Localization and Dualities in Three-dimensional Superconformal Field Theories

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Abstract

In this thesis we apply the technique of localization to three-dimensional $\mathcal{N} = 2$ superconformal field theories. We consider both theories which are exactly superconformal, and those which are believed to flow to nontrivial superconformal fixed points, for which we consider implicitly these fixed points. We find that in such theories, the partition function and certain supersymmetric observables, such as Wilson loops, can be computed exactly by a matrix model. This matrix model consists of an integral over $\mathfrak{g}$, the Lie algebra of the gauge group of the theory, of a certain product of 1-loop factors and classical contributions. One can also consider a space of supersymmetric deformations of the partition function corresponding to the set of abelian global symmetries.

In the second part of the thesis we apply these results to test dualities. We start with the case of ABJM theory, which is dual to M-theory on an asymptotically $AdS_4 \times S^7$ background. We extract strong coupling results in the field theory, which can be compared to semiclassical, weak coupling results in the gravity theory, and a nontrivial agreement is found. We also consider several classes of dualities between two three-dimensional field theories, namely, 3D mirror symmetry, Aharony duality, and Giveon-Kutasov duality. Here the dualities are typically between the IR limits of two Yang-Mills theories, which are strongly coupled in three dimensions since Yang-Mills theory is asymptotically free here. Thus the comparison is again very nontrivial, and relies on the exactness of the localization computation. We also compare the deformed partition functions, which tests the mapping of global symmetries of the dual theories. Finally, we discuss some recent progress in the understanding of general three-dimensional theories in the form of the F-theorem, a conjectured analogy to the a-theorem in four dimensions and c-theorem in two dimensions, which is closely related to the localization computation.

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Contents

Chapter 1

Introduction

Quantum field theory has proven to be an extremely rich subject, and our attempts to understand it have brought us deep insights, both physically and mathematically. As we learn more about it, we uncover new structure that indicates our lack of a comprehensive understanding. One of the most spectacular examples of such an unexpected discovery is the existence of dualities: two quantum field theories which appear utterly different in our conventional descriptions, but for which there is strong evidence that, at a basic level, they describe the same mathematical object.

An early example of this phenomenon is the equivalence between the sine-Gordon and massive Thirring models. These are a pair of two-dimensional quantum field theories which superficially look very different – in fact, their fundamental degrees of freedom are bosons in the first case and fermions in the second, and the potentials have quite different forms – but it was proven [1] that, at a quantum level, they are identical.

One interesting aspect of this equivalence concerns the existence of solitons in sine-Gordon theory. Classically, these are field configurations which are topologically stable because they interpolate between different vacua of the theory at spatial infinity, and therefore cost an infinite amount of energy to disassemble. At a quantum level, these can be shown to obey Fermi statistics, and one can rewrite the theory using the solitons as the fundamental degrees of freedom, recovering the Thirring model. Note that solitons can be seen classically in sine-Gordon theory, while the corresponding objects are quantum excitations of the fermionic field in the Thirring model. Moreover, the topological stability of the solitons translates to an ordinary particle number conservation law in the Thirring model. These phenomena – the exchange of solitons and fundamental degrees of freedom, classical and quantum effects, and topological and flavor symmetries – seem to be recurring themes in dualities, and we will see them play a prominent role in the examples explored in this thesis.

The nontrivial nature of dualities stems from our difficulty in understanding quantum field theory at strong coupling. Typically one can only perform perturbative calculations in an interacting theory, and nonperturbative effects like solitons will not be easy to see. The two models discussed above are very special in that they are exactly soluble, but such examples are very rare. To explore other dualities, even to find them in the first place, one needs some way to examine field theories at strong coupling. Understanding strongly coupled theories is important for other reasons as well. For example, QCD becomes strongly coupled at low energies, and the perturbative description in terms of gluons and quarks breaks down and is replaced by an effective theory of hadrons and glueballs, with a mass gap which is still not rigorously understood.

The most successful approach to understanding strong coupling dynamics has been to focus on theories with supersymmetry. This is a symmetry exchanging bosonic and fermionic degrees of freedom, and is the most general spacetime symmetry a theory may possess beyond the ordinary Poincare group [2]. There are reasons to believe that supersymmetry may play a role in realistic physical theories. For example, supersymmetry provides a promising solution to the hierarchy problem, gives natural candidates for dark matter particles, and is required for the only known consistent theory of quantum gravity, string theory. However, our reasons for working with supersymmetric theories will be more pragmatic: supersymmetry imposes strong enough constraints on the structure of a theory that one is able to study it at strong coupling, the regime where the interesting and poorly understood effects become prominent.

It is important to specify the amount of supersymmetry the theory possesses, for which we use the notation $\mathcal{N}$, with $\mathcal{N}=1$ denoting the minimal number of supercharges in a given number of spacetime dimensions, and larger values denoting multiples of this quantity. Typically, for too small a value of $\mathcal{N}$, the constraints imposed by supersymmetry are too weak to make powerful statements about strong coupling, and for too large a value, the constraints are too strong, and theory is rigid and exhibits less rich structure. It turns out that in three dimensions, the amount of supersymmetry where the most interesting behavior appears is $\mathcal{N}=2$, and this is the case we will focus on in this thesis. In addition, there are many interesting theories with $\mathcal{N}=3,4,5,6$ and even the maximal $\mathcal{N}=8$ symmetry, and our results will apply to these as well.

The precise way in which supersymmetry allows one to study a strongly coupled theory can be separated into two related, and somewhat complementary, ideas: nonrenormalization theorems and localization. Correspondingly, as we will see, the supersymmetric actions will typically split into two pieces: F-terms and D-terms. In the first case, the fact that the structure of the supersymmetry algebra must be preserved at all length scales allows one to argue that the F-terms do not experience renormalization group (RG) flow, or do so in a controlled way. This allows one, for example, to make precise statements about the moduli space of a theory. On the other hand, one can show that D terms can be rescaled until the theory is nearly free and computations become tractable, without affecting certain quantities one associates to a theory, such as correlation functions of certain operators. This is the basic idea behind localization.

Both techniques have been well-studied and applied to many interesting theories and dualities. In this thesis, we will focus on a relatively new application of localization, pioneered in the four-dimensional case by Pestun [3]. We will see that one can apply similar ideas in three dimensions, and the result is computationally more tractable. Just as the four-dimensional calculation was used to provide nontrivial tests of several dualities, both of the field theory-gravity type (AdS/CFT) and between two superconformal field theories, we will apply our results to test the matching of partition function and supersymmetric observables between dual three-dimensional theories.

This thesis will split naturally into two parts. In the first half, we discuss supersymmetry and the formalism of localization. In Chapter 2, we introduce supersymmetry in the context we will be most interested in: three-dimensional $\mathcal{N} = 2$ supersymmetry. We explore extensions of this supersymmetry algebra, to extended supersymmetry with larger values of $\mathcal{N}$, and to superconformal symmetry. We also introduce some of the theories we will be exploring in the second half. In Chapter 3, we discuss how superconformal symmetry allows one to place theories on curved manifolds, and describe some of the technical aspects of this process. As we will see, localization is most easily applied not on flat space, where the theories are originally defined, but on compact curved manifolds, such as $S^2 \times S^1$ and S^3. In Chapter 4 we go through the technique of localization. We describe some historical background, and then proceed to apply the idea to three-dimensional theories. We review some aspects of the superconformal index, and discuss how it is related to localization on $S^2 \times S^1$.

In the second half of the thesis we put our calculations of the first half to work, and make nontrivial statements about strongly coupled field theories. In Chapter 5 we discuss ABJM theory, and provide some tests AdS/CFT duality. In Chapter 6, we look at three-dimensional mirror symmetry, another duality motivated by string theory. We then briefly return in Chapter 7 to discuss some more formalism related to theories with only $\mathcal{N} = 2$ supersymmetry, which are not protected by the nonrenormalization theorems. We also discuss the F-theorem, a deep statement about the RG flow in general (i.e., not necessarily supersymmetric) field theories. Finally, in Chapter 8, we discuss some $\mathcal{N} = 2$ dualities due to Aharony and Giveon-Kutasov.

Part I

Localization Method

Chapter 2

Three-dimensional $\mathcal{N} = 2$ Superconformal Field Theories

In this chapter we review the class of theories we will be exploring in this thesis, namely, three-dimensional gauge theories with at least $\mathcal{N} = 2$ supersymmetry. This amounts to the existence of 4 real supercharges, and when we also have superconformal symmetry, which will be essential for localization, this is doubled to 8 supercharges. In this chapter we review theories in flat Minkowski space; we will consider more general backgrounds in the following chapter. Spinor conventions are reviewed in Appendix A.

2.1 Supersymmetry Algebra

In this section we review the supersymmetry algebra and some of its representations.[1] In flat Minkowski space, spinors come in representations of $Spin(2,1) \equiv SL(2,\mathbb{R})$, and a Majorana condition can be imposed, so that the least amount of supersymmetry is a single spinor supercharge with two real components. This gives the $\mathcal{N} = 1$ case, but holomorphy is not active, and we expect there are not enough constraints to allow us to extract much interesting information about $\mathcal{N} = 1$ theories.

The next case is the $\mathcal{N} = 2$ algebra, with two spinor charges, which we can organize into a single complex spinor Q_α, together with its conjugate $\tilde{Q}_\alpha$. The algebra also contains the bosonic generators of the Poincare group, the momentum P_μ and generator of Lorentz transformations $M_{\mu\nu}$[2]. The algebra here is the dimensional reduction of the $\mathcal{N} = 1$ algebra in 4 dimensions, and is given by

$$\begin{aligned} \{Q_\alpha, Q_\beta\} &= [P_\mu, Q_\alpha] = 0, \\ \{Q_\alpha, \tilde{Q}_\beta\} &= 2\gamma^\mu{}_{\alpha\beta} P_\mu + 2i\epsilon_{\alpha\beta} Z, \end{aligned} \tag{2.1}$$

[1] Many of the results quoted in this section follow [4].

[2] We will often omit commutation relations of $M_{\mu\nu}$ since they are implicitly determined by the index structure, although it will appear on the RHS of a commutation relation when we consider the superconformal algebra.

where Z is a central charge, which can be thought of as the dimensional reduction of the 4th component of the momentum, and we pick the gamma matrices to be real and symmetric. As in four dimensions, there is a $U(1)$ R-symmetry which acts as an automorphism of this algebra, rotating the charges Q_α and $\tilde{Q}_\alpha$ by opposite phases. This R-symmetry will play a more active role in the superconformal algebra we consider in a moment, and will be very important in the remainder of this thesis.

2.1.1 Multiplets

Representations of this algebra on fields can be most easily obtained by working with superfields. These can be obtained by dimensional reduction from the well-known four-dimensional formalism. As we will see later, it is important that we work with off-shell representations in order for localization to work. We review the superfield formalism in Appendix **??**. Here we simply state the result, that the representations relevant for us will come in three classes.

The first is a chiral multiplet, and consists of a complex scalar ϕ and a complex Weyl fermion ψ, along with a complex auxiliary scalar F. These are grouped into a chiral superfield X, which satisfies the constraint $\bar{\mathcal{D}}_\alpha X = 0$. There is also a corresponding anti-chiral field, satisfying $\mathcal{D}_\alpha \tilde{X} = 0$. It is convenient to introduce the notation $\delta_\epsilon = \epsilon Q$, for some complex spinor parameter ϵ, which we take to be anticommuting, so that δ_ϵ is a bosonic object.[3] Then we can write the supersymmetry transformations of the chiral as (we will suppress the subscript on δ_ϵ when the spinor parameter in question is clear from context)

$$\begin{aligned} &\delta\phi = 0, && \delta\phi^\dagger = \psi^\dagger\epsilon, \\ &\delta\psi = -i\gamma^\mu\partial_\mu\phi\epsilon, && \delta\psi^\dagger = \epsilon F^\dagger, \\ &\delta F = -i\epsilon\gamma^\mu\partial_\mu\psi, && \delta F^\dagger = 0. \end{aligned} \tag{2.2}$$

We can deduce the action of $\delta^\dagger = \epsilon^\dagger\tilde{Q}$ by taking the Hermitian conjugate of these expressions. Then it is straightforward to check that the supersymmetry algebra above is obeyed with $Z = 0$. A representation with nonzero Z can be obtained by adding a real mass, as we will see from a slightly different point of view when we consider gauge theories in a moment.

We can write an action for the chiral multiplet in the form

$$S_m = S_D + S_F = \int d^3x d^4\theta K(X, X^\dagger) + \int d^3x \left(d^2\theta W(X) + c.c. \right) \tag{2.3}$$

[3] By this we mean that, when computing the algebra, we should be considering commutators of δ_ϵ since

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = \epsilon_1 Q \epsilon_2 Q - \epsilon_2 Q \epsilon_1 Q = \epsilon_1^\alpha \epsilon_2^\beta \{Q_\alpha, Q_\beta\}$$

Using $\epsilon_1^\alpha\epsilon_2^\beta = -\epsilon_2^\beta\epsilon_1^\alpha$, this is necessary to recover the anticommutators of Q.

where $K(X, X^\dagger)$ is the Kahler potential, and $W(X)$ is the superpotential. We will denote the two contributions the D-term and F-term, respectively. The superpotential is protected by holomorphy, but the Kahler potential is not, and will in general be renormalized. There is a natural generalization of this action to multiple chiral multiplets. By analogy with four dimensions, we will call such a theory a Wess-Zumino model.

In general we will be interested in superconformal theories, or the IR fixed point of nonconformal theories, in which case the Kahler potential will generically flow to $K(X, X^\dagger) = XX^\dagger$, and the corresponding part of the action can be written in components as:

$$S_D = \int d^3x d^4\theta X X^\dagger = \int d^3x \Big(\partial_\mu \phi^\dagger \partial^\mu \phi + \psi^\dagger \not{\nabla} \psi + F^\dagger F \Big) \tag{2.4}$$

It is important to note that, as shown in Appendix C, since this is full superspace integral, it can be written as the total δ-variation of some quantity [5]. In this case, we find, for two linearly independent spinors ϵ_1, ϵ_2:

$$\delta_1 \delta_2 \int d^3x \phi^\dagger F = (\epsilon_1 \epsilon_2) \int \Big(- \phi^\dagger \nabla^2 \phi + \psi^\dagger \not{\nabla} \psi + F^\dagger F \Big). \tag{2.5}$$

which, up to a constant and total derivative, is equal to the D-term action. The fact that the D-term is a δ-total variation will be very important for localization.

Note that the parameter ϵ above should be assigned R-charge -1 to balance the R-charge of Q_α. Then we can see that $R(\phi) = R(\psi) + 1 = R(F) + 2$. The value of $R(\phi)$, which we will define to be the R-charge of the multiplet X, is somewhat convention dependent, since one can mix the R-charge with any flavor symmetry commuting with the supercharges. In our UV Lagrangian descriptions, we will use the convention $R(\phi) = \frac{1}{2}$. However, as we will see in the next section, the R-charge in a superconformal theory, such as the IR fixed point of these theories, is a physical quantity, and will have to be chosen more carefully.

The next multiplet we consider is the vector multiplet. It consists of a gauge field A_μ, real scalar σ, and complex Weyl spinor λ, along with an auxiliary real scalar D. In terms of the dimensional reduction from four dimensions, σ can be thought of as the fourth component of the 4D gauge field. The bosonic fields are uncharged under R-symmetry, while the gaugino λ has charge 1.

These fields can be organized into a real superfield, satisfying $V = V^\dagger$. To be more precise, the actions we will write down will be invariant under a gauge symmetry $V \to V + \Lambda + \Lambda^\dagger$ for chiral superfield Λ, and it is only in a particular choice of gauge, Wess-Zumino gauge, that one finds the above field content. The supersymmetry transformations on these components fields is given by

$$\delta A_\mu = -\frac{i}{2} \lambda^\dagger \gamma_\mu \epsilon, \quad \delta \sigma = -\frac{1}{2} \lambda^\dagger \epsilon, \quad \delta D = -\frac{i}{2} (D_\mu \lambda^\dagger) \gamma^\mu \epsilon - \frac{i}{2} \gamma [\lambda^\dagger, \sigma] \epsilon,$$

$$\delta\lambda = (i\gamma^\mu(-\frac{1}{2}\epsilon_{\mu\nu\rho}F^{\nu\rho} + D_\mu\sigma) - D)\epsilon, \quad \delta\lambda^\dagger = 0. \tag{2.6}$$

A natural gauge-invariant combination of these fields is:

$$W_\alpha = -\frac{1}{4}\bar{D}^2 e^{-V} D_\alpha e^V \tag{2.7}$$

which is a (spinorial) chiral superfield. Then the ordinary Yang-Mills action can be written as

$$\begin{aligned} S_{YM} &= \int d^3x d^2\theta \frac{1}{g^2}\mathrm{Tr} W_\alpha W^\alpha + c.c. \\ &= \int d^3x \frac{1}{g^2}\mathrm{Tr}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\sigma D^\mu\sigma + D^2 + \lambda^\dagger\gamma^\mu D_\mu\lambda\right) \end{aligned} \tag{2.8}$$

Note the coupling constant has negative mass dimension, and so the theory is infinitely strongly coupled in the infrared. We will see in a moment that it can also be written as a D-term, and so is a total δ-variation.

These two kinds of multiplets can be coupled by modifying the matter D-term to [6, 7]

$$\begin{aligned} S_m &= \int d^3x D^4\theta X^\dagger e^V X \\ &= \int d^3x \Big(D_\mu\phi^\dagger D^\mu\phi + \phi^\dagger\sigma^2\phi + i\phi^\dagger D\phi + i\psi^\dagger \not{D}\psi - i\psi^\dagger\sigma\psi + i\phi^\dagger\lambda^\dagger\psi - i\psi^\dagger\lambda\phi + F^\dagger F\Big) \end{aligned} \tag{2.9}$$

Upon fixing the Wess-Zumino gauge one finds the chiral transformations are modified to

$$\begin{aligned} &\delta\phi = 0, && \delta\phi^\dagger = \psi^\dagger\epsilon, \\ &\delta\psi = (-i\gamma^\mu D_\mu\phi - i\sigma\phi)\epsilon, && \delta\psi^\dagger = \epsilon F^\dagger, \\ &\delta F = \epsilon(-i\gamma^\mu D_\mu\psi + i\sigma\psi - i\lambda\phi), && \delta F^\dagger = 0, \end{aligned} \tag{2.10}$$

One can check that now [7]

$$\{\delta, \delta^\dagger\} = -i(\epsilon^\dagger\gamma^\mu\epsilon D_\mu + \epsilon^\dagger\epsilon\sigma), \tag{2.11}$$

so that the algebra is satisfied only up to gauge transformations.

An important class of actions we will consider are those containing dynamical chiral multiplets which are coupled to background gauge multiplets in a supersymmetric configuration. By this we mean that we formally write down the action of a gauge theory, but take the gauge fields to be frozen in a fixed configuration by effectively giving them very large kinetic terms, essentially treating them as parameters. We can couple the gauge field to any global symmetry of the theory, since we do not have to worry about anomalies in three dimensions.[4] This is sometimes called "weakly gauging" the flavor symmetry, since a large kinetic term corresponds to small gauge coupling.

An important point is that in order to preserve supersymmetry, this configuration must have the property that $\delta A_\mu = \delta\sigma = ... = 0$, i.e., all component field configurations must be invariant under the supersymmetry. This is because the total gauge-coupled action is typically only invariant if we vary all of the fields, but can not vary the background gauge fields. We say such a configuration is BPS. In all examples we consider, BPS configurations will have all fermions vanishing.

The simplest example of such a BPS configuration is to take σ to have a constant, nonzero value, while setting all the other fields to zero. Since it is only $D_\mu\sigma$ that appears on the RHS of the gauge multiplet transformations, this configuration is obviously supersymmetric. Inspecting the action 2.9 above, we see this corresponds to giving a nonzero mass to the chiral multiplet, called a real mass. Also, from the commutation relations 2.11, we see that this corresponds precisely to including a nonzero central charge in 2.1. Put another way, one can think of the central charge as living in a background BPS vector multiplet. Note that the BPS bound $m \geq |Z|$ is saturated for the chiral. In general, the chiral multiplet can also get a contribution to its mass from the superpotential, which will not contribute to Z, and then the inequality will be strict.

Finally, we can consider linear multiplets, satisfying $D^2\Sigma = \bar{D}^2\Sigma = 0$. One can show that, for a vector multiplet V, $\Sigma = D\bar{D}V$ defines a linear multiplet, which is gauge-invariant, with lowest component σ. We can write the Yang-Mills term alternatively as

$$S_{YM} = \int d^3x d^4\theta \Sigma^2. \tag{2.12}$$

As with the matter, the fact that it is a full superspace integral means it can be written as a total δ variation. Namely, one computes:

$$\delta_1\delta_2 \int d^3x \mathrm{Tr}(\lambda\lambda) = (\epsilon_1\epsilon_2) \int \sqrt{g} d^3x \mathrm{Tr}\Big((-\frac{1}{2}\epsilon^{\mu\nu\rho}F_{\nu\rho} + D^\mu\sigma)^2 + D^2 - i\lambda \not{D} \lambda^\dagger - i\lambda[\lambda^\dagger, \sigma] \Big) \tag{2.13}$$

Global currents live in linear multiplets, and the defining conditions $D^2 J = \bar{D}^2 J = 0$ are the supersymmetric generalization of their conservation laws. For the vector multiplet, this reduces to the Bianchi identity and the corresponding conserved current is the $U(1)_J$ topological current:

$$J_\mu = \star \mathrm{Tr} F. \tag{2.14}$$

There is such a symmetry for each $U(1)$ factor in the gauge group.

[4] In fact, there is the parity anomaly, where a gauge field with an odd number of chiral fermions must have a half-integral Chern-Simons term, but this will not arise in this context in the examples we consider, and we will not have much to say about this phenomenon until Chapter 8.

Before moving on, we note that in addition to the Yang-Mills action, we can consider the supersymmetric Chern-Simons action. This cannot straightforwardly be written as a superspace integral in general, but in components it takes the form:

$$S_{CS} = \int d^3x \mathrm{Tr}\Big(\epsilon^{\mu\nu\rho}(A_\mu \partial_\nu A_\rho + \frac{2i}{3} A_\mu A_\nu A_\rho) + 2D\sigma - \lambda^\dagger \lambda\Big) \tag{2.15}$$

Here the trace must be normalized properly to ensure gauge-invariance, e.g., for a $U(N)$ gauge group one can take it to be $\frac{k}{4\pi}$ times the trace in the fundamental representation, where k must be an integer. It is shown in Appendix C.2 that it is invariant under the transformations 2.6. We will review this action in more detail, along with bosonic Chern-Simons action to which it is closely related, in the next chapter.

Actually, in the abelian case, one can write a simple superspace expression involving the vector and linear multiplets for a gauge group:

$$S_{CS} = \int d^3x d^4\theta \Sigma V$$

Even though this is a superspace integral, unlike the cases above, the integrand is not (super-) gauge-invariant, and so it cannot be written as the total δ-variation of some functional of the fields. Similarly, one can define off-diagonal Chern-Simons terms, known as BF couplings, between two gauge fields:

$$S_{BF} = \int d^3x d^4\theta \Sigma_1 V_2 = \int d^3x d^4\theta \Sigma_2 V_1$$

If one takes one of these to be frozen in a BPS configuration, and the other to be a dynamical $U(1)$ gauge multiplet, or more generally the trace part of a $U(N)$ gauge group, then one obtains a Fayet Iliopoulos term:

$$S_{FI} = \int d^3x d^4\theta \xi V$$

where ξ is the scalar in a background linear multiplet Σ_b. Formally, this is the same result we would get for gauging the $U(1)_J$ current (by adding a term $V_b\Sigma$) and putting it in a BPS configuration with constant σ. Thus we can think of an FI term as bearing the same relationship to the corresponding $U(1)_J$ symmetry as a real mass does to a flavor symmetry.

2.1.2 Moduli Space

After integrating out the auxilliary fields D and F,[5] one finds a potential for the scalar fields of the theory. In the absense of a superpotential, this has the schematic form:

[5] In the case where the gauge kinetic term is only of Chern-Simons type, σ and λ are also auxilliary.

$$V \sim \sum |\sigma\phi|^2$$

Then, in order to obtain finite action configurations, we must pick asymptotic values for the scalar fields such that this vanishes. To achieve this, one typically either set $\sigma = 0$, leading to the Higgs branch of solutions parameterized by VEVs for the matter fields ϕ, or set $\phi = 0$, giving the Coulomb branch parameterized by VEV's for σ.[6] Quantum effects will modify this moduli space, e.g., by generating an effective superpotential which lifts some or all of it.

As we will see in some examples of the second part of the thesis, the exact moduli space of a theory can in many cases be precisely determined, especially in the case of extended supersymmetry. The matching of moduli spaces across dual theories can be used as a nontrivial test of duality, indeed, it was the main test available in many of these dualities before localization was discovered. We will see localization provides an independent test.

It is worth stressing that the moduli space of a theory is essentially the space of choices of asymptotic boundary conditions for the fields, and in particular depends on the manifold we are considering. In the case of compact manifolds which we consider later, there is no moduli space, as constant backgrounds must be integrated over. When we discuss coupling a superconformal flat space theory to curved space, we are then implicitly talking about the interacting fixed point at the origin of moduli space.

2.2 Superconformal Symmetry

There are important nonrenormalization theorems for theories with $\mathcal{N} = 2$ supersymmetry, which tightly constrain the form of the superpotential and central charge Z [4]. These can be thought of as arising from the constraints on representations of the supersymmetry algebra, which the theory must satisfy at all length scales. However, in the far IR, this algebra is enhanced further to the superconformal algebra, and one can make even more precise statements. This is also the domain where localization can be applied.

The $\mathcal{N} = 2$ superconformal algebra contains all the generators in the supersymmetry algebra, namely P_μ, $M_{\mu\nu}$, and Q_α . In addition, it contains two new bosonic generators, K_μ and D, the generators of special conformal transformations and dilatations, as well as the new fermionic generators S_α and $\bar{S}_\alpha$. The charge of a state or operator under D is called its conformal dimensions, and the operators in this algebra have the following dimensions:

[6] In addition, there may be mixed branches where some components of both σ and ϕ are nonzero. In fact, one can only rigorously distinguish Higgs and Coulomb branches in theories with extended supersymmetry, where they are classified by their different transformation properties under the nonabelian R-symmetry group. We will discuss this in more detail in Chapter 6.

$$P_\mu \to 1, \quad K_\mu \to -1, \quad Q_\alpha, \bar{S}_\alpha \to \frac{1}{2}, S_\alpha, \bar{Q}_\alpha \to -\frac{1}{2}. \tag{2.16}$$

The commutation relations must respect these charges, and in most cases this fixes them up to a constant. The most important relation for our purposes will be

$$\{Q_\alpha, S_\beta\} = M_{\mu\nu}[\gamma^\mu, \gamma^\nu]_{\alpha\beta} + 2D\epsilon_{\alpha\beta} + i\epsilon_{\alpha\beta}R \tag{2.17}$$

Here R is the R-charge. Note that the algebra involves the R-charge in a direct way, not merely as an automorphism as in the case of the supersymmetry algebra. In particular, one can show that, for any unitary representation of the algebra, the dimension of an operator is bounded by

$$D \geq |R|. \tag{2.18}$$

This is saturated with $D = R$ when the operator is chiral (i.e., annihilated by $\tilde{Q}$), and with the $D = -R$ when it is anti-chiral. Moreover, the dimension is bounded below by $\frac{1}{2}$, and attains this value only for a free scalar field. Thus, for example, the dimension of the scalar field ϕ in a chiral multiplet is equal to its R-charge. It is important to note that this relation cannot get quantum corrections, because the property of a multiplet being chiral means it has fewer degrees of freedom than a generic multiplet, and, in general, this can not change under continuous deformations of the theory. We will come back to this point when we discuss the superconformal index in Chapter 4.

It might seem strange that the R-charge is related to the generator of scale transformations, since the former is ambiguous up to mixing with a flavor symmetry, while the latter is physical. The point is that the R-charge appearing in the superconformal algebra is in the same supermultiplet as the stress-energy tensor, which shares a similar ambiguity, in that it can be redefined by correction terms, but in a superconformal theory there is always a unique stress-energy tensor which is traceless. It is the corresponding R-charge, now uniquely determined, for which the above inequality is obeyed.

Determining this R-charge among all the candidates is often nontrivial. Moreover, as one can explicitly check by computing anticommutator of the superconformal symmetries, as shown in Appendix **??**, the representation we will state below has $R(\phi)$ restricted to be $\frac{1}{2}$.[7] We will find more general representations later, but for now, if we want to use these supersymmetry transformations we are forced to restrict to theories for which the R-charge of all chiral multiplets is $\frac{1}{2}$. This will typically be guaranteed by the presence of extended supersymmetry, as we will see in a moment.

To construct explicit representations of the superconformal algebra on fields, we will again use the notation $\delta_\epsilon = \epsilon^\alpha Q_\alpha$, for a fermionic parameter ϵ. Then to include the superconformal symmetries S_α, we allow the spinor parameter ϵ to have nontrivial spacetime dependence. The transformations

[7] Note this does not imply it is free, as it is not, in general, a gauge-invariant operator.

are almost the same as before; one only needs to add a few extra terms proportional to the derivative of ϵ. For the gauge multiplet, we have

$$\delta A_\mu = -\frac{i}{2}\lambda^\dagger\gamma_\mu\epsilon, \quad \delta\sigma = -\frac{1}{2}\lambda^\dagger\epsilon, \quad \delta D = -\frac{i}{2}(D_\mu\lambda^\dagger)\gamma^\mu\epsilon - \frac{i}{2}\gamma[\lambda^\dagger,\sigma]\epsilon + \frac{i}{6}\lambda^\dagger\gamma^\mu\partial_\mu\epsilon,$$

$$\delta\lambda = (i\gamma^\mu(-\frac{1}{2}\epsilon_{\mu\nu\rho}F^{\nu\rho} + D_\mu\sigma) - D)\epsilon + \frac{2i}{3}\gamma^\mu\partial_\mu\epsilon\sigma, \quad \delta\lambda^\dagger = 0. \tag{2.19}$$

and for the matter, we get:

$$\begin{aligned} &\delta\phi = 0, && \delta\phi^\dagger = \psi^\dagger\epsilon, \\ &\delta\psi = -i\gamma^\mu\partial_\mu\phi\epsilon - \frac{i}{3}\phi\gamma^\mu\partial_\mu\epsilon, && \delta\psi^\dagger = \epsilon F^\dagger, \\ &\delta F = -i\epsilon\gamma^\mu\partial_\mu\psi, && \delta F^\dagger = 0. \end{aligned} \tag{2.20}$$

In addition, one must impose that ϵ take the special form

$$\epsilon = \epsilon_s + x^\mu\gamma_\mu\epsilon_c, \tag{2.21}$$

where ϵ_s, ϵ_c are constant spinors. Note that the ordinary supersymmetries we had before are parametrized $\epsilon_s = 0$, and the new superconformal generators are parametrized by ϵ_c. It is shown in Appendix **??** that the algebra above is satisfied with $R(\phi) = \frac{1}{2}$.

Recall that, as a result of them being defined by full superspace integrals, the D term for the chiral multiplet could be written as the total supersymmetry variation of some quantity, namely:

$$\delta_1\delta_2 \int d^3x\phi^\dagger F = (\epsilon_1\epsilon_2)\int\Big(- \phi^\dagger\nabla^2\phi + \psi^\dagger\nabla\!\!\!\!/\,\psi + F^\dagger F\Big). \tag{2.22}$$

This remains true also for the superconformal symmetries. Namely, as shown in Appendix C, one can take ϵ_1 and ϵ_2 to be superconformal symmetries, and one obtains precisely the same expression as above. Note the factor of $\epsilon_1\epsilon_2$ is spacetime dependent if either variation contains a superconformal transformation, but as long as it has no zeros we can divide by it and bring it to the other side, and the statement that the action is a total δ-variation still holds. In particular, this makes it manifest that these actions are invariant under superconformal symmetry.

For the Yang-Mills action, the story is a little more subtle. We do not expect it to preserve the full superconformal symmetry (indeed, it is not even scale-invariant), however, it will turn out it preserves a sufficiently large subalgebra that it will be useful for localization. We will explore this in more detail in the next chapter.

A more convenient kinetic term for the gauge field will be the Chern-Simons term. The pure

Chern-Simons action is topological, and so in particular has conformal symmetry, and this remains true if one couples to matter in some special cases. Firstly, it is always true classically, as one can verify by a direct computation [8]. As shown in [61], in the absence of any superpotential, an $\mathcal{N} = 2$ Chern-Simons-matter theory is exactly superconformal, i.e., even after accounting for quantum effects. In a theory with extended supersymemtry, this is also guaranteed. We will turn to such theories now.

2.3 Extended Supersymmetry

Next we consider the possibility of having more than four supercharges. To achieve this, we will first need to group our multiplets into what we will call $\mathcal{N} = 4$ multiplets. These will essentially be the reduction of the corresponding $\mathcal{N} = 2$ multiplets in four dimensions.

There will be two types of multiplets relevant to the theories we consider in this thesis. The first is the $\mathcal{N} = 4$ vector multiplet. This contains an $\mathcal{N} = 2$ vector multiplet V and a chiral multiplet Φ in the adjoint representation of the gauge group. The second is the hypermultiplet, which consists of two chiral multiplets X and $\tilde{X}$ in conjugate representations of all flavor and gauge groups. Then the basic matter action consistent with $\mathcal{N} = 4$ supersymmetry is:

$$\int d^3x d^4\theta X^\dagger e^V X + \int d^3x d^4\theta \tilde{X}^\dagger e^V \tilde{X} + \int d^3x \left(d^2\theta \tilde{X} \Phi X + c.c \right) \tag{2.23}$$

In theories with extended supersymmetry, $\mathcal{N} \geq 3$, the R-symmetry group is enhanced to a nonabelian group $SO(\mathcal{N})$. For example, for $\mathcal{N} = 4$ we find an R-symmetry group $SU(2)_L \times SU(2)_R$. This can also be seen by reducing from $\mathcal{N} = 1$ theories in six dimensions, where the first factor is the R-symmetry in the 6D theories, and the second comes from rotations in the reduced dimensions. We will discuss this R-symmetry and its role in dualities in more detail in the second half of the thesis.

