Localization of Gauge Theories on the Three-Sphere

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Itamar Yaakov

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Abstract

We describe the application of localization techniques to the path integral for supersymmetric gauge theories in three dimensions. The localization procedure reduces the computation of the expectation value of BPS observables to a calculation in a matrix model. We describe the ingredients of this model for a general quiver gauge theory and the incorporation of supersymmetric deformations and observables.

We use the matrix model expressions to test several duality conjectures for supersymmetric gauge theories. We perform tests of mirror symmetry of three-dimensional quiver gauge theories and of Seiberg-like dualities. Specifically, we explicitly show that the partition functions of the dual pairs, which are highly nontrivial functions of the deformations, agree. We describe extensions of these dualities which can be inferred from the form of the partition functions. We review the application of the matrix model to the study of renormalization group flow and the space of conformal field theories in three dimensions.

Contents

Forward

In this thesis we describe the application of localization formulas to supersymmetric gauge theories in three dimensions, and the applications thereof to the study of such theories. Localization is among the very few tools available for the study of strongly coupled gauge theories and for the extraction of exact results from path integrals for interacting theories in general. The study of supersymmetric theories, either as toy models for realistic models of particle physics, or as conjectural extensions of the Standard Model, has proven both fruitful and satisfying. We hope that the work described here can be used to advance the study of supersymmetric gauge theories and to further our understanding of strongly interacting theories.

In Chapter 1 we describe the construction of supersymmetric gauge theories in three dimensions. We review some of the dynamical aspects of the theories and show how they can be defined on a compact curved manifold: the three-sphere. The material in this chapter is collected from various sources. Where possible, the author has made an effort to include all relevant material and to provide references to the original derivations or to standard reviews and books. We will restrict ourselves to theories with extended supersymmetry in which localization computations, our main topic of interest, can be performed. Even with these restrictions, the amount of material is vast and one cannot hope for a self-contained explanation of much of anything within this modest framework. Some important and closely related subjects are mentioned only briefly or not at all: topological field theory, confinement, supersymmetry breaking, gauge/gravity correspondence, and many others.

In Chapter 2 we review path integral localization. We derive a matrix model which represents the localized path integral, and can be used to compute the expectation value of BPS observables. The matrix model is one of our main results. The derivation does not, in all cases, follow the original paper, but uses some of the simplifications and advances that have been provided by subsequent works. We note that there are additional, closely related localization computations for quantum field theories in three dimensions. Among these are the superconformal index and the computation on the squashed sphere. There is also an intimate relationship between the computation on the three-sphere and the four-dimensional superconformal index.

In Chapter 3 we apply the results of Chapter 2 to several problems. One of these is the study of dualities in gauge theories. We compare the partition functions of dual gauge theories, which are highly nontrivial functions of the possible supersymmetry-preserving deformations of the theories. The ability to make such exact comparisons is a direct consequence of the application of localization to these theories, and would not otherwise be possible except in very simple cases. The explicit computations also allow us to extend the duality conjectures to more complicated theories which, for example, do not have type IIB brane constructions. Holography and the AdS/CFT correspondence provide a vast playground in which localization computations can be used to provide interesting results. We will review one of these results related to ABJM theory. Finally, rather surprising use has been made of the localization computation on the three sphere to provide insight into the renormalization group flow and the space of conformal field theories in three dimensions. We include a show review of some of this ongoing work.

Chapter 1

Introduction

In this chapter we introduce supersymmetric theories in three dimensions and the supersymmetry algebra on which they are based. We will restrict ourselves to theories with at least $\mathcal{N} = 2$ supersymmetry, in the three-dimensional sense. The first section covers the algebra and the actions, as well as the global symmetries. In the second section we review some of the dynamical aspects of gauge theories in three dimensions and review their construction as low-energy theories on coincident D-branes in type IIB string theory. The third section describes the transition to gauge theories on the three-sphere.

1.1 Supersymmetry in $2 + 1$ Dimensions

In this section we summarize the $\mathcal{N} = 2$ supersymmetry algebra in three dimensions and supersymmetric actions. The section will mainly follow the exposition in [1, 2, 3, 4, 5, 6].

1.1.1 Spinor Conventions

The Clifford algebra in three dimensions is defined by

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu} \tag{1.1.1}$$

where $g_{\mu\nu}$ is either the flat Minkowski metric $\eta_{\mu\nu}$ or the flat Euclidean metric $\delta_{\mu\nu}$. We define the usual Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.1.2}$$

The conventions for supersymmetry in signature $(-++)$ can be found in [7]. We repeat them here for convenience. In Minkowski signature the Lorentz group is $SL(2,\mathbb{R})$ and we can impose a Majorana condition and choose a real set of generators for the Clifford algebra. The following is a possible choice

$$\gamma^1 = i\sigma_2, \gamma^2 = \sigma_3, \gamma^3 = \sigma_1 \tag{1.1.3}$$

Note that these matrices have the index structure $(\gamma^\mu)_\beta^\alpha$ where μ is the usual contravariant Lorentz index and α, β are indices in the defining representation of $SL(2,\mathbb{R})$. These indices are raise and lowered with

$$C_{\alpha\beta} = -C_{\beta\alpha} = -C^{\alpha\beta} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{1.1.4}$$

These matrices act on real, anticommuting, two-component Majorana spinors ψ^α. Spinors with lower indices are thus imaginary. A Hermitian, $SL(2,\mathbb{R})$ invariant inner product is defined by

$$\psi^2 \equiv \frac{1}{2}\psi^\alpha\psi_\alpha = \frac{1}{2}\psi^\alpha C_{\alpha\beta}\psi^\beta \tag{1.1.5}$$

We will later work in Euclidean space where the relevant rotation group is $SU(2)$. Our conventions are the same as in [8]. We make the simple choice

$$\gamma_\mu = \gamma^\mu = \sigma_\mu \tag{1.1.6}$$

under which $\gamma_\mu^\dagger = \gamma_\mu$. Fundamental $SU(2)$ indices are raised and lowered with

$$\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha} = -\varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{1.1.7}$$

Note that

$$S^i = \frac{1}{2}\gamma^i \tag{1.1.8}$$

satisfy the $SU(2)$ algebra

$$[S^i, S^j] = i\varepsilon^{ij}{}_k S^k \tag{1.1.9}$$

We will work with unrestricted complex, two-component spinors. There are two invariant inner products

$$\bar{\psi}\xi \equiv \psi^{\alpha}\varepsilon_{\alpha\beta}\xi^{\beta}, \psi^{\dagger}\xi \equiv \psi^{*}{}_{\alpha}\xi^{\alpha} \tag{1.1.10}$$

the second of which is anti-Hermitian for anticommuting spinors. With this choice of basis we have the following identity

$$\gamma_{\mu\nu} \equiv \frac{1}{2}[\gamma_{\mu}, \gamma_{\nu}] = i\varepsilon_{\mu\nu\rho}\gamma^{\rho} \tag{1.1.11}$$

and the following relation (Fierz identity) for anticommuting spinors

$$({\eta_1}^{\dagger}\eta_2)({\eta_3}^{\dagger}\eta_4) = -\frac{1}{2}({\eta_1}^{\dagger}\eta_4)({\eta_3}^{\dagger}\eta_2) - \frac{1}{2}({\eta_1}^{\dagger}\gamma_{\mu}\eta_4)({\eta_3}^{\dagger}\gamma^{\mu}\eta_2) \tag{1.1.12}$$

1.1.2 The $\mathcal{N} = 2$ Supersymmetry Algebra

The $\mathcal{N} = 2$ supersymmetry algebra has twice the minimal number of fermionic generators for a total of 4 supercharges. These can be combined into a complex supercharge and its adjoint (indices on the adjoint have been lowered)

$$Q_{\alpha}, \bar{Q}_{\alpha} \tag{1.1.13}$$

This is the minimal amount required for a theory in $2+1$ dimensions to have holomorphy properties [1]. The algebra can be written [1, 6]

$$\{Q_{\alpha}, Q_{\beta}\} = \{\bar{Q}_{\alpha}, \bar{Q}_{\beta}\} = 0 \tag{1.1.14}$$

$$\{Q_{\alpha}, \bar{Q}_{\beta}\} = 2\gamma^{\mu}_{\alpha\beta}P_{\mu} + 2C_{\alpha\beta}Z \tag{1.1.15}$$

where Z is a real central charge. The automorphism group of the $\mathcal{N} = 2$ algebra is $U(1)$. The $U(1)_R$ R-charge rotates the supercharges

$$[R, Q_{\alpha}] = -Q_{\alpha} \tag{1.1.16}$$

The various multiplets will be presented in 1.1.3. The BPS bound takes the form

$$M \geq |Z| \tag{1.1.17}$$

Irreducible representations contain two real bosonic and two real fermionic degrees of freedom. Z gets contributions only from global $U(1)$ symmetries and sits in a linear multiplet (see 1.1.3). Representations which saturate the BPS bound are charged and CPT dictates that they come in conjugate pairs, each with half the number of degrees of freedom.

The $\mathcal{N} = 4$ algebra can be written in terms of four real supercharges Q^i_α as

$$\{Q^i{}_\alpha, Q^j{}_\beta\} = 2\delta^{ij}\gamma^\mu_{\alpha\beta}P_\mu + 2C_{\alpha\beta}Z^{ij} \tag{1.1.18}$$

for a real, antisymmetric matrix of central charges Z^{ij}. The automorphism group is

$$SO(4) \simeq SU(2)_{\rm R} \times SU(2)_{\rm L} \tag{1.1.19}$$

The superconformal version of the $\mathcal{N} = 2$ algebra is [3]

$$SO(3,2)_{\rm Conformal} \times SO(2)_R \subset OSp(2|4) \tag{1.1.20}$$

It has a distinguished R-symmetry. This "IR" R-symmetry is a linear combination of the exact global $U(1)$ symmetries of the theory F_i and the UV R-symmetry $R_{\rm UV}$

$$R_{\rm IR} = R_{\rm UV} + \sum_{U(1)_i} a_i F_i \tag{1.1.21}$$

All operators satisfy the constraint

$$D \geq |R_{\rm IR}| \tag{1.1.22}$$

where D is the conformal dimension of the operator and $R_{\rm IR}$ its R-charge. The bound must be saturated for chiral operators. Unitarity demands that all gauge invariant operators in the theory satisfy

$$D \geqslant \frac{1}{2} \tag{1.1.23}$$

If the lowest component of a gauge invariant chiral superfield saturates this bound, then the scalar in the superfield satisfies the free Klein-Gordon equation, and the entire chiral multiplet is free.

In Euclidean signature the superconformal version of the $\mathcal{N} = 2$ algebra is [3]

$$SO(4,1)_{\rm Conformal} \times SO(2)_R \subset OSp(2|2,2) \tag{1.1.24}$$

with the real supercharges (including superconformal charges) Q^i_A with $A = 1...4$ an $SO(4,1)_{\text{Conformal}}$ spinor index and $i = 1...2$ an $SO(2)_{\text{R}}$ R-symmetry index. These satisfy

$$\{Q^i_A, Q^j_B\} = \delta^{ij} M_{AB} + i\omega_{AB}\varepsilon_{ij} R \tag{1.1.25}$$

$$\omega = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{1.1.26}$$

and the supercharges transform under the R-symmetry as

$$[R, Q^i_A] = \varepsilon^{ij} Q^j_A \tag{1.1.27}$$

1.1.3 Multiplets

The irreducible multiplets of the $\mathcal{N} = 2$ algebra are familiar from the four-dimensional $\mathcal{N} = 1$ supersymmetry algebra. Defining the supercovariant derivatives

$$D_\alpha = \frac{\partial}{\partial \theta_\alpha} - i\big(\gamma^\mu \bar{\theta}\big)_\alpha \partial_\mu \tag{1.1.28}$$

$$\bar{D}_\alpha = -\frac{\partial}{\partial \bar{\theta}_\alpha} + i(\gamma^\mu \theta)_\alpha \partial_\mu \tag{1.1.29}$$

The chiral multiplet is a spin 0 superfield satisfying

$$\bar{D}_\alpha \Phi = 0 \tag{1.1.30}$$

The antichiral superfield satisfies

$$D_\alpha \bar{\Phi} = 0 \tag{1.1.31}$$

The chiral superfield can be expanded in terms of the components: a complex scalar field ϕ, a complex Dirac fermion ψ and an auxiliary complex scalar F

$$\Phi = \phi + \sqrt{2}\theta\psi + \theta^2 F \tag{1.1.32}$$

The $\mathcal{N} = 2$ vector multiplet is a spin 0 supermultiplet satisfying

$$V^\dagger = V \tag{1.1.33}$$

The vector superfield can be expanded, in Wess-Zumino gauge, in terms of the components: a real scalar field σ, a vector field A_μ, a complex Dirac fermion λ and a real auxiliary scalar D

$$V^{(a)} = -\theta^\alpha(\sigma^\mu)_{\alpha\beta}\bar{\theta}^\beta A_\mu{}^{(a)}(x) - \theta\bar{\theta}\sigma^{(a)} + i\theta\theta\bar{\theta}\bar{\lambda}^{(a)}(x) - i\bar{\theta}\bar{\theta}\theta\lambda^{(a)}(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D^{(a)}(x) \tag{1.1.34}$$

The superscript (a) indicates that these fields are Lie algebra valued. The components are usually written in terms of the matrices of the adjoint representation. The gauge transformation parameter is a Lie algebra valued chiral superfield Λ (here in matrix notation) and the transformation takes the form

$$e^V \to e^{i\Lambda}e^V e^{-i\Lambda} \tag{1.1.35}$$

which for Abelian theories can be written more simply as

$$V \to V + i(\Lambda - \Lambda^\dagger) \tag{1.1.36}$$

The field-strength superfields are defined as

$$W_\alpha = \frac{1}{4}\bar{D}\bar{D}e^{-V}D_\alpha V \tag{1.1.37}$$

$$\bar{W}_\alpha = \frac{1}{4}DDe^{-V}D_\alpha V \tag{1.1.38}$$

and transform covariantly under the gauge transformations above. These superfields contain the normal non-Abelian field-strength $F^{(a)}_{\mu\nu}$. In three dimensions, there is an alternative multiplet containing the field-strength. This linear multiplet is defined by

$$\Sigma = \bar{D}^\alpha D_\alpha V \tag{1.1.39}$$

$$\Sigma^\dagger = \Sigma \tag{1.1.40}$$

$$D^\alpha D_\alpha \Sigma = \bar{D}^\alpha \bar{D}_\alpha \Sigma = 0 \tag{1.1.41}$$

Its lowest component is the scalar σ.

An $\mathcal{N} = 4$ hypermultiplet is a pair of $\mathcal{N} = 2$ chiral multiplets in conjugate representations. An on-shell hypermultiplet contains 4 bosonic and 4 fermionic degrees of freedom. The $\mathcal{N} = 4$ vector multiplet comprises an $\mathcal{N} = 2$ vector multiplet and an adjoint $\mathcal{N} = 2$ chiral multiplet. It has 4 bosonic and 4 fermionic degrees of freedom.

1.1.4 Components

The following are the component transformations for the multiplets in 1.1.3. We begin with the Minkowski version of the $\mathcal{N} = 2$ supersymmetry algebra. The component transformations for the gauged chiral multiplet are [4]

$$\delta\phi = \bar{\varepsilon}\psi \tag{1.1.42}$$

$$\delta\psi = (-i\gamma^\mu D_\mu \phi - \sigma\phi)\varepsilon + F\varepsilon^* \tag{1.1.43}$$

$$\delta F = \bar{\varepsilon}^*(-i\gamma^\mu D_\mu \psi + i\lambda\phi + \sigma\psi) \tag{1.1.44}$$

note that here $\bar{\varepsilon} = \gamma^0 \varepsilon$ and we define the gauge covariant derivative

$$D_\mu = \partial_\mu + i[A_\mu, .] \tag{1.1.45}$$

we will write the action of an adjoint valued scalar, a, on a field, X, in the representation R of the gauge group simply as aX. The transformations for the vector multiplet are

$$\delta A_\mu = \frac{i}{2}(\bar{\varepsilon}\gamma_\mu\lambda - \bar{\lambda}\gamma_\mu\varepsilon) \tag{1.1.46}$$

$$\delta\sigma = \frac{i}{2}(\bar{\varepsilon}\lambda - \bar{\lambda}\varepsilon) \tag{1.1.47}$$

$$\delta D = \frac{1}{2}(\bar{\varepsilon}\gamma^\mu D_\mu\lambda + D_\mu\bar{\lambda}\gamma^\mu\varepsilon) + \frac{i}{2}(\bar{\varepsilon}[\lambda, \sigma] + [\bar{\lambda}, \sigma]\varepsilon) \tag{1.1.48}$$

$$\delta\lambda = (-\frac{i}{2}\varepsilon^{\mu\nu\rho}F_{\mu\nu}\gamma_\rho - iD - \gamma^\mu D_\mu\sigma)\varepsilon \tag{1.1.49}$$

$$\delta\bar{\lambda} = \bar{\varepsilon}(\frac{i}{2}\varepsilon^{\mu\nu\rho}F_{\mu\nu}\gamma_\rho + iD - \gamma^\mu D_\mu\sigma) \tag{1.1.50}$$

The Euclidean version of the algebra involves two unrestricted complex spinors $\varepsilon, \bar{\varepsilon}$. The transformations for the vector multiplet, including the possibility of a nonconstant spinor which generates a conformal transformation (or any transformation on a curved manifold) are [8]

$$\delta A_\mu = \frac{i}{2}(\bar{\varepsilon}\gamma_\mu\lambda - \bar{\lambda}\gamma_\mu\varepsilon) \tag{1.1.51}$$

$$\delta\sigma = \frac{1}{2}(\bar{\varepsilon}\lambda - \bar{\lambda}\varepsilon) \tag{1.1.52}$$

$$\delta D = -\frac{i}{2}\bar{\varepsilon}\gamma^\mu D_\mu\lambda - \frac{i}{2}D_\mu\bar{\lambda}\gamma^\mu\varepsilon + \frac{i}{2}\bar{\varepsilon}[\lambda,\sigma] + \frac{i}{2}[\bar{\lambda},\sigma]\varepsilon - \frac{i}{6}\left(D_\mu\bar{\varepsilon}\gamma^\mu\lambda + \bar{\lambda}\gamma^\mu D_\mu\varepsilon\right) \tag{1.1.53}$$

$$\delta\lambda = (-\frac{i}{2}\varepsilon^{\mu\nu\rho}F_{\mu\nu}\gamma_\rho - D + i\gamma^\mu D_\mu\sigma)\varepsilon + \frac{2i}{3}\sigma\gamma^\mu D_\mu\varepsilon \tag{1.1.54}$$

$$\delta\bar{\lambda} = (-\frac{i}{2}\varepsilon^{\mu\nu\rho}F_{\mu\nu}\gamma_\rho + D - i\gamma^\mu D_\mu\sigma)\bar{\varepsilon} - \frac{2i}{3}\sigma\gamma^\mu D_\mu\bar{\varepsilon} \tag{1.1.55}$$

In flat space, constant spinors generate supersymmetry transformations, and spinors linear in the flat space coordinates generate superconformal transformations.

We will need the same sort of transformations for the chiral multiplet. In addition to the conformal supersymmetries, the chiral multiplet can have a nonstandard conformal dimension Δ. The transformations are

$$\delta\phi = \bar{\varepsilon}\psi \tag{1.1.56}$$

$$\delta\phi^\dagger = \varepsilon\bar{\psi} \tag{1.1.57}$$

$$\delta\psi = \left(i\gamma^\mu D_\mu\phi + i\sigma\phi\right)\varepsilon + \frac{2\Delta i}{3}\phi\gamma^\mu D_\mu\varepsilon + \bar{\varepsilon}F \tag{1.1.58}$$

$$\delta\bar{\psi} = \left(i\gamma^\mu D_\mu\phi^\dagger + i\sigma\phi^\dagger\right) + \frac{2\Delta i}{3}\phi^\dagger\gamma^\mu D_\mu\bar{\varepsilon} + \bar{F}\varepsilon \tag{1.1.59}$$

$$\delta F = \varepsilon\left(i\gamma^\mu D_\mu\psi - i\lambda\phi - i\sigma\psi\right) + \frac{i}{3}\left(2\Delta - 1\right)\left(\gamma^\mu D_\mu\varepsilon\right)\psi \tag{1.1.60}$$

$$\delta F^\dagger = \bar{\varepsilon}\left(i\gamma^\mu D_\mu\bar{\psi} + i\bar{\lambda}\phi^\dagger - i\sigma\bar{\psi}\right) + \frac{i}{3}\left(2\Delta - 1\right)\left(\gamma^\mu D_\mu\bar{\varepsilon}\right)\bar{\psi} \tag{1.1.61}$$

1.1.5 Supersymmetric Actions

We will not use $\mathcal{N} = 1$ superfields, but it is convenient to start with an action for a set of real scalars ϕ^i and Majorana fermions ψ^i. The $\mathcal{N} = 1$ supersymmetric sigma model has the action [6]

$$S_{\text{Sigma Model}} = \int d^3x \left(g_{ij}(\phi)\left(\partial_\mu\phi^i\partial^\mu\phi^j + \psi^i_\alpha(\gamma^\mu)^{\alpha\beta}\nabla_\mu\psi^j_\beta\right) + \frac{1}{6}R_{ijkl}\psi^{\alpha i}\psi^k_\alpha\psi^{\beta j}\psi^l_\beta\right) \tag{1.1.62}$$

$$\nabla_\mu \psi^i_\alpha = \partial_\mu \psi^i_\alpha + \Gamma^i_{jk} \left(\partial_\mu \phi^j\right) \psi^k_\alpha \tag{1.1.63}$$

where g_{ij} is the metric on the target space, a real manifold on which the ϕ^i are coordinates, and R_{ijkl} its curvature tensor. The fermions are vector fields on this manifold, which explains their kinetic term. This action is invariant under change of coordinates of the target space.

