Vanishing Integrals for Hall–Littlewood Polynomials

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Abstract

It is well-known that if one integrates a Schur function indexed by a partition λ over the symplectic (resp. orthogonal) group, the integral vanishes unless all parts of λ have even multiplicity (resp. all parts of λ are even). In a recent work of Rains and Vazirani, Macdonald polynomial generalizations of these identities and several others were developed and proved using Hecke algebra techniques. However at $q = 0$ (the Hall–Littlewood level), these approaches do not directly work; this obstruction was the motivation for this thesis. We investigate three related projects in chapters 2–4 (the first chapter consists of an introduction to the thesis). In the second chapter, we develop a combinatorial technique for proving the results of Rains and Vazirani at $q = 0$. This approach allows us to generalize some of those results in interesting ways and leads us to a finite-dimensional analog of a recent result of Warnaar, involving the Rogers–Szegő polynomials. In the third chapter, we provide a new construction for Koornwinder polynomials at $q = 0$, allowing these polynomials to be viewed as Hall–Littlewood polynomials of type BC. This is a first step in building the analogy between the Macdonald and Koornwinder families at the $q = 0$ limit. We use this construction in conjunction with the combinatorial technique of the previous chapter to prove some vanishing results of Rains and Vazirani for Koornwinder polynomials at $q = 0$. In the fourth chapter, we provide an interpretation for vanishing results for Hall–Littlewood polynomials using p-adic representation theory; it is an analog of the Schur case. This p-adic approach allows us to generalize our original vanishing results. In particular, we exhibit a t-analog of a classical vanishing result for Schur functions due to Littlewood and Weyl; our vanishing condition is in terms of Hall polynomials and Littlewood-Richardson coefficients.

Contents

Chapter 1

Introduction

1.1 Branching Rules for Classical Groups and Generalizations

Macdonald polynomials were first introduced by I. G. Macdonald in the late 1980s (see [14]) and continue to make important appearances in a variety of fields such as algebraic geometry, physics, representation theory, and combinatorics. They provide an example of a family of *symmetric functions*, that is, they are invariant under all permutations of the variables. For example, note that

$$f(x_1, x_2, x_3) = x_1 + x_3 + x_1^2 x_2 + x_2^2 x_1$$

is not a symmetric function since $f(x_1, x_2, x_3) \neq f(x_2, x_1, x_3)$, but

$$g(x_1, x_2, x_3) = x_1 + x_2 + x_3 + x_1^2 x_2 + x_2^2 x_1 + x_1^2 x_3 + x_3^2 x_1 + x_2^2 x_3 + x_3^2 x_2$$

is indeed a symmetric function. Macdonald polynomials are indexed by partitions (a decreasing string of nonnegative integers, only finitely many of which are nonzero), and have as arguments two parameters q, in addition to variables $x_1, \ldots, x_n$; they are denoted $P_\lambda(x_1, \ldots, x_n; q, t)$. These polynomials contain many other important and well-studied families of symmetric functions as limiting cases of the parameters. For example, at $q = t$, one obtains the Schur functions, and at $q = 0$ the Hall–Littlewood polynomials. Macdonald polynomials, and their degenerations, are important examples of *orthogonal polynomials*. In other words, these symmetric polynomials are uniquely determined by the following two requirements:

(i) $P_\lambda(x; q, t) = x^\lambda +$ lower-order terms,

(ii) $\langle P_\lambda, P_\mu \rangle_{q,t} = 0$ if $\lambda \neq \mu$, where the inner product in (ii) is with respect to a certain density on the n-torus. We refer the interested reader to [16] for an excellent introduction to these polynomials, and the theory of symmetric functions. We will now explain some connections between symmetric functions and representation theory that served as the motivation for this thesis.

A crucial problem in representation theory can be described in the following way: let G and H be complex algebraic groups, with an embedding $H \hookrightarrow G$. Also let V be a completely reducible representation of G, and W an irreducible representation of H. What information can one obtain about $[V, W] := \dim \mathrm{Hom}_H(W, V)$, the multiplicity of W in V? Here V is viewed as a representation of H by restriction. Such *branching rules*, as they are called in the literature, have important connections to physics as well as other areas of mathematics. There are often beautiful combinatorial objects describing these multiplicities. One prototypical example is that of the symmetric groups $G = S_n$ and $H = S_{n-1}$: the resulting rule has a particularly nice description in terms of Young tableaux, an important object in combinatorics.

The two motivating examples for this thesis are the restrictions of Gl_{2n} to Sp_n (the compact symplectic group) and Gl_{2n} to O_{2n} (the orthogonal group); the combinatorics of these branching rules was first developed by D. Littlewood and continue to be a well-studied and active area at the forefront of algebraic combinatorics and invariant theory. These pairs are also important because they are examples of *symmetric spaces*. That is, G is a reductive algebraic group and H is the fixed point set of an involution on G; $S = G/H$ is the resulting symmetric space. The multiplicities in these branching rules are given in terms of *Littlewood-Richardson coefficients*, another important entity described in terms of tableaux and and lattice permutations. In fact, since Schur functions are characters of irreducible polynomial representations of Gl_{2n}, one may rephrase these rules in terms of Schur functions and symplectic characters (respectively, orthogonal characters). A classical restriction rule of this flavor is the following: if one decomposes a Schur function $s_\lambda(x_1^{\pm 1}, \ldots, x_n^{\pm 1})$ in terms of symplectic characters, the coefficient on the trivial character is zero unless the indexing partition λ has all parts occurring with even multiplicity (there is a similar statement for the orthogonal group). That is,

Theorem 1.1. *[16] For any even integer $n \geq 0$, we have*

$$\int_{S \in Sp(n)} s_\lambda(S) dS = \begin{cases} 1, & \textit{if all parts of } \lambda \textit{ have even multiplicity,} \\ 0, & \textit{otherwise} \end{cases}$$

(where the integral is with respect to Haar measure on the symplectic group).

Theorem 1.2. *[16] For any integer $n \geq 0$ and partition λ with at most n parts, we have*

$$\int_{O\in O(n)} s_\lambda(O)dO = \begin{cases} 1, & \text{if all parts of } \lambda \text{ are even,} \\ 0, & \text{otherwise} \end{cases}$$

(where the integral is with respect to Haar measure on the orthogonal group).

Proofs of these identities may be found in [16]; they involve structure results for the Gelfand pairs $(GL_n(\mathbb{H}), U(n,\mathbb{H}))$ and $(G,K) = (GL_n(\mathbb{R}), O(n))$. Note that using the eigenvalue densities for the orthogonal and symplectic groups, we may rephrase the above identities in terms of random matrix averages. For example, the left-hand side of the symplectic integral above can be rephrased as

$$\frac{1}{2^n n!}\int_T s_\lambda(z_1, z_1^{-1}, z_2, z_2^{-1}, \dots, z_n, z_n^{-1}) \prod_{1\leq i\leq n} |z_i - z_i^{-1}|^2 \prod_{1\leq i<j\leq n} |z_i + z_i^{-1} - z_j - z_j^{-1}|^2 dT,$$

where

$$T = \{(z_1, \dots, z_n) : |z_1| = \dots = |z_n| = 1\},$$
$$dT = \prod_j \frac{dz_j}{2\pi\sqrt{-1}z_j}$$

are the n-torus and Haar measure, respectively.

In 2005, Rains [18] conjectured the existence of (q,t)-analogs of such restriction rules for Schur functions. That is, he conjectured choices of densities such that when one integrates a (suitably specialized) Macdonald polynomial against it over the n-torus, the result vanishes unless the indexing partition satisfies some explicit condition. Moreover, the values of the integral when the condition is satisfied are "nice," and at $q = t$ one recovers a Schur identity akin to those discussed in the previous paragraph. In 2007, Rains and Vazirani [20] developed affine Hecke algebra techniques that allowed them to prove almost all of these results. For example, in the symplectic case, their result is the following:

Theorem 1.3. *[20] For any integer $n \geq 0$, and partition λ with at most $2n$ parts, and any complex numbers q, t with $|q|, |t| < 1$, the integral*

$$\int P_\lambda(\dots, z_i^{\pm 1}, \dots; q, t) \prod_{1\leq i\leq n} \frac{(z_i^{\pm 1}; q)}{(tz_i^{\pm 1}; q)} \prod_{1\leq i<j\leq n} \frac{(z_i^{\pm 1} z_j^{\pm 1}; q)}{(tz_i^{\pm 1} z_j^{\pm 1}; q)} dT$$

vanishes unless $\lambda = \mu^2$ for some μ.

To prove these results, they studied nonsymmetric versions of these integrals and showed (1) that these are annihilated by a particular ideal of the affine Hecke algebra, and (2) that the partition corresponding to any such annihilated functional satisfies the appropriate vanishing condition. However, many of the relevant difference operators do not behave well under the specialization $q = 0$, so this method does not directly work in that case. This thesis originated from this particular obstruction and tries to make a systematic study of such vanishing results for Hall–Littlewood polynomials.

1.2 A Combinatorial Technique

In the first part of this thesis, we develop a direct approach for proving the identities of [20] at the Hall–Littlewood level; an important by-product of this approach is that it allows us to generalize some of those results in different ways. Our method is combinatorial in nature, and relies heavily on the structure of the Hall–Littlewood polynomial as a sum over the Weyl group:

Theorem 1.4. *[16] The Macdonald polynomial $P_\lambda(x_1, \dots, x_n; q, t)$ at $q = 0$ is given by*

$$\frac{1}{v_\lambda(t)} \sum_{w \in S_n} w\Big(x^\lambda \prod_{1 \le i < j \le n} \frac{x_i - tx_j}{x_i - x_j}\Big),$$

where we write x^λ for $x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ and w acts on the subscripts of the x_i. The normalization $1/v_\lambda(t)$ has the effect of making the coefficient of x^λ equal to unity.

In particular to prove the symplectic group result, we integrate each term directly and use induction, noting that there are only simple poles at zero. In fact, this argument shows that each individual term vanishes unless λ has all parts occurring with even multiplicity. One can then combine terms and use t-combinatorics to obtain the result. We state the theorem in the symplectic case:

Theorem 1.5. *We have the following identity (see [23] and theorem 4.1 of [20])*

$$\frac{1}{Z} \int P_\lambda(x_1^{\pm 1}, \dots, x_n^{\pm 1}; t) \tilde{\Delta}_K^{(n)}(\pm\sqrt{t}, 0, 0) dT = \frac{\phi_n(t^2)}{(1-t^2)^n v_\mu(t^2)},$$

when $\lambda = \mu^2$ for some μ (i.e., all parts of λ occur with even multiplicity) and 0 otherwise. Here Z is a normalization which makes the integral 1 when $\lambda = 0$.

The constants $\phi_n(t^2)$ and $v_\mu(t^2)$ are as defined in [16], and $\tilde{\Delta}_K$ is the $q = 0$ symmetric Koornwinder density [13]. Similar arguments can be used to give a new proof of the well-known fact that the Hall–Littlewood polynomials form an orthogonal basis with respect to the standard symmetric density, $\tilde{\Delta}_S^{(n)}(x; t)$. The orthogonal group cases are more complicated due to the existence of poles on the torus, so one needs some extra technical arguments, although the basic idea remains the same.

As mentioned above, there are several interesting features of this method. The first is that, in the four orthogonal group cases (one for each component, parity), we are able to introduce an extra parameter α and obtain a nice evaluation that becomes the original vanishing result at $\alpha = 0$. In these cases, the evaluations are in terms of Pfaffians of suitable matrices; moreover, the terms in the expansion of the Pfaffians are exactly the individual term integrals described above. For example, in the $O^+(2n)$ case, we first show the following:

Proposition 1.6.

$$\int_T R_\lambda(x_1^{\pm 1}, \dots, x_n^{\pm 1}; t)\tilde{\Delta}_K^{(n)}(\pm 1, \pm\sqrt{t})\prod_{i=1}^{n}(1-\alpha x_i^{\pm 1})dT = \frac{1}{2^n(1-t)^n}\mathrm{Pf}[a_{j,k}]^\lambda,$$

where the $2n \times 2n$ antisymmetric matrix $[a_{j,k}]^\lambda$ is defined by

$$a_{j,k}^\lambda = (1+\alpha^2)\chi_{(\lambda_j - j)-(\lambda_k - k)\ odd} + 2(-\alpha)\chi_{(\lambda_j - j)-(\lambda_k - k)\ even},$$

for $1 \le j < k \le 2n$.

In fact, this formula is a t-analog of a result obtained by Forrester and Rains [7] when studying the Hammersley process. Their evaluation of the above Pfaffian (note the t-independence) enables us to prove the following result:

Theorem 1.7. *[23] Let λ be a partition satisfying $l(\lambda) \le 2n$. Then*

$$\frac{1}{\int \tilde{\Delta}_K^{(n)}(\pm 1, \pm\sqrt{t})dT}\int_T P_\lambda^{(2n)}(x_i^{\pm 1}; t)\tilde{\Delta}_K^{(n)}(\pm 1, \pm\sqrt{t})\prod_{i=1}^{n}(1-\alpha x_i^{\pm 1})dT$$
$$= \frac{\phi_{2n}(t)}{v_\lambda(t)(1-t)^{2n}}\Big[(-\alpha)^{\#odd\ parts\ of\ \lambda} + (-\alpha)^{\#even\ parts\ of\ \lambda}\Big].$$

We mention that there are analogous results for the $O^-(2n), O^+(2n+1), O^-(2n+1)$ cases as well, and the $t = 0$ versions and related Pfaffians were studied in [7]; one can find the details in the first section of the thesis.

Another nice consequence of our technique involves a recent identity discovered by Warnaar for Hall–Littlewood polynomials [24]. He uses the Rogers–Szegő polynomials (denoted $H_i(x;t)$) to unify some Littlewood summation identities for Hall–Littlewood functions:

Theorem 1.8. *[24]*

$$\sum_\lambda P_\lambda(x;t)\Big(\prod_{i>0\ even} H_{m_i(\lambda)}(\alpha\beta; t)\prod_{i>0\ odd} H_{m_i(\lambda)}(\beta/\alpha; t)\Big)(-\alpha)^{\#odd(\lambda)}$$
$$= \prod_{j<k}\frac{(1-tx_jx_k)}{(1-x_jx_k)}\prod_j\frac{(1-\alpha x_j)(1-\beta x_j)}{(1-x_j)(1+x_j)}.$$

We find a two-parameter integral identity and, using a method of Rains [18], we show that in the limit $n \to \infty$ it becomes Warnaar's identity. Thus, the following identity may be viewed as a finite-dimensional analog of Warnaar's summation result:

Theorem 1.9. *[23] Let λ be a partition satisfying $l(\lambda) \leq 2n$. Then*

$$\frac{1}{Z}\int_T P_\lambda(x_1^{\pm 1},\ldots,x_n^{\pm 1};t)\tilde{\Delta}_K^{(n)}(\pm 1,\pm\sqrt{t})\prod_{i=1}^{n}(1-\alpha x_i^{\pm 1})(1-\beta x_i^{\pm 1})dT$$
$$= \frac{\phi_{2n}(t)}{v_\lambda(t)(1-t)^{2n}}\Bigg[\Big(\prod_{i\geq 0}H_{m_{2i}(\lambda)}(\alpha\beta;t)\prod_{i\geq 0}H_{m_{2i+1}(\lambda)}(\beta/\alpha;t)\Big)(-\alpha)^{\#odd(\lambda)}$$
$$+\Big(\prod_{i\geq 0}H_{m_{2i+1}(\lambda)}(\alpha\beta;t)\prod_{i\geq 0}H_{m_{2i}(\lambda)}(\beta/\alpha;t)\Big)(-\alpha)^{\#even(\lambda)}\Bigg],$$

where $Z = \int \tilde{\Delta}_K^{(n)}(\pm 1,\pm\sqrt{t})dT$.

Just as is the case with Warnaar's identity, special choices of the parameters α and β recover interesting integral identities. Finally, there are some identities which are not amenable to the Hecke algebra approach of [20] or where the values of the integral (when it does not vanish) are intractable; our method proves to be fruitful in those cases.

1.3 Koornwinder Polynomials

Motivated by the earlier work of Macdonald, in the 1990s T. Koornwinder introduced the so-called Macdonald-Koornwinder (or Koornwinder) polynomials. These (Laurent) polynomials have four additional parameters other than (q,t), and satisfy a slightly different type of symmetry than the Macdonald polynomials: they are invariant under permutations of variables, as well as taking inverses. It was later shown by van Diejen that Macdonald polynomials can be obtained from Koornwinder polynomials via suitable limits of the parameters (see [6]). Just as in the Macdonald polynomial case, standard constructions via difference operators do not allow one to control the polynomials when $q = 0$ (we note that the above construction in the Macdonald case is due to Hall and Littlewood, independently).

The second part of this thesis deals with an explicit construction for these polynomials. In particular, we prove the following result:

Theorem 1.10. *[22] Let λ be a partition with $l(\lambda) \leq n$ and $|t|,|t_0|,\ldots,|t_3| < 1$. Then the Koornwinder $q = 0$ polynomial $K_\lambda(z_1,\ldots,z_n;t;t_0,\ldots,t_3)$ indexed by λ is given by*

$$\frac{1}{v_\lambda(t;t_0,\ldots,t_3)}\sum_{\omega\in B_n}\omega\Bigg(\prod_{1\leq i\leq n}u_{\lambda_i}(z_i)\prod_{1\leq i<j\leq n}\frac{1-tz_i^{-1}z_j}{1-z_i^{-1}z_j}\frac{1-tz_i^{-1}z_j^{-1}}{1-z_i^{-1}z_j^{-1}}\Bigg),$$

where

$$u_{\lambda_i}(z_i) = \begin{cases} 1 & \text{if } \lambda_i = 0, \\ z_i^{\lambda_i} \frac{(1-t_0 z_i^{-1})(1-t_1 z_i^{-1})(1-t_2 z_i^{-1})(1-t_3 z_i^{-1})}{1-z_i^{-2}} & \text{if } \lambda_i > 0. \end{cases}$$

One immediately notes the structural similarity to the Hall–Littlewood polynomials, namely, as a sum over the associated Weyl group (the symmetric group in the Hall–Littlewood case, the hyperoctahedral group in the Koornwinder case). We show that these polynomials satisfy the defining properties of Koornwinder polynomials, namely that they are BC_n-symmetric Laurent polynomials that are triangular with respect to dominance order and orthogonal with respect to the Koornwinder density $\tilde{\Delta}_K^{(n)}(x; t; t_0, \ldots, t_3)$. To prove orthogonality, we use an adaptation of the methods used in [23] to the type BC case. We note that this construction is a first step in understanding Hall–Littlewood polynomials in the type BC case. We mention that when two of the parameters are equal to zero, our family becomes Macdonald's (BC_n, B_n) two-parameter family at $q = 0$. Finally, with this construction of the Koornwinder $q = 0$ polynomials, we may prove the results of [20] involving Koornwinder polynomials using a direct approach analogous to that of [23].

1.4 Connection to p-adic Representation Theory

The last part of the thesis deals with an interpretation of the results of [23] in terms of p-adic representation theory. The motivation for this connection stems from [15], [16, chapter 5], which we briefly discuss. Let $G = Gl_n(\mathbb{Q}_p)$, and let $K = Gl_n(\mathbb{Z}_p)$ be its maximal compact subgroup. Then G/K is the affine Grassmannian and the spherical Hecke algebra $\mathcal{H}(G, K)$ is the convolution algebra of compactly supported, K-bi-invariant, complex valued functions on G; it has a basis given by $\{c_\lambda\}_{l(\lambda) \leq n}$, where c_λ is the characteristic function of the double coset $Kp^\lambda K$ and $p^\lambda = diag.(p^{\lambda_1}, \ldots, p^{\lambda_n})$. Macdonald provides a Plancherel theorem in this context, where the zonal spherical functions are given in terms of Hall–Littlewood polynomials with $t = p^{-1}$. One consequence of this is another interpretation of Hall–Littlewood orthogonality:

Proposition 1.11. *[15], [16, chapter 5] For partitions λ, μ of length at most n, we have*

$$\int_T P_\lambda(z_1, \ldots, z_n; p^{-1}) P_\mu(z_1^{-1}, \ldots, z_n^{-1}; p^{-1}) \tilde{\Delta}_S^{(n)}(z; p^{-1}) dT = \frac{n!}{v_n(p^{-1})} p^{-\langle \lambda, \rho \rangle - \langle \mu, \rho \rangle} \int_{Gl_n(\mathbb{Q}_p)} c_\lambda(g) c_\mu(g) dg,$$

where $\rho = \frac{1}{2}(n-1, n-3, \ldots, 1-n)$ *and* $v_n(p^{-1}) = \left(\prod_{i=1}^n (1-p^{-i})\right)/(1-p^{-1})^n$.

Here $\tilde{\Delta}_S$ is the symmetric $q = 0$ Macdonald-Morris density [16]. Since $Kp^\lambda K \cap Kp^\mu K = \emptyset$ unless $\lambda = \mu$, the right hand side vanishes unless $\lambda = \mu$. One may also compute $meas.(Kp^\lambda K)$ using [15];

in particular, the nonzero value of the right-hand side agrees with that obtained by integrating over the torus. Given the structural similarity between orthogonality and the vanishing results of [23], we were lead to search for p-adic interpretations of the latter results. In [21], we show that the vanishing results for Hall–Littlewood polynomials have a p-adic interpretation analogous to that of the Schur identities.

To phrase our result for the symplectic case, let E be an unramified quadratic extension of $F = \mathbb{Q}_p$, and let $G = Gl_{2n}(\mathbb{Q}_p)$ and $H = Gl_n(E)$, where p is an odd prime. Then there is an involution on G that has H as its set of fixed points; $S := G/H$ is a p-adic symmetric space. For this symmetric space (and two others), relative zonal spherical functions and a Plancherel theorem are found in [17]. The method used is that of Casselman and Shalika [3, 4], who provide another derivation of Macdonald's formula for zonal spherical functions (see [15] for the general reductive group case) using the theory of admissible representations of p-adic reductive groups. We use this work to prove the following result:

Theorem 1.12. *[21] We have the following identity*

$$\frac{1}{Z}\int_T P_\lambda^{(2n)}(z_i^{\pm 1};p^{-1})\tilde{\Delta}_K^{(n)}(z;\pm p^{-1/2},0,0;p^{-1})dT = p^{-\langle \lambda,\rho_2\rangle}\int_H c_\lambda(h)dh,$$

where $\rho_2 = (n-1/2, n-3/2, \dots, 1/2-n)$ and $c_\lambda \in \mathcal{H}(G,K)$ is the characteristic function of $Kp^\lambda K$, where $K = Gl_{2n}(\mathbb{Z}_p)$ is the maximal compact subgroup of G.

In particular, this gives an interpretation of theorem 1.5 using p-adic representation theory. Note that we may evaluate the right-hand side by using the Cartan decompositions for G and H, along with some measure computations. We mention that there are similar interpretations for other identities in [23]; to prove those we use the Plancherel theorems from [17] and [9].

The method described above using integration over p-adic groups actually supports a generalization of the usual vanishing identities at the Hall–Littlewood level; we briefly discuss it here. Note first that the symmetric function interpretation of theorem 1.5 is that it gives the constant coefficient in the decomposition of $P_\lambda(x_1^{\pm 1},\dots,x_n^{\pm 1};t)$ into Koornwinder $q=0$ polynomials $\{K_\mu(x;t;\pm\sqrt{t},0,0)\}_{\mu\in\Lambda_n^+}$. A natural question, then, is whether there are interesting vanishing conditions for the other coefficients in this expansion. We note that the $t=0$ version of this question is addressed by a classical result of Weyl and Littlewood:

Theorem 1.13. *If $l(\lambda)\le n$, we have the branching rule*

$$s_\lambda^{(2n)}(x^{\pm 1}) = \sum_{l(\mu)\le n} sp_\mu(x_1,\dots,x_n)\left(\sum_{\substack{\beta\in\Lambda_{2n}^+\\ \beta=\nu^2}} c_{\mu,\beta}^\lambda\right),$$

where $c_{\mu,\beta}^\lambda$ are the Littlewood-Richardson coefficients and sp_μ is an irreducible symplectic character.

A consequence of this is that, for $l(\lambda) \leq n$, the integral

$$\int_{S \in Sp(2n)} s_\lambda(S) sp_\mu(S) dS$$

vanishes if and only if $c^\lambda_{\mu,\beta} = 0$ for all $\beta = \nu^2 \in \Lambda^+_{2n}$. We prove that the Hall–Littlewood polynomials satisfy the same vanishing condition:

Theorem 1.14. *[21] Let $\lambda, \mu \in \Lambda^+_n$. Then the following three statements are equivalent:*

(i) The integral

$$\frac{1}{\int_T \tilde{\Delta}^{(n)}_K(x; \pm\sqrt{t}, 0, 0; t) dT} \int_T P^{(2n)}_\lambda(x_i^{\pm 1}; t) K^{BC_n}_\mu(x; t; \pm\sqrt{t}, 0, 0) \tilde{\Delta}^{(n)}_K(x; \pm\sqrt{t}, 0, 0; t) dT$$

vanishes as a rational function of t.

(ii) The Hall polynomials

$$g^\lambda_{\mu,\beta}(t^{-1})$$

vanish as a function of t, for all $\beta \in \Lambda^+_{2n}$ with all parts occurring with even multiplicity.

(iii) The Littlewood-Richardson coefficients

$$c^\lambda_{\mu,\beta}$$

are equal to 0 for all $\beta \in \Lambda^+_{2n}$ with all parts occurring with even multiplicity.

The proof of this relies on p-adic arguments similar to those used to prove theorem 1.12, as well as some technical arguments involving Hall polynomials and their relationship to Littlewood-Richardson coefficients. As an interesting application of this result, consider the case where λ has all parts occurring with even multiplicity (and $l(\lambda) \leq n$), and $\mu = (r)$ has exactly one nonzero part. It is a fact that $c^\lambda_{\beta^2,(r)}$ vanishes unless $|\lambda| = |\beta^2| + r$ and $\lambda - \beta^2$ is a horizontal strip (see [16]). The latter condition is equivalent to the following interlacing condition:

$$\lambda_1 \geq (\beta^2)_1 \geq \lambda_2 \geq (\beta^2)_2 \ldots,$$

but since both λ and β^2 have all parts occurring with even multiplicity this happens if and only if $\lambda = \beta^2$. Thus $c^\lambda_{\beta^2,(r)} = 0$ for all β, so by the above theorem the integral

$$\frac{1}{\int_T \tilde{\Delta}^{(n)}_K(x; \pm\sqrt{t}, 0, 0; t) dT} \int_T P^{(2n)}_\lambda(x_i^{\pm 1}; t) K^{BC_n}_\mu(x; t; \pm\sqrt{t}, 0, 0) \tilde{\Delta}^{(n)}_K(x; \pm\sqrt{t}, 0, 0; t) dT$$

vanishes as a rational function of t. In other words, if one expands $P^{(2n)}_\lambda(x_i^{\pm 1}; t)$ in terms of Koornwinder polynomials $\{K^{BC_n}_\mu(x; t; \pm\sqrt{t}, 0, 0)\}_\mu$, the polynomials $K^{BC_n}_{(r)}(x; t; \pm\sqrt{t}, 0, 0)$ for $r \neq 0$ do not appear in the decomposition.

One may also investigate the values of the integral

$$\int_T P_\lambda^{(2n)}(x_i^{\pm 1};t) K_\mu^{BC_n}(x;t;\pm\sqrt{t},0,0)\tilde{\Delta}_K^{(n)}(x;\pm\sqrt{t},0,0;t)dT$$

in the case when it does *not* vanish; note that this would make explicit the decomposition of the specialized Hall–Littlewood polynomial in terms of the Koornwinder basis with parameters $(t;\pm\sqrt{t},0,0)$. In fact, one may use the p-adic theory discussed above to provide a characterization of these values; we discuss this in the last chapter of the thesis.

1.5 Outline of the Thesis

This thesis consists of three separate, but related, works. The beginning of each chapter contains some relevant notation and terminology that will be used throughout the chapter. In the second chapter, we develop a combinatorial technique for proving the results of Rains and Vazirani at $q = 0$. We first use this method to give another proof of Hall–Littlewood orthogonality. We then prove the theorems mentioned in section 1.2, as well as several other results of a similar flavor. This chapter appeared in [23]. The third chapter investigates Koornwinder polynomials at the $q = 0$ limit, and extends the ideas of chapter 2 to the type BC case. In particular, we prove theorem 1.10 mentioned in section 1.3. The fourth chapter interprets the results of chapter 2 using integration over p-adic groups and ideas from p-adic representation theory. Alternate proofs and some generalizations of the results of chapter 2 are given. This work relies on [3, 4, 17, 9, 15], among others.

Chapter 2

Vanishing Integrals for Hall–Littlewood Polynomials

2.1 Background and Notation

We will briefly review Hall–Littlewood polynomials; we follow [16]. We also set up the required notation.

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition, in which some of the λ_i may be zero. In particular, note that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. Let $l(\lambda)$, the length of λ, be the number of nonzero λ_i and let $|\lambda|$, the weight of λ, be the sum of the nonzero λ_i. We will write $\lambda = \mu^2$ if there exists a partition μ such that $\lambda_{2i-1} = \lambda_{2i} = \mu_i$ (equivalently all parts of λ occur with even multiplicity). Analogously, we write $\lambda = 2\mu$ if there exists a partition μ such that $\lambda_i = 2\mu_i$ (equivalently each part of λ is even). Also let $m_i(\lambda)$ be the number of λ_j equal to i for each $i \geq 0$.

Recall the t-integer $[i] = [i]_t = (1-t^i)/(1-t)$, as well as the t-factorial $[m]! = [m][m-1]\cdots[1]$, $[0]! = 1$. Let

$$\phi_r(t) = (1-t)(1-t^2)\cdots(1-t^r),$$

so that in particular $\phi_r(t)/(1-t)^r = [r]!$. Then we define

$$v_\lambda(t) = \prod_{i\geq 0}\prod_{j=1}^{m_i(\lambda)} \frac{1-t^j}{1-t} = \prod_{i\geq 0} \frac{\phi_{m_i(\lambda)}(t)}{(1-t)^{m_i(\lambda)}} = \prod_{i\geq 0}[m_i(\lambda)]!,$$

and

$$v_{\lambda+}(t) = \prod_{i\geq 1}\prod_{j=1}^{m_i(\lambda)} \frac{1-t^j}{1-t} = \prod_{i\geq 1} \frac{\phi_{m_i(\lambda)}(t)}{(1-t)^{m_i(\lambda)}} = \prod_{i\geq 1}[m_i(\lambda)]!,$$

so that the first takes into account the zero parts, while the second does not. The Hall–Littlewood polynomial $P_\lambda(x_1, \dots, x_n; t)$ indexed by λ is defined to be

$$\frac{1}{v_\lambda(t)} \sum_{w \in S_n} w\Big(x^\lambda \prod_{1 \le i < j \le n} \frac{x_i - tx_j}{x_i - x_j}\Big),$$

where we write x^λ for $x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ and w acts on the subscripts of the x_i. The normalization $1/v_\lambda(t)$ has the effect of making the coefficient of x^λ equal to unity. (We will also write $P_\lambda^{(n)}(x;t)$ and use $P_\lambda(x^{(m)}, y^{(n)}; t)$ to denote $P_\lambda(x_1, \dots, x_m, y_1, \dots, y_n; t)$ in the final section.) We define the polynomials $\{R_\lambda^{(n)}(x;t)\}$ by $R_\lambda^{(n)}(x;t) = v_\lambda(t) P_\lambda^{(n)}(x;t)$. For $w \in S_n$, we also define

$$R_{\lambda,w}^{(n)}(x;t) = w\Big(x^\lambda \prod_{1 \le i < j \le n} \frac{x_i - tx_j}{x_i - x_j}\Big), \tag{2.1}$$

so that $R_{\lambda,w}^{(n)}(x;t)$ is the term of $R_\lambda^{(n)}(x;t)$ associated to the permutation w.

There are two important degenerations of the Hall–Littlewood symmetric functions: at $t = 0$, we recover the Schur functions $s_\lambda(x)$ and at $t = 1$ the monomial symmetric functions $m_\lambda(x)$. We remark that the Macdonald polynomials $P_\lambda(x; q, t)$ do not have poles at $q = 0$, so there is no obstruction to specializing q to zero; in fact we obtain the Hall–Littlewood polynomials (see [16], chapter 6). Similarly, when $q = t$ (or $q = 0$ then $t = 0$), $P_\lambda(x; q, t)$ reduces to $s_\lambda(x)$.

Let

$$b_\lambda(t) = \prod_{i \ge 1} \phi_{m_i(\lambda)}(t) = v_{\lambda^+}(t)(1-t)^{l(\lambda)}.$$

Then we let $Q_\lambda(x;t)$ be multiples of the $P_\lambda(x;t)$:

$$Q_\lambda(x;t) = b_\lambda(t) P_\lambda(x;t);$$

these form the adjoint basis with respect to the t-analog of the Hall inner product. With this notation the Cauchy identity for Hall–Littlewood functions is

$$\sum_\lambda P_\lambda(x;t) Q_\lambda(x;t) = \prod_{i,j \ge 1} \frac{1 - tx_i y_j}{1 - x_i y_j}. \tag{2.2}$$

We recall the definition of Rogers–Szegő polynomials, which appear later in this chapter. Let m be a nonnegative integer. Then we let $H_m(z;t)$ denote the Rogers–Szegő polynomial (see [1], chapter 3, examples 3–9)

$$H_m(z;t) = \sum_{i=0}^{m} z^i \begin{bmatrix} m \\ i \end{bmatrix}_t, \tag{2.3}$$

where

$$\begin{bmatrix} m \\ i \end{bmatrix}_t = \begin{cases} \frac{[m]!}{[m-i]![i]!}, & \text{if } m \geq i \geq 0, \\ 0, & \text{otherwise} \end{cases}$$

is the t-binomial coefficient. It can be verified that the Rogers–Szegő polynomials satisfy the following second-order recurrence:

$$H_m(z;t) = (1+z)H_{m-1}(z;t) - (1-t^{m-1})zH_{m-2}(z;t).$$

Also, we recall the definition of the symmetric $q=0$ Macdonald-Morris density [16]:

$$\tilde{\Delta}_S^{(n)}(x;t) = \prod_{1 \leq i \neq j \leq n} \frac{1 - x_i x_j^{-1}}{1 - t x_i x_j^{-1}},$$

and the symmetric Koornwinder density [13]:

$$\tilde{\Delta}_K^{(n)}(x;a,b,c,d;t) = \frac{1}{2^n n!} \prod_{1 \leq i \leq n} \frac{1 - x_i^{\pm 2}}{(1 - a x_i^{\pm 1})(1 - b x_i^{\pm 1})(1 - c x_i^{\pm 1})(1 - d x_i^{\pm 1})} \prod_{1 \leq i < j \leq n} \frac{1 - x_i^{\pm 1} x_j^{\pm 1}}{1 - t x_i^{\pm 1} x_j^{\pm 1}}, \tag{2.4}$$

where we write $1-x_i^{\pm 2}$ for the product $(1-x_i^2)(1-x_i^{-2})$ and $1-x_i^{\pm 1}x_j^{\pm 1}$ for $(1-x_i x_j)(1-x_i^{-1}x_j^{-1})(1-x_i^{-1}x_j)(1-x_i x_j^{-1})$ etc. For convenience, we will write $\tilde{\Delta}_S^{(n)}$ and $\tilde{\Delta}_K^{(n)}(a,b,c,d)$ with the assumption that these densities are in $x_1, \ldots, x_n$ with parameter t when it is clear. We recall some notation for hypergeometric series from [20] and [18]. We define the q-symbol

$$(a;q) = \prod_{k \geq 0} (1 - aq^k),$$

and $(a_1, a_2, \ldots, a_l; q) = (a_1;q)(a_2;q)\cdots(a_l;q)$. Also, let

$$(a;q)_n = \prod_{j=0}^{n-1} (1 - aq^j),$$

for $n > 0$ and $(a;q)_0 = 1$. We also define the C-symbols, which appear in the identities of [20]. Let

$$C_\mu^0(x;q,t) = \prod_{1 \leq i \leq l(\mu)} \frac{(t^{1-i}x;q)}{(q^{\mu_i} t^{1-i} x;q)},$$

$$C_\mu^-(x;q,t) = \prod_{1 \leq i \leq l(\mu)} \frac{(x;q)}{(q^{\mu_i} t^{l(\mu)-i} x;q)} \prod_{1 \leq i < j \leq l(\mu)} \frac{(q^{\mu_i - \mu_j} t^{j-i} x;q)}{(q^{\mu_i - \mu_j} t^{j-i-1} x;q)},$$

$$C_\mu^+(x;q,t) = \prod_{1 \leq i \leq l(\mu)} \frac{(q^{\mu_i} t^{2-l(\mu)-i} x;q)}{(q^{2\mu_i} t^{2-2i} x;q)} \prod_{1 \leq i < j \leq l(\mu)} \frac{(q^{\mu_i + \mu_j} t^{3-j-i} x;q)}{(q^{\mu_i + \mu_j} t^{2-j-i} x;q)}.$$

We note that $C^0_\mu(x;q,t)$ is the q,t-shifted factorial. As before, we extend this by

$$C^{0,\pm}_\mu(a_1,a_2,\ldots,a_l;q,t) = C^{0,\pm}_\mu(a_1;q,t)\cdots C^{0,\pm}_\mu(a_l;q,t).$$

We note that for $q=0$ we have

$$\begin{aligned} C^0_\mu(x;0,t) &= \prod_{1\le i\le l(\mu)} (1-t^{1-i}x), \\ C^-_\mu(t;0,t) &= (1-t)^{l(\mu)} v_{\mu+}(t), \\ C^+_\mu(x;0,t) &= 1. \end{aligned}$$

Finally, we explain some notation involving permutations. Let $w \in S_n$ act on the variables $z_1,\ldots,z_n$ by

$$w(z_1\cdots z_n) = z_{w(1)}\cdots z_{w(n)},$$

as in the definition of Hall–Littlewood polynomials above. We view the permutation w as this string of variables. For example the condition "z_i is in the kth position of w" means that $w(k)=i$. Also we write

$$\text{“}z_i \prec_w z_j\text{”}$$

if $i=w(i')$ and $j=w(j')$ for some $i'<j'$, i.e., z_i appears to the left of z_j in the permutation representation $z_{w(1)}\cdots z_{w(n)}$. For $w \in S_{2n}$, we use $w(x_1^{\pm1},\ldots,x_n^{\pm1})$ to represent $z_{w(1)}\cdots z_{w(2n)}$, with $z_i = x_i$ for $1\le i\le n$ and $z_j = x_{j-n}^{-1}$ for $n+1\le j\le 2n$.