In theories with extended supersymmetry, the $U(1)_R$ symmetry discussed in the previous section sits inside this nonabelian group. As such, it cannot mix with flavor symmetries, and so is unambiguous. In particular, this means that the R-symmetry of a hypermultiplet in such a theory is protected under RG flow, and will always be $\frac{1}{2}$. Thus the representation of the superconformal algebra stated above will hold in these theories at all length scales.

One could also add a supersymmetric Chern-Simons term for the gauge multiplet. This restricts one to at most $\mathcal{N} = 3$ supersymmetry. However, in the presence of a Chern-Simons term alone (i.e., no Yang-Mills term), the gauge field is nonpropagating, and with a clever choice of matter content and superpotential, one can obtain very large amounts of supersymmetry, such as in [8], and in

[8]The invariance of the (gauged) matter action under the fermionic part of the superconformal group, which generates the whole group, was just argued, and that of the Chern-Simons matter is demonstrated in Appendix C.2

ABJM theory [9] which we will study in Chapter 5.

We will study these extended supersymmetry theories more in the second part of the thesis, but their extra structure will not play any direct role in the localization calculation itself.

Chapter 3

Superconformal Theories on Curved Manifolds

In the previous chapter, we have discussed the class of theories we will be interested in, and gave their actions in flat Minkowskian space. We have also discussed superconformal symmetry, which is essential for localization.

However, the partition function in flat space is not a good observable to compute, as it suffers from divergences coming from the infinite volume of space. To get around this, we will attempt to define the theory on a compact manifold. The manifold we will be most interested in is S^3, although we will try to keep our discussion as general as possible, and will also consider the case $S^2 \times S^1$ in some detail. We will motivate these actions mainly by conformal invariance and a little guesswork; a more careful analysis starting with three-dimensional supergravity may be possible, as in [10].

3.1 Euclidean Theories

Before putting our theories on general curved manifolds, it will be very useful to pass to Euclidean signature. Formally, this corresponds to Wick-rotating the path-integral to let the time coordinate t run over imaginary values. We define the Euclidean time to be $x^3 = it$, since it will appear in a symmetric way with the spatial variables x^1, x^2. This formal manipulation changes the signature of the metric appearing in the action from Minkowskian to Euclidean. Typically the action then becomes positive definite, and enters the with a minus sign, so that the Wick-rotation has the effect

$$Z = \int \mathcal{D}\Phi e^{iS[\Phi]} \rightarrow \int \mathcal{D}\Phi e^{-S_E[\Phi]},$$

which gives the path-integral better convergence properties, and will make the saddle-point approximation we use for localization more straightforward.

We also have to slightly modify our interpretation of the partition function. For an integral over $\mathbb{R}^2 \times S^1$, in passing from Minkowski to Euclidean signature, the object we are calculating becomes:

$$\mathrm{Tr} e^{i\beta H} \to \mathrm{Tr} e^{-\beta H}$$

where β is the radius of the circle. Thus the Euclidean path-integral reproduces the thermal partition function at temperature $T = \beta^{-1}$. In the limit $\beta \to \infty$, we get the $T = 0$ partition function, which, at least formally, counts the ground states of the system.

One can check that, in computing the trace above using the path-integral, any fermions in the theory should be given antiperiodic boundary conditions. This is typically not compatible with supersymmetry, as can be seen by looking at the supersymmetry transformations, for which the variations of the fermions are proportional to the bosons, and so are periodic. Thus we must insert an additional twisting operator, giving the Witten index:

$$Z = \mathrm{Tr}((-1)^F e^{-\beta H}).$$

Again, the $\beta \to \infty$ limit counts ground states, this time giving fermionic ground states a weight -1. However, as we will see in the next chapter, in many cases this quantity can be shown to be independent of β.

There are some slight subtleties when passing to Euclidean signature. Note that the Hermiticity condition $A(t) = A^\dagger(t)$ is modified to $A(ix^3) = A^\dagger(-ix^3)$, and so includes a reflection in Euclidean time. Invariance of the Euclidean action under this involution is known as reflection positivity, and corresponds to Hermiticity of the action in Minkowski space. In particular, a short argument shows the partition function of a reflection-positive Euclidean theory is real and positive. In addition, one must take special care with spinors. For example, in three dimensions, the Minkowskian spin group is $SL(2,\mathbb{R})$, which has Majorana representations, while the Euclidean group is $SU(2)$, which does not. The reality condition in Minkowski space translates under Wick rotation to a condition which is not $SU(2)$ covariant in Euclidean space.[1] In particular, the minimal amount of supersymmetry in three Euclidean dimensions is a single complex spinor. The story we found above for a single complex supercharge goes through nearly identically, with appropriately modified gamma matrices, which in particular can no longer be chosen to be real.

3.2 Conformal Theories on Conformally Flat Manifolds

Next we move on to general manifolds, and see how much of the above flat space story can be generalized there. If the manifold has the form $\Sigma \times S^1$ for some two-dimensional manifold Σ, we can assign an interpretation of the Euclidean partition function as computing a trace of $e^{-\beta H}$ for

[1] If one is interested in computing correlation functions in a theory with Majorana spinors by rotating to Euclidean space, one must break the $SU(2)$ covariance at some point in the calculation, although the $SO(2)$ subgroup untouched by the rotation can be maintained.

the Hamiltonian H generating time translations in the Hilbert space assigned to the theory on Σ. This will be the interpretation we adopt when we consider $S^2 \times S^1$. However, we will not restrict to such manifolds, and indeed the case of most interest will be S^3, which has no such decomposition.[2] Nevertheless, the path integral over the space of fields on such a manifold can be defined, and we will consider it, at least initially, as an object of its own interest. Later, we will see that we can assign a more physical interpretation to some of the quantities we compute.

Consider a three-dimensional spin manifold M, with metric g, and suppose one wants to define supersymmetry on it. As a first guess, one might try to look for supersymmetry parameters ϵ which are covariantly constant, generalizing ordinary supersymmetries in flat space. However, by a standard argument, this would imply the manifold admits a reduced holonomy group, and this cannot happen for nonflat manifolds in three dimensions. Thus we are forced to work with noncovariantly constant spinors, and we expect the story to be more analogous that of superconformal symmetry in flat space. It us therefore natural to look at the case of conformally flat manifolds.

Coupling a theory to a curved metric is typically done using a stress-energy tensor derived from the flat space theory. In general, the stress-energy tensor is ambiguous up to correction terms, and this leads to an ambiguity in the definition of the curved space theory. However, in the special case where the theory is conformal, and the metric is conformally flat (i.e., $g_{ij} = e^{-2\Omega}\delta_{ij}$ for some scalar function Ω, and in some coordinate system), there is a unique way to do this such that conformal invariance is maintained. In fact, as we will see in a moment, one can even maintain the superconformal invariance present in the theories we consider.

3.2.1 Chiral Multiplet

Let us start by determining how to write an superconformal action for the chiral multiplets of the theory. The free action of a chiral multiplet is that of a free scalar, free fermion, and auxiliary scalar. The prescriptions for coupling free scalars and fermions to conformally flat metrics is well known, and relies on the conformal covariance of the operators $\nabla^2 + \frac{(d-1)}{4(d-2)}R$, where R is the Ricci scalar (so that this reduces to just the Laplacian in flat space), and $\not{\nabla}$, as reviewed in Appendix B. This guarantees that the following action has a conformal symmetry on any conformally flat manifold (specializing now to three dimensions):

$$S = \int \sqrt{g} d^3x \Big(-\phi(\nabla^2 + \frac{1}{8}R)\phi + i\psi^\dagger \not{\nabla}\psi + F^\dagger F \Big) \tag{3.1}$$

To check superconformal invariance, we try to find spinors ϵ for which the action is invariant under the transformations 2.20 (appropriately covariantizing the derivatives):

[2]Although S^3 can be written as an S^1 bundle over S^2, this bundle is nontrivial and has no global sections, so it is not clear how to formulate a Hilbert space interpretation on it.

$$\begin{aligned} &\delta\phi = 0, && \delta\phi^\dagger = \psi^\dagger\epsilon \\ &\delta\psi = -i\gamma^\mu\nabla_\mu\phi\epsilon - \frac{i}{3}\phi\gamma^\mu\nabla_\mu\epsilon, && \delta\psi^\dagger = \epsilon F^\dagger \\ &\delta F = -i\epsilon\gamma^\mu\nabla_\mu\psi, && \delta F^\dagger = 0 \end{aligned} \tag{3.2}$$

Recall that on flat space we imposed the condition $\epsilon = \epsilon_s + x^\mu\gamma_\mu\epsilon_c$ for constant ϵ_s, ϵ_c. We can state this more conveniently by noting that this form of ϵ is precisely the most general flat space solution to the conformal Killing spinor, or twistor spinor, equation:

$$\nabla_\mu\epsilon = \gamma_\mu\epsilon' \tag{3.3}$$

for some arbitrary spinor ϵ'. This equation essentially says that $\nabla_\mu\epsilon \sim \nabla_{\alpha\beta}\epsilon_\delta$ contains only spin-$\frac{1}{2}$ components, as is natural from the point of view of the closure of the supersymmetry algebra. The more restrictive condition:

$$\nabla_\mu\epsilon = \lambda\gamma_\mu\epsilon$$

for some constant λ, identifies ϵ as a Killing spinor. We will see such spinors exist on S^3, but not on general manifolds (e.g., not even on $\mathbb{R}^3$).

The advantage of writing things this way is that, as shown in Appendix B, the conformal Killing spinor equation is conformally covariant. Given any solution in flat space, one can write a corresponding solution on any conformally flat manifold (at least locally), and vice versa. In particular there is always a four complex-dimensional space of solutions. Most importantly, for such an ϵ, the conformally coupled chiral multiplet action is invariant under the supersymmetry transformations above.

To see this, recall that in flat space we were able to express our D-term actions as total δ-variations of some quantity for any superconformal symmetry δ. The same is true in this setting. Specifically, let δ_1 and δ_2 be any two superconformal transformations. Then, as shown in Appendix C, one has:

$$\delta_1\delta_2 \int \sqrt{g}d^3x(\phi^\dagger F) = \int \sqrt{g}d^3x(\epsilon_1\epsilon_2)\Big(-\phi^\dagger\nabla^2\phi + \frac{R}{8}\phi^\dagger\phi + i\psi^\dagger \not{\nabla}\psi + F^\dagger F\Big) \tag{3.4}$$

Up to a field-independent function that we can ignore, this is precisely the conformally coupled action we found above. In this form, it is manifestly invariant under δ_1, since ${\delta_1}^2 = 0$, and since the form of the action is the same for any choice of δ_1, this demonstrates the complete superconformal symmetry.

In anticipation of coupling to a gauge field, let us use the natural guess for the gauge coupled SUSY transformations on this manifold:

$$\begin{aligned}
&\delta\phi = 0, && \delta\phi^\dagger = \psi^\dagger\epsilon \\
&\delta\psi = (-i\gamma^\mu D_\mu\phi - i\sigma\phi)\epsilon - \frac{i}{3}\phi\gamma^\mu\nabla_\mu\epsilon, && \delta\psi^\dagger = \epsilon F^\dagger \\
&\delta F = \epsilon(-i\gamma^\mu\nabla_\mu\psi + i\sigma\psi - i\lambda\phi), && \delta F^\dagger = 0
\end{aligned} \tag{3.5}$$

where D_μ is covariant with respect to the metric and gauge fields, i.e., $D_\mu = \nabla_\mu + iA_\mu$. Then we find the gauge-coupled D-term to be:

$$\delta_1\delta_2\int\sqrt{g}d^3x(\phi^\dagger F) = \int\sqrt{g}d^3x(\epsilon_1\epsilon_2)\Big(-\phi^\dagger D_\mu D^\mu\phi + \phi^\dagger(\sigma^2 + iD + \frac{R}{8})\phi + i\psi^\dagger(\gamma^\mu D_\mu - \sigma)\psi + F^\dagger F\Big) \tag{3.6}$$

Note that, in checking the invariance, there was no need to impose conformal flatness: on any manifold which admits conformal Killing spinors, the above action has such a symmetry. Still, it can be shown that the maximal number of solutions to the conformal Killing equation, four, is achieved only on conformally flat manifolds [11].

We note that this procedure only works if the conformal dimension of the scalar is $\frac{1}{2}$, or equivalently, if the R-charge is $\frac{1}{2}$. We will have to wait until Chapter 7 to understand how to write down an appropriate action for a chiral multiplet of general R-charge; until then we will restrict to theories with extended supersymmetry where this property is guaranteed.

3.2.2 Gauge Multiplet – Chern Simons Action

For the gauge multiplet, recall that the Yang-Mills term is not conformally invariant, so the above procedure will not work. We will return to the issue of the Yang-Mills term in a moment, but for now we recall that we also have the option of adding a supersymmetric Chern-Simons kinetic term. This is conformally invariant – in fact, being topological, it is completely independent of the metric – and so seems to be a viable option. We will review this action now.

Let us first briefly review the bosonic theory [12]. The Chern-Simons action is a functional on the connections A on a principal G bundle E over a manifold M, and is given by

$$S[A] = \int_M \mathrm{Tr}(A\wedge dA + \frac{2i}{3}A\wedge A\wedge A),$$

where the Tr is some trace on the Lie algebra of G. Under a large gauge transformation this action changes by some discrete amount, but if we take the Tr to have a n appropriate normalization this will not affect the path integral measure,

$$Z = \int \mathcal{D}A e^{iS[A]}.$$

For example, if $G = U(N)$, and we take the trace to be $k/4\pi$ times the trace in the fundamental representation for an integer k, one can check that large gauge transformations shift the action by an integer multiple of 2π, and so do not affect the path integral.

Note that the action does not explicitly involve a choice of metric and so, at least naively, depends only on the topology of M. More precisely one must check that it is possible to regularize the path integral without breaking this general covariance. This is nontrivial since one typically introduces a metric in order to define a gauge-fixing condition. Nevertheless, one can show that one can preserve general covariance up to a choice of framing, i.e., a choice of trivialization of the tangent bundle of M.

To get a supersymmetric action, we need to introduce a metric g and some auxiliary variables. Namely, there will be two auxiliary real scalars σ and D, and a complex auxiliary fermion λ, all taking values in the Lie algebra of G, and the action is given by:

$$S[A, \sigma, D, \lambda, \lambda^\dagger] = \int \sqrt{g}\mathrm{Tr}\Big(\sqrt{g}^{-1}\epsilon^{\mu\nu\rho}(A_\mu \partial_\nu A_\rho + \frac{2i}{3} A_\mu A_\nu A_\rho) + 2\sigma D - \lambda^\dagger \lambda\Big) \tag{3.7}$$

The supersymmetry transformations depend on the spinor parameter ϵ, which is a section of the spinor bundle on M associate to g, and are given by:

$$\delta A_\mu = -\frac{i}{2}\lambda^\dagger \gamma_\mu \epsilon, \quad \delta\sigma = -\frac{1}{2}\lambda^\dagger \epsilon, \quad \delta D = -\frac{i}{2}(D_\mu \lambda^\dagger)\gamma^\mu \epsilon - \frac{i}{2}\gamma[\lambda^\dagger, \sigma]\epsilon - \frac{i}{2}\beta\lambda^\dagger \gamma^\mu (\nabla_\mu \epsilon)$$

$$\delta\lambda = (i\gamma^\mu(-\frac{1}{2}\epsilon_{\mu\nu\rho}F^{\nu\rho} + D_\mu \sigma) - D)\epsilon + i\alpha\sigma\gamma^\mu \nabla_\mu \epsilon, \quad \delta\lambda^\dagger = 0 \tag{3.8}$$

Here α, β, and γ are some parameters, and the partition function is invariant for any choice of them.

It is clear from the fact that $\delta\lambda^\dagger = 0$ that $[\delta_1, \delta_2] = 0$ manifestly on all of the fields except λ. In fact, in general this does not vanish on λ, but gives a peculiar bosonic transformation acting only on λ. It will be useful to impose conditions that set it to zero, and, as shown in Appendix C.2, this gives the conditions $\alpha = \frac{2}{3}, \beta = \frac{1}{3}, \gamma = 1$, and:

$$\nabla_\mu \epsilon = \gamma_\mu \epsilon' \tag{3.9}$$

for some other spinor field ϵ'. Note that this is precisely the conformal Killing spinor equation we saw above, and we see we are led back to it by the supersymmetric Chern-Simons action. In addition, these fields and transformations agree with what we found for the $\mathcal{N} = 2$ vector multiplet in the previous chapter.

3.2.3 Wilson Loops

The natural observables in bosonic Chern-Simons theory are Wilson loops. These are operators labeled by a path γ and representation R of the gauge group, and they can be defined as insertions in the path integral of the quantity

$$\mathrm{Tr}_R \mathcal{P} \exp\left(\oint_\gamma A_\mu dx^\mu\right), \tag{3.10}$$

where "$\mathcal{P}\exp$" is the path-ordered exponential, which specifies that the order of terms in the power series expansion of this exponential should respect their ordering along the path.[3] Note this observable is independent of the metric, and so represent a good candidate for a topological observable.

In the $\mathcal{N}=2$ supersymmetric version of Chern-Simons theory, this operator gets modified to

$$\mathrm{Tr}_R \mathcal{P} \exp\left(\oint_\gamma (A_\mu dx^\mu - i\sigma d|x|)\right). \tag{3.11}$$

Note that this explicitly involves the metric. In pure Chern-Simons theory σ is auxiliary and is set to zero, so that we recover the original Wilson loop. However one can also consider this operator in more general theories where σ is dynamical.

Let us check that this operator preserves some supersymmetry. The δ variation of the integrand is (letting the path be parameterized by an arclength s)

$$-\frac{i}{2}\lambda^\dagger(\gamma^\mu \frac{dx^\mu}{ds} - 1)\epsilon,$$

which generically implies

$$\gamma^\mu \frac{dx^\mu}{ds}\epsilon = \epsilon.$$

This enforces that the path be along the integral curves of the vector field

$$v^\mu = \epsilon^\dagger \gamma^\mu \epsilon,$$

Typically such a choice of integral curve will restrict us to a four-dimensional subspace of the eight dimensional space of conformal Killing spinors, so that this operator is $\frac{1}{2}$ BPS.

We will see that, for the conformal Killings we use on the sphere, these integral curves will be great circles, and in fact will correspond to the fibers of the Hopf fibration. Thus any Wilson loop of the above type placed along one of these fibers will be supersymmetric, and its expectation value

[3] There is an ambiguity here canceled by the cyclicity of the trace.

can be computed by localization.

One technical point we should mention is that the Wilson loop needs to be regularized, since the path-ordered exponential forces us to consider the product of multiple copies of the integral over the loop, and so fields on the loop will approach each other and give rise to divergences. This is most naturally done by point-splitting, i.e., shifting the loops we integrate over in these copies to be slightly apart. However, there is some ambiguity in way this shifting is done, for example, one can pick everywhere along the loop a normal vector along which one shifts the various copies. This is called a "framing" of the loop, and typically the expectation value depends on the framing in a well-defined way, e.g., an overall phase. In many cases, there is a unique "trivial framing," in which the shifted loops have zero linking number with each other.

In the supersymmetric case, one needs the shifted loops to also preserve supersymmetry, which will typically fix a unique framing compatible with supersymmetry. In the case of the Wilson loops on S^3, which lie along fibers of the Hopf fibration, all having a mutual linking number of one, we see the framing will be nontrivial. We will have to account for the resulting phase if we wish to compare to the Wilson loop with trivial framing.

3.2.4 Yang-Mills Action

For the matter, we were able to generalize the fact that D-terms are δ-exact to the case of curved manifolds. We saw that the Yang-Mills term was also δ-exact in flat space, so we can attempt to repeat the procedure of Section 3.2.1 here, although since the Yang-Mills action is not conformal, we do not expect the story to go through as cleanly as above. To proceed, let us first define, for a given conformal Killing ϵ, a real scalar function a and real vector field b^μ such that we can express the spinor ϵ' on the RHS of the conformal Killing equation as

$$\epsilon' = \frac{1}{2}(ia + b^\mu \gamma_\mu)\epsilon. \tag{3.12}$$

Note that we can express an arbitrary spinor on the LHS in this form, provided ϵ is nonvanishing as it will be in the cases we consider.

This is merely a definition, but we now need to make an assumption. Let us assume there are two linearly independent conformal Killing spinors $\epsilon_i, i = 1, 2$ satisfying the above equation with the same choice of a and b^μ, i.e.:

$$\epsilon'_i = \frac{1}{2}(ia + b^\mu \gamma_\mu)\epsilon_i, \quad i = 1, 2 \tag{3.13}$$

We will find such spinors on the manifolds we are interested in, S^3 and $S^2 \times S^1$, and these can then be extended to a general manifold by conformal mapping. With this assumption, we can compute:

$$\delta_1\delta_2\int\sqrt{g}d^3x\mathrm{Tr}(\lambda\lambda)=\int\sqrt{g}d^3x(\epsilon_1\epsilon_2)\mathrm{Tr}\Big((-\frac{1}{2}\sqrt{g}^{-1}\epsilon^{\mu\nu\rho}F_{\nu\rho}+D^\mu\sigma+b_\mu\sigma)^2+ \\ +(D+a\sigma)^2+\lambda\Big(-i\gamma^\mu D_\mu\lambda^\dagger-i[\lambda^\dagger,\sigma]-a\lambda^\dagger\Big)\Big) \quad (3.14)$$

Again, this is manifestly invariant under linear combinations of δ_1 and δ_2, which span half of the algebra, but this time it is not invariant under the other generators. This must be the case, since the action is not conformally invariant, but the superconformal generators generate the entire algebra.

More precisely, the $\mathcal{N}=2$ superconformal algebra in three dimensions takes the form of the supergroup $OSp(2|4)$. This has bosonic part $Spin(4,1)\times U(1)$, where the first factor is the conformal group and the second is the $U(1)$ R symmetry. Then this choice of two spinors as above restricts us to an $OSp(2|2)$ subgroup. This subgroup preserves the R-symmetry, but not all of the conformal symmetry.

Note that this action vanishes when $\delta\lambda=0$, and it is not hard to show that the converse holds as well. Since it is written as a sum of squares, we can read off the BPS condition immediately:

$$0=-\frac{1}{2}\sqrt{g}^{-1}\epsilon^{\mu\nu\rho}F_{\nu\rho}+D^\mu\sigma+b_\mu\sigma \\ 0=D+a\sigma \quad (3.15)$$

3.3 Explicit Form of Actions on S^3 and $S^2\times S^1$

In this thesis we will be primarily interested in two manifolds, S^3 and $S^2\times S^1$. Let us then summarize the explicit form of the actions above in these cases. To do this, we will need explicit forms of the Killing spinors, as well as the form of the a and b_μ functions above.

3.3.1 S^3 Actions

Since S^3 is the group manifold of $SU(2)$, with the Killing metric corresponding to the usual round metric, it is convenient to work with a vielbein of left-invariant vector fields. Then the connection coefficients can be read off from the torsion (see Appendix B), which in turn comes directly from the structure constants of the Lie-algebra:[4]:

$$\omega_{ij}(e_k)=\frac{1}{r}\epsilon_{ijk}$$

Thus the spinor covariant derivative in this basis is simply:

[4] Here it is useful to make explicit a factor of the radius of the sphere.

$$\nabla_i = \partial_i + \frac{i}{8r}\epsilon_{ijk}[\gamma^j,\gamma^k]$$
$$= \partial_i + \frac{i}{2r}\gamma_i$$

But now we can immediately read off two solutions to the conformal Killing equation: just take the components to be constant in this basis. Then we see:

$$\nabla_i\epsilon = \frac{i}{2r}\epsilon \tag{3.16}$$

In particular, this shows that $a = \frac{1}{r}$ and $b_\mu = 0$ for these two spinors. One can find the remaining solutions to the conformal Killing equation in a right invariant basis, and they satisfy $a = -\frac{1}{r}$.

These results agree with the general result, stated in Appendix B, that on maximally symmetric spaces of constant nonzero curvature, i.e. S^n or $\mathbb{H}^n$, the conformal Killing spinors can always be refined to Killing spinors, with ϵ' proportional to ϵ. Specifically, as shown there, these spinors satisfy

$$\nabla_\mu\epsilon = \pm\sqrt{-\frac{R}{4d(d-1)}}\epsilon, \tag{3.17}$$

where R is the Ricci scalar of the space. Using $R = \frac{6}{r^2}$ for S^3, we see this agrees with our result above.

Thus we can immediately right down the δ-exact actions on S^3. For the matter, we have

$$S_\delta^m = \int \sqrt{g}d^3x\Big(- \phi^\dagger D_\mu D^\mu\phi + \phi^\dagger(\sigma^2 + iD + \frac{3}{4r^2})\phi + i\psi^\dagger(\gamma^\mu D_\mu - \sigma)\psi + F^\dagger F\Big), \tag{3.18}$$

and for the gauge multiplet we have

$$S_\delta^g = \int \sqrt{g}d^3x\Big((-\frac{1}{2}\sqrt{g}^{-1}\epsilon^{\mu\nu\rho}F_{\nu\rho} + D^\mu\sigma)^2 + (D + \frac{1}{r}\sigma)^2 + \lambda\Big(- i\gamma^\mu D_\mu\lambda^\dagger - i[\lambda^\dagger,\sigma] - \frac{1}{r}\lambda^\dagger\Big)\Big). \tag{3.19}$$

We can also read off the solutions to the BPS equation for the gauge multiplet:

$$0 = -\frac{1}{2}\sqrt{g}^{-1}\epsilon^{\mu\nu\rho}F_{\nu\rho} + D^\mu\sigma$$
$$0 = D + \frac{1}{r}\sigma \tag{3.20}$$

By acting on the first equation with a covariant derivative and using the Bianchi identity, one can show that σ must be covariantly constant and $F_{\mu\nu} = 0$. We may then pick a gauge where the gauge field is zero, so that $\sigma = \sigma_o$ for some fixed element in the Lie-algebra. Then $D = -\frac{1}{r}\sigma_o$. The matter action, on the other hand, is manifestly positive definite, and has only the trivial zero-mode where all fields vanish.

To summarize, the BPS configurations on S^3 are labeled by an element σ_o in the Lie algebra, and are given by

$$A_\mu = 0, \quad \sigma = -rD = \sigma_o, \tag{3.21}$$

with all other fields vanishing.

3.3.2 $S^2 \times S^1$

Let us first consider $S^2 \times \mathbb{R}$. Then, since this is a product of manifolds, we can decompose the spinors into tensor products of those on S^2 and those on $\mathbb{R}$. In the former space, we have seen above that there are Killing spinors satisfying, in this case,

$$\nabla^{S^2}_\mu \epsilon_\pm = \pm \frac{i}{2r}\gamma_\mu \epsilon,$$

where r is now the radius of S^2. Here ϵ is a two complex component Dirac spinor, and the chirality operator, which can be taken as γ_3, exchanges the solutions for the two signs in this equation. Then to construct conformal Killings on $S^2 \times \mathbb{R}$, we try a solution of the form:

$$\epsilon = A(t)\epsilon_+ + B(t)\gamma_3\epsilon_+$$

Then, for $\mu = 1, 2$,

$$\nabla_\mu \epsilon = \frac{i}{2r}\gamma_\mu \bigg(A(t)\epsilon_+ - B(t)\gamma_3\epsilon_+ \bigg),$$

implying

$$\epsilon' = \frac{i}{2r}\bigg(A(t)\epsilon_+ - B(t)\gamma_3\epsilon_+ \bigg).$$

For $\mu = 3$, we get

$$\nabla_3 \epsilon = \dot{A}(t)\epsilon_+ + \dot{B}(t)\gamma_3\epsilon_+.$$

Setting this equal to $\gamma_3\epsilon'$, we find the conditions

$$\dot{A}(t) = -\frac{i}{2r}B(t), \quad \dot{B}(t) = \frac{i}{2r}A(t),$$

with general solution

$$A(t) = \alpha e^{t/(2r)} + \beta e^{-t/(2r)}, \quad B(t) = i\alpha e^{t/(2r)} - i\beta e^{-t/(2r)}$$

Let us take the solution corresponding to $\beta = 0$:

$$\epsilon = e^{t/(2r)}(1 + i\gamma_3)\epsilon_+ \tag{3.22}$$

We see that

$$\epsilon' = \frac{1}{2r}\gamma_3\epsilon, \tag{3.23}$$

so that $a = 0$ and $b_\mu = \frac{1}{2r}u_\mu$, where u_μ is the covariantly constant unit vector field along $\mathbb{R}$. As with S^3, there are two solutions corresponding to this choice of a and b_μ, and a further two choices with the opposite sign.

Now let us return to $S^2 \times S^1$. At this point there appears to be a slight problem, because our conformal Killing spinor is not single valued for any choice of the radius of S^1. This is not just a problem with our particular choice of spinor, it is true of all the conformal Killing spinors on this space. In other words, the equation can be locally solved, but the solutions do not extend globally. Since the field variations are proportional to ϵ, this will mean the fields are not globally well-defined either.

However, we can get around this problem if we are willing to broaden slightly our interpretation of the partition function. Namely, recall that we define the Euclidean partition function on $\Sigma \times S^1$ with periodic boundary conditions for the spinors with the trace:

$$\mathrm{Tr}(-1)^F e^{\beta H}$$

We have already inserted an operator to modify the boundary conditions on the spinors as one goes around the S^1, so it seems not too much of a stretch to go further and allow fields which return to themselves only up to a prescribed factor as we go around the S^1. We will expand on this interpretation in the next chapter, and see this condition is also imposed on us directly by the superconformal algebra.

For now, let us proceed to write down the actions we will be considering. For the matter fields, we have (using $R = \frac{2}{r^2}$ on $S^2 \times S^1$):

$$S_m = \int \sqrt{g} d^3x \Big(- \phi^\dagger D_\mu D^\mu \phi + \phi^\dagger(\sigma^2 + iD + \frac{1}{4r^2})\phi + i\psi^\dagger(\gamma^\mu D_\mu - \sigma)\psi + F^\dagger F \Big) \tag{3.24}$$

and for the gauge multiplet we have (using $a = 0, b_\mu = u_\mu$):

$$S_g = \int \sqrt{g} d^3x \Big((-\frac{1}{2}\sqrt{g}^{-1}\epsilon^{\mu\nu\rho}F_{\nu\rho} + D^\mu\sigma + \frac{1}{2r}u_\mu\sigma)^2 + D^2 + \lambda\Big(- i\gamma^\mu D_\mu \lambda^\dagger - i[\lambda^\dagger, \sigma]\Big)\Big) \quad (3.25)$$

The BPS equations for the gauge multiplet here are given by $D = 0$ and:

$$0 = -\frac{1}{2}\sqrt{g}^{-1}\epsilon^{\mu\nu\rho}F_{\nu\rho} + D^\mu\sigma + \frac{1}{2r}u_\mu\sigma \quad (3.26)$$

One solution is to take $F_{\mu\nu}$ vanishing, which does not mean the gauge field is trivial here because this is not a simply connected space, and so the solutions are parametrized by the holonomy of the flat connection A_μ around the S^1. Another solution is given by taking A_μ to be in a Dirac monopole configuration on S^2, with magnetic flux:

$$\sqrt{g}\epsilon^{\mu\nu\rho}F_{\nu\rho} = su^\mu \quad (3.27)$$

for an integer s. This is made BPS by taking $\sigma = -2rs$. The most general solution to the BPS equation is a combination of these two, and is labeled by the holonomy eigenvalues α_i and monopole numbers s_i.

In the next chapter we will compute the partition functions for these field theories in the semi-classical limit, which we will argue corresponds to the exact result even for interacting theories.

Chapter 4

Localization

4.1 Background

Localization is a phenomenon that occurs in supersymmetric theories, whereby certain quantities can be computed exactly because they receive contributions only from a very small subset of the states and/or field configurations.

Let us introduce the general idea with the minimal possible number of ingredients. We take a Hilbert space $\mathcal{V}$, on which we pick mutually commuting Hermitian operators Q and G_a, and define:[1]

$$H = Q^2$$

Note then that $[H, G_a] = [H, Q] = 0$. The final condition we impose is that there is a $\mathbb{Z}_2$ grading on the space, such that Q has degree 1, and all the other operators have degree zero. We define a Hermitian operator $(-1)^F$ which take values 1 on degree zero states and -1 on degree 1 states.

Now consider the trace:

$$Z(\beta, \gamma_a) = \mathrm{Tr}((-1)^F e^{-\beta H + \gamma_a G_a}) \tag{4.1}$$

We claim this is β-independent. There are two ways of seeing this, which will naturally lead us to the two perspectives on the quantum-field-theoretic version of this quantity that we will see in this chapter:

- Let us decompose the Hilbert space into simultaneous eigenstates of H, G_a, and $(-1)^F$. First consider an eigenstate with nonzero eigenvalue E of H. Then since $<\psi|H|\psi>=<\psi|Q^2|\psi>= ||Q|\psi>||^2$, $Q|\psi>$ must be nonzero. But since Q commutes with H and G_a, $Q|\psi>$ is another

[1] Note that if we have two Hermitian supersymmetries Q_1, Q_2 which both square to H and anticommute with each other, we can define a single complex supersymmetry $Q = Q_1 + iQ_2$ which satisfies $Q^2 = 0$ and:

$$\{Q, Q^\dagger\} = 2H$$

This will be more closely analogous to the case of $\mathcal{N} = 2$ supersymmetry in three dimensions that we consider below.

eigenstate of opposite degree, and these two states furnish a two-dimensional representation of the supersymmetry algebra, whose contribution drops out because the states enter with opposite signs. Thus the nonzero eigenspaces do not contribute, and so we can only get contributions from the kernel of H, so that β drops out.

- Let us formally take a derivative of this expression with respect to β. Then we find:

$$
\begin{aligned}
-\frac{\partial}{\partial\beta} Z(\beta,\gamma_a) &= \mathrm{Tr}(He^{-\beta H+\gamma_a G_a}(-1)^F) \\
&= \mathrm{Tr}(Q^2(-1)^F e^{-\beta H+\gamma_a G_a})
\end{aligned}
$$

  But since Q commutes with H and G_a, and anticommutes with $(-1)^F$, we allows us to bring one of the factors of Q to the end of the trace at the cost of a sign, and then back to the beginning by the cyclicity of the trace, and so the expression is equal to its negative and so must vanish.

These arguments are admittedly quite heuristic, and may break down on infinite-dimensional spaces, where the spectrum of H may be continuous, and the various traces may not be well-defined. Nevertheless, it gives a flavor of the field theory arguments we will outline below.