A set of scalar components ϕ^i of chiral superfields Φ^i naturally act as complex coordinates on a complex target space. The most general two derivative action for a set of chiral superfields is the sigma model action given below

$$S^{\mathcal{N}=2}_{\text{Wess Zumino}} = \int d^3x d^2\theta d^2\bar{\theta} K(\Phi^i, \bar{\Phi}^i) + (\int d^3x d^2\theta W\left(\{\Phi\}\right) + h.c.) \tag{1.1.64}$$

where $\mathcal{N} = 2$ supersymmetry implies that the first term above, and the metric, $g_{ij}(\phi)$, for the kinetic terms in its component expansion come from a Kähler potential K for the target space manifold $\mathcal{M}$. The superpotential W is a holomorphic function on $\mathcal{M}$. The number of supercharges in fact determines the geometry of the target space $\mathcal{M}$ through the following classification [6]

Number of supersymmetries	Target space geometry
$\mathcal{N} = 1$	Riemannian
$\mathcal{N} = 2$	Kähler
$\mathcal{N} = 3, 4$	hyper-Kähler

We will consider mostly theories with flat target spaces. The relevant actions for chiral superfields are

$$S^{\mathcal{N}=2}_{\text{uncharged matter kinetic}} = -\int d^3x d^2\theta d^2\bar{\theta} \sum_i \left({\Phi_i}^\dagger \Phi_i\right) \tag{1.1.65}$$

$$= \sum_i \int d^3x \left(\left(\partial_\mu \phi\right)_i (\partial^\mu \phi)^i + i\bar{\psi}_i \gamma^\mu \partial_\mu \psi^i + F_i F^i \right) \tag{1.1.66}$$

$$S^{\mathcal{N}=2}_{\text{superpotential}} = \int d^3x d^2\theta W\left(\{\Phi\}\right) + h.c. \tag{1.1.67}$$

$$= \sum_i \int d^3x \left(\frac{\partial W}{\partial \phi^i} \left(\{\phi\}\right) F^i \right) + \sum_{i,j} \int d^3x \left(\frac{1}{2} \frac{\partial W}{\partial \phi^i \partial \phi^j} \left(\{\phi\}\right) \psi^i \psi^j \right) + h.c. \tag{1.1.68}$$

The superpotential must have R-charge 2 for the R-symmetry to be realized. We can couple these to vector superfields, thereby gauging a subset of the global symmetry group. The kinetic term becomes

$$S^{\mathcal{N}=2}_{\text{charged matter kinetic}} = -\int d^3x d^2\theta d^2\bar{\theta} \sum_i (\Phi_i{}^\dagger e^{2V} \Phi_i) \tag{1.1.69}$$

$$= \sum_i \int d^3x \left((D_\mu \phi)_i (D^\mu \phi)^i + i\bar{\psi}_i \gamma^\mu D_\mu \psi^i + F_i F^i - \phi_i \sigma^2 \phi^i + \phi_i D \phi^i - \bar{\psi}_i \sigma \psi^i + i\phi_i \bar{\lambda} \psi^i - i\bar{\psi}_i \lambda \phi^i \right) \tag{1.1.70}$$

where we have used some actions of adjoint valued fields $(A_\mu, \sigma, D, \lambda)$ on fields in the chiral multiplet. Denote a chiral multiplet field X in a representation R as X^a. We use a, b, c indices for the representation given by the matrices $(T_\alpha)^a{}_b$, and α, β, γ for the generators of the adjoint representation. Then the expressions above contain (for example)

$$(D_\mu X)^a = \partial_\mu X^a + i A^\alpha_\mu (T_\alpha)^a{}_b X^b \tag{1.1.71}$$

$$(\sigma X)^a = \sigma^\alpha (T_\alpha)^a{}_b X^b \tag{1.1.72}$$

and all representation indices are finally contracted in a gauge invariant manner. When gauging a background symmetry, we will work with a similar action where only the σ component of the vector superfield V are turned on. This will be referred to as a "real mass" term

$$S^{\mathcal{N}=2}_{\text{real mass}} = -\int d^3x d^2\theta d^2\bar{\theta} \sum_i (\Phi_i{}^\dagger e^{\theta\bar{\theta} m} \Phi_i) \tag{1.1.73}$$

$$= \sum_i \int d^3x \left((D_\mu \phi)_i (D^\mu \phi)^i + i\bar{\psi}_i \gamma^\mu D_\mu \psi^i + F_i F^i - \phi_i m^2 \phi^i - \bar{\psi}_i m \psi^i \right) \tag{1.1.74}$$

We can further add a kinetic term for the gauge fields. In three dimensions there are two equivalent ways of doing this in superspace

$$S^{\mathcal{N}=2}_{\text{Yang Mills}} = \frac{1}{g^2} \int d^3x d^2\theta d^2\bar{\theta} Tr_f \left(\frac{1}{4} \Sigma^2 \right) \tag{1.1.75}$$

$$= \frac{1}{g^2} \int d^3x Tr_f \left(\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + D_\mu \sigma D^\mu \sigma + D^2 + i\bar{\lambda} \gamma^\mu D_\mu \lambda \right) \tag{1.1.76}$$

$$S^{\mathcal{N}=2}_{\text{Yang Mills alternative}} = \frac{1}{g^2} \int d^2\theta \mathrm{Tr}_{\mathrm{f}} W_\alpha{}^2 + \text{c.c.} \tag{1.1.77}$$

An additional term involving only the vector multiplet that exists in three dimensions is the Chern-Simons density. Below, the notation Tr_f denotes a trace in the fundamental representation of a gauge group G. The coupling constant k must be an integer.

$$S^{\mathcal{N}=2}_{\text{Abelian Chern Simons}} = \frac{k}{4\pi}\int d^3x d^2\theta d^2\bar{\theta} Tr_f\left(V\Sigma\right) \tag{1.1.78}$$

$$= \frac{k}{4\pi}\int d^3x \mathrm{Tr}_f\left(\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho - \bar{\lambda}\lambda + 2D\sigma\right) \tag{1.1.79}$$

$$S_{\text{non-Abelian Chern Simons}} = \frac{k}{4\pi}\int d^3x d^2\theta d^2\bar{\theta}\left(\int_0^1 dt\left(Tr_f\left(V\bar{D}^\alpha\left(e^{-tV}D_\alpha e^{tV}\right)\right)\right)\right) \tag{1.1.80}$$

$$= \frac{k}{4\pi}\int d^3x Tr_f\left(\varepsilon^{\mu\nu\rho}\left(A_\mu\partial_\nu A_\rho + \frac{2i}{3}A_\mu A_\nu A_\rho\right) - \bar{\lambda}\lambda + 2D\sigma\right) \tag{1.1.81}$$

Two Abelian vector multiplets can be coupled by an off-diagonal Chern-Simons term (a "BF" coupling)

$$S^{\mathcal{N}=2}_{\text{BF}} = \frac{k_{ij}}{4\pi}\int d^3x d^2\theta d^2\bar{\theta} Tr_f\left(\Sigma^i V^j\right) \tag{1.1.82}$$

$$= \frac{k_{ij}}{4\pi}\int d^3x\left(\varepsilon^{\mu\nu\rho}A^j{}_\mu\partial_\nu A^i{}_\rho - \frac{1}{2}\bar{\lambda}^j\lambda^i + D^i\sigma^j\right) \tag{1.1.83}$$

This is related to the possibility of adding a Fayet-Iliopoulos term for every $U(1)$ factor of the gauge group

$$S^{\mathcal{N}=2}_{FI} = \zeta\int d^3x d^2\theta d^2\bar{\theta} V \tag{1.1.84}$$

$$= \zeta\int d^3x D \tag{1.1.85}$$

We write the action for an $\mathcal{N}=4$ gauge theory in terms of the $\mathcal{N}=2$ actions above. Let Φ_{adj} be the adjoint chiral superfield in an $\mathcal{N}=4$ vector multiplet. We must pick a set of hypermultiplets $\{\Phi,\tilde{\Phi}\}$, that is two chiral superfields in conjugate representations, the action for which is just the $\mathcal{N}=2$ action for chiral multiplets above. The following action produces a theory with $\mathcal{N}=4$ supersymmetry

$$S^{\mathcal{N}=4}_{\text{superpotential}} = -i\sqrt{2}\int d^3x d^2\theta\sum_i\tilde{\Phi}_i\Phi_{\text{adj}}\Phi_i + h.c. \tag{1.1.86}$$

$$S^{\mathcal{N}=4}_{\text{Yang Mills}} = S^{\mathcal{N}=2}_{\text{Yang Mills}} + \frac{1}{g^2} S^{\mathcal{N}=2}_{\text{charged matter kinetic}}[\Phi_{\text{adj}}] \tag{1.1.87}$$

We can add a Chern-Simons term, but this involves breaking the supersymmetry down to $\mathcal{N} = 3$

$$S^{\mathcal{N}=3}_{\text{non-Abelian Chern Simons}} = S^{\mathcal{N}=2}_{\text{non-Abelian Chern Simons}} + Tr_f \left(\int d^3x d^2\theta \Phi^2_{\text{adj}} + h.c. \right) \tag{1.1.88}$$

The off-diagonal Chern-Simons coupling changes to

$$S^{\mathcal{N}=4}_{\text{BF}} = S^{\mathcal{N}=2}_{\text{BF}} + \left(\int d^3x d^2\theta \Phi^i_{\text{adj}} \Phi^j_{\text{adj}} + h.c. \right) \tag{1.1.89}$$

which contributes an additional complex parameter to the $\mathcal{N} = 4$ version of the Fayet-Iliopoulos term

$$S^{\mathcal{N}=4}_{FI} = S^{\mathcal{N}=2}_{FI} + \zeta_{\mathbb{C}} \int d^3x d^2\theta \Phi_{\text{adj}} + \text{c.c.} \tag{1.1.90}$$

$$= S^{\mathcal{N}=2}_{FI} + \zeta_{\mathbb{C}} \int d^3x F + \text{c.c.} \tag{1.1.91}$$

1.1.6 Global Symmetries

Every superconformal $\mathcal{N} = 2$ theory in $2+1$ dimensions has a global $U(1)_{\text{R}}$ symmetry which rotates the supercharges 1.1.2. This symmetry is, in general, a linear combination of the global $U(1)$ symmetries visible in the UV action.

$$R_{\text{IR}} = R_{\text{UV}} + \sum_{U(1)_i} a_i F_i \tag{1.1.92}$$

Gauge theories with an Abelian factor have, in addition, a "topological" conserved current called $U(1)_{\text{J}}$. The current is

$$J_\mu = \varepsilon_{\mu\nu\rho} F^{\mu\nu} \tag{1.1.93}$$

it is identically conserved due to the Bianchi identity

$$\partial_{[\mu} F_{\nu\rho]} = 0 \tag{1.1.94}$$

where square brackets denote antisymmetrization. The current associated to non-Abelian theories is not gauge invariant. Fundamental fields are not charged under $U(1)_{\text{J}}$. Monopole operators, however, can have a nonvanishing $U(1)_{\mathrm{J}}$ charge, although this is not always the case. The $\mathcal{N}=2$ BF coupling 1.1.82 is equivalent to background gauging this symmetry in a supersymmetric manner.

Theories with additional matter multiplets can have flavor symmetries. For a $U(N)$ gauge theory with N_f fundamental hypermultiplets (that is, $2N_f$ chiral multiplets, half in the fundamental and half in the antifundamental representation), the flavor symmetry is $SU(N_f)\times SU(N_f)\times U(1)_{\mathrm{A}}$ which corresponds to arbitrary individual rotations of the chirals. The non-Abelian part of this group is broken to the diagonal $SU(N_f)$ by the $\mathcal{N}=4$ superpotential. The $U(1)_{\mathrm{A}}$ symmetry is similarly not present in such theories. The overall $U(1)_{\mathrm{B}}$ flavor symmetry, which rotates the fundamentals and antifundamentals in the opposite direction and by an overall phase, is gauged by the $U(1)$ factor in the gauge group. Real mass terms can be included though background vector multiplets for the flavor symmetries (see above). The scalar in the background multiplet must be a constant in the Cartan subalgebra of the gauge symmetry group. The Fayet-Iliopoulos term arises from a background vector multiplet which gauges the $U(1)_{\mathrm{J}}$ symmetry. A real mass for the $U(1)_{\mathrm{B}}$ symmetry can be shifted away by shifting the origin of the Coulomb branch.

$\mathcal{N}=4$ theories have a potential $SU(2)_L\times SU(2)_R$ R-symmetry. Choosing an $\mathcal{N}=2$ subalgebra picks out a single $U(1)_{\mathrm{R}}$. The following table summarizes the representations of this symmetry present in a gauge theory

field or charge	representation of $SU(2)_L\times SU(2)_R$
supercharges	$(2,2)$
λ	$(2,2)$
$\sigma, \mathrm{Re}(\phi_{\mathrm{Ad}}), \mathrm{Im}(\phi_{\mathrm{Ad}})$	$(3,1)$
hypermultiplet scalars	$(1,2)$
hypermultiplet fermions	$(2,1)$

For $\mathcal{N}=4$ supersymmetry, a triplet of background scalars in the Cartan subalgebra of the flavor group serve as deformations and contribute to the central charge. These transform as $(3,1)$ under $SU(2)_L\times SU(2)_R$. In $\mathcal{N}=2$ notation, one of these is the real mass and the two others combine into a complex superpotential mass. A triplet of Fayet-Iliopoulos terms in the $(1,3)$ representation exist for every $U(1)$ factor of the gauge group. The actions for these are given in 1.1.5. Theories with $\mathcal{N}=4$ matter content and superpotential, but with an additional $\mathcal{N}=3$ Chern-Simons term have only an $SO(3)$ R-symmetry.

Global symmetry currents sit in linear multiplets [1]. The linear multiplet containing the field strength tensor for an Abelian gauge theory contains the conserved current for the $U(1)_{\mathrm{J}}$ symmetry.

1.1.7 Abelian Duality

An important feature of $2+1$ dimensions is the duality between vector fields and scalars. A free vector is dual to a free periodic scalar via the relation

$$F_{\mu\nu} = \varepsilon_{\mu\nu\rho}\partial^{\rho}\gamma \tag{1.1.95}$$

One can consider the dual of an entire vector multiplet to a chiral multiplet satisfying

$$\Phi|_{\theta,\bar{\theta}=0} = \sigma + i\gamma \tag{1.1.96}$$

where γ is periodic with period g. If σ and γ are the light fields on the Coulomb branch of a theory with a non-Abelian gauge group G broken to its maximal torus, then the good single-valued chiral dual superfields are

$$Y_j \sim e^{\Phi\cdot\beta_j/g^2} \tag{1.1.97}$$

where the β_j are simple roots [5] and the identification is only valid semiclassically [1]. The $U(1)_{\mathrm{J}}$ symmetry acts by a shift on the dual photon γ.

Abelian duality can be carried out at the level of the action by considering the following steps [5, 6]. The most general low-energy effective action for a set of linear superfields Σ_i and neutral chiral multiplets $M, \bar{M}$ can be written

$$\int d^3x d^2\theta d^2\bar{\theta} f(\Sigma_i, M, \bar{M}) \tag{1.1.98}$$

Consider an action for a set of unconstrained real superfields G_i again with neutral chiral multiplets and a set of chiral multiplets $\Phi_i, \bar{\Phi}_i$

$$\int d^3x d^2\theta d^2\bar{\theta} \left(f(G_i, M, \bar{M}) - G_i(\Phi_i + \bar{\Phi}_i)\right) \tag{1.1.99}$$

Integrating out $\Phi_i, \bar{\Phi}_i$ results in an equation constraining the G_i to be linear superfields, and substituting this back into the action gives back 1.1.98. If, instead, we integrate out the G_i, we find the equation of motion

$$\Phi_i + \bar{\Phi}_i = \frac{\partial f(G_i, M, \bar{M})}{\partial G_i} \tag{1.1.100}$$

and a dual action for the chirals of the form

$$\int d^3x d^2\theta d^2\bar{\theta} K(\Phi_i + \bar{\Phi}_i, M, \bar{M}) \tag{1.1.101}$$

1.2 Dynamical Aspects

We now review some of the dynamical aspects of $\mathcal{N} = 2$ theories in $2+1$ dimensions. We will follow [1, 5]. Much can be learned about such theories by considering them as low-energy theories on stacks of D-branes in type IIB string theory. We will review this construction in 1.2.8.

1.2.1 Vacua

The set of space-time independent solutions of the equations of motion of a theory is called the set of vacua. In supersymmetric theories, the subset of vacua with vanishing vacuum energy is called the moduli space. For gauge theories in $2+1$ dimensions, the moduli space can take on a complicated form involving many distinct branches described by nonvanishing expectation values for different fields. Quantum effects, either perturbative or due to instantons, can lift part, or all, of the moduli space of vacua.

$\mathcal{N} = 2$ gauge theories have a Coulomb branch of vacua, so named because the gauge symmetry is spontaneously broken to its maximal torus at generic points on this space. The vacua are those that satisfy the two conditions [5]

$$\mathrm{Tr}([A_\mu, A_\nu])^2 = 0 \tag{1.2.1}$$

$$\mathrm{Tr}([A_\mu, \sigma])^2 = 0 \tag{1.2.2}$$

which restricts the photons and the adjoint scalar σ to sit in the Cartan subalgebra of the Lie algebra of the gauge group. For a gauge group G of rank r, the branch is r complex dimensional and parametrized by the expectation values of σ_i in the Cartan subalgebra and the dual photons γ_i. The complex structure is most easily seen by considering the duality 1.1.7. The fields Y_j are natural coordinates on this branch. One must actually divide by the remaining discrete gauge symmetry contained in the Weyl group $\mathcal{W}$ of G. The classical Coulomb branch is then isomorphic to

$$\mathbb{R}^r/\mathcal{W} \tag{1.2.3}$$

A Chern-Simons term for the gauge group removes the Coulomb branch.

Theories with charged chiral multiplets can have Higgs branches, where the gauge symmetry is partially or completely broken by the expectation values of the scalar components of the chiral superfields Φ_i

$$<\phi_i>\neq 0 \tag{1.2.4}$$

Coordinates on this branch are given by expectation values of gauge invariant combinations of the matter fields. For a theory with a set of flavors Φ^i and $\tilde{\Phi}_{\tilde{j}}$ in conjugate representations R and $\bar{R}$, these combinations are parametrized by the meson matrix

$$M^i{}_{\tilde{j}} = \Phi^i \tilde{\Phi}_{\tilde{j}} \tag{1.2.5}$$

which is subject to classical constraints. For $U(N_c)$ gauge groups with N_f flavors, the constraint, when the expectation values of fields in the vector multiplet vanish, is

$$\mathrm{rank}(M) \leq N_c \tag{1.2.6}$$

For $2N_f \geq N_c$ the Higgs branch is $N_c(2N_f - N_c)$ complex-dimensional. It is the Kähler reduction of the $2N_cN_f$ complex dimensional space of hypermultiplet scalars by the complexification of the gauge group G [6]. For $N_f < N_c$ the complex dimension is N_f^2. The Higgs branch is not subject to quantum corrections.

Besides these, there can be mixed branches where scalars from both types of multiplets acquire expectation values. The points at which different classical branches meet are often RG fixed points with interesting interacting conformal field theories. Quantum effects are especially important at these fixed points, which are often strongly interacting. In some situations, IR dualities relate different UV Lagrangian descriptions of the same fixed point. The theories are said to be in the same universality class. We will study various examples of this phenomenon in 3.1. The fact that the two types of branches meet can constrain the metric on the Coulomb branch. An example is SQED (see also 3.1.3) ($U(1)$ gauge theory with charge 1 and charge -1 chirals) in which the one complex dimensional Coulomb branch pinches off when it intersects the (one complex dimensional) Higgs branch. The effect can be argued to exist on the basis of the $U(1)_{\mathrm{J}}$ symmetry (see 1.1.6) which acts on the dual photon, but must act trivially at the contact point [1].

Quantum effects can, alternatively, smooth out a classically singular moduli space, joining two different branches in the process. Extended supersymmetry, specifically $\mathcal{N} = 3$ or $\mathcal{N} = 4$ in three dimensions, can often rule out such effects. Indeed, the analysis in 1.1.6 makes it clear that the different fields which gain an expectation value on a particular branch are acted upon, in these cases, by different global symmetry groups. For theories with only $\mathcal{N} = 2$ supersymmetry, the coulomb branch of a non-Abelian $U(N)$ gauge theory is generically lifted, by instantons, except for a one complex dimensional subspace corresponding to the diagonal $U(1)$.

Theories with $\mathcal{N} = 4$ supersymmetry in three dimensions have a non-Abelian R-symmetry group $SU(2)_L \times SU(2)_R$ (see 1.1.6). The vacua of the theory (see 1.2.1) can be invariantly split into a Higgs branch, which is a hyper-Kähler manifold ([6]) with a triplet of Kähler forms transforming in the adjoint representation of $SU(2)_R$, and a hyper-Kähler Coulomb branch with Kähler forms in the adjoint of $SU(2)_L$. The gauge coupling sits in a multiplet with scalars in the $(1+3, 1)$ representation ([9]) and can only affect the metric on the Coulomb branch perturbatively at one loop and through instanton corrections ([10] and see 1.2.4). The Higgs branch metric is not renormalized. For a single gauge group, G, the coulomb branch remains rank(G) dimensional after all quantum effects have been taken into account. For gauge group $U(N_c)$ and $N_f \geq N_c$ fundamental hypermultiplets, the Higgs branch has quaternionic dimension $N_c(N_f - N_c)$. It is the hyper-Kähler reduction of the N_f quaternionic dimensional manifold of hypermultiplet scalars by the group G [6]. The quantum corrected metric on the Coulomb branch can be inferred, for rank 1 examples, from the fermion zero modes in the instanton backgrounds and the constraints of constructing a hyper-Kähler manifolds with an $SU(2)$ action [10].

1.2.2 Renormalization group flow

A Wess-Zumino type Lagrangian has the superspace expression

$$S^{\mathcal{N}=2}_{\text{Wess Zumino}} = \int d^3x d^2\theta d^2\bar{\theta} K(\Phi, \bar{\Phi}) + \int d^3x d^2\theta W(\Phi) + h.c. \tag{1.2.7}$$

Unlike in four dimensions, some of these theories flow to interacting fixed points in the IR. Specifically, this applies to the theory with a cubic superpotential, which will be discussed later. The chiral superfields have engineering dimension 1/2. The quartic superpotential is thus classically marginal. The scaling dimension may be modified in the quantum theory, subject to the constraint

$$D = |R| \geq \frac{1}{2} \tag{1.2.8}$$

where R is the charge of the chiral multiplet under the distinguished R-symmetry which is part of the $\mathcal{N} = 2$ superconformal algebra and D its conformal dimension.

A gauge theory with $\mathcal{N} \geq 3$ cannot have wave function renormalization, as the coefficients in the superpotential are fixed by supersymmetry. The Chern-Simons term can only get an additive renormalization at one loop (see 1.2.3) [11]. The renormalization group flow is therefore controlled by the gauge coupling which has mass dimension 1/2. The flow to the infra-red is simply given by $g_{YM} \to \infty$. In the far IR, the Yang-Mills term with a $1/g_{YM}^2$ out front can be simply removed from the action. If the Chern-Simons level vanishes, the remaining action is singular. $\mathcal{N} = 2$ theories can have wave function renormalization. The effective superpotential, however, cannot depend on any of the coefficients appearing in the Kähler potential, specifically real mass or Fayet-Iliopoulos terms [1].

The low-energy action in the bulk of the Higgs or Coulomb branches is a sigma model with the target space geometry being that of the moduli space (see 1.1.5). For the Coulomb branch, one must first dualize the vector multiplet into a chiral multiplet as in 1.1.7. When two branches intersect, there can be an interesting interacting conformal fixed points. Some of these are discussed in 3.1.

1.2.3 Anomalies

In even dimensions, the path integral of a gauge theory with charged chiral fermions can pick up an anomalous transformation law due to the integration measure for the fermions. There are no such local gauge anomalies in $2+1$ dimensions. However, a similar situation arises when considering the transformation of the determinant of a gauged fermion under large gauge transformations. It has been shown that the determinant is not invariant, but that its anomalous transformation can be canceled by adding a Chern-Simons term to the theory. This is called the parity anomaly. Equivalently, one can consider the effective action resulting from integrating out a massive gauged fermion at one loop. In the Abelian theory, the bare Chern-Simons term has the superspace form

$$\sum_{ij} k_{ij} \int d^4\theta \Sigma_i V_j \tag{1.2.9}$$

The anomaly can renormalize the coefficients k_{ij} at one loop

$$(k_{\text{eff}})_{ij} = k_{ij} + \frac{1}{2} \sum_f (q_f)_i (q_f)_j \, \text{sgn}(M_f) \tag{1.2.10}$$

where the sum is over all fermions in the theory and $(q_f)_i$ is the charge of the fermion under the i'th $U(1)$ factor of the gauge group. Consistency of the quantum theory, which demands that the partition function be invariant under large gauge transformations, requires that

$$(k_{\text{eff}})_{ij} \in \mathbb{Z} \tag{1.2.11}$$

which puts a constraint on the k_{ij} subject to the matter content of the theory. Theories with charged fermions can therefore be inconsistent without a Chern-Simons term. In fact, since the anomaly depends on the effective mass of the fermion, the effective Chern-Simons level k_{ij} can be different at different points in the moduli space of the theory.

The one loop effect that leads to the anomaly can also induce off-diagonal Chern-Simons terms between the vector multiplet of a gauge symmetry and the linear multiplet which houses the conserved current of a global symmetry. if one introduced a background vector or linear multiplet for the global symmetry (V_{b} or Σ_{b}), this takes the form

$$\int d^4\theta \Sigma_b V = \int d^4\theta \Sigma V_b \tag{1.2.12}$$

This has the effect of mixing the topological symmetry $U(1)_J$, associated with the gauge field, with the global symmetry, such that the dual chiral superfield Y described in 1.1.7 is charged under the global $U(1)$.

A non-Abelian version of this effect shifts the (quantized) level of the Chern-Simons term of a non-Abelian gauge group G as

$$k \to k + \frac{1}{2}\sum_f d_3(R_f) \tag{1.2.13}$$

where R_f is the representation of the gauge group in which the fermion sits, and d_3 is the cubic index normalized so that the fundamental representation of $SU(N)$ has $d_3(N) = 1$ [1].

Note that in $\mathcal{N} \geq 3$ theories, the matter content comes in complete hypermultiplets, that is pairs of chiral multiplets in conjugate representations. The chiral multiplets are restricted to have opposite sign mass terms. Their contribution to the above anomaly therefore cancels.

1.2.4 Instantons

Gauge theories in 2+1 dimensions can have codimension three finite action Euclidean solutions to the equations of motion. These serve as instantons, similar to the self-dual field strength type instantons in four dimensions, and are responsible for many important effects. The relevant configurations for a gauge group G are classified by $\pi_2(G)$. This vanishes for all Lie groups, but is nonvanishing on the Coulomb branch, where the gauge group is broken to its maximal torus. For a rank$(G) = r$ gauge group, these are classified by

$$\pi_2(G/U(1)^r) = \mathbb{Z}^r \tag{1.2.14}$$

so there are r distinct types of configurations. The solutions are, in fact, identical to four-dimensional monopole solutions with the time direction removed.

The instantons are weighted, semiclassically, by

$$e^{-\sigma\cdot\beta_j/g^2} \tag{1.2.15}$$

where β_j are certain distinguished roots in the Lie algebra of G. These naturally generate an expectation value for the chiral dual fields Y_j on the Coulomb branch [1]. For $\mathcal{N} = 2$ $U(N)$ theories, these configurations generate a superpotential for the chiral multiplets which describe the Coulomb branch, thus lifting all but a one complex dimensional subspace.

For $\mathcal{N}$ =4 theories, the 1/2 BPS instantons correct the metric on the Coulomb branch without changing its dimension [10].

1.2.5 Vortices

Gauge theories in three dimensions can also have codimension two solutions of the equations of motion with finite energy. These are time-independent configurations that are analogous to monopole solutions in four dimensions. They are known as Abrikosov-Nielsen-Olesen vortices. The vorticity in question is the winding of the fields at infinity. These vortices show up on the Higgs branch of gauge theories.