2.2 Hall–Littlewood Orthogonality

It is a well-known result that Hall–Littlewood polynomials are orthogonal with respect to the density $\tilde{\Delta}_S$. We prove this result using our method below, to illustrate the technique in a simple case.

Theorem 2.1. *We have the following orthogonality relation for Hall–Littlewood polynomials:*

$$\int_T P_\lambda(x_1,\ldots,x_n;t)P_\mu(x_1^{-1},\ldots,x_n^{-1};t)\tilde{\Delta}_S^{(n)}(x;t)dT = \delta_{\lambda\mu}\frac{n!}{v_\mu(t)}.$$

Proof. Note first that by the definition of Hall–Littlewood polynomials, the left-hand side is a sum of $(n!)^2$ integrals in bijection with $S_n \times S_n$. Now, since the integral is invariant under inverting all variables, we may restrict to the case where $\lambda \geq \mu$ in the reverse lexicographic ordering (we assume this throughout). We will show that each of these terms vanish unless $\lambda = \mu$, and this argument will allow us to compute the normalization in the case $\lambda = \mu$. By symmetry and (2.1), we have

$$\int_T P_\lambda^{(n)}(x;t)P_\mu^{(n)}(x^{-1};t)\tilde{\Delta}_S^{(n)}dT = \frac{n!}{v_\lambda(t)v_\mu(t)} \sum_{\rho \in S_n} \int_T R_{\lambda,\mathrm{id}}^{(n)}(x;t)R_{\mu,\rho}^{(n)}(x^{-1};t)\tilde{\Delta}_S^{(n)}dT.$$

Claim 2.1.1. *We have the term evaluation*

$$\int_T R_{\lambda,id}^{(n)}(x;t)R_{\mu,\rho}^{(n)}(x^{-1};t)\tilde{\Delta}_S^{(n)}dT = t^{i(\rho)}$$

if $x_1^{\lambda_1}\cdots x_n^{\lambda_n}x_{\rho(1)}^{-\mu_1}\cdots x_{\rho(n)}^{-\mu_n} = 1$, *and is otherwise equal to 0. Here* $i(\rho)$ *is the number of inversions of* ρ *with respect to the permutation* $x_1^{-1}\cdots x_n^{-1}$.

Note that $i(\rho)$ is the Coxeter length and recall the distribution of this statistic: $\sum_\rho t^{i(\rho)} = [n]!$.

To prove the claim, we use induction on n. Note first that for $n = 1$, the only term is $\int x_1^{\lambda_1}x_1^{-\mu_1}dT$, which vanishes unless $\lambda_1 = \mu_1$. Now suppose the result is true for $n-1$. With this assumption we want to show that it holds true for n variables. One can compute, by integrating with respect to x_1 in the iterated integral, that the left-hand side above is equal to

$$\int_{T_{n-1}} \Big(\int_{T_1} x_1^{\lambda_1-\mu_{\rho^{-1}(1)}} \prod_{x_j^{-1}\prec_\rho x_1^{-1}} \frac{tx_j - x_1}{x_j - tx_1}\frac{dx_1}{2\pi\sqrt{-1}x_1}\Big) R_{\widehat{\lambda},\widehat{\mathrm{id}}}^{(n-1)}(x;t)R_{\widehat{\mu},\widehat{\rho}}^{(n-1)}(x^{-1};t)\tilde{\Delta}_S^{(n-1)}(x;t)dT,$$

where

$$\begin{aligned} \widehat{\mathrm{id}} &= \mathrm{id} \text{ with } x_1 \text{ deleted},\\ \widehat{\rho} &= \rho \text{ with } x_1^{-1} \text{ deleted},\\ \widehat{\lambda} &= \lambda \text{ with } \lambda_1 \text{ deleted},\\ \widehat{\mu} &= \mu \text{ with } \mu_{\rho^{-1}(1)} \text{ deleted}. \end{aligned}$$

Recall that $\lambda_1 \geq \mu_1 \geq \mu_i$ for all $1 \leq i \leq n$. Thus, the inner integral in x_1 is zero if $\lambda_1 > \mu_{\rho^{-1}(1)}$ and is $t^{|\{j: x_j^{-1}\prec_\rho x_1^{-1}\}|}$ if $\lambda_1 = \mu_{\rho^{-1}(1)}$. In the latter case, note that $\widehat{\lambda} \geq \widehat{\mu}$, so we may use the induction hypothesis on the resulting $(n-1)$-dimensional integral, and combining this with the contribution from x_1 gives the result of the claim.

Note that the claim implies each term is zero if $\lambda \neq \mu$, so consequently the entire integral in zero. Finally, we use the claim to compute the normalization value in the case $\lambda = \mu$. By the above remarks, we have

$$\int_T P_\lambda^{(n)}(x;t)P_\mu^{(n)}(x^{-1};t)\tilde{\Delta}_S^{(n)}dT = \frac{n!}{v_\mu(t)^2}\sum_{\substack{\rho\in S_n:\\ x_1^{\lambda_1}\cdots x_n^{\lambda_n}x_{\rho(1)}^{-\mu_1}\cdots x_{\rho(n)}^{-\mu_n}=1}} t^{i(\rho)}.$$

Note that the permutations in the index of the sum are in statistic-preserving bijection with $S_{m_0(\mu)} \times S_{m_1(\mu)} \times \cdots$ so, using the comment immediately following the claim, the above expression is equal to

$$\frac{n!}{v_\mu(t)^2}\sum_{\rho\in S_{m_0(\mu)}\times S_{m_1(\mu)}\times\cdots} t^{i(\rho)} = \frac{n!}{v_\mu(t)^2}\prod_{i\geq 0}[m_i(\mu)]! = \frac{n!}{v_\mu(t)},$$

as desired. □

2.3 An α-generalization

In this section, we prove the orthogonal group integrals with an extra parameter α. This gives four identities–one for each component of $O(l)$, depending on the parity of l. First, we use a result of Gustafson [8] to compute some normalizations that will be used throughout the chapter.

Proposition 2.2. *We have the following normalizations:*

(i) (symplectic)

$$\int_T \tilde{\Delta}_K^{(n)}(x;\pm\sqrt{t},0,0;t)dT = \frac{(1-t)^n}{(t^2;t^2)_n},$$

(ii)

$$\int_T \tilde{\Delta}_K^{(n)}(x;1,\sqrt{t},0,0;t)dT = \frac{(1-t)^n}{(\sqrt{t};\sqrt{t})_{2n}},$$

(iii) ($O^+(2n)$)

$$\int_T \tilde{\Delta}_K^{(n)}(x;\pm 1,\pm\sqrt{t};t)dT = \frac{(1-t)^n}{2(t;t)_{2n}},$$

(iv) ($O^-(2n)$)

$$\int_T \tilde{\Delta}_K^{(n-1)}(x;\pm t,\pm\sqrt{t};t)dT = \frac{(1-t)^{n-1}}{(t^3;t)_{2n-2}},$$

(v) *($O^+(2n+1)$)*

$$\int_T \tilde{\Delta}_K^{(n)}(x;t,-1,\pm\sqrt{t};t)dT = \frac{(1-t)^{n+1}}{(t;t)_{2n+1}},$$

(vi) *($O^-(2n+1)$)*

$$\int_T \tilde{\Delta}_K^{(n)}(x;1,-t,\pm\sqrt{t};t)dT = \frac{(1-t)^{n+1}}{(t;t)_{2n+1}}.$$

We omit the proof, but in all cases it follows from setting $q = 0$ and the appropriate values of (a, b, c, d) in the integral evaluation:

$$\int_T \tilde{\Delta}_K^{(n)}(x;a,b,c,d;q,t)dT = \prod_{0\leq j<n} \frac{(t, t^{2n-2-j}abcd; q)}{(t^{j+1}, t^j ab, t^j ac, t^j ad, t^j bc, t^j bd, t^j cd; q)},$$

which may be found in [8].

We remark that at $t = 0$ the above densities have special significance. In particular, (i) is the eigenvalue density of the symplectic group and (iii)–(vi) are the eigenvalue densities of $O^+(2n), O^-(2n), O^+(2n+1)$, and $O^-(2n+1)$ (in the orthogonal group case, the density depends on the component of the orthogonal group as well as whether the dimension is odd or even). The density in (ii) appears in corollary 2.14, and that result corresponds to a summation identity of Kawanaka [12] in the $n \to \infty$ limit (this connection is discussed in detail later in the chapter). One should also see [19], which conjectures an elliptic version of the integral identity.

In this section, we want to use a technique similar to the one used to prove Hall–Littlewood orthogonality. Namely, we want to break up the integral into a sum of terms, one for each permutation, and study the resulting term integral. The obstruction to this approach is that in many cases the poles lie on the contour, i.e., occur at ± 1, so the pieces of the integral are not well defined. However, since the overall integral does not have singularities, we may use the principal value integral which we denote by P.V. (see [10], section 8.3). This is basically an average of integrating along two contours: one is obtained by shrinking the contour, and the other is obtained by enlarging the contour (both by ϵ, as $\epsilon \to 0$). In other words,

$$\text{P. V.} \int_{|z|=1} f(z)\frac{dz}{2\pi\sqrt{-1}z} = \lim_{\epsilon\to 0^+} \frac{1}{2}\Big[\int_{|z|=1-\epsilon} f(z)\frac{dz}{2\pi\sqrt{-1}z} + \int_{|z|=1+\epsilon} f(z)\frac{dz}{2\pi\sqrt{-1}z}\Big].$$

We extend this to T_n by iterating this procedure for each copy of T_1. We first prove some results involving the principal value integrals.

Lemma 2.3. *Let $f(z)$ be a function in z such that $zf(z)$ is holomorphic in a neighborhood of the* *unit disk. Then*

$$\mathrm{P.V.}\int_T f(z)\frac{1}{1-z^{-2}}dT = \frac{f(1)+f(-1)}{4}.$$

Proof. We have

$$\mathrm{P.V.}\frac{1}{2\pi\sqrt{-1}}\int_{|z|=1} f(z)\frac{1}{1-z^{-2}}\frac{1}{z}dz = \lim_{\epsilon\to 0^+}\frac{1}{2}\Bigg[\frac{1}{2\pi\sqrt{-1}}\int_{|z|=1-\epsilon} zf(z)\frac{1}{z^2-1}dz + \frac{1}{2\pi\sqrt{-1}}\int_{|z|=1+\epsilon} zf(z)\frac{1}{z^2-1}dz\Bigg].$$

But now as $zf(z)$ is holomorphic in a neighborhood of the disk, and the singularities of $1/(z^2-1)$ lie outside of the disk, the first integral is zero by Cauchy's theorem. Using the residue theorem for the second integral (it has simple poles at ± 1) gives

$$\lim_{\epsilon\to 0}\frac{1}{2}\left[\mathrm{Res}_{z=1}\frac{zf(z)}{(z-1)(z+1)} + \mathrm{Res}_{z=-1}\frac{zf(z)}{(z-1)(z+1)}\right] = \frac{1}{2}\left[\frac{f(1)}{2}+\frac{f(-1)}{2}\right] = \frac{1}{4}\Big[f(1)+f(-1)\Big].$$

□

Lemma 2.4. *Let p be a function in $x_1,\dots,x_n$ such that x_ip is holomorphic in x_i in a neighborhood of the unit disk for all $1\le i\le n$ and $p(\pm1,\dots,\pm1)=0$ for all 2^n combinations. Let Δ be a function in $x_1,\dots,x_n$ such that $\Delta(\pm1,\dots,\pm1,x_{i+1},\dots,x_n)$ is holomorphic in x_{i+1} in a neighborhood of the unit disk for all $0\le i\le n-1$ (again for all 2^i combinations). Then*

$$\mathrm{P.V.}\int_T p\cdot\Delta\cdot\prod_{1\le i\le n}\frac{1}{1-x_i^{-2}}dT = 0.$$

Proof. We give a proof by induction on n. For $n=1$, since $x_1\cdot p\cdot\Delta$ is holomorphic in x_1 we may use lemma 2.3:

$$\mathrm{P.V.}\int_T p\cdot\Delta\cdot\frac{1}{1-x_1^{-2}}dT = \frac{1}{4}[p(1)\Delta(1)+p(-1)\Delta(-1)].$$

But then $p(1)=p(-1)=0$ by assumption, so the integral is zero as desired.

Now suppose the result holds in the case of $n-1$ variables. Consider the n variable case, and let p,Δ in $x_1,\dots,x_n$ satisfy the above conditions. Integrate first with respect to x_1 and note that $x_1\cdot p\cdot\Delta$ is holomorphic in x_1 so we can apply lemma 2.3:

$$\mathrm{P.V.}\int_T p\cdot\Delta\cdot\prod_{1\le i\le n}\frac{1}{1-x_i^{-2}}dT = \frac{1}{4}\mathrm{P.V.}\int_{T_{n-1}} p(1,x_2,\dots,x_n)\Delta(1,x_2,\dots,x_n)\prod_{2\le i\le n}\frac{1}{1-x_i^{-2}}dT + \frac{1}{4}\mathrm{P.V.}\int_{T_{n-1}} p(-1,x_2,\dots,x_n)\Delta(-1,x_2,\dots,x_n)\prod_{2\le i\le n}\frac{1}{1-x_i^{-2}}dT.$$

But now the pairs $p(1,x_2,\dots,x_n),\Delta(1,x_2,\dots,x_n)$ and $p(-1,x_2,\dots,x_n)$, $\Delta(-1,x_2,\dots,x_n)$ satisfy the conditions of the theorem for $n-1$ variables $x_2, \ldots, x_n$, so by the induction hypothesis each of the two integrals is zero, so the total integral is zero. □

For this section, we let $\rho_{2n} = (1, 2, \ldots, 2n)$. We also let $1^k = (1, 1, \ldots, 1)$ with exactly k ones. As above we will work with principal value integrals, as necessary. For simplicity, we will suppress the notation P.V.

Theorem 2.5. *Let $l(\lambda) \leq 2n$. We have the following integral identity for $O^+(2n)$:*

$$\frac{1}{\int \tilde{\Delta}_K^{(n)}(\pm 1, \pm\sqrt{t})dT} \int P_\lambda(x_1^{\pm 1}, \ldots, x_n^{\pm 1}; t)\tilde{\Delta}_K^{(n)}(\pm 1, \pm\sqrt{t}) \prod_{i=1}^{n}(1 - \alpha x_i^{\pm 1})dT$$
$$= \frac{\phi_{2n}(t)}{v_\lambda(t)(1-t)^{2n}}\left[(-\alpha)^{\#\text{ of odd parts of }\lambda} + (-\alpha)^{\#\text{ of even parts of }\lambda}\right]$$
$$= \frac{[2n]!}{v_\lambda(t)}\left[(-\alpha)^{\#\text{ of odd parts of }\lambda} + (-\alpha)^{\#\text{ of even parts of }\lambda}\right].$$

Proof. We will first show the following:

$$\int R_\lambda(x_1^{\pm 1}, \ldots, x_n^{\pm 1}; t)\tilde{\Delta}_K^{(n)}(\pm 1, \pm\sqrt{t}) \prod_{i=1}^{n}(1 - \alpha x_i^{\pm 1})dT = \frac{1}{2^n(1-t)^n}\mathrm{Pf}[a_{j,k}]^\lambda,$$

where Pf denotes the Pfaffian and the $2n \times 2n$ antisymmetric matrix $[a_{j,k}]^\lambda$ is defined by

$$a_{j,k}^\lambda = (1 + \alpha^2)\chi_{(\lambda_j - j) - (\lambda_k - k)\text{ odd}} + 2(-\alpha)\chi_{(\lambda_j - j) - (\lambda_k - k)\text{ even}},$$

for $1 \leq j < k \leq 2n$.

First, note that by symmetry we can rewrite the above integral as $2^n n!$ times the sum over all matchings w of $x_1^{\pm 1}, \ldots, x_n^{\pm 1}$, where a matching is a permutation in S_{2n} such that x_i occurs to the left of x_i^{-1} and x_i occurs to the left of x_j for $1 \leq i < j \leq n$. In particular, x_1 occurs first. Thus, we have

$$\int R_\lambda^{(2n)}(x^{\pm 1}; t)\tilde{\Delta}_K^{(n)}(\pm 1; \pm\sqrt{t}) \prod_{i=1}^{n}(1 - \alpha x_i^{\pm 1})dT$$
$$= 2^n n! \sum_w \int R_{\lambda,w}^{(2n)}(x^{\pm 1}; t)\tilde{\Delta}_K^{(n)}(\pm 1; \pm\sqrt{t}) \prod_{i=1}^{n}(1 - \alpha x_i^{\pm 1})dT,$$

where the sum is over matchings w in S_{2n}.

We introduce some notation for a matching $w \in S_{2n}$. We write $w = \{(i_1, i_1'), \ldots, (i_n, i_n')\}$ to indicate that x_k occurs in position i_k and x_k^{-1} occurs in position i_k' for all $1 \leq k \leq n$. Clearly we have $i_k < i_k'$ for all k and $i_j < i_k$ for all $j < k$.

Claim 2.5.1. *Let $\lambda = (\lambda_1, \dots, \lambda_{2n})$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2n} \in \mathbb{Z}$. Then we have the following term evaluation:*

$$2^n n! \,\mathrm{P.V.} \int_T R_{\lambda,w}(x_1^{\pm 1}, \dots, x_n^{\pm 1}; t) \tilde{\Delta}_K^{(n)}(\pm 1, \pm\sqrt{t}) \prod_{i=1}^{n} (1 - \alpha x_i^{\pm 1}) dT = \frac{\epsilon(w)}{2^n (1-t)^n} \prod_{1 \leq k \leq n} a^{\lambda}_{i_k, i'_k},$$

where $\epsilon(w)$ is the sign of w and $a^{\lambda}_{i_k,i'_k}$ is the (i_k, i'_k) entry of the matrix $[a_{j,k}]^{\lambda}$. In particular, the term integral only depends on the parity of the parts $\lambda_1, \dots, \lambda_{2n}$.

Let μ be such that $\lambda = \mu + \rho_{2n}$. We give a proof by induction on n, the number of variables. For $n = 1$, there is only one matching—in particular, x_1^{-1} must occur in position 2. The (principal value) integral is

$$\int_T x_1^{\lambda_1 - \lambda_2} \frac{(1 - t x_1^{-2})}{(1 - x_1^{-2})} \frac{(1 - \alpha x_1)(1 - \alpha x_1^{-1})}{(1 - t x_1^2)(1 - t x_1^{-2})} dT = \int_T x_1^{\lambda_1 - \lambda_2} \frac{(1 - \alpha x_1)(1 - \alpha x_1^{-1})}{(1 - x_1^{-2})(1 - t x_1^2)} dT$$
$$= \int_T x_1^{\lambda_1 - \lambda_2} \frac{(1 + \alpha^2) - \alpha(x_1 + x_1^{-1})}{(1 - t x_1^2)(1 - x_1^{-2})} dT,$$

and $\lambda_1 - \lambda_2 \geq 0$. Note that the conditions for lemma 2.3 are satisfied. Applying that result gives that the value of the integral is $2(-\alpha)/2(1-t)$ if $\lambda_1 - \lambda_2$ is odd, and $(1 + \alpha^2)/2(1-t)$ if $\lambda_1 - \lambda_2$ is even, which agrees with the above claim.

Now suppose the result is true for up to $n - 1$ variables and consider the n variable case. Note first that $i_1 = 1$. One can compute, by combining terms involving x_1 in the iterated integral, that

$$2^n n! \int R^{(2n)}_{\lambda,w}(x^{\pm 1}; t) \tilde{\Delta}_K^{(n)}(\pm 1, \pm\sqrt{t}) \prod_{i=1}^{n} (1 - \alpha x_i^{\pm 1}) dT$$
$$= \int_{T_{n-1}} \Big(\int_{T_1} x_1^{\lambda_1 - \lambda_{i'_1}} \frac{(1 - \alpha x_1)(1 - \alpha x_1^{-1})}{(1 - t x_1^2)(1 - x_1^{-2})} \prod_{\substack{x_j : \\ x_1 \prec_w x_j \prec_w x_1^{-1} \prec_w x_j^{-1}}} \frac{(t - x_1 x_j)}{(1 - t x_1 x_j)}$$
$$\prod_{\substack{x_j : \\ x_1 \prec_w x_j \prec_w x_j^{-1} \prec_w x_1^{-1}}} \frac{(t - x_1 x_j^{-1})(t - x_1 x_j)}{(1 - t x_1 x_j^{-1})(1 - t x_1 x_j)} dT \Big) F_{\widehat{\lambda}, \tilde{w}} dT,$$

where

$$F_{\widehat{\lambda}, \tilde{w}} = 2^{n-1}(n-1)! R_{\widehat{\lambda}, \tilde{w}}(x_2^{\pm 1}, \dots, x_n^{\pm 1}; t) \tilde{\Delta}_K^{(n-1)}(\pm 1, \pm\sqrt{t}) \prod_{i=2}^{n} (1 - \alpha x_i^{\pm 1}),$$

and $\widehat{\lambda}$ is λ with parts $\lambda_1, \lambda_{i'_1}$ deleted; $\tilde{w}$ is w with x_1, x_1^{-1} deleted.

In particular, the conditions for lemma 2.3 are satisfied for the inner integral in x_1. Note that the terms

$$\frac{(t - x_1 x_i)}{(1 - t x_1 x_i)} \frac{(t - x_1 x_i^{-1})}{(1 - t x_1 x_i^{-1})}$$

give 1 when evaluated at $x_1 = \pm 1$, so the above integral evaluates to

$$\frac{1}{4(1-t)} \int_{T_{n-1}} \Bigg[F_{\widehat{\lambda},\tilde{w}} \cdot (1+\alpha^2-2\alpha) \Big(\prod_{\substack{x_j: \\ x_1 \prec_w x_j \prec_w x_1^{-1} \prec_w x_j^{-1}}} \frac{t-x_j}{1-tx_j} \Big) + F_{\widehat{\lambda},\tilde{w}} \cdot (1+\alpha^2+2\alpha)(-1)^{\lambda_1-\lambda_{i_1'}} \Big(\prod_{\substack{x_j: \\ x_1 \prec_w x_j \prec_w x_1^{-1} \prec_w x_j^{-1}}} \frac{t+x_j}{1+tx_j} \Big) \Bigg] dT.$$

But now since $(t-x_i)/(1-tx_i)$ and $(t+x_i)/(1+tx_i)$ are power series in x_i, we may apply the inductive hypothesis to each part of the new integral: we reduce exponents on x_i modulo 2. We get

$$\frac{1}{4(1-t)} \int_{T_{n-1}} \Bigg[F_{\widehat{\lambda},\tilde{w}} \cdot (1+\alpha^2-2\alpha) \Big(\prod_{\substack{x_j: \\ x_1 \prec_w x_j \prec_w x_1^{-1} \prec_w x_j^{-1}}} (-x_j) \Big) + F_{\widehat{\lambda},\tilde{w}} \cdot (1+\alpha^2+2\alpha)(-1)^{\lambda_1-\lambda_{i_1'}} \Big(\prod_{\substack{x_j: \\ x_1 \prec_w x_j \prec_w x_1^{-1} \prec_w x_j^{-1}}} x_j \Big) \Bigg] dT.$$

But now note that

$$\prod_{\substack{x_j: \\ x_1 \prec_w x_j \prec_w x_1^{-1} \prec_w x_j^{-1}}} (-1) = \prod_{\substack{x_j: \\ x_1 \prec_w x_j \prec_w x_1^{-1} \prec_w x_j^{-1}}} (-1) \prod_{\substack{x_j: \\ x_1 \prec_w x_j \prec_w x_j^{-1} \prec_w x_1^{-1}}} (-1)^2 = (-1)^{i_1'-2},$$

since $i_1' - 2$ is the number of variables between x_1 and x_1^{-1} in the matching w. We can compute

$$\begin{aligned}(1+\alpha^2-2\alpha)(-1)^{i_1'-2} &+ (1+\alpha^2+2\alpha)(-1)^{\lambda_1-\lambda_{i_1'}} \\ &= (1+\alpha^2)[(-1)^{i_1'} + (-1)^{\lambda_1-\lambda_{i_1'}}] - 2\alpha[(-1)^{i_1'} + (-1)^{\lambda_1-\lambda_{i_1'}+1}] \\ &= \begin{cases} 2(-1)^{i_1'}(1+\alpha^2), & \text{if } \lambda_1 - \lambda_{i_1'} + i_1' - 1 \text{ is odd,} \\ -4(-1)^{i_1'}\alpha, & \text{if } \lambda_1 - \lambda_{i_1'} + i_1' - 1 \text{ is even.} \end{cases}\end{aligned}$$

Combining this with the factor $1/4(1-t)$ and noting that

$$F_{\widehat{\lambda},\tilde{w}} \cdot \Big(\prod_{\substack{x_j: \\ x_1 \prec_w x_j \prec_w x_1^{-1} \prec_w x_j^{-1}}} x_j \Big) = F_{\tilde{\lambda},\tilde{w}},$$

with

$$\tilde{\lambda} = (\lambda_2 + 1, \ldots, \lambda_{i_1'-1} + 1, \lambda_{i_1'+1}, \ldots, \lambda_{2n}),$$

gives that

$$2^n n! \int R^{(2n)}_{\lambda,w}(x^{\pm 1};t)\tilde{\Delta}^{(n)}_K(\pm 1, \pm\sqrt{t})\prod_{i=1}^{n}(1-\alpha x_i^{\pm 1})dT$$
$$= \frac{2^{n-1}(n-1)!}{2(1-t)} a^{\lambda}_{i_1,i_1'}(-1)^{i_1'}\int_T R_{\tilde{\lambda},\tilde{w}}(x_1^{\pm 1},\dots,x_{n-1}^{\pm 1};t)\tilde{\Delta}^{(n-1)}_K(\pm 1,\pm\sqrt{t})\prod_{i=1}^{n-1}(1-\alpha x_i^{\pm 1})dT.$$

Now set $\widehat{\mu} = (\mu_2, \dots, \mu_{i_1'-1}, \mu_{i_1'+1}, \dots, \mu_{2n})$, and note that $\tilde{\lambda}$ and $\widehat{\mu} + \rho_{2n-2}$ have equivalent parts modulo 2. Thus, using the induction hypothesis twice, the above is equal to

$$\frac{2^{n-1}(n-1)!}{2(1-t)} a^{\lambda}_{i_1,i_1'}(-1)^{i_1'}\int_T R_{\widehat{\mu}+\rho_{2n-2},\tilde{w}}(x_1^{\pm 1},\dots,x_{n-1}^{\pm 1};t)\tilde{\Delta}^{(n-1)}_K(\pm 1,\pm\sqrt{t})\prod_{i=1}^{n-1}(1-\alpha x_i^{\pm 1})dT$$
$$= \frac{a^{\lambda}_{i_1,i_1'}(-1)^{i_1'}}{2(1-t)}\frac{\epsilon(\tilde{w})}{2^{n-1}(1-t)^{n-1}}\prod_{2\le k\le n} a^{\widehat{\mu}+\rho_{2n-2}}_{i_k,i_k'} = \frac{\epsilon(w)}{2^n(1-t)^n}\prod_{1\le k\le n} a^{\lambda}_{i_k,i_k'},$$

as desired. This proves the claim.

Note in particular this result implies that the integral of a matching w is the term in the expansion of $\frac{1}{2^n(1-t)^n}\mathrm{Pf}[a_{j,k}]^{\lambda}$ corresponding to w.

Now using the claim, we have

$$\int_T R_{\lambda}(x_1^{\pm 1},\dots,x_n^{\pm 1};t)\tilde{\Delta}^{(n)}_K(\pm 1,\pm\sqrt{t})\prod_{i=1}^{n}(1-\alpha x_i^{\pm 1})dT$$
$$= 2^n n! \sum_{\substack{w \text{ a matching} \\ \text{in } S_{2n}}} \mathrm{P.V.}\int_T R_{\lambda,w}(x_1^{\pm 1},\dots,x_n^{\pm 1};t)\tilde{\Delta}^{(n)}_K(\pm 1,\pm\sqrt{t})\prod_{i=1}^{n}(1-\alpha x_i^{\pm 1})dT$$
$$= \frac{1}{2^n(1-t)^n}\mathrm{Pf}[a_{j,k}]^{\lambda},$$

since the term integrals are in bijection with the terms of the Pfaffian.

Now we use this to prove the theorem. Using proposition 2.2(iii), we have

$$\frac{1}{\int \tilde{\Delta}^{(n)}_K(\pm 1,\pm\sqrt{t})dT}\int P_{\lambda}(x_1^{\pm 1},\dots,x_n^{\pm 1};t)\tilde{\Delta}^{(n)}_K(\pm 1,\pm\sqrt{t})\prod_{i=1}^{n}(1-\alpha x_i^{\pm 1})dT$$
$$= \frac{2(1-t)(1-t^2)\cdots(1-t^{2n})}{(1-t)^n}\frac{1}{v_{\lambda}(t)2^n(1-t)^n}\mathrm{Pf}[a_{j,k}]^{\lambda}$$
$$= \frac{(1-t)(1-t^2)\cdots(1-t^{2n})}{(1-t)^{2n}}\frac{1}{v_{\lambda}(t)2^{n-1}}\mathrm{Pf}[a_{j,k}]^{\lambda}.$$

But now by [7, 5.17]

$$\mathrm{Pf}[a_{j,k}]^{\lambda} = 2^{n-1}\Big[(-\alpha)^{\sum_{j=1}^{2n}[\lambda_j \bmod 2]} + (-\alpha)^{\sum_{j=1}^{2n}[(\lambda_j+1) \bmod 2]}\Big],$$

which gives the result. □

Theorem 2.6. *Let $l(\lambda) \le 2n$. We have the following integral identity for $O^-(2n)$:*

$$\frac{(1-\alpha^2)}{\int \tilde{\Delta}_K^{(n-1)}(\pm t, \pm\sqrt{t})dT} \int P_\lambda(x_1^{\pm 1}, \ldots, x_{n-1}^{\pm 1}, 1, -1; t)\tilde{\Delta}_K^{(n-1)}(\pm t, \pm\sqrt{t}) \prod_{i=1}^{n-1}(1-\alpha x_i^{\pm 1})dT$$
$$= \frac{\phi_{2n}(t)}{v_\lambda(t)(1-t)^{2n}}\Big[(-\alpha)^{\#\text{ of odd parts of }\lambda} - (-\alpha)^{\#\text{ of even parts of }\lambda}\Big].$$

Proof. We will first show the following:

$$\int R_\lambda(x_1^{\pm 1}, \ldots, x_{n-1}^{\pm 1}, 1, -1; t)\tilde{\Delta}_K^{(n-1)}(\pm t, \pm\sqrt{t}) \prod_{i=1}^{n-1}(1-\alpha x_i^{\pm 1})dT = \frac{(1+t)}{2}\frac{1}{2^{n-1}(1-t)^{n-1}}\mathrm{Pf}[M]^\lambda,$$

where the $(2n+2)\times(2n+2)$ antisymmetric matrix $[M]^\lambda$ is defined by

$$\begin{cases} M_{1,2}^\lambda = 0, & \\ M_{1,k}^\lambda = (-1)^{\lambda_{k-2}-(k-2)}, & \text{if } k \ge 3, \\ M_{2,k}^\lambda = 1, & \text{if } k \ge 3, \\ M_{j,k}^\lambda = a_{j-2,k-2}^\lambda, & \text{if } 3 \le j < k \le 2n+2, \end{cases}$$

and the $2n \times 2n$ matrix $[a_{j,k}]^\lambda$ is as in theorem 2.5.

Note first that the integral is a sum of $(2n)!$ terms, but by symmetry we may restrict to the "pseudomatchings"—those with ± 1 anywhere, but x_i to the left of x_i^{-1} for $1 \le i \le n-1$ and x_i to the left of x_j for $1 \le i < j \le n-1$. There are $(2n)!/2^{n-1}(n-1)!$ such pseudomatchings, and each has $2^{n-1}(n-1)!$ permutations with identical integral.

