Topological Quantum Field Theory and the Geometric Langlands Correspondence

Thesis by

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Dedicated with love and gratitude to my parents.

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Abstract

In the pioneering work [1] of A. Kapustin and E. Witten, the geometric Langlands program of number theory was shown to be intimately related to duality of GL-twisted $\mathcal{N} = 4$ super Yang-Mills theory compactified on a Riemann surface. In this thesis, we generalize Kapustin-Witten by investigating compactification of the GL-twisted theory to three dimensions on a circle (for various values of the twisting parameter t). By considering boundary conditions in the three-dimensional description, we classify codimension-two surface operators of the GL-twisted theory, generalizing those surface operators studied [2] by S. Gukov and E. Witten. For $t = i$, we propose a complete description of the 2-category of surface operators in terms of module categories, and, in addition, we determine the monoidal category of line operators which includes Wilson lines as special objects. For $t = 1$ and $t = 0$, we discuss surface and line operators in the abelian case.

We generalize Kapustin-Witten also by analyzing a separate twisted version of $\mathcal{N} = 4$, the Vafa-Witten theory. After introducing a new four-dimensional topological gauge theory, the gauged 4d A-model, we locate the Vafa-Witten theory as a special case. Compactification of the Vafa-Witten theory on a circle and on a Riemann surface is discussed. Several novel two- and three-dimensional topological gauge theories are studied throughout the thesis and in the appendices.

In work unrelated to the main thread of the thesis, we conclude by classifying codimension-one topological defects in two-dimensional sigma models with various amounts of supersymmetry.

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Chapter 1

Introduction and overview

1.1 Electric-magnetic duality, geometric Langlands, and the work of Kapustin, Witten

Soon after Maxwell wrote his equations for electromagnetism, it was noted that exchanging electric fields with magnetic fields leaves the form of the equations invariant — provided one also exchanges electrically charged sources with magnetically charged sources. In mathematical terms, the duality operation simply exchanges the 2-form electromagnetic field strength F with its Hodge dual $*F$.

However, twentieth century physics developed in a direction that formally obscures this underlying symmetry of Maxwell's equations. Recasting electromagnetism as a gauge theory (that is, identifying the electromagnetic field as a connection A on a principal $U(1)$ bundle over spacetime), duality ceases to be manifest since the 2-form $F = dA$ and its Hodge dual $*dA$ play very dissimilar roles in the formalism. In particular, the inclusion of electric sources is handled very differently from that of magnetic sources: electric sources become *Wilson line* order operators, included as a source term in the action, while magnetic sources become *t'Hooft line* disorder operators, included by imposing a prescribed singularity in the space of field configurations. Electric-magnetic duality turns out to extend to the quantum theory of a $U(1)$ connection and the equivalence, though still provable [8], becomes even more hidden by the formalism.

For nonabelian gauge group G, a version of electric-magnetic duality extends to a particular 4d quantum gauge theory, the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (as we review in Chapter 2). In this setting, the equivalence (known as *S-duality*) is yet more nontrivial — to the degree that it acquires *mathematical* power. That is, the language used to describe dual structures take such different forms that their equivalence is interesting from a purely mathematical point of view. This thesis is devoted to a study of certain mathematical implications of S-duality.

In writing down such mathematical predictions, one encounters in the first instance a striking feature of nonabelian duality: the $\mathcal{N} = 4$ theory with gauge group G is mapped to an $\mathcal{N} = 4$

theory with gauge group ${}^LG$ (known as the Langlands dual group), where G and ${}^LG$ are related by interchange of their character and cocharacter lattices and will in general be distinct Lie groups. A comparison of dual theories therefore involves a comparison of mathematical structures labeled by Langlands dual pairs of Lie groups.

As it happens, pairs of Lie groups G and ${}^LG$ also appear in a set of deep conjectures and theorems of number theory which collectively have come to be known as the *Langlands correspondence*. These conjectures can be generalized and phrased in several ways, but, at heart, each version of the Langlands correspondence hypothesizes the existence of an isomorphism between a mathematical object labeled by a Lie group ${}^LG$ (what we may call the 'A-side' of the duality) with another object involving the dual group G (which we may call the 'B-side'). (See [7] and references therein for more on the Langlands program.)

For example, one guise of the Langlands correspondence (the one most directly relevant to physics) is known as *geometric Langlands* and asserts the equivalence of two collections of objects associated with a given (closed, oriented) Riemann surface C:

$$\boxed{\begin{array}{c}\text{Hecke eigensheaves}\\ \text{(of } \mathcal{D}\text{-modules) on } \mathrm{Bun}_{{}^LG}\, C\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{Flat (irreducible)}\\ G_{\mathbb{C}}\text{-bundles on } C\end{array}} \tag{1.1}$$

The B-side is the collection of principal $G_{\mathbb{C}}$-bundles on C equipped with flat connection, where $G_{\mathbb{C}}$ is the complexification of the Lie group, or, equivalently, it is the set of homomorphisms $\pi_1(C) \to G_{\mathbb{C}}$. Generically, this set forms a finite-dimensional moduli space $\mathcal{M}_{flat}(G_{\mathbb{C}}, C)$. The A-side is slightly more abstract: $\mathrm{Bun}_{{}^LG}C$ is the space [1] of holomorphic ${}^LG$-bundles on C, over which are defined modules for the sheaf of differential operators, or *$\mathcal{D}$-modules*. The conjecture is that to every flat $G_{\mathbb{C}}$-bundle on C, there is a corresponding sheaf of $\mathcal{D}$-modules — not just any sheaf, but an 'eigensheaf' of a 'Hecke transformation'. Without elaborating on the definitions of the quoted words in the preceding sentence, let us note a few features of the above mathematical statement of geometric Langlands that, as it turns out, will generalize to broad themes:

- In the right-hand box above, we meet flat, complexified connections on a Riemann surface C. Indeed, such connections emerge organically in several contexts throughout this thesis; their appearance is the hallmark of the B-side of the geometric Langlands correspondence.

- The structures in the left-hand box above require many more words and concepts to describe precisely than do the corresponding structures in the right-hand box above. Indeed, it is generally true that the A-side of the geometric Langlands correspondence is far more resistant to rigorous, simple mathematical definitions than is the B-side.

- Certain exotic mathematical definitions (such as the above notion of 'eigensheaf' and 'Hecke

[1] Actually, due to the singular nature of $\mathrm{Bun}_{{}^LG}C$, it is technically a 'moduli stack' rather than a 'space'.

transformation') turn out to have vivid physical realizations [1], suggesting that physics may supply the most natural language for phrasing the correspondence.

The appearance of dual pairs of Lie groups led M. Atiyah to speculate as early as 1977 [1] that S-duality of gauge theories might have something to do with geometric Langlands. This speculation was finally realized in the seminal paper [1] of A. Kapustin and E. Witten, who showed that the geometric Langlands correspondence emerges from S-duality applied to a very particular physical setup. Namely, they consider a certain topological field theory (TFT) constructed from $\mathcal{N}=4$ by a twisting procedure involving a complex twisting parameter t. (We describe this twisting procedure and the t parameter in the following chapter.) They focus on two particular values of t, corresponding to the two sides of the Langlands correspondence: on the A-side is the twisted theory at $t=1$ and gauge group ${}^LG$, while on the B-side is the twisted theory at $t=i$ and gauge group G. [2] Spacetime is chosen to be a product of two Riemann surfaces

$$M = \Sigma \times C$$

(where the genus g_C of the closed Riemann surface C is chosen to satisfy $g_C \geq 2$) and they consider the limit in which the size of C shrinks to zero.

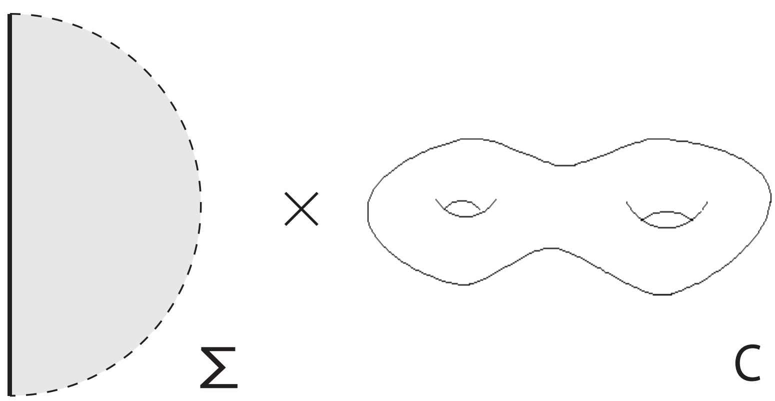

Figure 1.1: The twisted theories are placed on the product manifold $\Sigma \times C$ and the size of C is shrunk to zero

In this limit, the twisted theories reduce to effective 2d TFTs on Σ: on the A-side, the $t=1$ theory reduces to a topological A-model on Σ (reviewed in Section 7.1), a theory whose only bosonic field consists of a map from Σ into a symplectic target space. The target space in this case is the *Hitchin moduli space* $\mathcal{M}_H({}^LG, C)$, a hyperkähler manifold of dimension

$$\dim \mathcal{M}_H({}^LG, C) = 4(g_C - 1)\dim {}^LG$$

[2]For zero theta angle and inverted coupling strengths

equipped with three independent complex structures I, J, K and three independent symplectic forms $\omega_I, \omega_J, \omega_K$. Hence, $\mathcal{M}_H(^LG, C)$ can be viewed as a complex (resp. symplectic) manifold in different ways depending on which of I, J, K one chooses. The symplectic form relevant for $t = 1$ reduction is ω_K. (See Chapter 8 for more on the Hitchin moduli space.)

On the B-side, the $t = i$ twisted theory reduces to a topological B-model with target space $\mathcal{M}_H(G, C)$, viewed now as a complex manifold in complex structure J. This 2d manifestation of duality, between an A-model and B-model on a pair of target spaces, is known to physicists as *Mirror Symmetry* [22]. In order to make contact [1] with geometric Langlands, one takes Σ to be a Riemann surface with boundaries, and considers the collection of possible boundary conditions one can impose on the fields of the 2d theory along each component of the boundary. On the A-side, these are known to physicists as A-branes, and include Lagrangian submanifolds of the target space equipped with flat line bundles. (Branes of 2d theories are discussed extensively in Chapter 9.) On the B-side, boundary conditions are known as B-branes, and include complex submanifolds of the target.

Hence, S-duality implies the equivalence

A-branes on $\mathcal{M}_H(^LG, C)$, symplectic form ω_K	$\longleftrightarrow$	B-branes on $\mathcal{M}_H(G, C)$, complex structure J	(1.2)

The flat $G_{\mathbb{C}}$-bundles on the B-side of (1.1) correspond to points of $\mathcal{M}_H(G, C)$ in complex structure J. These, in turn, label a special class of B-branes, the 0-branes supported at individual points. Hence, we recognize the objects on the B-side of (1.1) as a subclass of those on the B-side of (1.2). Given a particular 0-brane supported at $p \in \mathcal{M}_H(G, C)$, Strominger, Yau, and Zaslow have given a proposal [19] for how to apply the map of p under Mirror Symmetry: it is mapped to an A-brane with flat, unitary line bundle wrapping a certain middle-dimensional submanifold of $\mathcal{M}_H(^LG, C)$, the so-called *dual Hitchin fiber* of p. Kapustin and Witten complete the argument by explaining how this A-brane is mapped to a Hecke eigensheaf on $\mathrm{Bun}_{^LG}\, C$.

In fact, (1.2) represents a richer equivalence than (1.1) in two respects. First, the boxes in (1.2) contain more objects that those of (1.1): there are many more B-branes than just the 0-branes. But, perhaps more interestingly from the point of view of comparing mathematical structures, the equivalence (1.2) is not merely an equivalence of sets, but of *categories*. A category is a set of objects, together with a collection of *morphisms*: arrows connecting the objects and satisfying a composition property. (The morphisms for the category of branes correspond to local operators sitting at the junction of two boundary conditions, as we discuss in Chapter 4.) Hence the equivalence (1.2) is a special kind of mapping respecting the 'rigidities' represented by the morphisms of the categories: i.e., a functor between categories.

1.2 Generalizing Kapustin-Witten

In this thesis, we generalize [1] in several directions. Generalizing the notion of compactification on a Riemann surface, we instead consider in Chapter 5 compactification on a *circle* to a 3d effective TFT. On the A-side, we will find that the $t = 1$ theory compactifies to a little-studied 3d TFT called the gauged 3d A-model. The 3d A-model TFT is a three-dimensional topological sigma model that can be defined for any Riemannian target manifold X; it was first studied for a general target in [6] (reviewed here in Section B.2) and can be regarded as a 3d analogue of the 2d A-model. When the target admits an action of a Lie group it can be coupled to a 3d topological gauge theory to form a gauged topological sigma model, as we shall discuss in greater detail in Chapter 3. (Gauged topological sigma models will play a central role in this thesis and we review certain mathematical aspects of these in Appendix C.) The target space of the 3d sigma model is the group manifold ${}^LG$, equipped with the action of conjugation on itself.

On the B-side, we have the good fortune that the 3d compactified theory is more familiar: it is a gauged version of the Rozansky-Witten (RW) 3d sigma model (the relevant gauging was written in an appendix to [4]; we review it in Section B.5). The target space is the total space $T^*G_{\mathbb{C}}$ of the cotangent bundle of the complexified group, equipped with the conjugation action of G on the base $G_{\mathbb{C}}$ and the induced coadjoint action on the fibers. Actually, it will turn out to be convenient to assign a 'ghost number' of two to the fibers and zero to the base; to indicate this we notate the target space as $T^*[2]G_{\mathbb{C}}$.

Duality predicts a highly nontrivial (and, except for the analysis in [4] and this thesis, completely untapped) equivalence between these two 3d gauged topological sigma models. In particular, they should have isomorphic sets of boundary conditions:

$$\boxed{\begin{array}{c}\text{Boundary conditions of}\\ \text{gauged 3d A-model, target } {}^LG\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{Boundary conditions of}\\ \text{gauged RW model, target } T^*G_{\mathbb{C}}\end{array}} \tag{1.3}$$

This '3d TFT' perspective on the Langlands correspondence is even richer than the 2d TFT perspective of (1.2). As we shall explain in Chapter 4, the boundary conditions of a 3d TFT have the structure of a *2-category*, including objects, morphisms, and morphisms-between-morphisms (i.e., *2-morphisms*). In summary, (1.1) is an equivalence of sets of objects, while (1.2) is an equivalence of categories (objects + morphisms), and (1.3) is an equivalence of 2-categories (objects + morphisms + 2-morphisms). At each step in this progression one adds in 'extra' objects and makes explicit rigidities that were only implicit at the previous step. In fact, (1.3) is certainly not the last word: by virtue of their 4d origin, the 3d boundary conditions 'secretly' possess a braided, monoidal structure preserved by duality; however, we stop at the 2-categorical level in this thesis and leave the braided, monoidal structure to future work.

Another direction in which we generalize [1] is to consider a wider class of observables than the line operators and 2d branes of [1]. In Chapter 6 we classify operators supported on codimension two submanifolds, i.e., *surface operators*. The simplest class of surface operators applied to geometric Langlands were studied by S. Gukov and E. Witten [2]; however, we shall find that there are many more general surface operators in the twisted theories. For the $t = i$ theory, we obtain a complete description of the 2-category of surface operators and, for the other twisted theories, we obtain a description only for abelian gauge group — this being one instance of the fact (noted above) that the B-side is always under better analytical control than the A-side.

In fact, the generalization to surface operators is closely related to the generalization to circle compactification, since, by virtue of the dimensional reduction trick discussed in Chapter 4, the 2-category of surface operators of a 4d TFT is precisely the same as the 2-category of boundary conditions of its 3d reduction.

Moreover, we will include in the story two other 4d TFTs obtainable from $\mathcal{N} = 4$ by a twisting procedure: the $t = 0$ twisted theory and an additional theory (not lying in the same family) that we call the Vafa-Witten (VW) twisted theory due to the role it played in [3]. These theories are self-dual (provided one suitably inverts couplings and exchanges the gauge group for its Langlands dual group). Certain aspects of the $t = 0$ theory were also considered in [35] and it is essentially equivalent to the $t = 1$ theory with nonzero instanton term (which case was also considered in [1]). It has been shown to provide a natural setting for the so-called quantum geometric Langlands correspondence (a generalization of (1.1) involving a quantum parameter).

In Chapter 3, we will define a new TFT, the gauged 4d A-model (closely analogous to the gauged 3d A-model) and will show that the Vafa-Witten theory can be thought of as a gauged 4d A-model with target space chosen to be the Lie algebra $\mathfrak{g}$ of the gauge group. In Chapter 5 we show that both the $t = 0$ theory and the Vafa-Witten theory compactify on S^1 to the same 3d TFT: a version of the gauged 3d A-model with target $\mathfrak{g}$. We will also obtain 3d descriptions of the $t = i$ and $t = 1$ theories compactified on a circle.

In Chapter 4, we will explain in greater detail why boundary conditions of 2d TFTs have the structure of a category and boundary conditions of 3d TFTs have the structure of a 2-category. In Chapter 6, we will analyze boundary conditions of the 3d theories, or, equivalently, surface operators of the 4d theories. In Chapter 7 we construct the *modified A-model*, a 2d TFT which can be defined for general Kähler target equipped with compatible $U(1)$ action. In Chapter 8, we show that the Vafa-Witten theory compactifies on a Riemann surface C to a modified A-model with target $\mathcal{M}_H(G, C)$.

Finally, in Chapter 9, we will obtain a geometric classification of topological defects of the 2d bosonic sigma model and 2d sigma models with various amounts of supersymmetry. (This final topic lies somewhat outside the main thread of the thesis.)

1.3 Results

We compile here a list of the main results presented in this thesis. Some results were first presented in the paper [4], some have been presented in the paper [5], and some are original work of this thesis.

- In Chapter 3, we define a new 4d topological gauge theory, the gauged 4d A-model with target space a general Riemannian manifold X equipped with an action of a Lie group G.
- In Chapter 3, we locate the Vafa-Witten twisted TFT as a particular member of the above family, for $X = \mathfrak{g}$ equipped with Ad action of G on $\mathfrak{g}$.
- In Chapter 5, we compactify on a circle the Vafa-Witten theory and GL-twisted theories at $t = 0$, $t = 1$, and $t = i$ (for nonabelian gauge group). (The abelian reductions of the GL-twisted theories were presented in [4].)
- In the Appendices A and B we write down and study several 2d and 3d TFTs with gauge fields and describe their properties. In particular, we show that the category of branes of the 2d gauged B-model is the equivariant-derived category of coherent sheaves. (These were first presented in appendices of [4].)
- In Chapter 6 we determine the full, monoidal category of line operators of the GL-twisted theory at $t = i$ (which contain the Wilson lines of [1] as a special case). (First presented in [4].)
- In Chapter 6 we determine the full 2-category of surface operators of the $t = i$ theory. (First presented in [4].)
- In Chapter 6 we determine, for abelian gauge group, the 2-category of surface operators of the $t = 0$ and $t = 1$ theories and discuss the action of duality. (First presented in [4].)
- In Chapter 7, we construct a modified A-model TFT for general Kähler target X equipped with Hamiltonian $U(1)$ action. We explore its properties.
- In Chapter 8, we compactify the Vafa-Witten theory on a genus $g_C \geq 2$ Riemann surface C to a modified A-model with target $\mathcal{M}_H(G, C)$. (Compactification of Vafa-Witten theory on a Riemann surface was briefly addressed in an appendix of [20]; in Chapter 8 we perform the compactification in full detail.)
- In Chapter 9, using notions of generalized geometry and neutral signature geometry, we classify topological defects of various 2d sigma models: the bosonic sigma model, the (0,1), (0,2), and (2,2) supersymmetric models. (First presented in [5].)

Additionally, we compile the following background material:

- In Chapter 2 we provide a review of relevant aspects of duality of $\mathcal{N} = 4$ SYM and its topologically twisted cousins.
- In Chapter 4, we provide a brief tutorial on categories, 2-categories, and their relation to TFTs.
- In Appendix C, we provide a review of the mathematics of associated fiber bundles and explain their connection to equivariant cohomology.

Chapter 2

Duality of four-dimensional gauge theories

We begin by reviewing $\mathcal{N} = 4$ super Yang-Mills (SYM) theory in four dimensions (4d), as well as its duality properties and the various topological field theories (TFTs) one can obtain by twisting this theory.

2.1 Nonabelian generalization of electric-magnetic duality

After the advent of Yang-Mills theories, it was natural to wonder whether some version of electric-magnetic duality extends to theories with nonabelian gauge group G. Indeed, since dualizing inverts the coupling parameter e^2, duality could potentially provide a powerful window into the strong-coupling behavior of nonabelian gauge theories: simply study a question about strong-coupling physics on the dual, weak-coupling side.

Such a nonabelian duality, if it exists, must exchange a Wilson line operator (the nonabelian generalization of an 'electric source') with a t'Hooft disorder line operator (the nonabelian generalization of a 'magnetic source'). But in looking into the classification of t'Hooft operators of nonabelian gauge theories, Goddard, Nuyts, and Olive [25] discovered a crucial twist in the story: while Wilson operators are labeled by the irreducible representations of the group G, t'Hooft operators are instead labeled by elements of a fundamental domain of G's coweight lattice, which, in turn, label irreducible representations of a different Lie group ${}^LG$.

As we have emphasized in the previous chapter, pairs of Lie groups G and ${}^LG$ related by exchange of coweight and weight lattices play a central role in the Langlands Program of number theory; for this reason, ${}^LG$ is known as the *Langlands dual* group to G.

These considerations inspired Montonen and Olive [23] to conjecture that, at the quantum level, Yang-Mills theory with gauge group G is dual to Yang-Mills theory with gauge group ${}^LG$ and inverted coupling.

However, it was quickly realized that this naïve mapping cannot quite be correct; for one thing, the simple mapping between couplings will not survive renormalization. Osborn [24], building on the work of Witten and Olive [26], refined the conjecture to say instead that a *supersymmetric* version of Yang-Mills — the $\mathcal{N} = 4$ super-YM with gauge group G — is dual to the $\mathcal{N} = 4$ super-YM with gauge group ${}^L G$ and inverted coupling. Since $\mathcal{N} = 4$ is a conformal field theory, the inverse mapping between couplings is left undisturbed by renormalization.

Although, strictly speaking, duality of $\mathcal{N} = 4$ remains unproven at the level of mathematical rigor, physicists have in the past three decades uncovered a wealth of evidence in support of the conjecture; by now, it is regarded as established fact and we shall treat it as such in this thesis.

We review in more detail the $\mathcal{N} = 4$ theory and its duality properties.

2.2 $\mathcal{N} = 4, d = 4$, gauge group G

The $\mathcal{N} = 4$ theory is the maximal supersymmetric extension of Yang-Mills theory in four dimensions. It is defined via an action on *spacetime*, a four-dimensional manifold M. In order to write down an action, let us first fix some gauge theory conventions. The remainder of this chapter will closely follow the notational conventions and exposition in [1].

2.2.1 Gauge theory conventions

Let G be a connected, reductive Lie group with Lie algebra $\mathfrak{g}$. We fix a faithful, N-dimensional 'defining' representation of G once and for all, thus identifying $\mathfrak{g}$ with a collection of $N \times N$ anti-Hermitian matrices. Let T_a for $a = 1, \ldots, \dim G$, be a basis for $\mathfrak{g}$, in terms of which the (real) structure constants $f^c_{\ ab}$ are defined by

$$[T_a, T_b] = f^c_{\ ab} T_c$$

Let P be a principal G-bundle on M, with trivializing neighborhoods $U^{(\alpha)} \subset M$ and local sections $s^{(\alpha)} : U^{(\alpha)} \to P$. The gauge field will be a connection A on P, which we represent locally as a $\mathfrak{g}$-valued 1-form

$$A^{(\alpha)} \equiv s^{(\alpha)*} A = A^{(\alpha)}_\mu dx^\mu = A^{(\alpha)a}_\mu T_a \, dx^\mu \in \Omega^1(U^{(\alpha)}, \mathfrak{g})$$

with curvature given by $F^{(\alpha)} = dA^{(\alpha)} + \frac{1}{2}[A^{(\alpha)}, A^{(\alpha)}]$. From now on, we tend to suppress the labels (α) on locally defined quantities.

We use the notation

$$\eta \in \Omega^k(M, ad\, P)$$

to indicate that η is a k-form on M taking values in $ad\, P$, the vector bundle associated to P by the adjoint representation of G on $\mathfrak{g}$. Its covariant derivative with respect to A is given by

$$d_A\eta = d\eta + [A, \eta]$$

An infinitesimal automorphism of P with parameter $\epsilon \in \Omega^0(M, ad\, P)$ transforms A and η as

$$\delta_\epsilon A = -d_A \epsilon$$
$$\delta_\epsilon \eta = [\epsilon, \eta]$$

Let 'Tr' be an Ad-invariant, negative definite metric on $\mathfrak{g}$, normalized such that

$$\frac{1}{8\pi^2}\int_M \mathrm{Tr}\, F \wedge F$$

takes integer values when $M = S^4$. When we have cause to refer to this metric directly, we denote it κ_{ab} and its inverse by κ^{ab}.

2.2.2 Action, supersymmetries

Initially, we take M to be flat $\mathbb{R}^{1,3}$ with Lorentz signature. In addition to the gauge field A (a connection on principal bundle P), the matter content consists of six $ad\, P$-valued scalar fields

$$\phi_1, \ldots, \phi_6 \in \Omega^0(M, ad\, P)$$

as well the eight fermionic fields

$$\lambda^1, \ldots, \lambda^4 \in \Omega^0(M, ad\, P \otimes \mathcal{S}_+)$$
$$\bar{\lambda}_1, \ldots, \bar{\lambda}_4 \in \Omega^0(M, ad\, P \otimes \mathcal{S}_-)$$

where $\mathcal{S}_\pm$ are the two spinor bundles on M. The action of $\mathcal{N} = 4$ super-YM is given by

$$S = \frac{1}{e^2}\int_M d^4x\, \mathrm{Tr}\Big\{\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + D_\mu\phi_i D^\mu\phi^i + \frac{1}{2}[\phi_i, \phi_j][\phi^i, \phi^j]\Big\} + \ldots \tag{2.1}$$

where $\ldots$ indicates terms in the action depending on the fermions (which we suppress here). In addition, we include a topological term measuring the second Chern class of the bundle P:

$$S_{top} = -\frac{\theta}{8\pi^2}\int_M \mathrm{Tr}\, F \wedge F \tag{2.2}$$

Since the theory is manifestly invariant under $\theta \to \theta + 2\pi$ (leaving the path integral weight invariant), it is natural to regard θ as an angular parameter. The parameters e and θ are usefully combined into a complexified coupling parameter:

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}$$

Besides the $Spin(1,3)$ Lorentz symmetry, the theory is also invariant under a $Spin(6)$ symmetry with respect to which which the six ϕs are rotated into each other as a $\mathbf{6}$, while the λs transform as a $\mathbf{4}$ and the $\bar{\lambda}$s transform as a $\bar{\mathbf{4}}$. We call this the *R-symmetry* group of the theory and denote it $Spin(6)_R \simeq SU(4)_R$.

Additionally, the theory is invariant under fermionic supersymmetries $Q^1_{\dot{A}}, \ldots, Q^4_{\dot{A}}$ transforming as $(\mathbf{2}, \mathbf{4})$ under the bosonic symmetry group

$$Spin(1,3) \times SU(4)_R \simeq SL(2,\mathbb{C}) \times SU(4)_R$$

and $\bar{Q}_{A1}, \ldots, \bar{Q}_{A4}$ transforming as $(\bar{\mathbf{2}}, \bar{\mathbf{4}})$. Here $A = 1,2$ are $\mathbf{2}$ of $SL(2,\mathbb{C})$ indices and $\dot{A} = 1,2$ are $\bar{\mathbf{2}}$ of $SL(2,\mathbb{C})$ indices.

2.3 S-duality

Let us examine the duality properties of $\mathcal{N} = 4$. In the first place, shifting the complexified coupling τ by $\tau \to \tau + 1$ corresponds to a 2π shift of the angle θ, which is manifestly a symmetry of the theory.

Less trivially, the extension of the Montonen-Olive conjecture to $\mathcal{N} = 4$ SYM states that, at the the quantum level, the theory with simply laced gauge group G and coupling τ is isomorphic to the theory obtained by replacing G by its Langlands dual group ${}^L G$, exchanging electric sources with magnetic sources, and replacing τ by

$${}^L\tau = -\frac{1}{\tau}$$

This *S-duality* combines with θ angle shifts to generate an $SL(2,\mathbb{Z})$ duality group.

In case G is non-simply laced, the coupling transforms under S-duality as

$${}^L\tau = -\frac{1}{n_{\mathfrak{g}}\tau}$$

where $n_{\mathfrak{g}} = 1$, 2, or 3 is the lacing number of the group, the ratio of the squared lengths of the long roots to the short roots in $\mathfrak{t}^*$, the dual of a Cartan subalgebra of $\mathfrak{g}$ (with respect to Ad-invariant metric Tr). When $n_{\mathfrak{g}} = 2$ or 3, the S-duality mapping on τ combines with θ angle shifts to generate a discrete subgroup of $SL(2,\mathbb{R})$ known as a Hecke group. See Appendix A of [2] for further details and examples of Langlands dual pairs of Lie groups.

As emphasized in [1], formulas relating coupling strengths in dual theories must be interpreted with care. Most simple Lie groups G have the same Lie algebra as their dual; i.e., $\mathfrak{g} = {}^L\mathfrak{g}$ for most simple G. In this case, it is natural to identify the two vector spaces $\mathfrak{g}$ and ${}^L\mathfrak{g}$ in writing down the actions for the two theories; indeed inspection of the above action shows that its form depends only on the data of the Lie algebra. In this way, one can compare couplings in the dual theories. On the other hand, the dual pair of Lie groups

$$G = Spin(2k+1) \quad \longleftrightarrow \quad {}^LG = Sp(k)/\mathbb{Z}_2$$

have non isomorphic Lie algebras: $\mathfrak{g} \neq {}^L\mathfrak{g}$. Therefore, a non canonical choice of normalization must be made in order to compare couplings; as in [1], we adopt the convention that the formula above for ${}^L\tau$ continues to hold, setting $n_{\mathfrak{g}} = 2$.

It will be important for us to know how S-duality transforms the supersymmetries of $\mathcal{N} = 4$. One can infer [1] from the transformations of the central charges of the supersymmetry algebra that the left-handed supersymmetries are multiplied by a phase and the right-handed supersymmetries are multiplied by the opposite phase, as follows:

$$\bar{Q}_{Ai} \longrightarrow e^{i\alpha/2}\,\bar{Q}_{Ai}, \qquad Q^i_{\dot{A}} \longrightarrow e^{-i\alpha/2}\,Q^i_{\dot{A}}, \qquad \text{with} \quad e^{i\alpha} \equiv \frac{|\tau|}{\tau}$$

2.4 Twisting

It is not the $\mathcal{N} = 4$ theory that will be our main interest, but, rather, a family of topological field theories obtained from $\mathcal{N} = 4$ via a *twisting* procedure. Twisting (introduced by E. Witten in the 1980s in his work [45] on the Donaldson invariants of 4-manifolds) is a way of generating a topological field theory starting with a given physical, supersymmetric field theory.

Our motivation for twisting is that, by exploiting metric-independence of correlators of the twisted theory, we will be able to compute certain quantities in the twisted theories exactly and with minimal work, the corresponding analysis of which, for untwisted $\mathcal{N} = 4$, would be far more intricate. For instance, in Chapter 6, we will exploit metric-independence to compute the spectrum of surface operators simply by compactifying to three dimensions and looking at boundary conditions.

A related motivation is to study duality of supersymmetric field theory on a more general 4-manifold than flat $\mathbb{R}^4$. Generically, the fermionic symmetries of a supersymmetric theory do not survive the transition from $\mathbb{R}^4$ to a more general curved manifold (aside from the restrictive cases where the geometry admits covariantly-preserved spinors). But, as we will see shortly, twisting produces at least one fermionic charge Q of the theory transforming as a 0-form from chart to chart and consequently will be preserved after taking spacetime to be a general 4-manifold. For instance, in later chapters, we will study duality on spacetimes of the form $\Sigma \times C$ (where Σ and C are Riemann surfaces) as well as spacetimes of the form $W \times S^1$ (where W is a 3-manifold and S^1 is a circle).

To twist $\mathcal{N}=4$, it is necessary to first Wick rotate to Euclidean signature. We will therefore assume that M is a Riemannian 4-manifold and that actions S we write down are understood as weighting factors of the form

$$e^{-S}$$

in path integrals. The Wick rotated Euclidean group is $Spin(4) \simeq SU(2)_l \times SU(2)_r$, with respect to which the Q's transform as $(\mathbf{2},\mathbf{1})$ and the $\bar{Q}$'s transform as $(\mathbf{1},\mathbf{2})$.

Twisting is a two-step trick. Starting with a physical [1], supersymmetric field theory (in our case, $\mathcal{N}=4$ super-YM), one first modifies the theory through a judicious redefinition of the spins of the matter fields. Specifically, one embeds the Euclidean rotation group $Spin(4)$ into the R-symmetry group $Spin(6)_R$ via a homomorphism $\aleph : Spin(4) \to Spin(6)_R$, declaring the *twisted rotation group* $Spin'(4)$ to consist of simultaneous rotations under the old $Spin(4)$ and its embedded image in $Spin(6)_R$; i.e.,

$$Spin'(4) \equiv (1 \times \aleph)(Spin(4)) \subset Spin(4) \times Spin(6)_R \ .$$

Under the twisted rotation group, the R-symmetry index i on the $Q^i_{\dot{A}}$ decomposes as some representation $\mathbf{r}$ of $Spin(4)$ with respect to the embedding $\aleph(Spin(4)) \subset Spin(6)_R$; the Q's therefore transform as $(\mathbf{2},\mathbf{1}) \otimes \mathbf{r}$ of $Spin'(4)$. Similarly, the $\bar{Q}$'s transform as $(\mathbf{1},\mathbf{2}) \otimes \mathbf{l}$ for some representation $\mathbf{l}$ of $Spin(4)$. Crucially, the embedding $\aleph$ is chosen such that there is a singlet in either $(\mathbf{2},\mathbf{1}) \otimes \mathbf{r}$ or $(\mathbf{1},\mathbf{2}) \otimes \mathbf{l}$, meaning that some fermionic symmetry Q transforms as a scalar under the twisted rotation group. A scalar Q transforms trivially from chart to chart, and therefore such a Q will remain a symmetry of the theory after taking spacetime to be a general curved manifold M. Having thus redefined spins of fields, the twisted theory is rather pathological in that it violates spin-statistics (in particular, it will no longer be unitary); this lack of unitarity does not pose a problem so long as we are careful to restrict which questions we ask of the theory and what uses we put it to.

The second step of the twisting trick is to pick exactly one of the singlets Q (necessarily nilpotent up to equations of motion) and to reinterpret the theory by restricting attention to states and operators annihilated by Q, modulo Q-exact states and operators. This is exactly by analogy with the BRST charge of gauge theories, for which reason we call our Q the *BRST charge* of the twisted theory. We will denote by δ_Q the symmetry variation of the fields corresponding to charge Q and formally declare it to have odd statistics.

The twisted action takes the form

$$S = \delta_Q \int_M V + S_{top}$$

[1] We take 'physical' to mean: satisfying the usual properties of relativistic quantum field theory on $\mathbb{R}^4$. Among these: unitarity; Euclidean-invariance; a positive definite action with conventional kinetic terms; free of anomalies.

where V is called the *gauge fermion* and S_{top} is a term depending only on connected component of the space of fields of the theory, e.g., the instanton term (2.2). We shall encounter this form of action over and over in this thesis. It guarantees that the energy-momentum tensor is Q-exact, which, in turn, guarantees that correlators of BRST-invariant functions of the fields are metric-independent. Indeed, consider a correlator of the form

$$< \mathcal{O}_1 \ldots \mathcal{O}_p >= \int [D\Phi] \ \mathcal{O}_1 \ldots \mathcal{O}_p \ e^{-S[\Phi]}$$

where Φ are the fields of the theory, and each of the operators $\mathcal{O}_i$ is annihilated by δ_Q (we call such operators *observables* of the theory). Computing the variation with respect to an arbitrary deformation $\delta g^{\mu\nu}$ of the metric on M, we find

$$\begin{aligned} \frac{\delta}{\delta g^{\mu\nu}} \int [D\Phi] \ \mathcal{O}_1 \ldots \mathcal{O}_p \ e^{-S[\Phi]} &= - \int [D\Phi] \ \mathcal{O}_1 \ldots \mathcal{O}_p \ T_{\mu\nu} \ e^{-S[\Phi]} \\ &= - \int [D\Phi] \ \mathcal{O}_1 \ldots \mathcal{O}_p \ \delta_Q \mathcal{V}_{\mu\nu} \ e^{-S[\Phi]} \\ &= - \int [D\Phi] \ \delta_Q \Big\{ \mathcal{O}_1 \ldots \mathcal{O}_p \ \mathcal{V}_{\mu\nu} \ e^{-S[\Phi]} \Big\} \\ &= 0 \end{aligned}$$

where we have used the definition $T_{\mu\nu} = \delta S / \delta g^{\mu\nu}$ of the energy-momentum tensor, BRST-invariance of the action, and have assumed metric-independence of the observables and the path integral measure. A theory of this type — with metric-independence of correlators after taking Q-cohomology — is called a *cohomological topological field theory.* [2]

It should be emphasized that, on flat $\mathbb{R}^4$, twisting amounts to simply reshuffling the labels on fields and therefore the twisted theory just describes a subsector of the original, physical theory.

2.4.1 Twists of $\mathcal{N} = 4$

So what are the possible ways of twisting $\mathcal{N} = 4$ into a TFT? This is the set of possible homomorphisms $\aleph : Spin(4) \to Spin(6)_R$, up to equivalence, which in turn can be characterized by how the $\mathbf{4}$ of $Spin(6)_R$ splits under the $Spin(4) \simeq SU(2)_l \times SU(2)_r$ embedding. The answer first given in [13] is that there are three possible twisted theories:

- *Donaldson-Witten (DW) twist.* The choice of $\aleph$ under which $\mathbf{4} \to (\mathbf{2}, \mathbf{1}) \otimes (\mathbf{1}, \mathbf{1}) \otimes (\mathbf{1}, \mathbf{1})$ yields a twisted theory with a single scalar supercharge. It corresponds to treating $\mathcal{N} = 4$ as an $\mathcal{N} = 2$ with matter, and twisting with the Donaldson-Witten twist [45] of $\mathcal{N} = 2$. We will not study the Donaldson-Witten twisted theory in this thesis.

[2] Technically, correlators can still depend on the smooth structure of the metric; they are merely invariant under smooth deformations.

- *Vafa-Witten (VW) twist.* The embedding under which $\mathbf{4} \to (\mathbf{2},\mathbf{1}) \otimes (\mathbf{2},\mathbf{1})$ yields a twisted theory with two supercharges. All linear combinations of these two are essentially equivalent choices for the BRST charge Q. This twisted theory was put to use in [3] in testing strong-coupling predictions of S-duality.

- *geometric Langlands (GL) twist.* The embedding under which $\mathbf{4} \to (\mathbf{2},\mathbf{1}) \otimes (\mathbf{1},\mathbf{2})$ yields a twisted theory with two scalar supercharges Q_l and Q_r; in contrast with the Vafa-Witten theory, different linear combinations of the two scalar supercharges are *not* equivalent. Hence, the GL-twist actually corresponds to a family of TFTs labeled by a parameter t. This twisted theory was shown in [1] to provide a natural setting for studying the geometric Langlands program from a gauge theory perspective, for which reason, we call it the GL twist.

In the subsequent chapters, we will study properties of the VW- and GL-twisted theories (at certain values of the twisting parameter t). We put off writing the explicit action and variations of the Vafa-Witten twisted theory until Chapter 3, where it will be shown to lie naturally within a wider family of 4d TFTs. Let us now discuss the GL-twisted theory in greater detail.

2.5 GL-twisted theory

2.5.1 t parameter

First, consider the choice of BRST charge. As we have discussed above, there are two (mutually anticommuting) scalar supercharges Q_l and Q_r corresponding to variations δ_l and δ_r, and therefore δ_Q can be any nonzero linear combination

$$\delta_Q = u\,\delta_l + v\,\delta_r \ ,$$

for two parameters $u, v \in \mathbb{C}$. Rescaling δ_Q by an overall complex constant does not change the properties of the theory, so (u, v) are to be regarded as homogenous coordinates for a $\mathbb{C}P^1$ of inequivalent choices. Alternatively, we parameterize the family of inequivalent choices by

$$t \equiv v/u \ , \qquad t \in \mathbb{C} \cup \{\infty\}$$

The GL-twisted theory at t is trivially related to the theory at $-t$ by an action of the center of the R-symmetry group $SU(4)_R$ [1] of the GL-twisted action; however, modulo $t \to -t$, all other choices of t are TFTs with distinct properties.[3] We have said above that the left-handed supercharges are

[3] As discussed in [1], when the 4-manifold M admits an orientation reversal symmetry, then the theory possesses a symmetry reversing the sign of the instanton number; this results in an additional symmetry between the theory at t and the theory at $1/t$.

multiplied by a phase under S-duality while the right-handed supercharges are multiplied by the opposite phase; from this, we can infer the action of S-duality on the t parameter to be

$$t \longrightarrow \frac{\tau}{|\tau|} t$$

For $\theta = 0$, one can imagine the action of S-duality as a $\pi/2$ rotation of the Riemann sphere in which t takes values:

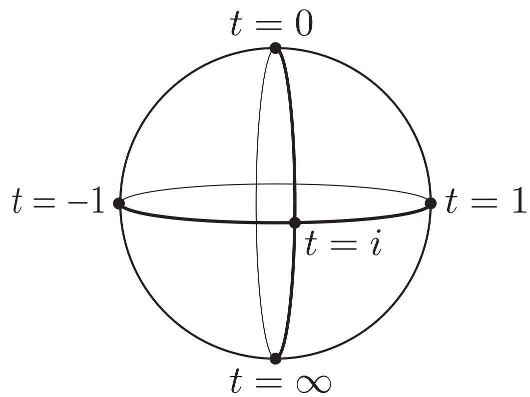

Figure 2.1: The twisting parameter t takes values in $\mathbb{C} \cup \{\infty\}$

We are particularly interested in GL-twisted theory at the three values: $t = i$, $t = 1$, and $t = 0$; in speaking of these three theories, we shall understand '$t = i$' as shorthand for 'the GL-twisted TFT with twisting parameter $t = i$'. The $t = i$ and $t = 1$ theories are exchanged by duality, and in [1] the duality between these two TFTs was shown to provide a natural setting for studying the geometric Langlands program. The $t = 0$ theory, on the other hand, is self-dual, and in [35] has been shown to provide a natural setting for studying quantum geometric Langlands (a generalization of geometric Langlands involving a quantum parameter).

2.5.2 Field content, action, and variations of GL-twisted theory

We now examine in detail the field content, BRST-variations, and action of the GL-twisted theory for the various values of the twisting parameter $t \in \mathbb{C} \cup \{\infty\}$.

Let P be a principal G-bundle over 4-manifold M. After modifying the spins of the $\mathcal{N} = 4$ fields according to the GL twist, the bosonic fields consist of a gauge field A (a connection on P), as well as the following forms taking values in the bundle $ad\, P$:

$$\sigma \in \Omega^0(M, ad\, P) \otimes \mathbb{C}$$
$$\phi \in \Omega^1(M, ad\, P)$$

where σ is a complex 0-form consisting of two real sections of $ad\, P$. The fermionic fields consist of the $ad\,P$-valued forms

$$\begin{aligned} \eta, \widetilde{\eta} &\in \Omega^0(M, ad\,P) \\ \psi, \widetilde{\psi} &\in \Omega^1(M, ad\,P) \\ \chi &\in \Omega^2(M, ad\,P) \quad . \end{aligned}$$

Twisting the $\mathcal{N} = 4$ variations, one obtains the following transformations of the fields under the BRST variation δ_Q:

$$\begin{aligned} \delta_Q A &= i(\psi + t\widetilde{\psi}), \\ \delta_Q \phi &= i(t\psi - \widetilde{\psi}), \\ \delta_Q \sigma &= 0, \\ \delta_Q \bar{\sigma} &= i(\eta + t\widetilde{\eta}), \\ \delta_Q \psi &= d_A\sigma + t[\phi, \sigma] \\ \delta_Q \widetilde{\psi} &= t d_A \sigma - [\phi, \sigma], \\ \delta_Q \eta &= t d_A^* \phi + [\bar{\sigma}, \sigma], \\ \delta_Q \widetilde{\eta} &= -d_A^* \phi + t[\bar{\sigma}, \sigma], \\ \delta_Q \chi &= \frac{1+t}{2}(F - \frac{1}{2}[\phi, \phi] + *d_A\phi) + \frac{1-t}{2}(*(F - \frac{1}{2}[\phi, \phi]) - d_A\phi) \end{aligned}$$

where $*$ is the 4d Hodge star operator with respect to the metric on M, we have defined $\bar{\sigma} = -\sigma^\dagger$ (recall that, according our conventions, $\mathfrak{g}$ elements are anti-Hermitian), and d_A^* is the differential operator

$$d_A^* \equiv *d_A* \quad .$$

The BRST-variation δ_Q, so defined, squares to zero only modulo the equations of motion (minimizing the twisted action) and, also, modulo a gauge transformation.

For $t \neq \pm i$, the action can be written as a BRST-exact term plus a term depending only on the topology of the bundle P:

$$S = \delta_Q \int_M V - \frac{\Psi}{4\pi i} \int_M \mathrm{Tr}\, F \wedge F \ , \tag{2.3}$$

where

$$\Psi = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2} \frac{t^2 - 1}{t^2 + 1} \ .$$

and we will record the explicit form of V for $t = 0$ and $t = 1$ in equations (5.6) and (5.7). The properties of the GL-twisted theory depend on the twisting parameter t, the coupling e^2 and the theta angle θ only through the combination Ψ, which in [1] has been called the *canonical parameter*.

For $t = \pm i$, the action can instead be written [40] as a sum of a BRST-exact term and a topological term depending on the fermionic fields. In order to write down the action and variations for $t = \pm i$, it is natural to form the complexified connections $\mathcal{A} = A + i\phi$ and $\bar{\mathcal{A}} = A - i\phi$ as well as the covariant derivatives $d_{\mathcal{A}}$, $d_{\bar{\mathcal{A}}}$ and curvatures $\mathcal{F}$, $\bar{\mathcal{F}}$ with respect to these connections. For instance, the combination $\mathcal{A}$ is BRST-invariant:

$$\delta_Q \mathcal{A} = 0 \qquad \text{at} \quad t = i$$

which fact will allows one to write down BRST-invariant line operators [1]. This is a first hint of a link with the flat $G_{\mathbb{C}}$ connections on Riemann surfaces that play a role in geometric Langlands. For vanishing theta angle, the action for GL-twisted theory at $t = i$ reads

$$S = \delta_Q \int_M V + S_{top}$$

where

$$\begin{aligned} V &= -\frac{1}{2e^2}\, \delta_Q \, \mathrm{Tr}\Big\{ (\chi^+ - i\chi^-) \wedge *\bar{\mathcal{F}} + d_{\bar{\mathcal{A}}}\bar{\sigma} \wedge *(\psi - i\widetilde{\psi}) \\ &\qquad + \frac{i}{2}(\eta - i\widetilde{\eta}) \wedge * \Big(i[\bar{\sigma}, \sigma] - d_A^* \phi \Big) \Big\}, \\ S_{top} &= \frac{i}{e^2} \int_M \mathrm{Tr}\Big\{ (\chi^+ - i\chi^-) \wedge \Big(d_{\mathcal{A}}(\psi - i\widetilde{\psi}) - [\chi^+ - i\chi^-, \sigma] \Big) \Big\} \end{aligned} \tag{2.4}$$

and χ^+, χ^- refer to the self-dual and anti-self-dual parts of the 2-form χ. The term S_{top} is BRST-inexact, but nonetheless BRST-invariant. It is an integral of a 4-form containing no dependence on the metric and, hence, it is topological.

The actions above for the GL-twisted theory for $t \neq \pm i$ and $t = \pm i$ possess a $U(1)$ symmetry which is a remnant of the original $Spin(6)_R$ symmetry of $\mathcal{N} = 4$ after twisting. Under this $U(1)$, the BRST variation δ_Q carries one unit of charge, for which reason we call it the *ghost number* symmetry (harkening again to the analogy between our δ_Q and the BRST variation of a physical gauge theory). The fields A and ϕ have ghost number 0, the fields ψ and $\widetilde{\psi}$ have ghost number 1, the fields $\eta, \widetilde{\eta}$, and χ have ghost number -1, and the field σ has ghost number 2.

Chapter 3

Vafa-Witten theory as a gauged 4d A-model

We have asserted in the preceding chapter that $\mathcal{N} = 4$ SYM in four dimensions can be twisted into a TFT in three inequivalent ways and we ended by describing one of these, the GL-twisted theory. In this chapter, we describe another of these twists — the Vafa-Witten twisted theory — from a novel point of view. Inspired by the construction of a *gauged A-model* TFT in three dimensions [6], we perform a similar construction of a gauged A-model TFT in four dimensions; this can be thought of as a family of 4d topological gauge theories (one for each choice of a target space X). We will then locate the Vafa-Witten theory as one particular member of this family (for the choice $X = \mathfrak{g}$, the Lie algebra of the gauge group).

Gauged topological sigma models

The original example of a cohomological TFT was Witten's twisted version [45] of $\mathcal{N} = 2$ SYM in 4d (which below we will call the *Donaldson-Witten* theory, not to be confused with the Donaldson-Witten twist of $\mathcal{N} = 4$, mentioned briefly in the preceding chapter). This is an example of a *topological gauge theory*, since the field content includes a gauge field transforming as a connection on a principal G-bundle P, as well as matter fields transforming as sections of associated vector bundles. Soon after Donaldson-Witten, there appeared [10] another class of cohomological field theories, the two-dimensional A- and B-models. These are examples of *topological sigma models*, with a bosonic field Φ corresponding to a map from M into some *target space* X, as well as matter fields transforming as sections of the pullback bundle Φ^*TX.