Under special conditions, vortex configurations can saturate the BPS bound in the supersymmetry algebra. An example of this is $\mathcal{N} = 2$ $U(1)$ gauge theory with N_f massless flavors [1]. When one includes a Fayet-Iliopoulos term ζ, there are vacua in which only one flavor gains an expectation value. Vortices in this vacuum have the profile

$$\phi \sim \sqrt{\zeta} e^{\pm i\theta} \tag{1.2.16}$$

$$A_\theta \sim \pm\frac{1}{r} \tag{1.2.17}$$

The central charge is

$$Z = \int r d\theta A_\theta = \zeta \int d^2 x \varepsilon^{0\mu\nu} F_{\mu\nu} = \zeta \int j^0{}_{U(1)_J} \tag{1.2.18}$$

so the configurations above have central charge given by

$$Z = \pm\zeta = \zeta q_{U(1)_J} \tag{1.2.19}$$

When the Fayet-Iliopoulos term is taken to zero, the vacuum in question is the origin of the Coulomb branch. The vortices are massless there. On the Coulomb branch one can identify the vortex solutions with the dual chiral superfields Y_j [1].

1.2.6 Monopole operators

Monopole operators are disorder operators where the dynamical fields appearing in the path integral are taken to have a prescribed singularity at a point [12, 13, 14]. The singularity for the gauge field takes the form [15]

$$F = \frac{a}{2} \star d \frac{1}{|\vec{x} - \vec{x}_0|} \tag{1.2.20}$$

where a is an integer. This implies that there is a nonvanishing flux for F through the two-sphere surrounding the insertion point. To define a BPS monopole operator, the scalar in the vector multiplet must have a corresponding singularity such that

$$d\sigma = \star F \tag{1.2.21}$$

Note that for $\mathcal{N} = 4$ theories, the scalar with this prescribed singularity can be taken to be any of the three scalars in the $\mathcal{N} = 4$ vector multiplet. The prescription can also be used to define monopole operators in non-Abelian theories by choosing a homomorphism from the algebra of $U(1)$ to that of G.

The conformal dimension of a monopole operator can be calculated at large N_f using radial quantization [12]. The R-charge of a BPS monopole operators, and therefore its conformal dimension in the CFT, can be calculated exactly using localization [16]. The dimensions thus calculated can sometimes violate the unitarity bound

$$D \geqslant \frac{1}{2} \tag{1.2.22}$$

Theories with $\mathcal{N} = 4$ supersymmetry and unitarity violating operators are called "bad" [15]. The explanation for the appearance of such operators is that the UV R-symmetry used to calculate the dimensions can mix with other global symmetries as the theory flows to the IR, thus changing the naive dimension of some operators. We will see an example of this in $\mathcal{N} = 2$ theories in 3.3.1. The effect can also take place for the non-Abelian R-symmetry of an $\mathcal{N} = 4$ theory [15]. Theories with dimension 1/2 monopole operators must have a free sector in the IR and are called "ugly". All others theories are called "good". For an $\mathcal{N} = 4$ quiver node with a $U(N_c)$ gauge group and N_f fundamental flavors the node is "good" if $N_c < 2N_f - 1$, ugly if $N_c = 2N_f - 1$ and "bad" if $N_c > 2N_f - 1$. For "good" theories with $\mathcal{N} = 4$ supersymmetry, one can identify the non-Abelian UV R-symmetry with the IR R-symmetry and thus all operators have their canonical UV conformal dimensions in the IR CFT.

For an $\mathcal{N} = 4$ theory, the requirement that a theory is either "good" (inequality) or "ugly" (equality) can be translated into a condition for the gauge group G and the representations of the hypermultiplets [17]

$$-\frac{1}{2}\sum_{\alpha}|\alpha(\tau)| + \frac{1}{2}\sum_{\rho}|\rho(\tau)| \geq \frac{1}{2}, \tag{1.2.23}$$

where the sum is over the roots α of the gauge group G and ρ goes over the weights of all representations (with multiplicity) of hypermultiplets charged under G. τ is an arbitrary nontrivial element of the cocharacter lattice of G that determines the magnetic charge of the monopole [17].

1.2.7 Chern-Simons Theory

In [18], Witten showed that the 3D quantum field theory with the non-Abelian Chern-Simons action is well defined and gives rise to interesting observables. The Euclidean action for this theory for gauge group $U(N)$ at level k is

$$S_{\text{Chern - Simons}} = \frac{k}{4\pi}\int d^3x \varepsilon^{\mu\nu\rho}\text{Tr}_f\left(A_\mu\partial_\nu A_\rho + \frac{2}{3}iA_\mu A_\nu A_\rho\right) \tag{1.2.24}$$

where Tr_f is the trace in the fundamental representation. The theory is topological, and the observables are Wilson loops in a representation of the gauge group G. As explained in [18], for the action to define a topological invariant at the quantum level, it must be regularized. A counter-term involving the gravitational Chern-Simons action for the background metric must be added to the action. This leads to the "framing" ambiguity. A particular framing corresponds to a trivialization of the tangent bundle of the manifold on which the theory is defined. Observables are not invariant, but transform simply under a change of framing: the partition function and Wilson loops have a framing-dependent phase. The different framings are encoded in an integer s. The framing with $s = 0$ is called the trivial framing. We will see that localization calculations, for the supersymmetric version of this theory, force us to use the "supersymmetric framing" $s = -1$ [19].

The partition function for the theory, and the expectation values for Wilson loops can be calculated exactly [18]. The results are topological invariants of three manifolds such as the knot invariants described by the Jones polynomial [18]. The S^3 partition function for a $U(N)$ gauge group is [18, 8]

$$Z_{\text{Chern - Simons}}\left(\mathbb{S}^3\right) = \frac{1}{(k+N)^{N/2}} \prod_{j=1}^{N-1} \left(2\sin\frac{\pi j}{k+N}\right)^{N-j} \tag{1.2.25}$$

The partition function is invariant under the exchange of N and k. This is known as level-rank duality and extends to the expectation values of Wilson loops. A Wilson loop in a representation given by a certain Young tableaux in the theory with N colors and Chern-Simons level k is mapped to the Wilson loop corresponding to a Young tableaux which is flipped along the diagonal in a theory with k colors and Chern-Simons level N.

1.2.8 Brane Constructions

The low-energy action on an infinite flat type IIB D-brane is a maximally supersymmetric gauge theory in d+1 dimensions. Some of the supersymmetry may be broken by suspending a D-brane segment between two other branes. The resulting d dimensional (compactified) theory will preserve a fraction of the original supersymmetry, providing one chooses correctly the orientation of the branes. We briefly summarize the rules of the game for constructing such a theory in three dimensions. The original derivation can be found in [20] with additional details in [21, 22, 23, 24]. Following [20], we denote

$$\begin{aligned} \vec{m} &= (x^3, x^4, x^5) \\ \vec{w} &= (x^7, x^8, x^9) \end{aligned} \tag{1.2.26}$$

Three types of branes enter into the construction

- D3 branes whose world volume spans the $(0, 1, 2, 6)$ directions. The low energy world volume action on these is $\mathcal{N} = 4$ SYM in 4 dimensions. Having the branes terminate on various 5-branes will reduce this to $\mathcal{N} = 2, 3, 4, 6, 8$ in 3 dimensions.

- NS5 branes spanning the $(0, 1, 2, 3, 4, 5)$ directions.

- NS5' branes spanning the $(0,1,2,3,8,9)$ directions.
- D5 branes spanning the $(0,1,2,7,8,9)$ direction.
- A bound state of 1 NS5 brane and k D5 branes, called a $(1,k)$ brane, spanning the $(0,1,2,3/7,4/8,5/9)$ directions, with the last three numbers indicating that the brane may be tilted in the corresponding plane.

A generic configuration of D3 brane segments stretching between 5-branes preserves 4 supercharges on the D3 brane world volume, and so $\mathcal{N}=2$ supersymmetry from the three-dimensional viewpoint, and has a supersymmetric vacuum provided the following restrictions are satisfied

- The D3 segments may form a line (linear quiver) or a circle (elliptic quiver). We consider only connected configurations. Disconnected configurations correspond to decoupled theories.
- At most one D3 brane may stretch from a specific solitonic (NS or $(1,k)$) 5-brane to a specific D5 brane. Only $n \leq k$ D3 branes may stretch from a specific NS5 brane to a $(1,k)$ brane. D3 brane segments ending on opposite sides of a 5 brane and coincident in the $(3,4,5,7,8,9)$ directions may be thought of as piercing the brane and are not counted for the purposes of this restriction. This is known as the "s rule" [20]. If a stack of branes can be arranged so as to satisfy the rule, by thinking of the various D3 branes as either piercing or beginning and ending on a 5 brane, then the theory has, potentially, a supersymmetric vacuum. Such a vacuum may correspond to part of a Coulomb branch, a Higgs branch or a mixture of the two.
- Configurations involving only NS5 and D5 branes preserve $\mathcal{N}=4$ supersymmetry, with the associated $SO(4) \simeq SO(3) \times SO(3)$ R-symmetry identified with independent rotations of $\vec{m}$ and $\vec{w}$. Distances in the $\vec{m}$ and $\vec{w}$ directions correspond to the triplet of mass and FI deformations, respectively, introduced in 1.1.6.
- Configurations involving NS5, D5 and (1,k) branes can preserve $\mathcal{N}=3$ supersymmetry if the $(1,k)$ brane is rotated by an appropriate angle.

The field content of the low energy $\mathcal{N}=2$ theory is read off a brane configuration using the following rules

- Every set of n coincident D3 brane segments stretching between two subsequent branes of type $\{\text{NS5}, \text{NS5'}, (1,k)\}$, whether piercing additional D5 branes or not, contributes a $U(n)$ $\mathcal{N}=2$ vector multiplet and an adjoint $\mathcal{N}=2$ chiral multiplet. The mass of the extra chiral multiplet, and its superpotential couplings, depend on the orientation of the branes.

- A D5 brane pierced by this type of segment contributes a fundamental hypermultiplet. This is the result of the $5-3$ string which has massless modes when the position of the D5 is adjusted so that it touches the D3s.

- $3-3$ strings stretching across solitonic 5 branes separating a segment of the type described above contribute bifundamental hypermultiplets of the neighboring gauge groups. If one of the segments ends on a D5 brane, the 3-3 strings contribute fundamental hypermultiplets to the other segment.

The action for the theory is that of minimally coupled $\mathcal{N}=2$ gauge theory with fundamental, antifundamental, bifundamental and adjoint flavors. If the right superpotential is produced, this may be enhanced to $\mathcal{N}=4$ by combining chiral multiplets into hypermultiplets and adjoint chirals with vector multiplets into $\mathcal{N}=4$ vector multiplets. The action is constructed according to the following rules

- The gauge coupling of the gauge theory related to a particular interval is determined by the distance in the x_6 direction between the pair of solitonic 5 branes. Specifically, for a pair of branes located at x_6 coordinates t_1 and t_2 we get [20]

$$\frac{1}{g^2} = |t_1 - t_2| \tag{1.2.27}$$

- When one of the branes is of $(1, k)$ type, the segments to the left and right get, in addition, a Chern-Simons term at levels k and $-k$, respectively [21]. There may be cancelations if the segment is bounded by two such branes.

- The triplet of FI terms described in 1.1.2 for a vector multiplet is determined by

$$\vec{D} = \vec{w}_1 - \vec{w}_2 \tag{1.2.28}$$

  where the coordinates are those of the solitonic branes bounding the interval.

- The triplet of mass terms described in 1.1.2 for a given hypermultiplet is determined by

$$\vec{m} = \vec{m}_{\mathrm{D5}} - \vec{m}_{\mathrm{D3}} \tag{1.2.29}$$

  only the difference of such parameters is physical as they are shifted together by shifting the triplet of scalars in the vector multiplet, thereby choosing the origin of the Coulomb branch.

- The superpotential involving the fundamental and bifundamental flavors, and the extra adjoint chiral, depends on the exact brane arrangement and tilt. We will need to consider the following cases
  - When an interval is bounded by two NS5 (or two NS5') branes the theory has the standard $\mathcal{N} = 4$ superpotential for the flavors.
  - When an interval is bounded by an NS5 and an NS5', the adjoint chiral is massive and can be integrated out, leaving an $\mathcal{N} = 2$ gauge theory [22].
  - When an interval is bounded by an NS5 and a $(1, k)$ brane, adjusted so that it preserves $\mathcal{N} = 3$ supersymmetry, we get both the $\mathcal{N} = 4$ Yang Mills and $\mathcal{N} = 3$ Chern-Simons action.

The effect of moving D5 branes past solitonic 5 branes was studied in [20]. Such moves may result in the creation or destruction of D3 brane segments. The low energy theory, however, remains unaffected — one mechanism for producing massless hypermultiplets having been traded for another. One may also try and move solitonic branes past each other. Such moves underlie the Seiberg-like duality proposals we intend to examine. In the absence of Chern-Simons interactions, such a maneuver necessarily involves a singularity where the gauge coupling becomes infinite. When one of the solitonic branes is of type $(1, k)$ or NS5', it seems that the situation is more mild. We will examine both scenarios.

For any of the quivers defined above, one can write down a set of invariant "linking numbers" associated with the 5-branes. These are are related to conservation of charge for the three form field strength H_{NS}. The total linking number associate to an NS5 brane is given by [20]

$$L_{\mathrm{NS}} = \frac{1}{2}(r - l) + (L - R) \tag{1.2.30}$$

where l,r are the number of D5 branes to the left and right of the NS5 brane respectively, and L,R are the number of D3 branes ending on the NS5 brane from the left and from the right. An identical formula holds for D5 branes, as long as we change the meaning of l,r to the number of NS5 branes to both sides

$$L_{\mathrm{D5}} = \frac{1}{2}(r - l) + (L - R) \tag{1.2.31}$$

When two 5-branes cross, additional D3 branes may be created or destroyed in such a way that the linking numbers above, for any given 5-brane, do not change. The same is true for a $(1, k)$ brane, regarded as a bound state of one NS5 brane and k D5 branes. In particular, when moving a D5 brane outside a segment bounded by two NS5 branes, a new D3 brane is created. The two ways of getting a fundamental hypermultiplet contribution to the action on the D3 branes of the segment bounded by the NS5 branes are thus related, and the low energy theory remains the same. Moving two NS5 branes past one another causes the gauge coupling in the D3 segment stretched between them to diverge. When one of the branes is replaced by a $(1, k)$ brane, the segment still has a Chern-Simons terms associated with it and the transition is, apparently, more mild. We examine the consequences of such transitions for duality of $\mathcal{N} \geq 3$ gauge theories in 3.1. Note that all the dualities described in 3.1 are valid at the IR fixed points of the theories. From the brane perspective, the IR fixed point is reached when all the branes are coincident in the t or x_6 direction.

1.3 Supersymmetric Lagrangians on S^3

The IR fixed points of the theories described above can be interesting interacting superconformal field theories. By a large conformal transformation, the path integral expression for the expectation value of an observable in such a theory can be mapped to a path integral for a theory on the three sphere S^3. Working on a compact space like the three sphere offers an advantage, as the finite volume provides an automatic IR cutoff and various observables, notably the partition function of the theory (the expectation value of the operator 1) are well defined. In this section, we follow the example of [25] and formulate supersymmetric gauge theories on S^3.

1.3.1 Coupling to Supergravity

Anticipating the use of localization techniques to evaluate the path integral (see 2), we want to put $\mathcal{N} = 2$ supersymmetric gauge theories on a compact manifold. That is, we wish to introduce a background metric without breaking the supersymmetry of the theory. There is an elegant way to describe such a change: by coupling the theory to a background supergravity multiplet. We review the procedure for doing this set down in [26]. To couple the theories one must first identify an appropriate current superfield which includes both the energy momentum tensor and the supersymmetry current of the flat space theory. These multiplets were written down in [27, 28].

We will follow the construction of the current multiplets for three-dimensional $\mathcal{N} = 2$ theories in [28]. The most general such multiplet is the $\mathcal{S}$ multiplet which a real vector superfield $\mathcal{S}_\mu$ defined by

$$\bar{D}^\alpha \mathcal{S}_{\alpha\beta} = \chi_\alpha + \mathcal{Y}_\alpha \tag{1.3.1}$$

$$\bar{D}_{\alpha}\chi_{\beta} = \frac{1}{2}C\varepsilon_{\alpha\beta} \tag{1.3.2}$$

$$D^{\alpha}\chi_{\alpha} = -\bar{D}^{\alpha}\bar{\chi}_{\alpha} \tag{1.3.3}$$

$$D_{\alpha}\mathcal{Y}_{\beta} + D_{\beta}\mathcal{Y}_{\alpha} = 0 \tag{1.3.4}$$

$$\bar{D}^{\alpha}\mathcal{Y}_{\alpha} = -C \tag{1.3.5}$$

where C is a complex constant. This multiplet is too general for our purposes. When the theory possesses a continuous R-symmetry one can construct an alternative: the $\mathcal{R}$-multiplet. It is defined by

$$\bar{D}^{\alpha}\mathcal{R}_{\alpha\beta} = \chi_{\alpha} \tag{1.3.6}$$

$$\bar{D}_{\alpha}\chi_{\beta} = 0 \tag{1.3.7}$$

$$D^{\alpha}\chi_{\alpha} = -\bar{D}^{\alpha}\bar{\chi}_{\alpha} \tag{1.3.8}$$

The superfield χ_{α} is of the same type as the field strength superfield W_{α}. The superfields $\mathcal{Y}_{\alpha}$appearing on the rhs of 1.3.1 has been "improved away" using the superspace improvement transformation [27, 28]

$$\mathcal{S}_{\alpha\beta} \to \mathcal{S}_{\alpha\beta} + \frac{1}{2}\left([D_{\alpha}, \bar{D}_{\beta}] + [D_{\beta}, \bar{D}_{\alpha}]\right)U \tag{1.3.9}$$

$$\chi_{\alpha} \to \chi_{\alpha} - \bar{D}^{2}D_{\alpha}U \tag{1.3.10}$$

$$\mathcal{Y}_{\alpha} \to \mathcal{Y}_{\alpha} - \frac{1}{2}D_{\alpha}\bar{D}^{2}U \tag{1.3.11}$$

using the real superfield U. All the theories we consider in this work will be superconformal. This allows a further simplification of the above, which yields the multiplet

$$\bar{D}^{\beta}\mathcal{J}_{\alpha\beta} = 0 \tag{1.3.12}$$

In [26], these multiplets were used to couple the theory to a background supergravity multiplet. Such a multiplet contains the metric, fermionic gravitino fields and a number of auxiliary fields. There is more than one such multiplet. The exact field content and couplings to the matter sector vary between formalisms and the dimension of the background manifold. We will follow the example given in [26], where a procedure resulting in the coupling of a Wess Zumino type action in three dimensions to a background metric for a round S^3 was written down explicitly. Including ordinary gauge fields in this action is not difficult. We will give an alternative way of constructing a supersymmetric theory on curved space in 2.2.2.

The procedure in [26] is analogous to the background gauging of global symmetries performed in 1.1.6. Specifically, there is no action added for the fields in the background multiplet except that which couples them to the dynamical fields already present in the theory. All fermions in the background multiplet are set to zero from the outset. Bosonic components, whether normally considered dynamical or auxiliary, are then set to arbitrary background values. Critically, the background values need not solve any equation of motion or even satisfy the correct reality conditions needed to define a sensible theory in which one integrates over these fields. The only criterion imposed on the background values is that the resulting coupled action preserves a minimal fraction of the supersymmetry of the flat-space theory.

To obtain an action for a theory in three Euclidean dimensions on a round S^3, one starts with a Lorentz signature four-dimensional theory on $S^3 \times \mathbb{R}$. Then one rotates the time direction associated with $\mathbb{R}$ to obtain a Euclidean version of the theory. Next, the $\mathbb{R}$ factor is compactified to S^1. The final step takes the radius of this circle to zero. Presumably, this chain can be shortened by starting with supergravity in $2+1$ dimensions, but the results are not available.The supersymmetry transformations for the (four dimensional) chiral multiplet coupled to the $\mathcal{R}$ mutiplet with R-charge q_i in the supergravity background are [26]

$$\delta\phi^i = -\sqrt{2}\zeta\psi^i \tag{1.3.13}$$

$$\delta\psi^i_\alpha = -\sqrt{2}\zeta_\alpha F^i - i\sqrt{2}\big(\sigma^\mu\bar{\zeta}\big)_\alpha \left(\partial_\mu - iq_i A_\mu\right)\phi^i \tag{1.3.14}$$

$$\delta F^i = -i\sqrt{2}\bar{\zeta}\bar{\sigma}^\mu \left(\nabla_\mu - i(q_i-1)A_\mu - \frac{i}{2}V_\mu\right)\psi^i \tag{1.3.15}$$

The background bosonic fields in the supergravity multiplet (in this case "new minimal supergravity") must satisfy conditions, independent of the matter fields, in order to preserve supersymmetry. The conditions on these fields, $A_\mu, V_\mu, g_{\mu\nu}$, to preserve four supercharges ($\mathcal{N}=1$ in four dimensions or $\mathcal{N}=2$ in three) are

$$\nabla_\mu V_\nu = 0 \tag{1.3.16}$$

$$\partial_{[\mu} A_{\nu]} = 0 \tag{1.3.17}$$

$$W_{\mu\nu\kappa\lambda} = 0 \tag{1.3.18}$$

$$\mathcal{R}_{\mu\nu} = -2\left(V_\mu V_\nu - g_{\mu\nu} V_\rho V^\rho\right) \tag{1.3.19}$$

so the metric is conformally flat, the one-form A_μ is closed and V_μ is Killing. This can be solved for $\mathbb{S}^3 \times \mathbb{S}^1$ by choosing

$$A_4 = V_4 = -\frac{i}{r} \tag{1.3.20}$$

where the $\mathbb{S}^1$ is along the 4 direction. The resulting action can then be evaluated in the limit where the size of the $\mathbb{S}^1$ goes to zero and the theory becomes three-dimensional. Choosing a flat target space for the matter fields yields the following action for chiral multiplet in three dimensions

$$S_{\text{chiral}} = \int d^3x \sqrt{g} \Bigg(D_\mu \phi^\dagger D^\mu \phi + i\psi^\dagger D\!\!\!/\, \psi - F^\dagger F - \phi^\dagger \sigma^2 \phi + i\phi^\dagger D\phi - i\psi^\dagger \sigma \psi + i\phi^\dagger \lambda^\dagger \psi - i\psi^\dagger \lambda \phi$$

$$+ 2i\left(\Delta - \frac{1}{2}\right)\phi^\dagger \sigma \phi + \Delta\left(2 - \Delta\right)\phi^\dagger \phi + \left(\Delta - \frac{1}{2}\right)\psi^\dagger \psi\Bigg) \tag{1.3.21}$$

where Δ is the R-charge of the ϕ component, which is a parameter in the action as a result of using the $\mathcal{R}$-multiplet. The choice of R-symmetry (see 1.1.6) thus affects the action of the theory on S^3. In fact, the construction in [26] makes it obvious that the action depends on the value of a complex background scalar v_s, the real part of which parameterizes the choice of R-current, and the imaginary part is the "real mass" described in 1.1.6. The dependence of the partition function of the theory on v_s is shown to be holomorphic [26]. This fact is used in 3.3.1.

There may also be a superpotential on S^3, but we will find that it will not play a role in the localization calculate, except to determine the set of global $U(1)$ symmetries which can mix with the UV R-symmetry.

