Elliptic Combinatorics and Markov Processes

Thesis by

Dan Betea

In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

California Institute of Technology
Pasadena, California

2012
(Defended May 22, 2012)

To my parents

Acknowledgements

I would like to thank my adviser Eric Rains for his support during my graduate career. His knowledge on the subject of special functions in particular and on various related areas in general (combinatorics, representation theory, algebraic geometry of elliptic curves, etc.) is encyclopedic, and the benefits can be seen on most pages of this document. In addition to helping correct many mistaken computations, he has provided guidance and intuition on the project as a whole and made pertinent comments aimed at improving, strengthening, or streamlining results and/or proofs.

I would also like to thank Alexei Borodin for many helpful conversations, for guiding me at the beginning of the project, and for suggesting the project in the first place. Many results provided in this thesis (and more not yet in print) are generalizations of results he has been involved with one way or another.

In preparation for this manuscript I have also benefited from conversations with Tom Alberts, Fokko van de Bult, Sunil Chhita, Anton Geraschenko, Vadim Gorin, Rick Kenyon, and Ben Young. They have answered numerous questions I had on various issues, and for that I am extremely grateful.

I would like to thank the Caltech math department staff and professors for making the graduate student experience here among the best anyone can hope for anywhere.

While a graduate student at Caltech, I have enjoyed the company of many wonderful friends. For all their support I will be eternally grateful. I am particularly indebted to Alexandra, Mihai, and Timea who in numerous private conversations have been forced to listen to my “complaints” about life and work in graduate school/academia.

Finally, I would have never started the nine-year “study abroad” program culminating in this document had it not been for the initial impulse from and support of my parents. Therefore this thesis is dedicated to them.

Abstract

We present combinatorial and probabilistic interpretations of recent results in the theory of elliptic special functions (due to, among many others, Frenkel, Turaev, Spiridonov, and Zhedanov in the case of univariate functions, and Rains in the multivariate case). We focus on elliptically distributed random lozenge tilings of the hexagon which we analyze from several perspectives. We compute the N-point function for the associated process, and show the process as a whole is determinantal with correlation kernel given by elliptic biorthogonal functions. We furthermore compute transition probabilities for the Markov processes involved and show they come from the multivariate elliptic difference operators of Rains. Properties of difference operators yield an efficient sampling algorithm for such random lozenge tilings. Simulations of said algorithm lead to new arctic circle behavior. Finally we introduce elliptic Schur processes on bounded partitions analogous to the Schur process of Reshetikhin and Okounkov (and to the Macdonald processes of Vuletic, Borodin, and Corwin). These give a somewhat different (and faster) sampling algorithm from these elliptic distributions, but in principle should encompass more than just tilings of a hexagon.

Contents

Chapter 1

Introduction

In this chapter we start by addressing the purpose and layout of the thesis. We give a description of each subsequent chapter. We continue by introducing most of the notation and auxiliary functions necessary throughout, and finish with a brief overview of the uniformization theorem for elliptic curves.

1.1 Foreword

In this thesis we present certain combinatorial and probabilistic interpretations of recent results in the theory of elliptic special (hypergeometric) functions. On the special functions side we are interested in the multivariate results of Rains [Rai10], [Rai06] and on some univariate results due to Spiridonov and Zhedanov [SZ00] and Frenkel and Turaev [FT97]. On the statistical mechanical and probabilistic side we try to generalize results of Borodin, Gorin and Rains [BG09], [BGR10].

We mostly restrict attention to random elliptically distributed lozenge tilings of a hexagon. For such tilings, we first study many properties of the elliptic measures and explain how they are many parameter deformations of natural measures already studied on tilings. Tilings can be viewed as Markov processes and we provide the N-point correlation function as well as transition probabilities via Kasteleyn theory (equivalently, via the Lindström-Gessel-Viennot theorem). We interpret the transition probabilities as multivariate elliptic difference operators due to Rains. Properties of such operators also have nice probabilistic interpretations (i.e., quasi-commutation and quasi-adjointness) and allow us to couple two "orthogonal" Markov chains using ideas of Borodin and Ferrari [BF10] and to obtain an efficient polynomial-time sampling algorithm for random (large) elliptically distributed tilings. We provide computer simulations of the algorithm that lead to interesting asymptotic behavior of such tilings.

We finally introduce elliptic Schur-like processes on partitions analogous to (and a generalization of) Reshetikhin and Okounkov's Schur process [OR03] and the Macdonald processes of Vuletic, Borodin and Corwin [Vul09], [BC11]. The key notions here are Rains' elliptic skew interpolation functions (generalizing skew Schur functions—see [Rai11]) and more precisely the elliptic binomial coefficients [Rai06]. Under certain specializations of the parameters such processes lead again to the elliptic measures we considered above. Identities for binomial coefficients and elliptic skew interpolation functions allow us to provide a second example of a polynomial-time efficient sampling algorithm (somewhat similar to the previous one, yet somewhat faster). We do this via a different coupling of two orthogonal Markov chains.

The layout of the thesis is as follows. For the rest of the first chapter we introduce the bulk of the notation used in all subsequent chapters. We also give some properties for some of the functions we introduce, restricting ourselves only to the necessary facts for our purposes. Finally we present a succinct description of the uniformization theorem that can oftentimes be useful when thinking about elliptic distributions. As a general note for the whole document, wherever our exposition follows one of the references closely, we make that point at the beginning of the respective section.

In Chapter 2 we collect a few recent results on elliptic hypergeometric functions. We are interested in univariate and multivariate evaluation formulas for integrals and summation formulas for terminating hypergeometric series. The most important results for our purposes are the Frenkel-Turaev summation formula and its multivariate analogue (see [FT97], [Rai06]) as well as a hypergeometric determinant evaluation of Warnaar [War02]. We introduce the combinatorial and probabilistic aspects of lozenge tilings of a hexagon in Chapter 3. We briefly discuss the Kasteleyn theory of bipartite planar dimers and then connect it to tilings of hexagons. We furthermore introduce the elliptic measures studied for the rest of the thesis, explain how one arrives at such measures, present some natural well-studied degenerations and study properties of the measures needed further on (like positivity since we want probability measures in the end). We present two disguises of the same measure: one symmetric, the other suited for Kasteleyn computations which we also perform after inverting the elliptic Kasteleyn matrix (see [BGR10]).

The bulk of Chapters 4, 5 and 6 follow the preprint [Bet11]. In Chapter 4 we compute the N-point correlation function and the transition probabilities for the Markov processes naturally associated to a lozenge tiling (via the non intersecting path interpretation). We interpret the latter as the multivariate elliptic difference operators of Rains [Rai10] and connect properties of operators to probabilistic and combinatorial statements. Based on these properties, in Chapter 5 we present an efficient sampling algorithm for elliptically distributed lozenge tilings of a hexagon. We follow the idea of Markov chain coupling laid out by Borodin and Ferrari (see [BF10]; also [BG09] and [BGR10]). We finally present computer simulations of the algorithm and sample large tilings that clearly exhibit new arctic circle phenomena. In Chapter 6 we show how the Markov point processes naturally associated to elliptically distributed lozenge tilings are determinantal point processes with correlation kernels given by the univariate elliptic biorthogonal functions of Spiridonov and Zhedanov [SZ00]. We give an integral representation for the biorthogonal functions that may be useful in obtaining asymptotics of the biorthogonal kernel, though we do not pursue asymptotics of the kernel any further.

In Chapter 7 we introduce the elliptic analogues of Schur processes of [OR03] and Macdonald processes of [Vul09], [BC11]. These are based on the elliptic skew interpolation functions of Rains [Rai11] which in turn rely on the elliptic binomial coefficients and abelian interpolation functions [Rai06]. We introduce the theory and then interpret the results combinatorially and probabilistically. Under certain specializations of the parameters such processes correspond to the previously mentioned elliptic measures on tilings of a hexagon. As a byproduct we obtain another sampling algorithm for elliptic distributions on lozenge tilings of a hexagon. This algorithm is somewhat faster than the one in Chapter 5, but is still based on coupling two appropriate Markov chains. We further believe the generality of the elliptic processes should be of use for covering other combinatorial and/or statistical mechanical models, but we do not pursue these ideas further in the present document.

1.2 Notation, θ, Γ, Δ

In this section we introduce notation used throughout the thesis. We introduce the most used functions and study some of their properties we will often use without explicit mention. Throughout the entire document we use LHS in place of *left hand side* and RHS in place of *right hand side*.

Assume $|p|, |q| < 1$. On $\mathbb{C}^*$ we define the theta and elliptic gamma functions (see [Rui97]) as follows:

$$\theta_p(x) := \prod_{k\geq 0}(1-p^k x)(1-\frac{p^{k+1}}{x}),$$
$$\Gamma_{p,q}(x) = \prod_{k,l\geq 0} \frac{1-p^{k+1}q^{l+1}/x}{1-p^k q^l x}.$$

We will make brief use of the following triple gamma function, which we also now define:

$$\Gamma_{p,q,t}(x) = \prod_{i,j,k\geq 0}(1-p^{i+1}q^{j+1}t^{k+1}/x)(1-p^i q^j t^k x).$$

Notice that θ_p has simple zeros in $p^{\mathbb{Z}}$ while $\Gamma_{p,q}$ has simple zeros in $pqp^{\mathbb{N}}q^{\mathbb{N}}$ and simple poles in $p^{-\mathbb{N}}q^{-\mathbb{N}}$. The triple gamma function only has zeros on $p^{-\mathbb{N}}q^{-\mathbb{N}}t^{-\mathbb{N}}$ and $pqtp^{\mathbb{N}}q^{\mathbb{N}}t^{\mathbb{N}}$. Also $\theta_0(x) = 1-x$.

The function θ_p just defined is indeed related to the usual Jacobi theta functions via the triple product identity. More precisely, let τ be a complex number with positive imaginary part such that $p = e^{2\pi i\tau}$ and let $x = e^{2\pi i u}$. Following the conventions set in [Spi08], the four Jacobi theta functions are defined by

$$\theta_{ab}(u) = \sum_{k\in\mathbb{Z}} e^{\pi i\tau(k+a/2)} e^{2\pi i(k+a/2)(u+b/2)},\ a,b \in \{0,1\}.$$

Then let $\theta_1(u) = -\theta_{11}(u)$. The triple product relation is exactly

$$\theta_1(u) = ip^{1/8}x^{-1/2}(p;p)\theta_p(x),$$

where

$$(z;p) = \prod_{i\geq 0}(1-p^i z).$$

The theta-Pochhammer symbol (a generalization of the q-Pochhammer symbol to which it degenerates as $p \to 0$) is defined, for m an integer, as

$$\theta_p(x;q)_m = \begin{cases} \prod_{0\leq i<m}\theta_p(q^i x),\ m>0, \\ 1,\ m=0, \\ \prod_{1\leq i\leq -m}\frac{1}{\theta_p(q^{-i}x)},\ m<0. \end{cases}$$

As is usual in this area, presence of multiple arguments before the semicolon (inside theta or elliptic gamma functions) will mean multiplication. To wit:

$$\theta_p(uz^{\pm 1};q)_m = \theta_p(uz;q)_m\theta_p(u/z;q)_m;\ \Gamma_{p,q}(a,b) = \Gamma_{p,q}(a)\Gamma_{p,q}(b).$$

We have the following important identities ($n \geq 0$ an integer):

$$\begin{aligned} &\theta_p(x) = \theta_p(p/x), \\ &\theta_p(px) = \theta_p(1/x) = -(1/x)\theta_p(x), \\ &\Gamma_{p,q}(q^n x) = \theta_p(x;q)_n\Gamma_{p,q}(x), \\ &\Gamma_{p,q}(pq/x)\Gamma_{p,q}(x) = 1, \\ &\Gamma_{p,q,t}(pqt/x) = \Gamma_{p,q,t}(x), \\ &\Gamma_{p,q,t}(tx) = \Gamma_{p,q}(x)\Gamma_{p,q,t}(x). \end{aligned} \tag{1.2.1}$$

The third identity in (1.2.1) can be extended for non integer n (via analytic continuation) to provide generalizations of the theta-Pochhammer symbol of non integral length. These identities also extend to the following among theta-Pochhammer symbols:

$$\theta_p(aq^n;q)_k = \frac{\theta_p(a;q)_k\theta_p(aq^k;q)_n}{\theta_p(a;q)_n} = \frac{\theta_p(a;q)_{n+k}}{\theta_p(a;q)_n},$$
$$\theta_p(a;q)_n = \theta_p(q^{1-n}/a;q)_n(-a)^n q^{\binom{n}{2}},$$
$$\theta_p(a;q)_{n-k} = \frac{\theta_p(a;q)_n}{\theta_p(q^{1-n}/a;q)_k}(-\frac{q}{a})^k q^{\binom{k}{2}-nk}, \tag{1.2.2}$$
$$\theta_p(aq^{-n};q)_k = \frac{\theta_p(a;q)_k\theta_p(q/a;q)_n}{\theta_p(q^{1-k}/a;q)_n}q^{-nk},$$
$$\theta_p(a;q)_{-n} = \frac{1}{\theta_p(aq^{-n};q)_n} = \frac{1}{\theta_p(q/a;q)_n}(-\frac{q}{a})^n q^{\binom{n}{2}}.$$

We will use the above identities throughout for simplifying computations without explicitly referring to them. Another very important identity involving theta functions is the Riemann addition formula which takes the following form:

$$\theta_1(u \pm a, v \pm v) - \theta_1(u \pm b, v \pm a) = \theta_1(a \pm b, u \pm b),$$

or

$$\theta_p(xw^{\pm 1}, yz^{\pm 1}) - \theta_p(xz^{\pm 1}, yw^{\pm 1}) = \frac{y}{w}\theta_p(xy^{\pm 1}, wz^{\pm 1}).$$

To prove the last equality, observe the ratio LHS/RHS is *elliptic* (see below) as a function of x and has no poles (the zeros $x = y^{\pm 1}$ of LHS are annihilated by the similar zeros on RHS), and thus by Liouville's theorem must be constant. Plugging in $x = w$ yields the result.

If $f(x_1, \dots, x_n)$ is a function of n variables defined on $(\mathbb{C}^*)^n$, we call it BC_n-*symmetric* if it is symmetric (does not change under permutation of the variables) and invariant under $x_k \to 1/x_k$ for all k. We will call it a BC_n-*symmetric theta function of degree* m if in addition, it satisfies the following:

$$f(px_1, \dots, x_n) = (\frac{1}{px_1^2})^m f(x_1, \dots, x_n).$$

The prototypical example of a BC_n-symmetric theta function of degree 1 is

$$f(x_1, \dots, x_n) = \prod_{1 \le k \le n} \theta_p(ux_k^{\pm 1}).$$

Notationally, for a function f of n variables, we will use the abbreviation $f(\dots x_k \dots)$ to stand for $f(x_1, \dots, x_n)$.

We now define the following function (which will play an important subsequent role):

$$\varphi(z, w) = z^{-1}\theta_p(zw, z/w). \tag{1.2.3}$$

Note φ is antisymmetric ($\varphi(z,w) = -\varphi(w,x)$) of degree 1 in each variable. It is a consequence of the addition formula for Riemann theta functions that

$$\varphi(x,y) = \left(\frac{\varphi(z,x)}{\varphi(w,x)} - \frac{\varphi(z,y)}{\varphi(w,y)}\right)\frac{\varphi(w,x)\varphi(w,y)}{\varphi(z,w)},$$

for arbitrary z, w. We observe that the expression in parentheses appearing above is a Vandermonde-like factor in transcendental coordinates $X = \frac{\varphi(z,x)}{\varphi(w,x)}, Y = \frac{\varphi(z,y)}{\varphi(w,y)}$, so $\varphi(z_i, z_j)$ is an "elliptic analogue" of the (Vandermonde) difference $z_i - z_j$. This is indeed the case if one takes the right limit and then a product over $i < j$. To wit:

$$\lim_{q\to 1}\frac{\lim_{p\to 0}\varphi(q^{x_i}, q^{x_j})}{q-q^{-1}} = x_i - x_j.$$

Throughout, we will denote by $\mathbb{E}$ the elliptic curve (written as a multiplicative group) $\mathbb{C}^*/\langle p\rangle$. This is of course isomorphic (via the exponential map) to the more familiar additive form for elliptic curves: $\mathbb{C}/\langle 1,\tau\rangle$ where $p = e^{2\pi i\tau}$.

An *elliptic* function is a function $f(x)$ defined on the curve $\mathbb{E}$. That is, a function invariant under $x \mapsto px$. It is a well-known theorem (see, e.g., [Sil09]) that elliptic functions must have the same number of poles as zeros in the fundamental domain (that is, mod p), that number must be greater than or equal to two for a nonconstant function, and a generic elliptic function f with poles at w_k and zeros at t_k is of the form

$$f(x) = const\prod_{1\le k\le n}\frac{\theta_p(z/t_k)}{\theta_p(z/w_k)},\text{ where }\prod t_k = \prod w_k.$$

Fix a *partition* λ with at most n parts (that is, a sequence $\lambda = \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n,\ \lambda_i \in \mathbb{N}$). We will write $\lambda \subset m^n$ for a partition with at most n parts such that the largest part $\lambda_1 \le m$ (and we say such a partition is contained in the rectangle m^n. Note there are $\binom{m+n}{n}$ such partitions).

We will now define the delta symbols introduced in [Rai10] (see also [Rai06]). We need an extra parameter t (usually taken to be of modulus less than one) for full generality. We start with

$$\mathcal{C}^0_\lambda(x;q;t;p) := \prod_{1\le i\le n}\theta_p(t^{1-i}x;q)_{\lambda_i},$$

$$\mathcal{C}^+_\lambda(x;q;t;p) := \prod_{1\le i\le j\le n}\frac{\theta_p(t^{j-i}x;q)_{\lambda_i-\lambda_{j+1}}}{\theta_p(t^{j-i}x;q)_{\lambda_i-\lambda_j}},$$

$$\mathcal{C}^-_\lambda(x;q;t;p) := \prod_{1\le i\le j\le n}\frac{\theta_p(t^{2-j-i}x;q)_{\lambda_i+\lambda_j}}{\theta_p(t^{2-j-i}x;q)_{\lambda_i+\lambda_{j+1}}}.$$

Next let

$$\Delta^0_\lambda(a|\dots b_i\dots;q;t;p) := \prod_i \frac{\mathcal{C}^0_\lambda(b_i;q;t;p)}{\mathcal{C}^0_\lambda(pqa/b_i;q;t;p)},$$

$$\Delta_\lambda(a|\dots b_i\dots;q;t;p) := \Delta^0_\lambda(a|\dots b_i\dots;q;t;p) \times \frac{\mathcal{C}^0_{2\lambda^2}(pqa;q;t;p)}{\mathcal{C}^-_\lambda(t,pq;q;t;p)\mathcal{C}^+_\lambda(a,pqa/t;q;t;p)},$$

where the partition $2\lambda^2$ is defined by $(2\lambda^2)_i = 2\lambda_{\lceil i/2\rceil}$.

We list here the following transformations for Δ_λ corresponding to involutions on partitions of length at most n (for the second, we need $\lambda \subset m^n$):

$$\Delta_{\lambda'}(a|\dots b_i\dots;1/t;1/q;p) = \Delta_\lambda(a/qt|\dots b_i\dots;q;t;p),$$
$$\frac{\Delta_{m^n-\lambda}(a|\dots b_i\dots;q;t;p)}{\Delta_{m^n}(a|\dots b_i\dots;q;t;p)} = \Delta_\lambda(\frac{t^{2n-2}}{q^{2m}a}|\dots\frac{t^{n-1}b_i}{q^m a}\dots,t^n,q^{-m},pqt^{n-1},pq/q^m t;q;t;p), \tag{1.2.4}$$

where $\lambda' \in n^m$ is the dual partition (the transpose of λ viewed as a Young diagram) and $m^n-\lambda$ is the complemented partition: $(m^n-\lambda)_i = m-\lambda_{n-i+1}$. We also list the following shift transformations:

$$\frac{\Delta_{k^n+\lambda}(a|\dots,b_i,\dots;q;t;p)}{\Delta_{k^n}(a|\dots,b_i,\dots;q;t;p)} = \Delta_\lambda(q^{2k}a|\dots,q^k b_i,\dots,t^n,pqt^{n-1},q^k t^{1-n}a,pqq^k t^{-n}a;q;t;p),$$
$$\frac{\Delta^0_{k^n+\lambda}(a|\dots,b_i,\dots;q;t;p)}{\Delta^0_{k^n}(a|\dots,b_i,\dots;q;t;p)} = \Delta^0_\lambda(q^{2k}a|\dots,q^k b_i,\dots;q;t;p). \tag{1.2.5}$$

Throughout we will mostly be interested in the above formulas when $t=q$. This simplifies the formulas considerably. Of particular interest will be the Δ-symbol with six parameters t_0,t_1,t_2,t_3,u_0,u_1 satisfying the balancing condition $q^{2n-2}t_0t_1t_2t_3u_0u_1 = pq$. The formula for Δ_λ becomes

$$\Delta_\lambda(q^{2n-2}t_0^2|q^n,q^{n-1}t_0t_1,q^{n-1}t_0t_2,q^{n-1}t_0t_3,q^{n-1}t_0u_0,q^{n-1}t_0u_1;q;p) = const\cdot\prod_{i<j}(\varphi(z_i,z_j))^2$$
$$\prod_{1\le i} p^{-l_i}q^{l_i^2+l_i(2n-1)}t_0^{2l_i}\theta_p(z_i^2)\frac{\theta_p(t_0^2,t_0t_1,t_0t_2,t_0t_3,t_0u_0,t_0u_1;q)_{l_i}}{\theta_p(q,q\frac{t_0}{t_1},q\frac{t_0}{t_2},q\frac{t_0}{t_3},q\frac{t_0}{u_0},q\frac{t_0}{u_1};q)_{l_i}},$$

where $l_i = n-i+\lambda_i$, $z_i = q^{l_i}t_0$.

Remark 1.2.1. The constant is independent of λ and present in the formula to make the Δ-symbol elliptic in all of its arguments (the value of the constant can be computed explicitly nevertheless). In addition to being invariant by multiplying the parameters by integer powers of p (so long as the balancing condition is maintained), the Δ-symbol above is also invariant by shifting the parameters by $p^{\pm 1/2}$ (as long as we shift half the parameters up and half down to maintain the balancing condition).

Throughout this thesis we will find the presence of p in the balancing condition unnatural from the combinatorial perspective, so we will be faced with 6 parameters multiplying to q instead. We then need to restore the balancing condition by multiplying one of said parameters by p (and we choose somewhat arbitrarily for that parameter to be u_1). We are then looking at

$$\Delta_\lambda(q^{2n-2}t_0^2|q^n, q^{n-1}t_0t_1, q^{n-1}t_0t_2, q^{n-1}t_0t_3, q^{n-1}t_0u_0, q^{n-1}t_0(pu_1); q; p) = const \cdot \prod_{i<j}(\varphi(z_i, z_j))^2 \tag{1.2.6}$$

$$\times \prod_{1\leq i} q^{l_i(2n-1)}\theta_p(z_i^2)\frac{\theta_p(t_0^2, t_0t_1, t_0t_2, t_0t_3, t_0u_0, t_0u_1; q)_{l_i}}{\theta_p(q, q\frac{t_0}{t_1}, q\frac{t_0}{t_2}, q\frac{t_0}{t_3}, q\frac{t_0}{u_0}, q\frac{t_0}{u_1}; q)_{l_i}}. \tag{1.2.7}$$

This *discrete elliptic Selberg density* is the weight function for the discrete elliptic multivariate biorthogonal functions defined in [Rai06]. Notice it can be written symmetrically in terms of the z_i's and the elliptic gamma functions as

$$const \cdot \prod_{i<j}(\varphi(z_i, z_j))^2 \cdot \prod_i z_i^{2n-1}\theta_p(z_i^2)\frac{\Gamma_{p,q}(t_0z_i, t_1z_i, t_2z_i, t_3z_i, u_0z_i, u_1z_i)}{\Gamma_{p,q}(\frac{q}{t_0}z_i, \frac{q}{t_1}z_i, \frac{q}{t_2}z_i, \frac{q}{t_3}z_i, \frac{q}{u_0}z_i, \frac{q}{u_1}z_i)}.$$

1.3 A note on uniformization

There are two ways one can think of elliptic curves, and both ways can have advantages depending on the context. Here we give a recipe to go from one to the other. We follow the books by Silverman [Sil09] and Husemöller [Hus04], and the reader is referred to either for more on the theory of elliptic curves.

One way involves complex tori (that is, lattices in the complex plane), and an elliptic curve is then just $\mathbb{C}/\langle 1, \tau\rangle$ with τ having strictly positive imaginary part. This is a one-dimensional complex Lie group with obvious addition law coming from the addition of complex numbers. It is isomorphic via the exponential map to the multiplicative group $\mathbb{C}^*/\langle p\rangle$ where $p = e^{2\pi i\tau}$ (a curve in this form is called a *Tate curve*). Let us denote the lattice by $\Lambda = \langle 1, \tau\rangle = \mathbb{Z} + \tau\mathbb{Z}$. There are a few quantities associated to Λ. One is the Weierstrass $\wp$ function, a doubly periodic meromorphic function on the complex plane with periods in Λ (that is, an elliptic function on $\mathbb{C}/\Lambda$):

$$\wp(z|\Lambda) = \frac{1}{z^2} + \sum_{\omega\in\Lambda-0}\left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right).$$

We also have the associated (modular) quantities $g_2(\Lambda) = 60G_4(\Lambda)$ and $g_3(\Lambda) = 140G_6(\Lambda)$ where $G_{2k}(\Lambda) = \sum_{\omega\in\Lambda-0}\omega^{-2k}$.

All the series above are of course convergent. It is furthermore not hard (but rather tedious) to check that

$$(\wp'(z))^2 = 4\wp(z)^3 - g_2\wp(z) - g_3. \tag{1.3.1}$$

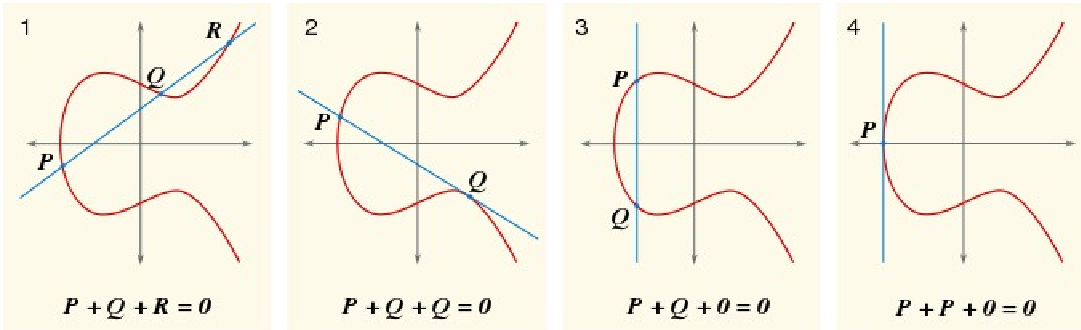

Figure 1.1: The addition law on an elliptic curve. The real locus of a real elliptic curve is pictured (horizontal axis is x, vertical y). In the first picture, $P+Q$ is the reflection of R in the horizontal x axis.

It is a well-known fact that every elliptic (doubly periodic with period lattice Λ) function on $\mathbb{C}/\Lambda$ belongs to the field $\mathbb{C}(\wp'(z), \wp(z))$ (which is the field of fractions of the polynomial ring $\mathbb{C}[X,Y]/Y^2 - X^3 - g_2X - g_3$).

Another way to think of an elliptic curve is the (complex) affine locus of points (x, y) satisfying a cubic equation of the form $y^2 = x^3 - Ax - B$. It is often useful to projectivize and look at the points $(x : y : z)$ in the complex projective plane $\mathbb{P}^2 := \mathbb{P}^2(\mathbb{C})$ satisfying $y^2z = x^3 - Axz^2 - Bz^3$. There is a way to define an addition on such points that makes the locus into a group with identity given by the point at infinity: $(0 : 1 : 0)$. To wit, take any two (distinct) points $(x : y : z), (u : v : w)$ on the curve. Let L be the line passing through them (it is unique). Since the curve is cubic, L will intersect it in 3 points, of which two we already know. Call the third one R. Then the group law is defined so that $(x : y : z) + (u : v : w)$ is $-R$. That is to say, if L' is the line connecting R to $(0 : 1 : 0)$ then $(x : y : z) + (u : v : w)$ is the third point of intersection of L' and the curve (we already know two of them: R and $(0 : 1 : 0)$). If the two points we started with are not distinct, we are looking at the line L that is tangent to the curve at $(x : y : z) = (u : v : w)$.

A corollary of the above is that three points on the curve add up to the identity if and only if they lie on the same line. We illustrate this in Figure 1.1 for a real elliptic curve (image available at http://en.wikipedia.org/wiki/File:ECClines.svg under the GPL license).

We can now finalize our discussion of uniformization. To go from an elliptic curve in the form $\mathbb{C}/\langle 1, \tau\rangle$ to a cubic equation, we use (1.3.1). That is $\mathbb{C}/\langle 1, \tau\rangle \cong E$ as complex Lie groups where E is the elliptic curve $\{(\wp(z), \wp'(z), 1) | (\wp'(z))^2 = 4\wp(z)^3 - g_2\wp(z) - g_3\}$ and the isomorphism is explicit $z \mapsto (\wp(z), \wp'(z), 1)$.

For the other way, given an elliptic curve with equation $y^2 = x^3 - Ax - B$ (and $A^3 - 27B^2 \neq 0$ so that the cubic in x has distinct roots), the two periods ω_1, ω_2 for the lattice Γ can be computed by the following *elliptic integrals*:

$$\omega_1 = \int_\alpha \frac{dx}{\sqrt{x^3 - Ax - B}},$$
$$\omega_2 = \int_\beta \frac{dx}{\sqrt{x^3 - Ax - B}},$$

where α, β are two paths in the complex manifold E generating the first homology group $H_1(E, \mathbb{Z})$. We then have $\mathbb{C}/\langle\omega_1, \omega_2\rangle \cong \{(x, y)|y^2 = x^3 - Ax - B\}$ as elliptic curves.

Chapter 2

Elliptic hypergeometric identities

In this chapter we state a couple of elliptic integral identities: a 6-parameter evaluation and an 8 parameter transformation under the Weyl group $W(E_7)$. We give proofs for the univariate case and mention the multivariate analogues we will use. We discuss the discrete (series) analogues of these identities, including the Frenkel-Turaev summation and a multivariate extension, as they will be central for the rest of the thesis. We finish with an elliptic determinantal identity due to Warnaar. For the proofs, we follow [Rai10], [Spi08] and [War02].

2.1 Elliptic beta integrals

We begin with the order 0 (evaluation) and order 1 (summation) elliptic beta integral identities.

For complex parameters $t_i, 0 \leq i \leq 2m+5, p, q$ such that $|p|, |q|, |t_i| < 1$ we define

$$I(t_0, \dots, t_{2m+5}) = \frac{(p;p)(q;q)}{2} \int_{\mathbb{T}} \frac{\prod_{0 \leq i \leq 2m+5} \Gamma_{p,q}(t_j z^{\pm 1})}{\Gamma_{p,q}(z_i^{\pm 2})} \frac{dz}{2\pi i z},$$

where $\mathbb{T}$ is the positively oriented unit circle. We will mostly be interested in $m = 0$ or $m = 1$ which we will call (following [Rai10]; see also [Spi08]) the *elliptic beta integrals* of order 0 and 1 respectively.

We have the following evaluation formula found by Spiridonov (see, e.g., [Spi02a]), whose proof we sketch following [Rai10].

Theorem 2.1.1. *With parameters as above, one has*

$$I(t_0, \dots, t_5) = \prod_{0 \leq i < j \leq 5} \Gamma_{p,q}(t_i t_j).$$

Proof. We observe that the contour separates the poles of the integrand converging to zero from those diverging to infinity (and contains all of the former), and this is in fact the only requirement on the contour needed to prove the result.

We divide the integral by the claimed evaluation and first show that the resulting function is invariant under the shifts:

$$(t_0,t_1,t_2,t_3,t_4,t_5) \to (p^{1/2}t_0, p^{1/2}t_1, p^{1/2}t_2, p^{-1/2}t_3, p^{-1/2}t_4, p^{-1/2}t_5),$$
$$(t_0,t_1,t_2,t_3,t_4,t_5) \to (q^{1/2}t_0, q^{1/2}t_1, q^{1/2}t_2, q^{-1/2}t_3, q^{-1/2}t_4, q^{-1/2}t_5),$$

and permutations of such shifts. Let us denote the integrand by $\rho(z) := \rho(z;t_0,\dots,t_5)$ and let

$$\rho_1(z;t_0,\dots,t_5) = \frac{\Gamma_{p,q}(pz/(t_0t_1t_2))\prod_{0\le i\le 5}\Gamma_{p,q}(t_iz)}{\Gamma_{p,q}(z^2, p/(zt_0t_1t_2))},$$
$$\rho_1'(z;t_0,\dots,t_5) = \frac{\Gamma_{p,q}(pz/(t_0't_1't_2'))\prod_{0\le i\le 5}\Gamma_{p,q}(t_i'z)}{\Gamma_{p,q}(z^2, p/(zt_0't_1't_2'))},$$

where

$$(t_0',t_1',t_2',t_3',t_4',t_5') \to (q^{-1/2}t_3, q^{-1/2}t_4, q^{-1/2}t_5, q^{1/2}t_0, q^{1/2}t_1, q^{1/2}t_2).$$

Clearly $\rho(z) = \rho_1(z)\rho_1(1/z)$. Moreover, we have

$$\frac{\rho_1'(q^{1/2}z)}{\rho_1(z)} = \frac{\theta_p(t_0z, t_1z, t_2z, pz/(t_0t_1t_2))}{\theta_p(z^2)},$$
$$\frac{\rho_1(q^{1/2}z)}{\rho_1'(z)} = \frac{\theta_p(t_3z, t_4z, t_5z, pz/(t_3t_4t_5))}{\theta_p(z^2)},$$

where in proving the above we use the balancing condition. The RHS above, after symmetrization, satisfies

$$\frac{\theta_p(uz, vz, wz, pz/(uvw))}{\theta_p(z^2)} + (z \mapsto (1/z)) = \theta_p(uv, uw, vw),$$

where by the term $(z \mapsto 1/z)$ we mean the previous summand with z replaced by its reciprocal. To show this, observe the LHS of the identity is elliptic in z and a careful limit shows it has no poles (at $\pm 1, \pm\sqrt{p}$), so it must be constant. Plugging in $z = u$ yields the result.

Now observe

$$\int_{\mathbb{T}} \rho_1'(q^{1/2}z)\rho_1(1/z)\frac{dz}{2\pi i z} = \int_{q^{1/2}\mathbb{T}} \rho_1(q^{1/2}z)\rho_1'(1/z)\frac{dz}{2\pi i z},$$

where to go from left to right we transform $z \mapsto 1/(q^{1/2}z)$. First, the integral on the left is over a contour that still separates (and contains) poles of the integrand converging to 0 from those diverging to infinity. Likewise on the right. Moreover, we can deform the contour on the right back into $\mathbb{T}$ without passing over any offending poles. Using the fact that $\rho_1'(q^{1/2}z)\rho_1(1/z) = \frac{\rho_1'(q^{1/2}z)}{\rho_1(z)}\rho(z)$

and that we can symmetrize both RHS and LHS above (since the contours are symmetrical under $z \mapsto 1/z$), we obtain

$$\theta_p(t_0t_1, t_0t_2, t_0t_3)I(t_0, \ldots, t_5) = \theta_p(t_3t_4, t_3t_5, t_4t_5)I(q^{1/2}t_0, q^{1/2}t_1, q^{1/2}t_2, q^{-1/2}t_3, q^{-1/2}t_4, q^{-1/2}t_5),$$

which means the integral divided by the claimed evaluation is invariant under the q shift depicted above, and permutations thereof (by the S_6 symmetry of the parameters). Since the elliptic gamma function is symmetric in p and q, we get invariance under p shifts as well, and so we get invariance of the quotient (integral/evaluation) under shifting any parameter by $p^i q^j$ for i, j half-integers. But such shifts are dense (at least for generic p, q), which means the quotient is invariant under changing the parameters. It must thus only depend on p and q, and to find such dependence we pass to the limit and use the residue calculus of [vDS00] (see next section) as follows: we first force the contour to pass over the poles of the integrand at t_0 (moving from the inside to the outside) and $1/t_0$ (moving from the outside to the inside). In the process we pick up two residues. We then let $t_0t_1 \to 1$. The new integral vanishes and we are left with only the two residues yielding the result. □

There is a multivariable generalization of the evaluation formula for the order 0 elliptic beta integral. The resulting integral is the *elliptic Selberg integral*, and the proof is very similar (essentially follows the same steps with more complicated notation) so we will skip it (but see [Rai10]). We will just state the result as obtained in the aforementioned reference:

Theorem 2.1.2. *Let p, q, t be complex parameters of modulus less than 1, $t_0, \ldots, t_6$ also complex satisfying the balancing condition $t^{2n-2}t_0t_1t_2t_3t_4t_5 = pq$. Let C be a positive contour around 0 invariant under reciprocation ($C = C^{-1}$) containing all points in $t_i p^{\mathbb{N}} q^{\mathbb{N}}$ (for all i), excluding the reciprocals of such points, and containing contours $p^{\mathbb{N}} q^{\mathbb{N}} tC$ (if all $|t_i| < 1$ we can take $C = \mathbb{T}$). We then have:*

$$\frac{(p;p)^n(q;q)^n\Gamma_{p,q}(t)^n}{2^n n!} \times \int_{C^n} \prod_{1\le i<j\le n} \frac{\Gamma_{p,q}(tz_i^{\pm 1}z_j^{\pm 1})}{\Gamma_{p,q}(z_i^{\pm 1}z_j^{\pm 1})} \prod_{1\le i\le n} \frac{\prod_{0\le s\le 5}\Gamma_{p,q}(t_s z_i^{\pm 1})}{\Gamma_{p,q}(z_i^{\pm 2})} \frac{dz_i}{2\pi\sqrt{-1}z_i} = \prod_{0\le j<n}\left(\Gamma_{p,q}(t^{j+1}) \prod_{0\le r<s\le 5} \Gamma_{p,q}(t^j t_r t_s)\right).$$

Clearly for $n = 1$ variable, the above theorem transforms into Theorem 2.1.1.

If instead of $m = 0$ we look at $m = 1$ (in the one variable case for now), we obtain a transformation satisfied by the integral. Under proper normalization, we obtain more: (proper) invariance under the Weyl group $W(E_7)$. We begin with the transformation (first discovered in [Spi03]; see also [Rai10]).

Theorem 2.1.3. *Let p, q, t_i, $0 \le i \le 7$ be complex parameters of modulus less than 1 satisfying the* *balancing condition* $\prod t_i = p^2q^2$. *Then we have*

$$I(t_0, \ldots, t_7) = \left(\prod_{0\le j<k\le 3} \Gamma_{p,q}(t_jt_k, t_{j+4}t_{k+4}) \right) I(s_0, \ldots, s_7),$$

where

$$s_i = ut_i, s_{i+4} = u^{-1}t_{i+4},\ 0 \le i \le 3,\ u = \sqrt{\frac{pq}{t_0t_1t_2t_3}} = \sqrt{\frac{t_4t_5t_6t_7}{pq}},$$

and the requirements are such that all s_i are of modulus less than 1 as well.

Remark 2.1.4. The requirements on the t_j's and s_j's can be lifted as long as we replace the unit circle with a contour that is invariant under reciprocation, contains all poles of the integrand converging to 0 (of the form $t_jp^{\mathbb{N}}q^{\mathbb{N}}$), and excludes all the poles diverging to infinity.

Proof. We start with the following integral:

$$\frac{(p;p)(q;q)}{2} \int_{\mathbb{T}^2} \frac{\Gamma_{p,q}(cz^{\pm1}w^{\pm1}) \prod_{0\le j\le 3} \Gamma_{p,q}(a_jz^{\pm1}, b_jw^{\pm1})}{\Gamma_{p,q}(z^{\pm1}, w^{\pm1})} \frac{dz}{2\pi iz} \frac{dw}{2\pi iw},$$

where a_j, b_j, c are complex numbers of modulus less than 1 such that $c^2 \prod a_j = c^2 \prod b_j = pq$. We can compute this integral in two ways (by first integrating over w or z respectively), and the first integration can be carried out using the evaluation formula in Theorem 2.1.1. The result follows. □

Remark 2.1.5. We used the 6-parameter evaluation formula to prove the 8 parameter transformation formula. In Section 2.3, we will go the other way, using a discrete transformation to prove a discrete evaluation formula for a certain elliptic hypergeometric series.

We make the connection with the root system of type E_7 now. Let $x = \sum x_ie_i$ be a vector in $\mathbb{R}^8$ satisfying $\sum x_i = 0$ where $\{e_i\}$ is the usual orthonormal basis of $\mathbb{R}^8$ under the usual inner product $\langle e_i, e_j\rangle = \delta_{i,j}$. The parameters t_j are connected to the coordinates of x via

$$t_j = e^{2\pi ix_j}(pq)^{1/4}.$$

The balancing condition on the t_j's is guaranteed via the condition on the x_j's. The root system A_7 consists of vectors $\{e_i - e_j, i \ne j\}$. Reflections $x \to S_v(x) = x - \frac{2v\langle v,x\rangle}{\langle v,v\rangle}$ in the aforementioned vectors generate the Weyl group S_8. Note these reflections act on the hyperplane in $\mathbb{R}^8$ perpendicular to the space generated by $\sum e_i$. Adding the extra reflection in the vector $w = (-\sum_{0\le i\le 3} e_i + \sum_{4\le i\le 7} e_i)/2$ to the group S_8 generated the Weyl group $W(E_7)$. The roots for the system E_7 (a total of 126) are those for A_7 together with vectors of the form $v = (\sum \epsilon_ie_i)/2$ where $\epsilon \in \{+, -\}$ and $\langle v, \sum e_i\rangle = 0$.