In recent years, several authors have studied a class of hybrid TFTs bridging the gap between these two lines of development in the sense that they incorporate both sigma model fields and a gauge field. These *gauged topological sigma models* are constructed by taking the target space to be a manifold X admitting an action of a Lie group G, which action is assumed to be a global symmetry of the sigma model theory. One then promotes the G-action to a local symmetry by coupling the sigma model fields to a gauge field transforming as a connection on a principal G-bundle. Finally, one includes in the action a copy of a topological gauge theory governing the dynamics of the gauge field.

For instance, in [38] a gauged version of the 3d topological Rozansky-Witten sigma model was constructed, with Chern-Simons gauge sector. In [4] and [15], alternative gaugings of the Rozansky-Witten model were constructed with B-type and BF gauge sectors, respectively. In [6], a three-dimensional analog of the A-model was constructed and then gauged by coupling it to an A-type gauge theory.

We construct a new gauged topological sigma model in four dimensions. The matter fields are sections of associated *fiber* bundles, with typical fiber a curved manifold X. The mathematics of associated fiber bundles is perhaps less familiar than that of associated vector bundles (in particular, the connection between topological terms in the action and equivariant cohomology classes on the curved fiber), so we review this mathematics in appendix C.

In Section 3.1 we begin with a review of the 4d A-model, a four-dimensional topological sigma model with target X, an arbitrary real manifold. Then, in Section 3.2, we gauge the 4d A-model by coupling it to a Donaldson-Witten gauge theory with gauge group G. Finally, we show in Section 3.3 that the Vafa-Witten twisted theory can be thought of as a gauged 4d A-model with a particular choice of target and G-action. This locates Vafa-Witten theory as one particular point in a moduli space of topological gauge theories.

3.1 4d A-model

Let X be a real manifold equipped with a Riemannian metric g. Let M be a four-dimensional manifold equipped with Riemannian metric h. We construct a topological sigma model with base manifold M and target space X; i.e., a topological field theory living on M whose fields consist of a map $\Phi : M \to X$, together with various forms on M taking values in the pullback bundle Φ^*TX. Following the terminology of [6], we refer to the theory as a *4d A-model with target X*. The designation 'A-model' is due to certain structural similarities with the familiar 2d A-model (reviewed in Section 7.1). In the first place, it is a cohomological topological sigma model. Also, like the 2d A-model (but unlike the 2d B-model), the 4d A-model localizes on a set of elliptic partial differential equations. Lastly, the 4d A-model reduces to the 2d A-model with target T^*X upon reduction on the cylinder $S^1 \times I$, where S^1 is a circle and I is an interval with certain prescribed boundary conditions imposed at either end.

Before launching into a detailed description of the 4d A-model, let us elaborate a bit on our discussion of cohomological TFTs in the preceding chapter. We have said that the action of a cohomological TFT is invariant under the fermionic symmetry δ_Q (the *BRST variation*), which squares to zero modulo equations of motion. In all the models we shall discuss, we choose to include auxiliary fields in order to make $\delta_Q^2 = 0$ hold even without using the equations of motion. Moreover, the action possesses a $U(1)$ *ghost number* symmetry, with respect to which δ_Q acts by adding one unit of charge to all fields.

Generically, a QFT correlator is computed as an integral over an infinite-dimensional field configuration space $\mathcal{C}$. TFTs, on the other hand, have the special feature that integrals over $\mathcal{C}$ localize on a finite-dimensional subspace $\mathcal{M} \in \mathcal{C}$, namely, the fixed point locus of the fermionic symmetry δ_Q; setting the right-hand side of the BRST variations δ_Q to zero for all fields yields a set of *localization equations*. (For topological gauge theories, the space $\mathcal{M}$ will be given by further quotienting the localization submanifold by an infinite-dimensional group of gauge transformations.) Hence, the study of TFT correlators reduces to the study of various integrals over the moduli space $\mathcal{M}$.

3.1.1 Field content, action, and variations

The bosonic fields of the 4d A-model [6] consist of a map

$$\Phi : M \to X$$

as well as the fields

$$B \in \Omega^{2+}(M, \Phi^* TX)$$
$$\widetilde{H} \in \Omega^1(M, \Phi^* TX)$$

where the notation $B \in \Omega^{2+}(M, \Phi^* TX)$ indicates that B is a self-dual 2-form on M taking values in the pullback vector bundle $\Phi^* TX$. The fermions consist of

$$\zeta \in \Omega^0(M, \Phi^* TX)$$
$$\widetilde{\chi} \in \Omega^1(M, \Phi^* TX)$$
$$\widetilde{\psi} \in \Omega^{2+}(M, \Phi^* TX)$$

The fields Φ, B, and $\widetilde{H}$ have ghost number 0, while ζ and $\widetilde{\psi}$ have ghost number 1, and $\widetilde{\chi}$ has ghost number -1. We describe configurations of Φ via local functions $\phi^I(x)$, where ϕ^I are local coordinates on X for $I = 1, \ldots, \dim X$ and x^μ are local coordinates on M for $\mu = 1, \ldots, \dim M$. Likewise, configurations of tangent-valued fields are written in components as

$$\zeta = \zeta^I \partial_I$$

where $\partial_I \equiv \partial/\partial\phi^I$ and $\zeta^I = \zeta^I(x)$.

The BRST-variations are given by

$$\begin{aligned}
\delta_Q \phi^I &= \zeta^I \\
\delta_Q \zeta^I &= 0 \\
\delta_Q B^I &= \widetilde{\psi}^I - \Gamma^I_{JK}\zeta^J B^K \\
\delta_Q \widetilde{\psi}^I &= \frac{1}{2} R^I{}_{JKL}\, B^J \zeta^K \zeta^L - \Gamma^I_{JK}\zeta^J \widetilde{\psi}^K \\
\delta_Q \widetilde{\chi}^I &= \widetilde{H}^I - \Gamma^I_{JK}\zeta^J \widetilde{\chi}^K \\
\delta_Q \widetilde{H}^I &= \frac{1}{2} R^I{}_{JKL}\, \widetilde{\chi}^J \zeta^K \zeta^L - \Gamma^I_{JK}\zeta^J \widetilde{H}^K
\end{aligned} \tag{3.1}$$

where Γ^I_{JK} are the Christoffel symbols and $R^I{}_{JKL}$ are the Riemann tensor components of the Levi-Civita connection with respect to the metric g_{IJ} (regarded as functions of the field Φ). The nilpotence condition $\delta_Q^2 = 0$ acting on all fields follows from using the definition of the Riemann tensor and Bianchi identity.

The action S for the 4d A-model is chosen to have the BRST-exact form

$$S = \frac{1}{e^2}\delta_Q \int_M d^4x\sqrt{h}\; g_{IJ}\widetilde{\chi}^I_\mu \Big(\widetilde{H}^{J\mu} - 2\partial^\mu\phi^J - 2D_\nu B^{J\nu\mu}\Big)$$

where

$$D_\nu B^{J\nu\mu} = \partial_\nu B^{J\nu\mu} + \gamma^\nu_{\nu\rho} B^{J\rho\mu} + \Gamma^J_{KL}\partial_\nu\phi^K B^{L\nu\mu}$$

refers to the connection on $\Lambda^2 TM \otimes \Phi^* TX$ induced from Levi-Civita with respect to $h_{\mu\nu}$ (with coefficients $\gamma^\mu_{\nu\rho}$) and the pullback of Levi-Civita with respect to g_{IJ}. We have introduced a parameter $1/e^2$ multiplying the overall action. Since it is buried inside a BRST-exact term, changing the value of e modifies the action by a BRST-exact amount, and correlators are therefore independent of the particular value of e. Acting with δ_Q, one finds that a kinetic term for $\widetilde{H}^I$ is absent (i.e., $\widetilde{H}^I$ is an auxiliary field) and the kinetic terms for the other bosonic fields take conventional forms.

The localization equations for this theory are given by setting the right-hand side of the δ_Q variations above to zero, which is the equation

$$\partial^\mu\phi^I + D_\nu B^{I\nu\mu} = 0$$

with fermions set to zero. This can be thought of as 4d analogue of the holomorphic instanton equation of the 2d A-model (see Section 7.1). For each value of I, this nonlinear PDE represents four equations in four variables since self-duality implies that $B^{I\nu\mu}$ has three independent components for each I while ϕ^I has one component for each I. The principal symbol of the equation depends only on the form of highest-order terms, which form is left invariant by coordinate changes on M. Choosing normal coordinates of a point in M and further using invariance of the equations under target space coordinate changes, ellipticity of the differential equations follows.

3.1.2 Local observables

Let us briefly discuss the spectrum of local observables of the 4d A-model, i.e., BRST-closed local operators modulo BRST-exact local operators. In order to write down a 0-form functional of the fields, one could optionally use the metric $h_{\mu\nu}$ to contract indices of the 1- and 2-form fields. However, the resulting observables would automatically be BRST-exact. Hence, in each BRST-cohomology class of local observables, one can choose a representative that is a (BRST-closed) functional $\mathcal{O}$ of the 0-form fields ϕ^I and ζ^I.

Given such a functional $\mathcal{O}$ of definite ghost number k, there is an obvious way to write down a corresponding differential k-form $W_{\mathcal{O}}$ on X, namely,

$$\mathcal{O} = \frac{1}{k!} f_{I_1 \ldots I_k}(\phi)\, \zeta^{I_1} \ldots \zeta^{I_k} \qquad \longleftrightarrow \qquad W_{\mathcal{O}} = \frac{1}{k!} f_{I_1 \ldots I_k}(\phi)\, d\phi^{I_1} \wedge \cdots \wedge d\phi^{I_k} \ .$$

From the BRST variations (3.1), one sees that the BRST variation of fields corresponds to exterior differentiation of forms on X in the sense that

$$W_{\delta_Q \mathcal{O}} = dW_{\mathcal{O}} \ .$$

Hence, BRST-cohomology classes of local observables correspond to the de Rham cohomology classes

$$H^{\bullet}_{\text{de Rham}}(X) \ .$$

3.2 Gauged 4d A-model

As indicated in the chapter introduction, we wish to gauge the 4d A-model with target X by choosing an action of a Lie group G on X and coupling it to a 4d topological gauge theory.

In general the gauging procedure is not unique. Our choice of gauging is guided by our desire to make contact with the Vafa-Witten theory; i.e., we wish to describe Vafa-Witten theory as a member of a class of gauged 4d A-models. To this end, we choose to couple the 4d A-model to a Donaldson-Witten topological gauge theory with gauge group G and, when writing down an action for the theory, we include certain 'nonminimal' BRST-exact terms.

3.2.1 Donaldson-Witten topological gauge theory

We recall the field content, variations and action of Donaldson-Witten topological gauge theory [45] with gauge group G, which is the sole twist of $\mathcal{N} = 2$ SYM in 4d.

The bosonic fields consist of the gauge field A as well as the following forms taking values in $ad\, P$:

$$\sigma \in \Omega^0(M, ad\, P) \otimes \mathbb{C}$$
$$H \in \Omega^{2+}(M, ad\, P)$$

where '2+' indicates that H is a self-dual 2-form with respect to the metric h on M and the complex 0-form σ consists of a pair of real 0-forms. The fermionic fields consist of

$$\eta \in \Omega^0(M, ad\, P)$$
$$\psi \in \Omega^1(M, ad\, P)$$
$$\chi \in \Omega^{2+}(M, ad\, P)$$

The fields A and H have ghost number 0, σ has ghost number 2, ψ has ghost number 1, and both η and χ have ghost number -1. The BRST variations are given by

$$\begin{aligned} &\delta_Q A = \psi, \quad \delta_Q \psi = -d_A \sigma, \\ &\delta_Q \sigma = 0, \\ &\delta_Q \bar{\sigma} = \eta, \quad \delta_Q \eta = [\sigma, \bar{\sigma}] \\ &\delta_Q \chi = H, \quad \delta_Q H = [\sigma, \chi] \end{aligned} \tag{3.2}$$

where $\bar{\sigma} = -\sigma^\dagger$. Unlike the variations of the 4d A-model, these variations are not nilpotent; rather, $\delta_Q^2 = \delta_\sigma$ acting on all fields. [1] The BRST-exact action is given by

$$S = \frac{1}{e^2} \delta_Q \int_M d^4x \sqrt{h}\, \mathrm{Tr} \Big\{ \chi_{\mu\nu} \big(H^{\mu\nu} - 2F^{+\mu\nu} \big) + 2\bar{\sigma}\, D_\mu \psi^\mu \Big\} \tag{3.3}$$

where F^+ is the self-dual part of F. In addition, we include a term in the action measuring the topology of the bundle P:

$$-\frac{i\tau}{4\pi} \int_M \mathrm{Tr}\, F \wedge F$$

[1] Since σ is complex, δ_σ is really an infinitesimal $G_{\mathbb{C}}$ transformation, where $G_{\mathbb{C}}$ is the complexified Lie group. The action will be invariant with respect to $G_{\mathbb{C}}$ transformations.

where recall that the coupling τ is defined in terms of an angular parameter θ and the coupling e as follows:

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2} \ .$$

3.2.2 Matter sector

Choose an action of Lie group G on the manifold X:

$$l : G \times X \to X \ .$$

For a fixed $x \in X$, we have a mapping $l(-, x) : G \to X$ sending the identity element of G to x. The differential of this mapping at the identity is an endomorphism $\mathfrak{g} \to T_x X$ or, equivalently, defines an element $V \in T_x X \otimes \mathfrak{g}^*$. We write this element as

$$V = V_a^I \frac{\partial}{\partial \phi^I} \widetilde{T}^a$$

where $\widetilde{T}^a$ is the basis for $\mathfrak{g}^*$ dual to the basis T_a for $\mathfrak{g}$. The vector fields V_a have Lie brackets

$$[V_a, V_b] = -f^c_{\ ab} V_c \ .$$

For a fixed $g \in G$, we have a mapping $l_g \equiv l(g, -) : X \to X$, which can be thought to act on the sigma model map by composition (i.e., $\Phi \to \Phi' = l_g \circ \Phi$). The differential of this mapping at x is an endomorphism $dl_g : T_x X \to T_{l_g(x)} X$, which can be thought to act on a $\Phi^* TX$-valued field ζ as

$$\zeta'^I = \frac{\partial \phi'^I}{\partial \phi^J} \zeta^J \ .$$

The infinitesimal form of these transformations on the fields is

$$\begin{aligned} \delta_\epsilon \phi^I &= \epsilon^a V_a^I \\ \delta_\epsilon \zeta^I &= \epsilon^a \zeta^J \partial_J V_a^I \end{aligned} \tag{3.4}$$

where $\epsilon \in \mathfrak{g}$ is the parameter of the G-action. The other sigma model fields take values in $\Phi^* TX$ and therefore obey the same transformation law as that for ζ above.

We take l to act on X by isometries, so that the variation δ_ϵ above is a global symmetry of the 4d A-model action, for a fixed parameter $\epsilon \in \mathfrak{g}$. In order to promote this to a local symmetry (with $\epsilon(x)$ varying over M), we make the derivatives appearing in the 4d A-model action gauge-covariant with respect to (3.4) as follows. We replace the field $\Phi : M \to X$ by a field $\mathbf{\Phi} : M \to E$, which is no longer merely a map into M, but, rather, a section of the associated fiber bundle $P \times_G X$ with typical fiber X associated to P via the action l. Here E is the total space. See appendix C for a detailed account of associated fiber bundles.

Likewise, we replace the field ζ, taking values in the pullback Φ^*TX, by a field $\boldsymbol{\zeta}$, taking values in the pullback $\mathbf{\Phi}^*VE$, where, as defined in appendix C, the vertical bundle VE consists of tangent vectors to E pointing along the fibers. Similar replacements are made for the remaining sigma model fields.

Triviality of the underlying principal bundle P over the neighborhood $U^{(\alpha)} \subset M$ induces a map $x^{(\alpha)} : E^{(\alpha)} \to X$, where $E^{(\alpha)} \equiv \pi_E^{-1}(U^{(\alpha)})$. The locally defined field $\Phi^{(\alpha)} \equiv x^{(\alpha)} \circ \mathbf{\Phi}$ is a map

$$\Phi^{(\alpha)} : U^{(\alpha)} \to X$$

while the locally defined field $\zeta^{(\alpha)} \equiv dx^{(\alpha)} \circ \boldsymbol{\zeta}$ is a section

$$\zeta^{(\alpha)} \in \Omega^0(U^{(\alpha)}, {\Phi^{(\alpha)}}^* TX)$$

and similarly for the other fields. Using local coordinates on the target, we can further trivialize the bundles in which the fields take values: if $V^{(\gamma)} \subset X$ is a coordinate chart with coordinates $\phi^{(\gamma)I}$, then on $U^{(\alpha,\gamma)} \equiv U^{(\alpha)} \cap \left[\Phi^{(\alpha)}\right]^{-1}\!\left(V^{(\gamma)}\right)$ we write the field $\zeta^{(\alpha)}$ in terms of its components

$$\zeta^{(\alpha)} = \zeta^{(\alpha,\gamma)I} \frac{\partial}{\partial \phi^{(\gamma)I}} \ .$$

As described in Appendix C, the connection A provides a notion of vertical projection $A_E^V : TE \to VE$, which in turn allows us to define a covariant derivative acting on sections of E as follows:

$$d_A \mathbf{\Phi} \equiv A_E^V \circ d\mathbf{\Phi} \in \Omega^1(\Sigma, \mathbf{\Phi}^*(VE)) \ .$$

Locally, the covariant derivative is represented by the tangent-valued 1-forms

$$d_A \Phi^{(\alpha)} \equiv dx^{(\alpha)} \circ d_A \mathbf{\Phi} \in \Omega^1(U^{(\alpha)}, {\Phi^{(\alpha)}}^* TX)$$

or, even more locally, by the ordinary 1-forms

$$\mathcal{D}\phi^I_{(\alpha,\gamma)} \equiv (d_A \Phi^{(\alpha)})^* \, d\phi^{(\gamma)I} \in \Omega^1(U^{(\alpha,\gamma)})$$

The covariant derivative of sections of $\mathbf{\Phi}^*VE$ is similarly defined. The local 1-forms are given explicitly by

$$\begin{aligned}\mathcal{D}\phi^I &= d\phi^I + A^a V_a^I \\ \mathcal{D}\zeta^I &= d\zeta^I + \Gamma^I_{JK}\mathcal{D}\phi^J\zeta^K + A^a\zeta^J\partial_J V_a^I\end{aligned} \tag{3.5}$$

where we go back to suppressing neighborhood labels (α, γ) with the understanding that the quantities ϕ^I and ζ^I are only defined locally on M. Gauge-covariant derivatives of the other matter fields are similarly defined.

We must contend with one more potential obstruction to gauging. As written in the previous section, the variations of the 4d A-model satisfy $\delta_Q^2 = 0$; however, if we are to successfully couple this theory to a copy of Donaldson-Witten theory, we must alter the variations in order that they satisfy $\delta_Q^2 = \delta_\sigma$, an infinitesimal gauge transformation with parameter σ. This can be easily achieved by adding appropriate V^I-dependent terms to the right-hand side of the variations for the fields ζ, $\widetilde{\psi}$, and $\widetilde{H}$. Having added these terms, the BRST variations for the matter sector of the gauged 4d A-model are given by

$$\begin{aligned}\delta_Q\phi^I &= \zeta^I \\ \delta_Q\zeta^I &= \sigma^a V_a^I \\ \delta_Q B^I &= \widetilde{\psi}^I - \Gamma^I_{JK}\zeta^J B^K \\ \delta_Q\widetilde{\psi}^I &= \frac{1}{2}R^I{}_{JKL}\, B^J\zeta^K\zeta^L - \Gamma^I_{JK}\zeta^J\widetilde{\psi}^K + \sigma^a B^J\nabla_J V_a^I \\ \delta_Q\widetilde{\chi}^I &= \widetilde{H}^I - \Gamma^I_{JK}\zeta^J\widetilde{\chi}^K \\ \delta_Q\widetilde{H}^I &= \frac{1}{2}R^I{}_{JKL}\,\widetilde{\chi}^J\zeta^K\zeta^L - \Gamma^I_{JK}\zeta^J\widetilde{H}^K + \sigma^a\widetilde{\chi}^J\nabla_J V_a^I\end{aligned} \tag{3.6}$$

where

$$\nabla_J V_a^I = \partial_J V_a^I + \Gamma^I_{JK}V_a^K \ .$$

The action S for the gauged 4d A-model is taken to be the following sum of terms:

$$S = \int_M d^4x\sqrt{h}\left(\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5\right) - \frac{i\tau}{4\pi}\int_M \mathrm{Tr}\, F\wedge F$$

where

$$
\begin{aligned}
\mathcal{L}_1 &= \frac{1}{e^2}\delta_Q \mathrm{Tr}\Big\{\chi_{\mu\nu}\big(H^{\mu\nu} - 2F^{+\mu\nu}\big) + 2\bar{\sigma}\, D_\mu\psi^\mu\Big\} \\
\mathcal{L}_2 &= \frac{1}{e^2}\delta_Q\Big\{g_{IJ}\widetilde{\chi}^I_\mu\Big(\widetilde{H}^{J\mu} - 2\mathcal{D}^\mu\phi^J - 2\mathcal{D}_\nu B^{J\nu\mu}\Big)\Big\} \\
\mathcal{L}_3 &= \frac{1}{e^2}\delta_Q\Big\{ - g_{IJ}\chi^a_{\mu\nu}\Big(V^I_a B^{J\mu\nu} + \frac{1}{2}(\nabla_K V^I_a)B^{J\mu\tau}B^{K\nu}_\tau\Big)\Big\} \\
\mathcal{L}_4 &= \frac{1}{e^2}\delta_Q\Big\{g_{IJ}\bar{\sigma}^a\Big(2V^I_a\zeta^J + \frac{1}{2}(\nabla_K V^I_a)\widetilde{\psi}^J_{\mu\nu}B^{K\mu\nu}\Big)\Big\} \\
\mathcal{L}_5 &= \frac{1}{e^2}\delta_Q \mathrm{Tr}\Big\{\eta[\sigma,\bar{\sigma}]\Big\}\ .
\end{aligned}
\tag{3.7}
$$

We have obtained this action by, first of all, including the action S_1 for Donaldson-Witten topological gauge theory (including a theta term) as well as the gauge-covariantized action S_2 for the matter sector. Since our goal is to achieve contact with Vafa-Witten theory, we have chosen to include the additional 'nonminimal' terms S_3, S_4, and S_5, which reduce to terms appearing in the Vafa-Witten action for a certain choice of target and G-action (as we shall demonstrate in the next section). In particular, the final term S_5 is a BRST-exact potential term, which does not affect the equations of motion of the auxiliary fields or aid in localization; it is included in the Vafa-Witten action strictly to make it formally equivalent to the twist of the $\mathcal{N}=4$ SYM action.

The localization equations of the gauged 4d A-model are given by

$$
\begin{aligned}
F^{+a}_{\mu\nu} + \frac{1}{2}g_{IJ}\kappa^{ab}V^I_b B^J_{\mu\nu} + \frac{1}{4}g_{IJ}\kappa^{ab}(\nabla_K V^I_b)B^J_\mu{}^\tau B^K_{\tau\nu} &= 0 \\
\mathcal{D}^\mu\phi^I + \mathcal{D}_\nu B^{I\nu\mu} &= 0
\end{aligned}
$$

in addition to the following equations involving the complex 0-form σ:

$$
\begin{aligned}
D_\mu\sigma &= 0 \\
[\sigma,\bar{\sigma}] &= 0 \\
\sigma^a V^I_a &= 0 \\
\sigma^a B^J_{\mu\nu}\nabla_J V^I_a &= 0\ .
\end{aligned}
$$

3.2.3 Local observables

Local observables of the gauged 4d A-model are gauge- and BRST-invariant functionals of the 0-form fields ϕ^I, ζ^I, and σ. (Inspection of the variations reveals that BRST cohomology class contains a representative free of the 0-form fields $\bar{\sigma}$ and η.)

Given such a functional $\mathcal{O}$, one can write down a corresponding *equivariant form* in $S(\mathfrak{g}^*)\otimes\Omega^\bullet(X)$, where $S(\mathfrak{g}^*)\approx\mathbb{C}[\mathfrak{g}]$ is the symmetric algebra on $\mathfrak{g}^*$, or equivalently, the algebra of polynomials on $\mathfrak{g}$. (See Section C.2 for more on equivariant forms.) For instance, we map

$$\mathcal{O} = \frac{1}{k!} f_{I_1 \ldots I_k}(\Phi)\, P(\sigma)\, \zeta^{I_1} \ldots \zeta^{I_k} \qquad \longleftrightarrow \qquad W_{\mathcal{O}} = P(\sigma)\, \otimes\, \frac{1}{k!} f_{I_1 \ldots I_k}(\Phi)\, d\phi^{I_1} \wedge \cdots \wedge d\phi^{I_k}$$

where $P(\sigma)$ is a homogenous polynomial in the σ^a. If the degree of P is j, then the ghost number of $\mathcal{O}$ is $k + 2j$. Gauge invariance restricts $W_{\mathcal{O}}$ to lie in the G-invariant subalgebra $\big(S(\mathfrak{g}^*) \otimes \Omega^\bullet(X)\big)^G$.

The BRST variation of the gauged 4d A-model can be verified to correspond to acting with the equivariant Cartan differential

$$d_C = 1 \otimes d + \sigma^a \otimes \iota_a$$

where ι_a denotes contraction of forms on X with the vector field $V_a = V_a^I \frac{\partial}{\partial \phi^I}$. Hence, BRST-cohomology of local observables corresponds to the cohomology of the $\mathbb{Z}$-graded complex

$$\Big(\big(S(\mathfrak{g}^*) \otimes \Omega^\bullet(X)\big)^G, d_C \Big)$$

which, by a theorem of Cartan [16], is precisely the *equivariant cohomology* $H_G^\bullet(X)$. The $\mathbb{Z}$-grading is achieved by assigning cohomological degree two to σ^a, which is consistent with our ghost number assignments.

3.3 Vafa-Witten theory as a gauged 4d A-model

We show that for a particular choice of target space and G-action, the gauged 4d A-model of the previous section is equivalent to the Vafa-Witten theory with gauge group G on the four-manifold M. Let X be the vector space $\mathfrak{g}$ and take the action l to be the Ad action of G on its Lie algebra. The basis T_a provides us with a global set of coordinates ϕ^a for $\mathfrak{g}$, which, in turn, collapses the distinction between tangent indices $I, J, \ldots$ and Lie algebra indices $a, b, \ldots$. The Riemannian metric on $\mathfrak{g}$ is taken to be

$$g_{ab} = -\kappa_{ab}$$

(recall that κ is *negative*-definite). Since X is a flat vector space, the Christoffel symbols Γ^a_{bc} vanish identically, thus simplifying the expressions for the variations and covariant derivatives of the previous section. For a fixed $\phi = \phi^a T_a \in \mathfrak{g}$, the differential $dl(-, \phi)$ is the ad representation of $\mathfrak{g}$ on itself. The corresponding element $V \in T\mathfrak{g} \otimes \mathfrak{g}^* \simeq \mathfrak{g} \otimes \mathfrak{g}^*$ has the components

$$V_a^c = f^c_{\ ab} \phi^b \ .$$

In matrix language, the infinitesimal gauge transformations (3.4) reduce to the usual form

$$\delta_\epsilon \phi = [\epsilon, \phi]$$
$$\delta_\epsilon \zeta = [\epsilon, \zeta]$$

for $ad\, P$-valued fields.

The gauge sector variations (3.2), together with the matter sector variations (3.6) reduce to following form:

$$\begin{aligned}
&\delta_Q A = \psi, \quad \delta_Q \psi = -d_A \sigma,\\
&\delta_Q \sigma = 0,\\
&\delta_Q \bar{\sigma} = \eta, \quad \delta_Q \eta = [\sigma, \bar{\sigma}],\\
&\delta_Q \chi = H, \quad \delta_Q H = [\sigma, \chi],\\
&\delta_Q \phi = \zeta, \quad \delta_Q \zeta = [\sigma, \phi],\\
&\delta_Q B = \widetilde{\psi}, \quad \delta_Q \widetilde{\psi} = [\sigma, B]\\
&\delta_Q \widetilde{\chi} = \widetilde{H}, \quad \delta_Q \widetilde{H} = [\sigma, \widetilde{\chi}]\ .
\end{aligned} \tag{3.8}$$

Indeed, these are the same field content and variations as those of the off-shell formulation of Vafa-Witten topological gauge theory studied in [3] and [9]. Moreover the action (3.7) reduces to the following form:

$$S = \int_M d^4x\sqrt{h}\,\Big(\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5\Big) - \frac{i\tau}{4\pi}\int_M \mathrm{Tr}\, F \wedge F$$

where

$$\begin{aligned}
\mathcal{L}_1 &= \frac{1}{e^2}\delta_Q\ \mathrm{Tr}\Big\{\chi_{\mu\nu}\big(H^{\mu\nu} - 2F^{+\mu\nu}\big) + 2\bar{\sigma}\, D_\mu \psi^\mu\Big\}\\
\mathcal{L}_2 &= \frac{1}{e^2}\delta_Q \mathrm{Tr}\Big\{\widetilde{\chi}_\mu\Big(\widetilde{H}^\mu - 2D^\mu\phi - 2D_\nu B^{\nu\mu}\Big)\Big\}\\
\mathcal{L}_3 &= \frac{1}{e^2}\delta_Q \mathrm{Tr}\Big\{-\chi_{\mu\nu}\Big([B^{\mu\nu}, \phi] + \frac{1}{2}[B^{\mu\tau}, B_\tau{}^\nu]\Big)\Big\}\\
\mathcal{L}_4 &= \frac{1}{e^2}\delta_Q \mathrm{Tr}\Big\{-\bar{\sigma}\Big(\frac{1}{2}[\widetilde{\psi}_{\mu\nu}, B^{\mu\nu}] + 2[\zeta, \phi]\Big)\Big\}\\
\mathcal{L}_5 &= \frac{1}{e^2}\delta_Q \mathrm{Tr}\Big\{\eta[\sigma, \bar{\sigma}]\Big\}
\end{aligned} \tag{3.9}$$

which is equivalent (when $M = \mathbb{R}^4$) to the action of the Vafa-Witten twist of $\mathcal{N} = 4$ SYM theory [9].

The localization equations of the gauged 4d A-model reduce to the form

$$\begin{aligned}
F^+_{\mu\nu} + \frac{1}{2}[B_{\mu\nu}, \phi] + \frac{1}{4}[B_\mu{}^\tau, B_{\tau\nu}] &= 0\\
D^\mu\phi + D_\nu B^{\nu\mu} &= 0
\end{aligned} \tag{3.10}$$

in addition to the equations

$$D_\mu \sigma = 0$$
$$[\sigma, \bar{\sigma}] = [\sigma, \phi] = 0 \tag{3.11}$$
$$[\sigma, B_{\mu\nu}] = 0 \ .$$

These are exactly equal to the equations (2.55) and (2.56) in [3], which define the Vafa-Witten moduli problem. (To make contact with the notation in [3], note that Vafa and Witten take $\mathfrak{g}$-valued fields to be Hermitian rather than anti-Hermitian and leave a factor of 'i' implicit in their commutators.) In [3] it is shown that the action possesses a symmetry under $\phi \to -\phi$, in consequence of which every solution of the above set of equations additionally obeys

$$D_\mu \phi = [\phi, B_{\mu\nu}] = 0 \ .$$

Chapter 4

Topological field theory, categories, and 2-categories

So far, we have been discussing a few specific examples of cohomological 4d TFTs. Their spectra of observables turn out to be endowed with rich algebraic structures, and duality relationships between different TFTs imply nontrivial equivalences between these algebraic structures. In this chapter, we set the stage for analyzing these structures by reviewing features shared by *all* 4d TFTs.

Indeed, when one places a 4d TFT on a manifold M with nontrivial boundary ∂M, one must impose boundary conditions on the fields along ∂M. The set of all such boundary conditions turns out to be endowed with a very rich algebraic structure known as a *3-category*, as we explain below. One can also consider defects of higher codimension; for instance, codimension two *surface operators*. These too are endowed with an algebraic structure, known as a *2-category*, and a natural way to study this 2-category is to compactify the 4d TFT to an effective 3d TFT on a circle. Similarly, one can study codimension three *line operators* by compactifying the 4d TFT to two dimensions on a 2-sphere.

Hence, in order to disentangle the information encoded in a 4d TFT, one is naturally led to studying lower-dimensional TFTs. In this chapter, we systematically review the categorical structures attached to TFTs in dimensions two, three, and four.

4.1 Two-dimensional TFT and categories of branes

For us, a category is a generalization of an algebra, 'an algebra with many objects'. That is, instead of one vector space V with a multiplication map $V \otimes V \to V$ we have a set Ob, a collection of vector spaces V_{AB}, $\mathsf{A}, \mathsf{B} \in$ Ob, and composition maps $V_{\mathsf{AB}} \otimes V_{\mathsf{BC}} \to V_{\mathsf{AC}}$. These composition maps must be associative, in an obvious sense. In particular, for each A the space V_{AA} is a (possibly noncommutative) algebra. We will assume in addition that all these algebras have unit elements.

The set Ob is called the set of objects, and the vector spaces V_{AB} are called spaces of morphisms.

An element of V_{AA} is called an endomorphism of A, and V_{AA} is called the endomorphism algebra of A. It is common to denote $V_{\mathsf{AB}} = \mathrm{Mor}(\mathsf{A}, \mathsf{B})$ and $V_{\mathsf{AA}} = \mathrm{Mor}(\mathsf{A}, \mathsf{A}) = \mathrm{End}(\mathsf{A})$. In physical applications the vector spaces are always complex and usually have integral grading (by some sort of $U(1)$ charge).

It is well known by now that the set of boundary conditions in a 2d TFT has the structure of a category. The set Ob of this category is the set of boundary conditions, and the vector space V_{AB} is the space of states of the TFT on an (oriented) interval with boundary conditions A and B. Composition of morphisms arises from the fact that the space V_{AB} can be interpreted as the space of local operators sitting at the junction of two segments of the boundary with boundary conditions A and B (figure 4.1), and from the fact that local operators can be fused together (figure 4.2).

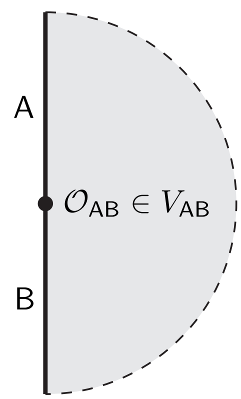

Figure 4.1: Morphisms in the category of boundary conditions correspond to local operators sitting at the junction of two segments of the boundary.

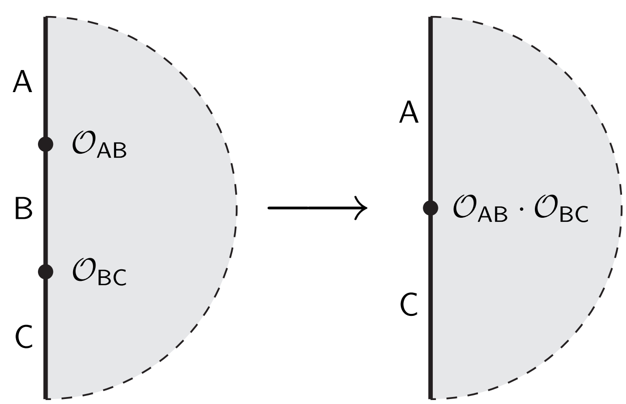

Figure 4.2: Composition of morphisms is achieved by merging the insertion points of the local operators. We use $\cdot$ to denote this operation

In the mathematical literature it is common to denote objects by marked points and elements of vector spaces V_{AB} by arrows connecting the points. From the physical viewpoint it is more natural to denote objects by marked segments of an oriented line, and morphisms by points sitting at the junction of two consecutive segments.

Let us recall two simple examples of 2d TFTs which will be important for us. The first one is a B-model with a target X, where X is a complex manifold with a holomorphic volume form (i.e., a possibly noncompact Calabi-Yau manifold). The corresponding category of boundary conditions has been argued to be equivalent to the bounded derived category of coherent sheaves on X, which is denoted $D^b(Coh(X))$. Its objects can be thought of as complexes of holomorphic vector bundles on X or complex submanifolds of X. The second one is an A-model with target Y, where Y is a symplectic manifold. The corresponding category of boundary conditions is believed to be equivalent to a version of the Fukaya-Floer category. Its simplest objects are Lagrangian submanifolds of Y equipped with unitary vector bundles with flat connections. In the 2d context one usually refers to boundary conditions as branes and talks about A-branes and B-branes.

A and B-models do not exhaust the possibilities even in two dimensions. In Chapter 7 we will encounter other, less familiar, 2d TFTs and their categories of branes.

4.2 Two-dimensional TFTs and 2-categories

A boundary of a 2d TFT can be regarded as a boundary between a nontrivial TFT and a trivial TFT. More generally, one can consider boundaries between arbitrary pairs of 2d TFTs. Such boundaries may be called defect lines, or walls. The set of all walls between a fixed pair of TFTs has the structure of a category. To see this, let $\mathbb{X}$ and $\mathbb{Y}$ denote our chosen pair of TFTs, and let $\bar{\mathbb{X}}$ denote the theory $\mathbb{X}$ with a reversed parity. By folding back the worldsheet at the wall location (see figure 4.3), we see that a wall between $\mathbb{X}$ and $\mathbb{Y}$ is the same as a boundary of the theory $\bar{\mathbb{X}} \times \mathbb{Y}$. Thus we may appeal

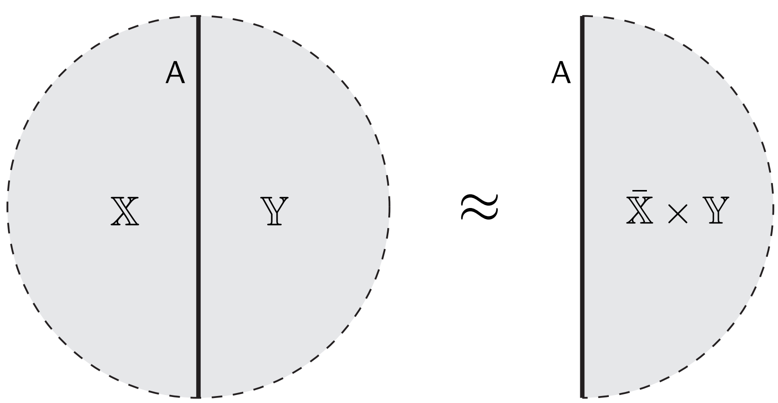

Figure 4.3: A wall separating theories $\mathbb{X}$ and $\mathbb{Y}$ is equivalent to a boundary of theory $\bar{\mathbb{X}} \times \mathbb{Y}$.

to the previous discussion and conclude that walls are objects of a category $\mathbb{V}_{\mathbb{X}\mathbb{Y}}$. Given any two walls $\mathsf{A}, \mathsf{B} \in \mathrm{Ob}(\mathbb{V}_{\mathbb{X}\mathbb{Y}})$, the space of morphisms from A to B is the space of local operators which can be inserted at the junction of A and B. Composition of morphisms is obtained by fusing local operators sitting on a defect line.

There is an obvious 'fusion' operation on the set of walls: given a wall between theories $\mathbb{X}$ and $\mathbb{Y}$ and a wall between theories $\mathbb{Y}$ and $\mathbb{Z}$ we may fuse them and get a wall between theories $\mathbb{X}$ and $\mathbb{Z}$ (figure 4.4). One can describe the situation mathematically by saying that the set of 2d TFTs has the structure of a 2-category. A 2-category has objects, morphisms, and 2-morphisms (morphisms between morphisms). In the present case, objects are 2d TFTs. The set of morphisms from an object $\mathbb{X}$ to an object $\mathbb{Y}$ is the set of walls between theories $\mathbb{X}$ and $\mathbb{Y}$. Fusion of walls gives rise to a way of composing morphisms. Given any two walls between the same pair of TFTs, the space of 2-morphisms between them is the space of local operators which can be inserted at the junction of these two walls.

One can put this slightly differently and say that a 2-category has a collection of objects (which are 2d TFTs in our case), and for any pair of objects $\mathbb{X}$ and $\mathbb{Y}$ one has a category of morphisms $\mathbb{V}_{\mathbb{XY}}$ (which is the category of walls in our case). Fusion of walls means that there is a way to 'compose' categories of morphisms. That is, given an object A of the category $\mathbb{V}_{\mathbb{XY}}$ and an object B of the category $\mathbb{V}_{\mathbb{YZ}}$ there is a rule which determines an object $\mathsf{A} \otimes \mathsf{B}$ of the category $\mathbb{V}_{\mathbb{XZ}}$. This is not all

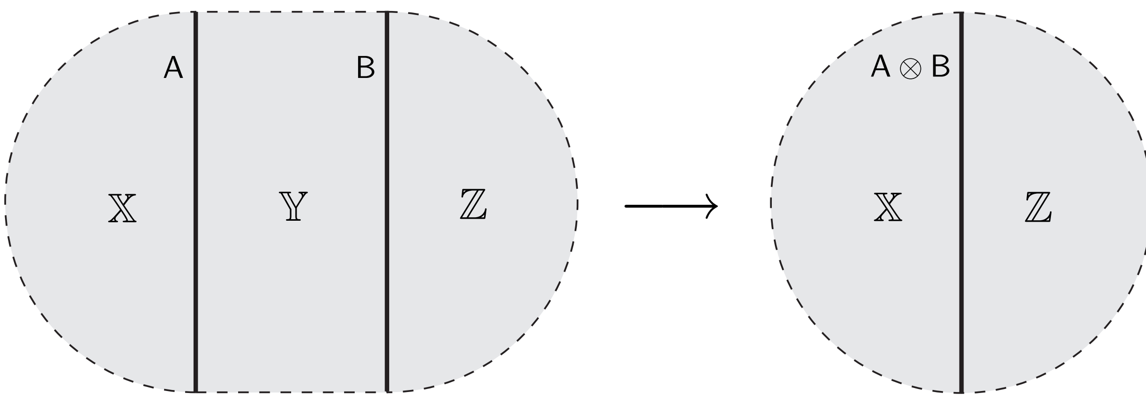

Figure 4.4: 1-morphisms of the 2-category of 2d TFTs correspond to walls, and composition of 1-morphisms corresponds to fusing walls. This operation is denoted $\otimes$.

though: one can fuse not only walls, but walls with local operators inserted on them (figure 4.4). This determines composition maps on 2-morphisms. This new composition is different from the composition of local operators regarded as morphisms in the category $\mathbb{V}_{\mathbb{XY}}$. The composition maps enjoy various properties which can be deduced by staring at the pictures of fusing walls and making use of the metric independence. For example, the old and new compositions commute, as illustrated in figure 4.5.

Even if one is interested in a particular TFT, the notion of a 2-category is useful. Namely, defect lines in a 2d TFT form a 2-category with a single object. In this case there is only one category of morphisms, $\mathbb{V}_{\mathbb{XX}}$, with additional structure coming from the fact that fusing two defect lines gives another defect line in the same theory. This structure allows one to define a rule for 'tensoring'

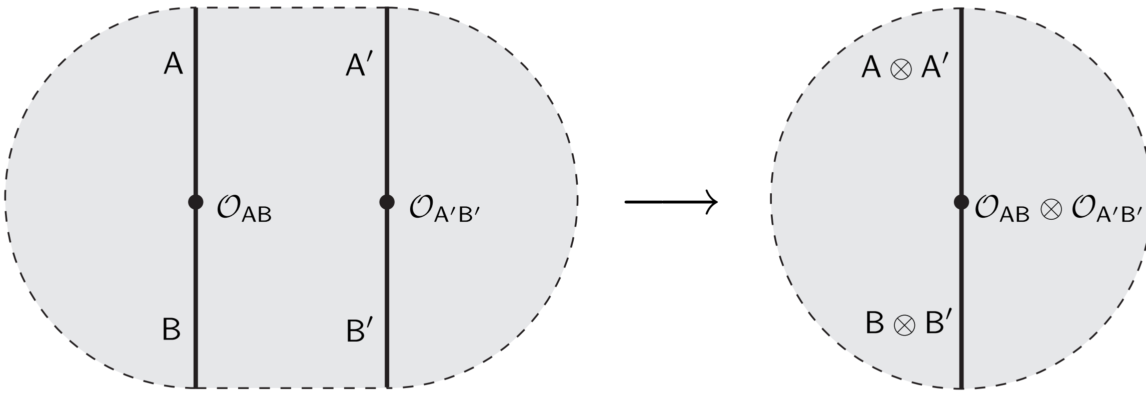

Figure 4.5: Composition of 2-morphisms of the 2-category of 2d TFTs is achieved by fusing the walls on which they are inserted. The corresponding operation is denoted ⊗.

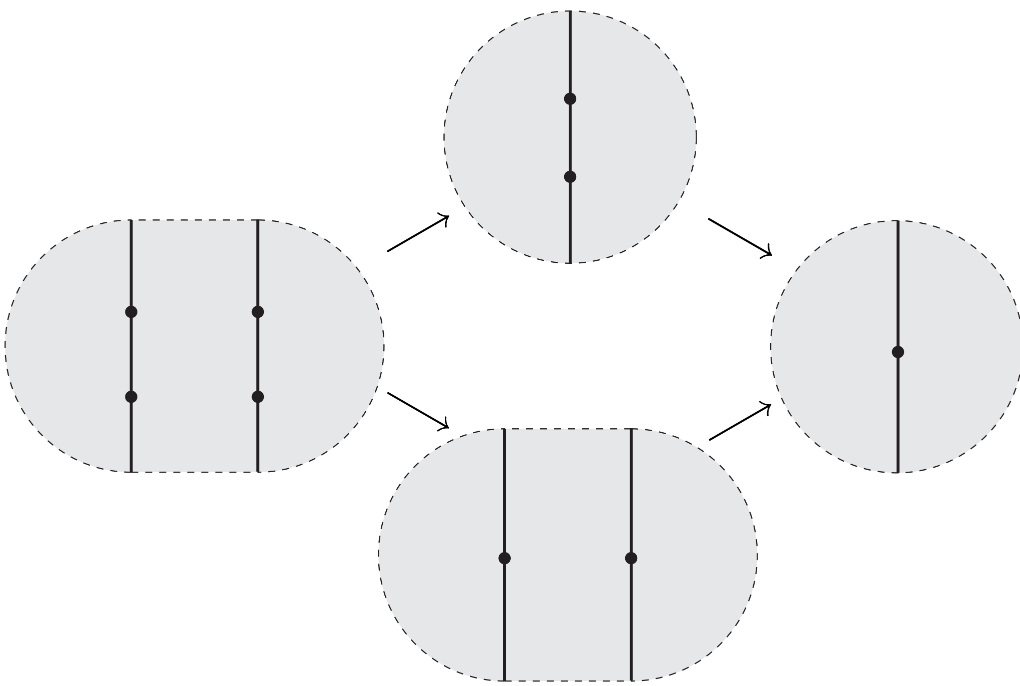

Figure 4.6: Local operators inserted on walls may be regarded either as 2-morphisms of the category of 2d TFTs, in which case composition corresponds to fusing 'horizontally', or they may be regarded as morphisms of the category of boundary conditions, in which case composition corresponds to fusing 'vertically'. These two operations commute.

objects of $\mathbb{V}_{\mathbb{XX}}$:

$$(\mathsf{A}, \mathsf{B}) \mapsto \mathsf{A} \otimes \mathsf{B} \in \mathrm{Ob}(\mathbb{V}_{\mathbb{XX}})$$

and morphisms

$$\mathrm{Mor}(\mathsf{A}, \mathsf{B}) \otimes \mathrm{Mor}(\mathsf{C}, \mathsf{D}) \to \mathrm{Mor}(\mathsf{A} \otimes \mathsf{C}, \mathsf{B} \otimes \mathsf{D})$$

This tensoring does not need to be commutative, in general. A category with such additional 'tensor' structure is called a monoidal category. It should be clear from the above that a monoidal category is the same thing as a 2-category with a single object.

Among all defect lines in a given 2d TFT there is a trivial defect line $\mathbf{1}$ which is equivalent to no defect at all. We may call it the invisible defect line. It is the identity object in the monoidal category of defect lines, in the sense that fusing it with any other defect line A gives back A. Endomorphisms of the trivial defect line (i.e., elements of the vector space $\mathrm{Mor}(\mathbf{1}, \mathbf{1})$) are the same as local operators in the bulk.

The simplest example of a monoidal category is the category of vector spaces, with the usual tensor product. It can be regarded as the 2-category of defect lines in a trivial 2d TFT (say, a topological sigma-model whose target is a point). Defect lines in Landau-Ginzburg TFTs have been studied in [31, 32].

One can fuse a defect line in a given 2d TFT with any boundary condition and get a new boundary condition in the same TFT. This defines an 'action' of the monoidal category of defect lines on the category of branes. Mathematically this can be described using the notion of a module category. Since a monoidal category is a categorification of the notion of an algebra, it is natural to define a module category over a monoidal category as a categorification of the notion of a module over an algebra. A definition of a module category $\mathbb{W}$ over a monoidal category $\mathbb{V}$ involves a rule for 'multiplying' an object on $\mathbb{W}$ by an object of $\mathbb{V}$:

$$(\mathsf{A}, \mathsf{C}) \mapsto \mathsf{A} \cdot \mathsf{C} \in \mathrm{Ob}(\mathbb{W}), \quad \forall \mathsf{A} \in \mathrm{Ob}(\mathbb{W}),\ \forall \mathsf{C} \in \mathrm{Ob}(\mathbb{V}),$$

as well as a rule for multiplying morphisms, i.e., a map

$$\mathrm{Mor}_{\mathbb{W}}(\mathsf{A}, \mathsf{B}) \otimes \mathrm{Mor}_{\mathbb{V}}(\mathsf{C}, \mathsf{D}) \to \mathrm{Mor}_{\mathbb{W}}(\mathsf{A} \cdot \mathsf{C}, \mathsf{B} \cdot \mathsf{D}).$$

The latter rule encodes the fact that we can fuse a junction of two defect lines with a junction of two boundary conditions and get a new junction of two new boundary conditions.