1.3.2 Coordinates on S^3

We will work with toroidal coordinates, in terms of which the line element on S^3 is

$$ds^2{}_{S^3} = d\theta^2 + \cos^2(\theta)d\tau^2 + \sin^2(\theta)d\phi^2 \tag{1.3.22}$$

and define a set of left invariant 1-forms

$$\begin{aligned}
\sigma_1 &= -\sin(\phi+\tau)d\theta + \cos(\phi+\tau)\cos(\theta)\sin(\theta)d\tau - \cos(\phi+\tau)\cos(\theta)\sin(\theta)d\phi \\
\sigma_2 &= \cos(\phi+\tau)d\theta + \sin(\phi+\tau)\cos(\theta)\sin(\theta)d\tau - \sin(\phi+\tau)\cos(\theta)\sin(\theta)d\phi \\
\sigma_3 &= \cos^2(\theta)d\tau + \sin^2(\theta)d\phi \\
\sigma_\pm &= \sigma_1 \pm i\sigma_2
\end{aligned} \tag{1.3.23}$$

which satisfy

$$d\sigma_i = \varepsilon_i^{jk}\sigma_j \wedge \sigma_k \tag{1.3.24}$$

These will serve as a vielbein for a noncoordinate basis

$$\begin{aligned}
\sigma^i_\mu &= e^i_\mu \\
e^\mu_i &= (e^i_\mu)^{-1}
\end{aligned} \tag{1.3.25}$$

$$ds^2{}_{S^3} = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 \tag{1.3.26}$$

Note that tangent space indices $\{i, j, k\}$ are raised and lowered with the flat metric δ^i_j. Define also the dual basis of vectors

$$l^\mu_i \partial_\mu \tag{1.3.27}$$

such that

$$\sigma^i(l_j) = \delta^i_j \tag{1.3.28}$$

These satisfy

$$[l_i, l_j] = -2\varepsilon_{ij}^k l_k \tag{1.3.29}$$

so that

$$L_i = -\frac{i}{2} l_i \tag{1.3.30}$$

$$[L_i, L_j] = i\varepsilon_{ij}^k L_k \tag{1.3.31}$$

are generators of the $SU(2)$ algebra. In terms of these operators, the scalar Laplacian is

$$\Delta = -\nabla^2 = -l^i l_i = 4L^i L_i \tag{1.3.32}$$

In this basis, the spin connection simplifies to

$$\omega_{\mu ij} = \varepsilon_{ijk} e_\mu^k \tag{1.3.33}$$

so that the spinor covariant derivative is

$$\nabla_\mu = \partial_\mu + \frac{1}{8} e_\mu^k \varepsilon_{ijk} [\gamma^i, \gamma^j] \tag{1.3.34}$$

$$= \partial_\mu + \frac{i}{2} e_\mu^k \gamma_k \tag{1.3.35}$$

and the Dirac operator satisfies

$$i \not{\nabla} = i\gamma^i l_i - \frac{3}{2} \tag{1.3.36}$$

$$= -4S^i L_i - \frac{3}{2} \tag{1.3.37}$$

We will use the usual Laplacian for p-forms

$$d^\dagger = *d* \tag{1.3.38}$$

$$\Delta = dd^\dagger + d^\dagger d \tag{1.3.39}$$

where $*$ is the Hodge dual. We will set Lorentz gauge for all vector fields (or 1-forms), thus projecting onto the space of coclosed 1-forms on S^3

$$d^\dagger v = 0 \Leftrightarrow \nabla^\mu v_\mu = 0 \tag{1.3.40}$$

1.3.3 The Spectrum

We will need the spectrum on S^3 for the scalar and vector Laplacians, and for the Dirac operator. We will follow [29, 30]. In [29] it was shown that the eigenvalues of the Laplacian on S^N acting on coclosed p-forms (scalars are included as 0-forms) are

$$\lambda_N(L,p) = (L+p)(L+N-p-1) \tag{1.3.41}$$

$$\begin{cases} L = 0, 1, ..., \infty & p = 0 \\ L = 1, 2..., \infty & p \geqslant 1 \end{cases} \tag{1.3.42}$$

with degeneracy

$$D_N(L,p) = \frac{(2L+N-1)(L+N-1)!}{p!(N-p-1)!(L-1)!(L+p)(L+N-p-1)} \tag{1.3.43}$$

with the understanding that $D_N(0,0) = D_N(0,N) = 1$. For S^3, the scalars sit in the $(\frac{L}{2}, \frac{L}{2})$ representations of the $SO(4)$ isometry group.

In [30] it was shown that the eigenvalues of the Dirac operator on S^N are

$$\lambda^D{}_N(n) = \pm i(n + N/2), \qquad n = 0, 1, ..., \infty \tag{1.3.44}$$

with degeneracy

$$D^D_N(n) = \begin{cases} \frac{2^{N/2}(N+n-1)!}{n!(N-1)!} & \text{N even} \\ \frac{2^{(N-1)/2}(N+n-1)!}{n!(N-1)!} & \text{N odd} \end{cases} \tag{1.3.45}$$

1.3.4 Killing Spinors

The fermionic symmetries of the actions in 1.3.1 are generated by Killing spinors. For our purposes, a Killing spinor will be one that satisfies

$$\nabla_\mu \varepsilon = \alpha \gamma_\mu \varepsilon \tag{1.3.46}$$

Recall the spinor covariant derivative from 1.3.2

$$\begin{aligned} \nabla_\mu &= \partial_\mu + \tfrac{1}{8} e^k_\mu \epsilon_{ijk} [\gamma^i, \gamma^j] \\ &= \partial_\mu + \tfrac{i}{2} e^k_\mu \gamma_k \end{aligned} \tag{1.3.47}$$

A Killing spinor can immediately be constructed by taking ε to be constant in the basis of left invariant one-forms, thus yielding [19]

$$\nabla_\mu \varepsilon = \frac{i}{2} e^k_\mu \gamma_k \varepsilon = \frac{i}{2} \gamma_\mu \varepsilon \tag{1.3.48}$$

which gives a two-dimensional space of spinors. There is a further two-dimensional space satisfying

$$\nabla_\mu \epsilon = -\frac{i}{2} \gamma_\mu \epsilon \tag{1.3.49}$$

These are all the Killing spinors on S^3, in agreement with the results in [31, **?**]. We will, specifically, take the linear combination which satisfies

$$(\gamma_3 - 1)\varepsilon = 0 \tag{1.3.50}$$

We will also need the bilinears

$$\varepsilon^\dagger \varepsilon = 1 \tag{1.3.51}$$

$$v_\mu = \varepsilon^\dagger \gamma_\mu \varepsilon \tag{1.3.52}$$

where we have normalized the spinor. The ability to do this can be inferred from the equations in [32]. We also get that [32]

$$\nabla_\mu v_\nu + \nabla_\nu v_\mu = 0 \tag{1.3.53}$$

so that v_μ is a Killing vector. The integral curves of v_μ are great circles on S^3.

1.3.5 Gauge Theories on S^3

To summarize, we have the following two supersymmetric actions on S^3. For the gauged chiral multiplets

$$S_{\text{chiral}} = \int d^3x\sqrt{g}\Bigg(D_\mu\phi^\dagger D^\mu\phi + i\psi^\dagger D\!\!\!/\,\psi - F^\dagger F - \phi^\dagger\sigma^2\phi + i\phi^\dagger D\phi - i\psi^\dagger\sigma\psi + i\phi^\dagger\lambda^\dagger\psi - i\psi^\dagger\lambda\phi$$

$$+ 2i\left(\Delta - \frac{1}{2}\right)\phi^\dagger\sigma\phi + \Delta\left(2-\Delta\right)\phi^\dagger\phi + \left(\Delta - \frac{1}{2}\right)\psi^\dagger\psi\Bigg) \tag{1.3.54}$$

A kinetic term for the fields in the vector multiplet can also be written down on S^3

$$S_{\text{Yang Mills}} = \frac{1}{g^2}\int d^3x\sqrt{g}\text{Tr}\left(\frac{1}{2}F^{\mu\nu}F_{\mu\nu} + D_\mu\sigma D^\mu\sigma + (D+\sigma)^2 + i\lambda^\dagger\gamma^\mu\nabla_\mu\lambda + i[\lambda^\dagger,\sigma]\lambda - \frac{1}{2}\lambda^\dagger\lambda\right) \tag{1.3.55}$$

This term is, in fact, a total supersymmetry variation of an odd functional on S^3 ([8]) and is thus guaranteed to preserve a subset of the supersymmetry (see 2.2.2). In fact, even the action for the chiral multiplets above can be written as such a total variation [8]. By "supersymmetric," we mean that these actions have a fermionic symmetry which involves a Killing spinor 1.3.4 with the usual Euclidean supersymmetry transformations given in 1.1.4.

Chapter 2

Localization

In this chapter we review localization of path integrals. This procedure was inspired by the localization computation in [25].

The basic building blocks for observables in a quantum field theory are correlators of the fundamental fields. Such correlators can, in principle, be computed by evaluating a path integral with appropriate insertions. In general, however, it is not possible to compute such a quantity exactly, unless the action is at most quadratic in all fields. In a theory with fermionic symmetries, some correlators may be computed exactly even for a nonquadratic action. This is made possible by deformation invariance. We will work through the logic of localization for a general theory. First, we review the basics of localization of path integrals and quote the localization formulas due to Duistermaat and Heckman and to Atiyah and Bott and Berline and Vergne 2.1. In the next section we specialize to $2+1$ dimensional gauge theories 2.2. Finally, we present our results for the localized partition function in terms of a matrix integral 2.3.

2.1 Localization Formulas

Let δ represent a fermionic symmetry of the action

$$\delta S = 0 \tag{2.1.1}$$

The symmetry is also preserved by a subset of the operators in the theory

$$\delta \mathcal{O} = 0 \tag{2.1.2}$$

Here $\mathcal{O}$ represents an arbitrary operator, local or otherwise, made of the fundamental fields. Consider the path integral expression for the expectation value of $\mathcal{O}$

$$< \mathcal{O} >= \int \mathcal{D}[\phi] e^{iS} \mathcal{O} \tag{2.1.3}$$

where we have set $\hbar = 1$ and $\mathcal{D}[\phi]$ represents a measure over all the dynamical fields. We can deform the action without changing the value of $< \mathcal{O} >$

$$< \mathcal{O} >_t \equiv \int \mathcal{D}[\phi] e^{iS + t\delta V} \mathcal{O} \tag{2.1.4}$$

where V is a fermionic functional such that $\delta^2 V = 0$ and t is an arbitrary real number. In general, δ squares to a bosonic symmetry of the theory. Examine the change for a small t

$$\frac{d}{dt} < \mathcal{O} >_t \equiv \int \mathcal{D}[\phi] (\delta V) e^{iS + t\delta V} \mathcal{O} = \delta(\int \mathcal{D}[\phi] V e^{iS + t\delta V} O) = 0 \tag{2.1.5}$$

which implies that $< \mathcal{O} >_t$ is independent of t. Consider, now, a Euclidean version of this computation with a positive definite functional δV and a large negative coefficient $-t$

$$< \mathcal{O} >_{Euc} \equiv \int \mathcal{D}[\phi] e^{-S - t\delta V} \mathcal{O} \tag{2.1.6}$$

Configurations for which V is nonzero, which we denote ϕ_β, are exponentially suppressed. In fact, in the limit in which $t \to \infty$, the integral over these modes can be evaluated exactly using the saddle point approximation. To see this, scale every field, ϕ, by

$$\phi \to \frac{\phi}{\sqrt{t}} \tag{2.1.7}$$

The quadratic part of the localizing action, δV, is then of order 1 for the modes ϕ_β. nonquadratic parts are of higher order in $t^{-1/2}$. Next, take the limit $t \to \infty$. The integral over these modes is then becomes Gaussian. Note that the appearance of these modes in the original action, S, cannot change this conclusion.

Zero modes of the functional V, which we denote ϕ_α, must be treated separately. The integral over such modes must be done in the usual manner. The general result is then

$$< \mathcal{O} >_{Euc} = \lim_{t \to \infty} \int \mathcal{D}[\phi] e^{-S - t\delta V} \mathcal{O} = \int \mathcal{D}[\phi_\alpha] J[\phi_\alpha] \frac{1}{\sqrt{\mathrm{sdet}_{\phi_\beta} V[\phi_\alpha]}} e^{-S[\phi_\alpha]} \mathcal{O}(\phi_\alpha) \tag{2.1.8}$$

Where we have denoted by $J[\phi_\alpha]$ the Jacobian for the change of integration measure, and by $(\mathrm{sdet}_{\phi_\beta}(V[\phi_\alpha]))^{-1/2}$ the result of evaluating the Gaussian integral over the modes ϕ_β.

The explanation above falls short of an actual derivation. We now review the general theory of localization as it applies to path integrals. We will closely follow [33] and [34].

2.1.1 The Duistermaat-Heckman Theorem

We review the Duistermaat-Heckman formula [35]. We will follow the definitions and formulas given in [33].

Definition 1. A vector field X on a manifold (or supermanifold) M is called compact if it generates the action of a one parameter subgroup of a compact group G which acts on M.

the fermionic symmetry δ used above will eventually be identified with such a vector field on the supermanifold of field of the theory.

Definition 2. Let (M, Ω) be a $2n$-dimensional symplectic manifold with symplectic form Ω. A vector field X is called *Hamiltonian* with a Hamiltonian H if for every vector field Y on M

$$dH(Y) = \Omega(X, Y) \tag{2.1.9}$$

The Duistermaat-Heckman theorem provides a simple expression for the integral

$$\int_M \Omega^n e^{iH} = \int_M e^{iH} dx \tag{2.1.10}$$

where $dx = \Omega^n/n!$ is the Liouville measure on M. At a point p where X vanishes, the Hessian of H is well defined as a matrix acting on the tangent space TM_p. Let $sgn(Hess(H(p))$ be the signature of this matrix at p. For the case when the vector field X has only isolated nondegenerate zeros (i.e. $det(Hess(H(p)) \neq 0)$, the theorem takes the following form

Theorem 1. *(Duistermaat-Heckman) Let X be a compact Hamiltonian vector field on (M, Ω) with an isolated nondegenerate zero set R, then*

$$\int_M \Omega^n e^{iH} = i^n \sum_{p \in R} e^{\frac{i\pi}{4} sgn(Hess(H(p)))} \frac{e^{iH(p)}}{\sqrt{det(Hess(H(p))}} \tag{2.1.11}$$

This integral can also be written in a different way. We denote by ΠTM the supermanifold obtained from the total space of the tangent bundle of M with the parity of the fibers reversed. Coordinates on ΠTM are a set of regular coordinates on M, $\{x^i\}$, and a set of Grassmann coordinates for the fibers ξ^i. The integral can then be rewritten as an integral over the supermanifold

$$\int_M \Omega^n e^{iH} = i^{-n} \int_{\Pi TM} \prod_{i=1}^{2n} dx^i d\xi^i e^{i(H(x)+\Omega_{ab}(x)\xi^a\xi^b)} \tag{2.1.12}$$

The exponential in the expression on the lhs can be identified with the inhomogeneous differential form

$$S(x,\xi) = H + \Omega \tag{2.1.13}$$

we can define an odd vector field on the supermanifold

$$Q = \xi^i \frac{\partial}{\partial x^i} + X^i(x)\frac{\partial}{\partial \xi^i} \tag{2.1.14}$$

and check that

$$QS = 0 \tag{2.1.15}$$

and the theorem states that the integral

$$\int_{\Pi TM} e^S dV \tag{2.1.16}$$

where dV is the volume element on ΠTM, gets contributions only from the zero locus of the vector field X.

In terms of differential forms on M, Q is an equivariant differential

$$Q = d + i_X \tag{2.1.17}$$

and we have

$$(d + i_X)(H + \Omega) = 0 \tag{2.1.18}$$

we also get that

$$Q^2 = (d + i_X)^2 = \mathcal{L}_X \tag{2.1.19}$$

where $\mathcal{L}_X$ is the Lie derivative with respect to X.

We will apply the more-general case considered in [33] to supersymmetric gauge theories. Specifically, we will rely on the following theorem which combines results from [33]

Theorem 2. *Let M be a compact supermanifold with volume form dV. Let Q be an odd vector field on M such that*

1. $div_{dV}Q = 0$ (the volume form is Q invariant)

2. Q^2 is an even compact Hamiltonian vector field on M.

Let $\mathcal{K}_Q$ be the zero set of Q and let S be an even Q-invariant function on M which is locally constant on $\mathcal{K}_Q$. Suppose that Q is nondegenerate in a neighborhood of $\mathcal{K}_Q$, then the stationary phase approximation for the following integral is exact and given by

$$Z_{Q,S} = \int_{\Pi TM} dV e^{is} = \sum_{p \in \mathcal{K}_Q} \frac{\rho(p) e^{iS(p)}}{\sqrt{sdet(Hess(S(p))}} \tag{2.1.20}$$

where $\rho(p)$ is the volume density at p, and "sdet" denotes the superdeterminant (Berezinian).

2.1.2 The Atiyah-Bott-Berline-Vergne theorem

The case where $\mathcal{K}_Q$ is not simply a set of points is covered by the Atiyah-Bott-Berline-Vergne localization formula [36, 37]. For a definition of the Cartan model of equivariant cohomology we refer the reader to [34].

Theorem 3. *Let Q be an equivariant differential and α a Q-closed equivariant form on a compact manifold M, then the following holds*

$$\int_M \alpha = \int_{\mathcal{K}_Q} \frac{i^*_{\mathcal{K}_Q} \alpha}{e(N_{\mathcal{K}_Q})} \tag{2.1.21}$$

where $\mathcal{K}_Q$ is the zero set of Q, which is now not necessarily discrete, $i^*_{\mathcal{K}_Q}$ is the pullback and $e(N_{\mathcal{K}_Q})$ is the equivariant Euler class of the normal bundle of $\mathcal{K}_Q$ in M.

In terms of the objects considered above, we can identify α with $e^{iS}dV$ and $e(N_{\mathcal{K}_Q})$ with $\sqrt{sdet(Hess(S(p))}$. Note that we can include in the definition of α any Q-closed observable of the theory, without changing the conclusions. Also note that the determinant factors appearing in the localization formulas do not depend on α. The factor $\rho(p)$ is the volume density at p. The analogue in 3 is the determinant factor for the change of variables implicit in the pullback $i^*_{\mathcal{K}_Q}\alpha$ of the form α to the submanifold $\mathcal{K}_Q$. This does not depend on the form α. Likewise, the equivariant Euler class appearing in 3 depends only on the embedding of $\mathcal{K}_Q$ in M. Specifically, a representative of this class can be computed by evaluating the superdeterminant of the Hessian of any suitable Q-closed function. By suitable we mean that the critical set of the function includes $\mathcal{K}_Q$ and that the Hessian is nonsingular after removing the "zero modes": those directions which parametrize $\mathcal{K}_Q$.

To sum up: the path integral computation of the expectation value of an observable of a supersymmetric theory which is invariant under a supercharge Q localizes to a subset $\mathcal{K}_Q$ of the entire field space. We can carry out the computation by parameterizing $\mathcal{K}_Q$ by a set of zero modes and computing the determinant for the change of variables. We also need to include a factor which takes into account the embedding of $\mathcal{K}_Q$, and which can be computed by choosing an appropriate Q-closed function and evaluating the determinant of its Hessian with the zero modes removed.

2.2 Localizing 3D Gauge Theories

In this section we apply the localization procedure to the supersymmetric gauge theories described in 1. We will work with gauge theories on S^3 that correspond to the Euclidean version of conformally invariant theories in $2+1$ dimensional Minkowski space.

2.2.1 The Supercharge

We are interested in using the localization procedure to evaluate correlation functions of the theories defined in 1.3.5. We begin by choosing an appropriate fermionic symmetry or, equivalently, a Killing spinor from those in 1.3.4. We will use the notation Q and δ interchangeably. We will choose to set $\bar{\varepsilon} = 0$ and take ε to be the spinor in 1.3.4. The transformations of the vector multiplet fields are then

$$\delta A_\mu = -\frac{i}{2}\lambda^\dagger \gamma_\mu \varepsilon \tag{2.2.1}$$

$$\delta\sigma = -\frac{1}{2}\lambda^\dagger \varepsilon \tag{2.2.2}$$

$$\delta D = -\frac{i}{2} D_\mu \lambda^\dagger \gamma^\mu \varepsilon + \frac{i}{2}[\lambda^\dagger, \sigma]\varepsilon + \frac{1}{4}\lambda^\dagger \varepsilon \tag{2.2.3}$$

$$\delta\lambda = (-\frac{i}{2}\varepsilon^{\mu\nu\rho} F_{\mu\nu}\gamma_\rho - D + i\gamma^\mu D_\mu \sigma - \sigma)\varepsilon \tag{2.2.4}$$

$$\delta\lambda^\dagger = 0 \tag{2.2.5}$$

and for the chiral multiplet fields

$$\delta\phi = 0 \tag{2.2.6}$$

$$\delta\phi^\dagger = \psi^\dagger\varepsilon \tag{2.2.7}$$

$$\delta F = \varepsilon^T(-i\gamma^\mu D_\mu\psi + i\sigma\psi + (\frac{1}{2} - \Delta)\psi + i\lambda\phi) \tag{2.2.8}$$

$$\delta F^\dagger = 0 \tag{2.2.9}$$

$$\delta\psi = (-i\gamma^\mu D_\mu\phi - i\sigma\phi + \frac{\Delta}{2}\phi)\varepsilon \tag{2.2.10}$$

$$\delta\psi^\dagger = \varepsilon^T F \tag{2.2.11}$$

The variations above correspond to choosing the generator

$$\delta = \frac{1}{\sqrt{2}}\left(Q_1^1 + iQ_1^2\right) \tag{2.2.12}$$

the would be Minkowski signature adjoint will also be conserved

$$\delta^\dagger = \frac{1}{\sqrt{2}}\left(Q_1^1 - iQ_1^2\right) \tag{2.2.13}$$

One can verify that [3, 19]

$$\delta^2 = \delta^{\dagger 2} = 0 \tag{2.2.14}$$

The anticommutator is a generator of the isometry group, with the Killing vector v_μ, and an R-symmetry transformation

$$\{\delta, \delta^\dagger\} = M_{12} + R \tag{2.2.15}$$

$$\{\delta, \delta^\dagger\}\phi = -i\left(v^\mu D_\mu + \sigma\right)\phi + \Delta\phi \tag{2.2.16}$$

$$\{\delta, \delta^\dagger\}\psi = -i\left(v^\mu D_\mu + \sigma\right)\psi + \left(\Delta - 1\right)\psi \tag{2.2.17}$$

$$\{\delta, \delta^\dagger\}F = -i\left(v^\mu D_\mu + \sigma\right)F + \left(\Delta - 2\right)F \tag{2.2.18}$$

In the presence of real mass terms this is modified to

$$\{\delta, \delta^\dagger\} = M_{12} + R + Z \tag{2.2.19}$$

2.2.2 The Functional

Next, we specify an odd functional V which will be acted upon by our supersymmetry. We will take V to be of the following form

$$V = V_{gauge} + V_{BRST} + V_{matter} \tag{2.2.20}$$

The first two terms are given by

$$V_{gauge} = \mathrm{Tr}(\delta\lambda)^{\dagger}\lambda \tag{2.2.21}$$

$$V_{BRST} = \bar{c}\nabla^{\mu}A_{\mu} \tag{2.2.22}$$

which serve to localize the fields in the vector multiplets, and to gauge fix the action. To do this, we need to include in δ the normal BRST charge δ_B. The fact that it is possible to do this all at once is a consequence of the fact that these two complexes commute. The action of the BRST supercharge is obviously zero on the gauge invariant term 2.2.21. We will choose its effect on 2.2.22 to yield the gauge fixing term

$$\delta_{\mathrm{B}}V_{\mathrm{BRST}} = \bar{c}\nabla_{\mu}D^{\mu}c + b\nabla^{\mu}A_{\mu} \tag{2.2.23}$$

The action of the supercharge δ on 2.2.22 can be absorbed into the definition of the ghost field c, as long as we choose

$$\delta\bar{c} = 0 \tag{2.2.24}$$

The resulting functional can be shown, with a little algebra, to be [19]

$$\delta V_{gauge} = \mathrm{Tr}'\left(\frac{1}{2}F^{\mu\nu}F_{\mu\nu} + D_{\mu}\sigma D^{\mu}\sigma + (D+\sigma)^2 + i\lambda^{\dagger}\gamma^{\mu}\nabla_{\mu}\lambda + i[\lambda^{\dagger},\sigma]\lambda - \frac{1}{2}\lambda^{\dagger}\lambda\right) \tag{2.2.25}$$

To localize the fields in the matter multiplets we could take

$$V_{matter} = (\delta\psi)^{\dagger}\psi + \psi^{\dagger}(\delta\psi^{\dagger})^{\dagger} \tag{2.2.26}$$

The functional resulting from this procedure is

$$\delta V_{matter} = \partial_{\mu}\phi^{\dagger}\partial^{\mu}\phi + i\phi^{\dagger}v^{\mu}\partial_{\mu}\phi + \phi^{\dagger}{\sigma_o}^2\phi + \frac{1}{4}\phi^{\dagger}\phi + F^{\dagger}F + \psi^{\dagger}\left(i\not{\nabla} - i\sigma_o + \left(\frac{1+\not{v}}{2}\right)\right)\psi \tag{2.2.27}$$

which explicitly depends on the vector field v_{μ} defined in 1.3.4. One could use this functional to compute the equivariant Euler class, but we will find it more convenient to use the action 1.3.54.

2.2.3 Zero Modes

The bosonic fields appear in the localizing action 2.2.25 as a sum of squares. This was achieved using integration by parts on S^3 [19]. One can then read off the zero modes: the bosonic configurations on which this action vanishes. We would prefer to take a more general approach. The zero locus of the supercharge used to localize the action coincides with the set of bosonic configurations for which the gaugino variation vanishes. Examination of 2.2.4 indicates that the conditions are

$$(\star F)_\mu = D_\mu \sigma \tag{2.2.28}$$

$$\mathrm{D} = -\sigma \tag{2.2.29}$$

dotting the first equation with D^μ and using the Bianchi identity

$$D_{[\mu} F_{\nu\rho]} = 0 \tag{2.2.30}$$

we get the condition

$$D^\mu D_\mu \sigma = 0 \tag{2.2.31}$$

This is a negative definite operator acting on σ so the vanishing implies

$$D_\mu \sigma = 0 \tag{2.2.32}$$

and so from above

$$F_{\mu\nu} = 0 \tag{2.2.33}$$

Since S^3 is connected and simply connected, this implies

$$A_\mu = 0, \qquad \sigma = -D = \sigma_0 = \text{const} \tag{2.2.34}$$

This is quite a simple zero locus, reminiscent of the one obtained for theories on S^4 in [25].

Note that the remaining gauge freedom of the theory allows us to diagonalize the zero mode σ_0. Henceforth, we will denote this diagonalized matrix as a. This matrix defines a point in the Cartan subspace of the algebra $\mathfrak{g}$ of the gauge group G into which σ_0 can be rotated. Its action on a field X, in a representation R, is diagonalized by choosing a basis for R in terms of its weights. We will denote this action as $\rho(a)$. The weights of the adjoint representation are the roots of the algebra $\mathfrak{g}$.

