsilon)|\ \psi(1,o,\epsilon) = O(\epsilon^k) \quad \text{for all } k, \tag{3.18}$$

and

$$\epsilon^n\ T_o^{-1} = O(1) \qquad \text{for some } n. \tag{3.19}$$

Then,

$$|u(t,\epsilon)| = [\,|L(t,\epsilon)|(|\ell_1^*| + \tilde{g}T_2) + |\ell_o^*|T_o\,\psi(t,o,\epsilon)$$

$$+ \tilde{g}\,\bar{u}(t,\epsilon)]\,O(1) \leq \quad T\tilde{u}(t,\epsilon) \quad \text{uniformly,} \tag{3.20}$$

where $T = T(\epsilon)$ is a constant multiple of

$$|\ell_1^*|\ T_1(1+T_o|b_1|) + |\ell_o^*|T_o + \tilde{g}\,[1+T_1T_2(1+T_o|b_1|). \tag{3.21}$$

It becomes clear now how to choose μ and $\sigma(t,\epsilon)$ in 2.57 and 2.58 so that $\mu \geq \sigma(t,\epsilon) \geq |u_2(t,\epsilon)|$ and $\mu \geq |u_1(t,\epsilon)|$ for all t in I and $0 < \epsilon < \epsilon_o \leq 1$. Let

$$\sigma(t,\epsilon) = [\epsilon + \psi(t,o,\epsilon)]\ \sigma_o \tag{3.22}$$

and

$$\mu > \sigma_o > \limsup_{\epsilon \to o+} T(\epsilon). \tag{3.23}$$

<u>Assumption 3iv.</u> There exists positive constants μ, ν, and σ_o such that for d and m defined by equations 2.57 and 2.58 with σ given by 3.22 and μ and σ_o satisfying 3.23, $f(t,y(t,\epsilon),\epsilon) \in C(I)$ when $y(t,\epsilon) \in \mathcal{D}$, $f(t,o,\epsilon) = 0$, and $||f(t,y,\epsilon)-f(t,z,\epsilon)|| \leq m^{\dagger}(t)|y-z|$ when (t,y) and (t,z) are in $I \times \mathcal{D}$ and $0 < \epsilon < \epsilon_o$.

<u>Theorem 3.1.</u> Let assumptions 3i to 3iv and relations 3.11, 3.18, and 3.19 hold. If

$$[1 + T_1T_2(1 + T_o\ |b_1|\ )]T = O(1) \tag{3.24}$$

and

$$\epsilon\ T_1T_2(1 + T_o\ |b_1|) = o(1), \tag{3.25}$$

then for sufficiently small ϵ there exists a unique solution $y = y(t,\epsilon)$ to 3.1, 3.2 such that

$$|y(t,\epsilon)| = O(\tilde{u}\,\Upsilon) \quad \text{uniformly} \tag{3.26}$$

and

$$|y(t,\epsilon)-u(t,\epsilon)| = \epsilon\Upsilon \; O[(1,1)^{\dagger} + \tilde{u}\,\Upsilon_2\Upsilon_1(1+\Upsilon_o|b_1|)]. \tag{3.27}$$

<u>Proof</u>. In the notation of section 2, we may choose κ, δ_1', and j to all be constants for the present problem. Since $f(t,o,\epsilon) = o$, the lipschitz condition satisfied by f gives

$$||f(t,u,\epsilon)|| \leq m^{\dagger}(t,\epsilon)\;|u| \;,$$

and hence,

$$||f(t,u,\epsilon)|| = O(\epsilon^{-1}\,\Upsilon\,\tilde{u}_2) = \delta_o'(t,\epsilon) \;.$$

3.14 implies that we may choose $\delta_2'(t,\epsilon)$ to be a constant multiple of $\epsilon^{-1}\,\Upsilon_1(1+\Upsilon_o|b_1|)\,\tilde{u}_2(t,\epsilon)$.

It follows that

$$\delta_o(t,\epsilon) = O(\Upsilon) \quad \text{uniformly},$$

$$\delta_2(t,\epsilon) = O[\Upsilon_1(1+\Upsilon_o|b_1|)] \quad \text{uniformly},$$

$$\bar{\delta}_o(t_1,\epsilon) = O(\Upsilon\Upsilon_2),$$

$$\bar{\delta}_2(t_1,\epsilon) = O[\Upsilon_1\Upsilon_2(1+\Upsilon_o|b_1|)].$$

Hence,

$$||(\Delta_o(t,\epsilon)-|u(t,\epsilon)|)|| = O(\epsilon\Upsilon) = O(\epsilon) \quad \text{uniformly}$$

and

$$\Delta_1(t,\epsilon) = O[\tilde{u}TT_1T_2(1+T_o|b_1|)] = O(1) \quad \text{uniformly.}$$

We get that $u(t,\epsilon) = O(1)$ uniformly from 3.20 and $T = O(1)$. 3.24 gives that $\epsilon\,\bar{\delta}_2(t_1,\epsilon) = o(1)$. We conclude that the corollary to Theorem 2.2 applies in the present case. Relations 3.26 and 3.27 follow from 2.36.

4. In this section we will consider the second order scalar differential equation given by

$$Tx = \epsilon x'' + x' + q(t,\epsilon)x + r(t,\epsilon) + \epsilon h(t,x,x',\epsilon) = 0, \qquad (4.1)$$

along with boundary conditions of the form

$$B[x] = b_1(\epsilon)x(o,\epsilon)+\epsilon b_2(\epsilon)x'(o,\epsilon) = \ell_o, \qquad (4.2)$$

$$C[x] = c_1(\epsilon)x(1,\epsilon)+\epsilon c_2(\epsilon)x'(1,\epsilon) = \ell_1. \qquad (4.3)$$

The notation in 4.2 and 4.3 is not meant to imply that $\epsilon b_2(\epsilon)$ and $\epsilon c_2(\epsilon)$ will later be assumed to be $O(\epsilon)$. It is only a convenience to allow us to identify vectors $b = (b_1,b_2)$ and $c = (c_1,c_2)$, which will be used in applying the results of section 3. Equation 4.1 is not a specialization of equation 1.6 but is the result of an elementary change of independent variable.

We will show how the existence of a solution to the complete system 4.1, 4.2, 4.3 depends on the existence of certain "approximate solutions" to systems that have a differential equation similar to 4.1 and only one boundary condition.