Claim 2.6.1. *Let w be a fixed pseudomatching with (-1) in position j and $(+1)$ in position k (here $1 \le j \ne k \le 2n$). Then we have the following:*

$$2^{n-1}(n-1)!\,\mathrm{P.V.}\int R_{\lambda,w}(x_1^{\pm 1}, \ldots, x_{n-1}^{\pm 1}, \pm 1; t)\tilde{\Delta}_K^{(n-1)}(\pm t, \pm\sqrt{t}) \prod_{i=1}^{n-1}(1-\alpha x_i^{\pm 1})dT$$
$$= 2^{n-1}(n-1)!(-1)^{\lambda_j+k-2+\chi_{j>k}}\frac{(1+t)}{2}$$
$$\times\,\mathrm{P.V.}\int R_{\tilde{\lambda},\tilde{w}}^{(2(n-1))}(x^{\pm 1}; t)\tilde{\Delta}_K^{(n-1)}(\pm 1, \pm\sqrt{t}) \prod_{i=1}^{n-1}(1-\alpha x_i^{\pm 1})dT,$$

where $\tilde{w}$ is w with ± 1 deleted (in particular, a matching in S_{2n-2}) and $\tilde{\lambda}$ is λ with parts λ_j, λ_k

deleted and all parts between λ_j *and* λ_k *increased by* 1*, so that (in the case* $j < k$*, for example)*

$$\tilde{\lambda} = (\lambda_1, \ldots, \lambda_{j-1}, \lambda_{j+1} + 1, \ldots, \lambda_{k-1} + 1, \lambda_{k+1}, \ldots, \lambda_{2n}).$$

We prove the claim. First, using (2.4), we have

$$2^{n-1}(n-1)!\tilde{\Delta}_K^{(n-1)}(\pm t, \pm\sqrt{t}) = \prod_{1 \le i \le n-1} \frac{1 - x_i^{\pm 2}}{(1 + tx_i^{\pm 1})(1 - tx_i^{\pm 1})(1 + \sqrt{t}x_i^{\pm 1})(1 - \sqrt{t}x_i^{\pm 1})} \prod_{1 \le i < j \le n-1} \frac{1 - x_i^{\pm 1}x_j^{\pm 1}}{1 - tx_i^{\pm 1}x_j^{\pm 1}}.$$

Define the set $X = \{(x_i^{\pm 1}, x_j^{\pm 1}) : 1 \le i \ne j \le n-1\}$, and let $u_{\lambda,w}^{(n-1)}(x;t)$ be defined by

$$R_{\lambda,w}(x_1^{\pm 1}, \ldots, x_{n-1}^{\pm 1}, \pm 1; t) = u_{\lambda,w}^{(n-1)}(x;t) \prod_{\substack{(z_i,z_j) \in X: \\ z_i \prec_w z_j}} \frac{z_i - tz_j}{z_i - z_j}.$$

Also define p_1 and Δ_1 by

$$u_{\lambda,w}^{(n-1)}(x;t) \prod_{1 \le i \le n-1} \frac{1 - x_i^{\pm 2}}{(1 + tx_i^{\pm 1})(1 - tx_i^{\pm 1})(1 + \sqrt{t}x_i^{\pm 1})(1 - \sqrt{t}x_i^{\pm 1})} \prod_{i=1}^{n-1}(1 - \alpha x_i^{\pm 1}) = p_1 \prod_{i=1}^{n-1} \frac{1}{1 - x_i^{-2}},$$

and

$$\prod_{1 \le i < j \le n-1} \frac{1 - x_i^{\pm 1}x_j^{\pm 1}}{1 - tx_i^{\pm 1}x_j^{\pm 1}} \prod_{\substack{(z_i,z_j) \in X: \\ z_i \prec_w z_j}} \frac{z_i - tz_j}{z_i - z_j} = \Delta_1.$$

Note that

$$R_{\lambda,w}(x_1^{\pm 1}, \ldots, x_{n-1}^{\pm 1}, \pm 1; t)\tilde{\Delta}_K^{(n-1)}(\pm t, \pm\sqrt{t}) \prod_{i=1}^{n-1}(1 - \alpha x_i^{\pm 1}) = p_1 \Delta_1 \prod_{i=1}^{n-1} \frac{1}{1 - x_i^{-2}}.$$

Define analogously p_2 and Δ_2 using $R_{\tilde{\lambda},\tilde{w}}(x_1^{\pm 1}, \ldots, x_{n-1}^{\pm 1}; t)$ and $\tilde{\Delta}_K^{(n-1)}(\pm 1, \pm\sqrt{t})$ instead of using $R_{\lambda,w}^{(2n)}(x^{\pm 1}, \pm 1; t)$ and $\tilde{\Delta}_K^{(n-1)}(\pm t, \pm\sqrt{t})$.

Then one can check $\Delta_1 = \Delta_2 =: \Delta$ and $\Delta(\pm 1, \ldots, \pm 1, x_{i+1}, \ldots, x_{n-1})$ is holomorphic in x_{i+1} for all $0 \le i \le n-2$ and all 2^i combinations. Also, the function $p = p_1 - (-1)^{\lambda_j + k - 2}\frac{(1+t)}{2}p_2$ (resp. $p = p_1 - (-1)^{\lambda_j + k - 1}\frac{(1+t)}{2}p_2$) satisfies the conditions of lemma 2.4 if $j < k$ (resp. $j > k$). So using that result, we have

$$\int p_1 \cdot \Delta \cdot \prod_{1 \le i \le n-1} \frac{1}{1 - x_i^{-2}} dT = (-1)^{\lambda_j + k - 2}\frac{(1+t)}{2} \int p_2 \cdot \Delta \cdot \prod_{1 \le i \le n-1} \frac{1}{1 - x_i^{-2}} dT,$$

if $j < k$ and

$$\int p_1 \cdot \Delta \cdot \prod_{1\le i\le n-1} \frac{1}{1-x_i^{-2}} dT = (-1)^{\lambda_j+k-1} \frac{(1+t)}{2} \int p_2 \cdot \Delta \cdot \prod_{1\le i\le n-1} \frac{1}{1-x_i^{-2}} dT,$$

if $j > k$. Thus, in the case $j < k$ we obtain

$$\int R_{\lambda,w}(x_1^{\pm 1}, \dots, x_{n-1}^{\pm 1}, \pm 1; t)\tilde{\Delta}_K^{(n-1)}(\pm t, \pm\sqrt{t}) \prod_{i=1}^{n-1}(1-\alpha x_i^{\pm 1}) dT$$
$$= (-1)^{\lambda_j+k-2} \frac{(1+t)}{2} \int R_{\tilde{\lambda},\tilde{w}}(x_1^{\pm 1}, \dots, x_{n-1}^{\pm 1}; t)\tilde{\Delta}_K^{(n-1)}(\pm 1, \pm\sqrt{t}) \prod_{i=1}^{n-1}(1-\alpha x_i^{\pm 1}) dT,$$

and analogously for the case $j > k$, which proves the claim.

As in theorem 2.5, we introduce notation for pseudomatchings. We will use the notation

$$\{(j,k),(i_1,i'_1),\dots,(i_{n-1},i'_{n-1})\}$$

for the pseudomatching with -1 in position j, 1 in position k and x_k in position i_k, x_k^{-1} in position i'_k for all $1 \le k \le n-1$. Note that we have $i_k < i'_k$ and $i_l < i_k$ for $l < k$. We may extend this to a matching in $S_{2(n+1)}$ by $\{(1, j+2), (2, k+2), (i_1+2, i'_1+2), \dots, (i_{n-1}+2, i'_{n-1}+2)\} = \{(j_1 = 1, j'_1 = j+2), (j_2 = 2, j'_2 = k+2), \dots, (j_{n+1}, j'_{n+1})\}$, with $i_k + 2 = j_{k+2}$ and $i'_k + 2 = j'_{k+2}$ for all $1 \le k \le n-1$.

Claim 2.6.2. *Let $w = \{(j,k),(i_1,i'_1),\dots,(i_{n-1},i'_{n-1})\}$ be a pseudomatching in S_{2n}, and extend it to a matching $\{(j_1 = 1, j'_1 = j+2), (j_2 = 2, j'_2 = k+2) \dots, (j_{n+1}, j'_{n+1})\}$ of $S_{2(n+1)}$ as discussed above. Let $\lambda = (\lambda_1, \dots, \lambda_{2n})$ with $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_{2n} \in \mathbb{Z}$. Then we have the following term evaluation:*

$$2^{n-1}(n-1)!\,\mathrm{P.V.}\int_T R_{\lambda,w}(x_1^{\pm 1}, \dots, x_{n-1}^{\pm 1}, \pm 1; t)\tilde{\Delta}_K^{(n-1)}(\pm t, \pm\sqrt{t}) \prod_{i=1}^{n-1}(1-\alpha x_i^{\pm 1}) dT$$
$$= \frac{1+t}{2} \frac{\epsilon(w)}{2^{n-1}(1-t)^{n-1}} \prod_{1\le k\le n+1} M^{\lambda}_{j_k,j'_k}.$$

We prove the claim. Let μ be such that $\lambda = \mu + \rho_{2n}$. By claim 2.6.1 the above left-hand side is equal to

$$\begin{cases} \frac{2^{n-1}(n-1)!(-1)^{\lambda_j+k-2}(1+t)}{2} \int R_{\tilde{\lambda},\tilde{w}}(x_1^{\pm 1},\ldots,x_{n-1}^{\pm 1};t)\tilde{\Delta}_K^{(n-1)}(\pm 1,\pm\sqrt{t})\prod_{i=1}^{n-1}(1-\alpha x_i^{\pm 1})dT & j<k, \\ \frac{2^{n-1}(n-1)!(-1)^{\lambda_j+k-1}(1+t)}{2} \int R_{\tilde{\lambda},\tilde{w}}(x_1^{\pm 1},\ldots,x_{n-1}^{\pm 1};t)\tilde{\Delta}_K^{(n-1)}(\pm 1,\pm\sqrt{t})\prod_{i=1}^{n-1}(1-\alpha x_i^{\pm 1})dT & j>k, \end{cases}$$

$$= 2^{n-1}(n-1)!\frac{1+t}{2}(-1)^{j_1'+j_2'-1-c_2(w)}M^{\lambda}_{1,j_1'}M^{\lambda}_{2,j_2'} \cdot \int_T R_{\tilde{\lambda},\tilde{w}}(x_1^{\pm 1},\ldots,x_{n-1}^{\pm 1};t)\tilde{\Delta}_K^{(n-1)}(\pm 1,\pm\sqrt{t})\prod_{i=1}^{n-1}(1-\alpha x_i^{\pm 1})dT,$$

where $c_2(w)$ is 0 if $j_1' > j_2'$ (i.e., $(1,j_1')$ and $(2,j_2')$ do not cross) and 1 if they do. Now we may use claim 2.5.1 on the $(n-1)$-dimensional integral: let $\widehat{\mu}$ be the partition μ with parts μ_j and μ_k deleted; note that $\tilde{\lambda}$ and $\widehat{\mu}+\rho_{2n-2}$ have equivalent parts modulo 2. Using this, we find that the above is equal to

$$\begin{aligned} &2^{n-1}(n-1)!\frac{1+t}{2}(-1)^{j_1'+j_2'-1-c_2(w)}M^{\lambda}_{1,j_1'}M^{\lambda}_{2,j_2'} \\ &\quad\cdot\int_T R_{\widehat{\mu}+\rho_{2n-2},\tilde{w}}(x_1^{\pm 1},\ldots,x_{n-1}^{\pm 1};t)\tilde{\Delta}_K^{(n-1)}(\pm 1,\pm\sqrt{t})\prod_{i=1}^{n-1}(1-\alpha x_i^{\pm 1})dT \\ &= \frac{1+t}{2}(-1)^{j_1'+j_2'-1-c_2(w)}M^{\lambda}_{1,j_1'}M^{\lambda}_{2,j_2'}\frac{\epsilon(\tilde{w})}{2^{n-1}(1-t)^{n-1}}\prod_{1\le k\le n-1} a^{\widehat{\mu}+\rho_{2n-2}}_{i_k,i_k'} \\ &= \frac{1+t}{2}\frac{\epsilon(w)}{2^{n-1}(1-t)^{n-1}}\prod_{1\le k\le n+1} M^{\lambda}_{j_k,j_k'}, \end{aligned}$$

as desired.

Note that in particular this result shows that the integral of a matching is a term in $\mathrm{Pf}[M]^{\lambda}(1+t)/2^n(1-t)^{n-1}$.

Now using the claim, we have

$$\int R_{\lambda}(x_1^{\pm 1},\ldots,x_{n-1}^{\pm 1},1,-1;t)\tilde{\Delta}_K^{(n-1)}(\pm t,\pm\sqrt{t})\prod_{i=1}^{n-1}(1-\alpha x_i^{\pm 1})dT = \frac{(1+t)}{2}\frac{1}{2^{n-1}(1-t)^{n-1}}\mathrm{Pf}[M]^{\lambda},$$

since the terms of the Pfaffian are in bijection with the integrals of the pseudomatchings.

Finally, to prove the theorem, we use proposition 2.2(iv) to obtain

$$\begin{aligned} &\frac{(1-\alpha^2)}{\int \tilde{\Delta}_K^{(n-1)}(\pm t,\pm\sqrt{t})dT}\int P_{\lambda}(x_1^{\pm 1},\ldots,x_{n-1}^{\pm 1},1,-1;t)\tilde{\Delta}_K^{(n-1)}(\pm t,\pm\sqrt{t})\prod_{i=1}^{n-1}(1-\alpha x_i^{\pm 1})dT \\ &\quad= \frac{(1-\alpha^2)(1-t)(1-t^2)\cdots(1-t^{2n})}{v_{\lambda}(t)(1-t)^{n+1}}\frac{1}{2^n(1-t)^{n-1}}\mathrm{Pf}[M]^{\lambda} = \frac{\phi_{2n}(t)}{v_{\lambda}(t)(1-t)^{2n}}\frac{(1-\alpha^2)}{2^n}\mathrm{Pf}[M]^{\lambda}. \end{aligned}$$

Following the computation in [7, 5.21] (but noting that they are missing a factor of 2), $\mathrm{Pf}[M]^{\lambda}$

may be evaluated as

$$\frac{2^n}{(1-\alpha^2)}\Big[(-\alpha)^{\sum_{j=1}^{2n}[\lambda_j \bmod 2]} - (-\alpha)^{\sum_{j=1}^{2n}[(\lambda_j+1) \bmod 2]}\Big],$$

which proves the theorem. □

Theorem 2.7. *Let $l(\lambda) \le 2n+1$. We have the following integral identity for $O^+(2n+1)$:*

$$\frac{(1-\alpha)}{\int \tilde{\Delta}_K^{(n)}(t,-1,\pm\sqrt{t})dT}\int P_\lambda(x_1^{\pm 1},\dots,x_n^{\pm 1},1;t)\tilde{\Delta}_K^{(n)}(t,-1,\pm\sqrt{t})\prod_{i=1}^n(1-\alpha x_i^{\pm 1})dT$$
$$= \frac{\phi_{2n+1}(t)}{v_\lambda(t)(1-t)^{2n+1}}\Big[(-\alpha)^{\#\ of\ odd\ parts\ of\ \lambda} + (-\alpha)^{\#\ of\ even\ parts\ of\ \lambda}\Big].$$

Proof. We use an argument analogous to the $O^-(2n)$ case. We will first show the following:

$$\int R_\lambda(x_1^{\pm 1},\dots,x_n^{\pm 1},1;t)\tilde{\Delta}_K^{(n)}(t,-1,\pm\sqrt{t})\prod_{i=1}^n(1-\alpha x_i^{\pm 1})dT = \frac{1}{2^n(1-t)^n}\mathrm{Pf}[M]^\lambda,$$

where the $2n+2\times 2n+2$ antisymmetric matrix $[M]^\lambda$ is given by

$$\begin{cases} M_{1,k}^\lambda = 1, & \text{if } 1 < k \le 2n+2, \\ M_{j,k}^\lambda = a_{j-1,k-1}^\lambda, & \text{if } 2 \le j \le k \le 2n+2, \end{cases}$$

and as usual $[a_{j,k}]^\lambda$ is the $2n+1\times 2n+1$ antisymmetric matrix specified by theorem 2.5. The integral is a sum of $(2n+1)!$ terms, one for each permutation in S_{2n+1}. But note that by symmetry we may restrict to pseudomatchings in S_{2n+1}: those with 1 anywhere but x_i to the left of x_i^{-1} for all $1 \le i \le n$, and x_i to the left of x_j for $1 \le i < j \le n$. There are $(2n+1)!/2^n n!$ such pseudomatchings, and for each there are exactly $2^n n!$ other permutations with identical integral value.

Claim 2.7.1. *Let w be a fixed pseudomatching with 1 in position k, for some $1 \le k \le 2n+1$. Then we have the following:*

$$2^n n!\,\mathrm{P.V.}\int R_{\lambda,w}(x_1^{\pm 1},\dots,x_n^{\pm 1},1;t)\tilde{\Delta}_K^{(n)}(t,-1,\pm\sqrt{t})\prod_{i=1}^n(1-\alpha x_i^{\pm 1})dT$$
$$= 2^n n!(-1)^{k-1}\,\mathrm{P.V.}\int R_{\tilde{\lambda},\tilde{w}}(x_1^{\pm 1},\dots,x_n^{\pm 1};t)\tilde{\Delta}_K^{(n)}(\pm 1,\pm\sqrt{t})\prod_{i=1}^n(1-\alpha x_i^{\pm 1})dT,$$

where $\tilde{w}$ is w with 1 deleted (in particular, a matching in S_{2n}) and $\tilde{\lambda}$ is λ with λ_k deleted and the parts to the left of λ_k increased by 1, i.e.,

$$\tilde{\lambda} = (\lambda_1+1,\dots,\lambda_{k-1}+1,\lambda_{k+1},\dots,\lambda_{2n+1}).$$

We prove the claim; note that this proof is very similar to claim 2.6.1 for the $O^-(2n)$ case. First, using (2.4), we have

$$2^n n!\tilde{\Delta}_K^{(n)}(t,-1,\pm\sqrt{t}) = \prod_{1\leq i\leq n} \frac{1-x_i^{\pm 2}}{(1-tx_i^{\pm 1})(1+x_i^{\pm 1})(1-\sqrt{t}x_i^{\pm 1})(1+\sqrt{t}x_i^{\pm 1})} \prod_{1\leq i<j\leq n} \frac{1-x_i^{\pm 1}x_j^{\pm 1}}{1-tx_i^{\pm 1}x_j^{\pm 1}}.$$

Define the set $X = \{(x_i^{\pm 1}, x_j^{\pm 1}) : 1 \leq i \neq j \leq n\}$, and let $u_{\lambda,w}^{(n)}(x;t)$ be defined by

$$R_{\lambda,w}(x_1^{\pm 1},\ldots,x_n^{\pm 1},1;t) = u_{\lambda,w}^{(n)}(x;t) \prod_{\substack{(z_i,z_j)\in X:\\ z_i \prec_w z_j}} \frac{z_i - tz_j}{z_i - z_j}.$$

Also define p_1 and Δ_1 by

$$u_{\lambda,w}^{(n)}(x;t) \prod_{1\leq i\leq n} \frac{1-x_i^{\pm 2}}{(1-tx_i^{\pm 1})(1+x_i^{\pm 1})(1-\sqrt{t}x_i^{\pm 1})(1+\sqrt{t}x_i^{\pm 1})} \prod_{i=1}^{n}(1-\alpha x_i^{\pm 1}) = p_1 \prod_{i=1}^{n} \frac{1}{1-x_i^{-2}}$$

and

$$\prod_{1\leq i<j\leq n} \frac{1-x_i^{\pm 1}x_j^{\pm 1}}{1-tx_i^{\pm 1}x_j^{\pm 1}} \prod_{\substack{(z_i,z_j)\in X:\\ z_i \prec_w z_j}} \frac{z_i - tz_j}{z_i - z_j} = \Delta_1.$$

Note that

$$R_{\lambda,w}(x_1^{\pm 1},\ldots,x_n^{\pm 1},1;t)\tilde{\Delta}_K^{(n)}(t,-1,\pm\sqrt{t}) \prod_{i=1}^{n}(1-\alpha x_i^{\pm 1}) = p_1\Delta_1 \prod_{i=1}^{n} \frac{1}{1-x_i^{-2}}.$$

Define analogously p_2 and Δ_2 using $R_{\tilde{\lambda},\tilde{w}}(x_1^{\pm 1},\ldots,x_n^{\pm 1};t)$, and $\tilde{\Delta}_K^{(n)}(\pm 1,\pm\sqrt{t})$, instead of using $R_{\lambda,w}^{(2n+1)}(x^{\pm 1},1;t)$, and $\tilde{\Delta}_K^{(n)}(t,-1,\pm\sqrt{t})$.

Then note that $\Delta_1 = \Delta_2 := \Delta$. Some computation shows that $\Delta(\pm 1,\ldots,\pm 1,x_{i+1},\ldots,x_n)$ is holomorphic in x_{i+1} for all $0 \leq i \leq n-1$ and all 2^i combinations. Further computations show that the function $p = p_1 - (-1)^{k-1}p_2$ satisfies the conditions of lemma 2.4, so we have

$$\int p \cdot \Delta \cdot \prod_{i=1}^{n} \frac{1}{1-x_i^{-2}} dT = 0,$$

or

$$\int p_1 \cdot \Delta_1 \cdot \prod_{i=1}^{n} \frac{1}{1-x_i^{-2}} dT = (-1)^{k-1} \int p_2 \cdot \Delta_2 \cdot \prod_{i=1}^{n} \frac{1}{1-x_i^{-2}} dT,$$

which proves the claim.

In keeping with the notation of the previous two theorems, we write $\{(k),(i_1,i'_1),\ldots,(i_n,i'_n)\}$ for the pseudomatching w with 1 in position k and x_k in position i_k, x_k^{-1} in position i'_k, for all $1 \leq k \leq n$. We can extend this to a matching in $S_{2(n+1)}$ by $\{(1,k+1),(i_1+1,i'_1+1),\ldots,(i_n+1,i'_n+1)\} =$

$\{(j_1 = 1, j'_1 = k+1), \ldots, (j_{n+1}, j'_{n+1})\}$, with $i_k + 1 = j_{k+1}, i_{k'} + 1 = j'_{k+1}$ for $1 \le k \le n$.

Claim 2.7.2. *Let $w = \{(k), (i_1, i'_1), \ldots, (i_n, i'_n)\}$ be a pseudomatching in S_{2n+1}, and extend it to a matching $\{(j_1 = 1, j'_1 = k+1), \ldots, (j_{n+1}, j'_{n+1})\}$ as discussed above. Let $\lambda = (\lambda_1, \ldots, \lambda_{2n+1})$ with $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_{2n+1} \in \mathbb{Z}$. Then we have the following term evaluation:*

$$2^n n!\,\mathrm{P.\,V.} \int_T R_{\lambda,w}(x_1^{\pm 1}, \ldots, x_n^{\pm 1}, 1; t)\tilde{\Delta}_K^{(n)}(t, -1, \pm\sqrt{t}) \prod_{i=1}^n (1 - \alpha x_i^{\pm 1}) dT$$
$$= \frac{\epsilon(w)}{2^n(1-t)^n} \prod_{1 \le k \le n+1} M^\lambda_{j_k, j'_k}.$$

We prove the claim. Let μ be such that $\lambda = \mu + \rho_{2n+1}$. By claim 2.7.1 the above left-hand side is equal to

$$2^n n! (-1)^{k-1} \int_T R_{\tilde{\lambda},\tilde{w}}(x_1^{\pm 1}, \ldots, x_n^{\pm 1}; t)\tilde{\Delta}_K^{(n)}(\pm 1, \pm\sqrt{t}) \prod_{i=1}^n (1 - \alpha x_i^{\pm 1}) dT$$
$$= 2^n n! (-1)^{j'_1 - j_1 + 1} M^\lambda_{j_1, j'_1} \int_T R_{\tilde{\lambda},\tilde{w}}(x_1^{\pm 1}, \ldots, x_n^{\pm 1}; t)\tilde{\Delta}_K^{(n)}(\pm 1, \pm\sqrt{t}) \prod_{i=1}^n (1 - \alpha x_i^{\pm 1}) dT.$$

Now we use claim 2.5.1: let $\widehat{\mu}$ be μ with part μ_k deleted; note $\tilde{\lambda} - 1^{2n} = \widehat{\mu} + \rho_{2n}$. Using that result, the above is equal to

$$2^n n! (-1)^{j'_1 - j_1 + 1} M^\lambda_{j_1, j'_1} \int_T R_{\widehat{\mu}+\rho_{2n},\tilde{w}}(x_1^{\pm 1}, \ldots, x_n^{\pm 1}; t)\tilde{\Delta}_K^{(n)}(\pm 1, \pm\sqrt{t}) \prod_{i=1}^n (1 - \alpha x_i^{\pm 1}) dT$$
$$= (-1)^{j'_1 - j_1 + 1} M^\lambda_{j_1, j'_1} \frac{\epsilon(\tilde{w})}{2^n(1-t)^n} \prod_{1 \le k \le n} a^{\widehat{\mu}+\rho_{2n}}_{i_k, i'_k} = \frac{\epsilon(w)}{2^n(1-t)^n} \prod_{1 \le k \le n+1} M^\lambda_{j_k, j'_k},$$

as desired.

Note that in particular this result shows that the integral of a matching is a term in the expansion of $\frac{1}{2^n(1-t)^n}\mathrm{Pf}[M]^\lambda$.

Now using the claim, we have

$$\int R_\lambda(x_1^{\pm 1}, \ldots, x_n^{\pm 1}, 1; t)\tilde{\Delta}_K^{(n)}(t, -1, \pm\sqrt{t}) \prod_{i=1}^n (1 - \alpha x_i^{\pm 1}) dT = \frac{1}{2^n(1-t)^n}\mathrm{Pf}[M]^\lambda,$$

since the terms of the Pfaffian are in bijection with the integrals of the pseudomatchings.

Finally, to prove the theorem, we use proposition 2.2(v) to obtain

$$\frac{(1-\alpha)}{\int \tilde{\Delta}_K^{(n)}(t, -1, \pm\sqrt{t}) dT} \int P_\lambda(x_1^{\pm 1}, \ldots, x_n^{\pm 1}, 1; t)\tilde{\Delta}_K^{(n)}(t, -1, \pm\sqrt{t}) \prod_{i=1}^n (1 - \alpha x_i^{\pm 1}) dT$$
$$= \frac{(1-\alpha)\phi_{2n+1}(t)}{v_\lambda(t)(1-t)^{n+1}} \frac{1}{2^n(1-t)^n}\mathrm{Pf}[M]^\lambda,$$

but by a change of basis $[M]^\lambda$ is equivalent to the one defined in [7, 5.24], and that Pfaffian was computed to be

$$\frac{2^n}{(1-\alpha)}\left[(-\alpha)^{\sum_{j=1}^{2n+1}[\lambda_j \bmod 2]} + (-\alpha)^{\sum_{j=1}^{2n+1}[(\lambda_j+1) \bmod 2]}\right],$$

which proves the theorem. □

Theorem 2.8. *Let $l(\lambda) \le 2n+1$. We have the following integral identity for $O^-(2n+1)$:*

$$\frac{(1+\alpha)}{\int \tilde{\Delta}_K^{(n)}(1,-t,\pm\sqrt{t})dT} \int P_\lambda(x_1^{\pm 1}, \ldots, x_n^{\pm 1}, -1; t)\tilde{\Delta}_K^{(n)}(1,-t,\pm\sqrt{t}) \prod_{i=1}^{n}(1-\alpha x_i^{\pm 1})dT$$
$$= \frac{\phi_{2n+1}(t)}{v_\lambda(t)(1-t)^{2n+1}}\left[(-\alpha)^{\#\text{ of odd parts of }\lambda} - (-\alpha)^{\#\text{ of even parts of }\lambda}\right],$$

Proof. We obtain the $O^-(2n+1)$ integral from the $O^+(2n+1)$ integral. See the discussion for the $O^-(2n+1)$ integral in the next section. The upshot is that the $O^-(2n+1)$ integral is $(-1)^{|\lambda|}$ times the $O^+(2n+1)$ integral with parameter $-\alpha$. Using theorem 2.7, we get

$$(-1)^{|\lambda|}\frac{\phi_{2n+1}(t)}{v_\lambda(t)(1-t)^{2n+1}}\left[\alpha^{\#\text{ of odd parts of }\lambda} + \alpha^{\#\text{ of even parts of }\lambda}\right].$$

But note that $(-1)^{\lambda_i}$ is -1 if λ_i is odd, and 1 if λ_i is even, so that $(-1)^{|\lambda|} = (-1)^{\#\text{ of odd parts of }\lambda}$. Also,

$$(-1)^{\#\text{ of odd parts of }\lambda}(-1)^{\#\text{ of even parts of }\lambda} = (-1)^{2n+1} = -1.$$

Combining these facts gives the result. □

We briefly mention some existing results related to theorems 2.5, 2.6, 2.7, and 2.8. First, note that these four results are t-analogs of the results of proposition 2 of [7]. For example, in the $O^+(2n)$ case, that result states

$$\langle \det(1_{2n} + \alpha U)s_\rho(U)\rangle_{U \in O^+(2n)} = \frac{1}{2^{n-1}}\mathrm{Pf}[a_{jk}] = \alpha^{\sum_{j=1}^{2n}[\rho_j \bmod 2]} + \alpha^{\sum_{j=1}^{2n}[(\rho_j+1) \bmod 2]},$$

where $\langle\cdot\rangle_{O^+(2n)}$ denotes the integral with respect to the eigenvalue density of the group $O^+(2n)$.

Also, note that the $\alpha = 0$ case of these identities gives that the four integrals

$$\frac{1}{Z}\int P_\lambda(x_1^{\pm 1},\dots,x_n^{\pm 1};t)\tilde{\Delta}_K^{(n)}(\pm 1,\pm\sqrt{t})dT,$$
$$\frac{1}{Z}\int P_\lambda(x_1^{\pm 1},\dots,x_{n-1}^{\pm 1},\pm 1;t)\tilde{\Delta}_K^{(n-1)}(\pm t,\pm\sqrt{t})dT,$$
$$\frac{1}{Z}\int P_\lambda(x_1^{\pm 1},\dots,x_n^{\pm 1},1;t)\tilde{\Delta}_K^{(n)}(t,-1,\pm\sqrt{t})dT,$$
$$\frac{1}{Z}\int P_\lambda(x_1^{\pm 1},\dots,x_n^{\pm 1},-1;t)\tilde{\Delta}_K^{(n)}(1,-t,\pm\sqrt{t})dT$$

vanish unless all $2n$ or $2n+1$ (as appropriate) parts of λ have the same parity (see theorem 4.1 of [20]). Here Z is the normalization: it makes the integral equal to unity when λ is the zero partition.

2.4 An α, β-generalization

In this section, we further generalize the identities of the previous section by using the Pieri rule to add an extra parameter β. The values are given in terms of Rogers–Szegő polynomials (2.3).

Theorem 2.9. *We have the following integral identities:*

(i) for $O(2n)$

$$\frac{1}{\int \tilde{\Delta}_K^{(n)}(\pm 1,\pm\sqrt{t})dT}\int P_\mu(x_1^{\pm 1},\dots,x_n^{\pm 1};t)\tilde{\Delta}_K^{(n)}(\pm 1,\pm\sqrt{t})\prod_{i=1}^{n}(1-\alpha x_i^{\pm 1})(1-\beta x_i^{\pm 1})dT +$$
$$\frac{(1-\alpha^2)(1-\beta^2)}{\int \tilde{\Delta}_K^{(n-1)}(\pm t,\pm\sqrt{t})dT}\int P_\mu(x_1^{\pm 1},\dots,x_{n-1}^{\pm 1},\pm 1;t)\tilde{\Delta}_K^{(n-1)}(\pm t,\pm\sqrt{t})\prod_{i=1}^{n-1}(1-\alpha x_i^{\pm 1})(1-\beta x_i^{\pm 1})dT$$
$$=\frac{2\phi_{2n}(t)}{v_\mu(t)(1-t)^{2n}}\Big[\Big(\prod_{i\geq 0}H_{m_{2i}(\mu)}(\alpha\beta;t)\prod_{i\geq 0}H_{m_{2i+1}(\mu)}(\beta/\alpha;t)\Big)(-\alpha)^{\#\text{ of odd parts of }\mu}\Big].$$

(ii) for $O(2n+1)$

$$\frac{(1-\alpha)(1-\beta)}{\int \tilde{\Delta}_K^{(n)}(t,-1,\pm\sqrt{t})dT}\int P_\mu(x_1^{\pm 1},\dots,x_n^{\pm 1},1;t)\tilde{\Delta}_K^{(n)}(t,-1,\pm\sqrt{t})\prod_{i=1}^{n}(1-\alpha x_i^{\pm 1})(1-\beta x_i^{\pm 1})dT$$
$$+\frac{(1+\alpha)(1+\beta)}{\int \tilde{\Delta}_K^{(n)}(1,-t,\pm\sqrt{t})dT}\int P_\mu(x_1^{\pm 1},\dots,x_n^{\pm 1},-1;t)\tilde{\Delta}_K^{(n)}(1,-t,\pm\sqrt{t})\prod_{i=1}^{n}(1-\alpha x_i^{\pm 1})(1-\beta x_i^{\pm 1})dT$$
$$=\frac{2\phi_{2n+1}(t)}{v_\mu(t)(1-t)^{2n+1}}\Big[\Big(\prod_{i\geq 0}H_{m_{2i}(\mu)}(\alpha\beta;t)\prod_{i\geq 0}H_{m_{2i+1}(\mu)}(\beta/\alpha;t)\Big)(-\alpha)^{\#\text{ of odd parts of }\mu}\Big].$$

Proof. The proof follows Warnaar's argument (see theorem 1.1 of [24]), with the only difference being that we take into account zero parts in the computation, whereas Warnaar's infinite version is concerned only with nonzero parts. The basic method is to use the Pieri rule for $P_\mu(x;t)e_r(x)$

in combination with the results of the previous section (the sum of the results of theorems 2.5, 2.6 for $O(2n)$ and similarly theorems 2.7, 2.8 for $O(2n+1)$). Note that Warnaar starts with the case $a = b = 0$ in his notation (the orthogonal group case) and successively applies the Pieri rule two times, introducing a parameter each time. Because we proved the α case in the previous section, we need only use the Pieri rule once. □

Theorem 2.10. *Write $\lambda = 0^{m_0(\lambda)}\ 1^{m_1(\lambda)}\ 2^{m_2(\lambda)}\cdots$, with total number of parts $2n$ or $2n+1$ as necessary. Then we have the following integral identities for the components of the orthogonal group:*

(i) for $O^+(2n)$

$$\frac{1}{Z}\int P_\lambda(x_1^{\pm 1},\dots,x_n^{\pm 1};t)\tilde{\Delta}_K^{(n)}(\pm 1,\pm\sqrt{t})\prod_{i=1}^{n}(1-\alpha x_i^{\pm 1})(1-\beta x_i^{\pm 1})dT$$
$$= \frac{\phi_{2n}(t)}{v_\lambda(t)(1-t)^{2n}}\Big[\Big(\prod_{i\geq 0}H_{m_{2i}(\lambda)}(\alpha\beta;t)\prod_{i\geq 0}H_{m_{2i+1}(\lambda)}(\beta/\alpha;t)\Big)(-\alpha)^{\#\text{ of odd parts of }\lambda}$$
$$+\Big(\prod_{i\geq 0}H_{m_{2i+1}(\lambda)}(\alpha\beta;t)\prod_{i\geq 0}H_{m_{2i}(\lambda)}(\beta/\alpha;t)\Big)(-\alpha)^{\#\text{ of even parts of }\lambda}\Big],$$

(ii) for $O^-(2n)$

$$\frac{(1-\alpha^2)(1-\beta^2)}{Z}\int P_\lambda(x_1^{\pm 1},\dots,x_{n-1}^{\pm 1},1,-1;t)\tilde{\Delta}_K^{(n-1)}(\pm t,\pm\sqrt{t})\prod_{i=1}^{n-1}(1-\alpha x_i^{\pm 1})(1-\beta x_i^{\pm 1})dT$$
$$= \frac{\phi_{2n}(t)}{v_\lambda(t)(1-t)^{2n}}\Big[\Big(\prod_{i\geq 0}H_{m_{2i}(\lambda)}(\alpha\beta;t)\prod_{i\geq 0}H_{m_{2i+1}(\lambda)}(\beta/\alpha;t)\Big)(-\alpha)^{\#\text{ of odd parts of }\lambda}$$
$$-\Big(\prod_{i\geq 0}H_{m_{2i+1}(\lambda)}(\alpha\beta;t)\prod_{i\geq 0}H_{m_{2i}(\lambda)}(\beta/\alpha;t)\Big)(-\alpha)^{\#\text{ of even parts of }\lambda}\Big],$$

(iii) for $O^+(2n+1)$

$$\frac{(1-\alpha)(1-\beta)}{Z}\int P_\lambda(x_1^{\pm 1},\dots,x_n^{\pm 1},1;t)\tilde{\Delta}_K^{(n)}(t,-1,\pm\sqrt{t})\prod_{i=1}^{n}(1-\alpha x_i^{\pm 1})(1-\beta x_i^{\pm 1})dT$$
$$= \frac{\phi_{2n+1}(t)}{v_\lambda(t)(1-t)^{2n+1}}\Big[\Big(\prod_{i\geq 0}H_{m_{2i}(\lambda)}(\alpha\beta;t)\prod_{i\geq 0}H_{m_{2i+1}(\lambda)}(\beta/\alpha;t)\Big)(-\alpha)^{\#\text{ of odd parts of }\lambda}$$
$$+\Big(\prod_{i\geq 0}H_{m_{2i+1}(\lambda)}(\alpha\beta;t)\prod_{i\geq 0}H_{m_{2i}(\lambda)}(\beta/\alpha;t)\Big)(-\alpha)^{\#\text{ of even parts of }\lambda}\Big],$$

(iv) for $O^-(2n+1)$

$$\frac{(1+\alpha)(1+\beta)}{Z}\int P_\lambda(x_1^{\pm1},\dots,x_n^{\pm1},-1;t)\tilde{\Delta}_K^{(n)}(1,-t,\pm\sqrt{t})\prod_{i=1}^{n}(1-\alpha x_i^{\pm1})(1-\beta x_i^{\pm1})dT$$
$$=\frac{\phi_{2n+1}(t)}{v_\lambda(t)(1-t)^{2n+1}}\Big[\Big(\prod_{i\geq0}H_{m_{2i}(\lambda)}(\alpha\beta;t)\prod_{i\geq0}H_{m_{2i+1}(\lambda)}(\beta/\alpha;t)\Big)(-\alpha)^{\#\text{ of odd parts of }\lambda}$$
$$-\Big(\prod_{i\geq0}H_{m_{2i+1}(\lambda)}(\alpha\beta;t)\prod_{i\geq0}H_{m_{2i}(\lambda)}(\beta/\alpha;t)\Big)(-\alpha)^{\#\text{ of even parts of }\lambda}\Big],$$

where Z is the normalization at $\alpha=0,\beta=0$ and $\lambda=0^{2n},0^{2n+1}$ as appropriate.

