Operational Calculus and the Finite Part of Divergent Integrals

Thesis by

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In Partial Fulfillment of the Requirements
For the Degree of
Doctor of Philosophy

California Institute of Technology
Pasadena, California

1960

Acknowledgements

The author wishes to thank Professor A. Erdélyi for suggesting the topic studied in this thesis, for encouragement and advice, and for many hours devoted to reading drafts of the thesis. While writing the thesis the author was supported financially by the California Institute of Technology and the National Science Foundation. The author wishes to thank Miss Evangeline Gibson for her efficiency and cheerfulness in typing the thesis.

ABSTRACT

In this thesis the operational calculus of J. Mikusiński is utilized to study the finite part of divergent convolution integrals.

In Chapters 2 and 3 the idea of an analytic operator function is utilized. An operator function $f(z)$ is said to be an analytic operator function on an open region S of the complex plane if there is an operator $a \neq 0$ such that $a f(z) = \{a f(z, t)\}$ has a partial derivative with respect to z which is continuous on $S \times [0, \infty)$. Let $f(z)$ be an analytic operator function and suppose that $\{f(z, t)\}$ is a continuous function on $S \times [0, \infty)$. Suppose also that for each $t \geq 0$ $f(z, t)$ is an analytic function of z on a larger region $S^* \supset S$. Let $f^*(z)$ be an analytic operator function on S^* which is such that $f^*(z) = f(z)$ on S. Then the operator function $f^*(z)$ is called $[FP\, f(z, t)]$ on S^*.

The relationship between the operator product $g[FP\, f(z, t)]$ and $\left\{FP \int_0^t g(t-u)\, f(z, u)\, du\right\}$ is studied for the case when $\{f(z, t)\} = \left\{\frac{m(t)}{t^z}\right\}$, where m is function which possesses continuous derivatives of some order on $[0, \infty)$.

In Chapter 4 the solutions to the singular integral equation

$$FP \int_0^t f(t-u) \frac{m(u)}{u^\alpha}\, du = g(t) \qquad \text{all } t \geq 0$$

are found from considering the operators $\left[FP \frac{m(t)}{t}\right]^{-1}$.

In Chapter 5 a type of generalized wave function is discussed.

Chapter 1

1.1. This thesis consists of a study of the finite part of divergent convolution integrals of one real variable, and some applications of the results of that study. The use of the finite parts of divergent integrals started with A. Cauchy (1)*, (2) who used what he called an "intégrale extraordinaire" to give a sense to the gamma function for negative values of the argument. Since that time the notion has been used and extended by various authors. J. Hadamard (3) was able to extend the concept to multiple integrals. He and F. Bureau (4) have used the finite part of divergent integrals as an important tool in solving partial differential equations. More lately L. Schwartz (5) and M. Lighthill (6) have applied the theory of Distributions to extend the idea of the finite part of divergent integrals.

P. L. Butzer (7) has used the operational calculus of J. Mikusiński to study the finite part of divergent convolution integrals. He has been able to show the extension in Mikusiński's operational calculus of the integral $\int_0^t g(t-u)\, u^{\alpha}\, du$ with $\alpha \geq -1$ to $FP\int_0^t g(t-u)\, u^{\alpha}\, du$ with α any real number. This extension, at least to the case when α is not a negative integer, is a very natural one. It is the work of Butzer which is followed up in this thesis.

In Chapter 2 we lay a foundation for what is to follow. The concept of an analytic operator function, which will be necessary in Chapter 3, is introduced, and the useful concept of the logarithm of an

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* Unless otherwise specified numbers in parentheses refer to the bibliography at the end of the thesis.

operator is duscussed.

The operator function $\left[FP\, f(z,t)\right]$ which is associated with the function $\left\{f(z,t)\right\}$ is defined in Chapter 3, and the relationship of this operator function to $\left\{FP\int_0^t g(t-u)\, f(z,u)\, du\right\}$ is discussed when $\left\{f(z,t)\right\}$ and g are such that $\left\{FP\int_0^t g(t-u)\, f(z,u)\, du\right\}$ is defined. It is found that for many functions $\left\{f(z,t)\right\}$ and g the operator product $g\left[FP\, f(z,t)\right]$ reproduces the finite part of the definite integral. The operator product has the advantage that it exists for all operators g whereas the finite part of the divergent integral exists only if g satisfies certain smoothness conditions. Also, the operator product is defined for certain functions $\left\{f(z,t)\right\}$ for which the finite part of the divergent integral has not been defined.

The theory developed in Chapter 3 is applied in two ways. In Chapter 4 a problem posed by Butzer is solved. The problem is to solve the singular integral equation

$$FP\int_0^t f(t-u)\frac{J_0(u)}{u}\, du = g(t) \qquad \text{all } t > 0$$

where J_0 is the Bessel function of the first kind of order zero. In Chapter 4 a condition is given which is necessary and sufficient for the existence of solutions, and a formula is given for calculating the solutions when they exist. In Chapter 5 the results of Chapter 3 are applied to partial differential equations. The application is to give some properties of a class of generalized wave functions.

Chapter 2

2.1. Let the interval $t \geq 0$ be denoted by I. The functions to be considered in what follows will unless otherwise noted be functions on I to the complex numbers. With the usual notion of multiplication by scalars and pointwise addition these functions form a vector space over the field of complex numbers. The symbols f and $\{f(t)\}$ will be used to denote an element of this vector space; the function whose value is one for all $t \in I$ will be denoted by h, and the function whose value is zero for all $t \in I$ will be denoted by 0. f(t) will denote the value of the function f at the point t. The function $|f|$ is related to the function f by the definition $|f|(t) = |f(t)|$ for all $t \in I$. Greek letters $\alpha, \beta, \zeta, \ldots$ will generally denote scalars; there will be certain exceptions to this, for example, the variable of integration in Cauchy's Integral Theorem (Theorem 4ii) is denoted by ζ.

If a function is absolutely continuous or integrable on each closed and bounded subinterval of I it will be said to be <u>locally absolutely continuous</u> or <u>locally integrable</u>. Thus, the function $f = \{t^2\}$, although not integrable on I, is locally integrable.

The space of continuous functions on I will be called C^*. Let the topology of C^* be defined by a countable number of semi-norms, $\|f\|_n$, $n = 1, 2, \ldots$, where the n^{th} norm of an element f in C^* is given by

$$\|f\|_n = \max_{0 \leq t \leq n} |f(t)| .$$

Thus, a sequence of elements (f_k) in C^* converges to an element f in C^* as $k \to \infty$ if $f_k \to f$ uniformly on every bounded subinterval of

I as $k \to \infty$. With this topology C^* is a locally convex topological vector space. A locally convex space whose topology is given by a countable number of semi-norms is metrisable (N. Bourbaki (8) p. 97, Prop. 6) and thus C^* is metrisable.

The topology of a topological vector space, E, defines a uniform structure on E (N. Bourbaki (9) p. 24). E is said to be complete if it is complete in this uniform structure (N. Bourbaki (8) p. 10). If E has a countable basis of neighborhoods of the origin (if for example the topology of E is defined by a countable number of semi-norms) E is complete if every Cauchy sequence in E converges to an element of E (N. Bourbaki (10) p. 25).

A sequence in C^* is a Cauchy sequence if for each $n > 0$ and each $\epsilon > 0$ there is a $p_n(\epsilon)$ such that

$$k_1, k_2 > p_n(\epsilon) \Rightarrow \|f_{k_1} - f_{k_2}\|_n < \epsilon.$$

It is easily seen that every Cauchy sequence in C^* is convergent, and thus C^* is complete.

When a sequence f_n is convergent in the C^* topology as $n \to \infty$ to an element f in C^* we will say $f_n \to f\ (C^*)$ as $n \to \infty$.

Besides the space C^* we shall utilize the following function spaces.

1. $\underline{C_k(E^p)}$

Let E^p be p dimensional Euclidean space. Let

$$x = (x_1, \ldots, x_p), \qquad m = (m_1, \ldots, m_p)$$

where the m_i, $i = 1, \ldots, p$ are non-negative integers, and

$$|x| = \left[\sum_{i=1}^{p} x_i^2\right]^{\frac{1}{2}}, \qquad |m| = \sum_{i=1}^{p} m_i ,$$

and

$$D^m f(x) = \frac{\partial^{|m|} f(x)}{\partial x_1^{m_1} \cdots \partial x_p^{m_p}} .$$

The function space $C_k(E^p)$ consists of all those complex valued functions, $f(x)$, on E^p which are such that $D^m f(x)$ exists and is a continuous function on E^p whenever $|m| \leq k$. The topology of $C_k(E^p)$ will be defined by a countable number of semi-norms as follows:

$$\| f(x) \|_n^m = \max_{|x| \leq n} |D^m f(x)|$$

$n = 1, 2, \ldots , \quad |m| \leq k.$

2. $\underline{C_k(I')}$

Let I' be a closed interval in E^1. $C_k(I')$ consists of those functions all of whose derivatives with respect to x, $f^{(m)}(x)$, $0 \leq m \leq k$, exist and are continuous on I'. By the derivative at an end point of I' is meant the appropriate one-sided derivative at that point. If I' is a bounded interval the topology of $C_k(I')$ is the normed topology given by

$$\| f \| = \max_{\substack{x \in I' \\ 0 \leq m \leq k}} |f^{(m)}(x)| .$$

If $I' = [a, \infty)$ the topology is given by the countable number of semi-norms

$$\| f \|_n = \max_{\substack{x \in [a, n] \\ 0 \leq m \leq k}} |f^{(m)}(x)| \qquad n > a .$$

If $I' = (-\infty, b]$ there is an analogous definition of the topology.

3. L^*

If f and g are locally Lebesque integrable functions f is said to be equivalent to g if $f(t) = g(t)$ almost everywhere on I. L^* denotes the space of equivalence classes of locally integrable functions. If f is contained in an equivalence class which is contained in L^* it will be said, for brevity, that f is in L^*. A topology is given to L^* by the countable number of semi-norms

$$\| f \|_n = \int_0^n |f(t)| \, dt \qquad n = 1, 2, \ldots .$$

4. $C_k(E^p)C^*$, $C_k(I')C^*$

If $f(x) = \{ f(x, t) \}$ is a function on $E^p \times I$ which is such that all the derivatives $D^m f(x)$, $|m| \leq k$, with respect to x are continuous on $E^p \times I$, $f(x)$ is said to belong to $C_k(E^p)C^*$. A topology on $C_k(E^p)C^*$ is defined by the countable number of semi-norms

$$\| f(x) \|_n^m = \max_{\substack{|x| \leq n \\ 0 \leq t \leq n}} |D^m f(x)|$$

where $|m| \leq k$ and $n = 1, 2, \ldots$. The space $C_k(I')C^*$ consists of those functions on $I' \times I$ to the complex numbers which possess k derivatives with respect to x, each continuous on $I' \times I$. The countable number of semi-norms on $C_k(I')C^*$ are found by taking the maximum over $0 \leq t \leq n$ of the $C_k(I')$ norm (or semi-norms, according to whether or not I' is bounded) for $n = 1, 2, \ldots$.

5. $C^2(S)C^*$, $C_k^2(S)C^*$

Let S be an open connected region in the complex plane.
$f(z) = \{ f(z, t) \}$ is said to be in $C^2(S)C^*$ if $\{ f(z, t) \}$ is continuous in

$S \times I$. $f(z)$ is in $C_k^2(S)C^*$ if each of the first k partial derivatives of $\{f(z,t)\}$ with respect to z are in $C^2(S)C^*$. Convergence in $C^2(S)C^*$ means uniform convergence on every compact subset of $S \times I$. Convergence in $C_k^2(S)C^*$ means uniform convergence of each of the partials of $f(z)$ with respect to z on each compact subset of $S \times I$.

Each of the spaces described in 1, 2, 3, and 4 are seen to be metrisable and complete.

We shall make use of the closed graph theorem in Chapter 5.

Theorem 1. Let E and F be two metrisable and complete vector spaces. In order for a linear mapping T of E into F to be continuous it is necessary and sufficient that the set $\left\{ (x, Tx) \middle| \; x \in E \right\}$ be closed in $E \times F$.

Proof. N. Bourbaki (8), p. 37.

2.2. We will give a brief survey of the foundations of Mikusiński's operational calculus. Most of what is in this section can be found in Mikusiński's Operational Calculus (11) and also in Erdélyi's Operational Calculus and Generalized Functions (12).

Let f and g be locally integrable. The function

$$k(t) = \int_0^t f(t-u)g(u)du \qquad \text{almost all } t \geq 0$$

is called the finite convolution of f and g. It is very well known that the finite convolution of two locally integrable functions is a locally integrable function and the finite convolution of two continuous functions is a continuous function. The finite convolution defines a multiplication which makes L^* and C^* into commutative rings. This multiplication will be denoted by juxtaposition, thus the above equation will be written

$$k = fg.$$

It is a corollary to a theorem of Titchmarsh that L^* and C^* have no divisors of zero. The ring C^* can be extended to a field F, its quotient field, whose elements $\frac{a}{b}$, $a, b \in C^*$, $b \neq 0$ are called Mikusiński operators or just operators. The ring L^* is isomorphically imbedded in F under the mapping $f \leftrightarrow \frac{fa}{a}$ where $f \in L^*$ and $a \in C^*$ and $a \neq 0$. The field of complex numbers is isomorphically imbedded in F by the mapping $\alpha \leftrightarrow \alpha \cdot 1$ where 1 is the unit element of F. The unit element of F will be written as 1; the zero element of F, like the function $\{0\}$, will be written as 0, and in general, operators of the form $\alpha \cdot 1$ where α is a scalar will be denoted merely by α.