Associated with the equation $Tx = o$ is another equation of a similar type which we propose to call an "adjoint equation" of 4.1. In order to define this adjoint equation of 4.1, let $Tx = Px + Qx$ where Px is linear in x and its derivatives and Qx contains no derivatives of x higher than the first order. Furthermore, suppose that for the vector $(x, \epsilon x')^{+}$ in $\mathcal{D}$, $Qx = r(t,\epsilon) + O(\epsilon+e^{-t/\epsilon})$ uniformly, where $r(t,\epsilon)$ is the function occurring in equation 4.1. It follows by the linearity of P that

$$v\,T(x + w) = v\,(Px + Pw) + v\,Q(x + w) =$$

$$v\,(Px + Qx) + v\,Pw + v\,[Q(x + w) - Qx] = v\,(Px + Qx) +$$

$$[B(v,w)]' + w\,P^{*}v + v\,[Q(x + w) - Qx],$$

where $B(v,w)$ is the bilinear concomitant of v and w and P^{*} is the formal adjoint to P. We set

$$B(v,w) = 0 \qquad (4.4)$$

and assume that w can be determined as a function $w(t,v)$ of t and v. The differential equation adjoint to $Tx = 0$ with respect to P is defined to be the equation in v given by

$$T_x^{*}v = w\,P^{*}v + v\,[Q(x + w) - Qx] = 0, \qquad (4.5)$$

where $w = w(t,v)$ and $Tx = Tx(t,\epsilon) = 0$.

For the considerations that follow, we choose

$$Px = \epsilon x'' + x' + q\,(t,\epsilon)\,x.$$

It follows that

$$w(t,v) = ve^{-t/\epsilon}$$

up to an arbitrary constant factor, which we have taken equal to unity. The equation $T_x^{*}v = 0$ is now determined uniquely by 4.5. Except for the arbitrary constant factor in w, it can be shown that $T_x^{*}v = 0$ in general depends only upon the coefficient of x' in $Px = 0$.

The significance of the operator T_x^{*} is indicated in the following equation:

$$T(x+w) = T(x+ve^{-t/\epsilon}) = Tx+e^{-t/\epsilon}\ T^*_x v\ . \qquad (4.6)$$

<u>Assumption</u> <u>4i.</u> $q(t,\epsilon)$, $r(t,\epsilon) \in C(I)$ and $q(t,\epsilon)$, $r(t,\epsilon) = O(1)$ uniformly.

Define

$$\Upsilon_o = \Upsilon_o(\epsilon) = |b_1 - b_2|^{-1} ,$$

$$\Upsilon_1 = \Upsilon_1(\epsilon) = |c_1 - c_2 \int_o^1 q(s,\epsilon) \exp\left(\frac{s-1}{\epsilon}\right) ds|^{-1} , \qquad (4.7)$$

$$\Upsilon_2 = \Upsilon_2(\epsilon) = |c_1| + \epsilon\ |c_2| ,$$

$$\Upsilon_3 = \Upsilon_3(\epsilon) = 1 + \Upsilon_1\Upsilon_2\ (1 + \Upsilon_o\ |b_1|),$$

and

$$\tilde{u}^\dagger(t,\epsilon) = (\tilde{u}_1(t,\epsilon),\ \tilde{u}_2(t,\epsilon)) = (1, \epsilon+e^{-t/\epsilon}). \qquad (4.8)$$

<u>Assumption</u> <u>4ii.</u> There exist functions $\tilde{x} = \tilde{x}(t,\epsilon)$ and $\tilde{v} = \tilde{v}(t,\epsilon)$ both in $C^2(I)$ such that for a given positive bounded function $\gamma(\epsilon)$, which satisfies

$$\gamma^{-1} = o(\epsilon e^{1/\epsilon}) , \qquad (4.9)$$

we have

$$T\tilde{x} = O(\gamma) \text{ uniformly}, \qquad (4.10)$$

$$T^*_{\tilde{x}}\ \tilde{v} = O[\epsilon^{-1}\gamma\ \tilde{u}_2(t,\epsilon)] \text{ uniformly}, \qquad (4.11)$$

$$\ell_1 - C[\tilde{x}] = O[\gamma\Upsilon_3\Upsilon_1^{-1}(1+\Upsilon_o|b_1|)^{-1}], \qquad (4.12)$$

and

$$\ell_o - B[\tilde{x} + \tilde{v}] + b_2 \tilde{v}(o,\epsilon) = O(\gamma T_3 T_0^{-1}). \qquad (4.13)$$

Suppose that

$$\tilde{z} = \tilde{z}(t,\epsilon) = \tilde{x}(t,\epsilon) + e^{-t/\epsilon} \tilde{v}(t,\epsilon) \qquad (4.14)$$

and the vector $d(t,\epsilon)$ is defined as in 2.57. However, assume that the domain $\mathcal{D}$ is now defined to be all the vectors $x(t,\epsilon)$ in $C(I)$ which satisfy $|x_1(t,\epsilon)-\tilde{z}(t,\epsilon)| \leq d_1(t,\epsilon)$ and $|x_2(t,\epsilon)-\epsilon\tilde{z}'(t,\epsilon)| \leq d_2(t,\epsilon)$ for all t in I and $o < \epsilon < \epsilon_o$. Choose

$$\sigma(t,\epsilon) = (\epsilon + e^{-t/\epsilon})\, \sigma_o , \qquad (4.15)$$

where σ_o is a positive constant, and suppose that

$$\mu > \sigma_o . \qquad (4.16)$$

For the vector $y^{\dagger} = (y_1, y_2)$, define a scalar function f by $f(t,y,\epsilon) = h(t,y_1,\epsilon^{-1}y_2, \epsilon)$.

<u>Assumption 4iii.</u> $f(t,y,\epsilon)$ satisfies Assumption 3iv only with $\mathcal{D}$, σ, and μ as just defined.

It is not necessary to assume $f(t,o,\epsilon) = h(t,o,o,\epsilon) = 0$, which is part of assumption 3iv. However, this is not an important point.

If we now substitute

$$x = w + \tilde{z} \qquad (4.17)$$

into $Tx = 0$, we get the new equation

$$Tx = Pw + T\tilde{z} + Q(w+\tilde{z}) - Q\tilde{z} = 0 \tag{4.18}$$

in the unknown w. Similarly, we may obtain two boundary conditions for w from the boundary conditions 4.2 and 4.3. Theorem 3.1 will be used to prove that this new problem in the unknown w has a solution, which has a bound proportional to $\gamma(\epsilon)$.

Theorem 4.1. Suppose that assumptions 4i to 4iii hold. If

$$T_o T_1 \; e^{-1/\epsilon} = o(1), \tag{4.19}$$

$$\epsilon \; T_1 T_2 \; (1 + T_o |b_1|) = o(1), \tag{4.20}$$

$$\epsilon^k \; T_o^{-1} = O(1), \text{ for some } k, \tag{4.21}$$

$$T_o T_1 \; |c_1 - c_2| \; e^{-1/\epsilon} = O(1), \tag{4.22}$$

$$T_3^2 \, \gamma = O(1), \tag{4.23}$$

and

$$\gamma T_3 = o(1), \tag{4.24}$$

then for ϵ sufficiently small the problem 4.1, 4.2, 4.3 has a unique solution x and

$$\begin{pmatrix} x - \tilde{z} \\ \\ \epsilon x' - \epsilon \tilde{z}' \end{pmatrix} = O \; [\gamma(\epsilon) T_3(\epsilon) \; \tilde{u}(t,\epsilon)] \;\; \text{uniformly}, \tag{4.25}$$

where $\tilde{z}$, T_3, and $\tilde{u}$ are defined in 4.13, 4.7, and 4.8 respectively.