An important idea which we systematically use in this thesis is that some properties of codimension-2 defects can be studied using dimensional reduction. We have already seen a simple example of this: the space of local operators sitting at the junction of two boundary conditions A and B can be thought of as the space of states of a 1d field theory (i.e., quantum mechanics) obtained by compactifying the 2d TFT on an interval with boundary conditions A and B. Another example is the space of local operators in the bulk. It is well known that it can be identified with the space of states of a 1d field theory obtained compactifying the 2d TFT on a circle. The argument is essentially the same in both cases. After one excises a tubular neighborhood of the local operator, the operator insertion is replaced by a boundary whose collar neighborhood looks like $\mathbb{R}_+ \times I$ in the first case and $\mathbb{R}_+ \times S^1$ in the second case. Then one uses the fact that in a TFT the size of the tubular neighborhood does not matter, and one can regard any boundary condition on the newly created boundary as a local operator.

It is important to note that this reinterpretation of codimension-2 defects in terms of a lower-dimensional theory causes 'information loss'. For example, we cannot compute the composition $V_{\mathsf{AB}} \otimes V_{\mathsf{BC}} \to V_{\mathsf{AC}}$ if we view the vector spaces involved as spaces of states of three 1d field theories. Similarly, we cannot see the commutative algebra structure on the space of bulk local operators if we view it as the space of states of a 1d field theory.

4.3 Three-dimensional TFT and 2-categories of boundary conditions

When we move to dimension three, we find that boundary conditions in a 3d TFT also form a 2-category. To see this, let us first consider a trivial 3d TFT (say, a 3d topological sigma-model whose target is a point). Even though there are no bulk degrees of freedom in this case, we may consider putting any 2d TFT on the boundary. Thus the set of all boundary conditions for the trivial 3d TFT is the set of all 2d TFTs, which form a 2-category. Walls between 2d TFTs now can be interpreted as defect lines on the 2d boundary of a 3d worldvolume. The monoidal category of defect lines for a given 2d TFT can be reinterpreted as the monoidal category of boundary line operators for a particular boundary condition.

If the 3d TFT in the bulk is nontrivial, we can still couple it to a 2d TFT on the boundary. Different boundary conditions are distinguished by the type of 2d TFT on the boundary and by its coupling to the bulk degrees of freedom. For a concrete example of how this works in the Rozansky-Witten 3d TFT, see [36]. Again one may consider boundary defect lines separating different boundary conditions, and their fusion and fusion of local operators on boundary defect lines can be described by a 2-category structure on the set of boundary conditions. If we focus on a particular boundary condition, then the set of defect lines on this boundary has the structure of a monoidal category.

Mimicking what we did in 2d, we may consider walls, or surface operators, between different 3d TFTs. The set of walls between any two 3d TFTs $\mathfrak{K}$ and $\mathfrak{L}$ has the structure of a 2-category. One way to see it is to fold the worldvolume along the defect surface and reinterpret the wall as a boundary condition for the theory $\bar{\mathfrak{K}} \times \mathfrak{L}$, where $\bar{\mathfrak{K}}$ is the parity-reversal of the theory $\mathfrak{K}$. Furthermore, just like in 2d, we can fuse walls with defect lines and local operators on them (figure 4.7). Altogether, one

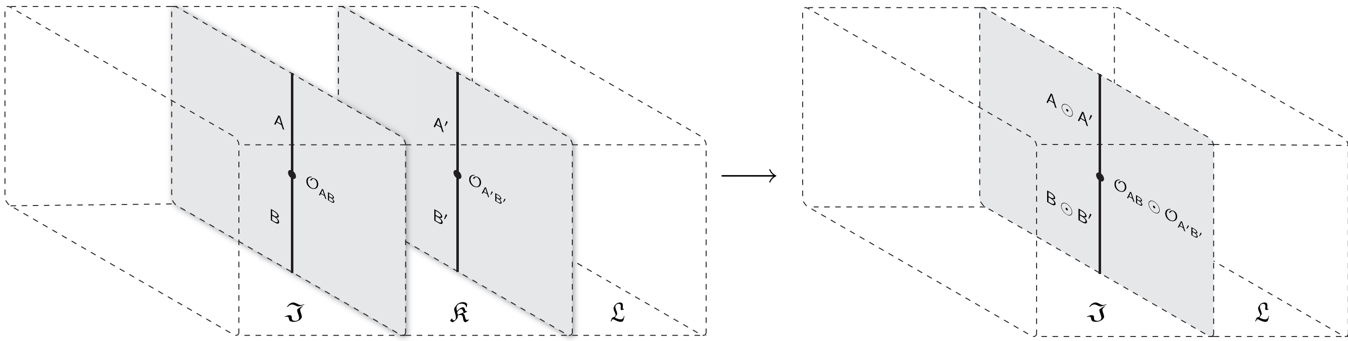

Figure 4.7: Regarding defect lines as 2-morphisms and local operators as 3-morphisms of the 3-category of 3d TFTs gives rise to yet another composition operation between them, which we denote $\odot$.

can summarize the situation by saying that 3d TFTs form a 3-category. Its objects are 3d TFTs, its morphisms are walls between 3d TFTs, its 2-morphisms (morphisms between morphisms) are defect lines on walls, and its 3-morphisms (morphisms between 2-morphisms) are local operators sitting on defect lines.

If we restrict attention to surface operators in a particular 3d TFT, we get a 3-category with a single object. Equivalently, the 2-category of surface operators in a 3d TFT has an extra structure which allows one to fuse objects, morphisms and 2-morphisms. In other words, it is a monoidal 2-category. It has an identity object (the trivial surface operator) whose endomorphisms can be thought of as defect lines in the bulk.

To determine the category of bulk defect lines in a given 3d TFT one may use the dimensional-reduction trick. To apply it, we excise a tubular neighborhood of a defect line and replace the defect line by a suitable boundary condition on its boundary. The collar neighborhood of the newly created boundary locally looks like $S^1 \times \mathbb{R}_+ \times \mathbb{R}$, where $\mathbb{R}_+$ corresponds to the 'radial' direction. Therefore we may identify the defect line with the boundary condition in the 2d TFT obtained by compactifying the 3d TFT on a circle. This trick allows one to determine the category of bulk defect lines by studying the category of branes in a 2d TFT. We lose some information in this way: the category of bulk defect lines in a 3d TFT is in fact a braided monoidal category (i.e., a category with a quasi-commutative tensor product), but this structure cannot be seen from the 2d viewpoint.

Similarly, the study of categories of boundary defect lines reduces to the study of the category of branes in a 2d TFT obtained by compactifying the 3d TFT on an interval. If the boundary defect line separates two boundary conditions $\mathbb{X}$ and $\mathbb{Y}$, then the boundary conditions on the endpoints of the interval should be $\mathbb{X}$ and $\mathbb{Y}$. This 2d viewpoint entails some information loss: for example, it does not allow one to compute the monoidal structure on the category $\mathbb{V}_{\mathbb{X}\mathbb{X}}$.

4.4 Four-dimensional TFT and 2-categories of surface operators

Boundary conditions in a 4d TFT form a 3-category. For example, if the 4d TFT is trivial, its 3-category of boundary conditions is the 3-category of all 3d TFTs. Similarly, walls (codimension-1 defects) in a given 4d TFT form a monoidal 3-category. In this thesis we will avoid dealing with such complicated structures and will focus instead on defects of codimension two, i.e., surface operators in a 4d TFT. Such surface operators form a 2-category. One way to see it is to apply the dimensional-reduction trick: excise a tubular neighborhood of a surface operator and replace the operator by a suitable boundary condition on the newly created boundary. The collar neighborhood of the boundary looks locally like $S^1 \times \mathbb{R}_+ \times \mathbb{R}^2$, so we may reinterpret a surface operator as a boundary condition in a 3d TFT obtained by compactifying the 4d theory on a circle. Then we can appeal to the known fact that boundary conditions in a 3d TFT form a 2-category.

Of course, one can also explain the meaning of this 2-category structure directly in 4d terms. The type of a surface operator may jump across a defect line, and one can regard defect lines on surface operators as morphisms in a 2-category. Local operators sitting at a junction point of two surface defect lines are 2-morphisms. The 4d viewpoint also makes it clear that the 2-category of surface operators has a rich extra structure. First of all, one may fuse surface operators together with defect lines and local operators sitting on them. This gives rise to a monoidal structure on the 2-category of surface operators. Second, by moving surface operators around one can easily see that the fusion operation is quasi-commutative, i.e., one gets a braided monoidal 2-category.[1]

In this thesis, we will use the dimensional-reduction trick to study the 2-category of surface operators of our twisted 4d TFTs in Chapter 6; we will not describe the braided monoidal structure on this 2-category.

[1] A possible mathematical definition of a braided monoidal 2-category is spelled out in [34]. However, it appears that braided monoidal structures which arise in 4d TFT are of a rather special kind. In particular, the braiding is always invertible.

Chapter 5

Compactifications of Vafa-Witten and GL-twisted theories on a circle

We begin the project of compactifying the various twisted versions of $\mathcal{N} = 4$, $d = 4$ SYM theory to lower-dimensional TFTs, starting with the case of compactification to three dimensions on a circle. Our aim is twofold: we wish to shed light on lower-dimensional TFTs (in particular, duality relationships between 4d TFTs extend to duality relationships between their dimensionally reduced versions); moreover, the lower-dimensional TFTs will help us understand aspects of the original 4d TFTs.

For instance, according to the dimensional-reduction trick discussed in Chapter 4, boundary conditions of the 3d theory obtained by compactifying on a circle will classify the spectrum of surface operators in the original 4d theory (we put off an analysis of these boundary conditions until the following chapter).

In the present chapter, we reduce the Vafa-Witten theory and the GL-twisted theories at $t = i$, $t = 1$, and $t = 0$ to three dimensions on a circle. The resulting 3d TFTs will be turn out to be topological, gauged sigma models: the reduction at $t = i$ yields a gauged version of the Rozansky-Witten model introduced in [14], while the reduction at $t = 1$ and $t = 0$ yield gauged versions of the 3d A-model TFT introduced in [6]; the Vafa-Witten theory is equivalent to the $t = 0$ theory when compactified on a circle.

5.1 Compactification of topological gauge theories

Before turning to the S^1 reductions in earnest, let us make a few general remarks about compactification of topological gauge theories. Consider one of our 4d TFTs on the product manifold $M = W \times S^1$ where W is a Riemannian, oriented 3-manifold and S^1 is a circle of circumference $2\pi R$. We wish to find an effective description in the $R \to 0$ limit. (More generally, one can compactify using a small compactification manifold of dimension two or higher.)

The usual Kaluza-Klein notion of compactification of a field theory is that one expands the fields of the 4d theory in terms of their Fourier modes along the circle and, in the limit $R \to 0$, discards all but the constant modes due to the fact that they become infinitely massive. In this manner, one obtains an effective 3d theory on W. (More precisely, one must integrate the nonzero modes out of the path integral, which can induce quantum corrections in the effective theory.)

But a TFT does not 'care' about the size of R, in the sense that changes in R alter the action by BRST-exact amounts and therefore leave topological correlators invariant. Hence (modulo the subtleties we discuss in a moment) the reduced theory — though it describes a simplified region of field space in which fields do not vary along the compact direction — is exactly equivalent to the original theory. This convenient fact has been put to use in [1], where the GL-twisted theories at $t = i$ and $t = 1$ were given 2d descriptions by compactifying on 2d surfaces. We set the circumference at the convenient value $2\pi R = 2\pi$ for the remainder of the chapter.

When our theory is a topological *gauge* theory, there are two important subtleties that complicate this picture. The first is that a given field configuration on the product manifold may leave some residual continuous gauge symmetry unbroken on the compactification manifold. Such field configurations represent singular points in the target space of the effective sigma model description: points where qualitatively new branches in the compactified theory open up. Practically, all one can do is to excise such configurations 'by hand' (for instance, by imposing appropriate boundary conditions that eliminate their occurrence). The 2d effective theories in [1] were truncated in just this manner. This problem does not arise in the present context since our compactification manifold is just S^1.

The second subtlety, however, is important for our analysis. Consider first an abelian gauge theory with gauge field A. It is not quite correct to simply declare A to be x^4-independent and the component A_4 to be a real scalar field, where x^4 is the coordinate for S^1, ranging from $x^4 = 0$ to $x^4 = 2\pi$. This is because the x^4-dependent gauge transformation $g(x^4) = \exp(imx^4)$ shifts the value of A_4 by an integer amount:

$$A_4 \mapsto A_4 + im, \quad m \in \mathbb{Z}.$$

Hence, the 3d effective theory is ill-defined if written directly in terms of the scalar A_4.

The solution to the problem is to write the effective theory instead in terms of the *periodic* scalar

$$h = \exp(-2\pi A_4)$$

which field is invariant with respect to x^4-dependent gauge transformations.

For nonabelian gauge group, an extra complication is that requiring the fields of the 4d theory to be independent of the x^4 coordinate is not a gauge-invariant condition. One can try to avoid dealing with this issue by first fixing a gauge such that A_4 does not depend on x^4, and, indeed, this works in the neighborhood of $A_4 = 0$, i.e., when the holonomy of A along S^1 is close to unity. But in general the condition that A_4 is x^4-independent does not fix the freedom to make x^4-dependent gauge transformations. For example, suppose A_4 is proportional to an element $\mu \in \mathfrak{g}$ satisfying

$$\exp(2\pi\mu) = 1.$$

Such μ are precisely those which lie in the G-orbits of the cocharacter lattice of G. Then the gauge transformation

$$g(x^4) = \exp(\mu x^4)$$

shifts A_4 by μ:

$$A_4 \mapsto A_4 + \mu \ .$$

Such a gauge transformation in general makes other fields x^4-dependent.

The solution again is to replace A_4 in the description of the theory by a group-valued field h, now defined to be the holonomy of A around the compact direction. Let us see how this works in detail. In computing a particular correlator of the 4d theory via compactification, it is convenient to first deform the support of all inserted operators to lie along the embedded hypersurface $W_0 \subset M$, defined as the locus of points in $W \times S^1$ with $x^4 = 0$. (Deforming in this manner does not affect the value of the correlator due to topological invariance.) Next, we define the 3d field h to be the holonomy of the connection around the loop beginning at a point $(\vec{x}, 0) \in W_0$ and winding once around the compact direction:

$$h(\vec{x}) = P\exp\Big(- \int_0^{2\pi} dx^4 A_4(\vec{x}, x^4)\Big)$$

for any $\vec{x} \in W$, where the holonomy is computed in the N-dimensional defining representation. Let P_W be the principal G bundle on W given by pulling back P under the embedding map

$$W \longrightarrow W_0 \longrightarrow M$$

For nonabelian G, the holonomy h is no longer gauge invariant; rather, it is gauge *covariant*, transforming as a section of the fiber bundle with typical fiber G associated to P_W by the conjugation action of G on itself; that is,

$$h \in \Omega^0(W,\ P_W \times_G G) \ .$$

In particular, h is a perfectly well-defined scalar field in three dimensions.

5.2 S^1 compactification of Vafa-Witten theory

Consider first the Vafa-Witten theory placed on the product manifold $M = W \times S^1$. We will show that, on this particular geometry for M, the theory is equivalent to a gauged version of the 3d A-model with target space $X = \mathfrak{g}$, where the gauge sector is a slight modification of the 3d A-type gauge theory discussed in Section B.2.1 (by replacing the algebra-valued scalar field ς by a group-valued field).

Reduced field content

We relabel the fields of the Vafa-Witten theory in a way that exposes the role they will play in the 3d reduced theory. In the following, the indices $r, s, t = 1, 2, 3$ refer to coordinates on W and the superscript $[VW]$ indicates the corresponding field of the Vafa-Witten theory, as we have written in Chapter 3. The 3d reduction of the field content will split into a gauge sector and a matter sector. The 3d gauge sector will come from the relabeled fields:

$$
\begin{aligned}
A_r &= A_r^{[VW]}, & \lambda_r &= \psi_r^{[VW]} \\
\varsigma &= A_4^{[VW]}, & \rho &= \psi_4^{[VW]} \\
\sigma &= \sigma^{[VW]}, & & \\
\bar{\sigma} &= \bar{\sigma}^{[VW]}, & \widetilde{\rho} &= \eta^{[VW]} \\
\widetilde{\lambda}_r &= \chi_{r4}^{[VW]}, & H_r &= H_{r4}^{[VW]} \ .
\end{aligned}
$$

The 3d matter sector will come from the relabeled fields:

$$
\begin{aligned}
\phi &= \phi^{[VW]}, & \eta &= \zeta^{[VW]} \\
b_r &= B_{r4}^{[VW]}, & \psi_r &= \widetilde{\psi}_{r4}^{[VW]} \\
\beta &= \widetilde{\chi}_4^{[VW]}, & P &= \widetilde{H}_4^{[VW]} \\
\chi_r &= \widetilde{\chi}_r^{[VW]}, & \widetilde{P}_r &= \widetilde{H}_r^{[VW]} \ .
\end{aligned}
$$

The BRST variations (3.8) of the Vafa-Witten theory, written in terms of the relabeled fields, become

$$
\begin{aligned}
\delta_Q A &= \lambda, \qquad \delta_Q \lambda = -d_A \sigma, \\
\delta \varsigma &= \rho, \qquad \delta_Q \rho = [\sigma, \varsigma] \\
\delta_Q \sigma &= 0, \\
\delta_Q \bar{\sigma} &= \widetilde{\rho}, \qquad \delta_Q \widetilde{\rho} = [\sigma, \bar{\sigma}] \\
\delta_Q \widetilde{\lambda} &= H, \quad \delta_Q H = [\sigma, \widetilde{\lambda}] \\
\delta_Q \phi &= \eta, \qquad \delta_Q \eta = [\sigma, \phi] \\
\delta_Q b &= \psi, \qquad \delta_Q \psi = [\sigma, b] \\
\delta_Q \beta &= P, \qquad \delta_Q P = [\sigma, \beta] \\
\delta_Q \chi &= \widetilde{P}, \quad \delta_Q \widetilde{P} = [\sigma, \chi] \ .
\end{aligned}
\tag{5.1}
$$

If we were to naïvely reduce to 3d by keeping only the constant Fourier modes of the above fields, then we would have the correct field content and variations for a gauged 3d A-model TFT with target space $\mathfrak{g}$ (parameterized by the field ϕ), as written in Section B.2. However, as we have discussed above, the field A_4 is ill-defined as a 3d scalar field, and must be replaced by its holonomy h in the description of the theory. Moreover, we must also replace its BRST partner ρ by an appropriately averaged field. We define

$$\widehat{\rho} = -\frac{1}{2\pi} \int_0^{2\pi} dx^4 H^{-1} \rho H$$

where

$$H(x^4) = P \exp\Big(- \int_0^{x^4} dt\, A_4 \Big)$$

is the holonomy of A along the arc of the circle from 0 to x^4, so that, in particular $H(2\pi) = h$. One can then verify, after some work, that the BRST-variations of h and $\widehat{\rho}$ are given by

$$
\begin{aligned}
h^{-1} \delta_Q h &= 2\pi \widehat{\rho} \\
\delta_Q (h \widehat{\rho}) &= (2\pi)^{-1} [\sigma, h] \ .
\end{aligned}
$$

Combining these, one confirms that $\delta_Q^2 h = [\sigma, h]$, i.e., a gauge transformation with parameter σ. The field $\widehat{\rho}$ transforms as a section

$$\widehat{\rho} \in \Omega^0(W, ad\, P_W) \ .$$

Reduced action

In order to write out the Vafa-Witten action in terms of the relabeled fields, it is helpful to use the following identities. Let $\chi_{\mu\nu}$ and $F_{\mu\nu}$ be 2-forms in 4d and let

$$\nu_r = \chi_{r4}, \quad \widetilde{\nu}_r = (*\chi)_{r4}$$
$$\omega_r = F_{r4}, \quad \widetilde{\omega}_r = (*F)_{r4}$$

be the 1-forms in 3d given by restricting F, $*F$, χ, and $*\chi$ to the 1,2,3 directions, where * indicates the 4d Hodge dual. We have

$$\sqrt{g}\left(\tfrac{1}{2}\chi_{\mu\nu}F^{\mu\nu}\right) = \sqrt{g^{44}}\sqrt{g_W}\Big(\nu_r\,\omega^r + \widetilde{\nu}_r\,\widetilde{\omega}^r\Big)$$

where $g_{\mu\nu}$ is the 4d metric with determinant g and $(g_W)_{rs}$ is the 3d metric with determinant g_W. In a similar vein, we have

$$F_{rs} = \sqrt{g^{44}}\sqrt{g_W}\epsilon_{rst}\widetilde{\omega}^t = \sqrt{g^{44}}\big(\star\,\widetilde{\omega}\big)_{rs}$$

where $\star$ is the 3d Hodge star. Finally, let $B_{\mu\nu}$ be a self-dual 2-form in 4d and let $C_{\mu\nu}$ be the following 2-form appearing in the Vafa-Witten localization equations:

$$C_{\mu\nu} = \tfrac{1}{2}g^{\rho\tau}[B_{\mu\rho}, B_{\nu\tau}]$$

One finds that

$$C_{r4} = (*C)_{r4} = \tfrac{1}{2}\sqrt{g^{44}}\sqrt{g_W}\,\epsilon_{rst}[b^s, b^t] = \tfrac{1}{2}\sqrt{g^{44}}\big(\star\,[b,b]\big)_r$$

where $b_r = B_{r4}$. For our choice of the circle circumference, $g_{44} = g^{44} = 1$.

The Vafa-Witten action (3.9), written in terms of the relabeled fields, becomes:

$$S = \int_M d^4x\sqrt{g}\,\Big(\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4\Big) - \frac{i\tau}{4\pi}\int_M \mathrm{Tr}\,F\wedge F$$

where

$$\begin{aligned}
\mathcal{L}_1 &= \frac{1}{e^2}\delta_Q\ \mathrm{Tr}\Big\{4\widetilde{\lambda}^r\big(H_r - D_rA_4 + \partial_4A_r - (\star F)_r\big) + 2\bar{\sigma}\,d_A^\star\lambda\Big\}\\
\mathcal{L}_2 &= \frac{1}{e^2}\delta_Q\mathrm{Tr}\Big\{\chi^r\Big(\widetilde{P}_r - 2D_r\phi + 2\big(\star\,d_Ab\big)_r\Big) + \beta\Big(P - 2d_A^\star b\Big)\Big\}\\
\mathcal{L}_3 &= \frac{1}{e^2}\delta_Q\mathrm{Tr}\Big\{4\widetilde{\lambda}^r\Big(-[b_r,\phi] + \tfrac{1}{2}\star[b,b]_r\Big) + 2\chi^rD_4b_r - 2\beta D_4\phi\Big\}\\
\mathcal{L}_4 &= \frac{1}{e^2}\delta_Q\mathrm{Tr}\Big\{2\bar{\sigma}\Big(-[\psi_r,b^r] - [\eta,\phi] - \tfrac{1}{2}[\sigma,\widetilde{\rho}] + D_4\,\rho\Big)\Big\}
\end{aligned} \tag{5.2}$$

where $d_A^\star\lambda = D_r\lambda^r$. Simply keeping constant modes in the above action would result in a 3d action containing commutator terms of the form, e.g.,

$$[A_4, \phi]$$

which (as we have been stressing) would not be well defined due to the explicit appearance of A_4. In order to eliminate the appearance of A_4, we exploit the wide freedom of choice available in writing down the action for a given cohomological TFT. Namely, we are free to continuously deform the BRST-exact portion of the action in any way without affecting the values of topological correlators, provided that the bosonic kinetic terms remain well defined, and that we do not alter the field content or variations of the theory. We write down a new action $\hat{S}$ by replacing the commutator $[A_4, \phi]$ inside the BRST-exact portion of S with

$$(2\pi)^{-1}\Big(Ad_{h^{-1}}\phi - \phi\Big) = -(2\pi)^{-1}h^{-1}[h, \phi] \ .$$

(The latter expression reduces to the former in the neighborhood of $A_4 \approx 0$.) One can smoothly interpolate between S and $\hat{S}$ with a family of actions

$$S_\alpha = (1-\alpha)S + \alpha\hat{S}$$

for $0 \leq \alpha \leq 1$ such that, as we dial the parameter α from zero to one, the action is smoothly deformed by a BRST-exact amount. (The effect of nonzero α is to 'turn on' higher-order BRST-exact terms involving A_4 commutators.) At $\alpha = 1$, we are left with the equivalent action $\hat{S}$, which, it should be emphasized, is not Lorentz-invariant due to the appearance of h. Unlike S, the action $\hat{S}$ can be safely be reduced to 3d by requiring fields to be independent of the x^4 coordinate, and one obtains

$$\widehat{S} = \int_W d^3x\sqrt{g_W}\,\Big(\widehat{\mathcal{L}}_1 + \widehat{\mathcal{L}}_2 + \widehat{\mathcal{L}}_3 + \widehat{\mathcal{L}}_4\Big) + S_{top}$$

where

$$\begin{aligned}
\widehat{\mathcal{L}}_1 &= \frac{1}{e^2}\delta_Q\ \mathrm{Tr}\Big\{4\widetilde{\lambda}^r\big(H_r + (2\pi)^{-1}h^{-1}D_rh - \star F_r\big) + 2\bar{\sigma}\, d_A^\star\lambda\Big\} \\
\widehat{\mathcal{L}}_2 &= \frac{1}{e^2}\delta_Q\mathrm{Tr}\Big\{\chi^r\Big(\widetilde{P}_r - 2D_r\phi + 2\big(\star\, d_Ab\big)_r\Big) + \beta\Big(P - 2d_A^\star b\Big)\Big\} \\
\widehat{\mathcal{L}}_3 &= \frac{1}{e^2}\delta_Q\mathrm{Tr}\Big\{4\widetilde{\lambda}^r\Big(-[b_r,\phi] + \star[b,b]_r\Big) - 2(2\pi)^{-1}\chi^r h^{-1}[h,b_r] + 2(2\pi)^{-1}\beta\, h^{-1}[h,\phi]\Big\} \\
\widehat{\mathcal{L}}_4 &= \frac{1}{e^2}\delta_Q\mathrm{Tr}\Big\{-2\bar{\sigma}\Big([\lambda_r,b^r] + [\eta,\phi] - \tfrac{1}{2}[\widetilde{\rho},\sigma] + (2\pi)^{-1}h^{-1}[h,\widehat{\rho}]\Big)\Big\}
\end{aligned} \tag{5.3}$$

and S_{top} is the coupling-dependent term measuring the 4d instanton number in the 4d action. After acting with δ_Q, one finds that the kinetic term of the bosonic field h takes the standard form for a group manifold sigma model map:

$$\mathrm{Tr}\{h^{-1}(D_r h)\, h^{-1}\, (D^r h)\}$$

where the metric on G is taken to be the left-invariant metric induced by Tr at the identity.

After the integrating out the auxiliary fields, one finds that the theory localizes on the BPS equations

$$\begin{aligned} F - (2\pi)^{-1} h^{-1} \star d_A h + [\star b, \phi] - [b, b] &= 0 \\ \star d_A b - d_A \phi - (2\pi)^{-1} h^{-1} [h, b] &= 0 \\ d_A^{\star} b - (2\pi)^{-1} h^{-1} [h, \phi] &= 0 \end{aligned} \tag{5.4}$$

as well as equations involving σ.

The above field content, action and variations are those of a gauged 3d A-model with target $\mathfrak{g}$. The gauge sector is not precisely the 3d A-type gauge theory as described in Section B.2.1 but, rather, a modification in which the Lie algebra valued scalar field ς is replaced by a Lie group valued field h.

Instanton term

We wish to interpret the instanton term in the above action in 3d terms. The analysis splits into two cases: when $G = U(1)$ and when G is simple.

In case $G = U(1)$, we write the reduction of the instanton term as

$$S_{top} = -\frac{i\tau}{2} \int_W F \wedge dA_4 \tag{5.5}$$

Note that it is the periodicity of A_4 that makes this topological term nontrivial in general.

In case G is simple, there are no such deformation terms available for measuring the topological class of the principal bundle on W. In fact, since we have taken W to be compact, the term automatically vanishes. More generally, we wish to consider the possibility of inserting disorder operators and allowing W to have boundaries. For instance, in Chapter 6, we will describe the 2-category of surface operators of the 4d theory as the 2-category of 3d boundary conditions. However, the lack of 3d deformations demonstrates that anything about the theory understandable in purely 3d terms will be τ-independent; e.g., the 2-categorical structure of surface operators. As we have mentioned in Chapter 4, this 2-category secretly possesses a braided, monoidal structure (corresponding to colliding surface operators), which extra structure could be dependent on τ. But, since we will not be examining the braided monoidal structure, we discard the instanton term from the action of the 3d compactified theory.

5.3 S^1 compactification of GL-twisted theory at $t = 0$

We turn now to the reduction of the GL-twisted theories, treating first the case $t = 0$. The analysis will proceed similarly for each value of t. The Vafa-Witten theory and the $t = 0$ theory turn out to reduce to the same theory in three dimensions.

The fact that the Vafa-Witten twisted theory and $t = 0$ are identical on the manifold $W \times S^1$ can be understood as follows. Recall from Section 2.4.1 that, after performing the Vafa-Witten twist of $\mathcal{N} = 4$, $d = 4$ SYM, the scalar supercharge Q is drawn from amongst the left-handed Q's. On the other hand, performing the GL-twist also results in a left-handed scalar supercharge Q_l precisely when $t = 0$; indeed, it can be taken to be the same one. The Vafa-Witten and $t = 0$ theories then only differ in how fields transform under the 4d rotation group; e.g., the Vafa-Witten theory has a self-dual 2-form $B_{\mu\nu}$ and 0-form ϕ, while the $t = 0$ theory has a 1-form ϕ_μ. However, the fields transform identically under the the 3d rotation subgroup; e.g., both 3d reduced theories have a 3d 1-form b_r and a 3d 0-form ϕ.

Explicitly, recall from Section 2.5.2 the action for the GL-twisted theory at $t = 0$:

$$S = \int_M d^4x\sqrt{g}\,\Big(\mathcal{L}_1 + \mathcal{L}_2\Big) - \frac{\bar{\tau}}{4\pi i}\int_M \mathrm{Tr}\, F \wedge F$$

where

$$\begin{aligned} \mathcal{L}_1 &= \frac{1}{e^2}\delta_Q \mathrm{Tr}\Big\{\chi_{\mu\nu}H^{\mu\nu} - 2\chi^+_{\mu\nu}\Big(F - \tfrac{1}{2}[\phi,\phi]\Big)^{\mu\nu} + 2\chi^-_{\mu\nu}\left(d_A\phi\right)^{\mu\nu}\Big\} \\ \mathcal{L}_2 &= \frac{1}{e^2}\delta_Q \mathrm{Tr}\Big\{ -\widetilde{\eta}(P - 2d_A^*\phi) + \bar{\sigma}\Big(2d_A^*\psi + 2[\widetilde{\psi}_\mu, \phi^\mu] + [\sigma,\eta]\Big)\Big\}\ . \end{aligned} \tag{5.6}$$

We relabel the fields of the GL-twisted theory in 3d terms as follows (the superscript [0] indicates the corresponding field of the GL-twisted theory at $t = 0$, as we have written in Chapter 2). The gauge sector fields are

$$\begin{aligned} A_r &= A_r^{[0]}, & \lambda_r &= i\psi_r^{[0]} \\ \varsigma &= A_4^{[0]}, & \rho &= i\psi_4^{[0]} \\ \sigma &= -i\sigma^{[0]}, & & \\ \bar{\sigma} &= -i\bar{\sigma}^{[0]}, & \widetilde{\rho} &= \eta^{[0]} \\ \widetilde{\lambda}_r &= \chi^{+[0]}_{r4}, & H_r &= H^{+[0]}_{r4} \end{aligned}$$

The matter sector fields are

$$\begin{aligned} \phi &= -\phi_4^{[0]}, & \eta &= i\widetilde{\psi}_4^{[0]}, \\ b_r &= -\phi_r^{[0]}, & \psi_r &= i\widetilde{\psi}_r^{[0]} \\ \beta &= \widetilde{\eta}^{[0]}, & P &= -P^{[0]} \\ \chi_r &= 2\chi^{-[0]}_{r4}, & \widetilde{P}_r &= 2H^{-[0]}_{r4}\ . \end{aligned}$$

It is easily verified that the BRST variations of the relabeled fields are given by (5.1) above.

If one exchanges A_4 for its holonomy h, ρ for $\hat{\rho}$ (as we have done for the Vafa-Witten theory) and reduce to three dimensions, then the 3d reduced action matches that (5.3) of the reduction of Vafa-Witten theory, except for the sign of the term

$$\frac{1}{e^2}\delta_Q \mathrm{Tr}\{4\widetilde{\lambda}^r[b_r,\phi]\}$$

which changes the sign of the term $[\star b, \phi]$ in (5.4). However, solutions to the BPS equations set this term to zero anyway, so this sign is irrelevant and we conclude that the reductions of the Vafa-Witten and $t=0$ theories are identical.

5.4 S^1 compactification of GL-twisted theory at $t=1$

The reduction of the $t=1$ theory again yields a gauged version of the 3d A-model. There are, however, a few important differences as compared with the Vafa-Witten and $t=0$ reductions. The field A_4 will now lie in the matter sector (instead of the gauge sector) of the 3d theory; indeed, the target space of the reduced theory will be the group G, parameterized by the holonomy h. Accordingly, the matter sector fields take values in the vector space T_hG rather than $\mathfrak{g} \simeq T_1G$.

Moreover, the coefficient of the topological term in the action is just θ instead of the complexified coupling τ, implying that we can avoid the complications of interpreting the instanton term by setting $\theta = 0$. Lastly, the gauge sector will be a conventional 3d A-type gauge theory, as described in Section B.2.1 (rather than an exotic version involving a group-valued scalar).

The action for the GL-twisted theory [1] at $t=1$ is given by

$$S = \int_M d^4x\sqrt{g}\,\Big(\mathcal{L}_1 + \mathcal{L}_2\Big) + \frac{i\theta}{8\pi^2}\int_M \mathrm{Tr}\, F\wedge F$$

where

$$\begin{aligned}\mathcal{L}_1 &= \frac{1}{e^2}\delta_Q \mathrm{Tr}\Big\{\chi_{\mu\nu}\Big(\tfrac{1}{2}H - F + \tfrac{1}{2}[\phi,\phi] - *d_A\phi\Big)^{\mu\nu}\Big\}\\ \mathcal{L}_2 &= \frac{1}{e^2}\delta_Q \mathrm{Tr}\Big\{-\widetilde{\eta}\big(P - 2d_A^*\phi\big) + \bar{\sigma}\Big(2d_A^*\psi + 2[\widetilde{\psi}_\mu,\phi^\mu] + [\sigma,\eta]\Big)\Big\}\ .\end{aligned} \tag{5.7}$$

Relabeling the fields, we have the gauge sector fields

$$\begin{aligned} A_r &= A_r^{[1]}, & \lambda_r &= i\big(\psi_r^{[1]} + \widetilde{\psi}_r^{[1]}\big)\\ \varsigma &= \phi_4^{[1]}, & \rho &= i\big(\psi_4^{[1]} - \widetilde{\psi}_4^{[1]}\big)\\ \sigma &= -2i\sigma^{[1]}, & &\\ \bar{\sigma} &= -2i\bar{\sigma}^{[1]}, & \widetilde{\rho} &= 2\big(\eta^{[1]} + \widetilde{\eta}^{[1]}\big),\\ \widetilde{\lambda}_r &= \big(*\chi\big)_{r4}^{[1]}, & H_r &= \big(*H\big)_{r4}^{[1]}, \end{aligned}$$

and the matter sector fields

$$\begin{aligned} \phi &= A_4^{[1]}, & \eta &= i\big(\psi_4^{[1]} + \widetilde{\psi}_4^{[1]}\big), \\ b_r &= \phi_r^{[1]}, & \psi_r &= i\big(\psi_r^{[1]} - \widetilde{\psi}_r^{[1]}\big), \\ \beta &= \tfrac{1}{2}\big(\eta^{[1]} - \widetilde{\eta}^{[1]}\big), & P &= P^{[1]}\ , \\ \chi_r &= \chi_{r4}^{[1]}, & \widetilde{P}_r &= H_{r4}^{[1]}\ . \end{aligned}$$

The BRST variations of the relabeled fields, once again, are given by (5.1). Writing out the $t = 1$ action in terms of the relabeled fields, we find

$$S = \int_M d^4x\sqrt{g}\,\Big(\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5\Big)$$

where

$$\begin{aligned} \mathcal{L}_1 &= \frac{1}{e^2}\delta_Q\ \mathrm{Tr}\Big\{\widetilde{\lambda}^r\big(H_r - 2D_r\varsigma - 2\star F_r\big) + \tfrac{1}{2}\bar{\sigma}\, d_A^\star\lambda\Big\} \\ \mathcal{L}_2 &= \frac{1}{e^2}\delta_Q\mathrm{Tr}\Big\{\chi^r\Big(\widetilde{P}_r - 2D_rA_4 - 2\big(\star\, d_Ab\big)_r\Big) + \beta\Big(P - 2d_A^\star b\Big)\Big\} \\ \mathcal{L}_3 &= \frac{1}{e^2}\delta_Q\mathrm{Tr}\Big\{\widetilde{\lambda}^r\Big(2D_4b_r + \star[b,b]_r\Big) + \chi^r\Big(2[b_r,\varsigma] + 2\partial_4A_r\Big)\Big\} \\ \mathcal{L}_4 &= \frac{1}{e^2}\delta_Q\mathrm{Tr}\Big\{2\beta\Big(-D_4\varsigma - \tfrac{1}{8}[\bar{\sigma},\sigma]\Big) - \tfrac{1}{2}\widetilde{\rho}\Big(\tfrac{1}{2}P - d_A^\star b - D_4\varsigma + \tfrac{1}{8}[\bar{\sigma},\sigma]\Big)\Big\} \\ \mathcal{L}_5 &= \frac{1}{e^2}\delta_Q\mathrm{Tr}\Big\{\tfrac{1}{2}\bar{\sigma}\Big(d_A^\star\psi + D_4\eta + D_4\rho + [\lambda_r,b^r] - [\psi_r,b^r] + [\eta,\varsigma] - [\rho,\varsigma]\Big)\Big\}\ . \end{aligned} \tag{5.8}$$

We again replace A_4 in the description of the theory by its holonomy h. The target space is the group manifold $X = G$, with sigma model map Φ given by the holonomy

$$\Phi = h : W \to G\ .$$

The 3d A-model fields $\eta, b_r, \beta, P, \chi_r, \widetilde{P}_r$ now take values in T_hG instead of the Lie algebra itself and it is convenient to replace them by $\mathfrak{g}$-valued averages, exactly as we have done for ρ in the Vafa-Witten reduction above, i.e.,

$$\widehat{\eta} \equiv -\frac{1}{2\pi}\int_0^{2\pi} dx^4 H^{-1}\eta H$$

and similarly for the other matter fields. The BRST variations of the averaged fields pick up an extra term (due to the curvature of the group manifold), as follows:

$$\begin{aligned} \delta_Q\widehat{b}_r &= \widehat{\psi}_r + < \widehat{\eta},\ \widehat{b}_r > \\ \delta_Q\widehat{\chi}_r &= \widehat{\widetilde{P}}_r + < \widehat{\eta},\ \widehat{\chi}_r > \end{aligned}$$

where we have defined

$$< \widehat{\eta},\ \widehat{b}_r > \equiv \int_0^{2\pi} dx^4 \Big[\widehat{\eta}, \quad H^{-1} b_r H\Big]\ \Big|_{x^4},$$
$$< \widehat{\eta},\ \widehat{\chi}_r > \equiv \int_0^{2\pi} dx^4 \Big\{\widehat{\eta}, \quad H^{-1} \chi_r H\Big\}\ \Big|_{x^4}\ .$$

The reduction procedure will be similar to that which we have described for reducing the Vafa-Witten action above: we smoothly deform the action by a BRST-exact amount to eliminate the appearance of A_4 in favor of h and the T_hG-valued matter fields by their $\mathfrak{g}$-valued averages. After reducing to 3d, one obtains

$$\widehat{S} = \int_W d^3x \sqrt{g_W}\ \Big(\widehat{\mathcal{L}}_1 + \widehat{\mathcal{L}}_2 + \widehat{\mathcal{L}}_3 + \widehat{\mathcal{L}}_4 + \widehat{\mathcal{L}}_5\Big)$$

where

$$\begin{aligned}
\widehat{\mathcal{L}}_1 &= \frac{1}{e^2}\delta_Q\ \mathrm{Tr}\Big\{\widetilde{\lambda}^r\big(H_r - 2D_r\varsigma - 2\star F_r\big) + \tfrac{1}{2}\bar{\sigma}\, d_A^\star \lambda\Big\} \\
\widehat{\mathcal{L}}_2 &= \frac{1}{e^2}\delta_Q \mathrm{Tr}\Big\{\widehat{\chi}^r\Big(\widehat{\widehat{P}}_r + 2(2\pi)^{-1}h^{-1}D_r h - 2\big(\star\, d_A\widehat{b}\big)_r\Big) + \widehat{\beta}\Big(\widehat{P} - 2d_A^\star\widehat{b}\Big)\Big\} \\
\widehat{\mathcal{L}}_3 &= \frac{1}{e^2}\delta_Q \mathrm{Tr}\Big\{\widetilde{\lambda}^r\Big(-2(2\pi)^{-1}h^{-1}[h,\widehat{b}_r] + \star[\widehat{b},\widehat{b}]_r\Big) + 2\widehat{\chi}^r[\widehat{b}_r,\varsigma]\Big\} \\
\widehat{\mathcal{L}}_4 &= \frac{1}{e^2}\delta_Q \mathrm{Tr}\Big\{2\widehat{\beta}\Big((2\pi)^{-1}h^{-1}[h,\varsigma] - \tfrac{1}{8}[\bar{\sigma},\sigma]\Big) - \tfrac{1}{2}\widetilde{\rho}\Big(\tfrac{1}{2}\widehat{P} - d_A^\star\widehat{b} + (2\pi)^{-1}h^{-1}[h,\varsigma] + \tfrac{1}{8}[\bar{\sigma},\sigma]\Big)\Big\} \\
\widehat{\mathcal{L}}_5 &= \frac{1}{e^2}\delta_Q \mathrm{Tr}\Big\{\tfrac{1}{2}\bar{\sigma}\Big(d_A^\star\widehat{\psi} - (2\pi)^{-1}h^{-1}[h,\widehat{\eta}] - (2\pi)^{-1}h^{-1}[h,\rho] + [\lambda_r,\widehat{b}^r] - [\widehat{\psi}_r,\widehat{b}^r] + [\widehat{\eta},\varsigma] - [\rho,\varsigma]\Big)\Big\}\ .
\end{aligned}$$

After integrating out the auxiliary fields, one finds that the theory localizes on the BPS equations

$$\begin{aligned}
F + \star d_A\varsigma - \tfrac{1}{2}[\widehat{b},\widehat{b}] + (2\pi)^{-1}h^{-1}[h,\star\widehat{b}] &= 0 \\
\star d_A\widehat{b} - [\widehat{b},\varsigma] - (2\pi)^{-1}h^{-1}D_r h &= 0 \\
d_A^\star\widehat{b} - (2\pi)^{-1}h^{-1}[h,\varsigma] &= 0\ .
\end{aligned}$$

We have obtained a gauged 3d A-model with target space G. This 3d TFT has not been much studied in the literature, but, as it is conjectured to be dual to the 3d reduction of the $t = i$ theory, a study of its properties is expected to yield many interesting mathematical predictions. In the subsequent chapter we will take a modest first step in this direction by analyzing its 2-category of boundary conditions for the case $G = U(1)$.

5.5 S^1 compactification of GL-twisted theory at $t = i$

The reduction of the GL-twisted theory at $t = i$ can be analyzed in a similar way; however, the resulting 3d reduced theory will have a very different character to the cases considered so far. One obtains a gauged version of a well-studied 3d TFT, the Rozansky-Witten (RW) model [14], [36]. (The gauging is accomplished by coupling to a 3d B-type gauge theory, as described in Section B.4.) The target space of the model is the cotangent bundle of the complexified Lie group, $X = T^*G_{\mathbb{C}}$, equipped with the conjugation action of G on the base $G_{\mathbb{C}}$, as well as the induced action on the fibers. This comes about as follows.

The BRST-invariant, complex fields σ and $A_4 + i\phi_4$ of the $t = i$ theory will descend to the matter sector of the 3d theory. As we have discussed, the field A_4 itself is not well-defined and must be exchanged for an appropriate group-valued scalar. We define

$$h_{\mathbb{C}} = P\exp\Big(- \int_0^{2\pi} dx^4 (A_4 + i\phi_4)\Big)$$

taking values in the associated bundle with fiber $G_{\mathbb{C}}$

$$h_{\mathbb{C}} \in \Omega^0(W,\ P_W \times_G G_{\mathbb{C}})\ .$$

Its conjugate is

$$\overline{h}_{\mathbb{C}} = P\exp\Big(- \int_0^{2\pi} dx^4 (A_4 - i\phi_4)\Big)\ .$$

The field $h_{\mathbb{C}}$ is a good coordinate for the base and σ is a good coordinate for the fiber of $T^*G_{\mathbb{C}}$. The manifold $T^*G_{\mathbb{C}}$ turns out to endowed with a hyperkähler metric [63], although it is nontrivial to describe explicitly.

Simply declaring the target to be $T^*G_{\mathbb{C}}$ obscures an important symmetry of the action. For a general hyperkähler target space X, the RW model has a $\mathbb{Z}_2$ ghost number symmetry, but, as explained in [36], when X is a cotangent bundle one can promote it to a $U(1)$ ghost number symmetry by letting the fiber coordinates have ghost number two. This accords nicely with the fact that σ has ghost number two already in the 4d theory. Hence, to emphasize that the fiber coordinate is formally assigned ghost number two we will denote the target manifold instead by $T^*[2]G_{\mathbb{C}}$.

The action of the GL-twisted theory at $t = i$ is written in (2.4). The relabelings of the fields are as follows (adopting the convention that a '$[i]$' superscript denotes fields of the $t = i$ theory, and

$r, s, t = 1, 2, 3$ refers to W coordinates). For the 3d gauge sector, we have

$$\begin{aligned}
A_r &= A_r^{[i]}, \\
\varphi_r &= \phi_r^{[i]}, \\
\mathcal{A}_r &= (A_r + i\phi_r)^{[i]} \\
\bar{\mathcal{A}}_r &= (A_r - i\phi_r)^{[i]} \\
\lambda_r &= i\left(\psi_r + i\widetilde{\psi}_r\right)^{[i]}, \\
\zeta_{rs} &= -i\left(\chi^+ - i\chi^-\right)_{rs}^{[i]}, \\
\rho &= \frac{1}{\sqrt{2}}\left(\psi_4 - i\widetilde{\psi}_4\right)^{[i]}, \\
\widetilde{\rho} &= \frac{1}{2}\left(\eta - i\widetilde{\eta}\right)^{[i]}.
\end{aligned}$$

and for the matter sector we have

$$\begin{aligned}
\sigma &= \sqrt{2}\sigma^{[i]}, \\
\tau &= (A_4 + i\phi_4)^{[i]}, \\
\eta^{\bar{\sigma}} &= \sqrt{2}i\left(\eta + i\widetilde{\eta}\right)^{[i]}, \\
\eta^{\bar{\tau}} &= 2i\left(\psi_4 + i\widetilde{\psi}_4\right)^{[i]}, \\
\chi_r^{\sigma} &= \frac{1}{\sqrt{2}}\left(\psi_r - i\widetilde{\psi}_r\right)^{[i]}, \\
\chi_r^{\tau} &= \left(\chi^+ - i\chi^-\right)_{r4}^{[i]} .
\end{aligned}$$

In addition, it is useful to introduce an auxiliary 0-form P in order to make the BRST variations nilpotent off-shell; P-dependent terms in the action are chosen to ensure that its equation of motion is

$$P = d_A^{\star}\varphi - \frac{i}{2}([\bar{\sigma}, \sigma] + [\bar{\tau}, \tau]) .$$

The BRST variations of the relabeled fields are given by

$$
\begin{aligned}
\delta_Q A &= \lambda, & \delta_Q \sigma &= 0, \\
\delta_Q \varphi &= i\lambda, & \delta_Q \tau &= 0, \\
\delta_Q \lambda &= 0, & \delta_Q \bar{\sigma} &= \eta^{\bar{\sigma}} \\
\delta_Q \zeta &= -i\mathcal{F}, & \delta_Q \bar{\tau} &= \eta^{\bar{\tau}} \\
\delta_Q \rho &= [\tau, \sigma], & \delta_Q \eta^{\bar{\sigma}} &= 0, \\
\delta_Q \widetilde{\rho} &= iP, & \delta_Q \eta^{\bar{\tau}} &= 0, \\
\delta_Q P &= 0, & \delta_Q \chi^{\sigma} &= d_{\mathcal{A}}\sigma, \\
& & \delta_Q \chi^{\tau} &= d_{\mathcal{A}}\tau.
\end{aligned}
$$

where $d_{\mathcal{A}}$ is the covariant derivative and $\mathcal{F}$ curvature with respect to complexified connection $\mathcal{A}$.