Definition 1. Take $f_n \in F$, $n = 1, 2, \ldots$. Then $f_n \to f(F)$ as $n \to \infty$ if and only if there is a $b \neq 0$ in C^* such that $bf_n \in C^*$ for $n = 1, 2, \ldots$ and $bf_n \to bf(C^*)$ as $n \to \infty$.

The F limit, when it exists, is unique.

$f(x)$ is said to be an operator function if $f(x)$ is a function whose range is in F. The space $C_k(E^p)F$ consists of all those operator functions having the property that there is an $a \neq 0$ in C^* such that $af(x) \in C_k(E^p)C^*$. A sequence of operator functions $f_n(x) \in C_k(E^p)F$, $n = 1, 2, \ldots$, is said to converge to $f(x)$ as $n \to \infty$ if there is an

$a \neq 0$ in C^* such that $a f_n(x) \in C_k(E^p)C^*$ and $a f_n(x)$ converges to $a f(x)$ in $C_k(E^p)C^*$.

Definition 2. Let $f(x)$ be in $C_k(E^p)F$ and suppose $a \neq 0$ is an element of C^* such that $a f(x) \in C_k(E^p)C^*$. We define

$$D^m f(x) = \frac{1}{a}\left\{D^m (a f(x))\right\}$$

when $|m| \leq k$.

There are spaces $C_k(I')F$, $C^2(S)F$, and $C_k^2(S)F$ which are defined in analogy to $C_k(E^p)F$. The derivative of an element of $C_k^2(S)F$ or $C_k(S)F$ is defined in analogy to Definition 2.

Definition 3. Let M be a bounded measurable region in E^p. Suppose the operator function $f(x)$ to be in $C(E^p)F$ (or $C(I')F$ where $I' \supset M$). Let the scalar function $\phi(x)$ be integrable over M. If $a \neq 0$ is such that $a f(x)$ is in $C(E^p)C^*$ $(C(I')C^*)$ the integral of $\phi(x) f(x)$ is defined by

$$\int_M \phi(x) f(x)\, dx = \frac{1}{a}\left\{ \int_M \phi(x)\, a f(x, t)\, dx \right\}. \qquad (1)$$

The surface integral over a sphere M' is defined by the equation analogous to equation (1) in which the volume integral is replaced by a surface integral over M'. If $f(z) \in C^2(S)F$ where S is a region in the complex plane and J is a rectifiable curve contained in S, the line integral over J is defined by

$$\int_J f(z)\, dx = \frac{1}{a}\left\{ \int_J a f(z, t)\, dz \right\} \qquad (1')$$

where $a \neq 0$ is such that $a\,f(z) \in C^2(S)C^*$.

When $E^p = E^1$ the parameter will usually be denoted by λ instead of x. Let the scalar function $\phi(\lambda)$ be locally integrable and let the operator function $f(\lambda)$ be in $C(0 \leq \lambda < \infty)F$. The integral of $f(\lambda)\phi(\lambda)$ on the infinite interval $(0, \infty)$ is defined by

$$\int_0^\infty f(\lambda)\phi(\lambda)\,d\lambda = \lim_{T \to \infty} \int_0^T f(\lambda)\phi(\lambda)\,d\lambda$$

when the limit on the right exists. The limit on the right hand side is taken in the sense of Definition 1.

The values of the integrals defined above and the derivative defined in Definition 2 do not depend on which particular element a is chosen to make $a\,f(x)$ a continuous function.

2.3.

Definition 4. Suppose $\alpha < 0 < \beta$, $f(\lambda) \in C_1[\alpha, \beta]F$, $w \in F$, $\frac{d}{d\lambda} f(\lambda) = w f(\lambda)$ for $\lambda \in (\alpha, \beta)$ and $f(0) = 1$. In such a case w is said to be a logarithm on $[\alpha, \beta]$ and $f(\lambda) = e^{\lambda w}$ on $[\alpha, \beta]$.

It is known (A. Erdélyi (12) p. 68) that an operator w which is a logarithm on an interval $\alpha \leq \lambda \leq \beta$ is in fact a logarithm on each finite interval $[\alpha_1, \beta_1]$ where $\alpha_1 < \alpha$, $\beta < \beta_1$, and the extension for $f(\lambda)$ from $[\alpha, \beta]$ to $(-\infty, \infty)$ is unique.

Every function in L^* is a logarithm. An example of a logarithm which is not a function is the operator $s = \frac{1}{h}$.

Example. Let $H(\lambda)$ be the operator function given by

$$H(\lambda, t) = 1 \qquad \text{when } t \geq \lambda,$$

$$H(\lambda, t) = 0 \qquad \text{when } t < \lambda.$$

The operator function $sH(\lambda)$ has the property that

$$\frac{d}{d\lambda} sH(\lambda) = s^3 \frac{d}{d\lambda} h^2 H(\lambda) = -s(sH(\lambda)).$$

Thus $sH(\lambda) = e^{-\lambda s}$.

The operators of Mikusiński are closely related to the Laplace transform. The relationship between the Laplace transform and that subspace of F which consists of all operators of the form $s^n a$, where $a \in C^*$, $a(t) = O(e^{kt})$ as $t \to \infty$ and n and k are positive numbers, has been investigated by J. D. Weston (see (13) and (14)). The following theorem of Mikusiński suggests the existence of some relationship.

Theorem 2. Let f be locally integrable, then,

$$\int_0^\infty e^{-s\lambda} f(\lambda)\, d\lambda = f.$$

Proof. J. Mikusiński (11) pp. 337 and 377.

This theorem will be extended in Chapter 3 to the case in which the integral exists only as a "finite part".

2.4. Mikusinski ((11) p. 412) makes the following definition.

Definition 5.

i) A function $a \in C^*$ is called real if $a(t)$ is a real number for each $t \geq 0$.

ii) an operator $w \in F$ is called real if there are real $a, b \in C^*$ such that $w = \frac{a}{b}$.

If $w = \frac{e}{f}$ is a real operator where $e, f \in C^*$ then e and f have a common factor $c \in F$ such that $a = \frac{e}{c}$ and $b = \frac{f}{c}$ are real elements of C^*. Thus any expression of a real operator in terms of elements of C^* is, except for common factors in the numerator and denominator, in terms of real functions. Every element $w \in F$ has a unique decomposition $w_1 + w_2 i$ where w_1 and w_2 are real. The real operators form a subfield of F. The following Lemma is due to Mikusinski ((4) p. 192).

<u>Lemma 1.</u> Let $F(\lambda) = e^{\lambda a}$ and $F(1) = 1$ then $a = 2\pi ki$ where k is an integer.

<u>Proof.</u> Since $x^n = 1$ has only n possible solutions it must be that $e^{\frac{a}{n}} = e^{\frac{2\pi k_n i}{n}}$ where k_n is an integer and $0 \leq k_n \leq n-1$. Thus when $r = \frac{m}{n}$ is rational $e^{ra} = e^{\frac{2\pi m k_n i}{n}}$. Letting $r \rightarrow \lambda$ it is seen that $e^{\lambda a}$ is a scalar, $A(\lambda)$, times the unit element of F. Since $A'(\lambda) = a A(\lambda)$, a is itself a scalar multiple of the unit element and finally, since $A(0) = A(1) = 1$, it is seen that $a = 2\pi ki$ where k is an integer.

<u>Definition 6.</u> Let w be a real logarithm and suppose $e^w = a$, then w is called the logarithm of a (i. e. $\ln a = w$).

In this case the operator function $e^{\lambda w}$ is alternatively denoted by a^{λ}. It is seen by Lemma 1 that the logarithm of an operator, if it exists, is uniquely defined. If α is a positive scalar the operator $\ln(\alpha \cdot 1) = (\ln \alpha) \cdot 1$ where 1 is the unit element of F and $\ln \alpha$ is the principal value of the logarithm of α. $\ln (\alpha \cdot 1)$ will be written as just $\ln \alpha$. The logarithm obeys most of the rules expected of a function with this name. The following statements are easily proved by referring to Definition 6.

Lemma 2.

i) $\ln a + \ln b = \ln a\, b$

ii) $\alpha > 0$; $\ln a + \ln \alpha = \ln \alpha\, a$

iii) α real; $\ln a^{\alpha} = \alpha \ln a$

iv) $\ln \frac{1}{a} = - \ln a$.

Statements i) and ii) are true in the sense that if any two of the quantities involved exist the third quantity exists and is given by the formula shown. Statements iii) and iv) are true in the sense that if one side of the equation exists the other does and is given by the formula shown. Thus the function given by $f(a) = \ln a$ is an isomorphism between the multiplicative group of operators which have logarithms and the additive group of real logarithms.

Example. When n is a positive integer $h^n = \left\{\frac{t^{n-1}}{\Gamma(n)}\right\}$. For $\lambda > 0$ define $h^{\lambda} = \left\{\frac{t^{\lambda-1}}{\Gamma(\lambda)}\right\}$ and for $\lambda < 0$ let $h^{\lambda} = \frac{1}{h^{-\lambda}}$. With this definition, and taking $h^0 = 1$, it is seen by means of Euler's integral of the second kind that $h^{\lambda} h^{\mu} = h^{\lambda+\mu}$ for all real λ, μ. The real operator $w = s\{\ln t\}$ is a logarithm and (Mikusiński (15))

$$h^{\lambda} = e^{\lambda(s\{\ln t\} + C)}$$

where λ is real and $C = .577...$ is Euler's constant. By definition of $\ln h$

$$\ln h = s\{\ln t\} + C$$

$$\ln s = -s\{\ln t\} - \ln \gamma \qquad \text{where } \gamma = e^C$$

or

$$\ln h = s\{\ln \gamma t\}$$

$$\ln s = -s\{\ln \gamma t\}.$$

Thus it results that $\{\ln t\} = \frac{-\ln \gamma s}{s}$. It is interesting to note that the Laplace transform of $\{\ln t\}$ is $\frac{-\ln \gamma z}{z}$.

Lemma 3. If $\lambda > -1$ then $\binom{\lambda}{n} \to 0$ as $n \to \infty$ and $\frac{\partial}{\partial \lambda}\binom{\lambda}{n} \to 0$ as $n \to \infty$.

Proof.

$$\binom{\lambda}{n} = \frac{\Gamma(\lambda+1)}{\Gamma(n+1)\,\Gamma(\lambda-n+1)} = \frac{\sin(n-\lambda)}{\pi}\,\frac{\Gamma(\lambda+1)\,\Gamma(n-\lambda)}{\Gamma(n+1)}. \qquad (2)$$

Stirling's formula shows that

$$\binom{\lambda}{n} = O((n-\lambda)^{-(\lambda+1)}) = O(n^{-\lambda-1})$$

as $n \to \infty$.

Differentiating Equation (2) and utilizing Stirling's formula again yields

$$\frac{\partial}{\partial \lambda}\binom{\lambda}{n} = O((n-\lambda)^{-(\lambda+1)}) + O\left(\frac{\Gamma'(n-\lambda)}{\Gamma(n+1)}\right).$$

To see that the second term on the right is $o(1)$ it may be noted that $\frac{\Gamma'(n-\lambda)}{\Gamma(n-\lambda)} = \ln(n-\lambda) + O(\frac{1}{n})$. Thus

$$\frac{\partial}{\partial\lambda}\binom{\lambda}{n} = O((n-\lambda)^{-(\lambda+1)}) + O((n-\lambda)^{-(\lambda+1)}\ln(n-\lambda)) = o(1)$$

as $n \to \infty$.

Lemma 4. Take $a \in C^*$ and take (α_n) to be a sequence of complex numbers such that $|\alpha_n| < M < \infty$ for all n. Then the sequence $b_m = \sum_{n=1}^{m} |\alpha_n a^n|$ is convergent (C^*) as $n \to \infty$.

Proof. Take $T > 0$. $|a(t)| \leq B$ on $[0, T]$ implies $|\alpha_n a^n|(t) \leq \frac{BMT^{n-1}}{(n-1)!}$ uniformly on $[0, T]$. Thus for $q > p > 1$ and $0 \leq t \leq T$, $\sum_p^q |\alpha_n a^n|(t) \leq \frac{MBT^{p-1}}{(p-1)!} e^T$ which tends to zero as $p \to \infty$.

Lemma 5. If $\lambda > -1$ then

$$\frac{\partial}{\partial\lambda}\binom{\lambda}{n} = \sum_{m=1}^{n} \frac{(-1)^m}{m}\binom{\lambda}{n-m}.$$

Proof. By Lemma 3 the series $\sum_{n=0}^{\infty}\binom{\lambda}{n} x^n$ can be differentiated term by term with respect to λ when $-1 < x < 1$ and $\lambda > -1$ to get

$$\frac{\partial}{\partial\lambda}(1+x)^\lambda = (1+x)^\lambda \ln(1+x) = \sum_{n=0}^{\infty} \frac{\partial}{\partial\lambda}\binom{\lambda}{n} x^n. \qquad (3)$$

The series expansions for $(1+x)^\lambda$ and for $\ln(1+x)$ can be multiplied together term by term when $-1 < x < 1$ and $\lambda > -1$ to get

$$(1+x)^\lambda \ln(1+x) = \sum_{n=0}^{\infty}\left(\sum_{m=1}^{n} \frac{(-1)^m}{m}\binom{\lambda}{n-m}\right) x^n. \qquad (4)$$

Equating the coefficients of like powers of x in Equations (3) and (4) gives the result.