In view of the above paragraph, it should now be clear how $W(E_7)$ acts on the parameters of the order 1 elliptic beta integral. For example, Theorem 2.1.3 corresponds to reflecting in the root $w = (-\sum_{0\le i\le 3} e_i + \sum_{4\le i\le 7} e_i)/2$.

We list two more consequences of the integral transformation above. We follow [Rai10] where these are proved in greater generality (in particular, as n-dimensional integrals).

Proposition 2.1.6.

$$(i)\quad I(t_0,\ldots,t_7) = \left(\prod_{0\le r\le 3, 4\le s\le 7} \Gamma_{p,q}(t_r t_s)\right) \times I(u/t_0, u/t_1, u/t_2, u/t_3, v/t_4, v/t_5, v/t_6, v/t_7),$$

where

$$u^2 = t_0t_1t_2t_3, v^2 = t_4t_5t_6t_7.$$

$$(ii)\quad I(t_0,\ldots,t_7) = \left(\prod_{0\le r<s\le 7} \Gamma_{p,q}(t_r t_s)\right) \times I(u/t_0, u/t_1, u/t_2, u/t_3, u/t_4, u/t_5, u/t_6, u/t_7),$$

where

$$u^2 = \sqrt{t_0t_1t_2t_3t_4t_5t_6t_7} = pq.$$

2.2 Some residue theory

In this section we present two results (one univariate, one multivariate) on residue theory. That is, we explain what happens to the elliptic beta and Selberg integrals as the contour is deformed so that it passes over some poles of the integrand. To be more precise, we deform the contour so that some poles that are outside move inside the contour and vice versa. The residue calculation is necessary because we will deal with (mostly) discrete phenomena, where two different parameters multiply in $p^{-\mathbb{N}}q^{-\mathbb{N}}$. In such a case the contour does not exist anymore (it gets pinched), and the integrals (order 0 or 1 elliptic beta or Selberg) become infinite. There is a way to make sense of this via the following results.

The univariate result is due to van Diejen and Spiridonov (see [vDS00], [vDS01]). The multivariate result, in addition to being derived in the aforementioned references from conjectural data, is part of a more general theory of taking residues of elliptic integrals due to Rains (see [Rai10]; see also the explicit computation in [vdBR11]). The univariate result is a consequence of the multivariate one, but we list it separately as it will appear prominently in the next section. Also in the univariate result, notice the balancing on the parameters: they multiply to q (for the order zero beta integral; replace by q^2 for order 1) as opposed to the more usual pq (or $(pq)^2$). In the univariate case we are not concerned with the exact hypotheses on the parameters, as these can mostly be lifted (see the multivariate generalization).

Theorem 2.2.1. *Choose parameters $|p| < 1, |q| < 1$ and $t_0,\ldots,t_5$ such that $t_0t_1t_2t_3t_4t_5 = q$. Let C be a smooth Jordan contour around 0 invariant under $z \mapsto 1/z$ such that every ray from 0 intersects* *it exactly once and such that it separates the poles of ρ at $t_i p^{\mathbb{N}} p^{\mathbb{N}}$ (which it contains inside) from the reciprocal poles. Assume $|t_0| > 1, |t_i| < 1,\ 1 \leq i \leq 5$, and that t_i are "generic" and p is small enough (see reference for details). Then*

$$\frac{(p;p)(q;q)}{2} \int_C \frac{\Gamma_{p,q}(t_0 z^{\pm 1}, t_1 z^{\pm 1}, t_2 z^{\pm 1}, t_3 z^{\pm 1}, t_4 z^{\pm 1}, pt_5 z^{\pm 1})}{\Gamma_{p,q}(z^{\pm 2})} \frac{dz}{2\pi i z} =$$
$$\frac{(p;p)(q;q)}{2} \int_{\mathbb{T}} \frac{\Gamma_{p,q}(t_0 z^{\pm 1}, t_1 z^{\pm 1}, t_2 z^{\pm 1}, t_3 z^{\pm 1}, t_4 z^{\pm 1}, pt_5 z^{\pm 1})}{\Gamma_{p,q}(z^{\pm 2})} \frac{dz}{2\pi i z} +$$
$$A \sum_{l \geq 0, |t_0 q^l| > 1} q^l \frac{\theta_p(t_0^2 q^{2l})}{\theta_p(t_0^2)} \prod_{r=0}^{5} \frac{\theta_p(t_0 t_r; q)_l}{\theta_p(q t_0 / t_r; q)_l},$$

where $A = \frac{\Gamma_{p,q}(t_1 t_0^{\pm 1}, t_2 t_0^{\pm 1}, t_3 t_0^{\pm 1}, t_4 t_0^{\pm 1}, pt_5 t_0^{\pm 1})}{\Gamma_{p,q}(t_0^{-2})}$.

Remark 2.2.2. This calculation also works for 8 t parameters (order 1 case) balanced so that they multiply to $(pq)^2$ or q^2 (in which case we absorb 2 p's into the parameters like above). In fact it works in more generality than that, but we will only be concerned with order 0 or 1 case presently.

Remark 2.2.3. The reason for choosing parameters multiply to q instead of pq is that although such a choice breaks symmetry, it is more natural from the combinatorial perspective we will develop in subsequent chapters. Also, the Frenkel-Turaev summation/transformation (next section) usually appears in the literature with this choice of balancing condition.

Remark 2.2.4. The summand that appears above in the RHS is a $\Delta_l(t_0^2|q, t_0 t)1, \ldots, t_0 t_5; q)$ (univariate) symbol as per the Introduction.

Remark 2.2.5. The importance of such a calculation is proving the Frenkel-Turaev summation formula, whose statement we defer to the next section (and give a slightly different proof). The main point is though that two parameters $t_0 t_j$ ($j \neq 0$) of the 6 mentioned will multiply to q^{-N} for N a positive integer. Then what happens is that the LHS of the equation in Theorem 2.2.1 becomes infinite as the contour gets pinched by the poles of the integrand approaching it (after all, the integral on the LHS is generically an explicit product of elliptic gamma functions evaluated at products of pairs of parameters), and so will A on the RHS. The integral on the RHS though will be finite (no contour violation). Upon dividing by A and canceling poles, we observe the summation on the RHS (without the prefactor; it contains $N + 1$ terms) will be equal to whatever is left on the LHS. This is made explicit in the next section.

For the case of more than one variable (elliptic Selberg integral) things get more complicated with taking residues. For an extended discussion see the Appendix of [Rai10]. The multivariate theorem we state follows [vDS00]. Again, all of the conditions on the parameters (which we do not state) can be lifted, at the cost of the conditions on both contours becoming more complicated.

Theorem 2.2.6. *In addition to the parameters introduced already, let t be an extra complex parameter of modulus less than 1. Replace the balancing condition by $t^{2n-2}\prod t_i = q$. Let C as before.*

$$\int_{C^n} \prod_{1\le i<j\le n} \frac{\Gamma_{p,q}(tz_i^{\pm 1}z_j^{\pm 1})}{\Gamma_{p,q}(z_i^{\pm 1}z_j^{\pm 1})} \prod_{j=1}^{n} \frac{\Gamma_{p,q}(t_0 z_j^{\pm 1}, t_1 z_j^{\pm 1}, t_2 z_j^{\pm 1}, t_3 z_j^{\pm 1}, t_4 z_j^{\pm 1}, pt_5 z_j^{\pm 1})}{\Gamma_{p,q}(z_j^{\pm 2})} \prod_{j=1}^{n} \frac{dz_j}{2\pi i z_j} =$$

$$\sum_{m=0}^{n} 2^m m! \binom{n}{m} \sum_{0\le\lambda_1\le\cdots\le\lambda_m, |\tau_m q^{\lambda_m}|>1} A_m B_{m,\lambda} \int_{\mathbb{T}^{n-m}} \prod_{1\le i\le n-m, 1\le j\le m} \frac{\Gamma_{p,q}(t(\tau_j q^{\lambda_j})^{\pm 1} z_i^{\pm 1})}{\Gamma_{p,q}((\tau_j q^{\lambda_j})^{\pm 1} z_i^{\pm 1})} \times$$

$$\prod_{1\le i<j\le n-m} \frac{\Gamma_{p,q}(tz_i^{\pm 1}z_j^{\pm 1})}{\Gamma_{p,q}(z_i^{\pm 1}z_j^{\pm 1})} \prod_{j=1}^{n-m} \frac{\Gamma_{p,q}(t_0 z_j^{\pm 1}, t_1 z_j^{\pm 1}, t_2 z_j^{\pm 1}, t_3 z_j^{\pm 1}, t_4 z_j^{\pm 1}, pt_5 z_j^{\pm 1})}{\Gamma_{p,q}(z_j^{\pm 2})} \prod_{j=1}^{n-m} \frac{dz_j}{2\pi i z_j},$$

where $\tau_j = t_0 t^{j-1}$,

$$A_m = \prod_{1\le i<j\le m} \frac{\Gamma_{p,q}(t\tau_i^{-1}\tau_j^{\pm 1})}{\Gamma_{p,q}(\tau_i^{-1}\tau_j^{\pm 1})} \prod_{1\le i\le m} \frac{\Gamma_{p,q}(t_0\tau_j^{\pm 1}, t_1\tau_j^{\pm 1}, t_2\tau_j^{\pm 1}, t_3\tau_j^{\pm 1}, t_4\tau_j^{\pm 1}, pt_5\tau_j^{\pm 1})}{(p;p)(q;q)\Gamma_{p,q}(z_j^{\pm 2})}$$

$$B_{m,\lambda} = \prod_{1\le i<j\le m} \frac{\theta_p((\tau_i q^{\lambda_i})^{\pm 1}\tau_j q^{\lambda_j})}{\theta_p(\tau_i^{\pm 1}\tau_j)} \frac{\theta_p(t\tau_i\tau_j;q)_{\lambda_i+\lambda_j}\theta_p(t\tau_i^{-1}\tau_j;q)_{\lambda_j-\lambda_i}}{\theta_p(qt^{-1}\tau_i\tau_j;q)_{\lambda_i+\lambda_j}\theta_p(qt^{-1}\tau_i^{-1}\tau_j;q)_{\lambda_j-\lambda_i}} \times$$

$$\prod_{1\le j\le m} q^{\lambda_j} t^{2(n-j)\lambda_j} \frac{\theta_p(\tau_j^2 q^{2\lambda_j})}{\theta_p(\tau_j^2)} \prod_{0\le r\le 5} \frac{\theta_p(\tau_j t_r;q)_{\lambda_j}}{\theta_p(q\tau_j/t_r;q)_{\lambda_j}}.$$

Remark 2.2.7. For the rest of the paper we will be interested in the case $t = q$, and this will simplify some of the terms above. Just as in the univariate case, this makes sense for more than 6 parameters as long as the balancing condition is changed appropriately.

Furthermore, this also gives rise to a multivariate (discrete elliptic Selberg) evaluation generalizing the Frenkel-Turaev summation discussed in the next section. The proof goes through a similar analysis as in the univariate case, except extra care is needed due to the presence of multiple variables being integrated over. We refer the reader to [vDS00] for details.

2.3 Elliptic hypergeometric series

In this section we will discuss discrete integral analogs of the elliptic hypergeometric integrals of the previous section. That is, we will be concerned with elliptic hypergeometric series. These were first introduced by Frenkel and Turaev [FT97] who proved most of the results in this section (using different methods than what we will present), but we will mostly follow [Spi02b] for the notation and [Spi08] for proofs. A textbook account of this and other q-hypergeometric identities (the limit $p \to 0$) can be found in the book by Gasper and Rahman [GR04].

A *theta hypergeometric series* is a formal series of the form:

$$
{}_rE_s\left(\begin{matrix} t_0,\ldots,t_{r-1} \\ w_1,..,w_s \end{matrix};q;p;z\right)=\sum_{k=0}^{\infty}\frac{\theta_p(t_0,\ldots,t_{r-1};q)_k}{\theta_p(q,w_0,\ldots,w_s;q)_k}q^{\binom{k}{2}}z^n.
$$

We will only be interested in ${}_{r+1}E_r$ which we call *balanced* if the top parameters balance the bottom parameters:

$$
\prod_{0\le i\le r}t_i=q\prod_{1\le i\le r}w_i.
$$

We want to add two additional restrictions on the parameters to obtain what are called *very-well-poised* (also balanced) series, which must also satisfy

$$
\begin{aligned}
&qt_0=t_1w_1=\cdots=t_rw_r,\\
&\{t_{r-3},t_{r-2}\}=\{\pm t_0^{1/2}q\},\ \{t_{r-3},t_{r-2}\}=\{\pm t_0^{1/2}qp^{\mp 1/2}\}.
\end{aligned}
$$

For a very-well-poised balanced elliptic hypergeometric series, the k-th summand becomes

$$
\frac{\theta_p(t_0q^{2n})}{\theta_p(t_0)}\prod_{0\le i\le r-4}\frac{\theta_p(t_m;q)_k}{\theta_p(qt_0/t_m;q)_k}(-qz)^k.
$$

We can write this more symmetrically by reparametrizing (changing the t parameters and $z\mapsto -z$). We will also downsize our notation, so a very-well-poised balanced elliptic hypergeometric series is one of the form

$$
{}_{r+1}E_r(t_0;t_1,\ldots,t_{r-4};q;p;z)=\sum_{k\ge 0}\frac{\theta_p(t_0^2q^{2n})}{\theta_p(t_0^2)}\prod_{0\le i\le r-4}\frac{\theta_p(t_0t_m;q)_k}{\theta_p(qt_0/t_m;q)_k}(qz)^k.
$$

Remark 2.3.1. The summands above are elliptic in the parameters t_i and q. Note they are also Δ_k-symbols. Also, the ratio of the $k+1$-st summand over the kth summand is an elliptic function of q^k—hence the name *elliptic hypergeometric*. This is in analogy with ordinary and q-hypergeometric series. A series $\sum c_k$ is called *hypergeometric* (*q-hypergeometric*) if c_{k+1}/c_k is a rational function of k (respectively q^k). Then c_k is a ratio of Pochhammer symbols $(a)_k=a(a+1)\cdot\cdots\cdot(a+k-1)$ (respectively q-Pochhammer symbols $(a;q)_k=(1-a)(1-aq)\cdot\cdots\cdot(1-aq^{k-1})$). In the elliptic case c_k is a ratio of theta-Pochhammer symbols, and letting $p\to 1$ degenerates such series into q-hypergeometric ones.

In general the convergence of such infinite theta series is difficult to study because of the quasi periodicity of theta functions, so one often imposes a termination condition. That is, if the product of two parameters $t_0t_i\in q^{-\mathbb{N}}$ the sum will only have finitely many nonzero terms due to the vanishing of high-order theta Pochhammer symbols. We will only be concerned with such series throughout.

Frenkel and Turaev discovered the following transformation satisfied by a very-well-poised balanced ${}_{12}E_{11}$ which we now state following notation in [Spi02b].

Theorem 2.3.2. *Let $t_0, \ldots, t_7$ be complex parameters such that $\prod t_i = q^2$ (balancing condition) and $t_0 t_6 = q^{-N}$ (termination condition). Then*

$$\begin{aligned} {}_{12}E_{11}(t_0; t_1, \ldots, t_7; q; p; 1) =& \frac{\theta_p(qt_0^2, qs_0/s_4, qs_0/s_5, q/t_4t_5; q)_N}{\theta_p(qs_0^2, qt_0/t_4, qt_0/t_5, q/s_4s_5; q)_N} \\ & \times {}_{12}E_{11}(s_0; s_1, \ldots, s_7; q; p; 1), \end{aligned}$$

where

$$s_0^2 = \frac{qt_0}{t_1t_2t_3},\ s_i = \frac{s_0 t_i}{t_0},\ 1 \leq i \leq 3,\ s_j = \frac{t_0 t_i}{s_0},\ 4 \leq j \leq 7.$$

Proof. Follows from the integral transformation of Theorem 2.1.3 along with the remarks of that section and a residue calculation similar to Theorem 2.2.1 and Remark 2.2.5 (with 8 parameters instead of 6 and the appropriate balancing condition). Note the termination condition in the t's (which forces the contour of 2.1.3 to blow up) is translated to a termination condition in the s's. □

We now state the following summation formula for a very-well-poised balanced ${}_{10}E_9$, due to Frenkel and Turaev [FT97]. The proof given here is an easy consequence of the transformation above (see for example [Spi02b], but also remarks in [Rai10]).

Theorem 2.3.3. *Let $t_0, \ldots, t_5$ satisfy the balancing condition $t_0t_1t_2t_3t_4t_5 = q$ and the (termination) condition $t_0t_4 = q^{-N}$. Then*

$${}_{10}E_9(t_0; t_1, \ldots, t_5; q; p; 1) = \frac{\theta_p(qt_0^2, \frac{q}{t_1t_2}, \frac{q}{t_1t_3}, \frac{q}{t_2t_3}; q)_N}{\theta_p(q/t_0t_1t_2t_3, qt_0/t_1, qt_0/t_2, qt_0/t_3;)_N}.$$

Proof. Plug in $t_2t_3 = q$ in Theorem 2.3.2, then decrease the labels of $t_4, \ldots, t_7$ by two. □

Remark 2.3.4. An alternative proof of this formula can be given via Theorem 2.2.1. This is sketched in Remark 2.2.5. Based on similar arguments (see [vDS00], [Rai10]; see also [Rai06] for an algebraic perspective), we can formulate and prove the following multivariate generalization:

Theorem 2.3.5. *For $t_0, \ldots, t_6, t, |p| < 1, |q| < 1$ complex parameters satisfying the balancing condition $t^{2n-2}\prod t_i = pq$ and the termination condition $t_0t_1 = q^{-N}t^{1-n}$ we have the following summation formula:*

$$\begin{aligned} \sum_{\lambda \in N^n} \Delta_\lambda(t^{2n-2}t_0^2 | t^n, t^{n-1}t_0t_1, \ldots, t^{n-1}t_0t_5; q; t; p) = \\ \Delta^0_{N^n}(t^{n-1}t_1/t_4 | t_1/t_0, pq/t_4t_2, pq/t_4t_3, pq/t_4t_5; q; t; p). \end{aligned}$$

2.4 An elliptic hypergeometric determinant

In this very short section we give yet another elliptic identity of the hypergeometric kind. In this case it is a determinantal identity discovered by Warnaar in [War02], which we follow for the proof. In fact we will not give the most general form of the identity but just a sufficiently powerful version needed for our purposes.

Theorem 2.4.1.

$$\det_{1\le i,j\le n}\left(\frac{\theta_p(az_i, ac/z_i;q)_{n-j}}{\theta_p(bz_i, bc/z_i;q)_{n-j}}\right) = a^{\binom{n}{2}}q^{\binom{n}{3}} \prod_{1\le i<j\le n} z_j\theta_p(z_i/z_j, c/z_iz_j)\prod_{i=0}^{n}\frac{\theta_p(b/a, abcq^{2n-2i};q)_{i-1}}{\theta_p(bz_i, bc/z_i;q)_{n-1}}.$$

Remark 2.4.2. If $c = 1$ the bivariate product appearing in the RHS is the elliptic Vandermonde-like product discussed in Section 1.2.

Proof. First, both LHS and RHS, viewed as a function of z_i, have the same multiplier (under the shift $z_i \mapsto pz_i$) via a direct computation. Thus their ratio, which we call f is an elliptic function in z_i. If we show it has no poles (or zeros), then it must be constant by Liouville's theorem. Both the LHS and RHS are also analytic in $\mathbb{C}^*$ (because θ_p is), so the only poles of f come from the zeros of the RHS, which are at $z_i = z_j$ and $z_i = c/z_j$ for $j \neq i$ (mod powers of p). Plugging either into the determinant will make two of the columns equal, and thus the determinant zero. Hence every zero of the RHS is canceled by one on the LHS, which means f has no poles and is therefore constant.

We now specialize at $z_i = q^{i-n}/a$ to obtain the constant. This will leave the LHS as a determinant of an upper triangular matrix (due to the vanishing of theta Pochhammer symbols in the numerator) which can be evaluated explicitly. This evaluation coincides with the RHS after the specialization and simplification of the theta Pochhammer symbols (and using the fact that $\sum_{1\le j\le n}(j-1)(n-j) = \binom{n}{3}$), so $f = 1$ as desired. $\square$

Chapter 3

Elliptic lozenge tilings

In this chapter we will set up the main statistical mechanical/combinatorial model studied for the rest of the thesis. We start with generalities on dimer coverings on bipartite graphs—for which a good reference are the lecture notes by Kenyon [Ken09] (but we will only be interested in detail in the honeycomb graph). We then introduce an elliptic measure on a certain class of dimer coverings in the honeycomb lattice. We finish by deriving some properties of the measure and by making some explicit computations.

3.1 On dimers and tilings

We start with a bipartite planar graph $G = (V, E)$ where the vertex set V (if finite) has an even number of elements, half of which we call black and half of which we call white: $V = B \cup W$ (that is, we impose that the bipartite structure on the graph leads to such half-half splitting). A *dimer covering* of G is a subset of edges in G such that each vertex is covered by (belongs to) exactly one edge and every vertex of G is covered. By the bipartite structure, every edge e in such a dimer covering will connect a black vertex to a white one, and we will denote it $e = (b, w)$ when we find such notation convenient. We can also talk about dimer coverings of infinite bipartite planar graphs (we focus on the honeycomb lattice—the dual of the triangular lattice).

In this thesis we will be concerned with dimer coverings of certain finite subsets of the honeycomb lattice. We will call them lozenge tilings of a hexagon because if one looks at the dual picture, it is a tiling of an $a \times b \times c$ hexagon in the triangular lattice tiled by 3 types of lozenges (rhombi with 2 acute angles of $\pi/3$). There are 3 types of lozenges because there are 3 ways of making a rhombus out of two adjacent triangles in the lattice. Such a tiling is depicted in Figure 3.1 (left), along with its interpretation as a three-dimensional *stepped surface* (right). This second interpretation will be important to us. It corresponds to packing unit cubes in a rectangular parallelepiped where the cubes gather in the (hidden) corner of the bounding box (physically, we can think of the cubes being acted on by gravity which points towards the origin in the perpendicular direction to the plane

$x+y+z=0$). To see that every such tiling gives rise to a dimer cover of a portion of the honeycomb lattice, proceed as follows: call all left-pointing triangles in the lattice black and all right-pointing ones white, and for each rhombus present in the tiling, draw a line between the centroid of its black triangle and the centroid of its white triangle. This will give a matching of the dual graph (whose vertices consist of all the centroids). Each edge in this matching joins a black vertex (triangle) to a white one. See Figure 3.2.

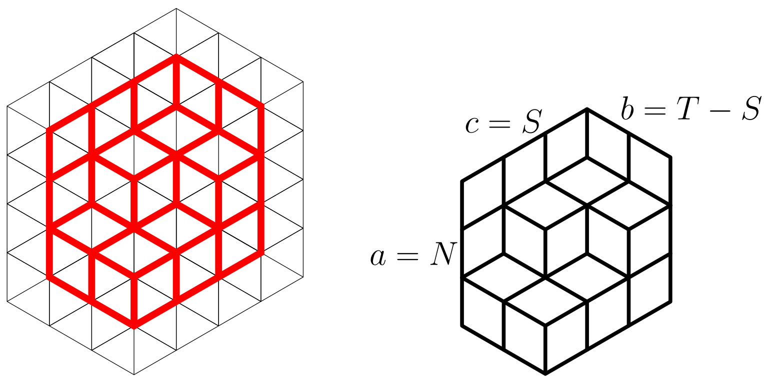

Figure 3.1: A tiling of a $3 \times 2 \times 3$ hexagon and the associated stepped surface.

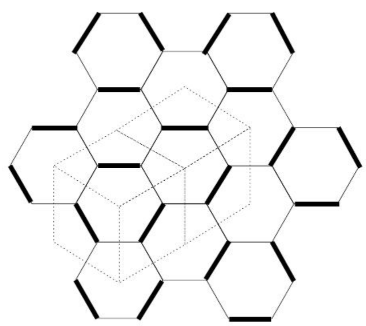

Figure 3.2: Duality between tilings and matchings, as appears in [Ken97] (figure used with permission).

A yet different way of viewing such tilings, important hereinafter, is as collections of nonintersecting paths in the square lattice. The paths start at consecutive points on the vertical axis (counting from the origin upwards) and end at consecutive points on a vertical line with displacement $b+c$ from the origin. Each path is composed of horizontal segments or diagonal (southwest to northeast, slope 1) segments, and the paths are required not to intersect. Given any such collection of paths, we can recover a hexagonal tiling (and hence a dimer cover) from it and vice versa. For reasons that will become clear as we progress, we will find it more convenient to encode the hexagon via the following three numbers:

$$N = a, T = b + c, S = c.$$

Figure 3.3 explains the above, and also introduces two coordinate systems useful later on: Cartesian coordinates (i, j) for the hexagon picture and (t, x) for the nonintersecting paths in the square lattice picture. They are related by

$$(i, j) = (t, x - t/2).$$

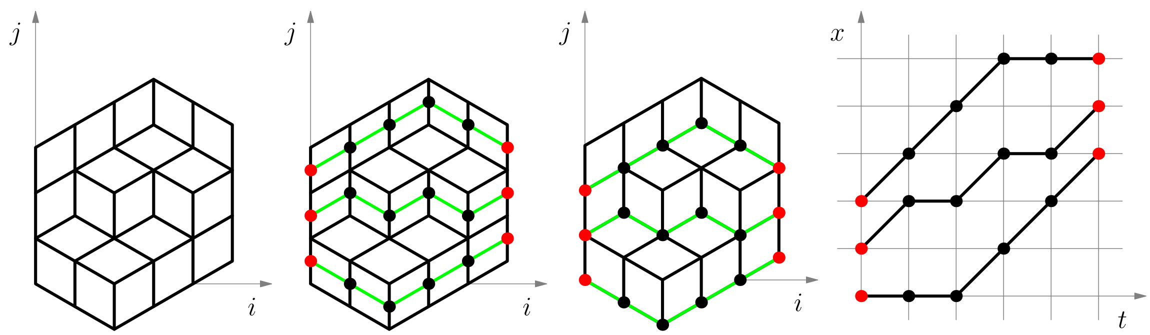

Figure 3.3: Duality between tilings and nonintersecting paths.

Following the notation in [BG09], let $\Omega(N, S, T)$ denote the set of N nonintersecting paths in the lattice $\mathbb{N}^2$ starting from positions $(0, 0), \ldots, (0, N-1)$ and ending at positions $(T, S), \ldots, (T, S + N - 1)$. Each path has segments of slope 0 or 1 (as explained above). Set

$$\mathfrak{X}_{N,T}^{S,t} = \{x \in \mathbb{Z} : \max(0, t + S - T) \leq x \leq \min(t + N - 1, S + N - 1)\},$$
$$\mathcal{X}_{N,T}^{S,t} = \{X = (x_1, \ldots, x_N) \in (\mathfrak{X}_{N,T}^{S,t})^N : x_1 < x_2 < \cdots < x_N\}.$$

$\mathfrak{X}_{N,T}^{S,t}$ is the set of all possible particle positions in a section vertical section of our hexagon with horizontal coordinate t (in (t, x) coordinates). $\mathcal{X}_{N,T}^{S,t}$ is the set of all possible N-tuples of particles in the same vertical section.

For $X \in \Omega(N, S, T)$, we have $X = (X(t))_{0 \leq t \leq T}$ and each $X(t) \in \mathcal{X}_{N,T}^{S,t}$. X is a discrete time Markov chain as it will be shown.

To any dimer cover of a planar bipartite graph one can associate a *height*, which is a function from dimer covers to (usually real, often rational) numbers. While we could do this in generality, we restrict attention to lozenge tilings of the hexagon (and the associated dimer covers). We want to formalize the three-dimensional "height" one sees in Figure 3.1. We will be skipping most homological details, trivial as they may be in the case of bipartite planar graphs.

Fixing a (bipartite planar) graph $G = (V, E)$, we define Λ_0 to be the space of (real-valued) functions on V, and Λ_1 to be the space of *flows* (*1-forms*) whereby a flow is a function on oriented edges (each $e \in E$ has two orientations) antisymmetric under changing orientation of edges. There is a natural linear map $d : \Lambda_0 \to \Lambda_1$ defined by $d(f)(vv') = f(v') - f(v)$ where by vv' we mean an edge in E oriented from v to v'. This map has a (linear algebraic) transpose $d^* : \Lambda_1 \to \Lambda_0$ defined by $d^*(\omega)(v) = \sum_e \omega(e)$ where ω is a flow and the sum is over all edges incident at v. This map is called the divergence. If $d^*(\omega)(v)$ is positive (negative), we say v is a source (sink) for the flow ω.

Given a dimer cover M of G, we associate the flow ω_M as follows: $\omega_M(e = (b, w)) = 1$ if (b, w) is an (oriented black to white) edge present in the cover, and $\omega(e) = 0$ if e is not in M. This flow has divergence $+1$ (-1) at black (white) vertices. For M_1, M_2 dimer covers of G, $\omega_{M_1} - \omega_{M_2}$ is divergence free.

Given a divergence free flow ω on G and a reference (fixed) face f_0, one defines a height function h on all other faces as follows. First $h(f_0) = 0$ (hence the name). For any other face f, pick a path p in the dual graph from f_0 to f. We define $h(f)$ as the net (signed sum) flow of p as it crosses the edges of G with the caveat that as p crosses an unoriented edge e, we add to the sum the contribution $w(e$: oriented left to right). h is independent of the path p chosen in the dual graph since ω is divergence free.

For G we can fix a base flow ω_0 with divergence 1 (-1) at black (white) vertices. Given M a dimer cover, we can construct the flow ω_M as above. Choosing a reference face f_0, we can associate to the difference flow (which is divergence free) a height function h_M as in the previous paragraph. We call it the *height of the dimer cover* (equivalently, of the lozenge tiling for the case of honeycomb dimers).

For the lozenge tilings dual to honeycomb dimers we are interested in, a natural base flow ω_0 is defined by $\omega_0(e = (b, w)) = 1/3$ where the edge (b, w) comes with black to white orientation. In this case, for a given dimer cover M, the height h_M is (up to an additive constant), when evaluated at the center of a face, the distance between that center and the plane $x + y + z = 0$ when we interpret M as a stepped surface. See Figure 3.4 where on the left we exhibit the height of the dimer cover and on the right, the distance (multiplied by $\sqrt{3}$) from the interior vertices of the tiling to the plane $x + y + z = 0$.

3.2 Kasteleyn theory

Given a bipartite planar graph $G = (V, E)$ that has dimer covers (in particular, if G is finite, V needs to be even), we can attach various weights $w : E \to \mathbb{C}$ to its edges. We can then consider the following Boltzmann measure on dimer covers M of G:

$$\mu(M) = \frac{1}{Z} \prod_{e \in M} w(E), \tag{3.2.1}$$

where $Z = \sum_M \prod_{e \in M} w(E)$ is the partition function. We will talk about probability measures hereinafter, so we want μ to take positive real values between 0 and 1 (this does not mean that w need to take such values). Because of the product nature of the Boltzmann distribution, we can

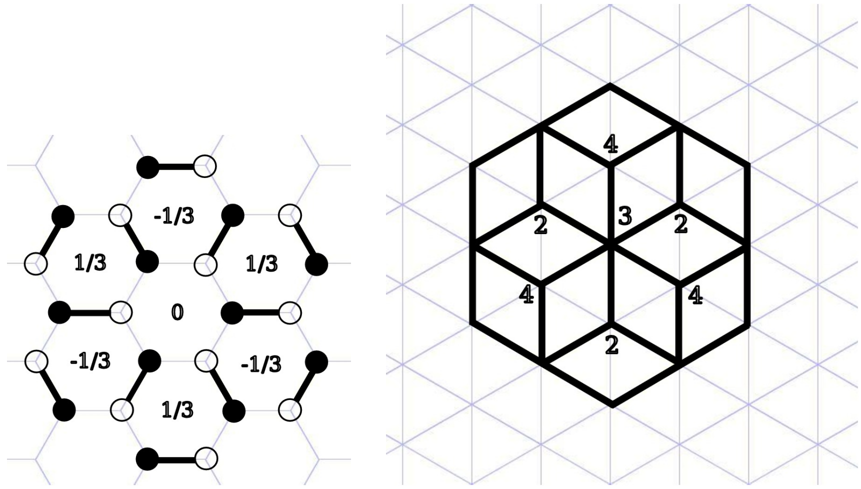

Figure 3.4: The height function on a matching (left) and on a stepped surface/lozenge tiling (multiplied by $\sqrt{3}$).

change the function w by multiplying weights of all edges incident to a single vertex u by a number c. Then both Z and the numerator in (3.2.1) will get multiplied by c (because in the sum over all matchings, the vertex u is going to be matched, so c will multiply every term in the partition function, as well as the denominator). Therefore, the measure μ will not change, though we are looking at a new Boltzmann weight w'. We call two such weights *gauge equivalent* if they differ by a finite number of such multiplications.

There is a simple criterion (see, e.g.,[Ken09]) to test whether two weights w and w' on the graph are gauge equivalent: they are so if and only if for every face bounded by edges $e_1, \ldots, e_{2k}$ (listed consecutively; note every face has an even number of edges due to the bipartite structure), we have

$$\frac{w(e_1)w(e_3)\ldots w(e_{2k-1})}{w(e_2)w(e_4)\ldots w(e_{2k})} = \frac{w'(e_1)w'(e_3)\ldots w'(e_{2k-1})}{w'(e_2)w'(e_4)\ldots w'(e_{2k})}.$$

We will now define a *Kasteleyn sign weight* on the graph G. It is a choice of signs assigned to every edge such that each face with 0 mod 4 edges around it has an odd number of $-$ signs around it, and each face bounded by 2 mod 4 edges has an even number of minus signs around it. Note for dimers of the honeycomb graph (or parts of it), there is a very convenient choice of Kasteleyn weight coming from the fact that $6 = 2$ mod 4: just put a $+$ sign on every edge. This is the only case we will be interested in the present work, but in general one can prove existence of such weights via spanning trees (see [Kas67] and [TF61]).

The *Kasteleyn matrix* K associated to a planar bipartite graph G whose vertex set $V = B \cup W$ consists of black and white vertices is defined by assigning to every edge (b, w) the weight of that edge times its Kasteleyn sign. To every pair (b, w) that does not share an edge, we set $K(b, w) = 0$. For the honeycomb lattice, it is just the weighted adjacency matrix since we can choose all $+$ signs for the Kasteleyn sign weight.

We have the following theorem from [Kas67], [TF61] for computing the partition function Z (the total weight of all matchings). Set $G(V, E)$ to be a finite bipartite planar graph with an even number of edges such that $V = B \cup W$.

Theorem 3.2.1.

$$Z = |\det K|.$$

Proof. We will only prove this for the honeycomb lattice, in which case the proof simplifies a lot. We follow [Ken09]. We first expand the determinant as

$$\det K = \sum_{\sigma \in S_n} (-1)^\sigma K(b_1, w_{\sigma(1)}) \ldots K(b_n, w_{\sigma(n)}).$$

Notice each term in the above sum is 0 unless each vertex b_i in the product is paired with an adjacent w_{σ_i}. Hence for each dimer cover we find a nonzero term in the sum and vice versa. The summand in question is indeed the total weight of that dimer cover by definition.

So all there is to check is that all signs appearing in front of the nonzero terms are the same. It suffices to show that given a reference cover (a choice of σ), all other covers are obtained my multiplying σ by even permutations. But this can be translated into the tiling picture of Figure 3.1 bijectively. There, it is easy to see that we can get from any tiling to any other tiling by changing unit cubes, one at a time, from empty to full (or vice versa). In particular we can reach any tiling from the empty box tiling with such moves. But every such move is local, on a unit cube alone. Such empty/full (or full/empty) swap corresponds to swapping the two dimer covers of a hexagonal in the honeycomb lattice - a $2\pi/3$ rotation. It corresponds to multiplying the initial permutation σ (corresponding to the initial cover) by a 3-cycle - an even permutation. Hence all the terms in the sum have the same sign as the term corresponding to the empty tiling (being obtained from it by multiplying the "empty" σ by even permutations). This concludes the proof. □

The next theorem, due to Kenyon [Ken97] will allow us to compute total weight of all matchings containing certain prescribed edges. We will omit the proof but see the reference (the proof uses the Jacobi lemma relating minors of a matrix with its inverse).

Theorem 3.2.2. *The total weight of matchings containing n fixed edges $(b_1, w_1), \ldots, (b_n, w_n)$ is equal to*

$$\left(\prod_{i=1}^{n} K(b_i, w_i)\right) \det_{1 \le i,j \le n} K^{-1}(w_i, b_j).$$

A version of this theorem which we find useful in computations is the following.

Theorem 3.2.3. *The total weight of matchings of the graph G' which is obtained from G by removing n black vertices b_i and n white vertices w_j (not necessarily adjacent) is, up to a constant independent* *of* b_i *and* w_j*:*

$$\det_{1\le i,j\le n} K^{-1}(w_i, b_j).$$

Remark 3.2.4. Theorem 3.2.2 can be deduced from 3.2.3 since fixing certain edges in a matching is equivalent to removing the corresponding vertices, computing the total weight, and then adding the vertices back in the matching with prescribed edges (in which case we have to multiply by the weights of those edges).

Remark 3.2.5. Both of the previous theorems follow from the Jacobi lemma relating minors of a matrix M with those of its inverse. Let M be nonsingular and $Adj(M)$ (the adjugate of M) be defined by $Adj(M)_{i,j} = (-1)^{i+j} M_{\hat{j},\hat{i}}$ where $M_{\hat{i},\hat{j}}$ is the determinant of the matrix obtained by removing the i-th row and j-th column from M. Then $M^{-1}_{i,j} = \frac{1}{\det M} Adj(M)_{i,j}$. In fact, more is true: any $k \times k$ minor in $Adj(M)$ is equal to the complementary signed minor in M^T (the transpose of M) times $(\det M)^{k-1}$. As a corollary, if N' is a proper square submatrix of M^{-1}, then $|\det N'| = |\det N|/|\det M|$ for some proper square submatrix N of M.

For the honeycomb graph, we can enumerate matchings differently. We look at the associated tiling, and then at the associated collection of nonintersecting paths. The total weight of all nonintersecting path collections (equivalently, all dimer covers) is then given by the Lindström-Gessel-Viennot lemma, which we now state in more generality (see [Ste90] and references therein):

Theorem 3.2.6. *For a planar weighted connected graph* G*, let* $(u_1, \dots, u_n)$ *be a tuple of starting points and* $(v_1, \dots, v_n)$ *be a tuple of ending points. Assume there are no nonintersecting path collections pairing (start-end)* u_i *to* v_{σ_i} *for a nontrivial* $\sigma \in S_n$*. Moreover, let* $T(u,v)$ *be the total weight of all paths from* u *to* v *in the graph (the weight of one path is the product of the edge weights over edges in the path). Then the total weight of all nonintersecting paths from the starting tuple to the ending one such that the* i*-th path starts at* u_i *and ends at* v_i *is:*

$$\det_{1\le i\le n} T(u_i, v_j).$$

3.3 Elliptic measure on tilings and canonical coordinates

We will now define the (Boltzmann) probability measure on $\Omega(N,S,T)$ (equivalently, on tilings of the $a\times b\times c$ hexagon) that will be the object of study. For a tiling $\mathcal{T}$ corresponding to an $X \in \Omega(N,S,T)$ we define its weight to be

$$w(\mathcal{T}) = \prod_{l\in\{\text{horizontal lozenges}\}} w(l),$$

where by a horizontal lozenge we mean a lozenge whose diagonals are parallel to the i and j axes. The probability of such a tiling would then simply be

$$Prob(\mathcal{T}) = \frac{w(\mathcal{T})}{\sum_{\mathcal{S}\in\Omega(N,S,T)} w(\mathcal{S})}.$$

The weight function w on horizontal lozenges l is defined by

$$\begin{aligned} w(l) := w(i,j) &= \frac{(u_1u_2)^{1/2}q^{j-1/2}\theta_p(q^{2j-1}u_1u_2)}{\theta_p(q^{j-3i/2-1}u_1, q^{j-3i/2}u_1, q^{j+3i/2-1}u_2, q^{j+3i/2}u_2)} \\ &= \frac{(v_1v_2)^{1/2}q^{j-S/2-1/2}\theta_p(q^{2j-S-1}v_1v_2)}{\theta_p(q^{j-3i/2-S-1}v_1, q^{j-3i/2-S}v_1, q^{j+3i/2-1}v_2, q^{j+3i/2}v_2)}, \end{aligned} \tag{3.3.1}$$

where (i,j) is the coordinate of the top vertex of the horizontal lozenge l, u_1, u_2, q, p are complex parameters ($|p| < 1$) and $u_1 = q^{-S}v_1, u_2 = v_2$ (the reason for the v parameters is that this break in symmetry will make other formulas throughout the thesis more symmetric).

Remark 3.3.1. Only considering weights of horizontal lozenges for a tiling of a hexagon is equivalent to considering all types of lozenges but assigning the other two types weight 1 (i.e., each lozenge that is not horizontal has weight 1). This is a break in symmetry that can easily be fixed. However, for most computations in this Chapter and next we prefer this non-symmetric weight assignment system as it makes some things easier. Nevertheless, we show in Section 3.8 that we can assign weights to the 3 types of lozenges in an S_3-invariant way (i.e., invariant under permuting the 3 types of lozenges or equivalently the 3 spatial directions) by changing gauge.

This weight on dimer coverings of a hexagon was derived in [BGR10] (see also [Sch07] for the nonintersecting paths derivation), and the derivation will be sketched in Section 3.6.