The action written in terms of the relabeled fields is given by

$$
S = \int_M d^4x\sqrt{g}\ (\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3) + S_{top}
$$

where

$$
\begin{aligned}
\mathcal{L}_1 &= -\frac{1}{2e^2}\ \delta_Q\ \mathrm{Tr}\Big\{ \tfrac{i}{2}\zeta_{rs}\bar{\mathcal{F}}^{rs} + i\widetilde{\rho}\Big(P - 2d_A^\star\,\varphi + i[\bar{\sigma}, \sigma] + i[\bar{\tau}, \tau]\Big)\Big\} \\
\mathcal{L}_2 &= -\frac{1}{2e^2}\ \delta_Q\ \mathrm{Tr}\Big\{ \chi^\tau_r (d_{\bar{\mathcal{A}}}\bar{\tau})^r + \chi^\sigma_r (d_{\bar{\mathcal{A}}}\bar{\sigma})^r \Big\} \\
\mathcal{L}_3 &= -\frac{1}{2e^2}\ \delta_Q\ \mathrm{Tr}\Big\{ \rho\,[\bar{\tau}, \bar{\sigma}]\Big\}, \\
S_{top} &= \frac{\sqrt{2}}{e^2}\int_W\ \mathrm{Tr}\Big\{ i\chi^\tau \wedge d_{\mathcal{A}}\chi^\sigma - \zeta \wedge \Big(\mathrm{d}_{\mathcal{A}}\,\rho + [\chi^\sigma, \tau] - [\chi^\tau, \sigma]\Big)\Big\}\ .
\end{aligned}
$$

After replacing τ by $h_{\mathbb{C}}$, certain matter fields are tangent to a nonidentity point of the (complexified) group and it is convenient to replace them by algebra-valued averages in the description of the theory (just as for the $t = 1$ reduction above). Namely, $\eta^{\bar{\tau}}$ (taking values in $T_{\overline{h}_{\mathbb{C}}}$) and χ^τ (taking values in $T_{h_{\mathbb{C}}}$) are exchanged for averaged fields

$$
\begin{aligned}
\widehat{\eta}^{\bar{\tau}} &\equiv -\frac{1}{2\pi}\int_0^{2\pi} dx^4\,\overline{H}_{\mathbb{C}}^{-1}\,\eta^{\bar{\tau}}\,\overline{H}_{\mathbb{C}} \\
\widehat{\chi}^{\tau} &\equiv -\frac{1}{2\pi}\int_0^{2\pi} dx^4\,H_{\mathbb{C}}^{-1}\,\chi^{\tau}\,H_{\mathbb{C}}\ .
\end{aligned}
$$

The BRST variations written in terms of the new variables are as follows

$$\begin{aligned}
&\delta_Q h_{\mathbb{C}} = 0\\
&\delta_Q \overline{h}_{\mathbb{C}} = 2\pi\, \overline{h}_{\mathbb{C}}\ \widehat{\eta}^{\bar\tau}\\
&\delta_Q \widehat{\chi}^{\tau} = (2\pi)^{-1} h_{\mathbb{C}}^{-1} d_{\mathcal{A}} h_{\mathbb{C}}\\
&\delta_Q \widehat{\eta}^{\bar\tau} = 0\ .
\end{aligned}$$

The reduction proceeds similarly to what we have done for the Vafa-Witten, $t = 0$, and $t = 1$ theories: we deform the action by a BRST-exact amount to write it in terms of the new variables and then reduce to 3d by taking fields to be independent of the x^4 coordinate. The resulting 3d action is given by

$$\widehat{S} = \int_W d^3x \sqrt{g_W}\ (\widehat{\mathcal{L}}_1 + \widehat{\mathcal{L}}_2 + \widehat{\mathcal{L}}_3) + S_{top}$$

where

$$\begin{aligned}
\widehat{\mathcal{L}}_1 &= -\frac{1}{2e^2}\ \delta_Q\ \mathrm{Tr}\Big\{ \tfrac{i}{2}\zeta_{rs}\bar{\mathcal{F}}^{rs} + i\widetilde{\rho}\Big(P - 2d_A^{\star}\,\varphi + i[\bar\sigma,\sigma] + i[\bar\tau,\tau] + \dots\Big)\Big\}\\
\widehat{\mathcal{L}}_2 &= -\frac{1}{2e^2}\ \delta_Q\ \mathrm{Tr}\Big\{ -(2\pi)^{-1}\widehat{\chi}_r^{\tau}\ \overline{h}_{\mathbb{C}}^{-1}(d_{\bar A}\overline{h}_{\mathbb{C}})^r + \chi_r^{\sigma}(d_{\bar A}\bar\sigma)^r\Big\}\\
\widehat{\mathcal{L}}_3 &= -\frac{1}{2e^2}\ \delta_Q\ \mathrm{Tr}\Big\{ -(2\pi)^{-1}\rho\,\overline{h}_{\mathbb{C}}^{-1}[\overline{h}_{\mathbb{C}},\bar\sigma]\Big\},
\end{aligned}$$

where the ... in $\widehat{\mathcal{L}}_1$ indicate higher-order terms that we do not write explicitly. The G-invariant symplectic form on the target space is proportional to

$$\Omega = \mathrm{Tr}\, d\sigma\ h_{\mathbb{C}}^{-1} dh_{\mathbb{C}}.$$

The G-invariant Kähler form on the target space corresponds to the hyperkähler metric described in [63] and does not admit a simple closed-form expression; in a neighborhood of the identity it is proportional to

$$J = \frac{i}{2}\mathrm{Tr}\Big(d\sigma d\bar\sigma + d\tau d\bar\tau\Big)\ .$$

The moment maps μ_3, μ_+, μ_- of the G-action with respect to symplectic forms J, Ω, and $\bar\Omega$ are proportional to the following functions of the coordinates:

$$\begin{aligned}
\mu_3 &= -\frac{i}{2}([\bar\sigma,\sigma] + [\bar\tau,\tau] + \dots)\\
\mu_+ &= -i h_{\mathbb{C}}^{-1}[h_{\mathbb{C}},\sigma]\\
\mu_- &= -i\overline{h}_{\mathbb{C}}^{-1}[\overline{h}_{\mathbb{C}},\bar\sigma]\ .
\end{aligned}$$

We have shown that the BRST-exact portion of the action and variations reduce to those of a gauged Rozansky-Witten model with target $T^*[2]G_{\mathbb{C}}$ written in Section B.5. We conjecture that the topological term S_{top} also reduces appropriately to the topological term of this model (although we do not show this here).

Chapter 6

Surface operators of 4d TFTs

Having written down the twisted versions of $\mathcal{N} = 4$ SYM and analyzed their 3d reductions, we are in a position to classify the spectrum of surface operators in these theories. As emphasized in Chapter 4, the surface operators of a given 4d TFT have the structure of a 2-category; namely, the 2-category of boundary conditions of the 3d reduced TFT.

In this chapter, we systematically employ this observation to classify surface operators of the GL-twisted theories at $t = i$, $t = 1$, and $t = 0$. (By what we have said in the preceding chapter, the Vafa-Witten theory will have the same spectrum of surface operators as that of the $t = 0$ theory.) The advantage of the 3d viewpoint is that the problem of classifying boundary conditions is more familiar.

The material in this chapter was presented in [4]. Much of the analysis of boundary conditions of the reduced theories relies on work on 2d and 3d TFTs relegated to the appendices A and B.

6.1 Gukov-Witten surface operators

The simplest surface operators of the GL-twisted theories have been introduced by S. Gukov and E. Witten [2]. They are disorder operators corresponding to a codimension-2 singularity in the gauge field A and bosonic 1-form ϕ of the form

$$A = \alpha d\theta, \quad \phi = \beta \frac{dr}{r} - \gamma d\theta.$$

Here α is an element of a maximal torus T of G, and β, γ are elements of the Lie algebra $\mathfrak{t}$ of T. For simplicity, let us assume that the triple (α, β, γ) breaks G down to T. Gauge transformations which preserve T form the Weyl group $\mathcal{W}$; the triplet (α, β, γ) is defined up to the action of $\mathcal{W}$ on $\mathsf{T} \times \mathfrak{t} \times \mathfrak{t}$. All fields other than A and ϕ are nonsingular.

The surface operator depends on an additional parameter η taking values in the torus $\mathrm{Hom}(\Lambda_{\mathrm{cochar}}, U(1))$. Here $\Lambda_{\mathrm{cochar}}$ is the lattice of magnetic charges $\mathrm{Hom}(U(1), \mathsf{T})$. Equivalently, as explained in [2], η can be thought of as taking values in ${}^L\mathsf{T}$, the maximal torus of the Langlands-dual group. The parameter η arises as follows. First, note that the above singularity in the fields breaks the gauge group down to T. Thus if D is the codimension-2 submanifold on which the surface operator is supported, the restriction of the gauge field to D has a first Chern class $c_1|_D$ taking values in Λ_{cochar}. Given η we can insert into the path-integral a phase factor

$$\eta(c_1(D)).$$

This factor depends only on the behavior of the gauge field on D and can be regarded as an η-dependent modification of the surface operator defined above.

Gukov-Witten surface operators are BRST-invariant for arbitrary t, but their properties depend on t. We will see below that there are many other surface operators. In what follows we will focus on the cases $t = i$, $t = 1$, and $t = 0$. The first two cases are exchanged by S-duality (at zero θ-angle) and play a prominent role in the physical approach to the geometric Langlands program [1]. The last case is self-dual and is the most natural starting point for understanding quantum geometric Langlands duality [35].

We shall find that, for each value of t, there exist much more general surface operators.

6.2 Surface operators at $t = i$ and $G = U(1)$

In Section 5.5, we found that the $t = i$ theory on $W \times S^1$ reduces to a gauged Rozansky-Witten model on W with target $T^*G_{\mathbb{C}}$. Initially, we take $G = U(1)$ and analyze boundary conditions of the gauged RW model with target $T^*\mathbb{C}^*$. The analysis of boundary conditions splits cleanly into boundary conditions of the matter sector and boundary conditions of the gauge sector; we discuss each in turn.

6.2.1 Boundary conditions of RW model, target $T^*\mathbb{C}^*$

According to [36] the simplest boundary conditions in the RW model correspond to complex Lagrangian submanifolds of X. If we want to preserve ghost number symmetry, these Lagrangian submanifolds must be invariant with respect to the rescaling $\sigma \mapsto \lambda^2\sigma$, $\lambda \in \mathbb{C}^*$. This requires the Lagrangian submanifold of $T^*\mathbb{C}^*$ to be the conormal bundle of a complex submanifold in $\mathbb{C}^*$. This means that a (closed) $\mathbb{C}^*$-invariant complex Lagrangian submanifold is either the zero section $\sigma = 0$ or one of the fibers of the cotangent bundle given by $\tau = \tau_0$. The zero section boundary condition plays a special role and will be denoted $\mathbb{X}_0$ in this subsection.

More general boundary conditions correspond to families of B-models or Landau-Ginzburg models parameterized by points in a complex Lagrangian submanifold. As mentioned in [36] and explained in more detail in [37] it is sufficient to restrict oneself to the case when the Lagrangian submanifold is the zero section $\sigma = 0$. One can describe these boundary conditions more algebraically as follows. Recall that the category of boundary line operators on the boundary $\mathbb{X}_0$ is a monoidal category which we denote $\mathbb{V}_{\mathbb{X}_0\mathbb{X}_0}$. Given any boundary condition $\mathbb{X}$ one may consider the category $\mathbb{V}_{\mathbb{X}\mathbb{X}_0}$ of boundary defect lines which may separate $\mathbb{X}$ from $\mathbb{X}_0$. This category is a module category over the monoidal category $\mathbb{V}_{\mathbb{X}_0\mathbb{X}_0}$. It was proposed in [36] that this module category completely characterizes the boundary condition $\mathbb{X}$. Concretely, in the case of the RW model with target $T^*[2]\mathbb{C}^*$ the category of boundary line operators $\mathbb{V}_{\mathbb{X}_0\mathbb{X}_0}$ is equivalent to $D^b(Coh(\mathbb{C}^*))$. One way to see it is to reduce the RW model on an interval with the boundary condition $\mathbb{X}_0$ on both boundaries. The resulting 2d TFT is a B-model with target $\mathbb{C}^*$, and its category of branes may be identified with $D^b(Coh(\mathbb{C}^*))$. The 2d viewpoint does not allow one to determine the monoidal structure, but one can show that it is given by the usual derived tensor product [36, 37].

It was further argued in [36, 37] that the 2-category of boundary conditions for the RW model with target $T^*[2]\mathbb{C}^*$ is equivalent to the derived 2-category of module categories over $D^b(Coh(\mathbb{C}^*))$. That is, it is the 2-category of derived categorical sheaves over $\mathbb{C}^*$ as defined by B. Toen and G. Vezzosi [44]. This provides an algebraic description of boundary line operators and their OPEs for all boundary conditions.

6.2.2 Boundary conditions of B-type 3d gauge theory, $G = U(1)$

Recall from Section 5.5 that the bosonic fields $A|_W$, $\phi|_W$ and the fermionic fields $\psi + i\widetilde{\psi}, \chi, \eta - i\widetilde{\eta}, \psi_4 - i\widetilde{\psi}_4$ reduce correctly to fields of the B-type 3d gauge theory (discussed in Section B.4), with action given by

$$S = -\frac{1}{2e^2}\,\delta_Q \int_W \left(\chi \wedge \star\mathcal{F} - \frac{i}{2}(\eta - i\widetilde{\eta}) \wedge \star d^\star\phi\right) + \frac{1}{2e^2}\int_W (\psi_4 - i\widetilde{\psi}_4) d\chi$$

In principle we should gauge-fix the theory and modify the BRST operator appropriately, but throughout this thesis we suppress this complication.

As in any gauge theory, the most natural boundary conditions are the Dirichlet and Neumann ones. The Dirichlet condition requires the restriction of $A + i\phi$ to the boundary to be trivial. In addition, one requires ϕ_3 (the component of ϕ orthogonal to the boundary) to satisfy the Neumann condition $\partial_3\phi_3 = 0$. BRST-invariance then fixes the boundary conditions for fermions: the restriction of the forms $\psi + i\widetilde{\psi}$, χ and $\eta - i\widetilde{\eta}$ to the boundary must vanish, The Neumann boundary condition leaves the restriction of $\mathcal{A}$ to the boundary unconstrained but requires the restriction of the 1-form $\star\mathcal{F} = \star d\mathcal{A}$ to vanish. In addition ϕ_3 must satisfy the Dirichlet boundary condition, i.e., it must take a prescribed value on the boundary. In the Neumann case BRST-invariance requires the restrictions of the fermions $\star\chi$, $\psi_3 + i\widetilde{\psi}_3$ and $\psi_4 - i\widetilde{\psi}_4$ to vanish. Note that in the Dirichlet case the gauge group is completely broken at the boundary, while in the Neumann case it is unbroken.

The Dirichlet condition does not have any parameters, while the Neumann condition seems to depend on a single real parameter β, the boundary value of ϕ_3. On the quantum level there is another parameter: we can add to the action a boundary topological term

$$\theta \int_{\partial W} \frac{\mathcal{F}}{2\pi}$$

In fact, both parameters are irrelevant, in the sense that topological correlators do not depend on them. The irrelevance of the parameter θ follows from the fact that the above topological term is BRST-exact and equal to

$$\frac{\theta}{2\pi} \delta_Q \int_{\partial W} \chi$$

To see the irrelevance of the parameter β, note that to shift β we need to add to the action a boundary term proportional to

$$\int_{\partial W} \partial_3 \phi_3$$

Since ϕ_1 and ϕ_2 vanish on the boundary, this is also BRST-exact and proportional to

$$\delta_Q \int_{\partial W} (\eta - i\widetilde{\eta}).$$

Following the same line of thought as in [36], one can try to describe the 2-category of boundary conditions in this theory by picking a distinguished boundary condition $\mathbb{X}_0$ and characterizing any other boundary condition $\mathbb{X}$ by the category $\mathbb{V}_{\mathbb{X}\mathbb{X}_0}$ of defect line operators between $\mathbb{X}$ and $\mathbb{X}_0$. That is, one attaches to any boundary condition $\mathbb{X}$ a module category $\mathbb{V}_{\mathbb{X}\mathbb{X}_0}$ over the monoidal category $\mathbb{V}_{\mathbb{X}_0\mathbb{X}_0}$.

An obvious guess for the distinguished boundary condition is the free (Neumann) one since it leaves the gauge group unbroken. To determine the category of boundary line operators $\mathbb{V}_{\mathbb{X}_0\mathbb{X}_0}$ for this boundary condition, one may reduce the 3d theory on an interval and study the category of branes in the resulting 2d TFT. In the Neumann case, reduction on an interval gives the following result: the bosonic fields are the gauge field A and the 1-form ϕ, the fermionic ones are the 0-form $\eta - i\widetilde{\eta}$, the 1-form $\psi + i\widetilde{\psi}$ and the 2-form χ. This is the field content of a B-type topological gauge theory in 2d, see Section A.2. It is easy to check that the BRST transformations of these fields are also the same as in the B-type 2d gauge theory. The category of branes for this 2d TFT is the category of graded finite-dimensional representations of $G = U(1)$, see Section A.2 for details. This is because the only boundary degrees of freedom one can attach are described by a vector space which carries a representation of the gauge group. The monoidal structure cannot be determined from 2d considerations, but it easy to see that it is given by the usual tensor product. Indeed, as described in Section A.2, a brane corresponding to a representation space V is obtained by inserting the holonomy of the complex connection $\mathcal{A} = A + i\phi$ in the representation V into the path-integral. From the 3d viewpoint this means that the corresponding boundary line operator is the Wilson line operator for $\mathcal{A}$ in the representation V. On the classical level, the fusion of two Wilson line operators in representations V_1 and V_2 gives the Wilson line in representation $V_1 \otimes V_2$, and clearly there can be no quantum corrections to this result (the gauge coupling e^2 is an irrelevant parameter).

To summarize, the monoidal category $\mathbb{V}_{\mathbb{X}_0\mathbb{X}_0}$ is the category of graded finite-dimensional representations of $\mathbb{C}^*$, or equivalently the equivariant derived category of coherent sheaves over a point which we denote $D^b_{\mathbb{C}^*}(Coh(\bullet))$. We propose that the 2-category of boundary conditions is equivalent to the 2-category of module categories over $D^b_{\mathbb{C}^*}(Coh(\bullet))$. To give a concrete class of examples of such a module category, consider a Calabi-Yau manifold Y with a $\mathbb{C}^*$ action. The corresponding B-model can be coupled to the boundary gauge field and provides a natural set of topological boundary degrees of freedom for the 3d gauge theory. The corresponding category of boundary-changing line operators is the $\mathbb{C}^*$-equivariant bounded derived category of Y which is obviously a module category over $D^b_{\mathbb{C}^*}(Coh(\bullet))$.

6.2.3 Putting the sectors together

It is fairly obvious how to combine boundary conditions for the two models. The most basic boundary condition in the full theory is $\sigma = 0$ in the RW sector and the free (Neumann) condition in the gauge sector. We will call this the distinguished boundary condition. The bosonic fields which are free on the boundary are the $\mathbb{C}^*$-valued scalar $h = \exp(-2\pi\tau)$ and the restriction of the complex gauge field $\mathcal{A} = A + i\phi$. More general boundary conditions involve a boundary B-model or a boundary Landau-Ginzburg model fibered over $\mathbb{C}^*$ and admitting a $\mathbb{C}^*$-action. The fibration over $\mathbb{C}^*$ determines the coupling to the boundary value of τ, while the $\mathbb{C}^*$-action determines the coupling to the boundary gauge field $\mathcal{A}$.

As in the RW model, we can give a more algebraic definition of the set of all boundary conditions in the full theory. This description is also useful because it suggests how to define the 2-category structure of the set of boundary conditions. We consider the monoidal category of boundary line operators for the distinguished boundary condition. This is the category of branes for the 2d TFT obtained by reducing the gauged RW model on an interval. Since the reduction of the B-type 3d gauge theory gives the B-type 2d gauge theory, and the reduction of the RW model gives the B-model with target $\mathbb{C}^*$, the effective 2d TFT is the gauged B-model with target $\mathbb{C}^*$, where the gauge group $U(1)$ acts trivially. As described in Section A.3, the corresponding category of branes is equivalent to $D^b_{\mathbb{C}^*}(Coh(\mathbb{C}^*))$. The monoidal structure cannot be determined from the 2d considerations, but it is easy to see (given the results for the RW model and the B-type gauge theory in 3d) that it is given by the derived tensor product.

Every boundary condition gives rise to a module category over this monoidal category. It is natural to conjecture that the converse is also true, i.e., every reasonable module category over this monoidal category can be thought of as a boundary condition for the full 3d TFT. For example, we may consider a family of Calabi-Yau manifolds parameterized by points of $\mathbb{C}^*$ such that each model in the family has a $\mathbb{C}^*$ symmetry. The corresponding module category is the $\mathbb{C}^*$-equivariant derived category of the total space of the fibration. This gives us a conjectural description of the 2-category of surface operators in the parent 4d gauge theory.

Let us describe how Gukov-Witten surface operators fit into this picture. Such operators depend on a complex parameter $h_0 = \exp(-2\pi(\alpha - i\gamma))$ taking values in $\mathbb{C}^*$. From the 3d viewpoint, h_0 determines the boundary value of the scalar $h = \exp(-2\pi\tau)$ in the RW sector. The other scalar σ is left free. Thus the boundary conditions for the RW sector correspond to a Lagrangian submanifold of $T^*[2]\mathbb{C}^*$ given by $h = h_0$ (the fiber over the point h_0). The gauge sector boundary conditions are of Neumann type and have no nontrivial parameters.There are no boundary degrees of freedom. From our algebraic viewpoint we may describe this as follows. In the usual RW theory the fiber over $h = h_0$ corresponds to a skyscraper sheaf of DG-categories over $\mathbb{C}^*$ whose 'stalk' over h_0 is the category of bounded complexes of vector spaces. We may denote it $D^b(Coh(\bullet))$. Including the gauge degrees of freedom means working with a sheaf of categories with a $\mathbb{C}^*$ action. Thus we simply consider a skyscraper sheaf of categories over $\mathbb{C}^*$ whose 'stalk' over h_0 is the category of $\mathbb{C}^*$-equivariant complexes of vector spaces $D^b_{\mathbb{C}^*}(Coh(\bullet))$. The monoidal category $D^b_{\mathbb{C}^*}(Coh(\mathbb{C}^*))$ acts on it in a fairly obvious manner: one simply tensors an object of $D^b_{\mathbb{C}^*}(Coh(\bullet))$ with the (derived) restriction of an object of $D^b_{\mathbb{C}^*}(Coh(\mathbb{C}^*))$ to the point $h = h_0$.

6.2.4 Line operators on Gukov-Witten surface operators

The category of morphisms between two different skyscraper sheaves of categories is trivial (the set of objects is empty). This corresponds to the fact that two different Gukov-Witten surface operators cannot join along a boundary-changing line operator. But the category of line operators sitting on a particular Gukov-Witten surface operator (i.e., the endomorphism category of a Gukov-Witten surface operator) is nontrivial. Its most obvious objects are Wilson lines for the complexified gauge field $\mathcal{A}$, which are obviously BRST-invariant. Such operators are labeled by irreducible representations of $\mathbb{C}^*$. One might guess therefore that the category of surface line operators is simply the category of representations of $\mathbb{C}^*$, or perhaps the category of $\mathbb{C}^*$-equivariant complexes of vector spaces which we denoted $D^b_{\mathbb{C}^*}(Coh(\bullet))$ above. However, this naive guess is wrong, which can be seen by inspecting BRST-invariant local operators which can be inserted into such a Wilson line operator. From the abstract viewpoint they form an algebra (the endomorphism algebra of an object in the category of line operators). It is clear that any power of the field σ gives such an operator, so the algebra of local operators on a line operator is the algebra of polynomial functions of a single variable of ghost number 2. In what follows we will denote the line parameterized by σ by $\mathbb{C}[2]$ to indicate that

σ sits in degree 2; thus $\mathbb{C}[2]$ is a purely even graded manifold. On the other hand, the algebra of endomorphisms of an irreducible representation of $\mathbb{C}^*$ is simply $\mathbb{C}$.

To determine what the category of line operators is it is convenient to take the 2d viewpoint and reduce the 3d theory on an interval with the Gukov-Witten-type boundary condition on both ends. Let x^3 denote the coordinate on the interval. Gukov-Witten boundary conditions eliminate the complex scalar h (which is now locked at the value h_0) and the field ϕ_3 but keep the complex scalar σ and the gauge field $\mathcal{A}$. Thus the effective 2d theory also has two sectors: the B-model with target $\mathbb{C}[2]$ and the B-type 2d topological gauge theory. According to A.3, the corresponding category of branes is equivalent to the $\mathbb{C}^*$-equivariant derived category of $\mathbb{C}[2]$: its objects can be regarded as $\mathbb{C}^*$-equivariant complexes of holomorphic vector bundles on $\mathbb{C}[2]$ (with a trivial $\mathbb{C}^*$ action on $\mathbb{C}[2]$).

This answer is independent of the parameter $h_0 = \exp(-2\pi(\alpha - i\gamma))$ of the Gukov-Witten surface operator. In particular, we can choose the trivial surface operator $h_0 = 1$, in which case we should get the category of bulk line operators in the GL-twisted theory at $t = i$.

It is not difficult to see that this answer for the category of bulk line operators agrees with the computation of the endomorphism algebra of a Wilson line explained above. Indeed, an insertion of a Wilson line does not put any constraints on σ and does not add any degrees of freedom, and therefore should correspond to a trivial line bundle over $\mathbb{C}[2]$. Its fiber carries a representation of $\mathbb{C}^*$ determined by the charge of the Wilson line. The endomorphism algebra of such an object of $D^b_{\mathbb{C}^*}(Coh(\mathbb{C}[2]))$ is simply the algebra of polynomial functions on $\mathbb{C}[2]$.

It is now clear that the category of line operators contains objects other than Wilson lines. For example, we may consider a skyscraper sheaf at the origin of $\mathbb{C}[2]$, whose stalk at the origin is a complex line V carrying some representation of $\mathbb{C}^*$. There are two different ways to define the corresponding line operator. First, we may consider a free resolution of the skyscraper:

$$V[-2] \otimes \mathcal{O} \rightarrow V \otimes \mathcal{O},$$

where $V[-2]$ means V placed in ghost degree -2, $\mathcal{O}$ is the algebra of polynomial functions on $\mathbb{C}[2]$, and the cochain map is multiplication by σ. The shift by -2 is needed so that the cochain map has total degree 1. The existence of such a resolution means that we can realize the 'skyscraper' line operator as a 'bound state' of two Wilson lines both associated with the representation V but placed in different cohomological degrees. The corresponding bulk line operator is obtained using the formulas of Section A.3, where the target of the gauged B-model is taken to be $\mathbb{C}[2]$, the vector bundle E on $\mathbb{C}[2]$ is trivial and of rank 2, with graded components in degrees 1 and 0, and the bundle morphism T from the former to the latter component is multiplication by σ. In accordance with the results of Section A.3, we consider a superconnection on $\sigma^* E$ of the form

$$\mathcal{N} = \begin{pmatrix} n\mathcal{A} & 0 \\ \frac{1}{2}(\psi - i\widetilde{\psi}) & n\mathcal{A} \end{pmatrix},$$

where $n \in \mathbb{Z}$ is the weight with which $\mathbb{C}^*$ acts on V. The bulk line operator corresponding to the skyscraper sheaf at the origin of $\mathbb{C}[2]$ is the holonomy of this superconnection along the insertion line ℓ.

Another (equivalent) way is to take seriously the fact that the skyscraper sheaf is localized at $\sigma = 0$ and require the field σ to vanish at the insertion line ℓ. To make this well defined, one needs to excise a small tubular neighborhood of ℓ and impose a suitable boundary condition on the resulting boundary. This condition must set $\sigma = 0$ and leave the components of $\mathcal{A}$ tangent to the boundary ℓ unconstrained. BRST-invariance determines uniquely the boundary conditions for all other fields.

6.3 Surface operators at $t = i$ and G nonabelian

It was shown in Section 5.5 that compactification of the $t = i$ theory for general G yields a gauged Rozansky-Witten model with target $T^*[2]G_{\mathbb{C}}$ (the '[2]' indicates that the fiber coordinates of the cotangent bundle are formally assigned ghost number 2). The G-action on $T^*[2]G_{\mathbb{C}}$ is induced by the conjugation action on the base.

6.3.1 Some simple boundary conditions

Let us consider some boundary conditions in the gauged Rozansky-Witten model with target $T^*[2]G_{\mathbb{C}}$. The most natural boundary condition in the gauge sector is the Neumann condition, which preserves full gauge-invariance on the boundary. In the matter sector one has to pick a G-invariant complex Lagrangian submanifold of $T^*[2]G_{\mathbb{C}}$ which is invariant with respect to the rescaling of the fiber. Such a Lagrangian submanifold can be constructed by picking a $G_{\mathbb{C}}$-invariant closed complex submanifold of $G_{\mathbb{C}}$ and taking its conormal bundle. For example, one can take the whole $G_{\mathbb{C}}$, and then the Lagrangian submanifold is given by $\sigma = 0$. We will call the resulting boundary condition in the gauged RW model the distinguished boundary condition. It is an analogue of the NN condition in the abelian case.

Another natural choice of a G-invariant Lagrangian submanifold is the conormal bundle of a complex conjugacy class in $G_{\mathbb{C}}$. In order for the submanifold to be closed take the conjugacy class to be semisimple. This boundary condition is a nonabelian analogue of the ND condition. The corresponding surface operator is a semisimple Gukov-Witten-type surface operator. Indeed, fixing a semisimple conjugacy class of $\exp(-2\pi(A_4 + i\phi_4))$ is the same as fixing a semisimple conjugacy class of the limiting holonomy of the complex connection $A + i\phi$ in the 4d gauge theory. More generally, if the conjugacy class is not closed, one needs to consider the conormal bundle of its closure.

It is easy to analyze boundary line operators for these boundary conditions. Reducing the 3d theory on an interval with the distinguished boundary conditions we get a B-type 2d gauge theory coupled to a B-model with target $G_{\mathbb{C}}$. The gauge group acts on $G_{\mathbb{C}}$ by conjugation. According to A.3, the corresponding category of branes is equivalent to $D^b_{G_{\mathbb{C}}}(Coh(G_{\mathbb{C}}))$. The monoidal structure cannot be deduced from the 2d considerations, but the same analysis as in the usual RW model shows that it is given by the derived tensor product.

In the Gukov-Witten case we need to fix a semisimple complex conjugacy class $\mathcal{C}$ in $G_{\mathbb{C}}$. Let $N^*\mathcal{C}$ denote the total space of its conormal bundle in $T^*G_{\mathbb{C}}$. Concretely, it is the space of pairs (g, σ), where $g \in \mathcal{C}$ and $\sigma \in \mathfrak{g}_{\mathbb{C}}$ satisfies $\mathrm{Tr}\,\sigma\, g^{-1}\delta g = 0$ for any δg tangent to $\mathcal{C}$ at g. The fiber coordinate σ has cohomological degree 2; to indicate this we will denote the corresponding graded complex manifold $N^*[2]\mathcal{C}$. Reduction on an interval in the Gukov-Witten case gives a B-type 2d gauge theory coupled to a B-model whose target is $N^*[2]\mathcal{C}$. Its category of branes is $D^b_{G_{\mathbb{C}}}(Coh(N^*[2]\mathcal{C}))$. The monoidal structure is given by the derived tensor product.

6.3.2 Bulk line operators

It is interesting to consider the special case of a Gukov-Witten surface operator corresponding to the trivial conjugacy class in $G_{\mathbb{C}}$ (i.e., the identity). This is the trivial surface operator, so the category of 3d boundary line operators in this case can be identified with the category of bulk line operators in the 4d TFT. The conormal bundle of the identity element is simply the dual of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$; the group $G_{\mathbb{C}}$ acts on it by the adjoint representation. Thus the category of 4d bulk line operators is equivalent to $D^b_{G_{\mathbb{C}}}(Coh(\mathfrak{g}^*_{\mathbb{C}}[2]))$. In other words, it is the $G_{\mathbb{C}}$-equivariant derived category of the graded algebra $\oplus_p \mathrm{Sym}^p\mathfrak{g}$ where the p^{th} component sits in cohomological degree $2p$.

In view of this result it is interesting to consider local operators sitting at the junction of two Wilson loops in representations V_1 and V_2 of G. The corresponding objects of the category $D^b_{G_{\mathbb{C}}}(Coh(\mathfrak{g}^*_{\mathbb{C}}[2]))$ are free modules over $\mathfrak{A} = \oplus_p \mathrm{Sym}^p\mathfrak{g}[2]$ of the form $V_1 \otimes_{\mathbb{C}} \mathfrak{A}$ and $V_2 \otimes \mathfrak{A}$, with the obvious $G_{\mathbb{C}}$ action. The space of morphisms between them is the space of $G_{\mathbb{C}}$-invariants in the infinite-dimensional graded representation

$$V_1^* \otimes V_2 \otimes \mathfrak{A}\ .$$

Indeed, a BRST-invariant and gauge-invariant junction of two Wilson lines should be an operator in representation $V_1^* \otimes V_2$ constructed out of the complex scalar σ taking values in $\mathfrak{g}_{\mathbb{C}}$. The space of such operators in ghost number $2p$ is $\mathrm{Hom}_G(\mathrm{Sym}^p\mathfrak{g}, V_1^* \otimes V_2)$, where Hom_G denotes the space of morphisms in the category of representations of G. Summing over all p we get the above answer.

6.3.3 More general surface operators

As in the abelian case, the above examples do not exhaust the set of objects in the 2-category of surface operators. For example, in [47] more complicated surface operators have been considered which involve higher-order poles for the complex connection $\mathcal{A} = A + i\phi$. By analogy with the Rozansky-Witten model we propose that the most general surface operator at $t = i$ (or equivalently, the most general boundary condition in the 3d theory) can be defined as a module category of the monoidal category of boundary line operators for the distinguished boundary condition $\mathbb{X}_0$. As explained above, this monoidal category is $D^b_{G_\mathbb{C}}(Coh(G_\mathbb{C}))$. Concretely, this means that the most general surface operator can be obtained by fibering a family of 2d TFTs over $G_\mathbb{C}$, so that the $G_\mathbb{C}$ action on the base (by conjugation) lifts to a $G_\mathbb{C}$ action on the whole family. For example, one may consider a complex manifolds X which is a fibration over $G_\mathbb{C}$, so that fibers are Calabi-Yau manifolds, and one is given a lift of the $G_\mathbb{C}$ action on the base (by conjugation) to a $G_\mathbb{C}$ action on the total space.

Given any surface operator $\mathbb{X}$, one may construct a module category over $D^b_{G_\mathbb{C}}(Coh(G_\mathbb{C}))$ by looking at the category of line operators sitting at the junction of $\mathbb{X}$ and the distinguished surface operator $\mathbb{X}_0$. This category is the category of branes in the 2d TFT obtained by compactifying the 3d TFT on an interval, with the boundary conditions on the two ends given by $\mathbb{X}$ and $\mathbb{X}_0$. Equivalently, one may compactify the 4d TFT on a twice-punctured 2-sphere, with surface operators $\mathbb{X}$ and $\mathbb{X}_0$ inserted at the two punctures.

For example, if we consider a surface operator defined, as in [47], by a prescribed singularity in the complex connection $\mathcal{A}$, and take into account that the distinguished surface operator is defined by allowing the holonomy of $\mathcal{A}$ to be free, we see that the space of vacua of the effective 2d TFT is the moduli space of connections on a punctured disc with the prescribed singularity at the origin. Let us denote this moduli space $\mathcal{M}$. If in the definition of $\mathcal{M}$ we divide by the group of gauge transformations which reduce to the identity at some chosen point on the boundary of the disk, then $\mathcal{M}$ is acted upon by $G_\mathbb{C}$ and is fibered over $G_\mathbb{C}$ (the holonomy of $\mathcal{A}$ along the boundary of the disk). It looks plausible that the effective 2d TFT is the B-model with target $\mathcal{M}$ coupled to a B-type gauge theory with gauge group $G_\mathbb{C}$. Its category of branes is a module category over $D^b_{G_\mathbb{C}}(Coh(G_\mathbb{C}))$.

6.4 Surface operators at $t = 1$ and $G = U(1)$

In Section 5.4, we found that, for $t = 1$, the GL-twisted theory compactifies to a gauged 3d A-model TFT with target space G. The analysis of the boundary conditions of the matter sector (a 3d A-model with target G, see Section B.1) has been carried out in [6]. However, boundary conditions of the full gauged 3d A-model have not previously been analyzed. In this section, we restrict ourselves to $G = U(1)$, in which case the gauge sector splits cleanly from the matter sector.

6.4.1 Boundary conditions in the gauge sector

The gauge sector is the dimensional reduction of the Donaldson-Witten 4d TFT [45] down to 3d; in Appendix B we call this the A-type 3d gauge theory. However, this by itself does not teach us very much, since boundary conditions in this theory have not been discussed previously. Without adding boundary degrees of freedom, the only choices are the Dirichlet and Neumann boundary conditions for gauge fields, with BRST-invariance fixing the conditions on all other fields.

6.4.1.1 The Dirichlet condition

Let us begin with the Dirichlet condition which says that the restriction of A to the boundary is trivial. Since $\delta_Q^2 = 2i\delta_g(\sigma)$, this makes sense only if σ also vanishes on the boundary. BRST-invariance then requires $\eta + \widetilde{\eta}$ and the restriction of the 1-form $\psi + \widetilde{\psi}$ to vanish. The fermionic equations of motion then require the restriction of χ to vanish, and the BRST-invariance implies that ϕ_4 must satisfy the Neumann condition $\partial_3\phi_4 = 0$, where we assumed that the boundary is given by $x^3 = 0$.

The Dirichlet boundary condition has the property that it has no nontrivial local BRST-invariant boundary observables. Indeed, the only nonvanishing BRST-invariant 0-form is $\psi_4 - \widetilde{\psi}_4$, but it is BRST-exact. To analyze boundary line operators, we use the dimensional reduction trick and compactify the 3d theory on an interval with the Dirichlet boundary conditions. The only bosonic fields in the effective 2d theory are the constant mode of ϕ_4 and the holonomy of A along the interval parameterized by x^3. That is, the bosonic fields are a real scalar and a periodic real scalar. The effective 2d TFT is therefore a sigma-model with target $\mathbb{R} \times S^1$. In fact, it can be regarded as an A-model with target T^*S^1. The easiest way to see this is to note that the path-integral of the 3d theory localizes on configurations given by solutions of the Bogomolny equations

$$F + \star d\phi_4 = 0.$$

Upon setting all fields to zero except A_3 and ϕ_4 and assuming that they are independent of x^3, this equation becomes

$$dA_3 + \star d\phi_4 = 0,$$

where $\star$ is the 2d the Hodge star operator. This is an elliptic equation which can be interpreted as the holomorphic instanton equation, provided we declare $A_3 + i\phi_4$ to be a complex coordinate on the target. Since the action of the 4d theory is BRST-exact, so is the action of the 2d model. This agrees with the well-known fact that the action of an A-model is BRST-exact if the symplectic form on the target space is exact.

The category of line operators on the Dirichlet boundary is therefore the Fukaya-Floer category of T^*S^1 whose simplest objects are Lagrangian submanifolds equipped with unitary vector bundles with flat connections. Since this category arises as the endomorphism category of an object in a 2-category, it must have a monoidal structure, which is not visible from the purely 2d viewpoint. In fact, we do not expect the Fukaya-Floer category of a general symplectic manifold to have a natural monoidal structure. We will argue below that the monoidal structure is induced by the mirror symmetry which establishes the equivalence of the Fukaya-Floer category of T^*S^1 with $D^b(Coh(\mathbb{C}^*))$ and the monoidal structure on the latter category. For now we just note that the base S^1 has a distinguished point corresponding to the trivial holonomy of A on the interval. The fiber over this point is a Lagrangian submanifold in T^*S^1 and is the identity object with respect to the monoidal structure. The distinguished point allows us to identity S^1 with the group manifold $U(1)$.

6.4.1.2 The Neumann condition

Now let us consider the Neumann condition for the 3d gauge field A. This means that the gauge symmetry is unbroken on the boundary and the restriction of the 1-form $\star F$ vanishes. Then the Bogomolny equation requires ϕ_4 to have the Dirichlet boundary condition $\phi_4 = a = const$, and by BRST-invariance $\psi_4 - \widetilde{\psi}_4$ must vanish at $x^3 = 0$. Fermionic equations of motion imply then that $\psi_3 + \widetilde{\psi}_3$ vanishes as well, and since $\delta_Q(\psi_3 + \widetilde{\psi}_3) = 2\partial_3\sigma$, the field σ satisfies the Neumann condition. Finally, the restriction of the 1-form $\star\chi$ to the boundary must vanish, in order for the fermionic boundary conditions to be consistent. Indeed, if x^1 is regarded as the time direction, then $(\star\chi)_2$ is canonically conjugate to $\psi_3 + \widetilde{\psi}_3$, so if one of them vanishes, so should the other. Similarly, if x^2 is regarded as time, then $(\star\chi)_1$ is canonically conjugate to $\psi_3 + \widetilde{\psi}_3$ and therefore must vanish too.

In the Neumann case the space of BRST-invariant local observables on the boundary is spanned by powers of the field σ. To determine the category of boundary line operators one has to reduce the 3d gauge theory on an interval with the Neumann boundary conditions. The bosonic fields of the effective 2d theory are the 2d gauge field and the constant mode of the scalar σ, the fermionic ones are the 0-form $\eta + \widetilde{\eta}$, the 1-form $\psi + \widetilde{\psi}$, and the 2-form χ. Their BRST transformations are

$$\begin{aligned}
\delta_Q A &= i(\psi + \widetilde{\psi}),\\
\delta_Q \sigma &= 0,\\
\delta_Q \bar{\sigma} &= i(\eta + \widetilde{\eta}),\\
\delta_Q(\eta + \widetilde{\eta}) &= 0,\\
\delta_Q(\psi + \widetilde{\psi}) &= 2d\sigma,\\
\delta_Q \chi &= F.
\end{aligned}$$

This 2d TFT can be obtained from the usual $N = (2, 2)$ $d = 2$ supersymmetric gauge theory by means of a twist which makes use of the $U(1)_V$ R-symmetry. Since this is the same R-symmetry as that used for constructing an A-type sigma-model, we might call this TFT an A-type 2d gauge theory. As far as we know, its boundary conditions have not been analyzed in the literature previously. It is shown in Section A.1 that its category of branes is equivalent to the bounded derived category of coherent sheaves on the graded line $\mathbb{C}[2]$. [1] Again, the 3d origin of this category means that it must have monoidal structure. Here it is given by the usual derived tensor product of complexes of coherent sheaves. The trivial line bundle on $\mathbb{C}[2]$ is the identity object. From the 3d viewpoint, it corresponds to the 'invisible' line operator on the boundary.

As mentioned above, the Neumann condition depends on a real parameter a, the boundary value of the scalar ϕ_4. On the quantum level there is another parameter which takes values in $\mathbb{R}/2\pi\mathbb{Z}$. It enters as the coefficient of a topological term in the boundary action:

$$\theta \int_{x^3=0} \frac{F}{2\pi}.$$

Thus overall the Neumann condition in the gauge sector has the parameter space $\mathbb{R} \times \mathrm{S}^1 \simeq \mathbb{C}^*$.

6.4.2 Boundary conditions in the matter sector

We may impose either Dirichlet or Neumann condition on the periodic scalar A_4. Let us discuss these two possibilities in turn.

6.4.2.1 The Dirichlet condition

If A_4 satisfies the Dirichlet condition, then BRST-invariance requires the 1-form ϕ to satisfy the Neumann condition. This means that the components of ϕ tangent to the boundary are free and satisfy $\partial_3\phi_1 = \partial_3\phi_2 = 0$, while the component ϕ_3 takes a fixed value $\phi_3 = a$ on the boundary. BRST-invariance also requires the following fermions to vanish on the boundary: $\psi_4 + \widetilde{\psi}_4$, $\psi_3 - \widetilde{\psi}_3$, ρ_1, ρ_2. The real parameter a together with the boundary value of A_4 combine into a parameter taking values in $S^1 \times \mathbb{R}$. These parameters are actually irrelevant, in the sense that topological correlators do not depend on them. To see this, note that shifting the boundary value of A_4 can be achieved by adding a boundary term to the action of the form

$$\int_{x^3=0} \partial_3 A_4 d^2x = \int_{x^3=0} \left(\delta_Q \rho_3 - (\partial_1\phi_2 - \partial_2\phi_1)\right) d^2x \ .$$

We see that up to a total derivative this boundary term is BRST-exact, hence does not affect the correlators. A similar argument can be made for the boundary value of ϕ_3.

Placing the theory on $W = \Sigma \times I$ (with Σ a 2-manifold and I an interval with Dirichlet boundary

[1] This category is equivalent to the $U(1)$-equivariant constructible derived category of sheaves over a point [28].

conditions) yields a novel 2d TFT on Σ that we shall call the *modified A-model* in Chapter 7 (where we will study its properties at greater length). For a target space X equipped with a $U(1)$ action, one can twist the ordinary 2d A-model to produce a modified A-model with target X. In our case, $X = \mathbb{C}$, endowed with rotations by a phase. The bosonic $\mathbb{C}$-valued field Z of the ordinary A-model becomes, after twisting, a real 1-form (the two components are the real and imaginary components of Z). This is the 1-form $\phi|_\Sigma$ that survives reduction.

Apart from the bosonic 1-form ϕ, the modified A-model has a fermionic 1-form $\psi - \widetilde{\psi}$ and a pair of fermionic 0-forms $\eta - \widetilde{\eta}$ and ρ (the latter comes from the component ρ_3 of the 1-form ρ in 3d). Their BRST transformations are

$$\begin{aligned} \delta_Q \phi &= i(\psi - \widetilde{\psi}), \\ \delta_Q(\psi - \widetilde{\psi}) &= 0, \\ \delta_Q(\eta - \widetilde{\eta}) &= 2d^\star \phi, \\ \delta_Q \rho &= \star d\phi \ . \end{aligned}$$

Here $\star$ is the 2d Hodge star operator, and $d^\star = \star d \star$.

To understand the category of boundary line operators in 3d, we need to describe the category of boundary conditions for the modified A-model. This is fairly straightforward. A natural class of boundary conditions is obtained by imposing on the boundary

$$(a\phi + b \star \phi)\,|_{\partial W} = 0.$$

The special cases $b = 0$ and $a = 0$ correspond to the 2d Dirichlet and Neumann conditions. Since the theory obviously has a symmetry rotating ϕ into $\star\phi$, it is sufficient to consider the Neumann condition $\star\phi| = 0$. BRST-invariance requires the restriction of $\star\psi$ and ρ to vanish on such a boundary. It is easy to see that there are no nontrivial BRST-invariant boundary observables (the only BRST-invariant fermion ψ is BRST-exact), so there is no possibility to couple boundary degrees of freedom in a nontrivial way. This implies that the category of boundary conditions is the same as for a trivial 2d TFT, i.e., the category of complexes of finite-dimensional vector spaces. We may denote it $D^b(Coh(\bullet))$.

There is an important subtlety here related to the fact that the scalar A_4 is periodic with period 1. When reducing on an interval, this means that there are 'winding sectors', where

$$\int dx^3 \partial_3 A_4 = n, \quad n \in \mathbb{Z}.$$

This winding is constant along a connected component of the boundary and does not affect the 2d theory in any way. We may incorporate it by introducing an additional integer label on each boundary component which serves as a conserved boundary charge. This is mathematically equivalent to saying that the category of boundary conditions is the category of $\mathbb{C}^*$-equivariant coherent sheaves over a point $D^b_{\mathbb{C}^*}(Coh(\bullet))$. Objects of this category are complexes of finite-dimensional vector spaces with a $\mathbb{C}^*$-action, such that the differentials in the complex commute with the $\mathbb{C}^*$ action. Morphisms are required to preserve the $\mathbb{C}^*$-action, i.e., to have zero $\mathbb{C}^*$-charge.

6.4.2.2 The Neumann condition

If A_4 satisfies the Neumann condition $\partial_3 A_4 = 0$, then BRST-invariance requires ϕ to satisfy the Dirichlet condition . That is, the restriction of ϕ to the boundary must vanish, and ϕ_3 must satisfy $\partial_3\phi_3 = 0$. This boundary condition does not have any parameters.

The reduction on an interval gives rise to the A-model with the bosonic fields A_4 and ϕ_3. This can be seen, for example, by looking at the 3d BPS equation $dA_4 + \star d\phi = 0$ and restricting to field configurations where $\phi_1 = \phi_2 = 0$ and A_4 and ϕ_3 are independent of x^3. For such field configuration the BPS equation becomes the holomorphic instanton equation with target $S^1 \times \mathbb{R} \simeq \mathbb{C}^*$. From the symplectic viewpoint, $\mathbb{C}^*$ with its standard Kähler form is isomorphic to T^*S^1. Thus the category of boundary line operators in this case is the Fukaya-Floer category of T^*S^1. Since this category arises as the category of boundary line operators in the 3d TFT, it must have a monoidal structure. Although the category appears to be the same as in the gauge sector with the Dirichlet boundary condition, we will see that the monoidal structure is completely different and is induced by the equivalence between (a version of) the Fukaya-Floer category of T^*S^1 and the constructible derived category of S^1 [43]. In particular, the identity object (i.e., the invisible boundary line operator) is different and corresponds to the zero section of T^*S^1 with a trivial rank-1 local system. This illustrates the fact that a monoidal structure on branes in a 2d TFT depends on the way this 2d TFT is realized as a compactification of a 3d TFT on an interval.

6.4.3 Electric-magnetic duality of surface operators at $t = i$ and $t = 1$

We are now ready to describe how the 4d electric-magnetic duality acts on various boundary conditions described above. Since for both gauge and matter sectors one can have either Dirichlet or Neumann conditions, there are four possibilities to consider.

From the 3d viewpoint, 4d electric-magnetic duality amounts to dualizing the 3d gauge field A into a periodic scalar, and simultaneously dualizing the periodic scalar A_4 into a 3d gauge field. It is easy to see that electric-magnetic duality applied to the A-type gauge theory gives the Rozansky-Witten model with target $T^*[2]\mathbb{C}^*$, i.e., it maps the A-type gauge sector to the B-type matter sector. Similarly, it maps the A-type matter sector into the B-type gauge theory (with gauge group $U(1)$). In other words, electric-magnetic duality reduces to particle-vortex duality done twice.

The dual of the Neumann condition for a periodic scalar is the Dirichlet condition for the gauge field, and vice versa. We will use this well-known fact repeatedly in what follows.

6.4.3.1 The DD condition

The first possibility is the Dirichlet condition in both gauge and matter sectors at $t = 1$. The Dirichlet condition in the A-type gauge sector maps into a boundary condition in the Rozansky-Witten model with target $T^*[2]\mathbb{C}^*$ which sets $\sigma = 0$ on the boundary and leaves the complex scalar τ free to fluctuate. The Dirichlet condition in the A-type matter sector is mapped to the Neumann condition in the B-type gauge theory. Note that the Dirichlet condition in the A-type matter sector has two real parameters taking values in S^1 and $\mathbb{R}$. The former one is mapped to a boundary theta-angle, i.e., a boundary term in the action of the form

$$\theta \int_{x^3=0} \frac{F}{2\pi} = \theta \int_{x^3=0} \frac{\mathcal{F}}{2\pi}.$$

The latter parameter is the boundary value of the field ϕ_3. Both of these parameters are irrelevant, as discussed in section 4.

As discussed above, the category of boundary line operators in the A-type 3d gauge theory is the Fukaya-Floer category of $T^*U(1)$. On the other hand, the category of boundary line operators in the Rozansky-Witten model is $D^b(Coh(\mathbb{C}^*))$, as explained in [36]. These categories are equivalent, by the usual 2d mirror symmetry.