Theorem 3. If $a \in C^*$ is real then $w = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} a^n$ is a logarithm and $\ln(1+a) = w$.

Proof. Let $f(\lambda) = \sum_{n=0}^{\infty} \binom{\lambda}{n} a^n$ when $\lambda \geq -1$; then $f(0) = 1$. Let

$$g(\lambda) = a\,f(\lambda) = \sum_{n=0}^{\infty} \binom{\lambda}{n} a^{n+1} \qquad \lambda \geq -1.$$

By Lemmas 3 and 4

$$\frac{\partial}{\partial\lambda} g(\lambda) = \sum_{n=0}^{\infty} \frac{\partial}{\partial\lambda}\binom{\lambda}{n} a^{n+1} = a\,f'(\lambda) \qquad \lambda \geq -1.$$

Thus on any interval $I' = [\lambda_o, \lambda_1]$, where $-1 \leq \lambda_o \leq 0 \leq \lambda_1 \leq \infty$, $f(\lambda) \in C_1(I')F$. By Lemmas 3 and 4 term by term multiplication is justified in the product

$$w\,a\,f(\lambda) = \left[\sum_{n=1}^{\infty} \frac{(-1)^n}{n} a^n\right]\left[\sum_{n=0}^{\infty} \binom{\lambda}{n} a^{n+1}\right] \qquad \lambda \in I',$$

and the resulting absolutely convergent series can by Lemma 5 be rearranged to get

$$w\,a\,f(\lambda) = \sum_{n=0}^{\infty} \frac{\partial}{\partial\lambda}\binom{\lambda}{n} a^{n+1} = a\,f'(\lambda) \qquad \lambda \in I'$$

or

$$w\,f(\lambda) = f'(\lambda)$$

$$f(\lambda) = e^{\lambda w} \qquad \lambda \in I'.$$

Since $f(1) = 1 + a$ we have

$$\ln(1+a) = w.$$

Corollary 1. If f is any continuously differentiable function on the interval $t \geq 0$ and $f(0) = \alpha \geq 0$ then $\ln f$ exists.

Proof. In fact, $f = \alpha h + \left\{ \int_0^t f'(u)du \right\} = \alpha h + f'h = \alpha h(1 + \frac{f'}{\alpha})$ so that $\ln f = \ln(1 + \frac{f'}{\alpha}) + \ln h + \ln \alpha$.

It is not necessary that $f(t)$ be positive for every or almost every $t \geq 0$ in order for f to possess a (real) logarithm. For example

$$\ln \{\cos t\} = \ln h + \ln(1 - \{\sin t\}) = s\{\ln \gamma t\} + \sum_{n=1}^{\infty} \frac{\{\sin t\}^n}{n}.$$

2.5.

Definition 7. Let S be a region in the complex plane. $f(z)$ is said to be an analytic operator function in S if $f(z) \in C_1^2(S) F$.

Examples.

i) Let $S_n = \{z \mid \operatorname{Re} z > -n\}$. $f(z) = h^z$ is analytic in S_n for every integer n.

ii) Let $R_n = \{z \mid \operatorname{Re} z > -n,\ z \neq -1, -2, \ldots, n-1\}$. $\Gamma(z+1)h^z$ is analytic in R_n for every integer n.

Theorem 4. Let $f(z)$ be an analytic operator function on S. Then

i) if $f(z) = 0$ on a set which has an accumulation point in S, $f(z) = 0$ everywhere on S;

ii) if J is a simple, closed, rectifiable curve in S and z is in the bounded region enclosed by J then $f^{(n)}(z) = \frac{n!}{2\pi i} \int_J \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$;

iii) if $K(z_o, p) = \{z \mid |z - z_o| < p\} \in S$ then for $z \in K(z_o, p)$

we have
$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_o)}{n!} (z-z_o)^n.$$

The series is convergent (F) to $f(z)$.

Proof. The proofs follow directly from the analogous theorems in complex variable theory. For example, to prove ii) it is noted that for $a \neq 0$ and $a f(z) \equiv \{g(z,t)\} \in C_1^2(S)C^*$

$$\left\{\frac{\partial^n g}{\partial z^n}(z,t)\right\} = \left\{\frac{n!}{2\pi i}\int_J \frac{g(\zeta,t)}{(\zeta-z)^{n+1}}\,d\zeta\right\}. \tag{5}$$

Since z is not on J,
$$\left\{\frac{\partial^n g}{\partial z^n}(z,t)\right\}$$

is continuous at $(z,t) \in S \times I$. Since each point $z \in S$ can be enclosed by a simple closed curve which does not pass through z it is seen that
$$\frac{\partial^n g(z)}{\partial z^n} = \frac{\partial^n}{\partial z^n}(a f(z)) \in C^2(S)C^*$$

and
$$f^{(n)}(z) = \frac{1}{a}\frac{\partial^n}{\partial z^n}(a f(z)) = \frac{n!}{2\pi i}\int_J \frac{f(\zeta)}{(\zeta-z)^{n+1}}\,d\zeta.$$

It is seen from Equation (5) that an analytic operator function has the property that if $a f(z) \in C_1^2(S)C^*$ then, in fact, $a f^{(n)}(z) \in C^2(S)C^*$ for $n = 0, 1, \ldots$.

Since h^z is analytic in every half plane $\operatorname{Re} z > -n$ it follows from Theorem 4 iii) that
$$h^{\lambda} = \sum_{n=0}^{\infty} \frac{s^n \{\ln \gamma t\}^n}{n!} \lambda^n$$

for each real λ. Furthermore, if $a = h^{1+\epsilon}$ we have $a h^z \in C_1^2(\mathrm{Re}\, z > -\epsilon)C^*$, and it must be that $h^{1+\epsilon} \frac{d}{d\lambda^n} h^\lambda \Big|_{\lambda=0}$ is in C^* for each $n \geq 0$. That is

$$h^\epsilon s^{n-1} \{\ln \gamma t\}^n \in C^* \qquad n = 0, 1, 2, \ldots .$$

If $f(z)$ is analytic in a region which includes the origin and $f(\lambda) = e^{\lambda w}$ when $z = \lambda$ is real the power series expansion,

$$e^{\lambda w} = \sum_{n=0}^{\infty} \frac{w^n}{n!} \lambda^n,$$

converges for sufficiently small λ. It is not true that every exponential $e^{\lambda w}$ is the restriction to the real axis of an analytic operator function.

Example. For $a = C^*$, $a e^{-\lambda s} = \begin{Bmatrix} a(t-\lambda) & t \geq \lambda \\ 0 & t < \lambda \end{Bmatrix}$. If $f(\lambda) = ae^{-\lambda s}$ is the restriction of an analytic function that function must be identically zero since $f(\lambda, t)$ is zero for all $\lambda > t$. This implies $a = 0$. Since there is no non-zero $a \in C^*$ such that $a e^{-\lambda s} \in C_1^2(S)C^*$, and thus $e^{-\lambda s} \notin C_1^2(S)F$ for any region S which intersects the axis of reals.

Theorem 5. Suppose that w and iw are both logarithms. Then $e^{zw} = e^{(\lambda + i\mu)w}$ is analytic in every bounded region of the complex plane.

Proof. Let $z = \lambda + i\mu$, $f(z) = e^{zw}$. Then $\frac{\partial f(z)}{\partial \lambda} = -i \frac{\partial f(z)}{\partial \mu}$. Take $a \neq 0$ and so that $a f(z)$ is in both $C_1(-n \leq \lambda \leq n)C^*$ and $C_1(-n \leq \mu \leq n)C^*$. Let $g(z) = \{g(z, t)\} = a f(z)$. $g(z + \Delta z, t) - g(z, t) = \frac{\partial g(z, t)}{\partial \lambda} \Delta \lambda + \frac{\partial g}{\partial \mu} \Delta \mu + 0(|\Delta z|)$.

Since $\frac{\partial f(z)}{\partial \lambda} = -i \frac{\partial f(z)}{\partial \mu}$ we have

$$a \frac{\partial f(z)}{\partial \lambda} = \frac{\partial g(z)}{\partial \lambda} = -i \frac{\partial g(z)}{\partial \mu} = -ia \frac{\partial f(z)}{\partial \mu}.$$

Thus

$$g(z+\Delta z, t) - g(z,t) = a \frac{\partial f(z)}{\partial \lambda}(\Delta \lambda + i\Delta \mu) + o(|\Delta z|)$$

$$\frac{g(z+\Delta z, t) - g(z,t)}{z} \rightarrow a \frac{\partial f(z)}{\partial \lambda} = a \frac{\partial f(z)}{\partial z}$$

as $|\Delta z| \rightarrow 0$. Thus $f(z)$ is analytic in every bounded region of the complex plane.

Theorem 5 together with the last example demonstrates that i s is not a logarithm.

Definition 8. Let S_1 and S_2 be non-empty regions in the complex plane and let $S_1 \supset S_2$. If $f_1(z)$ is an analytic operator function on S_1 and $f_2(z)$ is an analytic operator function on S_2 such that

$$f_1(z) = f_2(z)$$

whenever $z \in S_2$, $f_1(z)$ is said to be the analytic continuation of $f_2(z)$ to S_1.

Chapter 3

3.1. For certain functions $\{f(z,t)\}$ the finite part of the convolution integral $\int_0^t g(t-u)\,f(z,u)\,du$ has been defined by Hadamard (3) and Bureau (4) even though for some values of z the function $\{f(z,t)\}$ is not a Lebesgue integrable function. The definition is applicable only if g satisfies sufficient smoothness conditions. Butzer (7) has found an operator function $[FP\ t^z]$ which is such that under certain assumptions on g, $g[FP\ t^z] = \{FP \int_0^t g(t-u)\,f(z,u)\,du\}$. It is the purpose of this chapter to define operator functions $[FP\ f(z,t)]$ by a more or less general method. These operator functions will each be associated with a possibly non-integrable function $\{f(z,t)\}$. The purpose of introducing these operator functions is three-fold. First, the relationship between the integral $\{FP \int_0^t g(t-u)\,f(z,u)\,du\}$ and the operator product $g[FP\ f(z,t)]$ is shown in Theorem 6 and its corollaries to be quite close in the cases in which g and $\{f(z,t)\}$ are such that the finite part of the convolution integral is defined. In these cases the operator product $g[FP\ f(z,t)]$ is a convenient method of calculating the finite part of the divergent convolution integral. Secondly, the operator product $g[FP\ f(z,t)]$ exists as an operator for all locally integrable functions g and indeed for all operators g. The operator product may be used as an alternative to the finite part of the convolution integral and thus the smoothness condition on g can be eliminated. The third point in introducing these operator functions is that the operator function $[FP\ f(z,t)]$ is defined in the case of many functions $\{f(z,t)\}$ for which the finite part of the convolution integral has not previously been defined. In these cases the operator product provides a definition of the finite part of the convolution integral.

3.2.

<u>Definition 9</u>. Let S_1 and S_2 be non-empty regions in the complex plane such that $S_2 \supset S_1$. Let $g(z)$ be in $C_1^2(S_1)C^*$ and suppose that $g(z)$ can be continued analytically as an operator function to all of S_2. Suppose also that $\{g(z,t)\}$ can be continued analytically in z for each $t > 0$ so as to be defined on all of S_2. Denote these two continuations by $f(z)$ and $\{f(z,t)\}$ respectively. The operator function $[FP\, f(z,t)]$ is then defined on S_2 by

$$[FP\, f(z,t)] = f(z).$$

The operator function $[FP\, f(z,t)]$, when it exists, is unique. To see this, let $f_1(z) = [FP\, f(z,t)]$ on S_2 and $f_2(z) = [FP\, f(z,t)]$ on S_2. Then $f_1(z) = g(z) = f_2(z)$ on S_1 and by Theorem 4 i) it is seen that $f_1(z) - f_2(z) = 0$ on all of S_2.

An example of a function whose finite part can be defined is $\{g(z,t)\} = \left\{\frac{t^z}{\Gamma(z+1)}\right\}$ where $S_1 = \{z \mid \operatorname{Re} z > 0\}$. For each positive t the function $\{f(z,t)\} = \left\{\frac{t^z}{\Gamma(z+1)}\right\}$ is analytic in the entire z plane and the operator function $f(z) = h^{z+1}$ is equal on S_1 to $\{g(z,t)\}$. Since h^z is an analytic operator function in every half-plane $\operatorname{Re} z > -n$, Definition 9 gives

$$\left[FP\, \frac{t^z}{\Gamma(z+1)}\right] = h^{z+1}$$

in any half-plane $\operatorname{Re} z > -n$.