Proof. In the application of Theorem 3.1, let

$$y = \begin{pmatrix} w \\ \epsilon w' \end{pmatrix} \;, \tag{4.26}$$

$$A(t,\epsilon) = \begin{pmatrix} 0 & -1 \\ \epsilon q(t,\epsilon) & 1 \end{pmatrix}, \tag{4.27}$$

$$g(t,\epsilon) = \begin{pmatrix} 0 \\ T\tilde{z}(t,\epsilon) \end{pmatrix} \quad , \tag{4.28}$$

$$f(t,y,\epsilon) = \begin{pmatrix} 0 \\ h(t,y_1+\tilde{z},\epsilon^{-1}y_2+\tilde{z}',\epsilon)-h(t,\tilde{z},\tilde{z}',\epsilon) \end{pmatrix} \quad , \tag{4.29}$$

$$\ell_0^*(\epsilon) = \ell_0 - B[\tilde{z}], \tag{4.30}$$

$$\ell_1^*(\epsilon) = \ell_1 - C[\tilde{z}]. \tag{4.31}$$

In the notation of section 3, we get here that

$$\psi'(t,\epsilon) = \beta(t,\epsilon) = -1$$

and

$$\zeta'(t,\epsilon) = \alpha(t,\epsilon) = -\epsilon^{-1} \int_0^t q(s,\epsilon)\exp\left(\frac{s-t}{\epsilon}\right)ds \ .$$

Thus, T_i $(i = 0,1,2)$ and $\tilde{u}(t,\epsilon)$ defined in 4.7 and 4.8 are identified with $T_i (i = 0, 1, 2)$ and $\tilde{u}(t,\epsilon)$ of section 3. It follows that relations 4.19 through 4.22 correspond to relations 3.11, 3.25, 3.19, and 3.18 respectively.

We will show next that 4.23 corresponds to 3.24 by proving $\Upsilon = O(T_3\gamma)$. From 4.31, 4.12, and 4.9, we obtain

$$T_1(1+T_o|b_1|)\ell_1^* \leq O(\gamma T_3) + |(c_1+c_2)\tilde{u}(1,\epsilon) -$$

$$\epsilon c_2\, \tilde{u}'(1,\epsilon)\, e^{-1/\epsilon}\, T_1(1+T_o|b_1|) = O(\gamma T_3) +$$

$$O[\epsilon^{-1}e^{-1/\epsilon}\, T_1T_2\, (1+T_o|b_1|\ )] = O(\gamma T_3)\ .$$

4.30 and 4.13 imply

$$T_o\ \ell_o^* = O(\gamma T_3)\ .$$

For $g(t,\epsilon)$ as defined in 4.28, we get

$$||g(t,\epsilon)|| \leq |T\tilde{z}| \leq |T\tilde{x}| + e^{-t/\epsilon}\ |T_{\tilde{x}}^*\ \tilde{v}\ |\ ,$$

and hence, from 4.10 and 4.11

$$||g(t,\epsilon)|| = (1+\epsilon^{-1}\ e^{-2t/\epsilon})\ O(\gamma).$$

It follows that a constant multiple of $\gamma(\epsilon)$ may be taken for $\tilde{g}(\epsilon)$ in assumption 3iii. From the definition of T given by 3.21, we see that $T = O(T_3\gamma)$.

There remains to show that f as defined by 4.29 satisfies assumption 3iv. One obtains $f(t, o, \epsilon) = 0$ immediately. Assumption 4iii implies that

$$||f(t,y,\epsilon) - f(t,z,\epsilon)|| \leq \epsilon^{-1}\ \omega\ [(\epsilon\nu+\sigma)|y_1-z_1| + |y_2-z_2|],$$

when y and z are in $\mathcal{D}$ as defined in this section. Since $T(\epsilon) = O(\gamma T_3)$ and $\gamma T_3 = o(1)$ by 4.24, we have $T(\epsilon) = o(1)$, and hence, for sufficiently small ϵ,

$$\mu \geq \sigma(t,\epsilon) \geq T(\epsilon)\quad .$$

The assumptions of Theorem 3.1 hold for the present problem, and we conclude that there exists a unique vector

$$y = \begin{pmatrix} w \\ \epsilon w' \end{pmatrix} = \begin{pmatrix} x - \tilde{z} \\ \epsilon x' - \epsilon \tilde{z}' \end{pmatrix} ,$$

where x is a solution to the problem 4.1, 4.2, 4.3. Relation 4.25 follows from 3.26 of Theorem 3.1 and $T = O(\gamma T_3)$.

5. Theorem 4.1 reduces the problem of existence and asymptotic expansion of a solution of the second order equation $Tx = 0$ with two boundary conditions to a problem of finding asymptotic expansions $\tilde{x}$ and $\tilde{v}$ of certain solutions of the second order equations $Tx = 0$ and $T^*_{\tilde{x}} v = 0$ each with one boundary condition. We will determine analytic expressions that have the properties of $\tilde{x}$ and $\tilde{v}$ by a method involving only regular perturbation procedures for weakly non-linear first order equations with one boundary condition.

Because there exists only one boundary condition to satisfy, one is at first tempted to treat the higher order terms $\epsilon x''$ and $\epsilon v''$ as part of the perturbations in the successive approximation schemes. This will require that the weakly nonlinear term, which will also occur in the perturbation term, be analytic in all its variables. Furthermore, all the derivatives need to be bounded uniformly for $o < \epsilon < \epsilon_o$. Such cannot be the case for very general circumstances for the weakly non-linear term in the equation $T^*_{\tilde{x}} v = 0$, because of the presence of the factor $e^{-t/\epsilon}$.

Hence, we shall proceed in a different manner. Multiply $Tx = 0$ by an integrating factor for $\epsilon x'' + x'$ and then integrate between zero and t, setting $x'(o,\epsilon) = 0$. This gives

$$\epsilon e^{t/\epsilon} x' + \int_o^t e^{s/\epsilon} q(s,\epsilon)x(s,\epsilon)ds + \int_o^t e^{s/\epsilon} r(s,\epsilon)ds +$$

$$\epsilon \int_o^t e^{s/\epsilon} h(s,x(s,\epsilon),x'(s,\epsilon),\epsilon)ds = 0. \qquad (5.1)$$

Integration by parts of the second term in equation 5.1 and subsequent multiplication of the equation by $\epsilon^{-1}\, e^{-t/\epsilon}$ produces the integro-differential equation

$$x'+p(t,\epsilon)x+\epsilon^{-1}\int_0^t r(s,\epsilon)\exp\left(\frac{s-t}{\epsilon}\right) ds +$$

$$\int_0^t [h(s,x(s,\epsilon),x'(s,\epsilon),\epsilon) - p(s,\epsilon)x'(s,\epsilon)]\exp\left(\frac{s-t}{\epsilon}\right)ds = 0, \qquad (5.2)$$

where

$$p(t,\epsilon) = \epsilon^{-1}\int_0^t q(s,\epsilon)\exp\left(\frac{s-t}{\epsilon}\right)ds \ . \qquad (5.3)$$

Equation 5.2 will be called the perturbation equation for $Tx = 0$.