Proof. Note that the Hall–Littlewood polynomials satisfy the following property:

$$\Big(\prod_{i=1}^{l}z_i\Big)P_\lambda(z_1,\dots,z_l;t)=P_{\lambda+1^l}(z_1,\dots,z_l;t).$$

So in the case $O(2n)$, for example, we have

$$P_\mu(x_1^{\pm1},\dots,x_n^{\pm1};t)=P_{\mu+1^{2n}}(x_1^{\pm1},\dots,x_n^{\pm1};t)$$
$$P_\mu(x_1^{\pm1},\dots,x_{n-1}^{\pm1},1,-1;t)=-P_{\mu+1^{2n}}(x_1^{\pm1},\dots,x_{n-1}^{\pm1},1,-1;t).$$

Thus,

$$\frac{1}{\int\tilde{\Delta}_K^{(n)}(\pm1,\pm\sqrt{t})dT}\int P_\mu(x_1^{\pm1},\dots,x_n^{\pm1};t)\tilde{\Delta}_K^{(n)}(\pm1,\pm\sqrt{t})\prod_{i=1}^{n}(1-\alpha x_i^{\pm1})(1-\beta x_i^{\pm1})dT$$
$$-\frac{(1-\alpha^2)(1-\beta^2)}{\int\tilde{\Delta}_K^{(n-1)}(\pm t,\pm\sqrt{t})dT}\int P_\mu(x_1^{\pm1},\dots,x_{n-1}^{\pm1},1,-1;t)\tilde{\Delta}_K^{(n-1)}(\pm t,\pm\sqrt{t})\prod_{i=1}^{n-1}(1-\alpha x_i^{\pm1})(1-\beta x_i^{\pm1})dT$$
$$=\frac{1}{\int\tilde{\Delta}_K^{(n)}(\pm1,\pm\sqrt{t})dT}\int P_{\mu+1^{2n}}(x_1^{\pm1},\dots,x_n^{\pm1};t)\tilde{\Delta}_K^{(n)}(\pm1,\pm\sqrt{t})\prod_{i=1}^{n}(1-\alpha x_i^{\pm1})(1-\beta x_i^{\pm1})dT+$$
$$\frac{(1-\alpha^2)(1-\beta^2)}{\int\tilde{\Delta}_K^{(n-1)}(\pm t,\pm\sqrt{t})dT}\int P_{\mu+1^{2n}}(x_1^{\pm1},\dots,x_{n-1}^{\pm1},1,-1;t)\tilde{\Delta}_K^{(n-1)}(\pm t,\pm\sqrt{t})\prod_{i=1}^{n-1}(1-\alpha x_i^{\pm1})(1-\beta x_i^{\pm1})dT$$
$$=\frac{2\phi_{2n}(t)}{v_{\mu+1^{2n}}(t)(1-t)^{2n}}\Big[\Big(\prod_{i\geq0}H_{m_{2i}(\mu+1^{2n})}(\alpha\beta;t)\prod_{i\geq0}H_{m_{2i+1}(\mu+1^{2n})}(\beta/\alpha;t)\Big)(-\alpha)^{\#\text{odd parts of }\mu+1^{2n}}\Big],$$

where the last equality follows from theorem 2.9(i). Now note that $v_{\mu+1^{2n}}(t)=v_\mu(t)$, $m_i(\mu+1^{2n})=m_{i-1}(\mu)$ for all $i\geq1$, and the number of odd parts in $\mu+1^{2n}$ is the same as the number of even parts in μ. Thus the above is equal to

$$\frac{2\phi_{2n}(t)}{v_\mu(t)(1-t)^{2n}}\Big[\Big(\prod_{i\geq 0} H_{m_{2i+1}(\mu)}(\alpha\beta;t)\prod_{i\geq 0} H_{m_{2i}(\mu)}(\beta/\alpha;t)\Big)(-\alpha)^{\#\text{ of even parts of }\mu}\Big].$$

Then, taking the sum/difference of this equation and theorem 2.9(i), we obtain

$$\begin{aligned}\frac{2}{\int \tilde\Delta_K^{(n)}(\pm 1,\pm\sqrt{t})dT}&\int P_\mu(x_1^{\pm 1},\dots,x_n^{\pm 1};t)\tilde\Delta_K^{(n)}(\pm 1,\pm\sqrt{t})\prod_{i=1}^n(1-\alpha x_i^{\pm 1})(1-\beta x_i^{\pm 1})dT\\ &=\frac{2\phi_{2n}(t)}{v_\lambda(t)(1-t)^{2n}}\Big[\Big(\prod_{i\geq 0} H_{m_{2i}(\lambda)}(\alpha\beta;t)\prod_{i\geq 0} H_{m_{2i+1}(\lambda)}(\beta/\alpha;t)\Big)(-\alpha)^{\#\text{ of odd parts of }\lambda}\\ &\qquad+\Big(\prod_{i\geq 0} H_{m_{2i+1}(\lambda)}(\alpha\beta;t)\prod_{i\geq 0} H_{m_{2i}(\lambda)}(\beta/\alpha;t)\Big)(-\alpha)^{\#\text{ of even parts of }\lambda}\Big],\end{aligned}$$

and

$$\begin{aligned}\frac{2(1-\alpha^2)(1-\beta^2)}{\int \tilde\Delta_K^{(n-1)}(\pm t,\pm\sqrt{t})}&\int P_\lambda(x_1^{\pm 1},\dots,x_{n-1}^{\pm 1},1,-1;t)\tilde\Delta_K^{(n-1)}(\pm t,\pm\sqrt{t})\prod_{i=1}^{n-1}(1-\alpha x_i^{\pm 1})(1-\beta x_i^{\pm 1})dT\\ &=\frac{2\phi_{2n}(t)}{v_\lambda(t)(1-t)^{2n}}\Big[\Big(\prod_{i\geq 0} H_{m_{2i}(\lambda)}(\alpha\beta;t)\prod_{i\geq 0} H_{m_{2i+1}(\lambda)}(\beta/\alpha;t)\Big)(-\alpha)^{\#\text{ of odd parts of }\lambda}\\ &\qquad-\Big(\prod_{i\geq 0} H_{m_{2i+1}(\lambda)}(\alpha\beta;t)\prod_{i\geq 0} H_{m_{2i}(\lambda)}(\beta/\alpha;t)\Big)(-\alpha)^{\#\text{ of even parts of }\lambda}\Big],\end{aligned}$$

as desired. The $O(2n+1)$ result is analogous; use instead theorem 2.9(ii). Note alternatively that as in the α case, we can obtain the $O^-(2n+1)$ integral directly from the $O^+(2n+1)$ integral, since the change of variables $x_i\to -x_i$ gives

$$\begin{aligned}&\int P_\lambda(x_1^{\pm 1},\dots,x_n^{\pm 1},-1;t)\tilde\Delta_K^{(n)}(1,-t,\pm\sqrt{t})\prod_{i=1}^n(1-\alpha x_i^{\pm 1})(1-\beta x_i^{\pm 1})dT\\ &\quad=\int P_\lambda(-x_1^{\pm 1},\dots,-x_n^{\pm 1},-1;t)\tilde\Delta_K^{(n)}(-1,t,\pm\sqrt{t})\prod_{i=1}^n(1+\alpha x_i^{\pm 1})(1+\beta x_i^{\pm 1})dT\\ &\qquad=(-1)^{|\lambda|}\int P_\lambda(x_1^{\pm 1},\dots,x_n^{\pm 1},1;t)\tilde\Delta_K^{(n)}(-1,t,\pm\sqrt{t})\prod_{i=1}^n(1+\alpha x_i^{\pm 1})(1+\beta x_i^{\pm 1})dT,\end{aligned}$$

and $\int\tilde\Delta_K^{(n)}(1,-t,\pm\sqrt{t})dT=\int\tilde\Delta_K^{(n)}(-1,t,\pm\sqrt{t})dT$, so that

$$\begin{aligned}&\frac{(1+\alpha)(1+\beta)}{\int\tilde\Delta_K^{(n)}(1,-t,\pm\sqrt{t})dT}\int P_\lambda(x_1^{\pm 1},\dots,x_n^{\pm 1},-1;t)\tilde\Delta_K^{(n)}(1,-t,\pm\sqrt{t})\prod_{i=1}^n(1-\alpha x_i^{\pm 1})(1-\beta x_i^{\pm 1})dT\\ &=\frac{(-1)^{|\lambda|}(1+\alpha)(1+\beta)}{\int\tilde\Delta_K^{(n)}(-1,t,\pm\sqrt{t})dT}\int P_\lambda(x_1^{\pm 1},\dots,x_n^{\pm 1},1;t)\tilde\Delta_K^{(n)}(-1,t,\pm\sqrt{t})\prod_{i=1}^n(1+\alpha x_i^{\pm 1})(1+\beta x_i^{\pm 1})dT,\end{aligned}$$

which is $(-1)^{|\lambda|}$ times the $O^+(2n+1)$ integral with parameters $-\alpha, -\beta$. $\square$

We remark that theorem 2.10(i) may be obtained using the direct method of the previous section. One ultimately obtains a recursive formula, for which the Rogers–Szegő polynomials are a solution. However, this argument does not easily work for $O^-(2n), O^+(2n+1)$ and $O^-(2n+1)$. Thus, it is more practical to use the Pieri rule to obtain the $O(l)$ (l odd or even) integrals, and then solve for the components.

2.5 Special Cases

We will use the results of the previous section to prove some identities that correspond to particular values of α and β.

Corollary 2.11. *($\alpha = -1$) We have the following identity:*

$$\frac{1}{Z}\int P_\lambda^{(2n)}(x^{\pm 1};t)\tilde{\Delta}_K^{(n)}(\pm 1, \pm\sqrt{t})\prod_{i=1}^{n}(1+x_i^{\pm 1})(1-\beta x_i^{\pm 1})dT = \frac{2\phi_{2n}(t)}{v_\lambda(t)(1-t)^{2n}}\prod_{i\geq 0} H_{m_i(\lambda)}(-\beta;t),$$

where the normalization $Z = \int \tilde{\Delta}_K^{(n)}(\pm 1, \pm\sqrt{t})dT$.

Proof. Just put $\alpha = -1$ into theorem 2.10(i). $\square$

Corollary 2.12. *($\alpha = -\beta$) We have the following identity:*

$$\frac{1}{Z}\int P_\lambda^{(2n)}(x^{\pm 1};t)\tilde{\Delta}_K^{(n)}(\pm 1, \pm\sqrt{t})\prod_{i=1}^{n}(1-\alpha^2 x_i^{\pm 2})dT$$
$$= \frac{\phi_{2n}(t)}{v_\lambda(t)(1-t)^{2n}}\Big[\Big(\prod_{i\geq 0} H_{m_{2i}(\lambda)}(-\alpha^2;t)\prod_{i\geq 0} H_{m_{2i+1}(\lambda)}(-1;t)\Big)(-\alpha)^{\#\text{ of odd parts of }\lambda}$$
$$+\Big(\prod_{i\geq 0} H_{m_{2i+1}(\lambda)}(-\alpha^2;t)\prod_{i\geq 0} H_{m_{2i}(\lambda)}(-1;t)\Big)(-\alpha)^{\#\text{ of even parts of }\lambda}\Big],$$

where the normalization $Z = \int \tilde{\Delta}_K^{(n)}(\pm 1, \pm\sqrt{t})dT$. In particular, this vanishes unless all odd parts of λ have even multiplicity, or all even parts of λ have even multiplicity.

Proof. Just put $\alpha = -\beta$ into theorem 2.10(i). For the second part, we use [24, 1.10b]: $H_m(-1;t)$ vanishes unless m is even, in which case it is $(t;t^2)_{m/2} = (1-t)(1-t^3)\cdots(1-t^{m-1})$. $\square$

Corollary 2.13. *Symplectic integral (see theorem 4.1 of [20]). We have the following identity:*

$$\frac{1}{Z}\int P_\lambda(x_1^{\pm 1},\dots,x_n^{\pm 1};t)\tilde{\Delta}_K^{(n)}(\pm\sqrt{t},0,0)dT = \frac{\phi_n(t^2)}{(1-t^2)^n v_\mu(t^2)} = \frac{C_\mu^0(t^{2n};0,t^2)}{C_\mu^-(t^2;0,t^2)},$$

when $\lambda = \mu^2$ for some μ and 0 otherwise (here the normalization $Z = \int \tilde{\Delta}_K^{(n)}(\pm\sqrt{t},0,0)dT$).

Proof. Use the computation

$$\tilde{\Delta}_K^{(n)}(\pm\sqrt{t},0,0) = \tilde{\Delta}_K^{(n)}(\pm 1,\pm\sqrt{t}) \prod_{1\le i\le n} (1-\alpha x_i^{\pm 1})(1-\beta x_i^{\pm 1})\big|_{\alpha=-1,\beta=1},$$

and corollary 2.11 with $\beta = 1$. The result then follows from [24, 1.10b]: $H_{m_i(\lambda)}(-1;t)$ vanishes unless $m_i(\lambda)$ is even, in which case it is $(1-t)(1-t^3)\cdots(1-t^{m_i(\lambda)-1})$. □

We remark that this integral identity may also be proved directly, using techniques similar to those used for the orthogonal group integrals of section 4. In fact, in this case, there are no poles on the unit circle so the analysis is much more straightforward.

Corollary 2.14. *We have the following identity (see [19] for a conjectured elliptic version, [11], [12]):*

$$\frac{1}{Z}\int P_\lambda(x_1^{\pm 1},\dots,x_n^{\pm 1};t)\tilde{\Delta}_K^{(n)}(1,\sqrt{t},0,0) = \frac{\phi_{2n}(\sqrt{t})}{(1-\sqrt{t})^{2n} v_\lambda(\sqrt{t})} = \frac{C_\lambda^0(t^n;0,\sqrt{t})}{C_\lambda^-(\sqrt{t};0,\sqrt{t})}$$

(here the normalization $Z = \int \tilde{\Delta}_K^{(n)}(1,\sqrt{t},0,0)dT$).

Proof. Use the computation

$$\tilde{\Delta}_K^{(n)}(1,\sqrt{t},0,0) = \tilde{\Delta}_K^{(n)}(\pm 1,\pm\sqrt{t}) \prod_{1\le i\le n} (1-\alpha x_i^{\pm 1})(1-\beta x_i^{\pm 1})\big|_{\alpha=-1,\beta=-\sqrt{t}},$$

and corollary 2.11 with $\beta = -\sqrt{t}$. The result then follows from [24, 1.10d]: $H_m(\sqrt{t};t) = \prod_{j=1}^m (1 + (\sqrt{t})^j)$. □

2.6 Limit $n \to \infty$

In this section, we show that the $n \to \infty$ limit of theorem 2.10(i) in conjunction with the Cauchy identity gives Warnaar's identity ([24, theorem 1.1]). Thus, theorem 2.10(i) may be viewed as a finite dimensional analog of that particular generalized Littlewood identity.

Proposition 2.15. *(Gaussian result for $O^+(2n)$) For any symmetric function f,*

$$\lim_{n\to\infty} \frac{\displaystyle\int f(x^{\pm 1})\tilde{\Delta}_K^{(n)}(x;t;\pm 1,t_2,t_3)dT}{\displaystyle\int \tilde{\Delta}_K^{(n)}(x;t;\pm 1,t_2,t_3)dT} = I_G(f;m;s),$$

where $|t|, |t_2|, t_3| < 1$ and m and s are defined as follows:

$$m_{2k-1} = \frac{t_2^{2k-1} + t_3^{2k-1}}{1 - t^{2k-1}},$$
$$m_{2k} = \frac{t_2^{2k} + t_3^{2k} + 1 - t^k}{1 - t^{2k}},$$
$$s_k = \frac{k}{1 - t^k}.$$

Here $I_G(;m;s)$ is the Gaussian functional on symmetric functions defined by

$$\int_{\mathbb{R}^{\deg(f)}} f \prod_{j=1}^{\deg(f)} (2\pi s_j)^{-1/2} e^{-(p_j - m_j)^2/2s_j} dp_j.$$

Proof. This is formally a special case of [18, theorem 7.17]. That proof relies on theorem 6 of [5] and section 8 of [2]. The fact that two of the parameters $(t_0, \dots, t_3)$ are ± 1 makes that argument fail: however, replacing the symplectic group with $O^+(2n)$ resolves that issue. □

Note that a similar argument would work for the components $O^-(2n), O^+(2n+1)$ and $O^-(2n+1)$.

Proposition 2.16. *We have the following:*

$$\lim_{n\to\infty} \frac{\int \prod_{j,k} \frac{1 - t x_j y_k^{\pm 1}}{1 - x_j y_k^{\pm 1}} \prod_k (1 - \alpha y_k^{\pm 1})(1 - \beta y_k^{\pm 1}) \tilde{\Delta}_K^{(n)}(y; t; \pm 1, t_2, t_3) dT}{\int \tilde{\Delta}_K^{(n)}(y; t; \pm 1, t_2, t_3) dT}$$
$$= \frac{(t_2\alpha, t_3\alpha, t_2\beta, t_3\beta; t)}{(\alpha^2 t, \beta^2 t; t^2)(\alpha\beta; t)} \prod_{j<k} \frac{1 - t x_j x_k}{1 - x_j x_k} \prod_j \frac{(1 - t x_j^2)(1 - \alpha x_j)(1 - \beta x_j)}{(1 - t_2 x_j)(1 - t_3 x_j)(1 - x_j)(1 + x_j)}.$$

Proof. Put

$$f = \prod_{j,k} \frac{1 - t x_j y_k^{\pm 1}}{1 - x_j y_k^{\pm 1}} \prod_k (1 - \alpha y_k^{\pm 1})(1 - \beta y_k^{\pm 1}) = \exp\Big(\sum_{1 \le k} \frac{p_k(x) p_k(y)(1 - t^k)}{k} - \frac{p_k(y)(\alpha^k + \beta^k)}{k} \Big)$$

(see [16] for more details). Then use the previous result, and complete the square in the Gaussian integral. □

Corollary 2.17. *We have the following identity in the limit:*

$$\lim_{n\to\infty} \frac{\int \prod_{j,k} \frac{1 - t x_j y_k^{\pm 1}}{1 - x_j y_k^{\pm 1}} \prod_k (1 - \alpha y_k^{\pm 1})(1 - \beta y_k^{\pm 1}) \tilde{\Delta}_K^{(n)}(y; t; \pm 1, \pm\sqrt{t}) dT}{\int \tilde{\Delta}_K^{(n)}(y; t; \pm 1, \pm\sqrt{t}) dT}$$
$$= \frac{1}{(\alpha\beta; t)} \prod_{j<k} \frac{1 - t x_j x_k}{1 - x_j x_k} \prod_j \frac{(1 - \alpha x_j)(1 - \beta x_j)}{(1 - x_j)(1 + x_j)}.$$

Proof. Put $t_2, t_3 = \pm\sqrt{t}$ in the previous result. Also note that

$$(\sqrt{t}\alpha; t)(-\sqrt{t}\alpha; t) = (t\alpha^2; t^2),$$

so that

$$\frac{(\sqrt{t}\alpha, -\sqrt{t}\alpha, \sqrt{t}\beta, -\sqrt{t}\beta; t)}{(\alpha^2 t, \beta^2 t; t^2)} = 1.$$

□

Theorem 2.18. *We have the following formal identity ([24] theorem 1.1):*

$$\sum_\lambda P_\lambda(x;t)\Big[\Big(\prod_{i>0} H_{m_{2i}(\lambda)}(\alpha\beta;t)\prod_{i\geq 0} H_{m_{2i+1}(\lambda)}(\beta/\alpha;t)\Big)(-\alpha)^{\#\text{ of odd parts of }\lambda}\Big]$$
$$= \prod_{j<k}\frac{1-tx_jx_k}{1-x_jx_k}\prod_j\frac{(1-\alpha x_j)(1-\beta x_j)}{(1-x_j)(1+x_j)}.$$

Proof. We prove the result for $|\alpha|, |\beta| < 1$, then use analytic continuation to obtain it for all α, β. We start with the Cauchy identity for Hall–Littlewood polynomials (2.2). Using this in the left-hand side of corollary 2.17, and multiplying both sides by $(\alpha\beta; t)$ gives

$$(\alpha\beta;t)\sum_\lambda P_\lambda(x;t)\lim_{n\to\infty}\Big[\frac{b_\lambda(t)\int P_\lambda(y_1^{\pm1},\dots,y_n^{\pm1};t)\prod_k(1-\alpha y_k^{\pm1})(1-\beta y_k^{\pm1})\tilde{\Delta}_K^{(n)}(y;t;\pm1,\pm\sqrt{t})dT}{\int\tilde{\Delta}_K^{(n)}(y;t;\pm1,\pm\sqrt{t})dT}\Big]$$
$$= \prod_{j<k}\frac{1-tx_jx_k}{1-x_jx_k}\prod_j\frac{(1-\alpha x_j)(1-\beta x_j)}{(1-x_j)(1+x_j)}.$$

Now note that the quantity within the limit is the α, β version of the $O^+(2n)$ integral, see theorem 2.10(i). Using that result, the above equation becomes

$$(\alpha\beta;t)\sum_\lambda P_\lambda(x;t)\lim_{n\to\infty}\frac{b_\lambda(t)\phi_{2n}(t)}{v_\lambda(t)(1-t)^{2n}}\Big[\Big(\prod_{i\geq0} H_{m_{2i}(\lambda)}(\alpha\beta;t)\prod_{i\geq0} H_{m_{2i+1}(\lambda)}(\beta/\alpha;t)\Big)(-\alpha)^{\#\text{odd parts of }\lambda}$$
$$+\Big(\prod_{i\geq0} H_{m_{2i+1}(\lambda)}(\alpha\beta;t)\prod_{i\geq0} H_{m_{2i}(\lambda)}(\beta/\alpha;t)\Big)(-\alpha)^{\#\text{ of even parts of }\lambda}\Big]$$
$$= \prod_{j<k}\frac{1-tx_jx_k}{1-x_jx_k}\prod_j\frac{(1-\alpha x_j)(1-\beta x_j)}{(1-x_j)(1+x_j)}.$$

But note that

$$\frac{b_\lambda(t)}{v_\lambda(t)} = \frac{(1-t)^{2n}}{\phi_{m_0(\lambda)}(t)},$$

so that

$$\frac{b_\lambda(t)\phi_{2n}(t)}{v_\lambda(t)(1-t)^{2n}} = \frac{\phi_{2n}(t)}{\phi_{m_0(\lambda)}(t)} = (1-t^{m_0(\lambda)+1})\cdots(1-t^{2n}),$$

which goes to 1 as $m_0(\lambda), n \to \infty$. Moreover, as $m_0(\lambda) \to \infty$, we have

$$\begin{aligned} H_{m_0(\lambda)}(\alpha\beta;t) &= \sum_{j=0}^{m_0(\lambda)} \begin{bmatrix} m_0(\lambda) \\ j \end{bmatrix}_t (\alpha\beta)^j = \sum_{j=0}^{m_0(\lambda)} \frac{\phi_{m_0(\lambda)}(t)}{\phi_j(t)\phi_{m_0(\lambda)-j}(t)}(\alpha\beta)^j \\ &= \sum_{j=0}^{m_0(\lambda)} \frac{(1-t^{m_0(\lambda)-j+1})(1-t^{m_0(\lambda)-j+2})\cdots(1-t^{m_0(\lambda)})}{(1-t)(1-t^2)\cdots(1-t^j)}(\alpha\beta)^j \to \sum_{j=0}^{\infty} \frac{(\alpha\beta)^j}{(t;t)_j}. \end{aligned}$$

But for $|\alpha\beta| < 1$, it is an identity that this is $1/(\alpha\beta;t)$.

Finally, we show that the second term in the sum vanishes. We must look at

$$\lim_{m_0(\lambda),k\to\infty} (-\alpha)^k H_{m_0(\lambda)}(\beta/\alpha;t),$$

where k is the number of even parts, so in particular $k \geq m_0(\lambda)$. We have the following upper bound:

$$\lim_{m_0(\lambda)\to\infty} \alpha^{m_0(\lambda)} \sum_{j=0}^{m_0(\lambda)} \frac{(\beta/\alpha)^j}{(1-t)^j};$$

the sum is geometric with ratio $\beta/\alpha(1-t)$. Thus, this is equal to

$$\lim_{m_0(\lambda)\to\infty} \alpha^{m_0(\lambda)} \frac{1-\left(\frac{\beta}{\alpha(1-t)}\right)^{m_0(\lambda)+1}}{1-\frac{\beta}{\alpha(1-t)}} = \lim_{m_0(\lambda)\to\infty} \frac{\alpha^{m_0(\lambda)} - \frac{\beta^{m_0(\lambda)+1}}{\alpha(1-t)^{m_0(\lambda)+1}}}{1-\frac{\beta}{\alpha(1-t)}}.$$

But since α, β are sufficiently small (take $|\beta| < |1-t|$), this is zero, giving the result. □

2.7 Other Vanishing Results

We introduce notation for dominant weights with negative parts: if μ, ν are partitions with $l(\mu) + l(\nu) \leq n$ then $\mu\bar{\nu}$ is the dominant weight vector of GL_n, $\mu\bar{\nu} = (\mu_1, \ldots, \mu_{l(\mu)}, 0, \ldots, 0, -\nu_{l(\nu)}, \ldots, -\nu_1)$. Often, we will use λ for a dominant weight with negative parts, i.e., $\lambda = \mu\bar{\nu}$.

In this section, we prove four other vanishing identities from [20] and [18]. In all four cases, the structure of the partition that produces a nonvanishing integral is the same: opposite parts must add to zero ($\lambda_i + \lambda_{l+1-i} = 0$ for all $1 \leq i \leq l$, where l is the total number of parts). Note that an equivalent condition is that there exists a partition μ such that $\lambda = \mu\bar{\mu}$.

We comment that the technique is similar to that of previous sections: we first use symmetries of the integrand to restrict to the term integrals associated to specific permutations. Then, we obtain an inductive evaluation for the term integral, and use this to give a combinatorial formula for the total integral. We mention that the first result corresponds to the symmetric space $(U(m+n), U(m)\times U(n))$ in the Schur case $t=0$.

Theorem 2.19. *(see [18, conjecture 3]) Let m and n be integers with $0\le m\le n$. Then for a dominant weight $\lambda=\mu\bar{\nu}$ of $U(n+m)$,*

$$\frac{1}{Z}\int_T P_{\mu\bar{\nu}}(x_1,\ldots,x_m,y_1,\ldots,y_n;t)\frac{1}{n!m!}\prod_{1\le i\ne j\le m}\frac{1-x_ix_j^{-1}}{1-tx_ix_j^{-1}}\prod_{1\le i\ne j\le n}\frac{1-y_iy_j^{-1}}{1-ty_iy_j^{-1}}dT=0,$$

unless $\mu=\nu$ and $l(\mu)\le m$, in which case the integral is

$$\frac{C^0_\mu(t^n,t^m;0,t)}{C^-_\mu(t;0,t)C^+_\mu(t^{m+n-2}t;0,t)}.$$

Here the normalization Z is the integral for $\mu=\nu=0$.

Proof. Note first that the integral is a sum of $(n+m)!$ terms, one for each element in S_{n+m}. But by the symmetry of the integrand, we may restrict to the permutations with x_i (resp. y_i) to the left of x_j (resp. y_j) for $1\le i<j\le m$ (resp. $1\le i<j\le n$). Moreover, by symmetry we can deform the torus to

$$T=\{|y|=1+\epsilon;|x|=1\},$$

and preserve the integral. Thus, we have

$$\begin{aligned}&\int_T R_{\mu\bar{\nu}}(x^{(m)},y^{(n)};t)\frac{1}{n!m!}\prod_{1\le i\ne j\le m}\frac{1-x_ix_j^{-1}}{1-tx_ix_j^{-1}}\prod_{1\le i\ne j\le n}\frac{1-y_iy_j^{-1}}{1-ty_iy_j^{-1}}dT\\ &=\sum_{\substack{w\in S_{n+m}\\ x_i\prec_w x_j \text{ for } 1\le i<j\le m\\ y_i\prec_w y_j \text{ for } 1\le i<j\le n}}\int_T R_{\mu\bar{\nu},w}(x^{(m)},y^{(n)};t)\prod_{1\le i\ne j\le m}\frac{1-x_ix_j^{-1}}{1-tx_ix_j^{-1}}\prod_{1\le i\ne j\le n}\frac{1-y_iy_j^{-1}}{1-ty_iy_j^{-1}}dT.\end{aligned}$$

We first compute the normalization.

Claim 2.19.1. *We have*

$$Z=\int_T P_{0^{n+m}}(x^{(m)},y^{(n)};t)\frac{1}{n!m!}\prod_{1\le i\ne j\le m}\frac{1-x_ix_j^{-1}}{1-tx_ix_j^{-1}}\prod_{1\le i\ne j\le n}\frac{1-y_iy_j^{-1}}{1-ty_iy_j^{-1}}dT=\frac{(1-t)^{m+n}}{\phi_n(t)\phi_m(t)}.$$

Since

$$\frac{1}{v_{(0^{n+m})}(t)} = \frac{(1-t)^{m+n}}{\phi_{m+n}(t)},$$

this is equivalent to showing

$$\int R_{0^{n+m}}(x^{(m)}, y^{(n)}; t) \frac{1}{n!m!} \prod_{1\le i\ne j\le m} \frac{1 - x_i x_j^{-1}}{1 - t x_i x_j^{-1}} \prod_{1\le i\ne j\le n} \frac{1 - y_i y_j^{-1}}{1 - t y_i y_j^{-1}} dT = \frac{\phi_{m+n}(t)}{\phi_n(t)\phi_m(t)}.$$

We may use the above discussion to rewrite the left-hand side as a sum over suitable permutations. Let $w \in S_{n+m}$ be a permutation with the x, y variables in order and consider

$$\int_T R_{0^{n+m},w}(x^{(m)}, y^{(n)}; t) \prod_{1\le i\ne j\le m} \frac{1 - x_i x_j^{-1}}{1 - t x_i x_j^{-1}} \prod_{1\le i\ne j\le n} \frac{1 - y_i y_j^{-1}}{1 - t y_i y_j^{-1}} dT.$$

Integrating with respect to $x_1, \dots, x_m, y_1, \dots, y_n$ in order shows that this is $t^{\#\text{inversions of } w}$, where inversions are in the sense of the multiset $M = \{0^n, 1^m\}$, and we define $y_1 \cdots y_n x_1 \cdots x_m$ to have 0 inversions. But now by an identity of MacMahon

$$\sum_{\text{multiset permutations } w \text{ of } \{0^n,1^m\}} t^{\#\text{ inversions of } w} = \begin{bmatrix} m+n \\ n \end{bmatrix}_t = \frac{\phi_{m+n}(t)}{\phi_n(t)\phi_m(t)},$$

which proves the claim. Note that we could also prove the claim by observing that

$$\int_T P_{0^{n+m}}(x^{(m)}, y^{(n)}; t) \frac{1}{n!m!} \prod_{1\le i\ne j\le m} \frac{1 - x_i x_j^{-1}}{1 - t x_i x_j^{-1}} \prod_{1\le i\ne j\le n} \frac{1 - y_i y_j^{-1}}{1 - t y_i y_j^{-1}} dT$$
$$= \frac{1}{n!m!} \int_T \tilde{\Delta}_S^{(m)}(x; t) \tilde{\Delta}_S^{(n)}(y; t) dT$$

and using the results of theorem 2.1.

For convenience, from now on we will write

$$\Delta(x^{(m)}; y^{(n)}; t) = \prod_{1\le i\ne j\le m} \frac{1 - x_i x_j^{-1}}{1 - t x_i x_j^{-1}} \prod_{1\le i\ne j\le n} \frac{1 - y_i y_j^{-1}}{1 - t y_i y_j^{-1}} = \tilde{\Delta}_S^{(m)}(x; t) \tilde{\Delta}_S^{(n)}(y; t),$$

for the density function.

Claim 2.19.2. *Let $w \in S_{n+m}$ be a permutation of $\{x^{(m)}, y^{(n)}\}$ with $x_i \prec_w x_j$ for all $1 \le i < j \le m$ and $y_i \prec_w y_j$ for all $1 \le i < j \le n$. Suppose*

$$\int_T R_{\mu\bar{\nu},w}(x^{(m)}, y^{(n)}; t) \Delta(x^{(m)}; y^{(n)}; t) dT \ne 0.$$

Then w has $y_1 \dots y_{l(\mu)}$ in first $l(\mu)$ positions, and $x_{m-l(\nu)+1} \dots x_m$ in the last $l(\nu)$ positions. Consequently $l(\nu) \leq m$, $l(\mu) \leq n$.

We prove the claim. We will first show that if, in $w(x,y)^{\mu\bar{\nu}}$, x_1 has exponent a strictly positive part, the integral is zero. Indeed, one can compute that the integral restricted to the terms in x_1 is

$$\int_{T_1} x_1^{\mu_i} \prod_{1<i\leq m} \frac{x_i - x_1}{x_i - tx_1} \prod_{y_j \prec_w x_1} \frac{y_j - tx_1}{y_j - x_1} \prod_{x_1 \prec_w y_j} \frac{x_1 - ty_j}{x_1 - y_j} dT = 0,$$

since by assumption $\mu_i > 0$.

Dually if in $w(x,y)^{\mu\bar{\nu}}$, y_n has exponent a strictly negative part, we can show the integral is zero. The integral restricted to the terms in y_n is

$$\int_{T_1} y_n^{\bar{\nu}_i} \prod_{1\leq i<n} \frac{y_n - y_i}{y_n - ty_i} \prod_{x_j \prec_w y_n} \frac{x_j - ty_n}{x_j - y_n} \prod_{y_n \prec_w x_j} \frac{y_n - tx_j}{y_n - x_j} dT$$
$$= \int_{T:|x|>|y|} y_n^{-\bar{\nu}_i} \prod_{1\leq i<n} \frac{y_i - y_n}{y_i - ty_n} \prod_{x_j \prec_w y_n} \frac{y_n - tx_j}{y_n - x_j} \prod_{y_n \prec_w x_j} \frac{x_j - ty_n}{x_j - y_n} dT,$$

where in the second step we have inverted all variables which preserves the integral. But now by assumption $\bar{\nu}_i < 0$, so integrating with respect to y_n gives that the above integral is zero. This gives the desired structure of w to have nonvanishing associated integral.

Claim 2.19.3. *Let $w \in S_{n+m}$ be a permutation of $\{x^{(m)}, y^{(n)}\}$ with $x_i \prec_w x_j$ for all $1 \leq i < j \leq m$ and $y_i \prec_w y_j$ for all $1 \leq i < j \leq n$. Suppose also that $y_1, \dots, y_{l(\mu)}$ are in the first $l(\mu)$ positions and $x_{m-l(\nu)+1}, \dots, x_m$ are in the last $l(\nu)$ positions.*

Let $l(\mu) > 0$. Then we have the following formula for the term integral associated to w:

$$\int_T R_{\mu\bar{\nu},w}(x^{(m)}, y^{(n)}; t)\Delta(x^{(m)}; y^{(n)}; t)dT$$
$$= (1-t)\Big(\sum_{\substack{i:\\ \lambda_1+\lambda_i=0}} t^{n+m-i} \Big) \int R_{\widehat{\lambda},\widehat{w}}(x^{(m-1)}, y^{(n-1)}; t)\Delta(x^{(m-1)}; y^{(n-1)}; t)dT,$$

where $\widehat{w}$ is w with y_1, x_m deleted and $\widehat{\lambda}$ is λ with λ_1 and λ_i deleted (where index i is such that $\lambda_1 + \lambda_i = 0$).