It should be noted that a particular non-integrable function

$\{k(t)\}$, for example $\{k(t)\} = \left\{\frac{t^{-3/2}}{\Gamma(-1/2)}\right\}$ does not define a unique operator $[FP\, k(t)]$ having the property that $[FP\, f(z,t)]_{z=z_1} = [FP k(t)]$ whenever $f(z_1, t) = k(t)$ for all positive t. For example

$$\left[FP \frac{t^z}{\Gamma(z+1)} + \frac{t^{z+1/2}}{\Gamma(z+3/2)}\right]_{z=-3/2} = h^{-1/2} + 1$$

and

$$\left[FP \frac{t^z}{\Gamma(z+1)}\right]_{z=-3/2} = h^{-1/2}$$

however $\left\{\frac{t^z}{\Gamma(z+1)} + \frac{t^{z+1/2}}{\Gamma(z+1/2)}\right\}_{z=-3/2} = \left\{\frac{t^{-3/2}}{\Gamma(-1/2)}\right\} = \left\{\frac{t^z}{\Gamma(z+1)}\right\}_{z=-3/2}$.

Thus when speaking of the finite part of a particular function care must be used to show how it was calculated. In spite of this the notation $[FP\, f(\alpha, t)]$ will frequently be used in place of the more cumbersome $[FP\, f(z,t)]_{z=\alpha}$ when there is no possibility of confusion.

3.3.

Definition 10. Let n be a positive integer and suppose that $0 \leq \beta < 1$.

i) Let $\alpha = -n - \beta$ and let the integrable function f be n times differentiable at the point t. The quantities $I_\alpha(f,t)$ and $FP\int_0^t f(t-u)u^\alpha du$ are defined by the equation

$$I_\alpha(f,t) = FP\int_0^t f(t-u)u^\alpha du = \lim_{\epsilon \to 0}\left(\int_\epsilon^t f(t-u)u^\alpha du + Q(\epsilon)\right) \qquad (6)$$

where $Q(\epsilon)$ is that unique linear combination of $\ln \epsilon$ and negative powers of ϵ which causes the limit on the right hand side to exist.

ii) Let the function m possess n derivatives each continuous on $[0,t]$. Then for some continuous function g

$$\frac{m(u)}{u^n} = g(u) + \sum_{k=0}^{n-1} \frac{m^{(k)}(0)}{k!} u^{n-k}$$

on $[0,t]$. The quantity $FP\int_0^t f(t-u)\frac{m(u)}{u^{n+\beta}}du$ is defined by the equation

$$FP\int_0^t f(t-u)\frac{m(u)}{u^{n+\beta}}du = \int_0^t f(t-u)\frac{g(u)}{u^{\beta}}du + \sum_{k=0}^{n-1}\frac{m^{(k)}(0)}{k!} FP\int_0^t \frac{f(t-u)du}{u^{n-k+\beta}}$$

whenever f is such that the integrals on the right exist.

The continuous function $\{t^z\}$ is an analytic operator function in the region $\operatorname{Re} z > 0$ and in this region $\{t^z\} = \Gamma(z+1)h^{z+1}$. The function $\Gamma(z+1)h^{z+1}$ is an analytic operator function in any region $S_n = \{z \mid \operatorname{Re} z > -n,\ z \neq -1, -2, \ldots, -n-1\}$. Thus, by Definition 9, $[FP\, t^z] = \Gamma(z+1)h^{z+1}$ in any such region. In order to investigate the relationship between the finite part of the convolution integral, $FP\int_0^t f(t-u)u^{\alpha}du$, and the operator function $[FP\, t^z]$, it is convenient to introduce two new vector spaces. Denote by $L^*[FP\, t^z]$ that linear subspace of F obtained by adjoining finite linear combinations of $\Gamma(z+1)h^{z+1}$, $z \neq -1, -2, \ldots$, to that subspace of F which consists of all locally integrable functions. Every element of $L^*[FP\, t^z]$ is of the form $[g] = g_1 + \sum_{k=0}^{p} \beta_k \Gamma(z_k+1)h^{z_k+1}$ where g_1 is locally integrable, the z_k are distinct complex numbers, $\operatorname{Re} z_k < -1$ for $k = 0, 1, \ldots, p$, and none of the z_k are negative integers. The β_k's are complex numbers. The value of p and of course the complex numbers β_k and z_k may be different for different elements of

$L^*[FP\, t^z]$. The second linear space to be considered is the linear space over the complex numbers of equivalence classes of functions on $(0, \infty)$ obtained by adjoining to L^* finite linear combinations of $\{t^z\}$ with $z \neq -1, -2, \ldots$. This space will be denoted by $L^*(t^z)$. Each element of $L^*(t^z)$ can be represented as $g = g_1 + \sum_{k=0}^{p} \beta_k \{t^{z_k}\}$ where $g_1 \in L^*$ and the restrictions on the z_k are as in $L^*[FP\, t^z]$.

It is easy to see that an element of $L^*[FP\, t^z]$, $[g] = g_1 + \sum_{k=0}^{p} \beta_k \Gamma(z_k+1) h^{z_k+1}$, is zero only if g_1 is zero and if $\beta_k = 0$, $k = 0, \ldots p$. Thus each element of $L^*[FP\, t^z]$ has a unique representation. Again, it is easy to show that an element of $L^*(t^z)$ is zero only if $g_1 = 0$ and $\beta_k = 0, 1, \ldots, p$. Thus each element of $L^*(t^z)$ has a unique representation. It follows that the mapping of $L^*[FP\, t^z]$ onto $L^*(t^z)$ defined by $g_1 \leftrightarrow g_1$ and $\Gamma(z+1) h^{z+1} \leftrightarrow t^z$ is a vector space isomorphism. In what follows the fact that a function g is the image of an operator $[g]$ under this isomorphism will frequently be recognized by writing $g = [g]$.

Let f have $n-1$ absolutely continuous derivatives on $[0, t]$ and suppose $\beta \neq 0$. Then Equation (6) can be transformed by integration by parts to yield

$$I_\alpha(f,t) = \sum_{k=0}^{n-1} \frac{\Gamma(\alpha+1) f^{(k)}(0)}{\Gamma(\alpha+k+2)} t^{\alpha+k+1} + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+n+1)} \int_0^t f^{(n)}(t-u) u^{\alpha+n} du. \quad (7)$$

If each $f^{(k)}$ is locally absolutely continuous for $k = 0, 1, \ldots, n-1$ and the isomorphism between $L^*[FP\, t^z]$ and $L^*(t^z)$ is utilized Equation (7) can be written

$$I_\alpha(f) = \Gamma(\alpha+1) \sum_{k=0}^{n-1} f^{(k)}(0)h^{\alpha+k+2} + f^{(n)}\, h^{\alpha+n+1}$$

or

$$\left\{FP\int_0^t f(t-u)u^\alpha\, du\right\} = f\Gamma(\alpha+1)h^{\alpha+1}$$

$$\left\{FP\int_0^t f(t-u)u^\alpha\, du\right\} = f\left[FP\, t^\alpha\right] . \qquad (8)$$

Definition 9 does not yield a value for the operator function $\left[FP\, t^z\right]$ when $z = -n$. In analogy to what has been done by Hadamard (16) the value of the operator $\left[FP\, t^z\right]$ at $z = -n$ will be defined by means of residues. The operator function $\left[FP\, t^z\right]$ has a simple pole at $z = -n$ and the residue at $z = -n$ is given by

$$\lim_{z\to -n} (z+n)\left[FP\, t^z\right] = \operatorname*{Res}_{z=-n}\left[FP\, t^z\right]$$

where the limit is taken in the sense of convergence in F. It is seen that

$$\operatorname*{Res}_{z=-n}\left[FP\, t^z\right] = \frac{(-1)^{n-1}}{\Gamma(n)} h^{-n} .$$

The operators $\left[FP\, t^z\right]_{z=-n}$ will now be defined by the equation

$$\left[FP\, t^z\right]_{z=-n} = \lim_{z\to -n}\left(\left[FP\, t^z\right] - \frac{\operatorname*{Res}_{z=-n}\left[FPt^z\right]}{z+n}\right) . \qquad (9)$$

Evaluating this limit gives

$$\left[FP\, t^{-n}\right] = \frac{(-1)^{n-1}}{\Gamma(n)}\, s^n\left\{\ln t + \gamma_{n-1}\right\} \qquad n = 1,\ 2,\ \ldots (10)$$

where $\gamma_o = 0$ and $\gamma_{n-1} = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1}$ when $n > 1$. The operator function $\left[FP\, t^z\right]$ as defined by Definition 9 and Equation (9) is the same as that arrived at by Butzer although he utilized quite different considerations.

If f has n - 1 absolutely continuous derivatives on $[0, t]$ and $\beta = 0$, Equation (6) can be transformed by a rather lengthy integration by parts to obtain

$$I_{-n}(f, t) = \sum_{k=0}^{n-2} \frac{\Gamma(n-k-1)(-1)^{k+1}}{\Gamma(n)} f^{(k)}(0)\, t^{k-n-1} + \frac{(-1)^{n-1}}{\Gamma(n)} f^{(n-1)}(0) \ln t$$

$$+ \frac{(-1)^{n-1}}{\Gamma(n)} \int_0^t f^{(n)}(t-u) \ln u\, du + \frac{(-1)^{(n-1)}}{\Gamma(n)} \gamma_{n-1} f^{(n-1)}(t) \qquad (11)$$

where the γ_{n-1} are the same as those given in Equation (10).

To the space $L^*(t^z)$ adjoin all finite linear combinations of $\left\{t^{-n}\right\}$, n = 1, 2, ... and to $L^*\left[FP\, t^z\right]$ adjoin the finite linear combinations of $\left[FP\, t^{-n}\right]$, n = 1, 2, ... The cononical mapping which is defined by $g_1 \leftrightarrow g_1$ for g_1 locally integrable and $\left\{t^z\right\} \leftrightarrow \left[FP\, t^z\right]$ for all z defines an isomorphism between these two vector spaces. If in Equation (11) f and its first n - 1 derivatives are locally absolutely continuous and if the functions $\left\{t^{k-n+1}\right\}$ are replaced by their images $\left[FP\, t^{k-n+1}\right]$ Equation (11) can be written in operator notation as

$$I_{-n}(f) = \left(\sum_{k=0}^{n-2} f^{(k)}(0)\, s^{n-k-1} + f s\right) \frac{(-1)^{n-1}}{\Gamma(n)} \left\{\ln t\right\}$$

$$+ \frac{(-1)^{n-1}}{\Gamma(n)} \gamma_{n-1} f^{(n-1)} + \frac{(+1)^{n-1}}{\Gamma(n)} \left(\sum_{k=0}^{n-2} \gamma_{n-k-2} f^{(k)}(0) s^{n-k-2}\right)$$

$$I_{-n}(f) = f\left[FP\, t^{-n}\right] + \frac{(-1)^{n-1}}{\Gamma(n)} \left(\sum_{k=0}^{n-2} (\gamma_{n-k-2} - \gamma_{n-1}) f^{(k)}(0)\, s^{n-k-2}\right). \qquad (12)$$

Since for $n \geq 2$

$$\gamma_{n-k-2} - \gamma_{n-1} \neq 0$$

for any $k = 0, 1, \ldots, n-2$, it follows that

$$I_{-n}(f) = f\left[FP\, t^{-n}\right]$$

if and only if $f^{(k)}(0) = 0$, $k = 0, 1, \ldots, n-2$. The results of Equations (8) and (12) are summarized in the following theorem.

Theorem 6. Let $\alpha = -n-\beta$ where $n \geq 1$ is an integer and $0 \leq \beta < 1$. Let $f^{(k)}$ be locally absolutely continuous for $k = 0, 1, \ldots, n-1$, and let the functions $\left\{t^z\right\}$, $\operatorname{Re} z \leq -1$, be identified with the operators $\left[FP\, t^z\right]$. Then

i) if $\beta \neq 0$

$$\left\{FP \int_0^t f(t-u)\, u^{\alpha}\, du\right\} = f\left[FP\, t^{\alpha}\right]$$

ii) if $\beta = 0$

$$\left\{FP \int_0^t f(t-u)\, u^{-n}\, du\right\} = f\left[FP\, t^{-n}\right]$$

if and only if $f^{(k)}(0) = 0$ when $k = 0, 1, \ldots, n-2$.

A restatement of Theorem 6 ii) is given by

Corollary 1. Let the conditions of Theorem 6 hold and suppose that

$$g = \left\{FP \int_0^t f(t-u)\, u^{-n}\, du\right\};$$

then $g = f\left[FP\, t^{-n}\right]$ if and only if g is locally integrable.

Proof. Equation (11) shows that $g \in L^*$ if and only if $f^{(k)}(0) = 0$ for $k = 0, 1, \ldots, n-2$.

3.4. Let $\{m(t)\}$ be such that $m^{(k)} \in C^*$ for $k = 0, 1, \ldots, n$. The function $\left\{\frac{m(t)}{t^z}\right\}$ is in $C_1^2(S_1)C^*$ where $S_1 = \{z \mid \operatorname{Re} z > -0\}$. Let

$$g_n(t) = m(t) - \sum_{k=0}^{n-1} \frac{m^{(k)}(0)}{k!} t^k;$$

then the operator function $F(z)$ defined by

$$F(z) = \left\{\frac{g_m(t)}{t^z}\right\} + \sum_{k=0}^{n-1} \frac{m^{(k)}(0)}{k!} \left[FP\, t^{k-z}\right]$$

is such that

$$F(z) = \left\{\frac{m(t)}{t^z}\right\}$$

when $z \in S_1$ and $F(z) \in C_1^2(S_2)F$ where $S_2 = \{z \mid \operatorname{Re} z > -n, z \neq -1, -2, \ldots, -(n-1)\}$. Since for each $t > 0$, $\frac{m(t)}{t^z}$ is an analytic function of z in S_2, it follows by Definition 9 that

$$\left[FP\, \frac{m(t)}{t^z}\right] = \left\{\frac{g_n t}{t^z}\right\} + \sum_{k=0}^{m-1} \frac{m^{(k)}(0)}{k!} \left[FP\, t^{k-z}\right] \qquad (13)$$

in S_2. When $z = -1, -2, \ldots, -(n-1)$ let $\left[FP\, \frac{m(t)}{t^z}\right]$ be defined by means of residues. The following corollary to Theorem 6 is obtained immediately.