The connection with elliptic functions will now be explained. Fix a horizontal coordinate i, denote by $w(i,j)$ the weight of the horizontal lozenge with top vertex coordinates (i,j), and observe that for two consecutive vertical positions we have ($u_1u_2u_3 = 1$):

$$\begin{aligned} r(i,j) = \frac{w(i,j+1)}{w(i,j)} &= \frac{q^3\theta_p(q^{j-3i/2-1}u_1, q^{j+3i/2-1}u_2, q^{-2j-1}u_3)}{\theta_p(q^{j-3i/2+1}u_1, q^{j+3i/2+1}u_2, q^{-2j+1}u_3)} \\ &= \frac{q^3\theta_p(q^{j-3i/2-S-1}v_1, q^{j+3i/2-1}v_2, q^{-2j+S-1}/v_1v_2)}{\theta_p(q^{j-3i/2-S+1}v_1, q^{j+3i/2+1}v_2, q^{-2j+S+1}/v_1v_2)}. \end{aligned} \tag{3.3.2}$$

In the three-dimensional coordinates (x,y,z) pictured in Figure 3.5 (note we only consider surfaces in 3 dimensions that are *stepped*, meaning there is a one to one correspondence between the two-dimensional tiling picture and the three-dimensional surface picture) with $i = x-y, j = z-(x+y)/2$ (after shifting (i,j) so that origin is at the hidden corner of the box), the weight ratio is equal to:

$$r(x,y,z) = \frac{w(\text{full box})}{w(\text{empty box})} = \frac{q^3\theta_p(\tilde{u}_1/q, \tilde{u}_2/q, \tilde{u}_3/q)}{\theta_p(\tilde{u}_1q, \tilde{u}_2q, \tilde{u}_3q)}, \tag{3.3.3}$$

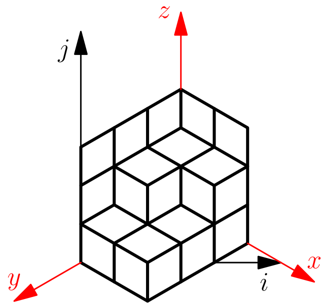

Figure 3.5: Going from 3 dimensions to 2 dimensions.

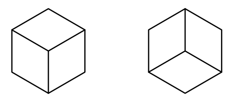

Figure 3.6: A full $1 \times 1 \times 1$ box (left) and an empty one (right).

where

$$\tilde{u}_1 = q^{y+z-2x} u_1,\ \tilde{u}_2 = q^{x+z-2y} u_2, \tilde{u}_3 = q^{x+y-2z} u_3,\ u_1 u_2 u_3 = 1,$$

and (x, y, z) is the three-dimensional centroid of the $1 \times 1 \times 1$ full cube (on the left in Figure 3.6) with top lid the horizontal lozenge having top vertex coordinate (i, j).

The word *elliptic* now becomes clear as r in (3.3.3) is an elliptic function of q (that is, defined on $\mathbb{E}$—see the Introduction for details). Moreover, r is the unique elliptic function of q with zeros at $\tilde{u}_1, \tilde{u}_2, \tilde{u}_3$ and poles at $1/\tilde{u}_1, 1/\tilde{u}_2, 1/\tilde{u}_3$ normalized such that $r(1) = 1$. Of interest is also that r is elliptic in $\tilde{u}_k$ for $k = 1, 2, 3$ subject to the condition that $\prod_{k=1}^{3} \tilde{u}_k = 1$.

Remark 3.3.2. The above paragraph can be restated by observing we can build everything by choosing 3 points q, u_1, u_2 on the elliptic curve $\mathbb{E}$.

Remark 3.3.3. The weight ratio r is invariant under the natural action of S_3 permuting the $\tilde{u}_k$'s (and of course the 3 axes: x, y, z).

We can view our tilings as stepped surfaces composed of $1 \times 1 \times 1$ cubes bounded by the 6 planes $x = 0, y = 0, z = 0, x = b, y = c, z = a$. Then the two-dimensional picture in Figure 3.1 can be viewed as a projection of the 3 dimensional stepped surface onto the plane $x + y + z = 0$.

For $\mathcal{T}$ a tiling, we have

$$wt(\mathcal{T}) = \prod_{\Diamond \in \mathcal{T}} w(i, j),$$

where (i, j) are the coordinates of the top vertex of a horizontal lozenge. Grouping all $1 \times 1 \times 1$

cubes into columns in the z direction with fixed (x, y) coordinates (see Figure 3.5), we obtain

$$wt(\mathcal{T}) = const \cdot \prod_{\text{cube}} \frac{w(i, j+1)}{w(i, j)},$$

where the product is taken over all cubes (visible and hidden) of the boxed plane partition and (i, j) is the top coordinate of the bounding hexagon of a $1 \times 1 \times 1$ cube. Note to get to this equality we have merely observed that $wt(\text{empty box})$ is a constant independent of i and j. We can further refine this (rearranging the terms in the product and gauging away more constants—see Section 2.3 of [BGR10] and also Section 3.1 above for more details) as

$$wt(\mathcal{T}) = const \cdot \prod_v \left(\frac{w(i, j+1)}{w(i, j)} \right)^{h(v)} = const \cdot \prod_v r(i, j)^{h(v)},$$

where $v = (x_0, y_0, z_0)$ ranges over all vertices on the border (but not on the bounding hexagon) of the stepped surface with x_0, y_0, z_0 integers (equivalently, v ranges over all vertices of the triangular lattice inside the hexagon, but we view v in 3 dimensions). $h(v)$ is the distance from v to the plane $x + y + z = 0$ divided by $\sqrt{3}$: $h(v) = (x_0 + y_0 + z_0)/3$.

For the remainder of the section, we will discuss the concept of *canonical coordinates* for the geometry of elliptic tilings. That is, it will be convenient for various computations to express the geometry of an elliptic lozenge tiling in terms of coordinates on a certain product of elliptic curves. First we will introduce 6 parameters A, B, C, D, E, F depending on q, t, S, T, N, v_1, v_2 (note we have listed, other than q, 6 parameters, of which 4 are discrete and dictate the geometry: t, S, T, N). t here is a (discrete) time parameter and ranges from 0 to T. It will be explained better in Chapter 4. It corresponds to the fact that we will be interested in distributions of particles on a certain vertical line: that is, tilings of hexagons that have prescribed positions of particles (or holes) on the vertical line with horizontal coordinate t.

The set of parameters of interest to us is

$$\begin{aligned} A &= q^{t/2+S/2-T+1/2}\sqrt{v_1 v_2}, & B &= q^{t/2+S/2+T+1/2}\sqrt{\frac{v_2}{v_1}}, \\ C &= q^{t/2-S/2-N+1/2}\frac{1}{\sqrt{v_1 v_2}}, & D &= q^{-t/2+S/2-N+1/2}\frac{1}{\sqrt{v_1 v_2}}, \\ E &= q^{-t/2-S/2+1/2}\sqrt{\frac{v_1}{v_2}}, & F &= q^{-t/2-S/2+1/2}\sqrt{v_1 v_2}. \end{aligned} \tag{3.3.4}$$

Observe that they satisfy a certain balancing condition (like the one in Section 2.3):

$$q^{2N-2}ABCDEF = q. \tag{3.3.5}$$

The weight function (to be more precise, the ratio of weights of a full to an empty $1 \times 1 \times 1$ box—see (3.3.3)) depends on the geometry of the hexagon via the three parameters $\tilde{u}_1, \tilde{u}_2, \tilde{u}_3$ ($\prod \tilde{u}_k = 1$) which in the (i, j) coordinates are

$$(\tilde{u}_1 = q^{j-3i/2-S} v_1, \tilde{u}_2 = q^{j+3i/2} v_2, \tilde{u}_3 = q^{-2j+S}/v_1 v_2).$$

What we want is to change coordinates from (i, j) (2-dimensional) or (x, y, z) (three-dimensional) to $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ via the above formula. We call these new coordinates *canonical*. Each line of interest in the geometry has an equation in the (i, j) plane which can then be translated in terms of the $\tilde{u}_k$'s by solving in (3.3.4) for t, S, T, N, v_1, v_2 in terms of A, B, C, D, E, F. We thus find the following equations for the relevant edges of our hexagon:

$$\begin{aligned}
&\text{Left vertical edge (corresp. equation : } i = 0) : \frac{\tilde{u}_1}{\tilde{u}_2} = q^{-S} v_1/v_2 = \left(\frac{ABC}{DEF}\right)^{1/2} E^3 q^{-3/2}, \\
&\text{Right vertical edge (corresp. equation : } i = T) : \frac{\tilde{u}_1}{\tilde{u}_2} = q^{-3T-S} v_1/v_2 = \left(\frac{ABC}{DEF}\right)^{1/2} B^{-3} q^{3/2}, \\
&\text{NW edge (corresp. equation : } j = i/2 + N) : \frac{\tilde{u}_3}{\tilde{u}_1} = q^{2S-3N} 1/v_1^2 v_2 = \left(\frac{ABC}{DEF}\right)^{1/2} D^3 q^{-3/2}, \\
&\text{SE edge (corresp. equation : } j = i/2 - (T - S)) : \frac{\tilde{u}_3}{\tilde{u}_1} = q^{3T-S} 1/v_1^2 v_2 = \left(\frac{ABC}{DEF}\right)^{1/2} A^{-3} q^{3/2}, \\
&\text{NE edge (corresp. equation : } j = -i/2 + S + N) : \frac{\tilde{u}_2}{\tilde{u}_3} = q^{2S+3N} v_1 v_2^2 = \left(\frac{ABC}{DEF}\right)^{1/2} C^{-3} q^{3/2}, \\
&\text{SW edge (corresp. equation : } j = -i/2) : \frac{\tilde{u}_2}{\tilde{u}_3} = q^{-S} v_1 v_2^2 = \left(\frac{ABC}{DEF}\right)^{1/2} F^3 q^{-3/2}, \\
&\text{Vertical particle line (corresp. equation : } i = t) : \frac{\tilde{u}_1}{\tilde{u}_2} = q^{-3t-S} v_1/v_2 = \frac{DEF}{ABC}.
\end{aligned} \tag{3.3.6}$$

Remark 3.3.4. We can see from above that there exists a bijection between the six bounding edges of our hexagon and the 6 parameters A, B, C, D, E, F. That is, to an edge we assign the parameter that appears to the power ± 3 above. The 6 parameters are not independent (they satisfy the balancing condition (3.3.5)), but then neither are the 6 edges (they must satisfy the condition that the hexagon they form is tilable by the three types of rhombi, which in this case tautologically means the edges form the 6 visible frame-edges of a rectangular parallelepiped). See Figure 3.7.

With (3.3.6) in mind we have a (local) map $\mathbb{R}^2 \to \mathbb{E}^2$ (where $\mathbb{E}^2$ is isomorphic to the subvariety of $\mathbb{E}^3$ with coordinates $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ and relation $\prod \tilde{u}_i = 1$) which embeds our hexagon in $\mathbb{E}^2$:

$$(i, j) \mapsto (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3).$$

Note that $\mathbb{E}^2$ is the square of a real elliptic curve if parameters are chosen so that the weight

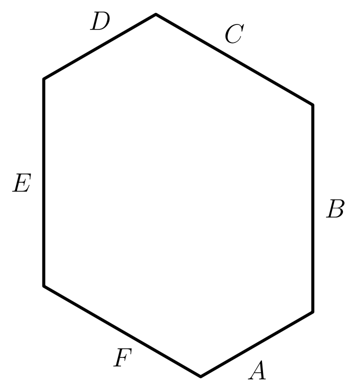

Figure 3.7: Correspondence between edges and the 6 parameters.

ratio (of full to empty $1 \times 1 \times 1$ box) is real (positive). Hence as $\mathbb{E}$ is homeomorphic to a circle or a disjoint union of two circles, the above embeds our hexagon in a 2-dimensional real torus (base field $= \mathbb{R}$).

Remark 3.3.5. In light of the discussion in this section, we can phrase the elliptic tiling model in such a way as to start with a (real Tate/multiplicative) elliptic curve $\mathbb{E}$ and points q, u_1, u_2, u_3 on it such that $u_1 u_2 u_3 = 1$ (they are on a line). From there we can derive the elliptic weight ratio as the unique function on the elliptic curve with prescribed divisor (3 poles and 3 zeros) which evaluates to 1 at $q = 1$. The hexagon with discrete parameters N, S, T is then the interior of the 6 edges described in terms of A, B, C, D, E, F (see (3.3.6)). While such an observation seems trivial at first, it nevertheless plays a pivotal role in many arguments in what follows.

3.4 Positivity of the measure

The content of the previous section shows that in order to make the whole model well defined as a probabilistic model, it suffices to establish positivity of the elliptic weight ratio $r(i,j) = w(i,j)/w(i,j-1)$ defined in (3.3.2) (where (i,j) is the location of a given horizontal tiling and ranges over all possible horizontal tilings inside the hexagon). Recall that

$$r(i,j) = \frac{q^3 \theta_p(\tilde{u}_1/q, \tilde{u}_2/q, \tilde{u}_3/q)}{\theta_p(q\tilde{u}_1, q\tilde{u}_2, q\tilde{u}_3)},$$

where $\tilde{u}_1 = q^{j-3i/2} u_1, \tilde{u}_2 = q^{j+3i/2} u_2, \tilde{u}_3 = q^{-2j} u_3$ and $u_1 u_2 u_3 = 1$. We recall that r is elliptic in $\tilde{u}_k$ for $k = 1, 2, 3$ as well as in q. In order to make r positive, we will first restrict ourselves to the case where r is real valued. This means r is defined over a real elliptic curve, and we have $-1 < p \neq 0 < 1$ (a priori, p is complex of modulus less than 1; $p \in (-1,1) - \{0\}$ is equivalent to $\mathbb{E}$ being defined over $\mathbb{R}$—for more on real elliptic curves, we refer the reader to Chapter 5 of [Sil94]). We then ensure positivity of r by an explicit computation. We will of course have two cases: $p < 0$ and $p > 0$. We deal with the case $p > 0$ throughout (and make remarks when necessary for $p < 0$).

Now that we have restricted ourselves to real elliptic curves $\mathbb{E}$, we first note that $q \in \mathbb{E}$ (i.e., r is elliptic as a function of q). For a chosen $0 < p < 1$ there are two nonisomorphic elliptic curves defined over $\mathbb{R}$ (since $Gal(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$), both homeomorphic to a disjoint union of two circles (every real elliptic curve is topologically homeomorphic to a circle if $p < 0$ or with a disjoint union of two circles if $p > 0$—one can just see this by plotting the Weierstrass equation in $\mathbb{R}^2$ and compactifying):

$$\mathbb{E} \cong_{\mathbb{R}} \mathbb{R}^*/p^{\mathbb{Z}}, \text{ and}$$
$$\mathbb{E} \cong_{\mathbb{R}} \{u \in \mathbb{C}^*/p^{\mathbb{Z}} : |u|^2 \in \{1, p\}\}.$$

We will call the first case real and the second trigonometric (abusing terminology, since both are real elliptic curves). We will analyze the trigonometric case, but the real case is similar (via a modular transformation). In the trigonometric case, the curve has two connected components (circles): the identity component (it contains the points 1 and -1) and another component that contains the other 2-torsion points: $\pm\sqrt{p}$. There will be 3 cases to be analyzed which we list now and motivate after (if $p < 0$ there is only one component so the 3 cases coalesce to only one—Case 2.):

- Case 1. q lies on the nonidentity component ($|q| = \sqrt{p}$).
- Case 2. q and all the u_k's (and so the $\tilde{u}_k$'s) lie on the identity component ($|q| = |u_1| = |u_2| = |u_3| = 1$).
- Case 3. q and one of the u_k's lies on the identity component, the other two u_k's lie on the nonidentity component.

To analyze positivity at a fixed site (i, j) inside the hexagon, we note that $r(q)$ has zeros at $\tilde{u}_k$ and poles at $1/\tilde{u}_k$ ($k = 1, 2, 3$). We note $r = \pm 1$ at $q = \pm 1$ so at least one u_k (along with its reciprocal/complex conjugate $1/u_k$) needs to be on the identity component (so that r can change signs on the identity component). Since $r = -1$ at $q = \pm\sqrt{p}$ and $u_1 u_2 u_3 = 1$, either exactly one or all three of the u's need to be on the identity component. This motivates the three choices above.

Case 1. will never lead to positivity for all four admissible sites (i, j) inside a $1 \times 2 \times 2$ hexagon (see Figure 3.8), so we can eliminate it (if a $1 \times 2 \times 2$ hexagon is never positive, much larger ones which are of interest to us will also never be as they contain the $1 \times 2 \times 2$ case). For a proof, we suppose that u_1 is on the identity component, and u_2, u_3 are (along with q) on the nonidentity component (the case where all three u's are on the identity component is handled similarly). The $\tilde{u}$'s differ from the u's by integer powers of q given in the last three columns of the following table (for the four admissible (i, j) pairs in the $1 \times 2 \times 2$ hexagon):

j	i	$j-3i/2$	$j+3i/2$	$-2j$
1/2	1	-1	2	-1
1	2	-2	4	-2
0	2	-3	3	0
1/2	3	-4	5	-1

Notice mod 2 (and we only care about mod 2 as q^2 is on the identity component), the four vectors (from the last 3 columns of the table) above are $(1,0,1),(0,0,0),(1,1,0),(0,1,1)$. The corresponding $\tilde{u}_k$'s we get by multiplying each u_k by q to the power coming from the vector $(0,1,1)$, $(\tilde{u}_1,\tilde{u}_2,\tilde{u}_3) = (q^{-4}v_1, q^5 v_2, q^{-1}v_3)$, will all be on the identity component, which means the elliptic weight ratio will be negative at the site $(i,j) = (1/2,3)$ as q is on the nonidentity component. This is a contradiction. The other cases are handled similarly, leading to contradictions. This proves q must be on the identity component, so only cases 2. and 3. above can lead to positive hexagons.

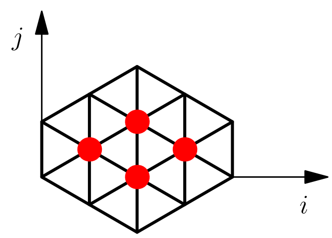

Figure 3.8: The admissible sites (i,j) inside a $1\times 2\times 2$ hexagon.

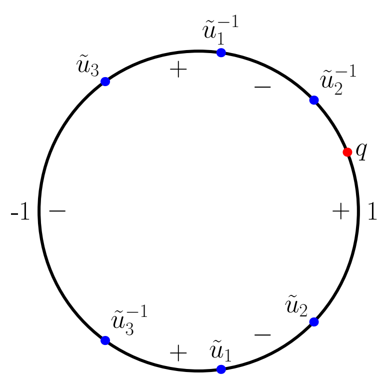

Figure 3.9: The identity component of $\mathbb{E} \cong_{\mathbb{R}} \{u \in \mathbb{C}^*/p^{\mathbb{Z}} : |u|^2 \in \{1,p\}\}$. For positivity of r throughout the hexagon (i.e., for all admissible $\tilde{u}_k$'s), q must always be closer to 1 than any $\tilde{u}_k^{\pm 1}$ as depicted.

I will next discuss the case where q and all u_k are on the identity component (case 2. above; for case 3. the reasoning is similar). For a fixed site (i,j) inside the hexagon, the 3 $\tilde{u}_k$'s and their reciprocals (complex conjugates) break down the unit circle into 6 arcs (see Figure 3.9) and q must be on one of the three arcs where r is positive (as depicted in the figure). If we want to ensure positivity of the ratio for all 4 admissible sites (i,j) within a given $1\times 2\times 2$ hexagon (Figure 3.8), we first observe that for $|x| = 1$:

$$\theta_p(x) = (1-x)\prod_{i\geq 1}|1-p^i x|^2.$$

So we reduce to positivity of the corresponding four functions $\prod_{i=1,2,3}\frac{1-\tilde{u}_i/q}{1-\tilde{u}_i q}$. Through standard trigonometric manipulations we thus want positivity of each of the following functions:

$$\begin{gathered}
\frac{\sin\pi(\alpha_1-\alpha)}{\sin\pi(\alpha_1+\alpha)}\cdot\frac{\sin\pi(\alpha_2-\alpha)}{\sin\pi(\alpha_2+\alpha)}\cdot\frac{\sin\pi(\alpha_3-\alpha)}{\sin\pi(\alpha_3+\alpha)},\\
\frac{\sin\pi(\alpha_1)}{\sin\pi(\alpha_1+2\alpha)}\cdot\frac{\sin\pi(\alpha_2-3\alpha)}{\sin\pi(\alpha_2-\alpha)}\cdot\frac{\sin\pi(\alpha_3)}{\sin\pi(\alpha_3+2\alpha)},\\
\frac{\sin\pi(\alpha_1-3\alpha)}{\sin\pi(\alpha_1-\alpha)}\cdot\frac{\sin\pi(\alpha_2)}{\sin\pi(\alpha_2+2\alpha)}\cdot\frac{\sin\pi(\alpha_3+\alpha)}{\sin\pi(\alpha_3+3\alpha)},\\
\frac{\sin\pi(\alpha_1+\alpha)}{\sin\pi(\alpha_1+3\alpha)}\cdot\frac{\sin\pi(\alpha_2-2\alpha)}{\sin\pi(\alpha_2)}\cdot\frac{\sin\pi(\alpha_3-2\alpha)}{\sin\pi(\alpha_3)},
\end{gathered}$$

where $2\pi\alpha_i = \arg u_i, \alpha_1+\alpha_2+\alpha_3 \in \{0,1,2\}, 2\pi\alpha = \arg q$ and $(\alpha, \alpha_1, \alpha_2)$ are defined on the three-dimensional unit torus $\mathbb{R}^3/\mathbb{Z}^3$. If we restrict to the fundamental domain $[0,1]^3$ and look at all the regions (polytopes) cut out by the planes (linear functions) in the arguments of the sines above (divided by π), we find (by solving the appropriate linear programs via Mathematica) that there exists only one region of positivity for all 4 functions. We can characterize the region best in terms of Figure 3.9. That is, as (i,j) range over all 4 sites inside a $1\times 2\times 2$ hexagon, there should not be any $\tilde{u}_k$ ($k = 1,2,3$) or any $\tilde{u}_k^{-1}$ on the arc subtended by 1 and q (and that does not contain -1).

Remark 3.4.1. In view of the above, for any choice of a reasonably large hexagon (say one that contains a $1\times 2\times 2$ hexagon) and parameters u_1, u_2, u_3 (satisfying the balancing condition), the set of q giving rise to nonnegative weights is a symmetric closed arc containing 1.

3.5 Degenerations of the measure

Certain degenerations of the weight have been studied before (among the relevant sources for our purposes are [BG09], [BGR10], [Joh05], [KO07], [Gor08]) from many angles. For example, when $q = 1$ the weight in (3.3.1) becomes a constant independent of the position of the horizontal lozenges, and so we are looking at uniformly distributed tilings of the appropriate hexagon. An exact sampling algorithm to sample from such a distribution was constructed in [BG09] and the theory behind this (as well as behind other results for such tilings) is closely connected to the theory of discrete Hahn orthogonal polynomials (see [Joh05], [BG09], [Gor08]). The frozen boundary phenomenon (the shape of a "typical boxed plane partition") was first proven in [CLP98] and then via alternate techniques (and generalized) in [CKP01] and [KO07].

A more general limit than the above is the following: in (3.3.1) we let $v_1 = v_2 = \kappa\sqrt{p}$ and then let $p \to 0$. This is the q-Racah limit (named after the discrete orthogonal polynomials that appear in the analysis). This limit is the most general limit that can be analyzed by orthogonal polynomials (as q-Racah polynomials sit at the top of the q-Askey scheme—see [KS96]). Up to gauge equivalence, we obtain the weight of a horizontal lozenge with top corner (i, j) as

$$w(i,j) = \kappa q^j - \frac{1}{\kappa q^j}. \tag{3.5.1}$$

This weight was studied in [BGR10]. If we take κ to 0 or ∞, we see the q-Racah weight is an interpolation between two types of weights:

$$w(i,j) = q^j \text{ and } w(i,j) = q^{-j}.$$

A direct alternative limit from the elliptic level is given by $v_1 = v_2 = p^{1/3}, p \to 0$ (and then replace q^2 by q or $1/q$). These two weights give rise to tilings weighted proportional to q^{Volume} or $q^{\text{-Volume}}$ (where Volume = number of $1 \times 1 \times 1$ cubes in the stepped surface representing a tiling). This is the q-Hahn weight (as the analysis leads to q-Hahn orthogonal polynomials). The frozen boundary phenomenon for this type of weight was first studied in [KO07], and then via alternative methods in [BGR10].

Finally, the Racah weight is the limit $q \to 1$ in (3.5.1) (we denote $k = \log_q(\kappa)$ and need $\kappa \to 1$ as $q \to 1$). The weight function becomes linear in the vertical coordinate:

$$w(i,j) = k + j.$$

Notice in all these limits the weight of a horizontal lozenge is independent of the horizontal coordinate of its top vertex.

Taking these limits corresponds to the hypergeometric hierarchy of special functions involved in the analysis via the orthogonal polynomial (OP) or biorthogonal elliptic functions (down arrows denote limits):

Elliptic hypergeometric (elliptic weights; elliptic biorthogonal ensembles)
$\downarrow$
q-hypergeometric (q-weights; q-OP ensembles)
$\downarrow$
Hypergeometric (uniform/Racah weight; Hahn/Racah OP ensemble)

As a final note, the most general degeneration of the weight is the top level trigonometric limit $p \to 0$, which gives rise to a 3 parameter family of weights (the use of the word *trigonometric* here should not be confused with its usage in Section 3.4). Being more general (more parameters) than the q-Racah limit, its analysis requires q rational biorthogonal functions rather than orthogonal polynomials. Taking $q \to 1$ in this top trigonometric limit yields the top rational limit. We will not use these limits hereinafter as we work directly with the elliptic level.

3.6 Deriving the weight

One way to derive the weight function (present in Section 3.3) is the following, obtained in [BGR10]. This section and the next are an expansion of the Appendix in the aforementioned reference and follow the notation therein.

We start by first looking for nice weights and nice partition function. That is, we want the partition function and individual weights of lozenges to be relatively simple products of theta functions.

Second, just like in the q^{Volume} case, we want the weights to vary in geometric progressions as we move inside the hexagonal tiling. That is to say, we want all meaningful univariate (and multivariate) partition functions to be of the (elliptic) hypergeometric series kind. Unlike the q^{Volume} case, the weights will vary in both i and j directions.

Third, we want the gauge invariant ratio of full/empty unit cube to have a reasonable group of symmetries

Fourth, because we are dealing with a projection of a three-dimensional picture onto a 2-dimensional plane, we want to capture that in the weight. A natural way to do that is to require, for the gauge-invariant ratio full/empty of an unit cube, that the following relation holds:

$$r(x, y, z) = r(x+1, y+1, z+1),$$

where (x, y, z) is the centroid of that cube (in other words, multiple points in 3-dimensions corresponding to the same projection should have the same weight).

Because of the first and second desiderata, we want to look at the Frenkel-Turaev hypergeometric series as some sort of partition function. The simplest case is to consider the univariate case of course. We can transform our three-dimensional stepped surface picture into a univariate case by looking at tilings of a $1 \times 1 \times n$ box. The partition function then is a univariate sum over the number of full cubes in the tiling (which ranges from 0 to n). Hence we want the sum

$$\sum_{0 \le l \le n} \prod_{0 \le k \le l} r(x, y, z+k)$$

to be hypergeometric, for any choice of half integer vector (x, y, z) (representing the centroid of the initial cube of the "univariate" hexagon just considered. Thus we can match the above sum with the following version of the Frenkel-Turaev summation formula (see Theorem 2.3.3 with $t_5 = \sqrt{q}, t_0^2 =$

$ab/q, \dots$):

$$\sum_{0\le l\le n} \frac{q^l \theta_p(q^{2l}ab/q)}{\theta_p(ab/q)} \frac{\theta_p(a,b,a/q,b/q)}{\theta_p(q^l a, q^l b, q^l a/q, q^l b/q)} = \frac{\theta_p(q^{n+1},a,b,q^n ab/q)}{\theta_p(q,q^n a, q^n b, ab/q)}$$

(note the summands are indeed ratios of theta-Pochhammer symbols - for example, $\theta_p(a)/\theta_p(q^l a) = \theta_p(a;q)_l/\theta_p(qa;q)_l$) to obtain

$$r(x,y,z+m) = \frac{q\theta_p(q^{m-1}a, q^{m-1}b, q^{2m+1}ab)}{\theta_p(q^{m+1}a, q^{m+1}b, q^{2m-1}ab)} = \frac{q^3\theta_p(q^{m-1}a, q^{m-1}b, q^{-2m-1}/ab)}{\theta_p(q^{m+1}a, q^{m+1}b, q^{1-2m}/ab)}.$$

Here a, b depend on (x,y,z), and we have used the simple fact that $\frac{\theta_p(a/q;q)_l}{\theta_p(aq;q)_l} = \frac{\theta_p(a,a/q)}{\theta_p(q^l a, q^l a/q)}$ above. By looking at $r(x,y,m)$ and substituting $z+m \mapsto m$, it follows that $q^{-z}a, q^{-z}b$ are independent of z. Thus $a = q^z a_1, b = q^z b_1, ab = q^{2z}c_1$. We can now impose the third desideratum and require rotational invariance of $r(x,y,z)$ (after first setting $m=0$). After rotation and matching, we find $a = q^{y+z-2x}u_1, b = q^{x+z-2y}u_2, 1/ab = q^{x+y-2z}u_3$ where u_1,u_2,u_3 are complex continuous parameters multiplying to 1. Hence we arrive at the gauge invariant weight ratio described in Section 3.3:

$$r(x,y,z) = q^3 \frac{\theta_p(q^{y+z-2x}u_1/q, q^{x+z-2y}u_2/q, q^{x+y-2z}u_3/q)}{\theta_p(q^{y+z-2x}u_1, q^{x+z-2y}u_2, q^{x+y-2z}u_3)}.$$

To aid future computations, we can choose a particularly nice gauge by breaking symmetry and requiring all lozenges that are not horizontal to have weight 1, and horizontal lozenges to have weight

$$w(i,j) = \frac{(u_1u_2)^{1/2}q^{j-1/2}\theta_p(q^{2j-1}u_1u_2)}{\theta_p(q^{j-3i/2-1}u_1, q^{j-3i/2}u_1, q^{j+3i/2-1}u_2, q^{j+3i/2}u_2)},$$

where we have switched to planar (i,j) coordinates. Note we can multiply $w(i,j)$ by any (nonvanishing) function of i alone, and it will not change the weight ratio.

3.7 Inverting the Kasteleyn matrix

In this section we compute the inverse Kasteleyn matrix in an appropriate domain and use it for certain auxiliary computations to be used later. Like the previous section, the contents of the present follows closely the Appendix of [BGR10].

The Kasteleyn K matrix associated to a tiling of a hexagon with these elliptic weights (equivalently, to a dimer cover) can be defined as follows. We divide every lozenge into two equilateral triangles and color the left (right) pointing triangle black (white). Then $K := K(b,w)_{b,w}$ has indexed rows indexed by black triangles and columns indexed by white triangles. Let (i,j) be the coordinate of the top vertex of any such triangle. We have

$$\begin{aligned}
&K((i,j),(i,j)) = \frac{(u_1u_2)^{1/2}q^{j-1/2}\theta_p(q^{2j-1}u_1u_2)}{\theta_p(q^{j-3i/2-1}u_1, q^{j-3i/2}u_1, q^{j+3i/2-1}u_2, q^{j+3i/2}u_2)},\\
&K((i,j),(i+1,j+1/2)) = 1,\\
&K((i,j),(i+1,j-1/2)) = 1,\\
&K((i,j),(i',j')) = 0 \text{ if } (i',j') \notin \{(i,j),(i+1,j+1/2),(i+1,j-1/2)\}.
\end{aligned}$$

In [BGR10] the inverse of this matrix is computed in a certain domain as follows. Let $P := P(x_0 < x_1, y_0 < y_1)$ be the parallelogram given by inequalities $x_0 \le i \le x_1, y_0 \le j + i/2 \le y_1$. In this domain K is a square matrix and we can compute its inverse which we denote by L (note that $L := L(w,b)_{w,b}$—that is, L has rows indexed by white and columns indexed by black vertices). We have that in P, the following is true.

Theorem 3.7.1. *The inverse Kasteleyn matrix is given by*

$$\begin{aligned}
L((i_0,j_0),(i_1,j_1)) =& \delta_{i_0<i_1}(u_1u_2)^{(i_1-i_0-1)/2}\\
&\times \delta_{j_1+i_1/2\le j_0+i_0/2}(-1)^{j_0-i_0/2-j_1+i_1/2-1}q^{(i_1-i_0-1)(i_1-i_0+4j_1-2)/4}\\
&\times \frac{\theta_p(q^{j_0+i_0/2-j_1-i_1/2+1}, q^{j_0+i_0/2+j_1-i_1/2}u_1u_2;q)_{i_1-i_0-1}}{\theta_p(q, q^{j_0-i_0/2-i_1}u_1, q^{j_1-3i_1/2+1}u_1, q^{j_1+i_0+i_1/2}u_2, q^{j_0+3i_0/2+1}u_2;q)_{i_1-i_0-1}}.
\end{aligned}$$

Proof. Up to a constant, $L(w,b)$ must be (by Kasteleyn's Theorem 3.2.3) the total weight of the tilings omitting the two triangles w and b (the total weight of tilings of a hexagon with two vertical sides of size 1). Thus $L((i_0,j_0),(i_1,j_1))$ vanishes if $i_0 \ge i_1$ or $j_0 + i_0/2 < j_1 + i_1/2$ because no such tilings exist. Moreover, when performing the multiplication LK, we see we are left with checking the relation:

$$\begin{aligned}
L((i_0,j_0),(i_1,j_1))K((i_1,j_1),(i_1,j_1)) + L((i_0,j_0),(i_1+1,j_1-1/2)) +\\
L((i_0,j_0),(i_1+1,j_1+1/2)) = \delta_{(i_0,j_0),(i_1,j_1)}.
\end{aligned}$$

This relation is easy to check (via the explicit formulas for L and K) in all cases except for $i_0 < i_1$ and $j_1 + i_1/2 < j_0 + i_0/2$ when all three summands above are nontrivial theta functions and the relation becomes an instance of the addition formula. □

Remark 3.7.2. Computing the inverse Kasteleyn matrix is usually hard, and the above formula was arrived at by guessing the form based on the fact that if the vertical size of the hexagon is equal to 1 then plane partitions become ordinary partitions. The partition function for such a hexagon (which is what we want to compute) is then a hypergeometric sum which we can hope to evaluate using the results from Section 2.3.

With the inverse Kasteleyn matrix in hand, we can now compute two important quantities for future purposes. We start by cutting a hexagon in half along a vertical line with i coordinate between 0 and $T = b+c$ and prescribing the position of horizontal lozenges on said line. We can then compute the total weight of the left and right halves (without the cutoff line with the prescribed horizontal lozenges). We thus have the following two lemmas.

Lemma 3.7.3. *The total weight to tile the left half-hexagon in Figure 3.10 is, up to a constant independent of the prescribed lozenges on the cutoff line*

$$\prod_{1\le l\le c} \frac{(-1)^{x_l}\theta_p(q^{I+1}, q^{I-j_0+3i_0/2}/u_1, q^{-I+2-j_0-3i_0/2}/u_2, q^{I+1-2j_0}/u_1u_2;q)_{x_l}}{\theta_p(q, q^{2I-j_0+3i_0/2}/u_1, q^{2-j_0-3i_0/2}/u_2, q^{1-2j_0}/u_1u_2;q)_{x_l}}$$
$$\times \frac{\prod_{1\le k<l\le c} q^{-x_k}\theta_p(q^{x_k-x_l}, q^{x_k+x_l+I-2j_0+1}/u_1u_2)}{\prod_{1\le l\le c}\theta_p(q^{1-I+j_0-3i_0/2-x_l}u_1, q^{x_l-j_0-3i_0/2+2}/u_2;q)_{c-1}}.$$

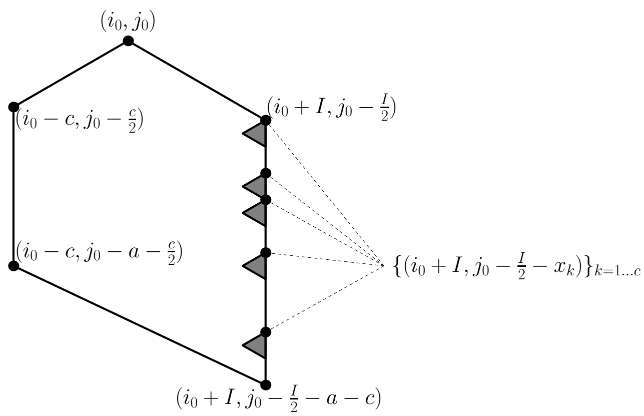

Figure 3.10: Tiling a left half-hexagon with prescribed triangles removed (corresponding to the prescribed lozenges on the cutoff line).

Proof. Tiling the domain in question is equivalent to tiling a parallelogram from where we removed an equal number of black (corresponding to lozenges) and white triangles as in Figure 3.11 since nonhorizontal lozenges have weight 1.

We can use Kasteleyn's Theorem 3.2.3 to reduce to computing the following determinant:

$$\det_{1\le l,k\le c} L((i_0-k, j_0-k/2+1), (i_0+I, j_0-I/2-x_l)).$$

We can factor

$$\frac{L((i_0-k, j_0-k/2+1), (i_0+I, j_0-I/2-x_l))}{L((i_0-1, j_0+1/2), (i_0+I, j_0-I/2))} =$$
$$\frac{L((i_0-1, j_0+1/2), (i_0+I, j_0-I/2-x_l))}{L((i_0-1, j_0+1/2), (i_0+I, j_0-I/2))} \times \frac{L((i_0-k, j_0-k/2+1), (i_0+I, j_0-I/2-x_l))}{L((i_0-1, j_0+1/2), (i_0+I, j_0-I/2-x_l))},$$

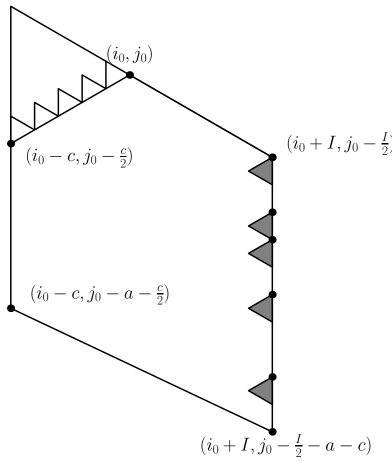

Figure 3.11: Tiling the left half-hexagon is equivalent to tiling the parallelogram with black and white triangles removed.

where the denominators are nonzero for generic values of the parameters. By multiplying the quotient

$$\frac{L((i_0-1,j_0+1/2),(i_0+I,j_0-I/2-x_l-1))}{L((i_0-1,j_0+1/2),(i_0+I,j_0-I/2-x_l))}$$
$$= -\frac{\theta_p(q^{x_l}q^{I+1}, q^{x_l}q^{I-j_0+3i_0/2}/u_1, q^{x_l}q^{-I+2-j_0-3i_0/2}/u_2, q^{x_l}q^{I+1-2j_0}/u_1u_2)}{\theta_p(q^{x_l}q, q^{x_l}q^{2I-j_0+3i_0/2}/u_1, q^{x_l}q^{2-j_0-3i_0/2}/u_2, q^{x_l}q^{1-2j_0}/u_1u_2)},$$

x_l times each time increasing the second j coordinate inside $L(\cdot,\cdot)$, we find

$$\frac{L((i_0-1,j_0+1/2),(i_0+I,j_0-I/2-x_l))}{L((i_0-1,j_0+1/2),(i_0+I,j_0-I/2))}$$
$$= (-1)^{x_l}\frac{\theta_p(q^{I+1}, q^{I-j_0+3i_0/2}/u_1, q^{-I+2-j_0-3i_0/2}/u_2, q^{I+1-2j_0}/u_1u_2;q)_{x_l}}{\theta_p(q, q^{2I-j_0+3i_0/2}/u_1, q^{2-j_0-3i_0/2}/u_2, q^{1-2j_0}/u_1u_2;q)_{x_l}}.$$

Note inside the determinant, this factor depends only on l and thus factors out to give the necessary univariate factor in the lemma. For the other factor, we similarly have

$$\frac{L((i_0-k,j_0-k/2+1),(i_0+I,j_0-I/2-x_l))}{L((i_0-1,j_0+1/2),(i_0+I,j_0-I/2-x_l))} \propto \frac{\theta_p(q^{-x_l}, q^{1+x_l-2j_0+I}/u_1u_2;q)_{k-1}}{\theta_p(q^{1-I+j_0-3i_0/2-x_l}u_1, q^{x_l-j_0-3i_0/2+2}/u_2;q)_{k-1}},$$

with the proportionality constant independent of x_l. We can now apply Warnaar's determinant (Theorem 2.4.1) with $z_l = q^{x_l+I/2-j_0+1/2}/\sqrt{u_1u_2}$ to obtain the interaction term:

$$\det_{1\le k,l\le c}\left(\frac{L((i_0-k,j_0-k/2+1),(i_0+I,j_0-I/2-x_l))}{L((i_0-1,j_0+1/2),(i_0+I,j_0-I/2-x_l))}\right)$$
$$\propto \frac{\prod_{1\le k<l\le c} q^{-x_k}\theta_p(q^{x_k-x_l}, q^{x_k+x_l+I-2j_0+1}/u_1u_2)}{\prod_{1\le l\le c}\theta_p(q^{1-I+j_0-3i_0/2-x_l}u_1, q^{x_l-j_0-3i_0/2+2}/u_2;q)_{c-1}}.$$

□

Lemma 3.7.4. *The total weight to tile the right half-hexagon in Figure 3.12 is, up to a constant* *independent of the prescribed lozenges on the cutoff line*

$$\prod_{1\le k\le c} \frac{(-1)^{x_k}\theta_p(q^{1-a-c}, q^{2I-j_0+2+3i_0/2}/u_1, q^{-b+c-j_0-3i_0/2}/u_2, q^{-2j_0+a+b+1}/u_1u_2;q)_{x_k}}{\theta_p(q^{I-a-b+1}, q^{I+2-j_0+3i_0/2+b-c}/u_1, q^{-I-j_0-3i_0/2}/u_2, q^{I-2j_0+a+c+1}/u_1u_2;q)_{x_k}}$$
$$\times \frac{\prod_{1\le k<l\le c} q^{-x_k}\theta_p(q^{x_k-x_l}, q^{x_k+x_l+I-2j_0+1}/u_1u_2)}{\prod_{1\le k\le c}\theta_p(q^{x_k+I-j_0+3i_0/2+2+b-c}/u_1, q^{j_0+3i_0/2+b+1-c-x_k}u_2;q)_{c-1}}.$$

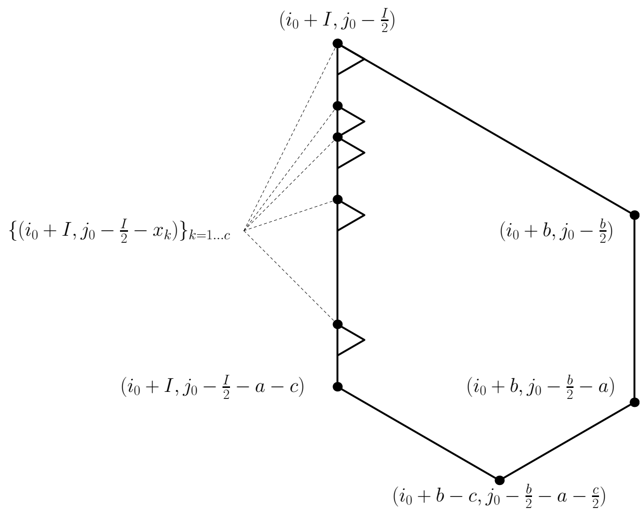

Figure 3.12: Tiling a right half-hexagon with prescribed triangles removed (corresponding to the prescribed lozenges on the cutoff line).