We formally substitute

$$x = \tilde{x}(t,\epsilon) = \sum_{k=0}^{\infty} \epsilon^k x_k(t,\epsilon)$$

into the perturbation equation 5.2, and set successively

$$x_k' + p(t,\epsilon)x_k + \epsilon^{-1}\int_0^t Q_{k-1}(s,\epsilon)\exp\left(\frac{s-t}{\epsilon}\right) ds = 0,$$

$$k = 0,\ 1,\ \dots , \qquad (5.4)$$

where

$$Q_{-1}(s,\epsilon) = r(s,\epsilon),$$

$$Q_k(s,\epsilon) = \epsilon^{-k}\,[h(s,\tilde{x}_k,\tilde{x}_k', \epsilon) - h(s,\tilde{x}_{k-1},\tilde{x}_{k-1}',\epsilon)] - p(s,\epsilon)x_k' ,$$

$$k = 0,\ 1,\ \dots , \qquad (5.5)$$

and

$$\tilde{x}_k = \sum_{i=0}^{k} \epsilon^i\, x_i \ . \qquad (5.6)$$

The constants of integration appearing in the general solutions of the equations comprising 5.4 represent exactly one degree of freedom for $\tilde{x}$. We will use this freedom to determine $\tilde{x}(t,\epsilon)$ at $t = 1$ so that

$$C[x_o] = \ell_1 \quad \text{and} \quad C[x_k] = 0, \quad k = 1, 2, \ldots \quad . \tag{5.7}$$

The solutions, if they exist, to the equations in 5.4 subject to the boundary conditions 5.7 are given by

$$x_k(t,\epsilon) = A_k \exp\left(\int_t^1 p(s,\epsilon)ds \right) +$$

$$\epsilon^{-1} \int_t^1 \exp\left(\int_t^s p(r,\epsilon)dr \right) \int_o^s Q_{k-1}(r,\epsilon)\exp\left(\frac{r-s}{\epsilon}\right)dr\, ds,$$

$$k = 0, 1, \ldots , \tag{5.8}$$

where

$$A_o = [\ell_1 + c_2 \int_o^1 r(s,\epsilon)\exp\left(\frac{s-1}{\epsilon}\right)ds][c_1 - \epsilon c_2 p(1,\epsilon)]^{-1}, \tag{5.9}$$

$$A_k = [c_1 - \epsilon c_2 p(1,\epsilon)]^{-1} \, c_2 \int_o^1 Q_{k-1}(s,\epsilon)\exp\left(\frac{s-1}{\epsilon}\right)ds,$$

$$k = 1, 2, \ldots \quad . \tag{5.10}$$

Also,

$$T\, \tilde{x}_n(t,\epsilon) = \epsilon^{n+1}\, Q_n(t,\epsilon), \quad n = o, 1, \ldots \quad . \tag{5.11}$$

Suppose that n is specified and that $\tilde{x}_n(t,\epsilon)$ exists. Then, a formal expansion

$$\tilde{v}_n(t,\epsilon) = \sum_{k=o}^{n} \epsilon^k\, v_{nk}(t,\epsilon)$$

for v in equation $T_{\tilde{x}_n} v = 0$ and boundary condition 4.2 may be constructed in the same way as $\tilde{x}_n$ was devised. For brevity, we will write

$$v_k(t,\epsilon) = v_{nk}(t,\epsilon).$$

If

$$\tilde{x}_\infty = \lim_{n \to \infty} \tilde{x}_n ,$$

exists, then

$$\tilde{v}_\infty = \lim_{n \to \infty} \sum_{k=0}^{n} v_{\infty k} (t,\epsilon)$$

will represent an asymptotic expansion. We will state conditions under which $\tilde{x}_n$ and $\tilde{v}_n$ exist for all n and assume the role of $\tilde{x}$ and $\tilde{v}$ for $\gamma = \epsilon^{n+1}$ in section 4. These same conditions will imply that $\tilde{x}_\infty$ and $\tilde{v}_\infty$ exist.

Suppose that $\mathcal{D}'$ is the set of continuous vectors $y(t,\epsilon) = (y_1,y_2)^\dagger$ on I such that

$$|y_1(t,\epsilon) - x_o(t,\epsilon)| \leq \mu ,$$

$$|y_2(t,\epsilon) - x_o'(t,\epsilon)| \leq \pi + \epsilon^{-1} \sigma(t,\epsilon) =$$

$$\pi + \sigma_o \epsilon^{-1} \tilde{u}_2 (t,\epsilon) \quad (5.12)$$

for given constants μ, π, and σ_o, which satisfy

$$\mu > \sigma_o > 0 \quad \text{and} \quad \pi > 0. \quad (5.13)$$

Assumption 5i. For $0 < \epsilon < \epsilon_o$, $h(t,x,x',\epsilon)$ is of class C^2 in x and x' and of class C in t, when $(x,x')^\dagger \in \mathcal{D}'$ and $t \in I$;

$h(t,x_o(t,\epsilon),x_o'(t,\epsilon),\epsilon) = O(1)$ uniformly; and $\frac{\partial h}{\partial x} = O(1)+O(x'-x_o)$, $\frac{\partial h}{\partial x'} = O(1)$, $\frac{\partial^2 h}{\partial x \partial x'} = O(1)$, and $\frac{\partial^2 h}{\partial x'^2} = O(\epsilon)$ uniformly when $(x,x')^{\dagger} \in \mathcal{B}'$ and $t \in I$.

Let μ_1 be a fixed positive constant not larger than $\min(\mu,\pi+\sigma_o)$. Two applications of the mean value theorem and use of assumption 5i give

$$|h(t,x,x',\epsilon) - h(t,z,z',\epsilon)| \leq \omega_1(|x-z| + |x'-z'|), \qquad (5.14)$$

when $|x-x_o| + |x' - x_o'|$ and $|z-z_o| + |z'-z_o'|$ are not greater than μ_1. ω_1 is a constant whose value depends in general on μ_1.