More generally, the independence of this trace on a more general class of sufficiently mild deformations of H can be argued. We have seen that the only states that contribute to the index are those at $H = 0$. Any states away from $H = 0$ must come in pairs which cancel out of the index. Thus if H is varied in such a way that new states enter or leave the $H = 0$ eigenspace, they must do so in such pairs, and so the index will not change. Again, this relies on the discreteness of the spectrum of H, and is more subtle (and in many cases, incorrect) in the presence of a continuous spectrum.

The independence of this expression on β can be very useful. For example, there are many examples of the above setup where the $\beta \to 0$ and $\beta \to \infty$ limits represent interesting and/or computable quantities, so that we may derive nontrivial identities and/or perform nontrivial computations. We will see an example of this in the next section.

In the physical contexts where such setups arise, H will correspond to the Hamiltonian of a quantum mechanical system, the G_a are some set of global symmetries, and Q is a supersymmetry. The trace above may be computed on the Hilbert space, using constraints from the representation theory of the superconformal algebra, or it may be translated into a Euclidean path integral. We will explore both methods.

4.1.1 Example: SUSY Quantum Mechanics

We now provide a simple example of the setup above which will demonstrate how the various ingredients arise in a simple system, and allow us to recover nontrivial mathematical results [13].

We will consider the quantum mechanics of a sigma model with target space a Riemannian manifold M with metric $g_{\mu\nu}$. This is defined by the following Lagrangian:

$$L = \frac{1}{2} g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu + \frac{i}{2} g_{\mu\nu}(x)\psi^{\mu\dagger} D_t \psi^\nu + \frac{1}{12} R_{\mu\nu\rho\sigma}\psi^{\mu\dagger}\psi^\nu\psi^{\rho\dagger}\psi^\sigma \tag{4.2}$$

Here x^μ are coordinates on M, which we will take as our degrees of freedom, along with their fermionic partners ψ^μ. $R_{\mu\nu\rho\sigma}$ is the Riemann curvature tensor, which, along with the metric $g_{\mu\nu}$, is a function of the x^μ. Also, we define $D_t\psi^\mu = \dot{\psi}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu\psi^\rho$.

This theory is invariant under the following supersymmetry transformation:

$$\begin{aligned} \delta x^\mu &= \epsilon\psi^\mu \\ \delta\psi^\mu &= -i\gamma^0 x^\mu \epsilon - \Gamma^\mu_{\nu\rho}\psi^\mu\psi^\nu \end{aligned} \tag{4.3}$$

The Hilbert space for the bosons is some suitable space of functions on M. Specifically, we can represent the fermion ψ^μ as a 1-form dx^μ on M, and then the entire Hilbert space can be represented as the exterior algebra $\Lambda^*(M)$. Then the Hamiltonian is represented by the Laplacian, and the supersymmetry operator is associated to $d + d^\dagger$, which indeed squares to the Hamiltonian. Note it also flips the value of $(-1)^F$ of a state if we associate F with the degree of the form.

Now let us see how localization can be applied to this system. In the Hilbert space picture, we are considering the quantity:

$$\mathrm{Tr}((-1)^F e^{-\beta H})$$

For β very large, the only states that contribute are those annihilated by $d + d^\dagger$, which are precisely the harmonic forms. The number of such linearly independent forms of degree p counts the rank of the cohomology group $H^p(M)$, and one has the result:

$$\lim_{\beta\to\infty} \mathrm{Tr}((-1)^F e^{-\beta H})\chi(M) = \sum_p (-1)^p \mathrm{dim} H^p(M) \tag{4.4}$$

where $\chi(M)$ is the Euler characteristic of the space. Note that this is independent of the metric on M, providing an example of our comment above that the partition function is insensitive to a large class of deformations of the system.

On the other hand, we can compute this quantity at very small β as well. This corresponds to the path integral on a very small circle, and we see that the contribution of nonconstant paths is highly suppressed. Thus we find the result is given by the 1-loop approximation around the saddle points of constant x^μ and ψ^μ. With some work, one can show that this is given precisely by:

$$\int_M Pf(R)$$

where R is the Riemann curvature tensor, and Pf is the Pfaffian. The equality of this expression with the Euler characteristic of the space is known as the Gauss-Bonnet theorem. One can generalize this in several directions to reproduce other well-known index theorems involving characteristic classes.

4.2 Localization on S^4

A much more sophisticated application of the localization was the recent work of Pestun [3] on the localization of $\mathcal{N} = 2$ supersymmetric gauge theories on S^4. The main idea is very similar to the one we will use in three dimensions, so we will not review in in detail now, instead just pointing out some important differences between the two calculations.

In the original setup, one considers either with $\mathcal{N} = 2, 2^*$, or 4 super Yang-Mills theory, the former two being deformations of the latter by adding infinite and finite mass, respectively, for the adjoint chiral hypermultiplet. Note that the least amount of supersymmetry one considers involves 8 supercharges, which is twice what we will need in three dimensions. The latter theory is superconformal, and so can be conformally mapped to the four-sphere. Then we construct an off-shell representation of at least one superconformal symmetry, which squares to a bosonic symmetry. By the general arguments, which we will review in more detail below, one can add a positive definite term which is a total variation under this symmetry, and the action is unchanged by its inclusion. Thus we scale the coefficient to be very large, where the calculations become tractable.

The calculation is technically involved because of the large number of fields in the supersymmetry complex, the fact that the supercharge squares to a non-zero bosonic symmetry, and the existence of point-like instanton zero-modes which must be dealt with using Nekrasov's instanton partition function. However, in the case of $\mathcal{N} = 4$ super Yang Mills theory, one finds a very simple result: the partition function is computed by a Gaussian matrix model, and Wilson loop insertions are computed by corresponding insertions of characters into the matrix integral. We will find a similar story in three dimensions.

4.3 Localization of $\mathcal{N} = 2$ Theories in Three Dimensions

In three dimensions, the localization is somewhat technically simpler than in four dimensions for a number of reasons. First of all, our supersymmetries square to zero, as opposed to a nontrivial bosonic symmetry. This means that any action which is a total δ-variation is automatically supersymmetric, as we have seen above. Moreover, at least on S^3, there are no instanton corrections. On $S^2 \times S^1$ there is an infinite discrete sum over monopole number, but this is still conceptually and calculationally simpler than the point-like instantons that had to be dealt with on S^4. Finally, as we will see below, the final answer we obtain, at least for theories with at least $\mathcal{N} = 3$ supersymmetry, can be expressed as integrals of hyperbolic functions, and as such are quite a bit easier to manipulate. Indeed, we will see in the second half of the thesis that many of the dualities reduce, at the level of the localized partition functions, to elementary identities of these functions.

To start, let us review the basic idea of localization in more detail. We start with a theory of the general type discussed in Chapter 2. Specifically, we will be initially interested theories which are explicitly superconformal at a quantum level. Then, as discussed in Chapter 3, we may study the partition function of these theories on an arbitrary conformally flat manifold. The flat space action uniquely determines the action on these manifolds.

We have also seen that these actions typically contain D-term-like pieces which can be written as the total variation of some functional V of the fields under a superconformal symmetry δ. Let us consider the dependence of the partition function on the coefficient of such a term. We write, schematically,

$$Z(t) = \int \mathcal{D}\Phi e^{-S[\Phi]+t\delta V}.$$

Then we compute:

$$\frac{d}{dt} Z(t) = \int \mathcal{D}\Phi e^{-S[\Phi]+t\delta V} \delta V$$

Thus the derivative can be seen to be formally equal to the expectation value of some δ-exact operator. But it is well known from the theory of chiral rings that such expectation values vanish. Indeed, since the rest of the action is δ-invariant, we can write:

$$< \delta V >= \int \mathcal{D}\Phi e^{-S[\Phi]+t\delta V} \delta V = \int \mathcal{D}\Phi \delta(e^{-S[\Phi]+t\delta V} V)$$

This is a total variation of some symmetry, and should vanish. Put another way, if we define a change of variables $\Phi \to \Phi + \delta\Phi$, then this term represents the corresponding change in the path integral, which should be zero if the symmetry is respected by the measure, as we will assume here. In other words, we see that the partition function is completely insensitive to the coefficient t on these δ-exact terms. This can be seen as analogous to the independence on β we saw in the last section.

The key use of localization will follow from the observation that, since these terms are positive semidefinite, in the limit $t \to \infty$, the path integral only picks up contributions from the region infinitesimally close to the zero-locus of the terms. This is where the term "localization" comes from. For each classical field configuration for which these terms vanish, the semiclassical, or one-loop, approximation in this background will be exact, and the integral of this result over all such zero-modes will give the exact result for the path integral.

4.3.1 Chern-Simons Theory

We start with the case of supersymmetric Chern-Simons theory, without any matter. As discussed above, the fields in the vector multiplet besides the gauge field are all auxiliary and can trivially be integrated out, and we expect this theory to be equivalent to bosonic Chern-Simons theory, which is topological. This theory is well studied, and in fact a matrix model computing its partition function is already known from studying topological strings [14, 15]. We will see that this matrix model is recovered from the different point of view of localization. We will see how matter is added in the next section.

Recall the action for supersymmetric Chern-Simons theory is given, on any manifold, by:

$$\int d^3x\sqrt{g}\,\mathrm{Tr}\bigg(\epsilon^{\mu\nu\rho}(A_\mu\partial_\nu A_\rho+\frac{2i}{3}A_\mu A_\nu A_\rho)+2D\sigma-\lambda^\dagger\lambda\bigg) \tag{4.5}$$

As discussed in the previous chapter, this action has an infinite-dimensional super-algebra of symmetries δ_ϵ parametrized by a spinor field ϵ, but we will specialize to the finite subset of those with ϵ a conformal Killing. For the remainder of this section, let us pick a spinor ϵ and the corresponding superconformal transformation, which we will denote simply by δ.

On a conformally flat manifold, we can add the δ-exact Yang-Mills term in 3.14:

$$S^g_\delta=\int\sqrt{g}d^3x\bigg((-\frac{1}{2}\sqrt{g}^{-1}\epsilon^{\mu\nu\rho}F_{\nu\rho}+D^\mu\sigma+b_\mu\sigma)^2+(D+a\sigma)^2+\lambda\Big(-i\gamma^\mu D_\mu\lambda^\dagger-i[\lambda^\dagger,\sigma]-a\lambda^\dagger\Big)\bigg) \tag{4.6}$$

Thus the total action for our system is

$$S_{C.S.}+tS^g_\delta \tag{4.7}$$

However, as argued above, the partition function will be independent of the value of t, and in particular, the result at $t=0$ corresponding to the pure Chern-Simons theory can be computed by taking t to be very large.

When t is large, the factor e^{-S} in the path integral will be strongly suppressed at all points where S^g_δ does not vanish. Since it is written as a sum of squares, the condition for it to vanish is read off as:

$$
\begin{aligned}
0 &= -\frac{1}{2}\sqrt{g}^{-1}\epsilon^{\mu\nu\rho}F_{\nu\rho} + D^{\mu}\sigma + b_{\mu}\sigma \\
0 &= D + a\sigma
\end{aligned} \tag{4.8}
$$

To explicitly solve these equations, we will need to choose our manifold and spinor ϵ, which we will do in the next section. However, let us first discuss how the rest of the argument runs.

Suppose we find a zero-mode for the gauge multiplet, which we can write in superfield notation as V_o. Let us then consider the integral in the neighborhood of this point in field space, which we can parametrize by:

$$
V = V_o + \frac{1}{\sqrt{t}}V' \tag{4.9}
$$

Here V' represents fluctuations of the gauge multiplet fields, and we should be careful not to include zero-mode fluctuations, since these will be accounted for when we integrate over the zero-modes in a moment. Plugging this form into the action and expanding in V', we find (here one should not confuse the functional derivatives with the supersymmetry transformation δ):

$$
\begin{aligned}
S_{C.S.}[V] &= S_{C.S}[V_o] + \frac{1}{\sqrt{t}}\frac{\delta S_{C.S.}}{\delta V}[V_o]V' + ... \\
S^g_\delta[V] &= tS^g_\delta[V] + \sqrt{t}\frac{\delta S^g_\delta[V_o]}{\delta V}V' + \frac{\delta^2 S^g_\delta[V_o]}{\delta V^2}V'^2 + ...
\end{aligned} \tag{4.10}
$$

However, since V_o is a zero-mode, as well as a saddle point, the first two terms in the second line vanish. The third can be thought of as the expansion of S^g_δ to quadratic order about the zero-mode V_o, and we will denote it by $S^{(2)}_{V_o}[V']$. The omitted terms are of order $t^{-1/2}$ or higher, and so can be ignored for sufficiently large t. Thus we are left with the quadratic action:

$$
S = S_{C.S.}[V_o] + S^{(2)}_{V_o}[V'] + ... \tag{4.11}
$$

We will refer to the first term as the classical contribution. It will contribute an overall factor to the partition function. The second term must be integrated over fluctuations V', which will give rise to a contribution we call the 1-loop determinant, since it is computed in the semiclassical or 1-loop approximation.

For each zero-mode, we can compute the partition function of this free theory defined by $S^{(2)}_{V_o}[V']$ straightforwardly. One then merely has to sum this contribution over all zero-modes, and we arrive at the leading approximation to the partition function in the large t limit. However, as discussed above, this leading approximation is in fact the exact answer for all t.

The careful reader may be worried about the infinite-dimensional gauge symmetry which must somehow be fixed. Our approach will be to first localize and expand to quadratic order, and then gauge-fix the resulting free action. As explained in [16], this can be justified by adding ghosts and forming an extended complex which involving $\delta + \delta_{BRST}$, where δ_{BRST} is the BRST symmetry.

Let us now turn to the specific case of S^3 and see how this procedure works in detail.

4.3.2 S^3

The explicit form of S^g_δ on S^3 is given by:

$$S^g_\delta = \int \sqrt{g} d^3x \bigg((-\frac{1}{2}\sqrt{g}^{-1}\epsilon^{\mu\nu\rho}F_{\nu\rho} + D^\mu\sigma)^2 + (D + \frac{1}{r}\sigma)^2 + \lambda\Big(- i\gamma^\mu D_\mu \lambda^\dagger - i[\lambda^\dagger, \sigma] - \frac{1}{r}\lambda^\dagger \Big)\bigg)$$

This can be integrated by parts to put in a slightly more standard form:

$$S^g_\delta = \int \sqrt{g} d^3x \bigg(\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + D_\mu\sigma D^\mu\sigma + (D + \frac{1}{r}\sigma)^2 + i\lambda^\dagger(\gamma^\mu D_\mu + i\mathrm{Ad}(\sigma) - \frac{1}{r})\lambda \bigg) \tag{4.12}$$

The zero-modes are labeled by an element $\sigma_o \in \mathfrak{g}$:

$$A_\mu = 0, \quad \sigma = -rD = \sigma_o \tag{4.13}$$

Let us now focus on the contribution from a specific σ_o. The classical contribution comes from the $2D\sigma$ term in $S_{C.S.}$:

$$S_{C.S.}[V_o] = \int \sqrt{g} d^3x \mathrm{Tr} 2D\sigma = -\frac{2}{r}\mathrm{vol}(S^3)\mathrm{Tr}{\sigma_o}^2 \tag{4.14}$$

The volume of S^3 is $2\pi^2 r^3$, and we will mostly be interested in the case of $U(N)$ theories, where the trace must be normalized as $\frac{k}{4\pi}$ times the trace in the fundamental representation, which we will denote tr. Then this becomes:

$$S_{C.S.}[V_o] = -i\pi k r^2 \mathrm{tr}{\sigma_o}^2 \tag{4.15}$$

Next we compute the 1-loop determinant. First note that we can use the residual gauge symmetry to ensure that σ_o is in the Cartan of the Lie algebra. Then the integral over the Lie algebra is replaced by:

$$Z = \int d\sigma_o Z_{cl}[\sigma_o] Z^g_{1-loop}[\sigma_o] \to \frac{1}{|\mathcal{W}|} \int d^r\lambda \left| \prod_{\alpha \in Ad(G)} \alpha(\sigma_o) \right| Z_{cl}[\mathrm{diag}(\lambda_i)] Z_{1-loop}[\mathrm{diag}(\lambda_i)] \quad (4.16)$$

where the we have picked up a factor of the Vandermonde determinant $\left|\prod_{\alpha \in Ad(G)} \alpha(\sigma_o)\right|$ from the change of variables, where α runs over the roots of the Lie algebra. In addition, we have accounted for the residual Weyl symmetry by dividing by the order of the Weyl group, $|\mathcal{W}|$. Thus in computing the 1-loop determinant, we may assume that σ_o lies in the Cartan and then use this expression for the partition function.

We will need to expand the action to quadratic order around σ_o. We find:

$$S^{(2)}_{V_o}[V'] = \int \sqrt{g} d^3x \left(\frac{1}{2} F^A{}_{\mu\nu} F^{A\mu\nu} + \partial_\mu \sigma' \partial^\mu \sigma' - [A_\mu, \sigma_o]^2 + (D' + \frac{1}{r}\sigma')^2 + i\lambda^\dagger(\gamma^\mu \nabla_\mu + i\mathrm{Ad}(\sigma_o) - \frac{1}{r})\lambda \right)$$

Here σ' and D' denote the fields with the zero-modes removed, since we do not want to integrate over them just yet. Also, F^A denotes the "abelian" field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, which we use even for nonabelian field because we are only working to quadratic order.

At this point we need to gauge-fix. We will add ghosts $c, \bar{c}$, and b, and add the gauge-fixing action:

$$S_{g.f} = \int \sqrt{g} d^3x \left(\partial_\mu \bar{c} D^\mu c + b \nabla^\mu A_\mu \right)$$

Let us solve the gauge-fixing condition by writing $A_\mu = B_\mu + \partial_\mu \phi$, where B_μ is transverse, i.e., $\nabla^\mu B_\mu = 0$. Then integrating over b imposes $\nabla^2 \phi = 0$. In addition, since the A_μ is suppressed by $\frac{1}{\sqrt{t}}$, we can replace the covariant derivative on c by an ordinary one. As a result, the contributions from c, ϕ, and σ give the determinant of ∇^2 on the sphere to various powers depending on the field statistics, and one can check that they all cancel. The integral over D is trivial as it is auxiliary. What remains is simply

$$S^{(2)}_{V_o}[V'] = \int \sqrt{g} d^3x \left(B_\mu ({\nabla_V}^2 + \mathrm{Ad}({\sigma_o}^2)) B_\mu + i\lambda^\dagger(\gamma^\mu \nabla_\mu + i\mathrm{Ad}(\sigma_o) - \frac{1}{r})\lambda \right), \quad (4.17)$$

where ${\nabla_V}^2$ is the vector Laplacian.

Note that the Lie algebra is spanned by a basis e_i of the Cartan, where $i = 1, ..., r$, up to the rank r of the group, together with the root vectors X_α, where α runs over the roots of the Lie algebra. It is convenient to combine these to define a basis e_a of the Lie algebra, where $a = 1, ..., \dim(G)$.[2] Then we can write

[2]More precisely, they form a complex basis for the complexification $\mathfrak{g}^{\mathbb{C}}$

$$B_\mu = \sum_a B^a{}_\mu e_\alpha,$$

and then, since σ_o is in the Cartan, we find

$$\mathrm{Ad}(\sigma_o) B_\mu = \sum_\alpha B^a{}_\mu \rho_a(\sigma_o) e_\alpha,$$

where $\rho_a(\sigma_o)$ is the eigenvalue of σ_o acting on e_a, i.e., zero for the components along the Lie algebra and $\alpha(\sigma_o)$ for the root vectors. Doing the same thing for λ, and assuming the e_a are normalized so $\mathrm{Tr}(e_a e_b) = \delta_{a+b}$, we can rewrite the action in terms of ordinary (as opposed to Lie algebra-valued) fields:

$$S^{(2)}_{V_o}[V'] = \sum_\alpha \int \sqrt{g} d^3x \Big(B^a{}_\mu ({\nabla_V}^2 + \rho_a(\sigma_o)^2) B^{-a}{}_\mu + i\lambda^{a\dagger}(\gamma^\mu \nabla_\mu + i\rho_a(\sigma_o) - \frac{1}{r})\lambda^{-a} \Big) \tag{4.18}$$

To evaluate this, we use the fact that the vector Laplacian acting on transverse vectors has eigenvalues $\frac{1}{r^2}(\ell+1)^2$ with degeneracy $2\ell(\ell+2)$, and the Dirac operator has eigenvalues $\pm\frac{1}{r}(\ell+\frac{1}{2})$ with degeneracy $\ell(\ell+1)$ for each sign, where in both cases ℓ runs over the positive integers. Let us focus on a single term in the sum over a. Then we can write:

$$Z^\alpha_{1-loop}[\sigma_o] = \frac{((\ell+\frac{1}{2}+i\alpha(\sigma_o)-\frac{1}{2})(-\ell-\frac{1}{2}+i\alpha(\sigma_o)-\frac{1}{2}))^{\ell(\ell+1)}}{((\ell+1)^2+\alpha(\sigma_o)^2))} \tag{4.19}$$

After some cancellation, and using the fact that the eigenvalues $\alpha(\sigma_o)$ will come in positive-negative pairs, we are left with:

$$\begin{aligned} Z_{1-loop}[V_o] &= \prod_\alpha \prod_{\ell=1}^\infty (\ell^2 + {\sigma_o}^2)) \\ &= \prod_{\alpha \in Ad(G)} \Big(\Big(\prod_{\ell=1}^\infty \ell^4 \Big) \prod_{\ell=1}^\infty (1 + \frac{{\sigma_o}^2}{\ell^2}) \Big) \end{aligned} \tag{4.20}$$

The first factor can be regularized using zeta functions to give 2 [17], and the second is the infinite product representation for the hyperbolic sinh function. Putting this together, we find

$$Z_{1-loop}[V_o] = 2^r \prod_{\alpha \in \mathrm{Ad}} \frac{2 \sinh \pi\alpha(\sigma_o)}{\alpha(\sigma_o)} \tag{4.21}$$

where the product now runs only over the roots in the Lie algebra, and we have extracted an overall factor of 2^r, where r is the rank of the gauge group, to account for the factors from the Cartan.

The last step is to combine this with the classical contribution, and integrate over the zero-modes. Then we see the denominator of the 1-loop determinant cancels (up to a sign) against the Vandermonde determinant, and we're left with:

$$Z = \frac{2^r}{|W|}\int d^r\lambda \exp\Big(-\frac{4\pi^2 i}{r}\mathrm{Tr}{\sigma_o}^2\Big)\Big|\prod_{\alpha\in\mathrm{Ad}} 2\sinh\pi\alpha(\lambda)\Big| \tag{4.22}$$

This gives the large t-approximation to the partition function, which, by the arguments above, is actually exact for all t.

As an example, for a theory with $U(N)$ gauge group, which will be the case we mainly consider in this thesis, we write:

$$\sigma_o = \mathrm{diag}(\lambda_1, ..., \lambda_N)$$

Then the roots are α_{ij}, $i \neq j$, with

$$\alpha_{ij}(\sigma_o) = \lambda_i - \lambda_j,$$

and the Weyl group is S_N, with order $N!$. Finally, the Chern-Simons term 4.15 will take the form:

$$-kpii\sum_i {\lambda_i}^2$$

Putting this together, we find the $U(N)$ Chern-Simons partition function is given by:

$$Z = \frac{2^n}{N!}\int d^N\lambda e^{-k\pi i\sum_i {\lambda_i}^2}\prod_{i<j}(2\sinh\pi(\lambda_i-\lambda_j))^2$$

This agrees with the result for bosonic Chern-Simons theory found in [14, 15].

4.3.3 Wilson Loop

The story above is only minimally modified if, instead of the partition function, one wishes to compute the expectation value of one of the supersymmetric Wilson loops discussed in the last chapter. It does not have any effect on the set of zero-modes of S^g_δ, neither does it contribute to the 1-loop determinant. The only contribution is through the classical contribution. Indeed, recall the operator is given by

$$\mathrm{Tr}\mathcal{P}\mathrm{exp}\Big(\oint_\gamma (A_\mu dx^\mu + \sigma d|x|)\Big),$$

and that, to be supersymmetric, this must run over a great circle on S^3. Since $A_\mu = 0$ and σ is constant, the integral over this curve in the zero-mode background is given simply by:

$$\mathrm{Tr} e^{2\pi\sigma_o} \tag{4.23}$$

Thus the only modification to the matrix integral above is the insertion of such a factor. For example, in the $U(N)$ case, for a fundamental Wilson loop, this gives an insertion of:

$$\sum_i e^{2\pi\lambda_i} \tag{4.24}$$

4.4 Matter

In this section we consider how localization works for theories with matter. Although we will ultimately be interested in adding matter to a theory with a dynamical gauge field, our approach will be to first consider the matter fields in isolation, or more generally, coupled to a fixed, background supersymmetric gauge multiplet configuration, as discussed in Chapter 2. The justification for this is that one is always free to add the localizing term for the gauge sector first, after which the entire path integral will only be nonnegligible in the regions of field space where the gauge fields are in a supersymmetric configuration.

Thus consider a general matter action S_m on a conformally flat manifold M, and let us add the δ-exact matter term:

$$S_\delta^m = \int \sqrt{g} d^3x \bigg(-\phi^\dagger D_\mu D^\mu \phi + \phi^\dagger(\sigma^2 + iD + \frac{R}{8})\phi + i\psi^\dagger(\gamma^\mu D_\mu - \sigma)\psi + F^\dagger F \bigg) \tag{4.25}$$

Our logic will follow that of the gauge sector closely, so we proceed more quickly. We consider the system defined by the action:

$$S = S_m + t S_\delta^m \tag{4.26}$$

The path integral defining the partition function and expectation value of δ-closed observables is then independent of t, so we calculate it in the large t limit.

We will see below that the matter action has no nontrivial zero-mode solutions, so localization is even easier here than in the gauge sector. The only saddle point is at the origin of field space, and so we simply rescale the fields in the chiral multiplet by $X \to \frac{1}{\sqrt{t}} X$. Then S_m drops out completely – there is no classical contribution here – and one merely computes the 1-loop determinant in S_δ^m, which is actually already quadratic.

In the case where there is no coupling to a gauge multiplet, one finds a somewhat trivial answer: the partition function is just that of a free, massless, conformally coupled chiral multiplet, regardless of the original action S_m. This may seem odd, but recall that we are restricting to superconformal theories, and the only examples of such theories with only chiral multiplets are indeed free theories.

Thus it will be important that our matter theory is actually embedded in a gauge theory, and we will need to compute the 1-loop determinant as a function of the background field configuration.

Let us specialize again to the case of S^3.

4.4.1 S^3

From the previous section, we have seen that the supersymmetric configurations for the gauge multiplet on S^3 have only σ and D nonvanishing. Plugging this form into the matter action gives:

$$S_\delta^m = \int \sqrt{g} d^3x \bigg(\phi^\dagger (-\nabla^2 + {\sigma_o}^2 - \frac{i}{r}\sigma_o + \frac{3}{4})\phi + i\psi^\dagger(\gamma^\mu \nabla_\mu - \sigma_o)\psi + F^\dagger F \bigg) \tag{4.27}$$

Up to constant terms, the operators in this action are just the scalar Laplacian, with eigenvalues $\ell(\ell+2)$ with degeneracy $(\ell+1)^2$, and, as in the last section, the Dirac operator, with eigenvalues $\pm(\ell+\frac{1}{2})$ with degeneracy $\ell(\ell+1)$. Let us first consider the case where the gauge group is $U(1)$, so that σ_o is simply a real number. Then we can read off the 1-loop determinant as:

$$\begin{aligned} Z_{1-loop}[\sigma_o] &= \prod_{\ell=1}^{\infty} \frac{(\ell+\frac{1}{2}-i\sigma_o)^{\ell(\ell+1)}(-\ell-\frac{1}{2}-i\sigma_o)^{\ell(\ell+1)}}{(\ell(\ell+2)+{\sigma_o}^2-i\sigma_o+\frac{3}{4})^{(\ell+1)^2}} \\ &= \prod_{\ell=1}^{\infty} \frac{(\ell+\frac{1}{2}-i\sigma_o)^{\ell(\ell+1)}(-\ell-\frac{1}{2}-i\sigma_o)^{\ell(\ell+1)}}{((\ell+1+i\sigma_o+\frac{1}{2})(\ell+1-i\sigma_o-\frac{1}{2}))^{(\ell+1)^2}} \end{aligned} \tag{4.28}$$

After some cancellation, this becomes:

$$Z_{1-loop}[\sigma_o] = \prod_{\ell=1}^{\infty} \left(\frac{\ell+\frac{1}{2}+i\sigma_o}{\ell-\frac{1}{2}-i\sigma_o} \right)^{\ell}$$

Following Jafferis, we define the function $\ell(z)$ by regularizing the following product using Hurwitz zeta functions:

$$e^{\ell(z)} = \prod_{\ell=1}^{\infty} \left(\frac{\ell+z}{\ell-z} \right)^{\ell}$$

The function can be related to the hyperbolic gamma function and double sine function. One simple way to characterize it is by $\ell(0)=0$ and:

$$\frac{d\ell}{dz} = -\pi z \cot(\pi z)$$

Thus the result for a single chiral multiplet is:

$$Z^{m}_{1-loop}[\sigma_o] = e^{\ell(\frac{1}{2}+i\sigma_o)}$$

This is straightforwardly modified for a general representation to give [16]:

$$Z^{m}_{1-loop}[\sigma_o] = \prod_{\rho \in R} e^{\ell(\frac{1}{2}+i\rho(\sigma_o))} \tag{4.29}$$

Recall that in order to ensure the extended supersymmetry typically required for the chiral multiplets to have R-charge $\frac{1}{2}$, the chiral multiplets must come in pairs in conjugate representations, forming hypermultiplets. Then the contribution for a hypermultiplet is

$$Z^{hyp}_{1-loop}[\sigma_o] = \prod_{\rho \in R} e^{\ell(\frac{1}{2}+i\rho(\sigma_o))+\ell(\frac{1}{2}-i\rho(\sigma_o))},$$

which, using an identity of the $\ell(z)$ function, simplifies to:

$$Z^{hyp}_{1-loop}[\sigma_o] = \prod_{\rho \in R} \frac{1}{2\cosh(\pi\rho(\sigma_o))} \tag{4.30}$$

4.4.2 General Result for Gauge Theory on S^3

Let us now complete the last step and put the results of the last two sections together. We consider a theory with gauge group G (possibly containing several factors), and matter in a (possibly reducible) representation R. In addition, there is a Chern-Simons term with a trace we will denote Tr, absorbing into the Chern-Simons levels. This will have an action of the form:

$$S = S_g[V] + S_m[X, V] \tag{4.31}$$

which represents a gauge kinetic term (of both Yang-Mills and Chern-Simons form) and a matter term, which will include a gauge-coupled kinetic term and possibly a superpotential term.

As mentioned above, the idea is that we first localize the gauge sector by adding a term S^g_δ, and then compute the resulting 1-loop determinant. With this term in place with a large coefficient t, that we may assume (to leading order in t^{-1}) that the gauge field is in a supersymmetric configuration, and then around each zero-mode of the gauge multiplet, we localize the matter action and compute its 1-loop determinant. The result is:

$$Z = \frac{1}{|\mathcal{W}|} \int d^r\lambda \frac{\prod_{\alpha \in G} 2\sinh(\pi\alpha(\sigma_o))}{\prod_{\rho \in R} 2\cosh(\pi\rho(\sigma_o))} \tag{4.32}$$

We will see many applications of this formula in Part 2.

4.5 Background Gauging of Global Symmetries

Although we did not make this explicit, the above results apply only when the superconformal transformations of the matter fields are the ones without a central charge Z, and with no FI term for the gauge field. This brings up a very important idea that will give our duality tests much of their power.

Recall that we have computed the partition function of the matter theory coupled to a supersymmetric background gauge configuration. The original motivation for this is that, after localizing the gauge multiplet, we can assume that the dynamical gauge fields are essentially constrained to such a configuration.

However, there is another possibility. Recall that in flat space one could think of a real mass term for a chiral multiplet as a coupling to a background BPS gauge multiplet, namely, one with constant σ and all other fields vanishing. Here the symmetry we are "gauging" is a flavor symmetry, and the gauge fields are not dynamical, and the symmetry remains a global symmetry. This coupling adds new parameters by which one can deform the action.

We can do the same thing in curved space, and the localization computation is modified in a simple way. Namely, in computing the 1-loop matter determinant, we simply reinterpret the gauge multiplet as a nondynamical background multiplet, and the space of its zero-modes should not be integrated over, but rather is a space of deformations of the theory.

For example, on S^3, the zero-modes are parametrized by σ_o. Suppose we consider a $U(1)$ vector multiplet coupled to some flavor symmetry under which a pair of chiral multiplets X has charge q. Then for a given value of $\sigma_o := m$,

$$S = \int \sqrt{g} d^3x \bigg(\phi^\dagger(-\nabla^2 + q^2 m^2 - \frac{i}{r} mq + \frac{3}{4})\phi + i\psi^\dagger(\gamma^\mu \nabla_\mu - mq)\psi + F^\dagger F \bigg) \tag{4.33}$$

This action is invariant under δ for any choice of the parameter m, and so this defines a family of supersymmetric actions. We can see from the results above that the partition function of this action is given by:

$$Z(m) e^{\ell(\frac{1}{2} + iqm)} \tag{4.34}$$

Note that we are not interpreting m as the zero-mode of some dynamical gauge field, and so there is no reason to integrate over it. Instead, as the notation suggests, the partition function is a function of this parameter. One can think of it as a chemical potential for the flavor symmetry it couples to, analogous to the γ_a in 4.1.

In general, one can have a theory with both gauge and flavor symmetries, and then both types of parameters – those which are integrated over and those which parametrize deformations of the theory – will be present. For example, in Chapter 6 we will consider $\mathcal{N}=2$ $U(1)$ SQED with two chirals of charge ± 1. The partition function of this theory is:

$$Z=\int d\lambda e^{\ell(\frac{1}{2}+i\lambda)+\ell(\frac{1}{2}-i\lambda)}=\int d\lambda \frac{1}{2\cosh(\pi\lambda)} \tag{4.35}$$

There is a global symmetry $U(1)_A$ which rotates the two chirals by the same phase, and if we couple a background gauge multiplet with $\sigma_o=m$ to it, we find:

$$Z(m)=\int d\lambda e^{\ell(\frac{1}{2}+i\lambda+im)+\ell(\frac{1}{2}-i\lambda+im)} \tag{4.36}$$

Note that if we were to naively attempt to couple such a background multiplet to the gauged $U(1)_V$ symmetry which rotates the chirals by opposite phases, we would get:

$$Z(m,\hat{m})=\int d\lambda e^{\ell(\frac{1}{2}+i\lambda+im+i\hat{m})+\ell(\frac{1}{2}-i\lambda+im-i\hat{m})} \tag{4.37}$$

But this is independent of $\hat{m}$, as it can be absorbed into a shift of λ, so such an operation is not meaningful.