The zero locus for the matter fields with the action 1.3.54 is clearly

$$\phi = 0, \qquad F = 0 \tag{2.2.35}$$

This is the result of the conformal mass term. Note that this means, in particular, that the form of the superpotential does not affect the localization computation directly. The superpotential does restrict the choice of global symmetries, and thus indirectly affects the action of the theory on S^3 (see 1.1.6 and 1.3.1).

The modes found above parametrize the space $\mathcal{K}_Q$. According to the prescription in 2.1 they must be integrated over. Evaluating the original action for the gauge theory on S^3 with only these modes turned on yields "classical" contributions to the localization calculation. These are discussed in 2.3.2.

2.2.4 Fluctuations

We now turn to the evaluation of the path integral for the nonzero modes. As explained above, the remaining action for such modes is quadratic in the fields. This is a free action for which the evaluation of the path integral reduces to the computation of a determinant. The relevant operator is the first- or second-order pseudodifferential operator acting on a field. This operator can depend on the supersymmetry employed, including any central charges, and on the zero modes.

Below is a list of the relevant operators for the localizing functional given in 2.2.2 after taking into account the space of zero modes in 2.2.3. These are the result or expanding the actions 1.3.54 and 1.3.55 around the zero modes.

$$D_{\text{vector}} = \Delta_{\text{vector}} - [\cdot, a]^2 \tag{2.2.36}$$

$$D_{\text{vector multiplet scalars}} = \Delta_{\text{scalar}} \tag{2.2.37}$$

$$D_{\text{vector multiplet fermions}} = i \not{\nabla} - i[\cdot, a] - \frac{1}{2} \tag{2.2.38}$$

$$D_{\text{vector multiplet ghosts}} = \Delta_{\text{scalar}} \tag{2.2.39}$$

$$D_{\text{chiral multiplet scalars}} = \Delta_{\text{scalar}} + \rho(a)^2 - i\rho(a) + 2i(\Delta - \frac{1}{2})\rho(a) + \Delta(2-\Delta) \tag{2.2.40}$$

$$D_{\text{chiral multiplet fermions}} = i \not{\nabla} - i\rho(a) + (\Delta - \frac{1}{2}) \tag{2.2.41}$$

We also have an alternative set of operators for the chiral multiplet which would be appropriate had we chosen to use 2.2.27([3, 19])

$$D^{\text{alt}}{}_{\text{chiral multiplet scalars}} = \Delta_{\text{scalar}} + 2i(1-\Delta)v^{\mu}\partial_{\mu} + \Delta^2 + \rho(a)^2 \tag{2.2.42}$$

$$= -l^i l_i - 2i(1-\Delta)l_3 + \Delta^2 + \rho(a)^2 \tag{2.2.43}$$

$$= 4L^i L_i - 4(1-\Delta)L_3 + \Delta^2 + \rho(a)^2 \tag{2.2.44}$$

$$D^{\text{alt}}{}_{\text{chiral multiplet fermions}} = i \not{\nabla} - i\rho(a) + \frac{1}{2} + (1-\Delta) \not{v} \tag{2.2.45}$$

$$= i\gamma^i l_i - i\rho(a) - 1 + (1-\Delta)\gamma^3 \tag{2.2.46}$$

$$= -4\vec{L}\cdot\vec{S} + 2(1-\Delta)S^3 - i\rho(a) - 1 \tag{2.2.47}$$

Note that, as explained in 2.1, the specific functional used for this part of the localization computation is irrelevant. The superpotential for the chiral superfields does not contribute to these operators.

In addition to these, we must take into account the gauge fixing condition and the delta function introduced by integrating over the field b. Since what appears in the path integral measure is A_μ not $\nabla^\mu A_\mu$, we incur an additional determinant from the change of variables. We will parametrize the one form A_μ as the sum of a coclosed one form and a divergence

$$A_\mu = \nabla_\mu \tilde{\phi} + B_\mu \tag{2.2.48}$$

The constraint from setting Lorentz gauge then forces

$$\nabla^2 \tilde{\phi} = 0 \tag{2.2.49}$$

which, considering the relationship to the one form field A_μ allows us to set $\tilde{\phi}$ to zero at the expense of a factor

$$\frac{1}{\sqrt{\text{Det}_{\text{BRST}}}} \tag{2.2.50}$$

$$\text{Det}_{\text{BRST}} = \text{Det}(\nabla^2) = \text{Det}(\Delta_{\text{scalar}}) \tag{2.2.51}$$

The auxiliary fields D and F can be integrated over by setting them to the values specified by their equations of motion

$$F = 0, \qquad D = -\sigma \tag{2.2.52}$$

thus eliminating them from the action. To compute the path integral we must therefore evaluate the following expressions: for the gauge multiplet

$$Z_{\text{1 - loop}}^{\text{gauge multiplet}}(a) = \frac{\text{Det}\left(D_{\text{gauge multiplet fermions}}\right)\text{Det}\left(D_{\text{gauge multiplet ghosts}}\right)}{\sqrt{\text{Det}\left(D_{\text{vector}}\right)\text{Det}\left(D_{\text{gauge multiplet scalars}}\right)\text{Det}(\nabla^2)}} \tag{2.2.53}$$

and for a chiral multiplet

$$Z_{\text{1 - loop}}^{\text{chiral multiplet}}(a, \Delta) = \frac{\text{Det}\left(D_{\text{chiral multiplet fermions}}\right)}{\sqrt{\text{Det}\left(D_{\text{chiral multiplet scalars}}\right)}} \tag{2.2.54}$$

Note that cancelations in the vector multiplet immediately yield

$$Z_{\text{1 - loop}}^{\text{gauge multiplet}}(a) = \frac{\text{Det}\left(D_{\text{gauge multiplet fermions}}\right)}{\sqrt{\text{Det}\left(D_{\text{vector}}\right)}} \tag{2.2.55}$$

The remaining calculation is performed in A, and the results presented in the next section.

2.3 The Matrix Model

Having assembled all the components entering the localization calculation 2.1.8, we are now ready to present the complete result. The matrix model described in this section can be used to calculate exact expectation values of supersymmetric operators in an $\mathcal{N} = 2$ superconformal gauge theory in $2+1$ dimensions assuming one can correctly identify the R-symmetry charges of all dynamical fields.

2.3.1 The Integration Measure

The analysis in 2.2.3 implies that for the class of $\mathcal{N} = 2$ theories under consideration, the zero locus of the supersymmetry generator Q is the set of constant modes on S^3 of the scalar and auxiliary scalar, $\sigma_0 = -D_0$, in every vector multiplet. This constant matrix is in the adjoint representation of the gauge group. If the vector multiplet in question is dynamical, the constant matrix must be integrated over. Denote the gauge group, which may be a direct product, for the dynamical vector multiplet by G. Then the integration measure dictated by the path integral is

$$\frac{1}{\mathrm{Vol}(G)} da|_{a \in \mathrm{Ad}(\mathfrak{g})} \tag{2.3.1}$$

We can use the residual gauge symmetry, the freedom to perform constant gauge transformations on S^3, to rotate a into the Cartan subalgebra. This is standard practice in analyzing matrix models. The resulting measure is

$$\frac{1}{|\mathcal{W}|} \left(\prod_{\rho \in \mathrm{roots}(\mathfrak{g})} \rho \right) \prod_{i=1}^{\mathrm{rank}(G)} d\lambda_i \tag{2.3.2}$$

where $\mathcal{W}$ is the Weyl group of G and the expression in parentheses is the Vandermonde determinant. The λ_i parametrize the Cartan subspace of $\mathfrak{g}$. One can identify them with the eigenvalues of the matrix a. For $G = U(N)$ the expressions are

$$|\mathcal{W}| = N! \tag{2.3.3}$$

$$\prod_{\rho \in \mathrm{roots}(\mathfrak{g})} \rho = \prod_{i<j} (\lambda_i - \lambda_j)^2 \tag{2.3.4}$$

2.3.2 Classical Contributions

The functionals for gauge theories on S^3 described in 1.3.5 vanish in the background of zero-modes found in 2.2.3. However, the matrix model can receive classical contributions, as defined in 2.1, from the Chern-Simons functional 1.1.80 and from the background Fayet-Iliopoulos term 1.1.84. Taking into account the volume of S^3, The contribution of a level k Chern-Simons term is

$$e^{-i\pi k \mathrm{Tr}\left(a^2\right)} = e^{-i\pi k \sum_i \lambda_i^2} \tag{2.3.5}$$

Note that the matrix model with this contribution and the one loop contribution of the vector multiplets in the next section can be used to compute expectation values in pure Chern-Simons theory, since in that case the additional fields in the $\mathcal{N} = 2$ vector multiplet are all auxiliary (see 1.1.5). A matrix model for pure Chern-Simons theory was written down in [38] and agrees with our results.

The Fayet-Iliopoulos term with coefficient η contributes

$$e^{2\pi i\eta \mathrm{Tr}(a)} \tag{2.3.6}$$

which, for gauge group $U(N)$ is

$$e^{2\pi i\eta \sum_i \lambda_i} \tag{2.3.7}$$

2.3.3 One-Loop Contributions

For a representation R of a group G, we use the notation

$$\det_R f(a) = \prod_{\rho\in \mathrm{weights}(R)} f\left(\rho(a)\right) \tag{2.3.8}$$

As shown in A, the expression for the equivariant Euler class appearing in the localization formulas in 2.1, or equivalently, the one-loop contribution to the path integral due to a dynamical vector multiplet with gauge group G is given by

$$\det_{\mathrm{Ad}(\mathfrak{g})}\left(\frac{2\sinh(\pi a)}{a}\right) = \prod_{\rho\in \mathrm{roots}(\mathfrak{g})} \frac{2\sinh(\pi\rho(a))}{\rho(a)} \tag{2.3.9}$$

For a $U(N)$ gauge group

$$\rho(a) = \lambda_i - \lambda_j, \qquad i,j = 1...N \tag{2.3.10}$$

where λ_i is the i'th eigenvalue of the matrix a. So the expression for the one loop contribution, in terms of eigenvalues, is

$$Z_{\text{1 - loop}}^{\text{gauge multiplet}}(a) = \prod_{i<j} \frac{4\sinh^2\left(\pi\left(\lambda_i - \lambda_j\right)\right)}{\left(\lambda_i - \lambda_j\right)^2} \tag{2.3.11}$$

Similarly, the expression for the one loop contribution from a dynamical chiral multiplet is

$$Z_{\text{1 - loop}}^{\text{chiral multiplet}}(a,\Delta) = \prod_{\rho\in R} \exp\left(\ell\left(z(\rho(a),\Delta)\right)\right) \tag{2.3.12}$$

Note that this implies that an $\mathcal{N} = 2$ chiral multiplet with $\Delta = 1$ does not contribute to the matrix model. Such a multiplet appears as part of the $\mathcal{N} = 4$ vector multiplet. Also, we include the possibility of adding real mass terms by shifting $\rho \to \rho + m$.

For the fundamental representation of $U(N)$ we have

$$\rho(a) = \lambda_i, \qquad i, j = 1 \ldots N \tag{2.3.13}$$

so the expression becomes

$$Z_{\text{1 - loop}}^{\text{fundamental chiral multiplet}}(a, \Delta) = \prod_{i=1}^{N} \exp\left(\ell\left(z(\lambda_i, \Delta)\right)\right) \tag{2.3.14}$$

When such multiplets appear in complete hypermultiplet pairs in representations R and $\bar{R}$, and the symmetries of the theory guarantee that the conformal dimension, Δ, is the classical one, 1/2, the expression simplifies to

$$Z_{\text{1 - loop}}^{\text{hypermultiplet}}(a, \Delta) = \prod_{\rho \in R} \frac{1}{2\cosh\left(\pi \rho(a)\right)} \tag{2.3.15}$$

This is the expression which is applicable for theories with $\mathcal{N} \geq 3$ supersymmetry. A real mass term changes this to

$$Z_{\text{1 - loop}}^{\text{hypermultiplet}}(a, \Delta, m) = \prod_{\rho \in R} \frac{1}{2\cosh\left(\pi(\rho(a) + m)\right)} \tag{2.3.16}$$

2.3.4 Observables

The simplest Q closed observable is the partition function of the theory. Note that this can be a very nontrivial function of the various classical contributions 2.3.2 and the deformation parameters 2.3.3. Note that the overall normalization for this calculation is defined by the zeta function prescription given in A.

$\mathcal{N} = 2$ theories have BPS observable supported on curves. The simplest of these is the Wilson line

$$W = \frac{1}{\dim R} \mathrm{Tr}_R \left(\mathcal{P}\exp\left(\oint_\gamma d\tau \left(i A_\mu \dot{x}^\mu + \sigma |\dot{x}| \right) \right) \right) \tag{2.3.17}$$

The data involved is a curve γ, parametrized by $x^\mu(\tau)$, and a representation R of the gauge group G. The symbol $\mathcal{P}$ denotes path ordering. The supersymmetry variation of the operator is

$$\delta W \propto -\frac{1}{2} \eta^\dagger \left(\gamma_\mu \dot{x}^\mu + |\dot{x}| \right) \lambda + \frac{1}{2} \lambda^\dagger \left(\gamma_\mu \dot{x}^\mu - |\dot{x}| \right) \varepsilon \tag{2.3.18}$$

W is invariant if the following conditions are satisfied

$$\eta^{\dagger}\left(\gamma_{\mu}\dot{x}^{\mu}+|\dot{x}|\right)=0 \tag{2.3.19}$$

$$\left(\gamma_{\mu}\dot{x}^{\mu}-|\dot{x}|\right)\varepsilon=0 \tag{2.3.20}$$

This system can be solved to yield a 1/2 BPS loop by choosing $\eta = 0$ and γ an integral curve of the vector field $\epsilon^{\dagger}\gamma_{\mu}\epsilon$ corresponding to the Killing spinor ϵ given in 1.3.4. All such integral curves are great circles on S^3. With our choice of spinor the equation is simply

$$(\gamma_3-1)\varepsilon=0 \tag{2.3.21}$$

Note that the formal product of any number of such operators W_i is invariant when these are supported on γ_i which are all integral curves corresponding to the same spinor. The matrix model expression is easily seen to be

$$W(a)=\frac{1}{\dim(R)}\mathrm{Tr}_R\left(e^{2\pi a}\right) \tag{2.3.22}$$

Note the similarity to the result for the Wilson loop in [25].

2.3.5 Convergence

The integrals obtained from the elements described above can be divergent. Specifically, if a gauge group factor for the theory has an associated Chern-Simons term, then the integrals can always be defined by analytically continuing from a similar integral with a small negative imaginary Chern-Simons level k. We will implicitly assume this continuation in the calculations in 3.

When no Chern-Simons term is present, the divergence can be interpreted as arising from a an incorrect identification of the IR R-charge for the fields. Specifically it was shown in [17] that for large values of the eigenvalues entering the matrix model integration, or alternatively for a specific direction τ in the Cartan subalgebra, the asymptotic behavior of the integrand is as $\exp(-ta)$ where t is a coordinate in the direction τ and a is given by

$$a=-\sum_{\alpha}|\alpha(\tau)|+\sum_{\rho}|\rho(\tau)| \tag{2.3.23}$$

where the sum is over the roots α of the gauge group G and ρ over the weights of all representations (with multiplicity) of hypermultiplets charged under G. The condition that this asymptotic behavior describes the exponential decay of the integrand in all directions (i.e., $a > 0$) was further shown to be equivalent to the Gaiotto-Witten criterion for $\mathcal{N} = 4$ gauge theories to be of either "good" or "ugly" type [17]. That is, that the theory supports only monopole operators of dimension greater than or equal to 1/2. Theories of type "bad", those which seem to have unitarity violating monopole operators, also have divergent partition functions (see 1.2.6). We will not deal with such theories.

Chapter 3

Applications

In this chapter we review some applications of the localization procedure described in 2. In 3.1, we show how localization and the matrix model can be used to check various field theory dualities. The dualities in question apply to strongly coupled filed theories and checking their validity requires tools beyond perturbation theory.

In 3.2 we show how localization is used to compare the UV and IR description of maximally supersymmetric gauge theories in three dimensions. We also show an example of the computation of the expectation value of an observable, a supersymmetric Wilson loop, using localization and the matrix model. Finally, we comment on how localization can be used to check holographic dualities.

In 3.3 we review some applications of localization to studying aspects of the renormalization group flow of gauge theories. We describe how the matrix model can be used to compute exact coefficients for the R-symmetry mixing phenomenon described in 1.1.6. We show how the partition function on the three sphere, computed using localization, can be used to constrain the renormalization group flow and investigate the space of conformal field theories.

3.1 Dualities

Duality of interacting quantum field theories is a fascinating phenomenon. The duality can exchange weakly coupled theories with strongly coupled ones, relate theories with different gauge groups and different matter content and create a map between regular observables built out of the fundamental fields and disorder operators. In this section we show how localization and the matrix model can be used to test some duality conjectures involving gauge theories in three dimensions. We note that aspects of these dualities can be probed using the superconformal index [39, 40, 41, 42, 43] and monopole operators [44, 16, 45, 46, 14, 13] (see 1.2.6). In 3.1.1, we discuss mirror symmetry in three dimensions. In 3.1.2, we show results for a set of dualities between gauge theories in three dimensions which resemble Seiberg duality of $\mathcal{N} = 1$ gauge theories in four dimensions. In 3.1.3, we extend the discussion to theories with less supersymmetry and richer dynamics.

3.1.1 Mirror Symmetry

Mirror symmetry of three-dimensional gauge theories was first proposed in [9]. The original construction was for $\mathcal{N} = 4$ theories of the type introduced by Kronheimer [47]. Since then there have been extensions to more general settings, including $\mathcal{N} = 2$ versions [48] and more general quivers [49]. An understanding of this phenomenon from the field theory perspective has been offered in [50, 5] and from the string theory D-brane perspective in [51, 20]. There have been many checks of the specifics of the duality relation using monopole operators [14, 13], the superconformal index [42] and the S^3 partition function [52, 53]. We will begin by defining mirror symmetry and reviewing the original construction of the dual theories in [9].

The vacuum structure of theories with $\mathcal{N} = 4$ supersymmetry is highly constrained (see 1.1.6,1.2.1). Mirror symmetry, as introduced in [9], is a duality between different $\mathcal{N} = 4$ theories with the following properties

- The duality exchanges the $SU(2)_L$ and $SU(2)_R$ R-symmetries.
- The Higgs and Coulomb branches are exchanged.
- Mass and Fayet-Iliopoulos terms map to one another.

The moniker "mirror symmetry" refers to the fact that the dual theories can be viewed as coming from string theory compactifications on $\mathcal{M} \times S^1$ and $\mathcal{M}' \times S^1$, where $\mathcal{M}, \mathcal{M}'$ are mirror pairs and the two S^1's have inverse radii [9]. There are also connections between the $\mathcal{N} = 2$ version of this duality and mirror symmetry of $1 + 1$ dimensional sigma models and gauged Landau-Ginsburg models [54].

Kronheimer gauge theories ([47]) are based on the extended Dynkin diagram of the Lie algebra $\mathfrak{g}$ of a Lie group G of rank r. The gauge group is

$$K_G = \left(\prod_{i=1}^{r} U(n_i) \right) / U(1)_c \tag{3.1.1}$$

where $U(1)_c$ is the diagonal sum of the r $U(1)$s and n_i is the Dynkin index of the node i in the extended Dynkin diagram of G. The matter comes in the bifundamental representations

$$\oplus_{ij} a_{ij} (n_i, n_j) \tag{3.1.2}$$

where a_{ij} is 1 if the nodes i, j of the extended Dynkin diagram are connected by a link and 0 otherwise. This gives $2n_i$ fundamental flavors for each gauge group factor $U(n_i)$.

All the theories have a one quaternionic dimensional Higgs branch. In the absence of Fayet-Iliopoulos terms the metric on the Higgs branch is

$$\mathbb{C}/\Gamma_G \tag{3.1.3}$$

where Γ_G is the discrete $SU(2)$ subgroup corresponding to the group G. Adding Fayet-Iliopoulos terms can resolve the singularity. The available terms are ζ_i where $i = 1...r$ in the Cartan of G (what does this mean exactly?). There is also a rank$(K_G) = C_2(G) - 1$ quaternionic dimensional Coulomb branch ($C_2$is the dual Coxeter number). Classically, this branch looks like

$$\left(\mathbb{R} \times S^1\right)^{\text{rank}(K_G)} \tag{3.1.4}$$

, but quantum mechanically it is modified to the moduli space of a G instanton with the $\mathbb{R}^4$ factor removed [9]. The origin of the Coulomb branch corresponds to zero size instantons.

An example given in [9] is the model based on $G = SU(2)$. The gauge group is

$$K_G = U(1)^2/U(1)_c \tag{3.1.5}$$

This model is self-dual. It has one (quaternionic) dimensional Higgs/Coulomb branches. The metric on the Higgs branch is the Eguchi-Hanson metric [55]

$$ds^2{}_{\text{Eguchi Hanson}} = g^2(\vec{x})(dt + \vec{w} \cdot d\vec{x})^2 + g^{-2}(\vec{x})d\vec{x} \cdot d\vec{x} \tag{3.1.6}$$

$$g^{-2}(\vec{x}) = \sum_{i=1}^{2} \frac{1}{\left|\vec{x} - \vec{\zeta}_i\right|} \tag{3.1.7}$$

$$\vec{\nabla}(g^{-2}) = \vec{\nabla} \times \vec{w} \tag{3.1.8}$$

and

$$\vec{\zeta} = \vec{\zeta}_1 - \vec{\zeta}_2 \tag{3.1.9}$$

with the two Fayet-Iliopoulos triplets those of the original $U(1)$'s. The Coulomb branch is corrected by instantons to a Taub-NUT metric [56, 57]

$$ds^2{}_{\text{Taub NUT}} = g_{\text{TN}}{}^2(\vec{x})(dt + \vec{w} \cdot d\vec{x})^2 + g_{\text{TN}}{}^{-2}(\vec{x}) d\vec{x} \cdot d\vec{x} \tag{3.1.10}$$

$$g_{\text{TN}}{}^{-2}(\vec{x}) = g_{\text{classical}}{}^{-2} + \sum_{i=1}^{2} \frac{1}{|\vec{x} - \vec{m}_i|} \tag{3.1.11}$$

In the far IR, the classical gauge coupling vanishes and $g_{\text{TN}} \to g$. Thus the two metrics agree, assuming we make the substitution

$$\vec{m}_i \leftrightarrow \vec{\zeta}_i \tag{3.1.12}$$

The duality implies that the $SU(2)_F$ flavor symmetry of the Higgs branch is related to a hidden $SU(2)_{\tilde{F}}$ symmetry of the Coulomb branch. The Cartan of this hidden symmetry group is visible classically: it is the $U(1)_J$ symmetry 1.1.6.

Mirror symmetry for Abelian theories was shown in [50] to be the result of a single path integral identity. Implies all other Abelian mirror. A special case is the constant background which can be incorporated into the matrix model.

At the level of the partition function, the simple Abelian mirror symmetry described above reduces to the statement

$$Z_{U(1),N_f=1}(\eta) \leftrightarrow Z_{\text{free hypermultiplet}}(m), \qquad m \leftrightarrow \eta \tag{3.1.13}$$

which can easily be checked by noting

$$Z_{U(1),N_f=1}(\eta) = \int d\sigma \frac{e^{2\pi i \eta \sigma}}{\cosh(\pi\sigma)} \tag{3.1.14}$$

$$Z_{\text{free hypermultiplet}}(m) = \frac{1}{\cosh(\pi m)} \tag{3.1.15}$$

The two partition function are related in the appropriate way due to the basic Fourier identity

$$\mathcal{F}\left(\frac{1}{\cosh(\pi x)}\right)(p) = \int dx \frac{e^{2\pi i p x}}{\cosh(\pi x)} = \frac{1}{\cosh(\pi p)} \tag{3.1.16}$$

The non-Abelian case involves integrals of the type specified in 2.3. The matching of parameters in the dual theories is more complicated and was explicitly given in [49]. A general setup and comparison of the partition functions was considered in [52]. The quiver theories considered there

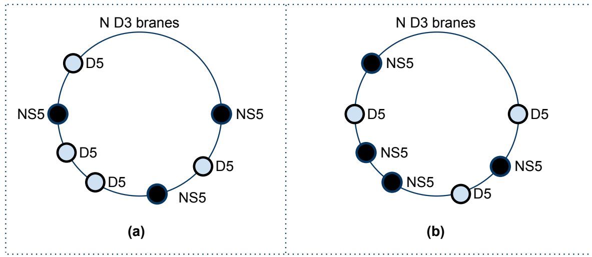

Figure 3.1: Mirror symmetry for an elliptical quiver, as realized in the brane construction. Figure (a) shows a $U(N)^3$ $\mathcal{N}=4$ gauge theory with 1,1 and 2 fundamental flavors for the three gauge groups respectively. There is also one set of bifundamental hypermultiplets for each gauge group pair. Figure (b) shows the dual theory with product gauge groups and fundamental hypermultiplets: $U(N), N_f = 1$,$U(N), N_f = 1$,$U(N), N_f = 1$,$U(N), N_f = 0$, again with bifundamental hypermultiplets, now for every adjacent pair of gauge groups.

are given by the following data

- The theory has a gauge group $G = U(N)^n$ Every factor $U(N)$ is associated with a set of N coincident D3 branes in Type IIB string theory. Branes associated to adjacent factors end on the same NS5 brane, of which there are n in total. The dimension along which the five-branes are spaced is compactified to a circle, and so the first and last factors are considered adjacent.