Corollary 2. Let $\alpha = n + \beta$ where $n \geq 1$ is an integer and $0 \leq \beta < 1$. Let $m^{(k)}$ be in C^* for $k = 0, \ldots, n$ and suppose that $m(0) \neq 0$. Let $f^{(k)}$ be locally absolutely continuous for $k = 0, 1, \ldots, n-1$. If the functions $\{t^z\}$, $\operatorname{Re} z \leq -1$, are identified with the operators $\left[FP\, t^z\right]$ then

i) if $\beta \neq 0$

$$\left\{ FP \int_0^t f(t-u)m(u)u^{\alpha}\, du \right\} = f \left[FP \frac{m(t)}{t^{\alpha}} \right] \tag{14}$$

whenever the lefthand side of the equation is defined by Definition 8,

ii) if $\beta = 0$ Equation (14) holds if and only if $f^{(k)}(0) = 0, \quad k = 0, 1, \ldots, n-2.$ These derivatives are zero at the origin if and only if the function $\left\{ FP \int_0^t f(t-u) \frac{m(u)}{u^n} du \right\}$ is locally integrable.

Proof. From Definition 10 it is seen that

$$\left\{ FP \int_0^t f(t-u) \frac{m(u)}{u^{\alpha}} du \right\} = f \left\{ \frac{g_n(t)}{t^{\alpha}} \right\} + \sum_{k=0}^{n-1} \frac{m^{(k)}(0)}{k!} \quad FP \int_0^t f(t-u)u^{k-\alpha} du \ .$$

Applying Theorem 6 to the above equation proves the corollary.

We specify $m(0) \neq 0$ in order to know the proper number of derivatives to specify for m and f.

3.4. Definition 3 tells how to form the integral with respect to a parameter of an operator function. In order to get an analogue of Theorem 2 in the case of divergent integrals we need the following definition.

Definition 11. Suppose $f(\lambda) \in C_n[0, \mu]F$. Take an element a in C^* such that $a f(\lambda) \in C_n[0, \mu]C^*$. The integral $FP \int_0^{\mu} \frac{f(\lambda)}{\lambda^{\alpha}} d\lambda$ where $\alpha = n + \beta$, $n \geq 1$ is an integer, and $0 \leq \beta < 1$ is defined by

$$FP\int_0^{\mu} \frac{f(\lambda)}{\lambda^{\alpha}}\, d\lambda = \frac{1}{a}\left\{FP\int_0^{\mu} \frac{a\, f(\lambda, t)}{\lambda^{\alpha}}\, d\lambda\right\}.$$

If the limit in the sense of convergence in F exists as $\mu \to \infty$ then

$$FP\int_0^{\infty} \frac{f(\lambda)}{\lambda^{\alpha}}\, d\lambda = \lim_{\mu\to\infty} FP\int_0^{\mu} \frac{f(\lambda)}{\lambda^{\alpha}}\, d\lambda.$$

We must show that $FP\int_0^{\mu} \frac{a\, f(\lambda, t)}{\lambda^{\alpha}}\, d\lambda \in C^*$. But $\left\{a\, f(\lambda, t)\right\} \in C_n\left[0, \mu\right]C^*$ so that for each $t \geq 0$

$$a\, f(\lambda, t) = \sum_{k=0}^{n-1} \frac{(a\, f)^{(k)}(0, t)}{k!}\, \lambda^k + g(\lambda, t)\,\lambda^n$$

where g is defined by the equation. But $\left\{(a\, f)^{(k)}(0, t)\right\} \in C^*$, $k = 0, 1, \ldots, n-1$ and $\left\{g(\lambda, t)\right\} \in C\left[0, \mu\right]C^*$. Thus

$$FP\int_0^{\mu} \frac{a\, f(\lambda, t)}{\lambda^{\alpha}}\, d\lambda = \sum_{k=0}^{n-1} \frac{\left\{(a\, f)^{(k)}(0, t)\right\}}{k!}\, FP\int_0^{\mu} \frac{d\lambda}{\lambda^{\alpha-k}} + \left\{\int_0^{\mu} \frac{g(\lambda, t)\, d\lambda}{\lambda^{\beta}}\right\}$$

$$= a\left(\sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!}\, FP\int_0^{\mu} \frac{d\lambda}{\lambda^{\alpha-k}}\right) + \left\{\int_0^{\mu} \frac{g(\lambda, t)\, d\lambda}{\lambda^{\beta}}\right\}$$

and this is in C^* since $(a\, f)^{(k)}(0) \in C^*$, $k = 0, 1, \ldots, n-1$ and $\left\{g(\lambda, t)\right\} \in C\left[0, \mu\right]C^*$. Also, since

$$\left\{\int_0^{\mu} \frac{g(\lambda, t)}{\lambda^{\beta}}\, d\lambda\right\} = a\int_0^{\mu} \frac{f(\lambda) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!}\lambda^k}{\lambda^{\alpha}}\, d\lambda,$$

where the last integral is defined by Definition 3, it is seen that the value of

$$\frac{1}{a}\left\{FP\int_0^{\mu} \frac{a\, f(\lambda, t)}{\lambda^{\alpha}}\, d\lambda\right\}$$

does not depend on a.

The integral we have defined is linear in $f(\lambda)$, that is, for complex β_1 and β_2

$$FP\int_0^{\mu}\frac{\beta_1 f_1(\lambda)}{\lambda^{\alpha}}\,d\lambda + FP\int_0^{\mu}\frac{\beta_2 f_2(\lambda)}{\lambda^{\alpha}}\,d\lambda = FP\int_0^{\mu}\frac{\beta_1 f_1(\lambda)+\beta_2 f_2(\lambda)}{\lambda^{\alpha}}\,d\lambda$$

whenever two of the three integrals exist.

Theorem 7. Let $\alpha = n+\beta$ where $n \geq 1$ is an integer and $0 \leq \beta < 1$. Suppose that $m^{(k)} \in C^*$ for $k = 0, 1, \ldots, n$ then

$$FP\int_0^{\infty} e^{-\lambda s}\frac{m(\lambda)}{\lambda^{\alpha}}\,d\lambda = \left[FP\ \frac{m(t)}{t^{\alpha}}\right].$$

Proof. Let $\{m(t)\} = h$. Suppose $\beta = 0$. Then

$$FP\int_0^{\mu}\frac{e^{-\lambda s}}{\lambda^{n}}\,d\lambda = s^{n+1}\ FP\int_0^{\mu}\frac{h^{n+1}e^{-\lambda s}}{\lambda^{n}}\,d\lambda = \frac{s^{n+1}}{\Gamma(n+1)}I(\mu)$$

where

$$I(\mu, t) = FP\int_0^{\mu}\frac{(t-\lambda)^n}{\lambda^n}\,d\lambda \qquad \text{when } \mu < t$$

$$I(\mu, t) = FP\int_0^{t}\frac{(t-\lambda)^n}{\lambda^n}\,d\lambda \qquad \text{when } \mu > t.$$

Thus,

$$I(\mu t) = (-1)^{n-1}\left(n\,t\ln t + t\sum_{k=2}^{n}\frac{(-1)^{n-k-1}}{k-1}\binom{n}{k}\right) \qquad \mu > t$$

which after some computation is seen to be

$$I(\mu, t) = (-1)^{n-1}\,n\left(t\ln t - t + \gamma_{n-1}t\right) \qquad \mu > t$$

so that $I(\mu) \in C^*$ for each $\mu > 0$ and $I(\mu) \to (-1)^{n-1}nh\{\ln t + \gamma_{n-1}\}$ (F) as $\mu \to \infty$. Thus

$$\mathrm{FP}\int_0^\infty \frac{e^{-\lambda s}}{\lambda^n}\, d\lambda = \frac{(-1)^{n-1}}{\Gamma(n)}\, s^n\left\{\ln t + \gamma_{n-1}\right\}$$

which proves the theorem in the case $m = h$, $\beta = 0$.

When $m = h$, $\beta \neq 0$ the proof is similar. Since

$$\frac{m(\lambda)}{\lambda^\alpha} = \frac{g_n(\lambda)}{\lambda^\beta} + \sum_{k=0}^{n-1} \frac{m^{(k)}(0)}{k!}\, \lambda^{k-\alpha}$$

where $\left\{\dfrac{g_n(t)}{t^\beta}\right\}$ is locally integrable, Theorem 2 together with the special case of Theorem 7 just proved shows that

$$\mathrm{FP}\int_0^\infty e^{-\lambda s}\,\frac{m(\lambda)}{\lambda^\alpha}\, d\lambda = \left\{\frac{g_n(t)}{t^\beta} + \sum_{k=0}^{n-1} \frac{m^{(k)}(0)}{k!}\left[\mathrm{FP}\, t^{k-\alpha}\right]\right\}$$

and this is the statement of Theorem 7.

3.5. We give a short table of some operator functions which are finite parts.

TABLE OF FINITE PARTS OF FUNCTIONS

	$\{f(z, t)\}$	$[FP\, f(z, t)]$	Region of Validity
1.	$\left\{\frac{t^{z-1}}{\Gamma(z)}\right\}$	h^z	all z
2.	$\{t^z\}$	$\Gamma(z+1)h^{z+1}$	$z \neq -1, -2, \ldots$
	$\{t^{-n}\}$	$\frac{(-1)^{n-1} s^n}{\Gamma(n)} \{\ln t + \gamma_{n-1}\}$	$z = -n$ $n = 1, 2, \ldots$
3.	$\{t^z \ln t\}$	$(\psi(z+1) + s\{\ln \gamma t\})\, [FP\, t^z]$	$z \neq -1, -2, \ldots$
	$\{t^{-n} \ln t\}$	$(\gamma_n + s\{\ln t\})\, [FP\, t^{-n}]$	$z = -n$ $n = 1, 2, \ldots$
4.	$\left\{\frac{J_o(t)}{t^z}\right\}$	$\left\{\frac{J_o(t) - \sum_{n=0}^{n_1} \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2}}{t^z}\right\}$ $+ \sum_{n=0}^{n_1} \frac{(-1)^n [FP\, t^{2n-z}]}{2^{2n}(n!)^2}$	$Re(2n_1 - z) > -3$
5.	$\left\{\frac{e^{-z^2/4t}}{t^{3/2}}\right\}$	$\frac{\sqrt{\pi}}{z} e^{-z\sqrt{s}}$	$z \neq 0$

$\gamma_o = 0 \qquad \gamma_{n-1} = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1}$ when $n \geq 1$.

$\gamma = e^C \qquad C = .577\ldots$ is Euler's constant.

$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$

Table 1

The first entry in table one has been discussed in Section 3. 2. The second entry has been discussed at length in Section 3. 3. The third entry consists of two cases. It is not difficult to verify that for $\operatorname{Re} z > -1$

$$\left\{ t^{z} \ln t \right\} = (\psi(z+1) + s\{\ln \gamma t\})\Gamma(z+1)h^{z+1}.$$

An application of Definition 10 gives the operator function $\left[FP\, t^{z} \ln t \right]$ everywhere except where z is a negative integer. Where $z = -n$ let the operators be defined by residues as follows

$$\left[FP\, t^{-n} \ln t \right] = \lim_{z \to -n} \left(\left[FP\, t^{z} \ln t \right] + \frac{\operatorname{Res}_{\zeta=-n} \left[FP\, t^{\zeta} \ln t \right]}{z+n} \right).$$

The entry in the table for $\left[FP\, t^{-n} \ln t \right]$ gives the value of this limit.

It is shown in Mikusiński's book ((11) chapter VIII) that both $i\sqrt{s}$ and $\sqrt{s}$ are logarithms. Thus by Theorem 5 we know that $e^{z\sqrt{s}} = e^{x\sqrt{s}}\, e^{y\, i\sqrt{s}}$ is an analytic operator function in every bounded region of the complex plane and thus $\frac{1}{z}\, e^{-z\sqrt{s}}$ is an analytic operator function in every bounded region which does not contain the origin.

In Mikusiński ((11) pp. 221-2) it is shown that

$$\frac{e^{-z\sqrt{s}}}{z} = \frac{e^{-x\sqrt{z}}}{x} = \left\{ \frac{1}{\sqrt{\pi t^{3}}}\, e^{-x^{2}/4t^{2}} \right\}$$

when $z = x > 0$. The function

$$\left\{ \frac{1}{\sqrt{\pi z^{3}}}\, e^{-z^{2}/4t^{2}} \right\}$$

is in C_1^2 $(|\arg z| < \frac{\pi}{4})$ C^* and since it is equal to $\frac{e^{-z\sqrt{s}}}{z}$ for an infinite number of z (all positive z) we know by Theorem 4i) that

$$\frac{e^{-z\sqrt{s}}}{z} = \left\{ \frac{1}{\sqrt{\pi t^3}} \quad e^{-z^2/4t^2} \right\}$$

for $|\arg z| < \frac{\pi}{4}$.