Finally, one can also couple to the $U(1)_J$ symmetry discussed in Chapter 2, which, in flat space, produces an FI term η. Since this is essentially an off-diagonal Chern-Simons term, we can see that it enters the partition function as a classical contribution:

$$e^{2\pi i\eta Tr(\sigma_o)} \tag{4.38}$$

In the case of SQED above, we find that the most general deformed partition function, with $U(1)_A$ deformation m and $U(1)_J$ deformation η, is:

$$Z(m,\eta)=\int d\lambda e^{2\pi i\eta\lambda} e^{\ell(\frac{1}{2}+i\lambda+im+i\hat{m})+\ell(\frac{1}{2}-i\lambda+im-i\hat{m})} \tag{4.39}$$

4.6 Localization on $S^2\times S^1$ and the Superconformal Index

Although this thesis will mainly focus on the S^3 partition function, much of the discussion of superconformal field theories and localization above has been for a more general manifold. This was partly in order to describe the potential generality of the approach, and to provide a starting point for future generalizations, but also so that we may mention another manifold which has been studied in detail: $S^2\times S^1$. In this case, there are two complementary interpretations of the localized partition function, one which agrees with our formalism above, and one which is based on the representation theory of the superconformal algebra. In this section will briefly review both interpretations.

4.6.1 Localization on $S^2 \times S^1$

In the previous chapter we wrote down the δ-exact actions for both the gauge and matter multiplets on $S^2 \times S^1$, so let us summarize them here. For the matter, we have

$$S_\delta^m = \int \sqrt{g} d^3x \Big(-\phi^\dagger D_\mu D^\mu \phi + \phi^\dagger (\sigma^2 + iD + \frac{1}{4r^2})\phi + i\psi^\dagger (\gamma^\mu D_\mu - \sigma)\psi + F^\dagger F \Big), \tag{4.40}$$

and for the gauge multiplet we have

$$S_\delta^g = \int \sqrt{g} d^3x \Big((-\frac{1}{2}\sqrt{g}^{-1}\epsilon^{\mu\nu\rho}F_{\nu\rho} + D^\mu \sigma + \frac{1}{2r}u_\mu \sigma)^2 + D^2 + \lambda \Big(-i\gamma^\mu D_\mu \lambda^\dagger - i[\lambda^\dagger, \sigma] \Big) \Big). \tag{4.41}$$

As on S^3, there are no matter zero-modes, but there are gauge zero-modes, and on $S^2 \times S^1$ they are slightly more complicated than on S^3. Namely, in the abelian case, we find [19, 20]:

$$A_\mu = \frac{h}{\beta r} u_\mu + 2sB_\mu, \quad \sigma = \frac{s}{r} \tag{4.42}$$

where h is the holonomy of the gauge field around the S^1, and s is an integer representing the magnetic flux through the S^2, with B_μ the Dirac monopole configuration along S^2. Then the sum over zero-modes will include an integral over a, somewhat analogous to σ_o in the S^3 case, as well as a discrete sum over s. There is a straightforward generalization to the nonabelian case.

First one must compute the classical contribution and 1-loop determinant of the gauge and matter multiplets in each of these zero-mode backgrounds. Just as for S^3, the only classical contribution comes from the Chern-Simons term, and is given by:

$$e^{S_o[V_o]} = e^{2i\mathrm{Tr}(hs)} \tag{4.43}$$

The 1-loop determinant is still represented by a Gaussian path-integral, although it is somewhat complicated by the monopole background. Nevertheless, one can diagonalize the operators appearing in the actions using monopole spherical harmonics. For the matter, one finds a 1-loop determinant of:

$$Z_{1-loop}^m = \prod_{\rho \in R} \left(x^{1/2} \prod_j e^{i\rho(h)} \prod_a {t_a}^{-f_a(\Phi)} \right)^{|\rho(s)|} \frac{(e^{-i\rho(h)} {t_a}^{-f_a(\Phi)} x^{2|\rho(s)|+3/2}; x^2)_\infty}{(e^{i\rho(h)} {t_a}^{f_a(\Phi)} x^{2|\rho(s)|+1/2}; x^2)_\infty} \tag{4.44}$$

where $(z;q)_n = \prod_{j=0}^{n-1}(1 - zq^j)$.

For the gauge multiplet, one finds a 1-loop determinant of:

$$Z_{gauge} = \prod_{\alpha \in ad(G)} x^{-|\alpha(s)|} \left(1 - e^{i\alpha(h)} x^{2|\alpha(s)|}\right) \tag{4.45}$$

Then the total index is given by multiplying the 1-loop factors and classical contributions and summing over the saddle points. Defining $z = e^{ih}$, we find:

$$I(t_a; x) = \sum_{s_j} \frac{1}{Sym} \int e^{2i\mathrm{Tr}(hs)} Z_{gauge}(z_j, s_j; x) \prod_{\Phi} Z_\Phi(z_j, s_j; x) \prod_j \frac{dz_j}{2\pi i z_j} \tag{4.46}$$

where Sym is a symmetrization factor for nonabelian groups, as discussed in [19, 20].

We also note that, as shown in [18] the procedure of assigning BPS values to background gauge fields and treating them as parameters, as discussed in the previous section, can also be applied to the partition function on $S^2 \times S^1$, leading to the so-called generalized superconformal index.

4.7 Superconformal Index

The quantity above was actually known before localization had been applied to any three-dimensional field theories [21]. In the approach used there, it was known as the superconformal index, and was found by studying representations of the superconformal algebra, which we now review.

Recall from Chapter 2 that the superconformal generator Q satisfies:

$$H := \{Q, Q^\dagger\} = \Delta - R - j_3 \tag{4.47}$$

By the general arguments at the start of this chapter, the index:

$$\mathrm{Tr}((-1)^F e^{\beta H} {t_i}^{G_i}) \tag{4.48}$$

will be independent of β for G_i which commute with Q and $Q^\dagger$. A convenient choice will be:

$$\mathrm{Tr}((-1)^F e^{\beta H} x^{\Delta + j_3} \prod_a {t_a}_a^F) \tag{4.49}$$

where F_a runs over flavor symmetries of the theory.

Now recall the index is insensitive the continuous deformations of the theory. Thus one may compute the index in the UV and then argue that it should be unchanged for the IR fixed point. For a three-dimensional theory which is free in the UV, such as Yang-Mills theory, the UV calculation can be done exactly.[3] Note that the computation in the IR fixed point involves a very different set of states than in the UV, but the particular combination of states that the index counts is unchanged.

[3] Alternatively, flowing to the UV corresponds to turning on the Yang-Mills term with a large coefficient, which is essentially what we have done above.

The computation in the UV is performed by first computing the so-called single-particle index, which counts the index of the BPS states associated to single-particle excitations of a given multiplet. Then, since the theory is free, the full Hilbert space is a Fock space built out of these single-particle states, and one computes the total contribution by taking the plethyistic exponential. Finally, in order to isolate gauge-invariant states, an integral over the parameter z is performed. In summary, we end up with precisely the same expression as 4.46. This is not surprising, since ultimately they both correspond to a computation in the same free field theory, although the argument for why this quantity is unchanged for the IR fixed point takes a somewhat different form in the two cases.

Part II

Applications

Chapter 5

ABJM Theory

In this chapter we will discuss an example of AdS/CFT duality involving a three-dimensional superconformal gauge theory with $\mathcal{N} = 6$ supersymmetry, known as the ABJM theory [9]. We will be able to use the results of the first part of this thesis to make exact calculations in the CFT, for all values of the effective coupling constant λ. These can be compared to the results from the string theory side, computed at weak coupling in the supergravity approximation,[1] to provide a nontrivial test of the duality. We will see that both the free energy and the expectation values of Wilson loops in the CFT have string theory duals which can be computed perturbatively, and we will find agreement.

5.1 AdS/CFT

To begin, let us give a very brief review of the general idea of AdS/CFT dualities [22, 23, 24]. The general principle goes back to an observation of 't Hooft that the large N limit of a gauge theory looks like a string theory. Specifically, suppose one computes vacuum diagrams in an ordinary $U(N)$ gauge theory, where we take N to be large. Then one can naturally arrange the expansion in powers of $\frac{1}{N}$ in terms of diagrams of different Euler characteristic, $\chi = V - E + F$. Planar diagrams have a contribution scaling as N^2, genus one diagrams as N^0, and so on, with each higher-genus being suppressed by an additional power of N^{-2}. Now compare this to string theory, where there is also a sum over Riemann surfaces, and the higher genus diagrams are suppressed by powers of the string coupling ${g_s}^2$. This suggests there is correspondence between string theories and gauge theories, with $g_s \sim \frac{1}{N}$, so that the large N limit corresponds to a weakly coupled string theory. Insertions of sources in the gauge theory would correspond to the insertion of vertex operators in the string theory.

This idea was made more precise after the discovery of D-branes in supersymmetric string theory. These are solitons in string theory on which open strings can end. Consider a stack of N D branes.

[1] There is no known way to compute these quantities exactly in the string theory, e.g., by something like localization, although this is an interesting area to pursue.

Then the effective loop expansion parameter for open strings on the brane is $g_s N$, and when this is small, and the D-brane description in terms of Dirichlet boundary conditions in open string perturbation theory is a good one. The effective field theory for the massless excitations of this system is a supersymmetric Yang-Mills theory. In the case of N $D3$ branes in flat space, one obtains maximally symmetric $\mathcal{N} = 4$ super Yang-Mills with gauge group $U(N)$, which is the most well-studied case.

On the other hand, it is known that D branes source Ramond-Ramond fields. In the supergravity approximation to string theory, applicable when both the string coupling is small and the radius of curvature of the background is much greater than the string length, these correspond to classical p-form gauge fields, and the configuration with N $D3$ branes has the form [25]:

$$ds^2 = h(r)^{-1/2}(-dt^2 + dx_1{}^2 + dx_2{}^2 + dx_3{}^2) + h(r)^{1/2}(dr^2 + r^2 d\Omega_5{}^2) \tag{5.1}$$

$$g_s F_5 = (1 + \star) d^4x \wedge dh^{-1}(r) \tag{5.2}$$

where $h(r) = 1 + \frac{(4\pi g_s N \alpha'^2)^4}{r^4}$ and $d\Omega_5{}^2$ is the metric on S^5. We see this approximation is a good one when $g_s N >> 1$, which is the opposite limit as the Yang-Mills description above.

Since they both arise as approximations to the same underlying theory, these two descriptions should be related in some way. Note that in the near-horizon limit, the metric above approaches that of $AdS_5 \times S^5$. On the other hand, an excitation sent into this near-horizon region will get red-shifted to infinitesimal energy. Thus we might expect this region to be well described by the low-energy theory of the branes, which is the $\mathcal{N} = 4$ SYM theory. This leads to the conjecture that type IIB string theory on a space that is asymptotically $AdS_5 \times S^5$ is equivalent to $\mathcal{N} = 4$ super Yang-Mills theory.

One would like to perform tests to check this duality. For example, the energy of BPS strings stretching between separated branes can be shown to equal the mass of W-bosons in the corresponding gauge theory vacuum on the Coulomb branch. In general, since the correspondence is between a strong coupling on one side and weak on the other, such protected quantities are among the few one can compare. We will see localization provides a new class of such quantities, some of which were tested using the S^4 localization of the $\mathcal{N} = 4$ theory in [3].

Another useful object to consider in this correspondence is a Wilson loop [27, 28]. This is a natural operator in the gauge theory, and as discussed above, can be written as:

$$\mathrm{Tr}_R \mathcal{P}\exp\left(\oint_\gamma A_\mu dx^\mu \right) \tag{5.3}$$

It is labeled by the representation R and loop γ. In the case of supersymmetric theories, one often inserts additional bosonic fields into the integral to ensure this operator preserves some fraction of the supersymmetry, as we saw in Chapter 3. In that case there may be additional parameters.

We will focus on the case where the representation is the fundamental. Then this operator can be thought of as computing the amplitude for a very heavy quark in the fundamental representation to go around the loop γ. On the string theory side, such a heavy quark can be represented by a single D brane which we have moved far away from the stack of N D branes, all the way to the AdS boundary. Then the expectation value of the Wilson loop corresponds, in the supergravity limit, to the classical action of a string ending on the loop γ at the AdS boundary. The classical equations of motion dictate that the string worldsheet should have extremal area, so one must find a minimal surface in AdS space bounding this loop. Typically the result will be divergent and must be regularized in some way.

There are many other examples of this type of duality, typically always between string theory on an asymptotically AdS_{d+1} space and a d-dimensional conformal field theory. In the remainder of this chapter, we will focus on an example with $d = 3$, where the certain quantities on the field theory side will be susceptible to the calculational methods we developed in Part 1.

5.2 ABJM Theory

To motivate the case of interest, we will focus on M theory, and consider a stack of N $M2$ in flat 11D spacetime. By compactifying one of the transverse dimensions we should get a stack of $D2$ branes in type IIA string theory, whose low-energy field theory is maximally supersymmetric $\mathcal{N} = 8$ Yang-Mills theory in three dimensions. Unlike the corresponding maximally symmetric SYM theory in four dimensions, this theory is not superconformal, and in fact, like all Yang-Mills theories in three dimensions, the coupling constant blows up in the infrared. Thus we expect it to have a nontrivial, strongly interacting IR fixed point, which should agree with the low-energy theory on the $M2$ branes.

This implicit description of the low-energy theory is not very useful, and for a long time an explicit Lagrangian description for this superconformal theory was sought. An important point is that such a description need not exist for a strongly coupled CFT, since there is no parameter choice or length scale at which the theory is weakly coupled and semiclassical. We do not expect tunable parameters because in M theory there is no dimensionless coupling constant like g_s. Nevertheless, in [6] it was proposed that a natural place to look for such a theory is among the Chern-Simons matter theories, which can be made superconformal by an appropriate choice of superpotential.

An initial candidate was found by Bagger, Lambert, and Gustaavson [29, 30], which had $\mathcal{N} = 8$ superconformal symmetry, but the gauge group was fixed to be $SU(2) \times SU(2)$, and so there was no parameter N to parametrize the number of $M2$ branes.[2] This was later found to be closely related to a more general construction of Aharony, Bergman, Jafferis, and Maldacena [9], now called the ABJM theory, which is believed to give the correct answer to this question.

The ABJM theory has gauge group $U(N) \times U(N)$, with the kinetic term a Chern-Simons term at level k and $-k$ for the two gauge group factors, whose $\mathcal{N} = 4$ vector multiplets we will denote (V, Φ) and $(\hat{V}, \hat{\Phi})$, each consisting of an $\mathcal{N} = 2$ vector multiplet and an adjoint chiral multiplet. In addition, there are two hypermultiplets in the fundamental representation of the gauge group, i.e., the tensor product of the fundamental of the first $U(N)$ factor and the antifundamental of the second. We will write them in terms of their chiral multiplets as $(A, \tilde{A})$ and $(B, \tilde{B})$. In addition, we include the superpotential required by $\mathcal{N} = 3$ supersymmetry, as discussed in Chapter 3:

$$W = -\frac{k}{8\pi}(\Phi^2 - \hat{\Phi}^2) + A(\Phi - \hat{\Phi})\tilde{A} + B(\Phi - \hat{\Phi})\tilde{B} \tag{5.4}$$

Let us first summarize a few basic properties of this theory. We can see from the general arguments of Chapter 3 that it has at least $\mathcal{N} = 3$ supersymmetry. However, note that after integrating out the auxiliary fields Φ and $\hat{\Phi}$, one can see the superpotential becomes:

$$W = \frac{4\pi}{k}(A\tilde{A}B\tilde{B} - A\tilde{B}B\tilde{A}) \tag{5.5}$$

In addition to the $SU(2)_L \times SU(2)_R$ symmetry under which $(A, \tilde{A})$ and $(B, \tilde{B})$ are both in the $(2, 1)$ representation, this superpotential (along with the rest of the action) is invariant under rotating A and $\tilde{A}$ (or B and $\tilde{B}$) alone, giving rise to a new $SU(2)$ symmetry which does not commute with the others, and so there must be an extended R-symmetry group. One can check that it is $SU(4) \cong SO(6)$, implying the theory has $\mathcal{N} = 6$ supersymmetry.

Although this extended supersymmetry is quite interesting, we are looking for a theory with $\mathcal{N} = 8$ supersymmetry, so it seems this is not good enough. However, recall there is an extra parameter k whose interpretation on the M-theory side we have not yet seen. In fact, it turns out that the correct way to interpret this theory is as the low-energy theory of M2 branes on a $\mathbb{C}^4/\mathbb{Z}_k$ singularity, where $\mathbb{Z}_k$ acts as $z_a \to e^{2\pi i/k} z_a, a = 1, ..., 4$. This generically breaks supersymmetry to $\mathcal{N} = 6$. However, for $k = 1$, where we recover $M2$ branes in flat space, the supersymmetry should be enhanced to $\mathcal{N} = 8$ (this also occurs for $k = 2$). This enhancement cannot be seen at the level of the Lagrangian since k only appears as the overall coefficient of the action, and so does not enter the analysis above. Nevertheless, the extra supersymmetry currents can be realized by monopole operators, as found in [31, 32].[3]

[2] There is an integer parameter k, the Chern-Simons level, but this cannot do the job, as one can see in the large N or k limit, which behave very differently.

[3] There is also a related version of this theory with $SU(N) \times SU(N)$ gauge group and retaining the $\mathcal{N} = 6$ supersymmetry, which agrees with the BLG theory for $N = 2$ and so the supersymmetry is enhanced classically there to $\mathcal{N} = 8$.

Another important consequence of this parameter is that, for large k, the theory is weakly coupled. In fact, the natural 't Hooft limit here is the limit of large N and k while holding $\lambda = \frac{N}{k}$ fixed, with small λ corresponding to weak coupling. The existence of such a "coupling constant," even one which takes discrete values, is presumably what allows there to be a Lagrangian description at all.

5.2.1 Supergravity Dual

By the standard AdS/CFT arguments, the field theory describing the low-energy excitations of the stack of N $M2$ branes should have a supergravity dual. The relevant 11D SUGRA background in this case is [25]:

$$ds^2 = h(r)^{-2/3}(-dt^2 + dx_1{}^2 + dx_2{}^2 + dx_3{}^2) + h(r)^{1/3}(dr^2 + r^2 d\Omega_7{}^2) \tag{5.6}$$

$$F_4 = d^3x \wedge dh(r)^{-1} \tag{5.7}$$

where $h(r) = 1 + \frac{32\pi^2 N \ell_p{}^6}{r^6}$, and $d\Omega_r{}^2$ is the metric on S^7 for the case of $M2$ branes on flat space, or more generally $S^7/\mathbb{Z}_k$ for the orbifold. Just like in the $AdS_5/\mathcal{N} = 4$ duality above, as $r \to 0$ the asymptotic form of this metric picks up an AdS factor, namely $AdS_4 \times S^7/\mathbb{Z}_k$. Thus we expect there to be a dual three-dimensional conformal field theory, which will be the ABJM theory at the appropriate N and k.

We now outline two quantities that can be computed in the supergravity theory, and then compared to the gauge theory using the matrix model. The first is the free energy on S^3, which is simply

$$F = -\log |Z|,$$

where Z is the Euclidean partition function. In the supergravity approximation, the field theory partition function is:

$$Z \approx e^{-S_{SUGRA}}$$

this is divergent but can be regularized using holographic renormalization. The result one finds is [17]:

$$F \approx -S_{SUGRA} \approx \frac{4\pi^3\sqrt{2}}{3}\lambda^{3/2} \tag{5.8}$$

Note that the free energy is, in some sense, supposed to count the degrees of freedom of the theory (see Chapter 7 for a more precise statement). Thus we see here that, for large λ, the number of degrees of freedom scales as $N^{3/2}$. This is counterintuitive, since the fundamental fields are all $N \times N$ matrices, and so we naively expect an N^2 scaling. However, at strong coupling, as we will see, one indeed finds such a scaling of F from the matrix model.

Another nontrivial test will involve the supersymmetric Wilson loop. Note that the Wilson loop we found in Chapter 3 preserves 4 of the 8 superconformal generators in the $\mathcal{N} = 2$ algebra, which amounts to being only $\frac{1}{6}$ BPS in the $\mathcal{N} = 6$ ABJM theory. A circular string ending on the boundary, however, would be expected to preserve $\frac{1}{2}$ of the supersymmetries. Thus this Wilson loop is not quite the correct thing to look at. However, in [33, 34] the correct, $\frac{1}{2}$ BPS Wilson loop in ABJM was constructed. Although it involves fermionic operators, it is in the same δ-cohomology class as a difference of two $\frac{1}{6}$ BPS Wilson loops, one in each gauge group factor. Thus we will compare the AdS calculation to the expectation value of this operator.

5.3 Testing the Duality

To start, let us summarize the prescription for writing down the matrix model for ABJM theory. The integral will be over the Cartan of the gauge group, which in this case can be spanned by the eigenvalues λ_i and $\hat{\lambda}_i$, $i = 1, ..., N$, for the two $U(N)$ gauge group factors. The integration measure is given by the product (here and below i and j run over the $1, ..., N$ unless otherwise noted):

$$\int d^N\lambda d^N\hat{\lambda} \prod_{i \neq j} (2\sinh\pi(\lambda_i - \lambda_j))(2\sinh\pi(\hat{\lambda}_i - \hat{\lambda}_j))$$

The Chern-Simons term for the gauge group gives an additional contribution of:

$$\prod_i e^{-ik\pi({\lambda_i}^2 - \hat{\lambda}_i^2)}$$

In addition, there is the contribution of the matter. The weights of the fundamental representation have $\rho_{ij} = \lambda_i - \hat{\lambda}_j$ for all $i, j = 1, ..., N$, so each hypermultiplet contributes:

$$\prod_{i,j=1}^{N} \frac{1}{2\cosh\pi(\lambda_i - \lambda_j)}$$

Note that the superpotential of the theory does not enter the matrix model explicitly, although it it necessary for the extended supersymmetry which ensures that the R-charge of the matter is not renormalized.

Putting this all together, we see that the partition function of the $U(N) \times U(N)$ ABJM theory at level k is given by:

$$Z_{ABJM}^{N,k} = \int d^N\lambda d^N\hat{\lambda} \prod_i e^{-ik\pi({\lambda_i}^2-\hat{\lambda}_i^2)} \frac{\prod_{i\neq j}(2\sinh\pi(\lambda_i-\lambda_j))^2(2\sinh\pi(\hat{\lambda}_i-\hat{\lambda}_j))^2}{\prod_{i,j}(2\cosh\pi(\lambda_i-\hat{\lambda}_j))^2} \tag{5.9}$$

Although it is still nontrivial to evaluate this integral, we have achieved an enormous simplification from the original infinite-dimensional path integral expression we started with.

Additionally, one may consider the insertion of a Wilson loop operator. If we take on in the fundamental representation of the first gauge group, for example, we find it is computed by an insertion in the matrix model of:

$$\frac{1}{N}\sum_i e^{2\pi\lambda_i} \tag{5.10}$$

5.3.1 Matrix Model at Large N

In principle, the partition functions and Wilson loops can be computed exactly at all N and k using the matrix model. For large N, where comparison to the AdS side is possible, this is somewhat difficult, since the integral is over a many-dimensional space. However, there are techniques to evaluate an integral of this form as a series in $\frac{1}{N}$, as we now briefly review (see [35] for more details).

Let us first rewrite the integral in the form:

$$Z = \int d^N\lambda d^N\hat{\lambda} \exp\bigg(-i\pi k\sum_i({\lambda_i}^2-\hat{\lambda}_i^2) - 2\sum_{i,j}\log(\cosh\pi(\lambda_i-\hat{\lambda}_j))+ \tag{5.11}$$

$$+2\sum_{i<j}\log(\sinh\pi(\lambda_i-\lambda_j)) + 2\sum_{i<j}\log(\sinh\pi(\hat{\lambda}_i-\hat{\lambda}_j))\bigg) \tag{5.12}$$

We now introduce the 't Hooft parameter $t = \frac{N}{k}$, and define:

$$S_{eff}(\lambda,\hat{\lambda}) = i\pi\frac{t}{N}\sum_i({\lambda_i}^2-\hat{\lambda}_i^2) + 2\frac{t^2}{N^2}\sum_{i,j}\log(\cosh\pi(\lambda_i-\hat{\lambda}_j))- \tag{5.13}$$

$$-2\frac{t^2}{N^2}\sum_{i<j}\log(\sinh\pi(\lambda_i-\lambda_j)) - 2\frac{t^2}{N^2}\sum_{i<j}\log(\sinh\pi(\hat{\lambda}_i-\hat{\lambda}_j)) \tag{5.14}$$

Note that in the 't Hooft limit of large N and k with t finite, S_{eff} is of order 1, since the sums and double sums scale as N and N^2. Then the matrix integral can be written:

$$Z = \int d^N\lambda d^N\hat{\lambda} e^{-k^2 S_{eff}(\lambda,\hat{\lambda})} \tag{5.15}$$

Since the exponent is multiplied by k, which we are taking very large, we can approximate this using the saddle point approximation. A further simplification comes from the fact that, for large N, we can approximate the eigenvalue distributions in terms of continuous (normalized) densities $\rho(\lambda)$ and $\hat{\rho}(\lambda)$. Thus all that remains is to find the eigenvalue densities which extremize the functional S_{eff}, and compute the leading contribution to the free energy as:

$$\frac{1}{k^2}F_o = S_{eff}[\rho;\hat{\rho}] \tag{5.16}$$

$$= i\pi t\int d\lambda(\rho(\lambda)\lambda^2 - \hat{\rho}(\lambda)\lambda^2) + 2t^2\int d\lambda d\lambda'\Big(\rho(\lambda)\hat{\rho}(\lambda')\log(\cosh\pi(\lambda-\hat{\lambda}))- \tag{5.17}$$

$$-\rho(\lambda)\rho(\lambda')\log(\sinh\pi(\lambda-\lambda)) - \hat{\rho}(\lambda)\hat{\rho}(\lambda')\log(\sinh\pi(\hat{\lambda}-\hat{\lambda}))\Big) \tag{5.18}$$

In addition, the $\frac{1}{6}$-BPS Wilson loop is given by an insertion of $\frac{1}{N}\sum_i e^{\lambda_i}$, which is computed at the saddle point by:

$$\int d\lambda\rho(\lambda)e^{\lambda} \tag{5.19}$$

with an appropriate modification for the $\frac{1}{2}$ BPS Wilson loop.

The determination of the extremal eigenvalue densities is somewhat involved, and the resulting expressions are quite complicated. We state here the results of [34, 17] for the leading behavior at large and small λ:

$$F_o(\lambda) \approx \begin{cases} N^2\Big(\log(2\pi\lambda) - \frac{3}{2} - 2\log 2\Big) & \lambda << 1 \\ \frac{\pi N^2\sqrt{2}}{3\sqrt{\lambda}} & \lambda >> 1 \end{cases} \tag{5.20}$$

$$< W_{1/6} >\approx \begin{cases} e^{\pi i\lambda}\Big(1 + \frac{5\pi^2\lambda^2}{6} + ...\Big) & \lambda << 1 \\ \frac{i}{2\pi\sqrt{2\lambda}}e^{\pi\sqrt{2\lambda}} & \lambda >> 1 \end{cases} \tag{5.21}$$

The small λ behavior can be matched to perturbative field theory computations (e.g., [36]), a nontrivial check of the localization procedure. The large λ behavior are seen to agree with the AdS calculations above. Note that the naive N^2 scaling of the free energy seen at weak coupling is corrected by a factor of $\lambda^{-1/2}$ at strong coupling, which reproduces the correct $N^{3/2}$ behavior. The matching of the field theory and supergravity results provides a nontrivial test of the duality.

Chapter 6

3D Mirror Symmetry

We will now move away from the AdS/CFT story and focus on dualities between two field theories. Nevertheless, as we will see below, these dualities are motivated in many cases by dualities of certain string theories, of which they are the low-energy effective field theories.

The first example, which will be the subject of the present chapter, is three-dimensional mirror symmetry [37, 38, 39, 40]. This relates the superconformal infrared fixed points of a certain class of quiver gauge theories in three dimensions. These theories also appear as the low-energy theories of certain brane configurations in type IIB string theory, and the field theory duality descends from the action of S-duality here. Previously, the only direct evidence for the field theory dualities came from the comparison of moduli spaces, as we will briefly review. The matching of partition functions provides a nontrivial independent test of the duality. Moreover, we can deform the partition functions by weakly gauge flavor symmetries on both sides, which will provide a test of the mapping of global symmetries between the theories.

An important aspect of this duality is that the theories involved are not superconformal. The meaning of duality in such cases is that the theories flow to equivalent IR superconformal fixed points. Since we will typically be considering gauge theories with a Yang-Mills term, these theories are strongly coupled, and so it is very difficult to find evidence for the dualities.

6.1 $\mathcal{N} = 4$ $U(1)$ SQED and a Free Hypermultiplet

To motivate the duality, consider the case of a nonsupersymmetric free abelian gauge theory in flat space. Since there is no matter, the field strength $F_{\mu\nu}$ satisfies $d \star F = 0$ on shell, and so we can write $\star F = d\gamma$. The field γ is called the dual photon, and satisfies the equations of motion of a free scalar, although gauge-invariance restricts it to be periodic with period g, the gauge coupling. Thus, in a somewhat trivial sense, the free $U(1)$ gauge theory is "dual" to a free scalar theory.

In particular, this means the $U(1)$ gauge theory has a one-dimensional moduli space S^1 with period g. Note that this moduli space is not visible at the level of the Lagrangian. In fact, there is a symmetry of the theory which shifts the dual photon, and so acts nontrivially on this moduli space, but can not be seen as a transformation of the gauge field, although the corresponding conserved current can be written as $J = \star F$, which is conserved by the Bianchi identity. We call this the $U(1)_J$ symmetry. Note that in the IR, where the gauge coupling flows to infinity, the free scalar becomes nonperiodic and gains conformal symmetry, although this is not visible in the gauge field description.

In the presence of matter, the above story only holds in the far UV, where the gauge coupling flows to zero. In the IR, where the gauge field is no longer free, the conserved current $J = \star F$ cannot be a descendant, and so by general arguments it must have dimension 2, whereas in the UV it has dimension $\frac{3}{2}$, being the derivative of a free scalar. Thus we see the theory changes significantly as we flow from the UV fixed point to the IR one. In general, there is not a simple description of the IR fixed point, although it is believed to be nontrivial for sufficiently large N_f, and some results can be obtained in $N_f \to \infty$ limit [41]. Note that the $U(1)_J$ symmetry still exists, since $J = \star F$ is always conserved, but its interpretation as a shift symmetry of some scalar is lost here.

More can be said in the supersymmetric case. Let us first consider the case of $\mathcal{N} = 2$ supersymmetry. In the pure gauge case, the action can be written:

$$\int d^3x \frac{1}{g^2}\left(\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \partial_\mu\sigma\partial^\mu\sigma + D^2 + i\lambda^\dagger \not{\partial}\lambda\right) \tag{6.1}$$

This theory is free, and has a moduli space $\mathbb{R} \times S^1$ given by expectation values of σ and the dual photon.

Now consider adding matter. To preserve parity, we will arrange the matter in N_f hypermultiplets, each consisting of two charge ± 1 chiral multiplets $(Q_i, \tilde{Q}_i)$, with an action which can be written out in components as:

$$\sum_{i=1}^{N_f} \int d^3x \left(D_\mu\phi_i{}^\dagger D^\mu\phi_i + \phi_i{}^\dagger\sigma^2\phi_i + i\phi_i{}^\dagger D\phi_i + i\psi_i{}^\dagger \not{D}\psi_i - i\psi_i{}^\dagger\sigma\psi_i + i\phi_i{}^\dagger\lambda^\dagger\psi_i - i\psi_i{}^\dagger\lambda\phi_i + F_i{}^\dagger F_i\right) \tag{6.2}$$

To start, we will not add a superpotential. Then, we get a D-term potential of the form:

$$\sum_i \phi_i{}^\dagger\phi_i\sigma^2 \tag{6.3}$$

Thus the classical moduli space has two branches, the Higgs branch with $< \sum_i \phi_i{}^\dagger\phi_i > \neq 0$ but $< \sigma >= 0$, and the Coulomb branch with $< \sigma > \neq 0$ and $< \sum_i \phi_i{}^\dagger\phi_i >= 0$. The Higgs branch does not receive quantum corrections, and can be parametrized by $2N_f - 1$ chiral multiplets of the form $M \sim Q\tilde{Q}$.

Far out on the Coulomb branch, this potential acts as a very large mass term for the matter, effectively decoupling it, and we get a free $\mathcal{N}=2$ gauge theory, so that the Coulomb branch should asymptotically look like $\mathbb{R}\times S^1$. Out here we can approximately dualize the gauge field into a dual photon, and one can show that it combines with σ as $\sigma+i\gamma$ to form the scalar component of a chiral multiplet we will denote Φ. It is convenient to define $Y_\pm = e^{\pm\Phi/g^2}$, which are chiral multiplets which transforms with charge ± 1 under the $U(1)_J$ (at least approximately, at large $<Y>$).

Note that the $U(1)_J$ symmetry shifts the S^1 factor of the Coulomb branch, but does not act on the Higgs branch, so quantum corrections must cause the radius of this S^1 to shrink to zero at $<\sigma>=0$. There are no instanton corrections in the abelian theory, but such a correction can appear in perturbation theory since the size of the circle is of order g. We can parametrize the two branches by V_+ and V_-, which we take to run over all of $\mathbb{C}$. Thus the moduli space looks like three cones intersecting at a point.

In the case $N_f=1$, the moduli space looks like three copies of $\mathbb{C}$, parametrized by V_+, V_-,and $M=Q\tilde{Q}$, which intersect at a point. This is the same moduli space as the Wess-Zumino model with three chirals X,Y,Z and superpotential $W=XYZ$, called the XYZ theory, and one is led to propose their equivalence. One can perform other checks of the duality, such as matching of $\mathbb{Z}_2$ global anomalies, and the duality checks out.