Proof. As in the previous lemma, we need to compute the number of ways to tile the domain in Figure 3.13 (with white and black triangles removed). Again we use Kasteleyn's theorem to equate this with the determinant:

$$\det_{1\le k,l\le c} L((i_0+I, j_0-I/2-x_k),(i_0+b-c+l, j_0-a-b/2-c/2+l/2)).$$

We proceed as before, by factoring:

$$\frac{L((i_0+I, j_0-I/2-x_k),(i_0+b-c+l, j_0-a-b/2-c/2+l/2))}{L((i_0+I, j_0-I/2),(i_0+b-c+1, j_0-a-b/2-c/2+1/2))} =$$
$$\frac{L((i_0+I, j_0-I/2-x_k),(i_0+b-c+1, j_0-a-b/2-c/2+1/2))}{L((i_0+I, j_0-I/2),(i_0+b-c+1, j_0-a-b/2-c/2+1/2))}\times$$
$$\frac{L((i_0+I, j_0-I/2-x_k),(i_0+b-c+l, j_0-a-b/2-c/2+l/2))}{L((i_0+I, j_0-I/2-x_k),(i_0+b-c+1, j_0-a-b/2-c/2+1/2))}\propto$$
$$\frac{\theta_p(q^{x_k-a-c+1}, q^{2j_0-I-a-c-x_k}u_1u_2;q)_{l-1}}{\theta_p(q^{I-j_0+3i_0/2+2+b-c}/u_1, q^{j_0+3i_0/2+b+1-c-x_k}u_2;q)_{l-1}}\times$$
$$(-1)^{x_k}\frac{\theta_p(q^{1-a-c}, q^{2I-j_0+2+3i_0/2}/u_1, q^{-b+c-j_0-3i_0/2}/u_2, q^{-2j_0+a+b+1}/u_1u_2;q)_{x_k}}{\theta_p(q^{I-a-b+1}, q^{I+2-j_0+3i_0/2+b-c}/u_1, q^{-I-j_0-3i_0/2}/u_2, q^{I-2j_0+a+c+1}/u_1u_2;q)_{x_k}}.$$

Finishing the argument as in the case of the left half-hexagon yields the result. □

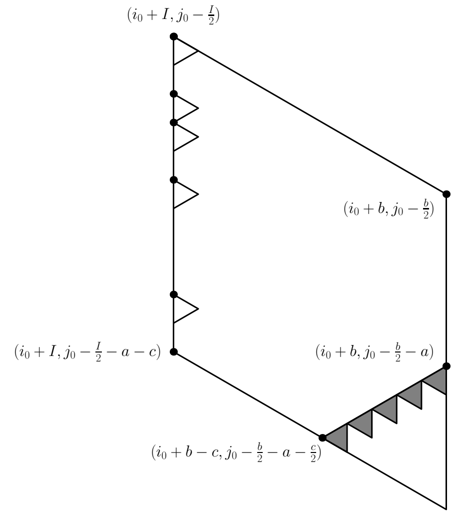

Figure 3.13: Tiling the right half-hexagon is equivalent to tiling the parallelogram with black and white triangles removed.

We can also compute the total partition function for tilings of the hexagon with such weights. That is, we have the following elliptic Macmahon identity.

Theorem 3.7.5. *Given an $a \times b \times c$ hexagon and $|p| < 1, q, u_1, u_2, u_3$ complex with $u_1 u_2 u_3 = 1$, we have*

$$\frac{\sum_{\mathcal{T}} wt(\mathcal{T})}{wt(0)} = q^{abc} \prod_{(1,1,1) \leq (i,j,k) \leq (a,b,c)} \frac{\theta_p(q^{i+j+k-1}, q^{j+k-i-1} u_1, q^{i+k-j-1} u_2, q^{i+j-k-1} u_3)}{\theta_p(q^{i+j+k-2}, q^{j+k-i} u_1, q^{i+k-j} u_2, q^{i+j-k} u_3)},$$

where 0 *denotes the empty-box tiling and the sum is over all tilings $\mathcal{T}$.*

Remark 3.7.6. When $p \to 0$ such that all theta functions containing any of the u_i tend to 1 (in essence, we are in the q-Hahn limit described in Section 3.5), we recover the usual q-Macmahon formula. That is, the partition function for q^{Volume}-type weights is equal to

$$\prod_{(1,1,1) \leq (i,j,k) \leq (a,b,c)} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}},$$

which becomes the usual Macmahon partition function when $q \to 1$. This formula counts the number of tilings of such a hexagon:

$$\prod_{(1,1,1) \leq (i,j,k) \leq (a,b,c)} \frac{i+j+k-1}{i+j+k-2}.$$

Proof. We are looking at tiling Figure 3.11 glued to Figure 3.13 where we have removed the restriction of having prescribed lozenges on the gluing line. The total weight of such tilings is, by Kasteleyn's theorem, the following determinant:

$$\det_{1\le l,k\le c} L((i_0-k, j_0-k/2+1),(i_0+b-c+l, j_0-a-b/2-c/2+l/2)).$$

We can compute this determinant using the same method (Warnaar's determinant) used in the previous two lemmas. This yields a gauge-dependent weight, which, after dividing by the weight of the empty tiling, gives the result. □

3.8 S_3-symmetric weight

In this section we show how one can assign S_3-invariant weights to the three types of rhombi (lozenges) that make up a tiling of a hexagon. We start with the $2\times 2\times 2$ triangle (inside the triangular lattice) depicted in Figure 3.14 that contains an overlapping of the 3 types of rhombi considered for our tilings.

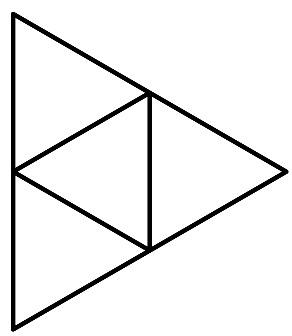

Figure 3.14: 3 overlapping lozenges of each type.

To each such type of rhombus we assign a label from the set $\{\tilde{u}_1, \tilde{u}_2, \tilde{u}_3\}$ (see Figure 3.15) such that if the rhombi are as described overlapping inside a $2\times 2\times 2$ triangle we have

$$\tilde{u}_1\tilde{u}_2\tilde{u}_3 = 1.$$

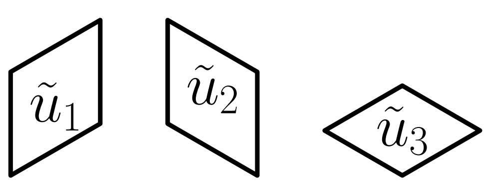

Figure 3.15: The 3 types of rhombi (lozenges) and their labels.

Each $\tilde{u}_i$ will eventually be a power of q times u_i (see Section 3.3). First, we can obviously shift any of such rhombi along the directions given by their edges, either upwards or downwards. If we shift the horizontal lozenge labeled $\tilde{u}_3$ upwards-right or upwards-left, the label of the new lozenges will be multiplied by q^{-1}. If we shift it downwards-right/left, the label will get multiplied by q. Naturally, if we shift directly upwards, the label will be multiplied by q^{-2} as a composite of an upwards-right and and upwards-left shift. A similar rule is used for lozenges with labels $\tilde{u}_2$ and $\tilde{u}_3$. The process is depicted in Figure 3.16, with the caveat that for labels $\tilde{u}_1$ and $\tilde{u}_2$ we only show the directions in which the label gets multiplied by q (it gets multiplied by q^{-1} in the opposite two directions than the ones depicted). Clearly translating any lozenge along its long diagonal does not change its label.

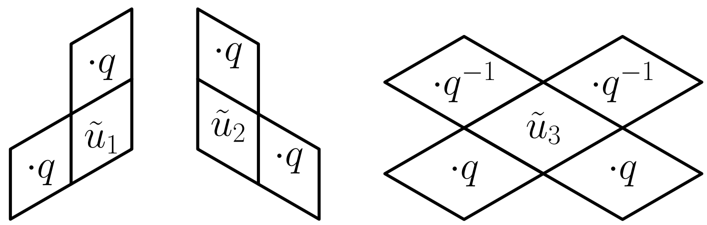

Figure 3.16: Shifting lozenges in the triangular lattice, we shift the labels by q or q^{-1} as depicted.

To a lozenge with label $\tilde{u}_i$ $(i = 1, 2, 3)$ we assign the following weight:

$$wt(\text{lozenge with label } \tilde{u}_i) = \tilde{u}_i^{-1/2}\theta_p(\tilde{u}_i),\ i = 1, 2, 3,$$

where

$$\tilde{u}_1 = q^{y+z-2x}u_1, \tilde{u}_2 = q^{x+z-2y}u_2, \tilde{u}_3 = q^{x+y-2z}u_3, u_1u_2u_3 = 1,$$

u_1, u_2, u_3 are three complex numbers that multiply to 1 and (x, y, z) is the three-dimensional coordinate of the center (intersection of the diagonals) of a lozenge. At this point we need to fix a choice of square roots: $\sqrt{q}, \sqrt{u_1}, \sqrt{u_2}, \sqrt{u_3}$ such that $\sqrt{u_1}\sqrt{u_2}\sqrt{u_3} = 1$. Note the three-dimensional coordinates are only defined up to the diagonal action of $\mathbb{Z}$. Figure 3.17 depicts the 3 lozenges with labels u_i $(x = y = z = 0)$ in the chosen coordinate system.

This way of assigning weights is manifestly S_3-invariant. To recover the same probability distribution as in Section 3.3 (i.e., a gauge-equivalent weight for tilings) we again require that the weight of a tiling of a hexagon is the product of weights of lozenges inside it. To check this, one can simply check the weight ratio of a full $1 \times 1 \times 1$ box to an empty $1 \times 1 \times 1$ box (this is a gauge-invariant quantity) under the present assumptions and observe the result is the same as in (3.3.2).

The S_3 invariance can be viewed at the level of the partition function (the sum of weights of all tilings in a hexagon written in this gauge) as follows. We start with an $\alpha \times \beta \times \gamma$ hexagon. The origin is at the hidden corner of the three-dimensional box. In the canonical coordinates,

$$(\tilde{u}_1 = q^{y+z-2x}u_1, \tilde{u}_2 = q^{x+z-2y}u_2, \tilde{u}_3 = q^{x+y-2z}u_3),$$

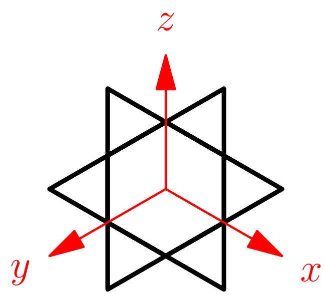

Figure 3.17: The 3 lozenges corresponding to u_1, u_2, u_3.

the six bounding edges have the following equations (see Figure 3.18 for correspondence between edges and L_i's):

$$\begin{aligned}
\tilde{u}_1/\tilde{u}_2 &:= L_0 := q^{3\beta}u_1/u_2, \\
\tilde{u}_3/\tilde{u}_1 &:= L_1 := q^{-3\gamma}u_3/u_1, \\
\tilde{u}_2/\tilde{u}_3 &:= L_2 := q^{3\gamma}u_2/u_3, \\
\tilde{u}_1/\tilde{u}_2 &:= L_3 := q^{-3\alpha}u_1/u_2, \\
\tilde{u}_3/\tilde{u}_1 &:= L_4 := q^{3\alpha}u_3/u_1, \\
\tilde{u}_2/\tilde{u}_3 &:= L_5 := q^{-3\beta}u_2/u_3.
\end{aligned} \tag{3.8.1}$$

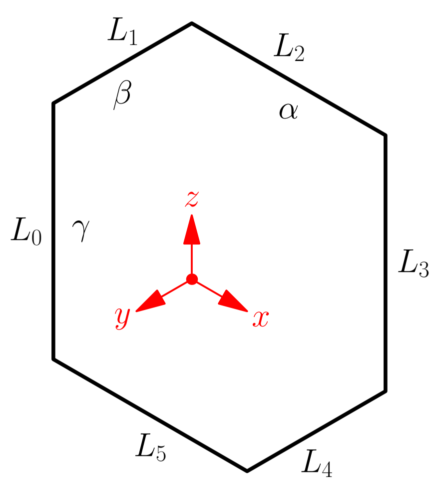

Figure 3.18: An $\alpha \times \beta \times \gamma$ hexagon with canonical coordinates of the edges on the outside and edge lengths on the inside.

We then have the following proposition. Throughout, the S_3-invariant weights are assumed.

Proposition 3.8.1. *The partition function for an $\alpha \times \beta \times \gamma$ hexagon is equal to*

$$P \times \lim_{\rho \to 1} \frac{\Gamma_{p,q,q}(q^{1+\alpha+\beta+\gamma}\rho, q^{1+\alpha}\rho, q^{1+\beta}\rho, q^{1+\gamma}\rho)}{\Gamma_{p,q,q}(q\rho, q^{1+\alpha+\beta}\rho, q^{1+\alpha+\gamma}\rho, q^{1+\beta+\gamma}\rho)} \times$$
$$\frac{\Gamma_{p,q,q}(q^{1-\alpha+\beta+\gamma}u_1, q^{1-\alpha}u_1, q^{1-\beta+\alpha+\gamma}u_2, q^{1-\beta}u_2, q^{1-\gamma+\alpha+\beta}u_3, q^{1-\gamma}u_3)}{\Gamma_{p,q,q}(q^{1-\alpha+\beta}u_1, q^{1-\alpha+\gamma}u_1), q^{1-\beta+\alpha}u_2, q^{1-\beta+\gamma}u_2, q^{1-\gamma+\alpha}u_3, q^{1-\gamma+\beta}u_3)} =$$
$$P \times \lim_{\rho \to 1} \frac{\Gamma_{p,q,q}(q(L_0L_2L_4)^{1/3}\rho, q(L_0L_4L_5)^{1/3}\rho, q(L_0L_1L_2)^{1/3}\rho, q(L_2L_3L_4)^{1/3}\rho)}{\Gamma_{p,q,q}(q\rho, q(L_0/L_3)^{1/3}\rho, q(L_4/L_1)^{1/3}\rho, q(L_2/L_5)^{1/3}\rho)} \times$$
$$\frac{\Gamma_{p,q,q}(q(L_0L_2L_3)^{1/3}, q(L_0L_3L_5)^{1/3}, q(L_2L_4L_5)^{1/3}, q(L_1L_2L_5)^{1/3}, q(L_0L_1L_4)^{1/3}, q(L_1L_3L_4)^{1/3})}{\Gamma_{p,q,q}(q(L_0/L_4)^{1/3}, q(L_3/L_1)^{1/3}, q(L_5/L_3)^{1/3}, q(L_2/L_0)^{1/3}, q(L_4/L_2)^{1/3}, q(L_1/L_5)^{1/3})},$$

where

$$P = q^{\alpha\beta\gamma - \frac{\alpha\beta^2+\beta\alpha^2+\alpha\gamma^2+\gamma\alpha^2+\beta\gamma^2+\gamma\beta^2}{4}} u_1^{-\frac{\beta\gamma}{2}} u_2^{-\frac{\alpha\gamma}{2}} u_3^{-\frac{\alpha\beta}{2}}.$$

It is left invariant by S_3 permuting the coordinates $\tilde{u}_i$; equivalently, the tuple

$$((x, \alpha, u_1), (y, \beta, u_2), (z, \gamma, u_3)).$$

Furthermore, this invariance can be expanded to the group $W(G_2) = S_3 \times \mathbb{Z}_2 = Dih_6$ (the symmetry group of a regular hexagon) with the missing involution being the transformation:

$$(u_1, u_2, u_3) \to (\frac{1}{q^A u_1}, \frac{1}{q^B u_2}, \frac{1}{q^C u_3}),$$

where $A = -2\alpha + \beta + \gamma, B = \alpha - 2\beta + \gamma, C = \alpha + \beta - 2\gamma$.

Proof. We start with the elliptic Macmahon identity derived in the Appendix of [BGR10] and in Section 3.7:

$$\frac{\sum_{\text{tilings } T} wt(T, G)}{wt(0, G)} = q^{\alpha\beta\gamma} \prod_{1 \le x \le \alpha, 1 \le y \le \beta, 1 \le z \le \gamma} \frac{\theta_p(q^{x+y+z-1}, q^{y+z-x-1}u_1, q^{x+z-y-1}u_2, q^{x+y-z-1}u_3)}{\theta_p(q^{x+y+z-2}, q^{y+z-x}u_1, q^{x+z-y}u_2, q^{x+y-z}u_3)},$$

where 0 denotes the empty tiling (box) and G is any gauge equivalent to the ones used in this paper (that is to say, both sides are gauge-independent). For G the S_3 invariant gauge herein discussed, the formula for the empty tiling multiplied by the RHS above simplifies the partition function via straightforward computations. We arrive at the desired result using the following transformations for Γ functions:

$$\Gamma_{p,q}(qx) = \theta_p(x)\Gamma_{p,q}(x),$$
$$\Gamma_{p,q,t}(tx) = \Gamma_{p,q}(x)\Gamma_{p,q,t}(x).$$

The limit $\rho \to 1$ is needed for technical reasons to avoid zeros of triple Γ functions.

For the S_3-invariance, it suffices to show how the edges transform under the 3-cycle $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) \to (\tilde{u}_2, \tilde{u}_3, \tilde{u}_1)$ (a 120° clockwise rotation) and the transposition $\tilde{u}_1 \leftrightarrow \tilde{u}_2$ (a reflection in the z axis). For the 3-cycle, the new edges (denoted with primes) have equations:

$$L'_i = L_{i+2},$$

where $+2$ is taken modulo 6, while for the transposition we have

$$L'_0 = 1/L_3, L'_1 = 1/L_2, L'_2 = 1/L_1, L'_3 = 1/L_0, L'_4 = 1/L_5, L'_5 = 1/L_4.$$

Both these transformations leave the partition function invariant. The extra involution giving the group $W(G_2)$ is a reflection through the centroid of the hexagon having coordinates:

$$(q^{A/2}u_1, q^{B/2}u_2, q^{C/2}u_3).$$

The edges transform as

$$L'_i = 1/L_{i+3},$$

where addition is mod 6. We look at the first form of the partition function written in the statement. We use the following two difference equations to simplify the calculations and arrive at the original form:

$$\Gamma_{p,q,q}(q/x) = \Gamma_{p,q,q}(pqx) = \Gamma_{q,q}(qx)\Gamma_{p,q,q}(qx),$$
$$\frac{\Gamma_{q,q}(q^l q^m x, x)}{\Gamma_{q,q}(q^l x, q^m x)} = (-x)^{ml} q^{-(l\binom{m}{2}+m\binom{l}{2})}.$$

□

Chapter 4

Relevant distributions

In this chapter we compute probabilistic quantities of interest which follow from the application of Kasteleyn's theorem in Section 3.7. The N-point distribution is a (discrete elliptic) generalization of many distributions appearing in random matrix theory (i.e., the GUE N-point distribution) and interacting particle systems. We then introduce the elliptic difference operators of Rains [Rai10] and show how they capture the dynamics of the particle system under certain specializations. This allows us to sample exactly from such elliptic distributions as described in Chapter 5.

4.1 Stationary and transitional distributions

In the present section we compute the N-point correlation function (an instance of the discrete elliptic Selberg density) and transitional probabilities for the model under study. We refer the reader to Section 3.7 and references therein for the relevant application of Kasteleyn's theorem which makes these computations possible.

Take a collection of N nonintersecting lattice paths in $\Omega(N,S,T)$. Fix a vertical line inside the hexagon with horizontal integer coordinate t $(0 \leq t \leq T)$. This vertical line will contain N particles $X = (x_1 < \cdots < x_N) \in \mathcal{X}^{S,t}_{N,T}$. Depending on the geometry of our hexagon, there are four ways in which we can fix a vertical line with horizontal coordinate t inside a collection of N nonintersecting paths in $\Omega(N,S,T)$. They are described below (see also Figure 4.1 in which the four cases are depicted—we only depict the outside bounding hexagon and the middle vertical line that is the desired particle line).

$$\begin{aligned}
&\text{Case 1. } t < S,\ t < T-S,\ 0 \leq x_k \leq t+N-1,\\
&\text{Case 2. } S \leq t \leq T-S,\ 0 \leq x_k \leq S+N-1,\\
&\text{Case 3. } T-S \leq t < S,\ t+S-T \leq x_k \leq t+N-1,\\
&\text{Case 4. } t \geq T-S,\ t \geq S,\ t+S-T \leq x_k \leq S+N-1.
\end{aligned} \tag{4.1.1}$$

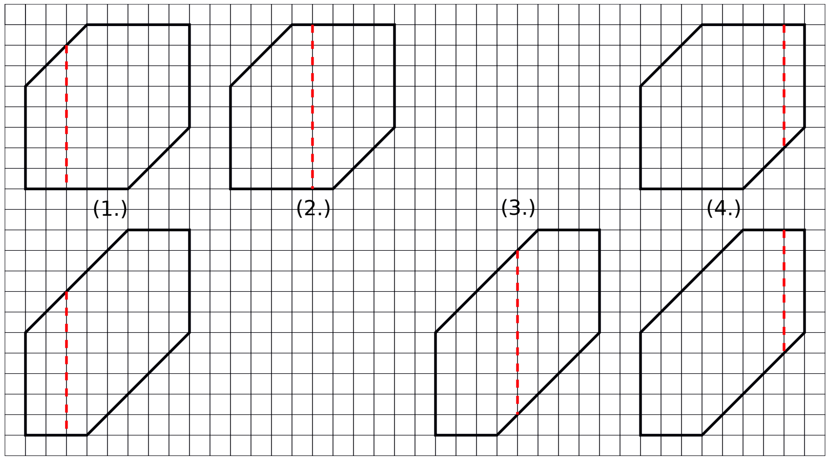

Figure 4.1: The four ways of choosing a vertical particle line (dotted) inside a hexagon. In all cases $N = 5$ particles, $T = 8, S \in \{3, 5\}$. The middle vertical line in any hexagon is the particle line.

We make use of the following notations:

$L_t(X)$ = sum of products of weights corresponding to holes (horizontal lozenges) to the left of the vertical line with coordinate t. The sum is taken over all possible ways of tiling the region to the left of this line. Equivalently, it is taken over all families of paths starting at $((0,0),\ldots,(0,N-1))$ and ending at $((t,x_1),\ldots,(t,x_N))$.

$R_t(X)$ = sum of products of weights corresponding to holes to the right of the vertical line with coordinate t. The sum is taken over all possible ways of tiling the region to the right of this line. Equivalently, it is taken over all families of paths starting at $((t,x_1),\ldots,(t,x_N))$ and ending at $((T,S),\ldots,(T,S+N-1))$.

$C_t(X)$ = product of weights corresponding to the holes on this vertical line.

Let

$$\varphi_{t,S}(x_k,x_l) = q^{-x_k}\theta_p(q^{x_k-x_l}, q^{x_k+x_l+1-t-S}v_1v_2). \tag{4.1.2}$$

Remark 4.1.1. As observed in Section 1.2, $\varphi_{t,S}(x,y) = -\varphi_{t,S}(y,x)$ so the product $\prod_{k<l}\varphi_{t,S}(x_k,x_l)$ is the "elliptic" analogue of the Vandermonde product $\prod_{k<l}(x_k-x_l)$ (to which it tends in the limit $p \to 0, q \to 1$ as explained in Section 1.2).

Proposition 4.1.2. *We have*

$$L_t(X=(x_1,\ldots,x_N)) = const \cdot \prod_{k<l}\varphi_{t,S}(x_k,x_l)\times$$
$$\prod_{1\le k\le N} q^{Nx_l}\theta_p(q^{2x_l+1-t-S}v_1v_2)\frac{\theta_p(q^{1-N-t},q^{1-t-S}v_1,q^tv_2,q^{1-t-S}v_1v_2;q)_{x_l}}{\theta_p(q,q^{2-2t-S}v_1,qv_2,q^{1+N-S}v_1v_2;q)_{x_l}}.$$

Proof. This follows from an elaborate calculation and Lemma 3.7.3.

First, as is in the case of the aforementioned lemma, we restrict ourselves to the case $S < t < T-S$ (Case 2. in (4.1.1); computations are similar for the other 3 cases). Note in such a case we have N particles and S holes on the line with abscissa t. We then apply the particle-hole involution (as the weight in Lemma 3.7.3 is given in terms of the positions of the holes = horizontal lozenges on the t-line). There are two types of products appearing in the total weight in question: a univariate one over the holes and a bivariate Vandermonde-like (again over the holes). For the first product, we just reciprocate to turn it into a product over particles (as the total product over holes and particles of the functions involved is a constant dependent only on $t, S, T, N, q, p, v_1, v_2$). For the Vandermonde-like product, we note for a function f satisfying $f(y_i, y_j) = -f(y_j, y_i)$ we have (up to a possible sign not depending on holes or particles):

$$\prod_{1\le i<j\le S} f(y_i,y_j) = \prod_{1\le i<j\le N} f(x_i,x_j) \times \prod_{0\le u<v\le S+N-1} f(u,v)\times \prod_{1\le i\le N} \frac{1}{\prod_{0\le u<x_i} f(x_i,u)\prod_{x_i<u\le S+N-1} f(u,x_i)}, \tag{4.1.3}$$

where y's represent locations of holes (top vertices of horizontal lozenges) and x's locations of particles. We take $f = \varphi_{t,S}$ as defined in (4.1.2). Finally, in Section 3.7 the convention is that particles and holes are counted from the top going down. For convenience, we count from the bottom up, so we substitute $x_l \mapsto S+N-1-x_l$. After standard manipulations with theta-Pochhammer symbols we arrive at the desired result. □

Proposition 4.1.3. *We have*

$$R_t(X=(x_1,\ldots,x_N)) = const\cdot\prod_{k<l}\varphi_{t,S}(x_k,x_l)\times \prod_{1\le k\le N} q^{Nx_l}\theta_p(q^{2x_l+1-t-S}v_1v_2)\frac{\theta_p(q^{1-N-S},q^{-2t-S}v_1,q^{1+T}v_2,q^{1-T}v_1v_2;q)_{x_l}}{\theta_p(q^{1-S-t+T},q^{1-t-S-T}v_1,q^{2+t}v_2,q^{1+N-t}v_1v_2;q)_{x_l}}.$$

Proof. Similar to the previous proof except we use Lemma 3.7.4. □

Proposition 4.1.4. *We have*

$$C_t(X=(x_1,\ldots,x_N)) = const\cdot\prod_{1\le k\le N}\frac{\theta_p(q^{x_l-2t-S}v_1,q^{x_l-2t-S+1}v_1,q^{x_l+t}v_2,q^{x_l+t+1}v_2)}{q^{x_l}\theta_p(q^{2x_l+1-t-S}v_1v_2)}.$$

Proof. This weight is (up to a constant not depending on holes or particles) the reciprocal of the total weight of the S holes (horizontal lozenges) on the t line and the latter is readily computed from the definition (3.3.1) of the weight function. □

Theorem 4.1.5.

$$Prob(X(t)=(x_1,\dots,x_N)) = const\cdot\prod_{k<l}(\varphi_{t,S}(x_k,x_l))^2\times\prod_{1\le k\le N}q^{(2N-1)x_k}\theta_p(q^{2x_k+1-t-S}v_1v_2)\times$$
$$\prod_{1\le k\le N}\frac{\theta_p(q^{1-N-t},q^{1-N-S},q^{1-t-S}v_1,q^{1+T}v_2,q^{1-T}v_1v_2,q^{1-t-S}v_1v_2;q)_{x_k}}{\theta_p(q,q^{1-S-t+T},q^{1-t-T-S}v_1,qv_2,q^{1+N-S}v_1v_2,q^{1+N-t}v_1v_2;q)_{x_k}}$$
$$= const\cdot\prod_{k<l}(\varphi_{t,S}(x_k,x_l))^2\times\prod_{1\le k\le N}q^{(2N-1)x_k}\theta_p(q^{2x_k}F^2)\frac{\theta_p(AF,BF,CF,DF,EF,F^2;q)_{x_k}}{\theta_p(q,q\frac{A}{F},q\frac{B}{F},q\frac{C}{F},q\frac{D}{F},q\frac{E}{F};q)_{x_k}}. \tag{4.1.4}$$

Proof.

$$Prob(X(t)=(x_1,\dots,x_N))\propto L_t(X)C_t(X)R_t(X).$$

□

Remark 4.1.6. The above distribution is what was called in Section 1.2 the discrete elliptic Selberg density. That is to say,

$$Prob(X(t)=(x_1,\dots,x_N))\propto\Delta_\lambda(q^{2N-2}F^2|q^N,q^{N-1}AF,q^{N-1}(pB)F,q^{N-1}CF,q^{N-1}DF,q^{N-1}EF), \tag{4.1.5}$$

where $\lambda\in m^N$ $(m=S+N-1)$ and $\lambda_i+N-i=x_{N+1-i}$ (to account for the fact that $x_1<x_2<\cdots<x_N$ whereas partitions are always listed in nonincreasing order). The constant of proportionality is given by Theorem 2.3.5. The particle-hole involution invoked in Proposition 4.1.2 then takes the following form. If λ_p is the partition associated to the particle positions (at time t) via the above equation and λ_h is the partition associated to the whole positions at the same time (in the case above, there are S holes), then

$$\lambda_h=(m^n-\lambda_p)',$$

where we recall from Section 1.2 $m^n-\lambda$ denotes the complemented partition corresponding to $\lambda\in m^n$ $((m^n-\lambda)_i=m-\lambda_{n+1-i})$ and λ' denotes the dual (transposed) partition (λ'_i = number of parts of λ that are $\ge i$). The fact that both probabilities (in terms of holes and in terms of particles) are Δ-symbols can be observed directly as shown in Proposition 4.1.2 or using the following relations mentioned in Section 1.2:

$$\Delta_{\lambda'}(a|\dots b_i\dots;1/q)=\Delta_\lambda(a/q^2|\dots b_i\dots;q),$$
$$\frac{\Delta_{m^n-\lambda}(a|\dots b_i\dots;q)}{\Delta_{m^n}(a|\dots b_i\dots;q)}=\Delta_\lambda(\frac{q^{2m-2}}{q^{2n}a}|\dots\frac{q^{n-1}b_i}{q^m a}\dots,q^n,pq^n,q^{-m},pq^{-m};q).$$

We will for brevity denote the measure described in Theorem (4.1.5) by $\rho_{S,t}$ (note it also depends on N,T,v_1,v_2,p,q, but it is the dependence on S and t that will be of most interest to us). Observe we can transform the factor

$$q^x q^{(2N-2)x} \frac{\theta_p(q^{1-t-S}v_1, q^{1+T}v_2)}{\theta_p(q^{1-t-S-T}v_1, qv_2)}$$

appearing in the univariate product of the above probability into something proportional to

$$q^x \frac{\theta_p(q^{N-t-S}v_1, q^{N+T}v_2)}{\theta_p(q^{2-N-t-S-T}v_1, q^{2-N}v_2)} \cdot \frac{1}{\theta_p(q^{x+1-t-S}v_1, q^{-x+t+S+T}/v_1, q^{x+1+T}v_2, q^{-x}/v_2)_{N-1}}$$

by using

$$\theta_p(Aq^{N-1};q)_x = \frac{\theta_p(A;q)_x \theta_p(Aq^x;q)_{N-1}}{\theta_p(A;q)_{N-1}} \text{ and } \theta_p(Aq^{1-N};q)_x = \frac{q^{(1-N)x}\theta_p(A;q)_x \theta_p(q/A;q)_{N-1}}{\theta_p(q^{1-x}/A;q)_{N-1}}$$

and absorbing into the initial constant anything independent of x (of the particle positions x_k). After using (3.3.4) our probability distribution becomes

$$\begin{aligned} &Prob(X(t) = (x_1, \ldots, x_N)) = \\ &const \cdot \prod_{k<l} (\varphi_{t,S}(x_k, x_l))^2 \times \prod_{1 \le k \le N} \frac{1}{\theta_p(B(Fq^{x_k})^{\pm 1}, E(Fq^{x_k})^{\pm 1};q)_{N-1}} \times \prod_{1 \le k \le N} w(x_k), \end{aligned} \tag{4.1.6}$$

where

$$\begin{aligned} w(x) &= \frac{q^x \theta_p(q^{2x+1-t-S}v_1v_2)\theta_p(q^{1-N-t}, q^{1-N-S}, q^{N-t-S}v_1, q^{N+T}v_2, q^{1-T}v_1v_2, q^{1-t-S}v_1v_2;q)_x}{\theta_p(q^{1-t-S}v_1v_2)\theta_p(q, q^{1-S-t+T}, q^{2-N-t-T-S}v_1, q^{2-N}v_2, q^{1+N-S}v_1v_2, q^{1+N-t}v_1v_2;q)_x} \\ &= \frac{q^x \theta_p(F^2q^{2x})\theta_p(AF, BF\left(\frac{q}{ABCDEF}\right)^{\frac{1}{2}}, CF, DF, EF\left(\frac{q}{ABCDEF}\right)^{\frac{1}{2}}, F^2;q)_x}{\theta_p(F^2)\theta_p(\frac{F}{A}q, \frac{F}{B}q\left(\frac{ABCDEF}{q}\right)^{\frac{1}{2}}, \frac{F}{C}q, \frac{F}{D}q, \frac{F}{E}q\left(\frac{ABCDEF}{q}\right)^{\frac{1}{2}}, q;q)_x}. \end{aligned}$$

We have that w is the weight function for the discrete elliptic univariate biorthogonal functions discovered by Spiridonov and Zhedanov (see [SZ00], [SZ01]). It is of course also the discrete elliptic Selberg density for $N = 1$ (hence a Δ-symbol in $n = 1$ variable as seen in (1.2.6)). Notice in (4.1.6) above B and E play a special role, as does F. This will become more transparent in Section 6.3. Note w is elliptic in q, v_1, v_2 and $q^{\{t,S,T,N\}}$ (or, analogously, in A, B, C, D, E, F, q).

Remark 4.1.7. Note that in the definition of w above, the first line is given in terms of the geometry of the hexagon and the choice of the particular particle line (Case 2. in (4.1.1) as previously discussed), while the second line is intrinsic and the geometry of the hexagon only comes in after using (3.3.4). We can also define the equivalent of (3.3.4) in the other 3 cases described in (4.1.1) (and the three other choices of 6 parameters differ from (3.3.4) by (a): interchanging S ant t, (b): shifting the 6 parameters in (3.3.4) by $q^{\pm(t+S-T)}$), or (c): a combination of both (a) and (b)). We will not use this any further, as all calculations will be done in Case 2. from (4.1.1).

Remark 4.1.8. The limit $v_1 = v_2 = \kappa\sqrt{p},\ p \to 0$ gives the distributions present in [BGR10] at the q-Racah level. Also, as will be seen in Section 6.3, such probabilities are structurally a product of a "Vandermonde-like" determinant squared (the first two products in (4.1.6)) and a product over the particles of univariate weights of elliptic biorthogonal functions. Indeed, under the appropriate limits, one can arrive from (4.1.6) to a much simpler (prototypical) such N-point function: the joint density of the N eigenvalues of a GUE $N \times N$ random matrix.

The transition and co-transition probabilities for the Markov chain $X(t)$ are given by the next two statements.

Theorem 4.1.9. *If* $Y = (y_1, \dots, y_N)$ *and* $X = (x_1, \dots, x_N)$ *such that* $y_k - x_k \in \{0, 1\}\ \forall k$*, then*

$$Prob(X(t+1) = Y | X(t) = X) = const \cdot \prod_{k<l} \frac{\varphi_{t+1,S}(y_k, y_l)}{\varphi_{t,S}(x_k, x_l)} \times \prod_{k: y_k = x_k + 1} w_1(x_k) \prod_{k: y_k = x_k} w_0(x_k),$$

where

$$w_0(x) = \frac{q^{-x-N+1}\theta_p(q^{x+T-t-S}, q^{x-T-t-S}v_1, q^{x+t+1}v_2, q^{x+N-t}v_1v_2)}{\theta_p(q^{2x+1-t-S}v_1v_2)}$$
$$w_1(x) = -\frac{q^{-x}\theta_p(q^{x+1-N-S}, q^{x-2t-S}v_1, q^{x+T+1}v_2, q^{x-T+1}v_1v_2)}{\theta_p(q^{2x+1-t-S}v_1v_2)}.$$

Proof. The formula

$$\begin{aligned} Prob(X(t+1) = Y | X(t) = X) &= \frac{L_t(X)C_t(X)C_{t+1}(Y)R_{t+1}(Y)}{L_t(X)C_t(X)R_t(X)} \\ &= \frac{C_{t+1}(Y)R_{t+1}(Y)}{R_t(X)}, \end{aligned}$$

along with the formulas for L, R and C yield the result. □

Theorem 4.1.10. *If* $Y = (y_1, \dots, y_N)$ *and* $X = (x_1, \dots, x_N)$ *such that* $y_k - x_k \in \{0, -1\}\ \forall k$*, then*

$$Prob(X(t-1) = Y | X(t) = X) = const \cdot \prod_{k<l} \frac{\varphi_{t-1,S}(y_k, y_l)}{\varphi_{t,S}(x_k, x_l)} \times \prod_{k: y_k = x_k - 1} w'_1(x_k) \prod_{k: y_k = x_k} w'_0(x_k),$$

where

$$w'_0(x) = -\frac{q^{-x}\theta_p(q^{x-N-t+1}, q^{x-t-S+1}v_1, q^{x+t}v_2, q^{x-t-S+1}v_1v_2)}{\theta_p(q^{2x+1-t-S}v_1v_2)},$$
$$w'_1(x) = \frac{q^{-x-N+1}\theta_p(q^{x}, q^{x-2t-S+1}v_1, q^{x}v_2, q^{x+N-S}v_1v_2)}{\theta_p(q^{2x+1-t-S}v_1v_2)}.$$

Proof.

$$Prob(X(t-1)=Y|X(t)=X) = \frac{L_{t-1}(X)C_{t-1}(X)C_t(Y)R_t(Y)}{L_t(X)C_t(X)R_t(X)} = \frac{L_{t-1}(Y)C_{t-1}(Y)}{L_t(X)}.$$

□

We are now in a position to define six stochastic matrices (Markov chains) needed in what follows. Their stochasticity along with other properties will be proven in Section 4.2 (the first two are already stochastic as they represent the transition probabilities obtained in this section). To condense notation, we denote $z_k = Fq^{x_k}$. Let

$$P_{t\pm}^{S,t} : \mathcal{X}^{S,t} \times \mathcal{X}^{S,t\pm1} \to [0,1],$$
$${}_{t+}P_{S\pm}^{S,t} : \mathcal{X}^{S,t} \times \mathcal{X}^{S\pm1,t} \to [0,1],$$
$${}_{t-}P_{S\pm}^{S,t} : \mathcal{X}^{S,t} \times \mathcal{X}^{S\pm1,t} \to [0,1],$$

be defined by

$$P_{t+}^{S,t}(X,Y) = \begin{cases} const \cdot \prod_{k<l} \frac{\varphi_{t+1,S}(y_k,y_l)}{\varphi_{t,S}(x_k,x_l)} \cdot \prod_{k:y_k=x_k+1} -\frac{q^{-x_k}\theta_p(Az_k,Bz_k,Cz_k,q^{1-N}z_k/ABC)}{\theta_p(z_k^2)} \times \\ \prod_{k:y_k=x_k} \frac{q^{-x_k-N+1}\theta_p(z_k/A,z_k/B,z_k/C,q^{N-1}z_kABC)}{\theta_p(z_k^2)} \text{ if } \ y_k - x_k \in \{0,1\} \ \forall k \\ 0, \quad \text{otherwise.} \end{cases} \tag{4.1.7}$$

$$P_{t-}^{S,t}(X,Y) = \begin{cases} const \cdot \prod_{k<l} \frac{\varphi_{t-1,S}(y_k,y_l)}{\varphi_{t,S}(x_k,x_l)} \cdot \prod_{k:y_k=x_k-1} \frac{q^{-x_k-N+1}\theta_p(z_k/D,z_k/E,z_k/F,q^{N-1}z_kDEF)}{\theta_p(z_k^2)} \times \\ \prod_{k:y_k=x_k} -\frac{q^{-x_k}\theta_p(Dz_k,Ez_k,Fz_k,q^{1-N}z_k/DEF)}{\theta_p(z_k^2)} \text{ if } \ y_k - x_k \in \{0,-1\} \ \forall k \\ 0, \quad \text{otherwise.} \end{cases} \tag{4.1.8}$$

$${}_{t+}P_{S+}^{S,t}(X,Y) = \begin{cases} const \cdot \prod_{k<l} \frac{\varphi_{t,S+1}(y_k,y_l)}{\varphi_{t,S}(x_k,x_l)} \cdot \prod_{k:y_k=x_k+1} -\frac{q^{-x_k}\theta_p(Az_k,Bz_k,Dz_k,q^{1-N}z_k/ABD)}{\theta_p(z_k^2)} \times \\ \prod_{k:y_k=x_k} \frac{q^{-x_k-N+1}\theta_p(z_k/A,z_k/B,z_k/D,q^{N-1}z_kABD)}{\theta_p(z_k^2)} \text{ if } \ y_k - x_k \in \{0,1\} \ \forall k \\ 0, \quad \text{otherwise.} \end{cases} \tag{4.1.9}$$

$$ {}_{t+}P^{S,t}_{S-}(X,Y) = \begin{cases} const \cdot \prod_{k<l} \frac{\varphi_{t,S-1}(y_k,y_l)}{\varphi_{t,S}(x_k,x_l)} \cdot \prod_{k:y_k=x_k+1} -\frac{q^{-x_k}\theta_p(Bz_k,Cz_k,Fz_k,q^{1-N}z_k/BCF)}{\theta_p(z_k^2)} \times \\ \prod_{k:y_k=x_k} \frac{q^{-x_k-N+1}\theta_p(z_k/B,z_k/C,z_k/F,q^{N-1}z_kBCF)}{\theta_p(z_k^2)} \text{ if } \; y_k - x_k \in \{0,-1\} \; \forall k \\ 0, \quad \text{otherwise.} \end{cases} \tag{4.1.10} $$

$$ {}_{t-}P^{S,t}_{S+}(X,Y) = \begin{cases} const \cdot \prod_{k<l} \frac{\varphi_{t,S+1}(y_k,y_l)}{\varphi_{t,S}(x_k,x_l)} \cdot \prod_{k:y_k=x_k-1} \frac{q^{-x_k-N+1}\theta_p(z_k/D,z_k/E,z_k/A,q^{N-1}z_kDEA)}{\theta_p(z_k^2)} \times \\ \prod_{k:y_k=x_k} -\frac{q^{-x_k}\theta_p(Dz_k,Ez_k,Az_k,q^{1-N}z_k/DEA)}{\theta_p(z_k^2)} \text{ if } \; y_k - x_k \in \{0,1\} \; \forall k \\ 0, \quad \text{otherwise.} \end{cases} \tag{4.1.11} $$

$$ {}_{t-}P^{S,t}_{S-}(X,Y) = \begin{cases} const \cdot \prod_{k<l} \frac{\varphi_{t,S-1}(y_k,y_l)}{\varphi_{t,S}(x_k,x_l)} \cdot \prod_{k:y_k=x_k-1} \frac{q^{-x_k-N+1}\theta_p(z_k/E,z_k/F,z_k/C,q^{N-1}z_kEFC)}{\theta_p(z_k^2)} \times \\ \prod_{k:y_k=x_k} -\frac{q^{-x_k}\theta_p(Ez_k,Fz_k,Cz_k,q^{1-N}z_k/EFC)}{\theta_p(z_k^2)} \text{ if } \; y_k - x_k \in \{0,-1\} \; \forall k \\ 0, \quad \text{otherwise.} \end{cases} \tag{4.1.12} $$

The normalizing constants are independent of the x_k's and the y_k's. They will become explicit in Section 4.2.