For each $t > 0$, $\left\{ \frac{1}{\sqrt{\pi t^3}} \quad e^{-z^2/4t^2} \right\}$ is an entire function of z;

thus, by Definition 9 we have

$$\frac{e^{-z\sqrt{s}}}{z} = \left[FP \frac{1}{\sqrt{\pi t^3}} \; e^{-z^2/4t^2} \right]$$

in every bounded region of the complex plane which does not include the origin, and this is the last statement in the table.

Chapter 4

4.1. The inverses of several of the operators which represent finite parts have been found by Butzer (7). If $\alpha = -n-\beta$, $0 < \beta < 1$ and n is a positive integer the operator

$$\left[FP\, t^{\alpha}\right]^{-1} = \frac{h^{n+\beta-1}}{\Gamma(\alpha+1)} \qquad (15)$$

is a locally integrable function. The inverse of $\left[FP\, t^{-1}\right] = s\left\{\ln t\right\}$ is $\left\{\int_o^{\infty} \frac{t^{u-1}}{\gamma^u \Gamma(u)}\, du\right\}$ which is also a locally integrable function. For $n \geq 1$

$$\left[FP\, t^{-n}\right]^{-1} = (-1)^{n-1}\Gamma(n)\left\{\int_o^{\infty} \frac{t^{u-1}}{\Gamma(u)\,\alpha_{n-1}^{u}}\, du\right\} h^{n-1} \qquad n = 1, 2, \ldots \qquad (16)$$

where $\alpha_{n-1} = \gamma e^{-\gamma_{n-1}}$, and for $n > 1$ these functions are locally absolutely continuous. These facts enable one to find the inverse of $\left[FP\, \frac{m(t)}{t^{\alpha}}\right]$.

Lemma 6. Suppose $m^{(k)} \in C^*$ when $k = 0, 1, \ldots, n$, $\alpha = n+\beta$ where $0 \leq \beta < 1$. Suppose that at least one of the quantities $m^{(k)}(0)$, $k = 0, 1, \ldots, n-1$ is not zero. Then $\left[FP\, \frac{m(t)}{t^{\alpha}}\right]^{-1}$ is a locally integrable function.

Proof. Let k_1 be the first integer such that $m^{(k_1)}(0) \neq 0$. From section 3.4 we know that

$$\left[FP\, \frac{m(t)}{t^{\alpha}}\right] = g_n + \sum_{k=k_1}^{n-1} \frac{m^{(k)}(0)}{k!}\left[FP\, t^{k-\alpha}\right]$$

where $g_n \in C^*$. Let

$$f = \left(g_n + \sum_{k=k_1+1}^{n-1} \frac{m^{(k)}(0)}{k!} \left[FP\, t^{k-\alpha}\right]\right)\left[FP\, t^{k_1-\alpha}\right]^{-1}$$

and

$$\beta = \frac{m^{(k)}(0)}{k_1!} .$$

Then $f \in C^*$ and $\beta \neq 0$ is a scalar. We have

$$\left[FP \frac{m(t)}{t^\alpha}\right]^{-1} = \frac{\left[FP\, t^{k_1-\alpha}\right]^{-1}}{f+\beta} .$$

The inverse of $f+\beta$ is $\frac{1}{\beta}\sum_{n=0}^{\infty}(-1)^n\left(\frac{f}{\beta}\right)^n = \frac{1}{\beta} + \sum_{n=1}^{\infty}\frac{(-1)^n f^n}{\beta^{n+1}}$.

Since $f \in C^*$ the last sum is likewise in C^*. Thus,

$$\left[FP \frac{m(t)}{t^\alpha}\right]^{-1} = \frac{1}{\beta}\left[FP\, t^{k_1-\alpha}\right]^{-1} + \left[FP\, t^{k_1-\alpha}\right]^{-1}\left(\sum_{k=1}^{\infty}\frac{(-1)^n f^n}{\beta^{n+1}}\right)$$

is locally integrable since it is the sum of a locally integrable function and a continuous function.

The knowledge of the inverse operator functions allows one to solve singular integral equations involving the finite part of the convolution integral.

Lemma 7. Suppose $\alpha = -n-\beta$, n is a positive integer, $0 \leq \beta < 1$, $t > 0$, and f possesses n derivatives at the point t. Then

i) if $\beta \neq 0$

$$FP\int_0^t f(t-u)\, u^\alpha\, du = \frac{\Gamma(\alpha+1)}{\Gamma(1-\beta)} \frac{d^n}{dt^n}\int_0^t f(t-u)\, u^{-\beta}\, du, \qquad (17)$$

ii) if $\beta = 0$

$$FP\int_0^t f(t-u)\, u^{-n}\, du = \frac{(-1)^{n-1}}{\Gamma(n)} \frac{d^n}{dt^n}\int_0^t f(t-u)(\gamma_{n-1} + \ln u)\, du. \qquad (18)$$

Proof. The proof of this is given by Bureau in reference (6).

In certain cases an operator function which is not a function can be said to be equal to a function on a particular subinterval of $[0, \infty)$. For the following definition see Erdélyi (12), Appendix, or Mikusiński (11) Part VI, Chapter III.

Definition 12. Let there be an n such that $h^n f \in C^*$. Suppose that $g = h^n f$ is locally integrable and the function g is n times differentiable on the open interval (a, b). Then the operator f is said to be equal on (a, b) to $g^{(n)}$. This is written

$$f(t) = g^{(n)}(t) \qquad \text{on } (a, b).$$

The following lemma in the case of non-integer α is an exercise in Erdélyi (12) p. 133. For α a negative integer the proof follows easily from Lemma 7 ii).

Lemma 8. Suppose that

$$FP \int_0^t f(t-u)\, u^{\alpha}\, du = g_1(t) \qquad \text{on } (0, \infty)$$

and

$$f\left[FP\, t^{\alpha}\right] = g_2.$$

In the sense of Definition 12

$$g_2(t) = g_1(t) \qquad \text{on } (0, \infty).$$

It can now be seen how to use operators in order to solve singular integral equations.

Theorem 8. Let $\alpha = -n - \beta$, where $0 \leq \beta < 1$ and n is a positive integer. A necessary and sufficient condition that there exist a function f such that

$$FP \int_0^t f(t-u)\, u^{\alpha}\, du = g(t) \qquad \text{all } t > 0 \tag{19}$$

is that both of the following conditions be satisfied:

i) g is the n^{th} derivative on $(0, \infty)$ of a function k which is locally integrable;

ii) the operator $s^n k\left[FP\, t^{\alpha}\right]^{-1}$ is a locally integrable function which is n times differentiable at each point of the interval $(0, \infty)$.

If conditions i) and ii) are satisfied all of the solutions to Equation (19) are given by

$$f = f_1 + f_o \tag{20}$$

where

$$f_1(t) = (s^n k\left[FP\, t^{\alpha}\right]^{-1})(t) \qquad \text{on } (0, \infty)$$

and f_o is any function such that

$$f_o(t) = ((\alpha_o + \alpha_1 s + \cdots + \alpha_{n-1} s^{n-1})\left[FP\, t^{\alpha}\right]^{-1})(t) \quad \text{on } (0, \infty) \tag{21}$$

where the α_i's are complex numbers. The functions are equal to the operators in the sense of Definition 12.

Proof. First we show that the set of all solutions to the homogeneous equation

$$FP \int_0^t f(t-u)\, u^{\alpha}\, du = 0 \qquad \text{all } t > 0$$

is given by Equation (21).

Let f be a solution to the homogeneous equation. By Lemma 8 the operator $f[FP\, t^{\alpha}]$ is zero on the interval $(0, \infty)$ and thus is a polynomial in s. We have

$$f = [FP\, t^{\alpha}]^{-1} \left(\sum_{k=0}^{p} \alpha_k s^k \right)$$

for some integer p. f must be at least locally integrable in order that $FP \int_0^t f(t-u)\, u^{\alpha}\, du$ exist according to Definition 10. If f is locally integrable it must be that $p \leq n-1$. Thus every solution to the homogeneous equation must be of the form given in Equation (21).

From Equations (15) and (16) it is seen that the operator specified in Equation (21) is indeed a locally integrable function which is infinitely differentiable for each positive t. Thus $FP \int_0^t f_0(t-u)\, u^{\alpha}\, du$ is defined by Definition 10 for each positive t. The operator function $f_0[FP\, t^{\alpha}] = \sum_0^{n-1} \alpha_i s^i$ is equal to zero on $(0, \infty)$ in the sense of Definition 12. Thus by Lemma 8

$$FP \int_0^t f_0(t-u)\, u^{\alpha}\, du = 0 \qquad \text{all } t > 0.$$

Thus we have proved that the set of all solutions to the homogeneous equation is given by Equation (21). Let $\bar{f}$ be any solution to Equation (19). A function f is then a solution to Equation (19) if and only if $f = \bar{f} + f_0$ where f_0 is a solution to the homogeneous equation. Thus, if there is one solution we can find all the solutions. We will now show that conditions i) and ii) together are necessary and sufficient conditions in order that there exist one solution to Equation (19).

Suppose that i) and ii) are satisfied. Let $f_1 = s^n k [FPt^{\alpha}]^{-1}$. Since f_1 is n times differentiable the integral $FP \int_0^t f_1(t-u)\, u^{\alpha}\, du$

exists for each $t > 0$. Define g_1 by

$$FP \int_o^t f_1(t-u)\, u^{\alpha}\, du = g_1(t) \qquad \text{all } t > 0.$$

Since

$$f_1\left[FP\, t^{\alpha}\right] = s^n k,$$

it is seen by Lemma 8 that $s^n k(t) = g_1(t)$ on $(0, \infty)$ in the sense of Definition 12. But $s^n k(t) = g(t)$ on $(0, \infty)$ so that

$$g(t) = g_1(t) \qquad \text{all } t > 0.$$

Thus f_1 is a solution to Equation (19).

On the other hand if Equation (19) has a solution, Lemma 7 shows that i) is satisfied, that is $g(t) = k^{(n)}(t)$ for each $t > 0$ for some locally integrable function k; this solution f is such that

$$f\left[FP\, t^{\alpha}\right] = g_2$$

for some operator g_2 and

$$g_2(t) = k^{(n)}(t)$$

on $(0, \infty)$ in the sense of Definition 12. Thus

$$\frac{d^n}{dt^n}(h^n g_2)(t) = k^{(n)}(t)$$

at each point of $(0, \infty)$. Solving this differential equation we get

$$h^n g_2 = k + \sum_{r=1}^{n} \alpha_{n-r} h^r.$$

Thus

$$g_2 = s^n k + \sum_{r=1}^{n} \alpha_{n-r} h^{k-n}$$

and

$$g_2\left[FP\,t^{\alpha}\right]^{-1} - s^n k\left[FP\,t^{\alpha}\right]^{-1} = f - s^n k\left[FP\,t^{\alpha}\right]^{-1} = \left(\sum_0^{n-1} \alpha_r s^r\right)\left[FP\,t^{\alpha}\right]^{-1}.$$

f satisfies Equation (20) and the right hand side of the above equation satisfies the homogeneous equation, thus the operator $s^n k\left[FP\,t^{\alpha}\right]^{-1}$ satisfies condition ii) and Equation (19). This completes the proof of the theorem.

These same methods may be used to solve singular convolution equations where the convolution involves $\left\{\frac{m(t)}{t^{n+\beta}}\right\}$ rather than $\left\{\frac{1}{t^{n+\beta}}\right\}$. The fact that the operator $\left[FP\,\frac{m(t)}{t^{\alpha}}\right]$ can be expressed in terms of the operators $\left[FP\,t^{k-\alpha}\right]$ by the representation

$$\left[FP\,\frac{m(t)}{t^{\alpha}}\right] = g_n + \sum_{k=0}^{n-1} \frac{m^{(k)}(0)}{k!}\left[FP\,t^{k-\alpha}\right] \qquad (22)$$

allows us to state a lemma analogous to Lemmas 7 and 8 but involving $\left[FP\,\frac{m(t)}{t^{\alpha}}\right]$ rather than $\left[FP\,t^{\alpha}\right]$.

Lemma 9. Let $\alpha = n + \beta$ where n is a positive integer and $0 \leq \beta < 1$. Let m have n derivatives each of which is in C^*. Suppose $m(0) \neq 0$. Then if the locally integrable function f is n times differentiable on $(0, \infty)$

$$FP\int_0^t f(t-u)\,\frac{m(u)}{u^{\alpha}}\,du = k^{(n)}(t) \qquad t > 0$$

where k is a locally integrable function. The operator

$$f\left[FP\,\frac{m(t)}{t}\right] = g$$

is such that

$$g(t) = k^{(n)}(t) \qquad \text{on } (0, \infty)$$

in the sense of Definition 12.

Proof. Express $\left[FP\ \frac{m(t)}{t^\alpha}\right]$ in the form (22) and apply Lemmas 7 and 8.

The requirement that $m(0)$ be non-zero is made only in order to know the number of derivatives to require of m.

Theorem 9. Let α and the function m be as in Lemma 9. A necessary and sufficient condition that there exist a solution f to the equation

$$FP \int_0^t f(t-u)\ \frac{m(t)}{t^\alpha}\ du = g(t) \qquad \text{all} \quad t > 0$$

is that both of the following conditions be satisfied:

i) g is the n^{th} derivative on $(0, \infty)$ of a locally integrable function k;

ii) the operator $s^n k \left[FP\ \frac{m(t)}{t^\alpha}\right]^{-1}$ is a locally integrable function which is n times differentiable on $(0, \infty)$.