Let us recall how 2d mirror symmetry acts on some objects in this case. The trivial line bundle on $\mathbb{C}^*$ is mapped to the fiber over a distinguished point of the base S^1. This distinguished point allows us to identify S^1 with the group manifold $U(1)$. More generally, we may consider a holomorphic line bundle on $\mathbb{C}^*$ with a $\bar{\partial}$-connection of the form

$$\bar{\partial} + i\lambda \frac{d\bar{z}}{\bar{z}}, \quad \lambda \in \mathbb{C}.$$

We will denote such a line bundle $\mathcal{L}_\lambda$. Gauge transformations can be used to eliminate the imaginary part of λ. They also can shift the real part of λ by an arbitrary integer. Thus we may regard the parameter λ as taking values in $\mathbb{R}/\mathbb{Z} \simeq S^1$. Mirror symmetry maps $\mathcal{L}_\lambda$ to a Lagrangian submanifold in $T^*U(1)$ which is a fiber over the point $\exp(2\pi i\lambda) \in U(1)$.

Applying mirror symmetry to the obvious monoidal structure on $D^b(Coh(\mathbb{C}^*))$ given by the derived tensor product we get a monoidal structure on the Fukaya category of $T^*U(1)$. The trivial holomorphic line bundle on $\mathbb{C}^*$, which serves as the identity object in $D^b(Coh(\mathbb{C}^*))$, is mapped to the Lagrangian fiber over the identity element of $U(1)$. If we consider two Lagrangian fibers over the points $\exp(2\pi i\lambda_1), \exp(2\pi i\lambda_2) \in U(1)$, their mirrors are line bundles $\mathcal{L}_{\lambda_1}$ and $\mathcal{L}_{\lambda_2}$. Their tensor product is a line bundle $\mathcal{L}_{\lambda_1+\lambda_2}$ whose mirror is the Lagrangian fiber over the point $\exp(2\pi i(\lambda_1 + \lambda_2)) \in U(1)$. Clearly, this rule for tensoring objects of the Fukaya category makes use of the group structure of $U(1)$, i.e., it is a convolution-type tensor product.

Another natural class of Lagrangian submanifolds to consider are constant sections of $T^*U(1)$, i.e., submanifolds given by the equation $\phi_4 = const$. These submanifolds are circles and may carry a nontrivial flat connection. Thus such A-branes are labeled by points of $\mathbb{R} \times U(1) \simeq \mathbb{C}^*$. The mirror objects are skyscraper sheaves on $\mathbb{C}^*$. The derived tensor product of two skyscrapers supported at different points is obviously the zero object. The derived tensor product of a skyscraper with itself can be shown to be isomorphic to the sum of the skyscraper and the skyscraper shifted by -1. That is, it is a skyscraper sheaf over the same point whose stalk is a graded vector space $\mathbb{C}[-1] \oplus \mathbb{C}$. Applying mirror symmetry, we see that the tensor product of a section of $T^*U(1)$ with itself must be the sum of two copies of the same section, but with the Maslov grading of one of them shifted by -1. We do not know how to reproduce this result without appealing to mirror symmetry, i.e., by computing the product of boundary line operators in the A-type gauge theory.

As discussed above, the category of boundary line operators in the A-type matter sector is the category of branes in a somewhat unusual 2d TFT which is a modification of the A-model with target $T^*\mathbb{R}$. It was argued above that this category is equivalent to $D^b_{\mathbb{C}^*}(Coh(\bullet))$. This agrees with the B-side, where the reduction on an interval gives a B-type 2d gauge theory.

Putting the gauge and matter sectors together, we see that the DD boundary condition on the A-side is mapped to what we called the distinguished boundary condition on the B-side. The category of boundary line operators for such a boundary condition is the $\mathbb{C}^*$-equivariant derived category of coherent sheaves $D^b_{\mathbb{C}^*}(Coh(\mathbb{C}^*))$ with its obvious monoidal structure. On the A-side we get a graded version of the Fukaya-Floer category of $T^*U(1)$ where a flat vector bundle over a Lagrangian submanifold has an additional integer grading and morphisms are required to have degree zero with respect to it. This grading arises from the winding number of the periodic scalar A_4.

We can also interpret the duality in 4d terms. Indeed, it is easy to see that the DD boundary condition on the A-side arises from a 4d Dirichlet boundary condition at $t = 1$, while its dual on the B-side arises from the 4d Neumann condition at $t = i$. Thus electric-magnetic duality exchanges Dirichlet and Neumann boundary conditions in 4d, as expected. The surface operators corresponding to such 4d boundary conditions can be interpreted as follows: we excise a tubular neighborhood of the support of the surface operator and impose the 4d boundary condition on the resulting boundary. In a TFT, such a procedure gives a surface operator (i.e., there is no need to take the limit where the thickness of the tubular neighborhood goes to zero).

6.4.3.2 The NN condition

This condition is the distinguished boundary condition on the A-side, since the gauge group is unbroken on the boundary, and the periodic scalar A_4 is free to explore the whole circle. It is mapped by electric-magnetic duality to the Dirichlet boundary condition for the B-type gauge theory and the boundary condition in the RW model with target $\mathbb{C}^*$ which fixes the $\mathbb{C}^*$-valued scalar τ and leaves σ free. Note that both the Neumann boundary condition in the A-type gauge theory and the corresponding boundary condition in the RW model have a parameter taking values in $\mathbb{C}^* \simeq \mathbb{R} \times U(1)$.

Let us compare the categories of boundary line operators. The category of boundary line operators in the A-type gauge theory is the bounded derived category of coherent sheaves $D^b(Coh(\mathbb{C}[2]))$. The category of boundary line operators in the RW model is also $D^b(Coh(\mathbb{C}[2]))$. The category of boundary line operators in the A-type matter sector is the Fukaya-Floer category of T^*S^1. The category of boundary line operators in the B-type gauge sector is $D^b(Coh(\mathbb{C}^*))$. Their equivalence is a special case of the usual 2d mirror symmetry.

But there is more: we expect that the categories of boundary line operators are equivalent as monoidal categories. This is easy to see directly for the RW model with target $\mathbb{C}^*$ and A-type gauge theory with gauge group $U(1)$. Indeed, in both cases typical objects in the category of boundary line operators are complexes of holomorphic vector bundles which can be represented by Wilson line operators on the boundary for some superconnection on the pullback vector bundle. In the classical approximation, fusing two such boundary line operators corresponds to the tensor product of complexes, and there can be no quantum corrections to this result.

It is more complicated to compare the monoidal structures for the other pair of dual theories (B-type gauge theory and A-type matter). We will not attempt to do an independent computation on the A-side but instead describe the monoidal structure on the B-side and then explain what it corresponds to on the A-side.

Note that since $\mathbb{C}^*$ is a complex Lie group, the category $D^b(Coh(\mathbb{C}^*))$ has two natural monoidal structures: the derived tensor product, and the convolution-type product. The former one does not make use of the group structure, while the latter one does. The identity object of the former one is the sheaf of holomorphic functions on $\mathbb{C}^*$, while for the latter structure it is the skyscraper sheaf at the identity point $1 \in \mathbb{C}^*$. It is the latter monoidal structure which describes the fusion of boundary line operators on the B-side. Indeed, the 3d meaning of the coordinate on $\mathbb{C}^*$ is the holonomy of the connection $A + i\phi$ along a small semicircle with both ends on the boundary and centered at the boundary line operator (see figure 6.1). Skyscraper sheaves correspond to boundary line operators for which this holonomy is fixed. In particular, the skyscraper sheaf at $1 \in \mathbb{C}^*$ corresponds to the 'invisible' boundary line operator for which this holonomy is trivial. By definition, this is the identity object in the monoidal category of boundary line operators.

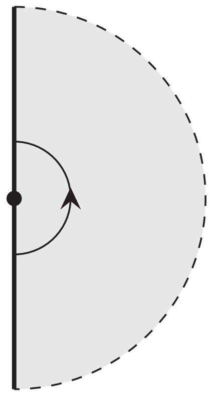

Figure 6.1: A skyscraper sheaf corresponds to a boundary line operator for which the holonomy of $A + i\phi$ along a small semi-circle around it is fixed. The dot marks the location of the boundary line operator, which we view here in cross section.

Mirror symmetry maps a skyscraper sheaf on $\mathbb{C}^*$ to a Lagrangian submanifold of T^*S^1 which is a graph of a closed 1-form α on S^1. Topologically this submanifold is a circle and is equipped with a trivial line bundle with a flat unitary connection. The moduli space of such an object is $\mathbb{C}^*$: for $\lambda \in \mathbb{C}^*$ the phase of λ determines the holonomy of the unitary connection, while the absolute value determines the integral of α on S^1. Thus the identity object on the B-side is mirror to the zero section of T^*S^1 with a trivial flat connection. To describe the monoidal structure on the A-side it is best to recall a theorem of Nadler [42] according to which (a version of) the Fukaya-Floer category of T^*X is equivalent to the constructible derived category of X. Recall that a constructible sheaf on a real manifold X is a sheaf which is locally constant on the strata of a Whitney stratification of X; such sheaves can be regarded as generalizations of flat connections. Objects of the constructible derived category are bounded complexes of sheaves whose cohomology sheaves are constructible. The constructible derived category has an obvious monoidal structure arising from the tensor product of complexes of sheaves. The sheaf of locally constant functions is the identity object with respect to this monoidal structure. According to [43, 42], this object corresponds to the zero section of T^*S^1 with a trivial flat connection. This suggests that the monoidal structure on the A-side is given by the tensor product on the constructible derived category. It is easy to check that this is compatible with the way mirror symmetry acts on the skyscraper sheaves on $\mathbb{C}^*$.

We can try put the gauge and matter sectors together. On the B-side, we have the B-model with target $\mathbb{C}^* \times \mathbb{C}[2]$ whose category of branes is $D^b(Coh(\mathbb{C}^* \times \mathbb{C}[2]))$. On the A-side, we have an A-model with target T^*S^1 tensored with an A-type 2d gauge theory with gauge group $U(1)$. One could guess that the corresponding category of branes is a $U(1)$-equivariant version of the Fukaya-Floer category of T^*S^1. More generally, one could guess that the category of branes in an A-model with target T^*X tensored with the A-type 2d $U(1)$ gauge theory is a $U(1)$-equivariant version of the Fukaya-Floer category of T^*X. It is not clear to us how to define such an equivariant Fukaya-

Floer category mathematically. Given the results of [43, 42], a natural guess is the equivariant constructible derived category of sheaves on X. As a check, note that when X is a point, the $U(1)$-equivariant constructible derived category is equivalent to $D^b(Coh(\mathbb{C}[2]))$ [28]. As mentioned above and explained in Appendix A, this is indeed the category of branes for the A-type 2d gauge theory. The monoidal structure seems to be the standard one (derived tensor product). On the B-side, on the other hand, the monoidal structure is a combination of the tensor product of coherent sheaves on $\mathbb{C}[2]$ and the convolution product on $\mathbb{C}^*$.

6.4.3.3 The DN condition

Next consider the boundary condition on the A-side which is a combination of the Dirichlet condition in the gauge sector and the Neumann condition for A_4 in the matter sector. It is dual to the Dirichlet condition for the B-type gauge sector and a boundary condition for the RW model with target $T^*[2]\mathbb{C}^*$ which sets $\sigma = 0$ and leaves the complex scalar $\tau = A_4 + i\phi_4$ free to fluctuate.

On the B-side reduction on an interval gives a B-model with target $\mathbb{C}^* \times \mathbb{C}^*$, therefore the category of boundary line operators is $D^b(Coh(\mathbb{C}^* \times \mathbb{C}^*))$. On the A-side reduction gives an A-model with target $T^*U(1) \times T^*U(1)$, therefore the category of boundary line operators is the Fukaya-Floer category. The two categories are equivalent by the usual 2d mirror symmetry. The monoidal structure is easiest to determine on the B-side. It is neither the derived tensor product, nor the convolution, but a combination of both. This happens because the two copies of $\mathbb{C}^*$ have a very different origin: one of them arises from a 3d B-type gauge theory, and the other one arises from the Rozansky-Witten model with target $T^*[2]\mathbb{C}^*$.

6.4.3.4 The ND condition

Finally we consider the boundary condition on the A-side which is a combination of the Neumann condition in the gauge sector and the Dirichlet condition for A_4. This is the case which corresponds to the Gukov-Witten surface operator at $t = 1$. Indeed, the Dirichlet conditions for A_4, ϕ_4 and ϕ_3 mean that the holonomy of A is fixed, while the 1-form ϕ has a singularity of the form

$$\beta \frac{dr}{r} - \gamma d\theta,$$

where $-\gamma$ is the boundary value of ϕ_4 and β is the boundary value of ϕ_3. The boundary value of A_4 is the Gukov-Witten parameter α. The Neumann condition in the gauge sector also depends on the boundary theta-angle which corresponds to the Gukov-Witten parameter η. As explained above, the boundary values of A_4 and ϕ_3 are actually irrelevant. This agrees with the results of [2], where it is shown that at $t = 1$ the parameters α and β are irrelevant. Thus the true parameter space of the surface operator on the A-side is $\mathbb{C}^*$.

Electric-magnetic duality maps the DD condition to the Neumann condition for the B-type gauge theory and the boundary condition in the RW model which fixes τ and leaves σ free to fluctuate. The latter boundary condition depends on the boundary value of the field $\tau = A_4 + i\phi_4$. From the 4d viewpoint this boundary value encodes the Gukov-Witten parameters α and γ. These are the relevant parameters at $t = i$, as explained in [2]. The Neumann boundary condition in the B-type gauge theory also has two parameters (the boundary value of ϕ_3 and the boundary theta-angle) which correspond to the Gukov-Witten parameters β and η. But as explained above and from a different viewpoint in [2], these parameters are irrelevant at $t = i$.

Let us compare the categories of 3d boundary line operators, which from the 4d viewpoint are interpreted as categories of line operators sitting on Gukov-Witten surface operators. On the B-side reduction on an interval gives a B-model with target $\mathbb{C}[2]$ tensored with a B-type 2d gauge theory, therefore the category of boundary line operators is $D^b_{\mathbb{C}^*}(Coh(\mathbb{C}[2]))$. On the A-side reduction on an interval gives an A-type 2d gauge theory tensored with a modified A-model with target $T^*\mathbb{R}$. Its category of branes is a modification of the category of boundary conditions for the A-type 2d gauge theory where the space of boundary degrees of freedom has additional integer grading coming from the winding of the periodic scalar A_4, and morphisms are required to have degree zero with respect to it. Since branes in the A-type 2d gauge theory can be identified with objects of $D^b(Coh(\mathbb{C}[2]))$, the category of boundary conditions in the combined system is equivalent to $D^b_{\mathbb{C}^*}(Coh(\mathbb{C}[2]))$, in agreement with what we got on the B-side.

6.4.4 A proposal for the 2-category of surface operators at $t = 1$

By analogy with the Rozansky-Witten model, one may conjecture that the 2-category of surface operators at $t = 1$ can be described in terms of module categories over the monoidal category of boundary line operators for the distinguished boundary condition (the NN condition). We have argued above that this monoidal category is the $U(1)$-equivariant constructible derived category of S^1, where the $U(1)$ action on S^1 is trivial. It is probably better to think about it as a sheaf of $U(1)$-equivariant monoidal DG-categories over S^1. To each surface operator we may associate a sheaf of $U(1)$-equivariant module categories over this sheaf of $U(1)$-equivariant monoidal categories, and we conjecture that this map is an equivalence of 2-categories. Gukov-Witten-type operators correspond to skyscraper sheaves on S^1.

Electric-magnetic duality then implies that there is an equivalence between this 2-category and the 2-category of coherent $\mathbb{C}^*$-equivariant derived categorical sheaves over $\mathbb{C}^*$.

6.5 Surface operators at $t = 0$ and $G = U(1)$

In Section 5.2, we showed that the 3d reduction of the GL-twisted theory at $t = 0$ is identical to that of Vafa-Witten theory. Hence, the classification of surface operators is identical in both of these theories. We consider boundary conditions of the 3d reduced theory for abelian gauge group, treating first the gauge sector and then the matter sector.

6.5.1 The gauge sector

As for the $t = 1$ case, we may consider either Dirichlet or Neumann conditions for the gauge field, and then BRST-invariance determines the rest. The category of boundary line operators is determined by compactifying the theory on an interval with the appropriate boundary conditions and analyzing branes in the resulting 2d TFT.

In the Neumann case the effective 2d TFT is the A-type 2d gauge theory, just as for $t = 1$. As explained above, its category of boundary conditions is equivalent to $D^b(Coh(\mathbb{C}[2]))$.

In the Dirichlet case the effective 2d TFT is a topological sigma-model with two bosonic fields, A_4 and the holonomy of the 3d gauge field A along the interval. Both are periodic scalars, so the target of the sigma-model is T^2. The BPS equations reduce to a holomorphic instanton equation

$$dA_3 + \star dA_4 = 0,$$

which means that we are dealing with an A-model with target T^2. Its category of branes is the Fukaya-Floer category of T^2, which is fairly nontrivial (and by mirror symmetry equivalent to the bounded derived category of coherent sheaves on an elliptic curve). The A-model depends on the symplectic form on T^2 which can be read off the topological piece of the action. Setting A_1 and A_2 to zero and reducing on an interval of length 2π it becomes

$$\frac{-4\pi^2}{e^2} \int_{M_2} dA_3 \wedge dA_4 \ .$$

We may regard this expression as an integral of the pull-back of a symplectic 2-form

$$\frac{4\pi^2}{e^2} dx \wedge dy$$

on the 2-torus with periodic coordinates x, y, both with period one. The symplectic area of this 2-torus is $4\pi^2/e^2$.

We do not know how to describe the monoidal structure on this category arising from the fusion of boundary line operators.

6.5.2 The matter sector

As for $t = 1$, we may consider either the Dirichlet or Neumann conditions for the scalars ϕ_3 and ϕ_4 (BRST-invariance requires them to be of the same type). In the Dirichlet case reduction on an interval gives the modified A-model whose only bosonic field is a real 1-form ϕ in two dimensions. As discussed above, it category of branes is the same as for a trivial TFT, i.e., it is equivalent to $D^b(Coh(\bullet))$. Unlike in the $t = 1$ case, there are no 'winding sectors', since the scalars ϕ_3 and ϕ_4 are not periodic. So the category of boundary line operators in this case is $D^b(Coh(\bullet))$, with its standard monoidal structure.

If ϕ_3 and ϕ_4 satisfy the Neumann condition, then the restriction of the 1-form ϕ to the 2d boundary must vanish. Reducing on an interval, we get an A-model whose only bosonic fields are ϕ_3 and ϕ_4, namely an A-model with target $T^*\mathbb{R}$. Its category of branes is the Fukaya-Floer category of $T^*\mathbb{R}$. Since this should be thought as the category of boundary line operators in a 3d TFT, it should have a monoidal structure. Since the only difference compared to the $t = 1$ matter sector is the noncompactness of ϕ_4, we expect that after we apply the equivalence of [42], this monoidal structure becomes the standard monoidal structure on the constructible derived category of $\mathbb{R}$.

6.5.3 Putting the sectors together

6.5.3.1 The DD condition

The DD boundary condition corresponds to a surface operator such that the 1-form ϕ has a fixed singularity of the form

$$\beta\frac{dr}{r} - \gamma d\theta,$$

while the holonomy of the gauge field A is allowed to fluctuate, and the scalar field σ vanishes at the insertion surface. To define such an operator properly, one has to excise a tubular neighborhood of the insertion surface and impose suitable conditions on the newly created boundary.

Since the matter sector in the Dirichlet case does not have interesting boundary conditions, the category of boundary line operators is the same as in the gauge sector, i.e., the Fukaya-Floer category of T^2 with the symplectic area $\mathfrak{S} = 4\pi^2/e^2$. From the 4d viewpoint, this is the category of line operators on the surface operator.

Electric-magnetic duality maps the DD condition to itself. Indeed, it does not affect the matter sector, while in the gauge sector it maps the periodic scalar A_4 into a gauge field and maps the gauge field to a periodic scalar. Since in the DD case A_4 satisifies the Neumann condition, the dual gauge field satisfies the Dirichlet condition. Contrariwise, the Dirichlet condition for the gauge field is mapped by duality to the Dirichlet condition for the new periodic scalar. The only effect of duality is to replace e^2 with $4\pi^2/e^2$. Therefore the symplectic area of the T^2 is also inverted:

$$\mathfrak{S} \mapsto \mathfrak{S}' = \frac{4\pi^2}{\mathfrak{S}} \ .$$

The Fukaya-Floer categories of two tori whose symplectic areas are related as above are equivalent by the usual T-duality. Moreover, we expect that the monoidal structure (which we have not determined!) is preserved by T-duality.

6.5.3.2 The NN condition

The NN condition corresponds to the surface operator such that A has a fixed singularity of the form

$$\alpha d\theta,$$

while the singularity for the 1-form ϕ is allowed to fluctuate. To define such a surface operator properly, one has to impose suitable conditions on a boundary of a tubular neighborhood of the insertion surface.

Upon reduction on an interval with NN boundary conditions on both ends, we get a 2d TFT which is a product of an A-type 2d gauge theory and an A-model with target $T^*\mathbb{R}$. Its category of branes is an equivariant version of the Fukaya-Floer category of $T^*\mathbb{R}$. It was conjectured above that it is equivalent to the equivariant constructible derived category of $\mathbb{R}$, with the standard monoidal structure (derived tensor product).

Electric-magnetic duality maps the NN condition to itself, for the same reason as in the DD case. It acts trivially on the category of line operators, because the bosonic fields which survive the reduction on an interval (that is, σ, ϕ_3 and ϕ_4) are not involved in the duality.

6.5.3.3 The DN condition

The DN condition corresponds to a surface operator such that both A and ϕ are allowed to have fluctuating singularities, while σ has to vanish at the surface operator. Upon reduction on an interval with DN boundary conditions on both ends, we get a product of an A-model with target T^2 and an A-model with target $T^*\mathbb{R}$. Its category of branes is the Fukaya-Floer category of $T^2 \times T^*\mathbb{R}$. Electric-magnetic duality maps the DN condition to itself. Its action on the category of line operators amounts to a T-duality on T^2 (duality acts trivially on the matter sector). The monoidal structure (which we have not determined) must be preserved by T-duality.

6.5.3.4 The ND condition

This case corresponds to the Gukov-Witten surface operator where the holonomy of A is fixed, and the 1-form ϕ has a fixed singularity of the form

$$\beta\frac{dr}{r} - \gamma d\theta.$$

Reduction on an interval with ND boundary conditions gives a 2d TFT which is a product of an A-type 2d gauge theory and a modified A-model whose only bosonic field is a real 1-form. Since there are no interesting boundary conditions in the latter theory, the category of boundary conditions in this case is the same as in the former theory. That is, it is the $U(1)$-equivariant constructible derived category of sheaves over a point, or equivalently $D^b(Coh(\mathbb{C}[2]))$ [28]. This is therefore the category of line operators sitting on the Gukov-Witten surface operator. The monoidal structure is the standard one (derived tensor product).

In particular, since the trivial surface operator is a special case of the Gukov-Witten surface operator, we conclude that the category of bulk line operators in the GL-twisted theory at $t = 0$ is $D^b(Coh(\mathbb{C}[2]))$. In 4d terms, this can be interpreted as saying that all bulk line operators can be constructed by taking a sum of several copies of the trivial line operator and deforming it using the descendants of the BRST-invariant field σ and its powers. This agrees with the results of [35], where it was argued that neither Wilson nor 't Hooft line operators are allowed at $t = 0$.

Electric-magnetic duality maps the ND condition to itself. It acts trivially on the category of line operators since the field σ is not involved in the duality.

Chapter 7

Modified A-model

This chapter is devoted to a detailed study of the 2d TFT one obtains upon compactification of Vafa-Witten theory from 4d down to 2d on a genus $g_C \geq 2$ Riemann surface C. We call this 2d TFT the *modified A-model* by virtue of the fact that it is obtainable from an ordinary 2d A-model with Kähler target X by twisting with a $U(1)$ isometry of X. To our knowledge, the modified A-model has been little studied outside of this thesis and we therefore discuss it in detail.

Note that in Appendix A, we present an additional series of 2d TFTs arising in the analysis of the dimensional reduction of the various twisted versions of $d = 4$, $\mathcal{N} = 4$ SYM (among these are the A-type and B-type 2d topological gauge theories, as well as the gauged 2d B-model, which have made an appearance in the surface operator analysis of the previous chapter).

Twisting the 2d A-model

We have noted a close resemblance between GL-twisted theory at $t = 0$ and Vafa-Witten theory. In [35] it was shown that the $t = 0$ theory on a product of Riemann surfaces $\Sigma \times C$ compactifies to an A-model on Σ with target $\mathcal{M}_H(G, C)$ as the size of C shrinks to zero, where $\mathcal{M}_H(G, C)$ is the moduli space of solutions to the *Hitchin equations* on the Riemann surface C. (The space $\mathcal{M}_H(G, C)$ can roughly be thought of as the cotangent bundle of the moduli space of flat G bundles on C; we will discuss it further in the following chapter.) Hence, we expect to obtain a similar description of the compactification of Vafa-Witten theory on C, and indeed we do – with a slight twist.

Instead of an A-model, we obtain a theory that we shall call the *modified A-model*, which is a twisted version of the A-model.

We construct from scratch the modified 2d A-model with target space X, any Kähler manifold equipped with a Hamiltonian action of the group $U(1)$, which action is assumed to preserve both the Kähler form and complex structure of X (and, hence, also the Kähler metric). Starting with an A-model on Σ with target X, one twists the theory by replacing the rotation group $U(1)_E$ of Σ by the twisted rotation group

$$U(1)_E \to U(1)'_E = diag(U(1)_E \times U(1)_X), \tag{7.1}$$

where $U(1)_X$ is the transformation of the fields induced by the $U(1)$ action on the target and *diag* indicates the diagonal subgroup. Alternatively, one could start with a $(2,2)$ nonlinear sigma model on Σ with target X and make the replacement

$$U(1)_E \to U(1)'_E = diag(U(1)_E \times U(1)_V \times U(1)_X),$$

where $U(1)_V$ is the vector R-symmetry of the untwisted theory.

Locally on Σ (i.e., confining attention to a single trivializing chart of the principal $U(1)$ frame bundle of Σ), the modified A-model and the ordinary A-model are nearly identical. The difference between the two theories is felt once one considers overlaps between charts, since rotations on Σ are accompanied by an action of $U(1)_X$ on the fields.

The modified A-model considered in this chapter is a close relative of gauged sigma models with general gauge bundles and fluctuating gauge fields (for instance, the gauged 4d A-model considered in Chapter 3). In particular, a gauged 2d A-model has been discussed in [12] and elsewhere. The modified 2d A-model is a specialization thereof in which the gauge bundle is taken to be the principal $U(1)$ frame bundle P on Σ, and the connection is fixed to be the (metric-compatible, torsion-free) spin connection.

In Section 7.1 we review the ordinary 2d A-model. In Section 7.2, we describe the modified A-model twist in detail and write down the field content, variations, and action of the twisted theory. We will find it necessary to augment the usual topological term of the A-model with an extra term. In Section 7.3 we offer an interpretation of this equivariant topological term. In Section 7.4, we analyze the spectrum of local observables of the modified A-model and the ghost number anomaly of the path integral measure. Then, in Section 7.5 we consider the modified A-model defined on a Riemann surface with boundary and discuss the category of branes. Finally, in Section 7.6 we analyze the condition for the ghost number symmetry to be nonanomalous when the boundary is nontrivial.

7.1 2d A-model

We begin by reviewing the field content, action and variations of the familiar 2d A-model. Let Σ be a genus g_Σ, compact, oriented Riemann surface without boundary, and let h be a Riemannian metric on Σ. Let $z, \bar{z}$ be local complex coordinates on Σ adapted to h in the sense that

$$h_{zz} = h_{\bar{z}\bar{z}} = 0, \qquad h_{z\bar{z}} = h_{\bar{z}z} = \tfrac{1}{2}e^{2f}, \qquad h^{z\bar{z}} = h^{\bar{z}z} = 2e^{-2f},$$

where $f = f(z, \bar{z})$ is a local, real function. We write $\sigma^1 = \operatorname{Re} z$ and $\sigma^2 = \operatorname{Im} z$ for real coordinates on Σ adapted to $z, \bar{z}$. The real components of h are $h_{\mu\nu} = e^{2f}\delta_{\mu\nu}$ with $\mu, \nu = 1, 2$. The Kähler (volume) form on Σ is

$$\varepsilon = ih_{z\bar{z}}dz \wedge d\bar{z} = \frac{i}{2}e^{2f}dz \wedge d\bar{z} = e^{2f}d\sigma^1 \wedge d\sigma^2 = \sqrt{h}\, d^2\sigma$$

the final expression being valid for any real coordinates σ^μ on Σ.

The target space X is taken to be a Kähler manifold with Kähler form

$$\Omega = ig_{i\bar{j}}d\phi^i \wedge d\phi^{\bar{j}} = \frac{1}{2}\Omega_{IJ}d\phi^I \wedge d\phi^J$$

where ϕ^I are local real coordinates and $\phi^i, \phi^{\bar{j}}$ are local complex coordinates on X, and g is the Kähler metric, with

$$g_{ij} = g_{\bar{i}\bar{j}} = 0, \qquad g_{i\bar{j}} = g_{\bar{j}i} = (g_{j\bar{i}})^* = (g_{\bar{i}j})^* \ .$$

The bosonic fields of the A-model consist of a map

$$\Phi : \Sigma \to X$$

described locally via functions $\phi^I(\sigma)$, and also auxiliary fields

$$P \in \Omega^{0,1}(\Sigma, \Phi^*T^{1,0}X)$$
$$\widetilde{P} \in \Omega^{1,0}(\Sigma, \Phi^*T^{0,1}X)$$

where $T^{1,0}X$ and $T^{0,1}X$ are the holomorphic and antiholomorphic subbundles of the complexified tangent bundle of X with respect to the complex structure J_X on X; similarly, the notation $\Omega^{1,0}$ and $\Omega^{0,1}$ indicates projection onto the (1,0) and (0,1) parts of 1-forms on Σ, with respect to the complex structure J_Σ on Σ.

The fermionic fields of the A-model consist of the tangent-valued forms

$$\chi \in \Omega^0(\Sigma, \Phi^*TX)$$
$$\rho \in \Omega^{0,1}(\Sigma, \Phi^*T^{1,0}X)$$
$$\widetilde{\rho} \in \Omega^{1,0}(\Sigma, \Phi^*T^{0,1}X) \ .$$

Since the A-model is obtained by twisting a (2,2) supersymmetric nonlinear sigma model by its $U(1)_V$ vector R-symmetry, the $U(1)_A$ axial R-symmetry remains as a symmetry in the twisted theory, where instead we call it the ghost number symmetry. The bosonic fields transform with ghost number 0, the fermionic field χ with ghost number 1, and the fields ρ and $\widetilde{\rho}$ with ghost number -1. The BRST variation δ_Q itself carries one unit of ghost number; it acts on the fields as

$$\begin{aligned} \delta_Q \phi^i &= \chi^i, & \delta_Q \phi^{\bar{i}} &= \chi^{\bar{i}} \\ \delta_Q \chi^i &= 0, & \delta_Q \chi^{\bar{i}} &= 0 \\ \delta_Q \rho_{\bar{z}}{}^i &= P_{\bar{z}}{}^i - \Gamma^i{}_{jk}\chi^j \rho_{\bar{z}}{}^k, & \delta_Q \widetilde{\rho}_z{}^{\bar{i}} &= \widetilde{P}_z{}^{\bar{i}} - \Gamma^{\bar{i}}{}_{\bar{j}\bar{k}}\chi^{\bar{j}}\widetilde{\rho}_z{}^{\bar{k}} \\ \delta_Q P_{\bar{z}}{}^i &= -R^i{}_{jk\bar{l}}\,\rho_{\bar{z}}{}^j\chi^k\chi^{\bar{l}} - \Gamma^i{}_{jk}\chi^j P_{\bar{z}}{}^k, & \delta_Q \widetilde{P}_z{}^{\bar{i}} &= -R^{\bar{i}}{}_{\bar{j}\bar{k}l}\,\widetilde{\rho}_z{}^{\bar{j}}\chi^{\bar{k}}\chi^l - \Gamma^{\bar{i}}{}_{\bar{j}\bar{k}}\chi^{\bar{j}}\widetilde{P}_z{}^{\bar{k}} \end{aligned} \tag{7.2}$$

where $\Gamma^i_{jk}, \Gamma^{\bar{i}}{}_{\bar{j}\bar{k}}, R^i{}_{jk\bar{l}}, R^{\bar{i}}{}_{\bar{j}\bar{k}l}$ are the nonvanishing Christoffel symbols and Riemann tensor coefficients of the Kähler metric on X (regarded as functions of Φ). It is easy to verify that the off-shell form of the variations given above square to zero.

The A-model action is a sum of a BRST-exact term and a metric-independent term:

$$S = \delta_Q V + t\int_\Sigma \Phi^*(\Omega) \tag{7.3}$$

where

$$V = -\frac{1}{4}t\int_\Sigma g_{i\bar{j}}\Big\{\rho_{\bar{z}}{}^i(\widetilde{P}_z{}^{\bar{j}} + 4i\partial_z\phi^{\bar{j}}) + \widetilde{\rho}_z{}^{\bar{j}}(P_{\bar{z}}{}^i - 4i\partial_{\bar{z}}\phi^i)\Big\}(idz\wedge d\bar{z})$$

and t is a coupling parameter multiplying the overall action. The pullback of the Kähler form under the sigma model map is given in local coordinates by

$$\int_\Sigma \Phi^*(\Omega) = \int_\Sigma \frac{1}{2}\Omega_{IJ}d\phi^I\wedge d\phi^J = \int_\Sigma g_{i\bar{j}}\Big\{\partial_z\phi^i\partial_{\bar{z}}\phi^{\bar{j}} - \partial_{\bar{z}}\phi^i\partial_z\phi^{\bar{j}}\Big\}(idz\wedge d\bar{z}) \tag{7.4}$$

After eliminating the auxiliary fields by their equations of motion

$$P_{\bar{z}}{}^i = 2i\partial_{\bar{z}}\phi^i, \qquad \widetilde{P}_z{}^{\bar{i}} = -2i\partial_z\phi^{\bar{i}}$$

one finds that the path integral of the A-model localizes on solutions of the *holomorphic instanton equation*

$$\partial_{\bar{z}}\phi^i = 0$$

corresponding to holomorphic maps between the complex manifolds $\Phi : \Sigma \to X$.

7.2 Modified 2d A-model field content, action, variations

We now specialize to Kähler target manifolds X equipped with a left action of the group $U(1)$, which action is assumed to preserve both the symplectic form and complex structure of X in the sense that

$$\mathcal{L}_V\Omega = \mathcal{L}_V J_X = 0,$$

where $V \in \Gamma(TX)$ is the generating vector field of the $U(1)$ action. It follows that $U(1)$ acts by isometries since $\mathcal{L}_V g = 0$ is implied by the above. With these assumptions, the induced action $U(1)_X$ on fields of the A-model leaves the action invariant; however, for reasons that will become apparent shortly, we make the additional assumption that $U(1)$ is a *Hamiltonian* group action, meaning that it is generated by a globally defined, equivariant moment map on X. This is the statement that there exists a globally defined, scalar function $H : X \to \mathbb{R}$ satisfying

$$dH = -\iota_V\Omega \tag{7.5}$$

at every point of X, where ι_V indicates contraction on the vector V. In components, $\partial_J H = \Omega_{IJ}V^I$. The 1-form $\iota_V\Omega$ is automatically closed by virtue of

$$d\iota_V\Omega = \mathcal{L}_V\Omega - \iota_V d\Omega = 0,$$

implying that a locally defined function satisfying (7.5) exists whenever Ω is closed and preserved by V; hence, the nontrivial part of the Hamiltonian assumption is that the function H is globally defined. Moreover, for the group $U(1)$, equivariance of the moment map H is automatic since $\iota_V dH = -\iota_V\iota_V\Omega = 0$.

Field content of the modified A-model

As described above, we wish to twist the theory by modifying the spins of fields according to equation (7.1). After twisting, the bosonic field Φ is no longer simply a map from Σ to X, since it transforms nontrivially under rotations on Σ. Rather, it is replaced by a section $\mathbf{\Phi}$ of the associated fiber bundle

$$E = P \times_{U(1)} X$$

with typical fiber X associated to the principal $U(1)$ frame bundle P on Σ by the group action l. That is to say, $\mathbf{\Phi}$ is a map from Σ to the total space E which serves as a right inverse of the projection map $\pi_E : E \to X$. Likewise, the fermionic fields in the twisted theory take values in the pullback vector bundle $\mathbf{\Phi}^*VE$, where recall that $VE = \ker d\pi_E \subset TE$ is the subbundle of vertical vectors pointing along the fibers. As discussed in Appendix C, this bundle canonically inherits a complex structure $J^V : VE \to VE$ from that of X, so it makes sense to project vertical vectors onto their (1,0) and (0,1) parts.

In summary, the bosonic fields of the modified A-model consist of a section

$$\mathbf{\Phi} \in \Omega^0(\Sigma, E)$$

as well as auxiliary fields

$$\begin{aligned} \mathbf{P} &\in \Omega^{0,1}(\Sigma, \mathbf{\Phi}^* V^{1,0} E) \\ \widetilde{\mathbf{P}} &\in \Omega^{1,0}(\Sigma, \mathbf{\Phi}^* V^{0,1} E) \ . \end{aligned}$$

The fermionic fields of the modified A-model consist of the forms

$$\begin{aligned} \boldsymbol{\chi} &\in \Omega^0(\Sigma, \mathbf{\Phi}^* V E) \\ \boldsymbol{\rho} &\in \Omega^{0,1}(\Sigma, \mathbf{\Phi}^* V^{1,0} E) \\ \widetilde{\boldsymbol{\rho}} &\in \Omega^{1,0}(\Sigma, \mathbf{\Phi}^* V^{0,1} E) \ . \end{aligned}$$

See Appendix C for a detailed discussion of the geometry of E and its attendant structures.

Just as for the gauged 4d A-model discussed in Chapter 3, a section $\mathbf{\Phi} : \Sigma \to E$ is represented locally by maps

$$\Phi^{(\alpha)} : U^{(\alpha)} \to X$$

and a section $\boldsymbol{\chi} \in \Omega^0(\Sigma, \mathbf{\Phi}^* V E)$ is represented locally by sections

$$\chi^{(\alpha)} \in \Omega^0(U^{(\alpha)}, {\Phi^{(\alpha)}}^* TX)$$

on trivializing neighborhoods $U^{(\alpha)} \subset \Sigma$ for the underlying principal bundle P. Hence, locally on Σ, the field content of the modified A-model is equivalent to that of an ordinary A-model.

Action of the modified A-model

The local expression (7.2) for the off-shell form of the BRST variations remains valid after twisting. However, the expression for the action requires modification in order to ensure that in the twisted theory, it is globally well-defined on Σ and BRST-invariant. To make the action globally well-defined on Σ, we must replace ordinary derivatives with covariant derivatives acting on sections of the bundles E and $\mathbf{\Phi}^* V E$. This, in turn, requires us to pick a connection on the underlying principal bundle P. We choose this to be the unique metric-compatible and torsion-free spin connection ω on P; indeed, we regard this choice as part of the definition of a 'modified A-model'. In our adapted local coordinates on Σ, the spin connection is given explicitly by

$$\omega = (\partial_2 f)d\sigma^1 - (\partial_1 f)d\sigma^2 = i(\partial_z f)dz - i(\partial_{\bar{z}} f)d\bar{z} \ . \tag{7.6}$$

As discussed in Appendix C, the connection ω allows us to define covariant derivatives of sections of E and $\mathbf{\Phi}^* VE$. Just as for the gauged 4d A-model, we represent these covariant derivatives locally by the ordinary 1-forms

$$\begin{aligned}
\mathcal{D}\phi^I &= d\phi^I + \omega\, V^I \\
\mathcal{D}\chi^I &= d\chi^I + \Gamma^I_{JK} d\phi^J \chi^K + \omega\, \chi^K \nabla_K V^I
\end{aligned}$$

on neighborhoods $U^{(\alpha,\gamma)} \subset \Sigma$.

Let S_0 be the *minimal coupling* action obtained by simply replacing derivatives with covariant derivatives in the A-model action (7.3). The BRST-exact portion of S_0 remains BRST-invariant; however, substituting $\mathcal{D}\phi^I$ for $d\phi^I$ in the term (7.4) disturbs its BRST-invariance:

$$\begin{aligned}
\delta_Q S_0 &= \delta_Q\Big(t \int_\Sigma \frac{1}{2}\Omega_{IJ}\mathcal{D}\phi^I \wedge \mathcal{D}\phi^J\Big) \\
&= t\int_\Sigma \Big\{(\mathcal{L}_V\Omega)_{IJ}(d\phi^I \wedge \omega)\,\chi^J - d(\Omega_{IJ}\mathcal{D}\phi^I\chi^J) + \Omega_{IJ}V^I\chi^J d\omega\Big\} \\
&= t\int_\Sigma \Omega_{IJ}V^I\chi^J d\omega
\end{aligned}$$

where we have used $d\Omega = 0$. In going from the second to third line, we have used $U(1)$ invariance of the Kähler form Ω, and have dropped the total derivative term since Σ is boundaryless by assumption.

In order to restore BRST-invariance we add an extra (nonminimal) term S_1 to the action whose BRST variation cancels the above. As will become clear in Section 7.4, BRST-exactness of the term $\Omega_{IJ}V^I\chi^J$ is equivalent to exactness (in de Rham cohomology) of the 1-form $\iota_V\Omega$ on X. Indeed, we recognize this as the assumption we made at the outset that the group action is Hamiltonian. The existence of global moment map H allows us to write down the expression

$$S_1 = -t\int_\Sigma H(\Phi)d\omega$$

which has the proper BRST variation to cancel that of S_0. This term is globally well-defined on Σ since the curvature $d\omega$ of the spin connection is a globally defined 2-form and the function H is constant along orbits of the group action; that is,

$$H(\Phi^{(\alpha)})d\omega^{(\alpha)} = H(\Phi^{(\beta)})d\omega^{(\beta)}$$

on overlaps $U^{(\alpha)} \cap U^{(\beta)}$. In total, the action of the modified A-model is given by

$$S = \delta_Q V + t \int_\Sigma \frac{1}{2} \Omega_{IJ} \mathcal{D}\phi^I \wedge \mathcal{D}\phi^J - t \int_\Sigma H(\Phi) d\omega \tag{7.7}$$

where

$$V = -\frac{1}{4} t \int_\Sigma g_{i\bar{j}} \Big\{ \rho_{\bar{z}}{}^i (\widetilde{P}_z{}^{\bar{j}} + 4i\mathcal{D}_z \phi^{\bar{j}}) + \widetilde{\rho}_z{}^{\bar{j}} (P_{\bar{z}}{}^i - 4i\mathcal{D}_{\bar{z}} \phi^i) \Big\} (idz \wedge d\bar{z}) \ . \tag{7.8}$$

Taking the BRST variation and writing this out in complex coordinates, we find

$$\begin{aligned} S = t \int_\Sigma \Big\{ & g_{i\bar{j}} \Big(\mathcal{D}_z \phi^i \mathcal{D}_{\bar{z}} \phi^{\bar{j}} + \mathcal{D}_{\bar{z}} \phi^i \mathcal{D}_z \phi^{\bar{j}} + i \rho_{\bar{z}}{}^i \mathcal{D}_z \chi^{\bar{j}} - i \widetilde{\rho}_z{}^{\bar{j}} \mathcal{D}_{\bar{z}} \chi^i \Big) \\ & - \frac{1}{2} R_{i\bar{j}k\bar{l}} \rho_{\bar{z}}{}^i \widetilde{\rho}_z{}^{\bar{j}} \chi^k \chi^{\bar{l}} + 2(\partial_z \partial_{\bar{z}} f) H(\Phi) \Big\} (idz \wedge d\bar{z}) \ . \end{aligned}$$

The equations of motion of the auxiliary fields are given by

$$P_{\bar{z}}{}^i = 2i\mathcal{D}_{\bar{z}} \phi^i, \qquad \widetilde{P}_z{}^{\bar{i}} = -2i\mathcal{D}_z \phi^{\bar{i}} \ .$$

The modified A-model localizes on solutions of a covariantized version of the holomorphic instanton equation of the A-model:

$$\mathcal{D}_{\bar{z}} \phi^i = \partial_{\bar{z}} \phi^i + \omega_{\bar{z}} V^i = 0 \ .$$

This local expression is equivalent to the condition

$$d\mathbf{\Phi} \circ J_\Sigma = J_E \circ d\mathbf{\Phi}$$

where J_Σ is the complex structure on Σ and J_E is the complex structure on the total space E determined by ω. (See Appendix C for the construction of complex structure $J_E : TE \to TE$ using a connection on the underlying principal bundle.) This equation, in turn, says that the map $\mathbf{\Phi} : \Sigma \to E$ is a holomorphic map of complex manifolds.

Finally, note that the action of the modified A-model reduces to that of an ordinary A-model when Σ is a flat torus (in which case, the spin connection is trivial).

7.3 Equivariant topological term

Consider the BRST-inexact portion

$$S_{top} \equiv \int_\Sigma \frac{1}{2} \Omega_{IJ} \mathcal{D}\phi^I \wedge \mathcal{D}\phi^J - \int_\Sigma H(\Phi) d\omega$$

of the modified A-model action. By construction, S_{top} is annihilated by the BRST variation δ_Q; a quick inspection of the logic of the preceding section reveals that it is also left invariant by arbitrary infinitesimal variations of the map $\mathbf{\Phi}$ and the connection ω. In this sense, it represents a topological term in the action analogous to the topological term in (7.3).

We can interpret this 'equivariant' topological term as measuring the pullback of a certain equivariant cohomology class on the target space. Let $\eta \in \Omega^2_{U(1)}(X)$ be the equivariant 2-form given by

$$\eta = \Omega - H\zeta$$

where ζ is the generator of $S(\mathfrak{g}^*)$ with $\mathfrak{g} = \mathfrak{u}(1)$. The assumption that the $U(1)$ action on X is Hamiltonian with moment map H implies that η is closed with respect to the Cartan differential d_C acting on equivariant forms:

$$d_C\eta = (d - \iota_V\zeta)(\Omega - H\zeta) = -(dH)\zeta - (\iota_V\Omega)\zeta = 0 \ .$$

As described in Appendix C, a section $\mathbf{\Phi} : \Sigma \to E$ and connection ω can be used to pull back equivariant 2-form η on X to an ordinary 2-form $\widetilde{\eta}$ on Σ, then integrated over Σ to yield a topological term.

To be precise, we first pull η back to ordinary 2-form $\eta(\varphi,\omega)$ on P using the covariant derivative $d_\omega\varphi$, where $\varphi : P \to X$ is the G-equivariant map corresponding to section $\mathbf{\Phi}$. The form $\eta(\varphi,\omega)$ is basic, so it descends to a globally defined, closed 2-form $\widetilde{\eta}(\varphi,\omega)$ on Σ given by $\widetilde{\eta}(\varphi,\omega) = s^{(\alpha)*}\eta(\varphi,\omega)$ on each neighborhood $U^{(\alpha)}$. By the discussion in Appendix C, we are assured that integration of this form produces a number

$$\int_\Sigma \widetilde{\eta}(\varphi,\omega)$$

depending only on the homotopy classes of $\mathbf{\Phi}$ and ω, and the equivariant cohomology class of η. On the local neighborhood $U^{(\alpha,\gamma)}$, we can write $\widetilde{\eta}$ as

$$\begin{aligned}
\widetilde{\eta}(\varphi,\omega) &= s^{(\alpha)*}\eta(\varphi,\omega) \\
&= s^{(\alpha)*}\big[\ (d_\omega\varphi)^*\Omega_X - (H\circ\varphi)F_\omega\ \big] \\
&= \big[d_\omega\varphi \circ ds^{(\alpha)}\big]^*\Omega_X - \big(H\circ\varphi\circ s^{(\alpha)}\big)\ s^{(\alpha)*}F_\omega \\
&= \big[d_\omega\Phi^{(\alpha)}\big]^*\Big(\frac{1}{2}\,\Omega^{(\gamma)}_{IJ}\,d\phi^{(\gamma)I}\wedge d\phi^{(\gamma)J}\Big) - \big(H\circ\Phi^{(\alpha)}\big)d\omega^{(\alpha)} \\
&= \frac{1}{2}\big(\Omega^{(\gamma)}_{IJ}\circ\Phi^{(\alpha)}\big)\mathcal{D}\phi^I_{(\alpha,\gamma)}\wedge\mathcal{D}\phi^J_{(\alpha,\gamma)} - \big(H\circ\Phi^{(\alpha)}\big)d\omega^{(\alpha)}\ .
\end{aligned}$$

We recognize the integral of this expression as S_{top} written above.

7.4 Local observables

The spectrum of local observables of the modified A-model is quite straightforward. It is given simply by restricting the de Rham cohomology to its $U(1)$ invariant subcomplex.

More precisely, consider the space of closed differential k-forms ρ on the target space that are $U(1)$ invariant, in the sense that $\mathcal{L}_V\rho = 0$. Using the Cartan identity for the Lie derivative, one immediately finds that the exterior derivative $d\rho$ is $U(1)$ invariant as well. Hence, we can consistently restrict the de Rham complex on X to its $U(1)$ invariant subcomplex. The cohomology of this subcomplex is called the $U(1)$ invariant cohomology of X and is denoted $(H^*X)^{U(1)}$.

Ghost number anomaly

Let us now compute the quantum mechanical anomaly of the $U(1)$ ghost number symmetry, the vanishing of which will impose a selection rule on which of the above observables can have nonzero expectation values. The anomaly arises due to a mismatch in the path integral measure between the number of zero modes of fermions with ghost number 1 ($\boldsymbol{\chi}$ and $\boldsymbol{\rho}$) and the number of zero modes of fermions with ghost number -1 ($\widetilde{\boldsymbol{\chi}}$ and $\widetilde{\boldsymbol{\rho}}$). (Note that here the symbol $\boldsymbol{\chi}$ indicates the $(1,0)$ part of the field we previously called $\boldsymbol{\chi}$ and the symbol $\widetilde{\boldsymbol{\chi}}$ indicates its $(0,1)$ part.)