Similarly, if $l(\nu) > 0$, we have

$$\int_T R_{\mu\bar{\nu},w}(x^{(m)}, y^{(n)}; t)\Delta(x^{(m)}; y^{(n)}; t)dT$$
$$= (1-t)\Big(\sum_{\substack{i:\\ \lambda_i+\lambda_{n+m}=0}} t^{i-1} \Big) \int R_{\widehat{\lambda},\widehat{w}}(x^{(m-1)}, y^{(n-1)}; t)\Delta(x^{(m-1)}; y^{(n-1)}; t)dT,$$

where $\widehat{w}$ is w with y_1, x_m deleted and $\widehat{\lambda}$ is λ with λ_i and λ_{n+m} deleted (where index i is such that $\lambda_i + \lambda_{n+m} = 0$).

For the first statement, integrate with respect to y_1. We have the following integral restricted to the terms involving y_1:

$$\int_{T_1} y_1^{\lambda_1} \prod_{1<i\leq n} \frac{y_i - y_1}{y_i - ty_1} \prod_{1\leq j\leq m} \frac{y_1 - tx_j}{y_1 - x_j} dT,$$

with $\lambda_1 = \mu_1 > 0$. Evaluating gives a sum of m terms, one for each residue $y_1 = x_j$. We consider one of these residues: suppose x_j is in position i, then the resulting integral in x_j is

$$\begin{aligned}(1-t)\int_{T_1} x_j^{\lambda_1+\lambda_i} \prod_{1<i\leq n} \frac{y_i - x_j}{y_i - tx_j} \prod_{i\neq j} \frac{x_j - tx_i}{x_j - x_i} \prod_{\substack{y_i \prec_w x_j \\ y_i \neq y_1}} \frac{y_i - tx_j}{y_i - x_j} \\ \times \prod_{x_j \prec_w y_i} \frac{x_j - ty_i}{x_j - y_i} \prod_{i<j} \frac{x_j - x_i}{x_j - tx_i} \prod_{j<i} \frac{x_i - x_j}{x_i - tx_j} dT \\ = (1-t)\int_{T_1} x_j^{\lambda_1+\lambda_i} \prod_{x_j \prec_w y_i} (-1)\frac{x_j - ty_i}{y_i - tx_j} \prod_{j<i} (-1)\frac{x_j - tx_i}{x_i - tx_j} dT,\end{aligned}$$

where we may assume $\lambda_i \leq 0$, by the structure of w. Note first that if $\lambda_1 + \lambda_i > 0$, the integral is zero. One can similarly argue that the term integral is zero if $\lambda_1 + \lambda_i < 0$ (use $\lambda_{n+m} + \lambda_k < 0$ for any $1 \leq k < n+m$ and integrate with respect to x_m, and take the residue at any $x_m = y_i$). Thus for a nonvanishing residue term we must have $\lambda_1 = -\lambda_i$, and in this case one can verify that the above integral evaluates to

$$(1-t)t^{|\{z : x_j \prec_w z\}|} = (1-t)t^{n+m-i},$$

as desired.

The second statement is analogous, except integrate with respect to x_m instead of y_1, and invert all variables. This proves the claim.

Thus,

$$\int_T R_{\mu\bar{\nu},w}(x^{(m)}, y^{(n)}; t)\Delta(x^{(m)}; y^{(n)}; t) dT = 0,$$

unless $\mu = \nu$ and $l(\mu) \leq m$, which gives the vanishing part of the theorem. For the second part, suppose $\mu = \nu$ and $l(\mu) \leq m$. Then by the above claims,

$$\int_T R_{\mu\bar{\mu},w}(x^{(m)}, y^{(n)}; t)\Delta(x^{(m)}; y^{(n)}; t)dT$$
$$= (1-t)^{l(\mu)} v_{\mu+}(t) \int R_{0^{(n-l(\mu))+(m-l(\nu))},\delta}(x^{(m-l(\nu))}, y^{(n-l(\mu))}; t)\Delta(x^{(m-l(\nu))}; y^{(n-l(\mu))}; t)dT,$$

if $w = y_1 \dots y_{l(\mu)}\delta x_{m-l(\nu)+1} \dots x_m$ for some permutation δ of $\{y_{l(\mu)+1}, \dots, y_n, x_1, \dots, x_{m-l(\nu)}\}$, and 0 otherwise.

By claim 2.19.1, we have

$$\int R_{0^{(n-l(\mu))+(m-l(\mu))}}(x^{(m-l(\mu))}, y^{(n-l(\mu))}; t)\frac{\Delta(x^{(m-l(\mu))}; y^{(n-l(\mu))}; t)}{(m-l(\mu))!(n-l(\mu))!}dT = \begin{bmatrix} m+n-2l(\mu) \\ n-l(\mu) \end{bmatrix}_t.$$

So we have

$$\int_T P_{\mu\bar{\mu}}(x^{(m)}, y^{(n)}; t)\frac{1}{n!m!}\Delta(x^{(m)}; y^{(n)}; t)dT = \frac{1}{v_{\mu\bar{\mu}}(t)}(1-t)^{l(\mu)} v_{\mu+}(t) \begin{bmatrix} m+n-2l(\mu) \\ n-l(\mu) \end{bmatrix}_t.$$

Noting that $v_{\mu\bar{\mu}}(t) = v_{\mu+}(t)^2 v_{(0^{m+n-2l(\mu)})}(t)$ and multiplying by the reciprocal of the normalization gives

$$\frac{1}{Z}\int_T P_{\mu\bar{\mu}}(x^{(m)}, y^{(n)}; t)\frac{1}{n!m!}\Delta(x^{(m)}; y^{(n)}; t)dT$$
$$= \frac{\phi_n(t)\phi_m(t)}{(1-t)^{m+n}} \frac{(1-t)^{l(\mu)}}{v_{\mu+}(t)v_{(0^{m+n-2l(\mu)})}(t)} \begin{bmatrix} m+n-2l(\mu) \\ n-l(\mu) \end{bmatrix}_t$$
$$= (1-t^{n-l(\mu)+1})\cdots(1-t^n)(1-t^{m-l(\mu)+1})\cdots(1-t^m)\frac{\phi_{m+n-2l(\mu)}(t)}{(1-t)^{m+n-l(\mu)}v_{\mu+}(t)v_{(0^{m+n-2l(\mu)})}(t)}$$
$$= \frac{(1-t^{n-l(\mu)+1})(1-t^{n-l(\mu)+2})\cdots(1-t^n)(1-t^{m-l(\mu)+1})(1-t^{m-l(\mu)+2})\cdots(1-t^m)}{(1-t)^{l(\mu)}v_{\mu+}(t)},$$

where the last equality follows from the definition of $v_{(0^{m+n-2l(\mu)})}$. One can check from the definition of the C-symbols that

$$C^+_\mu(t^{m+n-2}t; 0, t) = 1,$$
$$C^-_\mu(t; 0, t) = v_{\mu+}(t)(1-t)^{l(\mu)},$$
$$C^0_\mu(t^n, t^m; 0, t) = \prod_{1\leq i\leq l(\mu)}(1-t^{n+1-i})(1-t^{m+1-i}),$$

so that our formula gives

$$\frac{C^0_\mu(t^n, t^m; 0, t)}{C^-_\mu(t; 0, t)C^+_\mu(t^{m+n-2}t; 0, t)},$$

as desired. ☐

Theorem 2.20. *(see [18, conjecture 5]) Let $n \geq 0$ be an integer and $\lambda = \mu\bar{\nu}$ a dominant weight of $U(2n)$. Then*

$$\frac{1}{Z}\int_T P_{\mu\bar{\nu}}(x_1,\ldots,x_n,y_1,\ldots,y_n;t)\frac{1}{(n!)^2} \times \prod_{1\leq i,j\leq n}\frac{1}{(1-tx_iy_j^{-1})(1-ty_ix_j^{-1})}\prod_{1\leq i\neq j\leq n}(1-x_ix_j^{-1})(1-y_iy_j^{-1})dT,$$

is equal to 0 unless $\mu = \nu$, in which case the integral is

$$\frac{C^0_\mu(t^n,-t^n;0,t)}{C^-_\mu(t;0,t)C^+_\mu(t^{2n-2}t;0,t)}.$$

Here the normalization Z is the integral for $\mu = \nu = 0$.

Proof. Note first that the integral is a sum of $(2n)!$ terms, one for each element in S_{2n}. But by the symmetry of the integrand, we may restrict to the permutations with x_i (resp. y_i) to the left of x_j (resp. y_j) for $1 \leq i < j \leq n$. By symmetry, we can deform the torus to

$$T = \{|y| = 1+\epsilon; |x| = 1\}.$$

For convenience, we will write $\Delta(x^{(n)};y^{(n)};t)$ for the density

$$\prod_{1\leq i,j\leq n}\frac{1}{(1-tx_iy_j^{-1})(1-ty_ix_j^{-1})}\prod_{1\leq i\neq j\leq n}(1-x_ix_j^{-1})(1-y_iy_j^{-1}).$$

We first compute the normalization.

Claim 2.20.1. *We have*

$$Z = \int_T P_{0^{2n}}(x^{(n)},y^{(n)};t)\frac{1}{(n!)^2}\Delta(x^{(n)};y^{(n)};t)dT = \frac{1}{\phi_n(t^2)}.$$

By the definition of $v_{(0^{2n})}(t)$, this is equivalent to showing

$$\int_T R_{0^{2n}}(x^{(n)},y^{(n)};t)\frac{1}{(n!)^2}\Delta(x^{(n)};y^{(n)};t)dT = \frac{\phi_{2n}(t)}{(1-t)^{2n}\phi_n(t^2)}.$$

We prove this statement by induction on n. For $n = 1$, we have

$$\int_T \frac{x_1y_1}{(x_1-y_1)(y_1-tx_1)}dT = 0,$$

and

$$\int_T \frac{x_1y_1}{(y_1-x_1)(x_1-ty_1)}dT = \frac{1}{1-t} = \frac{\phi_2(t)}{(1-t)^2\phi_1(t^2)},$$

as desired. Now suppose the claim holds for $n-1$; with this assumption we show that it holds for n.

Consider permutations w with x_1 first. We claim $\int_T R_{\mu\bar{\nu},w}(x^{(n)}, y^{(n)}; t)\Delta(x^{(n)}; y^{(n)}; t)dT = 0$. Indeed, we have the following integral restricting to the terms in x_1:

$$\begin{aligned}\int_{T_1} \prod_{1\le i\le n} \frac{x_1-ty_i}{x_1-y_i} \prod_{1<i\le n} \frac{x_1-tx_i}{x_1-x_i} \prod_{1\le j\le n} \frac{x_1y_j}{(y_j-tx_1)(x_1-ty_j)} \prod_{1<j\le n} \frac{(x_j-x_1)(x_1-x_j)}{x_1x_j}dT \\ = \int_{T_1} \prod_{1\le j\le n} \frac{x_1y_j}{(x_1-y_j)(y_j-tx_1)} \prod_{1<j\le n} \frac{(x_1-tx_j)(x_j-x_1)}{x_1x_j}dT \\ = \int_{T_1} x_1 \prod_{1\le j\le n} \frac{1}{(x_1-y_j)(y_j-tx_1)} \prod_{1<j\le n} (x_1-tx_j)(x_j-x_1)dT = 0.\end{aligned}$$

Thus, we may suppose y_1 occurs first in w. A similar calculation for the integral restricting to terms in y_1 yields:

$$\int_{T_1} y_1 \prod_{1<j\le n} (y_1-ty_j)(y_j-y_1) \prod_{1\le i\le n} \frac{1}{(y_1-x_i)(x_i-ty_1)}dT.$$

We may evaluate this as the sum of n residues, one for each $y_1 = x_i$ for $1 \le i \le n$. We compute the residue at $y_1 = x_i$, and look at the resulting integral in x_i:

$$\begin{aligned}\frac{1}{1-t}\int_{T_1} \prod_{1<j\le n} (x_i-ty_j)(y_j-x_i) \prod_{j\ne i} \frac{1}{(x_i-x_j)(x_j-tx_i)} \prod_{i'<i} (x_{i'}-tx_i)(x_i-x_{i'}) \\ \cdot \prod_{i<i''} (x_i-tx_{i''})(x_{i''}-x_i) \prod_{x_i\prec_w y_j} \frac{1}{(x_i-y_j)(y_j-tx_i)} \prod_{\substack{y_j\prec_w x_i \\ y_j\ne y_1}} \frac{1}{(y_j-x_i)(x_i-ty_j)}dT \\ = \frac{1}{1-t}\int_{T_1} \prod_{i<i''} \frac{(tx_{i''}-x_i)}{(x_{i''}-tx_i)} \prod_{x_i\prec_w y_j} \frac{(ty_j-x_i)}{(y_j-tx_i)}dT.\end{aligned}$$

But, letting $2 \le k \le 2n$ be the position of x_i in w, this evaluates to

$$\frac{1}{1-t}\prod_{i<i''} t \prod_{x_i\prec_w y_j} t = \frac{t^{2n-k}}{1-t}.$$

Thus, varying over all such permutations with y_1 first gives a factor of

$$\frac{1}{1-t}(t^{2n-2}+t^{2n-3}+\cdots+t+1) = \frac{(1-t^{2n-1})}{(1-t)^2}.$$

Note that permutations of $\{y_1, \ldots, y_n, x_1, \ldots, x_n\}$ with y_1 in position 1 and x_i in position k are in bijection with permutations of $\{y_2, \ldots, y_n, x_1, \ldots, \widehat{x_i}, \ldots, x_n\}$. So using the induction hypothesis, the total integral evaluates to

$$\frac{(1-t^{2n-1})}{(1-t)^2} \frac{\phi_{2(n-1)}(t)}{(1-t)^{2(n-1)}\phi_{n-1}(t^2)} = \frac{\phi_{2n}(t)}{(1-t)^{2n}\phi_n(t^2)},$$

as desired.

Note that the density is not of a standard form (i.e., as a product of Koornwinder or Macdonald-Morris densities), so we cannot appeal to an earlier result (compare with claim 2.19.1).

Claim 2.20.2. *Let $w \in S_{2n}$ a permutation of $\{x^{(n)}, y^{(n)}\}$ with $x_i \prec_w x_j$ for all $1 \le i < j \le n$ and $y_i \prec_w y_j$ for all $1 \le i < j \le n$. Suppose*

$$\int_T R_{\mu\bar{\nu},w}(x^{(n)}, y^{(n)}; t)\Delta(x^{(n)}; y^{(n)}; t)dT \neq 0.$$

Then w has $y_1 \ldots y_{l(\mu)}$ in the first $l(\mu)$ coordinates, and $x_{n-l(\nu)+1} \ldots x_n$ in the last $l(\nu)$ coordinates. Consequently $l(\nu) \le n, l(\mu) \le n$.

The proof is analogous to claim 2.19.2 of the previous theorem.

Claim 2.20.3. *Let $w \in S_{2n}$ be a permutation of $\{x^{(n)}, y^{(n)}\}$ with $x_i \prec_w x_j$ for all $1 \le i < j \le n$ and $y_i \prec_w y_j$ for all $1 \le i < j \le n$. Suppose also that $y_1, \ldots, y_{l(\mu)}$ are in the first $l(\mu)$ coordinates, and $x_{n-l(\nu)+1} \ldots x_n$ in the last $l(\nu)$ coordinates.*

Let $l(\mu) > 0$. Then we have the following formula for the term integral associated to w:

$$\int_T R_{\mu\bar{\nu},w}(x^{(n)}, y^{(n)}; t)\Delta(x^{(n)}; y^{(n)}; t)dT = \frac{1}{1-t}\Big(\sum_{\substack{i:\\ \lambda_1+\lambda_i=0}} t^{2n-i}\Big) \int R_{\widehat{\lambda},\widehat{w}}(x^{(n-1)}, y^{(n-1)}; t)\Delta(x^{(n-1)}; y^{(n-1)}; t)dT,$$

where $\widehat{w}$ is w with y_1, x_n deleted and $\widehat{\lambda}$ is λ with λ_1 and λ_i deleted (where the index i is such that $\lambda_1 + \lambda_i = 0$).

Similarly, if $l(\nu) > 0$, we have

$$\int_T R_{\mu\bar{\nu},w}(x^{(n)}, y^{(n)}; t)\Delta(x^{(n)}; y^{(n)}; t)dT = \frac{1}{1-t}\Big(\sum_{\substack{i:\\ \lambda_i+\lambda_{2n}=0}} t^{i-1}\Big) \int R_{\widehat{\lambda},\widehat{w}}(x^{(n-1)}, y^{(n-1)}; t)\Delta(x^{(n-1)}; y^{(n-1)}; t)dT,$$

where $\widehat{w}$ is w with y_1, x_n deleted and $\widehat{\lambda}$ is λ with λ_i and λ_{2n} deleted (where the index i is such that $\lambda_i + \lambda_{2n} = 0$).

The proof is analogous to the proof of claim 2.19.3 of the previous theorem.

Thus,

$$\int_T R_{\mu\bar{\nu},w}(x^{(n)}, y^{(n)}; t)\Delta(x^{(n)}; y^{(n)}; t)dT = 0,$$

unless $\mu = \nu$. Moreover, if $\mu = \nu$, the integral is

$$\frac{1}{(1-t)^{l(\mu)}} v_{\mu+}(t) \int R_{0^{2n-2l(\mu)},\delta}(x^{(n-l(\mu))}, y^{(n-l(\mu))}; t)\Delta(x^{(n-l(\mu))}; y^{(n-l(\mu))}; t)dT,$$

if $w = y_1 \dots y_{l(\mu)}\delta x_{n-l(\nu)+1} \dots x_n$ for some permutation δ of $\{y_{l(\mu)+1}, \dots, y_n, x_1, \dots, x_{n-l(\nu)}\}$ and 0 otherwise.

By claim 2.20.1, we have

$$\int_T R_{0^{2n-2l(\mu)}}(x^{(n-l(\mu))}, y^{(n-l(\mu))}; t)\frac{\Delta(x^{(n-l(\mu))}; y^{(n-l(\mu))}; t)}{\Big((2n-2l(\mu))!\Big)^2}dT = \frac{\phi_{2n-2l(\mu)}(t)}{(1-t)^{2n-2l(\mu)}\phi_{n-l(\mu)}(t^2)}.$$

Thus,

$$\begin{aligned}
\frac{1}{Z}\int_T P_{\mu\bar{\mu}}(x^{(n)}, y^{(n)}; t)\frac{1}{(n!)^2}\Delta(x^{(n)}; y^{(n)}; t)dT \\
= \frac{\phi_n(t^2)}{v_{\mu+}(t)^2 v_{(0^{2n-2l(\mu)})}(t)}\frac{v_{\mu+}(t)}{(1-t)^{l(\mu)}}\frac{\phi_{2n-2l(\mu)}(t)}{(1-t)^{2n-2l(\mu)}\phi_{n-l(\mu)}(t^2)} \\
= \frac{(1-(t^2)^{n-l(\mu)+1})\cdots(1-(t^2)^n)}{v_{\mu+}(t)(1-t)^{2n-l(\mu)}}\frac{\phi_{2n-2l(\mu)}(t)}{v_{(0^{2n-2l(\mu)})}(t)} = \frac{(1-(t^2)^{n-l(\mu)+1})\cdots(1-(t^2)^n)}{v_{\mu+}(t)(1-t)^{l(\mu)}},
\end{aligned}$$

where the last equality follows from the definition of $v_{(0^{2n-2l(\mu)})}(t)$. Finally, one can check from the definition of the C-symbols that

$$\begin{aligned}
C^+_\mu(t^{2n-2}t; 0, t) &= 1, \\
C^0_\mu(t^n, -t^n; 0, t) &= \prod_{1\le i\le l(\mu)}(1 - t^{2(n+1-i)}), \\
C^-_\mu(t; 0, t) &= (1-t)^{l(\mu)}v_{\mu+}(t),
\end{aligned}$$

so that our formula gives

$$\frac{C^0_\mu(t^n, -t^n; 0, t)}{C^-_\mu(t; 0, t)C^+_\mu(t^{2n-2}t; 0, t)},$$

as desired. ∎

Theorem 2.21. *(see [20, theorem 4.4]) Let λ be a weight of the double cover of GL_{2n}, i.e., a* *half-integer vector such that $\lambda_i - \lambda_j \in \mathbb{Z}$ for all i, j. Then*

$$\frac{1}{Z}\int P_{\lambda}^{(2n)}(\cdots t^{\pm 1/2}z_i \cdots ; t)\frac{1}{n!}\prod_{1\le i<j\le n}\frac{(1-z_i/z_j)(1-z_j/z_i)}{(1-t^2z_i/z_j)(1-t^2z_j/z_i)}dT = 0,$$

unless $\lambda = \mu\bar{\mu}$. In this case, the nonzero value is

$$\frac{\phi_n(t^2)}{(1-t)^n v_\mu(t)(1+t)(1+t^2)\cdots(1+t^{n-l(\mu)})} = \frac{C_\mu^0(t^n, -t^n; 0, t)}{C_\mu^-(t; 0, t)C_\mu^+(t^{2n-2}t; 0, t)}.$$

Proof. As usual, note that $P_{\lambda}^{(2n)}(\cdots t^{\pm 1/2}z_i \cdots ; t)$ is a sum of $(2n)!$ terms, one for each permutation in S_{2n}. We first note that many of these have vanishing integrals:

Claim 2.21.1. *Let $w \in S_{2n}$ be a permutation of $(t^{\pm 1/2}z_1, \ldots, t^{\pm 1/2}z_n)$, such that for some $1 \le i \le n$ $\sqrt{t}z_i$ appears to the left of $\frac{z_i}{\sqrt{t}}$ in w. Then*

$$\int R_{\lambda,w}^{(2n)}(\cdots t^{\pm 1/2}z_i \cdots ; t)\tilde{\Delta}_S^{(n)}(z; t^2)dT = 0.$$

To prove the claim note that $R_{\lambda,w}^{(2n)}(\cdots t^{\pm 1/2}z_i \cdots ; t) = 0$ in this case. Indeed, we have the term

$$\frac{\sqrt{t}z_i - tz_i/\sqrt{t}}{\sqrt{t}z_i - z_i/\sqrt{t}} = \frac{tz_i - tz_i}{z_i(t-1)} = 0$$

appearing in the product defining the Hall–Littlewood polynomial.

Thus, we may restrict our attention to those permutations w with $z_i/\sqrt{t}$ to the left of $\sqrt{t}z_i$ for all $1 \le i \le n$. Moreover, we may order the variables so that $z_i/\sqrt{t}$ appears to the left of $z_j/\sqrt{t}$ for all $1 \le i < j \le n$. We compute the normalization first.

Claim 2.21.2. *We have*

$$Z = \int_T P_{0^{2n}}^{(2n)}(\cdots t^{\pm 1/2}z_i \cdots ; t)\frac{1}{n!}\tilde{\Delta}_S^{(n)}(z; t^2)dT = \frac{1}{v_{(0^n)}(t^2)} = \frac{(1-t^2)^n}{(1-t^2)(1-t^4)\cdots(1-t^{2n})}.$$

The proof follows by noting that $P_{0^{2n}}^{(2n)}(\cdots t^{\pm 1/2}z_i \cdots ; t) = 1$ and applying theorem 2.1.

Claim 2.21.3. *Let $w \in S_{2n}$ be a permutation with $z_i/\sqrt{t}$ to the left of $\sqrt{t}z_i$ for all $1 \le i \le n$ and $z_i/\sqrt{t}$ to the left of $z_j/\sqrt{t}$ for all $1 \le i < j \le n$, and $\sqrt{t}z_1$ in position k for some $2 \le k \le 2n$. Then*

$$\int_T R_{\lambda,w}^{(2n)}(\cdots t^{\pm 1/2}z_i \cdots ; t)\tilde{\Delta}_S^{(n)}(z; t^2)dT$$
$$= \chi_{\lambda_1+\lambda_k=0}(1+t)t^{2n-k}\int_T R_{\widehat{\lambda},\widehat{w}}^{(2(n-1))}(\cdots t^{\pm 1/2}z_i \cdots ; t)\tilde{\Delta}_S^{(n-1)}(z; t^2)dT,$$

where $\widehat{w}$ is the permutation w with $z_1/\sqrt{t}$ and $\sqrt{t}z_1$ deleted, and $\widehat{\lambda}$ is the partition λ with parts λ_1 and λ_k deleted.

To prove the claim, integrate with respect to z_1. Note that if $\lambda_1 + \lambda_k > 0$, the integral vanishes. If $\lambda_1 + \lambda_k < 0$, note that $\lambda_{2n} + \lambda_j < 0$ for all $1 \leq j \leq 2n-1$. Integrate with respect to the last variable in w, and invert all variables to find the integral vanishes, as desired.

The above claim implies that the integral $\int_T R^{(2n)}_{\lambda,w}(\cdots t^{\pm 1/2} z_i \cdots; t)\tilde{\Delta}^{(n)}_S(z;t^2)dT$ vanishes unless $\lambda = \mu\bar{\mu}$ for some μ. Moreover, if $\lambda = \mu\bar{\mu}$, the term integral vanishes unless

$$w(\cdots t^{\pm 1/2} z_i \cdots)^{\lambda}$$

is a constant in t (i.e., independent of z_i). Thus, in the case $\lambda = \mu\bar{\mu}$, a computation gives that the total integral

$$\begin{aligned}
&\int_T R^{(2n)}_{\lambda}(\cdots t^{\pm 1/2} z_i \cdots; t)\frac{1}{n!}\tilde{\Delta}^{(n)}_S(z;t^2)dT \\
&\quad = (1+t)^{l(\mu)} v_{\mu+}(t) \int_T R^{(2(n-l(\mu)))}_{0^{2(n-l(\mu))}}(\cdots t^{\pm 1/2} z_i \cdots; t)\frac{1}{(n-l(\mu))!}\tilde{\Delta}^{(n-l(\mu))}_S(z;t^2)dT \\
&\qquad = (1+t)^{l(\mu)} v_{\mu+}(t)\frac{(1-t^2)^{n-l(\mu)}}{(1-t^2)(1-t^4)\cdots(1-t^{2(n-l(\mu))})} v_{(0^{2(n-l(\mu))})}(t).
\end{aligned}$$

Multiplying this by $1/Zv_{\lambda}(t) = 1/Zv_{\mu+}(t)^2 v_{(0^{2(n-l(\mu))})}(t)$ and simplifying gives the result. □

Theorem 2.22. *(see [20, corollary 4.7(ii)]) Let λ be a partition with $l(\lambda) \leq n$. Then the integral*

$$\int P_{\lambda}(x_1,\ldots,x_n;t^2)P_{m^n}(x_1^{-1},\ldots,x_n^{-1};t)\frac{1}{n!}\tilde{\Delta}^{(n)}_S(x;t)dT$$

vanishes unless $\lambda = (2m)^n - \lambda$.

Note that the above integral gives the coefficient of $P_{m^n}(x;t)$ in the expansion of $P_{\lambda}(x;t^2)$ as Hall–Littlewood polynomials with parameter t.

Proof. Since $P_{m^n}(x_1^{-1},\ldots,x_n^{-1};t) = (x_1^{-1}\cdots x_n^{-1})^m$, an equivalent statement is the following:

Let λ be a weight of GL_n with possibly negative parts. Then the integral

$$\frac{1}{Z}\int P_{\lambda}(x_1,\ldots,x_n;t^2)\frac{1}{n!}\tilde{\Delta}^{(n)}_S(x;t)dT$$

vanishes unless $\lambda = \mu\bar{\mu}$, and in this case it is

$$\frac{(1-t^{n-2l(\mu)+1})\cdots(1-t^n)t^{|\mu|}}{(1-t^2)^{l(\mu)} v_{\mu+}(t^2)}.$$

We first compute the normalization $Z = \frac{1}{n!}\int P_{0^n}^{(n)}(x;t^2)\tilde{\Delta}_S^{(n)}(x;t)dT$. Note that $P_{0^n}(x;t^2) = 1$, so we have

$$Z = \frac{1}{n!}\int \tilde{\Delta}_S^{(n)}(x;t)dT = \frac{1}{n!}\int P_{0^n}^{(n)}(x;t)P_{0^n}^{(n)}(x^{-1};t)\tilde{\Delta}_S^{(n)}dT = \frac{1}{n!}\frac{n!}{v_{(0^n)}(t)}$$
$$= \frac{(1-t)^n}{(1-t)(1-t^2)\cdots(1-t^n)}$$

using theorem 2.1.

Now we look at $\frac{1}{n!}\int R_\lambda(x_1,\dots,x_n;t^2)\tilde{\Delta}_S^{(n)}(x;t)dT$, which is a sum of $n!$ integrals—one for each $w \in S_n$. By symmetry we have

$$\frac{1}{n!}\int R_\lambda^{(n)}(x;t^2)\tilde{\Delta}_S^{(n)}(x;t)dT = \int R_{\lambda,\mathrm{id}}^{(n)}(x;t^2)\tilde{\Delta}_S^{(n)}(x;t)dT,$$

so we may restrict to the case $w = \mathrm{id}$. We assume $\lambda_1 > 0$: note that if $\lambda_1 \leq 0$ we have $\lambda_n < 0$ (we are assuming λ is not the zero partition) and we can invert all variables and make a change of variables to reduce to the case $\lambda_1 > 0$. Then the integral restricted to terms in x_1 is

$$\int_{T_1} x_1^{\lambda_1}\prod_{j>1}\frac{x_1-t^2x_j}{x_1-x_j}\prod_{j>1}\frac{(x_1-x_j)(x_j-x_1)}{(x_1-tx_j)(x_j-tx_1)}\frac{dx_1}{2\pi\sqrt{-1}x_1}$$
$$= \int_{T_1} x_1^{\lambda_1}\prod_{j>1}\frac{(x_1-t^2x_j)(x_j-x_1)}{(x_1-tx_j)(x_j-tx_1)}\frac{dx_1}{2\pi\sqrt{-1}x_1}$$
$$= \sum_{j>1}\frac{t^{\lambda_1}(1-t)^2}{(1-t^2)}x_j^{\lambda_1}\prod_{i\neq 1,j}\frac{(tx_j-t^2x_i)(x_i-tx_j)}{(tx_j-tx_i)(x_i-t^2x_j)} = \sum_{j>1}\frac{t^{\lambda_1}(1-t)^2}{(1-t^2)}x_j^{\lambda_1}\prod_{i\neq 1,j}\frac{(x_j-tx_i)(x_i-tx_j)}{(x_j-x_i)(x_i-t^2x_j)},$$

where the second line follows by evaluating the residues at $x_1 = tx_j$ for $j > 1$. For each $j > 1$, we can combine this with the terms in x_j from the original integrand. The integral restricted to terms in x_j is

$$\frac{t^{\lambda_1}(1-t)^2}{(1-t^2)}\int_{T_1} x_j^{\lambda_1}\prod_{i\neq 1,j}\frac{(x_j-tx_i)(x_i-tx_j)}{(x_j-x_i)(x_i-t^2x_j)}x_j^{\lambda_j}\prod_{1\neq i<j}\frac{x_i-t^2x_j}{x_i-x_j}\prod_{j<i}\frac{x_j-t^2x_i}{x_j-x_i}$$
$$\cdot\prod_{i\neq 1,j}\frac{(x_i-x_j)(x_j-x_i)}{(x_i-tx_j)(x_j-tx_i)}\frac{dx_j}{2\pi\sqrt{-1}x_j} = \frac{t^{\lambda_1}(1-t)^2}{(1-t^2)}\int x_j^{\lambda_1+\lambda_j}(-1)^{n-j}\prod_{j<i}\frac{x_j-t^2x_i}{x_i-t^2x_j}\frac{dx_j}{2\pi\sqrt{-1}x_j}.$$

Now, this is 0 if $\lambda_1 + \lambda_j > 0$ and

$$\frac{t^{\lambda_1}(1-t)(t^2)^{n-i}}{(1+t)},$$

if $\lambda_1 + \lambda_j = 0$. Finally, if $\lambda_1 + \lambda_j < 0$ note that $\lambda_n + \lambda_i < 0$ for all $1 \leq i < n$. We can invert all variables and make a change of variables to arrive at the case $\lambda_1 + \lambda_j > 0$, so the integral is zero by the above argument.

Iterating this argument shows that the partition λ must satisfy $\lambda_i + \lambda_{n+1-i} = 0$ for the integral to be nonvanishing. Thus $\lambda = \mu\bar{\mu}$ for some μ. In this case, we compute from the above remarks:

$$\frac{1}{Z}\int P^{(n)}_{\lambda}(x;t^2)\frac{1}{n!}\tilde{\Delta}^{(n)}_S(x;t)dT = \frac{1}{Z}\frac{1}{v_\lambda(t^2)}\int R^{(n)}_{\lambda,\mathrm{id}}(x;t^2)\tilde{\Delta}^{(n)}_S(x;t)dT$$
$$= \frac{\phi_n(t)}{(1-t)^n}\frac{t^{|\mu|}}{v_{\mu+}(t^2)^2 v_{(0^{n-2l(\mu)})}(t^2)}\frac{(1-t)^{l(\mu)}}{(1+t)^{l(\mu)}}v_{\mu+}(t^2)\int R_{0^{n-2l(\mu)}}(x;t^2)\frac{1}{(n-2l(\mu))!}\tilde{\Delta}^{(n)}_S(x;t)dT.$$

Using the computation of Z, this is equal to

$$\frac{\phi_n(t)}{(1-t)^n}\frac{t^{|\mu|}}{v_{\mu+}(t^2)}\frac{(1-t)^{l(\mu)}}{(1+t)^{l(\mu)}}\int P^{(n-2l(\mu))}_{0^{n-2l(\mu)}}(x;t^2)\frac{1}{(n-2l(\mu))!}\tilde{\Delta}^{(n)}_S(x;t)dT$$
$$= \frac{\phi_n(t)}{(1-t)^n}\frac{t^{|\mu|}}{v_{\mu+}(t^2)}\frac{(1-t)^{l(\mu)}}{(1+t)^{l(\mu)}}\frac{(1-t)^{n-2l(\mu)}}{\phi_{n-2l(\mu)}(t)} = \frac{\phi_n(t)}{\phi_{n-2l(\mu)}(t)}\frac{t^{|\mu|}}{(1-t^2)^{l(\mu)}v_{\mu+}(t^2)}$$
$$= \frac{(1-t^{n-2l(\mu)+1})\cdots(1-t^n)t^{|\mu|}}{(1-t^2)^{l(\mu)}v_{\mu+}(t^2)},$$

as desired. □

Chapter 3

Hall–Littlewood Polynomials of Type BC

3.1 Background and Notation

In this section, we set up notation that will be used throughout the chapter. We also define the relevant polynomials that are the subject of this chapter.

Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition, in which some of the λ_i may be zero. In particular, note that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. Let $l(\lambda) \leq n$ be the number of nonzero parts of λ (the "length"), and $|\lambda|$ the sum of the nonzero parts (the "weight").

Let $m_i(\lambda)$ be the number of λ_j equal to i for each $i \geq 0$. Then we define

$$v_\lambda(t; a, b; t_0, \dots, t_3) = \left(\prod_{i \geq 0} \prod_{j=1}^{m_i(\lambda)} \frac{1 - t^j}{1 - t} \right) \left(\prod_{i=1}^{m_1(\lambda)} 1 - t_0 t_1 t_2 t_3 t^{i-1+2m_0(\lambda)} \right) \left(\prod_{i=1}^{m_0(\lambda)} 1 - abt^{i-1} \right), \quad (3.1)$$

and

$$v_{\lambda+}(t; t_0, \dots, t_3) = \left(\prod_{i \geq 1} \prod_{j=1}^{m_i(\lambda)} \frac{1 - t^j}{1 - t} \right) \left(\prod_{i=1}^{m_1(\lambda)} (1 - t_0 t_1 t_2 t_3 t^{i-1+2m_0(\lambda)} \right). \quad (3.2)$$

Note the comparison with the factors making the Hall–Littlewood polynomials monic in [16, chapter 3].