Every solution is of the form

$$f = f_1 + f_o$$

where $f_1(t) = (s^n k \left[FP \frac{m(t)}{t^\alpha}\right]^{-1})(t)$ on $(0, \infty)$ and f_o is a solution to the homogeneous equation

$$FP \int_0^t f(t-u)\ \frac{m(t)}{t^\alpha}\ du = 0 \qquad \text{all} \quad t > 0.$$

Proof. The proof is essentially the same as the proof of Theorem 7 and will not be given again.

By Lemma 6 $\left[FP\ \frac{m(t)}{t^\alpha}\right]^{-1}$ is locally integrable and if it is n times differentiable on $(0, \infty)$ there will be non-zero solutions to the homogeneous equation. The functions $f = \alpha_o \left[FP\ \frac{m(t)}{t}\right]^{-1}$ with α_o

a complex number are such that

$$f\left[FP\,\frac{m(t)}{t^{\alpha}}\right] = \alpha_o$$

and the operator α_o is equal to zero on $(0, \infty)$ in the sense of Definition 12. Thus by Lemma 9 these functions satisfy the homogeneous equation. The dimension of the vector space of solutions to the homogeneous equation in dependent on the order to which m' vanishes at the origin. If $m'(0) \neq 0$ the dimension will be one. As we have seen when $m = h$ (i.e. $m'(0), \ldots, m^{(n-1)}(0) = 0$) the dimension is n.

4.2. The results discussed in the preceding section will be applied in this section to solve the particular singular integral equation

$$FP \int_o^t f(t-u)\,\frac{J_o(u)}{u}\,du = g(t) \qquad \text{all } t > 0.$$

Here J_o is the Bessel function of the first kind and of order zero.

The operator $r = s^2 + 1$ is defined by the power series expansion of $(s^2+1)^{1/2}$, and from Mikusiński (11) p. 456 it is known that

$$(r - s)^{2n} = \left\{\frac{2n}{t}\, J_{2n}(t)\right\} \qquad n = 1, 2, \ldots$$

where J_{2n} is the Bessel function of order $2n$. In the remainder of this section we shall use the symbol r to denote $s^2 + 1$ and the symbol ρ to denote $z^2 + 1$ where z is a complex number. If a is a locally integrable function which has a Laplace transform we shall denote its Laplace transform, $\int_o^{\infty} e^{-zt}\,a(t)\,dt$, by $\bar{a}(z)$.

Lemma 10.

i) $\left[FP\,\frac{J_o(t)}{t}\right] = \ln\frac{2}{\gamma}\,(r - s).$

ii) Let a be the locally integrable function $\left[FP\ \frac{J_o(t)}{t}\right]^{-1}$. We have

$$\overline{a}(z) = \frac{1}{\ln \frac{2}{\gamma} (\rho - z)}$$

when $\operatorname{Re} z > 0$

Proof.

i) It is shown in Erdélyi (17) p. 26 that

$$J_o(t) + 2 \sum_{n=1}^{\infty} J_{2n}(t) = 1 \qquad t \geq 0.$$

Thus

$$\left\{ \frac{J_o(t) - 1}{t} \right\} = - \left\{ \sum_{n=1}^{\infty} \frac{2}{t} J_{2n}(t) \right\}$$

$$= - \sum_{n=1}^{\infty} \frac{1}{n} \left\{ \frac{2n}{t} J_{2n}(t) \right\}$$

$$= - \sum_{n=1}^{\infty} \frac{(r-s)^{2n}}{n}$$

and the series is convergent (C^*) since $(r-s)^2 \in C^*$. Thus

$$\left\{ \frac{J_o(t) - 1}{t} \right\} = \ln (1 - (r-s)^2) = \ln 2s\,(r-s).$$

Now

$$\left[FP\ \frac{J_o(t)}{t}\right] = \left\{ \frac{J_o(t) - 1}{t} \right\} + \left[FP\, t^{-1}\right]$$

$$= \ln 2s\,(r-s) - \ln \gamma s$$

$$= \ln \frac{2}{\gamma} (r-s)$$

which proves i).

ii) Let $a = \left[FP\,\frac{J_o(t)}{t}\right]^{-1}$, $b = \left\{\frac{J_o(t) - 1}{t}\right\}$, and $c = \left[FP\,t^{-1}\right]^{-1} = \left\{\int_o^\infty \frac{t^{u-1}}{\Gamma(u)\,\gamma^u}\,du\right\}$. From the proof of Lemma 6 we know that

$$a = \frac{c}{1+cb} = c + c\sum_{n=1}^{\infty} (-1)^n (cb)^n.$$

The function c possesses a Laplace transform (Erdélyi (18) p. 251, Eq. (11)); thus if the sum on the right (which represents a continuous function) is exponentially bounded a possesses a Laplace transform. We will first show this sum to be exponentially bounded.

Since $|J_o(t)| \leq 1$ for all t and $b \in C^*$ there is a constant B such that $|b|(t) \leq B$ for all $t \geq 0$. Since $c(t)$ is positive for $t > 0$

$$|cb|(t) \leq B\int_o^t c(\xi)\,d\xi = B\int_o^t\int_o^\infty \frac{\xi^{u-1}}{\Gamma(u)\,\gamma^u}\,du\,d\xi.$$

The integral $\int_o^\infty \frac{\xi^{u-1}}{\Gamma(u)\,\gamma^u}\,du$ converges uniformly on each interval $0 \leq \xi \leq t$ so that

$$\int_o^t\int_o^\infty \frac{\xi^{u-1}}{\Gamma(u)\,\gamma^u}\,du\,d\xi = \int_o^\infty\int_o^t \frac{\xi^{u-1}}{\Gamma(u)\,\gamma^u}\,d\xi\,du = \int_o^\infty \frac{t^u}{\Gamma(u+1)\gamma^u}\,du.$$

This is a well known function $\nu(\frac{t}{\gamma})$, and from Erdélyi (19) p. 219

$$0 \leq \nu(\tfrac{t}{\gamma}) = \int_o^\infty \frac{t^u}{\Gamma(u)\,\gamma^u}\,du \leq e^{t/\gamma} \qquad t > 0.$$

Thus

$$|cb|(t) \leq B\,e^{t/\gamma}, \qquad t > 0$$

and

$$\sum_{n=1}^{\infty} |c\,b|^{n}(t) \leq \sum_{n=1}^{\infty} \frac{B^n t^{n-1}}{\Gamma(n)} e^{t/\gamma} = B\, e^{\frac{\gamma B+1}{\gamma} t}.$$

Thus the function a possesses a Laplace transform.

In order to evaluate $\bar{a}(z)$ we note that $\bar{J}_o(z) = \frac{1}{\rho}$ and the Laplace transform of $\{J_o(t) - 1\}$ is $\frac{1}{\rho} - \frac{1}{z}$. Since $b = \left\{\frac{J_o(t) - 1}{t}\right\}$

$$\bar{b}(z) = \int_z^{\infty} \frac{1}{\sqrt{u^2+1}} - \frac{1}{u}\, du = \ln \frac{2z}{z+\rho}.$$

Thus the transform of $h\left[FP \frac{J_o(t)}{t}\right] = h\,b + \{\ln t\}$ is

$$\overline{h\left[FP \frac{J_o(t)}{t}\right]}(z) = \frac{1}{z} \ln \frac{2z}{z+\rho} - \frac{\ln \gamma z}{z} = \frac{1}{z} \ln \frac{2}{\gamma} (\rho - z).$$

Now

$$\bar{a}(z)\, \overline{h\left[FP \frac{J_o(t)}{t}\right]}(z) = \overline{a\, h\left[FP \frac{J_o(t)}{t}\right]}(z) = \bar{h}(z) = \frac{1}{z} \qquad \mathrm{Re}\, z > 0$$

and we have

$$\bar{a}(z) = \frac{1}{\ln \frac{2}{\gamma} (\rho - z)} \qquad \mathrm{Re}\, z > 0$$

which proves ii).

A more explcit representation of $\left[FP \frac{J_o(t)}{t}\right]^{-1}$ can be obtained from the inversion formula for Laplace transforms. To get convergence of the integral which occurs we will use the representation for $h^2\left[FP \frac{J_o(t)}{t}\right]^{-1}$. Thus

$$\left[FP \frac{J_o(t)}{t}\right]^{-1} = \left\{\frac{d^2}{dt^2}\left(h^2\left[FP \frac{J_o(t)}{t}\right]^{-1}(t)\right)\right\}$$

$$= \left\{\frac{d^2}{dt^2} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{zt}\, dz}{z^2 \ln \frac{2}{\gamma} (\rho - z)}\right\}$$

whenever $c > 0$.

We already know that a necessary and sufficient condition that the equation

$$FP \int_0^t f(t-u) \frac{J_0(u)}{u} du = g(t) \qquad \text{all } t > 0 \qquad (23)$$

possess a solution is that both the following conditions hold: i) there is a locally integrable function k which is differentiable on $(0, \infty)$ and is such that $k(t) = g(t)$ for all $t > 0$, and ii) $s \left[FP \frac{J_0(t)}{t} \right]^{-1} k$ is locally integrable and differentiable on $(0, \infty)$.

Corollary 3. Let the conditions i) and ii) hold. Let k be as specified in i), let $c > 0$ and

$$a(t) = \frac{d^2}{dt^2} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{zt}\, dz}{z^2 \ln \frac{2}{\gamma} \, (\rho - z)} \qquad \text{all } t > 0.$$

A necessary and sufficient condition that f be a solution to Equation (23) is that for some complex number α_0

$$f(t) = (a\,k)'(t) + \alpha_0\, a(t) \qquad \text{all } t > 0.$$

Proof. Since $\left[FP \frac{J_0(t)}{t} \right]^{-1} = \left\{ a(t) \right\}$ it follows from Corollary 3 that $\left\{ f(t) \right\}$ satisfies Equation (23). It only remains to show that the solutions to the homogeneous equation form a vector space of exactly one dimension. The function

$$a = \frac{1}{\left\{ \frac{J_0(t) - 1}{t} \right\} + \left[FP\, t^{-1} \right]}$$

is the sum of a locally integrable function $\left[FP\, t^{-1} \right]^{-1}$ and an infinite series which represents a continuous function. Since $\left[FP\, t^{-1} \right]^{-1}$

$= \left\{ \int_0^\infty \frac{t^{u-1}}{\Gamma(u)\, \gamma^u}\, du \right\}$ is discontinuous at the origin, $s^n \mathbf{a}$ is not a function for any $n \geq 1$, and the only solutions to the homogeneous equation are scalar multiples of a.

Chapter 5

5.1. The main applications of the finite part of divergent integrals occur in solving partial differential equations. Besides the finite part of divergent integrals another useful concept which is closely related to the finite part of divergent integrals occurs in partial differential equations. If f possesses n derivatives at a point t the logarithmic part of the divergent convolution integral $\int_0^t \frac{f(t-u)}{u^n}\,du$ is defined to be $\frac{(-1)^n}{\Gamma(n)} f^{(n-1)}(t)$. Thus the logarithmic part of this integral (written $LP\int_0^t \frac{f(t-u)}{u^n}\,du$) is the negative of the residue at $z = -n$ of $FP\int_0^t f(t-u)\,u^z\,du$. From section 3.3 it is known that the $\operatorname{Res}_{z=-n}\left[FP\,t^z\right]$ is $\frac{(-1)^{n-1}}{\Gamma(n)} s^{n-1}$. We will define the negative of this operator to be $\left[L\,t^{-n}\right]$ and combine the finite part and logarithmic part of an operator function in what we will call the improper part of $\{t^z\}$.

Definition 13. Let n be a positive integer. The operator $\left[LP\,t^{-n}\right]$ is defined to be $\frac{(-1)^n}{\Gamma(n)} s^{n-1}$ when $n = 1, 2, \ldots$.
The operator function which will be called the improper part of $\{t^z\}$ is defined by

$$\left[P\,t^z\right] = \left[FP\,t^z\right] \qquad \text{when } z \neq -1, -2, \ldots$$

$$\left[P\,t^z\right] = \left[LP\,t^z\right] \qquad \text{when } z = -1, -2, \ldots .$$

When f is such that $LP\int_0^t \frac{f(t-u)}{u^n}\,du$ is defined for some positive n and all $t > 0$ the operator $f\left[LP\,t^{-n}\right]$ is equal to this integral on the interval $(0, \infty)$ in the sense of Definition 12. If f has

$n-1$ locally absolutely continuous derivatives and $f^{(k)}(0) = 0$ for $k = 0, 1, \ldots, n-2$ the function $\left\{LP\int_0^t \frac{f(t-u)}{u^n}\,du\right\}$ is in F, in fact it is in C^*, and is equal to the operator $f\left[LP\, t^{-n}\right]$.