Note that ${}_{t-}P^{S,t}_{S-}$, under interchanging t and S, becomes $P^{S,t}_{t-}$. Under the same procedure ${}_{t+}P^{S,t}_{S+}$ becomes $P^{S,t}_{t+}$. We can think of $P^{S,t}_{t+}$ ($P^{S,t}_{t-}$) as a Markov chain that increases (decreases) t, while ${}_{t\pm}P^{S,t}_{S+}$ (${}_{t\pm}P^{S,t}_{S-}$) increases (decreases) S.

Remark 4.1.11. In the q-Racah limit $v_1 = v_2 = \kappa\sqrt{p},\ p \to 0$, the chains ${}_{t\pm}P^{S,t}_{S+}$ coalesce into one ($P^{S,t}_{S+}$ in [BGR10]). Likewise for ${}_{t\pm}P^{S,t}_{S-}$.

4.2 Elliptic difference operators

In the next two sections we explain how recent results on elliptic special functions and elliptic difference operators intrinsically capture the model we described thus far. In the present section we introduce certain multivariate elliptic difference operators. We give the probabilistic and combinatorial interpretations of the difference operators in the next section. The main two references are [Rai10] and [Rai06] and we will state results from these without going into the proofs (with a few exceptions where the proofs are short and revealing of common techniques employed in the area). The focus will be on certain normalization, quasi-commutation and quasi-adjointness relations satisfied by the difference operators. A univariate version of these difference operators has already appeared in the proof of Theorem 2.1.1.

In [Rai10] (see also [Rai06] for an algebraic description) Rains has introduced a family of difference operators acting nicely on various classes of BC_n-symmetric functions. To define them, we start with $r_0, r_1, r_2, r_3 \in \mathbb{C}^*$ satisfy $q^n r_0 r_1 r_2 r_3 = pq$. Following [Rai10], let $A^n(r_0)$ be the space of BC_n-symmetric abelian functions f such that

$$\prod_{1 \leq i \leq n} \theta_p(pqz_i^{\pm 1}/r_0; q)_m f(\dots z_i \dots)$$

is analytic for m large enough, and at the points $(q^l/r_0)^{\pm 1}, 1 \leq l \leq m$ (and p shifts thereof) it has at most simple poles. Then define $\mathcal{D}(r_0, r_1, r_2, r_3) : A^n(\sqrt{q} r_0) \to A^n(r_0)$ by (note also the dependence on q, p, n) by

$$(\mathcal{D}(r_0, r_1, r_2, r_3) f)(\dots z_k \dots) = \sum_{\sigma \in \{\pm 1\}^n} \prod_{1 \leq k \leq n} \frac{\prod_{0 \leq s \leq 3} \theta_p(r_s z_k^{\sigma_k})}{\theta_p(z_k^{2\sigma_k})} \prod_{1 \leq k < l \leq n} \frac{\theta_p(q z_k^{\sigma_k} z_l^{\sigma_l})}{\theta_p(z_k^{\sigma_k} z_l^{\sigma_l})} f(\dots q^{\sigma_k/2} z_k \dots). \tag{4.2.1}$$

Remark 4.2.1. The difference operator above described is the special case $t = q$ of the more general elliptic (q, t) elliptic difference operator introduced in the references.

In view of the balancing condition $q^n r_0 r_1 r_2 r_3 = pq$ we break symmetry and denote the difference operator by $\mathcal{D}(r_0, r_1, r_2)$, the fourth parameter being implied.

By letting $\mathcal{D}$ act on the function $f \equiv 1$, we obtain the following important lemma, whose proof we sketch following [Rai10].

Lemma 4.2.2. *For $r_0 r_1 r_2 r_3 = pq^{1-n}$ we have*

$$\sum_{\sigma \in \{\pm 1\}^n} \prod_{1 \leq k \leq n} \frac{\prod_{0 \leq s \leq 3} \theta_p(r_s z_k^{\sigma_k})}{\theta_p(z_k^{2\sigma_k})} \prod_{1 \leq k < l \leq n} \frac{\theta_p(q z_k^{\sigma_k} z_l^{\sigma_l})}{\theta_p(z_k^{\sigma_k} z_l^{\sigma_l})} = \prod_{0 \leq k < n} \theta_p(q^k r_0 r_1, q^k r_0 r_2, q^k r_1 r_2).$$

Proof. By direct computation the LHS above is invariant under $z_k \to p z_k$ for all k (this is insured by the fact $r_0 r_1 r_2 r_3 = pq^{1-n}$). It is also BC_n-symmetric (invariant under permutations of z_k's and under $z_k \to 1/z_k$). Finally, by multiplying LHS by $R = \prod_k z_k^{-1} \theta_p(z_k^2) \prod_{k<l} \varphi(z_k, z_l)$ we will have cleared potential poles of the LHS. Because R is BC_n-antisymmetric the result will end up being a multiple of R: $R \cdot \text{LHS} = const \cdot R$ showing LHS has no singularities in the variables and is thus independent of the z_i's. Evaluating then at $z_i = r_0 q^{n-i}$ yields the result. Observe the main point here was to prove the LHS is elliptic and has no poles in the variables, and indeed any analysis that shows this will prove the result. □

Hereinafter we will use $\mathcal{D}$ for the "normalized" difference operator (so that $\mathcal{D}(r_0, r_1, r_2)1 = 1$) following Lemma 4.2.2.

The difference operators described above satisfy a number of identities, including a series of quasi-commutation relations. For an elegant proof which relies on the action of such operators on a suitably large space of functions (more precisely, on BC_n-symmetric interpolation abelian functions), see [Rai10] or [Rai06]. We will not give a proof of quasi-commutation here, but rather postpone it to Chapter 7 where the proof will fit in more naturally (more precisely see Remark 7.2.4). Note nevertheless that the univariate instance can be proven by a direct computation.

Lemma 4.2.3. *If U, V, W, Z are 4 parameters, then*

$$\mathcal{D}(U,V,W)\mathcal{D}(q^{1/2}U, q^{1/2}V, q^{-1/2}Z) = \mathcal{D}(U,V,Z)\mathcal{D}(q^{1/2}U, q^{1/2}V, q^{-1/2}W).$$

Next we look at the action of the difference operators on special classes of functions. For $\lambda \in m^n$ a partition, let

$$\mathscr{d}_\lambda(\dots x_k \dots) = \prod_{1\le k\le n} \frac{\prod_{1\le l\le m+n}\theta_p(uq^{l-1}x_k^{\pm 1})}{\prod_{1\le l\le n}\theta_p(uq^{\lambda_l+n-l}x_k^{\pm 1})}.$$

By direct computation, we see that $\mathscr{d}_\lambda(\dots uq^{\mu_k+n-k}\dots) = \delta_{\lambda,\mu}c_\lambda$.

Remark 4.2.4. $\mathscr{d}_\lambda$ is a special version of the interpolation theta functions,

$$P_\lambda^{*(m,n)}(\dots x_k \dots; a, b; q; p),$$

defined in [Rai06] (matching the notation in the reference with ours, $a = u, b = q^{-m-n+1}/a$). They are defined (up to normalization) by two properties: being BC_n-symmetric of degree m (which happens for $\mathscr{d}_\lambda$'s) and vanishing at $\mu \neq \lambda$ (which trivially happens in our case). We will return to interpolation (theta) functions in Chapter 7.

If we now define $\mathfrak{d}_\lambda = \frac{\mathscr{d}_\lambda}{c_\lambda}$ we see that

$$\mathfrak{d}_\lambda(\dots uq^{\mu_k+n-k}\dots) = \delta_{\lambda,\mu}, \tag{4.2.2}$$

so in a precise way, $\mathfrak{d}_\lambda$ is an interpolation Kronecker-delta theta-function. We then immediately have the following proposition.

Proposition 4.2.5. *Fix $\tau \in \{\pm 1\}^n$. Let $z_k = uq^{\lambda_k+n-k}$. Then*

$$(\mathcal{D}(r_0,r_1,r_2)\mathfrak{d}_\lambda)(\dots q^{-\tau_k/2}z_k\dots) = \prod_k \frac{\theta_p(r_0 z_k^{\tau_k}, r_1 z_k^{\tau_k}, r_2 z_k^{\tau_k}, (pq^{1-n}/r_0r_1r_2)z_k^{\tau_k})}{\theta_p(z_k^{2\tau_k})} \prod_{k<l}\frac{\theta_p(qz_k^{\tau_k}z_l^{\tau_l})}{\theta_p(z_k^{\tau_k}z_l^{\tau_l})}.$$

Proof. Immediate by substituting into equation (4.2.1). For any $\sigma \neq \tau$, $q^{\sigma_k/2-\tau_k/2}z_k$ will be of the form uq^{μ_k+n-k} with $\mu \neq \lambda$ and the corresponding summand will be 0. □

A useful final property of the difference operators is their quasi-adjointness. It was shown in [Rai10] that the $\mathcal{D}$'s satisfy a certain "adjointness" relation that we will need in the next section. We start with 6 parameters $t_0, t_1, t_2, t_3, u_0, u_1$ satisfying the balancing condition $q^{2n-2}t_0t_1t_2t_3u_0u_1 = pq$.

We fix the number of variables at n and λ a partition in m^n. As in the introduction, we denote $l_i = \lambda_i + n - i$. We define the discrete Selberg inner product $\langle , \rangle$ (depending on p, q and the six parameters) by

$$\begin{aligned} \langle f, g \rangle = \frac{1}{Z} \sum_{\lambda \subseteq m^n} f(\dots t_0 q^{l_i} \dots) g(\dots t_0 q^{l_i} \dots) \times \\ \Delta_\lambda(q^{2n-2}t_0^2 | q^n, q^{n-1}t_0t_1, q^{n-1}t_0t_2, q^{n-1}t_0t_3, q^{n-1}t_0u_0, q^{n-1}t_0u_1; q), \end{aligned} \tag{4.2.3}$$

where f, g belong to the appropriate spaces of BC_n symmetric functions (the spaces will become obvious below) and Z is given in Theorem 2.3.5 (it makes $\langle 1, 1 \rangle = 1$). This is a discrete analogue of the continuous inner product introduced in [Rai10]:

$$\begin{aligned} \langle f, g \rangle = \frac{1}{Z'} \int_{C^n} f(\dots z_k \dots) g(\dots z_k \dots) \prod_{1 \leq k < l \leq n} \frac{\Gamma_{p,q}(t z_k^{\pm 1} z_l^{\pm 1})}{\Gamma_{p,q}(z_k^{\pm 1} z_l^{\pm 1})} \times \\ \prod_{1 \leq k \leq n} \frac{\Gamma_{p,q}(t_0 z_k^{\pm 1}, t_1 z_k^{\pm 1}, t_2 z_k^{\pm 1}, t_3 z_k^{\pm 1}, u_0 z_k^{\pm 1}, u_1 z_k^{\pm 1})}{\Gamma_{p,q}(z_k^{\pm 2})} \frac{dz_k}{2\pi i z_k}, \end{aligned}$$

and can be obtained from that by residue calculus (Z' can be obtained from Theorem 2.1.2; we refer the reader to Section 2.1 and [Rai10] for the contour conditions). We have the following proposition, which we only prove for the discrete (also $t = q$) inner product.

Proposition 4.2.6.

$$\langle \mathcal{D}(u_0, t_0, t_1) f, g \rangle = \langle f, \mathcal{D}(u_1', t_2', t_3') g \rangle', \tag{4.2.4}$$

where

$$(t_0', t_1', t_2', t_3', u_0', u_1') = (q^{1/2}t_0, q^{1/2}t_1, q^{-1/2}t_2, q^{-1/2}t_3, q^{1/2}u_0, q^{-1/2}u_1),$$

and $\langle , \rangle'$ *is the inner product defined in* (4.2.3) *with primed parameters inserted throughout.*

Proof. The LHS is a double sum. Interchanging the order of summation and a change of variables yields the RHS. □

4.3 Interpreting difference operators

We show in this section how the difference operators and their properties discussed in the previous section can be given probabilistic interpretations.

Remark 4.3.1. Observe from (3.3.4) that $q^{2n-3}ABCDEF = 1$.

In what follows h_k (h'_k) is the location of the k-th particle on the vertical line $i = t$ ($i = t+1$) in the (i, j) frame (note according to the $t \to t+1$ dynamics the particles move either up or down by $1/2$). We can prove the following proposition.

Proposition 4.3.2. *For A, B, C, D, E, F and $z_k = Fq^{h_k}$ given by* (3.3.4)*, the summands in*

$$(\mathcal{D}(A, B, C)1)(\dots z_k \dots)$$

(see (4.2.1)*), appropriately normalized using* (4.2.2)*, are equal to the transition probabilities (entries in the stochastic matrix) $P^{S,t}_{t+}(H, H')$ defined in* (4.1.7) *(after switching coordinates from (x, y) back to (i, j)). This statement also holds for*

$$\begin{aligned}
&\mathcal{D}(D, E, F) \ \textit{and} \ \ P^{S,t}_{t-}, \\
&\mathcal{D}(A, B, D) \ \textit{and} \ \ {}_{t+}P^{S,t}_{S+}, \\
&\mathcal{D}(B, C, F) \ \textit{and} \ \ {}_{t+}P^{S,t}_{S-}, \\
&\mathcal{D}(D, E, A) \ \textit{and} \ \ {}_{t-}P^{S,t}_{S+}, \\
&\mathcal{D}(E, F, C) \ \textit{and} \ \ {}_{t-}P^{S,t}_{S-}.
\end{aligned}$$

Proof. We will only prove the statement for $\mathcal{D}(A, B, C)$ and the $t+$ Markov chain. The other cases are similar. The proof is immediate in view of (3.3.4), the change of variables $(X, Y) \mapsto (H, H')$ in (4.1.7) (from (t, x) to (i, j) coordinates) and the following observations.

First, a choice of $\sigma_k \in \{\pm 1\}$ for all k in the definition of $\mathcal{D}(A, B, C)$ is equivalent to a choice of which particles move up/down from the position vector H (at vertical line t) to the position vector H' (at vertical line $t+1$). If $\sigma_k = 1$, the corresponding k-th particle at vertical position h_k moves up to $h'_k = h_k + 1/2$ (and if $\sigma_k = -1$, the k-th particle moves down). Next observe that in the univariate product appearing in any term of $(\mathcal{D}(A, B, C)1)(\dots z_k \dots)$, we can change $\theta_p(uz_i^{-b})$ ($b = 1, 2$) to $\theta_p(z_i^b/u)$ by the reflection formula for theta functions and it will now match with the univariate product appearing in $P^{S,t}_{t+}$. The product $\prod_{k: y_k = x_k + 1}(\dots) \prod_{y_k = x_k}(\dots)$ now indeed is identical (modulo constants independent of the particle positions) to $\prod_{k: h'_k = h_k + 1/2}(\dots) \prod_{k: h'_k = h_k - 1/2}(\dots)$ which is nothing more than $\prod_{k: \sigma_k = 1}(\dots) \prod_{k: \sigma_k = -1}(\ dots)$ in (4.2.1).

The elliptic Vandermonde product $\prod_{k<l}$ appearing in (4.1.7) is the same product (modulo constants independent of the particles) as the Vandermonde-like product in any term of

$$(\mathcal{D}(A, B, C)1)(\dots z_k \dots)$$

once we have transformed (in the latter product) $\theta_p(z_l/z_k)$ into $\theta_p(z_k/z_l)$ and $\theta_p(1/z_k z_l)$ into $\theta_p(z_k z_l)$

(picking up appropriate multipliers in front that will be powers of q appearing the Vandermonde-like product in (4.1.7)). The extra powers of q appearing in (4.1.7) will also surface in the difference operator once we have performed the aforementioned transformations. Finally observe that the ratio $\frac{\varphi_{t+1,S}(h'_k,h'_l)}{\varphi_{t,S}(h_k,h_l)}$ reduces (modulo the power of q up front already accounted for) to a ratio of only 2 theta functions (of the 4 initially present) because either $h'_k - h'_l = h_k - h_l$ or $h'_k + h'_l = h_k + h_l$ (depending whether particles k and l moved both in the same or in different directions). □

Remark 4.3.3. We describe how the difference operators capture the particle interpretation of the model intrinsically. In their definition specialized appropriately as in the statement of the above proposition, if two consecutive particles $k, k+1$ are 1 unit apart ($h_{k+1} - h_k = 1$), the bottom one cannot move up and the top one down to collide because the summand in the difference operator is 0 (indeed $\theta_p(qz_k z_{k+1}^{-1}) = \theta_p(1) = 0$ in the cross terms). Thus, the nonintersecting condition on the paths is intrinsically built into the difference operator. A similar reasoning shows that top-most and bottom-most particles are not allowed to leave the bounding hexagon either. To exemplify, for the difference operator $\mathcal{D}(A, B, C)$ corresponding to the $t \to t+1$ transition (particles moving from left most vertical line to the right), we observe that the restriction on top (bottom) particle is not to cross the NE (SE) edge labeled C (A) in Figure 3.7 (or indeed not to "walk too far" to the right by crossing the B edge). However A and C are two of the parameters of the difference operator, and the corresponding terms in the univariate product in the appropriate summand in (4.2.1) become 0 once the top (bottom) particle tries to leave the hexagon. Similar reasoning applies to the particles not being able to "walk too far right". Hence the difference operators intrinsically capture the boundary constraints of our model.

Remark 4.3.4. Proposition 4.3.2 is even more general, as we obtain $\binom{6}{3} = 20$ different stochastic matrices (Markov chains) from the 20 different difference operators (six of them already described).

We are now in a position to prove that the 6 matrices defined in Section 4.1 are indeed stochastic and measure preserving.

Theorem 4.3.5.

$$\sum_Y P_{t\pm}^{S,t}(X,Y) = 1,$$

$$\sum_Y {}_{t\pm}P_{S\pm}^{S,t}(X,Y) = 1,$$

$$\rho_{S,t\pm1}(Y) = \sum_X P_{t\pm}^{S,t}(X,Y) \cdot \rho_{S,t}(X),$$

$$\rho_{S\pm1,t}(Y) = \sum_X {}_{t\pm}P_{S\pm}^{S,t}(X,Y) \cdot \rho_{S,t}(X).$$

Proof. There is one way to prove these statements which works for 4 of the 6 matrices. Observe that the results for $t\pm$ follow from Theorems 4.1.9 and 4.1.10, and then to observe that under $t \leftrightarrow S$, we have

$$\mathcal{X}^{S,t} = \mathcal{X}^{t,S}, \text{ and } \rho_{S,t} = \rho_{t,S}$$

and then under interchanging S and t, $P_{t+}^{S,t}$ becomes ${}_{t+}P_{S+}^{S,t}$ (and $P_{t-}^{S,t}$ becomes ${}_{t-}P_{S-}^{S,t}$, respectively). This idea worked both at the q-Racah level and Hahn level (see [BGR10] and [BG09]).

Alternatively we can observe that the first two equalities are, by using (3.3.4) and Proposition 4.3.2, restatements of Lemma 4.2.2 for difference operators corresponding to parameters (A, B, C) (for $P_{t+}^{S,t}$), (D, E, F) (for $P_{t-}^{S,t}$), (A, B, D) (for ${}_{t+}P_{S+}^{S,t}$), (B, C, F) (for ${}_{t+}P_{S-}^{S,t}$), (D, E, A) (for ${}_{t-}P_{S+}^{S,t}$), (E, F, C) (for ${}_{t-}P_{S-}^{S,t}$). Moreover, the normalizing constants that we omitted in defining the transition matrices can be recovered easily from Proposition 4.3.2.

The last two statements are a special case of the adjointness relation. We will prove the third statement for the $t+$ operator. Similar results exist for the other 5 operators. We recall that $\rho_{S,t}(X)$ is nothing more than the discrete elliptic Selberg density,

$$\Delta_{\lambda_X}(q^{2N-2}F^2|q^N, q^{N-1}AF, q^{N-1}(pB)F, q^{N-1}CF, q^{N-1}DF, q^{N-1}EF),$$

defined in the Introduction, with $\lambda_{X,k} + n - k = x_{n+1-k}$. We also define the partition λ_Y to be the one corresponding to vertical line $t+1$ and particle positions given by Y: $\lambda_{Y,k} + n - k = y_{n+1-k}$. Then one sees $\rho_{S,t+1}(Y) = \sum_X P_{t+}^{S,t}(X, Y) \cdot \rho_{S,t}(X)$ is equivalent to

$$\langle \mathcal{D}(A, B, C)\mathfrak{d}_{\lambda_Y}, 1\rangle = \langle \mathfrak{d}_{\lambda_Y}, \mathcal{D}(D', E', F')1\rangle', \tag{4.3.1}$$

where the prime parameters and $\langle , \rangle'$ are defined in the previous section. The above equality (4.3.1) is only "morally correct" as we encounter the following issue: the (summands in the) difference operators $\mathcal{D}$ correspond to transitional probabilities in the (i, j) coordinates where particles move up or down by 1/2 from the t vertical line to the $t + 1$ vertical line (from Proposition 4.3.2). $\mathfrak{d}_\lambda$, Δ_λ as well as the definition of the inner product (4.2.3) correspond to coordinates (t, x) where particles either move horizontally 1 step to the right or diagonally up by 1 from vertical line t to vertical line $t+1$ (see the previous subsection and recall $\lambda_k + n - k = x_{n+1-k}$). But this can be easily fixed since $(i, j) = (t, x - t/2)$.

With the previous comment in mind, the RHS in (4.3.1) equals

$$\sum_\mu \mathfrak{d}_{\lambda_Y}(\ldots Fq^{\mu_k+n-k}\ldots)\Delta'_\mu = \Delta'_{\lambda_Y} = \rho_{S,t+1}(Y)$$

(observe $\Delta' = \Delta$ with prime parameters corresponds to the distribution of particles at the line $t+1$), while the LHS equals $\sum_{\lambda_X} Prob(\lambda_Y|\lambda_X)\cdot\Delta_{\lambda_X} = \sum_X P_{t+}^{S,t}(Y|X)\cdot\rho_{S,t}(X)$. The result follows. $\square$

We finish this section with a graphical description of the 6 Markov processes described thus far. The key is to look at the domain and codomain of the difference operators in canonical coordinates. We will exemplify with the difference operator $\mathcal{D}(A,B,D)$, corresponding to Markov chain ${}_{t+}P_{S+}^{S,t}$. Recall this Markov chain quasi-commutes with the $t\to t+1$ chain. The key is the following relation (a restatement of Theorem 4.3.5):

$$\sum_X Prob(Y|X;A,B,D)Prob(X;A,B,C,D,E,F) = Prob(Y;A',B',C',D',E',F') \text{ where}$$
$$(A',B',C',D',E',F') = (q^{\frac{1}{2}}A, q^{\frac{1}{2}}B, q^{-\frac{1}{2}}C, q^{\frac{1}{2}}D, q^{-\frac{1}{2}}E, q^{-\frac{1}{2}}F).$$

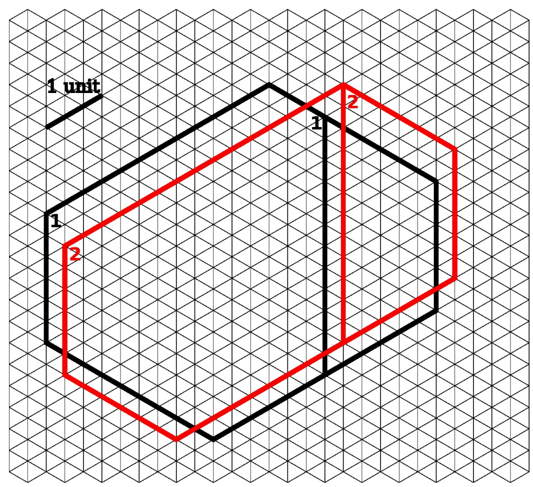

Figure 4.2: Action of the difference operator $\mathcal{D}(A,B,D)$ on a tiling of a $N=2, S=4, T=7$ hexagon drawn in canonical coordinates. The source is marked 1 and the destination 2. Only edges relevant to the model are considered: the 6 bordering edges and the particle line at horizontal displacement t from the leftmost vertical edge. Note the slight shifting, the increase in S by 1, and the fact that the particle line's displacement from the left vertical edge ($=t$) is kept constant (though particle positions are shifted by a third step).

We note ${}_{t+}P_{S+}^{S,t}$ corresponding to difference operator $\mathcal{D}(A,B,D)$ maps marked random tilings of hexagons determined by parameters (A,B,C,D,E,F) to random tilings of hexagons determined by parameters (A',B',C',D',E',F') (marked here refers to the particle line corresponding to parameter t). We figure what happens to the edges of such hexagons when parameters get shifted by $q^{\pm1/2}$ by using (3.3.6) (canonical coordinates). Figure 4.2 is a graphical description. In particular, we observe ${}_{t+}P_{S+}^{S,t}$ increases S by 1. Similarly for the other difference operators: they increase (decrease) S or t by 1 while leaving the other constant.

Chapter 5

Exact sampling algorithms

In this chapter we describe a polynomial-time algorithm for exact sampling of elliptically distributed lozenge tilings of a hexagon. It is a generalization of the shuffling algorithm for boxed plane partitions introduced in [BG09] and based on formalism from [BF10] (which formalism we introduce first). We finish by providing computer simulations (in JAVA) of our algorithm showing an arctic circle phenomenon for large tilings with features that are new to the elliptic level (and can be partially explained through the multitude of parameters we have at our disposal).

5.1 Intertwining two Markov chains

In this section we briefly discuss two ways to obtain a new Markov chain out of two quasi-commuting ones following (almost verbatum) Section 2.2 of [BF10]. A similar coupling (though not as general) between Markov chains was introduced in [DF90]. We will not provide the proofs, as we will prove the needed results in Section 5.2 (and this will really be without loss of generality). We will further aim for simplicity rather than full generality in stating the facts. The notation follows [BF10] closely. This formalism will come into play again in Chapter 7.

We start with (for simplicity finite) state spaces $\mathcal{S}_1, \dots, \mathcal{S}_n$, and stochastic matrices $P_1, \dots, P_n$ defining Markov chains on them: $P_k : \mathcal{S}_k \times \mathcal{S}_k \to [0, 1]$. We identify as before $P(x, y)$ with $Prob(y|x)$. We also assume the existence of *Markov links* $\Lambda_1^2, \dots, \Lambda_{n-1}^n$. These are stochastic matrices changing the state space: $\Lambda_{k-1}^k : \mathcal{S}_k \times \mathcal{S}_{k-1} \to [0, 1]$.

We further assume the following quasi-commutation relations between the Λ's and the P's:

$$\Delta_{k-1}^k := \Lambda_{k-1}^k P_{k-1} = P_k \Lambda_{k-1}^k, \qquad k = 2, \dots, n.$$

We define a new state space

$$\mathcal{S}_\Lambda^{(n)} = \Big\{ (x_1, \dots, x_n) \in \mathcal{S}_1 \times \cdots \times \mathcal{S}_n \mid \prod_{k=2}^{n} \Lambda_{k-1}^k (x_k, x_{k-1}) \neq 0 \Big\}$$

on which we define a Markov chain given by transition probabilities $P_\Lambda^{(n)}(X,Y)$ (where $X = (x_1,\dots,x_n)$, $Y = (y_1,\dots,y_n)$)

$$P_\Lambda^{(n)}(X_n,Y_n) = \begin{cases} P_1(x_1,y_1)\prod_{k=2}^{n} \dfrac{P_k(x_k,y_k)\Lambda_{k-1}^k(y_k,y_{k-1})}{\Delta_{k-1}^k(x_k,y_{k-1})}, & \prod_{k=2}^{n}\Delta_{k-1}^k(x_k,y_{k-1}) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

The fact that $P_\Lambda^{(n)}$ is a stochastic matrix is not apriori obvious and will be given a proof in the next section (though in the case $n = 2$ the proof is quite simple).

We can think of $P_\Lambda^{(n)}$ in the following way. Starting from $X = (x_1,\dots,x_n)$, we choose $Y = (y_1,\dots,y_n)$ *sequentially.* We first choose y_1 according to the transition matrix $P_1(x_1,y_1)$, then we choose y_2 sampling from $\frac{P_2(x_2,y_2)\Lambda_1^2(y_2,y_1)}{\Delta_1^2(x_2,y_1)}$. This is the the conditional distribution of the middle point in the successive application of P_2 and Λ_1^2 provided that we start at x_2 and finish at y_1. We then choose y_3 based on y_2 and x_3 as above and so on. We call this procedure a *sequential update.*

The next theorem (quoting from [BF10]) describes an important projection property of $P_\Lambda^{(n)}$.

Theorem 5.1.1. *Let $m_n(x_n)$ be a probability measure on $\mathcal{S}_n$. Consider the evolution of the measure*

$$m_n(x_n)\Lambda_{n-1}^n(x_n,x_{n-1})\cdots\Lambda_1^2(x_2,x_1)$$

on $\mathcal{S}_\Lambda^{(n)}$ under the Markov chain $P_\Lambda^{(n)}$, and denote by $(x_1(j),\dots,x_n(j))$ the result after $j = 0,1,2,\dots$ steps. Then for any $k_1 \geq k_2 \geq \cdots \geq k_n \geq 0$ the joint distribution of

$$(x_n(0),\dots,x_n(k_n),x_{n-1}(k_n),x_{n-1}(k_n+1),\dots,x_{n-1}(k_{n-1}),$$
$$x_{n-2}(k_{n-1}),\dots,x_2(k_2),x_1(k_2),\dots,x_1(k_1))$$

coincides with the stochastic evolution of m_n under transition matrices

$$(\underbrace{P_n,\dots,P_n}_{k_n},\Lambda_{n-1}^n,\underbrace{P_{n-1},\dots,P_{n-1}}_{k_{n-1}-k_n},\Lambda_{n-2}^{n-1},\dots,\Lambda_1^2,\underbrace{P_1,\dots,P_1}_{k_1-k_2}).$$

Remark 5.1.2. So far there is an obvious asymmetry between the Λ chains and the P chains: the source and target for the P chains is the same, while for the Λ chains it is not. This can be remedied by making the state spaces $\mathcal{S}$ depend on an additional (time) parameter t. Then the P's can be made to change t and the Λ's to (still) change k and symmetry is restored. Theorem 5.1.1 still applies in this time-inhomogeneous setting. This will be the approach pursued (along with proofs) in the next section where we apply the formalism to exact sampling of lozenge tilings distributed according to the elliptic measure.

5.2 Exact sampling: $S \mapsto S+1$

In this section, which follows closely the notation and proofs of [BG09] (see also [BGR10]), we will define a stochastic matrix

$$P^S_{S\mapsto S+1} : \Omega(N,S,T) \times \Omega(N,S+1,T) \to [0,1]$$

that preserves the elliptic measure $\mu(N,S,T)$—by definition, the total mass of a hexagon tiling (collection of N nonintersecting lattice paths) in $\Omega(N,S,T)$. Recall μ is defined (after fixing a gauge) as a product of the weights of the individual horizontal lozenges inside the hexagon. This stochastic matrix is defined by coupling two quasi-commuting existing stochastic matrices. We then show how this coupling leads to an algorithm for sampling the measure $\mu(N,S,T)$ in polynomial time.

Remark 5.2.1. $P^S_{S\mapsto S+1}$ corresponds to $P^{(n)}_\Lambda$ of the previous section. The P and the Λ chains of the previous section correspond (in this and the next section) to (one of each) $P^{S,t}_{t\pm}$ and $P^{S,t}_{S\pm}$ defined in Section 4.1.

Viewed as a Markov chain, the input for $P^S_{S\mapsto S+1}$ is a hexagon of size $a\times b\times c$ (N,S,T) and the output a hexagon of size $a\times(b-1)\times(c+1)$ $(N,S+1,T)$. If the input is distributed with measure $\mu(N,S,T)$, then the output will be distributed with measure $\mu(N,S+1,T)$.

Given a collection of nonintersecting paths $X=(X(0),\ldots,X(T))\in\Omega(N,S,T)$, we will construct a (random) new collection $Y=(Y(0),\ldots,Y(T))\in\Omega(N,S+1,T)$ by defining a stochastic transition matrix $P^S_{S\mapsto S+1}(X,Y)$. Observe that $Y(0)\in\mathcal{X}^{S+1,0}=(0,\ldots,N-1)$ is unambiguously defined. Next we perform the *sequential (inductive) update*. That is, the procedure which produces a random $Y(t+1)$ given knowledge of $Y(0),\ldots,Y(t)$ and X which we assume to have already been obtained. $Y(t+1)$ will be defined according to the distribution

$$\begin{aligned} Prob(Y(t+1)=Z) &= \frac{P^{S+1,t}_{t+}(Y(t),Z)\cdot {}_{t+}P^{S+1,t+1}_{S-}(Z,X(t+1))}{(P^{S+1,t}_{t+}\cdot {}_{t+}P^{S+1,t+1}_{S-})(Y(t),X(t+1))} \\ &= \frac{{}_{t-}P^{S,t+1}_{S+}(X(t+1),Z)\cdot P^{S+1,t+1}_{t-}(Z,Y(t))}{({}_{t-}P^{S,t+1}_{S+}\cdot P^{S+1,t+1}_{t-})(X(t+1),Y(t))}, \end{aligned} \tag{5.2.1}$$

where the last equality follows from the fact that

$$\rho_{S+1,t+1}(A)P^{S+1,t+1}_{t-}(A,B) = \rho_{S+1,t}(B)P^{S+1,t}_{t+}(B,A)$$

(this is nothing more than the equality $Prob(A\cap B)=Prob(A)Prob(B|A)=Prob(B)Prob(A|B)$).

We define the matrix $P_{S\mapsto S+1} : \Omega(N,S,T) \times \Omega(N,S+1,T) \to [0,1]$ by

$$P_{S\mapsto S+1} = \begin{cases} \prod_{t=0}^{T-1} \frac{P_{t+}^{S+1,t}(Y(t),Y(t+1)) \cdot {}_{t+}P_{S-}^{S+1,t+1}(Y(t+1),X(t+1))}{(P_{t+}^{S+1,t} \cdot {}_{t+}P_{S-}^{S+1,t+1})(Y(t),X(t+1))}, \\ \text{if } \ \prod_{t=0}^{T-1}(P_{t+}^{S+1,t} \cdot {}_{t+}P_{S-}^{S+1,t+1})(Y(t),X(t+1)) > 0, \\ 0, \ \text{otherwise.} \end{cases} \tag{5.2.2}$$

Theorem 5.2.2. *The matrix $P_{S\mapsto S+1}$ is stochastic and μ-measure preserving, in the sense that*

$$\mu(N,S+1,T)(Y) = \sum_{X\in\Omega(N,S,T)} P_{S\mapsto S+1}(X,Y)\mu(N,S,T)(X). \tag{5.2.3}$$

Proof. (following [BG09]) We want to show that

$$\sum_Y P_{S\mapsto S+1}(X,Y) = \sum_Y \prod_{t=0}^{T-1} \frac{P_{t+}^{S+1,t}(Y(t),Y(t+1)) \cdot {}_{t+}P_{S-}^{S+1,t+1}(Y(t+1),X(t+1))}{(P_{t+}^{S+1,t} \cdot {}_{t+}P_{S-}^{S+1,t+1})(Y(t),X(t+1))} = 1,$$

where the sum is taken over all $Y = (Y(0),\ldots,Y(T)) \in \Omega(N,S+1,T)$ such that

$$\prod_{t=0}^{T-1}(P_{t+}^{S+1,t} \cdot {}_{t+}P_{S-}^{S+1,t+1})(Y(t),X(t+1)) > 0. \tag{5.2.4}$$

We first sum over $Y(T)$ and because $Y(T)$ is distributed according to a singleton measure, the respective sum is 1. Next we deal with the sum

$$\sum_{Y(T-1)} \frac{P_{t+}^{S+1,T-2}(Y(T-2),Y(T-1)) \cdot {}_{t+}P_{S-}^{S+1,T-1}(Y(T-1),X(T-1))}{(P_{t+}^{S+1,T-2} \cdot {}_{t+}P_{S-}^{S+1,T-1})(Y(T-2),X(T-1))}$$

over $Y(T-1)$ satisfying $(P_{t+}^{S+1,T-1} \cdot {}_{t+}P_{S-}^{S+1,T})(Y(T-1),X(T)) > 0$ (because of (5.2.4)). Because of the quasi-commutation relations from Theorem 4.2.3, we have

$$\begin{aligned} &(P_{t+}^{S+1,T-1} \cdot {}_{t+}P_{S-}^{S+1,T})(Y(T-1),X(T)) = (P_{S-}^{S+1,T-1} \cdot {}_{t+}P_{S-}^{S,T-1})(Y(T-1),X(T)) \\ &\geq P_{S-}^{S+1,T-1}(Y(T-1),X(T-1))P_{t+}^{S,T-1}(X(T-1),X(T)). \end{aligned}$$

We are summing over $Y(T-1)$ such that the LHS above is nonvanishing, but if it vanishes, then by the above inequality so does ${}_{t+}P_{S-}^{S+1,T-1}(Y(T-1),X(T))$ (one of the numerator terms in the sum over $Y(T-1)$ considered). This means we can drop the condition that $(P_{t+}^{S+1,T-1} \cdot {}_{t+}P_{S-}^{S+1,T})(Y(T-1),X(T)) > 0$ and sum over all $Y(T-1)$. We obtain 1 for this sum (the denominator is independent of the summation variable, and summing the numerator over $Y(T-1)$ we obtain the denominator). We next sum inductively over $Y(T-2)$ and so on until we are left over with a sum over $Y(0)$. This sum only has 1 term, so we obtain the desired result.