Also, we define the symmetric Koornwinder density [13]:

$$\tilde{\Delta}_K^{(n)}(x; t; t_0, t_1, t_2, t_3)$$
$$= \frac{1}{2^n n!} \left(\prod_{1 \leq i \leq n} \frac{1 - x_i^{\pm 2}}{(1 - t_0 x_i^{\pm 1})(1 - t_1 x_i^{\pm 1})(1 - t_2 x_i^{\pm 1})(1 - t_3 x_i^{\pm 1})} \right) \left(\prod_{1 \leq i < j \leq n} \frac{1 - x_i^{\pm 1} x_j^{\pm 1}}{1 - t x_i^{\pm 1} x_j^{\pm 1}} \right), \quad (3.3)$$

where we write $(1 - x_i^{\pm 2})$ for the product $(1 - x_i^2)(1 - x_i^{-2})$ and $(1 - x_i^{\pm 1} x_j^{\pm 1})$ for $(1 - x_i x_j)(1 - x_i^{-1} x_j^{-1})(1 - x_i^{-1} x_j)(1 - x_i x_j^{-1})$, etc. For convenience we will write $\tilde{\Delta}_K^{(n)}(t_0, \dots, t_3)$ with the assumption that the density is in variables $x_1, \ldots, x_n$ with parameter t when it is clear. We define the q-symbol

$$(a;q) = \prod_{k \geq 0} (1 - aq^k),$$

and let $(a_1, a_2, \ldots, a_l; q)$ denote $(a_1;q)(a_2;q)\cdots(a_l;q)$.

For simplicity of notation, we will write $v_\lambda(t), v_{\lambda+}(t), N_\lambda, \tilde{\Delta}_K^{(n)}$, etc. when the parameters $(a, b; t_0, \ldots, t_3)$ are clear.

Finally, put

$$\begin{aligned} N_\lambda(t; t_0, \ldots, t_3) &= \frac{1}{v_{\lambda+}(t)} \int_T \tilde{\Delta}_K^{(m_0(\lambda))}(z; t; t_0, \ldots, t_3) dT \\ &= \frac{1}{v_{\lambda+}(t)} \prod_{0 \leq j \leq m_0(\lambda)} \frac{(t, t^{2n-2-j} t_0 t_1 t_2 t_3; 0)}{(t^{j+1}, t^j t_0 t_1, t^j t_0 t_2, t^j t_0 t_3, t^j t_1 t_2, t^j t_1 t_3, t^j t_2 t_3; 0)}, \end{aligned} \tag{3.4}$$

where the explicit evaluation for the integral is a result of Gustafson [8].

Finally, we explain some notation involving elements of the hyperoctahedral group, B_n. An element in B_n is determined by specifying a permutation $\rho \in S_n$ as well as a sign choice $\epsilon_\rho(i)$, for each $1 \leq i \leq n$. Thus, ρ acts on the subscripts of the variables, for example by

$$\rho(z_1 \cdots z_n) = z_{\rho(1)}^{\epsilon_\rho(1)} \cdots z_{\rho(n)}^{\epsilon_\rho(n)}.$$

If $\rho(i) = 1$, we will say that z_1 occurs in position i of ρ. We also write

$$\text{“} z_i \prec z_j \text{”},$$

if $i = \rho(i')$ and $j = \rho(j')$ for some $i' < j'$, i.e., z_i appears to the left of z_j in the permutation $z_{\rho(1)}^{\epsilon_\rho(1)} \cdots z_{\rho(n)}^{\epsilon_\rho(n)}$. We also define $\epsilon_\rho(z_i)$ to be $\epsilon_\rho(i')$ if $i = \rho(i')$, i.e., it is the exponent (± 1) on z_i in $z_{\rho(1)}^{\epsilon_\rho(1)} \cdots z_{\rho(n)}^{\epsilon_\rho(n)}$.

We now define the Koornwinder polynomials at $q = 0$.

Definition 3.1. Let λ be a partition with $l(\lambda) \leq n$ and $|t|, |t_0|, \ldots, |t_3| < 1$. Then $K_\lambda(z_1, \ldots, z_n; t; a, b; t_0, \ldots, t_3)$, indexed by λ, is defined by

$$\frac{1}{v_\lambda(t; a, b; t_0, \ldots, t_3)} \sum_{w \in B_n} w \left(\prod_{1 \leq i \leq n} u_{\lambda_i}(z_i) \prod_{1 \leq i < j \leq n} \frac{1 - t z_i^{-1} z_j}{1 - z_i^{-1} z_j} \frac{1 - t z_i^{-1} z_j^{-1}}{1 - z_i^{-1} z_j^{-1}} \right), \tag{3.5}$$

where

$$u_{\lambda_i}(z_i) = \begin{cases} \frac{(1-az_i^{-1})(1-bz_i^{-1})}{1-z_i^{-2}}, & \text{if } \lambda_i = 0, \\ z_i^{\lambda_i} \frac{(1-t_0 z_i^{-1})(1-t_1 z_i^{-1})(1-t_2 z_i^{-1})(1-t_3 z_i^{-1})}{1-z_i^{-2}}, & \text{if } \lambda_i > 0. \end{cases}$$

Remarks. We note that the K_λ are actually independent of a, b–this is a scaling factor accounted for in v_λ. In particular, the arguments below for showing this is indeed the Koornwinder polynomial at $q = 0$ work for any choice of a, b. However, we leave in arbitrary a, b (as opposed to the choice ± 1) because the resulting form is useful for proving the vanishing identities.

We will also write $K_\lambda^{(n)}(z; t; a, b; t_0, \dots, t_3)$ for convenience. Also define

$$R_\lambda^{(n)}(z; t; a, b; t_0, \dots, t_3) = v_\lambda(t) K_\lambda^{(n)}(z; t; a, b; t_0, \dots, t_3), \tag{3.6}$$

and for $w \in B_n$, let

$$R_{\lambda,w}^{(n)}(z; t; a, b; t_0, \dots, t_3) = w\Bigg(\prod_{1 \le i \le n} u_{\lambda_i}(z_i) \prod_{1 \le i < j \le n} \frac{1 - t z_i^{-1} z_j}{1 - z_i^{-1} z_j} \frac{1 - t z_i^{-1} z_j^{-1}}{1 - z_i^{-1} z_j^{-1}} \Bigg)$$

be the associated term in the summand.

Remarks. When $(t_0, t_1, t_2, t_3) = (a, b, 0, 0)$, we obtain

$$K_\lambda(z_1, \dots, z_n; t; a, b) = \frac{1}{v_\lambda(t)} \sum_{w \in B_n} w\Big(\prod_{1 \le i \le n} z_i^{\lambda_i} \frac{(1 - a z_i^{-1})(1 - b z_i^{-1})}{1 - z_i^{-2}} \prod_{1 \le i < j \le n} \frac{1 - t z_i^{-1} z_j}{1 - z_i^{-1} z_j} \frac{1 - t z_i^{-1} z_j^{-1}}{1 - z_i^{-1} z_j^{-1}} \Big),$$

which gets rid of the difference in zero and nonzero parts in the univariate terms. In particular, this is Macdonald's 2-parameter family $(BC_n, B_n) = (BC_n, C_n)$ polynomials at $q = 0$.

3.2 Main Results

In this section, we will show that the $K_\lambda^{(n)}$ (we write this for $K_\lambda^{(n)}(z; t; a, b; t_0, \dots, t_3)$ when the parameter values are clear) satisfy the defining properties for Koornwinder polynomials.

Theorem 3.2. *The function $K_\lambda^{(n)}(z; t; a, b; t_0, \dots, t_3)$ is a BC_n-symmetric Laurent polynomial (i.e., invariant under permuting variables $z_1, \dots, z_n$ and inverting variables $z_i \to z_i^{-1}$).*

Proof. Recall the fully BC_n-antisymmetric Laurent polynomials:

$$\Delta_{BC} = \left(\prod_{1\le i\le n} z_i - z_i^{-1}\right)\left(\prod_{1\le i<j\le n} z_i^{-1} - z_j - z_j^{-1} + z_i\right)$$
$$= \left(\prod_{1\le i\le n} \frac{z_i^2-1}{z_i}\right)\left(\prod_{1\le i<j\le n} \frac{1-z_iz_j}{z_iz_j}(z_j-z_i)\right). \tag{3.7}$$

Then we have

$$K_\lambda^{(n)}(z;a,b;t_0,\ldots,t_3;t)\cdot\Delta_{BC} = \frac{1}{v_\lambda(t)}\sum_{w\in B_n}\epsilon(w)w\Big(\prod_{1\le i\le n} u'_{\lambda_i}(z_i)\prod_{1\le i<j\le n}(1-tz_i^{-1}z_j^{-1})(z_i-tz_j)\Big), \tag{3.8}$$

where

$$u'_{\lambda_i}(z_i) = \begin{cases} z_i(1-az_i^{-1})(1-bz_i^{-1}), & \text{if } \lambda_i = 0, \\ z_i^{\lambda_i+1}(1-t_0z_i^{-1})\cdots(1-t_3z_i^{-1}), & \text{if } \lambda_i > 0. \end{cases}$$

Notice that $K_\lambda^{(n)}\cdot\Delta_{BC}$ is a BC_n-antisymmetric Laurent polynomial, so in particular Δ_{BC} divides $K_\lambda^{(n)}\cdot\Delta_{BC}$ as polynomials. Consequently, $K_\lambda^{(n)}$ is a BC_n-symmetric Laurent polynomial, as desired. □

Theorem 3.3. *The functions $K_\lambda^{(n)}(z;t;a,b;t_0,\ldots,t_3)$ are triangular with respect to dominance ordering:*

$$K_\lambda^{(n)}(z;t;a,b;t_0,\ldots,t_3) = m_\lambda + \sum_{\mu<\lambda} c_\mu^\lambda m_\mu.$$

Remarks. Here $\{m_\lambda\}_\lambda$ is the monomial basis with respect to Weyl group of type BC:

$$m_\lambda = \sum_{w\in B_n} w(z_1^{\lambda_1}\cdots z_n^{\lambda_n}).$$

Proof. We show that when $K_\lambda^{(n)}$ is expressed in the monomial basis, the top degree term in m_λ; moreover, it is monic. First note that from (3.7) in the previous proof, we have

$$\Delta_{BC} = m_\rho + (\text{dominated terms}),$$

where $\rho = (n\ n-1\ \cdots\ 2\ 1)$. We compute the dominating monomial in $K_\lambda^{(n)}\cdot\Delta_{BC}$; see (3.8) in the previous proof for the formula. Note that if $\lambda_i = 0$, we have highest degree λ_i+1 in $u'_{\lambda_i}(z_i)$. Similarly, if $\lambda_i > 0$, we note that $\lambda_i + 1 \ge -\lambda_i + 3$ (with equality if and only if $\lambda_i = 1$) so we have highest degree $\lambda_i + 1$ in $u'_{\lambda_i}(z_i)$. Moreover,

$$\prod_{1 \le i < j \le n} (1 - tz_i^{-1} z_j^{-1})(z_i - tz_j) = \prod_{1 \le i < j \le n} (z_i - tz_j^{-1} - tz_j + t^2 z_i^{-1})$$

has highest degree term $z^{\rho - 1}$. Thus, the dominating monomial in $K_\lambda^{(n)} \cdot \Delta_{BC}$ is $z^{\lambda + \rho}$, so that the dominating monomial in $K_\lambda^{(n)}$ is z^λ.

We now show that the coefficient on $z^{\lambda+\rho}$ in $R_\lambda^{(n)} \cdot \Delta_{BC}$ (see (3.6) for the definition of $R_\lambda^{(n)}$) is $v_\lambda(t)$, so that $K_\lambda^{(n)}$ is indeed monic. Note first that by the above argument the only contributing w are those such that (1) $z_1^{\lambda_1} \cdots z_n^{\lambda_n} = z_{w(1)}^{\lambda_1} \cdots z_{w(n)}^{\lambda_n}$ and (2) $\epsilon_w(z_i) = 1$ for all $1 \le i \le n - m_0(\lambda) - m_1(\lambda)$; let the set of these special permutations be denoted by $P_{\lambda,n}$. Now fix $w \in P_{\lambda,n}$, we compute the coefficient on $z_1^{\lambda_1 + n}$. Using (3.8) and the arguments of the previous paragraph, one can check that the coefficient is

(i) If $\lambda_1 > 1$:

$$t^{\#\{z_i \prec_w z_1\}}.$$

(ii) If $\lambda_1 = 1$:

$$\begin{cases} t^{\#\{z_i \prec_w z_1\}}, & \text{if } \epsilon_w(z_1) = 1, \\ -t_0 \cdots t_3 (t^2)^{\#\{z_1 \prec_w z_i\}} t^{\#\{z_i \prec_w z_1\}}, & \text{if } \epsilon_w(z_1) = -1. \end{cases}$$

(iii) If $\lambda_1 = 0$:

$$\begin{cases} t^{\#\{z_i \prec_w z_1\}}, & \text{if } \epsilon_w(z_1) = 1, \\ -ab (t^2)^{\#\{z_1 \prec_w z_i\}} t^{\#\{z_i \prec_w z_1\}}, & \text{if } \epsilon_w(z_1) = -1. \end{cases}$$

Note that we have used the contribution of (-1) factors from $\epsilon(w)$ in $K_\lambda^{(n)} \cdot \Delta_{BC}$.

Now define the following subsets of the variables $z_1, \ldots, z_n$:

$$N_{w,\lambda}^1 = \{z_i : n - m_0(\lambda) - m_1(\lambda) < i \le n - m_0(\lambda) \text{ and } \epsilon_w(z_i) = -1\},$$
$$N_{w,\lambda}^0 = \{z_i : n - m_0(\lambda) < i \le n \text{ and } \epsilon_w(z_i) = -1\},$$
$$N_{w,\lambda} = N_{w,\lambda}^1 + N_{w,\lambda}^0.$$

Finally, define the following statistics of w:

$$n(w) = |\{(i,j) : 1 \le i < j \le n \text{ and } z_j \prec_w z_i\}|,$$
$$c_\lambda(w) = |\{(i,j) : 1 \le i < j \le n \text{ and } z_i \prec_w z_j \text{ and } z_i \in N_{w,\lambda}\}|.$$

Then by iterating the coefficient argument above, we get that the coefficient on $z^{\lambda+\rho}$ is given by

$$\sum_{w \in P_{\lambda,n}} t^{n(w)} t^{2c_\lambda(w)} (-t_0 \dots t_3)^{|N^1_{w,\lambda}|} (-ab)^{|N^0_{w,\lambda}|}.$$

Since $P_{\lambda,n} = B_{m_0(\lambda)} B_{m_1(\lambda)} \prod_{i \ge 2} S_{m_i(\lambda)}$, it is enough to show the following three cases:

$$\sum_{w \in S_m} t^{n(w)} = \prod_{j=1}^{m} \frac{1-t^j}{1-t}, \tag{3.9}$$

$$\sum_{w \in B_m} t^{n(w)} t^{2c_{1^m}(w) + 2m_0(\lambda)} (-t_0 \dots t_3)^{\left|N^1_{w,1^m}\right|} = \prod_{j=1}^{m} \frac{1-t^j}{1-t} (1 - t_0 \cdots t_3 t^{j-1+2m_0(\lambda)}), \tag{3.10}$$

$$\sum_{w \in B_m} t^{n(w)} t^{2c_{0^m}(w)} (-ab)^{\left|N^0_{w,0^m}\right|} = \prod_{j=1}^{m} \frac{1-t^j}{1-t} (1 - abt^{j-1}). \tag{3.11}$$

To show (3.9), we note that the left-hand side is exactly enumerated by the terms of

$$(1 + t + t^2 + \cdots + t^{m-1})(1 + t + t^2 + \cdots + t^{m-2}) \cdots (1+t)(1),$$

which is equal to the right-hand side. Also refer to [16, chapter 3, proof of (1.2) and (1.3)]. We now show (3.10); (3.11) is analogous. One can verify that the left-hand side of (3.10) is exactly enumerated by the terms of

$$\prod_{k=1}^{m} \Big[\sum_{i=1}^{k} \big(t^{i-1} + t^{i-1} (t^2)^{m_0(\lambda)+k-i} (-t_0 \cdots t_3) \big) \Big]. \tag{3.12}$$

But we also have

$$\begin{aligned} \sum_{i=1}^{k} \Big(t^{i-1} + t^{i-1} (t^2)^{m_0(\lambda)+k-i} (-t_0 \cdots t_3) \Big) &= \sum_{i=1}^{k} \big(t^{i-1} - t_0 \cdots t_3 t^{k+2m_0(\lambda)-1} t^{k-i} \big) \\ &= (1 - t_0 \cdots t_3 t^{k+2m_0(\lambda)-1})(1 + t + \cdots t^{k-1}) = (1 - t_0 \cdots t_3 t^{k+2m_0(\lambda)-1}) \frac{1-t^k}{1-t}; \end{aligned}$$

substituting this into (3.12) gives the right-hand side of (3.10) as desired.

Multiplying these functions together for each distinct part i of λ (put $m = m_i(\lambda)$ in (3.9), (3.10), and (3.11), depending on whether $i \ge 2, i = 1,$ or $i = 0$, respectively), and using (3.1) shows that the coefficient on $z^{\lambda+\rho}$ in $R_\lambda^{(n)} \cdot \Delta_{BC}$ is indeed $v_\lambda(t)$, as desired. ☐

Theorem 3.4. *The family of polynomials* $\{K_\lambda^{(n)}(z;a,b;t_0,\ldots,t_3;t)\}_\lambda$ *satisfy the following orthogonality result:*

$$\int_T K_\lambda(z_1,\ldots,z_n;a,b;t_0,\ldots,t_3;t)K_\mu(z_1,\ldots,z_n;a,b;t_0,\ldots,t_3;t)\tilde{\Delta}_K^{(n)}(z;t_0,\ldots,t_3;t)dT = N_\lambda\delta_{\lambda\mu}$$

(refer to (3.3) and (3.4) for the definitions of $\tilde{\Delta}_K^{(n)}(z;t_0,\ldots,t_3;t)$ *and* N_λ*, respectively).*

Proof. By symmetry of λ,μ, we may restrict to the case where $\lambda \geq \mu$ in the reverse lexicographic ordering. We assume $\lambda_1 > 0$, so we are not in the situation where both partitions are trivial; these assumptions hold throughout the proof. By definition of $K_\lambda^{(n)}(z;a,b;t_0,\ldots,t_3;t)$ as a sum over B_n, the above integral is equal to

$$\sum_{w,\rho\in B_n}\int_T K_{\lambda,w}^{(n)}(z;a,b;t_0,\ldots,t_3;t)K_{\mu,\rho}^{(n)}(z;a,b;t_0,\ldots,t_3;t)\tilde{\Delta}_K^{(n)}(z;t_0,\ldots,t_3;t)dT.$$

Consider an arbitrary term in this sum over $B_n \times B_n$ indexed by (w,ρ). Note that using a change of variables in the integral and inverting variables (which preserves the integral), we may assume w is the identity permutation, and all sign choices are 1 (and ρ is arbitrary). That is, we have

$$\begin{aligned}&\int_T K_\lambda(z_1,\ldots,z_n;a,b;t_0,\ldots,t_3;t)K_\mu(z_1,\ldots,z_n;a,b;t_0,\ldots,t_3;t)\tilde{\Delta}_K^{(n)}(z;t_0,\ldots,t_3;t)dT\\ &\quad= 2^n n!\sum_{\rho\in B_n}\int_T K_{\lambda,\mathrm{id}}^{(n)}(z;a,b;t_0,\ldots,t_3;t)K_{\mu,\rho}^{(n)}(z;a,b;t_0,\ldots,t_3;t)\tilde{\Delta}_K^{(n)}(z;t_0,\ldots,t_3;t)dT\\ &= 2^n n!\frac{1}{v_\lambda(t)v_\mu(t)}\sum_{\rho\in B_n}\int_T R_{\lambda,\mathrm{id}}^{(n)}(z;a,b;t_0,\ldots,t_3;t)R_{\mu,\rho}^{(n)}(z;a,b;t_0,\ldots,t_3;t)\tilde{\Delta}_K^{(n)}(z;t_0,\ldots,t_3;t)dT,\end{aligned}$$

where $R_\lambda^{(n)}$ is as defined in (3.6).

We study an arbitrary term in this sum. In particular, we give an iterative formula that shows that each of these terms vanishes unless $\lambda = \mu$.

Claim 3.4.1. *Fix an arbitrary* $\rho \in B_n$ *and let* $\rho(i) = 1$ *for some* $1 \leq i \leq n$*. Then we have the following formula:*

$$2^n n!\int_T R_{\lambda,id}^{(n)}(z;a,b;t_0,\ldots,t_3;t)R_{\mu,\rho}^{(n)}(z;a,b;t_0,\ldots,t_3;t)\tilde{\Delta}_K^{(n)}dT =$$

$$\begin{cases} t^{i-1}2^{n-1}(n-1)!\int R_{\widehat{\lambda},\widehat{id}}^{(n-1)}R_{\widehat{\mu},\widehat{\rho}}^{(n-1)}\tilde{\Delta}_K^{(n-1)}dT, & \mu_i = \lambda_1, \epsilon_\rho(z_1) = -1,\\ t^{i-1}(t^2)^{m_0(\mu)+m_1(\mu)-i}(-t_0\cdots t_3)2^{n-1}(n-1)!\int R_{\widehat{\lambda},\widehat{id}}^{(n-1)}R_{\widehat{\mu},\widehat{\rho}}^{(n-1)}\tilde{\Delta}_K^{(n-1)}dT, & \mu_i = \lambda_1 = 1, \epsilon_\rho(z_1) = 1,\\ 0, & \textit{otherwise,}\end{cases}$$

where $\widehat{\lambda}$ and $\widehat{\mu}$ are the partitions λ and μ with parts λ_1 and μ_i deleted (respectively), and $\widehat{id}$ and $\widehat{\rho}$ are the permutations id and ρ with z_1 deleted (respectively) and signs preserved.

To prove the claim, we integrate with respect to z_1 in the iterated integral, using the definition of $R^{(n)}_{\lambda,\mathrm{id}}, R^{(n)}_{\mu,\rho}$ and $\tilde{\Delta}^{(n)}_K$.

First suppose $\mu_i > 0$. The univariate terms in z_1 are

$$\begin{aligned} &z_1^{\lambda_1}\frac{(1-t_0z_1^{-1})\cdots(1-t_3z_1^{-1})}{(1-z_1^{-2})}z_1^{\mu_i}\frac{(1-t_0z_1^{-1})\cdots(1-t_3z_1^{-1})}{(1-z_1^{-2})}\frac{(1-z_1^{\pm 2})}{(1-t_0z_1^{\pm 1})\cdots(1-t_3z_1^{\pm 1})} \\ &= z_1^{\lambda_1+\mu_i}\frac{(-z_1^2)(1-t_0z_1^{-1})\cdots(1-t_3z_1^{-1})}{(1-t_0z_1)\cdots(1-t_3z_1)}, \end{aligned}$$

if $\epsilon_\rho(z_1)=1$, and

$$\begin{aligned} &z_1^{\lambda_1}\frac{(1-t_0z_1^{-1})\cdots(1-t_3z_1^{-1})}{(1-z_1^{-2})}z_1^{-\mu_i}\frac{(1-t_0z_1)\cdots(1-t_3z_1)}{(1-z_1^{2})}\frac{(1-z_1^{\pm 2})}{(1-t_0z_1^{\pm 1})\cdots(1-t_3z_1^{\pm 1})} \\ &= z_1^{\lambda_1-\mu_i}, \end{aligned}$$

if $\epsilon_\rho(z_1)=-1$.

Now suppose $\mu_i = 0$. The univariate terms in z_1 are

$$\begin{aligned} &z_1^{\lambda_1}\frac{(1-t_0z_1^{-1})\cdots(1-t_3z_1^{-1})}{(1-z_1^{-2})}\frac{(1-az_1^{-1})(1-bz_1^{-1})}{(1-z_1^{-2})}\frac{(1-z_1^{\pm 2})}{(1-t_0z_1^{\pm 1})\cdots(1-t_3z_1^{\pm 1})} \\ &= z_1^{\lambda_1}\frac{(-z_1^2)(1-az_1^{-1})(1-bz_1^{-1})}{(1-t_0z_1)\cdots(1-t_3z_1)}, \end{aligned}$$

if $\epsilon_\rho(z_1)=1$, and

$$\begin{aligned} &z_1^{\lambda_1}\frac{(1-t_0z_1^{-1})\cdots(1-t_3z_1^{-1})}{(1-z_1^{-2})}\frac{(1-az_1)(1-bz_1)}{(1-z_1^{2})}\frac{(1-z_1^{\pm 2})}{(1-t_0z_1^{\pm 1})\cdots(1-t_3z_1^{\pm 1})} \\ &= z_1^{\lambda_1}\frac{(1-az_1)(1-bz_1)}{(1-t_0z_1)\cdots(1-t_3z_1)}, \end{aligned}$$

if $\epsilon_\rho(z_1)=-1$.

Notice that for the cross terms in z_1 (those involving z_j for $j\neq 1$), we have

$$\prod_{j>1}\frac{1-tz_1^{-1}z_j^{-1}}{1-z_1^{-1}z_j^{-1}}\frac{1-tz_1^{-1}z_j}{1-z_1^{-1}z_j}\times\prod_{j>1}\frac{1-z_1^{\pm 1}z_j^{\pm 1}}{1-tz_1^{\pm 1}z_j^{\pm 1}},$$

from the corresponding terms in z_1 of $R_{\lambda,\mathrm{id}}$ and the density. Combining this with the cross terms of $R_{\mu,\rho}$ in z_1 (and taking into account the various sign possibilities for ρ), we obtain

$$\prod_{\substack{z_i \prec_\rho z_1 \\ \text{sign } 1 \text{ for } z_i}} \frac{t - z_1 z_i}{1 - t z_1 z_i} \prod_{\substack{z_i \prec_\rho z_1 \\ \text{sign } -1 \text{ for } z_i}} \frac{t - z_1 z_i^{-1}}{1 - t z_1 z_i^{-1}} \prod_{z_1 \prec_\rho z_j} \frac{(t - z_1 z_j^{-1})(t - z_1 z_j)}{(1 - t z_1 z_j^{-1})(1 - t z_1 z_j)},$$

if $\epsilon_\rho(z_1) = 1$, and

$$\prod_{\substack{z_i \prec_\rho z_1 \\ \text{sign } 1 \text{ for } z_i}} \frac{t - z_1 z_i}{1 - t z_1 z_i} \prod_{\substack{z_i \prec_\rho z_1 \\ \text{sign } -1 \text{ for } z_i}} \frac{t - z_1 z_i^{-1}}{1 - t z_1 z_i^{-1}},$$

if $\epsilon_\rho(z_1) = -1$.

Thus, the integral in z_1 is

$$\begin{cases} \int_{T_1} z_1^{\lambda_1 + \mu_i} \frac{(-z_1^2)(1 - t_0 z_1^{-1}) \cdots (1 - t_3 z_1^{-1})}{(1 - t_0 z_1) \cdots (1 - t_3 z_1)} \cdot & \\ \prod_{\substack{z_k \prec_\rho z_1 \\ \epsilon_\rho(z_k) = 1}} \frac{t - z_1 z_k}{1 - t z_1 z_k} \prod_{\substack{z_k \prec_\rho z_1 \\ \epsilon_\rho(z_k) = -1}} \frac{t - z_1 z_k^{-1}}{1 - t z_1 z_k^{-1}} \prod_{z_1 \prec_\rho z_j} \frac{(t - z_1 z_j^{-1})(t - z_1 z_j)}{(1 - t z_1 z_j^{-1})(1 - t z_1 z_j)} dT & \mu_i > 0,\ \epsilon_\rho(z_1) = 1, \\ \int_{T_1} z_1^{\lambda_1} \frac{(-z_1^2)(1 - a z_1^{-1})(1 - b z_1^{-1})}{(1 - t_0 z_1) \cdots (1 - t_3 z_1)} \cdot & \\ \prod_{\substack{z_k \prec_\rho z_1 \\ \epsilon_\rho(z_k) = 1}} \frac{t - z_1 z_k}{1 - t z_1 z_k} \prod_{\substack{z_k \prec_\rho z_1 \\ \epsilon_\rho(z_k) = -1}} \frac{t - z_1 z_k^{-1}}{1 - t z_1 z_k^{-1}} \prod_{z_1 \prec_\rho z_j} \frac{(t - z_1 z_j^{-1})(t - z_1 z_j)}{(1 - t z_1 z_j^{-1})(1 - t z_1 z_j)} dT & \mu_i = 0,\ \epsilon_\rho(z_1) = 1, \\ \int_{T_1} z_1^{\lambda_1 - \mu_i} \prod_{\substack{z_k \prec_\rho z_1 \\ \epsilon_\rho(z_k) = 1}} \frac{t - z_1 z_k}{1 - t z_1 z_k} \prod_{\substack{z_k \prec_\rho z_1 \\ \epsilon_\rho(z_k) = -1}} \frac{t - z_1 z_k^{-1}}{1 - t z_1 z_k^{-1}} dT & \mu_i > 0,\ \epsilon_\rho(z_1) = -1, \\ \int_{T_1} z_1^{\lambda_1} \frac{(1 - a z_1)(1 - b z_1)}{(1 - t_0 z_1) \cdots (1 - t_3 z_1)} \prod_{\substack{z_k \prec_\rho z_1 \\ \epsilon_\rho(z_k) = 1}} \frac{t - z_1 z_k}{1 - t z_1 z_k} \prod_{\substack{z_k \prec_\rho z_1 \\ \epsilon_\rho(z_k) = -1}} \frac{t - z_1 z_k^{-1}}{1 - t z_1 z_k^{-1}} dT & \mu_i = 0,\ \epsilon_\rho(z_1) = -1. \end{cases}$$

In particular, the first integral vanishes unless $\lambda_1 = \mu_i = 1$; the second integral always vanishes; the third integral vanishes unless $\lambda_1 = \mu_i$; the fourth integral always vanishes. Thus, we obtain the vanishing conditions of the claim. To obtain the nonzero values, use the residue theorem and evaluate at the simple pole $z_1 = 0$ in the cases $\lambda_1 = \mu_i = 1$ and $\lambda_1 = \mu_i$. Finally, combine with the original integrand involving terms in $z_2, \dots, z_n$ to obtain the result of the claim.

Note that in particular the claim implies that if $\lambda \neq \mu$, each term vanishes and consequently the total integral is zero. This proves the vanishing part of the orthogonality statement.

Next, we compute the norm when $\lambda = \mu$. The claim shows that only certain $\rho \in B_n$ give nonvanishing term integrals. Such permutations must satisfy

$$z_1^{\lambda_1} \cdots z_n^{\lambda_n} z_{\rho(1)}^{-\lambda_1} \cdots z_{\rho(n)}^{-\lambda_n} = 1,$$

and $\epsilon_\rho(z_i) = -1$ for all $1 \leq i \leq n - m_0(\lambda) - m_1(\lambda)$. For simplicity of notation, define $B_{\lambda,n}$ to be the set of such permutations $\rho \in B_n$. Then we have

$$\int_T K_\lambda^{(n)}(z;a,b;t_0,\dots,t_3;t) K_\lambda^{(n)}(z;a,b;t_0,\dots,t_3;t) \tilde{\Delta}_K^{(n)} dT = \frac{2^n n!}{v_\lambda(t)^2} \sum_{\rho \in B_n} \int_T R_{\lambda,\mathrm{id}}^{(n)} R_{\lambda,\rho}^{(n)} \tilde{\Delta}_K^{(n)} dT$$
$$= \frac{2^n n!}{v_\lambda(t)^2} \sum_{\rho \in B_{\lambda,n}} \int_T R_{\lambda,\mathrm{id}}^{(n)} R_{\lambda,\rho}^{(n)} \tilde{\Delta}_K^{(n)} dT,$$

since only these permutations give nonvanishing terms.

Then, using the formula of the claim, we have

$$2^n n! \sum_{\rho \in B_{\lambda,n}} \int_T R_{\lambda,\mathrm{id}}^{(n)} R_{\lambda,\rho}^{(n)} \tilde{\Delta}_K^{(n)} dT$$
$$= \begin{cases} C_1 2^{n-m_{\lambda_1}(\lambda)} (n - m_{\lambda_1}(\lambda))! \displaystyle\sum_{\rho \in B_{\tilde{\lambda}, n-m_{\lambda_1}(\lambda)}} \int_T R_{\tilde{\lambda},\mathrm{id}}^{(n-m_{\lambda_1}(\lambda))} R_{\tilde{\lambda},\rho}^{(n-m_{\lambda_1}(\lambda))} \tilde{\Delta}_K^{(n-m_{\lambda_1}(\lambda))} dT & \lambda_1 > 1, \\ C_2 2^{n-m_{\lambda_1}(\lambda)} (n - m_{\lambda_1}(\lambda))! \displaystyle\sum_{\rho \in B_{\tilde{\lambda}, n-m_{\lambda_1}(\lambda)}} \int_T R_{\tilde{\lambda},\mathrm{id}}^{(n-m_{\lambda_1}(\lambda))} R_{\tilde{\lambda},\rho}^{(n-m_{\lambda_1}(\lambda))} \tilde{\Delta}_K^{(n-m_{\lambda_1}(\lambda))} dT & \lambda_1 = 1, \end{cases}$$

where

$$C_1 = \prod_{k=1}^{m_{\lambda_1}(\lambda)} \Big(\sum_{i=1}^{k} t^{i-1} \Big),$$
$$C_2 = \prod_{k=1}^{m_1(\lambda)} \Big[\sum_{i=1}^{k} \big(t^{i-1} + t^{i-1} (t^2)^{m_0(\lambda)+k-i} (-t_0 t_1 t_2 t_3) \big) \Big],$$

and $\tilde{\lambda}$ is the partition λ with all $m_{\lambda_1}(\lambda)$ occurrences of λ_1 deleted. Iterating this argument gives that

$$2^n n! \sum_{\rho \in B_{\lambda,n}} \int_T R_{\lambda,\mathrm{id}}^{(n)} R_{\lambda,\rho}^{(n)} \tilde{\Delta}_K^{(n)} dT$$
$$= \Bigg(\prod_{j>1} \prod_{k=1}^{m_j(\lambda)} \Big(\sum_{i=1}^{k} t^{i-1} \Big) \Bigg) \Bigg(\prod_{k=1}^{m_1(\lambda)} \sum_{i=1}^{k} \Big(t^{i-1} + t^{i-1} (t^2)^{m_0(\lambda)+k-i} (-t_0 \cdots t_3) \Big) \Bigg)$$
$$\times 2^{m_0(\lambda)} m_0(\lambda)! \sum_{\rho \in B_{m_0(\lambda)}} \int_T R_{0^{m_0(\lambda)},\mathrm{id}}^{(m_0(\lambda))} R_{0^{m_0(\lambda)},\rho}^{(m_0(\lambda))} \tilde{\Delta}_K^{(m_0(\lambda))} dT;$$

note that the expression on the final line is exactly $\int_T {R_{0^{m_0(\lambda)}}^{(m_0(\lambda))}}^2 \tilde{\Delta}_K^{(m_0(\lambda))} dT$.