Note that in this theory, there is an S_3 symmetry permuting the chirals which ensures they all have IR R-charge $\frac{2}{3}$. In particular, the matrix model above does not apply, and we will have to wait until Chapter 7 to test the duality.

However, let us now consider an $\mathcal{N}=4$ version of this duality. We add a free chiral superfield Φ to complete the $\mathcal{N}=4$ vector multiplet, and a superpotential $X\Phi\tilde{X}$ to ensure $\mathcal{N}=4$ supersymmetry. The operator $Q\tilde{Q}$ maps to one of the chirals in the XYZ theory, so we are left on the dual side with a theory of four chirals X,Y,Z,Φ and superpotential:

$$W = XYZ + \Phi Z \tag{6.4}$$

The effect is to give a mass to Φ and Z and remove the potential for X and Y, so that in the IR we are left with the theory of a free hypermultiplet.

The $\mathcal{N}=4$ supersymmetry ensures that the hypermultiplet on the SQED side has R-charge $\frac{1}{2}$. Thus we can apply the results of the localization procedure. To make this less trivial, let us also add background SUSY gauge fields for the global symmetries on both sides. For the free hypermultiplet, there is a $U(1)_V$ symmetry, which rotates X and Y in opposite directions. Since these fields came from $V_\pm$, we expect this to correspond to the $U(1)_J$ symmetry on the dual side. The corresponding deformations give rise to real mass terms and FI terms. We find, for $\mathcal{N}=4$ SQED:

$$Z_{SQED}(\eta) = \int d\lambda e^{2\pi i\eta\lambda}\frac{1}{2\cosh(\pi\lambda)} \tag{6.5}$$

Meanwhile, for the hypermultiplet, we get:

$$Z_{hyp}(m) = \frac{1}{2\cosh(\pi m)} \tag{6.6}$$

One can check that these expressions are equal when one sets $\eta = m$, a consequence of $\mathrm{sech}(x)$ being fixed under the Fourier transform operation.[1] The identification of m and η is a test of the identification of the corresponding $U(1)$ global symmetries, $U(1)_V$ and $U(1)_J$, as argued above.

Let us now return to the case of general N_f. In [37], it was conjectured that an $\mathcal{N} = 4$ $U(1)$ theory with N_f hypermultiplets should be dual to a theory called the A_n model, with gauge group $U(1)^{N_f}/U(1)_{tot}$, with N_f hypermultiplets of charges $(1, -1, 0, ..., 0), (0, 1, -1, ..., 0), ..., (-1, 0, 0, ..., 1)$. Note we divide out by the overall $U(1)$, which has no matter charged under it and decouples. Specifically, this was a part of a more general ADE classifications of such dualities, corresponding to the A_n series, which is all we will focus on here.

To give evidence for such a duality, we study the moduli spaces of both sides. As discussed in Chapter 2, these will consist of a Higgs and Coulomb branch, both hyper-Kahler manifolds. The Higgs branch of the A_n model is known to be an ALE space, a quotient $\mathbb{C}^2/\Gamma$ by a discrete subgroup Γ of $SU(2)$. Turning on FI terms resolves some of the singularities in a well-defined way by reducing Γ to a subgroup, but does not lift the Higgs branch. Actually this form of the Higgs branch is found in any dimension, dimensionally reducing the corresponding $\mathcal{N} = 1$ theory in six dimensions. On the dual side, the Higgs branch of is known to be the moduli space of an $SU(N_f)$ instanton, modulo translation, which has quaternionic dimension $N_f - 1$. Both receive no quantum corrections.

The Coulomb branches of both theories have semiclassically the form $(\mathbb{R}^3 \times S^1)^r$ coming from expectation values of the three scalars and the dual photon in each vector multiplet, with $r = N_f - 1$ in the A_{N_f} model and $r = 1$ in the dual. However, these are subject to quantum corrections, and it was argued in [37] that these are such that the Higgs branch of one theory is the same as the Coulomb branch of the dual. As a very basic check, we see that the have the correct dimension for this correspondence to hold.

There is a heuristic way to obtain this general abelian duality from the first example, following [42]. Let us consider N_f uncoupled copies of the basic $\mathcal{N} = 4$ pairs, i.e., a $U(1)$ theory with one hypermultiplet and a free twisted hypermultiplet. The symmetry group on both sides is $U(1)^{N_f}$. Now consider the overall sum of these symmetries, and suppose we add a dynamical gauge field coupled to this symmetry. In the case of the free hypers, we get a theory of SQED with N_f hypermultiplets. On the dual side, we are gauging a $U(1)_J$ symmetry. In [43] it was shown that this is the same as ungauging the corresponding gauge symmetry. Thus we have a theory with gauge group $U(1)^{N_f}/U(1)_{tot}$ and, after redefining the matter charges, hypers with charges $(1, -1, 0, ..., 0), (0, 1, -1, ..., 0)$, as ex-

[1] See [42] for a field theory interpretation of this Fourier transform property.

pected. This operation of gauging symmetries is a little suspect, since we perform it in the UV, while the duality really only holds in the IR fixed point.

This manipulation can also be performed at the level of the matrix model. To start, we take the product of N_f copies of the free hypermultiplet partition function with deformations m_i corresponding to their $U(1)_V$ charges:

$$\prod_i Z_{hyp}(m_i) = \prod_i \frac{1}{2\cosh(\pi m_i)}$$

To gauge the sum of the $U(1)_V$ symmetries, define $m_i = \hat{m}_i + \lambda$, and integrate over λ. This recovers the partition function of SQED with N_f flavors:

$$Z_{N_f} = \int d\lambda \prod_i Z_{hyp}(\lambda + \hat{m}_i) = \int d\lambda \prod_i \frac{1}{2\cosh \pi(\lambda + \hat{m}_i)}$$

On the other hand, if we perform the same procedure starting with the dual theory, we obtain:

$$\int d\lambda \prod_i Z_{SQED}(\lambda + \hat{m}_i) = \int d\lambda \prod_i \int d\lambda_i \frac{e^{2\pi i(\lambda+\hat{m}_i)\lambda_i}}{2\cosh(\pi\lambda_i)}$$

Then the integral over λ gives a delta function constraint $\delta(\sum_i \lambda_i)$ which effectively ungauges the $U(1)_{diag}$ symmetry, as in [43]. We obtain precisely the partition function of the A_{N_f} theory, with mass terms having mapped to FI terms:

$$Z_{A_{N_f}}(\hat{m}_i) = \int d^{N_f}\lambda \prod_i \delta(\sum_i \lambda_i) \prod_i \frac{e^{2\pi i(\hat{m}_i)\lambda_i}}{2\cosh(\pi\lambda_i)} \tag{6.7}$$

This argument was also carried out for the $S^2 \times S^1$ partition function in [18].

6.2 General Case

6.2.1 Brane Construction

This duality between $\mathcal{N} = 4$ SQED and a free (twisted) hypermultiplet will follow as a special case of the string theory construction we will consider below, but we have so far motivated it completely in field theory. For the general, nonabelian case, it will be very useful to use a string theory construction of the theories due to Hanany and Witten [38]. Then the duality will correspond to S-duality in string theory.

Let us start by constructing the corresponding brane setup in type IIB string theory. To start, consider a single infinite $D3$ brane. This preserves half the supersymmetry of type IIB, and its low-energy theory is $U(1)$ $\mathcal{N} = 4$ Super Yang-Mills, and for N $D3$ branes one finds the $U(N)$ theory. From the three-dimensional point of view, such a theory has $\mathcal{N} = 8$ supersymmetry.

Table 6.1: Orientations of Branes in Hanany-Witten setup

	0	1	2	3	4	5	6^c	7	8	9
D3	×	×	×				×			
D5	×	×	×	×	×	×				
NS5	×	×	×					×	×	×

Next consider adding five-branes. Specifically, to start we will consider $D5$ branes and $NS5$ branes. These will be oriented so that they share three spacetime dimensions with each other and the $D3$ branes, and are perpendicular in the remaining directions. These must break at least half the remaining supersymmetry, and if they are oriented as in Table 6.2.1, then no additional supersymmetry is broken, and the resulting setup has $\mathcal{N} = 4$ supersymmetry from the three-dimensional point of view. These are the configurations we will investigate.

In [38] the low-energy effective theory of such a configuration was deduced by studying the boundary conditions imposed by the two types of five-branes on the $D3$ brane fields. If a single $D3$ brane is stretched between two NS5 branes, the resulting low-energy theory on the brane is three-dimensional $\mathcal{N} = 4$ super Yang-Mills theory with gauge group $U(1)$, and with coupling constant $\frac{1}{g^2}$ proportional to the separation between the branes. This fact can be seen by integrating out the modes along the x_6 direction, which leaves a factor of this separation in the action. In the low-energy limit, where the $NS5$ branes are taken to zero separation, corresponding to $g \to \infty$ in the field theory. As usual, for N coincident $D3$ branes, the gauge group is enhanced to $U(N)$. brane.

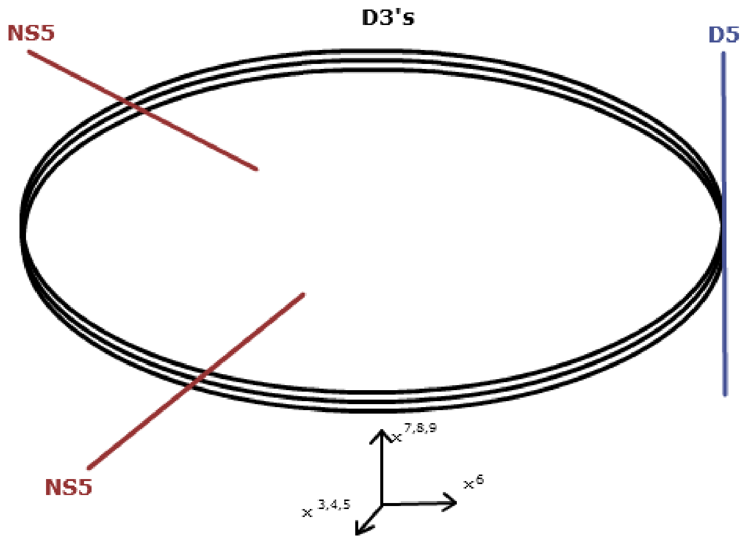
For a $D3$ brane stretched between two $D5$ branes, on the other hand, the contribution is a (twisted) hypermultiplet. One can imagine there is a magnetic gauge group with coupling g_m, related to the separation between the five-branes as above, however, here it is not possible to see the effect of finite g_m in the field theory, and one is restricted to infinite g_m where one finds a twisted hypermultiplet.

The basic example of mirror symmetry from the last section then corresponds to taking a single $D3$ brane between either two $NS5$ branes or two $D5$ branes.

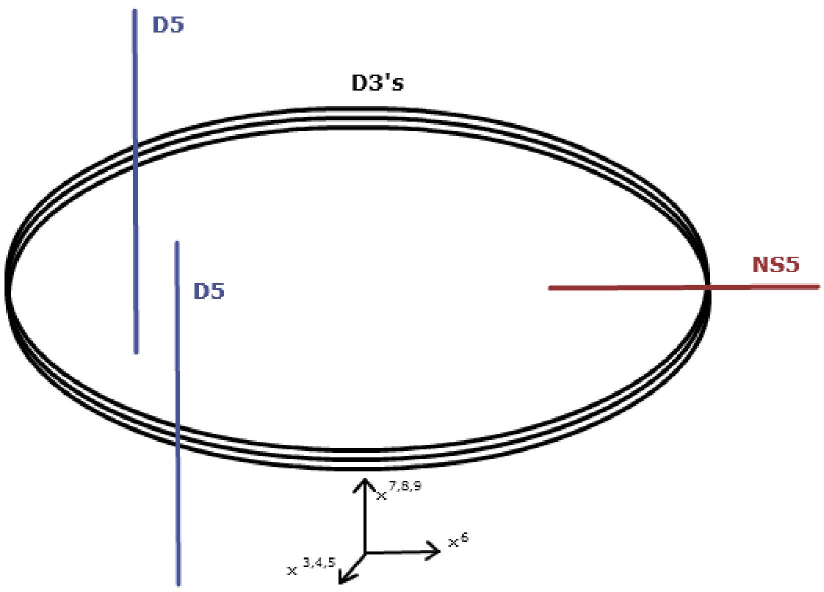
For the configurations we are interested in, we will let the 6th direction be compactified on a circle. We then wrap N D3 branes on this circle, and five-branes are placed at various points along the 6 direction as above. In the IR limit the 6 direction shrinks to zero size, and an effective three-dimensional gauge theory living on the 012 directions will remain. Two examples of such a configuration are shown in figure 6.4.2.

Let us assume there are M five-branes. We will label them $\alpha_a, a = 1, ..., M$ where $\alpha_a \in \{NS5, D5\}$, with r $D5$ branes and s $NS5$ branes, where $r + s = M$. We will be interested in the low-energy effective theory of this configuration. This will be the IR limit of a $U(N)$ quiver gauge theory. We now summarize the prescription for reading off this theory:

- For each set of N $D3$ brane segments between consecutive $NS5$ branes, there is a $U(N)$ factor



[(a)]



[(b)]

Figure 6.1: Examples of brane configuration. In the first figure, we read off the field theory as a $U(N)$ gauge group with one adjoint and two fundamental hypermultiplets, while in the second the gauge group is $U(N) \times U(N)$ with two bifundamental hypermultiplets and a fundamental in one of the gauge groups. These configurations are exchanged under S-duality, and so the corresponding quiver gauge theories are dual.

in the gauge group. In our case, N is the same for each segment, so the gauge group is $U(N)^s$, but one could also consider a more general configuration.

- For each $NS5$ brane, which bounds two such segments, there is a bifundamental hypermultiplet between the corresponding gauge group factors. This comes from massless modes coming of fundamental strings stretching between the $D3$ branes on either side of the $NS5$ branes.
- On each such segment, any $D5$ branes intersecting the $D3$ brane segments contribute a fundamental hypermultiplet in the corresponding gauge group factor, and is associated to fundamental strings stretching between the $D3$ and $D5$ branes.

We can also consider deformations of this setup, corresponding to moving the various branes:

- The transverse positions of the $D3$ branes are parametrized by scalar fields in the gauge theory, so moving the branes corresponds to changing the position in moduli space. As usual, we will be interested in the origin of moduli space, where the $D3$ branes are coincident
- Moving the $NS5$ branes gives rise to an FI term $\int d^4\theta(D_i - D_j)$.
- Moving the $D5$ branes corresponds to a real mass to the corresponding hypermultiplet, i.e., real masses of opposite signs to the two chiral multiplets it contains.

With this construction, mirror symmetry follows directly from S-duality in type IIB string theory. This duality exchanges $NS5$ branes with $D5$ branes, while preserving $D3$ branes. Thus it maps one configuration of the type above to another. In the IR, the two gauge theories must be equivalent. Typically the UV description of dual theories will look very different from the point of view of gauge theory, and there is not convenient IR description, so the duality is very nontrivial.

6.3 Proof of Duality in Matrix Model

We now proceed to prove that dual theories have equal partition functions, following [44]. We will also deform them by mass and FI terms, and show they map in the appropriate way. First we review the various contributions to the matrix model:

- Each $U(N)$ gauge group factor contribute N integration variables λ^i along with the measure:

$$\frac{1}{N!}\prod_{i<j}(2\sinh\pi(\lambda_i - \lambda_j))^2 \tag{6.8}$$

- A fundamental hypermultiplet in the $U(N)$ factor corresponding to λ^i contributes:

$$\prod_i \frac{1}{2\cosh(\pi\lambda^i)} \tag{6.9}$$

- A bifundamental hypermuliplet between the $U(N)$ factor corresponding to λ^i and another factor corresponding to $\hat{\lambda}^i$ contributes:

$$\prod_{i,j} \frac{1}{2\cosh\pi(\lambda^i - \hat{\lambda}^j)} \tag{6.10}$$

One multiplies all the appropriate 1-loop factors, and integrates over the λ^i for each $U(N)$ factor in the gauge group.

In principle, this prescription allows us to write down the matrix model for any of the theories considered above, given only the sequence α_a. However, to prove the duality, it will be convenient to organize these in a coherent way.

- For each $D5$ brane, there is a factor of:

$$\prod_i \frac{1}{2\cosh(\pi\lambda^i)} \tag{6.11}$$

  where λ^i corresponds to the gauge group in which the hypermultiplet sits. Note that there is a set of integration variables λ^i for each $NS5$ brane, but not for the $D5$ branes. To try to make the situation more symmetric, we can introduce variables ${\lambda^i}_a$ for each five-brane, and then if the ath five-brane is a $D5$ brane, we can simply write its contribution as:[2].

$$\int d^N\lambda_a \prod_i \frac{\delta({\lambda_a}^i - {\lambda_{a+1}}^i)}{2\cosh(\pi\lambda^i)} \tag{6.12}$$

  We will find it convenient to introduce an auxiliary variable τ_i to enforce the delta function constraint:

$$\int d^N\lambda_a d^n\tau_a \prod_i \frac{e^{2\pi i {\tau_a}^i ({\lambda_a}^i - {\lambda_{a+1}}^i)}}{2\cosh(\pi\lambda^i)} \tag{6.13}$$

- If the ath five-brane is an $NS5$ brane, it will contribute a bifundamemental hypermultiplet between the two neighboring gauge groups, giving a contribution:

$$\prod_{i,j} \frac{1}{2\cosh\pi({\lambda_a}^i - {\lambda_{a+1}}^j)} \tag{6.14}$$

[2] Although we write this as an integral, this expression is to be inserted into a larger integral, with other factors that may depend on λ_a and λ_{a+1}, so the integral should not be evaluated at this point

In addition, we must account for gauge group 1-loop determinants. Since the gauge groups are associated to a pair of consecutive branes, we can use the fact that the 1-loop determinant is a perfect square to split it up into two factors, and associate each factor to the two bounding $NS5$ branes. Thus the total contribution of the ath $NS5$ brane can be written:

$$\frac{1}{N!}\int d^N\lambda_a \frac{\prod_{i<j}(2\sinh\pi({\lambda_a}^i-{\lambda_a}^j))(2\sinh\pi({\lambda_{a+1}}^i-{\lambda_{a+1}}^j))}{\prod_{i,j}2\cosh\pi({\lambda_a}^i-{\lambda_{a+1}}^j)} \tag{6.15}$$

The real reason for writing the contribution this way is because this can be rewritten using the Cauchy determinant formula as:

$$\frac{1}{N!}\int d^N\lambda_a \sum_\sigma(-1)^\sigma\prod_i \frac{1}{2\cosh\pi({\lambda_a}^i-{\lambda_{a+1}}^{\sigma(i)})} \tag{6.16}$$

where σ runs over the permutations in S_N. Again, we introduce an auxiliary variable τ_a and use the fact that $\frac{1}{2\cosh(\pi x)}$ is its own Fourier transform, as we saw in the abelian duality above, to write this as:

$$\frac{1}{N!}\int d^N\lambda_a d^N\tau_a \sum_\sigma(-1)^\sigma\prod_i \frac{e^{2\pi i{\tau_a}^i({\lambda_a}^i-{\lambda_{a+1}}^{\sigma(i)})}}{2\cosh(\pi{\tau_a}^i)} \tag{6.17}$$

With this in mind, let us define

$$I_\alpha({\lambda_a}^i,{\tau_a}^i)=\begin{cases}\displaystyle\prod_i\frac{1}{2\cosh(\pi{\sigma_a}^i)} & \alpha=D5\\ \displaystyle\prod_i\frac{1}{2\cosh(\pi{\tau_a}^i)} & \alpha=NS5\end{cases} \tag{6.18}$$

Then we claim the partition function of the brane configuration corresponding to the sequence α_a is given by:

$$\prod_a\int d^N\lambda_a d^N\tau_a\sum_\sigma(-1)^\sigma\prod_i e^{2\pi i{\tau_a}^i({\lambda_a}^i-{\lambda_{a+1}}^{\sigma(i)})}I_{\alpha_a}({\lambda_a}^i,{\tau_a}^i) \tag{6.19}$$

We have essentially shown this above. The only thing to check is that the antisymmetrization we have added in the $D5$ contribution does not affect anything. But, provided there is at least one $NS5$ brane[3], one can see that the integral will always be antisymmetric in all of the ${\lambda_a}^i$, so this is true.

In this form, the duality is essentially manifest. Namely, upon exchanging the variables λ_a and τ_a, the contributions of $NS5$ and $D5$ branes are exchanged, and we get the partition function of the

[3] We can assume this without loss, since the matter content of a theory without $NS5$ branes is the same as one with a single $NS5$ brane added. The naive dual of the original, pure $D5$ brane theory would have only $NS5$ branes, and is not well-defined for the same reason as $\mathcal{N}=8$ SYM, which we will see below.

dual theory.[4]

Finally, we consider the effect of adding deformations.

- For each $D5$ brane, we can add a real mass m for the corresponding hypermultiplet, which simply gives:

$$\prod_i \frac{1}{2\cosh\pi(\lambda^i + m)} \tag{6.20}$$

- For each $NS5$ brane, the corresponding operation is to add an FI term η to the two adjacent gauge groups with opposite signs, so that the contribution is modified to:

$$\frac{1}{N!}\int d^N\lambda_a d^N\tau_a \sum_\sigma (-1)^\sigma \prod_i \frac{e^{2\pi i({\tau_a}^i+\eta)({\lambda_a}^i-{\lambda_{a+1}}^{\sigma(i)})}}{2\cosh(\pi{\tau_a}^i)} \tag{6.21}$$

  By shifting ${\tau_a}^i$, one can move this factor inside the cosh.

Thus we see the effect of the two deformations is identical: they shift the argument inside the cosh in I_α. Thus the deformed partition function will be unchanged provided that when we exchanged λ and τ, we simultaneously exchange m and η. This demonstrates the correct mapping of the symmetries.

6.4 Including $(1,1)$ Branes

We now present a slight generalization of the above construction, in which in addition to $NS5$ and $D5$ branes, one allows $(1,1)$ branes, a bound state of a single $NS5$ and $D5$ brane which is invariant under the S-duality above.

The contribution to the gauge theory of this brane is the same as an $NS5$ brane, with the added effect of including Chern-Simons terms at levels 1 and -1 in the two adjacent gauge groups [45]. We can thus write the contribution as:

$$\frac{1}{N!}\int d^N\lambda_a \sum_\sigma (-1)^\sigma \prod_i \frac{e^{-\pi i({\lambda_a}^2-{\lambda_{a+1}}^2)}}{2\cosh\pi({\lambda_a}^i-{\lambda_{a+1}}^{\sigma(i)})} \tag{6.22}$$

Now let us define a function:

$$I_{(1,1)}(\lambda,\tau) = \int d\kappa \frac{e^{\pi i\kappa^2+2\pi i\kappa(\lambda+\tau)}}{2\cosh(\pi\kappa)} \tag{6.23}$$

Then, as the notation suggests, we claim the contribution of a $(1,1)$ brane can be written as:

[4] One also needs to check that the exponential factor is symmetric under this exchange, which is straightforward and shown in [44].

$$\sum_\sigma (-1)^\sigma \int d^N\lambda_a d^N\tau_a \prod_i e^{2\pi i \tau_a{}^i(\lambda_a{}^i - \lambda_{a+1}{}^{\sigma(i)})} I_{(1,1)}(\lambda_i, \tau_i) \tag{6.24}$$

To see this, let us plug in the form of $I_{(1,1)}$ and integrate the auxiliary variables:

$$\sum_\sigma (-1)^\sigma \int d^N\lambda_a d^N\tau_a d^N\kappa \prod_i \frac{e^{\pi i\kappa^2 + 2\pi i(\kappa^i(\lambda_a{}^i + \tau_a{}^i) + \tau_a{}^i(\lambda_a{}^i - \lambda_{a+1}{}^{\sigma(i)}))}}{2\cosh(\pi\kappa_i)} \tag{6.25}$$

$$= \sum_\sigma (-1)^\sigma \int d^N\lambda_a d^N\kappa \prod_i \delta(\kappa^i + \lambda_a{}^i - \lambda_{a+1}{}^{\sigma(i)}) \frac{e^{\pi i\kappa^2 + 2\pi i\kappa^i\lambda_a{}^i}}{2\cosh(\pi\kappa_i)} \tag{6.26}$$

$$= \sum_\sigma (-1)^\sigma \int d^N\lambda_a \prod_i \frac{e^{-\pi i(\lambda_a{}^{i2} - \lambda_{a+1}{}^{i2})}}{2\cosh\pi(\lambda_a{}^i - \lambda_{a+1}{}^{\sigma(i)})} \tag{6.27}$$

which is the correct contribution.

In this form, the invariance of the $(1,1)$ brane contribution under the duality is manifest, since it contribution is invariant under exchanging the λ and τ variables.

Interestingly, following [46], we note that so far, all contributions can be written in the form:[5]

$$I_{(p,q)}(\lambda, \tau) = \int d\kappa \frac{e^{\pi pqi\kappa^2 + 2\pi i\kappa(p\lambda + q\tau)}}{2\cosh(\pi\kappa)} \tag{6.28}$$

In fact, as shown in [46], such a contribution transforms covariantly under $SL(2,\mathbb{Z})$. Thus a natural conjecture is that the contribution to the partition function of a general (p,q) brane should have this form. For $q = 0, 1$, this agrees with the prescriptions above, but for $q > 1$, after integrating out the auxiliary variables one finds a contribution that does not naturally come out of the matrix model, so it is not clear how to interpret this formula physically.

6.4.1 T-Transformation

The introduction of $(1,1)$ branes allows us to consider a new duality. Recall that the S-duality sits inside a large $SL(2,\mathbb{Z})$ group of duality transformations. This group is generated by two elements, S and T:

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{6.29}$$

As the name suggests, S-duality corresponds to the action of the S-generator. Similarly, one can discuss the T generator. As we can see from above, this take (p,q) branes to $(p+1,q)$ branes.

Note that we have only considered the inclusion of (p,q) branes for $p, q = 0, 1$. The S generator preserves this property, but the T-generator in general does not. Thus to consider the action of

[5]The coefficient on the κ^2 term can be motivated by the contribution of a $(1,k)$ brane and $SL(2,\mathbb{Z})$ covariance.

this generator, we will need to restrict to brane configurations containing only $(0,1)$ branes (i.e., $D5$ branes) and $(1,1)$ branes. But in this case, the duality is trivial already at the level of the field theory. This is because, when there are only $(1,1)$ and $D5$ branes present, the Chern-Simons terms cancel pairwise between adjacent branes, and one finds the same action as if the $(1,1)$ branes were replaced by $NS5$ branes. One can consider other T-like dualities by combining with the S duality, but they can always be reduced to this basic case. Note also that T can be applied only once. Thus we are only able to look at a relatively small subset of the set of brane configurations and $SL(2,\mathbb{Z})$ actions, although as noted above, see [46] for a generalization.

6.4.2 ABJM Theory

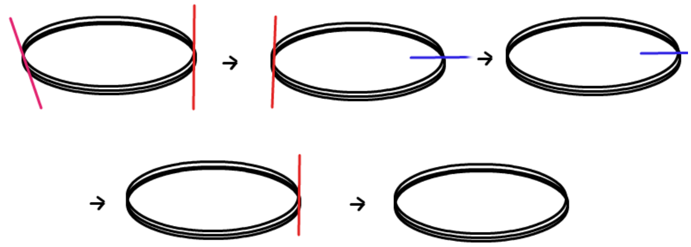
In the presence of $(1,1)$ branes, some of the gauge groups gain Chern-Simons terms. This means that the IR fixed point can be written explicitly: one simply strikes out the Yang-Mills terms in those gauge groups with Chern-Simons terms. Performing this operation in the remaining gauge groups leaves them with no kinetic term at all, which is not very well-defined, so we will usually only do this if all gauge groups have a Chern-Simons term.

It turns out that the simplest example of this is ABJM theory. Namely, if one takes the configuration with a single $NS5$ brane and $(1,k)$ brane, one finds the field content of ABJM theory, namely, a $U(N)\times U(N)$ gauge multiplet together with two bifundamental hypermultiplets. In the UV, the gauge field has a Yang Mills term, but since it also has a Chern-Simons term in all factors, one can simply remove the Yang-Mills term and find the IR fixed point to be precisely the action of ABJM theory.

Let us consider the effect of dualities on such a configuration. As above, we are forced to restrict to the case $k=1$. Then an S-duality takes us to a configuration with a $D5$ and $(1,1)$ brane. As argued above, T duality is trivial in this case, and this is manifestly the same as the contribution of a $D5$ and $NS5$ brane. Then, as noted above, a single $NS5$ brane can always be removed, and one obtains the theory of a single $D5$ brane. This can be further dualized to a single $NS5$ brane, which can also be removed, and we are left with no five-branes at all. The result is simply $\mathcal{N}=8$ super Yang-Mills theory, and the chain of dualities has shown us that this should be equivalent at low energies to the ABJM theory, as noted in the previous chapter.

Our general arguments above prove the matching of the partition function of ABJM with the dual theory with an $NS5$ brane and $D5$ brane, which is $\mathcal{N}=4$ $U(N)$ Yang-Mills with an adjoint hypermultiplet and a fundamental hypermultiplet. This in turn should be dual to $\mathcal{N}=8$, as just argued, but this cannot be seen at the level of the partition function, since our arguments only applied to the case where at least one $NS5$ brane is present at all times. In fact, we claim the naive matrix model one writes down for the $\mathcal{N}=8$ theory must be incorrect. This can be seen from the fact that, by $\mathcal{N}=8$ supersymmetry, the IR superconformal R-charge of the scalars in the adjoint

Figure 6.2: Sequence of dualities which lead from ABJM theory at level 1 to $\mathcal{N} = 8$ SYM. Here $NS5$ branes are in red, $D5$ in blue, and $(1,1)$ in purple.



hypermultiplet must be the same as that of the scalars in the vector multiplet, but we have assumed the former have R-charge $\frac{1}{2}$ and the latter have R-charge 1. Indeed, the naive matrix model does not even converge:

$$Z_{\mathcal{N}=8} \stackrel{?}{=} \int d^N\lambda \frac{\prod_{i\neq j} \sinh \pi(\lambda_i - \lambda_j)}{\prod_{i\neq j} \cosh \pi(\lambda_i - \lambda_j)} = \infty \tag{6.30}$$

Thus we cannot directly compare this theory to ABJM or its $\mathcal{N} = 4$ dual SYM dual. The nonrenormalization theorems that usually apply to theories with at least $\mathcal{N} = 4$ supersymmetry break down here, ironically, because there is too much supersymmetry, and the $\mathcal{N} = 4$ $SU(2) \times SU(2)$ subgroup may rotate in an unspecified way inside the full $SO(8)$ R-symmetry as we flow to the IR.

Chapter 7

General R-Charges

Up until now, with the exception of pure Chern-Simons theory, we have considered only theories with extended supersymmetry, i.e., at least $\mathcal{N} = 3$. There is no inherent reason why the localization calculation requires this, since in our derivation of the matrix model we used only $\mathcal{N} = 2$ supersymmetry. However, we wrote a specific form for the superconformal transformations of the matter fields, and although this form is necessary in theories with extended supersymmetry, it is not the only one compatible with $\mathcal{N} = 2$ supersymmetry. Generically, the matter may come in some other representation of the algebra – specifically, with an R-charge other than than $\frac{1}{2}$ – and then the δ-exact terms above cannot be used. In this chapter, following [7, 5], we attempt to find these more general representations, and use them to localize arbitrary $\mathcal{N} = 2$ theories. We will see that in addition to opening up the interesting realm of $\mathcal{N} = 2$ theories to investigation via localization, one is led to insights into RG flow that apply even to nonsupersymmetric three-dimensional theories.

7.1 Representations of $\mathcal{N} = 2$ Superconformal Symmetry

In Chapters 2 and 3, we found representations of the superconformal algebra in terms of multiplets of fields on conformally flat manifolds. In the case of the chiral multiplet, these were motivated in part by the conformal covariance of the Laplacian and Dirac operator, as discussed in Appendix B. These operators are only covariant provided one assigns ϕ a scaling dimension $\frac{1}{2}$ and ψ dimension 1. However, it is known that this is not, in general, the correct dimension in the IR fixed point.

For a simple example, consider the XYZ theory we discussed in the previous chapter. Since this has a superpotential XYZ, which is not renormalized and must have R-charge 2 in the IR, then given the symmetry under exchanging the fields, we see they must each be given R-charge $\frac{2}{3}$. Then by general arguments of Chapter 2, this must also be the scaling dimension of the scalars in the chiral multiplets. This is not consistent with the actions for the scalars we have written above, which are only conformally invariant if we assign them scaling dimension $\frac{1}{2}$. It is also not consistent with the superconformal transformations δ we wrote above.

Before attempting to address this issue, we should note that for the vector multiplet, one typically does not run into this problem. This is because the operator $\star F$ is a conserved current, by the Bianchi identity, and so in an interacting conformal theory it must have dimension 2, consistent with the superconformal transformations for the vector multiplet we discussed above. Actually, one should really only consider gauge-invariant operators like $\mathrm{Tr}(\star F)$, but still, it seems reasonable that the vector multiplet does not receive corrections (in particular, it is real and uncharged under R-symmetry).

The transformations with general R-charge Δ were obtained in [7, 5]. In principle, they can be obtained by commuting the flat space, ordinary supersymmetries with conformal generators, whose action on fields of arbitrary dimension is known. We summarize them here:

$$\begin{aligned} &\delta\phi = 0, && \delta\phi^\dagger = \psi^\dagger\epsilon \\ &\delta\psi = (-i\gamma^\mu D_\mu\phi - i\sigma\phi)\epsilon - 2i\Delta\phi\epsilon', && \delta\psi^\dagger = \epsilon F^\dagger \\ &\delta F = \epsilon(-i\gamma^\mu D_\mu\psi - i\lambda\phi + i\sigma\psi) + i(2\Delta - 1)\epsilon'\psi && \delta F^\dagger = 0 \end{aligned} \tag{7.1}$$

One can check that these give rise to the correct anticommutators, specifically, with the R-charge Δ.

In principle, one now simply needs to compute the new δ-exact actions using these supersymmetries, and see how the 1-loop determinants are modified. This can be done following the procedure outlined in Chapter 4, and involves diagonalizing these new operators. However, there is a trick which slightly simplifies this procedure.