<u>Lemma 5.1</u> Let assumptions 4i and 5i hold. If

$$T_1(1+\epsilon|c_2|)(|\ell_1| + \epsilon\, |c_2|) = O(1), \qquad (5.15)$$

then for sufficiently small ϵ and $n = 0, 1, \ldots$, $\tilde{x}_n(t,\epsilon)$ exists on I, $\tilde{x}_n(t,\epsilon) = \sum_{k=0}^{n} \epsilon^k x_k(t,\epsilon)$ where $x_k(t,\epsilon)$ is defined by equation 5.8, a function $\tilde{x}(t,\epsilon)$ in $C^2(I)$ exists so that $\lim_{n\to\infty} \tilde{x}_n(t,\epsilon)= \tilde{x}(t,\epsilon)$ and $\lim_{n\to\infty} \tilde{x}_n'(t,\epsilon) = \tilde{x}'(t,\epsilon)$ uniformly for all t in I,

$$|\tilde{x}_n(t,\epsilon)-x_o(t,\epsilon)|+|\tilde{x}_n'(t,\epsilon)-x_o'(t,\epsilon)| \leq \mu_1, \qquad (5.16)$$

$$|\tilde{x}(t,\epsilon)-x_o(t,\epsilon)|+|\tilde{x}'(t,\epsilon)-x_o'(t,\epsilon)| \leq \mu_1, \qquad (5.17)$$

$$T\tilde{x}_n(t,\epsilon) = O(\epsilon^{n+1}) \text{ uniformly for } t \text{ in } I, \qquad (5.18)$$

and

$$T\tilde{x}(t,\epsilon) = 0. \qquad (5.19)$$

<u>Proof</u>. For any function $Q(t,\epsilon)$ define $\bar{Q}$ by

$$\bar{Q} = \sup_{t\in I} ||Q(t,\epsilon)||. \tag{5.20}$$

From equation 5.8 and 5.9 we obtain

$$|x_o(t,\epsilon)| \leq (|\ell_1|+|\epsilon c_2|)\ RT_1 \tag{5.21}$$

for some constant R. Hence,

$$|x_o'(t,\epsilon)| \leq (|\ell_1|+|\epsilon c_2|)RT_1\ \bar{p} + \bar{r}$$

follows from equation 5.4, and so

$$\bar{Q}_o \leq \bar{p}\ \bar{x}_o' + \bar{h}(t,x_o,x_o',\epsilon) = O(1) \tag{5.22}$$

by equation 5.15 and $h(t,x_o,x_o',\epsilon) = O(1)$ uniformly.

Define

$$Q = Q(\epsilon) = 1 + (1+\bar{p})(1+T_1|\epsilon c_2|) \sup_{o\leq t\leq s\leq 1} \exp(\int_t^s p(r,\epsilon)dr). \tag{5.23}$$

We will show by induction that for ϵ sufficiently small and $n = 0,1,\ldots$

$$\bar{Q}_n \leq Q^n(\omega_1+\bar{p})^n\ \bar{Q}_o, \tag{5.24}$$

and that equation 5.16 holds.

Suppose that equations 5.16 and 5.24 hold for $n \leq k$ and $k \geq 0$. Then,

$$|x_{i+1}|+|x_{i+1}'| \leq Q\bar{Q}_i \leq (\omega_1+\bar{p})^i\ Q^{i+1}\ \bar{Q}_o,\ i = 0, \cdots ,k, \tag{5.25}$$

and

$$|\tilde{x}_{k+1}-x_o| + |\tilde{x}_{k+1}'-x_o'| \leq \sum_{i=1}^{k+1} \epsilon^i(|x_i|+|x_i'|) \leq$$

$$\epsilon\bar{Q}_o\ Q\ [1-\epsilon(\omega_1 + \bar{p})Q]^{-1} \leq \mu_1\ ,$$

if ϵ_o is fixed small enough to make

$$2\epsilon Q \max(\omega_1+\bar{p},\ \mu_1^{-1}Q_o) \leq 1,\ o < \epsilon < \epsilon_o .$$

ϵ_o can be so chosen because ϵQ and $\epsilon Q\bar{Q}_o$ are $O(\epsilon)$ by equation 5.15 and $h(t,x_o,x_o',\epsilon) = O(1)$ uniformly. We conclude that $Q_{k+1}(t,\epsilon)$ is defined, and

$$|Q_{k+1}| \leq (\omega_1+|p|)(|x_{k+1}| + |x'_{k+1}|).$$

Equation 5.24 for $n = k + 1$ follows now from equation 5.14. This completes the induction proof.

$T\,\tilde{x}_n = O(\epsilon^{n+1})$ uniformly for t in I follows from equation 5.11 and $Q_n = O(1)$ uniformly. We obtain

$$|\tilde{x}_{i+1} - \tilde{x}_i| + |\tilde{x}'_{i+1} - \tilde{x}'_i| \leq \epsilon^{i+1}(|x_{i+1}| + |x'_{i+1}|) \leq$$

$$(\omega_1+\bar{p})^i\ (\epsilon Q)^{i+1}\ \bar{Q}_o$$

from equation 5.25. Because $2\ \epsilon\ Q(\omega_1+p) \leq 1$ for $o < \epsilon < \epsilon_o$, the infinite series $\sum_{i=o}^{+\infty} (|\tilde{x}_{i+1}-\tilde{x}_i|+|\tilde{x}'_{i+1}-\tilde{x}'_i|)$ converges uniformly for t in I and $o < \epsilon < \epsilon_o$ to a sum less than μ_1. It follows that there exists a function $\tilde{x}(t,\epsilon)$ in $C^1(I)$ such that

$$\lim_{n\to\infty} \tilde{x}_n(t,\epsilon) = \tilde{x}(t,\epsilon) \text{ and } \lim_{n\to\infty} \tilde{x}'_n(t,\epsilon) = \tilde{x}'(t,\epsilon) \text{ uniformly.}$$

From equation 5.11, we see that $\epsilon\tilde{x}_n''$ is a linear combination of $\tilde{x}'_n$, $\tilde{x}_n$, and $\epsilon^{n+1}\,Q_n$. Because $\lim_{n\to\infty} \epsilon^{n+1}\,\bar{Q}_n = 0$ uniformly for $o < \epsilon < \epsilon_o$, we conclude that $\epsilon x_n''$ converges uniformly for t in I and $o < \epsilon < \epsilon_o$ to a function which must be $\tilde{x}''(t,\epsilon)$. Hence,

$$\lim_{n\to\infty} T\tilde{x}_n(t,\epsilon) = T\lim_{n\to\infty}\tilde{x}_n(t,\epsilon) = T\tilde{x}(t,\epsilon) = 0.$$

The proof of the lemma is now complete.

For n fixed, $n = 0, 1, \ldots, \infty$, we will consider next $\tilde{v}_n(t,\epsilon)$. The convention $\tilde{x}_\infty(t,\epsilon) = \tilde{x}(t,\epsilon)$ is used in what follows. We write $T^*_{\tilde{x}_n}$ in the form

$$\epsilon v'' - v' + q^*(t,\epsilon)v + \epsilon h^*(t,v,v',\epsilon),$$

where

$$q^*(t,\epsilon) = q(t,\epsilon) - \frac{\partial h}{\partial x'}(t,x_o(t,\epsilon), x_o'(t,\epsilon),\epsilon),$$

$$h^*(t,v,v',\epsilon) = e^{t/\epsilon}[h(t,\tilde{x}_n+ve^{-t/\epsilon},\tilde{x}_n' +$$

$$v'e^{-t/\epsilon}-\epsilon^{-1}ve^{-t/\epsilon},\epsilon) - h(t,\tilde{x}_n,\tilde{x}_n',\epsilon)] +$$

$$\epsilon^{-1}v\frac{\partial h}{\partial x'}(t,x_o,x_o',\epsilon). \tag{5.26}$$