Thus,

$$\frac{2^n n!}{v_\lambda(t)^2}\sum_{\rho\in B_{\lambda,n}}\int_T R^{(n)}_{\lambda,\mathrm{id}}R^{(n)}_{\lambda,\rho}\tilde{\Delta}^{(n)}_K dT$$

$$= \frac{1}{v_{\lambda+}(t)^2}\left(\prod_{j>1}\prod_{k=1}^{m_j(\lambda)}\Big(\sum_{i=1}^{k}t^{i-1}\Big)\right)\left(\prod_{k=1}^{m_1(\lambda)}\sum_{i=1}^{k}\Big(t^{i-1}+t^{i-1}(t^2)^{m_0(\lambda)+k-i}(-t_0\cdots t_3)\Big)\right)$$

$$\times\frac{1}{v_{0^{m_0(\lambda)}}(t)^2}\int_T R^{(m_0(\lambda))^2}_{0^{m_0(\lambda)}}\tilde{\Delta}^{(m_0(\lambda))}_K dT,$$

since by (3.1) and (3.2) we have $v_{\lambda+}(t)\cdot v_{0^{m_0(\lambda)}}(t)=v_\lambda(t)$. Now using

$$\prod_{k=1}^{m_j(\lambda)}\Big(\sum_{i=1}^{k}t^{i-1}\Big)=\prod_{k=1}^{m_j(\lambda)}\frac{1-t^k}{1-t},$$

and

$$\begin{aligned}\sum_{i=1}^{k}\Big(t^{i-1}+t^{i-1}(t^2)^{m_0(\lambda)+k-i}(-t_0\cdots t_3)\Big) &= \sum_{i=1}^{k}\big(t^{i-1}-t_0\cdots t_3 t^{k+2m_0(\lambda)-1}t^{k-i}\big)\\ &=(1-t_0\cdots t_3t^{k+2m_0(\lambda)-1})(1+t+\cdots t^{k-1})\\ &=(1-t_0\cdots t_3t^{k+2m_0(\lambda)-1})\frac{1-t^k}{1-t},\end{aligned}$$

the above expression can be simplified to

$$\frac{1}{v_{\lambda+}(t)^2}\Big(\prod_{j\geq 1}\prod_{k=1}^{m_j(\lambda)}\frac{1-t^k}{1-t}\Big)\prod_{k=1}^{m_1(\lambda)}(1-t_0\cdots t_3t^{k+2m_0(\lambda)-1})\int_T K^{(m_0(\lambda))^2}_{0^{m_0(\lambda)}}\tilde{\Delta}^{(m_0(\lambda))}_K dT$$

$$=\frac{1}{v_{\lambda+}(t)}\int_T\tilde{\Delta}^{(m_0(\lambda))}_K dT = N_\lambda(t_0,\ldots,t_3;t),$$

since $K^{(m_0(\lambda))}_{0^{m_0(\lambda)}}=1$, by theorem 3.3. □

3.3 Application

In this section, we use the closed formula (3.5) for the Koornwinder polynomials at $q=0$ to prove a result from [20] in this special case. The idea is the same as in [23]: we use the structure of $K^{(n)}_\lambda$ as a sum over the Weyl group and the symmetry of the integral to restrict to one particular term. We obtain an explicit formula for the integral of this particular term by sequentially integrating with respect to one variable at a time.

Theorem 3.5. *[20, theorem 4.10] For partitions λ with $l(\lambda) \le n$, the integral*

$$\int_T K_\lambda(z_1,\ldots,z_n;t^2;a,b;a,b,ta,tb)\tilde{\Delta}_K^{(n)}(z;t;\pm\sqrt{t},a,b)dT$$

vanishes if λ is not an even partition (i.e., $\lambda \neq 2\mu$ for any μ). If λ is an even partition, the integral is equal to

$$\frac{(\sqrt{t})^{|\lambda|}}{(1+t)^{l(\lambda)}}\frac{N_\lambda(t;\pm\sqrt{t},a,b)v_{\lambda+}(t;\pm\sqrt{t},a,b)}{v_{\lambda+}(t^2;a,b,ta,tb)}.$$

Proof. We have

$$\begin{aligned}
&\int_T K_\lambda(z_1,\ldots,z_n;t^2;a,b;a,b,ta,tb)\tilde{\Delta}_K^{(n)}(z;t;\pm\sqrt{t},a,b)dT\\
&\quad=\frac{1}{v_\lambda(t^2;a,b;a,b,ta,tb)}\sum_{w\in B_n}\int_T R_{\lambda,w}^{(n)}(z;t^2;a,b;a,b,ta,tb)\tilde{\Delta}_K^{(n)}(z;t;\pm\sqrt{t},a,b)dT\\
&\qquad=\frac{2^n n!}{v_\lambda(t^2;a,b;a,b,ta,tb)}\int_T R_{\lambda,\mathrm{id}}^{(n)}(z;t^2;a,b;a,b,ta,tb)\tilde{\Delta}_K^{(n)}(z;t;\pm\sqrt{t},a,b)dT,
\end{aligned}$$

where in the last equation we have used the symmetry of the integral. We assume $\lambda_1 > 0$ so that $\lambda \neq 0^n$. Next, we restrict to terms involving z_1 in the integrand, and integrate with respect to z_1. Doing this computation gives the following:

$$\begin{aligned}
&\int_{T_1} z_1^{\lambda_1}\frac{(1-az_1^{-1})(1-bz_1^{-1})(1-taz_1^{-1})(1-tbz_1^{-1})}{(1-z_1^{-2})}\frac{(1-z_1^{\pm1})}{(1+\sqrt{t}z_1^{\pm1})(1-\sqrt{t}z_1^{\pm1})(1-az_1^{\pm1})(1-bz_1^{\pm1})}\\
&\qquad\times\prod_{j>1}\frac{(1-t^2z_1^{-1}z_j)(1-t^2z_1^{-1}z_j^{-1})}{(1-z_1^{-1}z_j)(1-z_1^{-1}z_j^{-1})}\prod_{j>1}\frac{(1-z_1^{\pm1}z_j^{\pm1})}{(1-tz_1^{\pm1}z_j^{\pm1})}dT\\
&\quad=\frac{1}{2\pi i}\int_C z_1^{\lambda_1-1}\frac{(z_1-ta)(z_1-tb)(1-z_1^2)}{(1-tz_1^2)(z_1+\sqrt{t})(z_1-\sqrt{t})(1-az_1)(1-bz_1)}\\
&\qquad\times\prod_{j>1}\frac{(z_1-t^2z_j)(z_1-t^2z_j^{-1})(1-z_1z_j)(1-z_1z_j^{-1})}{(z_1-tz_j)(z_1-tz_j^{-1})(1-tz_1z_j)(1-tz_1z_j^{-1})}dz_1.
\end{aligned}$$

Note that this integral has poles at $z_1 = \pm\sqrt{t}$ and $z_1 = tz_j, tz_j^{-1}$ for each $j > 1$.

We first compute the residue at $z_1 = \sqrt{t}$. It is equal to

$$\begin{aligned}
&(\sqrt{t})^{\lambda_1-1}\frac{(\sqrt{t}-ta)(\sqrt{t}-tb)(1-t)}{(1-t^2)2\sqrt{t}(1-a\sqrt{t})(1-b\sqrt{t})}\prod_{j>1}\frac{(\sqrt{t}-t^2z_j)(\sqrt{t}-t^2z_j^{-1})(1-\sqrt{t}z_j)(1-\sqrt{t}z_j^{-1})}{(\sqrt{t}-tz_j)(\sqrt{t}-tz_j^{-1})(1-t\sqrt{t}z_j)(1-t\sqrt{t}z_j^{-1})}\\
&\quad=(\sqrt{t})^{\lambda_1}\frac{1}{2(1+t)}\prod_{j>1}\frac{(1-t\sqrt{t}z_j)(1-t\sqrt{t}z_j^{-1})(1-\sqrt{t}z_j)(1-\sqrt{t}z_j^{-1})}{(1-\sqrt{t}z_j)(1-\sqrt{t}z_j^{-1})(1-t\sqrt{t}z_j)(1-t\sqrt{t}z_j^{-1})}=\frac{(\sqrt{t})^{\lambda_1}}{2(1+t)}.
\end{aligned}$$

Similarly, we can compute the residue at $z_1 = -\sqrt{t}$. It is equal to

$$
\begin{aligned}
&(-\sqrt{t})^{\lambda_1 - 1} \frac{(-\sqrt{t} - ta)(-\sqrt{t} - tb)(1-t)}{(1-t^2)(-2\sqrt{t})(1+a\sqrt{t})(1+b\sqrt{t})} \\
&\qquad \times \prod_{j>1} \frac{(-\sqrt{t} - t^2 z_j)(-\sqrt{t} - t^2 z_j^{-1})(1+\sqrt{t}z_j)(1+\sqrt{t}z_j^{-1})}{(-\sqrt{t} - tz_j)(-\sqrt{t} - tz_j^{-1})(1+t\sqrt{t}z_j)(1+t\sqrt{t}z_j^{-1})} \\
&= (-\sqrt{t})^{\lambda_1} \frac{1}{2(1+t)} \prod_{j>1} \frac{(1+t\sqrt{t}z_j)(1+t\sqrt{t}z_j^{-1})(1+\sqrt{t}z_j)(1+\sqrt{t}z_j^{-1})}{(1+\sqrt{t}z_j)(1+\sqrt{t}z_j^{-1})(1+t\sqrt{t}z_j)(1+t\sqrt{t}z_j^{-1})} = \frac{(-\sqrt{t})^{\lambda_1}}{2(1+t)}.
\end{aligned}
$$

The residues at tz_j, tz_j^{-1} can be computed in a similar manner. One can then combine these residues (at tz_j, tz_j^{-1}) with the terms from the original integrand and integrate with respect to z_j. Some computations show the resulting integral is zero; the argument is similar that used in [23, theorem 23].

Finally, we add the residues at $z_1 = \pm\sqrt{t}$ to get

$$
\frac{(\sqrt{t})^{\lambda_1}}{2(1+t)} + \frac{(-\sqrt{t})^{\lambda_1}}{2(1+t)} = \begin{cases} \frac{(\sqrt{t})^{\lambda_1}}{(1+t)}, & \text{if } \lambda_1 \text{ is even,} \\ 0, & \text{if } \lambda_1 \text{ is odd.} \end{cases}
$$

Thus,

$$
\begin{aligned}
&2^n n! \int_T R^{(n)}_{\lambda,\mathrm{id}}(z; t^2; a, b; a, b, ta, tb) \tilde{\Delta}^{(n)}_K(z; t; \pm\sqrt{t}, a, b) dT \\
&= \begin{cases} \frac{(\sqrt{t})^{\lambda_1}}{(1+t)} 2^{n-1}(n-1)! \int_T R^{(n-1)}_{\widehat{\lambda},\widehat{\mathrm{id}}}(z; t^2; a, b; a, b, ta, tb) \tilde{\Delta}^{(n-1)}_K(z; t; \pm\sqrt{t}, a, b) dT, & \text{if } \lambda_1 \text{ is even,} \\ 0, & \text{otherwise,} \end{cases}
\end{aligned}
$$

where $\widehat{\lambda}$ is the partition λ with the part λ_1 deleted, and $\widehat{\mathrm{id}}$ is the permutation id with z_1 deleted and signs preserved.

Consequently, the entire integral vanishes if any part is odd and if λ is even, it is equal to

$$
\begin{aligned}
&\frac{2^n n!}{v_\lambda(t^2; a, b; a, b, ta, tb)} \int_T R^{(n)}_{\lambda,\mathrm{id}}(z; t^2; a, b; a, b, ta, tb) \tilde{\Delta}^{(n)}_K(z; t; \pm\sqrt{t}, a, b) dT \\
&= \frac{2^{n-l(\lambda)}(n - l(\lambda)!}{v_{\lambda+}(t^2; a, b, ta, tb) v_{0^{n-l(\lambda)}}(t^2; a, b; a, b, ta, tb)} \frac{(\sqrt{t})^{|\lambda|}}{(1+t)^{l(\lambda)}} \\
&\qquad \times \int_T R^{(n-l(\lambda))}_{0^{n-l(\lambda)},\mathrm{id}}(z; t^2; a, b; a, b, ta, tb) \tilde{\Delta}^{(n-l(\lambda))}_K(z; t; \pm\sqrt{t}, a, b) dT,
\end{aligned}
$$

where, by abuse of notation in the last line, we use id to denote the identity element in $B_{n-l(\lambda)}$. By (3.6), the last line is equal to

$$
\begin{aligned}
&\frac{2^{n-l(\lambda)}(n-l(\lambda)!}{v_{\lambda+}(t^2;a,b,ta,tb)}\frac{(\sqrt{t})^{|\lambda|}}{(1+t)^{l(\lambda)}}\int_T K^{(n-l(\lambda))}_{0^{n-l(\lambda)},\mathrm{id}}(z;t^2;a,b;a,b,ta,tb)\tilde{\Delta}_K^{(n-l(\lambda))}(z;t;\pm\sqrt{t},a,b)dT \\
&= \frac{1}{v_{\lambda+}(t^2;a,b,ta,tb)}\frac{(\sqrt{t})^{|\lambda|}}{(1+t)^{l(\lambda)}}\int_T K^{(n-l(\lambda))}_{0^{n-l(\lambda)}}(z;t^2;a,b;a,b,ta,tb)\tilde{\Delta}_K^{(n-l(\lambda))}(z;t;\pm\sqrt{t},a,b)dT \\
&= \frac{1}{v_{\lambda+}(t^2;a,b,ta,tb)}\frac{(\sqrt{t})^{|\lambda|}}{(1+t)^{l(\lambda)}}\int_T \tilde{\Delta}_K^{(n-l(\lambda))}(z;t;\pm\sqrt{t},a,b)dT \\
&= \frac{(\sqrt{t})^{|\lambda|}}{(1+t)^{l(\lambda)}}\frac{N_\lambda(t;\pm\sqrt{t},a,b)v_{\lambda+}(t;\pm\sqrt{t},a,b)}{v_{\lambda+}(t^2;a,b,ta,tb)},
\end{aligned}
$$

since $K^{(l)}_{0^l}(z;t;a,b;t_0,\ldots,t_3)=1$ by theorem 3.3 and $n-l(\lambda)=m_0(\lambda)$.

□

Chapter 4

An Interpretation Using p-adic Representation Theory

4.1 Background and Notation

Let F be a non-archimedean local field with residue field of odd characteristic. Let E be an unramified quadratic extension of F. We will refer to the following cases throughout this chapter:

Case 1: $G = Gl_{2n}(F), H = Gl_n(E)$.

Case 2: $G = Gl_{2n}(E), H = Gl_{2n}(F)$.

Case 3: $G = Gl_{2n}(F), H = Sp_{2n}(F)$.

For simplicity, from now on we will assume $F = \mathbb{Q}_p$ and $E = \mathbb{Q}_p(\sqrt{a})$, for p an odd prime and a prime to p and without a square root. However, the argument applies to *any* F, E as described above. Note that the number of elements in the residue field of F is p, and for E it is p^2. Throughout, we will use K to denote the maximal compact subgroup of G (for example $K = Gl_{2n}(\mathbb{Z}_p) \subset Gl_{2n}(\mathbb{Q}_p)$) and K' the maximal compact subgroup of H.

Define

$$\Lambda_n^+ = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n | \lambda_1 \geq \cdots \geq \lambda_n \geq 0\},$$

and

$$\Lambda_n = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n | \lambda_1 \geq \cdots \geq \lambda_n\},$$

so that Λ_n^+ is the set of partitions with length at most n, while $\lambda \in \Lambda_n$ is allowed to have negative parts.

We set up some notation following [17], [9]. Let $g \mapsto g^*$ denote the involution on G given by

Case 1: $g^* = g^{-1}$.

Case 2: $g^* = \bar{g}^{-1}$.

Case 3: $g^* = g^t$.

Fix the element $s_0 \in G$ to be

$$s_0 = \begin{cases} \begin{pmatrix} 0 & w_n \\ aw_n & 0 \end{pmatrix}, & \text{Case 1,} \\ I_{2n}, & \text{Case 2,} \\ J_n, & \text{Case 3,} \end{cases}$$

where $J_n = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix} \right)$ and w_n is the $n \times n$ matrix with ones on the antidiagonal and zeroes everywhere else. Define $S = G \cdot s_0$, where the action is $g \cdot s_0 = g s_0 g^*$. Let H be the stabilizer of s_0 in G; then

In **Case 1**:

$$H = \left\{ \begin{pmatrix} i & j \\ aw_n j w_n & w_n i w_n \end{pmatrix} \in G | i, j \in Gl_n(\mathbb{Q}_p) \right\} \cong Gl_n(\mathbb{Q}_p(\sqrt{a})).$$

In **Case 2**:

$$H = GL_{2n}(F).$$

In **Case 3**:

$$H = Sp_{2n}(F).$$

Note that in case 1, the maximal compact subgroup $K' = Gl_n(\mathbb{Z}_p(\sqrt{a})) \subset Gl_n(\mathbb{Q}_p(\sqrt{a}))$ maps to $K \cap H$. In the other two cases, $K' = Gl_{2n}(\mathbb{Z}_p)$ and $Sp_{2n}(\mathbb{Z}_p)$, respectively. The map $\theta : G \to S$ defined by $\theta(g) = g s_0 g^* = g \cdot s_0$ induces a bijection between G/H and S.

Now let $\mathcal{H}(G, K)$ be the Hecke algebra of G with respect to K, i.e., the convolution algebra of compactly supported, K-bi-invariant, complex valued functions on G. Let $C^\infty(K \setminus S)$ be the space of K-invariant complex valued functions on S. Put a $\mathcal{H}(G, K)$-module structure on $C^\infty(K \setminus S)$ via the convolution:

$$f \star \phi(s) = \int_G f(g) \phi(g^{-1} \cdot s) dg,$$

where $f \in \mathcal{H}(G, K)$ and $\phi \in C^\infty(K \setminus S)$ and dg is the Haar measure on G normalized so $\int_K dg = 1$. Then a **relative spherical function** on S is an eigenfunction $\Omega \in C^\infty(K \setminus S)$ of $\mathcal{H}(G, K)$ under this convolution, normalized so that $\Omega(s_0) = 1$. Also define $\mathcal{S}(K \setminus S)$ to be the $\mathcal{H}(G, K)$-submodule of K-invariant functions on S with compact support.

Define the elements d_λ in G as follows:

Case 1:

$$d_\lambda = \text{antidiag.}(p^{\lambda_1}, \ldots, p^{\lambda_n}, ap^{-\lambda_n}, \ldots, ap^{-\lambda_1}).$$

Case 2:

$$d_\lambda = \text{antidiag.}(p^{\lambda_1}, \ldots, p^{\lambda_n}, p^{-\lambda_n}, \ldots, p^{-\lambda_1}).$$

Case 3:

$$d_\lambda = \text{antidiag.}(p^{\lambda_1}, \ldots, p^{\lambda_n}, -p^{\lambda_n}, \ldots, -p^{\lambda_1}).$$

In particular, we have $d_0 = s_0$ in each case. By [17, proposition 3.1], [9] the K-orbits of S are given by the disjoint union

$$S = \cup K \cdot d_\lambda,$$

varying over $l(\lambda) \leq n$, and λ has all nonnegative parts. Let ch_λ denote the characteristic function for the K-orbit $K \cdot d_\lambda$, then the space $\mathcal{S}(K \setminus S)$ is spanned by the functions $\{ch_\lambda | \lambda \in \Lambda_n^+\}$.

By the Cartan decomposition for G, we have

$$G = \cup K p^\lambda K,$$

disjoint union varying over $\lambda \in \Lambda_{2n}$. Throughout this chapter, we use the notation p^λ to refer to the diagonal matrix diag.$(p^{\lambda_1}, p^{\lambda_2}, \ldots, p^{\lambda_{2n}})$. Let c_λ, with $\lambda \in \Lambda_{2n}$, be the characteristic function for the double coset $Kp^\lambda K$ inside G. These functions form a basis for $\mathcal{H}(G, K)$.

Let the constant V_λ (for any $l(\lambda) \leq n$) be the following constants:

In **Case 1:** it is the reciprocal of the normsquared of $K_\lambda^{BC_n}(x; p^{-1}; \pm p^{-1/2}, 0, 0)$, i.e.,

$$\frac{1}{V_\lambda} = \int_T K_\lambda^{BC_n}(x; p^{-1}; \pm p^{-1/2}, 0, 0)^2 \tilde{\Delta}_K^{(n)}(x; \pm p^{-1/2}, 0, 0; p^{-1}) dT.$$

In **Case 2:**

$$\frac{1}{V_\lambda} = \int_T K_\lambda^{BC_n}(x; p^{-2}; 1, p^{-1}, 0, 0)^2 \tilde{\Delta}_K^{(n)}(x; p^{-2}; 1, p^{-1}, 0, 0) dT.$$

In **Case 3:**

$$\frac{1}{V_\lambda} = \int_T P_\lambda^{(n)}(x; p^{-2}) P_\lambda^{(n)}(x^{-1}; p^{-2}) \tilde{\Delta}_S^{(n)}(x; p^{-2}) dT.$$

In particular, V_0 is the reciprocal of the integral of the density function in each of the three cases. Note that for the first two cases, the V_λ are determined explicitly in [22] for general parameters $t_0, \ldots, t_3$ of the Koornwinder $q = 0$ polynomials and in [15] for these choices of parameters. In the third case, the norm is computed in [23], for example.

Also let $m_i(\lambda)$ be the number of λ_j equal to i for each $i \geq 0$. Let

$$\phi_r(t) = (1-t)(1-t^2) \cdots (1-t^r).$$

Then we define

$$v_\lambda(t) = \prod_{i\geq 0}\prod_{j=1}^{m_i(\lambda)} \frac{1-t^j}{1-t} = \prod_{i\geq 0} \frac{\phi_{m_i(\lambda)}(t)}{(1-t)^{m_i(\lambda)}};$$

this is the factor that makes the Hall–Littlewood polynomials monic. Also, let

$$v_n(t) = \frac{\phi_n(t)}{(1-t)^n}.$$

Finally, for $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $f \in \mathcal{H}(G,K)$, define (for **case 1** and **case 2**)

$$\tilde{f}(z) = \hat{f}(z_1, \ldots, z_n, -z_1, \ldots, -z_n),$$

where $\hat{}$ denotes the Satake transform on $\mathcal{H}(G,K)$. For **case 3**, define

$$\tilde{f}(z) = \hat{f}(z_1 + 1/2, z_1 - 1/2, \ldots, z_n + 1/2, z_n - 1/2).$$

In fact by [17, lemma 4.2] and [9, lemma 2.1], $f \to \tilde{f}(z)$ is the eigenvalue map, that is

$$(f \star \Omega_z)(s) = \tilde{f}(z)\Omega_z(s),$$

where $\Omega_z(s)$, $z \in \mathbb{C}^n$ are the relative spherical functions, as determined in [17] and [9].

Also for $f \in \mathcal{H}(G,K), g \in G$ put $\check{f}(g) := f(g^{-1})$.

4.2 Main Results

Proposition 4.1. *Let $l(\lambda) \leq 2n$ and $l(\mu) \leq n$. Then we have*

$$\int_S (c_\lambda \star ch_0)(s) ch_\mu(s) ds$$
$$= \begin{cases} \frac{p^{\langle \mu,\rho_1\rangle + \langle\lambda,\rho_2\rangle}}{Z} \int_T P_\lambda^{(2n)}(x_i^{\pm 1}; p^{-1}) K_\mu^{BC_n}(x; p^{-1}; \pm p^{-1/2}, 0, 0) \tilde{\Delta}_K^{(n)}(x; p^{-1}; \pm p^{-1/2}, 0, 0) dT, \\ \frac{p^{2\langle \mu,\rho_1\rangle + 2\langle\lambda,\rho_2\rangle}}{Z} \int_T P_\lambda^{(2n)}(x_i^{\pm 1}; p^{-2}) K_\mu^{BC_n}(x; p^{-2}; 1, p^{-1}, 0, 0) \tilde{\Delta}_K^{(n)}(x; p^{-2}; 1, p^{-1}, 0, 0) dT, \\ \frac{p^{\langle \mu,\rho_3\rangle + \langle\lambda,\rho_2\rangle}}{Z} \int_T P_\lambda^{(2n)}(p^{\pm 1/2} x_i; p^{-1}) P_\mu^{(n)}(x^{-1}; p^{-2}) \tilde{\Delta}_S^{(n)}(x; p^{-2}) dT, \end{cases}$$

in cases 1, 2 and 3, respectively, where $\rho_1 = (n-1/2, n-3/2, \ldots, 1/2) \in \mathbb{C}^n$, $\rho_2 = (n-1/2, n-3/2, \ldots, 1/2-n) \in \mathbb{C}^{2n}$, $\rho_3 = (n-1, n-3, \ldots, 1-n) \in \mathbb{C}^n$ and the normalization Z is the evaluation of the integral at $\lambda = \mu = 0$.

Proof.

Case 1. We use the spherical Fourier transform on $\mathcal{S}(K \setminus S)$:

$$\int_S f_1(s)\overline{f_2(s)}ds = \int_T \hat{f}_1(z)\overline{\hat{f}_2(z)}d_\mu(z);$$

here $d_\mu(z)$ is the Plancherel measure on $\mathcal{S}(K \setminus S)$. We apply this to

$$\int_S (c_\lambda \star ch_0)(s)ch_\mu(s)ds.$$

Note that the spherical Fourier transform satisfies (by lemma 4.4 [17])

$$(c_\lambda \star ch_0)\hat{(}z) = \tilde{c}_\lambda(z)\hat{ch}_0(z) = \tilde{c}_\lambda(z),$$

since $\hat{ch}_0(s) = 1$. Here

$$\tilde{c}_\lambda(z) = \hat{c_\lambda}(z_1, \ldots, z_n, -z_1, \ldots, -z_n),$$

where $\hat{c_\lambda}$ denotes here the usual Satake transform on $\mathcal{H}(Gl_{2n}(\mathbb{Q}_p), Gl_{2n}(\mathbb{Z}_p))$. But [16, chapter 5] , this is equal to

$$p^{\langle\lambda,\rho_2\rangle}P_\lambda^{(2n)}(p^{-z_1}, \ldots, p^{-z_n}, p^{z_1}, \ldots, p^{z_n}; p^{-1}).$$

Also, using [17] theorem 1.2 and proposition 5.15, we have

$$\begin{aligned}\hat{ch}_\mu(z) = \Big\{\int_{K\cdot d_\mu} ds\Big\}\Omega_z(d_\mu) = \Big\{p^{2\langle\mu,\rho_1\rangle}\frac{V_0}{V_\mu}\Big\}p^{-\langle\mu,\rho_1\rangle}\frac{V_\mu}{V_0}K_\mu^{BC_n}(p^{z_i}; p^{-1}; \pm p^{-1/2}, 0, 0) \\ = p^{\langle\mu,\rho_1\rangle}K_\mu^{BC_n}(p^{z_i}; p^{-1}; \pm p^{-1/2}, 0, 0).\end{aligned}$$

Finally, by [17] theorem 1.3 the Plancherel density is

$$\frac{\tilde{\Delta}_K^{(n)}(p^{z_i}; p^{-1}; \pm p^{-1/2}, 0, 0)}{\int_T \tilde{\Delta}_K^{(n)}(p^{z_i}; p^{-1}; \pm p^{-1/2}, 0, 0)dT}.$$

Combining these, and putting $x_i = p^{z_i}$ gives the result.

Case 2. The argument is the same as case 1, but the Plancherel measure and zonal spherical functions are different. We indicate the differences (see the above references of [17] but for case 2, and [16, chapter 5] for the group $Gl_{2n}(E)$):

$$\tilde{c}_\lambda(z) = \hat{c_\lambda}(z_1, \ldots, z_n, -z_1, \ldots, -z_n) = p^{2\langle\lambda,\rho_2\rangle}P_\lambda^{(2n)}(p^{-2z_1}, \ldots, p^{-2z_n}, p^{2z_1}, \ldots, p^{2z_n}; p^{-2}).$$

We also have

$$\hat{ch_\mu}(z) = \Big\{\int_{K\cdot d_\mu} ds\Big\}\Omega_z(d_\mu) = \Big\{p^{4\langle \mu,\rho_1\rangle}\frac{V_0}{V_\mu}\Big\}p^{-2\langle \mu,\rho_1\rangle}\frac{V_\mu}{V_0}K_\mu^{BC_n}(p^{2z_i};p^{-2};1,p^{-1},0,0)$$
$$= p^{2\langle \mu,\rho_1\rangle}K_\mu^{BC_n}(p^{2z_i};p^{-2};1,p^{-1},0,0).$$

Finally, the Plancherel density is

$$\frac{\tilde{\Delta}_K^{(n)}(p^{2z_i};p^{-2};1,p^{-1},0,0)}{\int_T \tilde{\Delta}_K^{(n)}(p^{2z_i};p^{-2};1,p^{-1},0,0)dT}.$$

Combining these, and putting $x_i = p^{2z_i}$ gives the result.

Case 3. The argument is the same as in the above cases, but the Plancherel measure and zonal spherical functions are different. We indicate the differences (see [9]):

$$\tilde{c}_\lambda(z) = \hat{c_\lambda}(z_1+1/2, z_1-1/2,\dots,z_n+1/2,z_n-1/2)$$
$$= p^{\langle\lambda,\rho_2\rangle}P_\lambda^{(2n)}(p^{-z_1-1/2},p^{-z_1+1/2},\dots,p^{-z_n-1/2},p^{-z_n+1/2};p^{-1}).$$

We also have

$$\hat{ch_\mu}(z) = \Big\{\int_{K\cdot d_\mu} ds\Big\}\Omega_z(d_\mu) = \Big\{p^{2\langle \mu,\rho_3\rangle}\frac{V_0}{V_\mu}\Big\}p^{-<\mu,\rho_3>}\frac{V_\mu}{V_0}P_\mu(p^{z_1},\dots,p^{z_n};p^{-2})$$
$$= p^{\langle \mu,\rho_3\rangle}P_\mu(p^{z_1},\dots,p^{z_n};p^{-2}).$$

Finally, the Plancherel density is

$$\frac{\tilde{\Delta}_S^{(n)}(p^{z_i};p^{-2})}{\int_T \tilde{\Delta}_S^{(n)}(p^{z_i};p^{-2})dT}.$$

Combining these, and putting $x_i = p^{z_i}$ gives the result.

□

Proposition 4.2. *We have*

$$\int_S (\check{c}_\lambda \star ch_0)(s)ch_\mu(s)ds = \begin{cases} p^{2\langle\mu,\rho_1\rangle}\frac{V_0}{V_\mu}\int_H c_\lambda(g_\mu h)dh, & \textit{in } \textbf{\textit{case 1,}} \\ p^{4\langle\mu,\rho_1\rangle}\frac{V_0}{V_\mu}\int_H c_\lambda(g_\mu h)dh, & \textit{in } \textbf{\textit{case 2,}} \\ p^{2\langle\mu,\rho_3\rangle}\frac{V_0}{V_\mu}\int_H c_\lambda(g_\mu h)dh, & \textit{in } \textbf{\textit{case 3,}} \end{cases}$$

where $g_\mu = diag.(1,\dots,1,p^{-\mu_n},\dots,p^{-\mu_1}) \in \mathbb{C}^{2n}$. *In particular, when* $\mu = 0$*, the right-hand side is* $\int_H c_\lambda(g)dg$.

Proof. We have

$$\int_S (c_\lambda \star ch_0)(s) ch_\mu(s) ds = \int_{K\cdot d_\mu} (c_\lambda \star ch_0)(s) = meas.(K \cdot d_\mu)(c_\lambda \star ch_0)(d_\mu),$$

where the first equality follows since $ch_\mu(s)$ vanishes off of $K \cdot d_\mu$, and the second follows since $(c_\lambda \star ch_0)$ is K-invariant. Now by definition of the convolution action, we have

$$(c_\lambda \star ch_0)(d_\mu) = \int_G c_\lambda(g^{-1}) ch_0(g \cdot d_\mu) dg.$$

Letting $H_\mu = \{g \in G | g \cdot d_0 = d_\mu\}$, we have

$$g \cdot d_\mu \in K \cdot d_0 \Leftrightarrow (kg) \cdot d_\mu = d_0 \text{ for some } k \in K \Leftrightarrow g \in KH_\mu^{-1}.$$

Now one can check that $g_\mu \cdot d_0 = d_\mu$, so that $H_\mu = g_\mu H$ (clearly $g_\mu H \subset H_\mu$, for the other direction let $g \in H_\mu$ then $g \cdot d_0 = d_\mu = g_\mu \cdot d_0$, so $g_\mu^{-1} g \in H$) and so $KH_\mu^{-1} = KHg_\mu^{-1}$. Thus, the above integral can be rewritten as

$$\int_{KHg_\mu^{-1}} c_\lambda(g^{-1}) dg = \int_{KH} c_{-\lambda}(gg_\mu^{-1}) dg.$$

Finally, write

$$KH = \cup K x_i,$$

a disjoint union and $x_i \in H$. Then we claim $H = \cup K' x_i$, again a disjoint union. That the union is contained inside H is clear, suppose next that $h \in H$. But then $h = kx_i$ for some $k \in K$ and x_i. But since $h, x_i \in H$ we have $k \in H$, i.e., $k \in K'$. Clearly the union is disjoint, since $K'x_i \subset Kx_i$ for all i. Thus,

$$\begin{aligned}\int_{KH} c_{-\lambda}(gg_\mu^{-1}) dg = \sum_{x_i} \int_{Kx_i} c_{-\lambda}(gg_\mu^{-1}) dg = \sum_{x_i} c_{-\lambda}(x_i g_\mu^{-1}) dg \\ = \sum_{x_i} \int_{K'x_i} c_{-\lambda}(x_i g_\mu^{-1}) dg = \int_H c_{-\lambda}(hg_\mu^{-1}) dh.\end{aligned}$$

Finally, we have to multiply this by $meas.(K \cdot d_\mu)$, see the previous proof for these values in each case. □

Proposition 4.3. *We have the following:*

Case 1:

$$\int_H c_\lambda(h) dh = \begin{cases} 0, & \text{if } \lambda \neq \mu^2 \text{ for any } \mu, \\ p^{2\langle \mu, \rho_3 \rangle} \frac{v_n(p^{-2})}{v_\mu(p^{-2})}, & \text{if } \lambda = \mu^2 \text{ for some } \mu. \end{cases}$$

Case 2:

$$\int_H c_\lambda(h)dh = p^{2\langle\lambda,\rho_2\rangle}\frac{v_{2n}(p^{-2})}{v_\lambda(p^{-2})}.$$

Case 3:

$$\int_H c_\lambda(h)dh = \begin{cases} 0, & \textit{if } \lambda \neq \mu\bar{\mu} \textit{ for any } \mu, \\ p^{2<\mu,\rho_1>}\frac{\phi_n(p^{-2})}{\phi_{n-l(\mu)}(p^{-2})(1-p^{-1})^{l(\mu)}v_{\mu^+}(p^{-1})}, & \textit{if } \lambda = \mu\bar{\mu} \textit{ for some } \mu. \end{cases}$$

Proof.

Case 1: Note first that the integral of the left-hand side is the measure of the intersection $H \cap Kp^\lambda K$. We recall the Cartan decomposition of $G = Gl_{2n}(\mathbb{Q}_p)$:

$$Gl_{2n}(\mathbb{Q}_p) = \cup Kp^\lambda K \text{ (disjoint union)},$$

where p^λ is the element diag.$(p^{\lambda_1}, \ldots, p^{\lambda_{2n}})$ in G. Similarly, we have the Cartan decomposition for $Gl_n(\mathbb{Q}_p(\sqrt{a}))$:

$$Gl_n(\mathbb{Q}_p(\sqrt{a})) = \cup K'p^\mu K' \text{ (disjoint union)},$$

where p^μ is the element diag.$(p^{\mu_1}, \ldots, p^{\mu_n})$ in $Gl_n(\mathbb{Q}_p(\sqrt{a}))$ and $K' = Gl_n(\mathbb{Z}_p(\sqrt{a}))$. Note that under the isomorphism $Gl_n(\mathbb{Q}_p(\sqrt{a})) \to H$, K' is mapped to $K \cap H$, which is contained in K. Also note that the element diag.$(p^{\mu_1}, \ldots, p^{\mu_n}) \in Gl_n(\mathbb{Q}_p(\sqrt{a}))$ is mapped to the diagonal matrix diag.$(p^{\mu_1}, \ldots, p^{\mu_n}, p^{\mu_n}, \ldots, p^{\mu_1})$, which is an element of $Kp^\lambda K$, where $\lambda = \mu_1\mu_1\mu_2\mu_2\ldots\mu_n\mu_n$. Thus, H may be realized inside G as the disjoint union of the double cosets $\{(K\cap H)p^{(\mu_1,\ldots,\mu_n,\mu_n,\ldots,\mu_1)}(K \cap H)\}$, where μ is a partition of length at most n.