The zero mode mismatch corresponds to (twice) the index of the elliptic operator $\mathcal{D}$ appearing in the fermion kinetic term:

$$\begin{aligned} k &= 2\dim\ker\left\{\mathcal{D}_{\bar{z}} : \Omega^0(\Sigma, \boldsymbol{\Phi}^*V^{1,0}E) \to \Omega^{0,1}(\Sigma, \boldsymbol{\Phi}^*V^{1,0}E)\right\} \\ &- 2\dim\ker\left\{\mathcal{D}_{z} : \Omega^0(\Sigma, \boldsymbol{\Phi}^*V^{0,1}E) \to \Omega^{1.0}(\Sigma, \boldsymbol{\Phi}^*V^{0,1}E)\right\} \\ &= 2\dim\ker\mathcal{D}_{\bar{z}} \quad - \quad 2\dim\ker\mathcal{D}_{\bar{z}}^{\dagger} \\ &= 2\,\mathrm{ind}\ \mathcal{D}_{\bar{z}}\ . \end{aligned}$$

By the Hirzebruch-Riemann-Roch theorem, the index of this Cauchy-Riemann operator on the holomorphic vector bundle $\boldsymbol{\Phi}^*V^{1,0}E$ is given in terms of its first Chern number by the formula

$$\mathrm{ind}\ \mathcal{D}_{\bar{z}} = n(1-g_\Sigma) + \int_\Sigma c_1(\boldsymbol{\Phi}^*V^{1,0}E)\ .$$

The first Chern class can be further interpreted [12] as the pullback of the equivariant first Chern class on tangent bundle TX under the section $\boldsymbol{\Phi}$:

$$c_1(\boldsymbol{\Phi}^*V^{1,0}E) = \widetilde{\eta}(\varphi, \omega)$$

where

$$\eta = \frac{1}{2\pi}\Omega - \frac{1}{4\pi}\zeta\ \Delta H \quad \in \Omega^2_{U(1)}(X)$$

is the equivariant first Chern class and $\widetilde{\eta}(\varphi, \omega)$ is its pullback (in the notation of the preceding section). Here ΔH is the Laplacian of the Hamiltonian function and Ω is the Kähler form on X.

7.5 Boundary conditions

Consider now the modified A-model placed on a Riemann surface Σ with nontrivial boundary $\partial\Sigma$, consisting of h_Σ components diffeomorphic to h_Σ circles, which canonically inherit orientations from that of Σ. As we shall discuss further in Chapter 9, we must impose local boundary conditions on the fields along $\partial\Sigma$ to ensure that the boundary term in the variation of the action vanishes identically for all allowed variations. Moreover, demanding that the boundary condition preserves the symmetry δ_Q of the bulk theory imposes an extra constraint on boundary conditions, as we now see in detail.

$U(1)$ invariant boundary conditions

We write a set of sufficient local boundary conditions on the fields (leaving aside, for the time being, the question of necessity). The conditions we write are implicitly understood to be imposed along $\partial\Sigma$.

We are free to work with any compatible set of trivializing charts of the principal bundle P on Σ; for simplicity, let us choose each component of $\partial\Sigma$ to lie within exactly one chart $U^{(\alpha)}$. Just as for the ordinary A-model, we impose the condition that the sigma model map $\Phi^{(\alpha)}$ takes values in a submanifold $Y^{(\alpha)} \subset X$, where $Y^{(\alpha)}$ is Lagrangian with respect to the the symplectic form Ω.

We demand, moreover, that $Y^{(\alpha)}$ be *U(1) invariant*, in the sense that the generating Killing vector field V be tangent to $Y^{(\alpha)}$ at all points of $Y^{(\alpha)}$. Since $Y^{(\alpha)}$ is isotropic, the condition of $U(1)$ invariance implies that the Hamiltonian function H is locally constant over the submanifold. (This is because the rate of change of H along a tangent vector $W \in TY$ is given by $W^I\partial_I H = \Omega_{IJ}V^I W^J = 0$.) Conversely, the fact that $Y^{(\alpha)}$ is coisotropic means that constancy of H implies $U(1)$ invariance. Hence, the assumption of $U(1)$ invariance is equivalent to assuming that H takes a constant value on each component of the boundary $\partial\Sigma$. For such submanifolds, it is no longer necessary to include a label (α).

Allowed variations $\delta\Phi$ of the sigma model map are those that are tangent to the submanifold, in the sense that

$$\delta\phi^I \frac{\partial}{\partial\phi^I} \in \Omega^0(\partial\Sigma, \Phi^*TY) \subset \Omega^0(\partial\Sigma, \Phi^*TX)\ .$$

Looking ahead to BRST-invariance, we want $\delta = \delta_Q$ to be an allowed variation, so we impose the condition that the fermionic field χ be also tangent to the submanifold.

Taking the variation of the bulk action under an infinitesimal variation of the fields and imposing the bulk equations of motion, one is left with a boundary term

$$\delta S = \int_{\partial\Sigma} \star g_{IJ} d\sigma^{\mu} \Big\{ \mathcal{D}_{\mu}\phi^I \delta\phi^J + 2i\rho_{\mu}^I \delta^{(\mathrm{c})}\chi^J - 2i\widetilde{\rho}_{\mu}^I \delta^{(\mathrm{c})}\chi^J \Big\}$$

where $\star$ is the Hodge star on Σ and

$$\delta^{(\mathrm{c})}\chi^J \equiv \delta\chi^J + \Gamma^J_{KL}\delta\phi^K \chi^L$$

are the *covariant variations* of the fermions. Compatibility with the bulk equation of motion demands that this term vanish identically for all allowed variations. To ensure this vanishing, we impose the covariant Neumann condition that $n^{\mu}\mathcal{D}_{\mu}\phi^I$ takes values in the normal subbundle $(TY)^{\perp}$ (with respect to metric g), where n is normal to the boundary (with respect to metric h). Likewise, we demand that the $n^{\mu}(\rho_{\mu} - \widetilde{\rho}_{\mu})$ be normal to the submanifold. In summary, we impose the following boundary conditions:

$$\begin{aligned} \Phi &\in Y \subset X \\ \iota_n(d_{\omega}\Phi) &\in \Omega^0(\partial\Sigma, \Phi^*(TY)^{\perp}) \\ \chi &\in \Omega^0(\partial\Sigma, \Phi^*(TY)) \\ \iota_n(\rho - \widetilde{\rho}) &\in \Omega^0(\partial\Sigma, \Phi^*(TY)^{\perp}) \ . \end{aligned} \tag{7.9}$$

Working off-shell, one simply replaces $d_{\omega}\Phi$ by $P - \widetilde{P}$ in the second line above.

Let us now verify that these boundary conditions are BRST-invariant. Working off-shell, and using $\delta_Q^2 = 0$, one finds the following boundary term in the variation of the action:

$$\delta_Q S = \delta_Q S_{top} = -\int_{\partial\Sigma} \Omega_{IJ} \mathcal{D}_{\mu}\phi^I \chi^J$$

which vanishes by virtue of the fact that V, χ, and $t^{\mu}\partial_{\mu}\phi^I$ are all tangent to the submanifold (where t is tangent to the boundary).

Finally, consider the effect of varying the worldsheet metric by an amount $h_{\mu\nu} \to h_{\mu\nu} + \delta h_{\mu\nu}$. In order to continue to have a topological field theory, the action must vary under δ_h by a BRST-exact amount. A quick inspection of the off-shell variations (7.2) reveals that the metric h makes no appearance, and therefore δ_h commutes with the BRST variation δ_Q. Hence, we find

$$\delta_h S = \delta_Q \delta_h V + \delta_h S_{top} = (\text{Q-exact}) \ - \int_{\partial\Sigma} H(\Phi)\delta_h \omega$$

where $\delta_h \omega$ is the variation of the spin connection under variations of the metric. The latter term does not vanish under arbitrary variations of the metric, and therefore we introduce the additional boundary term

$$S_{\partial\Sigma} = \int_{\partial\Sigma} H(\Phi)\,\omega$$

in the modified A-model action to ensure Q-exactness of $\delta_h S$. This boundary term does not disturb compatibility with bulk equations of motion since it does not contain any derivatives of fields. It does not disturb BRST-invariance of the action since the boundary condition on χ and local constancy of H on Y implies $\delta_Q H = 0$.

7.6 Equivariant Maslov index

Let us revisit the analysis of the ghost number anomaly of the modified A-model path integral measure when the Riemann surface Σ has nontrivial boundary.

Bundles on a bordered Riemann surface

We formulate a bit more precisely the bundles where the fermions live in case Σ has nontrivial boundary with boundary conditions (7.9) imposed, where submanifold $Y \subset X$ is assumed to be Lagrangian and $U(1)$ invariant.

The $U(1)$ invariance of Y means that there is an associated fiber subbundle

$$E_Y = (P|_{\partial\Sigma}) \times_{U(1)} Y \qquad \subset \qquad E|_{\partial\Sigma}$$

with total space

$$E_Y \equiv \{\ [p,y] \quad | \quad p \in P, \quad y \in Y\}$$

and projection map $\pi_{E_Y}[p,y] = \pi_E[p,y] = \pi(p)$ (see Appendix C for notation). The bosonic field $\mathbf{\Phi}$ is a section of the pair (E, E_Y) on the bordered Riemann surface $(\Sigma, \partial\Sigma)$, i.e., a map $\mathbf{\Phi} : \Sigma \to E$ that is a right inverse of the projection map π_E and such that $\mathbf{\Phi}|_{\partial\Sigma} : \partial\Sigma \to E_Y$.

The vertical subbundles $(VE, VE_Y) \subset (TE, TE_Y)$ are the kernels of $(d\pi_E, d\pi_{E_Y})$. As discussed in Appendix C, VE inherits a metric, symplectic form, and complex structure from those of X, with respect to which we decompose $VE \otimes \mathbb{C} = V^{(1,0)}E \oplus V^{(0,1)}E$.

The fermionic field $\boldsymbol{\chi}$ is a section of the pair of bundles $(F, F_{\mathbb{R}})$ over the bordered Riemann surface $(\Sigma, \partial\Sigma)$, where

$$F \equiv \mathbf{\Phi}^* V^{(1,0)} E$$

$$F_{\mathbb{R}} \equiv \mathbf{\Phi}^* V^{(1,0)} E_Y$$

with real dimensions $(2n, n)$. Indeed, $F_{\mathbb{R}}$ is a *totally real* and Lagrangian subbundle $F_{\mathbb{R}} \subset F|_{\partial\Sigma}$ (in the terminology of [22]), satisfying

$$F|_{\partial\Sigma} = F_{\mathbb{R}} \otimes \mathbb{C} \ .$$

The complex double of a holomorphic vector bundle

As before, the ghost number anomaly will be measured by a mismatch between the number of zero modes of fermions with ghost number 1 ($\boldsymbol{\chi}$ and $\boldsymbol{\rho}$) and the number of zero modes of fermions with ghost number -1 ($\widetilde{\boldsymbol{\chi}}$ and $\widetilde{\boldsymbol{\rho}}$). This, in turn, corresponds to the index of the Cauchy-Riemann operator $\bar{\partial}_F$ of the holomorphic vector bundle pair $(F, F_{\mathbb{R}})$ over $(\Sigma, \partial\Sigma)$ (holomorphicity in this context is with respect to the interior of Σ).

The computation of the index for a pair is discussed in [21] and proceeds by first forming the *complex double* $\Sigma_{\mathbb{C}}$ of Σ, i.e., a closed Riemann surface $\Sigma_{\mathbb{C}}$ equipped with an antiholomorphic involution $\tau : \Sigma_{\mathbb{C}} \to \Sigma_{\mathbb{C}}$ and double covering $\pi_{\Sigma} : \Sigma_{\mathbb{C}} \to \Sigma$ as well as a holomorphic inclusion $\Sigma \to \Sigma_{\mathbb{C}}$, such that

$$\frac{\Sigma_{\mathbb{C}}}{<\tau>} = \Sigma, \qquad \text{and} \qquad (\Sigma_{\mathbb{C}})^{\tau} = \partial\Sigma$$

where $(\Sigma_{\mathbb{C}})^{\tau}$ is the fixed point set of the involution. This doubled surface has genus

$$g_{\Sigma_{\mathbb{C}}} = 2g_{\Sigma} + h_{\Sigma} - 1$$

where h_{Σ} is the number of boundary components and g_{Σ} is the number of holes. (For instance, the complex double of the disk is a 2-sphere equipped with antipodal map τ fixing the equator.) A holomorphic vector bundle F over Σ can be doubled to yield a holomorphic vector bundle $F_{\mathbb{C}}$ over $\Sigma_{\mathbb{C}}$ together with an involution $\widetilde{\tau} : F_{\mathbb{C}} \to F_{\mathbb{C}}$ covering τ and such that

$$F_{\mathbb{C}}|_{\Sigma} = F, \qquad \text{and} \qquad (F_{\mathbb{C}})^{\widetilde{\tau}} = F_{\mathbb{R}}$$

(see Theorem 3.3.8 of [21]). Indeed, there exists a holomorphic atlas on the closed Riemann surface $F_{\mathbb{C}}$ in which the action of $\widetilde{\tau}$ is simply τ acting on the base and complex conjugation acting on the fiber $\mathbb{C}^n$ (so that, in particular, the fiber of $F_{\mathbb{R}}$ above a point of $\partial\Sigma$ consists of the real subspace $\mathbb{R}^n \subset \mathbb{C}^n$ in this atlas).

The Maslov index of a pair $(F, F_{\mathbb{R}})$

As a complex bundle, one can trivialize F over Σ with a single trivialization $F \simeq \Sigma \times \mathbb{C}^n$. Then, we can represent the fiber of $F_{\mathbb{R}}$ over a point $\sigma \in \partial\Sigma$ by a complex $n \times n$ matrix $u(\sigma) \in GL(n, \mathbb{C})$ (the fiber being the real span of the columns). Assuming that the trivialization has been adapted to the metric on F (and therefore also the symplectic form) in the sense that the corresponding Hermitian inner product is the standard one on $\mathbb{C}^n$, it is quite easy to characterize the condition that $F_{\mathbb{R}} \subset F|_{\partial\Sigma}$ is a Lagrangian subbundle. Namely, the Lagrangian condition is that

$$u(\sigma) \in U(n) \subset GL(n, \mathbb{C})$$

for each point $\sigma \in \partial\Sigma$. Matrices related by $O(n)$ elements represent the same real subspace of $\mathbb{C}^n$, so, more properly, $[u(\sigma)] \in U(n)/O(n) \equiv \Lambda(n)$ (the *Lagrangian Grassmanian* manifold of $\mathbb{C}^n$). The square of the determinant is a well-defined map from $\Lambda(n) \to U(1)$, measuring how the real subspace is situated within $\mathbb{C}^n$.

We therefore have a map

$$\partial\Sigma \to \Lambda(n) \to U(1)$$

and since each component of $\partial\Sigma$ is an oriented circle, one can associate a winding number of the above map for each component. The sum of these winding numbers for each of the boundary circles is called the *Maslov index* of the pair $(F, F_{\mathbb{R}})$ and denoted

$$\mu(F, F_{\mathbb{R}}) \ .$$

Although we made use of a particular trivialization of F in defining this index, it can be shown [21] that the integer so obtained is trivialization independent.

Index calculation

The index of the Cauchy-Riemann operator $\bar{\partial}_F$ acting on the pair $(F, F_{\mathbb{R}})$ is given in terms of the first Chern number of the doubled holomorphic bundle on the doubled Riemann surface and the doubled genus by

$$\text{ind } \bar{\partial}_F = c_1(F_{\mathbb{C}}) + n(1 - g_{\Sigma_{\mathbb{C}}}) = c_1(F_{\mathbb{C}}) + n(2 - 2g_{\Sigma} - h_{\Sigma}) \ .$$

The integer $c_1(F_{\mathbb{C}})$ is precisely the Maslov index of the pair $\mu(F, F_{\mathbb{R}})$. The index

$$\mu(\mathbf{\Phi}^* V^{(1,0)} E, \quad \mathbf{\Phi}^* V^{(1,0)} E_Y)$$

can be given a further interpretation in case X is equivariantly Calabi-Yau; in this case, there is a $U(1)$ invariant holomorphic n-form on X and its restriction to Y is therefore a nonzero complex multiple $c \in \mathbb{C}^\times$ of the volume form vol_Y at each point $y \in Y$. By $U(1)$ invariance, the function c on Y induces a well defined function $E_Y \to \mathbb{C}^\times$, which we also denote c. The Maslov index is then the winding number of the map $c^2 \circ \mathbf{\Phi} : \partial\Sigma \to \mathbb{C}^\times$.

Chapter 8

Vafa-Witten theory compactified on a genus $g \geq 2$ Riemann surface

Previously, we considered the compactification of Vafa-Witten theory to three dimensions on a circle. Now we discuss the compactification of Vafa-Witten theory to two dimensions on a Riemann surface C. The details of the reduction depend strongly on the genus g_C of C due to the fact that the virtual dimension of the moduli space of solutions to the Hitchin equations is positive for $g_C \geq 2$, zero for $g_C = 1$, and negative for $g_C = 0$, roughly corresponding to the fact that a generic solution to the low action equations of the theory breaks all gauge symmetry when $g_C \geq 2$ but not for $g_C = 0, 1$. In order to avoid the complications introduced by unbroken gauge symmetry we treat the $g_C \geq 2$ case in the present chapter and shall restrict our analysis to irreducible configurations. (This amounts to an excision of singular points in the target space of the 2d effective theory.)

Our analysis will be quite similar to that adopted in [1] for the reduction of GL-twisted theory to two dimensions. The effective 2d TFT at $t = i$ was found there to be a B-model and the effective 2d TFT at $t = 1$ was found to be an A-model. However, we will find the reduction of Vafa-Witten theory to be a novel 2d TFT: the modified A-model of Chapter 7. (This theory also briefly made an appearance in the analysis of surface operators in Section 6.4.) The statement that Vafa-Witten theory reduces on a Riemann surface to a modified A-model with target $\mathcal{M}_H(G, C)$ was first obtained in Appendix B of [20].

8.1 Reducing the Vafa-Witten action

Consider Vafa-Witten theory on $\Sigma \times C$, where Σ and C are both closed Riemann surfaces of genus g_Σ and g_C, respectively, and we assume that $g_C \geq 2$. We wish to find an effective theory in the limit as the size of C shrinks to zero. More precisely, we wish to find a 2d TFT on Σ along with a dictionary between 2d field configurations on Σ and low-action 4d field configurations on $\Sigma \times C$ such that the

2d and 4d actions approximately agree. [1] By the assumption of irreducibility of the connection along C, all gauge symmetry will be broken and the 2d TFT will simply be a (nongauged) sigma model of maps $\Sigma \to \mathcal{M}_H(G, C)$.

Rescale the metric h_C on C by a small parameter ϵ

$$h_C \to h'_C = \epsilon\, h_C$$

and consider the limit $\epsilon \to 0$ in which the volume of C is tiny compared with the volume of Σ. In terms of real coordinates σ^1, σ^2 on Σ and σ^3, σ^4 on C (adapted to the complex structures on Σ and C as in Section 7.1), the rescaled metric takes the form

$$h_\Sigma \oplus h_C = diag\Big(e^{2f_\Sigma},\ e^{2f_\Sigma},\ \epsilon\ e^{2f_C},\ \epsilon\ e^{2f_C}\Big)$$

where the real functions $f_\Sigma(\sigma^1, \sigma^2)$ and $f_C(\sigma^3, \sigma^4)$ are conformal factors.

Recall from Chapter 3 that the Vafa-Witten theory contains bosonic 0-forms ϕ and σ, as well as a self-dual 2-form B and connection A. The field B_{12} is a 0-form with respect to rotations on Σ and C. The fields B_{13} and B_{14} transform as the components of a 1-form on C and also — crucially — as the components of a 1-form on Σ. The components B_{34}, B_{24}, and B_{23} are fixed in terms of B_{12}, B_{13} and B_{14} by self-duality, and, for ease of exposition, we neglect the fermions.

Different terms in the Vafa-Witten action (3.9) will be multiplied by different powers of ϵ after rescaling. In the $\epsilon \to 0$ limit, low action field configurations will be those that set to zero the terms multiplying the most negative power of ϵ. One thus obtains the following *low-ϵ equations*:

$$\begin{aligned}\big(e^{2f_\Sigma}\big)F_{34} - [B_{13}, B_{14}] &= 0 \\ D_3(e^{-2f_C}B_{13}) + D_4(e^{-2f_C}B_{14}) &= 0 \\ D_3(e^{-2f_C}B_{14}) - D_4(e^{-2f_C}B_{13}) &= 0\ .\end{aligned} \tag{8.1}$$

Additionally, the scalars B_{12}, ϕ, and σ are required to be covariantly constant along C; e.g.,

$$D_3 B_{12} = D_4 B_{12} = 0\ .$$

Covariant constancy of the 0-forms has the interpretation that they generate infinitesimal gauge transformations leaving the field A_C fixed. However, by the assumption that the configuration of A_C breaks all gauge symmetry, this in turn requires covariantly constant 0-forms to vanish. Hence, the fields σ, ϕ, and B_{12} do not survive as dynamical fields of the effective theory.

[1] Since we are working in Euclidean signature, the effective theory describes configurations of low action rather than low energy.

As for the fields A_3, A_4, B_{13} and B_{14}, the above equations can be put into the following form

$$\begin{aligned} F_{34} - [\varphi_3, \varphi_4] &= 0 \\ D_3\varphi^3 + D_4\varphi^4 &= 0 \\ D_3\varphi_4 - D_4\varphi_3 &= 0 \end{aligned} \tag{8.2}$$

where φ is the rescaled 1-form on C given by

$$\varphi = e^{-f_\Sigma}(B_{13}\, d\sigma^3 + B_{14}\, d\sigma^4)\ .$$

These are precisely the equations $F_C = \frac{1}{2}[\varphi, \varphi] = D_C^*\varphi = D_C\varphi = 0$ for a connection A_C on a principal G-bundle P_C over Riemann surface C and section $\varphi \in \Omega^1(C, ad\, P_C)$, which have been studied by N. Hitchin [64]. The space of solutions to these equations modulo gauge transformations is a finite dimensional space $\mathcal{M}_H(G, C)$, the *Hitchin moduli space*. Since the (σ^1, σ^2)-dependence of A_C and φ has been left arbitrary after imposing the above equations, these fields define a map $\Phi : \Sigma \to \mathcal{M}_H(G, C)$. Actually, this is not quite right since φ is not a scalar on Σ: under the action of the $U(1)$ structure group of Σ, the field φ_w is rotated into $\varphi_{\bar{w}}$. Hence, the pair (A_C, φ) defines a section

$$\boldsymbol{\Phi} \in \Omega^0\Big(\Sigma,\ P_\Sigma \times_{U(1)} \mathcal{M}_H(G, C)\Big)$$

of the fiber bundle associated to the $U(1)$ frame bundle P_Σ by the $U(1)$ action

$$\mathcal{U}_1 : \mathcal{M}_H(G, C) \to \mathcal{M}_H(G, C)$$

given by

$$\mathcal{U}_1 : (A_w, A_{\bar{w}}, \varphi_w, \varphi_{\bar{w}}) \to (A_w, A_{\bar{w}}, e^{i\alpha}\varphi_w, e^{-i\alpha}\varphi_{\bar{w}})$$

for $e^{i\alpha} \in U(1)$.

The effective action governing the fields A_Σ, A_C, and φ will be given by the subleading term in the ϵ^{-1} expansion of the Vafa-Witten action; i.e., we impose the low-ϵ equations and discard terms that become negligible as $\epsilon \to 0$. One finds the following truncated action

$$\begin{aligned} S_{\text{trunc.}} = -\frac{1}{e^2}\int_M d^4\sigma\, \text{Tr}\Big\{ & (F_{13} - F_{24})^2 + (F_{14} + F_{23})^2 \\ & + \Big(\partial_1\varphi_3 + (\partial_1 f_\Sigma)\varphi_3 + \partial_2\varphi_4 + (\partial_2 f_\Sigma)\varphi_4\Big)^2 \\ & + \Big(\partial_1\varphi_4 + (\partial_1 f_\Sigma)\varphi_4 - \partial_2\varphi_3 - (\partial_2 f_\Sigma)\varphi_3\Big)^2 \\ & + 2F_{13}F_{24} - 2F_{14}F_{23}\Big\}\Big|_{(A_3, A_4, \varphi_3, \varphi_4)\ \text{satisfy Hitchin equations on}\ C} \end{aligned} \tag{8.3}$$

where the last line is the contribution of the instanton term in the Vafa-Witten action with θ set to zero and we have suppressed fermionic terms. The truncated action does not contain any kinetic terms (with derivatives along Σ) for the fields A_1 and A_2; i.e., they can be treated as auxiliary fields in the effective action on Σ and integrated out via their equations of motion. This sets them equal to combinations of fermionic modes, but, since we are suppressing fermionic contributions, we can simply discard A_1 and A_2 from the effective action.

8.2 Modified A-model with target $\mathcal{M}_H(G,C)$

Let us now interpret the field content and effective action that we have obtained as that of a modified A-model on Σ. It is convenient to work with the space of pairs (A_C, φ) prior to imposing the Hitchin equations and quotienting by gauge transformations; this is the infinite-dimensional, flat variety

$$\mathcal{W} = \mathcal{A}(G,C) \times \Omega^1(C, ad\, P_C)$$

where $\mathcal{A}(G,C)$ is the space of connections on P_C. Let $z = \sigma^1 + i\sigma^2$ and $w = \sigma^3 + i\sigma^4$ be the complex coordinates on Σ and C, respectively. Adopting the notation of Section 4.1 of [1], we take the flat metric on $\mathcal{W}$ to be given by

$$ds^2 = -\frac{1}{4\pi}\int_C (idw \wedge d\bar{w})\,\mathrm{Tr}\Big\{\delta A_w \otimes \delta A_{\bar{w}} + \delta A_{\bar{w}} \otimes \delta A_w + \delta\varphi_w \otimes \delta\varphi_{\bar{w}} + \delta\varphi_{\bar{w}} \otimes \delta\varphi_w\Big\}$$

where δ indicates the exterior derivative on the space of fields. This metric is hyperkähler with respect to three independent complex structures I, J, and K on $\mathcal{W}$ satisfying the quaternion relations. We denote the corresponding three Kähler forms by ω_I, ω_J, and ω_K. (See Section 4.1 of [1] for the explicit forms of the complex structures and Kähler forms; we write ω_I below.)

The Hitchin moduli space is also hyperkähler. This fact follows from its description as a hyperkähler quotient of $\mathcal{W}$ with respect to the group of gauge transformations of the bundle P_C, as follows. Consider an infinitesimal gauge transformation generated by the element $\varepsilon \in \Omega^0(C, ad\, P_C)$, where we identify the Lie algebra of the infinite-dimensional group of gauge transformations as the vector space of sections of $ad\, P_C$. The three moment maps corresponding to symplectic forms ω_I, ω_J, and ω_K are given by

$$\begin{aligned}
\mu_I(\varepsilon) &= \frac{1}{2\pi}\int_C (idw \wedge d\bar{w})\,\mathrm{Tr}\,\varepsilon\Big\{F - \frac{1}{2}[\varphi,\varphi]\Big\} \\
\mu_J(\varepsilon) &= \frac{1}{2\pi}\int_C (idw \wedge d\bar{w})\,\mathrm{Tr}\,\varepsilon\Big\{D_w\varphi_{\bar{w}} + D_{\bar{w}}\varphi_w\Big\} \\
\mu_K(\varepsilon) &= \frac{i}{2\pi}\int_C (idw \wedge d\bar{w})\,\mathrm{Tr}\,\varepsilon\Big\{D_w\varphi_{\bar{w}} - D_{\bar{w}}\varphi_w\Big\}
\end{aligned}$$

where we have evaluated the moment maps on the Lie algebra element ε. For instance, one has

$$\delta\mu_I(\varepsilon) = -\iota_{V(\varepsilon)}\,\omega_I$$

where $V(\varepsilon)$ is the vector field on $\mathcal{W}$ corresponding to the action of the gauge transformation generated by ε. The Hitchin moduli space $\mathcal{M}_H(G,C)$ is precisely the quotient of the level set $\mu_I(\varepsilon) = \mu_J(\varepsilon) = \mu_K(\varepsilon) = 0$ (for all ε) by the group of gauge transformations, and is therefore itself a hyperkähler manifold. At irreducible points (which, by assumption, are the only regions we consider here), the dimension of $\mathcal{M}(G,C)$ agrees with its virtual dimension, defined as the index of a certain elliptic differential complex constructed by linearizing the Hitchin equations around the point. In [64], this was shown to be related to the dimension of the gauge group and the genus of C by

$$\dim \mathcal{M}_H(G,C) = 4(g_C - 1)\dim G\ .$$

The complex structure relevant for compactification of Vafa-Witten theory is the one we have called I, with corresponding Kähler form given by

$$\omega_I = -\frac{i}{2\pi}\int_C (idw\wedge d\bar{w})\,\mathrm{Tr}\left\{\delta A_{\bar{w}}\wedge\delta A_w - \delta\varphi_{\bar{w}}\wedge\delta\varphi_w\right\}\ .$$

We emphasize again that δA in the above is regarded as a 1-form on the space of fields $\mathcal{W}$ and $\wedge$ indicates the wedge product on both C and $\mathcal{W}$. Note that this Kähler form is preserved by the $U(1)$ action $\mathcal{U}_1 : \mathcal{W}\to\mathcal{W}$ defined above and, indeed, that this action is Hamiltonian with respect to the globally defined moment map

$$H = -\frac{1}{2\pi}\int_C (idw\wedge d\bar{w})\,\mathrm{Tr}\left\{\varphi_w\varphi_{\bar{w}}\right\}$$

regarded as a real function on $\mathcal{W}$. It is easy to check that $\mathcal{U}_1$ maps the zero set of the μ's into itself and therefore descends to a Hamiltonian $U(1)$ action on $\mathcal{M}_H(G,C)$.

Let us write down the action of the modified A-model with target space $\mathcal{M}_H(G,C)$, Kähler form ω_I, and Hamiltonian $U(1)$ action $\mathcal{U}_1$. In terms of complex coordinates X^i and $X^{\bar{j}}$ on the target, we write the action (7.7) as

$$S = t\int_\Sigma (idz\wedge d\bar{z})\left\{2g_{i\bar{j}}\mathcal{D}_{\bar{z}}X^i\mathcal{D}_z X^{\bar{j}}\ +\ \text{fermions}\right\} + t\int_\Sigma \mathbf{\Phi}^*(\omega_I)_{eq}$$

where t is the coupling of the 2d theory, and $\mathbf{\Phi}^*(\omega_I)_{eq}$ is the pullback (under the section $\mathbf{\Phi}$ of the associated fiber bundle) of the equivariant symplectic form

$$(\omega_I)_{eq} = \omega_I - \zeta H \qquad \in \quad \Omega^2_{U(1)}(\mathcal{M}_H)$$

where ζ is the degree two, generating element of $\mathfrak{u}(1)$, which gets replaced by the curvature of the spin connection under pullback. (See Appendix C for a detailed discussion of the notion of pullback of an equivariant form under a section of an associated fiber bundle.) The covariant derivatives are given by

$$\begin{aligned}\mathcal{D}_{\bar{z}}X^i &= \partial_{\bar{z}}X^i + \omega_{\bar{z}}V^i = \partial_{\bar{z}}X^i - i(\partial_{\bar{z}}f_\Sigma)V^i \\ \mathcal{D}_z X^{\bar{j}} &= \partial_z X^{\bar{j}} + \omega_z V^{\bar{j}} = \partial_z X^{\bar{j}} + i(\partial_z f_\Sigma)V^{\bar{j}}\end{aligned}$$

where we have substituted (7.6) for the components $\omega_{\bar{z}}$ and ω_z of the spin connection on Σ and V is the vector field generating the $U(1)$ action.

In complex structure I, the linear, holomorphic and antiholomorphic functions on $\mathcal{W}$ are given by

$$\begin{aligned}X^i &\longrightarrow (A_{\bar{w}}, \varphi_w)_c \\ X^{\bar{j}} &\longrightarrow (A_w, \varphi_{\bar{w}})_c\end{aligned}$$

where $c \in C$ is a point of evaluation. The covariant derivatives are

$$\begin{aligned}\mathcal{D}_{\bar{z}}X^i &\longrightarrow \Big(\partial_{\bar{z}}A_{\bar{w}}, \quad \partial_{\bar{z}}\varphi_w + (\partial_{\bar{z}}f_\Sigma)\varphi_w\Big)_c \\ \mathcal{D}_z X^{\bar{j}} &\longrightarrow \Big(\partial_z A_w, \quad \partial_z\varphi_{\bar{w}} + (\partial_z f_\Sigma)\varphi_{\bar{w}}\Big)_c .\end{aligned}$$

Substituting these expressions (as well as the metric written above) into our expression for S, one finds

$$\begin{aligned}S = -\frac{t}{2\pi}\int_\Sigma (idz\wedge d\bar{z})\int_C (idw\wedge d\bar{w})\,\mathrm{Tr}\Big\{2\,\partial_z A_w \partial_{\bar{z}}A_{\bar{w}} + 2\Big(\partial_z\varphi_{\bar{w}} + (\partial_z f_\Sigma)\varphi_{\bar{w}}\Big)\Big(\partial_{\bar{z}}\varphi_w + (\partial_{\bar{z}}f_\Sigma)\varphi_w\Big) \\ + \ \text{fermions}\Big\} + t\int_\Sigma \mathbf{\Phi}^*(\omega_I)_{eq} \ .\end{aligned}$$

In order to make contact with the effective action obtained in the previous section, we massage the equivariant topological term slightly by using

$$(\omega_I)_{eq} = \omega_I' + \delta_C\lambda_I$$

where the 2-form ω_I' and 1-form λ_I are given by

$$\begin{aligned}\omega_I' &= -\frac{i}{2\pi}\int_C (idw\wedge d\bar{w})\,\mathrm{Tr}\Big\{\delta A_{\bar{w}}\wedge\delta A_w\Big\}, \\ \lambda_I &= -\frac{i}{4\pi}\int_C (idw\wedge d\bar{w})\,\mathrm{Tr}\Big\{\varphi_w\,\delta\varphi_{\bar{w}} - \varphi_{\bar{w}}\,\delta\varphi_w\Big\}\end{aligned}$$

and $\delta_C = \delta - \zeta\, \iota_V$ is the Cartan equivariant differential on $\mathcal{W}$. That is to say, the equivariant symplectic form $(\omega_I)_{eq}$ is equivariantly cohomologous to the 2-form ω'_I; by a theorem in Section C.2, the numbers produced by pulling back under section $\mathbf{\Phi}$ and integrating over Σ will be identical:

$$\int_\Sigma \mathbf{\Phi}^*(\omega_I)_{eq} = \int_\Sigma \mathbf{\Phi}^*(\omega'_I)\ .$$

Having made this replacement, we arrive finally at the following action for modified A-model with target $\mathcal{M}_H(G,C)$:

$$S = -\frac{t}{2\pi}\int_\Sigma (idz\wedge d\bar{z})\int_C (idw\wedge d\bar{w})\,\mathrm{Tr}\Big\{\partial_z A_w \partial_{\bar{z}} A_{\bar{w}} + \partial_{\bar{z}} A_w \partial_z A_{\bar{w}} \\ + 2\Big(\partial_z\varphi_{\bar{w}} + (\partial_z f_\Sigma)\varphi_{\bar{w}}\Big)\Big(\partial_{\bar{z}}\varphi_w + (\partial_{\bar{z}} f_\Sigma)\varphi_w\Big)\ +\ \text{fermions}\Big\}\ .$$

If we identify the 2d coupling t as

$$t = \frac{4\pi}{e^2}$$

then the above action is precisely what we have written in (8.3) for the truncated Vafa-Witten action. The effective 2d TFT localizes on sections $\mathbf{\Phi}$ (i.e., configurations of (A_C, φ)) satisfying

$$\partial_{\bar{z}} A_{\bar{w}} \quad = \quad \partial_z A_w \quad = \quad \partial_{\bar{z}}\varphi_w + (\partial_{\bar{z}} f_\Sigma)\varphi_w \quad = \quad \partial_z \varphi_{\bar{w}} + (\partial_z f_\Sigma)\varphi_{\bar{w}} \quad = \quad 0\ .$$

Branes

In case Σ has nontrivial boundary, one must impose boundary conditions on the fields along each component of the boundary. The simplest boundary conditions correspond to Lagrangian submanifolds Y of the target space which, in addition, are preserved by the hamiltonian $U(1)$-action. [2] (See Section 7.5 for details.) As discussed in [35], the Hitchin moduli space $\mathcal{M}_H(G,C)$ admits several simple classes of Lagrangian submanifolds with respect to the symplectic form ω_I. However, none of these examples are invariant with respect to the action of the $\mathcal{U}_1$ symmetry.

[2] One also requires vanishing of the equivariant Maslov index of the embedded submanifold, see 7.5.

Chapter 9

Defects of two-dimensional sigma models

It is well-known that two-dimensional sigma models with differing amounts of left- and right-moving supersymmetry (e.g., the (0,1) and (0,2) supersymmetric sigma models) cannot be consistently defined on worldsheets with nontrivial boundary. This is essentially because a boundary condition must set to zero a linear combination of left- and right-moving fermions.

This does not however rule out the possibility of defining these models on a worldsheet equipped with *defects*, i.e., one-dimensional submanifolds D along which the fields of two distinct CFTs are glued together consistently. (A boundary condition is a special case in which one of the CFTs is trivial.) Our goal is to study supersymmetric defects in the (0,1) and (0,2) sigma models.

Defects of two-dimensional CFTs have attracted much interest recently, especially in the context of rational conformal field theory and as a means of elegantly implementing dualities [55], [53], [51], [52]. In this chapter, we will analyze criteria for preserving superconformal symmetry, disregarding any enhanced chiral symmetry which the theories may possess; moreover, we will focus on the target space geometry of defects, and shall only briefly comment on their role in implementing dualities.

Consider a two-dimensional worldsheet disconnected into two domains Σ and $\widehat{\Sigma}$ by a defect line D. On the domain Σ, one defines a sigma model of maps $\Phi : \Sigma \to X$ and on the domain $\widehat{\Sigma}$ one defines a sigma model of maps $\widehat{\Phi} : \widehat{\Sigma} \to \widehat{X}$, where X and $\widehat{X}$ are two compact, Riemannian target spaces. The restriction of these maps to D defines a product map $\Phi \times \widehat{\Phi}|_D : D \to X \times \widehat{X}$. As observed in [50], gluing conditions will require the product map to takes values in some submanifold $Y \subseteq X \times \widehat{X}$ of the product of the targets.

One may also include a term in the action coupling the bulk fields to a line bundle with connection living on the worldvolume of Y (equipping it with a closed 2-form F), exactly by analogy with Chan-Paton bundles on branes for the case of sigma model boundary conditions. Borrowing the terminology of [50], we refer to the pair (Y, F) as a *bibrane*.

To ensure that the defect theory preserves a specified set of symmetries of the bulk theories, we impose certain requirements on the geometry of the embedding Y, as well as on the choice of the 2-form F. Let us describe the relevant set of symmetries we wish to preserve by listing the associated Noëther charges. The (0,1) supersymmetry algebra in 1+1 dimensions reads

$$\{Q_+, Q_+\} = H + P$$

where Q_+ is the single, real, right-moving supercharge, H is the worldsheet energy, and P is the worldsheet momentum. We wish to find (Y, F) such that H, P, and Q_+ remain as conserved charges of the defect theory; since H and P will be separately conserved, we will be studying examples of what are known in the literature as *topological defects* [52].

Already in the case of the bosonic sigma model (i.e., with no fermionic fields present) it is an interesting question which bibranes (Y, F) define topological defects, and it is to this that we will first direct our attention. It turns out to be natural to choose the neutral signature metric $G = g \oplus -\widehat{g}$ on the (pseudo-Riemannian) product manifold $M = X \times \widehat{X}$, where g and $\widehat{g}$ are the positive-definite metrics on X and $\widehat{X}$.

Those (Y, F) that supply topological defects turn out to bear a structural resemblance to A-branes of symplectic manifolds (exchanging the symplectic form on the ambient target space for the neutral signature metric defined above); for instance, just as A-branes are coisotropic submanifolds with respect to the symplectic form on the target, Y will be required to be 'coisotropic' with respect to G, in a sense that we will explain in Section 9.3. Moreover, we will describe two special classes of topological defects (the 'graphs of isometries' and 'half para-Kähler' defects), which are the analogs of Lagrangian and space-filling A-branes.

Alternatively, it turns out to be natural to employ the language of Hitchin's generalized geometry [58], in terms of which we obtain the following simple characterization: (Y, F) define a topological defect of the bosonic sigma model if and only if the 'F-rotated generalized tangent bundle' of Y is stabilized by the 'generalized metric' on $X \times \widehat{X}$. We will explain what these terms mean in greater depth in Section 9.3.

We refer to (Y, F) satisfying the above stabilization condition as *topological bibranes*; they can be defined for any neutral signature manifold M and are, perhaps, mathematically interesting in their own right. However, we should point out that the neutral signature manifolds relevant to the current physics discussion are of a very restricted type: namely, manifolds that can be expressed as a global product $M = X \times \widehat{X}$ such that the metric restricted to TX (resp. $T\widehat{X}$) is positive (resp. negative) definite.

In Section 9.1 we discuss defect gluing conditions in general and say what it means for a gluing condition to preserve a symmetry of the bulk theories. In Section 9.2 we write down gluing conditions on the fields corresponding to the choice of (Y, F) and analyze the topological defect requirement. In Section 9.3, we explain the analogy with A-branes and reformulate the topological bibrane condition on (Y, F) terms of generalized geometry. In Section 9.4 we analyze defects of the (0,1) supersymmetric sigma model, supplementing the bosonic gluing conditions with an additional fermionic gluing condition and studying the geometry of a certain middle dimensional subbundle of TY. In Section 9.5 we treat topological defects of the (0,2) sigma model, which we will describe as those (Y, F) that are simultaneously A-branes and B-branes with respect to a certain symplectic form and complex structure. Finally, in Section 9.6 we briefly comment on a subset of our topological defects which implement dualities relating the sigma model theories on Σ and $\widehat{\Sigma}$.

Proofs of selected propositions discussed in the text of this chapter will be offered in Appendix D.

9.1 Defect gluing conditions

Before discussing defects in specific theories, let us discuss defect gluing conditions in general and say what it means for a defect to preserve a symmetry of the bulk theories.

The variation of the action will, in general, consist of three types of terms:

$$\delta S = (\delta S)_\Sigma + (\delta S)_{\widehat{\Sigma}} + (\delta S)_D$$

where $(\delta S)_\Sigma$ and $(\delta S)_{\widehat{\Sigma}}$ are integrals over the domains Σ and $\widehat{\Sigma}$, respectively, which vanish for field configurations solving the bulk equations of motion, and the third term, $(\delta S)_D$, is an integral over D, which in general, will not vanish unless we impose a gluing condition on the fields along D. A gluing condition constrains the values of the fields along D and hence also the set of *allowed variations*, by which we mean variations mapping one solution of the gluing condition to another solution. Therefore, the first requirement of a good gluing condition is that it sets $(\delta S)_D$ to zero identically for all allowed variations and that it does so 'minimally' (i.e., without overconstraining the data along D).

Additionally, we may wish for the defect theory to preserve a certain symmetry of the bulk. Let (σ^0, σ^1) be worldsheet coordinates in which D is described locally as the set of points with $\sigma^1 = 0$. In these coordinates, we say that a gluing condition classically preserves a symmetry of the bulk if and only if the 1- component of the associated Noëther current glues continuously across D, thereby ensuring the existence of a conserved Noëther charge in the composite theory. (The quantum theory may develop an anomaly, but we will confine our discussion to the classical problem.) This being satisfied, the symmetry variation will automatically be among the allowed variations.

For instance, conservation of the worldsheet energy H requires that the off-diagonal components of the stress-energy tensors glue continuously:

$$T^1{}_0 - \widehat{T}^1{}_0 = 0$$

at points of D. Defects satisfying this condition are said to be *conformal defects* [49].

If, in addition, the defect gluing condition ensures that the diagonal components of the stress-energy tensor glue continuously:

$$T^1{}_1 - \widehat{T}^1{}_1 = 0$$

then worldsheet momentum P is also a conserved charge. Defects satisfying this condition are called *topological defects* [52], due to the fact that the location of D on the worldsheet can be deformed smoothly without affecting the values of correlators (so long as it does not cross through the location of a local operator insertion).

In subsequent sections, we will write down an action, vary the action, and then systematically analyze what gluing conditions set $(\delta S)_D$ minimally and ensure continuous gluing across D of the 1-components of relevant Noëther currents.

9.2 Topological defects of the bosonic sigma model

To begin, we analyze topological defects of the bosonic sigma model.

Let us fix notation. As above, let (σ^0, σ^1) be worldsheet coordinates in which the defect line D is given locally by $\sigma^1 = 0$, and the worldsheet metric is taken to be flat, with signature (-, +). Let Σ be the domain given by $\sigma^1 \geq 0$, and $\widehat{\Sigma}$ be the domain given by $\sigma^1 \leq 0$. The fields of the bosonic sigma model with defect consist of maps $\Phi : \Sigma \to X$ and $\widehat{\Phi} : \widehat{\Sigma} \to \widehat{X}$; in terms of local coordinates ϕ^i on X and $\hat{\phi}^i$ on $\widehat{X}$, we can describe these maps by functions $\phi^i(\sigma)$ for $\sigma^1 \geq 0$ and $\hat{\phi}^i(\sigma)$ for $\sigma^1 \leq 0$. (Note that the target spaces are assumed to have the same dimensionality n since we are looking for topological – not merely conformal – defects.) Likewise, the product map $\Phi \times \widehat{\Phi}|_D : D \to X \times \widehat{X}$ can be described by the functions $\phi^I = (\phi^i, \hat{\phi}^i)|_{(\sigma^0,0)}$. Indices i, j range from $1, \ldots n$ and I, J range from $1, \ldots 2n$.

The total action for the theory with defect is the sum of bulk terms and a term coupling the bulk fields to a connection on a rank one vector bundle living on the submanifold $Y \subseteq X \times \widehat{X}$:

$$\begin{aligned} S = &\int_\Sigma d^2\sigma \Big(-\frac{1}{2} g_{ij}\partial_\mu\phi^i\partial^\mu\phi^j - \frac{1}{2} b_{ij}\epsilon^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j \Big) \\ &+ \int_{\widehat{\Sigma}} d^2\sigma \Big(-\frac{1}{2}\widehat{g}_{ij}\partial_\mu\hat{\phi}^i\partial^\mu\hat{\phi}^j - \frac{1}{2}\widehat{b}_{ij}\epsilon^{\mu\nu}\partial_\mu\hat{\phi}^i\partial_\nu\hat{\phi}^j \Big) + \int_D d\sigma^0 \mathcal{A}_I \partial_0 \phi^I \end{aligned}$$

where $\mathcal{A} = \mathcal{A}_I(\phi) d\phi^I$ is the connection 1-form, $g_{ij}(\phi)$ and $b_{ij}(\phi)$ are the metric and B-field on X, and $\widehat{g}_{ij}(\phi)$ and $\widehat{b}_{ij}(\phi)$ are the metric and B-field on $\widehat{X}$. For simplicity we assume that both b and $\widehat{b}$

are closed 2-forms.

Varying this action and picking out the term localized on D, one finds

$$(\delta S)_D = \int_D d\sigma^0 \Big(-(g_{ij}\partial_1\phi^i + b_{ij}\partial_0\phi^i)\,\delta\phi^j + (\widehat{g}_{ij}\partial_1\hat{\phi}^i + \widehat{b}_{ij}\partial_0\hat{\phi}^i)\,\delta\hat{\phi}^j \\ - (\partial_I\mathcal{A}_J - \partial_J\mathcal{A}_I)\,\partial_0\phi^I\,\delta\phi^J \Big) . \tag{9.1}$$

Let $\sigma = (\sigma^0, 0)$ be a point on D and let us write down gluing conditions on the fields at σ sufficient to ensure vanishing of this expression for all allowed variations.

As mentioned in the introduction, a choice of gluing condition corresponds to a choice of a submanifold $Y \subseteq X \times \widehat{X}$. Let $y = \Phi \times \widehat{\Phi}|_D(\sigma)$ be the image of σ under the product map. We impose first the Dirichlet condition

$$y \in Y \ .$$

If k is the dimension of Y, with $0 \leq k \leq 2n$, then the above represents $2n - k$ independent gluing constraints on the $2n$ bosonic fields ϕ^I. In order to ensure vanishing of (9.1), we supplement these Dirichlet conditions with a set of k Neumann conditions on the derivatives $\partial_1\phi^I$. It is easiest to describe these in terms of three target space tangent vectors $u, v, w \in T_y(X \times \widehat{X})$, with

$$\begin{aligned} u &= u^I\partial_I = (\partial_0\phi^I)\partial_I \\ v &= v^I\partial_I = (\partial_1\phi^I)\partial_I \\ w &= w^I\partial_I = (\delta\phi^I)\partial_I \end{aligned}$$

where $\partial_I \equiv \frac{\partial}{\partial\phi^I}$. Constraining points of D to be mapped to the submanifold Y requires $u \in TY$, the subspace of vectors tangent to Y; moreover, allowed variations $\delta\phi^I$ are those such that $w \in TY$ as well.

Here and throughout, we shall only have occasion to discuss vectors that are evaluated at points of Y; hence, we regard all vectors as lying in the pullback bundle e^*TM, where $e : Y \to M = X \times \widehat{X}$ is the embedding map, and shall regard TY as a subbundle $TY \subseteq e^*TM$.

Since the target space metrics g and $\widehat{g}$ enter $(\delta S)_D$ with opposite signs, we find it useful to define a *neutral signature* metric and B-field on $X \times \widehat{X}$ by

$$G_{IJ} = \begin{pmatrix} g_{ij} & 0 \\ 0 & -\widehat{g}_{ij} \end{pmatrix} , \quad B_{IJ} = \begin{pmatrix} b_{ij} & 0 \\ 0 & -\widehat{b}_{ij} \end{pmatrix} .$$

Moreover, we write $F = -\mathcal{F} - e^*B$ for closed 2-form on Y obtained by combining the curvature $\mathcal{F} = d\mathcal{A}$ of the line bundle and the pullback of the B-field. Using these definitions, we may state the

Neumann conditions as follows:

$$G(u,w) = F(v,w)$$

for all $w \in TY$. As promised this represents k independent conditions, one for each linearly independent tangent vector w.