Set

$$v_k'-p^*(t,\epsilon)v_k - \epsilon^{-1}\int_t^1 Q^*_{k-1}(s,\epsilon)\exp\left(\frac{t-s}{\epsilon}\right)ds = 0,$$

$$k = 0, 1, \ldots, n, \tag{5.27}$$

and

$$B[v_o] - b_2v_o(o,\epsilon) = \ell_o - B[\tilde{x}_n], \; B[v_k]-b_2v_k(o,\epsilon) = 0,$$

$$k = 1, \ldots, n, \tag{5.28}$$

where

$$p^*(t,\epsilon) = \epsilon^{-1} \int_t^1 q^*(s,\epsilon)\exp(\tfrac{t-s}{\epsilon})\, ds,$$

$$Q^*_{-1}(t,\epsilon) = 0,$$

$$Q^*_k(t,\epsilon) = p^*(t,\epsilon)\, v_k'(t,\epsilon)+\epsilon^{-k}[h^*(t,\tilde{v}_{nk},\tilde{v}'_{nk},\epsilon)$$

$$- h^*(t,\tilde{v}_{n,k-1},\tilde{v}'_{n,k-1},\epsilon)], \quad k = 0, 1, \ldots , n.$$

Here

$$\tilde{v}_{nk} = \sum_{i=o}^{k} \epsilon^i v_{ni} = \sum_{i=o}^{k} \epsilon^i v_i, \; k = 0, \ldots , n; \text{ and } \tilde{v}_{n,-1} = 0.$$

This will produce

$$T^*_{\tilde{x}_n} \tilde{v}_n = \epsilon^{n+1} Q^*_n \quad . \tag{5.29}$$

The solution of the system of differential equations 5.27 subject to the boundary conditions 5.28 is given by

$$v_k(t,\epsilon)=B_k \exp(\int_o^t p^*(s,\epsilon)ds) +$$

$$\epsilon^{-1}\int_o^t [\exp(\int_s^t p^*(r,\epsilon)dr)] \int_s^1 Q^*_{k-1} (r,\epsilon)\exp(\tfrac{s-r}{\epsilon})dr\, ds ,$$

$$k = 0, 1, \ldots , n, \tag{5.30}$$

where

$$B_o = [b_1-b_2+\epsilon b_2 p^*(o,\epsilon)]^{-1}[\ell_o-b_1\tilde{x}_n(o,\epsilon)],$$

$$B_k = [b_1-b_2+\epsilon b_2 p^*(o,\epsilon)]^{-1} \int_o^1 Q^*_{k-1}(s,\epsilon)e^{-s/\epsilon}\, ds,$$

$$k = 1, \ldots , n. \tag{5.31}$$

Because of assumption 5i we obtain upon two applications of the mean value theorem the existence of a constant ω_2 so that

$$|h^*(t,v,v',\epsilon)-h^*(t,w,w',\epsilon)| \leq \omega_2[|v'-w'|$$
$$+|v-w|\epsilon^{-1}\tilde{u}_2(t,\epsilon)], \qquad (5.32)$$

when $v, w \in C^1(I)$ and v, w satisfy

$$|\tilde{x}_n-x_o+xe^{-t/\epsilon}| \leq \mu , \quad |\tilde{x}'_n-x'_o +$$
$$(\epsilon x'-x)\epsilon^{-1}e^{-t/\epsilon}| \leq \pi + \epsilon^{-1}\sigma_o\tilde{u}_2(t,\epsilon) \qquad (5.33)$$

for x. $|\tilde{x}_n-x_o|$ and $|\tilde{x}'_n-x'_o|$ may be made arbitrarily small uniformly in n by choosing μ_1 of Lemma 5.1 small. Hence, the inequalities 5.33 hold for all ϵ and μ_1 sufficiently small, when v and w are $C^1(I)$ and satisfy for x,

$$|xe^{-t/\epsilon}| \leq \mu-\mu_1 \quad \text{and} \quad |\epsilon x'-x|\ \epsilon^{-1}e^{-t/\epsilon} \leq \pi-\mu_1 + \epsilon^{-1}\sigma_o\tilde{u}_2. \qquad (5.34)$$

<u>Assumption</u> <u>5ii</u>. Let $v_o(t,\epsilon)$ be defined by equation 5.30. For any positive number α, there exists a number $\epsilon_o(\alpha)$ such that

$$|v_o(t,\epsilon)| + \alpha\epsilon \leq \min\ [(\mu-\mu_1)e^{t/\epsilon},\ \sigma_o],$$

when $t \in I$ and $o < \epsilon < \epsilon_o\ (\alpha)$.

Define

$$T_4 = T_4(\epsilon) = |b_1-b_2+\epsilon b_2\ p^*(o,\epsilon)|^{-1}\ . \qquad (5.35)$$

<u>Lemma</u> <u>5.2.</u> Suppose that assumptions 4i, 5i, and 5ii and equation 5.15 hold. If

$$(1+|\ell_o| + |b_1|)\ T_4 = O(1), \qquad (5.36)$$

then for sufficiently small ϵ, $\tilde{v}_n(t,\epsilon)$ exists on I, $\tilde{v}_n(t,\epsilon) = \sum_{k=o}^{n} \epsilon^k v_k(t,\epsilon)$ where $v_k(t,\epsilon)$ depends upon n and is given by 5.30, and

$$T^*_{\tilde{x}_n} \tilde{v}_n (t,\epsilon) = O[\epsilon^m \tilde{u}_2(t,\epsilon)] \text{ uniformly.} \tag{5.37}$$

Proof. One can show from the definition of h* and from equations 5.15 and 5.36, which imply that $v_o(t,\epsilon)$ and $v_o'(t,\epsilon)$ are $O(1)$ uniformly, that

$$h^*(t,v_o(t,\epsilon),\ v_o'(t,\epsilon),\epsilon) = O[\epsilon^{-1}\tilde{u}_2(t,\epsilon)] \text{ uniformly,}$$

by using the mean value theorem twice and assumption 5i. It follows from the definition of Q_o^* that $Q_o^* \leq \tilde{Q}\,\epsilon^{-1}\,\tilde{u}_2$ uniformly, where $\tilde{Q}$ is a constant independent of ϵ and n.

Define

$$\psi^* = \sup_{o\leq s\leq t\leq 1} \exp \int_s^t p^*(r,\epsilon)dr$$

and

$$Q^* = Q^*(\epsilon) = (1+\bar{p}^*)(2T_4+3)\psi^* .$$

The bar in $\bar{p}^*$ and in what follows has the meaning attached to it in equation 5.20. $Q^* = O(1)$, because $T_4 = O(1)$ by assumption.

We will sketch the induction proof that for ϵ sufficiently small and $k = 0, \dots , n$

$$|\tilde{v}_{nk}(t,\epsilon)e^{-t/\epsilon}| \leq \mu - \mu_1,\ |\epsilon\tilde{v}'_{nk}(t,\epsilon) - \tilde{v}_{nk}(t,\epsilon)|e^{-t/\epsilon} \leq (\pi-\mu_1)\epsilon + \sigma_o\,\tilde{u}_2(t,\epsilon), \text{ and}$$

$$|Q_k^*(t,\epsilon)| \leq (\omega_2+\bar{p}^*)^k\, Q^{*k}\,\tilde{Q}\,\epsilon^{-1}\,\tilde{u}_2(t,\epsilon). \tag{5.38}$$

The existence of $\tilde{v}_n = \tilde{v}_{nn}$ follows from equation 5.38. Furthermore, $Q^*_n(t,\epsilon) = O(Q^{*n}\, \tilde{Q}\, \epsilon^{-1}\, \tilde{u}_2) = O(\epsilon^{-1}\tilde{u}_2)$ uniformly for t in I and equation 5.29 imply equation 5.37 of the lemma.