- For every gauge group factor there are v_i fundamental hypermultiplets, $v_i \geq 0$. These are associated with v_i D5 branes intersecting the i'th set of D3 branes.

- There is an additional bifundamental hypermultiplet for every adjacent pair of gauge group factors. These come from fundamental strings crossing the NS5 branes.

In the mirror theory, the D5 and NS5 branes are exchanged. The gauge group is $G = U(N)^v$ where $v = \sum_i v_i$. For every i there is a fundamental hypermultiplet associated to the j'th gauge group factor, where $j = \sum_{l=1}^{i-1} v_l$ and for the first factor we sum l from 1 to n. Note that some of the v_i's may vanish, so two i's may contribute a fundamental hypermultiplet to the same gauge group factor. As before, there is an additional bifundamental hypermultiplet for every adjacent pair of gauge group factors. Mirror symmetry for these theories can be inferred from the $SL(2,\mathbb{Z})$ duality of the type IIB string theory used in the brane construction. Specifically, the S generator exchanges D5 and NS5 branes and leaves D3 branes invariant, thus changing the gauge group and matter content of the theory in the manner described above. The T generator (or rather ST) can be used to turn D5 branes into $(1,1)$ branes (see 1.2.8) without acting on NS5 branes. A demonstration is provided in 3.1.

A derivation of the equality between the partition functions for gauge theories related by mirror symmetry was given in [52]. We now review the elements of this proof. The Fourier identity 3.1.16 made possible the comparison of partition functions related to Abelian mirror symmetry. In fact, all partition functions of Abelian mirror pairs can be shown to be equivalent using this one identity [50, 52]. The key to relating partition function of non-Abelian theories is the following identity for hyperbolic functions [52]

$$\frac{\prod_{i<j}\sinh(x_i - x_j)\sinh(y_i - y_j)}{\prod_{i,j}\cosh(x_i - y_j)} = \sum_\rho (-1)^\rho \prod_i \frac{1}{\cosh(x_i - y_{\rho(i)})} \tag{3.1.17}$$

where ρ is the set of permutations on the index i and $(-1)^\rho$ its signature. This is a version of the Cauchy determinant formula.

Let α be an index enumerating the D3 brane segments described above, and denote

$$d^N\sigma_\alpha = d{\sigma_\alpha}^1 ... d{\sigma_\alpha}^N \tag{3.1.18}$$

Since a pair of NS5 branes contribute a vector multiplet and bifundamental hypermultiplets to the theory, it is convenient to write the contribution to the partition function of such a brane on which the α'th D3 segment terminates as

$$\frac{1}{N!}\frac{\prod_{i<j}\sinh\pi({\sigma_\alpha}^i - {\sigma_\alpha}^j)\sinh\pi({\sigma_{\alpha+1}}^i - {\sigma_{\alpha+1}}^j)}{\prod_{i,j}\cosh\pi({\sigma_\alpha}^i - {\sigma_{\alpha+1}}^j)} = \frac{1}{N!}\sum_\rho (-1)^\rho \prod_i \frac{1}{\cosh\pi({\sigma_\alpha}^i - {\sigma_{\alpha+1}}^{\rho(i)})} \tag{3.1.19}$$

$$= \frac{1}{N!}\sum_\rho (-1)^\rho \int d^N\tau_\alpha \prod_i \frac{e^{2\pi i {\tau_\alpha}^i ({\sigma_\alpha}^i - {\sigma_{\alpha+1}}^{\rho(i)})}}{\cosh(\pi {\tau_\alpha}^i)} \tag{3.1.20}$$

The contribution of a D3 brane to such a segment can be written as

$$\prod_i \frac{1}{\cosh(\pi\sigma^i)} = \int d^N\hat{\sigma} \prod_i \frac{\delta(\hat{\sigma}^i - \sigma^i)}{\cosh(\pi\sigma^i)} \tag{3.1.21}$$

$$= \int d^N\hat{\sigma} d^N\tau \prod_i \frac{e^{2\pi i \tau^i(\hat{\sigma}^i - \sigma^i)}}{\cosh(\pi\hat{\sigma}^i)} \tag{3.1.22}$$

where in both case additional auxiliary integration variables have been introduced in the second line. The complete partition function now takes the form [52]

$$Z = \int \prod_{a=1}^n \frac{1}{N!} d^N\sigma_a d^N\tau_a \sum_{\rho_a} (-1)^{\rho_a} \prod_i \frac{e^{2\pi i {\tau_a}^i ({\sigma_a}^i - {\sigma_{a+1}}^{\rho_a(i)})}}{I_{\alpha_a}({\sigma_a}^i, {\tau_a}^i)} \tag{3.1.23}$$

$$I_\alpha(\sigma,\tau) = \begin{cases} \cosh(\pi\sigma) & \alpha = \text{D5} \\ \cosh(\pi\tau) & \alpha = \text{NS5} \end{cases} \tag{3.1.24}$$

which can be shown to be invariant under the exchange of D5 and NS5 brane contributions by relabeling indices [52].

A stronger check involves including all deformation parameters: mass and Fayet-Iliopoulos terms, to the two partition functions and evaluating how these map. This can be done in terms of a deformed D5 brane contribution which includes both "mass" parameters ω and Fayet-Iliopoulos parameters η

$$\int d^N\sigma_a d^N\tau_a \prod_i \frac{e^{2\pi i {\tau_a}^i({\sigma_a}^i - {\sigma_{a+1}}^i)}}{\cosh\pi({\sigma_a}^i + \omega_a)} e^{2\pi i \eta_a {\sigma_a}^i} \tag{3.1.25}$$

The dual contribution of an NS5 brane is

$$\frac{1}{N!}\int d^N\sigma_a d^N\tau_a \sum_\rho (-1)^\rho \prod_i \frac{e^{2\pi i {\tau_a}^i({\sigma_a}^i - {\sigma_{a+1}}^{\rho(i)})}}{\cosh\pi({\tau_a}^i + \eta_a)} e^{2\pi i \omega_a {\tau_a}^i} \tag{3.1.26}$$

Integrating out the auxiliary variables one finds

$$Z(\{\omega\},\{\eta\}) = \frac{1}{N!}e^{-2\pi i \eta_a \omega_a}\int d^N\sigma \frac{\prod_{i<j}\text{sinh}\pi({\sigma_a}^i - {\sigma_a}^j)\sinh\pi({\sigma_{a+1}}^i - {\sigma_{a+1}}^j)}{\prod_{i,j}\text{cosh}\pi({\sigma_a}^i - {\sigma_{a+1}}^j + \omega_a)} e^{2\pi i \eta_a \sum_i ({\sigma_{a+1}}^i - {\sigma_a}^i)} \tag{3.1.27}$$

From which one can identify physical Fayet-Iliopoulos parameters ξ_α

$$\xi_\alpha = \eta_{\alpha-1} - \eta_\alpha + \sum_{a_\alpha} \eta_{a_\alpha} \tag{3.1.28}$$

and mass parameters m

$$m_\alpha^{\text{bifundamental}} = \omega_\alpha \tag{3.1.29}$$

$$m_{a_\alpha}^{\text{fundamental}} = \omega_{a_\alpha} \tag{3.1.30}$$

Which are exchanged in the correct way under mirror symmetry [52, 49]. The phase in front of the integration in 3.1.27 can be though of as a BF type coupling for the vector and linear multiplets containing ω and η, respectively. These are indeed induced in the mirror transformation, although that fact is hard to see without evaluating the partition functions as was done above.

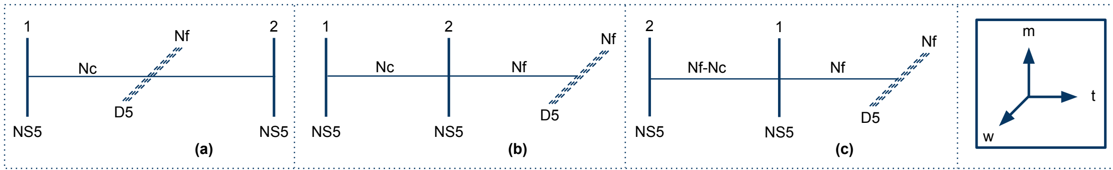

Figure 3.2: Brane manipulations in type IIB string theory which yield a naive dual. Solid vertical lines are NS5 branes. Horizontal lines are coincident D3 branes. Dashed lines are D5 branes. The legend indicates the compactification direction (t or x_6) and the directions of possible triplet mass (m) terms (3,4,5), and possible triplet FI (w) terms (7 8 9). Directions (0 1 2) are common to the world volume of all branes and are suppressed. We first move N_f D5 branes through the right NS5 brane, creating N_f D3 branes in the process. We then exchange the two NS5 branes, changing the number of suspended D3 branes in the interval.

3.1.2 Seiberg-Like Dualities

In [58], Seiberg proposed that the IR fixed point at the origin of moduli space of SQCD in four dimensions with gauge group $SU(N_c)$ and N_f massless flavors has dual descriptions in terms of "electric" and "magnetic" variables. For $N_f > 3N_c$ the theory is not asymptotically free and the IR fixed point is Gaussian. For $N_f < 3/2N_c$ the theory is infinitely strongly coupled in the IR, but there exists a dual IR free description in terms of "magnetic" variables, which are supersymmetric solitons in the original theory. In the window $3/2N_c < N_f < 3N_c$ the theory has a nontrivial RG fixed point and flows to an interacting supersymmetric CFT. This CFT has a dual description in terms of $SU(N_f - N_c)$ SQCD with N_f massless flavors, additional uncharged meson fields transforming in the $(N_f, \bar{N}_f)$ of the flavor symmetry and a superpotential coupling the quarks to the meson fields.

We will study several duality proposals for three-dimensional theories which resemble Seiberg duality. The similarities lie in the connection between the "electric" and "magnetic" gauge groups, such that the number of fundamental flavors appears in the rank of the "magnetic" gauge group, and in the fact that the flavor symmetries in the "electric" and "magnetic" theories are identified. This may be contrasted with mirror symmetry in three dimensions where flavor symmetries are realized as topological symmetries in the dual theory. For the Seiberg-like dualities, there are constraints relating the number of fundamental flavors and the rank of the gauge group. These constraints will also include the Chern-Simons level. Although $\mathcal{N} = 1$ in four dimensions corresponds to $\mathcal{N} = 2$ in three dimensions, in this section we will only analyze theories with at least $\mathcal{N} = 3$ supersymmetry in the three-dimensional sense. This is a necessary condition, but not a sufficient one, for identifying the conformal dimensions of the fields of a generic theory at the IR fixed point (see 1.2.6). The theories of interest all have brane constructions of the type introduced in 1.2.8.

Theory	$Z(\zeta)$
$U(1), N_f = 1$	$\frac{1}{2}\text{sech}[\pi\zeta]$
$U(1), N_f = 3$	$\frac{1}{16}(1+4\zeta^2)\text{sech}[\pi\zeta]$
$U(2), N_f = 3$	$\frac{1}{32}(1+4\zeta^2)\text{sech}[\pi\zeta]^2$
$U(2), N_f = 5$	$\frac{(1+4\zeta^2)^2(9+4\zeta^2)\text{sech}[\pi\zeta]^2}{36864}$
$U(3), N_f = 5$	$\frac{(1+4\zeta^2)^2(9+4\zeta^2)\text{sech}[\pi\zeta]^3}{73728}$

Table 3.1: Exact result of the matrix integral for a partition function deformed by an FI term ζ.

3.1.2.1 Naive Duality

Following the results of [20], one can try to manipulate a type IIB brane configuration like the ones described in 1.2.8 to obtain, from a given three-dimensional theory, a gauge theory with a gauge group of different rank. The basic manipulation, which was introduced in [20], is shown in figure 3.2 above. The constraints taken into account in this manipulation are preservation of the various “linking numbers” and the “s-rule” [20]. The critical step, moving two NS5 branes past each other, turns out to destroy the naive IR duality one would expect by reading off the gauge theories given by the initial and final brane configurations. In this section, we explore what the calculation of the deformed partition function implies for these theories. We write down a prescription for possible dual theories. We relate our findings to previous observations regarding such theories [15][16] and find that they concur.

The initial and final brane configurations depicted in figure 3.2 naively suggest an IR duality between a pair of $\mathcal{N} = 4$ quiver gauge theories in three dimensions. The putative dual pair is

1. $\mathcal{N} = 4, U(N_c)$ gauge theory with N_f hypermultiplets in the fundamental representation.
2. $\mathcal{N} = 4, U(N_f - N_c)$ gauge theory with N_f hypermultiplets in the fundamental representation.

We note that this pair resembles the $\mathcal{N} = 2$ dual pair suggested in [59]. The difference is in the amount of supersymmetry.

The integrals involved in the calculation of the partition functions, deformed by FI parameters and real mass terms, can be done exactly in this case [17]. Some examples are given in table 3.1. All these examples are “good” or “ugly”, since otherwise the partition function does not converge. It is clear that the results contradict the naive duality presented above. We can try and correct the statement of the duality “by hand”. The two sets of results suggest the following possible identification

- $U(1), N_f = 3 \oplus U(1), N_f = 1 \Leftrightarrow U(2), N_f = 3$
- $U(2), N_f = 5 \oplus U(1), N_f = 1 \Leftrightarrow U(3), N_f = 5$

where $\oplus$ indicates the sum of two decoupled theories. More generally, the partition function can be calculated with arbitrary FI (η) and mass terms (m_j). The result, derived in [52], is the following:

$$Z_{N_f}^{(N_c)}(\eta;m_j) = \binom{N_f}{N_c}\left(\frac{i^{N_f-1}e^{\pi\eta}}{1+(-1)^{N_f-1}e^{2\pi\eta}}\right)^{N_c}\left(\prod_{j=1}^{N_c} e^{2\pi i\eta m_j}\right)\left(\prod_{j=1}^{N_c}\prod_{k=N_c+1}^{N_f} 2\sinh\pi(m_j-m_k)\right)^{-1}\Bigg|_{\{m_j\}} \tag{3.1.31}$$

where the bar at the end denotes symmetrization over the m_j. As shown in there, the equivalence noted above continues to hold in general. Namely:

$$Z_{2N-1}^{(N)}(\eta;m_j) = Z_1^{(1)}(-\eta;m_1+...+m_{2N-1})Z_{2N-1}^{(N-1)}(-\eta;m_j) \tag{3.1.32}$$

Note that a $U(1)$ theory with a single charge 1 hypermultiplet is equivalent to a *free theory* of a single twisted hypermultiplet [50]. The appearance of decoupled sectors might seem like a surprising result, especially in light of the fact that the other proposed dualities, discussed later, have no such subtleties associated with them. However, we stress that brane manipulations do not provide a proof of the types of IR dualities we have been analyzing. Furthermore, the appearance of decoupled sectors in the IR theory has previously been predicted using the analysis of monopole operators [15, 16]. Namely, the $U(N_c)$ theory with $N_f = 2N_c - 1$ fundamental multiplets is "ugly", and contains a decoupled free sector generated by BPS monopole operators of dimension 1/2. It was argued in [15] that the "remainder" is dual to the IR-limit of a "good" theory, namely $U(N_c - 1)$ gauge theory with $N_f = 2N_c - 1$. The above computation of the partition functions provides a check of this duality. The analysis of monopole operators provides some understanding of why the naive $\mathcal{N} = 4$ duality cannot be true in general. The naive dual of a "good" theory ($N_f \geq 2N_c$) is either "bad", when $N_f > 2N_c + 1$, "ugly", when $N_f = 2N_c + 1$ (giving the examples above), or self-dual, when $N_f = 2N_c$. We can never get a duality between a distinct pair of "good" theories. If the naive dual is "ugly", we can try to correct the naive duality by adding some free fields to the original "good" theory; we have seen that this works. If the naive dual of a "good" theory is "bad", there is no way to correct the naive duality.

3.1.2.2 Giveon-Kutasov Duality

A duality very similar to the one considered in the previous section was suggested in [60]. The dual pair proposed there is

1. $\mathcal{N} = 2$ $U(N_c)_k$ gauge theory with N_f hypermultiplets in the fundamental representation (that is, N_f fundamental chiral multiplets Q_i and N_f antifundamental chiral multiplets $\tilde{Q}^j$) and no

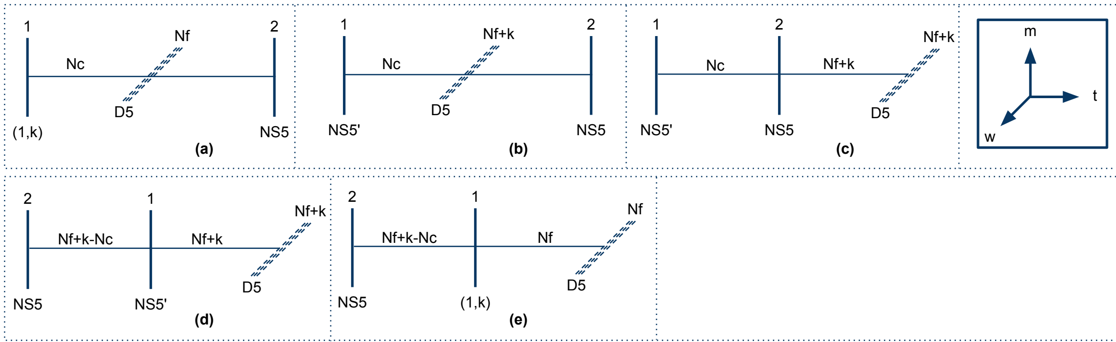

Figure 3.3: Brane manipulations in type IIB string theory which yield a duality between Chern-Simons theories. Panels (b) through (d) relate a pair of theories without CS terms. The deformations of the theory needed to go from (b) to (a) and from (d) to (e) are identified.

Yang-Mills term.

2. $\mathcal{N}=2$ $U(|k|+N_f-N_c)_{-k}$ gauge theory with N_f hypermultiplets in the fundamental representation (q_i and $\tilde{q}^j$), no Yang-Mills term and an $N_f \times N_f$ matrix of uncharged chiral fields, M^i_j, coupled via a superpotential of the form $M^i_j q_i \tilde{q}^j$.

where the subscript k denotes the level of the Chern-Simons term associated to the gauge group. It has been argued that $\mathcal{N}=2$ Chern-Simons theories with $N_f+|k|<N_c$ do not have a supersymmetric ground state. This is a consequence of the "s rule" 1.2.8. The dual theory would, in that case, have a negative rank gauge group. We will not consider such theories. A further consequence of the "s rule" is the equivalence of the Higgs branches of the two theories [60]. The Coulomb branch is, of course, absent in theories with a Chern-Simons term. In order to compare the partition functions, we use a version of the duality that preserves $\mathcal{N}=3$ supersymmetry by adding the corresponding superpotential to the electric theory (1). This has the effect of giving mass to the matrix M^i_j and producing the correct superpotential on the magnetic side. Figure 3.3 shows the brane manipulations that lead to the dual configurations. The naive version of the duality described in the previous section is the "$k=0$" version of this proposal (assuming we start with an $\mathcal{N}=3$ gauge theory with both a Yang-Mills and a Chern-Simons term). However, we will find that the calculation of the partition function supports the dualities suggested in [60] without alteration. Note that the $N_f=0$ case is level-rank duality of pure Chern-Simons theory 1.2.7.

To compare the partition functions, we would like to show that

$$Z^{(N_c)}_{k,N_f}(\eta) = Z^{(|k|+N_f-N_c)}_{-k,N_f}(-\eta) \tag{3.1.33}$$

where the LHS represents the partition function of a theory with N_c colors, N_f fundamental hypermultiplets, Chern-Simons level k, and an FI term η. This was proved for the case $N_f = 1$ in [17]. Numerical evidence for some other small N_f is presented in B. A more general formula was conjectured in [17], based on numerical evidence

$$Z^{(N_c)}_{k,N_f}(\eta; m_a) = e^{\mathrm{sgn}(k)\pi i(c_{|k|,N_f} - \eta^2)} e^{\sum_a (k\pi i {m_a}^2 + 2\pi i\eta m_a)} Z^{(|k|+N_f-N_c)}_{-k,N_f}(-\eta; m_a) \tag{3.1.34}$$

where:

$$c_{k,N_f} = -\frac{1}{12}(k^2 + 3(N_f - 2)k + a_{N_f}) \tag{3.1.35}$$

with:

$$a_{N_f} = \begin{cases} -1 & N_f = 1(\mathrm{mod}\ 4) \\ 2 & N_f = 2, 4(\mathrm{mod}\ 4) \\ -13 & N_f = 3(\mathrm{mod}\ 4) \end{cases} \tag{3.1.36}$$

which describes the background couplings induced in the duality as well as the overall phase. It was also shown that

$$Z^{(N_c)}_{k,N_f}(\eta) = 0, \qquad N_c > k + N_f \tag{3.1.37}$$

This is in line with the analysis of the brane picture which suggests that in such theories supersymmetry is spontaneously broken.

3.1.2.3 Fractional M-Brane (ABJ) Duality

A similar duality in the context of $\mathcal{N} = 6$ theories of fractional M2 branes was proposed in [61]. The relevant brane moves are shown in figure 3.4. These dual pairs are

1. $U(N + \ell)_k \times U(N)_{-k}$ with two bifundamental flavors.

2. $U(N)_k \times U(N + k - \ell)_{-k}$ with two bifundamental flavors.

for any $k \geq l$. This is nothing more than the duality studied in the last section, performed on only one of the factors in the gauge group. The fundamental flavors in the first gauge group retain their charge under the second gauge group after the duality transformation. Said differently, ignoring

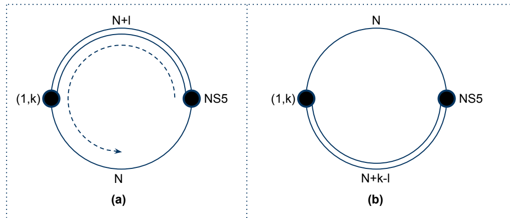

Figure 3.4: Brane manipulations in type IIB string theory which yield a duality between Chern-Simons theories of an elliptical quiver. An NS5 brane moves past a $(1, k)$ brane creating k and destroying l D3 branes in the process. Reproduced from [61].

the second gauge group, the flavor symmetry associated with having N fundamental flavors maps to itself under the duality transformation, and the theories where this symmetry is gauged by the second gauge group should also be equivalent.

The partition functions for a generalized version of this duality we considered in [17]. The partition function of a $U(N_1)_{k_1} \times U(N_2)_{k_2}$ theory with N_b bifundamental flavors was shown to satisfy

$$Z^{(N_1,N_2)}_{k_1,k_2,N_b}(\eta_1, \eta_2) = e^{\text{sgn}(k_1)\pi i(c_{|k_1|,N_bN_2} - {\eta_1}^2)} Z^{(N_1',N_2)}_{-k_1,k_2+N_bk_1,N_b}(-\eta_1, \eta_2 + N_b\eta_1). \tag{3.1.38}$$

This suggests that this theory is dual to a $U(|k_1| + N_bN_2 - N_1)_{-k_1} \times U(N_2)_{k_2+N_bk_1}$ gauge theory with N_b bifundamental flavors. If we consider the special case $N_1 = N + \ell$, $N_2 = N$, $k_1 = -k_2 = k$, and $N_b = 2$, the above equation becomes:

$$Z^{(N+\ell,N)}_{k,-k,2}(\eta_1, \eta_2) = e^{\text{sgn}(k)\pi i(c_{|k|,2N} - {\eta_1}^2)} Z^{(N+k-\ell,N)}_{-k,k,2}(-\eta_1, \eta_2 + 2\eta_1), \tag{3.1.39}$$

which is just the ABJ duality. One can continue to play the same game, performing the duality node-wise, for larger quivers and more complicated matter representations. Giveon-Kutasov duality therefore extends to a much larger class of dualities between $\mathcal{N} = 3$ Chern-Simons matter theories. Note that knowing the induced background Chern-Simons couplings is crucial for identifying the dual. These background couplings are an additional contribution to the bare Chern-Simons terms for the neighboring nodes.

The

3.1.3 Duality in $\mathcal{N} = 2$ Theories

Dualities involving $\mathcal{N} = 2$ gauge theories are often complicated by the possibility of anomalous dimensions for the chiral superfields. The simplest example of such a duality is the equivalence of

$\mathcal{N}=2$ SQED ($U(1)$ gauge theory with charge 1 and charge -1 chiral multiplets) and the theory of three chiral multiplets X,Y,Z with a superpotential XYZ (henceforth the XYZ theory). The accidental S_3 symmetry which acts by permutations of X,Y and Z guarantees that the R-charge of each of these chiral multiplets is 2/3 yielding the appropriate charge 2 for the superpotential [1].