To show $P_{S\mapsto S+1}$ preserves the measure μ, observe first that

$$\mu(N,S,T)(X) = m_0(X(0)) \cdot P_{t+}^{S,0}(X(0),X(1)) \dots P_{t+}^{S,T-1}(X(T-1),X(T)),$$

where m_0 is the unique probability measure on any singleton set (in this case $\mathcal{X}^{S,0}$). Then the RHS of (5.2.3) becomes

$$\sum_X m_0(X(0)) \prod_{t=0}^{T-1} P_{t+}^{S,t}(X(t),X(t+1)) \prod_{t=0}^{T-1} \frac{P_{t+}^{S+1,t}(Y(t),Y(t+1)) \cdot {}_{t+}P_{S-}^{S+1,t+1}(Y(t+1),X(t+1))}{(P_{t+}^{S+1,t} \cdot {}_{t+}P_{S-}^{S+1,t+1})(Y(t),X(t+1))}. \tag{5.2.5}$$

Pulling out factors independent of the summation variables, replacing $1 = m_0(X(0))$ with $1 = m_0(Y(0))$, using ${}_{t+}P_{S-}^{S+1,T}(Y(T),X(T)) = {}_{t+}P_{S-}^{S+1,0}(Y(0),X(0)) = 1$ and $P_{t+}^{S,t} \cdot {}_{t+}P_{S-}^{S,t+1} = {}_{t+}P_{S-}^{S,t} \cdot P_{t+}^{S-1,t}$, we transform (5.2.5) into

$$m_0(Y(0)) \prod_{t=0}^{T-1} P_{t+}^{S+1,t}(Y(t),Y(t+1)) \sum_X \prod_{t=0}^{T-1} \frac{{}_{t+}P_{S-}^{S+1,t}(Y(t),X(t)) \cdot P_{t+}^{S,t}(X(t),X(t+1))}{({}_{t+}P_{S-}^{S,t} \cdot P_{t+}^{S+1,t})(Y(t),X(t+1))}.$$

Now we sum first over $X(T)$, then over $X(T-1)$ and so on like in the previous argument to finally obtain on the LHS the desired result:

$$m_0(Y(0)) \prod_{t=0}^{T-1} P_{t+}^{S+1,t}(Y(t),Y(t+1)) = \mu(N,S+1,T)(Y).$$

$\square$

We now explain the sampling algorithm. For $x \in \mathbb{N}$ we define

$$p(x) = \frac{q\theta_p(q^{x-t-S+T-1}, q^{x-t-T-1}v_1, q^{x+t+1}v_2, q^{x-t-S-1}v_1v_2)}{\theta_p(q^{x+1}, q^{x-2t-S-1}v_1, q^{x-S+T+1}v_2, q^{x-T+1}v_1v_2)} \times \frac{\theta_p(q^{2x-t-S+1}v_1v_2)}{\theta_p(q^{2x-t-S-1}v_1v_2)}.$$

Note p also depends on S, T, v_1, v_2, q, p, but we omit these for simplicity of notation. Also note p is an elliptic function of $q, q^S, q^T, q^t, v_1, v_2, q^x$. Consider (again omitting most parameter dependence)

$$P(x;s) = \prod_{i=1}^{s} p(x+i-1).$$

P is just a ratio of 5 length s theta-Pochhammer symbols over 5 others (multiplied by q^{s-1} to make everything elliptic). We define the following probability distribution on the set $\{0,1,\dots,n\}$:

$$Prob(s) = D(x;n)(s) = \frac{P(x;s)}{\sum_{j=0}^{n} P(x;j)}. \tag{5.2.6}$$

For the exact sampling algorithm, given $X = (X(0), \dots, X(T)) \in \Omega(N, S, T)$, we construct $Y = (Y(0), \dots, Y(T)) \in \Omega(N, S+1, T)$ by first observing that $Y(0) = (0, \dots, N-1)$ is uniquely defined. We then perform T sequential updates. At step $t+1$ we obtain $Y(t+1)$ based on $Y(t)$ and $X(t+1)$. Suppose $X(t+1) = (x_1, \dots, x_N) \in \mathcal{X}^{S,t+1}$ and $Y(t) = (y_1, \dots, y_N) \in \mathcal{X}^{S+1,t}$. We want to define/sample $Y(t+1) = (z_1, \dots, z_N) \in \mathcal{X}^{S+1,t+1}$. $Y(t)$ and $X(t+1)$ satisfy $x_i - y_i \in \{0, -1, 1\}$ (follows by construction from $(P_{t+}^{S+1,t} \cdot {}_{t+}P_{S-}^{S+1,t+1})(Y(t), X(t+1)) > 0$). We thus have three cases, and we describe how to choose z_i in each:

- Case 1. Consider all i such that $x_i - y_i = 1$. Then $z_i = x_i$ is forced.
- Case 2. Consider all i such that $x_i - y_i = -1$. Then $z_i = y_i$ is forced.
- Case 3. For the remaining indices, group them in blocks and consider one such called a (k, l) block (where k is the smallest particle location in the block, and l is the number of particles in the block). That is, we have $y_{i-1} < k-1$, $y_{i+l} > k+l$ and the block consists of

$$x_i = y_i = k,\ x_{i+1} = y_{i+1} = k+1, \dots, x_{i+l-1} = y_{i+l-1} = k+l-1.$$

  For each such block independently, we sample a random variable ξ according to the distribution $D(k; l)$. We set $z_i = x_i$ for the first ξ consecutive positions in the block, and we set $z_i = x_i + 1$ for the remainder of the $l - \xi$ positions. We provide an example in Figure 5.1 below.

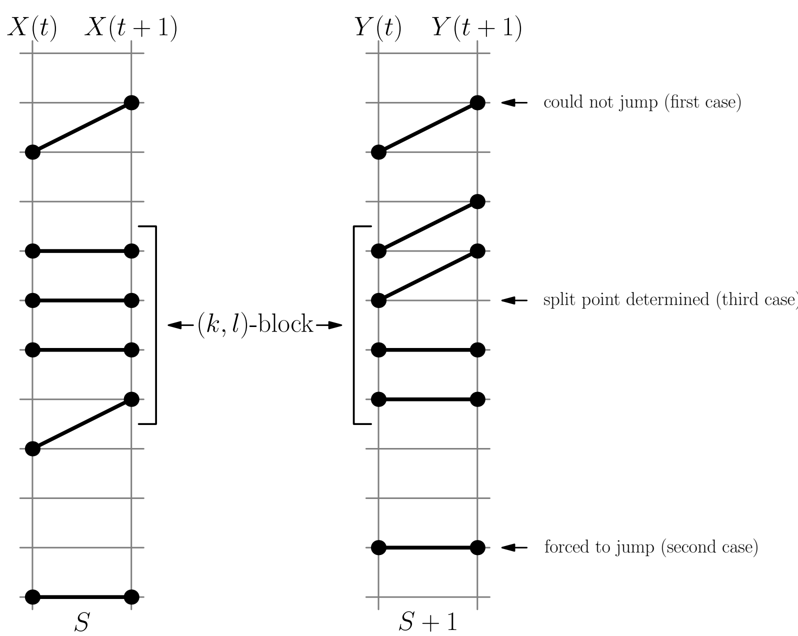

Figure 5.1: Sample block split. See also [BG09].

Theorem 5.2.3. *By constructing Y this way, we have simulated a $S \mapsto S+1$ step of the Markov chain $P_{S\mapsto S+1}$.*

Proof. We perform the following computation (and are interested in Case 3. described above, that is on how to split a (k,l) block; note $x_i = y_i$ in the case of interest):

$$
\begin{aligned}
&Prob(Y(t+1)=Z) = \frac{P_{t+}^{S+1,t}(Y(t),Z)\cdot {}_{t+}P_{S-}^{S+1,t+1}(Z,X(t+1))}{(P_{t+}^{S+1,t}\cdot {}_{t+}P_{S-}^{S+1,t+1})(Y(t),X(t+1))} = (\text{factors independent of} Z)\\
&\times \prod_{i:z_i=y_i} q^{-y_i-N+1}\frac{\theta_p(q^{y_i+T-S-t-1}, q^{y_i-T-S-t-1}v_1, q^{y_i+t+1}v_2, q^{y_i+N-t}v_1v_2)}{\theta_p(q^{2y_i-t-S}v_1v_2)}\\
&\times \prod_{i:z_i=y_i+1} q^{-y_i}\frac{\theta_p(q^{y_i-S-N}, q^{y_i-2t-S-1}v_1, q^{y_i+T+1}v_2, q^{y_i-T+1}v_1v_2)}{\theta_p(q^{2y_i-t-S}v_1v_2)}\\
&\times \prod_{i:z_i=x_i} q^{-x_i}\frac{\theta_p(q^{x_i-S-N}, q^{x_i-t-T-1}v_1, q^{x_i+T+1}v_2, q^{x_i-t-S-1}v_1v_2)}{\theta_p(q^{2x_i-t-S-1}v_1v_2)}\\
&\times \prod_{i:z_i=x_i+1} q^{-x_i-N}\frac{\theta_p(q^{x_i+1}, q^{x_i-T-S-t-1}v_1, q^{x_i-S+T+1}v_2, q^{x_i+N-t}v_1v_2)}{\theta_p(q^{2x_i-t-S+1}v_1v_2)}.
\end{aligned}
\tag{5.2.7}
$$

We thus see the blocks split independently. The probability that the first j particles in a (k,l) block stay put from $Y(t)$ to $Y(t+1)$ (and the rest of $l-j$ jump by 1) is, by using the above formula:

$$
\begin{aligned}
&\prod_{i=0}^{j-1}\frac{q\theta_p(q^{k+i-t-S+T-1}, q^{k+i-t-T-1}v_1, q^{k+i+t+1}v_2, q^{k+i-t-S-1}v_1v_2)}{\theta_p(q^{2k+2i-t-S-1}v_1v_2)}\\
&\times\prod_{i=j}^{l-1}\frac{\theta_p(q^{k+i+1}, q^{k+i-2t-S-1}v_1, q^{k+i-S+T+1}v_2, q^{k+i-T+1}v_1v_2)}{\theta_p(q^{2k+2i-t-S+1}v_1v_2)}\times(\text{factors independent of } j),
\end{aligned}
$$

where in (5.2.7) we have gauged away everything independent of the split position j. This probability is nothing more than the distribution D we defined in (5.2.6). This finishes the proof. □

5.3 Exact sampling: $S \mapsto S-1$

Similar to the $P_{S\mapsto S+1}$ matrix described in the previous two sections, we can construct a $P_{S\mapsto S-1}$ measure preserving Markov chain that takes random tilings in $\Omega(N,S,T)$ and maps them to random tilings in $\Omega(N,S-1,T)$. We proceed exactly as in Section 5.2 and will omit most details and theorems as they transfer verbatim from Section 5.2 and the previous section. Given $X\in\Omega(N,S,T)$ and $Y(0), Y(1), \ldots, Y(t)$ already defined inductively, we choose $Y(t+1)$ from the distribution:

$$
Prob(Y(t+1)=Z) = \frac{P_{t+}^{S-1,t}(Y(t),Z)\cdot {}_{t+}P_{S+}^{S-1,t+1}(Z,X(t+1))}{(P_{t+}^{S-1,t}\cdot {}_{t+}P_{S+}^{S-1,t+1})(Y(t),X(t+1))}. \tag{5.3.1}
$$

We define

$$P_{S\mapsto S-1} = \begin{cases} \prod_{t=0}^{T-1} \frac{P_{t+}^{S-1,t}(Y(t),Y(t+1))\cdot_{t+} P_{S+}^{S-1,t+1}(Y(t+1),X(t+1))}{(P_{t+}^{S-1,t}\cdot_{t+} P_{S+}^{S-1,t+1})(Y(t),X(t+1))}, \\ \text{if } \prod_{t=0}^{T-1}(P_{t+}^{S-1,t}\cdot_{t+} P_{S+}^{S-1,t+1})(Y(t),X(t+1)) > 0, \\ 0, \text{ otherwise.} \end{cases} \tag{5.3.2}$$

We will also sketch the algorithm for sampling using $P_{S\mapsto S-1}$. We need to define the equivalent for p from the previous section. For $x \in \mathbb{N}$ we define

$$p'(x) = \frac{q\theta_p(q^{x-t-N-1}, q^{x-t-2S}v_1, q^{x+t}v_2, q^{x-t+N-1}v_1v_2)}{\theta_p(q^{x-S-N+1}, q^{x-2t-S}v_1, q^{x+S}v_2, q^{x-S+N+1}v_1v_2)} \times \frac{\theta_p(q^{2x-t-S+1}v_1v_2)}{\theta_p(q^{2x-t-S-1}v_1v_2)}.$$

As before, p' is an elliptic in $q, q^S, q^N, q^t, v_1, v_2, q^x$. We also have $P'(x;s) = \prod_{i=1}^{s} p'(x+i-1)$ and the following distribution on $\{0,1,\dots,n\}$:

$$Prob(s) = D'(x;n)(s) = \frac{P'(x;s)}{\sum_{j=0}^{n} P'(x;j)}. \tag{5.3.3}$$

Assuming we have $X \in \Omega(N,S,T)$ with $X(t+1) = (x_1 < \cdots < x_N)$ and inductively $Y(0)$, $\dots, Y(t) = (y_1 < \cdots < y_N)$), we sample $Y(t+1) = (z_1 < \cdots < z_N)$ by first observing that $x_i - y_i \in \{0,1,2\}$ (because $(P_{t+}^{S-1,t}\cdot_{t+} P_{S+}^{S-1,t+1})(Y(t),X(t+1)) > 0$) and then performing appropriate updates for the following three simple cases:

- Case 1. For all i with $x_i - y_i = 0$ we set $z_i = x_i$.
- Case 2. For all i with $x_i - y_i = 2$ we set $z_i = y_i + 1$.
- Case 3. For the remaining indices (for which $x_i - y_i = 1$), group them in blocks and consider one such called a (k,l) block (where k is the smallest particle location in the block, and l is the number of particles in the block). That is, we have $y_{i-1} < k-1$, $y_{i+l} > k+l$ and the block consists of

  $$x_i = y_i + 1 = k,\ x_{i+1} = y_{i+1} + 1 = k+1, \dots, x_{i+l-1} = y_{i+l-1} + 1 = k+l-1.$$

  For each such block independently, we sample a random variable ξ according to the distribution $D'(k;l)$. We set $z_i = y_i$ for the first ξ consecutive positions in the block, and we set $z_i = y_i + 1$ for the remainder of the $l - \xi$ positions. See Figure 5.1.

An analogue of Theorem 5.2.3 exists and is proved in a similar way to show the above 3 steps are all that is necessary to simulate the Markov chain $P_{S\mapsto S-1}$.

5.4 Computer simulations

In this section we present computer simulations of the exact sampling algorithm from Section 5.2. We are (with one exception) looking at $200 \times 200 \times 200$ hexagons, and parameters are chosen so the elliptic measure sampled is positive throughout the range of the algorithm (recall that the algorithm starts with a $200 \times 400 \times 0$ box and increases c while decreasing b by 1, until it reaches the desired size—after 200 iterations in our case). Under each figure we list the values of the four parameters p, q, v_1, v_2. Computations and simulations are done using double precision, the $S \mapsto S+1$ algorithm polynomial algorithm described above, and a custom program written in Java and that can handle large hexagons (in excess of $N = 1000$ particles) fast enough on modern CPUs.

In Figure 5.2 we observe that the sample looks like one from the uniform measure with the arctic ellipse theoretically predicted in [CLP98] clearly visible.

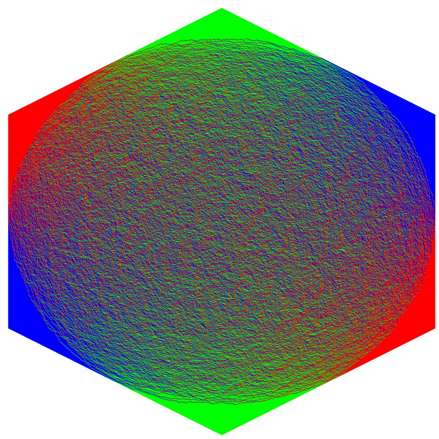

Figure 5.2: $p = 10^{-7}, q = 0.999999995, v_1 = 0.0000214, v_2 = 1.00675$. $400 \times 400 \times 400$. Because q is *very* close to 1, the limit shape looks uniform (recall that $q = 1$ gives rise to the uniform measure).

Figures 5.3 and 5.4 exhibit a new behavior for the arctic circle: the curve seems to acquire 3 nodes at the 3 vertices of the hexagon seen in the pictures. To obtain these shapes, the parameters have been tweaked so that the elliptic weight ratio vanishes (or $= \infty$) at the respective corners (in other words, the weight ratio (3.3.2) is "barely positive" as described in Section 3.4). To be more precise, we have

$$q = e^{\frac{2\pi i}{T-1}},$$

$$v_1 = q^{2T-1},$$

$$v_2 = 1/q.$$

This fixes 3 of the 4 parameters of the measure and we have the extra degree of freedom p and so we obtain a 1-parameter family of trinodal arctic boundaries. All simulations are taken from the trigonometric positivity case (q, v_1, v_2 are of unit modulus—see Section 3.4). While the first arctic boundary looks like an equilateral "flat" triangle, the second looks like an equilateral "thin/hyperbolic" triangle. The change from Figure 5.3 to 5.4 is an increase in p (and indeed if we increase p further the triangle will get thinner and thinner, until it will degenerate into a union of the 3 coordinate axes as $p \to 1$). The limit $p \to 0$ yields the same "thinning behavior" in the real positivity case.

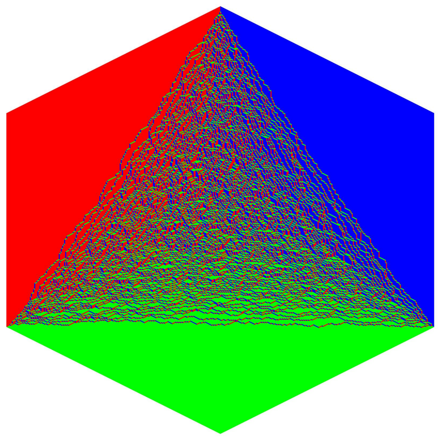

Figure 5.3: An instance of a trinodal arctic boundary. $p = 0.00186743, \arg q = 0.000835422, \arg v_1 = 0.667502, \arg v_2 = -0.000835422$.

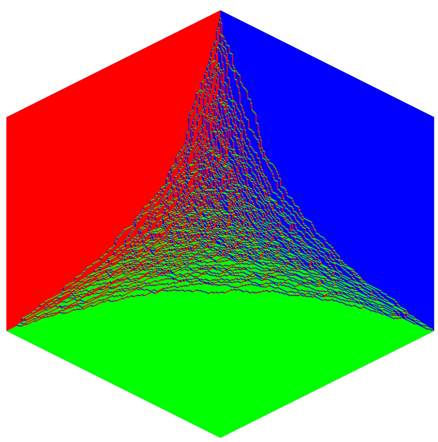

Figure 5.4: Another instance of a trinodal arctic boundary. $p = 0.2, \arg q = 0.000835422, \arg v_1 = 0.667502, \arg v_2 = -0.000835422$. Note p is larger in this case than in the previous.

Finally in Figure 5.5 we exhibit a trinodal case in the top level trigonometric case $p = 0$ when q, v_1, v_2 are of unit modulus (in the case q and v_i are real, the arctic boundary is the union of the coordinate axes as stated above).

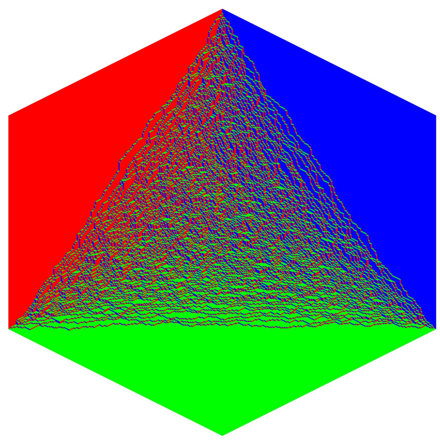

Figure 5.5: Top level trigonometric $p = 0$ case. As above, $\arg q = 0.000835422, \arg v_1 = 0.667502, \arg v_2 = -0.000835422$.

Chapter 6

Correlation kernel

In this chapter we look at the Markov process(es) coming from any elliptically distributed random lozenge tiling of a hexagon and show it is a determinantal point process with correlation kernel given by the univariate elliptic biorthogonal functions of Spiridonov and Zhedanov [SZ00]. We recall some facts about determinantal point processes and elliptic biorthogonal functions in the first and second sections respectively, and devote the rest to the specific case of lozenge tilings.

6.1 Determinantal point processes

In this section we define determinantal point processes and state the Eynard-Mehta theorem. We follow the exposition and quote the necessary results for our purposes from [BR05] and [Bor11] (the latter is a broad review of the subject).

Let $\mathcal{S}$ be a finite set. Subsets of $\mathcal{S}$ will be called *point configurations*. A *random point process* on $\mathcal{S}$ is a probability measure on $2^{\mathcal{S}}$. Such a process is called *determinantal* if there exists a $|\mathcal{S}| \times |\mathcal{S}|$ matrix K with rows and columns indexed by elements of $\mathcal{S}$ (called a *correlation kernel*) such that the following functions (called *correlation functions*):

$$\rho(Y) = Prob(X \in 2^{\mathcal{S}} | Y \subseteq X)$$

are determinantal for all $Y \in \mathcal{S}$. To wit: $\rho(Y) = \det K_Y$ where K_Y is the submatrix of K with rows and columns indexed by elements of Y. We note the correlation kernel K is not unique. We can conjugate the matrix by any diagonal matrix on either side and obtain the same determinantal point process.

An example of such a process is given by any $|\mathcal{S}| \times |\mathcal{S}|$ positive definite matrix J (which we consider indexed by elements of $\mathcal{S}$). For $X \in 2^{\mathcal{S}}$ we can consider the probability distribution given by

$$Prob(X) = \frac{\det J_X}{\det (Id + L)}.$$

It turns out (see for example [DVJ88]) that this random point process is a determinantal point process with kernel given by $K = J(Id + J)^{-1}$.

We now aim at stating the Eynard-Mehta theorem (see [EM98]; also see [BR05] for an elementary proof). We follow the narrative of [Bor11] and use the notation therein. We start with finite sets $\mathcal{S}_1, \dots, \mathcal{S}_m$ and we denote $\mathcal{S} = \mathcal{S}_1 \sqcup \dots \sqcup \mathcal{S}_m$. We fix a positive integer N and consider functions

$$F_i : \mathcal{S}_1 \to \mathbb{C},\ G_i : \mathcal{S}_m \to \mathbb{C},\ 1 \le i \le N,$$
$$T_{j,j+1} : \mathcal{S}_j \times \mathcal{S}_{j+1} \to \mathbb{C},\ 1 \le j \le m-1.$$

To any $X \in 2^{\mathcal{S}}$ we assign probability 0 if the number of points in the intersection with at least one of the $\mathcal{S}_i$ is not equal to N. Otherwise, let us denote $X \cap \mathcal{S}_i = \{x_1^i, \dots, x_N^i\}$. To such a configuration we assign a probability proportional to

$$\det_{i,j}(F_i(x_j^1)) \det_{i,j}(T_{1,2}(x_i^1, x_j^2)) \dots \det_{i,j}(T_{m-1,m}(x_i^{m-1}, x_j^m)) \det_{i,j}(G_i(x_j^m)), \tag{6.1.1}$$

where all indices in all determinants range from 1 to N. The partition function for this weighting of configurations is equal to $\det M$ where

$$M = F T_{1,2} \dots T_{m-1,m} G,$$

a fact that follows from the Cauchy-Binet identity for determinants

$$\sum_{Z=(\dots,z_i,\dots)} \det_{1\le i,j\le N} P(x_i, z_j) \det_{1\le i,j\le N} Q(z_i, y_j) = \det_{1\le i,j\le N} (PQ)(x_i, y_j).$$

We are of course interested in the case $\det M \neq 0$ but can otherwise allow complex probabilities. We denote by $\star$ the following product operation (a combination of matrix and scalar product):

$$(f \star g)(x,y) = \sum_z f(x,z) g(z,y),\ h \star k = \sum_z h(z) k(z),$$
$$(h \star f)(y) = \sum_z h(z) f(z,y),\ (g \star k)(x) = \sum_z g(x,z) k(z).$$

The following theorem is the main result of this section (we follow [Bor11] for the statement):

Theorem 6.1.1. *The point process defined in* (6.1.1) *is determinantal with correlation kernel* $K_{k,l}$:

$\mathcal{S}_k \times \mathcal{S}_l \to \mathbb{C}$ *(for any $1 \leq k, l \leq N$) given by*

$$K_{k,l}(x^k, y^l) = -\,\delta_{k>l}(T_{l,l+1} \star \cdots \star T_{k-1,k})(y^l, x^k) + \sum_{i,j=1}^{N} (W^{-t})_{i,j} (F_i \star T_{1,2} \star \cdots \star T_{k-1,k})(x^k)(T_{l,l+1} \star \cdots \star T_{m-1,m} \star G_j)(y^l),$$

where the matrix W is defined by $W_{i,j} = F_i \star T_{1,2} \star \cdots \star T_{m-1,m} \star G_j$.

Suppose now that we have a *biorthonormal basis* $\{\mathcal{F}_i^j, \mathcal{G}_i^j\}_{i\geq 1}$ for each $L^2(\mathcal{S}_j)$ (such a set is called a biorthonormal basis if $\{\mathcal{F}_i^j\}_{i\geq 1}$ and $\{\mathcal{G}_i^j\}_{i\geq 1}$ are both bases of $L^2(\mathcal{S}_j)$ and $\langle \mathcal{F}_k^j, \mathcal{G}_l^j \rangle = \delta_{k,l}$ for the inner product $\langle \cdot, \cdot \rangle$ on $L^2(\mathcal{S}_j)$). Furthermore suppose that

$$T_{j,j+1}(x,y) = \sum_{i \geq 1} c_{j,j+1} \mathcal{F}_i^j(x) \mathcal{G}_i^{j+1}(y),\ 1 \leq j \leq m-1,$$

and that

$$Span\{\mathcal{F}_1^1, \dots, \mathcal{F}_N^1\} = Span\{F_1, \dots, F_N\},$$
$$Span\{\mathcal{G}_1^m, \dots, \mathcal{G}_N^m\} = Span\{G_1, \dots, G_N\}.$$

If we then denote $c_{k,l;i} := \prod_j^{l-k-1} c_{k+j,k+j+1;i}$ we then obtain the following form for the kernel on $\mathcal{S}_k \times \mathcal{S}_l$:

$$K_{k,l}(x^k, y^l) = \begin{cases} \sum_{i=1}^N c_{k,l;i}^{-1} \mathcal{F}_i^k(x^k) \mathcal{G}_i^l(y^l),\ k \leq l, \\ -\sum_{i>N} c_{l,k;i} \mathcal{F}_i^k(x^k) \mathcal{G}_i^l(y^l),\ k > l. \end{cases}$$

6.2 Elliptic biorthogonal functions

In this section we gather together a few results about univariate discrete elliptic biorthogonal functions. The notation and exposition will mostly be following [Rai06].

We need to make brief use of univariate *interpolation abelian functions*. Such functions will be introduced in more detail (and in the multivariate case) in Chapter 7. The multivariate analogues first appeared in [Rai10] (see also [Rai06] for a description closer to our purposes). Univariately they are, for a fixed integer l, BC_1-symmetric (i.e., symmetric under $x \mapsto 1/x$) ratios of BC_1-symmetric theta functions of degree l with prescribed poles and zeros. To wit the univariate abelian interpolation function of degree l is defined by

$$R_l^*(x; a, b) = \frac{\theta_p(ax^{\pm 1}; q)_l}{\theta_p(bq^{-l}x^{\pm 1}; q)_l},$$

where a, b are parameters (for brevity, we omit the q, p dependence for all functions present in this section). Observe R_l^* has zeros at finitely many q-shifts of a and poles at finitely many q-shifts of b (up to taking reciprocals and shifting by p).

The univariate elliptic biorthogonal functions $R_l(x; t_0 : t_1, t_2, t_3; u_0, u_1)$ discovered by Spiridonov and Zhedanov [SZ00] (see also [Rai10] and [Rai06] for multivariate analogues) can be defined in terms of the interpolation functions as follows (see [Rai06]). Fix $|p| < 1, q$ as well as six complex parameters $t_0, t_1, t_2, t_3, u_0, u_1$ satisfying the balancing condition $t_0t_1t_2t_3u_0u_1 = pq$ (the same condition as for the order 0 elliptic beta integral of Chapter 2). Then we define

$$R_l(x; t_0 : t_1, t_2, t_3; u_0, u_1) = \sum_{0 \leq k \leq l} d_k R_k^*(x; t_0, u_0) = d_l R_l^*(x; t_0, u_0) + \text{lower-order terms},$$

where

$$d_k = \frac{\binom{l}{k}_{[1/u_0u_1, 1/t_0u_1]}}{\Delta_k^0(t_0/u_0 | t_0t_1, t_0t_2, t_0t_3, t_0u_1)},$$

and we have used $\binom{l}{k}_{[a,b]} := \Delta_k(a/b|q, 1/b) R_k^*(\sqrt{a}q^l; \sqrt{a}, b/\sqrt{a})$ to denote the univariate *elliptic binomial coefficient* (we will not use elliptic binomial coefficients anymore in this section; we postpone their study to Chapter 7).

By simplifying everything in the definition of R_l to ratios of theta Pochhammer symbols, we can express R_l as the following explicit terminating order 1 elliptic hypergeometric series:

$$R_l(x; t_0 : t_1, t_2, t_3; u_0, u_1) = {}_{12}E_{11}\left(\frac{t_0}{u_0}; \frac{pq^l}{u_0u_1}, \frac{q}{t_1u_0}, \frac{q}{t_2u_0}, \frac{q}{t_3u_0}, t_0x, t_0/x, q^{-l}; 1\right).$$

Remark 6.2.1. $R_l(x; t_0 : t_1, t_2, t_3; u_0, u_1)$ has poles at shifts of $u_0^{\pm 1}$ (we will say u_0 *controls the poles*). Moreover, we have the following special value (immediate from the terminating series representation):

$$R_l(t_0; t_0 : t_1, t_2, t_3; u_0, u_1) = 1,$$

and we say parameter t_0 is used for *normalization.*

The biorthogonal functions are elliptic in the six parameters (provided the balancing condition is satisfied) as well as in the variable x. Following [Rai10], we have the following theorem.

Theorem 6.2.2.

$$R_l(p^{1/2}z; p^{1/2}t_0 : p^{1/2}t_1, p^{-1/2}t_2, p^{-1/2}t_3; p^{1/2}u_0, p^{-1/2}u_1) = R_l(z; t_0 : t_1, t_2, t_3; u_0, u_1),$$
$$R_l(z; p^{k_0}t_0 : p^{k_1}t_1, p^{k_2}t_2, p^{k_3}t_3; p^{l_0}u_0, p^{l_1}u_1) = R_l(z; t_0 : t_1, t_2, t_3; u_0, u_1),$$

where the k_i's and the l_j's are integers respecting the balancing condition: $k_0+k_1+k_2+k_3+l_0+l_1 = 0$.

One of the main results of [SZ00] (but also of [Rai10], [Rai06] in the multivariate case) is that the elliptic biorthogonal functions with poles controlled by u_0 and those with poles controled by u_1 are biorthogonal.

Theorem 6.2.3. *If in addition to the balancing condition on the parameters we also have* $t_0 t_1 = q^{-m}$ *(for some* $m > 0$ *an integer), the functions with poles controlled by* u_0 *and those with poles controlled by* u_1 *are biorthogonal on* $\{0, \ldots, m\}$*:*

$$\sum_{0 \le s \le m} R_l(t_0 q^s; t_0 : t_1, t_2, t_3; u_0, u_1) R_k(t_0 q^s; t_0 : t_1, t_2, t_3; u_1, u_0) \Delta_s(t_0^2 | q, t_0 t_1, t_0 t_2, t_0 t_3, t_0 u_0, t_0 u_1)$$

vanishes unless $k = l$ *in which case it is equal to*

$$c_l := \frac{\Delta_m^0(t_1/u_0 | t_1/t_0, pq/u_0 t_2, pt/u_0 t_3, pq/u_0 u_1)}{\Delta_l(1/u_0 u_1 | q, t_0 t_1, t_0 t_2, t_0 t_3, 1/t_0 u_0, 1/t_0 u_1)} \propto \Delta(\hat{t}_0^2 | q, \hat{t}_0 \hat{t}_1, \hat{t}_0 \hat{t}_2, \hat{t}_0 \hat{t}_3, \hat{t}_0 \hat{u}_0, \hat{t}_0 \hat{u}_1)^{-1}, \tag{6.2.1}$$

where

$$\hat{t}_0 = \sqrt{\frac{t_0 t_1 t_2 t_3}{pq}}, \ \hat{t}_0 \hat{t}_i = t_0 t_i, \ i \in \{1, 2, 3\}, \ \frac{\hat{u}_j}{\hat{t}_0} = \frac{u_j}{t_0}, \ j \in \{0, 1\}. \tag{6.2.2}$$

The "hat" is an involution. Also the hat parameters satisfy the same balancing conditions the original parameters satisfy. They are important because by hatting we can exchange the variable and the index of the biorthogonal functions as follows (see [Rai06]):

$$R_l(t_0 q^s; t_0 : t_1, t_2, t_3; u_0, u_1) = R_s(\hat{t}_0 q^l; \hat{t}_0 : \hat{t}_1, \hat{t}_2, \hat{t}_3; \hat{u}_0, \hat{u}_1). \tag{6.2.3}$$

The normalized (in this case univariate) difference operators of Section 4.2 act on the biorthogonal functions as follows (note u_0 is special—it controls the poles, and t_0 is also special as the choice of normalization):

$$\begin{aligned} &\mathcal{D}(u_0, t_0, t_1) R_l((q^{1/2} t_0) q^s; q^{1/2} t_0 : q^{1/2} t_1, q^{-1/2} t_2, q^{-1/2} t_3; q^{1/2} u_0, q^{-1/2} u_1) \\ &\qquad = R_l(t_0 q^s; t_0 : t_1, t_2, t_3; u_0, u_1). \end{aligned} \tag{6.2.4}$$

We can renormalize and exchange t_0 with another t_j at the choice of picking up a factor:

$$R_l(x; t_1 : t_0, t_2, t_3; u_0, u_1) = \frac{R_l(x; t_0 : t_1, t_2, t_3; u_0, u_1)}{R_l(t_1; t_0 : t_1, t_2, t_3; u_0, u_1)}. \tag{6.2.5}$$

We finish by providing an integral representation for univariate biorthogonal functions following [Rai10].

Theorem 6.2.4. *For parameters t_1, t_2, t_3, u_0, u_1 multiplying to pq we have*

$$\frac{1}{Z}\int_C R_l^*(z;t_0v,u_0)R_k^*(z;t_0,u_1)\frac{\Gamma_{p,q}(t_0z^{\pm1},t_1z^{\pm1},t_2z^{\pm1},t_3z^{\pm1},u_0z^{\pm1},u_1z^{\pm1})}{\Gamma_{p,q}(z^{\pm2})}\frac{dz}{2\pi iz}=$$

$$\Delta_l^0(t_0v/u_0|vt_0t_1,vt_0t_2,vt_0t_3,t_0u_1;q)\Delta_k^0(t_0/u_1|t_0t_1,t_0t_2,t_0t_3,t_0u_0;q)R_l(t_0'q^k;t_0'v:t_1',t_2',t_3';u_0',u_1'/v)$$

where $Z=\frac{2\prod_{0\leq i<j\leq 7}\Gamma_{p,q}(t_it_j)}{(p;p)(q;q)}$, $t_4=u_0, t_5=u_1$, *v is a parameter and the primed parameters are defined by:*

$$t_0't_i'=t_0t_i,\ 1\leq i\leq 3,\ t_0'u_0'=t_0u_0,\ t_0'u_1'=1/t_0u_1,\ t_0'^2=t_0/u_1.$$

Observe the prime transformation on the parameters is involutive much like the hat transformation (6.2.2).

6.3 Biorthogonal correlation kernel

In this section we will show the processes $t\mapsto t\pm1$ of Section 4.1 are determinantal point processes. We do the calculation for the $t\mapsto t-1$ Markov process as it leads to less complicated formulas, but analogous results hold for $t\mapsto t+1$.

For the remainder, it is now convenient to relabel and rescale the parameter set $\{A,B,C,D,E,F\}$ from equation (3.3.4) as $\{t_0,t_1,t_2,t_3,u_0,u_1\}$ in order for certain symmetries to become more prominent (and in doing so, we will use the notation set forth in the previous section). To wit:

$$A=t_2,\ q^{N-1}B=u_1,\ C=t_3,\ D=t_1,\ q^{N-1}E=u_0,\ F=t_0. \tag{6.3.1}$$

Note these parameters depend on t (the time parameter), and such dependence will be made more explicit when it becomes important. Notation is as in the previous section. Note $u_0u_1t_0t_1t_2t_3=q$. Since the balancing condition for the biorthogonal functions requires a pq on the right hand side, we will again multiply u_1 by p. These are the parameters of the univariate biorthogonal functions discussed in the previous section. Parameters u_0 and u_1 control the poles of the pair of biorthogonal functions.

At the core of the computations is the Eynard-Mehta 6.1.1, which we now state in a “decreasing-time” form convenient for our purposes.

Theorem 6.3.1. *Assume we are given the following:*

- *a discrete biorthonormal system $(f_l^t,g_l^t)_{l\geq0}$ on $l_2(\{0,1,\dots,L\})$ for each time $t=0,\dots,T$,*

- *a matrix*

$$v_{t\to t-1}(x,y)=\sum_{l\geq 0} f_l^{t-1}(t_0^{t-1}q^x)g_l^t(t_0^t q^y),$$

for $n\geq 0$, $t=1,\dots,T$ *and a parameter* t_0 *changing with time,*

- *a discrete time Markov chain* $X(t)$ *(with time decreasing from* T *to 0) taking values in state spaces* $\mathcal{X}^t$ *(set of possible particle positions at time* t*) with one-dimensional distributions proportional to*

$$\det_{1\leq k,l\leq N}(f_{k-1}^t(t_0^t q^{x_l}))\det_{1\leq k,l\leq N}(g_{k-1}^t(t_0^t q^{x_l})),$$

and transition probabilities proportional to

$$\frac{\det_{1\leq k,l\leq N}(v_{t\to t-1}(x_k,y_l))\det_{1\leq k,l\leq N}(f_{k-1}^{t-1}(t_0^{t-1}q^{y_l}))}{\det_{1\leq k,l\leq N}(f_{k-1}^t(t_0^t q^{x_l}))}.$$

Then

$$Prob(x_1\in X(\tau_1),\dots,x_s\in X(\tau_s))=\det_{1\leq k,l\leq s}(K(\tau_k,x_k;\tau_l,x_l)),$$

where

$$K(\tau_1,x_1;\tau_2,x_2)=\begin{cases}\sum_{s\geq 0} f_s^{\tau_1}(t_0^{\tau_1}q^{x_1})g_s^{\tau_2}(t_0^{\tau_2}q^{x_2}), \text{ if } \tau_1\geq\tau_2,\\ \\ -\sum_{s\geq N} f_s^{\tau_1}(t_0^{\tau_1}q^{x_1})g_s^{\tau_2}(t_0^{\tau_2}q^{x_2}), \text{ if } \tau_1<\tau_2.\end{cases}$$

The first step in showing the required determinantal formulas needed to apply the Eynard-Mehta theorem is the following determinantal formula, a version of which was already stated and proved in Section 2.4.

Lemma 6.3.2.

$$\det_{1\leq k,l\leq n} R_{l-1}(z_k;t_0:t_1,t_2,t_3;u_0,pu_1)=const\cdot\prod_{k<l}\varphi(z_k,z_l)\prod_k\frac{1}{\theta_p(q^{1-n}u_0z_k^{\pm 1};q)_{n-1}},$$

where $z_k=t_0q^{x_k}$*, the constant is independent of the* z_k*'s and nonzero.*

Proof. This proof is essentially the same as that of Lemma 5.3 in [War02] (also proved as Theorem 2.4.1), so we only make a few observations and refer the reader to the proof of 2.4.1. If we denote the LHS by L and the right hand side by R, we notice both L and R are elliptic in the z_k's. Fixing a variable z_k, we see poles for L/R come from the zeros of R or the poles of L. For the latter, the poles are controlled by u_0 but are exactly canceled by the zeros of $1/R$ appearing in the univariate product (one can see this from the definition of biorthogonal functions in terms of abelian interpolation functions). For the former the zeros of R are $z_k = z_l, z_k = 1/z_l$ for $l \neq k$ (and p shifts thereof). Plugging in $z_k = z_l$ into L makes two columns the same, so L vanishes. Since univariate biorthogonal functions are BC_1-symmetric in the variable (a fact made explicit in the previous section), L also vanishes if $z_k z_l = 1$ for some $l\ nek$. Hence all the poles of L/R are removable, and so L/R is constant. We are not interested in the constant explicitly (which makes this lemma easier to prove) but we show it is nonzero by noticing that the functions inside the determinant are linearly independent, so the columns of the determinant are linearly independent. This concludes the proof. □

Remark 6.3.3. A more convoluted way to arrive at such determinantal representations (but the way that nevertheless suggested the formula above) would be to take the RHS of the above formula and observe that it appears in Waarnar's determinant (Theorem 2.4.1). What appear in the determinant on the left are the abelian interpolation functions R_l^* discussed in the previous section. The above formula in fact allows us to compute the constant explicitly by expanding the biorthogonal functions in terms of abelian interpolation functions (only leading coefficient is of interest for the determinant).