Definition 1. *Together, the pair (u,v) satisfying the above gluing conditions are said to be an* allowed pair *of tangent vectors.*

Having written down good gluing conditions on the fields, let us now analyze what additional constraints the topological defect condition places on allowed pairs; this will constrain the choice of submanifold Y as well as the choice of 2-form F on its worldvolume. First of all, the conformal defect condition ${T^1}_0 - {\widehat{T}^1}_0 = 0$ is equivalent to

$$G(u,v) = 0$$

for all allowed pairs (u,v). This is an automatic consequence of the antisymmetry of F since, if (u,v) is an allowed pair, then $G(u,v) = G(v,u) = F(u,u) = 0$, where we have applied the definition of allowed pair, setting $w = u$. Therefore, (Y,F) automatically defines a conformal defect of the bosonic sigma model.

Further requiring the topological condition ${T^1}_1 - {\widehat{T}^1}_1 = 0$ is equivalent to requiring

$$G(u,u) = -G(v,v)$$

for all allowed pairs (u,v). This condition turns out to be equivalent to an apparently stronger condition, as follows.

Proposition 1. *If (Y,F) are such that $G(u,u) = -G(v,v)$ for all allowed pairs (u,v), then $v \in TY$ and (v,u) is an allowed pair as well.*

(See Appendix D for the proof of this proposition and other proofs which are not immediate.) The converse is true as well, since if (Y,F) satisfy the conditions of this proposition, then by setting $w = u$, one has $G(u,u) = F(v,u) = -F(u,v) = -G(v,v)$. Hence, we arrive at the following:

Proposition 2. *Suppose $Y \subseteq X \times \widehat{X}$ is a submanifold and F is a closed 2-form on its worldvolume. Then (Y,F) supplies a topological defect of the bosonic sigma model if and only if the following condition is met:*

> *If (u,v) is a pair of vectors such that $u \in TY, v \in e^*TM$ and $G(v,w) = F(u,w)$ for all $w \in TY$, then*
>
> *$v \in TY$ as well, and $G(u,w) = F(v,w)$ for all $w \in TY$.*

Definition 2. *A pair* (Y, F) *satisfying the conditions of the preceding theorem are said to define a* topological bibrane.

9.3 Geometry of topological bibranes

Having obtained a characterization of topological bibranes (Y, F) above, we reformulate this condition slightly to put it in a more understandable form.

We have chosen to equip the manifold $M = X \times \widehat{X}$ with a neutral signature metric G; let us therefore record some basic facts from the theory of submanifolds of indefinite signature spaces. See [60] for a more complete discussion.

Definition 3. *Let* Y *be an embedded submanifold of a* $2n$*-dimensional pseudo-Riemannian manifold* M *equipped with a nondegenerate, symmetric metric* G*. The orthogonal subbundle* $(TY)^\perp$ *is defined to the be the set of vectors that are* G*-orthogonal to all of* TY*:*

$$(TY)^\perp = \{u \in e^*TM : G(u, v) = 0 \qquad \textit{for all } v \in TY\} \ .$$

If TY is k-dimensional, then $(TY)^\perp$ is $(2n - k)$-dimensional. The main difference between the theory of pseudo-Riemannian submanifolds as compared with the theory of Riemannian submanifolds is that the restriction of the metric G to Y can develop degenerate directions. Hence, the subbundle $\Delta = TY \cap (TY)^\perp$ will in general be nontrivial. Borrowing terminology from symplectic geometry, we define the following three special classes of submanifolds:

Definition 4. *Depending on whether the bundle* TY*, regarded as a subbundle of* e^*TM*, contains (or is contained by) its orthogonal bundle* $(TY)^\perp$*, we say that*

TY *is* isotropic *if* $TY \subseteq (TY)^\perp$

TY *is* coisotropic *if* $TY \supseteq (TY)^\perp$

TY *is* Lagrangian *if* $TY = (TY)^\perp$.

Hence, the dimension k lies in the range $0 \le k \le n$ for isotropic subbundles, $n \le k \le 2n$ for coisotropic subbundles, and all Lagrangian subbundles are n dimensional. (Lagrangian subbundles exist only when the signature of G is (n, n).) In the proof of proposition 1 it was shown that the submanifolds Y corresponding to topological bibranes have the property that orthogonal vectors are also tangent to the submanifold; in our classification, they are coisotropic submanifolds. Therefore, in the following we focus on coisotropic submanifolds (which include Lagrangian submanifolds as a special case).

It is convenient to define a particular frame for e^*TM adapted to the submanifold Y. This is not quite as straightforward as in the Riemannian case, since a canonical splitting $e^*TM = TY \oplus (TY)^\perp$ is no longer available. However, for coisotropic submanifolds there exists [59] a splitting of the tangent bundle of the form

$$e^*TM = \underbrace{(TY)^\perp \oplus SY}_{TY} \oplus NY$$

where SY is a complementary *screen* distribution to $(TY)^\perp$ within TY and NY is a complementary *transverse* distribution to TY in e^*TM. We have $\dim (TY)^\perp = \dim NY = 2n - k$ and $\dim SY = 2k - 2n$.

Making use of this adapted frame, let us now write down an equivalent characterization of topological bibranes:

Proposition 3. *The pair (Y, F) define a topological bibrane if and only if $Y \subseteq X \times \widehat{X}$ is a coisotropic submanifold such that $\ker F = (TY)^\perp$ (i.e., the degenerate directions of F and $G|_{TY}$ coincide) and, additionally,*

$$(\widetilde{G}^{-1}\widetilde{F})^2 = +1$$

on SY, where $\widetilde{G} \equiv G|_{SY}$ and $\widetilde{F} \equiv F|_{SY}$.

This is remarkably similar to the characterization of A-brane boundary conditions of the $(2, 2)$ supersymmetric sigma model given in [56], where the role of the antisymmetric symplectic form Ω on the target is exchanged for a neutral signature, symmetric metric G. Just as A-branes are required to be coisotropic with respect to Ω, topological bibranes are required to be coisotropic with respect to G. Moreover, for A-branes the quotient bundle $TY/(TY)^\Omega$ is equipped with an endomorphism given by $\Omega^{-1}F$ and squaring to -1; similarly, for topological bibranes the quotient bundle $TY/(TY)^\perp$ is equipped with an endomorphism $G^{-1}F$ squaring to $+1$.

We consider two important special classes of topological bibranes. When $F = 0$, the condition that the degenerate directions of F and $G|_{TY}$ agree implies that $TY = (TY)^\perp$, i.e., Y is a Lagrangian submanifold of $X \times \widehat{X}$. This in turn implies that, locally, Y is the graph of an isometry $f : X \to \widehat{X}$. These *graph-of-isometry* type bibranes are the analogs of Lagrangian A-branes.

The other special class of topological bibranes are the *space-filling* bibranes with $Y = X \times \widehat{X}$. In this case, $(TY)^\perp$ is trivial and the screen distribution SY is all of TY. The condition in proposition 3 then implies that F is a symplectic form such that $(G^{-1}F)^2 = 1$. Space-filling bibranes are the topological bibrane analog of space-filling A-branes.

Indeed, if a manifold M carries a symplectic form F, an almost product structure R, and a neutral signature metric G with the compatibility requirement

$$F(u, v) = G(Ru, v)$$

for all vector fields u, v, and if in addition one demands that

$$dF = 0$$

then M is said to be an *almost para-Kähler manifold* [62]. Alternatively, the triplet of structures (F, R, G) satisfying the above is known as an *almost bi-Lagrangian structure* on the manifold [61], which terminology is inspired by the fact that the +1 and -1 eigenbundles of R form two complementary Lagrangian subbundles with respect to the symplectic form F. In case one of these subbundles is integrable, we call the manifold *half para-Kähler* and in case both subbundles are integrable, we call the manifold *para-Kähler*. Integrability of one of the eigenbundles need not imply integrability of the other eigenbundle. Indeed, in Section 9.4 we will require integrability of just the positive eigenbundle in order to ensure supersymmetry. The study of para-Kähler manifolds is a rich and developing subbranch of indefinite signature geometry; they have appeared in an unrelated physical context in [48].

Another useful way to reformulate the topological bibrane condition is in terms of the language of *generalized geometry* (see [58] for a review). In the framework of generalized geometry one only works with the sum $TM \oplus T^*M$ as well as sections thereof (rather than TM or T^*M in isolation). In particular, one speaks of *generalized tangent vectors* of M, defined as pairs (u, ξ) with $u \in TM$, $\xi \in T^*M$.

The objects of ordinary of geometry have generalized counterparts; for instance, one defines the *generalized tangent subbundle* of a submanifold $Y \subseteq M$ to be sum of its tangent bundle and conormal bundle (regarded as a subbundle of T^*M):

$$\tau Y = TY \oplus N^*Y \ .$$

Generalized geometry provides a prescription for incorporating a nonzero 2-form F:

Definition 5. *The* F-rotated generalized tangent subbundle *of a submanifold $Y \subseteq M$ is the set of generalized tangent vectors (u, ξ) such that*

$$\tau^F Y = \{(u, \xi) : u \in TY, \xi \in T^*M \quad \textit{such that } Fu = e^*\xi\} \ .$$

Here $e : Y \to M$ is the embedding map and Fu represents the 1-form produced by contracting the 2-form F on the vector u.

What we have been calling allowed pairs of tangent vectors (u, v) are nothing but sections of $\tau^F Y$ (after lowering the vector v to a 1-form $\xi = Gv$).

The generalized counterparts of complex structures are the *generalized complex structures*: endomorphisms $\mathcal{J} : TM \oplus T^*M \to TM \oplus T^*M$ with $\mathcal{J}^2 = -1$. Symplectic manifolds with symplectic form Ω carry a generalized complex structure given by

$$\mathcal{J}_\Omega = \begin{pmatrix} & -\Omega^{-1} \\ \Omega & \end{pmatrix}$$

(referring coordinates on T^*M to a dual basis) and the A-branes are elegantly characterized as those (Y, F) such that the F-rotated generalized tangent subbundle is stabilized by the $\mathcal{J}_\Omega$:

Proposition 4. *(in reference [58]) (Y,F) define an A-brane of a symplectic manifold if and only if $\mathcal{J}_\Omega$ acting on a section of $\tau^F Y$ gives back another section of $\tau^F Y$, i.e.,*

$$\mathcal{J}_\Omega(\tau^F Y) = \tau^F Y \ .$$

Motivated by our theme of comparing topological bibranes of the bosonic sigma model with A-branes of the $(2,2)$ sigma model by replacing Ω with the neutral signature G and replacing almost complex structures with almost product structures, let us define a *generalized (almost) product structure* to be an endomorphism $\mathcal{R} : TM \oplus T^*M \to TM \oplus T^*M$ with $\mathcal{R}^2 = 1$. Indeed, the generalized geometry object encoding the metric is a generalized almost product structure:

Definition 6. *As defined in [58], the* generalized metric[1] *is the particular generalized almost product structure given by*

$$\mathcal{R}_G = \begin{pmatrix} & G^{-1} \\ G & \end{pmatrix} .$$

Finally, we can compactly describe topological bibranes as follows.

Proposition 5. *The pair (Y, F) with $Y \subseteq M$ and F a closed 2-form on Y, satisfy the conditions for a topological bibrane (as defined in the previous section) if and only if $\mathcal{R}_G$ acting on a section of $\tau^F Y$ gives back another section of $\tau^F Y$, i.e.,*

$$\mathcal{R}_G(\tau^F Y) = \tau^F Y \ .$$

9.4 Topological defects of the (0,1) supersymmetric sigma model

We now proceed to our main topic of interest: supersymmetric, topological defects of the (0,1) supersymmetric sigma model. In addition to the bosonic fields ϕ^i and $\hat{\phi}^i$ we now have right-moving fermionic fields ψ^i_+ on Σ and $\widehat{\psi}^i_+$ on $\widehat{\Sigma}$.

[1] Various authors disagree on what object is to be given the name 'generalized metric'; we follow the convention of [58].

The bosonic gluing conditions described in Section 9.2 must be supplemented with fermionic gluing conditions on the fields $\psi^I = (\psi^i, \widehat{\psi}^i)$, setting half of the total fermionic degrees of freedom to zero the defect line D; we accomplish this by constraining ψ^I to take values in an n-dimensional subbundle – which we denote $\mathcal{R}_+$ – of the $2n$-dimensional bundle e^*TM.

Indeed, topological bibranes (Y, F) are equipped with a natural middle dimensional subbundle, which we describe as follows

Definition 7. *Let (Y, F) be a topological bibrane, as characterized in propositions 2, 3, and 5. Define $\mathcal{R}_+$ be the following bundle on Y:*

$$\mathcal{R}_+ = \{u \in TY : G(u, w) = F(u, w) \quad \textit{for all } w \in TY\} \; . \tag{9.2}$$

Let us see what this subbundle corresponds to in the two special classes of topological bibranes we discussed previously. In case (Y, F) is space-filling, $\mathcal{R}_+$ is the $+1$ eigenbundle of the almost product structure $G^{-1}F$. On the other hand, for graph-of-isometry type bibranes, this subbundle is simply the tangent bundle TY itself.

Proposition 6. *The subbundle $\mathcal{R}_+$, as defined above, is Lagrangian with respect to e^*TM. In particular, it is n-dimensional.*

Given (Y, F) satisfying the geometric topological bibrane conditions, we impose the bosonic gluing conditions written previously, and we augment these with condition that the fermions take values in $\mathcal{R}_+$; moreover, we require that $\mathcal{R}_+$ be an integrable distribution on Y. For completeness, we record here the full set of gluing conditions:

$$\phi \in Y$$
$$G(v, w) = F(u, w)$$
$$G(s, w) = F(s, w)$$

for all $w \in TY$, where $u^I \equiv \partial_0 \phi^I$, $v^I \equiv \partial_1 \phi^I$, and $s^I \equiv \psi^I$.

Let us now show that these gluing conditions define a topological, supersymmetry-preserving defect. We write an explicit action for the $(0, 1)$ supersymmetric sigma model with defect (see [57]):

$$\begin{aligned} S = &\int_\Sigma d^2\sigma \Big(-\frac{1}{2} g_{ij} \partial_\mu \phi^i \partial^\mu \phi^j - \frac{1}{2} b_{ij} \epsilon^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j + \frac{i}{2} g_{ij} \psi_+^i D_- \psi_+^j \Big) \\ &+ \int_{\widehat{\Sigma}} d^2\sigma \Big(-\frac{1}{2} \widehat{g}_{ij} \partial_\mu \hat{\phi}^i \partial^\mu \hat{\phi}^j - \frac{1}{2} \widehat{b}_{ij} \epsilon^{\mu\nu} \partial_\mu \hat{\phi}^i \partial_\nu \hat{\phi}^j + \frac{i}{2} \widehat{g}_{ij} \widehat{\psi}_+^i \widehat{D}_- \widehat{\psi}_+^j \Big) + \int_D d\sigma^0 \mathcal{A}_I \partial_0 \phi^I \end{aligned}$$

where $\partial_\pm = \partial_0 \pm \partial_1$ and the covariant derivatives are given by

$$D_\pm \psi_+^i = \partial_\pm \psi_+^i + \partial_\pm \phi^j \Gamma^i_{jk} \psi_+^k$$
$$\widehat{D}_\pm \widehat{\psi}_+^i = \partial_\pm \widehat{\psi}_+^i + \partial_\pm \hat{\phi}^j \hat{\Gamma}^i_{jk} \widehat{\psi}_+^k \ .$$

Varying the action and picking out the term localized on D, one obtains

$$(\delta S)_D = \int_D d\sigma^0 \Big(- G_{IJ} \partial_1 \phi^I \delta\phi^J + F_{IJ} \partial_0 \phi^I \delta\phi^J - \frac{i}{2} G_{IJ} \psi_+^I \delta^{\rm cov} \psi_+^J \Big)$$

where G and F are as before and we have introduced the notation

$$\delta^{\rm cov} \psi_+^I = \delta\psi_+^I + \delta\phi^J \Gamma^I_{JK} \psi_+^K$$

for the *covariant variations*, which, unlike $\delta\psi^I$, transform covariantly under target space coordinate changes. Here Γ^I_{JK} are the coefficients of the Levi-Civita connection of the neutral signature metric G on $X \times \widehat{X}$.

The first two terms in $(\delta S)_D$ vanish identically since the bosonic gluing conditions are identical to those in Section 9.2. Moreover, the third term in $(\delta S)_D$ can be shown to vanish by appealing to the following

Proposition 7. *Let (M, G) be a (pseudo-)Riemannian manifold with Levi-Civita connection ∇. Let $Y \subseteq M$ be a submanifold equipped with closed 2-form F on its worldvolume, and let $\mathcal{R}_+$ be the subbundle defined in* (9.2)*. If $\mathcal{R}_+$ is integrable, then we have the following:*

$$(e^*\nabla)_u s \in \Gamma(\mathcal{R}_+)$$

for all $u \in \Gamma(TY)$ and $s \in \Gamma(\mathcal{R}_+)$. Here $e^*\nabla$ is the pullback of ∇ under the embedding map $e : Y \to M$. In words, the subbundle $\mathcal{R}_+$ is ∇-parallel with respect to TY.

Proposition 8. *The term $G_{IJ} \psi_+^I \delta^{\rm cov} \psi_+^J$ vanishes for all allowed variations.*

Note that in case $F = 0$, the subbundle $\mathcal{R}_+$ equals TY, and the above proposition reduces to the surprising result that the covariant derivative of sections of TY are again in TY, i.e., Lagrangian submanifolds are automatically *totally geodesic*, a result also obtained in [59], [62].

We now check that the above gluing conditions define a topological defect and that they preserve (0,1) supersymmetry of the theory. The bosonic portions of the stress-energy tensors automatically glue smoothly by the analysis of Section 9.2. It remains to check that the contributions involving fermions glue smoothly; i.e.,

$$(T^\mu{}_\nu)^f - (\widehat{T}^\mu{}_\nu)^f = 0$$

for all μ, ν. This is equivalent to the condition

$$G(\nabla_u s, s) = 0$$

where the vector fields u, s are as we have defined previously. This equation is satisfied, once again by appeal to our parallelness proposition.

Finally, the condition for preservation of supersymmetry is that the 1-component of the supercurrents glue smoothly. This is equivalent to

$$G(s, t) = 0$$

where $s \equiv \psi_+$ and $t \equiv \partial_+ \phi$. This is satisfied by virtue of the fact that both $\partial_+ \phi$ and ψ_+ are constrained to take values in $\mathcal{R}_+$ by the gluing conditions, and the fact that $\mathcal{R}_+$ is a Lagrangian subbundle.

9.5 Topological defects of the $(0, 2)$ supersymmetric sigma model

Consider now the case when the target spaces X and $\widehat{X}$ are Kähler manifolds with Kähler forms ω and $\widehat{\omega}$, complex structures j and $\widehat{j}$. In this case the supersymmetric sigma models on either side of the defects possess $(0, 2)$ supersymmetry; let us geometrically classify those topological bibranes (Y, F) that preserve both supercharges.

We impose the gluing conditions written in the previous section and assume that the conditions for $(0, 1)$ supersymmetry are met. Then $(0, 2)$ supersymmetry will follow if

$$G'^1 - \widehat{G}'^1 = 0$$

along D, where G' and $\widehat{G}'$ are the second supercurrents. This is the condition

$$\Omega(s, t) = 0 \tag{9.3}$$

for all $s, t \in \mathcal{R}_+$, where

$$\Omega = GJ$$

is the Kähler form on $X \times \widehat{X}$ with respect to complex structure $J = j \oplus \widehat{j}$. We can say the following about when (9.3) is satisfied:

Proposition 9. *The following conditions on topological bibrane (Y, F) are equivalent:*

1. *$\mathcal{R}_+$ is an Ω-Lagrangian subbundle.*

2. *$J(\mathcal{R}_+) = \mathcal{R}_+$, i.e., the leaves of the foliation by subbundle $\mathcal{R}_+$ are complex submanifolds with respect to complex structure $J = j \oplus \widehat{j}$.*

3. *Y is a complex submanifold of $X \times \widehat{X}$ and F has type $(1,1)$ with respect to the induced complex structure on Y.*

4. *In generalized geometry terms, $\mathcal{J}_J$ acting on a section of $\tau^F Y$ gives back another section of $\tau^F Y$, where*

$$\mathcal{J}_J = \begin{pmatrix} J & \\ & -J^{\mathsf{T}} \end{pmatrix}$$

is the generalized complex structure associated with complex structure J.

5. *(Y, F) is a B-brane with respect to complex structure J.*

The generalized metric $\mathcal{R}_G$ and the generalized complex structures $\mathcal{J}_\Omega$ and $\mathcal{J}_J$ are related by

$$\mathcal{R}_G = -\mathcal{J}_\Omega \mathcal{J}_J \ .$$

This shows that if the bundle $\tau^F Y$ is stabilized by any two of these, then it is automatically stabilized by the third. In particular, we have the following

Proposition 10. *The bibrane (Y, F) is a topological, supersymmetry preserving defect of the $(0,2)$ sigma model when it is both an A-brane with respect to symplectic form $\Omega = \omega \oplus -\widehat{\omega}$ and a B-brane with respect to complex structure $J = j \oplus \widehat{j}$ on $X \times \widehat{X}$ (with suitable integrability conditions).*

9.6 T-duality

We briefly comment on the role our topological defects play in implementing dualities of the bosonic sigma models with targets X and $\widehat{X}$. For concreteness let X and $\widehat{X}$ be flat tori and let $Y = X \times \widehat{X}$ be a space-filling defect equipped with a line bundle with curvature $\mathcal{F} = d\mathcal{A}$; we define the 2-form $F = -\mathcal{F} - B$, as in Section 2.

Topological defects separating theories X and $\widehat{X}$ can be fused with those separating $\widehat{X}$ and X' to yield topological defects separating X and X'. The invisible defect (choosing the diagonal $Y \subset X \times X$ and $F = 0$) is a unital element with respect to this operation of fusion and an *invertible defect* is one that can be fused with another defect to yield the invisible defect. On general grounds, an invertible topological defect implements a duality[2].

[2]See [53] for a recent discussion of this point in the context of rational CFTs.

Hence, we expect that if the 2-form $\mathcal{F}$ is chosen such that (Y, F) is an invertible topological defect, then the sigma model with target X will be related to the sigma model with target $\widehat{X}$ by a duality transformation. For tori, the group of duality transformations is $O(n, n; \mathbb{Z})$ (the group of automorphisms of a lattice of signature (n, n)) [54].

Let us fix the background data (g, b) and take

$$\mathcal{F} = \begin{pmatrix} f & h \\ -h^{\mathsf{T}} & \widehat{f} \end{pmatrix}$$

where $f, \widehat{f}$ are antisymmetric $n \times n$ matrices. Since $\mathcal{F}$ is the curvature of a line bundle on the torus $X \times \widehat{X}$, its entries are constrained to be integers. Generically, the determinant of the off-diagonal matrix h can take any integer value; however, let us now restrict the form of $\mathcal{F}$ by assuming that $\det h = \pm 1$, or equivalently $h \in GL(n, \mathbb{Z})$.

The topological defect condition for space-filling defects states that $(G^{-1}F)^2 = 1$; writing this out in terms of the blocks, this is equivalent to the following relationship between the background data (g, b) for X and $(\widehat{g}, \widehat{b})$ for $\widehat{X}$:

$$\begin{aligned} \widehat{g} &= h^{\mathsf{T}} \big(g - (b+f) g^{-1} (b+f)\big)^{-1} h \\ 0 &= (b+f) g^{-1} h + h \widehat{g}^{-1} (\widehat{b} - \widehat{f}) \ . \end{aligned}$$

We recognize this as the duality transformation consisting of a shift of the B-field by the matrix f, followed by a T-duality (inverting the data $g+b$), followed by a basis change of the lattice generating the torus (parametrized by unimodular matrix h), followed by another B-field shift (this time by $\widehat{f}$) (see [54], eqs. (2.4.25), (2.4.26), (2.4.39)). The corresponding duality element is

$$\begin{pmatrix} \widehat{f} h^{-1} & h^{\mathsf{T}} + \widehat{f} h^{-1} f \\ h^{-1} & h^{-1} f \end{pmatrix} \in O(n, n; \mathbb{Z}) \ .$$

(A similar calculation with $f = \widehat{f} = 0$ and $h = 1$ is performed in [51].) It is evident from this expression that quantization of the entries is implied by quantization of the entries of $\mathcal{F}$.

Appendix A

2d TFTs

A.1 A-type 2d topological gauge theory

In this section we discuss a topological gauge theory in 2d which can be obtained by twisting $d = 2$, $N = (2, 2)$ supersymmetric gauge theory by means of the $U(1)_V$ current [46]. This theory is somewhat analogous to the Donaldson-Witten theory in 4d, but is much simpler, because its path-integral does not get nonperturbative contributions.

The bosonic fields of the theory consist of connection A on a principal G-bundle P over an oriented 2-manifold Σ, as well as a complex 0-form

$$\sigma \in \Omega^0(\Sigma, ad\, P) \otimes \mathbb{C} \ .$$

The fermionic fields are

$$\begin{aligned} &\beta \in \Omega^0(\Sigma, ad\, P) \\ &\lambda \in \Omega^1(\Sigma, ad\, P) \\ &\chi = \Omega^2(\Sigma, ad\, P) \ . \end{aligned}$$

The BRST transformations are

$$\begin{aligned} &\delta_Q A = \lambda, \\ &\delta_Q \sigma = 0, \\ &\delta_Q \bar{\sigma} = \eta, \\ &\delta_Q \beta = 2i[\bar{\sigma}, \sigma], \\ &\delta_Q \lambda = 2i d_A \sigma, \\ &\delta_Q \chi = F \ . \end{aligned}$$

The relation $\delta_Q^2 = 2i\delta_\sigma$ holds off-shell for all fields except χ, where one has instead

$$\delta_Q^2 \chi = d_A \lambda \ .$$

If there is a fermionic equation of motion

$$d_A \lambda = 2i[\chi, \sigma] \ ,$$

the relation $\delta_Q^2 = 2i\delta_\sigma$ holds on-shell. To achieve closure off-shell we introduce an auxiliary bosonic 2-form $P \in \Omega^2(\Sigma, ad\, P)$ and redefine the transformation law for χ:

$$\delta_Q \chi = P, \quad \delta_Q P = 2i[\chi, \sigma] \ .$$

The action is constructed to ensure that on-shell $P = F$. One can take a BRST-exact action:

$$S = \frac{1}{2e^2}\, \delta_Q \int_\Sigma \left(\frac{i}{2} \lambda \wedge \star d_A \bar{\sigma} + \chi \wedge \star (P - 2F) \right) \ .$$

The gauge coupling enters only as the coefficient of a BRST-exact term and therefore is irrelevant. The path-integral of the theory localizes on configurations with $F = 0$ and constant σ, so there is no room for nonperturbative contributions.

Category of branes of the A-type 2d gauge theory for $G = U(1)$

In Chapter 6, we need to understand the category of branes of this 2d TFT, at least in the abelian case. We will now argue that, for $G = U(1)$, the theory is isomorphic to the B-model with target $\mathbb{C}[2]$, and therefore the category of branes is equivalent to $D^b(\mathbb{C}[2])$.

One approach to this problem is to construct an embedding of $D^b(Coh(\mathbb{C}[2]))$ into the category of branes. This does not prove that the two categories are equivalent, but it does show that the former is a full subcategory of the latter. The basic boundary condition in the A-type 2d gauge theory is the Neumann condition which leaves the restriction of A and σ to the boundary free and requires $\star F$ and the normal derivative of σ to vanish on the boundary. BRST-invariance also requires $\star\chi$ and the restriction of $\star\lambda$ to vanish on the boundary, while β and the restriction of λ remain unconstrained. The algebra of BRST-invariant observables on the Neumann boundary is spanned by powers of σ, i.e., it is the algebra $\mathcal{O}$ of holomorphic functions on $\mathbb{C}[2]$. One can construct a more general boundary condition by placing additional degrees of freedom on the boundary which live in a graded vector space V. The BRST operator gives rise to a degree-1 differential $T : V \to V$ which may depend polynomially on σ. Thus we may attach a brane to any free DG-module $M = (V \otimes \mathcal{O}, T)$ over the graded algebra $\mathcal{O}$. The space of morphisms between any two such branes $M_1 = (V_1 \otimes \mathcal{O}, T_1)$ and

$M_2 = (V_2 \otimes \mathcal{O}, T_2)$ is the cohomology of the complex $\mathrm{Hom}_{\mathcal{O}}(M_1, M_2)$, which agrees with the space of morphisms in the category $D^b(Coh(\mathbb{C}[2]))$.

In the A-type 2d gauge theory, one may also consider the Dirichlet boundary condition which sets $\sigma = 0$ on the boundary and requires the restriction of the gauge field to be trivial. One might guess that it corresponds to the skyscraper sheaf at the origin of $\mathbb{C}[2]$, and indeed one can verify that the space of morphisms from any of the branes considered above to the Dirichlet brane agrees with the space of morphisms from the corresponding complex of vector bundles on $\mathbb{C}[2]$ to the skyscraper sheaf.

Category of branes from equivalence with B-model, target $\mathbb{C}[2]$

Another way to approach the problem is construct an isomorphism between the A-type 2d gauge theory and the B-model with target $\mathbb{C}[2]$. From the physical viewpoint, an isomorphism of two 2d TFTs $\mathbb{X}$ and $\mathbb{Y}$ is an invertible topological defect line A between them. In the present case, there is a unique candidate for such a defect line. Recall that a B-model with target $\mathbb{C}[2]$ has a bosonic scalar ϕ, a fermionic 1-form ρ, and fermionic 0-forms θ and ξ. The BRST transformations read

$$\begin{aligned}
&\delta_Q \phi = 0, \\
&\delta_Q \bar{\phi} = \eta, \\
&\delta_Q \eta = 0, \\
&\delta_Q \theta = 0, \\
&\delta_Q \rho = d\phi \ .
\end{aligned}$$

The field σ has ghost number 2, the fields ρ has ghost number 1, and the fields η and θ have ghost number -1. The action of the B-model is

$$S = -\frac{1}{2}\delta_Q \int_\Sigma \rho \wedge \star d\bar{\phi} + \int_\Sigma \theta \wedge d\rho \ .$$

The ghost-number 2 bosons σ and ϕ must be identified on the defect line, up to a numerical factor which can be read of the action. Similarly, the fermionic 1-forms λ and ρ must be identified, as well as the fermionic 0-forms β and η. Finally, one must identify $\star\chi$ and θ. BRST invariance then requires $\star F$ to vanish on the boundary, which means that the gauge field obeys the Neumann boundary condition.

Note that this defect line is essentially the trivial defect line for the fermionic fields and σ. Since the zero-energy sector of the bosonic $U(1)$ gauge theory is trivial, the invertibility of the defect line is almost obvious. Let us show this more formally. First, consider two parallel defect lines with a sliver of the A-type 2d gauge theory between them. The sliver has the shape $\mathbb{R} \times I$, where $\mathbb{R}$

parameterizes the direction along the defect lines. The statement that the product of two defect lines is the trivial defect line in the B-model is equivalent to the statement that the $U(1)$ gauge theory on an interval with Neumann boundary conditions on both ends has a unique ground state. This is obviously true, because the space-like component of A in the sliver can be gauged away by a time-independent gauge transformation, and therefore the physical phase space of the $U(1)$ gauge theory on an interval is a point.

Second, consider the opposite situation where a sliver of the B-model is sandwiched between two defect lines. We would like to show that this is equivalent to the trivial defect line in the A-type 2d gauge theory. The sliver has the shape $S^1 \times I$. For simplicity we will assume that the worldsheet with a sliver removed consists of two connected components. Each component is an oriented manifold with a boundary isomorphic to S^1, and the path-integral of the A-type 2d gauge theory defines a vector in the Hilbert space V corresponding to S^1. Any topological defect line in the A-type gauge theory defines an element in $V^* \otimes V \simeq \mathrm{End}(V)$. We would like to show that the B-model sliver corresponds to the identity element in $V^* \otimes V$. First we note that V can be identified with the tensor product of the Hilbert space of the zero-energy gauge degrees of freedom and the Hilbert space of the zero-energy degrees of freedom of σ and the fermions. As mentioned above, the defect line separating the A-type 2d gauge theory and the B-model acts as the trivial defect line on σ and the fermions, so in this sector the statement is obvious. As for the gauge sector, the corresponding Hilbert space of zero-energy states is one-dimensional, so the B-model sliver is proportional to the identity operator. The argument of the preceding paragraph shows that its trace is one, so the sliver must be the identity operator.

Category of branes of gauged 2d A-model

The A-type 2d gauge theory with gauge group G can be coupled to an A-model with target X admitting a symplectic G-action (we do not write down this gauged 2d A-model here). It is natural to conjecture that the corresponding category of branes is some sort of G-equivariant version of the Fukaya-Floer category of X. To make this conjecture more precise, let us consider a special case where $X = T^*Y$ with its canonical symplectic form. It was shown in [43, 42] that a suitable version of the Fukaya-Floer category of T^*Y is equivalent to the constructible derived category of sheaves over Y. Now let Y admit a G-action. We conjecture that the category of branes for the A-type 2d gauge theory coupled to an A-model with target T^*Y is equivalent to the G-equivariant constructible derived category of sheaves over Y. In the special case when Y is a point and $G = U(1)$, the latter category is known to be equivalent to $D^b(Coh(\mathbb{C}[2]))$, in agreement with the fact that the A-type 2d gauge theory with $G = U(1)$ is isomorphic to the B-model with target $\mathbb{C}[2]$.

A.2 B-type 2d topological gauge theory

In this section we discuss a topological gauge theory in 2d which can be obtained by twisting $d = 2$, $N = (2, 2)$ supersymmetric gauge theory by means of a $U(1)_A$ current. This theory is a 2d analog of the GL-twisted theory at $t = i$.

The fields of the B-type 2d gauge theory are a connection A on a principal G-bundle P over an oriented 2-manifold Σ, as well as

$$\varphi \in \Omega^1(\Sigma, ad\, P)$$

and the fermionic fields

$$\beta \in \Omega^0(\Sigma, ad\, P)$$
$$\lambda \in \Omega^1(\Sigma, ad\, P)$$
$$\zeta \in \Omega^2(\Sigma, ad\, P)\ .$$

The BRST transformations are

$$\delta_Q A = \lambda,$$
$$\delta_Q \varphi = i\lambda,$$
$$\delta_Q \lambda = 0,$$
$$\delta_Q \beta = i d_A^\star \varphi,$$
$$\delta_Q \zeta = -i\mathcal{F}\ .$$

Here $\mathcal{F}$ is the curvature of the complex connection $\mathcal{A} = A + i\varphi$. We will denote the covariant derivative with respect to $\mathcal{A}$ by $d_{\mathcal{A}}$, the covariant derivative with respect to $\bar{\mathcal{A}} = A - i\varphi$ by $d_{\bar{\mathcal{A}}}$, and the curvature of $\bar{\mathcal{A}}$ by $\bar{\mathcal{F}}$. The differential operator $d_A^\star$ is $\star d_A \star$, where $\star$ is the 2d Hodge star.

The theory has a $U(1)$ ghost number symmetry with respect to which the fields A, φ are neutral, the field λ has charge 1, and the fields β, ζ have charge -1.

The BRST transformation satisfy $\delta_Q^2 = 0$ on all fields except β:

$$\delta_Q^2 \beta = -d_{\mathcal{A}}^\star \lambda\ .$$

If one uses the fermionic equation of motion $d_{\mathcal{A}}^\star \lambda = 0$, then the BRST transformations are nilpotent on-shell. It is more convenient to have $\delta_Q^2 = 0$ off-shell, so we introduce an auxiliary 0-form P and define

$$\delta_Q \beta = iP, \quad \delta_Q P = 0\ .$$

When constructing an action, we need to ensure that the equation of motion for P sets $P = d_A^\star \varphi$. A suitable action is BRST-exact:

$$S = -\frac{1}{2e^2}\delta_Q \int_\Sigma \mathrm{Tr}\, \left(i\zeta \wedge \star\bar{\mathcal{F}} + i\beta \wedge \star(P - 2d_A^\star \varphi)\right) \; .$$

The coupling constant e^2 enters only as the coefficient of a BRST-exact term, therefore the topological correlators do not depend on it, and the semiclassical approximation is exact. The topological nature of the theory is also apparent, since the metric enters only through BRST-exact terms.

Observables of the B-type 2d gauge theory

Usually local observables are defined as BRST-invariant and gauge-invariant scalar functions of fields, modulo BRST transformations. In the present case, there are no nontrivial local observables of this kind. However, there are nontrivial BRST-invariant local disorder operators which are defined by allowing certain singularities in the fields. For example, one can require the connection $\mathcal{A}$ to have a nontrivial holonomy around the insertion point. Such local operators are analogous to Gukov-Witten surface operators in 4d gauge theory. More systematically, to determine what kind of local operators are allowed one can reduce the 2d gauge theory theory on a circle and study the space of the states of the resulting 1d TFT. In the present case, this 1d TFT is a gauged sigma-model with target $G_{\mathbb{C}}$. From the 2d viewpoint, the target space parameterizes the holonomy of $\mathcal{A}$. BRST-invariant wavefunctions are holomorphic functions on $G_{\mathbb{C}}$ invariant with respect to conjugation, i.e., characters of $G_{\mathbb{C}}$. More generally, one may consider nonnormalizable wavefunctions, such as delta-functions supported on closed $G_{\mathbb{C}}$-invariant complex submanifolds of $G_{\mathbb{C}}$. For example, the identity operator can be thought of as a delta-function supported at the identity element, while Gukov-Witten-type local operators are delta-functions supported on closed conjugacy classes in $G_{\mathbb{C}}$.

There are also BRST-invariant and gauge-invariant line observables, the most obvious of which are Wilson line operators for the complex BRST-invariant connection $\mathcal{A}$. To define them, one needs to pick a finite-dimensional graded representation V of G and consider the holonomy of $\mathcal{A}$ in the representation V.

Category of branes of the B-type 2d gauge theory

The category of branes for this 2d TFT is the category of finite-dimensional graded representations of G. To see this, consider the Neumann boundary condition for the gauge field, that is, leave the restriction of $\mathcal{A}$ to the boundary free and require the restriction of $\star\phi$ to vanish. BRST-invariance then requires ζ and the restriction of $\star\lambda$ to vanish on the boundary. Since the gauge field $\mathcal{A}$ on the boundary is unconstrained and BRST-invariant, we may couple to it an arbitrary finite-dimensional graded representation V of G. That is, we may include into the path-integral the holonomy of $\mathcal{A}$

in the representation V. Thus boundary conditions are naturally labeled by representations of G. Given any two irreducible representations V_1 and V_2 one can form a junction between them only if V_1 and V_2 are isomorphic (because there are no nontrivial BRST-invariant local operators on the Neumann boundary). Further, if $V_1 \simeq V_2$, the space of morphisms between them is $\mathrm{Hom}_G(V_1, V_2)$ (for the same reason).

A.3 Gauged 2d B-model

In this section we describe how to couple a B-type 2d gauge theory to a B-model. We show that the category of branes for the resulting 2d TFT is closely related to the equivariant derived category of coherent sheaves.

Let X be a Calabi-Yau manifold (i.e., a Kähler manifold with a holomorphic volume form) which admits a G-action preserving both the Kähler structure and the holomorphic volume form. The bosonic field of the B-model matter sector is a section $\mathbf{\Phi}$ of the bundle

$$E = P \times_G X$$

associated to gauge bundle P by the holomorphic G-action. The fermionic fields of the B-model matter sector consist of the following vertical subbundle-valued forms:

$$\begin{aligned}
&\boldsymbol{\eta} \in \Omega^0(\Sigma, \mathbf{\Phi}^* V^{0,1} E) \\
&\boldsymbol{\theta} \in \Omega^0(\Sigma, \mathbf{\Phi}^* (V^{1,0} E)^\vee) \\
&\boldsymbol{\rho} \in \Omega^1(\Sigma, \mathbf{\Phi}^* V^{1,0} E)\ .
\end{aligned}$$

The BRST transformations are given by gauge-covariantized versions of the usual B-model transformations:

$$\begin{aligned}
&\delta_Q \phi^i = 0, \\
&\delta_Q \phi^{\bar{i}} = \eta^{\bar{i}}, \\
&\delta_Q \eta^{\bar{i}} = 0, \\
&\delta_Q \theta_i = 0, \\
&\delta_Q \rho^i = \mathcal{D}\phi^i
\end{aligned}$$

where $\mathcal{D}\phi^i = d\phi^i + V^i(\mathcal{A})$ is the gauge-covariant derivative with respect to complexified connection $\mathcal{A}$ and V is the holomorphic vector field (with values in $\mathfrak{g}^*$) corresponding to the G-action. Since $\mathcal{A}$

is BRST-invariant, these BRST transformations still satisfy $\delta_Q^2 = 0$.

To construct a BRST-invariant action, we start with the usual action of the B-model and gauge-covariantize all derivatives. The covariantized action is not BRST-invariant, but this can be corrected by adding a new term proportional to $\theta_i V^i(\zeta)$, where ζ is the fermionic, $ad\,P$-valued 2-form of the B-type 2d gauge theory. The full matter action is

$$S = \int_\Sigma \delta_Q \left(g_{i\bar{j}} \rho^i \wedge \star \bar{\mathcal{D}} \phi^{\bar{j}} \right) + \int_\Sigma \left(-i\theta_i V^i(\zeta) + \theta_i \mathcal{D}\rho^i + \frac{1}{2} R^i_{jk\bar{l}} \theta_i \rho^j \rho^k \eta^{\bar{l}} \right) \ .$$

Here g is the Kähler metric, R is its curvature tensor, and the covariant derivative of ρ includes both the Levi-Civita connection and the gauge connection:

$$\mathcal{D}\rho^i = d\rho^i + \Gamma^i_{jk} d\phi^j \rho^k + \nabla_j V^i(\mathcal{A})\rho^j, \quad \nabla_j V^i = \partial_j V^i + \Gamma^i_{jk} V^k \ .$$

The covariant derivative $\bar{\mathcal{D}}\phi^{\bar{j}}$ is defined so as to make the bosonic part of the action positive-definite:

$$\bar{\mathcal{D}}\phi^{\bar{j}} = d\phi^{\bar{j}} - V^{\bar{j}}(\bar{\mathcal{A}}), \quad \bar{\mathcal{A}} = A - i\varphi = -\mathcal{A}^\dagger \ .$$

Category of branes of the gauged 2d B-model

Since the category of branes for the B-model with target X is $D^b(Coh(X))$, a natural guess for the category of branes for the gauged B-model is $D^b_{G_\mathbb{C}}(Coh(X))$. We will now describe a construction of the boundary action corresponding to an equivariant complex of holomorphic vector bundles on X. Let E be a graded complex vector bundle over X with a holomorphic structure $\bar{\partial}^E : E \to E \otimes \Omega^{0,\bullet}(X)$, $(\bar{\partial}^E)^2 = 0$, and a holomorphic degree-1 endomorphism $T : E \to E$, $\bar{\partial}^E T = 0$ satisfying $T^2 = 0$. To write down a concrete boundary action we will assume that we are also given a Hermitian metric on each graded component of E, so that $\bar{\partial}^E$ gives rise to a connection ∇^E on E. We will denote the corresponding connection 1-form by ω and its curvature by F^E. We assume that we are given a lift of the G-action on X to a G-action on the total space of E which is fiberwise-linear and compatible with $\bar{\partial}^E$, T, and the Hermitian metric. Infinitesimally, the Lie algebra $\mathfrak{g}$ acts on a section s of E as follows:

$$(f, s) \mapsto f(s) = V^i(f)\nabla^E_i s + V^{\bar{i}}(f)\nabla^E_{\bar{i}} s + R(f)s, \quad f \in \mathfrak{g} \ .$$

Here $\nabla^E = d + \omega$, and R is a degree-0 bundle morphism $R : E \to E \otimes \mathfrak{g}^*$. The condition that the G-action commutes with ∇^E implies

$$\nabla^E R = \iota_V F^E \ .$$

The condition that the G-action commutes with T implies

$$V^i\nabla^E_i T + [R, T] = 0 \ .$$

Consider now the following field-dependent connection 1-form on the pullback bundle $\mathbf{\Phi}^*E$:

$$\mathcal{N} = \omega_i d\phi^i + \omega_{\bar{i}} d\phi^{\bar{i}} - R(\mathcal{A}) + \rho^i\eta^{\bar{j}} F^E_{i\bar{j}} + \rho^i\nabla^E_i T \ .$$

With some work one can check that its BRST variation satisfies

$$\delta_Q\mathcal{N} = d(\omega_{\bar{i}}\eta^{\bar{i}} + T) + [\mathcal{N}, \omega_{\bar{i}}\eta^{\bar{i}} + T].$$

Therefore the supertrace of its holonomy is BRST-invariant and can be used as a boundary weight factor in the path-integral associated. By definition the boundary action is minus the logarithm of the boundary weight factor.

Let us consider a ghost-number zero boundary observable $\mathcal{O}$ in the presence of a such a weight factor. It is an element of End(E) depending on the fields $\mathbf{\Phi}, \boldsymbol{\eta}$ and of total degree zero. More invariantly, we may think of it as a section of End$(E) \otimes \Omega^{0,\bullet}(X)$. The BRST-variation of the boundary weight factor in the presence of $\mathcal{O}$ is proportional to

$$\eta^{\bar{i}}\nabla^E_{\bar{i}}\mathcal{O} + [T, \mathcal{O}].$$

Hence BRST-invariant boundary observables are sections of End$(E)\otimes\Omega^{0,\bullet}(X)$ which are annihilated by $\bar{\partial}^E$ and commute with T. Further, a BRST-invariant $\mathcal{O}$ it is gauge-invariant iff it satisfies

$$V^i\nabla^E_i\mathcal{O} + [R, \mathcal{O}] = 0 \ .$$

Together these conditions mean that $\mathcal{O}$ represents an endomorphism of the equivariant complex $(E, \bar{\partial}^E, T)$ regarded as an object of $D^b_{G_\mathbb{C}}(Coh(X))$. It is also easy to see that such an observable $\mathcal{O}$ is a BRST-variation of a gauge-invariant observable iff it is homotopic to zero. In some cases this implies that the category of branes in the gauged B-model of the kind we have constructed is equivalent to $D^b_{G_\mathbb{C}}(Coh(X))$. This happens if any G-equivariant coherent sheaf on X has a G-equivariant resolution by G-equivariant holomorphic vector bundles. Such an X is said to have a G-resolution property. An example of such X is $\mathbb{C}^n$ with a linear action of G, or more generally a smooth affine variety with an affine action of G. Note that for a general complex manifold X the resolution property may fail even if G is trivial. But for trivial G the cure is known: one has to replace complexes of holomorphic vector bundles with more general DG-modules over the Dolbeault DG-algebra of X [30]. These more general DG-modules also arise naturally from the physical viewpoint [27, 36]. We expect that for any complex Lie group G with a complex-analytic action on X a G-equivariant coherent sheaf on X has a G-equivariant resolution by these more general DG-modules. This would imply that the category of B-branes for the gauged B-model is equivalent to $D^b_{G_{\mathbb{C}}}(Coh(X))$.

Appendix B

3d TFTs

B.1 3d A-model

We describe a 3d topological sigma model similar in form to the 4d A-model TFT discussed in Section 3.1. This 3d A-model TFT was studied first in [6]; it can be regarded as a 3d analogue of the 2d A-model.

Let X be a real manifold equipped with Riemannian metric g and let W be a 3d base manifold. The bosonic fields consist of a map

$$\Phi : W \to X$$

as well as the 1-form

$$b \in \Omega^1(W, \Phi^* TX),$$

and the auxiliary fields

$$P \in \Omega^0(W, \Phi^* TX),$$
$$\widetilde{P} \in \Omega^1(W, \Phi^* TX) \ .$$

The fermionic fields consist of the 0-forms

$$\beta, \eta \in \Omega^0(W, \Phi^* TX)$$

as well as the 1-forms

$$\psi, \chi \in \Omega^1(W, \Phi^* TX) \ .$$

The BRST-variations are given by

$$\begin{aligned}
\delta_Q \phi^I &= \eta^I \\
\delta_Q \eta^I &= 0 \\
\delta_Q b^I &= \psi^I - \Gamma^I_{JK}\eta^J b^K \\
\delta_Q \psi^I &= \frac{1}{2} R^I{}_{JKL}\, b^J \eta^K \eta^L - \Gamma^I_{JK}\eta^J \psi^K \\
\delta_Q \beta^I &= P^I - \Gamma^I_{JK}\eta^J \beta^K \\
\delta_Q P^I &= \frac{1}{2} R^I{}_{JKL}\, \beta^J \eta^K \eta^L - \Gamma^I_{JK}\eta^J P^K \\
\delta_Q \chi^I &= \widetilde{P}^I - \Gamma^I_{JK}\eta^J \chi^K \\
\delta_Q \widetilde{P}^I &= \frac{1}{2} R^I{}_{JKL}\, \chi^J \eta^K \eta^L - \Gamma^I_{JK}\eta^J \widetilde{P}^K
\end{aligned} \tag{B.1}$$

where Γ^I_{JK} are the Christoffel symbols and $R^I{}_{JKL}$ are the Riemann tensor components of the Levi-Civita connection with respect to the metric g_{IJ}. The BRST-exact action is given by

$$S = \frac{1}{e^2}\delta_Q \int_W d^3x\sqrt{h}\, g_{IJ}\Big\{\chi^{Ir}\Big(\widetilde{P}^J_r - 2D_r\phi^J + 2\sqrt{h}\,\epsilon_{rst}\, D^s b^{Jt}\Big) + \beta^I\Big(P^J - 2D_r b^{Jr}\Big)\Big\}$$

where $r, s, t = 1, 2, 3$ are form indices on W and D is the pullback to W of the Levi-Civita connection on X. As described in [6], local observables correspond to cohomology classes of the target X and line observables correspond to holonomies of flat connections on X pulled back under the sigma model map. Moreover, the simplest class of 'geometric' boundary conditions correspond to closed submanifolds $Y \subset X$. Note that, as we have not assumed X to be endowed with any extra geometric structures (such as a symplectic form or complex structure), there are no extra conditions on the embedding of Y within X.