The first step in getting from $i \leq k$ to $i \leq k+1$ $(k \geq 0)$ in the induction consists of using equation 5.30 and the induction hypothesis to obtain

$$|v_{i+1}| \leq \psi^* T_4 \int_0^1 |Q^*_i(s,\epsilon)| e^{-s/\epsilon}\, ds + \psi^* \int_0^t \sup_{s \leq r \leq 1} |Q^*_i(u,\epsilon)|\, ds$$

$$\leq (2T_4 + 2)\psi^* \tilde{Q}\, Q^{*i}\, (\omega_2 + \bar{p})^i, \qquad i = 0, \ldots, k. \qquad (5.39)$$

Thus, by assumption 5ii

$$|\tilde{v}_{n,k+1}| \leq |v_o| + \sum_{i=o}^{k} \epsilon^{i+1} |v_{i+1}| \leq |v_o| +$$

$$2(T_4+1)\psi^* \tilde{Q}\ \epsilon\ [1 - \epsilon Q^*(\omega_2 + \bar{p}^*)]^{-1} \leq v_o +$$

$$O(\epsilon) \leq e^{t/\epsilon}(\mu - \mu_1) \qquad (5.40)$$

for sufficiently small ϵ independent of k and n. The bound obtained in equation 5.39 for v_{i+1} and the first order differential equation satisfied by v_{i+1} give

$$|v'_{i+1}| \leq \bar{p}^* |v_{i+1}| + \sup_{t \leq s \leq 1} |Q^*_i(s,\epsilon)| \leq$$

$$(\omega_2 + \bar{p}^*)^i\ Q^{*i}\ \tilde{Q}\ [\epsilon^{-1}\tilde{u}_2(t,\epsilon) + 2(1+T_4)\bar{p}^*\ \psi^*],$$

$$i = 0, 1, \ldots, k. \qquad (5.41)$$

It follows by assumption 5ii that

$$| \epsilon\tilde{v}'_{n,k+1} - \tilde{v}_{n,k+1}| \leq |\tilde{v}_{n,k+1}| + |\epsilon\tilde{v}'_{n,k+1}| \leq$$

$$|v_o| + O(\epsilon) \leq \sigma_o ,$$

for sufficiently small ϵ independent of n and k. Hence, $Q^*_{k+1}(t,\epsilon)$ exists, and

$$|Q^*_{k+1}(t,\epsilon)| \leq (\omega_2+\bar{p}^*)(1+\bar{p}^*)\ \psi^*\ [\epsilon^{-1} \int_t^1 |Q^*_k(s,\epsilon)| \exp(\frac{t-s}{\epsilon})\ ds$$

$$+ \epsilon^{-1}\ \tilde{u}_2(t,\epsilon)(T_4 \int_o^1 |Q^*_k(s,\epsilon)|\ e^{-s/\epsilon}\ ds +$$

$$\int_o^t \sup_{s\leq r\leq 1} |Q^*_k(r,\epsilon)| ds)] \leq (\omega_2+\bar{p}^*)(1+$$

$$\bar{p}^*)\ \psi^*[\sup_{t\leq s\leq 1} |Q^*_k(s,\epsilon)| + \epsilon^{-1}\ \tilde{u}_2(t,\epsilon)(T_4\ \epsilon\ \bar{Q}^*_k +$$

$$\int_o^t \sup_{s\leq r\leq 1} |Q^*_k(s,\epsilon)| e^{-s/\epsilon}\ ds)] .$$

The complete induction statement for $k + 1$ follows.

The lemma is essentially proven if n is finite. For $n = \infty$ the additional problem of considering the convergence of $\tilde{v}_\infty (t,\epsilon)$ gives no difficulty, because equations 5.40 and 5.41 are then valid for $i = 0,1,...$. Both $\sum_{i=o}^{+\infty} \epsilon^i\ v_i(t,\epsilon)$ and $\sum_{i=o}^{+\infty} \epsilon^i\ v_i'(t,\epsilon)$ are uniformly convergent in I. Hence, equation 5.29 gives that $\epsilon\ \tilde{v}''_{\infty k}(t,\epsilon)$ converges uniformly in I as $k \to \infty$. The end result is that there exists a function $\tilde{v}(t,\epsilon) = \tilde{v}_\infty(t,\epsilon)$ in $C^2(I)$ such that

$$T^*_{\tilde{x}_\infty} \tilde{v} = 0 .$$

Theorem 5.1. Suppose that assumptions 4i, 5i, 5ii and equations 4.19 to 4.22, 5.15, 5.36 hold. Let n be specified, $n = 0, 1, \ldots, +\infty$, and $\tilde{x}_n(t,\epsilon)$ and $\tilde{v}_n(t,\epsilon)$ be given by Lemmas 5.1 and 5.2 respectively. Choose $\gamma(\epsilon) = \epsilon^{n+1}$ if n is finite and $\gamma(\epsilon) = e^{-1/\epsilon}$ if n is infinite. If

$$(1+T_o\ |b_1|)(|c_1| + T_oT_2\ |b_1|)\ T_1^2\ T_2\ \gamma = O(1), \tag{5.42}$$

then for ϵ sufficiently small a unique solution $x = x(t,\epsilon)$ exists for equation 4.1 and boundary conditions 4.2 and 4.3, and

$$x(t,\epsilon) = \tilde{x}_n(t,\epsilon) + e^{-t/\epsilon}\ \tilde{v}_n(t,\epsilon) + O(\gamma T_3)$$

uniformly for t in I, (5.43)

$$x'(t,\epsilon) = \tilde{x}'_n(t,\epsilon) + e^{-t/\epsilon}\ \tilde{v}'_n(t,\epsilon) - \epsilon^{-1}e^{-t/\epsilon}\ \tilde{v}_n(t,\epsilon)$$

$+ O[\epsilon^{-1}\ \gamma T_3\ \tilde{u}_2(t,\epsilon)]$ uniformly for t in I. (5.44)

Proof. This theorem is an immediate consequence of Lemmas 5.1 and 5.2 and Theorem 4.1. The choice for $\gamma(\epsilon)$ satisfies equation 4.9 and makes equation 4.24 a consequence of equation 4.20. Using equations 4.20 and 5.15 and the definition of T_2, we get that equation 5.42 implies the validity of equation 4.23 in the present case. Assumption 4ii holds here because of the conclusions of Lemmas 5.1 and 5.2 and the choice for γ. Assumption 4iii holds because of assumption 5i. This is shown by equations 5.32 and 5.34.