To simplify notation hereinafter we let

$$\Phi_l^t(t_0 q^s) := R_l(t_0 q^s; t_0 : t_1, t_2, t_3; u_0, pu_1),$$
$$\Psi_l^t(t_0 q^s) := R_l(t_0 q^s; t_0 : t_1, t_2, t_3; pu_1, u_0).$$

The t superscript for these functions stands for the fact their arguments, as it will become apparent in the next proposition, are essentially locations of the particles at time t. Likewise the parameters depend on t (t_i and u_j are implicit for t_i^t, u_j^t respectively; see (6.3.1) and (3.3.4)). We will also denote

$$\tilde{\Psi}_l(t_0 q^s) = \Psi_l(t_0 q^s)\Delta_s(t_0^2|q, t_0t_1, t_0t_2, t_0t_3, t_0u_0, pt_0u_1)/c_l, \text{ so that}$$
$$\sum_{s\geq 0} \Phi_k(t_0 q^s)\tilde{\Psi}_l(t_0 q^s) = \delta_{k,l}. \tag{6.3.2}$$

Thus Lemma 6.3.2 along with (4.1.6) and (6.3.1) yields the following proposition.

Proposition 6.3.4.

$$\begin{aligned} Prob(X(t) = (x_1, \ldots, x_N)) &= const \cdot \det_{1\leq k,l\leq n} \Phi_{l-1}^t(t_0 q^{x_k}) \cdot \det_{1\leq k,l\leq n} \Psi_{l-1}^t(t_0 q^{x_k}) \cdot \prod_k \Delta_{x_k} \\ &= const \cdot \det_{1\leq k,l\leq n} \Phi_{l-1}^t(t_0 q^{x_k}) \cdot \det_{1\leq k,l\leq n} \tilde{\Psi}_{l-1}^t(t_0 q^{x_k}), \end{aligned}$$

where $\Delta_{x_k} := \Delta_{x_k}(t_0^2|q, t_0t_1, t_0t_2t_0t_3, t_0u_0, pt_0u_1)$.

Proposition 6.3.5. *We have*

$$v_{t\to t-1}(k,l) := \sum_{s\geq 0} \Phi_s^{t-1}(t_0^{t-1}q^k)\tilde{\Psi}_s^t(t_0^tq^l) = \frac{1}{Z}(w_0'\delta_{k,l} + w_1'\delta_{k+1,l}), \tag{6.3.3}$$

with w_0' *and* w_1' *as in Theorem 4.1.10 and* $Z = \frac{1}{\theta_p(u_0^{t-1}t_0^{t-1}, u_0^{t-1}t_1^{t-1}, t_0^{t-1}t_1^{t-1})}$.

Proof. We observe that

$$\sum_{s\geq 0} \Phi_s^t(t_0^tq^k)\tilde{\Psi}_s^t(t_0^tq^l) = \delta_{k,l},$$

which expresses the relation $BA = 1$ where $A(k,l) = \Phi_k^t(t_0q^l), B(k,l) = \tilde{\Psi}_l^t(t_0q^k)$ and we know $AB = 1$ from biorthogonality (see (6.3.2)). We now apply the difference operator $\mathcal{D}(u_0^{t-1}, t_0^{t-1}, t_1^{t-1})$ (corresponding to the Markov transition $t \mapsto t-1$) to both sides and observe the parameters at time t are the required q shifts of the parameters at time $t-1$ (see (6.2.4)). Finally on the RHS we have a delta function which is acted upon by the difference operator to produce the desired result (see Proposition 4.2.5). $\square$

Remark 6.3.6. In [BGR10] and [BG09] formulas like the one in the above proposition are written in terms of discrete orthogonal polynomials (q-Racah and Hahn respectively) and are proven via the three-term recurrence relation satisfied by these polynomials (which is an identity between hypergeometric or q-hypergeometric series). Such a relation exists for biorthogonal functions as well (we refer the reader to [SZ00] for an explicit form, though with different notation) and can be used to prove the above proposition, but the computations are more involved.

Remark 6.3.7. A similar result holds if we apply the transition $t \mapsto t+1$ which corresponds to the operator $\mathcal{D}(u_1, t_2, t_3)$. For that though, we have to renormalize the biorthogonal functions at either t_2 or t_3 (see (6.2.4) and (6.2.5)), so the bidiagonal matrix that appears on the RHS will be of the above form conjugated by two diagonal matrices (coming from the renormalization coefficients). This is an artifact of our choice of coordinates (we are counting particles going up from the bottom left edge of the hexagon).

Finally, in applying Theorem 6.3.1 to the $t \to t-1$ Markov chain $X(t)$ we need to check that the transition probabilities have the required determinantal form. This is a consequence of Theorem 4.1.10, Lemma 6.3.2 and the following computation (the proof of which is immediate from Theorem 4.1.10 and Proposition 6.3.5; we use the notation from 4.1.10 for w_0', w_1', X, Y):

$$\det_{1\leq k,l\leq N}(v_{t\to t-1}(x_k, y_l)) = const \cdot \prod_{k:y_k=x_k-1} w_1'(x_k) \prod_{k:y_k=x_k} w_0'(x_k). \tag{6.3.4}$$

We thus obtain the following.

Proposition 6.3.8.

$$Prob(X(t-1)=Y|X(t)=X) = const \cdot \frac{\det_{1\le k,l\le N}(v_{t\to t-1}(x_k,y_l))\det_{1\le k,l\le N}(\Phi^{t-1}_{k-1}(t_0^{t-1}q^{y_l}))}{\det_{1\le k,l\le N}(\Phi^{t}_{k-1}(t_0^{t}q^{x_l}))}. \tag{6.3.5}$$

Theorem 6.3.9. *The Markov processes $t\mapsto t\pm 1$ discussed in Section 4.1 meet the assumptions of Theorem 6.3.1 and are therefore determinantal with correlation kernels given in terms of elliptic biorthogonal functions.*

Proof. This follows from all the results gathered in this Section for the $t-$ Markov chain with $f=\Phi$ and $g=\tilde{\Psi}$ in the notation of Theorem 6.3.1. For $t+$ see Remark 6.3.7. □

Remark 6.3.10. For obtaining quantitative results about the artic boundary, one can try to look at the asymptotics of the diagonal of the correlation kernel of the process (which is of course the probability that a particle is present at that site):

$$K(x,x) = \sum_{i=0}^{S+N-1} R_i^t(t_0q^x|t_0:t_1,t_2,t_3;u_0,pu_1)R_i^t(t_0q^x|t_0:t_1,t_2,t_3;pu_1,u_0)\times$$
$$\Delta_x(t_0^2|q,t_0t_1,t_0t_2,t_0t_3,t_0u_0,pt_0u_1)\Delta_i(1/(pu_0u_1)|q,t_0t_1,t_0t_2,t_0t_3,1/(t_0u_0),1/(pt_0u_1)).$$

Obtaining a good integral representation of this term can allow for complex analytic asymptotic methods to be used (such as steepest descent). We have the following partial result.

Proposition 6.3.11.

$$c_1R_l(t_0q^x;t_0:t_1,t_2,t_3;u_0,pu_1) = c_1R_l(p^{1/2}t_0q^x;p^{1/2}t_0:p^{1/2}t_1,p^{1/2}t_2,p^{1/2}t_3;p^{-1/2}u_0,p^{-1/2}u_1) =$$
$$c_2I(p\sqrt{q}q^{l/2}q^xt_0,\frac{1}{t_0}\sqrt{q}q^{l/2}q^{-x},\frac{1}{t_0}\sqrt{q}q^{-l/2},\frac{1}{t_1}\sqrt{q}q^{-l/2},\frac{1}{t_2}\sqrt{q}q^{-l/2},\frac{1}{t_3}\sqrt{q}q^{-l/2},\frac{1}{u_0}p\sqrt{q}q^{l/2},\frac{1}{u_1\sqrt{q}}q^{l/2})$$

where c_1,c_2 are constants such that both sides are holomorphic and I is the order 1 elliptic beta integral of Section 2.1.

Proof. Follows from 6.2.2, 6.2.4, and the $W(E_7)$ symmetry of the order 1 elliptic beta integral 2.1.3. The constants can be computed by setting the argument of the biorthogonal function to t_0 so that it evaluates to 1 ($x=0$) and clearing the poles on the LHS. □

Remark 6.3.12. The important feature of such an integral representation is that the eight parameters of the above order 1 elliptic beta integral have the property that no two distinct ones multiply in $p^{-\mathbb{N}}q^{-\mathbb{N}}$, and hence the integral (the contour for it) exists and is finite. Thus the integral does not degenerate to a sum of residues. The disadvantage is that the kernel itself is a sum of many such double integrals (arising from each biorthogonal function in the kernel), so one would need to control all of them at once.

Chapter 7

Elliptic processes

In this chapter we present the elliptic tilings of the hexagon discussed so far in a new light. The motivation comes from the Schur process introduced by Okounkov and Reshetikhin in [OR03]. This is a process on partitions where transition probabilities are given by skew Schur functions (or polynomials). The authors of that paper realized that using this formalism (along with some representation theory) they can explicitly compute correlation kernels and asymptotics. A different approach to the same calculation, sidestepping the representation theory, was given in [BR05]. The Schur process was also studied in [Bor10] where sampling algorithms were discussed at length in connection to Schur functions and intertwined Markov chains. A generalization to Macdonald polynomials (replacing the Schur functions) was introduced by Vuletic in [Vul09] and discussed at length with applications in [BC11]. Here we give a generalization of Macdonald processes where elliptic skew interpolation functions replace skew Macdonald polynomials (to which they degenerate in the right limit—see [Rai06], [Rai11]). We call them *elliptic processes*. With the appropriate specialization of parameters, such processes encompass the lozenge tilings of a hexagon discussed so far. The first three sections of this chapter present some results obtained by Rains in [Rai06] and [Rai11] that we will use in later sections. We follow the notation of the cited references for the most part. We mostly present the necessary definitions and collect the appropriate results without proofs, and without striving for the greatest generality. The list of results is far from comprehensive. The main identities will be a transfer and two Cauchy-type identities for skew interpolation functions generalizing identities well-known for Schur functions. We refer the reader to [Rai06], [Rai11], and [Rai10] for more details. In the fourth section we define elliptic processes. Lastly, we specialize the elliptic processes to the case of elliptic tilings of a hexagon. We also present a different (yet somewhat similar) sampling algorithm for such tilings based on the quasi-commutation relations satisfied by two “orthogonal” (quasi-commuting) Markov processes.

7.1 Abelian interpolation functions

In this section we introduce a class of functions which generalize Okounkov's interpolation polynomials [Oko98] as well as Macdonald polynomials (see. e.g., [Mac95]) to the elliptic level. They were introduced by Rains in two different papers from two different perspectives. In [Rai10] they were introduced in a continuous (integral) setting as special cases of the multivariate elliptic biorthogonal functions. In [Rai06] they were introduced via the notion of interpolation theta functions (which are certain functions associated to elliptic bigrids in the plane) as ratios of two such interpolation theta functions. We will pursue here the exposition in [Rai06] without going into the details of elliptic bigrids and refer the reader to op. cit. for proofs and more details.

We begin with a theorem about the existence and uniqueness of a certain class of functions which we call *interpolation theta functions*. We fix positive integers m, n and a partition $\lambda \subset m^n$. We also fix a, b, q, t, v nonzero complex.

Theorem 7.1.1. *There exists a unique n-variable (defined on $(\mathbb{C}^*)^n$) function*

$$P^{*(m,n)}_{\lambda}(x_1, \dots, x_n; a, b; q; t; p)$$

(also depending on a, b, q, t) with the following properties:

- *(symmetry) $P^{*(m,n)}_{\lambda}$ is a holomorphic BC_n-symmetric theta function of degree m in each variable x_i.*
- *(vanishing) For any partition $\mu \subset m^n$, $\mu \neq \lambda$, let l_0 be the largest index such that $\mu_l \neq \lambda_l$, let l_1 be the largest index such that $\mu_l = m$ (0 if none exists). If $\mu_{l_0} < \lambda_{l_0}$, then set $l = l_1$, otherwise set $l = l_0$. Then*

$$P^{*(m,n)}_{\lambda}(bq^{m-\mu_1}, \dots, bq^{m-\mu_l}t^{l-1}, aq^{\mu_{l+1}}t^{n-l-1}, \dots, aq^{\mu_n}; a, b; q; t; p) = 0.$$

- *(normalization)*

$$P^{*(m,n)}_{\lambda}(\dots vt^{n-i} \dots; a, b; q; t; p) = C^0_{\lambda}(t^{n-1}av, a/v; q; t; p)C^0_{m^n-\lambda}(t^{n-1}bv, b/v; q; t; p).$$

The interpolation theta functions are holomorphic and independent of v. A version of these functions was already introduced in Section 4.2. Among the key properties these functions exhibit, we mention the following extra vanishing result.

Proposition 7.1.2. *For parameters satisfying the balancing $q^m t^{n-1} ab = 1$ we have*

$$P_\lambda^{*(m,n)}(\dots aq^{\mu_i}t^{n-i}\dots;a,b;q;t;p) \\ = \delta_{\lambda\mu}\mathcal{C}^0_{m^n-\lambda}(pqt^{n-1};q;t;p)\mathcal{C}^+_{m^n-\lambda}(\frac{1}{q^{2m}a^2};q;t;p)\frac{\mathcal{C}^-_\lambda(pq,t;q;t;p)\mathcal{C}^+_\lambda(t^{2n-2}a^2;q;t;p)}{\mathcal{C}^0_\lambda(t^n;q;t;p)}.$$

When $\lambda = 0$, the value of the theta interpolation function is very nice:

$$P_0^{*(m,n)}(\dots x_i\dots;a,b;q;t;p) = \prod_{1\le i\le n}\theta_p(bx_i^{\pm 1};q)_m.$$

We now define the *abelian interpolation functions* (abelian here will stand for *elliptic* in the variables cf. [Rai06]) as renormalized theta interpolation functions:

$$R_\lambda^{*(n)}(;a,b;q;t;p) = \frac{P_\lambda^{*(m,n)}(;a,q^{-m}b;q;t;p)}{P_0^{*(m,n)}(;a,q^{-m}b;q;t;p)},$$

for $m \ge \lambda_1$.

Remark 7.1.3. These functions are elliptic in the variables. It can be shown the formula given above is independent of m (hence the notation; see [Rai06]). Moreover, they functions have poles whenever the denominator has zeros: at points of the form $x_i = b^{\pm 1}$. ever the denominator has zeros: at points of the form $x_i = b^{\pm 1}$.

Remark 7.1.4. For $n = 1$, the abelian interpolation functions are just explicit ratios of theta Pochhammer symbols, and were introduced in Section 6.2. To wit:

$$R_l^{*(1)}(x;a,b;q;p) = \frac{\theta_p(ax^{\pm 1};q)_l}{\theta_p(bq^{-l}x^{\pm 1};q)_l}.$$

7.2 Binomial coefficients and skew interpolation functions

In this section we introduce elliptic binomial coefficients (multivariate partition-based elliptic generalizations of regular binomial coefficients, and also elliptic analogues of q-binomial coefficients appearing in the theory of Macdonald and Koornwinder polynomials). Based on them we define elliptic skew interpolation functions necessary for transition probabilities in what will become the elliptic processes. These are elliptic generalizations of skew Schur functions. Finally we discuss vanishing conditions for such functions, which are important for the termination of multivariate elliptic hypergeometric sums.

We begin with the *elliptic binomial coefficients.* They are defined as follows:

$$\binom{\lambda}{\mu}_{[a,b];q;t;p} := \Delta_\mu(\frac{a}{b}|t^n, 1/b; q; t; p) R^{*(n)}_\mu(\dots \sqrt{a} q^{\lambda_i} t^{1-i} \dots; t^{1-n}\sqrt{a}, b/\sqrt{a}; q; t; p), \tag{7.2.1}$$

for any integer $n \geq \ell(\lambda), \ell(\mu)$.

Remark 7.2.1. The binomial coefficient thus described is elliptic in a and b, as well as symmetric under $(a, b, t, q) \mapsto (1/a, 1/b, 1/t, 1/q)$.

We list the following evaluations (and transformation) we will find useful:

$$\begin{aligned}
\binom{\lambda}{0}_{[a,b];q;t;p} &= 1, \\
\binom{\lambda}{\lambda}_{[a,b];q;t;p} &= \frac{\mathcal{C}^+_\lambda(a; q; t; p)\Delta^0_\lambda(a/b|1/b; q; t; p)}{\mathcal{C}^+_\lambda(\frac{a}{b}; q; t; p)\Delta^0_\lambda(a|b; q; t; p)}, \\
\binom{m^n}{\lambda}_{[a,b];q;t;p} &= \Delta_\lambda(\frac{a}{b}|t^n, q^{-m}, t^{1-n}q^m a, 1/b; q; t; p), \\
\frac{\binom{m^n+\lambda}{m^n+\mu}_{[a,b];q;t;p}}{\binom{m^n}{m^n}_{[a,b];q;t;p}} &= \frac{\Delta^0_\lambda(q^{2m}a|b, pqaq^m/b, pqt^{n-1}q^m, t^{1-n}q^{2m}a; q; t; p)}{\Delta^0_\mu(q^{2m}a/b|1/b, pqaq^m, pqt^{n-1}q^m, t^{1-n}q^{2m}a/b; q; t; p)}\binom{\lambda}{\mu}_{[q^{2m}a,b];q;t;p}.
\end{aligned} \tag{7.2.2}$$

Because we want to specialize b at negative powers of q ($b = 1/q$) and the round parentheses binomial coefficients just described have poles there, we will need the following angle bracket renormalization of the binomial coefficients:

$$\left\langle \begin{matrix}\lambda \\ \mu\end{matrix} \right\rangle_{[a,b](v_1,\dots,v_k);q;t;p} := \frac{\Delta^0_\lambda(a|b, v_1, \dots, v_k; q; t; p)}{\Delta^0_\mu(a/b|1/b, v_1, \dots, v_k; q; t; p)}\binom{\lambda}{\mu}_{[a,b];q;t;p}.$$

If both $\lambda = l$ and $\mu = m$ consist of only one part, we have the following expression for the generalized angle-bracket binomial coefficient:

$$\left\langle \begin{matrix} l \\ m\end{matrix} \right\rangle_{[a,b];q;p} := \frac{\Gamma_{p,q}(\frac{b}{a}q^{-l-m}, q^m\frac{a}{b}, aq^{l+m}, bq^{l-m})}{\Gamma_{p,q}(\frac{b}{a}q^{-2m}, q^{2m}\frac{a}{b}, aq^l, b)}\prod_{j=1}^m \frac{\theta_p(q^{j-l-1})}{\theta_p(q^{-j})}.$$

The last product of theta functions can be written as

$$\prod_{j=1}^m \frac{\theta_p(q^{j-l-1})}{\theta_p(q^{-j})} = q^{m(m-l)}\frac{\theta_p(q;q)_l}{\theta_p(q;q)_{l-m}\theta_p(q;q)_m}.$$

Taking the limit $p \to 0$, this degenerates to a version of the q-binomial coefficient $q^{m(m-l)}\binom{l}{m}_q$. Taking $q \to 1$ yields the usual binomial coefficient.

When $b = 1$, the binomial angle-bracket coefficients reduce to Kronecker delta symbols. We also have the following important proposition (see [Rai06]).

Proposition 7.2.2. *If $b = 1/q$, the binomial coefficient $\left\langle {\lambda \atop \mu} \right\rangle_{[a,1/q];q;t;p}$ vanishes unless $\lambda_i - 1 \leq \mu_i \leq \lambda_i$ (μ is obtained from λ by removing a vertical strip).*

When $t = q$ the formula for the renormalized (angle bracket) binomial coefficient becomes determinantal by an application of Warnaar's formula from Section 2.4 and using the fact that interpolation functions can themselves be written as determinants.

Proposition 7.2.3.

$$\left\langle {\lambda \atop \mu} \right\rangle_{[a,b];q;p} = \prod_{1 \leq i < j \leq n} \frac{q^{-\mu_i} \theta(q^{\mu_i - i - \mu_j + j}, q^{\mu_i + \mu_j + 2 - i - j} a/b; p)}{q^{-\lambda_i} \theta(q^{\lambda_i - i - \lambda_j + j}, q^{\lambda_i + \lambda_j + 2 - i - j} a; p)} \det_{1 \leq i,j \leq n} \left(\left\langle {\lambda_i + n - i \atop \mu_j + n - j} \right\rangle_{[\frac{a}{q^{2n-2}}, b];q;p} \right).$$

Remark 7.2.4. The reader should compare the above formula with the formula for transition probabilities for the Markov chains introduced in Chapter 4. To be more precise, the binomial coefficient with $t = q$ and b-parameter $1/q$ is indeed a transition probability like the ones defined in Section 4.1. That is, such a binomial coefficient is a coefficient of an appropriate difference operator from Section 4.2 where $z_i \propto q^{\mu_i + n - i}$. We discuss the appropriate specializations of parameters corresponding to the transition probabilities of Section 4.1 in Theorems 7.5.1 and 7.5.5. This observation allows us to prove the quasi-commutation for difference operators (Lemma 4.2.3) in the following way. In 4.2.3 we first specialize the z_i's to be proportional to $q^{\lambda_i + n - i}$. We express the coefficients of difference operators as determinants and sum everything using the Cauchy-Binet determinantal identity (written for M, N arbitrary square matrices we can compose):

$$\sum_{Z = (\dots, z_i, \dots)} \det_{1 \leq i,j \leq n} M(x_i, z_j) \det_{1 \leq i,j \leq n} N(z_i, y_j) = \det_{1 \leq i,j \leq n} (MN)(x_i, y_j).$$

The multivariate quasi-commutation relation reduces then to the univariate commutation which is proven directly. This proves it for $z_i \propto q^{\lambda_i + n - i}$ but for generic q this is sufficient as all such powers are dense (in other words, we analytically continue in $q^{\lambda_i + n - i}$).

Remark 7.2.5. If $b = 1/q$ then $\lambda_i - \nu_i \in \{0, 1\}$ and the determinant is of a block diagonal matrix with each block either upper or lower triangular (2-diagonal even). Hence in this case the determinant evaluates to the product of the diagonal elements.

The following identity allows us to flip the top and bottom of binomial coefficients when λ is of length at most n. It is also useful in computing binomial coefficients explicitly.

$$\left\langle {1^n + \lambda \atop \mu} \right\rangle_{[a,q^{-1}];q;t;p} = \frac{C^0_{1^n}(q^{-1}; q; t; p)}{C^0_{1^n}(pqaq; q; t; p)} \frac{\Delta_\mu(qa|t^n, t^{1-n}qa; q; t; p)}{\Delta_\lambda(q^2a|t^n, t^{1-n}q^a; q; t; p)} \left\langle {\mu \atop \lambda} \right\rangle_{[qa,q^{-1}];q;t;p}. \tag{7.2.3}$$

We are now in a position to introduce the *elliptic skew interpolation functions*. They are defined by the following hypergeometric formula:

$$R^*_{\lambda/\kappa}([v_0,\dots,v_{2n-1}];a,b;q;t;p) := \sum_{\kappa\subset\mu\subset\lambda} \left\langle {\lambda \atop \mu} \right\rangle_{[a/b,ab/pq];q;t;p} \left\langle {\mu \atop \kappa} \right\rangle_{[pq/b^2,pq\prod_{0\le r<2n} v_r/ab];q;t;p}$$
$$\Delta^0_\mu(pq/b^2|pq/bv_0,pq/bv_1,\dots,pq/bv_{2n-1};q;t;p). \quad (7.2.4)$$

Remark 7.2.6. The skew interpolation functions are invariant under permutations of the v_i's as well as under insertion/deletion of $(v',1/v')$ pairs among the v parameters.

With 0, 2 and 4 v arguments, the interpolation functions have the following simple expressions:

$$R^*_{\lambda/\kappa}([];a,b;q;t;p) = \delta_{\lambda\kappa},$$
$$R^*_{\lambda/\kappa}([v_0,v_1];a,b;q;t;p) = \left\langle {\lambda \atop \kappa} \right\rangle_{[a/b,v_0v_1](a/v_0,a/v_1);q;t;p},$$
$$R^*_{\lambda/\kappa}([v_0,v_1,v_2,v_3];a,b;q;t;p) =$$
$$\sum_\mu \left\langle {\lambda \atop \mu} \right\rangle_{[a/b,v_0v_1](a/v_0,a/v_1);q;t;p} \left\langle {\mu \atop \kappa} \right\rangle_{[a/v_0v_1b,v_2v_3](a/v_0v_1v_2,a/v_0v_1v_3);q;t;p}.$$

We will, in some sense, only make use of such interpolation functions, though at first they will appear with an arbitrary number of v parameters. When the bottom partition is equal to 0, the skew interpolation functions are indeed a version of the usual interpolation functions defined in Section 7.1:

$$R^{*(n)}_\lambda(z_1,\dots,z_n;a,b;q;t;p) =$$
$$\Delta^0_\lambda(t^{n-1}a/b|pqa/tb;q;t;p)R^*_{\lambda/0}([t^{1/2}z_1^{\pm1},\dots,t^{1/2}z_n^{\pm1}];t^{n-1/2}a,t^{1/2}b;q;t;p).$$

We can combine the last equality of (7.2.2) with (7.2.3) and rewrite in terms of skew interpolation functions to obtain (following [Rai11]):

Lemma 7.2.7. *Let $v_0,\dots,v_{2k-1}$ be such that $v_{2r}v_{2r+1} = q^{-m_r}$ with m_r nonnegative integers, for $0 \le r < k$. Assume m, m', n are also nonnegative integers satisfying $m' = m + \sum_r m_r$ and let λ,κ be partitions of length at most n. Then*

$$\frac{\Delta^0_{m^n+\kappa}(a/b\prod_{0\le r<2k} v_r|ab/pq\prod_{0\le r<2k} v_r;q;t;p)}{\Delta^0_{m'^n+\lambda}(a/b|ab/pq;q;t;p)} R^*_{m'^n+\lambda/m^n+\kappa}([v_0,\dots,v_{2k-1}];a,b;q;t;p)$$
$$\frac{\prod_{0\le r<k}\Delta^0_{m_r^n}(Q'a/Q_rb|Q'a/Q_rv_{2r},Q'a/Q_rv_{2r+1};q;t;p)}{\Delta^0_{m'-m^n}(Qa/b|ab/pq,pqQQ'a/b;q;t;p)}$$
$$\times \frac{\Delta_\kappa(aQQ'/b|t^n,atQ'/t^nb,abQ'/pq,pqQ/Q'ab;q;t;p)}{\Delta_\lambda(aQ'^2/b|t^n,atQ'/t^nb,abQ'/pq,pq/ab;q;t;p)}$$
$$\times \frac{\Delta^0_\kappa(a'/b'\prod_{0\le r<2k} v_r|a'b'/pq\prod_{0\le r<2k} v_r;q;t;p)}{\Delta^0_\lambda(a'/b'|a'b'/pq;q;t;p)} R^*_{\kappa/\lambda}([v_0,\dots,v_{2k-1}];a',b';q;t;p),$$

where $a' = pqQ/b$, $b' = pq/Q'b$ *and* $Q = q^m$, $Q' = q^{m'}$.

We end this section by listing a couple of vanishing properties of skew interpolation functions following [Rai11]. They will be used in the next section to ensure termination of certain hypergeometric sums. We first list the general results and then specialize in the case of interest to us.

Theorem 7.2.8. *If the* $2k$ v *parameters of the skew interpolation functions can be arranged in such a way that* $v_{2i}v_{2i+1} = t^{n_i}q^{-m_i}$ *for positive integers* $m_i, n_i 0$ *and* $0 \leq i < k$ *then for any partition* κ,

$$R^*_{\lambda/\kappa}([v_0, v_1, \ldots, v_{2k-1}]; a, b; q; t; p) = 0,$$

unless

$$\kappa_i \leq \lambda_i \leq \kappa_{i-N} + M,$$

where $M = \sum_i m_i$, $N = \sum_i n_i$ *and by definition* $\kappa_i = 0$ *for* $i \leq 0$.

Remark 7.2.9. This result is of interest to us for $\kappa = 0$. Note for $N = 0, \kappa = 0$ we have that all parts of λ for which the function is nonzero are bounded by M. Alternatively, for $M = 0$ we have that partitions λ for which the function is nonzero have at most N parts.

Theorem 7.2.10. *Fix notation as in Theorem 7.2.8 and assume that the* v *parameters have the property that* $v_{2i}v_{2i+1} = t^{n_i}q^{-m_i}$ *for* $1 \leq i < k$ *and that*

$$a/\prod_{1 \leq i < 2k} v_i = t^{n_0}q^{-m_0}.$$

If $\kappa_{n_0+1} \leq m_0$, *then*

$$R^*_{\lambda/\kappa}([v_0, v_1, \ldots, v_{2k-1}]; a, b; q; t; p) = 0,$$

unless $\lambda_{N+1} \leq M$.

Remark 7.2.11. For $\kappa = 0$ (our interest), this coincides with Theorem 7.2.8. The reason for stating both is that we will use both when $\kappa = 0$ (or rather use the common version twice). If in one we take $M = 0$, in the other $N = 0$ (and in both $\kappa = 0$), then skew interpolation functions will vanish unless $\lambda \in M^N$.

7.3 Transfer and Cauchy identities

In this section we state three important identities involving skew interpolation functions and generalizing known identities for Schur and skew Schur functions. We call these elliptic *transfer* and *Cauchy* identities. They were proven in [Rai11] which we follow for the rest of the section and to which we refer the reader for the proofs (though they appear in some detail also in [Rai06]).

We start with the transfer identity (the name *transfer* is not standard in the literature even at the level of Schur functions).

Theorem 7.3.1.

$$R^*_{\lambda/\kappa}([v_0,\dots,v_{2k-1},w_0,\dots,w_{2l-1}];a,b;q;t;p)=\sum_{\mu}R^*_{\lambda/\mu}([v_0,\dots,v_{2k-1}];a,b;q;t;p)$$
$$R^*_{\mu/\kappa}([w_0,\dots,w_{2l-1}];a/v_0\cdots v_{2k-1},b;q;t;p).$$

Remark 7.3.2. This identity will mostly be used when both interpolation functions in the sum have only two parameters (hence each is a binomial coefficient).

Remark 7.3.3. The above transfer identity is the skew interpolation analogue of a well-known transfer identity involving skew Schur functions (see, e.g., [Mac95])):

$$s_{\lambda/\kappa}(x,y)=\sum_{\mu}s_{\lambda/\mu}(x)s_{\mu/\kappa}(y),$$

where $x=(x_1,\dots),y=(y_1,\dots)$ are specializations of the algebra of symmetric functions.

We now state the Cauchy and skew Cauchy identities for interpolation functions. The nonskew version allow us to compute certain partition functions explicitly. To state the results, we need to make sure certain sums only contain a finite number of nonzero terms (due to convergence problems for infinite elliptic hypergeometric series—both univariate and multivariate).

Theorem 7.3.4.

$$\sum_{\mu}\Delta_{\mu}(a/b|;q;t;p)R^*_{\mu/0}([v_0,\dots,v_{2n-1}];a,b;q;t;p)$$
$$R^*_{\mu/0}([w_0,\dots,w_{2m-1}];\sqrt{pqt}/b,\sqrt{pqt}/a;q;t;p)=\prod_{\substack{0\le i<2n+2\\0\le j<2m+2}}\frac{\Gamma_{p,q,t}((pqt)^{1/2}v_i/w_j)}{\Gamma_{p,q,t}((pqt)^{1/2}v_iw_j)},$$

where

$$v_{2n}=a/\prod_{0\le r<2n}v_r,\quad v_{2n+1}=1/a,\quad w_{2m}=(pqt)^{1/2}/b\prod_{0\le r<2m}w_r,\quad w_{2m+1}=b/(pqt)^{1/2},$$

and the parameters are subject to technical termination conditions.

The skew elliptic Cauchy identity is the following result.

Theorem 7.3.5.

$$\sum_\mu \frac{\Delta_\mu(\frac{a}{b}|;q;t;p)}{\Delta_\lambda(\frac{a}{bV}|;q;t;p)} R^*_{\mu/\lambda}([v_0,\dots,v_{2n-1}];a,b;q;t;p) R^*_{\mu/\kappa}([w_0,\dots,w_{2m-1}];\frac{\sqrt{pqt}}{b},\frac{\sqrt{pqt}}{a};q;t;p) \propto$$
$$\sum_\mu R^*_{\lambda/\mu}([w_0,\dots,w_{2m-1}];\frac{\sqrt{pqt}}{b},\frac{\sqrt{pqt}V}{a};q;t;p)\frac{\Delta_\kappa(\frac{a}{bW}|;q;t;p)}{\Delta_\mu(\frac{a}{bVW}|;q;t;p)} R^*_{\kappa/\mu}([v_0,\dots,v_{2n-1}];a,bW;q;t;p),$$

where $V=\prod v_r$, $W=\prod w_r$, and the parameters are such that the RHS terminates. The constant of proportionality is independent of λ and κ, and is thus equal to the RHS of Theorem 7.3.4.

Remark 7.3.6. Of the three theorems mentioned in this section, 7.3.4 and 7.3.5 may involve nonterminating hypergeometric sums. One way to ensure termination in Theorem 7.3.4 is to use Theorems 7.2.8 and 7.2.8 to bound the partitions in the summation both *vertically and horizontally* (bound the number of parts and the magnitude of the first part). We argue similarly for the LHS appearing in Theorem 7.3.5 (assuming the constant of proportionality is already terminating). In Section 7.5 we will be more explicit with such conditions.

We again find it convenient to use the version of the above Cauchy identity where all 4 skew interpolation functions are binomial coefficients. We thus have the following corollary:

Corollary 7.3.7. *Assuming the LHS sum terminates and the balancing condition $b_0b_1v_0v_1v_2v_3 = pqta^2$, one has*

$$\sum_\mu \frac{\Delta_\mu(a|v_0,v_1,v_2,v_3;q;t;p)}{\Delta_\lambda(a/b_0|v_0,v_1,v_2,v_3;q;t;p)} \left\langle \begin{matrix}\mu\\ \lambda\end{matrix}\right\rangle_{[a,b_0];q;t;p} \left\langle \begin{matrix}\mu\\ \kappa\end{matrix}\right\rangle_{[a,b_1];q;t;p} \propto$$
$$\sum_\mu \frac{\Delta_\kappa(a/b_1|v_0,v_1,v_2,v_3;q;t;p)}{\Delta_\mu(a/b_0b_1|v_0,v_1,v_2,v_3;q;t;p)} \left\langle \begin{matrix}\lambda\\ \mu\end{matrix}\right\rangle_{[a/b_0,b_1];q;t;p} \left\langle \begin{matrix}\kappa\\ \mu\end{matrix}\right\rangle_{[a/b_1,b_0];q;t;p}.$$

The constant of proportionality is given by

$$\sum_\mu \Delta_\mu(a|b_0,b_1,v_0,v_1,v_2,v_3;q;t;p).$$

Remark 7.3.8. We can use two of the 6 parameters in the Δ-symbol in the sum

$$\sum_\mu \Delta_\mu(a|b_0,b_1,v_0,v_1,v_2,v_3;q;t;p)$$

for termination. If one of the parameters is equal to t^N for $N\in\mathbb{N}$ then $\Delta_\mu(a|t^n,\dots;q;t;p)=0$ unless μ has at most N parts. If another is equal to q^{-M} for $M\in\mathbb{N}$ then $\Delta_\mu(a|q^{-M},\dots;q;t;p)=0$ unless $\mu_1\le M$. Enforcing both conditions makes the sum finite since then it is a sum over partitions in M^N. If we use b_0 and b_1 to enforce termination, it is also the case that the sum on the LHS in the corollary terminates due to the vanishing of binomial coefficients with upper partition μ. With this in mind, Theorem 7.3.5 can be proved by an inductive application of MN instances of the above corollary. The termination conditions stated therein are exactly conditions needed for each successive application of the corollary.

Remark 7.3.9. The constant of proportionality in the above corollary is a multivariate discrete elliptic Selberg sum if parameters are specialized appropriately (see Section 2.3). Hence it has an evaluation.

Remark 7.3.10. These elliptic Cauchy identities are analogues of the well known Cauchy identities for Schur functions (see, e.g., [Mac95]):

$$\sum_{\mu} s_{\mu}(x)s_{\mu}(y) = \prod_{i,j} \frac{1}{1-x_i y_j},$$
$$\sum_{\mu} s_{\mu/\lambda}(x)s_{\mu/\kappa}(y) = \prod_{i,j} \frac{1}{1-x_i y_j} \sum_{\mu} s_{\kappa/\mu}(x)s_{\lambda/\mu}(y),$$

where $x = (x_1, \dots), y = (y_1, \dots)$ are specializations as before.

7.4 Elliptic Schur processes

In this section we define elliptic processes analogous to the Schur processes of Okounkov and Reshetikhin (see [OR03]). We specialize parameters appropriately and recover some of the results of Chapter 4 from a different perspective.

We begin by choosing $a, b, t, q, p, v_0, \dots, w_0, \dots$ complex numbers with $|p| < 1$. Define

$$a_k := a \prod_{0 \le i < 2k} v_i,$$
$$b_k := b / \prod_{0 \le i < 2k} w_i.$$

Let us denote

$$Z_{k,l}(v, w; a, b) :=$$
$$\sum_{\lambda} R^*_{\lambda/0}([v_0, \dots, v_{2k-1}]; a_k, b_l; q; t; p)\Delta_{\lambda}(\frac{a_k}{b_l}|; q; t; p)R^*_{\lambda/0}([w_0, \dots, w_{2l-1}]; \frac{\sqrt{pqt}}{b_l}, \frac{\sqrt{pqt}}{a_k}; q; t; p), \tag{7.4.1}$$

assuming the sum terminates (we will deal with the termination condition when we specialize the parameters).

The *elliptic processes*, much like the Schur processes, are processes on partitions. For now, unless explicitly otherwise specified, the probabilities involved are allowed to be complex. We fix a bounded region R in $\mathbb{N}^2$ such that the closure is a (Russian-style) Young diagram (that is, the region is bounded by the two axis and a path consisting of vertical-up and horizontal-left steps joining from the horizontal axis to the vertical axis)—though in principle a finite initial segment of the horizontal axis union a similar segment of the vertical axis is also allowed. Assuming termination conditions so that necessary sums are finite, there is a process that to every pair of nonnegative integers (k,l) inside R associates a random partition $\lambda^{(k,l)}$ with the following properties:

(i) The marginal distribution of $\lambda^{(k,l)}$ is

$$Prob(\lambda^{(k,l)}) = \frac{1}{Z_{k,l}(v,w;a,b)} R^*_{\lambda^{(k,l)}/0}([v_0,\dots,v_{2k-1}];a_k,b_l;q;t;p)\Delta_{\lambda^{(k,l)}}(a_k/b_l|;q;t;p)\times \\ R^*_{\lambda^{(k,l)}/0}([w_0,\dots,w_{2l-1}];\sqrt{pqt}/b_l,\sqrt{pqt}/a_k;q;t;p). \tag{7.4.2}$$

This is clearly a probability measure on partitions and it generalizes Okounkov's Schur measure [Oko01].

(ii) If $k_1 \leq k_2$, then the conditional probability of $\lambda^{(k_1,l)}$ given $\lambda^{(k_2,l)}$ is

$$P_{k-,k_2\to k_1}(\lambda^{(k_2,l)},\lambda^{(k_1,l)}) = Prob(\lambda^{(k_1,l)}|\lambda^{(k_2,l)}) = \\ \frac{R^*_{\lambda^{(k_2,l)}/\lambda^{(k_1,l)}}([v_{2k_1},\dots,v_{2k_2-1}];a_{k_2},b_l;q;t;p)R^*_{\lambda^{(k_1,l)}/0}([v_0,\dots,v_{2k_1-1}];a_{k_1},b_l;q;t;p)}{R^*_{\lambda^{(k_2,l)}/0}([v_0,\dots,v_{2k_2-1}];a_{k_2},b_l;q;t;p)}. \tag{7.4.3}$$

(ii') We can also increase k:

$$P_{k+,k_1\to k_2}(\lambda^{(k_1,l)},\lambda^{(k_2,l)}) = Prob(\lambda^{(k_2,l)}|\lambda^{(k_1,l)}) = \frac{\Delta_{\lambda^{(k_2,l)}}(a_{k_2}/b_l|;q;t;p)}{\Delta_{\lambda^{(k_1,l)}}(a_{k_1}/b_l|;q;t;p)}\times \\ \frac{R^*_{\lambda^{(k_2,l)}/\lambda^{(k_1,l)}}([v_{2k_1},\dots,v_{2k_2-1}];a_{k_2},b_l;q;t;p)R^*_{\lambda^{(k_2,l)}/0}([w_0,\dots,w_{2l-1}];\sqrt{pqt}/b_l,\sqrt{pqt}/a_{k_2};q;t;p)}{R^*_{\lambda^{(k_1,l)}/0}([w_0,\dots,w_{2l-1}];\sqrt{pqt}/b_l,\sqrt{pqt}/a_{k_1};q;t;p)}. \tag{7.4.4}$$

(iii) If $l_1 \leq l_2$, then the conditional probability of $\lambda^{(k,l_1)}$ given $\lambda^{(k,l_2)}$ is

$$P_{l-,l_2\to l_1}(\lambda^{(k,l_2)},\lambda^{(k,l_1)}) = Prob(\lambda^{(k,l_1)}|\lambda^{(k,l_2)}) = \\ R^*_{\lambda^{(k,l_2)}/\lambda^{(k,l_1)}}([w_{2l_1},\dots,w_{2l_2-1}];\sqrt{pqt}/b_{l_2},\sqrt{pqt}/a_k;q;t;p)\times \\ \frac{R^*_{\lambda^{(k,l_1)}/0}([w_0,\dots,w_{2l_1-1}];\sqrt{pqt}/b_{l_1},\sqrt{pqt}/a_k;q;t;p)}{R^*_{\lambda^{(k,l_2)}/0}([w_0,\dots,w_{2l_2-1}];\sqrt{pqt}/b_{l_2},\sqrt{pqt}/a_k;q;t;p)}. \tag{7.4.5}$$

(iii') We can also increase l:

$$P_{l+,l_1\to l_2}(\lambda^{(k,l_1)},\lambda^{(k,l_2)}) = Prob(\lambda^{(k,l_2)}|\lambda^{(k,l_1)}) = \frac{\Delta_{\lambda^{(k,l_2)}}(a_k/b_{l_2}|;q;t;p)}{\Delta_{\lambda^{(k,l_1)}}(a_k/b_{l_1}|;q;t;p)}\times$$
$$\frac{R^*_{\lambda^{(k,l_2)}/\lambda^{(k,l_1)}}([w_{2l_1},\dots,w_{2l_2-1}];\sqrt{pqt}/b_{l_2},\sqrt{pqt}/a_k;q;t;p)R^*_{\lambda^{(k,l_2)}/0}([v_0,\dots,v_{2k-1}];a_k,b_{l_2};q;t;p)}{R^*_{\lambda^{(k,l_1)}/0}([v_0,\dots,v_{2k-1}];a_k,b_{l_1};q;t;p)}. \tag{7.4.6}$$

(iv) The marginal process along any path with k nonincreasing and l nondecreasing (we call such a path an *anti-diagonal path*) is Markov in either direction.