B.2 Gauged 3d A-model

Suppose now that the target space X admits an action of a Lie group G by isometries. We wish to gauge the 3d A-model with target X by coupling it to a 3d topological gauge theory, in much the same way that we gauged the 4d A-model in Section 3.2. Indeed, we will couple the 3d A-model to what we will call the *A-type 3d gauge theory.*

B.2.1 A-type 3d topological gauge theory

There are two different topological gauge theories in 3d which can be obtained by twisting $\mathcal{N} = 4$, $d = 3$ SYM theory. The first one is the dimensional reduction of the Donaldson-Witten twist of

$\mathcal{N}=2$, $d=4$ SYM theory (see Section 3.2.1). The second one is intrinsic to 3d and has been first discussed by Blau and Thompson [29]. We will refer to them as the A-type and B-type topological gauge theories, respectively. (The reason for this terminology is that the BPS equations in the former theory are elliptic, as in the usual A-model, while in the latter theory they are overdetermined, as in the usual B-model.)

The bosonic field content of the A-type 3d gauge theory consists of a connection A on a principal G-bundle P over 3-manifold W, as well as 0-forms

$$\sigma \in \Omega^0(W, ad\, P) \otimes \mathbb{C}$$
$$\varsigma \in \Omega^0(W, ad\, P)$$

and auxiliary 1-form

$$H \in \Omega^1(W, ad\, P) \ .$$

The fermionic field content consists of the forms

$$\rho, \widetilde{\rho} \in \Omega^0(W, ad\, P)$$
$$\lambda, \widetilde{\lambda} \in \Omega^1(W, ad\, P) \ .$$

The BRST transformations are given by [6]:

$$\begin{aligned}
&\delta_Q A = \lambda, \qquad \delta_Q \lambda = -d_A \sigma, \\
&\delta_Q \sigma = 0, \\
&\delta_Q \bar{\sigma} = \widetilde{\rho}, \qquad \delta_Q \widetilde{\rho} = [\sigma, \bar{\sigma}] \\
&\delta_Q \widetilde{\lambda} = H, \quad \delta_Q H = [\sigma, \widetilde{\lambda}] \\
&\delta_Q \varsigma = \rho, \qquad \delta_Q \rho = [\sigma, \varsigma] \ .
\end{aligned}$$

The action is given by

$$S = \frac{1}{e^2} \delta_Q \int_W d^3x \sqrt{h}\, \mathrm{Tr} \Big\{ 4 \widetilde{\lambda}^r \big(H_r - D_r \varsigma - (\star F)_r \big) + 2 \bar{\sigma}\, D_r \lambda^r \Big\}$$

where F is the curvature of A and $\star$ is the 3d Hodge star.

B.2.2 Gauging the 3d A-model

Let V_a, $a = 1, 2, \cdots, \dim G$, be the vector fields on X corresponding to the generators of the G-action. The modification of the matter sector variations closely resembles that of the gauged 4d

A-model in Section 3.2:

$$\begin{aligned}
\delta_Q \phi^I &= \eta^I \\
\delta_Q \eta^I &= \sigma^a V_a^I \\
\delta_Q b^I &= \psi^I - \Gamma^I_{JK}\eta^J b^K \\
\delta_Q \psi^I &= \frac{1}{2} R^I{}_{JKL}\, b^J \eta^K \eta^L - \Gamma^I_{JK}\eta^J \psi^K + \sigma^a b^J \nabla_J V_a^I \\
\delta_Q \beta^I &= P^I - \Gamma^I_{JK}\eta^J \beta^K \\
\delta_Q P^I &= \frac{1}{2} R^I{}_{JKL}\, \beta^J \eta^K \eta^L - \Gamma^I_{JK}\eta^J P^K + \sigma^a \beta^J \nabla_J V_a^I \\
\delta_Q \chi^I &= \widetilde{P}^I - \Gamma^I_{JK}\eta^J \chi^K \\
\delta_Q \widetilde{P}^I &= \frac{1}{2} R^I{}_{JKL}\, \chi^J \eta^K \eta^L - \Gamma^I_{JK}\eta^J \widetilde{P}^K + \sigma^a \chi^J \nabla_J V_a^I
\end{aligned} \tag{B.2}$$

where

$$\nabla_J V_a^I = \partial_J V_a^I + \Gamma^I_{JK} V_a^K \ .$$

The action is given by adding the matter sector action to the gauge sector action and gauge-covariantizing derivatives:

$$\begin{aligned}
S = \frac{1}{e^2}\delta_Q \int_W d^3x \sqrt{h}\ \mathrm{Tr}\Big\{ 4\widetilde{\lambda}^r \big(H_r - D_r\varsigma - (\star F)_r\big) + 2\bar{\sigma}\, D_r \lambda^r \Big\} \\
+ \chi^{Ir}\Big(\widetilde{P}^J_r - 2\mathcal{D}_r \phi^J + 2\sqrt{h}\,\epsilon_{rst}\,\mathcal{D}^s b^{Jt}\Big) + \beta^I\Big(P^J - 2\mathcal{D}_r b^{Jr}\Big)\Big\}
\end{aligned}$$

where $\mathcal{D}$ is the gauge-covariant derivative, as discussed for the 4d A-model in Chapter 3. We remark that this action for the gauged 3d A-model is far from unique, as nonminimal BRST-exact terms could be added without affecting the theory. Moreover, the 3d A-model can be gauged by coupling to other gauge sectors besides the 3d A-type gauge theory; for instance, in Section 5.2 we write down a theory with gauge sector given by replacing the Lie algebra-valued scalar ς by a a Lie group-valued scalar.

B.3 Rozansky-Witten model

We describe a 3d topological sigma model which is a 3d analog of the 2d B-model. The Rozansky-Witten sigma model [14] is defined for any hyperkähler target space X and 3d base manifold W.

Its bosonic field consists of a map

$$\Phi : W \to X$$

which we describe using local complex coordinates ϕ^i and $\phi^{\bar{i}}$ on X. Its fermionic fields consist of the 0-form

$$\eta \in \Omega^0(W, \Phi^* T^{0,1} X)$$

as well as the 1-form

$$\chi \in \Omega^1(W, \Phi^* T^{1,0} X) \ .$$

The BRST variations are given by

$$\begin{aligned}
\delta_Q \phi^i &= 0 \\
\delta_Q \phi^{\bar{i}} &= \eta^{\bar{i}} \\
\delta_Q \eta^{\bar{i}} &= 0 \\
\delta_Q \chi^i &= d\phi^i
\end{aligned}$$

The BRST-exact action is given by

$$S = \delta_Q \int_W g_{i\bar{j}} \chi^i \wedge \star d\phi^{\bar{j}} + \frac{1}{2} \Omega_{ij} \chi^i \wedge D\chi^j + \frac{1}{6} \Omega_{ij} R^j_{kl\bar{m}} \chi^i \wedge \chi^j \wedge \chi^k \wedge \chi^i \wedge \eta^{\bar{m}}$$

where $D\chi^i = d\chi^i + \Gamma^i_{jk} d\phi^j \wedge \chi^k$ is the pullback of the Levi-Civita connection on X and $\star$ is the 3d Hodge star.

As discussed in [36], local observables of the Rozansky-Witten model correspond to the Dolbeault cohomology classes of X.

The simplest boundary conditions [36] of the Rozansky-Witten model correspond to complex, Lagrangian submanifolds of the target space X. Additionally, one can couple the Rozansky-Witten model to a B-model defined on the 2d boundary; such boundary conditions correspond to Calabi-Yau fibrations over complex, Lagrangian submanifolds. See [36] for further details.

B.4 B-type 3d topological gauge theory

We describe the B-type 3d gauge theory which was first constructed by Blau and Thompson [29]. The bosonic field content consists of a connection A on a principal G-bundle P over 3-manifold W, as well as 1-form

$$\varphi \in \Omega^1(W, ad\, P)$$

and auxiliary field

$$P \in \Omega^0(W, ad\, P) \ .$$

The fermionic field content consists of the forms

$$\begin{aligned} \rho,\, \widetilde{\rho} &\in \Omega^0(W, ad\, P) \\ \lambda &\in \Omega^1(W, ad\, P) \\ \zeta &\in \Omega^2(W, ad\, P)\ . \end{aligned}$$

The BRST variations are given by

$$\begin{aligned} \delta_Q A &= \lambda, \\ \delta_Q \varphi &= i\lambda, \\ \delta_Q \lambda &= 0, \\ \delta_Q \zeta &= -i\mathcal{F}, \\ \delta_Q \rho &= 0, \\ \delta_Q \widetilde{\rho} &= iP, \\ \delta_Q P &= 0 \end{aligned} \tag{B.3}$$

where $\mathcal{F}$ is the curvature of the complexified connection $\mathcal{A} = A + i\varphi$. We will denote the covariant derivative with respect to $\mathcal{A}$ by $d_{\mathcal{A}}$ and the covariant derivative with respect to $\bar{\mathcal{A}} = A - i\varphi$ by $d_{\bar{\mathcal{A}}}$, and the curvature of $\bar{\mathcal{A}}$ by $\bar{\mathcal{F}}$. The differential operator $d^\star_{\mathcal{A}}$ is $\star d_{\mathcal{A}} \star$, where $\star$ is the 3d Hodge star.

The action for this theory is BRST-exact up to a metric independent term:

$$\widetilde{S} = -\frac{1}{2e^2}\delta_Q \int_W \mathrm{Tr}\Big(i\zeta \wedge \star\bar{\mathcal{F}} + i\widetilde{\rho} \wedge \star\big(P - 2d^\star_{\mathcal{A}}\varphi\big)\Big) + \int_W \mathrm{Tr}\big(\rho \wedge d_{\mathcal{A}}\zeta\big)\ . \tag{B.4}$$

This action possesses a ghost number symmetry with respect to which ρ and λ have charge 1 and ζ and $\widetilde{\rho}$ have charge -1. The equation of motion for P reads $P = d^\star_{\mathcal{A}}\varphi$; if we substitute this value into the BRST transformations, the BRST operator is only nilpotent modulo the fermionic equations of motion.

It is more natural to regard the fermionic 0-form ρ as taking values in $\mathfrak{g}^*$ rather than $\mathfrak{g}$, because then the non-BRST-exact piece in the action takes the form

$$\int_W \rho_a \wedge d_{\mathcal{A}}\zeta^a,$$

which is manifestly independent of the choice of metric on $\mathfrak{g}$.[1]

[1] For a simple Lie algebra $\mathfrak{g}$ there is a canonical identification of $\mathfrak{g}$ and $\mathfrak{g}^*$ by means of the Killing form, but if $\mathfrak{g}$ has an abelian subalgebra, the metric is not uniquely determined.

Observables of the B-type 3d gauge theory

Local observables in this topological gauge theory are gauge invariant functions of ρ, which correspond to elements in the exterior algebra $\Lambda^\bullet(\mathfrak{g})$ invariant with respect to the Ad action. Unlike in the 2d case, there are no disorder local operators.

The simplest line operators can be constructed as Wilson lines for the BRST-invariant connection $\mathcal{A} = A + i\varphi$. Such operators are labeled by finite-dimensional representations of G. More generally, we may consider coupling a 1d TFT living on the line to the 3d gauge theory. The space of states for this 1d TFT is a $\mathbb{Z}$-graded vector space V. Endomorphisms of V are naturally graded as well. Let us denote the degree-1 endomorphism that generates the BRST symmetry in this theory as $T(\Phi) \in \mathrm{End}(V)$ and the degree-0 endomorphisms that generate the gauge symmetry as $R_a(\Phi) \in \mathrm{End}(V)$, where Φ represents the fields in the topological gauge theory. Since ρ is the only BRST-invariant 0-form in the 3d gauge theory, it is sufficient to assume that T is a function of ρ alone. It also must be nilpotent:

$$T(\Phi) = T(\rho), \tag{B.5}$$

$$T^2 = 0. \tag{B.6}$$

Since the gauge symmetry preserves the grading of V, the generators of the gauge symmetry R_a may be assumed to be ρ-independent:

$$R_a(\Phi) = R_a. \tag{B.7}$$

Since the gauge symmetry δ_g and BRST symmetry δ_Q commute in the 3d gauge theory, T and R_a must satisfy the following relation:

$$\begin{aligned} 0 &= \left[\delta_g(f), \delta\right] \\ &= \left[f, \rho\right]^a \frac{\partial T}{\partial \rho^a} + \left[f^a R_a, T\right] \end{aligned} \tag{B.8}$$

where $f \in \mathfrak{g}$. To construct the line observable associated to the triple (V, T, R), we apply the descent procedure to T to get a connection 1-form on the graded vector bundle with fiber V. By definition, the descendant connection $\mathcal{N}$ is defined by the equation

$$\delta\mathcal{N} = dT + [\mathcal{N}, T].$$

Using the relation (B.8) and the fermionic equations of motion, we find

$$\mathcal{N} = \frac{i}{2e^2} \star \bar{\mathcal{F}}^a \frac{\partial T}{\partial \rho^a} + \mathcal{A}^a R_a. \tag{B.9}$$

The supertrace of the holonomy of $\mathcal{N}$ along a curve γ in W is therefore a BRST-invariant, gauge-invariant loop operator in the topological gauge theory. The holonomy itself defines a line operator.

Braided monoidal category of line operators

Line operators in any 3d TFT form a braided monoidal category. The subcategory formed by line operators described above is the G-equivariant derived category of DG-modules over the DG-algebra $\Lambda^\bullet(\mathfrak{g})$ (with zero differential). To see this, consider a local operator inserted at the junction of two Wilson lines corresponding to the triples (V_1, T_1, R_1) and (V_2, T_2, R_2). Since we are looking for BRST-invariant operators, one may assume that it is a function $\mathcal{O}$ of ρ valued in $\mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$, or in other words an element of $\mathrm{Hom}_{\mathbb{C}}(V_1, V_2 \otimes \Lambda^\bullet(\mathfrak{g}))$. The BRST-operator acts on $\mathcal{O}$ by

$$\delta_Q \mathcal{O} = T_2 \mathcal{O} \pm \mathcal{O} T_1$$

where the sign is plus or minus depending on whether the total degree of $\mathcal{O}$ is odd or even. Gauge transformations act on $\mathcal{O}$ in the obvious way and commute with the BRST operator. The space of morphisms between the line operators is the cohomology of δ_Q on the G-invariant part of $\mathrm{Hom}_{\mathbb{C}}(V_1, V_2 \otimes \Lambda^\bullet(\mathfrak{g}))$.

The monoidal structure is obvious on the classical level and given by the tensor product. There can be no quantum corrections to this result since the gauge coupling e^2 is an irrelevant parameter. The braiding is trivial for the same reason.

There exist yet more general line operators. To see this, we may use the dimensional reduction trick and identify the category of line operators in the 3d theory with the category of branes in the 2d theory obtained by compactifying the 3d theory on a circle. One can show that reduction gives a B-model with target $G_{\mathbb{C}}$ coupled to a B-type gauge theory with gauge group G. From the 3d viewpoint, $G_{\mathbb{C}}$ parameterizes the holonomy of the connection $\mathcal{A}$ along the compactification circle. The gauge group G acts on $G_{\mathbb{C}}$ by conjugation. As explained in A.3, the category of branes for this TFT is the equivariant derived category of coherent sheaves $D^b_{G_{\mathbb{C}}}(Coh(G_{\mathbb{C}}))$. Line operators considered above correspond to coherent sheaves supported at the identity element of $G_{\mathbb{C}}$. Physically, this follows from the fact that the gauge field $\mathcal{A}$ is nonsingular for such line operators, and therefore the holonomy along the circle linking the line operator must be trivial. From the mathematical viewpoint, we may note that the equivariant derived category of the DG-algebra $\Lambda^\bullet(\mathfrak{g})$ is equivalent, by Koszul duality, to a full subcategory of the equivariant derived category of the DG-algebra $\mathrm{Sym}^\bullet(\mathfrak{g}^*)$ (regarded as sitting in degree zero) consisting of finite-dimensional DG-modules. The latter category can also be thought as the full subcategory of the equivariant derived category of $G_{\mathbb{C}}$ 'supported' at the identity element of $G_{\mathbb{C}}$. That is, focusing on line operators which are of Wilson type (i.e., are not disorder operators) is equivalent to focusing on equivariant sheaves on $G_{\mathbb{C}}$ supported at the identity element. More generally, one may also consider Gukov-Witten-type line operators for which the conjugacy class of the holonomy of $\mathcal{A}$ is fixed; such line operators can be thought of as objects of $D^b_{G_{\mathbb{C}}}(Coh(G_{\mathbb{C}}))$ supported at nontrivial conjugacy classes in $G_{\mathbb{C}}$.

B.5 Gauged Rozansky-Witten model

We gauge the Rozansky-Witten model with target X by coupling it to the B-type 3d gauge theory. Suppose that hyperkähler target space X admits a compatible G-action.

Let V_a, $a = 1, 2, \cdots, \dim G$, be the vector fields on X corresponding to the generators of the G-action. Let μ_+, μ_-, and μ_3 be the moment maps corresponding to the holomorphic symplectic form Ω, the antiholomorphic symplectic form $\bar{\Omega}$, and the Kähler form J, respectively,

$$\begin{aligned} d\mu_{+a} &= -i_{V_a}(\Omega), \\ d\mu_{-a} &= -i_{V_a}(\bar{\Omega}), \\ d\mu_{3a} &= i_{V_a}(J)\ . \end{aligned}$$

The BRST variations of the fields are

$$\begin{aligned} &\delta_Q A = \lambda, && \delta_Q \phi^i = 0, \\ &\delta_Q \varphi = i\lambda, && \delta_Q \phi^{\bar{i}} = \eta^{\bar{i}}, \\ &\delta_Q \lambda = 0, && \delta_Q \eta^{\bar{i}} = 0, \\ &\delta_Q \zeta = -i\mathcal{F}, && \delta_Q \chi^i = \mathcal{D}\phi^i, \\ &\delta_Q \rho = i\mu_+, && \\ &\delta_Q \widetilde{\rho} = iP, && \\ &\delta_Q P = 0 && \end{aligned} \tag{B.10}$$

where $\mathcal{D}\phi^i = d\phi^i + \mathcal{A}^a V_a^i$ is the gauge-covariant derivative with respect to the complexified connection.

The action of the gauged Rozansky-Witten model is given by

$$S = \int_W \left(\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4\right)$$

with

$$\mathcal{L}_1 = -\frac{1}{2e^2}\delta_Q \operatorname{Tr}\Big(i\zeta \wedge \star\bar{\mathcal{F}} + i\widetilde{\rho} \wedge \star\big(P - 2d_A^{\star}\varphi - 2e^2\mu_3\big)\Big), \tag{B.11}$$

$$\mathcal{L}_2 = \delta_Q\Big(g_{i\bar{j}}\chi^i \wedge \star\bar{\mathcal{D}}\phi^{\bar{j}}\Big), \tag{B.12}$$

$$\mathcal{L}_3 = \delta_Q\Big(\frac{i}{2}e^2 \star \rho^a\mu_{-a}\Big), \tag{B.13}$$

$$\begin{aligned}\mathcal{L}_4 = {} & \rho_a \wedge d_{\mathcal{A}}\zeta^a + i\Omega_{ij}\chi^i \wedge \zeta^a V_a^j + \frac{1}{2}\Omega_{ij}\chi^i \wedge \mathcal{D}\chi^j \\ & + \frac{1}{6}\Omega_{ij}\mathcal{R}^j_{kl\bar{m}}\chi^i \wedge \chi^k \wedge \chi^i \wedge \eta^{\bar{m}}\end{aligned} \tag{B.14}$$

where $\mathcal{D}\chi^i = d\chi^i + \Gamma^i_{jk}d\phi^j \wedge \chi^k + \mathcal{A}^a\partial_j V_a^i \wedge \chi^j + \mathcal{A}^a\Gamma^i_{jk}V_a^k \wedge \chi^j$.

Local observables in the gauged Rozansky-Witten model are BRST and gauge-invariant functions of ρ_a, ϕ^i, $\phi^{\bar{i}}$, and $\eta^{\bar{i}}$, which correspond to elements in the cohomology of $\Lambda^\bullet(\mathfrak{g})\otimes\Omega^{0,\bullet}(X)$ with respect to the following nilpotent operator [11]:

$$\delta = i\mu_{+a}T^a + (-1)^l\bar{\partial}_X$$

where T^a are elements of a basis for $\mathfrak{g}$ and l is the degree of an element of $\Lambda^\bullet(\mathfrak{g})$.

Appendix C

Associated fiber bundles and equivariant cohomology

Gauged sigma models play a significant role in this thesis: the gauged 4d A-model is discussed in Chapter 3, the gauged 3d A-model and gauged Rozansky-Witten model are discussed in Chapter 5, and the modified 2d A-model and gauged 2d B-model are discussed in Chapter 7.

Since the notion of an associated fiber bundle provides the geometric setting for a gauged sigma model, we review here the mathematics of associated fiber bundles and, in particular, we explain how equivariant cohomology classes on the target space can be used to write down topological terms in a gauged sigma model action. Theorems are mostly stated without proof.

Let P be a principal G-bundle over base manifold M with right G-action $r : P \times G \to P$, and let X be a manifold equipped with a left G-action

$$l : G \times X \to X.$$

We use a dot $\cdot$ to indicate both l and r actions. In physics applications, M is the spacetime, P is the gauge bundle over M, and X is the sigma model target space. The quintuple (M, P, G, l, X) determines a particular fiber bundle over M with typical fiber X and structure group G, called the *associated fiber bundle*. We discuss in turn two complementary points of view on what this associated fiber bundle is: the total space picture and the principal bundle picture.

C.1 The total space of an associated fiber bundle

We assume that M is Kähler of dimension $2m$, X is Kähler of dimension $2n$, and that the G-action on X preserves its metric g_X, complex structure J_X, and Kähler form Ω_X. Most of what we shall say holds true in the absence of Kähler and complex structures on M and X; these are necessary only in constructing a Kähler and complex structure on the total space. We describe the total space and projection map of the associated fiber bundle as follows.

Definition. *The* total space E *of the associated fiber bundle* (M, P, G, l, X) *is the* $(2m+2n)$ *dimensional manifold* $E = P \times_G X$*, defined as the quotient space*

$$E = P \times_G X = (P \times X)/G$$

where the right action of G *on* $P \times X$ *is given by* $(p, x) \cdot g \equiv (p \cdot g, g^{-1} \cdot x)$ *for all* $p \in P$*,* $x \in X$ *and* $g \in G$*.*

Definition. *The (surjective) projection map* $\pi_E : E \to M$ *is the map given by*

$$\pi_E[p, x] = \pi(p)$$

for all $p \in P$ *and* $x \in X$*, where* $\pi : P \to M$ *is the projection map of the underlying principal bundle.*

It will also be useful to have an explicit description of the tangent space TE:

Theorem. *The total space* TE *of the tangent bundle of* E *is simply given by the twisted product*

$$TE = TP \times_G TX$$

consisting of equivalence classes of tangent vectors

$$TE = \{ [u_p, w_x] \quad | \quad u_p \in T_p P, \quad w_x \in T_x X \},$$

where

$$[u_p, w_x] = [dr_g u_p, dl_{g^{-1}} w_x] = [u'_{p \cdot g}, w'_{g^{-1} \cdot x}]$$

The tangent vectors $u'_{p \cdot g} \in T_{p \cdot g} P$ *and* $w'_{g^{-1} \cdot x} \in T_{g^{-1} \cdot x} X$ *are the pushforwards of* u_p *and* w_x *under right-action by* g *on* P *and left action by* g^{-1} *on* X*, respectively. Moreover,* TE *is equipped with the obvious projection* $TE \to E$ *sending* $[u_p, w_x]$ *to* $[p, x]$*; it is a vector bundle over* E *with each fiber* $T_{[p,x]}E$ *a vector space of dimension* $2m + 2n$*.*

Definition. *Let* $E^{(\alpha)} \equiv \pi_E^{-1}(U^{(\alpha)})$ *and* $E_\sigma \equiv \pi_E^{-1}(\sigma)$*, where* $U^{(\alpha)} \subset M$ *is a trivializing neighborhood of* P *and* $\sigma \in U^{(\alpha)}$ *is a point in this neighborhood. That is,* $E^{(\alpha)}$ *is the portion of* E *above* $U^{(\alpha)}$ *and* E_σ *is the* fiber *above* σ*.*

Definition. *Let* $x^{(\alpha)} : E^{(\alpha)} \to X$ *be the trivializing map for the bundle* E *defined in terms of the trivializing map* $g^{(\alpha)} : P^{(\alpha)} \to G$ *for the bundle* P *as follows*

$$x^{(\alpha)}[p, x] = (g^{(\alpha)}(p))^{-1} \cdot x \ .$$

Let $x_\sigma^{(\alpha)} : E_\sigma \to X$ *be the restriction of* $x^{(\alpha)}$ *to* $E_\sigma \subset E^{(\alpha)}$*.*

The map $x^{(\alpha)}$ provides a trivialization of E over $U^{(\alpha)}$:

$$E^{(\alpha)} \simeq U^{(\alpha)} \times X,$$

via map $\simeq$: $[p,x] \longmapsto (\pi_E[p,x], x^{(\alpha)}[p,x]) = (\pi(p), (g^{(\alpha)}(p))^{-1} \cdot x)$. The restriction map

$$x^{(\alpha)}_\sigma \equiv x^{(\alpha)}|_{E_\sigma} : E_\sigma \to X$$

is a diffeomorphism; i.e., each fiber E_σ is diffeomorphic to a copy of X under diffeomorphism $x^{(\alpha)}_\sigma$. As the label (α) indicates, the particular diffeomorphism is induced by the trivialization of the underlying principal bundle P.

By analogy with the notion of vertical vectors to P, we define

Definition. *The* vertical subbundle $VE \subset TE$ *consists of tangent vectors* $[u_p, w_x] \in T_{[p,x]}E$ *that are projected to zero by the pushforward of the projection map:*

$$VE \equiv \ker\, d\pi_E \ .$$

The space VE is a vector bundle over E with typical fiber $V_{[p,x]}E$, a vector space of dimension $2n$; namely, $V_{[p,x]}E$ consists of the tangent vectors along the fiber E_σ, with $\sigma = \pi_E[p,x] = \pi(p)$:

$$V_{[p,x]}E = T_{[p,x]}(E_{\pi(p)}) \subset T_{[p,x]}E.$$

We can use the diffeomorphism $x^{(\alpha)}_\sigma$ to pull back the metric, complex structure, and Kähler form on X to a corresponding metric g^V, complex structure J^V, and symplectic 2-form Ω^V on the fibers of VE:

Definition. *The tensors*

$$g^V|_e \in S^2(V_eE), \quad J^V|_e \in V_e^*E \otimes V_eE, \quad \Omega^V|_e \in \Lambda^2(V_eE)$$

are given by pullback under $dx^{(\alpha)}_\sigma : V_eE \to T_{x^{(\alpha)}_\sigma(e)}X$*, where* $\sigma = \pi_E(e)$ *and* $e = [p,x] \in E$*. We need not include a label* (α) *on* $g^V|_e$ *precisely because* g_X *is* G*-invariant by assumption.*

So far, we have a notion of metric, (almost) complex structure, and Kähler form, acting strictly on vertical vectors. To extend the action of these structures to all directions in TE, we need to choose a connection ω on the underlying principal bundle P. Such a connection gives us a notion of vertical and horizontal projection on P, which in turn provides a notion of vertical and horizontal projection on E:

Definition. *Let* $\omega_E^V : TE \to VE$ *be the* vertical projection *given by*

$$\omega_E^V[u_p, w_x] \equiv [\omega^V u_p, w_x] \in V_{[p,x]}E$$

where $u_p \in T_pP$ *and* $w_x \in T_xX$*, and* $\omega^V : TP \to VP$ *is the vertical projection on* P *determined by the connection* ω*.*

Definition. *The* horizontal subbundle $HE \subset TE$ *is given by*

$$HE \equiv \ker \omega_E^V.$$

The horizontal projection $\omega_E^H : TE \to HE$ *is the map* $\omega_E^H \equiv \mathbf{1}_{TE} - \omega_E^V$*.*

The pushforward projection map $d\pi_E : TE \to TM$ has kernel VE; since HE is transverse to VE in TE it follows that the restriction of $d\pi_E$ to HE is an isomorphism.

Definition. *The map*

$$\pi_\omega^H \equiv d\pi_E|_{HE} : HE \to TM$$

is the horizontal isomorphism *at each point.*

Definition. *The tensors*

$$g^H|_e \in S^2(H_eE), \quad J^H|_e \in H_e^*E \otimes H_eE, \quad \Omega^H|_e \in \Lambda^2(H_eE)$$

are given by pullback of the metric, complex structure, and Kähler form on M *under* π_ω^H*.*

Definition. *Let* g_E*,* J_E*, and* Ω_E *be defined by*

$$\begin{aligned} g_E(v, v') &= g^V(v, v'), && \text{for } v, v' \in VE \\ g_E(h, h') &= g^H(h, h'), && \text{for } h, h' \in HE \\ g_E(v, h) &= 0 && \text{for } v \in VE,\ h \in HE \end{aligned}$$

and similarly for J_E *and* Ω_E*. It can be shown that the complex structure* J_E*, so defined, is integrable.*

In conclusion, the total space E is endowed with the structure of a Kähler manifold with metric g_E, complex structure J_E, and Kähler form Ω_E defined above. Sections and their covariant derivatives are the following globally defined maps on M:

Definition. *A* section $\mathbf{\Phi}$ *of the associated fiber bundle* $E = P \times_G X$ *is a map* $\mathbf{\Phi} : M \to E$ *that is a right inverse of the projection mapping* $\pi_E : E \to M$*.*

Definition. *The* covariant derivative *of a section* $\mathbf{\Phi}$ *induced by connection* ω *in the underlying principle bundle* P *is the element* $d_\omega \mathbf{\Phi} \in \Omega^1(M, \mathbf{\Phi}^* VE)$ *given by the pushforward composed with vertical projection:*

$$d_\omega \mathbf{\Phi} = \omega_E^V \circ d\mathbf{\Phi} : TM \to TE \to VE \ .$$

Note that we used an explicit choice of connection ω on P to define the complex structure J_E above. So, for instance, the condition that a given section $\mathbf{\Phi} : M \to E$ is a holomorphic map of complex manifolds will depend on the choice of ω.

C.2 Pulling back an equivariant cohomology class on X

A complementary point of view on associated fiber bundles is to regard sections and their covariant derivatives as globally defined quantities on P rather than M:

Theorem. *Given a section* $\mathbf{\Phi} : M \to E$*, there exists a unique map* $\varphi : P \to X$ *such that*

$$\mathbf{\Phi}(\pi(p)) = [p, \varphi(p)]$$

for all $p \in P$*. This map is* G*-equivariant in the sense that*

$$\varphi(p \cdot g) = g^{-1} \cdot \varphi(p)$$

for all $p \in P$ *and* $g \in G$*. Conversely, a* G*-equivariant map* φ *defines a map* $\mathbf{\Phi}$*, so we can equivalently think of sections of the associated fiber bundle as* G*-equivariant maps from* P *to* X*.*

Definition. *The* covariant derivative *of a* G*-equivariant map* $\varphi : P \to X$ *with respect to connection* ω *on* P *is the element* $d_\omega \varphi \in \Omega^1(P, \varphi^* TX)$ *given by*

$$d_\omega \varphi \equiv d\varphi \circ \omega^H : TP \to TX \ .$$

It is related to the covariant derivative $d_\omega \mathbf{\Phi}$ *defined above by*

$$d_\omega \mathbf{\Phi}(d\pi(u_p)) = [u_p, d_\omega \varphi]$$

for all $u_p \in TP$*.*

Theorem. *The covariant derivative* $d_\omega \varphi$ *is given by the formula*

$$d_\omega \varphi|_p = d\varphi|_p + \omega^a|_p \ V_a|_{\phi(p)} : T_p P \to T_{\varphi(p)} X$$

where $V_a \in TX$ *is the vector field on* X *corresponding to Lie algebra basis element* t_a *for* $a = 1, \dots, \dim G$*, and* $\omega \in \Omega^1(P, \mathfrak{g})$ *is the connection 1-form on* P*.*

Proof. We write

$$d_\omega \varphi|_p = d\varphi \circ \omega^H|_p = (d\varphi - d\varphi \circ \omega^V)|_p = (d\varphi - d\varphi \circ \xi \circ \omega)|_p$$

where $\xi|_p : \mathfrak{g} \to T_p P$ is the map corresponding to infinitesimal right G-actions. However, due to G-equivariance of φ, we have

$$(d\varphi \circ \xi)|_p : t_a \longmapsto -V_a|_{\varphi(p)}$$

where the minus sign is on account of the fact that G acts on P by the right, but X by the left. Substituting into the equation above proves the formula. □

Topological terms in the action of gauged sigma models correspond to equivariant cohomology classes of the target manifold X, as we now show. We choose to work in the Cartan model of equivariant cohomology for X. An *equivariant* k*-form* on X is a degree k element η of the complex

$$\eta \in \Omega^k_G(X) = S(\mathfrak{g}^*) \otimes \left[\Omega^\bullet(X)\right]^G|_{\text{degree } k}$$

where $S(\mathfrak{g}^*)$ is the symmetric algebra on $\mathfrak{g}^*$ and $\left[\Omega^\bullet(X)\right]^G$ is the subspace of all forms on X that are annihilated by the N Lie derivatives

$$\mathcal{L}_a \equiv \mathcal{L}_{V_a} = d_X \circ \iota_{V_a} + \iota_{V_a} \circ d_X \ .$$

This is a $\mathbb{Z}$-graded vector space, with each generator ζ^a of $S(\mathfrak{g}^*)$ formally assigned degree two and each element of $\Omega^j(X)$ assigned degree j. The most general form of η is

$$\eta = \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{i_N=0}^{\infty} \eta_{(i_1, i_2, \dots, i_N)} (\zeta^1)^{i_1} (\zeta^2)^{i_2} \dots (\zeta^N)^{i_N} \equiv \sum_I \eta_I \zeta^I$$

where the coefficients $\eta_I = \eta_{(i_1, \dots i_N)}$ are labeled by N-tuples of nonnegative integers, and

$$\eta_I \in \left[\Omega^{k-2|I|}(X)\right]^G,$$

with $|I| = i_1 + \cdots + i_N$. Given such an equivariant k-form η on X, one can 'pull back' η to an ordinary k-form on the principle bundle P using a section $\varphi : P \to X$ and a connection ω on P:

Definition. *Let* $\eta \in \Omega^k_G(X)$ *be an equivariant* k*-form as above,* $\varphi : P \to X$ *be the equivariant map determined by a section and* $\omega \in \Omega^1(P, \mathfrak{g})$ *be a connection on* P*, with curvature* $F_\omega = d\omega + \frac{1}{2}[\omega, \omega] \in$

$\Omega^2(P, \mathfrak{g})$. *The triple* (η, φ, ω) *defines an ordinary* k*-form* $\eta(\varphi, \omega) \in \Omega^k(P)$ *as follows:*

$$\eta(\varphi, \omega) = \sum_I \left[(d_\omega \varphi)^* \eta_I\right] \wedge F^I_\omega \ .$$

That is, we pull back the coefficients of η to P using the covariant derivative $d_\omega \varphi$ and replace the N generators $\zeta^1, \ldots, \zeta^N$ with the N 2-forms $F^1_\omega, \ldots, F^N_\omega$. The k-form $\eta(\varphi, \omega)$ has the following properties:

Theorem. Proposition 2.2 of [18].

1. $\eta(\varphi, \omega)$ *is invariant and horizontal; i.e.,*

$$\mathcal{L}_u \eta(\varphi, \omega) = \iota_u \eta(\varphi, \omega) = 0$$

for all vertical vectors $u \in VP = \ker d\pi$. *In the language of differential graded algebras, we say that* $\eta(\varphi, \omega)$ *is a* basic k*-form on* P.

2. *Suppose that* $\rho \in \Omega^{k+1}_G(X)$ *is the equivariant* $(k+1)$*-form produced by acting with the Cartan differential* d_C *on* η*:*

$$\rho = d_C \eta = d_X \eta - \zeta^a \iota_{V_a} \eta \ .$$

Then their respective pullbacks $\eta(\varphi, \omega)$ *and* $\rho(\varphi, \omega)$ *are related by the ordinary exterior derivative on* P*:*

$$\rho(\varphi, \omega) = d\big(\eta(\varphi, \omega)\big) \ .$$

3. *Suppose that the map* $\varphi : P \to X$ *is 'equivariantly homotopic' to the map* $\varphi' : P \to X$ *(i.e.,* φ *can be smoothly deformed into* φ' *in such a way that it remains an equivariant map at every stage of the homotopy). Moreover, suppose that the connection* ω *is homotopic to* ω' *in the space of connections. Then* $\eta(\varphi, \omega)$ *and* $\eta(\varphi', \omega')$ *differ by an exact* k*-form on* P*:*

$$\eta(\varphi', \omega') - \eta(\varphi, \omega) = d\Upsilon$$

where $\Upsilon \in \Omega^{k-1}(P)$*. Moreover,* Υ *too is basic.*

The implication of this theorem is the following: by property 1), the form $\eta(\varphi, \omega)$ descends to a k-form $\widetilde{\eta}(\varphi, \omega)$ on the base manifold. (Local descendants agree in the sense that $s^{(\alpha)*}\eta(\varphi, \omega) = s^{(\beta)*}\eta(\varphi, \omega)$ on overlaps $U^{(\alpha)} \cap U^{(\beta)}$; hence they piece together into a globally defined k-form on M.)

By property 2), if η is a d_C-closed equivariant k-form on X, then $\widetilde{\eta}(\varphi, \omega)$ is a closed k-form on M (in the ordinary sense), and if two equivariant forms on X differ by a d_C-exact amount, then their pullbacks differ by a d-exact amount on M, for a fixed pair (φ, ω). Therefore, if $S \subset M$

is a k-dimensional, oriented, boundaryless submanifold of the base, then the number produced by integrating

$$S(\eta, \varphi, \omega) \equiv \int_S \widetilde{\eta}(\varphi, \omega)$$

is independent of the representative of the equivariant cohomology class of η.

By property 3), if we allow φ and ω to vary continuously over a connected component of the space of sections and connections, then the number $S(\eta, \varphi, \omega)$ does not change; we write

$$S(\eta, \varphi, \omega) = S([\eta], [\varphi], [\omega])$$

to indicate that it depends only on the equivariant cohomology class of η and the homotopy classes of φ and ω; for this reason, we say the number is 'topological'. In case $k = m$ and $S = M$, this gives us a topological term to possibly include in the action of a gauged sigma model.

Appendix D

Proofs of propositions in Chapter 9

In this appendix we offer proofs of selected propositions discussed in the text of Chapter 9.

Proposition (1)**.** *If* (Y, F) *are such that* $G(u,u) = -G(v,v)$ *for all allowed pairs* (u,v)*, then* $v \in TY$ *and* (v,u) *is an allowed pair as well.*

Proof. First, we show that Y is necessarily a coisotropic submanifold (in the terminology of Section 9.3). This is because $v_0 \in (TY)^\perp$ implies that $(0, v_0)$ is an allowed pair. Then

$$G(v_0, v_0) = -G(0,0) = 0 \ .$$

This, in turn, implies that $(TY)^\perp$ is an isotropic subspace, since

$$G(v_0, v_0') = \frac{1}{2} G(v_0 + v_0', v_0 + v_0') = 0$$

for all $v_0, v_0' \in (TY)^\perp$. That is to say, $(TY)^\perp \subseteq (TY^\perp)^\perp = TY$, which is what it means for Y to be coisotropic.

Next, we show that if (u,v) is an allowed pair, then v is in the subspace TY. (The definition of allowed pair requires $u \in TY$ but not, a priori, $v \in TY$.) Let $v_0 \in (TY)^\perp$. For any value of a real parameter t, it is easy to see that $(u, v + tv_0)$ is also an allowed pair. Hence,

$$-G(u,u) = G(v + tv_0, v + tv_0)$$

for all t. Differentiating both sides of this equation with respect to t, we have

$$0 = \frac{d}{dt} G(v + tv_0, v + tv_0) = 2G(v, v_0)$$

(since $G(v,v)$ is t independent and $G(v_0, v_0) = 0$). The choice of $v_0 \in (TY)^\perp$ was arbitrary, so

$$v \in ((TY)^\perp)^\perp = TY \ .$$

Finally, we show that (u, v) an allowed pair implies (v, u) is an allowed pair as well. By linearity, if (u, v) and (u', v') are allowed pairs, then $(u + u', v + v')$ is also an allowed pair. By assumption,

$$G(u + u', u + u') = -G(v + v', v + v')$$

which, using $G(u, u) = -G(u', u')$ and $G(u', u') = -G(v', v')$, implies that

$$G(u, u') = -G(v, v'). \tag{D.1}$$

Let (u, v) be an allowed pair and $w \in TY$ be arbitrary. It is not hard to see that there exists a $v' \in TY$ such that (w, v') is an allowed pair. Setting $u' = w$ in (D.1) above, one finds that

$$G(u, w) = -G(v, v') = -G(v', v) = -F(w, v) = F(v, w) \ .$$

We conclude that (v, u) is also an allowed pair. □

Proposition (3)**.** *The pair* (Y, F) *define a topological bibrane if and only if* $Y \subseteq X \times \widehat{X}$ *is a coisotropic submanifold such that* $\ker F = (TY)^{\perp}$ *(i.e., the degenerate directions of* F *and* $G|_{TY}$ *coincide) and, additionally,*

$$(\widetilde{G}^{-1}\widetilde{F})^2 = +1$$

on SY*, where* $\widetilde{G} \equiv G|_{SY}$ *and* $\widetilde{F} \equiv F|_{SY}$*.*

Proof. To prove "only if," suppose that (Y, F) is a topological bibrane. Theorem 1 tells us that if $(u, 0)$ is an allowed pair then $(0, u)$ is an allowed pair as well, and vice versa; this implies that $\ker F = (TY)^{\perp}$. Hence, the degenerate directions of F and $G|_{TY}$ coincide and $\widetilde{F}$ and $\widetilde{G}$ (the restrictions of G and F to the screen distribution SY) are both nondegenerate.

Moreover, if $\widetilde{u}, \widetilde{v} \in SY$, then by definition, $(\widetilde{u}, \widetilde{v})$ is an allowed pair if and only if $\widetilde{v} = (\widetilde{G}^{-1}\widetilde{F})\widetilde{u}$. Since $(u, (\widetilde{G}^{-1}\widetilde{F})\widetilde{u})$ is an allowed pair, $((\widetilde{G}^{-1}\widetilde{F})\widetilde{u}, u)$ is an allowed pair as well (again, by theorem 1). This means that $\widetilde{u} = (\widetilde{G}^{-1}\widetilde{F})^2\widetilde{u}$ and since $\widetilde{u} \in SY$ was arbitrary, we have

$$(\widetilde{G}^{-1}\widetilde{F})^2 = 1 \ .$$

To prove 'if', let (u, v) be an allowed pair; we wish to show that $v \in TY$ and (v, u) is an allowed pair as well.

Let $v_0 \in (TY)^{\perp}$ be arbitrary. We then have

$$G(v, v_0) = F(v, v_0) = 0$$

where the first equality is by the definition of allowed pair and the second equality is by the fact that $\ker F = (TY)^{\perp}$. Hence $v \in ((TY)^{\perp})^{\perp} = TY$.

Now, we write $u = u_0 + \widetilde{u}$ and $v = v_0 + \widetilde{v}$ for the decomposition of TY vectors with respect to the splitting $TY = (TY)^{\perp} \oplus SY$. Since $F(u, w) = G(v, w)$ for all $w \in TY$, we have $F(\widetilde{u}, \widetilde{w}) = G(\widetilde{v}, \widetilde{w})$ for all $\widetilde{w} \in SY$, or $\widetilde{v} = (\widetilde{G}^{-1}\widetilde{F})\widetilde{u}$. But $(\widetilde{G}^{-1}\widetilde{F})^2 = 1$, so $\widetilde{u} = (\widetilde{G}^{-1}\widetilde{F})\widetilde{v}$. Hence, $F(\widetilde{v}, \widetilde{w}) = G(\widetilde{u}, \widetilde{w})$ for all $\widetilde{w} \in SY$, and finally $F(v, w) = G(u, w)$ for all $w \in TY$. □

Proposition (6). *The subbundle $\mathcal{R}_+$, as defined above, is Lagrangian with respect to e^*TM. In particular, it is n-dimensional.*

Proof. It is convenient to first refine the description of the tangent bundles of topological bibranes given in proposition 3. Let $\widetilde{R}$ be the endomorphism $\widetilde{G}^{-1}\widetilde{F} : SY \to SY$, which by proposition 3, squares to the identity. Hence the screen distribution admits a splitting $SY = \widetilde{\mathcal{R}_+} \oplus \widetilde{\mathcal{R}_-}$ into the +1 and -1 eigenbundles of $\widetilde{R}$. Indeed, since $\widetilde{G}^{-1}\widetilde{F}$ is the product of a symmetric and an antisymmetric matrix, it is traceless, and in particular, exactly half of its eigenvalues are +1 and half are -1. Hence,

$$\dim \widetilde{\mathcal{R}_+} = \dim \widetilde{\mathcal{R}_-} = \frac{1}{2} \dim SY = \frac{1}{2}(2k - 2n) = k - n$$

where $\dim M = 2n$ and $\dim Y = k$ (recall that since Y is coisotropic, $k \geq n$). From the definition of the bundle $\mathcal{R}_+$, it follows straightforwardly that

$$\mathcal{R}_+ = (TY)^{\perp} \oplus \widetilde{\mathcal{R}_+} \ .$$

Hence, its dimension is

$$\dim \mathcal{R}_+ = \dim (TY)^{\perp} + \dim \widetilde{\mathcal{R}_+} = (2n - k) + (k - n) = n \ .$$

Moreover, if $s, t \in \mathcal{R}_+$, then

$$G(s, t) = F(s, t) = -F(t, s) = -G(t, s) = -G(s, t) = 0$$

so $\mathcal{R}_+$ is an isotropic subbundle of e^*TM. (In the first and third equalities above, we have used the definition of $\mathcal{R}_+$ and in the second and fourth equalities we have used the symmetries of F and G.) Since it is middle dimensional and isotropic, it follows that $\mathcal{R}_+$ is Lagrangian with respect to e^*TM. □

Proposition (7). *Let (M, G) be a (pseudo-)Riemannian manifold with Levi-Civita connection ∇. Let $Y \subseteq M$ be a submanifold equipped with closed 2-form F on its worldvolume, and let $\mathcal{R}_+$ be the* *subbundle defined in* (9.2)*. If* $\mathcal{R}_+$ *is integrable, then we have the following:*

$$(e^*\nabla)_u s \in \Gamma(\mathcal{R}_+)$$

for all $u \in \Gamma(TY)$ *and* $s \in \Gamma(\mathcal{R}_+)$*. Here* $e^*\nabla$ *is the pullback of* ∇ *under the embedding map* $e : Y \to M$*. In words, the subbundle* $\mathcal{R}_+$ *is* ∇*-parallel with respect to* TY*.*

Proof. We will use the involutivity of $\mathcal{R}_+$ (which follows from integrability) to establish the formula

$$2G((e^*\nabla)_u s, t) = dF(u, s, t) \tag{D.2}$$

for all $u \in \Gamma(TY)$ and $s, t \in \Gamma(\mathcal{R}_+)$, keeping in mind that we regard sections of TY and $\mathcal{R}_+$ as sections of the pullback bundle e^*TM, equipped with pullback connection $e^*\nabla$. Since $dF = 0$ by assumption, this formula will imply that

$$(e^*\nabla)_u s \in \Gamma({\mathcal{R}_+}^{\perp}) = \Gamma(\mathcal{R}_+)\ .$$

In the space-filling case, we have noted that our manifold is (half) para-Kähler, and this formula reduces to an equation appearing in Theorem 1 in [48]; the proof will follow in a similar vein.

To show (D.2), we first write down the Kosczul formula

$$2G((e^*\nabla)_u s, t) = u(G(s,t)) + s(G(u,t)) - t(G(u,s)) + G([u,s],t) - G([u,t],s) - G([s,t],u)\ .$$

Since $s, t \in \Gamma(\mathcal{R}_+)$. we can replace G by $-F$ in all of the terms on the right-hand side above except for the last one. However, as $\mathcal{R}_+$ is involutive, $[s,t] \in \mathcal{R}_+$, so that we can replace G by $+F$ in the last term:

$$2G((e^*\nabla)_u s, t) = -u(F(s,t)) - s(F(u,t)) + t(F(u,s)) - F([u,s],t) + F([u,t],s) - F([s,t],u)\ .$$

On the right-hand side, we have the right form for the exterior derivative of F, except for the sign of the first term $-u(F(s,t))$. However, note that $\mathcal{R}_+$ is an isotropic subbundle with respect to both G and F, so this term is zero and we can flip its sign. The equation (D.2) is proved. □

Proposition (8)**.** *The term* $G_{IJ}\psi_+^I \delta^{\text{cov}} \psi_+^J$ *vanishes for all allowed variations.*

Proof. We write the fermionic boundary conditions as

$$R(\phi)^I{}_J \psi_+^J = \psi_+^I$$

where $R : e^*TM \to e^*TM$ is a ϕ-dependent matrix squaring to the identity. Varying the above and writing in terms of the covariant variations, one has

$$(\nabla_K R^I{}_J)\delta\phi^K \psi_+^J + R^I{}_J \delta^{\rm cov}\psi_+^J = \delta^{\rm cov}\psi_+^I$$

or, since $\delta\phi$ is tangent for allowed variations and ψ_+ takes values in $\mathcal{R}_+$,

$$(\nabla_K R^I{}_J)\pi^K{}_L P^J{}_M \delta\phi^L \psi_+^M + R^I{}_J \delta^{\rm cov}\psi_+^J = \delta^{\rm cov}\psi_+^I$$

where π is a projector onto TY and $P = \frac{1}{2}(1+R)$ is the projector onto $\mathcal{R}_+$. However, by proposition 7, the first term vanishes. We then have

$$G_{IJ}\psi_+^I \delta^{\rm cov}\psi_+^J = (G_{IJ}P^I{}_K P^J{}_L)\psi_+^K \delta^{\rm cov}\psi_+^L = 0$$

since $\mathcal{R}_+$ is an isotropic subbundle. □

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