7.1.1 General R Charge as Coupling to Supersymmetric Vector Multiplet

We now present a result which will allow us to perform this computation using only the tools we have already found above. Recall that we can write:[1]

$$\epsilon' = \frac{1}{2}(a + ib^\mu\gamma_\mu)\epsilon \tag{7.2}$$

Let us plug this form into the action above:

[1] Note this differs by a factor of i from the form used above.

$$
\begin{aligned}
&\delta\phi = 0, && \delta\phi^\dagger = \psi^\dagger\epsilon \\
&\delta\psi = (-i\gamma^\mu(D_\mu + i(\Delta - \frac{1}{2})b_\mu)\phi - i(\sigma + a\hat{\Delta}\phi)\epsilon - i\phi\epsilon', && \delta\psi^\dagger = \epsilon F^\dagger \\
&\delta F = \epsilon(-i\gamma^\mu(D_\mu + i(\Delta - \frac{1}{2})b_\mu\psi - i\lambda\phi + i(\sigma + \hat{\Delta}a)\psi), && \delta F^\dagger = 0
\end{aligned}
\tag{7.3}
$$

where we have defined $\hat{\Delta} = \Delta - \frac{1}{2}$. But now we see that this is precisely the transformation law we would obtain when coupling to a background vector multiplet with $\sigma = a$ and $A_\mu = b_\mu$, with the fields having charge $\hat{\Delta}$. Of course, in order for this fact to be useful, such a configuration must be supersymmetric. In Appendix D, it is checked that this is the case provided that one also sets:

$$
D = -\frac{i}{4}R - \frac{i}{2}a^2 + \nabla^\mu b_\mu - \frac{i}{2}b_\mu b^\mu \tag{7.4}
$$

Thus to find the supersymmetric actions for this chiral multiplet, one simply plugs these forms of the background multiplets into the action 3.6.

For S^3, using $a = \frac{i}{r}, b_\mu = 0$, and $R = \frac{6}{r^2}$, where r is the radius of S^3, we get:

$$
S_\delta^{S^3} = \int \sqrt{g}d^3x\Big(-\phi^\dagger D_\mu D^\mu\phi + \phi^\dagger\Big(\sigma^2 + iD + \frac{2i\tilde{\Delta}}{r}\sigma - \frac{1}{r^2}(\tilde{\Delta} - \frac{3}{2})(\tilde{\Delta} + \frac{1}{2})\Big)\phi + i\psi^\dagger(\gamma^\mu D_\mu - \sigma - \frac{i\tilde{\Delta}}{r})\psi + F^\dagger F\Big)
$$

One sees that this is precisely the action found in [7, 5].

Next consider $S^2 \times S^1$. Here one can find spinors ϵ_1, ϵ_2 with:

$$
\nabla_\mu\epsilon_i = \frac{1}{2r}\gamma_\mu \not{u}\,\epsilon_i
$$

where u is the unit vector along the S^1, and r is the radius of the S^2, so that $a = 0, b_\mu = -\frac{i}{r}u_\mu$. Noting that $R = 2/r^2$ on this space, we get the action:

$$
S_\delta^{S^2\times S^1} = \int \sqrt{g}d^3x\Big(-\phi^\dagger D_\mu D^\mu\phi + \phi^\dagger\Big(\sigma^2 + iD - \frac{2\tilde{\Delta}}{r}u^\mu D_\mu - \frac{1}{r^2}(\tilde{\Delta}^2 - \frac{1}{4})\Big)\phi + i\psi^\dagger(\gamma^\mu D_\mu - \sigma + \frac{i\tilde{\Delta}}{r}u_\mu\gamma^\mu)\psi + F^\dagger F\Big)
$$

which agrees with the action used in [20].

7.2 Trial R-Charge

Let us specialize to S^3. In general, a chiral multiplet comes in a certain representation of both the gauge group and the global flavor symmetry group of the theory. After we localize and reduce the gauge and global symmetry groups to their maximal torii, we can list the charges q_a of this multiplet under each $U(1)$ factor. Then, if λ_a denotes the corresponding eigenvalue, we have just seen that the 1-loop determinant is given by:

$$e^{\ell(1-\Delta+i\sum_a q_a\lambda_a)} \tag{7.5}$$

where Δ is the R-charge of the field at the IR fixed point, where there is a unique choice of R-symmetry whose current lies in the same multiplet as the stress-energy tensor. We can write this R charge as:

$$R_{IR} = R_{UV} + \sum_a c_a F_a \tag{7.6}$$

where F_a runs over the abelian global symmetries of the theory. Thus we see:

$$\Delta = \frac{1}{2} + \sum_a c_a q_a \tag{7.7}$$

and the 1-loop determinant becomes:[2]

$$e^{\ell(\frac{1}{2}+i\sum_a q_a(\lambda_a+ic_a))} \tag{7.8}$$

In other words, shifting the R-symmetry by a flavor symmetry is equivalent to coupling that symmetry to a background vector multiplet and giving the scalar a complex expectation value.

In addition, in both cases one can in principle consider the possibility that the R-charge mixes with any $U(1)_J$ topological symmetries that may be present. Since weakly gauging this symmetry is equivalent to adding an FI term, we expect that such a contribution would enter as an imaginary FI term. Such a mixing will often be prohibited by parity.

7.3 F-Maximization

We now have a prescription for localizing a theory whose R-charge is an arbitrary combination of the flavor symmetries. However, in order to get the correct result for the partition function we must choose the c_a correctly. In [7], a prescription for doing this was given.

To motivate it, note that in any conformal field theory, the 1-point function of an operator other than the identity must be zero. This is because such an operator must have nontrivial scaling dimension, and so its expectation value in the scale-invariant vacuum vanishes. In practice, the 1-point function one computes for renormalized operators may be nonzero because they can mix

[2] For ease of notation, we let the index a run over all symmetries of the theory, although for gauge symmetries there is no contribution to the R-symmetry, and so the corresponding c_a are zero.

with the identity operator under RG flow.

With this in mind, let us consider $Z(c_a)$, the partition function as a function of the trial R-symmetry. Since c_a appears in the action as an (imaginary) real mass, taking a derivative with respect to c_a computes the one point function of the corresponding flavor charge. As noted above, this need not be precisely zero, but if it is nonzero it must be imaginary, since it is i times something invariant under parity. Thus if we consider the real part, it must be zero. This gives the condition:

$$\frac{\partial}{\partial c_a}|Z(c_a)| = 0 \tag{7.9}$$

This will not be true for arbitrary c_a. Only for the specific choice of c_a which appears in the superconformal multiplet will one obtain the correct conformal theory, and the condition provides enough constraints to determine all the c_a. For example, it was shown in [7] that this condition reproduces the result $R = \frac{2}{3}$ for the chirals in the XYZ model, as well as the dimensions in some $\mathcal{N} = 2$ Chern-Simons matter theories, which can be computed in the field theory at large k [61].

7.4 F-Theorem

The procedure of F maximization is quite reminiscent of a related story in four dimensions, that of a-maximization. Recall that in even dimensional conformal field theories there may have a conformal anomaly, that is, when one couples to a curved metric the conformal symmetry becomes anomalous, with the divergence of the stress-energy tensor proportional to c-numbers times certain invariants assigned to the metric. In four dimensions, one of these terms is proportional to the Euler density, and the coefficient is called a. In supersymmetric theories, a can be related to an anomaly in the $U(1)_R$ symmetry, and it was shown in [47] that the correct choice of the IR R-symmetry is determined by a set of equations which are equivalent to the maximization of a over trial R-symmetries, much like we found above.

The a anomaly appears in another, more general context as well. Recall that in two dimensions, Zamolodchikov [48] proved the c-theorem, that the conformal anomaly c is strictly decreasing under RG flow. Cardy [49] conjectured a similar statement is true of the a anomaly in four-dimensional theories.

Note that we are no longer restricting to theories with supersymmetry. However, it is possible to motivate the a theorem in the subclass of supersymmetric theories as follows. When one deforms a conformal theory to get a new fixed point, one typically adds a relevant term which breaks some of the global symmetries. Thus when performing a-maximization in this new theory, one is maximizing over a smaller space of trial R-symmetries, and so generically one expects to get a smaller maximum. Thus the a coefficient in the new theory should be smaller. This argument is very heuristic, and has loopholes; for example, there may be extra, accidental symmetries in the IR.

Nevertheless, given the analogies between F maximization and a maximization, as well as the link between a maximization and the a theorem above, in [50] the authors were led to propose the F theorem; that the free energy F of a conformal theory on S^3 is a a good measure of the degrees of freedom, and is a decreasing quantity under RG flow. Note that F is a global quantity, not related to any anomalies, so this theorem has a somewhat different character than the theorems in two and four dimensions. For example, F may be nontrivial even for topological theories, such as Chern-Simons theory, even though, in principle, they contain no local degrees of freedom.

Several preliminary tests of the F theorem were carried out in [50, 51]. Nevertheless, theorem is still open. It is worth noting that the a-theorem, an open conjecture for many years, was recently proven in [52], although it is difficult to see how ideas from that proof can be applied to this problem due to the qualitative differences between the two theorems discussed above.

Chapter 8

Aharony and Giveon-Kutasov Duality

In this chapter we discuss a set of dualities between three-dimensional gauge theories which are reminiscent of Seiberg duality in four dimensions. Recall that in Seiberg duality, a theory with $U(N_c)$ gauge group and N_f hypermultiplets in the fundamental representation is dual (in the IR sense) to one with $U(N_f - N_c)$ gauge group, N_f hypermultiplets, and some additional uncharged matter. In the dualities we consider here, we will find a very similar story. One important difference is that, in three dimensions, there is the new possibility of adding a Chern-Simons term for the gauge group, and we will see this plays an interesting role in the duality. As with 3D mirror symmetry, these can also be motivated by a brane construction, and will have both $\mathcal{N} = 2$ versions and versions with extended supersymmetry. We will test these dualities by comparing partition functions deformed by real masses and FI terms in both the $\mathcal{N} = 2$ case, using the results of the previous chapter, and the extended supersymmetry case, which will turn out to follow as a special case. For simplicity, we will focus on the case of a unitary gauge group, although there is also a version of these dualities with symplectic [53, 54] and orthogonal [55, 56] gauge groups.

8.1 Aharony Duality

To start, we will discuss a duality due to Aharony [53] involving $\mathcal{N} = 2$ gauge theories with unitary gauge groups, but no Chern-Simons terms. The first theory is simply $\mathcal{N} = 2$ $U(N_c)$ Yang-Mills theory with N_f fundamental hypermultiplets $(Q_a, \tilde{Q}^a)$ and no superpotential. As discussed in [4], the Coulomb branch of this nonabelian theory is lifted by instanton corrections to two components of one complex dimension each, parametrized by the monopole operators $V_\pm$, as we saw in the abelian case in Chapter 6. The Higgs branch may be parameterized by $M_a^b = Q_a \tilde{Q}^b$, and is $2N_f - 1$ dimensional, not receiving any quantum corrections.

The dual theory is $\mathcal{N} = 2$ $U(N_f - N_c)$ gauge theory with N_f fundamental hypermultiplets $(\hat{q}_a, \tilde{\hat{q}}^a)$.

Table 8.1: Global Symmetries of the Dual Theories

Field	$SU(N_f) \times SU(N_f)$	$U(1)_A$	$U(1)_J$	$U(1)_{R-UV}$
q_a	$(N_f, 1)$	1	0	$\frac{1}{2}$
$\tilde{q}^a$	$(1, \bar{N}_f)$	1	0	$\frac{1}{2}$
$\hat{q}^a$	$(\bar{N}_f, 1)$	-1	0	$\frac{1}{2}$
$\hat{\tilde{q}}_a$	$(1, N_f)$	-1	0	$\frac{1}{2}$
$\hat{M}^a_b$	$(N_f, \bar{N}_f)$	2	0	1
$\hat{V}_\pm$	$(1, 1)$	$-N_f$	± 1	$\frac{N_f}{2} - N_c + 1$

In addition, there are ${N_f}^2$ uncharged chiral multiplets $\hat{M}^b_a$ and two uncharged chiral multiplets $\hat{V}_\pm$, which couple via the following superpotential

$$\tilde{\hat{q}}_a \hat{M}^a_b q^b + \hat{V}_+ \hat{\tilde{V}}_- + \hat{V}_- \hat{\tilde{V}}_+, \tag{8.1}$$

where $\hat{\tilde{V}}_\pm$ are monopole operators, parameterizing the Coulomb branch of this theory. We emphasize that $\hat{V}_\pm$ are elementary (i.e., noncomposite) fields, while $\tilde{V}_\pm$ are monopole operators, so can in principle be expressed in terms of the other fields. Since this contains the effective Coulomb branch parameters $\hat{\tilde{V}}_\pm$, it does not give a complete microscopic definition of the theory, but it will suffice for our purposes.

As the notation suggests, the conjectured mapping of chiral multiplets identifies M^b_a with $\hat{M}^b_a$ and $V_\pm$ with $\hat{V}_\pm$. The Coulomb branch of the dual theory is lifted by the superpotential, which sets $\hat{\tilde{V}}_\pm = 0$, so all that remains is the Higgs branch. Naively $\hat{V}_\pm$ is also set to zero, but we must remember that $\hat{\tilde{V}}_\pm$ are only effective variables and do not apply everywhere, and in fact $\hat{V}_\pm$ parameterize a moduli space which maps to the Coulomb branch of the original theory. As argued in [4], after some somewhat subtle lifting of parts of the Higgs branch, one can show that it maps to Higgs branch of the original theory.

We can also consider the flavor symmetries of these two theories, and how they are mapped under the duality. Both theories have $SU(N_f) \times SU(N_f)$ global symmetry rotating the two sets of chiral multiplets, as well as a $U(1)_A$ rotating all the chirals by the same phase. In addition, there is the $U(1)_J$ topological symmetry, and a UV $U(1)_R$ symmetry (the IR R-symmetry will get mixed with the other global symmetries). Note that this symmetry group is the same for both theories, so we can summarize how the duality acts by thinking of a single symmetry group which acts on both theories, and listing the charges of the fields of both theories under this group, as shown in Table 8.1.

8.1.1 Mapping of Partition Functions

We now consider the partition function for these theories, and test if they are equal for dual theories. The undeformed partition function for a $U(N_c)$ theory with N_f fundamental hypermultiplets of R-

charge Δ is:

$$\frac{1}{N_c!}\int d^{N_c}\lambda\prod_i e^{N_f\ell(1-\Delta+i\lambda_i)+N_f\ell(1-\Delta-i\lambda_i)}\prod_{i<j}(2\sinh\pi(\lambda_i-\lambda_j)^2) \tag{8.2}$$

In the dual theory, the contribution of the extra fields just give factors $e^{{N_f}^2\ell(1-\Delta_M)}$ and $e^{\ell(1-\Delta_{V_\pm})}$. Note these do not depend on the integration variables λ_i since these fields are not charged.

Corresponding to the two $SU(N_f)$ factors, we can add masses for the two chiral multiplets in each flavor, m_a and $\tilde{m}_a$, which are each constrained to sum to zero. In addition, for $U(1)_A$ there is an total axial mass μ, and for $U(1)_J$ there is the FI term η. We can also set the R-charges for the fields to their UV values, since a mixing of the R-charge with some flavor symmetry can be accounted for by shifting the corresponding deformation parameter by an imaginary value.

Including all of these deformations the partition function for the first theory can be written as:

$$Z^{(U)}_{N_f,N_c}(\eta;m_a;\tilde{m}_a;\mu)=\frac{1}{N_c!}\int\prod_{j=1}^{N_c}\left(d\lambda_j\prod_{a=1}^{N_f}e^{\ell(\frac{1}{2}+i\lambda_j+im_a+i\mu)+\ell(\frac{1}{2}-i\lambda_j-i\tilde{m}_a+i\mu)}\right)\prod_{i<j}(2\sinh\pi(\lambda_i-\lambda_j))^2 \tag{8.3}$$

For the second theory, we see that the representation of $SU(N_f)\times SU(N_f)\times U(1)_A$ in which the quarks lie is replaced by its conjugate, so all mass terms should come in with the opposite sign. Inspecting the table above, we see that the 1-loop partition function for ${M^a}_b$ is:

$$e^{\ell(i(m_a-\tilde{m}_b+2\mu)} \tag{8.4}$$

while that of $V_\pm$ is:

$$e^{\ell(N_c-\frac{N_f}{2}-iN_f\mu\pm i\eta)} \tag{8.5}$$

Thus the dual partition function is given by:

$$Z^{(U)}_{N_f,N_f-N_c}(\eta;-m_a;-\tilde{m}_a;-\mu)e^{\ell(N_c-\frac{N_f}{2}-iN_f\mu+i\eta)+\ell(N_c-\frac{N_f}{2}-iN_f\mu-i\eta)}\prod_{a,b}e^{\ell(2i\mu+im_a-i\tilde{m}_b)} \tag{8.6}$$

We wish to show that these two expressions are equal for all complex values of the deformations. Here we can simply quote the work of [57, 58]. Specifically, the integrals considered in those papers involved the hyperbolic gamma function $\Gamma_h(z;\omega_1,\omega_2)$, a generalization of the ordinary gamma function, but as shown in [54], this is related to the 1-loop determinant by

$$\Gamma_h(z;i,i)=e^{\ell(1+iz)}, \tag{8.7}$$

and after translating theorem 5.5.11 of [58] using this identity, we find we get precisely the equality of the dual partition functions. It would be interesting to study in more detail the origin of these integral identities, and see if they give new insight into the field theory dualities.

Note that we have not yet determined the IR R-charge of the theory. Nevertheless, as discussed above, the duality holds for any possible trial R-charge, and so in particular for the correct one. Thus the duality appears to be completely independent of the issue of the superconformal R-charge. This is not completely true since, as shown in [54], the duality may sometimes imply that a field on one side or the other must have dimension violating the unitarity bound $\Delta \geq \frac{1}{2}$. This signals that there must be new symmetries arising in the IR, possibly from some of the fields becoming decoupled, with which the R-symmetry may mix.

8.2 Giveon-Kutasov Duality

There is another duality, due to Giveon and Kutasov [60], which is very similar to Aharony duality. The main difference is that now there is a Chern-Simons term, and the duality is between groups $U(N_c)$ and $U(|k| + N_f - N_c)$, where k is the Chern-Simons level. Specifically, the theories are:

- $\mathcal{N} = 2$ $U(N_c)$ gauge theory with N_f fundamental hypermultiplets and a Chern-Simons term at level k.
- $\mathcal{N} = 2$ $U(|k| + N_f - N_c)$ gauge theory with N_f fundamental hypermultiplets and a Chern-Simons term at level $-k$. In addition, there are ${N_f}^2$ uncharged chiral multiplets ${M_a}^b$, which couple through a superpotential $\tilde{q}^a {M_a}^b q_b$. There is no $V_\pm$ field.

It turns out one can derive this duality from the duality of the previous section as follows. As mentioned in [4], integrating out a massive charged fermion generates a Chern-Simons term at level $\pm\frac{1}{2}$, whose sign is the same as the sign of the mass of the fermion. Thus if we take a $U(N_c)$ theory with some fundamental hypermultiplets, and add a large positive axial mass to one of them, we generate a level one Chern-Simons term for the gauge group.

Let us now consider an Aharony dual pair with with $N_f + k$ for $k > 0$ hypermultiplets and, respectively, N_c and $N_f + k - N - c$ colors. Suppose we give large positive axial masses to k of the flavors on one side, which corresponds to a negative mass on the dual. In the first theory we generate a level k Chern-Simons term and in the second one at level $-k$ is generated. This procedure also gives a large mass to $V_\pm$ and to some of the M fields, which can then be integrated out. One can see that we obtain precisely the duality described above.

The considerations above can actually be applied at the level of the matrix model to derive the expected mapping of the partition functions of Giveon-Kutasov duals. Specifically, we need to look at the asymptotic behavior of the 1-loop partition function for large mass. In addition to generating a Chern-Simons term, one finds a constant phase, which one can interpret as being due to the fact that we are computing a Chern-Simons partition function using a nonstandard framing of S^3, as discussed in [59]. In fact, a general formula for the mapping of the partition function, including the relative phase, was conjectured in that paper, and we will see that the results here reduce that conjecture to the identity of the partition functions in Section 3.1.

As shown in [54], if we take the 1-loop partition function for a flavor with axial mass M, then for $M \to \pm\infty$:

$$e^{\ell(\frac{1}{2}+i\lambda+iM)+\ell(\frac{1}{2}-i\lambda+iM)} \approx \exp\left(\pm\left(-i\pi\lambda^2 - i\pi M^2 - \pi M + \frac{i\pi}{12}\right)\right) \tag{8.8}$$

where we have ignored terms exponentially small in M. Note that, up to a λ-independent factor, this is precisely the contribution to the matrix model of a level-1 Chern-Simons term, as expected. One can repeat this argument for k hypermultiplets embedded in a $U(N_c)$ theory, and one finds precisely the two partition functions of the Giveon Kutasov duals.

8.3 Extended Supersymmetry

As discussed in [60], adding the $\mathcal{N} = 4$ superpotential to the Giveon-Kutasov theories gives an $\mathcal{N} = 3$ version of this duality. As we saw with some of the mirror pairs involving $(1,1)$ branes, for such theories the flow to the IR only has the effect of removing the Yang-Mills term. Thus we obtain a duality between two superconformal theories for which we can explicitly write down the Lagrangian on both sides.

At the level of the matrix model, this simply corresponds to taking the R-charges fixed to be their UV values, and disallowing axial masses, which are incompatible with the extended supersymmetry. Taking this specialization, we find the map between the dual theories to be:

$$\log(Z^{(U)}_{N_f,N_c,k}(\eta; m_a)) = \log(Z^{(U)}_{N_f,k+N_f-N_c,-k}(\eta; -m_a)) +$$

$$+\frac{\pi i}{12}(k^2 + 3(k+N_f)(N_f-2)+2) + \pi i\eta^2 - k\pi i \sum_a {m_a}^2 \tag{8.9}$$

This agrees with the results of [59], where it was proved in the cases $N_f = 0, 1$ using the original matrix model, but only conjectured for larger N_f.

Appendix A

Spinor and Superspace Conventions

A.1 Spinors in Three Dimensions

In three dimensions, spinors have two components, and transform in the two-dimensional representation of $SL(2,\mathbb{R})$ or $SU(2)$ according to whether the signature is Minkowskian or Euclidean, respectively. We define a spinor with lower components, ψ_α, to transform in the fundamental (e.g., multiplication on the left by a matrix $A_\beta{}^\alpha$), while a spinor ψ^α with upper components transforms in the antifundamental (multiplication on the right). Note $\psi_\alpha{}^* \cong \psi^{\dagger\alpha}$, i.e., taking the conjugate switches how the spinor transforms.

Of course, these representations are actually equivalent, and exchanged by the invariant symbols $\epsilon_{\alpha\beta}$ or $\epsilon^{\alpha\beta}$, which can both be taken as:

$$\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{A.1}$$

We define $\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta$, so that ψ_α can be recovered as $\psi_\alpha = \psi^\beta\epsilon_{\beta\alpha}$. In this way we can simply denote a spinor by ψ, without specifying how it transforms, and reinstate the upper or lower indices as required.

We define the product of spinors as:

$$\psi_1\psi_2 = \psi_1{}^\alpha\psi_{2\alpha} = \epsilon^{\beta\alpha}\psi_{1\alpha}\psi_{2\beta} \tag{A.2}$$

Note that when we reinstate indices, we always do it in such a way that upper indices come before the lower indices they are contracted with. It is important to make this convention clear, because the opposite one differs by a sign. Using this convention we will often suppress spinor indices unless it would cause unnecessary confusion.

We will use the following gamma matrices for Euclidean signature:

$$(\gamma_1)_\alpha{}^\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (\gamma_2)_\alpha{}^\beta = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad (\gamma_3)_\alpha{}^\beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{A.3}$$

For Minkowski signature, one can take $\gamma_0 = i\gamma_2$, and we obtain a real representation. One can check that these satisfy:

$$\gamma_i \gamma_j = \delta_{ij} + i\epsilon_{ijk}\gamma_k \tag{A.4}$$

When we write an equality that is only true numerically in the basis we are using (i.e., the index structure is not the same on both sides), we will use the symbol "$\cong$". Thus, for example

$$\epsilon_{\alpha\beta} \cong \epsilon^{\alpha\beta} \cong i(\gamma_2)_\alpha{}^\beta \tag{A.5}$$

Using this, one can compute:

$$(\gamma_i)^\alpha{}_\gamma = \epsilon^{\alpha\beta}(\gamma_i)_\beta{}^\delta \epsilon_{\delta\gamma} \cong \begin{cases} (\gamma_i)_\alpha{}^\gamma & i \neq 2 \\ -(\gamma_i)_\alpha{}^\gamma & i = 2 \end{cases} \tag{A.6}$$

It is sometimes useful to take the transpose of a spinor equation, e.g.:

$$\psi_1\psi_2 = \psi_1{}^\alpha \psi_{2\alpha} = \epsilon^{\alpha\beta}\psi_{1\beta}\psi_2{}^\gamma \epsilon_{\gamma\alpha} = -\psi_{1\alpha}\psi_2{}^\alpha \tag{A.7}$$

Depending on what object we are considering, we will sometimes treat the components of a spinor as ordinary (commuting) complex numbers, and sometimes as anticommuting Grassman numbers. Thus this becomes:

$$\psi_1\psi_2 = \mp\psi_2\psi_1 \tag{A.8}$$

where the top and bottom sign are for commuting and anticommuting spinors, respectively. Similarly, using the result above and the fact that γ_i is symmetric for $i \neq 2$ and antisymmetric for $i = 2$, we find:

$$\psi_1\gamma_i\psi_2 = \pm\psi_2\gamma_i\psi_1 \tag{A.9}$$

Note also that when taking the conjugate one should also reverse the order of the terms (this only makes a difference for anticommuting spinors). So, for example:

$$(\psi_1\psi_2)^* = \psi_2^\dagger\psi_1^\dagger, \quad (\psi_1\gamma_i\psi_2)^* = (\psi_2^\dagger\gamma_i\psi_1^\dagger) \tag{A.10}$$

where we have used the fact that the gamma matrices are Hermitian.

Fierz identities follow from the basic completeness relation:

$$\delta_\alpha{}^\beta \delta_\gamma{}^\delta = \frac{1}{2}\Big(\delta_\alpha{}^\delta \delta_\gamma{}^\beta + \sum_i (\gamma_i)_\alpha{}^\delta (\gamma_i)_\gamma{}^\beta\Big) \tag{A.11}$$

For example, we have:

$$(\psi_1\psi_2)(\psi_3\psi_4) = \pm\frac{1}{2}\Big((\psi_1\psi_4)(\psi_3\psi_2) + (\psi_1\gamma_i\psi_4)(\psi_3\gamma_i\psi_2)\Big) \tag{A.12}$$

A.2 $\mathcal{N} = 2$ $d = 3$ Superspace

As discussed in the text, $\mathcal{N} = 2$ supersymmetry in three dimensions is just the dimensional reduction of $\mathcal{N} = 1$ supersymmetry in four dimensions. More precisely, suppose we take a representation of the four-dimensional gamma matrices, letting $m \in \{\mu, 4\}$, where $\mu \in \{1, 2, 3\}$:

$$\sigma_{m\alpha}{}^{\dot\beta} \cong (\gamma_{\mu\alpha}{}^\beta, i\delta_\alpha^\beta,)$$

$$\bar\sigma_{m\dot\alpha}{}^\beta \cong (\gamma_{\mu\alpha}{}^\beta, -i\delta_\alpha^\beta)$$

which satisfy:

$$\sigma_m\bar\sigma_n + \sigma_m\bar\sigma_n = \bar\sigma_m\sigma_n + \bar\sigma_m\sigma_n = g_{mn}$$

Then, since we do not allow rotations involving the 0 direction, $\sigma_{4\alpha}{}^{\dot\beta} \cong \delta_\alpha^\beta$ becomes an invariant tensor, which identifies the two Weyl representations and gives the single spinor representation in three dimensions. Thus we can drop the distinction between dotted and undotted tensors. Then it is a simple matter to translate the conventions of Wess and Bagger. We will find a few new features, however, so it is worthwhile to spell out this procedure in detail.

Consider the supersymmetry algebra:

$$\{Q_\alpha, Q_\beta\} = [P_\mu, Q_\alpha] = 0 \tag{A.13}$$

$$\{Q_\alpha, \bar Q_\beta\} = 2\gamma^\mu{}_{\alpha\beta}P_\mu + \epsilon_{\alpha\beta}Z \tag{A.14}$$

Let us attempt to find representations with $Z = 0$. We start by formally defining superspace, parametrized by bosonic coordinates x^μ and fermionic coordinates $\theta^\alpha, \bar\theta^\alpha$. We identify a point in superspace with the operator:

$$e^{ix^\mu P_\mu + i\theta^\alpha Q_\alpha + i\bar{\theta}^\alpha \bar{Q}_\alpha} \tag{A.15}$$

In the special case $\theta = \bar{\theta} = 0$, this gives the translation operator, which can be naturally identified with $\mathbb{R}^3$, but more generally we get an object parametrized by the bosonic coordinate x^μ and fermionic coordinates θ^α and $\bar{\theta}^\alpha$. On this space we can define operators generating left and right translations in θ and $\bar{\theta}$, which we denote $\mathcal{Q}_\alpha, \bar{\mathcal{Q}}_\alpha$ for the left translations and $\mathcal{D}_\alpha$ and $\bar{\mathcal{D}}_\alpha$ for the right. Explicitly, these take the form:

$$\mathcal{Q}_\alpha = \frac{\partial}{\partial \theta^\alpha} + i(\gamma^\mu \bar{\theta})_\alpha \frac{\partial}{\partial x^\mu} \tag{A.16}$$

$$\bar{\mathcal{Q}}_\alpha = \frac{\partial}{\partial \bar{\theta}^\alpha} + i(\gamma^\mu \theta)_\alpha \frac{\partial}{\partial x^\mu} \tag{A.17}$$

$$\mathcal{D}_\alpha = \frac{\partial}{\partial \theta^\alpha} - i(\gamma^\mu \bar{\theta})_\alpha \frac{\partial}{\partial x^\mu} \tag{A.18}$$

$$\bar{\mathcal{D}}_\alpha = \frac{\partial}{\partial \bar{\theta}^\alpha} - i(\gamma^\mu \theta)_\alpha \frac{\partial}{\partial x^\mu} \tag{A.19}$$

These all anticommute, except for:

$$\{\mathcal{Q}_\alpha, \bar{\mathcal{Q}}_\beta\} = -\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_\beta\} = 2i\gamma^\mu{}_{\alpha\beta} \frac{\partial}{\partial x^\mu} \tag{A.20}$$

Thus we can form a representation of the supersymmetry algebra by assigning $Q_\alpha \to \mathcal{Q}_\alpha, \bar{Q}_\alpha \to \bar{\mathcal{Q}}_\alpha$, and $P_\mu \to i\frac{\partial}{\partial x^\mu}$ (or alternatively with the $\mathcal{D}_\alpha$ and $-i\frac{\partial}{\partial x^\mu}$.)

This representation acts on the coordinates of superspace, and by extension also on arbitrary functions of them. Thus we define a superfield, a general function on superspace. Since the components of θ and $\bar{\theta}$ all anticommute and square to zero, this can be written as a finite power series in these parameters, each term multiplying an arbitrary function of x^μ:

$$\Phi(x, \theta, \bar{\theta}) = \phi(x) + \theta^\alpha \psi_\alpha(x) + \bar{\theta}^\alpha \bar{\psi}_\alpha(x) + ... \tag{A.21}$$

This gives a reducible representation of the supersymmetry algebra. One way of reducing it is to impose:

$$\mathcal{D}_\alpha \Phi = 0 \tag{A.22}$$

A field satisfying this condition is called a chiral superfield, and one satisfying the complementary condition:

$$\bar{\mathcal{D}}_\alpha \Phi = 0 \tag{A.23}$$

is called an anti-chiral superfield. Note that, since $\mathcal{D}_\alpha$ and $\bar{\mathcal{D}}_\alpha$ (anti-)commute with the group generators, this condition defines an invariant subspace in the space of superfields. Imposing both of these conditions forces Φ to be constant. The Hermitian conjugate of a chiral field is an anti-chiral one. One can write the most general chiral field as:

$$\Phi = \phi(y) + \theta^\alpha \psi_\alpha(y) + \theta^2 F(y) \tag{A.24}$$

where $y^\mu = x^\mu - i\theta\gamma^\mu\bar{\theta}$. One can check by acting with $\mathcal{Q}_\alpha$ and $\bar{\mathcal{Q}}_\alpha$ on the superfield that the action of the $\epsilon\mathcal{Q}$ on the component fields is given by:

$$\begin{aligned} \delta\phi &= 0, & \delta\phi^\dagger &= \psi^\dagger\epsilon, \\ \delta\psi &= -i\gamma^\mu\partial_\mu\phi\epsilon, & \delta\psi^\dagger &= \epsilon F^\dagger, \\ \delta F &= -i\epsilon\gamma^\mu\partial_\mu\psi, & \delta F^\dagger &= 0. \end{aligned} \tag{A.25}$$

Another multiplet we can define is the vector multiplet, which satisfies:

$$V = V^\dagger \tag{A.26}$$

Note that this condition is preserved under the operation $V \to V + \Lambda + \Lambda^\dagger$ for chiral Λ, and, for reasons that will be clear in a moment, we consider this a gauge symmetry. In a particular choice of gauge, we can expand V as:

$$V = -\theta\gamma^\mu\bar{\theta}A_\mu + i\theta^2\bar{\theta}\lambda - i\bar{\theta}^2\theta\lambda^\dagger + \frac{1}{2}\theta^2\bar{\theta}^2(D - \nabla^\mu A_\mu) \tag{A.27}$$

Note that fixing the gauge breaks supersymmetry; however, performing a supersymmetry transformation followed by a suitable gauge transformation brings us back into this gauge.