One such duality was conjectured in [59]. The dual pairs are

- $\mathcal{N}=2$ $U(N_c)$ gauge theory with N_f fundamental flavors: N_f chiral multiplets Q_a and N_f antifundamental chiral multiplets $\tilde{Q}^a$.
- $\mathcal{N}=2$ $U(N_f-N_c)$ gauge theory with N_f fundamental flavors; ${N_f}^2$ uncharged chirals ${M_a}^b$ and two uncharged chiral multiplets $V_\pm$. The dual theory also has the following superpotential

$$\tilde{q}_a {M^a}_b q^b + V_+\tilde{V}_- + V_-\tilde{V}_+ \tag{3.1.40}$$

Both theories have a Yang-Mills term for the gauge groups and the duality holds in the IR limit where the theory is strongly coupled. The chiral fields $\tilde{V}_\pm$ are monopole operators 1.2.6. These parametrize the Coulomb branch of the dual theory. $V_\pm$ are mapped under the duality to the monopole operators of the first theory, while ${M^a}_b$ is mapped to $Q^a\tilde{Q}_b$. The latter identification is similar to Seiberg duality in four dimensions [58].

Both theories have the global symmetry group $SU(N_f)\times SU(N_f)\times U(1)_A\times U(1)_J$. The non-Abelian factors rotate the fundamental chiral flavors, as does the $U(1)_A$. The possible $U(1)_B$ symmetry which rotates the fundamental and ant-fundamental chirals in opposite directions is actually gauged, the gauge group being $U(N)$ not $SU(N)$ as in [58]. The charges of the fields are summarized below [59, 53]

Field	$SU(N_f)\times SU(N_f)$	$U(1)_A$	$U(1)_J$	$U(1)_{R-UV}$
Q_a	$(N_f,1)$	1	0	$\frac{1}{2}$
$\tilde{Q}^a$	$(1,\bar{N}_f)$	1	0	$\frac{1}{2}$
q^a	$(\bar{N}_f,1)$	-1	0	$\frac{1}{2}$
$\tilde{q}_a$	$(1,N_f)$	-1	0	$\frac{1}{2}$
${M^a}_b$	$(N_f,\bar{N}_f)$	2	0	1
$V_\pm$	$(1,1)$	$-N_f$	±1	$\frac{N_f}{2}-N_c+1$

The partition function of the first theory can be written [53]

$$Z^{(U)}_{N_f,N_c}(\eta;m_a;\tilde{m}_a;\mu)=\frac{1}{N_c!}\int\prod_{j=1}^{N_c}\left(d\lambda_j\prod_{a=1}^{N_f}e^{\ell(\frac{1}{2}+i\lambda_j+im_a+i\mu)+\ell(\frac{1}{2}-i\lambda_j-i\tilde{m_a}+i\mu)}\right)\prod_{i<j}(2\sinh\pi(\lambda_i-\lambda_j))^2 \tag{3.1.41}$$

where η is the coefficient for the FI term, the constants m_a and $\tilde{m}_a$ are the parameters in the Cartan of the background vector multiplets for the non-Abelian part of the flavor symmetry and μ is the parameter for $U(1)_A$. Note that the possibility of mixing between the UV R-symmetry and the global $U(1)_A$ symmetry, which could change the dimensions of the chiral superfields, is actually taken into account completely by promoting μ to a complex parameter. This is not surprising from the point of view of the supergravity construction of the action 1.3.1. The comparison of the partition functions for the dual theories is insensitive to this mixing, being an equality between two functions which depend on μ in a holomorphic fashion. The dual partition function is given by

$$Z^{(U)}_{N_f,N_f-N_c}(\eta;-m_a;-\tilde{m}_a;-\mu)e^{\ell(N_c-\frac{N_f}{2}-iN_f\mu+i\eta)+\ell(N_c-\frac{N_f}{2}-iN_f\mu-i\eta)}\prod_{a,b}e^{\ell(2i\mu+im_a-i\tilde{m}_b)} \tag{3.1.42}$$

where the factors outside of Z take into account the gauge neutral superfields M^a_b and $V_\pm$.

The equality between the two partition functions can be proven using identities for hyperbolic gamma functions [62, 63]. Specifically, it was shown in [53] that

$$\Gamma_h(z;i,i)=e^{\ell(1+iz)} \tag{3.1.43}$$

with the hyperbolic gamma function (at second and third argument fixed to be i)

$$\begin{aligned}\Gamma_h(z+\omega_1)&=2\sin(\frac{\pi z}{\omega_2})\Gamma_h(z)\\ \Gamma_h(z+\omega_2)&=2\sin(\frac{\pi z}{\omega_1})\Gamma_h(z)\\ \Gamma_h(z)\Gamma_h(\omega_1+\omega_2-z)&=1\end{aligned} \tag{3.1.44}$$

The integral defined in [62]

$$I^m_{n,(2,2)}(\mu;\nu;\lambda)=\frac{1}{\sqrt{-\omega_1\omega_2}^n n!}\int_{C^n}\prod_{1\le j<k\le n}\frac{1}{\Gamma_h(\pm(x_j-x_k))}\prod_{j=1}^{n}\left(e^{\frac{\pi i\lambda x_j}{\omega_1\omega_2}}\prod_{a=1}^{n+m}\Gamma_h(\mu_a-x_j)\Gamma_h(\nu_a+x_j)dx_j\right) \tag{3.1.45}$$

and the identity

$$I^m_{n,(2,2)}(\mu_a;\nu_a;\lambda) = I^n_{m,(2,2)}(\omega-\mu_a;\omega-\nu_a;-\lambda)\prod_{a,b=1}^{n+m}\Gamma_h(\mu_a+\nu_b)\times \tag{3.1.46}$$

$$\times\,\Gamma_h((m+1)\omega-\frac{1}{2}\sum_{a=1}^{n+m}(\mu_a+\nu_a)\pm\lambda)c(\lambda\sum_{a=1}^{n+m}(\mu_a-\nu_a))$$

with the identification

$$\Gamma_h(\pm z) = (2\sinh(\pi z))^{-2} \tag{3.1.47}$$

are sufficient to show the equality of the partition functions if one sets [53]

$$n = N_c,\quad m = N_f - N_c,\quad \mu_a = \frac{i}{2}-\tilde{m}_a+\mu,\quad \nu_a = \frac{i}{2}+m_a+\mu\quad \lambda = -2\eta \tag{3.1.48}$$

This also defines the analytic continuation of the partition functions to complex values of the background parameters needed when R-symmetry mixing is possible.

The $\mathcal{N}=2$ version of the Giveon-Kutasov duality 3.1.2.2 is between the following pair

- $\mathcal{N}=2$ $U(N_c)_k$ gauge theory with N_f flavors with no superpotential.
- $\mathcal{N}=2$ $U(|k|+N_f-N_c)_{-k}$ gauge theory with N_f flavors and ${N_f}^2$ uncharged chiral multiplets ${M_a}^b$, which couple through a superpotential $\tilde{q}^a {M_a}^b q_b$.

where the subscript for the unitary groups denotes a Chern-Simons level and the duality holds in the IR limit where the Yang-Mills terms can be ignored. These theories have no Coulomb branch, due to the presence of a Chern-Simons term, and the dual therefore does not have the addition superfields $V_\pm$. This duality can be inferred from the one in [59] by considering the latter with k additional hypermultiplets in the fundamental representation. When the additional hypermultiplets are given a large axial mass (the one associated with $U(1)_A$, they can be integrated out to leave the matter content above. The resulting theories also have a nonvanishing Chern-Simons level induced by the parity anomaly 1.2.3. This can be checked at the level of the partition function. For a flavor with large axial mass $M\to\pm\infty$ the contribution to the partition function behave as [53]

$$e^{\ell(\frac{1}{2}+i\lambda+iM)+\ell(\frac{1}{2}-i\lambda+iM)} \approx \exp\Bigg(\pm\Bigg(-i\pi\lambda^2-i\pi M^2-\pi M+\frac{i\pi}{12}\Bigg)\Bigg) \tag{3.1.49}$$

which exhibits the induced Chern-Simons level for the gauge group associated to the integration variable λ as well as an additional phase associated with framing 1.2.7. The integrals above can then be used to relate the two partition functions [53].

Both dualities have a version where unitary gauge groups are replaced by symplectic ones. The Aharony dual pairs are [59, 53]

- $\mathcal{N} = 2$ $Sp(2N_c)$ gauge theory with $2N_f$ chiral multiplets Q_a in the defining $(2N_c)$ representation.
- $\mathcal{N} = 2$ $Sp(2(N_f - N_c - 1))$ gauge theory with $2N_f$ fundamental chiral multiplets q_a and $N_f(2N_f - 1)$ uncharged chiral multiplets M^{ab} and a chiral multiplet Y, which couple through the superpotential

$$M^{ab} q_a q_b + Y\tilde{Y} \tag{3.1.50}$$

The superfield $\tilde{Y}$ parametrizes the Coulomb branch of the dual theory and Y is identified with the Coulomb branch of the original theory. There is a corresponding symplectic version of the Giveon-Kutasov duality. The partition functions for both dualities were compared using the same methods as used for the unitary groups and were found to agree [53].

3.2 Maximally Supersymmetric Gauge Theory

In this section we show how localization can be used to test the equivalence, in the IR limit, between two different gauge theories with maximal supersymmetry. In 3.2.1 we compare the partition functions of the two theories. In 3.2.2 we show an example of the computation of a supersymmetric Wilson loop in ABJM theory. In 3.2.3 we review some of the ways that localization has been used to test holographic dualities.

The maximally supersymmetric gauge theory in $2 + 1$ dimensions is $\mathcal{N} = 8$ super-Yang-Mills (SYM). This theory arises as the low energy action on a stack of coincident D2 branes in type IIA string theory. The theory has 16 conserved supercharges and a global $SO(7)$ symmetry which rotates the scalars in the directions transverse to the brane worldvolume. In terms of the $\mathcal{N} = 4$ multiplets described in 1.1.3, it comprises an $\mathcal{N} = 4$ vector multiplet, with a gauge group $U(N)$, and an adjoint $\mathcal{N} = 4$ hypermultiplet. The IR limit of $\mathcal{N} = 8$ SYM is described by the strong coupling limit of type IIA string theory - M-theory. The strong coupling limit of the D2 brane is the M-theory M2 brane. The action describing the fluctuations of an M2 brane is a superconformal $2 + 1$ dimensional theory with 16 supercharges (and 16 conformal supercharges) and an $SO(8)$ global symmetry rotating the scalars corresponding, as before, to the embedding of the brane in the 8 transverse directions. The theory corresponding to just one M2 brane is free. The action describing N M2 branes in flat space is a $U(N) \times U(N)$ gauge theory with Chern-Simons levels $k = 1$ and $\tilde{k} = -1$ for the two gauge groups and a set of two bifundamental hypermultiplets [64] (henceforth ABJM). This action is superconformal and has manifest $\mathcal{N} = 6$ supersymmetry. This is further enhanced to $\mathcal{N} = 8$ by monopole operators [16] (see 1.2.6). The generalization to $k = -\tilde{k} \geq 2$ describes a stack of M2 branes probing a $\mathbb{C}^4/\mathbb{Z}_k \times \mathbb{R}^3$ singularity.

3.2.1 The ABJM Partition Function

Under the classification in 1.2.6, $\mathcal{N} = 8$ SYM theory is "bad" and, therefore, cannot be directly analyzed use localization. However, one can construct a mirror dual theory using the brane configurations in 1.2.8. The mirror dual is identical to $\mathcal{N} = 8$ SYM, with the addition of a single hypermultiplet in the fundamental representation. This is dual to $\mathcal{N} = 8$ SYM because the latter can be built using an elliptical quiver with one NS5 brane. The S generator of the $SL(2, \mathbb{Z})$ duality group of type IIB string theory turns this configuration into one where the NS5 brane has been transformed into a D5 brane. This brane arrangement corresponds to the theory with an additional hypermultiplet. The additional hypermultiplet breaks the supersymmetry down to $\mathcal{N} = 4$, but maximal supersymmetry is restored in the IR limit. The supersymmetric deformations of the partition function for this theory are a Fayet-Iliopoulos term η and a real mass term ω for the adjoint hypermultiplet. The real mass for the additional fundamental hypermultiplet can be absorbed into η. The partition function, in terms of the matrix model is

$$Z_{\mathcal{N}=8 \text{ SYM}+1 \text{ fundamental}}(\eta, \omega) = \frac{1}{N!} \int d^N\sigma \frac{\prod_{i<j} \sinh^2(\pi(\sigma_i - \sigma_j)) e^{2\pi i \eta \sum_i \sigma_i}}{\prod_{i,j} \cosh(\pi(\sigma_i - \sigma_j + \omega)) \prod_i \cosh(\pi \sigma_i)} \tag{3.2.1}$$

The partition function for ABJM, including a Fayet-Iliopoulos term ζ and a real mass term ξ can be written down in terms of the matrix model

$$Z_{\text{ABJM}}(\eta, \omega) = \frac{1}{(N!)^2} \int d^N\sigma d^N\tilde{\sigma} \frac{\prod\limits_{i<j} \sinh^2(\pi(\sigma_i - \sigma_j))\sinh^2(\pi(\tilde{\sigma}_i - \tilde{\sigma}_j)) e^{2\pi i \zeta \sum\limits_i (\sigma_i + \tilde{\sigma}_i) + \pi i \sum\limits_i (\sigma_i^2 - \tilde{\sigma}_i^2)}}{\prod\limits_{i,j} \cosh(\pi(\sigma_i - \tilde{\sigma}_j + \xi)) \cosh(\pi(\sigma_i - \tilde{\sigma}_j - \xi))} \tag{3.2.2}$$

Note that there are two gauge groups with opposite Chern-Simons levels 1 and -1 respectively. There is also a contribution from the two sets of bifundamental hypermultiplets. This partition function was shown in [52] to be equal to the partition function for $\mathcal{N} = 8$ SYM, with the additional hypermultiplet, given above, provided one makes the identifications

$$\eta = \xi + 2\zeta, \quad \omega = \xi - 2\zeta \tag{3.2.3}$$

The proof utilizes only the identity 3.1.17 and Gaussian integration [52]. This is a powerful check of the IR equivalence of these two theories, and therefore of the identification of ABJM theory with the low energy theory on coincident M2 branes. Note that the version of ABJM considered above, the one describing M2 branes in flat space, is strongly coupled. ABJM with higher Chern-Simons levels can be perturbative. Calculations in the large N limit are described in 3.2.3.

3.2.2 A Supersymmetric Wilson Loop

A version of the supersymmetric Wilson loop described in 2.3.4 exists in ABJM theory. The operator is [65, 66, 67]

$$W = \frac{1}{N}\mathrm{Tr}\left(\mathrm{Pexp}\left(\oint d\tau\left(iA_\mu \dot{x}^\mu + M_A^B X^A X_B |\dot{x}|\right)\right)\right) \tag{3.2.4}$$

Here X^A are the scalar fields of the theory, of which there are four, and X_A are their adjoints. M_A^B is a constant Hermitian matrix which can be taken as $\mathrm{diag}(1, 1, -1, -1)$. In those papers, it was shown that this choice of M renders the Wilson loop 1/6 BPS, i.e., it preserves one real supersymmetry and one superconformal symmetry (in flat Minkowski space). That this is the same operator follows from the identification of σ with the second term above [68].

The matrix model expression for the general Wilson loop operator is

$$< W >= \frac{1}{Z|\mathcal{W}|\dim R}\int da\; e^{-i\pi k Tr a^2}\mathrm{Tr}_R(e^{2\pi a})\frac{\det_{Ad}(2\sinh(\pi a))}{\det_R 2\cosh(\pi a)} \tag{3.2.5}$$

and for ABJM, in terms of the eigenvalues λ_i and $\hat{\lambda}_i$

$$\rho_{i,j}^{(N,\bar{N})}(a) = \lambda_i - \hat{\lambda}_j \tag{3.2.6}$$

$$\rho_{i,j}^{(\bar{N},N)}(a) = -\lambda_i + \hat{\lambda}_j \tag{3.2.7}$$

The matrix model expression for the partition function is

$$Z = \int\left(\prod_i e^{-ik\pi({\lambda_i}^2 - \hat{\lambda}_i^2)} d\lambda_i d\hat{\lambda}_i\right)\frac{\prod_{i\neq j}\left(2\sinh\pi(\lambda_i - \lambda_j) 2\sinh\pi(\hat{\lambda}_i - \hat{\lambda}_j)\right)}{\prod_{i,j}(2\cosh\pi(\lambda_i - \hat{\lambda}_j))^2} \tag{3.2.8}$$

and for the Wilson loop in the fundamental representation

$$\langle W\rangle = \frac{1}{NZ}\int\left(\prod_i e^{-ik\pi({\lambda_i}^2-\hat{\lambda}_i^2)}d\lambda_i d\hat{\lambda}_i\right)\frac{\prod_{i\neq j}\left(2\sinh\pi(\lambda_i-\lambda_j)2\sinh\pi(\hat{\lambda}_i-\hat{\lambda}_j)\right)}{\prod_{i,j}(2\cosh\pi(\lambda_i-\hat{\lambda}_j))^2}\left(\sum_{i=1}^{N}e^{2\pi\lambda_i}\right) \tag{3.2.9}$$

To compare to perturbative calculations, care must be taken to choose the correct framing (see 1.2.7). Using the trivial framing, the expression for the Wilson loop above can be expanded in large N and small N/k to yield [19]

$$\langle W\rangle = 1+\left(\frac{5}{6}+\frac{1}{6N^2}\right)\frac{\pi^2N^2}{k^2}-\left(\frac{1}{2}-\frac{1}{2N^2}\right)\frac{i\pi^3N^3}{k^3}+... \tag{3.2.10}$$

which compares well with perturbative results.

3.2.3 Holography

ABJM theory has a large N holographic dual description in terms of M-theory on $AdS_4\times S^7/\mathbb{Z}_k$. In the 't Hooft limit of large N and fixed $\lambda = N/k$, this can be interpreted in terms of type IIA string theory on $\mathrm{AdS}_4\times\mathbb{CP}^3$ [64]. Operators of the ABJM CFT at large N therefore correspond to string/M-theory configurations in $9+1$ or $10+1$ dimensions. The S^3 partition function at strong 't Hooft coupling corresponds to the regularized supergravity action in AdS_4 [8]. One puzzling aspect of this statement is that the latter action is known to scale like $N^{3/2}$ at large N, whereas the partition function of a gauge theory is naively expected to scale as N^2. One can perform perturbative calculations to extract the leading order behavior of the planar (leading order at large N) partition function at weak coupling. In order to compare to the gravity result, however, a strong coupling calculation must be performed. Such a calculation is feasible using localization and the matrix model.

In [69], the authors solved the ABJM matrix model at large N and arbitrary λ. The S^3 free energy is related to the S^3 partition function

$$F_{S^3} = -\log|Z_{S^3}| \tag{3.2.11}$$

The solution in [69] provides an exact function for the free energy which interpolates between the weak and strong coupling results. The function is given implicitly using the following [8]

$$\lambda(\kappa) = \frac{\kappa}{8\pi}\,{}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,\frac{3}{2};-\frac{\kappa^2}{16}\right) \tag{3.2.12}$$

$$\partial_\lambda F_0(\lambda) = \frac{\kappa}{4} G^{2,3}_{3,3}\left(\begin{array}{ccc|c} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{-\kappa^2}{16} \\ 0 & 0 & -\frac{1}{2} & \end{array}\right) + \frac{\pi^2 i\kappa}{2}\,{}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,\frac{3}{2};-\frac{\kappa^2}{16}\right) \tag{3.2.13}$$

where F_0 denotes the planar partition function. This complicated relationship (G is a Meijer function) can be expanded at weak and strong coupling to yield [8] (here a factor of N^2 has been factored out of the exact free energy and the limit taken to recover F_0)

$$-\lim_{N\to\infty}\frac{1}{N^2}F_{S^3} \approx \begin{cases} -\log(2\pi\lambda) + \frac{3}{2} + 2\log(2), & \lambda \to 0 \\ \frac{\pi\sqrt{2}}{3\sqrt{\lambda}}, & \lambda \to \infty \end{cases} \tag{3.2.14}$$

which agrees with perturbative results at weak coupling, and supergravity results at strong coupling. One can explicitly see the $N^{3/2}$ scaling at large λ (recall $\lambda \sim N$).

The free energy can, in fact, be evaluated exactly for arbitrary N [8, 69]. This amounts to an "all genus" calculation in type IIA string theory [8]. Furthermore, the expectation value of Wilson loops, like the one described in 3.2.2, can be computed and compared to gravity results. The Wilson loop in 3.2.2 is 1/2 BPS from the $\mathcal{N} = 2$ point of view, but there exists an operator which preserves 6 supercharges (and 6 conformal supercharges) and is therefore 1/2 BPS from the $\mathcal{N} = 6$ point of view [70]. Its expectation value at strong coupling, indeed the entire interpolating function, can be computed using the same methods as for the partition function. The result for the 1/2 BPS Wilson loop in the fundamental representation (denoted with a single box) at strong coupling is [71]

$$\langle W_\square \rangle \sim \frac{i}{2\pi\sqrt{2\lambda}} e^{\pi\sqrt{2\lambda}} \tag{3.2.15}$$

Note that this operator preserves the right amount of supersymmetry to be dual to a fundamental type IIA string (or an M2 brane ending on the ABJM stack of branes in the M-theory description) [70] and the asymptotic behavior in 3.2.15 matches gravity calculations for the macroscopic fundamental string with worldsheet ending on a circle on the boundary of AdS_4 in $\mathrm{AdS}_4 \times \mathbb{CP}^3$ [70, 71].

3.3 Superconformal Fixed Points

In this section we show how localization can be used to study aspects of the renormalization group flow of gauge theories. In 3.3.1, we show how the matrix model was used to solve the long standing problem of computing the coefficients in the R-symmetry mixing of $\mathcal{N} = 2$ theories described in 1.1.6. In **??**, we review how the three sphere partition function, computed from the matrix model, can be used to put constraints on renormalization group flow. These constraints are similar in nature to those provided by the c-theorem in two dimensions and the a-theorem in four dimension. The validity of the constraints goes beyond the class of theories considered in this work, but the ability to check them using exact computations has led to renewed interest.

3.3.1 R-Symmetry Mixing

Recall that the superconformal algebra contains a distinguished $U(1)_{\mathrm{R}}$ R-symmetry, which is, in general, a linear combination of the global $U(1)$ symmetries of the theory, including the UV R-charge.

$$R_{\mathrm{IR}} = R_{\mathrm{UV}} + \sum_{U(1)_i} a_i F_i \tag{3.3.1}$$

The charge of the various chiral multiplets in the theory under this special $U(1)_{\mathrm{R}}$ determines their conformal dimension. The exact linear combination entering 3.3.1 is a dynamical question, which can be explored using perturbation theory, but which cannot be answered by examining the data available in the UV action alone. When the global symmetries entering 3.3.1 are all flavor symmetries, which are visible in the UV action, one can make some exact statements. An example of this is the result that the correct linear combination minimizes the two point correlation function of R-currents in the CFT [72]. This correlation function receives quantum corrections and cannot be evaluated using localization. We will review the alternative approach pioneered in [3]. Note that when the mixing involves accidental symmetries, which are not visible in the UV action, no explicit way of evaluating the coefficients in 3.3.1 is known. Indeed, identifying such accidental symmetries is a task in itself (see 1.2.6).

For the choice of supercharge in 2.2.1 the algebra on the three sphere is [3]

$$\{\delta, \delta^\dagger\} = M_{12} + R + Z \tag{3.3.2}$$

with the central charge Z getting contributions from the real mass deformations for the global $U(1)$

flavor symmetries F_i.

$$Z = \sum_{U(1)_i} (a_i - im_i) F_i \tag{3.3.3}$$

This deformation has an imaginary part corresponding to the usual deformation parameter m and a real part, a, which is the contribution of the global $U(1)$ symmetry to the IR superconformal R-symmetry. We previously called the sum of the real part of Z and the UV R-charge Δ. It is the conformal dimension of the scalar in the chiral multiplet on which the commutator acts. The parameters Δ_i and m_i enter the localization calculation only through the one loop contribution of the chiral multiplet, where they appear in the same combination as in 3.3.3 (see A). This holomorphic property of the partition function Z implies [3]

$$\partial_\Delta Z = i\partial_m Z \tag{3.3.4}$$

A similar conjecture could be made about the Fayet-Iliopoulos parameter ζ [3]. This would imply that the contribution of the $U(1)_{\mathrm{J}}$ current associated with the topological symmetry in 1.1.6 to the IR R-symmetry, which could change the conformal dimension of monopole operators, enters the partition function in the same sort of holomorphic combination (i.e., as a complex extension of ζ in the classical contribution noted in 2.3.2).