6. The results of the previous section will be reformulated in terms of the singular perturbation problem with differential equation

$$\epsilon y'' + F(t,y,y',\epsilon) = 0 \qquad (6.1)$$

and boundary conditions

$$B[y] = \tilde{\ell}_o, \quad C[y] = \tilde{\ell}_1 . \qquad (6.2)$$

We will state sufficient conditions to guarantee the existence of a solution y of the form

$$y = w + x\,\xi(\epsilon)$$

where the existence of x is given by the theory of section 5, and $\xi = \xi(\epsilon)$ is a positive function of ϵ such that

$$\xi = O(\epsilon). \qquad (6.3)$$

In what follows let the full argument $(t,y(t,\epsilon),y'(t,\epsilon),\epsilon)$ of F and its derivatives be denoted by $[y] = [y(t,\epsilon)]$.

<u>Assumption 6i</u>. There exists a function $w = w(t,\epsilon)$ of t and ϵ such that $w(t,\epsilon) \in C^2(I)$ and $\epsilon w'' + F[w] = O(\xi)$ uniformly.

<u>Assumption 6ii</u>. F is of class C^2 in y and y' when $|y-w(t,\epsilon)| + |y' - w(t,\epsilon)| \leq \tilde{\mu}$ for some constant $\tilde{\mu}$; $\frac{\partial^2 F}{\partial y^2}$, $\frac{\partial^2 F}{\partial y \partial y'}$, and $\epsilon^{-1}\frac{\partial^2 F}{\partial y'^2}$ $= O(\epsilon\xi^{-1})$ uniformly for $|y-w(t,\epsilon)|+|y'-w(t,\epsilon)| \leq \tilde{\mu}$ and t in I; $\frac{\partial F}{\partial y}[w] \in C(I)$ and $\frac{\partial F}{\partial y'}[w] \in C^1(I)$; $\frac{\partial F}{\partial y'}[w]$, $\frac{\partial F}{\partial y}[w]$, and $(\frac{\partial F}{\partial y'}[w])'$ $= O(1)$ uniformly; and $\frac{\partial F}{\partial y'}[w] \geq \eta > 0$ when $t \in I$ and $o < \epsilon < \epsilon_o$, for some constant η.

The functions in equations 4.1, 4.2, and 4.3 become in the notation of the present problem the following:

$$q(t,\epsilon) = (\frac{\partial F}{\partial y'}[w])^{-2} \frac{\partial F}{\partial y}[w],$$

$$r(t,\epsilon) = (\frac{\partial F}{\partial y'}[w])^{-2} (\epsilon w'' + F[w]) \xi^{-1},$$

$$h(t,x,x',\epsilon) = (\epsilon^{1/2} \xi^{1/2} \frac{\partial F}{\partial y'}[w])^{-2} \left\{ (\frac{\partial F}{\partial y'}[w])' \epsilon \xi x' + \right.$$

$$\left. F[w+\xi x] - F[w] - \frac{\partial F}{\partial y'}[w] \xi x' - \frac{\partial F}{\partial y}[w] \xi x \right\},$$

$$\ell_o = (\tilde{\ell}_o - B[w]) \xi^{-1},$$

$$\ell_1 = (\tilde{\ell}_1 - C[w]) \xi^{-1} . \qquad (6.4)$$

In the context of the present problem, functions q^*, p^*, x_o, v_o, and T_i $(i = 0, \ldots, 4)$ can be defined from the equations of 6.4 in the same way as found in sections 4 and 5.

Assumption 6iii. For ℓ_o and ℓ_1 defined in 6.4 and depending on w, equations 5.15 and 5.36 hold. Also, equations 4.19 to 4.22 hold.

Assumption 6iv. There exists a constant σ_o such that $|v_o(t,\epsilon)| < \sigma_o < \tilde{\mu} \epsilon \xi^{-1}$ when $t \in I$ and $o < \epsilon < \epsilon_o$.

Assumption 6iv is a requirement that $\tilde{\mu}$ is not too small. From this condition we get for any positive number α another number $\epsilon_o(\alpha)$ such that

$$|v_o(t,\epsilon)| + \alpha\epsilon \leq \sigma_o \qquad (6.5)$$

and

$$\sigma_o \leq (\tilde{\mu}-\alpha\xi) \epsilon \xi^{-1} e^{t/\epsilon}, \qquad (6.6)$$

when $t \in I$ and $o < \epsilon < \epsilon_o(\alpha)$.

Equation 6.5 is part of assumption 5ii, which must be verified to hold for the present problem. Assumption 5i holds for h defined in 6.4, if

$$|x| + |x'| \leq \tilde{\mu}\, \xi^{-1} .$$

Thus, assumption 5i holds for

$$|x-x_o| \leq \mu \qquad \text{and} \quad |x'-x_o'| \leq \pi + \epsilon^{-1}\sigma ,$$

if μ, π, and σ_o are determined so that

$$\mu + |x_o| + \pi + \epsilon^{-1}\sigma + |x_o'| \leq \tilde{\mu}\, \xi^{-1} \qquad (6.7)$$

when $t \in I$ and $o < \epsilon < \epsilon_o$. Let σ_o be determined by assumption 6iv, and then fix $\mu > \sigma_o$ and $\pi > 0$. Inequality 6.7 will result from inequality 6.6 when $\alpha \geq \mu + |x_o| + \pi + \sigma_o + |x_o'|$, because x_o and x_o' are bounded for t in I and $o < \epsilon < \epsilon_o$. This follows from equation 5.15, which is assumed to hold for the present problem. Assumptions 5i and 5ii have been shown to be valid.

We conclude that all the assumptions of Theorem 5.1 are true here. Hence, the following theorem has been proven.

Theorem 6.1. Suppose that assumptions 6i to 6iv hold, and let n be given ($n = o, 1, \ldots, \infty$). If n is finite, define $\gamma = \epsilon^{n+1}$, and if n is infinite, define $\gamma = e^{-1/\epsilon}$. Let $\tilde{z}_n(t,\epsilon) = \tilde{x}_n(t,\epsilon) + e^{-t/\epsilon}\, \tilde{v}_n(t,\epsilon)$ be given by Theorem 5.1 where the functions of the equations 4.1, 4.2, 4.3

are given in 6.4. Then, if $(1 + T_o |b_1|)(|c_1| + T_o T_2 |b_1|) T_1^2 T_2 \gamma = O(1)$, the problem consisting of equation 6.1 and boundary conditions 6.2 has a solution $y(t,\epsilon)$ for t in I and ϵ sufficiently small, and

$$y(t,\epsilon) = w(t,\epsilon) + \xi(\epsilon)\, \tilde{z}_n(t,\epsilon) + O(\xi\, \gamma\, T_3) \quad \text{uniformly,}$$

$$y'(t,\epsilon) = w'(t,\epsilon) + \xi(\epsilon)\, \tilde{z}_n'(t,\epsilon) + O[\epsilon^{-1} \xi\, \gamma\, T_3\, \tilde{u}_2(t,\epsilon)] \quad \text{uniformly.}$$

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