Then gauge-invariant actions can be written as:

$$\int d^3x d^4\theta \Phi^\dagger e^{qV} \Phi \tag{A.28}$$

which is invariant under the gauge symmetry $\Phi \to e^{-q\Lambda}\Phi$, $V \to V + \Lambda + \Lambda^\dagger$. Note that because of the gauge-fixing condition, which requires us to perform a gauge transformation after each supersymmetry transformation, the transformation law of the chiral multiplet gets modified to:

$$\begin{aligned}
&\delta\phi = 0, && \delta\phi^\dagger = \psi^\dagger\epsilon, \\
&\delta\psi = (-i\gamma^\mu D_\mu\phi - i\sigma\phi)\epsilon, && \delta\psi^\dagger = \epsilon F^\dagger, \\
&\delta F = \epsilon(-i\gamma^\mu D_\mu\psi + i\sigma\psi - i\lambda\phi), && \delta F^\dagger = 0,
\end{aligned} \tag{A.29}$$

A new possibility in three dimensions is the linear multiplet, satsifying:

$$\mathcal{D}^2\Sigma = \bar{\mathcal{D}}^2\Sigma = 0$$

Conserved currents live in linear multiplets. For example, the current $\star F$ is the $\theta\gamma^\mu\bar{\theta}$ component of the linear multiplet $\Sigma_V = \mathcal{D}\bar{\mathcal{D}}V$, whose lowest component is the gauge multiplet scalar σ.

Lastly, we mention that one can replace the fermionic integrals in the superspace integrals with fermionic derivatives and in fact, at the expense of an extra total derivative term, by the action of $\mathcal{Q}_\alpha$ and $\bar{\mathcal{Q}}_\alpha$, followed by extraction of the $\theta = \bar{\theta} = 0$ term. In other words, we have, for example, for the chiral multiplet action:

$$\int d^3x d^4\theta X^\dagger X = \int d^3x \mathcal{Q}_1\mathcal{Q}_2\bar{\mathcal{Q}}_1\bar{\mathcal{Q}}_2 X^\dagger X\Big|_{\theta=\bar{\theta}=0} \tag{A.30}$$

$$= \int d^3x Q_1 Q_2 \bar{Q}_1 \bar{Q}_2 \phi^\dagger\phi \tag{A.31}$$

Using:

$$\delta_1{}^\dagger\delta_2{}^\dagger(\phi^\dagger\phi) = \epsilon_1^\dagger\epsilon_2^\dagger(\phi^\dagger F + \psi\psi)$$

We recover the expression **??**:

$$S_m = \delta_1\delta_2\int d^3x(\phi^\dagger F + \psi\psi)$$

Similarly, the Yang-Mills action for the gauge multiplet is:

$$S_g = \int d^3x d^4\theta \mathrm{Tr}\Sigma^2$$

Using the same argument as above and the fact that the lowest component of Σ is σ, we can write:

$$S_g = \int d^3x \delta_1\delta_2\delta_1^\dagger\delta_2^\dagger \mathrm{Tr}\sigma^2$$

$$= \delta_1\delta_2 \int d^3x \mathrm{Tr}\lambda\lambda$$

Using the commutators of these operators, we can arrange for any of them to appear first, and so we see that these term are exact under all of the supersymmetries. This is true for any expression that can be written an integral over all of superspace[1] For the F terms, which are only integrated over half of superspace, this is typically not the case.

Note that in these expressions, δ is an ordinary supersymmetry in flat space. However, in Appendix C we will start with this expression but let the δ_i be superconformal symmetries inside an $OSp(2|2)$ subalgebra on a general conformally flat manifold. This guarantees the correct flat space limit as well as the essential property that they are exact under some δ.

A.2.1 Superconformal Algebra

Consider twistor spinors, satisfying:

$$\nabla_\mu \epsilon = \gamma_\mu \epsilon'$$

Let's start with the simplest representation of the superconformal algebra, an ungauged chiral multiplet, with transformation laws:

$$\delta\phi = 0, \quad \delta\phi^\dagger = \psi^\dagger\epsilon$$

$$\delta\psi = (-i\gamma^\mu\partial_\mu\phi)\epsilon - i\phi\epsilon', \quad \delta\psi^\dagger = \epsilon F^\dagger$$

$$\delta F = \epsilon(-i\gamma^\mu\nabla_\mu\psi), \quad \delta F^\dagger = 0$$

One computes:

$$\delta_1\delta_2\phi = 0, \qquad \delta_1\delta_2\phi^\dagger = (\epsilon_1\epsilon_2)F^\dagger,$$

$$\delta_1\delta_2\psi = 0, \qquad \delta_1\delta_2\psi^\dagger = 0,$$

$$\delta_1\delta_2 F = (\epsilon_1\epsilon_2)(-\nabla^2\phi + \frac{R}{8}\phi), \qquad \delta_1\delta_2 F^\dagger = 0$$

[1]An important exception is the FI term, which is a full superspace integral, but the integrand is a nongauge-invariant quantity, and so this argument does not apply.

From which it follows that $[\delta_1, \delta_2] = 0$. Next we have:[2]

$$
\begin{aligned}
[\delta_1, \delta_2^\dagger]\phi &= v^\mu \partial_\mu \phi + \frac{1}{2}\alpha\phi + \frac{i}{2}\beta\phi \\
[\delta_1, \delta_2^\dagger]\psi &= v^\mu \nabla_\mu \psi + \alpha\psi - \frac{i}{2}\beta\psi - \frac{i}{8}\nabla^\mu v^\nu [\gamma_\mu, \gamma_\nu]\psi \\
[\delta_1, \delta_2^\dagger]F &= v^\mu \partial_\mu F + \frac{3}{2}\alpha F - \frac{3i}{2}\beta F
\end{aligned}
\tag{A.32}
$$

where we have defined:

$$v^\mu = -i\epsilon_2^\dagger \gamma^\mu \epsilon_1$$

$$\alpha = \frac{1}{3}\nabla_\mu v^\mu = -i(\epsilon_2^\dagger \epsilon_1' + \epsilon_2^{\dagger\prime}\epsilon_1)$$

$$\beta = \epsilon_2^{\dagger\prime}\epsilon_1 - \epsilon_2^\dagger \epsilon_1'$$

The terms involving v^μ and $\alpha = \frac{1}{3}\nabla_\mu v^\mu$ are the variations expected under the conformal transformations corresponding to v^μ, which we check is a conformal Killing vector:

$$\nabla_\mu v_\nu + \nabla_\nu v_\mu = -i\epsilon_2^{\dagger\prime}\gamma_\mu\gamma_\nu\epsilon_1 - i\epsilon_2^\dagger\gamma_\nu\gamma_\mu\epsilon_1' + \mu \leftrightarrow \nu$$

$$= -2ig_{\mu\nu}(\epsilon_2^{\dagger\prime}\epsilon_1 + \epsilon_2^\dagger\epsilon_1)$$

which is proportional to the metric as required.

The term involving β are related to the R-symmetry.

A.2.2 Flat Space

To extract a better understanding of the algebra, it is useful to specialize to flat space, where we can write the most general conformal Killing spinor as:

$$\epsilon = \epsilon_s + x^\mu \gamma_\mu \epsilon_c$$

for constant spinors ϵ_s, ϵ_c. Then we compute:

$$v^\mu = a^\mu + b^{\mu\nu}x_\nu + cx^\mu + 2x^\mu d_\nu x^\nu - d^\mu x^2$$

[2]Here we do not bother writing the separate expressions for $\delta_1\delta_2^\dagger$ and $\delta_2^\dagger\delta_1$, as they will not have much use for us. On the other hand, $\delta_1\delta_2$ will be relevant below.

$$\alpha = \frac{1}{3}\nabla_\mu v^\mu = c + 2d_\mu x^\mu$$

$$\beta = \epsilon_{c2}^\dagger \epsilon_{s1} - \epsilon_{s2}^\dagger \epsilon_{c1}$$

where we have defined:

$$a^\mu = -i\epsilon_{s2}^\dagger \gamma^\mu \epsilon_{s1}$$

$$b^{\mu\nu} = -i(\epsilon_{s2}^\dagger \gamma^{\mu\nu} \epsilon_{c1} + \epsilon_{c2}^\dagger \gamma^{\nu\mu} \epsilon_{s1})$$

$$c = -i(\epsilon_{s2}^\dagger \epsilon_{c1} + \epsilon_{c2}^\dagger \epsilon_{s1})$$

$$d^\mu = -i\epsilon_{c2}^\dagger \gamma^\mu \epsilon_{c1}$$

We recognize the form of v^μ as the most general conformal Killing vector in flat space.

With this in mind, let us define the operators Q_α, S_α by associating the transformations δ to the action of $\epsilon_s Q + \epsilon_c S$, and conformal generators $P^\mu, M_{\mu\nu}, D$, and K^μ by assigning them to $a^\mu, b^{\mu\nu}, c$, and d^μ, respectively. Finally, we associate β to the operator R. From this we obtain the algebra:

$$\{Q_\alpha, Q_\beta\} = \{Q_\alpha, S_\beta\} = \{S_\alpha, S_\beta\} = 0$$

$$\{Q_\alpha, \tilde{Q}_\beta\} = -i{\gamma_\mu}_{\alpha\beta} P^\mu$$

$$\{S_\alpha, \tilde{S}_\beta\} = -i{\gamma_\mu}_{\alpha\beta} K^\mu$$

$$\{Q_\alpha, \tilde{S}_\beta\} = -i{\gamma_{\mu\nu}}_{\alpha\beta} M^{\mu\nu} - i\epsilon_{\alpha\beta} D - R$$

$$\{\tilde{Q}_\alpha, S_\beta\} = -i{\gamma_{\mu\nu}}_{\alpha\beta} M^{\mu\nu} - i\epsilon_{\alpha\beta} D + R$$

A.2.3 Gauge Multiplet

When we allow gauge fields, things can get a little more complicated because the field transformations can involve additional terms which are gauge transformations, and so do not affect the physical supersymmetry algebra. Recall that in the presence of a gauge field, the matter transformations are modified to:

$$\delta\phi = 0, \quad \delta\phi^\dagger = \psi^\dagger\epsilon$$

$$\delta\psi = (-i\gamma^\mu D_\mu\phi - i\sigma\phi)\epsilon - i\phi\epsilon', \quad \delta\psi^\dagger = \epsilon F^\dagger$$

$$\delta F = \epsilon(-i\gamma^\mu D_\mu\psi - i\lambda\phi + i\sigma\psi), \quad \delta F^\dagger = 0$$

This modifies the computation above slightly:

$$\delta_1\delta_2\phi = 0, \qquad \delta_1\delta_2\phi^\dagger = (\epsilon_1\epsilon_2)F^\dagger,$$

$$\delta_1\delta_2\psi = -i(\epsilon_1\epsilon_2)\lambda^\dagger\psi, \qquad \delta_1\delta_2\psi^\dagger = 0,$$

$$\delta_1\delta_2 F = (\epsilon_1\epsilon_2)(-D^2\phi + \sigma^2\phi + iD\phi + \frac{R}{8}\phi + i\lambda^\dagger\psi), \qquad \delta_1\delta_2 F^\dagger = 0$$

For the gauge sector, we have:

$$\delta A_\mu = -\frac{i}{2}\lambda^\dagger\gamma_\mu\epsilon, \quad \delta\sigma = -\frac{1}{2}\lambda^\dagger\epsilon, \quad \delta D = -\frac{i}{2}(D_\mu\lambda^\dagger)\gamma^\mu\epsilon - \frac{i}{2}\gamma[\lambda^\dagger,\sigma]\epsilon + \frac{i}{2}\lambda^\dagger\epsilon'$$

$$\delta\lambda = (i\gamma^\mu(-\frac{1}{2}\epsilon_{\mu\nu\rho}F^{\nu\rho} + D_\mu\sigma) - D)\epsilon + 2i\epsilon'\sigma, \quad \delta\lambda^\dagger = 0$$

Since $\delta\lambda^\dagger = 0$, it is clear that $\delta_1\delta_2 = 0$ for all fields except λ. As shown in Appendix C.2, we find:

$$\delta_1\delta_2\lambda = -i(\epsilon_1\epsilon_2)(\gamma^\mu D_\mu\lambda^\dagger + [\lambda^\dagger,\sigma]) - \frac{i}{2}\bigg((\lambda^\dagger\epsilon_1)\epsilon_2' + (\lambda^\dagger\epsilon_1)\epsilon_2'\bigg)$$

Appendix B

Conformal Transformations

In this appendix we review some basic properties of certain differential operators under conformal mappings.

B.1 Setup

Consider a Riemannian manifold M with a metric g. We pick a local orthonormal basis of tangent vectors e_i, satisfying $g(e_i, e_j) = \delta_{ij}$. We denote the dual basis of 1-forms by $\tilde{e}_i$, satisfying $\tilde{e}_i \cdot e_j = \delta_{ij}$.

We want to write the Levi-Cevita connection ∇ in this basis. Let us define a matrix of (locally defined) 1-forms ω_{ij} by:

$$\nabla e_i = \omega_{ij} e_j \tag{B.1}$$

The Levi-Cevita connection can be derived by imposing that it preserves the metric, and is torsion free. The first condition gives $0 = g(\nabla e_i, e_j) + g(e_i, \nabla e_j) = \omega_{ij} + \omega_{ji}$, implying ω_{ij} is antisymmetric. The second gives:

$$\nabla_{e_i} e_j - \nabla_{e_j} e_i = [e_i, e_j] \tag{B.2}$$

It will be convenient to express the connection in terms of the Lie brackets of the basis vectors. Let us write:

$$[e_i, e_j] = \tau_{ij}^k e_k \tag{B.3}$$

Then the torsion-free condition can be rewritten as:

$$\omega_{jk}(e_i) - \omega_{ik}(e_j) = \tau_{ij}^k \tag{B.4}$$

Together with the antisymmetry, this allows us to solve for ω_{ij} as:

$$\omega_{ij}(e_k) = -\frac{1}{2}\left(\tau^k_{ij} + \tau^j_{ik} - \tau^i_{jk}\right) \tag{B.5}$$

Next consider performing a conformal transformation. This amounts to defining a new metric g' on M by $g' = e^{-2\Omega}g$. We associate to this a new orthonormal basis $e'_i = e^{\Omega}e_i$. From above, we see we can express the connection in this new basis in terms of the new Lie brackets, and these are computed straightforwardly:

$$[e'_i, e'_j] = [e^{\Omega}e_i, e^{\Omega}e_j] = e^{2\Omega}\left([e_i, e_j] + (\partial_i\Omega)e_j - (\partial_j\Omega)e_i\right) \tag{B.6}$$

$$= e^{\Omega}\left(\tau^k_{ij} + (\partial_i\Omega)\delta_{jk} - (\partial_j\Omega)\delta_{ik}\right)e'_k \tag{B.7}$$

where ∂_i is the derivative along e_i. Then we can read off:

$$\tau^{k\,\prime}_{ij} = e^{\Omega}\left(\tau^k_{ij} + (\partial_i\Omega)\delta_{jk} - (\partial_j\Omega)\delta_{ik}\right) \tag{B.8}$$

and so the connection is given by:

$$\omega'_{ij}(e'_k) = e^{\Omega}\left(\omega_{ij}(e_k) - (\partial_i\Omega)\delta_{jk} + (\partial_j\Omega)\delta_{ik}\right) \tag{B.9}$$

B.2 Curvature

First we will determine how the curvature changes under this mapping. We can write the curvature in terms of the connection as follows:

$$R_{klij}e_l = (\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i} - \nabla_{[e_i,e_j]})e_k \tag{B.10}$$

$$= \nabla_i(\omega_{kl}(e_j)e_l) - \nabla_j(\omega_{kl}(e_i)e_l) - \tau^m_{ij}\omega_{kl}(e_m)e_l \tag{B.11}$$

$$= \partial_i\omega_{kl}(e_j)e_l + \omega_{km}(e_j)\omega_{ml}(e_i)e_l - \partial_j\omega_{kl}(e_i)e_l + \omega_{km}(e_i)\omega_{ml}(e_j)e_l - (\omega_{jm}(e_i) - \omega_{im}(e_j))\omega_{kl}(e_m)e_l \tag{B.12}$$

$$\Rightarrow R_{klij} = \partial_i\omega_{kl}(e_j) + \omega_{km}(e_i)\omega_{lm}(e_j) + \omega_{im}(e_j)\omega_{kl}(e_m) - i \leftrightarrow j \tag{B.13}$$

Now we can compute the change in this under a conformal transformation. After a tedious but straightforward calculation, one finds:

$$R'_{klij} = e^{2\Omega}\Big(R_{klij} - (\nabla_i\nabla_k\Omega + \partial_i\Omega\partial_k\Omega)\delta_{jl} + (\nabla_i\nabla_l\Omega + \partial_i\Omega\partial_l\Omega)\delta_{jk} + \tag{B.14}$$

$$+(\nabla_j\nabla_k\Omega + \partial_j\Omega\partial_k\Omega)\delta_{il} - (\nabla_j\nabla_l\Omega + \partial_j\Omega\partial_l\Omega)\delta_{ik} + (\partial\Omega)^2(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\Big) \tag{B.15}$$

where we have written:

$$\nabla_i\nabla_j\Omega = \partial_i\partial_j - \omega_{jk}(e_i)\partial_k\Omega \tag{B.16}$$

ie, the second covariant derivative of Ω.

Contracting the jk indices, we get the change in the Ricci curvature:

$$R'_{li} = e^{2\Omega}\Big(R_{li} + (d-2)(\nabla_i\nabla_l\Omega + \partial_i\Omega\partial_l\Omega) - (\nabla^2\Omega - (d-2)(\partial\Omega)^2)\delta_{il}\Big) \tag{B.17}$$

And finally, contracting the remaining il indices, we get the Ricci scalar:

$$R' = e^{2\Omega}\Big(R + 2(d-1)\Big(\nabla^2\Omega - \frac{d-2}{2}(\partial\Omega)^2\Big)\Big) \tag{B.18}$$

B.3 Spinor Covariant Derivative

Using this expression, we can determine how spinors should transform under conformal transformations. The spinor covariant derivative can be written:

$$\nabla_i\psi = \partial_i\psi + \frac{i}{8}[\gamma_j, \gamma_k]\omega_{jk}(e_i)\psi \tag{B.19}$$

Let us see how this transforms under a conformal transformation. We expect ψ to get multiplied by some power of the conformal scale factor:

$$\psi' = e^{\Delta\Omega}\psi \tag{B.20}$$

where Δ is the dimension of the spinor, which we will determine shortly. Then, in the new spin connection:

$$\nabla'_i\psi' = \partial'_i\psi' - \frac{1}{8}[\gamma_j, \gamma_k]\omega'_{jk}(e'_i)\psi' \tag{B.21}$$

$$= e^{\Omega}\partial_i e^{\Delta\Omega}\psi - \frac{1}{8}[\gamma_j, \gamma_k]e^{\Omega}\Big(\omega_{jk}(e_i) - (\partial_j\Omega)\delta_{ki} + (\partial_k\Omega)\delta_{ji}\Big)e^{\Delta\Omega}\psi \tag{B.22}$$

$$= e^{(\Delta+1)\Omega}\left(\nabla_i\psi + \Delta(\partial_i\Omega)\psi - \frac{1}{2}(\gamma_k\gamma_j - \delta_{jk})(\partial_j\Omega)\delta_{ki}\psi\right) \tag{B.23}$$

$$= e^{(\Delta+1)\Omega}\left(\nabla_i\psi + (\Delta + \frac{1}{2})(\partial_i\Omega)\psi - \frac{1}{2}\gamma_i\gamma_j(\partial_j\Omega)\psi\right) \tag{B.24}$$

Now we do not expect the spinor covariant derivative itself to transform in any particularly nice way. However, there are two related quantities that do. First, consider the Killing spinor equation:

$$\nabla_i\epsilon = \gamma_i\tilde{\epsilon} \tag{B.25}$$

This equation essentially says that the derivative of ϵ contains only spin 1/2 components, no spin 3/2 ones. One can see that Killing spinors map to Killing spinors under a conformal transformation provided we assign them dimension $\Delta = -\frac{1}{2}$, for then, in the new metric:

$$\nabla'_i\epsilon' = e^{\Omega/2}\left(\nabla_i\epsilon - \frac{1}{2}\gamma_i\gamma_j(\partial_j\Omega)\epsilon\right) \tag{B.26}$$

$$= \gamma_i e^{\Omega/2}\left(\tilde{\epsilon} - \frac{1}{2}(\partial_j\Omega)\gamma_j\epsilon\right) \tag{B.27}$$

so that ϵ' is a Killing spinor in the new metric, with associated $\tilde{\epsilon}'$ related by:

$$\tilde{\epsilon}' = e^{\Omega/2}\left(\tilde{\epsilon} - \frac{1}{2}(\partial_j\Omega)\gamma_j\epsilon\right) \tag{B.28}$$

Note that if we define a vector v by:

$$v^\mu = \epsilon^\dagger\gamma^\mu\epsilon \tag{B.29}$$

Then since $\gamma^\mu \to e^\Omega\gamma^\mu$, this quantity does not transform under a conformal transformation. However the associated 1-form v_μ will pick up a factor of $e^{-2\Omega}$.

Next consider the Dirac operator, $\not{\nabla}' = \gamma_i\nabla_i$. Using the transformation of the spinor covariant derivative above, we see:

$$\not{\nabla}'\psi' = e^{(\Delta+1)\Omega}\gamma_i\left(\nabla_i\psi + (\Delta + \frac{1}{2})(\partial_i\Omega)\psi - \frac{1}{2}\gamma_i\gamma_j(\partial_j\Omega)\psi\right) \tag{B.30}$$

$$= e^{(\Delta+1)\Omega}\left(\not{\nabla}'\psi + (\Delta - \frac{d-1}{2})(\partial_i\Omega)\gamma_i\psi\right) \tag{B.31}$$

So that, if we take $\Delta = \frac{d-1}{2}$, we find:

$$\not{\nabla}'\psi' = e^{(\Delta+1)\Omega}\not{\nabla}\psi \tag{B.32}$$

This is consistent with the fact that $\sqrt{g}\psi^\dagger\not{\nabla}\psi$ is a scalar density and should not transform under conformal mappings[1].

Before moving on, we mention another useful result:

$$([\nabla_{e_i}, \nabla_{e_j}] - \nabla_{[e_i,e_j]})\psi = \frac{1}{8}R_{ijkl}[\gamma_k, \gamma_l]\psi$$

This can be shown by an explicit computation, but also follows from the fact that the operator acting on ψ on the LHS is just a rotation corresponding to the kl indices in R_{ijkl}, regardless of the quantity it is acting on, and the RHS just represents this rotation in the appropriate way for a spinor.

Using this, we can derive a few more useful results. First, we compute:

$$\not{\nabla}\not{\nabla}\psi = \gamma_i\gamma_j\nabla_i\nabla_j\psi$$

$$= (\delta_{ij} + \frac{1}{2}[\gamma_i, \gamma_j])\nabla_i\nabla_j\psi$$

$$= (\nabla^2 + \frac{1}{4}[\gamma_i, \gamma_j][\nabla_i, \nabla_j])\psi$$

$$= (\nabla^2 + \frac{1}{32}[\gamma_i, \gamma_j][\gamma_k, \gamma_l]R_{ijkl})\psi$$

Using the first Bianchi idnetity ($R_{ijkl} + R_{iklj} + R_{iljk} = 0$) and gamma matrix identities, one can show:

$$[\gamma_i, \gamma_j][\gamma_k, \gamma_l]R_{ijkl} = 8R$$

where $R = \delta_{ik}\delta_{jl}R_{ijkl}$ is the Ricci scalar. Thus we get:

$$\not{\nabla}^2\psi = (\nabla^2 + \frac{1}{4}R)\psi$$

This is called the Lichnerowicz formula.

Next we return to Killing spinors. Assume we have a Killing spinor, satisfying:

[1]Specifically, under constant conformal mappings. As we will see, $\sqrt{g}\phi\nabla^2\phi$ is also a scalar density, but is invariant under transformations only up to some extra terms involving the change in curvature of the metric

$$\nabla_i \epsilon = \gamma_i \epsilon'$$

Now suppose we apply another derivative to this equation:

$$\nabla_j \nabla_i \epsilon = \gamma_i \nabla_j \epsilon'$$

$$\Rightarrow \nabla_i \epsilon' = \frac{1}{d} \gamma_j \nabla_i \nabla_j \epsilon$$

$$= \frac{1}{d} \gamma_j (\nabla_j \nabla_i + [\nabla_j, \nabla_j]) \epsilon$$

$$= \frac{1}{d} \gamma_j (\nabla_j \gamma_i \epsilon' + \frac{1}{8} R_{ijkl} [\gamma_k, \gamma_l] \epsilon)$$

Using the first Bianchi identity again, the second term can be rewritten in terms of the Ricci curvature $R_{ik} = \delta_{jl} R_{ijkl}$, and we get:

$$\nabla_i \epsilon' = \frac{1}{d} (-\gamma_i \gamma_j \nabla_j \epsilon' + 2 \nabla_i \epsilon' + \frac{1}{4} R_{ij} \gamma_j \epsilon)$$

$$\Rightarrow \nabla_i \epsilon' = \frac{1}{d-2} (-\gamma_i \gamma_j \nabla_j \epsilon' + \frac{1}{4} R_{ij} \gamma_j \epsilon)$$

We can do a little better. Note that if we contract this with γ_i, we get:

$$\gamma_i \nabla_i \epsilon' = \frac{1}{d-2} (-d \gamma_j \nabla_j \epsilon' + \frac{1}{4} R_{ij} \gamma_i \gamma_j \epsilon)$$

$$\Rightarrow \gamma_i \nabla_i \epsilon' = \frac{R}{8(d-1)} \epsilon$$

But now we can plug this expression for $\gamma_j \nabla_j \epsilon$ back in above, and we get:

$$\nabla_i \epsilon' = \frac{1}{d-2} (-\gamma_i \frac{R}{8(d-1)} + \frac{1}{4} R_{ij} \gamma_j) \epsilon$$

Now note that if the manifold is an Einstein mainfold, which means the Ricci curvature is pure trace, ie, $R_{ij} = \frac{1}{d} R \delta_{ij}$, then we get:

$$\Rightarrow \nabla_i \epsilon' = -\frac{R}{8d(d-1)} \gamma_i \epsilon$$

In other words, ϵ' itself is a Killing spinor, and the corresponding ϵ'' is just proportional to the Ricci scalar times ϵ. This is not the case on general manifolds. If moreover R is a constant (so that we are on a sphere or hyperbolic space), then the pair of equations:

$$\nabla_i \epsilon = \gamma_i \epsilon'$$

$$\nabla_i \epsilon' = -\frac{R}{8d(d-1)} \gamma_i \epsilon$$

can be diagonalized, and we see there are Killing spinors with ϵ' proportional to ϵ, namely:

$$\nabla_i \epsilon_\pm = \pm i \sqrt{\frac{R}{8d(d-1)}} \gamma_i \epsilon_\pm$$

B.4 Scalars

The scalar Laplacian is given by:

$$\nabla^2 \phi = \partial_i \partial_i \phi - \omega_{ik}(e_i) \partial_k \phi \tag{B.33}$$

We write $\phi' = e^{\Delta\Omega}\phi$, and compute:

$$\nabla^{2\prime} \phi' = e^{\Omega} \partial_i e^{\Omega} \partial_i e^{\Delta\Omega} \phi - e^{\Omega} (\omega_{ik}(e_i) - (\partial_i \Omega)\delta_{ik} + d(\partial_k \Omega)) e^{\Omega} \partial_k e^{\Delta\Omega} \phi \tag{B.34}$$

$$= e^{(\Delta+2)\Omega} \left(\nabla^2 \phi + (2\Delta + 2 - d)(\partial_i \Omega)\partial_i \phi + (\Delta \nabla^2 \Omega + \Delta(\Delta - d + 2)(\partial\Omega)^2)\phi \right) \tag{B.35}$$

We can eliminate the $\partial_i \phi$ term if we set $\Delta = \frac{d-2}{2}$. Then we find:

$$\nabla^{2\prime} \phi' = e^{(\Delta+2)\Omega} \left(\nabla^2 \phi + \frac{d-2}{2} (\nabla^2 \Omega - \frac{d-2}{2} (\partial\Omega)^2) \phi \right) \tag{B.36}$$

But we recognize the extra term as being proportional to the change in the Ricci scalar computed above. Thus we see that:

$$\left(\nabla^{2\prime} - \frac{1}{4} \left(\frac{d-2}{d-1} \right) R' \right) \phi' = e^{(\Delta+2)\Omega} \left(\nabla^2 - \frac{1}{4} \left(\frac{d-2}{d-1} \right) R \right) \phi \tag{B.37}$$

B.5 Killing Spinors

The Killing spinor equation can be stated on any d-dimensional spin manifold, as:

$$\nabla_\mu \epsilon = \gamma_\mu \epsilon' \tag{B.38}$$

Note that ϵ' is determined by the condition $\epsilon' = \frac{1}{d}\nabla\!\!\!/\, \epsilon$, and the above equation imposes $d-1$ conditions on ϵ. Intuitively, one can think of it as imposing that the spin-$\frac{3}{2}$ combination of the supersymmetry parameters vanishes, which is necessary since we do not expect bosonic operators of spin higher than 1.

In $\mathbb{R}^3$, general solution is:

$$\epsilon = \epsilon_s + x^\mu \gamma_\mu \epsilon_c \tag{B.39}$$

where ϵ_s, ϵ_c are constant spinors with two complex components each, so that the space of solution has four complex dimensions. One can check the Killing spinor equation is satisfied with $\epsilon' = \epsilon_c$.

The Killing spinor equation has nice properties under conformal mappings, as discussed above. In particular, for any conformally flat manifold M with metric $g_{ij} = e^{-2\Omega}\delta_{ij}$, if one takes the natural vielbein $e_i = e^{\Omega}\frac{\partial}{\partial x^i}$, one finds a 4 complex dimensional space of Killing spinors obtained by taking any of the $\mathbb{R}^3$ Killing spinors ϵ and multiplying it by $e^{-\Omega/2}$. In particular, on S^3, with metric:

$$g^{S^3}_{ij} = \frac{4}{1+(x/R)^2}\delta_{ij} \tag{B.40}$$

The Killing spinors have the form:

$$\epsilon = (\frac{1}{2}(1+(x/R)^2))^{-1/2}(\epsilon_s + x^\mu \gamma_\mu \epsilon_c) \tag{B.41}$$

for constant ϵ_s, ϵ_c. It will be convenient to set $\epsilon_s = \pm i \epsilon_c$, then we get:

$$\epsilon = \sum_{\pm} \frac{(1 \pm i x^\mu \gamma_\mu)}{(1+(x/R)^2)^{1/2}}\epsilon_\pm \tag{B.42}$$

for constant $\epsilon_\pm$. For reasons that will become clear when we interpret S^3 as the group manifold $SU(2)$, we call the spinors with the plus and minus sign left- and right-invariant, respectively. We see that a left- or right-invariant spinor has constant norm, and one can check that:

$$\nabla_\mu \epsilon_\pm = \pm \frac{i}{2R}\gamma_\mu \epsilon_\pm \tag{B.43}$$

ie, ϵ' is proportional to ϵ in this case. As discussed above, one can find such Killing spinors on Einstein manifolds with constant, nonzero Ricci scalar, i.e., S^n or $\mathbb{H}^n$.

Appendix C

Supersymmetric Actions

C.1 Matter Case

First we compute the δ-exact term for the matter. Let us take two Killing spinors ϵ_1 and ϵ_2. We compute:

$$S_\delta^m = \delta_1\delta_2(\phi^\dagger F) \tag{C.1}$$

For completeness, we will proceed directly to the case of a gauge theory, so that the SUSY transformations become:

$$\begin{aligned}
&\delta\phi = 0, && \delta\phi^\dagger = \psi^\dagger\epsilon, \\
&\delta\psi = (-i\gamma^\mu D_\mu\phi - i\sigma\phi)\epsilon, && \delta\psi^\dagger = \epsilon F^\dagger, \\
&\delta F = \epsilon(-i\gamma^\mu D_\mu\psi + i\sigma\psi - i\lambda\phi), && \delta F^\dagger = 0,
\end{aligned} \tag{C.2}$$

However, for simplicity, we will assume the gauge multiplet is in a BPS configuration, i.e., its component fields are all annihilated by δ_1.