The utility of the observation above is that, in the CFT, the quantity given by

$$\frac{1}{Z}\,\partial_m Z|_{m=0,\Delta=\Delta_{\mathrm{IR}}} \tag{3.3.5}$$

is a one-point function, corresponding to the insertion of the operator multiplying m. This one point function is integrated over the three sphere. The expectation value of this operator can be nonzero in a CFT only if it mixes with the identity operator [3]. In [3], it was shown that this cannot happen for theories in which parity remains unbroken (since the operator in question is parity odd). Even when parity is broken, the expectation value of the identity operator is a real number and therefore [3]

$$\mathrm{Im}\left(\frac{1}{Z}\,\partial_m Z|_{m=0,\Delta=\Delta_{\mathrm{IR}}}\right) = 0 \tag{3.3.6}$$

which leads to the conclusion that,

$$\partial_{\Delta_i} |Z|^2\big|_{\Delta=\Delta_{\mathrm{IR}}} = 0 \tag{3.3.7}$$

For a theory with n global symmetry currents, these are n conditions, which, generically, suffices to determine the linear combination entering the IR R-symmetry exactly. For all known examples, the extremum implied in 3.3.7 is a (local) minimum.

Examination of the XYZ theory 3.1.3 in light of the above criterion trivially yields the correct dimensions for the chiral multiplets (which can be inferred from symmetry considerations). The conjectured criterion has also been tested for Abelian Chern-Simons theory at level k with N_f flavors where one gets for all chiral multiplets

$$\Delta_{U(1)\ \mathrm{CS}}(N_f) = \frac{1}{2} - \frac{N_f + 1}{2k^2} + O\left(\frac{1}{k^4}\right) \tag{3.3.8}$$

For $SU(2)$ Chern-Simons theory

$$\Delta_{SU(2)\ \mathrm{CS}}(N_f) = \frac{1}{2} - \frac{3\,(N_f + 1)}{8k^2} + O\left(\frac{1}{k^4}\right) \tag{3.3.9}$$

and for $SU(2)$ Chern-Simons theory with g adjoint flavors

$$\Delta^{\mathrm{Ad}}_{SU(2)\ \mathrm{CS}}(g) = \frac{1}{2} - \frac{4\,(g + 1)}{k^2} + O\left(\frac{1}{k^4}\right) \tag{3.3.10}$$

All of which agree with perturbative (large k) calculations [3].

3.3.2 The F-Theorem

In $1+1$ dimensions the central charge in the superconformal algebra plays a special role in constraining the renormalization group flow. In [73], Zamolodchikov showed that there is a quantity, $C(g_i, \mu)$, which depends on the running coupling constants, and the renormalization group scale, which is monotonically decreasing along the flow to the IR. The beta functions for the coupling constants are determined by the gradient of C and a metric determined by the two point functions. At fixed points, the function coincides with the central charge c. The quantity c also appears in the Weyl anomaly [74]

$$g^{\mu\nu}\langle T_{\mu\nu}\rangle = \frac{c}{24\pi}R \tag{3.3.11}$$

where R is the Ricci scalar. It was later suggested that the corresponding coefficient of the "a-type" Weyl anomaly, that is the coefficient of the Euler density, should play the same role for theories in four dimensions [75]. There is an increasing amount of evidence for the validity of this conjecture (a recent example is [76]). There is no Weyl anomaly in three dimensions, but it has been suggested that the free energy on the three sphere

$$F_{S^3} = -\log |Z_{S^3}| \tag{3.3.12}$$

should play the same role [77, 78, 3]. This conjecture, now known as the F-theorem, is based on the following observations

- The free energy is derived from the finite part of the partition function, which is unambiguous and can be calculated using localization. The quantity F_{S^3} appears to be always positive ([78]), which matches the behavior of Euclidean gravity calculations for AdS_4

$$F_{\mathrm{AdS}_4} = \frac{\pi L^2}{2G_N} \tag{3.3.13}$$

  where G_N is the effective Newton's constant and L the radius of AdS_4 [78]. It has been shown that L decreases along holographic RG flow in the leading supergravity approximation [79, 78, 80].

- As described in 3.3.1, the free energy (or equivalently the partition function) determines the R-charges of chiral multiplets at the IR fixed point. The free energy is (locally) maximized at the correct values. This resembles the result in four dimensions that the "a-type" anomaly determines the R-charges of chiral multiplets by a-maximization [81].

- The free energy is constant under exactly marginal deformations of the CFT [78]. This is also true for the "a-type" anomaly in four dimensions and the central charge in two dimensions [76]. This property of the free energy can be argued by noting that such a marginal deformation is, to first order, an insertion of the corresponding (integrated) local operator. The expectation value of such an insertion vanishes on S^3, as was argued in 3.3.1.

In [78] the authors showed that given a CFT in three dimensions with a gravity dual of the form $\mathrm{AdS}_{4\times}\mathrm{Y}$, and where the cone over Y is a toric Calabi-Yau fourfold (a four complex dimensional manifold with vanishing first Chern class and a torus action), the maximization of the free energy with respect to the trial R-charge Δ is the holographic dual of the Z-minimization procedure used to determine the volume of Y [82]. This, again, mirror four-dimensional results for the "a-type" anomaly [82]. They also find the $N^{3/2}$ scaling discussed in 3.2.3 for a larger class of quivers: non-chiral theories with an equal number of fundamental and antifundamental flavors and Chern-Simons levels which sum to zero. The behavior changes to $N^{5/3}$ when the levels do not sum to zero, also in agreement with the gravity dual [83, 78]. Finally, explicit tests of the F-theorem done in [78], which compare the free energy in the UV and in the IR, confirm the general statement

$$F^{IR}_{S^3} \leqslant F^{UV}_{S^3} \tag{3.3.14}$$

The calculation of the free energy using localization is, of course, restricted to $\mathcal{N} \geq 2$ theories. The F-theorem, however, is thought to hold for a general quantum field theory in three dimensions, as do the analogous conjectures in two and four dimensions.

Discussion and Summary

We have shown that localization can be used to reduce the calculation of BPS observable in $\mathcal{N} = 2$ SCFTs in three dimensions to matrix model computations. This was achieved, in part, by mapping the conformal theories to the three-sphere. A supersymmetric action on the three-sphere was constructed, first by guessing the appropriate deformation of the flat space action, and later by considering a particular supergravity coupling procedure. A supercharge and the corresponding Killing spinor were identified and used to deform the theory in such a way that the semi-classical approximation became exact. This allowed us to write the entire path integral calculation as a matrix model involving the remaining zero modes.

The partition function is well defined on the three-sphere and this allowed us to compare the highly nontrivial deformed partition functions of dual theories. We were able to extend some of the duality conjectures using node-wise duality with the background couplings visible in the matrix model. We saw that the partition function calculation also gives a concrete way of evaluating the R-symmetry mixing induced by renormalization group flow and to determine the IR dimensions of chiral operators.

The localization performed here is part of a large set of calculations for theories in various dimensions. The three-dimensional calculation has been performed on $S^2 \times S^1$ (the superconformal index) and on the squashed sphere. In principle, a similar calculation can be performed on any manifold where one can write down an action which preserves enough supersymmetry. The work presented here was inspired by the calculations on S^4. That calculation was for $\mathcal{N} = 2$ and $\mathcal{N} = 4$ theories in four dimensions. It would be interesting to see if that can be extended to $\mathcal{N} = 1$ theories. Such theories are relevant as extensions of the Standard Model.

Defect operators, which play a role in duality, can also be incorporated into the localization calculation. Monopole operators corresponding to such defects appear in the chiral ring of theories in three dimensions. It is possible that localization can be used to probe the structure of the chiral ring as it was to compute the conformal dimensions of these operators. There are also defect operators supported on curves and 2-manifolds. The former may be related by duality to the usual Wilson loop operators. Localization could be used to evaluated the expectation values of such operators and to provide evidence for their role in duality.

Finally, there has recently been increased interest in using the three sphere partition function to constrain the renormalization group flow. The F-theorem states that this quantity plays a role similar to the central charge in two dimensions and the a-type conformal anomaly in four dimensions. Related quantities, such as Renyi entropy, can also be calculated using localization. This is work in progress.

Appendix A

Regularized Determinants

In this appendix we compute various regularized determinants of differential operators that enter into the localization calculation 2. We begin by defining the zeta regularized determinant of a Hermitian pseudodifferential operator. We will follow the exposition in [84].

Let D be a first or second order Hermitian elliptic pseudodifferential operator acting on the space of sections of some bundle over S^3. We will consider only the Laplacian acting on scalars or vectors, and the Dirac operator acting on the complex spinors defined in 1.1.1. These operators have a discrete spectrum of real eigenvalues λ_i with degeneracy d_i. We may choose an orthonormal basis for the eigensections. The inner product is defined by the action on S^3. Define the generalized zeta function (here each eigenvalue appears with its degeneracy)

$$\zeta_\lambda(s) = \sum_i \lambda_i^{-s} \tag{A.0.1}$$

the zeta regularized determinant of D is defined to be [84]

$$\det(D) = \exp(-\frac{d\zeta_\lambda}{ds}|_{s=0}) \tag{A.0.2}$$

For the operators in question, this can be evaluated in terms of the Hurwitz zeta function (25.11 in [85])

$$\zeta_H(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \tag{A.0.3}$$

We will need the determinant of operators with eigenvalues $n+a$ and degeneracies $\alpha n^2 + \beta n + \gamma$.

These are given by

$$\mathrm{Det}(D(a)_{\{\alpha,\beta,\gamma\}}) = \exp\left(\left(-\gamma+\beta a+\alpha a^2\right)\zeta_H{}'(0,a) + \left(-\beta+2\alpha a\right)\zeta_H{}'(-1,a) - \alpha\zeta_H{}'(-2,a)\right) \tag{A.0.4}$$

Define

$$H(\alpha,\beta,\gamma,\eta) = \left(-\gamma+\beta a+\alpha a^2\right)\zeta_H{}'(0,a) + \left(-\beta+2\alpha a\right)\zeta_H{}'(-1,a) - \alpha\zeta_H{}'(-2,a) \tag{A.0.5}$$

The expression above can often be simplified using special function identities. It is often simpler, however, to simplify the 1-loop factor appearing in 2 by explicitly canceling modes between the bosonic and fermionic contributions. In one some situations, it is also useful to pair eigenvalues that contain radicals before writing down the expression for the determinant. We now carry out these procedures for the relevant operators. We will perform the calculation for fields with a given weight ρ.

A.1 Vector Multiplet

We would like to evaluate the following expression for the Gaussian path integral in the gauge sector 2.2.4

$$Z_{\text{1 - loop}}^{\text{gauge multiplet}}(a) = \frac{\mathrm{Det}\left(D_{\text{gauge multiplet fermions}}\right)}{\sqrt{\mathrm{Det}\left(D_{\text{vector}}\right)}} \tag{A.1.1}$$

The relevant operators are 2.2.4

$$D_{\text{vector}} = \Delta_{\text{vector}} - [\cdot,a]^2 \tag{A.1.2}$$

$$D_{\text{vector multiplet fermions}} = i\,\not\nabla - i[\cdot,a] - \frac{1}{2} \tag{A.1.3}$$

and the modes 1.3.3

$$\lambda_3(L,1) = (L+1)^2 \tag{A.1.4}$$

$$L = 1,2...,\infty \tag{A.1.5}$$

with degeneracy

$$D_3(L,1) = \frac{(2L+2)(L+2)!}{(L-1)!(L+1)(L+1)} = 2L(L+1) \tag{A.1.6}$$

and

$$\lambda^D{}_3(n) = \pm i(n+3/2), \qquad n = 0,1,...,\infty \tag{A.1.7}$$

with degeneracy

$$D_3^D(n) = (n+1)(n+2) \tag{A.1.8}$$

In the notation with $\rho(a)$ replacing $i[\cdot, a]$ we get the following result

$$Z_{\text{1 - loop}}^{\text{gauge multiplet}}(a)|_\rho = \frac{\prod_{n=0}^{\infty}\left(\left(n+1-i\rho(a)\right)\left(-n-2-i\rho(a)\right)\right)^{(n+1)(n+2)}}{\sqrt{\prod_{l=1}^{\infty}\left(\left(l+1\right)^2+\rho(a)^2\right)^{2l(l+2)}}} \tag{A.1.9}$$

$$= \prod_{k=0}^{\infty}\frac{\left(\left(k+i\rho(a)\right)\left(k+1-i\rho(a)\right)\right)^{k(k+1)}}{\left(k+i\rho(a)\right)^{(k-1)(k+1)}\left(k+1-i\rho(a)\right)^{k(k+2)}} \tag{A.1.10}$$

$$= \prod_{k=1}^{\infty}\frac{\left(k+i\rho(a)\right)^{k+1}}{\left(k-i\rho(a)\right)^{k-1}} \tag{A.1.11}$$

anticipating the fact that for the adjoint representation the weights (now roots) come in positive/negative pairs, we write the product of such a pair

$$Z_{\text{1 - loop}}^{\text{gauge multiplet}}(a)|_{\pm\rho} = \prod_{k=0}^{\infty}\frac{\left(k+1+i\rho(a)\right)^{k+2}}{\left(k+1-i\rho(a)\right)^{k}}\frac{\left(k+1-i\rho(a)\right)^{k+2}}{\left(k+1+i\rho(a)\right)^{k}} \tag{A.1.12}$$

$$= \exp\left(H(0,1,2,1+i\rho(a)) - H(0,1,0,1+i\rho(a)) + \text{c.c.}\right) \tag{A.1.13}$$

$$= \frac{4\sinh^2(\pi\rho(a))}{\rho(a)^2} \tag{A.1.14}$$

We conclude that

$$Z_{\text{1 - loop}}^{\text{gauge multiplet}}(a)|_\rho = \frac{2\sinh(\pi\rho(a))}{\rho(a)} \tag{A.1.15}$$

A.2 Chiral Multiplet

We would like to evaluate the following expression for the Gaussian path integral in the matter sector 2.2.4

$$Z^{\text{chiral multiplet}}_{\text{1 - loop}}(a,\Delta) = \frac{\text{Det}\left(D_{\text{chiral multiplet fermions}}\right)}{\sqrt{\text{Det}\left(D_{\text{chiral multiplet scalars}}\right)}} \tag{A.2.1}$$

The relevant operators are 2.2.4

$$D_{\text{chiral multiplet scalars}} = \Delta_{\text{scalar}} + \rho(a)^2 - i\rho(a) + 2i(\Delta - \frac{1}{2})\rho(a) + \Delta(2-\Delta) \tag{A.2.2}$$

$$D_{\text{chiral multiplet fermions}} = i \not{\nabla} - i\rho(a) + (\Delta - \frac{1}{2}) \tag{A.2.3}$$

and the modes 1.3.3

$$\lambda_3(L,0) = L(L+2) \tag{A.2.4}$$

$$L = 0, 1..., \infty \tag{A.2.5}$$

with degeneracy

$$D_3(L,0) = 2(L+1)^2 \tag{A.2.6}$$

and

$$\lambda^D{}_3(n) = \pm i(n+3/2), \qquad n = 0, 1, ..., \infty \tag{A.2.7}$$

with degeneracy

$$D^D_3(n) = (n+1)(n+2) \tag{A.2.8}$$

Define

$$z(\rho(a),\Delta) = i\rho(a) - \Delta + 1 \tag{A.2.9}$$

then

$$Z^{\text{chiral multiplet}}_{\text{1 - loop}}(a)|_\rho = \frac{\prod_{n=1}^{\infty}\left((n+\frac{1}{2}-i\sigma+(\Delta-\frac{1}{2}))(n+\frac{1}{2}+i\sigma-(\Delta-\frac{1}{2}))\right)^{n(n+1)}}{\sqrt{\prod_{l=0}^{\infty}\left(l(l+2)+\sigma^2-i\sigma+2i(\Delta-\frac{1}{2})\sigma+\Delta(2-\Delta)\right)^{2(l+1)^2}}} \tag{A.2.10}$$

$$= \prod_{k=1}^{\infty}\frac{((k-z+1)(k+z))^{k(k+1)}}{((k+z)(k-z))^{k^2}} \tag{A.2.11}$$

$$= \prod_{k=0}^{\infty}\left(\frac{k+1+z}{k+1-z}\right)^{k+1} \tag{A.2.12}$$

$$= \exp\left(H(0,1,1,1+z)-H(0,1,1,1-z)\right) \tag{A.2.13}$$

$$= \exp\left(\ell(z)\right) \tag{A.2.14}$$

where, following [3], we define

$$\ell(z) = -z\log\left(1-e^{2\pi iz}\right)+\frac{i}{2}\left(\pi z^2+\frac{1}{\pi}\mathrm{Li}_2\left(e^{2\pi iz}\right)\right)-\frac{i\pi}{12} \tag{A.2.15}$$

The last two expressions can be seen to be equal by using identities for special functions [85]. Note that when $\Delta = 1$, z is pure imaginary and the one loop contribution is a pure phase. Moreover, the roots ρ of a Lie algebra come in positive/negative pairs, which implies, from examination of A.2.12, that the total one loop contribution vanishes for an adjoint chiral multiplet with $\Delta = 1$.

Appendix B

Partition Functions of Seiberg-Like Duals

Some of the integrals resulting from the localization procedure can be challenging to evaluate. In some cases, specifically in the presence of Chern-Simons terms and $N_f > 1$, we have used numerical integration to compare the partition functions of dual theories. Where numerical results are provided, the integrals were performed using the CUHRE numerical integration routine available in the CUBA library [86] and using the Mathematica interface. The calculation for large rank gauge groups becomes increasingly numerically demanding and only low-rank results are provided.

The figures B.1 and B.2 summarize the comparison of the absolute value and phase of the partition functions for the dualities in 3.1.2.2.

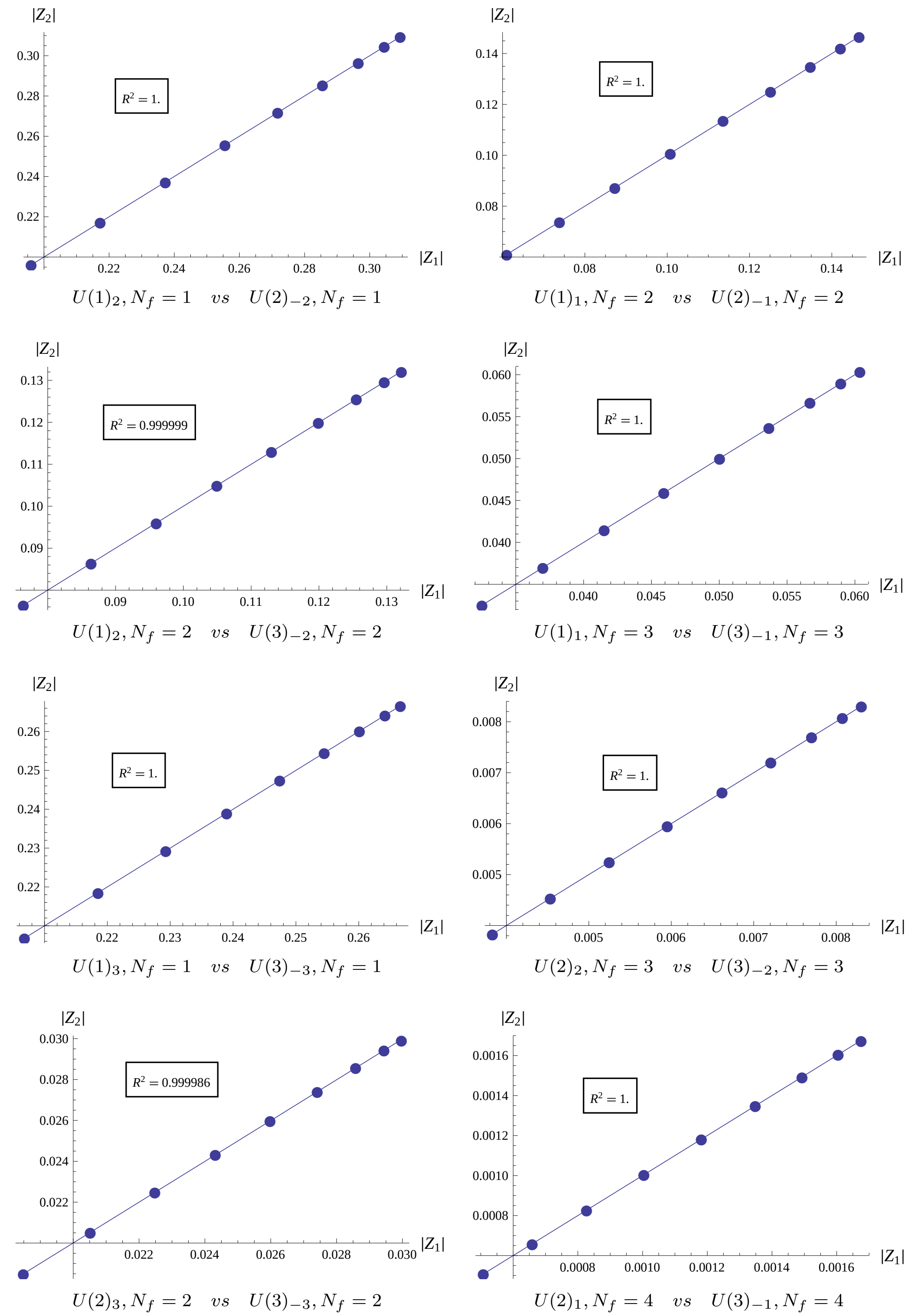

Figure B.1: A comparison of the magnitude of the partition functions with FI deformation (η) for 8 dual pairs and values of η from .1 to .9 and a best fit line, which, to the accuracy of the numerical evaluation, is of slope 1 and intercept 0.

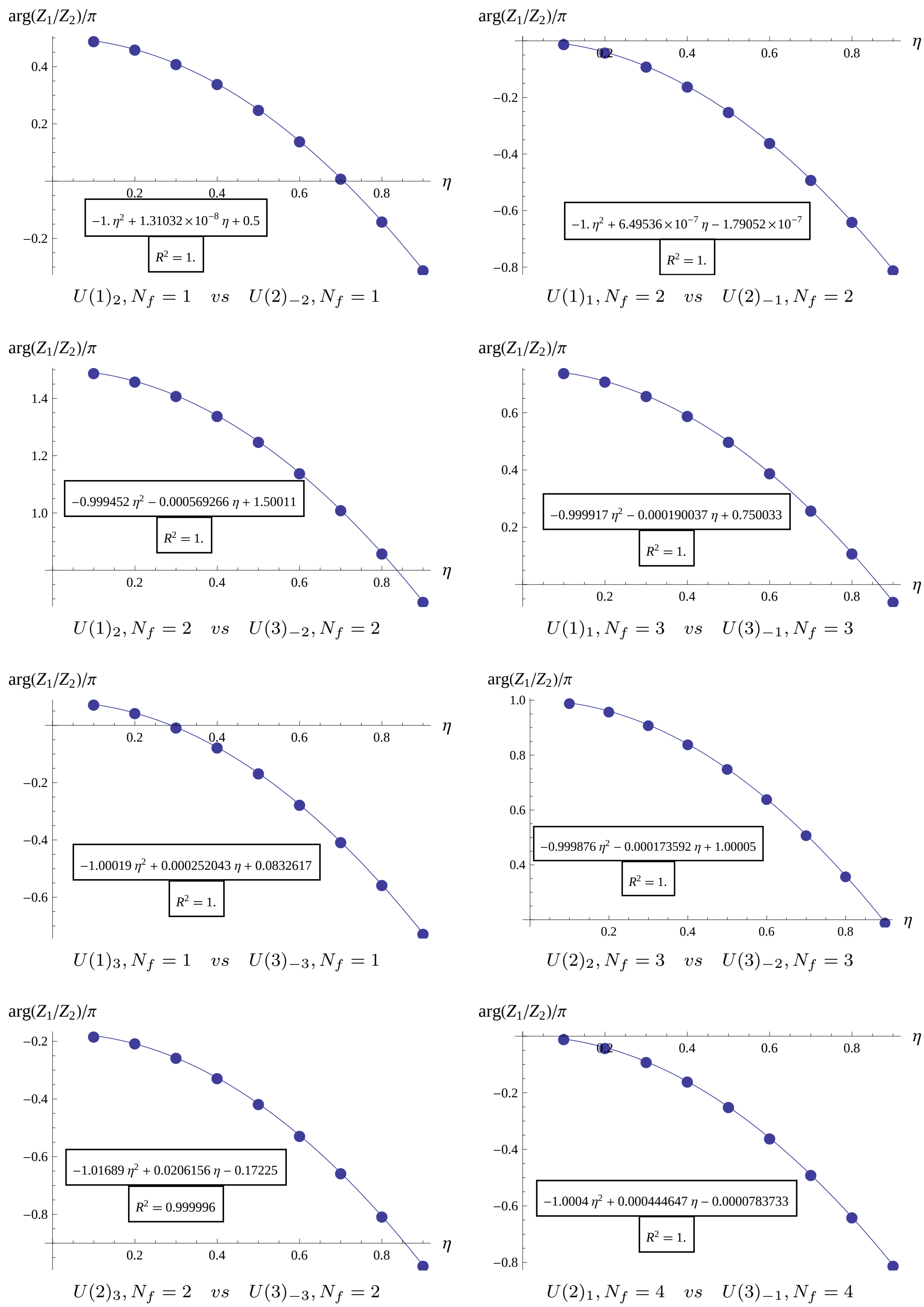

Figure B.2: A plot of the phase difference of the partition functions with FI deformation (η) for 8 dual pairs and values of η from .1 to .9 and a best fit parabola.

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