This implies $H \cap Kp^\lambda K$ is empty unless $\lambda = \mu^2$ for some partition μ, which gives the vanishing part of the claim. If $\lambda = \mu^2$, the integral is equal to $meas.((K\cap H)p^{(\mu_1,\ldots,\mu_n,\mu_n,\ldots,\mu_1)}(K\cap H))$, which is equivalent to $meas.(K'p^\mu K')$ inside $Gl_n(\mathbb{Q}_p(\sqrt{a}))$. We can compute this last quantity using [16, chapter 5]. Applying that result to the group $Gl_n(\mathbb{Q}_p(\sqrt{a}))$, and noting that p^2 is the size of the residue field of $\mathbb{Q}_p(\sqrt{a})$ gives

$$|K'p^\mu K'| = (p^2)^{\langle\mu,\rho_3\rangle}\frac{\left(\prod_{i=1}^n(1-p^{-2i})\right)/(1-p^{-2})^n}{\left(\prod_{j\geq 0}\prod_{i=1}^{m_j(\mu)}(1-p^{-2i})\right)/(1-p^{-2})^n} = p^{2\langle\mu,\rho_3\rangle}\frac{v_n(p^{-2})}{v_\mu(p^{-2})}.$$

Case 2: Note that we have the following Cartan decompositions:

$$G = Gl_{2n}(\mathbb{Q}_p(\sqrt{a})) = \bigcup_{\lambda\in\Lambda_{2n}}\left(Gl_{2n}(\mathbb{Z}_p(\sqrt{a}))p^\lambda Gl_{2n}(\mathbb{Z}_p(\sqrt{a}))\right),$$

and

$$H = Gl_{2n}(\mathbb{Q}_p) = \bigcup_{\lambda \in \Lambda_{2n}} \Big(Gl_{2n}(\mathbb{Z}_p) p^{\lambda} Gl_{2n}(\mathbb{Z}_p) \Big),$$

where in both cases the unions are disjoint. Note also that $Gl_{2n}(\mathbb{Z}_p) = K' \subset K = Gl_{2n}(\mathbb{Z}_p(\sqrt{a}))$. Thus, the intersection $Kp^{\lambda}K \cap H$ is exactly $K'p^{\lambda}K'$. Finally, from [16, chapter 5 (2.9)], we have

$$\text{measure of } K'p^{\lambda}K' = p^{2\langle \lambda, \rho_2 \rangle} \frac{v_{2n}(p^{-2})}{v_{\lambda}(p^{-2})} = p^{2\langle \lambda, \rho_2 \rangle} \frac{\phi_{2n}(p^{-2})}{\prod_{i \geq 0} \phi_{m_i(\lambda)}(p^{-2})},$$

as desired.

Case 3: Note that we have the following Cartan decompositions:

$$G = Gl_{2n}(\mathbb{Q}_p) = \bigcup_{\lambda \in \Lambda_{2n}} \Big(Gl_{2n}(\mathbb{Z}_p) p^{\lambda} Gl_{2n}(\mathbb{Z}_p) \Big),$$

and

$$H = Sp_{2n}(\mathbb{Q}_p) = \bigcup_{\substack{\lambda = \mu\bar{\mu} \\ \text{in } \Lambda_{2n}}} \Big(Sp_{2n}(\mathbb{Z}_p) p^{\lambda} Sp_{2n}(\mathbb{Z}_p) \Big),$$

where in both cases the unions are disjoint. This implies that the intersection $Kp^{\lambda}K \cap H$ is zero if $\lambda \neq \mu\bar{\mu}$ for some μ giving the vanishing part of the result. If $\lambda = \mu\bar{\mu}$, the intersection is $K'p^{\lambda}K'$. We use [15] (which deals with the general reductive p-adic group case) to compute

$$\text{measure of } K'p^{\mu\bar{\mu}}K' = p^{2\langle \mu, \rho_1 \rangle} \frac{\phi_n(p^{-2})}{\phi_{n-l(\mu)}(p^{-2})(1-p^{-1})^{l(\mu)} v_{\mu^+}(p^{-1})},$$

as desired. □

We are now prepared to provide p-adic proofs of the following theorems (recall the combinatorial proofs provided in chapter 2):

Theorem 4.4. *Symplectic identity (see [20], [23]). Let λ be a partition of length at most $2n$. Then we have*

$$\frac{1}{Z} \int_T P_{\lambda}^{(2n)}(x_i^{\pm 1}; t) \tilde{\Delta}_K^{(n)}(x; t; \pm\sqrt{t}, 0, 0) dT = \begin{cases} 0, & \text{if } \lambda \neq \mu^2 \text{ for any } \mu, \\ \frac{v_n(t^2)}{v_{\mu}(t^2)}, & \text{if } \lambda = \mu^2 \text{ for some } \mu \end{cases}$$

(here the normalization $Z = \int_T \tilde{\Delta}_K^{(n)}(x; t; \pm\sqrt{t}, 0, 0) dT$).

Theorem 4.5. *We have the following identity (see [19], [11], [12]):*

$$\frac{1}{Z} \int_T P_{\lambda}(x_1^{\pm 1}, \ldots, x_n^{\pm 1}; t) \tilde{\Delta}_K^{(n)}(x; t; 1, \sqrt{t}, 0, 0) = \frac{v_{2n}(\sqrt{t})}{v_{\lambda}(\sqrt{t})}$$

(here the normalization $Z = \int_T \tilde{\Delta}_K^{(n)}(x;t;1,\sqrt{t},0,0)dT$).

Theorem 4.6. *(see [20, theorem 4.4]) Let λ be a weight of the double cover of GL_{2n}, i.e., a half-integer vector such that $\lambda_i - \lambda_j \in \mathbb{Z}$ for all i, j. Then*

$$\frac{1}{Z}\int_T P_\lambda^{(2n)}(\cdots t^{\pm 1/2}z_i \cdots ; t)\frac{1}{n!}\prod_{1\le i<j\le n}\frac{(1-z_i/z_j)(1-z_j/z_i)}{(1-t^2z_i/z_j)(1-t^2z_j/z_i)}dT = 0,$$

unless $\lambda = \mu\bar{\mu}$. In this case, the nonzero value is

$$\frac{\phi_n(t^2)}{(1-t)^n v_\mu(t)(1+t)(1+t^2)\cdots(1+t^{n-l(\mu)})}$$

(here the normalization $Z = \int_T \tilde{\Delta}_S(z;t^2)dT$).

Proof of theorem 4.4. Using propositions 4.1, 4.2 with $\mu = 0^n$ gives

$$\frac{p^{\langle\lambda,\rho_2\rangle}}{Z}\int_T P_\lambda^{(2n)}(x_i^{\pm 1};p^{-1})\tilde{\Delta}_K^{(n)}(x;\pm p^{-1/2},0,0;p^{-1})dT = \int_H c_\lambda(h)dh,$$

and by proposition 4.3 this is equal to

$$\begin{cases} 0, & \text{if } \lambda \neq \mu^2 \text{ for any } \mu, \\ p^{2\langle\mu,\rho_3\rangle}\frac{v_n(p^{-2})}{v_\mu(p^{-2})}, & \text{if } \lambda = \mu^2 \text{ for some } \mu. \end{cases}$$

To obtain the nonzero value, let $\lambda = \mu^2$. Then we can compute

$$\begin{aligned} 2\langle\mu,\rho_3\rangle &= 2\Big((n-1)\mu_1 + (n-3)\mu_2 + \cdots + (1-n)\mu_n\Big) \\ &= (n-1)(\mu_1+\mu_1) + (n-3)(\mu_2+\mu_2) + \cdots + (1-n)(\mu_n+\mu_n) \\ &= (n-1)(\lambda_1+\lambda_2) + (n-3)(\lambda_3+\lambda_4) + \cdots + (1-n)(\lambda_{2n-1}+\lambda_{2n}) \\ &= \lambda_1(n-1/2) + \lambda_2(n-3/2) + \cdots + \lambda_{2n}(1/2-n) = \langle\lambda,\rho_2\rangle. \end{aligned}$$

Thus, we obtain

$$\frac{1}{Z}\int_T P_\lambda^{(2n)}(x_i^{\pm 1};p^{-1})\tilde{\Delta}_K^{(n)}(x;\pm p^{-1/2},0,0;p^{-1})dT = \begin{cases} 0, & \text{if } \lambda \neq \mu^2 \text{ for any } \mu, \\ \frac{v_n(p^{-2})}{v_\mu(p^{-2})}, & \text{if } \lambda = \mu^2 \text{ for some } \mu. \end{cases}$$

Thus the equation in the statement of theorem 4.4 holds for all $t = p^{-1}$, for p an odd prime. This provides an infinite sequence of values for t for which the equation holds, so in particular it holds for all values of t as desired. □

Proof of theorem 4.5. The identity follows from **case 2** of propositions 4.1, 4.2, and 4.3, as in the proof of theorem 4.4 above. Note that these arguments show that the theorem holds for all $t = p^{-1}$. This provides an infinite sequence of values for t for which the equation holds, so in particular it holds for all values of t as desired. □

Proof of theorem 4.6. The identity follows from **case 3** of propositions 4.1, 4.2, and 4.3, as in the proof of theorem 4.4 above. If $\lambda = \mu\bar{\mu}$ for some μ, the integral is non vanishing. The evaluation follows by noting that $2\langle\mu, \rho_1\rangle = \langle\lambda, \rho_2\rangle$. Note that these arguments show that the theorem holds for all $t = p^{-1}$. This provides an infinite sequence of values for t for which the equation holds, so in particular it holds for all values of t as desired. □

Remarks. In **case 1**, the involution is $g \to g^{\star} = g^{-1}$ and the action is $g \cdot x = gxg^{*}$. Then H is the stabilizer in G of s_0 under this action. But $H = \{g \in G | gs_0g^{*} = s_0\} = \{g \in G | g = s_0{g^{*}}^{-1}s_0^{-1}\}$. So H is the set of fixed points of the order 2 homomorphism $g \to s_0{g^{*}}^{-1}s_0^{-1}$. This provides an analog of theorem (1.1), where one restricts s_λ to the subgroup of fixed points of a suitable involution. The other two cases are analogous.

Note that theorem 4.5 is *not* a vanishing identity. As described in Chapter 2, this identity is a finite-dimensional analog of a result of Kawanaka (see [12], [11], [23]). Kawanaka's identity has an interesting representation-theoretic significance for general linear groups over finite fields: it encodes the fact that the symmetric space $Gl_n(\mathbb{F}_{p^2})/Gl_n(\mathbb{F}_p)$ is multiplicity free.

4.3 A p-adic Generalization

In this section, we deal only with the symplectic case, **case 1**; the notation is as in that case. We will prove some stronger results by extending the methods above.
Let $l(\lambda) \leq 2n$ and $l(\mu) \leq n$. Then, by the first two propositions, we have

$$\frac{1}{Z}\int_T P_\lambda^{(2n)}(x_i^{\pm 1}; p^{-1})K_\mu^{BC_n}(x; p^{-1}; \pm p^{-1/2}, 0, 0)\tilde{\Delta}_K^{(n)}(x; \pm p^{-1/2}, 0, 0; p^{-1})dT$$
$$= \frac{1}{p^{\langle\mu,\rho_1\rangle+\langle\lambda,\rho_2\rangle}}\int_S (\check{c}_\lambda \star ch_0)(s)ch_\mu(s)ds = p^{\langle\mu,\rho_1\rangle-\langle\lambda,\rho_2\rangle}\frac{V_0}{V_\mu}\int_H c_\lambda(hg_\mu^{-1})dh.$$

Using the Cartan decomposition for $(Gl_n(\mathbb{Q}_p(\sqrt{a})), Gl_n(\mathbb{Z}_p(\sqrt{a})))$ and the embedding into $Gl_{2n}(\mathbb{Q}_p)$, we have

$$\int_H c_\lambda(hg_\mu^{-1})dh = \sum_{\substack{\beta\in\Lambda_{2n} \\ \beta=\nu_1\ldots\nu_n\nu_n\ldots\nu_1 \\ \text{for some } \nu}} \int_{K'p^\beta K'} c_\lambda(hg_\mu^{-1})dh;$$

also note that

$$\int_{K'p^\beta K'} c_\lambda(hg_\mu^{-1})dh = meas.(Kp^\lambda Kg_\mu \cap K'p^\beta K'),$$

where the measure is with respect to the measure on H. Thus,

$$\frac{1}{Z}\int_T P_\lambda^{(2n)}(x_i^{\pm 1};p^{-1})K_\mu^{BC_n}(x;p^{-1};\pm p^{-1/2},0,0)\tilde{\Delta}_K^{(n)}(x;\pm p^{-1/2},0,0;p^{-1})dT$$
$$= p^{\langle \mu,\rho_1\rangle - \langle \lambda,\rho_2\rangle}\frac{V_0}{V_\mu}\sum_{\substack{\beta\in\Lambda_{2n}\\ \beta=\nu_1\dots\nu_n\nu_n\dots\nu_1\\ \text{for some } \nu}} meas.(Kp^\lambda K g_\mu \cap K'p^\beta K'). \quad (4.1)$$

Lemma 4.7. *Let $\beta = \nu_1\dots\nu_n\nu_n\dots\nu_1 \in \Lambda_{2n}$ have at least one negative part. Then $meas.(Kp^\lambda K g_\mu \cap K'p^\beta K') = 0$.*

Proof. Note that if $Kp^\lambda K \cap K'p^\beta K' g_\mu^{-1} \neq \emptyset$, then $Kp^\lambda K \cap p^\beta K' g_\mu^{-1} \neq \emptyset$. We will show that $Kp^\lambda K \cap p^\beta K' g_\mu^{-1} = \emptyset$, which proves the claim.

Note first that $g_\mu^{-1} = \text{diag.}(1,\dots,1,p^{\mu_n},\dots,p^{\mu_1})$. We will write $\bar{\mu} = (\mu_n,\dots,\mu_1)$. Suppose for contradiction that

$$k' = \begin{pmatrix} i & j \\ aw_njw_n & w_niw_n \end{pmatrix}$$

is an element in K' such that $p^\beta k' g_\mu^{-1} \in Kp^\lambda K$. By a direct computation we have

$$p^\beta k' g_\mu^{-1} = \begin{pmatrix} p^\nu & 0 \\ 0 & p^{\bar\nu} \end{pmatrix}\begin{pmatrix} i & j \\ aw_njw_n & w_niw_n \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & p^{\bar\mu} \end{pmatrix} = \begin{pmatrix} p^\nu i & p^\nu j p^{\bar\mu} \\ p^{\bar\nu}aw_njw_n & p^{\bar\nu}w_niw_np^{\bar\mu} \end{pmatrix}.$$

Now noting that $p^{\bar\nu}w_n = w_np^\nu$, the above becomes

$$\begin{pmatrix} p^\nu i & p^\nu j p^{\bar\mu} \\ aw_np^\nu jw_n & w_np^\nu ip^\mu w_n \end{pmatrix}.$$

Since $p^\beta k' g_\mu^{-1} \in Kp^\lambda K \subset M_{2n}(\mathbb{Z}_p)$, it follows that $p^\nu i$ and $p^\nu j$ are in $M_n(\mathbb{Z}_p)$. Since $\nu_n < 0$, it follows that the nth row of k' has entries all of which are divisible by p in $\mathbb{Z}_p$. Let B be the matrix obtained from k' by dividing the nth row by p; note that $B \in M_{2n}(\mathbb{Z}_p)$. Then

$$\det(k') = p\det(B) \in p\cdot\mathbb{Z}_p,$$

which contradicts $|\det(k')| = 1$. □

Thus, using the previous lemma, (4.1) now becomes

$$\frac{1}{Z}\int_T P_\lambda^{(2n)}(x_i^{\pm 1};p^{-1})K_\mu^{BC_n}(x;p^{-1};\pm p^{-1/2},0,0)\tilde{\Delta}_K^{(n)}(x;\pm p^{-1/2},0,0;p^{-1})dT$$
$$= p^{\langle \mu,\rho_1\rangle - \langle \lambda,\rho_2\rangle}\frac{V_0}{V_\mu}\sum_{\substack{\beta\in\Lambda_{2n}^+\\ \beta=\nu_1\ldots\nu_n\nu_n\ldots\nu_1\\ \text{for some } \nu}} meas.(Kp^\lambda K g_\mu \cap K'p^\beta K'). \quad (4.2)$$

We briefly recall the Hall-polynomials $g_{\mu\nu}^\lambda(p)$ [16, chapters 2 and 5]: they are the structure constants for the ring $\mathcal{H}(G^+,K)$. In other words, for $\mu,\nu\in\Lambda_{2n}^+$, we have

$$c_\mu \star c_\nu = \sum_{\lambda\in\Lambda_{2n}^+} g_{\mu\nu}^\lambda(p)c_\lambda. \quad (4.3)$$

Note that, in particular,

$$g_{\mu\nu}^\lambda(p) = (c_\mu\star c_\nu)(p^\lambda) = \int_G c_\mu(p^\lambda y^{-1})c_\nu(y)dy = meas.(p^\lambda K p^{-\nu}K\cap Kp^\mu K).$$

Lemma 4.8. *Let $\lambda,\mu,\beta\in\Lambda_{2n}$. Then we have*

$$\int_G c_{-\mu}(g')\int_G c_\beta(g)c_\lambda(gg')dgdg' = meas.(Kp^{-\mu}K)\int_G c_{-\lambda}(p^\mu g^{-1})c_\beta(g)dg.$$

Proof. Write $Kp^{-\mu}K$ as the disjoint union $\cup k_ip^{-\mu}K$, where $k_i\in K$. Then

$$\int_G c_{-\mu}(g')\int_G c_\beta(g)c_\lambda(gg')dgdg' = \int_{Kp^{-\mu}K}\int_G c_\beta(g)c_\lambda(gg')dgdg'$$
$$= \sum_{k_ip^{-\mu}}\int_K\int_G c_\beta(g)c_\lambda(gk_ip^{-\mu}k)dgdk = \sum_{k_ip^{-\mu}}\int_G c_\beta(g)c_\lambda(gk_ip^{-\mu})dg$$
$$= \sum_{k_ip^{-\mu}}\int_G c_\beta(yk_i^{-1})c_\lambda(yp^{-\mu})dy = \sum_{k_ip^{-\mu}}\int_G c_\beta(y)c_\lambda(yp^{-\mu})dy$$
$$= meas.(Kp^{-\mu}K)\int_G c_\beta(g)c_\lambda(gp^{-\mu})dg = meas.(Kp^{-\mu}K)\int_G c_{-\lambda}(p^\mu g^{-1})c_\beta(g)dg.$$

□

Proposition 4.9. *Let $\lambda\in\Lambda_{2n}^+$ and $\mu\in\Lambda_n^+$ and fix a prime $p\neq 2$. Suppose $g_{\mu,\beta}^\lambda(p)=0$ for all $\beta\in\Lambda_{2n}^+$ with all parts occurring with even multiplicity. Then the integral*

$$\frac{1}{Z}\int_T P_\lambda^{(2n)}(x_i^{\pm 1};p^{-1})K_\mu^{BC_n}(x;p^{-1};\pm p^{-1/2},0,0)\tilde{\Delta}_K^{(n)}(x;\pm p^{-1/2},0,0;p^{-1})dT,$$

with $Z=\int_T\tilde{\Delta}_K^{(n)}(x;\pm p^{-1/2},0,0;p^{-1})dT$, vanishes.

Proof. The starting point is (4.2) from the discussion above, recall that we have

$$\frac{1}{Z}\int_T P^{(2n)}_\lambda(x_i^{\pm 1};p^{-1})K^{BC_n}_\mu(x;p^{-1};\pm p^{-1/2},0,0)\tilde{\Delta}^{(n)}_K(x;\pm p^{-1/2},0,0;p^{-1})dT$$
$$= p^{\langle \mu,\rho_1\rangle-\langle\lambda,\rho_2\rangle}\frac{V_0}{V_\mu}\sum_{\substack{\beta\in\Lambda^+_{2n}\\ \beta=\nu_1\dots\nu_n\nu_n\dots\nu_1\\ \text{for some }\nu}} meas.(Kp^\lambda Kg_\mu\cap K'p^\beta K').$$

Now if we write

$$(Kp^\lambda Kg_\mu\cap K'p^\beta K') = \cup K'x_i,$$

a disjoint union and $x_i\in p^\beta K'$, then the above measure is the number of x_i's. But we also have

$$\cup Kx_i\subset (Kp^\lambda Kg_\mu\cap Kp^\beta K),$$

and the union is disjoint ($k_1x_i = k_2x_j$ implies $k_2^{-1}k_1x_i = x_j$, but $x_i, x_j\in H$ so $k_2^{-1}k_1\in K'$, a contradiction to the definition of the x_j's). Thus,

$$meas.(Kp^\lambda Kg_\mu\cap K'p^\beta K') = \#\{x_i\} = meas.(\cup Kx_i)\le meas.(Kp^\lambda Kg_\mu\cap Kp^\beta K),$$

so that

$$\int_{K'p^\beta K'} c_\lambda(hg_\mu^{-1})dh\le\int_{Kp^\beta K} c_\lambda(gg_\mu^{-1})dg = \int_G c_\beta(g)c_\lambda(gg_\mu^{-1})dg = \int_G c_{-\lambda}(g_\mu g^{-1})c_\beta(g)dg.$$

Recall that $g_\mu = p^{(0^n,-\mu_n,\dots,-\mu_1)}$. By lemma (4.8), we have

$$\int_G c_{-\lambda}(g_\mu g^{-1})c_\beta(g)dg = \frac{1}{meas.(Kp^{\mu 0^n}K)}\int_G c_{\mu 0^n}(g')\int_G c_\beta(g)c_\lambda(gg')dgdg'.$$

But, using a change of variables, we have

$$\int_G c_{\mu 0^n}(g')\int_G c_\beta(g)c_\lambda(gg')dgdg' = \int_G c_{\mu 0^n}(g')\int_G c_\beta(yg'^{-1})c_\lambda(y)dydg'$$
$$= \int_G c_{\mu 0^n}(g')\int_G c_\beta(y^{-1}g'^{-1})c_{-\lambda}(y)dydg' = \int_G c_{-\lambda}(y)\int_G c_{\mu 0^n}(g')c_{-\beta}(g'y)dg'dy$$
$$= meas.(Kp^{-\lambda}K)\int_G c_\beta(p^\lambda g^{-1})c_{\mu 0^n}(g)dg = meas.(Kp^{-\lambda}K)g^\lambda_{\beta,\mu 0^n}(p),$$

where $g^\lambda_{\beta,\mu 0^n}(p)$ is the Hall polynomial for $\mathcal{H}(G,K)$, see [16, chapter 2 section 4]. Thus,

$$\int_G c_{-\lambda}(g_\mu g^{-1})c_\beta(g)dg = \frac{meas.(Kp^{-\lambda}K)}{meas.(Kp^{\mu 0^n}K)}g^\lambda_{\beta,\mu 0^n}(p) = \frac{meas.(Kp^{-\lambda}K)}{meas.(Kp^{\mu 0^n}K)}g^\lambda_{\mu 0^n,\beta}(p),$$

where the last equality follows by [16, chapter 2, (4.3)(iv)].

To summarize, we have

$$\int_H c_\lambda(hg_\mu^{-1})dh = \sum_{\substack{\beta\in\Lambda_{2n}^+\\ \beta=\nu_1\ldots\nu_n\nu_n\ldots\nu_1\\ \text{for some }\nu}} meas.(Kp^\lambda Kg_\mu \cap K'p^\beta K') \leq \sum_{\substack{\beta\in\Lambda_{2n}^+\\ \beta=\nu^2\\ \text{for some }\nu}} \frac{meas.(Kp^{-\lambda}K)}{meas.(Kp^{\mu 0^n}K)} g^\lambda_{\mu 0^n,\beta}(p).$$

Since by assumption $g^\lambda_{\mu,\beta}(p) = 0$ for all $\beta = \nu^2 \in \Lambda_{2n}^+$, the result follows. □

Theorem 4.10. *Let $\lambda, \mu \in \Lambda_n^+$. Then the integral*

$$\frac{1}{\int_T \tilde{\Delta}_K^{(n)}(x;0,0,0,0;t)dT} \int_T s_\lambda^{(2n)}(x_i^{\pm 1}) sp_\mu(x_1,\ldots,x_n) \tilde{\Delta}_K^{(n)}(x;0,0,0,0;t)dT$$

vanishes if and only if the integral

$$\frac{1}{\int_T \tilde{\Delta}_K^{(n)}(x;\pm\sqrt{t},0,0;t)dT} \int_T P_\lambda^{(2n)}(x_i^{\pm 1};t) K_\mu^{BC_n}(x;t;\pm\sqrt{t},0,0) \tilde{\Delta}_K^{(n)}(x;\pm\sqrt{t},0,0;t)dT$$

vanishes as a rational function of t.

Proof. The "if" direction follows by setting $t = 0$ in the Hall-polynomial integral to obtain the Schur case. We consider the other direction: suppose the integral involving Schur polynomials vanishes. We will show the integral involving the Hall-polynomial vanishes.

Fix an odd prime p. We recall the classical branching rule of Littlewood and Weyl:

$$s_\lambda^{(2n)}(x^{\pm 1}) = \sum_{l(\mu)\leq n} sp_\mu(x_1,\ldots,x_n) \left(\sum_{\substack{\beta\in\Lambda_{2n}^+\\ \beta \text{ has even columns}}} c^\lambda_{\mu,\beta} \right),$$

where $c^\lambda_{\mu,\beta}$ are the Littlewood-Richardson coefficients (note: this requires $l(\lambda) \leq n$). Thus, since the above Schur integral vanishes, we must have $c^\lambda_{\mu,\beta} = 0$ for all $\beta \in \Lambda_{2n}^+$ with all parts occurring with even multiplicity.

By [16, chapter 2, (4.3)(i)], this implies $g^\lambda_{\mu,\beta}(p) = 0$ for all $\beta \in \Lambda_{2n}^+$ with all parts occurring with even multiplicity. Thus, by the previous proposition, the following integral is zero:

$$\frac{1}{Z}\int_T P_\lambda^{(2n)}(x_i^{\pm 1};p^{-1}) K_\mu^{BC_n}(x;p^{-1};\pm p^{-1/2},0,0) \tilde{\Delta}_K^{(n)}(x;\pm p^{-1/2},0,0;p^{-1})dT,$$

where $Z = \int_T \tilde{\Delta}_K^{(n)}(x;\pm p^{-1/2},0,0;p^{-1})dT$. This shows that the integral in question vanishes for all values $t = p^{-1}$, p an odd prime. Thus it vanishes for all values of t. □

Corollary 4.11. *Let $\lambda, \mu \in \Lambda_n^+$. Then the following are equivalent:*

(i) The integral

$$\frac{1}{\int_T \tilde{\Delta}_K^{(n)}(x;\pm\sqrt{t},0,0;t)dT}\int_T P_\lambda^{(2n)}(x_i^{\pm 1};t)K_\mu^{BC_n}(x;t;\pm\sqrt{t},0,0)\tilde{\Delta}_K^{(n)}(x;\pm\sqrt{t},0,0;t)dT$$

vanishes as a rational function of t*.*

(ii) The Hall polynomials

$$g^\lambda_{\mu,\beta}(t^{-1})$$

vanish as a function of t*, for all* $\beta\in\Lambda^+_{2n}$ *with all parts occurring with even multiplicity.*

(iii) The Littlewood-Richardson coefficients

$$c^\lambda_{\mu,\beta}=0,$$

for all $\beta\in\Lambda^+_{2n}$ *with all parts occurring with even multiplicity.*

Example. Let λ have all parts occurring with even multiplicity, and $\mu=(r)$ only one part (assume $r\neq 0$). Let β have all parts occurring with even multiplicity. We have $g^\lambda_{\beta,(r)}(t^{-1})=0$ unless $\lambda-\beta$ is a horizontal r-strip [16]. But $\lambda-\beta$ is a horizontal strip if and only if $\lambda_1\geq\beta_1\geq\lambda_2\geq\beta_2\cdots$ (interlaced), so $\lambda=\beta$. Thus $g^\lambda_{(r),\beta}(t^{-1})=0$ for all β with all parts occurring with even multiplicity. So for these conditions on λ, μ, the integral of the above corollary vanishes.

Finally, we provide a characterization of the measures

$$\int_{K'p^\beta K'} c_\lambda(hg_\mu^{-1})dh = meas.(Kp^\lambda Kg_\mu\cap K'p^\beta K') \tag{4.4}$$

appearing in (4.2) in terms of modified Hall polynomials. Recall $\beta=(\nu_1,\dots,\nu_n,\nu_n,\dots,\nu_1)=\nu\bar{\nu}\in\Lambda^+_{2n}$.

We will use the following pairing on lattices obtained by the theory of elementary divisors:

Definition 4.12. Let E be a local field and let o be its ring of integers. Also let q denote the number of elements in the residue field of o. Let L,M be rank n o-lattices in E^n. Denote by $\{L;M\}$ the set of elementary divisors of M in L. That is, if $\{L;M\}=\{p^{\lambda_1},\dots,p^{\lambda_n}\}$ this means there exists an o-basis $\{e_1,\dots,e_n\}$ of L so that $\{p^{\lambda_1}e_1,\dots,p^{\lambda_n}e_n\}$ is an o-basis of M.

We will use $\{L;M\}=\lambda$ to denote the above situation. We will restrict to the case $E=\mathbb{Q}_p$ and will write o' for the ring of integers of $\mathbb{Q}_p(\sqrt{a})$.

In (4.3), we have defined Hall polynomials as structure constants for a particular ring; one notes that they also have a more combinatorial description in terms of modules. We briefly recall this interpretation, see [16, chapter 2] for the relevant notation, and more information. Let M be a finite

o-module of type λ. Then the Hall polynomial $g^{\lambda}_{\mu,\nu}(q)$ is the number o-submodules N of M which have type ν and cotype μ.

To relate Hall polynomials to our measure (4.4), we first rewrite the latter as follows

$$\mathrm{meas}(Kp^{\lambda}K \cap K'p^{\nu\bar{\nu}}K'g_{\mu}^{-1}) = \int_H c_{\lambda}(hg_{\mu}^{-1})c_{K'p^{\nu\bar{\nu}}K'}(h)dh.$$

Now we choose r sufficiently large such that $\nu\bar{\nu} - r^{2n}$ has all parts negative. Then one notes that the above measure is equal to

$$\mathrm{meas}(Kp^{\lambda}K \cap K'p^{\nu\bar{\nu}-r^{2n}}K'g_{\mu}^{-1}p^{r^{2n}}) = \int_H c_{\lambda}(hp^{0^n\bar{\mu}+r^{2n}})c_{K'p^{\nu\bar{\nu}-r^{2n}}K'}(h)dh.$$

Now consider the following right coset decompositions

$$K'p^{\nu\bar{\nu}-r^{2n}}K' = \sqcup K'y_j$$
$$Kp^{\lambda}K = \sqcup Kx_i$$

(here $y_j \in p^{\nu\bar{\nu}-r^{2n}}K'$ and $x_i \in p^{\lambda}K$ respectively). Then we can rewrite the above integral as

$$\sum_j \int_{K'} c_{\lambda}(k'y_jp^{0^n\bar{\mu}+r^{2n}})dh = \sum_j c_{\lambda}(y_jp^{0^n\bar{\mu}+r^{2n}}),$$

which is the number of pairs (i, j) such that

$$y_jp^{0^n\bar{\mu}+r^{2n}} = kx_i,$$

for some $k \in K$. Let L denote the lattice o^n in the vector space $\mathbb{Q}_p^n$. Then, in terms of lattices, this is equivalent to counting the number of lattices $M = Ly_j$ such that

$$\{Lp^{-(0^n\bar{\mu}+r^{2n})}; M\} = \{L; Mp^{0^n\bar{\mu}+r^{2n}}\} = \lambda.$$

Rephrasing, this is the number of o-modules L'/L such that

$$L'/L \subset Lp^{-(0^n\bar{\mu}+r^{2n})}/L \subset Lp^{-(\mu\bar{\mu}+r^{2n})}/L$$

with the additional conditions that (1) L'/L is an o'-module of type $r^n - \nu$ (o-type is $r^{2n} - \nu\bar{\nu}$) inside the ambient o'-module $Lp^{-\mu\bar{\mu}+r^{2n}}/L$, and (2) it has cotype λ with respect to $Lp^{-(0^n\bar{\mu}+r^{2n})}/L$. Note that without the extra o'-condition, this is the Hall polynomial

$$g^{\mu+r^{2n}}_{r^{2n}-\nu\bar{\nu},\lambda}(q) = g^{-\lambda}_{r^{2n}-\nu\bar{\nu},-\mu-r^{2n}}(q) = g^{\lambda}_{\nu\bar{\nu},\mu}(q).$$

Bibliography

[1] G. E. Andrews, The Theory of Partitions, Cambridge Mathematical Library, Cambridge: Cambridge University Press, 1998. Reprint of the 1976 original.

[2] J. Baik and E. M. Rains, Algebraic aspects of increasing subsequences, *Duke Math. J.*, **109** (2001), pp. 1–65.

[3] W. Casselman, The unramified principal series of p-adic groups. I. The spherical function, *Compositio Math.*, **40(3)** (1980), 387–406.

[4] W. Casselman and J. Shalika, The unramified principal series of p-adic groups. II. The Whittaker function, *Compositio Math.*, **41(2)** (1980), 207–231.

[5] P. Diaconis and M. Shahshahani, On the eigenvalues of random matrices, *J. Appl. Probab.*, **31A** (1994), pp. 49–62.

[6] J. F. van Diejen, Commuting difference operators with polynomial eigenfunctions, *Compositio Math.*, **95**, pp. 183–233.

[7] P. J. Forrester and E. M. Rains, Symmetrized models of last passage percolation and non-intersecting lattice paths, *J. Stat. Phys.*, **129** (2007), 833–855.

[8] R. A. Gustafson, A generalization of Selberg's beta integral, *Bull. Amer. Math. Soc.*, **22** (1990), pp. 97–105.

[9] Y. Hironaka and F. Satō, Spherical functions and local densities of alternating forms, *Amer. J. Math.*, **110** (1988), 473–512.

[10] R. P. Kanwal, Linear Integral Equations: Theory and Technique, Boston: Birkhäuser, 1996.

[11] N. Kawanaka, On subfield symmetric spaces over a finite field, *Osaka J. Math.*, **28** (1991), 759–791.

[12] N. Kawanaka, A q-series identity involving Schur functions and related topics, *Osaka J. Math.*, **36** (1999), 157–176.

[13] T. H. Koornwinder, Askey-Wilson polynomials for root systems of type BC, in *Hypergeometric Functions on Domains of Positivity, Jack Polynomials, and Applications* (Tampa, FL, 1991), vol. 138 of Contemp. Math., Amer. Math. Soc., Providence, RI, 1992, 189–204.

[14] I. G. Macdonald, Orthogonal polynomials associated with root systems, *Sem. Lothar. Combin.*, **45**(2000/01), Art. B45a, 40 pp.

[15] I. G. Macdonald, Spherical functions on a group of p-adic type, Ramanujan Institute, Centre for Advanced Study in Mathematics,University of Madras, Madras, 1971. Publications of the Ramanujan Institute, No. 2.

[16] I. G. Macdonald, Symmetric Functions and Hall Polynomials, Oxford Mathematical Monographs, New York: Oxford University Press, second edition, 1995.

[17] O. Offen, Relative spherical functions on p-adic symmetric spaces (three cases), *Pacific J. Math.*, **215** (2004), 97–149.

[18] E. M. Rains, BC_n-symmetric polynomials, *Transform. Groups*, **10** (2005), 63–132.

[19] E. M. Rains, Elliptic Littlewood identities, http://arxiv.org/abs/0806.0871.

[20] E. M. Rains and M. Vazirani, Vanishing integrals of Macdonald and Koornwinder polynomials, *Transform. Groups*, **12** (2007), 725–759.

[21] V. Venkateswaran, A p-adic interpretation of some vanishing results for Hall-Littlewood polynomials, in preparation.

[22] V. Venkateswaran, Hall-Littlewood polynomials of type BC, in preparation.

[23] V. Venkateswaran, Vanishing integrals for Hall-Littlewood polynomials, *Transform. Groups*, **17** (2012), 259–302.

[24] S. O. Warnaar, Rogers-Szegö polynomials and Hall-Littlewood symmetric functions, *J. Algebra*, **303** (2006), 810–830.