We compute the fermionic part first. This is given by:

$$\begin{aligned}
S_\delta^{mF} &= \delta_1\phi^\dagger\delta_2 F + \delta_2\phi^\dagger\delta_1 F \\
&= (\psi^\dagger\epsilon_2)(\epsilon_1(-i\gamma^\mu D_\mu\psi + i\sigma\psi)) + (\psi^\dagger\epsilon_1)(\epsilon_2(-i\gamma^\mu D_\mu\psi + i\sigma\psi))
\end{aligned} \tag{C.3}$$

This can be Fierz rearranged using the identity (for anticommuting spinors):

$$(\eta_1\eta_2)(\eta_3\eta_4) = -\frac{1}{2}((\eta_1\eta_4)(\eta_3\eta_2) + (\eta_1\gamma_\mu\eta_4)(\eta_3\gamma^\mu\eta_2)) \tag{C.4}$$

If we also note $\epsilon_1\epsilon_2 = \epsilon_2\epsilon_1$ and $\epsilon_1\gamma_\mu\epsilon_2 = -\epsilon_2\gamma_\mu\epsilon_1$, we see that this becomes:

$$(\epsilon_1\epsilon_2)(\psi^\dagger(i\gamma^\mu D_\mu - i\sigma)\psi) \tag{C.5}$$

Next consider the bosonic part:

$$\begin{aligned} S_\delta^{mB} &= (\delta_1\delta_2\phi^\dagger)F + \phi^\dagger(\delta_1\delta_2 F) \\ &= (\epsilon_1\epsilon_2)F^\dagger F + \phi^\dagger\bigg(\epsilon_2(-i\gamma^\mu D_\mu + i\sigma)[(-i\gamma^\mu D_\mu\phi - i\sigma\phi)\epsilon_1 - i\phi\epsilon_1']\bigg) \end{aligned} \tag{C.6}$$

where we have used the fact that the variations of the gauge fields vanish by assumption. Using $\nabla\!\!\!/\epsilon' = -(R/8)\epsilon$ and some other manipulations, this can be simplified to:

$$(\epsilon_1\epsilon_2)(-\phi^\dagger D_\mu D^\mu\phi + \phi^\dagger(\sigma^2 + \frac{R}{8})\phi + F^\dagger F) + (\epsilon_2\gamma^\mu\epsilon_1)\phi^\dagger(\frac{1}{2}\sqrt{g}\epsilon_{\mu\nu\rho}F^{\nu\rho} - D_\mu\sigma)\phi - 2(\epsilon_2\epsilon_1')\phi^\dagger\sigma\phi \tag{C.7}$$

But recall we are allowed to use the BPS equation on the gauge multiplet. Specifically, the vanishing of $\delta\lambda$ gives:

$$0 = (i\gamma^\mu(-\frac{1}{2}\sqrt{g}\epsilon_{\mu\nu\rho}F^{\nu\rho} + D_\mu\sigma) - D)\epsilon_1 - \frac{i}{2}\sigma\epsilon_1' \tag{C.8}$$

This allows us to rewrite the $\epsilon_2\gamma^\mu\epsilon_1$ term, which contributes a piece proportional to $\phi^\dagger\sigma\phi\epsilon_2\epsilon_1'$ which cancels the similar term above, and we are left with:

$$(\epsilon_1\epsilon_2)\bigg(-\phi^\dagger D_\mu D^\mu\phi + \phi^\dagger(\sigma^2 + iD + \frac{R}{8})\phi + F^\dagger F\bigg) \tag{C.9}$$

The overall factor of $\epsilon_1\epsilon_2$ can be discarded, and we are left with:

$$S_\delta^m = \int \sqrt{g}d^3x\bigg(-\phi^\dagger D_\mu D^\mu\phi + \phi^\dagger(\sigma^2 + iD + \frac{R}{8})\phi + i\psi^\dagger(\gamma^\mu D_\mu - \sigma)\psi + F^\dagger F\bigg) \tag{C.10}$$

C.1.1 Gauge Case

Now we look for an action of the form:

$$S_\delta^g = \delta_1\delta_2\int\sqrt{g}d^3x\mathrm{Tr}(\lambda\lambda) \tag{C.11}$$

The supersymmetry transformations are:

$$\delta A_\mu = -\frac{i}{2}\lambda^\dagger\gamma_\mu\epsilon, \quad \delta\sigma = -\frac{1}{2}\lambda^\dagger\epsilon, \quad \delta D = -\frac{i}{2}(D_\mu\lambda^\dagger)\gamma^\mu\epsilon - \frac{i}{2}\gamma[\lambda^\dagger,\sigma]\epsilon,$$

$$\delta\lambda = (i\gamma^\mu(-\frac{1}{2}\epsilon_{\mu\nu\rho}F^{\nu\rho} + D_\mu\sigma) - D)\epsilon, \quad \delta\lambda^\dagger = 0. \tag{C.12}$$

As discussed in the text, it we will write:

$$\epsilon_i' = \frac{1}{2}(ia_i + b_i{}^\mu\gamma_\mu)\epsilon_i \tag{C.13}$$

for some real functions $a_i, b_i{}^\mu$. This is possible for an arbitrary spinor on the RHS, provided ϵ_i does not vanish, as will always be the case. Moreover, in the cases of interest in this thesis, one can find linearly independent spinors ϵ_1, ϵ_2 which have the same a and b_μ, so that we can drop the subscript. Then the supersymmetry transformations involving ϵ' can be rewritten in terms of ϵ alone:

$$\delta D = -\frac{i}{2}(D_\mu + \frac{1}{2}b_\mu)\lambda^\dagger\gamma^\mu\epsilon + \frac{i}{2}[\lambda^\dagger,\sigma]\epsilon + \frac{1}{4}a\lambda^\dagger\epsilon \tag{C.14}$$

$$\delta\lambda = (i\gamma_\mu(-\frac{1}{2}\sqrt{g}^{-1}\epsilon^{\mu\nu\rho}F_{\nu\rho} + D^\mu\sigma + b_\mu\sigma) - (D + a\sigma))\epsilon \tag{C.15}$$

Now the bosonic part of the action is given by:

$$S_\delta^{gB} = \delta_1\lambda\delta_2\lambda = \epsilon_2\bigg(- i\gamma_\mu(-\frac{1}{2}\sqrt{g}^{-1}\epsilon^{\mu\nu\rho}F_{\nu\rho} + D^\mu\sigma + b_\mu\sigma) - (D + a\sigma)\bigg)\times \tag{C.16}$$

$$\times\bigg(i\gamma_\mu(-\frac{1}{2}\sqrt{g}^{-1}\epsilon^{\mu\nu\rho}F_{\nu\rho} + D^\mu\sigma + b_\mu\sigma) - (D + a\sigma)\bigg)\epsilon_1 \tag{C.17}$$

The cross-terms vanish, and we find:

$$S_\delta^{gB} = (\epsilon_2\epsilon_1)\bigg((-\frac{1}{2}\sqrt{g}^{-1}\epsilon^{\mu\nu\rho}F_{\nu\rho} + D^\mu\sigma + b_\mu\sigma)^2 + (D + a\sigma)^2\bigg) \tag{C.18}$$

Next consider the fermionic part:

$$S_\delta^{gF} = 2\lambda\delta_1\delta_2\lambda \tag{C.19}$$

From C.52 below, we have:

$$\delta_1\delta_2\lambda = -i(\epsilon_1\epsilon_2)(\gamma^\mu D_\mu\lambda^\dagger + [\lambda^\dagger,\sigma]) - \frac{i}{2}\bigg((\lambda^\dagger\epsilon_1)\epsilon_2' + (\lambda^\dagger\epsilon_1)\epsilon_2'\bigg) \tag{C.20}$$

Using the proposed form of ϵ_i' and Fierz rearranging, the last term simply gives $-a(\epsilon_1\epsilon_2)\lambda^\dagger$. Thus we can write:

$$S_\delta^{gF} = 2(\epsilon_1\epsilon_2)\lambda\Big(- i\gamma^\mu D_\mu \lambda^\dagger - i[\lambda^\dagger, \sigma] - a\lambda^\dagger\Big) \tag{C.21}$$

To summarize, out δ-exact gauge action is given by (dropping the overall factor of $\epsilon_1\epsilon_2$):

$$S_\delta^g = \int \sqrt{g} d^3x \Big((-\frac{1}{2}\sqrt{g}^{-1}\epsilon^{\mu\nu\rho}F_{\nu\rho} + D^\mu\sigma + b_\mu\sigma)^2 + (D + a\sigma)^2 + \lambda\Big(- i\gamma^\mu D_\mu\lambda^\dagger - i[\lambda^\dagger, \sigma] - a\lambda^\dagger\Big)\Big) \tag{C.22}$$

C.2 Supersymmetric Chern-Simons Theory

The supersymmetric Chern-Simons action is given by:

$$S[A, \sigma, D, \lambda, \lambda^\dagger] = \int \sqrt{g}\mathrm{Tr}\Big(\sqrt{g}^{-1}\epsilon^{\mu\nu\rho}(A_\mu\partial_\nu A_\rho + \frac{2i}{3}A_\mu A_\nu A_\rho) + 2\sigma D - \lambda^\dagger\lambda\Big) \tag{C.23}$$

The supersymmetry transformations depend on spinor parameters ϵ, η (which are sections of some spinor bundle on M, more on this later), and are given by:

$$\delta A_\mu = \frac{i}{2}(\eta^\dagger\gamma_\mu\lambda - \lambda^\dagger\gamma_\mu\epsilon), \quad \delta\sigma = -\frac{1}{2}(\eta^\dagger\lambda + \lambda^\dagger\epsilon) \tag{C.24}$$

$$\delta D = \frac{i}{2}(\eta^\dagger\gamma^\mu(D_\mu\lambda) - (D_\mu\lambda^\dagger)\gamma^\mu\epsilon) - \frac{i}{2}\gamma(\eta^\dagger[\lambda, \sigma] - [\lambda^\dagger, \sigma]\epsilon) + \frac{i}{2}\beta((\nabla_\mu\eta^\dagger)\gamma^\mu\lambda - \lambda^\dagger\gamma^\mu(\nabla_\mu\epsilon)) \tag{C.25}$$

$$W = W_\mu dx^\mu = (-\frac{1}{2}\sqrt{g}\epsilon_{\mu\nu\rho}F^{\nu\rho} + D_\mu\sigma)dx^\mu = -\frac{1}{2}\star F + \mathcal{D}\sigma \tag{C.26}$$

We have also introduced parameters α, β, γ, which we will determine in a moment. Note that this transformation is generically complex, with the conjugate obtained by exchanging ϵ with η, and real transformations are obtained by setting $\epsilon = \eta$.

First we check that the action is invariant under this transformation. We have:

$$\delta S \sim \sqrt{g}^{-1}\epsilon^{\mu\nu\rho}((\delta A_\mu F_{\nu\rho}) + \nabla_\mu(\delta A_\nu A_\rho)) + 2\delta\sigma D + 2\sigma\delta D - \delta\lambda^\dagger\lambda - \lambda^\dagger\delta\lambda \tag{C.27}$$

Where the first two terms come from the variation of the original Chern-Simons action. We will ignore the total derivative term for now, and reinstate it later.

Let us define δ_ϵ to be the supersymmetry with $\eta = 0$ and $\delta_\epsilon^\dagger$ to be its conjugate, so that the most general supersymmetry described above corresponds to $\delta_\epsilon + \delta_\eta^\dagger$. Then, since the action is real, it suffices to check its invariance under δ_ϵ

$$\delta_\epsilon S \sim \sqrt{g}^{-1}\epsilon^{\mu\nu\rho}((-\frac{i}{2}\lambda^\dagger\gamma_\mu\epsilon)F_{\nu\rho}) + 2(-\frac{1}{2}\lambda^\dagger\epsilon)D+ \tag{C.28}$$

$$+2\sigma(-\frac{i}{2}(D_\mu\lambda^\dagger)\gamma^\mu\epsilon + \frac{i}{2}\gamma[\lambda^\dagger,\sigma]\epsilon - \frac{i}{2}\beta\lambda^\dagger\gamma^\mu\nabla_\mu\epsilon) - \lambda^\dagger((i\gamma^\mu W_\mu - D)\epsilon + i\alpha\sigma\gamma^\mu(\nabla_\mu\epsilon)) \tag{C.29}$$

The terms involving D cancel, and the one proportional to $\sigma[\lambda^\dagger,\sigma]$ will vanish under the trace. The term involving $F_{\mu\nu}$ partially cancels against the term involving W_μ, and we are left with:

$$-i\sigma(D_\mu\lambda^\dagger)\gamma^\mu\epsilon - i(\beta+\alpha)\sigma\lambda^\dagger\gamma^\mu(\nabla_\mu\epsilon) - i\lambda^\dagger\gamma^\mu\epsilon(D_\mu\sigma) \tag{C.30}$$

We see that if we set $\alpha+\beta=1$, this becomes:

$$-i\nabla_\mu(\sigma\lambda^\dagger\gamma^\mu\epsilon) = 2\nabla_\mu(\sigma\delta A^\mu) \tag{C.31}$$

Putting this all together, we see that, for $\alpha+\beta=1$ and any γ, the variation of the action is a total derivative, namely:

$$\delta S = \int \sqrt{g}\nabla_\mu \mathrm{Tr}((2g^{\mu\nu}\sigma + \sqrt{g}^{-1}\epsilon^{\mu\nu\rho}A_\rho)\delta A_\nu) \tag{C.32}$$

where δA_ν is the supersymmetry variation given above. This holds for $\delta^\dagger$ as well. On a manifold with a boundary, one must impose boundary conditions such that this total derivative vanishes. For example, in addition to the condition that one component of the gauge field tangent to the boundary vanishes, we need to impose a condition on λ,ϵ and/or η that preserves this when we apply a supersymmetry, and then a condition on the normal component and σ.

Note that we found no special conditions need to be imposed on ϵ or η, except possibly some boundary conditions. This is related to the fact that this theory is topological, and has an infinite-dimensional diffeomorphism symmetry, which should be related to the anticommutator of these supersymmetries. We will check this in the next two subsections.

C.2.1 Computation of $[\delta,\delta]$

Since δ acting on the bosons only produces $\lambda^\dagger$'s (as opposed to λ's), which are annihilated by δ, $[\delta_1,\delta_2]=0$ on the bosons, and also on $\lambda^\dagger$. However, this is not immediate for λ, and in fact, as we will see it does not automatically vanish, but only if we impose certain conditions on the parameters α,β, and γ.

We will approach this problem by decomposing $\delta_1\delta_2\lambda$ into a symmetric and antisymmetric part under $1 \leftrightarrow 2$, and then impose that the latter part vanishes. To start, we have:

$$\delta_1\delta_2\lambda = \delta_1((i\gamma^\mu W_\mu - D)\epsilon_2 + i\alpha\sigma\gamma^\mu\nabla_\mu\epsilon_2) \tag{C.33}$$

$$= (i\gamma^\mu(-\frac{1}{2}\sqrt{g}\epsilon_{\mu\nu\rho}\delta_1 F^{\nu\rho} + \delta_1(D_\mu\sigma)) - \delta_1 D)\epsilon_2 + i\alpha\delta_1\sigma\gamma^\mu(\nabla_\mu\epsilon_2) \tag{C.34}$$

$$= \Big(i\gamma^\mu(-\frac{1}{2}\sqrt{g}\epsilon_{\mu\nu\rho}(-i(D^\nu\lambda^\dagger)\gamma^\rho\epsilon_1 - i\lambda^\dagger\gamma^\rho(\nabla^\nu\epsilon_1)) - \frac{1}{2}(D_\mu\lambda^\dagger)\epsilon_1 - \frac{1}{2}\lambda^\dagger(\nabla_\mu\epsilon_1) + \frac{1}{2}[\lambda^\dagger,\sigma]\gamma_\mu\epsilon_1) - \tag{C.35}$$

$$-(-\frac{i}{2}(D_\mu\lambda^\dagger)\gamma^\mu\epsilon_1 + \frac{i}{2}\gamma[\lambda^\dagger,\sigma]\epsilon_1 - \frac{i}{2}\beta\lambda^\dagger\gamma^\mu(\nabla_\mu\epsilon_1))\Big)\epsilon_2 + i\alpha(-\frac{1}{2}\lambda^\dagger\epsilon_1)\gamma^\mu(\nabla_\mu\epsilon_2) \tag{C.36}$$

We see there are three types of terms here: those involving $D_\mu\lambda^\dagger$, those involving $[\lambda^\dagger,\sigma]$, and those involving $\nabla_\mu\epsilon_1$ (in the last term, we can add a term symmetric in $1 \leftrightarrow 2$ to trade $\nabla_\mu\epsilon_2$ for $\nabla_\mu\epsilon_1$; we will reinstate it at the end). The first group gives:

$$(i\gamma^\mu\epsilon_2)(\frac{i}{2}\sqrt{g}\epsilon_{\mu\nu\rho}(D^\nu\lambda^\dagger)\gamma^\rho\epsilon_1 - \frac{1}{2}(D_\mu\lambda^\dagger)\epsilon_1) + (\epsilon_2)(\frac{i}{2}(D_\mu\lambda^\dagger)\gamma^\mu\epsilon_1) \tag{C.37}$$

Using $[\gamma_\mu,\gamma_\nu] = 2i\sqrt{g}\epsilon_{\mu\nu\rho}\gamma^\rho$, we can rewrite this as:

$$\frac{i}{2}\Big((\epsilon_2)((D^\nu\lambda^\dagger)\gamma_\nu\epsilon_1) + (\gamma_\mu\epsilon_2)((D^\nu\lambda^\dagger)(\frac{1}{2}[\gamma_\mu,\gamma_\nu] - g_{\mu\nu})\epsilon_1)\Big) \tag{C.38}$$

$$= \frac{i}{2}\Big((\epsilon_2)((D^\nu\lambda^\dagger)\gamma_\nu\epsilon_1) - (\gamma_\mu\epsilon_2)((D^\nu\lambda^\dagger)\gamma_\nu\gamma_\mu\epsilon_1)\Big) \tag{C.39}$$

$$= -\frac{i}{2}\Big((\epsilon_2)(\epsilon_1\gamma_\nu(D^\nu\lambda^\dagger) + (\gamma_\mu\epsilon_2)(\epsilon_1\gamma_\mu\gamma_\nu(D^\nu\lambda^\dagger))\Big) \tag{C.40}$$

Now note the following Fierz identity for anticommuting spinors:

$$({\eta_1}^\dagger\eta_2)({\eta_3}^\dagger\eta_4) = -\frac{1}{2}\Big(({\eta_1}^\dagger\eta_4)({\eta_3}^\dagger\eta_2) + ({\eta_1}^\dagger\gamma_\mu\eta_4)({\eta_3}^\dagger\gamma^\mu\eta_2)\Big) \tag{C.41}$$

Using this above, we see that the first group of terms gives:

$$-i(\epsilon_1\epsilon_2)\gamma^\mu D_\mu\lambda^\dagger \tag{C.42}$$

This is symmetric in $1 \leftrightarrow 2$, as required (note this amounts to the absence of a term of involving

$\epsilon_1\gamma^\mu\epsilon_2$, which is antisymmetric).

Next consider the group of terms involving $[\lambda^\dagger, \sigma]$:

$$\frac{i}{2}(\gamma^\mu\epsilon_2)([\lambda^\dagger,\sigma]\gamma_\mu\epsilon_1) - \frac{i}{2}\gamma(\epsilon_2)([\lambda^\dagger,\sigma]\epsilon_1) \tag{C.43}$$

We see that if $\gamma = 1$, this is of the same form as above, and so can be rewritten:

$$-i(\epsilon_1\epsilon_2)[\lambda^\dagger,\sigma] \tag{C.44}$$

If $\gamma \neq 1$, one finds a contribution to $[\delta_1,\delta_2]\lambda$ proportional to $[\sigma,\lambda^*]$, and the variation of the action under this transformation vanishes because of the cyclicity of the trace. This symmetry is not physically very interesting, so we will set $\gamma = 1$.

Finally, there are the terms involving $\nabla_\mu\epsilon_1$. These give:

$$(i\gamma^\mu\epsilon_2)(\frac{1}{4}\lambda^\dagger[\gamma_\mu,\gamma_\nu](\nabla^\nu\epsilon_1) - \frac{1}{2}\lambda^\dagger(\nabla_\mu\epsilon_1)) + (\epsilon_2)(\frac{i}{2}\beta\lambda^\dagger\gamma^\mu(\nabla_\mu\epsilon_1)) + \frac{i}{2}\alpha(\gamma^\mu(\nabla_\mu\epsilon_1))(\lambda^\dagger\epsilon_2) \tag{C.45}$$

Using a Fierz identity on the last term, we can rewrite this as:

$$-i(1-\frac{3\alpha}{2})(\epsilon_2\gamma^\mu\epsilon_1)(\gamma_\mu\lambda^\dagger\gamma^\mu(\nabla_\mu\epsilon_1)) - \frac{i}{2}(\gamma^\mu\epsilon_2)(\lambda^\dagger\gamma_\nu\gamma_\mu(\nabla^\nu\epsilon_1)) - \frac{i}{4}\alpha(\gamma^\mu\epsilon_2)(\lambda^\dagger\gamma_\mu\gamma_\nu(\nabla^\nu\epsilon_1)) \tag{C.46}$$

where we have used $\alpha + \beta = 1$ to eliminate β.

To get further, it will be useful to write:

$$\nabla_\mu\epsilon = \gamma_\mu\epsilon' + S_\mu \tag{C.47}$$

where S_μ is an object with one vector and one spinor index, with the latter being suppressed as usual. If we set:

$$\epsilon' = \frac{1}{3}\gamma^\mu\nabla_\mu\epsilon \tag{C.48}$$

then we see $\gamma^\mu S_\mu = 0$. Plugging this in and Fierz rearranging, we find:

$$\Big(-i(1-\frac{3\alpha}{2})(\epsilon_2\gamma^\mu\epsilon_1) + 2iS_1{}^\nu\gamma^\mu\epsilon_2\Big)(\gamma_\mu\lambda^\dagger) \tag{C.49}$$

If it does not vanish, it gives a contribution to $[\delta_1,\delta_2]$ which shifts λ by an amount proportional to $\lambda\dagger$, and the action is invariant under this transformation because the gamma matrices (with two lower indices) are symmetric, so $\delta S \sim \lambda^\dagger\gamma_\mu\lambda^\dagger = 0$.

However, as argued above, it will be convenient to impose conditions such that $[\delta_1, \delta_2] = 0$ identically. This will be guaranteed provided we set:

$$\alpha = \frac{2}{3}, \beta = \frac{1}{3} \tag{C.50}$$

as well as $S_\mu = 0$, or:

$$\nabla_\mu \epsilon = \gamma_\mu \epsilon' \tag{C.51}$$

ie, ϵ is a Killing spinor. Then, putting this all together, and, reinstating the symmetric piece in we dropped before, we find:

$$\delta_1 \delta_2 \lambda = -i(\epsilon_1 \epsilon_2)(\gamma^\mu D_\mu \lambda^\dagger + [\lambda^\dagger, \sigma]) - \frac{i}{2}\Big((\lambda^\dagger \epsilon_1)\epsilon_2' + (\lambda^\dagger \epsilon_1)\epsilon_2' \Big) \tag{C.52}$$

Appendix D

R-charge as a BPS Vector Multiplet

As discussed in the text, we consider spinor pairs satisfying

$$\epsilon_i' = \frac{1}{2}(ia + b^\mu \gamma_\mu)\epsilon_i \tag{D.1}$$

Then we wish to write the superconformal transformations for matter of noncanonical R-charge in 7.1 in terms of coupling to a vector multiplet with $\sigma = a$, $b_\mu = A_\mu$.

If we want to use this point of view to do localization, then this background vector multiplet must be BPS. This would follow from the equation:

$$\begin{aligned} 0 = \delta\lambda &= (i\gamma^\mu(-\frac{1}{2}\sqrt{g}\epsilon_{\mu\nu\rho}F^{\nu\rho} + D_\mu\sigma) - D)\epsilon + 2i\sigma\epsilon' \\ &= (i\gamma^\mu(-\sqrt{g}\epsilon_{\mu\nu\rho}\partial^\nu b^\rho + \partial_\mu a + ib_\mu a) - D + ia^2)\epsilon \end{aligned} \tag{D.2}$$

We are free to pick a value of D, although it is not clear that we can do this in such a way that this expression vanishes. However, consider the following identity, which holds for any Killing spinor:

$$\gamma^\mu \nabla_\mu \epsilon' = -\frac{R}{8}\epsilon \tag{D.3}$$

Plugging in the proposed form for ϵ', we get:

$$
\begin{aligned}
-\frac{R}{4}\epsilon &= \gamma^\mu \nabla_\mu (a + ib_\nu \gamma^\nu)\epsilon \\
&= (\gamma^\mu \partial_\mu a + i\nabla_\mu b_\nu \gamma^\mu \gamma^\nu)\epsilon + \gamma^\mu (a + ib_\nu \gamma^\nu)\gamma_\mu \epsilon' \\
&= (\gamma^\mu \partial_\mu a + i\nabla^\mu b_\mu - \sqrt{g}\epsilon_{\mu\nu\rho}\partial^\mu b^\nu \gamma^\rho)\epsilon + \frac{1}{2}(3a - ib_\nu \gamma^\nu)(a + ib_\rho \gamma^\rho)\epsilon \\
&= (\gamma^\mu (\partial_\mu a - \sqrt{g}\epsilon_{\mu\nu\rho}\partial^\nu b^\rho + iab_\mu) + i\nabla^\mu b_\mu + \frac{3}{2}a^2 + \frac{1}{2}b_\mu b^\mu)\epsilon
\end{aligned}
$$

Notice that the quantity contracted with the gamma matrices is precisely the one that appears in the BPS condition. We can rearrange this to write:

$$
\gamma^\mu (\partial_\mu a - \sqrt{g}\epsilon_{\mu\nu\rho}\partial^\nu b^\rho + iab_\mu)\epsilon = (-\frac{R}{4} - i\nabla^\mu b_\mu - \frac{3}{2}a^2 - \frac{1}{2}b_\mu b^\mu)\epsilon \tag{D.4}
$$

Plugging this into the BPS condition, we arrive at:

$$
0 = (-i(\frac{R}{4} - i\nabla^\mu b_\mu - \frac{3}{2}a^2 - \frac{1}{2}b_\mu b^\mu) - D + ia^2)\epsilon \tag{D.5}
$$

So that we can make this vanish by setting:

$$
D = -\frac{i}{4}R - \frac{i}{2}a^2 + \nabla^\mu b_\mu - \frac{i}{2}b_\mu b^\mu \tag{D.6}
$$

Note we did not impose that a or b_μ be real, as would usually be the case for a vector multiplet.

Bibliography

[1] S. Coleman, “Quantum sine-Gordon equation as the massive Thirring model,” Phys. Rev. D 11, 20882097 (1975).

[2] R. Haag, J. T. Lopuszanski and M. Sohnius, “All Possible Generators Of Supersymmetries Of The S Matrix,” Nucl. Phys. B 88 (1975) 257.

[3] V. Pestun, “Localization of gauge theory on a four-sphere and supersymmetric Wilson loops,” arXiv:0712.2824 [hep-th].

[4] O. Aharony, A. Hanany, K. A. Intriligator, N. Seiberg and M. J. Strassler, Nucl. Phys. B **499**, 67 (1997) [hep-th/9703110].

[5] N. Hama, K. Hosomichi and S. Lee, “Notes on SUSY Gauge Theories on Three-Sphere,” JHEP **1103**, 127 (2011) [arXiv:1012.3512 [hep-th]].

[6] J. H. Schwarz, “Superconformal Chern-Simons theories,” JHEP **0411**, 078 (2004) [hep-th/0411077].

[7] D. L. Jafferis, “The Exact Superconformal R-Symmetry Extremizes Z,” arXiv:1012.3210 [hep-th].

[8] D. Gaiotto and E. Witten, “Janus Configurations, Chern-Simons Couplings, And The Theta-Angle in N=4 Super Yang-Mills Theory,” JHEP **1006**, 097 (2010) [arXiv:0804.2907 [hep-th]].

[9] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, “N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals,” JHEP **0810**, 091 (2008) [arXiv:0806.1218 [hep-th]].

[10] G. Festuccia and N. Seiberg, “Rigid Supersymmetric Theories in Curved Superspace,” JHEP **1106**, 114 (2011) [arXiv:1105.0689 [hep-th]].

[11] Baum, Helga, “Conformal Killing spinors and special geometric structures in Lorentzian geometry – a survey” eprint arXiv:math/0202008

[12] Witten, Edward, "Quantum field theory and the Jones polynomial" Comm. Math. Phys. Volume 121, Number 3 (1989), 351-399.

[13] Alvarez-Gaum, Luis, "Supersymmetry and the Atiyah-Singer index theorem" Comm. Math. Phys. Volume 90, Number 2 (1983), 161-173.

[14, 15] [14]

[14] M. Marino, "Chern-Simons theory, matrix integrals, and perturbative three-manifold invariants," Commun. Math. Phys. **253**, 25 (2004) [arXiv:hep-th/0207096].

[15] M. Aganagic, A. Klemm, M. Marino and C. Vafa, "Matrix model as a mirror of Chern-Simons theory," JHEP **0402**, 010 (2004) [arXiv:hep-th/0211098].

[16] A. Kapustin, B. Willett and I. Yaakov, "Exact Results for Wilson Loops in Superconformal Chern-Simons Theories with Matter," JHEP **1003**, 089 (2010) [arXiv:0909.4559 [hep-th]].

[17] N. Drukker, M. Marino and P. Putrov, "From weak to strong coupling in ABJM theory," Commun. Math. Phys. **306**, 511 (2011) [arXiv:1007.3837 [hep-th]].

[18] A. Kapustin and B. Willett, arXiv:1106.2484 [hep-th].

[19] S. Kim, Nucl. Phys. B **821**, 241 (2009) [arXiv:0903.4172 [hep-th]].

[20] Y. Imamura and S. Yokoyama, JHEP **1104**, 007 (2011) [arXiv:1101.0557 [hep-th]].

[21] J. Bhattacharya, S. Bhattacharyya, S. Minwalla and S. Raju, JHEP **0802**, 064 (2008) [arXiv:0801.1435 [hep-th]].

[22] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)] [hep-th/9711200].

[23] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2**, 253 (1998) [hep-th/9802150].

[24] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, "Large N field theories, string theory and gravity," Phys. Rept. **323**, 183 (2000) [hep-th/9905111].

[25] I. R. Klebanov and G. Torri, "M2-branes and AdS/CFT," Int. J. Mod. Phys. A **25**, 332 (2010) [arXiv:0909.1580 [hep-th]].

[26] C. P. Herzog, I. R. Klebanov and P. Ouyang, "D-branes on the conifold and N=1 gauge / gravity dualities," hep-th/0205100.

[27] S. J. Rey, T. Suyama and S. Yamaguchi, "Wilson Loops in Superconformal Chern-Simons Theory and Fundamental Strings JHEP **0903**, 127 (2009) [arXiv:0809.3786 [hep-th]].

[28] N. Drukker, J. Plefka and D. Young, "Wilson loops in 3-dimensional N=6 supersymmetric Chern-Simons Theory and their string theory duals," JHEP **0811**, 019 (2008) [arXiv:0809.2787 [hep-th]].

[29] J. Bagger and N. Lambert, "Comments on multiple M2-branes," JHEP **0802**, 105 (2008) [arXiv:0712.3738 [hep-th]].

[30] A. Gustavsson, "Algebraic structures on parallel M2-branes," Nucl. Phys. B **811**, 66 (2009) [arXiv:0709.1260 [hep-th]].

[31] A. Gustavsson and S. -J. Rey, arXiv:0906.3568 [hep-th].

[32] D. Bashkirov and A. Kapustin, JHEP **1105**, 015 (2011) [arXiv:1007.4861 [hep-th]].

[33] N. Drukker and D. Trancanelli, "A Supermatrix model for N=6 super Chern-Simons-matter theory," JHEP **1002**, 058 (2010) [arXiv:0912.3006 [hep-th]].

[34] M. Marino and P. Putrov, "Exact Results in ABJM Theory from Topological Strings," JHEP **1006**, 011 (2010) [arXiv:0912.3074 [hep-th]].

[35] M. Marino, "Lectures on localization and matrix models in supersymmetric Chern-Simons-matter theories," J. Phys. A A **44**, 463001 (2011) [arXiv:1104.0783 [hep-th]].

[36] B. Chen and J. -B. Wu, "Supersymmetric Wilson Loops in N=6 Super Chern-Simons-matter theory," Nucl. Phys. B **825**, 38 (2010) [arXiv:0809.2863 [hep-th]].

[37] K. A. Intriligator and N. Seiberg, "Mirror symmetry in three-dimensional gauge theories," Phys. Lett. B **387**, 513 (1996) [hep-th/9607207].

[38] A. Hanany and E. Witten, "Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics," Nucl. Phys. B **492**, 152 (1997) [hep-th/9611230].

[39] J. de Boer, K. Hori, H. Ooguri and Y. Oz, "Mirror symmetry in three-dimensional gauge theories, quivers and D-branes," Nucl. Phys. B **493**, 101 (1997) [hep-th/9611063].

[40] J. de Boer, K. Hori, H. Ooguri, Y. Oz and Z. Yin, "Mirror symmetry in three-dimensional theories, SL(2,Z) and D-brane moduli spaces," Nucl. Phys. B **493**, 148 (1997) [hep-th/9612131].

[41] V. Borokhov, A. Kapustin and X. -k. Wu, "Topological disorder operators in three-dimensional conformal field theory," JHEP **0211**, 049 (2002) [hep-th/0206054].

[42] A. Kapustin and M. J. Strassler, "On mirror symmetry in three-dimensional Abelian gauge theories," JHEP **9904**, 021 (1999) [hep-th/9902033].

[43] E. Witten, “SL(2,Z) action on three-dimensional conformal field theories with Abelian symmetry,” In *Shifman, M. (ed.) et al.: From fields to strings, vol. 2* 1173-1200 [hep-th/0307041].

[44] A. Kapustin, B. Willett and I. Yaakov, JHEP **1010**, 013 (2010) [arXiv:1003.5694 [hep-th]].

[45] K. Jensen and A. Karch, “ABJM Mirrors and a Duality of Dualities,” JHEP **0909**, 004 (2009) [arXiv:0906.3013 [hep-th]].

[46] D. R. Gulotta, C. P. Herzog and S. S. Pufu, JHEP **1112**, 077 (2011) [arXiv:1105.2817 [hep-th]].

[47] K. A. Intriligator and B. Wecht, “The Exact superconformal R symmetry maximizes a,” Nucl. Phys. B **667**, 183 (2003) [hep-th/0304128].

[48] Zamolodchikov, A. B. “Irreversibility” of the Flux of the Renormalization Group in a 2-D Field Theory,” JETP Lett 43, pp 730-732, 1986

[49] Cardy, J. L., Phys. Lett. B 215, 749752 (1988).

[61]

[50] D. L. Jafferis, I. R. Klebanov, S. S. Pufu and B. R. Safdi, “Towards the F-Theorem: N=2 Field Theories on the Three-Sphere,” JHEP **1106**, 102 (2011) [arXiv:1103.1181 [hep-th]].

[51] I. R. Klebanov, S. S. Pufu and B. R. Safdi, “F-Theorem without Supersymmetry,” JHEP **1110**, 038 (2011) [arXiv:1105.4598 [hep-th]].

[52] Z. Komargodski and A. Schwimmer, “On Renormalization Group Flows in Four Dimensions,” JHEP **1112**, 099 (2011) [arXiv:1107.3987 [hep-th]].

[53] O. Aharony, “IR duality in d = 3 N=2 supersymmetric USp(2N(c)) and U(N(c)) gauge theories,” Phys. Lett. B **404**, 71 (1997) [hep-th/9703215].

[54] B. Willett and I. Yaakov, “N=2 Dualities and Z Extremization in Three Dimensions,” arXiv:1104.0487 [hep-th].

[55] C. Hwang, K. -J. Park and J. Park, “Evidence for Aharony duality for orthogonal gauge groups,” JHEP **1111**, 011 (2011) [arXiv:1109.2828 [hep-th]].

[56] O. Aharony and I. Shamir, “On $O(N_c)$ $d = 3$ $\mathcal{N} = 2$ supersymmetric QCD Theories,” JHEP **1112**, 043 (2011) [arXiv:1109.5081 [hep-th]].

[57] E. Rains, “Transformations of elliptic hypergeometric integrals,” Annals of Mathematics, 171 (2010), 169243

[58] F. van de Bult, “Hyperbolic Hypergeometric Functions,” Thesis (2008)

[59] A. Kapustin, B. Willett and I. Yaakov, arXiv:1012.4021 [hep-th].

[60] A. Giveon and D. Kutasov, Nucl. Phys. B **812**, 1 (2009) [arXiv:0808.0360 [hep-th]].

[61] D. Gaiotto and X. Yin, "Notes on superconformal Chern-Simons-Matter theories," JHEP **0708**, 056 (2007) [arXiv:0704.3740 [hep-th]].