(v) $\lambda^{(k,0)} = \lambda^{(0,l)} = 0$ for $k, l \geq 0$.

Remark 7.4.1. Any anti-diagonal path in any instantiation of such a process (the 5 conditions above do not uniquely determine a process) will be called an *elliptic process*, but we may abuse verbiage and use the term for the process as a whole (not just an anti-diagonal path) if we find it convenient. The main feature of an elliptic process is that the transitional probabilities are given by skew interpolation functions, while the stationary (marginal) ones by interpolation functions (which often simplify to Δ-symbols).

Remark 7.4.2. As k (respectively l) increases and l (respectively k) stays fixed, our partitions increase.

Theorem 7.4.3. *Let $l_1 < l_2, k_1 < k_2$ be integers. Assuming all elliptic hypergeometric sums involved are finite, we have that $P_{k-,k_2\to k_1}, P_{k+,k_1\to k_2}, P_{l-,l_2\to l_1}, P_{l+,l_1\to l_2}$ are stochastic matrices compatible with the marginals given by the elliptic Schur measures defined in (i) above.*

Proof. If we expand the sums $\sum_\lambda Prob(\lambda|\mu)$ and $\sum_\mu Prob(\lambda|\mu)Prob(\mu)$ arising in the calculation and appeal to Theorems 7.3.5 and 7.3.1 we immediately obtain the result. □

Such a process may still not exist due to condition (iv), because given two anti-diagonal points with integral coordinates in the first quadrant (k_2, l_1) and (k_1, l_2) $(l_1 < l_2, k_1 < k_2)$, there are many anti-diagonal paths from one to the other, and the probabilities along each have to coincide. However, the following proposition assures us this is the case on every 1×1 square (that is, $k_2-k_1 = l_2-l_1 = 1$).

Proposition 7.4.4. *Assuming termination conditions for the hypergeometric series involved, we have the following quasi-commutation relation:*

$$P_{k-,k_2\to k_1}(\lambda^{(k_2,l_2)},\lambda^{(k_1,l_2)})P_{l+,l_1\to l_2}(\lambda^{(k_2,l_1)},\lambda^{(k_2,l_2)}) =$$
$$P_{l+,l_1\to l_2}(\lambda^{(k_1,l_1)},\lambda^{(k_1,l_2)})P_{k-,k_2\to k_1}(\lambda^{(k_2,l_1)},\lambda^{(k_1,l_1)}).$$

Remark 7.4.5. This is an elliptic process version of the quasi-commutation of elliptic difference operators.

Proof. Direct application of Theorem 7.3.5. □

This leads to the following consistency condition.

Proposition 7.4.6. *Given any two points (k_1, l_1) and (k_2, l_2) in the allowable region between which there is an anti-diagonal path, the probabilities on all anti-diagonal paths between the two points are the same.*

Proof. Follows immediately from Proposition 7.4.4 and the transfer identity of Theorem 7.3.1 (only the latter is needed if $k_1 = k_2$ or $l_1 = l_2$). □

Remark 7.4.7. Note an anti-diagonal path in the proposition need not be comprised of only length 1 steps.

This leads to the following *2-sided* Markov property.

Theorem 7.4.8. *Given any anti-diagonal path in the allowed region for the elliptic process comprised of vertices v_i and a nonterminal vertex v_I on the path, the probability of λ^{v_I} given the rest of the path only depends on the two (nearest) neighbors of v_I on the path. In the case v_I is an endpoint on the path, the mentioned probability only depends on the nearest predecessor/successor on the path.*

Proof. The result follows by construction of the elliptic process together with the fact that the conditional probability of λ^{v_I} given any one of its two neighbors is well defined by Proposition 7.4.6 (same argument applies if v_I is an endpoint). Note that, as before, the distance (number of lattice edges) between two neighbors need not necessarily be 1. □

One way to construct an elliptic process is to inductively sample $\lambda^{(k,l)}$ given the distributions of $\lambda^{(k-1,l)}$ and $\lambda^{(k,l-1)}$ much like the construction in Section 5.2. We thus define the following conditional distribution:

$$\begin{aligned}
&Prob(\lambda^{(k,l)}|\lambda^{(k-1,l)}, \lambda^{(k,l-1)}) = \frac{P_{k+,k-1\to k}(\lambda^{(k-1,l)}, \lambda^{(k,l)}) P_{l-,l\to l-1}(\lambda^{(k,l)}, \lambda^{(k,l-1)})}{(P_{l-,l\to l-1} P_{k+,k\to k+1})(\lambda^{(k-1,l)}, \lambda^{(k,l-1)})} = \\
&\frac{1}{Z} R^*_{\lambda^{(k,l)}/\lambda^{(k-1,l)}}([v_{2k-2}, v_{2k-1}]; a_k, b_l; q; t; p) \Delta_{\lambda^{(k,l)}}(a_k/b_l|; q; t; p) \times \\
&R^*_{\lambda^{(k,l)}/\lambda^{(k,l-1)}}([w_{2l-2}, w_{2l-1}]; \sqrt{pqt}/b_l, \sqrt{pqt}/a_k; q; t; p),
\end{aligned} \tag{7.4.7}$$

where Z is the partition function (a complicated elliptic hypergeometric sum, assumed terminating). We list the following theorem for completeness. The proof is immediate in view of the discussion in this section up to this point.

Theorem 7.4.9. *If we proceed inductively and sample $\lambda^{(k,l)}$ based on $\lambda^{(k,l-1)}$ and $\lambda^{(k-1,l)}$ as in 7.4.7, we will have sampled from an elliptic process satisfying (i) through (v).*

7.5 A different sampling algorithm

In this section we apply the formalism of Section 7.4 to lozenge tilings of a hexagon. We describe how elliptic distributions on tilings can be viewed through skew interpolation functions in two equivalent ways. One way is based on Section 7.4 and leads to an efficient sampling algorithm for lozenge tilings similar to that of Section 5.2, though slightly faster.

We fix K, L, N positive integers. Throughout N will be used to denote the maximum length of a partition or the number of particles on a given vertical slice of a hexagonal tiling (as depicted in Figure 3.3). Moreover, we fix $t = q$ throughout (so all binomial coefficients are determinants) and suppress t altogether from the notation.

We start with a sequence of $K + L + 1$ partitions

$$\Pi = (0 = \lambda^0, \lambda^1, \ldots, \lambda^{K+L-1}, K^N = \lambda^{K+L})$$

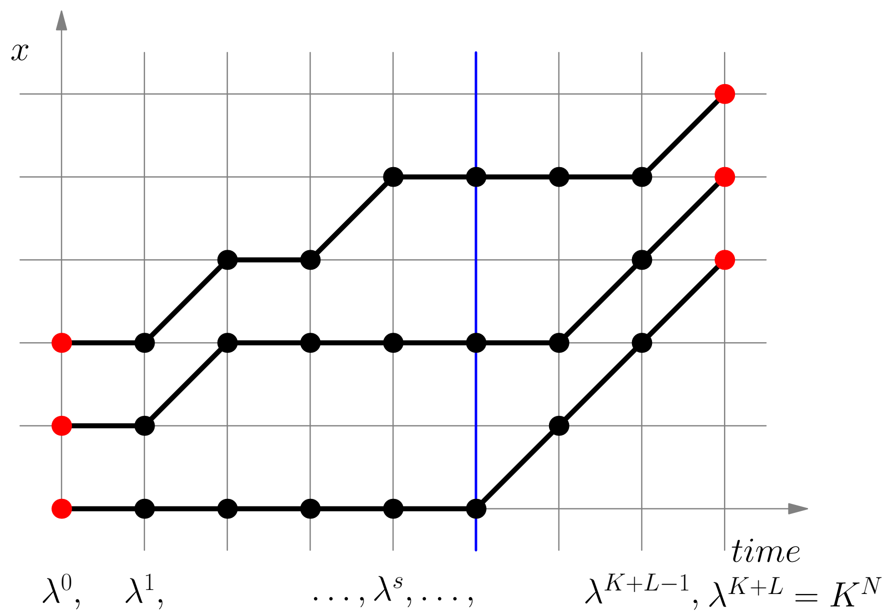

Figure 7.1: $K + L + 1$ partitions forming a tiling of a hexagon. $K = 3, L = 5, N = 3$. For example, $\lambda^1 = (1, 1), \lambda^7 = (2, 2, 2)$.

corresponding to a tiling of a hexagon as in Figure 7.1 where the partitions correspond to particle positions via

$$\{\text{particle positions}\} = \{\lambda_i + N - i\},$$

and we count particles from the bottom horizontal edge up.

Set the probability of such a sequence to

$$Prob(\Pi) = \frac{\prod_{s=0}^{K+L} R^*_{\lambda^{s+1}/\lambda^s}([W_{2s}, W_{2s+1}]; c_{s+1}, d; q; p)}{R^*_{K^N/0}([W_0, \ldots, W_{2(K+L)-1}]; c_{K+L}, d; q; p)}, \tag{7.5.1}$$

where

$$c_s = c_1 \prod_{1<i<2s} W_i.$$

The fact that the numerator, summed over all partitions, yields the denominator is a consequence of Theorem 7.3.1. As the partitions in Π correspond to a tiling of a hexagon, we must have that each differs from the previous by a vertical strip. This forces the following condition on the W's:

$$W_{2i}W_{2i+1} = \frac{1}{q},\ 0 \le i \le 2(K+L).$$

Moreover, we wish to make the denominator in 7.5.1 as simple as possible so we impose an additional constraint:

$$W_{2i+1}W_{2i+2} = 1,\ 1 \le i \le 2(K+L-1),$$

so that $R^*_{K^N/0}([W_0, \ldots, W_{2(K+L)-1}]; c_{K+L}, d; q; p) = R^*_{K^N/0}([W_0, W_{2(K+L)-1}]; c_{K+L}, d; q; p)$. This immediately leads to the following:

$$W_{2i} = q^i W_0,\ W_{2i-1} = \frac{1}{q^i W_0}, \tag{7.5.2}$$

$$c_s = qc_{s+1} = q^{1-s}c_1. \tag{7.5.3}$$

With parameters specialized as above, we have the following result:

Theorem 7.5.1. *The probability measure in 7.5.1, viewed as a measure on tilings of a $N \times K \times L$ hexagon, is the same as the one introduced in Section 3.3.*

Proof. For the proof it suffices to look at the ratio of a full unit box to an empty one in the bulk (inside the hexagon) and compare it to equation (3.3.2). We thus fix a time slice τ $(0 < \tau < K+L)$ and take the ratio of $Prob(\Pi')$ to $Prob(\Pi)$ where Π' differs from Π by a single square. That is, all partitions in Π' are the same as in Π with the exception of position τ, where we replace λ^τ by $\lambda^{\tau,0}$ with $\lambda^{\tau,0}_i = \lambda^\tau_i$ for all $i \neq I$ and for I we have $\lambda^{\tau,0}_I = \lambda^\tau_I - 1$ (for some index $I \le N$). All skew interpolation functions involved are elliptic binomial coefficients that can be expressed as determinants via Proposition 7.2.3. Remark 7.2.5 also helps simplify the calculations. If we denote $\lambda^\tau_I + N - I = x$ (x is the particle position corresponding to the part of λ^τ that changes), the ratio can be expressed as

$$\frac{q^3\theta_p(q^{x-2\tau}\frac{c_1}{q^N w_0}, q^{x+\tau}\frac{w_0}{q^N d}, q^{-2x+\tau-3}\frac{q^{2N}d}{c_1})}{\theta_p(q^{x-2\tau+2}\frac{c_1}{q^N w_0}, q^{x+\tau+2}\frac{w_0}{q^N d}, q^{-2x+\tau-1}\frac{q^{2N}d}{c_1})}.$$

This is nothing more than the weight ratio of (3.3.2) (after switching from (i,j) coordinates to (τ,x) coordinates as in Figure 3.3) where we identify (in the notation of Section 3.3):

$$u_1 = \frac{c_1}{q^{N-1}W_0},\ u_2 = \frac{W_0}{q^{N-1}d}.$$

□

Fix l,k such that $0 \le l \le L-1$ and $0 \le k \le K-1$. Let $k' = L+K-k$. The partitions $\lambda^{k'}$ and $\lambda^{k'+1}$ in the above process have all parts at least $K-k$ and $K-k+1$ respectively. That is, we can write $\lambda^{k'} = (K-k)^N + \mu^k, \lambda^{k'+1} = (K-k+1)^N + \mu^{k-1}$ for appropriate partitions $\mu^k, \mu^{k-1} \subseteq K^N$. We have the following lemma.

Lemma 7.5.2. *For parameters defined in* (7.5.2), $0 \le k \le K-1$, $0 \le l \le L+K-1$ *we have*

$$Prob(\lambda^{l+1}|\lambda^l) = \frac{R^*_{\lambda^{l+1}/\lambda^l}([W_{2l}, W_{2l+1}]; c_{l+1}, d; q; p) R^*_{K^N/\lambda^{l+1}}([W_{2l+2}, W_{2(K+L)-1}]; c_{K+L}, d; q; p)}{R^*_{K^N/\lambda^l}([W_{2l}, W_{2(K+L)-1}]; c_{K+L}, d; q; p)} \propto$$
$$\frac{\Delta_{\lambda^{l+1}}(q^{-l}c_1/d|q^N, q^{-K}, q^{2-N-L}c_1/d, q^{-2l}c_1/W_0, pq^{1+K+L}W_0/d; q; p)}{\Delta_{\lambda^l}(q^{1-l}c_1/d|q^N, q^{-K}, q^{2-N-L}c_1/d, q^{-2l}c_1/W_0, pq^{1+K+L}W_0/d; q; p)} \times \left\langle \begin{matrix} \lambda^{l+1} \\ \lambda^l \end{matrix} \right\rangle_{[q^{1-l}c_1/d, 1/q]},$$

$$Prob(\lambda^{k'+1}|\lambda^{k'}) \propto$$
$$\frac{\Delta^0_{\mu^k}(q^{1-k+K-L}c_1/d|pq^{2-2k+2K+L}W_0/d, pq^{1-L}/dW_0; q; p)}{\Delta^0_{\mu^{k-1}}(q^{2-k+K-L}c_1/d|pq^{2-2k+2K+L}W_0/d, pq^{1-L}/dW_0; q; p)} \times$$
$$\frac{\Delta^0_{\mu^{k-1}}(q^{2-k+K-L}c_1/d|pq^{2-k+2K+L}W_0/d, q^{-k+1}; q; p)}{\Delta^0_{\mu^k}(q^{1-k+K-L}c_1/d|pq^{2-k+2K+L}W_0/d, q^{-k1}; q; p)} \times \left\langle \begin{matrix} \mu^k \\ \mu^{k-1} \end{matrix} \right\rangle_{[q^{1-k+K-L}c_1/d, 1/q]},$$

$$Prob(\lambda^l) = \frac{R^*_{\lambda^l/0}([W_0, W_{2l-1}]; c_l, d; q; p)}{R^*_{K^N/\lambda^l}([W_{2l}, W_{2(K+L)-1}]; c_{K+L}, d; q; p)} \propto$$
$$\Delta_{\lambda^l}(q^{1-l}c_1/d|q^N, q^{-K}, q^{-l}, q^{2-N-L}c_1/d, q^{1-l}c_1/W_0, pq^{1+K+L}W_0/d; q; p).$$

Proof. The first and third equalities follow by direct computation when expanding the skew interpolation functions as binomial coefficients and simplifying the remaining Δ-symbols. For the first equality we also use equation (7.2.2). The second equality follows from equation (7.2.7) (or again from a combination of equations (7.2.2) and (7.2.3)). □

Remark 7.5.3. The first equality gives rise to the same transitional probability derived in Section 4.1 via alternative methods. This follows of course from Theorem 7.5.1.

We now connect the process described above with the elliptic Schur process of Section 7.4. To do that we first isolate the necessary features of the elliptic Schur process first.

Fix v_0, w_0, a, b and let

$$\begin{aligned} &v_{2i} = q^i v_0, v_{2i+1} = q^{-i-1}/v_0, \\ &w_{2j} = q^j w_0, w_{2j+1} = q^{-j-1}/w_0, \text{ so that} \\ &a_k = q^{-k} a, b_l = q^l b. \end{aligned}$$

These are the parameters of our elliptic Schur process as described in Section 7.4. We are interested in the $K \times L$ positive quadrant. That is we are interested in random partitions $\lambda^{(k,l)}, 0 \leq k \leq K, 0 \leq l \leq L$ satisfying the conditions (i) through (v) of sec. cit. See Figure 7.2.

Since as before $v_{2i-1}v_{2i} = 1, w_{2j-1}w_{2j} = 1$, most parameters disappear from the interpolation functions. To wit $R^*_{\lambda^{(k,l)}/0}([v_0, \ldots, v_{2k-1}]; a_k, b_l; q; p) = R^*_{\lambda^{(k,l)}/0}([v_0, v_{2k-1}]; a_k, b_l; q; p)$ and similarly for the w's.

For all the necessary sums which we assumed terminating in Sections 7.4 and 7.3 to actually terminate, we use Theorem 7.2.10: $R^*_{\lambda/0}([v, a/q^N]; a, b; q; p)$ vanishes unless $\lambda_{N+1} \leq 0$ (that is, unless λ has at most N parts). We call this the *vertical termination* condition (partitions can be no larger than N rows vertically). There are four ways this can be accomplished and each leads to a different efficient sampling algorithm. They come from the fact that we are interested in $R^*_{\lambda^{(K,L)}/0}([v_0, v_{2k-1}]; a_K, b_L; q; p)$ or $R^*_{\lambda^{(K,L)}/0}([w_0, w_{2L-1}]; \sqrt{p}q/b_L, \sqrt{p}q/a_K; q; p)$ to vanish at the point (K, L) if λ has more than N parts. We thus have the following four choices:

$$\begin{aligned} &a_K/q^N = v_0, \\ &a_K/q^N = v_{2K-1}, \\ &\frac{\sqrt{p}q}{b_L q^N} = w_0, \\ &\frac{\sqrt{p}q}{b_L q^N} = w_{2L-1}. \end{aligned}$$

To simplify things, we will only discuss the choice $a_K/q^N = v_{2K-1}$ in detail. That is, we set

$$av_0 = q^N.$$

We now describe the sampling algorithm we alluded to at the beginning of the section. Based on the above discussion, we start with the empty partitions at positions (k, l) with $k = 0, 0 \leq l \leq L$ or $l = 0, 0 \leq k \leq K$ and we inductively sample new partitions at every (k, l) until we have constructed the whole $K \times L$ rectangle. Then if we look at the path Λ described above, it corresponds to a random hexagon tiling of the kind we are after. As described in Section 7.4, we build $\lambda^{(k,l)}$ based

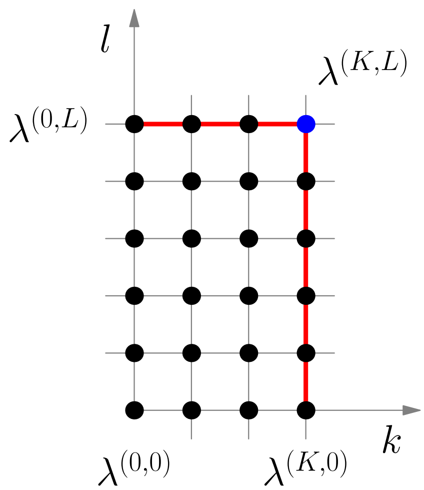

Figure 7.2: The partitions (dots) on the red bolded line correspond to a tiling of an $N \times K \times L$ hexagon. The partitions on the axes are all 0.

on (given the fact that we already have samples for) $\lambda^{(k-1,l)}$ and $\lambda^{(k,l-1)}$ by sampling from the distribution given in 7.4.7. We sample vertical slices from bottom to top, one slice at the time from left to right. That is, we first sample $\lambda^{(1,1)}$, then $\lambda^{(1,2)}$, ... until $\lambda^{(1,L)}$. Next we sample $\lambda^{(2,1)}$, $\lambda^{(2,2)}$, ..., $\lambda^{(2,L)}$ in this order and so on. The reader should consult Section 5.2 to compare the two sampling algorithms.

To make the things clearer in describing an atomic step, let us use the following shorthand notation for the three partitions involved:

$$\lambda := \lambda^{(k,l)},\ \mu := \lambda^{(k-1,l)},\ \nu := \lambda^{(k,l-1)}.$$

We are interested how to sample λ based on μ and ν. Following Section 5.2, let us define the following function:

$$p(j) = X^{-1} q^{-(j+2)} \frac{\theta_p(q^{2j+2}X)}{\theta_p(q^{2j+4}X)} \times \frac{\theta_p(q^{j+1}\mathcal{V}_1, q^{j+1}\mathcal{V}_2, q^{j+1}\mathcal{W}_1, q^{j+1}\mathcal{W}_2)}{\theta_p(pq^{j+2}\frac{X}{\mathcal{V}_1}, pq^{j+2}\frac{X}{\mathcal{V}_2}, pq^{j+2}\frac{X}{\mathcal{W}_1}, pq^{j+2}\frac{X}{\mathcal{W}_2})},$$

where

$$X = \frac{a_k}{b_l},$$
$$\mathcal{V}_1 = \frac{a_k}{v_{2k-2}},\ \mathcal{V}_2 = \frac{a_k}{v_{2k-1}},$$
$$\mathcal{W}_1 = \frac{\sqrt{p}q}{b_l w_{2l-2}},\ \mathcal{W}_2 = \frac{\sqrt{p}q}{b_l w_{2l-1}}.$$

As before, we omit parameter dependence in this notation to keep it simple. Note though the above parameters depend on (k, l) as well as on a, b, v_0, w_0, p, q. Moreover we define

$$P(\mathcal{k}, j; s) = \prod_{i=1}^{s} p(\mathcal{k} - (j + i - 1)).$$

On the set $\{0, 1, \dots, n\}$ we define the following discrete distribution:

$$Prob(s) = D(\mathcal{k}, j; n)(s) = \frac{P(\mathcal{k}, j; s)}{\sum_{s'=0}^{n} P(\mathcal{k}, j; s')}.$$

Because λ must differ from both ν and μ by a vertical strip, we have the three cases given below. Only one leads to nontrivial sampling, much like in Section 5.2.

- Case 1. For all i with $\nu_i - \mu_i = 1$ we set $\lambda_i = \nu_i$.
- Case 2. For all i with $\nu_i - \mu_i = -1$ we set $\lambda_i = \mu_i$.
- Case 3. For the remaining indices, we have $\nu_i = \mu_i$. Group the indices into blocks (such that the parts in each block are adjacent) and consider one such called a $(\mathcal{k}, j, \ell)$ block. Here $\mathcal{k} := \nu_i = \mu_i$ (for all indices i within this block), j is the first index in the block, and ℓ is the number of parts in the (length of the) block. That is, we have

  $$\mu_j = \nu_j = \mu_{j+1} = \nu_{j+1} = \cdots = \mu_{j+\ell-1} = \nu_{j+\ell-1} = \mathcal{k}$$

  and $\mu_{j-1} \neq \nu_{j-1}, \mu_\ell \neq \nu_\ell$. For each such block independently, we sample a random variable ξ according to the distribution $D(\mathcal{k}, j; \ell)$. We set $\lambda_i = \mathcal{k} + 1$ for the first ξ consecutive positions in the block, and we set $\lambda_i = \mathcal{k}$ for the remainder of the $\ell - \xi$ positions.

Similar to Theorem 5.2.3, we have the following theorem that makes the 3 cases described above necessary and sufficient for sampling λ based on μ and ν.

Theorem 7.5.4. *By using the procedure described in Cases 1., 2., and 3. above, we have constructed a random partition $\lambda := \lambda^{(k,l)}$ from an elliptic Schur process with appropriately specialized parameters.*

Proof. The proof is essentially the same as that of Theorem 5.2.3, the difference being in the complexity of computations. Using the results of Section 7.2 and 1.2, we make the following computation:

$$
\begin{aligned}
&Prob(\lambda|\mu,\nu) \propto \text{(factors independent of}\\
&R^*_{\lambda/\mu}([v_{2k-2},v_{2k-1}];a_k,b_l;q;p)\Delta_\lambda(a_k/b_l|;q;p)R^*_{\lambda/\nu}([w_{2l-2},w_{2l-1}];\sqrt{p}q/b_l,\sqrt{p}q/a_k;q;p) =\\
&\left\langle \begin{matrix}\lambda\\ \mu\end{matrix}\right\rangle_{[X,1/q](\mathcal{V}_1,\mathcal{V}_2)} \left\langle \begin{matrix}\lambda\\ \nu\end{matrix}\right\rangle_{[X,1/q](\mathcal{W}_1,\mathcal{W}_2)} \Delta_\lambda(X|;q;p) = \text{(factors independent of } \lambda)\times\\
&\prod_i q^{-2\lambda_i^2+\lambda_i(-7+2N+4i)}X^{-3\lambda_i}\theta_p(q^{2\lambda_i-2i+2}X;q)_{2\lambda_i}\left(\frac{\theta_p(q^{2-N-i}X;q)_{\lambda_i}}{\theta_p(q^{N-i+1};q)_{\lambda_i}}\right)^2\times\\
&\Delta^0_\lambda(X|\mathcal{V}_1,\mathcal{V}_2,\mathcal{W}_1,\mathcal{W}_2;q;p)\times \det_{1\le i,j\le N}\left\langle \begin{matrix}\lambda_i+N-i\\ \mu_j+N-j\end{matrix}\right\rangle_{[X_0,1/q]} \times \det_{1\le i,j\le N}\left\langle \begin{matrix}\lambda_i+N-i\\ \nu_j+N-j\end{matrix}\right\rangle_{[X_0,1/q]} =\\
&\text{(factors indep. of } \lambda)\times\prod_i q^{-2\lambda_i^2+\lambda_i(-7+2n+4i)}X^{-3\lambda_i}\theta_p(q^{2\lambda_i-2i+2}X;q)_{2\lambda_i}\left(\frac{\theta_p(q^{2-i-n}X;q)_{\lambda_i}}{\theta_p(q^{n-i+1};q)_{\lambda_i}}\right)^2\\
&\times\prod_i \frac{\theta_p(q^{1-i}\mathcal{V}_1,q^{1-i}\mathcal{V}_2,q^{1-i}\mathcal{W}_1,q^{1-i}\mathcal{W}_2;q)_{\lambda_i}}{\theta_p(pq^{2-i}\frac{X}{\mathcal{V}_1},pq^{2-i}\frac{X}{\mathcal{V}_2},pq^{2-i}\frac{X}{\mathcal{W}_1},pq^{2-i}\frac{X}{\mathcal{W}_2};q)_{\lambda_i}}\times\\
&\prod_{i:\lambda_i=\mu_i}\left\langle \begin{matrix}\mu_i+N-i\\ \mu_i+N-i\end{matrix}\right\rangle_{[X_0,1/q]}\times\prod_{i:\lambda_i=\mu_i+1}\left\langle \begin{matrix}\mu_i+N-i+1\\ \mu_i+N-i\end{matrix}\right\rangle_{[X_0,1/q]}\times\\
&\prod_{i:\lambda_i=\nu_i}\left\langle \begin{matrix}\nu_i+N-i\\ \nu_i+N-i\end{matrix}\right\rangle_{[X_0,1/q]}\times\prod_{i:\lambda_i=\nu_i+1}\left\langle \begin{matrix}\nu_i+N-i+1\\ \nu_i+N-i\end{matrix}\right\rangle_{[X_0,1/q]},
\end{aligned}
$$

where $X_0 = X/q^{2N-2}$. The fact that the determinants evaluate to the stated products follows from Remark 7.2.5. The product nature of this probability shows that blocks (which recall are groups of adjacent indices i for which $\nu_i = \mu_i := \mathscr{k}$) split independently. The probability for a split in a block of length ℓ is then obtained much like in the proof of Theorem 5.2.3. That is, for j the first index in such a block, we take the above formula evaluated at partition λ^0 (where λ^0 agrees with λ except $\lambda^0_j = \lambda_j + 1$) and divide it by the formula evaluated at λ. This leads us to the formula for $\mathscr{p}$ which in turn leads us to the distribution $D(\mathscr{k}, j; \ell)$. This finishes the proof. □

We finally can state how partitions sampled using this algorithm on the $K \times L$ positive quadrant grid indeed correspond to elliptically distributed lozenge tilings of a hexagon. We fix W_0, c_1, d. We furthermore assume $av_0 = q^N$ (for termination) and set

$$
\begin{aligned}
a &= \frac{pq^{1+K}}{d},\\
b &= \frac{p}{c_1},\\
v_0 &= \frac{1}{q^{L+K}W_0},\\
w_0 &= \frac{W_0}{\sqrt{p}}.
\end{aligned}
\tag{7.5.4}
$$

Theorem 7.5.5. *Let W_0, c_1, d be parameters and Π be the sequence of partitions*

$$\Pi = (0 = \nu^0, \nu^1, \dots, \nu^{K+L-1}, K^N = \nu^{K+L})$$

distributed according to (7.5.1) *corresponding to an elliptically distributed lozenge tiling of an $N \times K \times L$ hexagon with weight given in Section 3.3 and parameters $u_1 = \frac{c_1}{q^{N-1}W_0}$, $u_2 = \frac{W_0}{q^{N-1}d}$. Assume parameters a, b, v_0, w_0 are nspecialized according to* (7.5.4)*. Let*

$$\Lambda = (0 = \lambda^{(K,0)}, \lambda^{(K,1)}, \dots, \lambda^{(K,L-1)}, \lambda^{(K,L)}, \lambda^{(K-1,L)}, \dots, \lambda^{(0,L)} = 0)$$

be a sequence of partitions corresponding to the east and north boundary partitions of an elliptic Schur process with parameters a, b, v_0, w_0 in the rectangular region $0 \le k \le K,\ 0 \le l \le L$. Π and Λ have the same distribution via the measure preserving bijection

$$\begin{aligned} &\nu^l \mapsto \lambda^{(K,l)},\ 0 \le l \le L, \\ &\nu^{K+L-k} \mapsto (K-k)^N + \lambda^{(k,L)}, 0 \le k \le K-1. \end{aligned}$$

Remark 7.5.6. The theorem can be summarized as follows: given a sequence Λ as in the statement (the red bolded line in Figure 7.2), we can shift the last K partitions up (shift $\lambda^{k,L} \to (K-k)^N + \lambda^{k,L}$) to obtain a sequence Π corresponding to an elliptically distributed lozenge tiling of an $N \times K \times L$ hexagon (see Figure 7.1). Conversely, shifting the last K partitions of such a tiling down yields the NE boundary of an elliptic Schur process on the $K \times L$ grid subject to termination conditions.

Proof. In the first step we check that the marginals agree under the parameter specializations. The proof is then completed by checking the transitional probabilities. That is, we use Lemma 7.5.2 and expand the "decrease k" and "increase l" transitional probabilities of the elliptic Schur process of Section 7.4 (equations 7.4.3 and 7.4.6). We find that under the parameter specializations everything matches. □

Bibliography

[BC11] A. Borodin and I. Corwin, *Macdonald processes*, arXiv:1111.4408v2 [math.PR] (2011).

[Bet11] D. Betea, *Elliptically distributed lozenge tilings of a hexagon*, arXiv:1110.4176v1 [math-ph] (2011).

[BF10] A. Borodin and P. L. Ferrari, *Anisotropic growth of random surfaces in 2+1 dimensions*, arXiv:0804.3035v2 (2010).

[BG09] A. Borodin and V. Gorin, *Shuffling algorithm for boxed plane partitions*, Adv. Math. **220** (2009), no. 6, 1739–1770. MR MR2493180

[BGR10] A. Borodin, V. Gorin, and E. M. Rains, *q-distributions on boxed plane partitions*, Selecta Math. (N.S.) **16** (2010), no. 4, 731–789. MR 2734330

[Bor10] A. Borodin, *Schur dynamics of the Schur processes*, arXiv:1001.3442v1 [math.CO] (2010).

[Bor11] ______, *Determinantal point processes*, arXiv:0911.1153 (2011).

[BR05] A. Borodin and E. M. Rains, *Eynard-Mehta theorem, Schur process, and their Pfaffian analogs*, J. Stat. Phys. **121** (2005), no. 3-4, 291–317. MR 2185331 (2006k:82039)

[CKP01] H. Cohn, R. Kenyon, and J. Propp, *A variational principle for domino tilings*, J. Amer. Math. Soc. **14** (2001), no. 2, 297–346 (electronic). MR 1815214 (2002k:82038)

[CLP98] H. Cohn, M. Larsen, and J. Propp, *The shape of a typical boxed plane partition*, New York J. Math. **4** (1998), 137–165 (electronic). MR 1641839 (99j:60011)

[DF90] P. Diaconis and J. A. Fill, *Strong stationary times via a new form of duality*, Ann. Probab. **18** (1990), no. 4, 1483–1522. MR MR1071805 (91m:60127)

[DVJ88] D. J. Daley and D. Vere-Jones, *An introduction to the theory of point processes*, Springer Series in Statistics, Springer-Verlag, New York, 1988. MR 950166 (90e:60060)

[EM98] B. Eynard and M. L. Mehta, *Matrices coupled in a chain. I. Eigenvalue correlations*, J. Phys. A **31** (1998), no. 19, 4449–4456. MR 1628667 (99g:82028)

[FT97] I. B. Frenkel and V. G. Turaev, *Elliptic solutions of the Yang-Baxter equation and modular hypergeometric functions*, The Arnold-Gelfand mathematical seminars, Birkhäuser Boston, Boston, MA, 1997, pp. 171–204. MR 1429892 (98k:33034)

[Gor08] V. Gorin, *Nonintersecting paths and the Hahn orthogonal polynomial ensemble*, Funktsional. Anal. i Prilozhen. **42** (2008), no. 3, 23–44, 96. MR 2454474 (2010a:60027)

[GR04] G. Gasper and M. Rahman, *Basic hypergeometric series*, second ed., Encyclopedia of Mathematics and its Applications, vol. 96, Cambridge University Press, Cambridge, 2004, With a foreword by Richard Askey. MR 2128719 (2006d:33028)

[Hus04] D. Husemöller, *Elliptic curves*, second ed., Graduate Texts in Mathematics, vol. 111, Springer-Verlag, New York, 2004, With appendices by Otto Forster, Ruth Lawrence and Stefan Theisen. MR 2024529 (2005a:11078)

[Joh05] K. Johansson, *Non-intersecting, simple, symmetric random walks and the extended Hahn kernel*, Ann. Inst. Fourier (Grenoble) **55** (2005), no. 6, 2129–2145. MR 2187949 (2006k:60081)

[Kas67] P. Kasteleyn, *Graph theory and crystal physics*, Graph theory and theoretical physics, Academic Press, London, 1967, pp. 43–110.

[Ken97] R. Kenyon, *Local statistics of lattice dimers*, Ann. Inst. H. Poincaré Probab. Statist. **33** (1997), no. 5, 591–618. MR 1473567 (99b:82039)

[Ken09] ______, *Lectures on dimers*, Statistical mechanics, IAS/Park City Math. Ser., vol. 16, Amer. Math. Soc., Providence, RI, 2009, pp. 191–230. MR 2523460 (2010j:82023)

[KO07] R. Kenyon and A. Okounkov, *Limit shapes and the complex Burgers equation*, Acta Math. **199** (2007), no. 2, 263–302. MR MR2358053 (2008h:60038)

[KS96] R. Koekoek and R. F. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue*, arXiv:math/9602214v1 [math.CA] (1996).

[Mac95] I. G. Macdonald, *Symmetric functions and Hall polynomials*, second ed., Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1995, With contributions by A. Zelevinsky, Oxford Science Publications. MR 1354144 (96h:05207)

[Oko98] A. Okounkov, BC-*type interpolation Macdonald polynomials and binomial formula for Koornwinder polynomials*, Transform. Groups **3** (1998), no. 2, 181–207. MR 1628453 (99h:33061)

[Oko01] ______, *Infinite wedge and random partitions*, Selecta Math. (N.S.) **7** (2001), no. 1, 57–81. MR 1856553 (2002f:60019)

[OR03] A. Okounkov and N. Reshetikhin, *Correlation function of Schur process with application to local geometry of a random 3-dimensional Young diagram*, J. Amer. Math. Soc. **16** (2003), no. 3, 581–603 (electronic). MR 1969205 (2004b:60033)

[Rai06] E. M. Rains, *BC_n-symmetric Abelian functions*, Duke Math. J. **135** (2006), no. 1, 99–180. MR 2259924 (2008e:33043)

[Rai10] ______, *Transformations of elliptic hypergeometric integrals*, Ann. of Math. (2) **171** (2010), no. 1, 169–243. MR 2630038

[Rai11] ______, *Elliptic Littlewood identities*, arXiv:0806.0871v2 [math.CO] (2011).

[Rui97] S. N. M. Ruijsenaars, *First order analytic difference equations and integrable quantum systems*, J. Math. Phys. **38** (1997), no. 2, 1069–1146. MR 1434226 (98m:58065)

[Sch07] M. Schlosser, *Elliptic enumeration of nonintersecting lattice paths*, J. Combin. Theory Ser. A **114** (2007), no. 3, 505–521. MR 2310747 (2008j:05028)

[Sil94] J. H. Silverman, *Advanced topics in the arithmetic of elliptic curves*, Graduate Texts in Mathematics, vol. 151, Springer-Verlag, New York, 1994. MR 1312368 (96b:11074)

[Sil09] ______, *The arithmetic of elliptic curves*, second ed., Graduate Texts in Mathematics, vol. 106, Springer, Dordrecht, 2009. MR 2514094 (2010i:11005)

[Spi02a] V. P. Spiridonov, *An elliptic beta integral*, New trends in difference equations (Temuco, 2000), Taylor & Francis, London, 2002, pp. 273–282. MR 2016068 (2004j:33024)

[Spi02b] ______, *Theta hypergeometric series*, Asymptotic combinatorics with application to mathematical physics (St. Petersburg, 2001), NATO Sci. Ser. II Math. Phys. Chem., vol. 77, Kluwer Acad. Publ., Dordrecht, 2002, pp. 307–327. MR 2000728 (2004j:33012)

[Spi03] ______, *Theta hypergeometric integrals*, Algebra i Analiz **15** (2003), no. 6, 161–215. MR 2044635 (2005m:33030)

[Spi08] ______, *Essays on the theory of elliptic hypergeometric functions*, Uspekhi Mat. Nauk **63** (2008), no. 3(381), 3–72. MR MR2479997

[Ste90] J. R. Stembridge, *Nonintersecting paths, Pfaffians, and plane partitions*, Adv. Math. **83** (1990), no. 1, 96–131. MR 1069389 (91h:05014)

[SZ00] V. P. Spiridonov and A. S. Zhedanov, *Spectral transformation chains and some new biorthogonal rational functions*, Comm. Math. Phys. **210** (2000), no. 1, 49–83.

[SZ01] ______, *Generalized eigenvalue problem and a new family of rational functions biorthogonal on elliptic grids*, Special functions 2000: current perspective and future directions (Tempe, AZ), NATO Sci. Ser. II Math. Phys. Chem., vol. 30, Kluwer Acad. Publ., Dordrecht, 2001, pp. 365–388. MR MR2006295 (2004j:33028)

[TF61] H. N. V. Temperley and M. E. Fisher, *Dimer problem in statistical mechanics—an exact result*, Philos. Mag. (8) **6** (1961), 1061–1063. MR 0136398 (24 #B2436)

[vdBR11] F. van de Bult and E. Rains, *Limits of multivariate elliptic beta integrals and related bilinear forms*, arXiv:1110.1460v1 (2011).

[vDS00] J. F. van Diejen and V. P. Spiridonov, *An elliptic Macdonald-Morris conjecture and multiple modular hypergeometric sums*, Math. Res. Lett. **7** (2000), no. 5-6, 729–746. MR 1809297 (2002m:33022)

[vDS01] ______, *Modular hypergeometric residue sums of elliptic Selberg integrals*, Lett. Math. Phys. **58** (2001), no. 3, 223–238 (2002). MR 1892922 (2003g:11046)

[Vul09] M. Vuletić, *A generalization of MacMahon's formula*, Trans. Amer. Math. Soc. **361** (2009), no. 5, 2789–2804. MR 2471939 (2009m:05192)

[War02] S. O. Warnaar, *Summation and transformation formulas for elliptic hypergeometric series*, Constr. Approx. **18** (2002), no. 4, 479–502. MR 1920282 (2003h:33018)