5.2. Let

$$x = (x_1, \ldots, x_p) \qquad \nabla^2 U(x,t) = \sum_{i=1}^{p} \frac{\partial^2 U}{\partial x_i^2}(x,t)$$

$$U_t(x,0) = \frac{\partial U}{\partial t}(x,t)\Big|_{t=0} \qquad U_{tt}(x,t) = \frac{\partial^2 U}{\partial t^2}(x,t).$$

The statement of the Cauchy problem for the wave equation in a half space is as follows: to find a function $\{U(x,t)\}$ on the half space $E^p \times I$ such that

$$\nabla^2 U(x,t) = U_{tt}(x,t) \qquad U(x,0) = f(x) \qquad U_t(x,0) = g(x). \tag{24}$$

The solution to the system

$$\nabla^2 V(x,t) = V_{tt}(x,t) \qquad V(x,0) = f(x) \qquad V_t(x,0) = 0$$

can be found from the solution to the system

$$\nabla^2 W(x,t) = W_{tt}(x,t) \qquad W(x,0) = 0 \qquad W_t(x,0) = f(x)$$

and is in fact $V(x,t) = \frac{\partial W}{\partial t}(x,t)$, as is easily verified. Thus we shall only be concerned with the special case in which $U(x,0) = f(x) = 0$. Since the cases $p = 1$ and $p = 2$ do not give rise to divergent integrals we shall only be concerned with the case in which p is greater than 2.

For the remainder of this chapter k shall denote the integral part of $\frac{p+3}{2}$. If $g(x)$ is k times differentiable in E^p the solution to our problem can be expressed in terms of the improper part of a divergent convolution integral. This solution is given by F. Bureau in (4) pp. 154-7 to be

$$U(x, t) = A_p \omega_p \, P \int_o^t \overline{g}(x, u) u^{p-1} (t^2 - u^2)^{-\frac{p-1}{2}} du \qquad (25)$$

where

$$A_p = \begin{cases} \dfrac{(-1)^{k-2}}{2\pi^{k-1/2}} \Gamma(\frac{p-1}{2}) & \text{for } p \text{ even} \\ \\ \dfrac{(-1)^{k-2}}{2\pi^{k-2}} \Gamma(\frac{p-1}{2}) & \text{for } p \text{ odd,} \end{cases}$$

and $\omega_p = \dfrac{2\pi^{p/2}}{\Gamma(p/2)}$ is the surface area of the unit sphere in E^p, and

$$\overline{g}(x, t) = \frac{1}{\omega_p} \int_{\omega_p} g(x + t\eta) \, d\omega_p$$

is the mean value of $g(x)$ on a sphere with center x and radius t. A typical point in this sphere is $x + t\eta$ where η is a point in the unit sphere.

The improper integral in Equation (25) is not of the convolution type, however Equation (25) can be written as

$$U(x, \sqrt{t}) = \frac{A_p \omega_p}{2} \, P \int_o^t \overline{g}(x, \sqrt{u}) \, u^{\frac{p-2}{2}} (t-u)^{-\frac{p-1}{2}} du \qquad (25')$$

and the improper integral involved in Equation (25') is of the convolution type.

We shall now consider the transformations which arise in solving the Cauchy problem for the wave equation. From now on the symbol E^p will be omitted when speaking of spaces of functions on E^p and we shall merely say C, C_k, $C_k C^*$, etc. when speaking of $C(E^p)$,

$C_k(E^p)$, $C_k(E^p)C^*$, etc. If $A \subset B$ and B is a topological space then B induces a topology on A. We shall always denote by $A \subset B$ the set A endowed with the topology induced by B and willdistinguish this if necessary from the space A. Thus $C_k \subset C$ is a subspace of C but C_k is the space described in Chapter 2.

We shall use a symbol $T, T_1, \ldots,$ etc. to represent a mapping of one set into another. Then the transformation of a space A into a space C is continuous or not continuous depending on the topologies of A and C. Thus $T \,|\, A \to C$ and $T \,|\, A \subset B \to C \subset D$ are the same mappings of the set A into the set C but different transformations since A and $A \subset B$ are different spaces as are C and $C \subset D$.

The transformation $T\,g(x) = U(x) = \{U(x,t)\}$ defined by quation (25') is a transformation on C_k to C_2C^*. T is a composition of two transformations.

$$T = T_2\, T_1 \tag{26}$$

where

$$T_1 \,\big|\, C_k \to C_2 C^*$$

is such that

$$T_1\, g(x) = U_1(x) = \left\{ U_1(x,t) \right\}$$

$$U_1(x) = \left\{ \frac{A_p\, \omega_p}{2}\; P \int_0^t \overline{g}(x, \sqrt{u})\, u^{\frac{p-2}{2}} (t-u)^{-\frac{p-1}{2}}\, du \right\}$$

and

$$T_2 \,\big|\, C_k C^* \to C_k C^*$$

is such that

$$T_2\, U_1(x) = U(x) = \left\{ U_1(x, t^2) \right\}.$$

Also let

$$T_o \,\big|\, C_k \to C_k C^*$$

be such that

$$T_o g(x) = \frac{1}{\omega_p} \left\{ \int_{\omega_p} t^{\frac{p-2}{2}} g(x+\sqrt{t}\,\eta)\, d\omega_p \right\}.$$

Theorem 10. The transformation $T \,\Big|\, C_k \to C_2\, C^*$ defined by Equation (26) is continuous.

Proof.

i) It is trivial to show T_2 is continuous.

ii) Let

$$T_o g(x) = \left\{ \frac{t^{\frac{p-2}{2}}}{\omega_p} \int_{\omega_p} g(x+\sqrt{t}\,\eta)\, d\omega_p \right\}.$$

Since g is in C_k, differentiation with respect to x may be carried out under the integral sign. Suppose that $g_r(x)$ is a sequence in C_k and $g_r(x) \to g(x)\ (C_k)$ as $r \to \infty$. When $|x| \leq n$

$$\left| D^m T_o(g_r(x) - g(x)) \right| (t) \leq t^{\frac{p-2}{2}} \int_{\omega_p} \left| D^m(g_r(x+\sqrt{t}\,\eta) - g(x+\sqrt{t}\,\eta)) \right| d\omega_p$$

$$\left| D^m T_o(g_r(x) - g(x)) \right| (t) \leq t^{\frac{p-2}{2}} \left\| g_r(x) - g(x) \right\|^m_{n+\sqrt{t}} .$$

Since $n \geq 1$ and $p > 2$, when $t \leq n$ we have

$$\left\| T_o(g_r(x) - g(x)) \right\|^m_n \leq n^{\frac{p-2}{2}} \left\| g_r(x) - g(x) \right\|^m_{2n} ,$$

where the semi-norm on the left is on $C_k C^*$ and the semi-norm on the right is on C_k. Thus $T_o g_r(x) \to T_o g(x)\ (C_k C^*)$ and T_o is a continuous transformation on C_k to $C_k C^*$.

iii) To show that T_1 is continuous we will show its graph is closed and apply Theorem 1.

Suppose

$$g_r(x) \to g(x) \; (C_k)$$

and

$$T_1 \, g_r(x) \to V(x) \quad (C_2 \, C^*)$$

as $r \to \infty$. Let $k_1 = k - 3$. Since $g_r(x) \in C_k$,

$$\frac{1}{\omega_p} \int_{\omega_p} t^{\frac{p-2}{2}} \, g(x + \sqrt{t}\,\eta) \, d\omega_p$$

can be differentiated at least k_1 times with respect to t under the integral sign and all these derivatives are zero at $t = 0$. Thus

$$T_1 \, g_r(x) = \frac{A_p \, \omega_p}{2} \left\{ P \int_o^t (T_o \, g_r(x)) \, (u) \, (t-u)^{-\frac{p-1}{2}} \, du \right\}$$

is

$$T_1 \, g_r(x) = \beta_p \, s^{\frac{p-3}{2}} \; T_o \, g_r(x)$$

where β_p is a complex number. Since $g_r(x) \to g(x) \; (C_k)$ as $n \to \infty$ and T_o is continuous we have

$$T_o \, g_r(x) \to T_o \, g(x) \quad (C_k \, C^*)$$

as $n \to \infty$. Thus

$$T_1 \, g_r(x) = \beta_p \, s^{\frac{p-3}{2}} \; T_o \, g_r(x) \to \beta_p \, s^{\frac{p-3}{2}} \; T_o \, g(x) \; (C_k \, F)$$

as $n \to \infty$. But

$$T_1 \, g_r(x) \to V(x) \quad (C_k \, C^*)$$

as $n \to \infty$, so that

$$V(x) = \beta_p \, s^{\frac{p-3}{2}} \; T_o \, g(x)$$

is in $C_k \, C^*$ and thus

$$V(x) = \beta_p s^{\frac{p-3}{2}} T_o g(x) = \beta_p \left\{ P \int_o^t (T_o g(x))(u)(t-u)^{\frac{p-1}{2}} du \right\}$$

or

$$V(x) = T_1 g(x).$$

Thus $T = T_2 T_1$ is the composition of two continuous mappings and so is continuous.

5.3. We would like to increase the size of the class of functions with which a solution to the wave equation can be associated. One way to do this might be as follows.

Let $D^k = \left\{ s^k g \middle| g \epsilon C^* \right\}$. This is a subspace of F. There is a unique extension of the transformation.

$$T_1 \Big| C_k \subset C \rightarrow C_2 \; C^* \subset C_2 \; F$$

to a transformation

$$\hat{T}_1 \Big| C \rightarrow C \; D^k \subset C \; F$$

which is continuous in the sense that it transforms convergent sequences into convergent sequences. This extension is given by

$$\hat{T}_1 \, g(x) = \beta_p \, s^{\frac{p-3}{2}} \, T_o \, g(x)$$

where β_p and T_o are as in the proof of Theorem 10. If we could find an extension of T_2 to a transformation $\hat{T}_2 \Big| CD^k \rightarrow C\,D^k$ which takes convergent sequences into convergent sequences we could extend $T = T_2 T_1$ to a transformation $\hat{T} = \hat{T}_2 \hat{T}_1$ with domain C and $\hat{T}$ would be continuous in the sense of taking convergent sequences into convergent sequences. Unfortunately, there exists no such extension for T_2, since there are sequences which are convergent in $C^* \subset F$ which are transformed by T_2 into divergent sequences in $C^* \subset F$.

Example. Let $f_n = \left\{ n^{3/2} e^{-n\sqrt{t}} \right\}$. Then $f_n \in C^*$ for $n = 1, 2, \ldots$ and $f_n \to 0$ (F) as $n \to \infty$. However, the sequence $T_2 f_n = \left\{ n^{3/2} e^{-nt} \right\}$ diverges in F.

In view of this we settle for extending T to a closed transformation whose domain is contained in C and whose range is contained in $C C^*$.

Let
$$W = \left\{ g(x) \,\middle|\, g(x) \in C, \quad s^{\frac{p-3}{2}} \left\{ \bar{g}(x, \sqrt{t})\, t^{\frac{p-2}{2}} \right\} \in C C^* \right\}.$$

Let
$$\hat{T}_1 \,\Big|\, W \subset C \to C C^*$$

be defined by
$$\hat{T}_1 \, g(x) = \beta_p \, s^{\frac{p-3}{2}} \left\{ \bar{g}(x, \sqrt{t}) \, t^{\frac{p-2}{2}} \right\} \tag{27}$$

where β_p is as in the proof of Theorem 10. Define
$$\hat{T} \,\Big|\, W \subset C \to C C^*$$

by
$$\hat{T} = T_2 \hat{T}_1 . \tag{27'}$$

The domain W of $\hat{T}$ contains C_k and thus W is dense in C. However, in the next theorem we show that any $g(x) \in C$ for which there is a sequence $g_n(x)$ in W such that
$$g_n(x) \to g(x) \quad (C)$$

and
$$T \, g_n(x) \to V(x) \quad (C C^*)$$

as $n \to \infty$ is in fact in W and we have $T \, g(x) = V(x)$.

Theorem 11. Let the transformation $\hat{T} \mid W \subset C \to C C^*$ be defined by Equation (27'). The set $\{(g(x), Tg(x)) \mid g(x) \in W\}$ is closed in $C \times C C^*$.

Proof. Suppose $g_n(x) \in W$, $n = 1, 2, \ldots$, $g_n(x) \to g(x)$ (C) and $\hat{T} g_n(x) \to V(x)$ $(C C^*)$ as $n \to \infty$. We will show that $g(x) \in W$ and $\hat{T} g(x) = V(x)$.

The transformation $T_2 \mid C C^* \to C C^*$ given by $T_2\{U(x,t)\} = \{U(x,t^2)\}$ has a unique two-sided inverse T_2^{-1}. It is such that $T_2^{-1}\{U(x,t)\} = \{U(x,\sqrt{t})\}$. In the semi-norms on $C C^*$

$$\| T_2 U(x) \|_n \leq \| U(x) \|_{n^2}$$

and

$$\| T_2^{-1} U(x) \|_n \leq \| U(x) \|_n$$

for $n = 1, 2, \ldots$. Thus not only T_2 but also T_2^{-1} is continuous. Since $\hat{T} g_n(x) \to V(x)$ $(C C^*)$ as $n \to \infty$ we have, also, $T_2^{-1} \hat{T} g_n(x) \to T_2^{-1} V(x)$ $(C C^*)$ as $n \to \infty$.

Since the $g_n(x)$ are in W, $\hat{T}_1 g_n(x) = \beta_p s^{\frac{p-3}{2}} (T_o g_n(x))$ where β_p and T_o are as in the proof of Theorem 10. Since $g_n(x) \to g(x)$ (C) as $n \to \infty$, we have $T_o g_n(x) \to T_o g(x)$ $(C C^*)$ as $n \to \infty$. Thus, $\hat{T}_1 g_n(x) \to \beta_p s^{\frac{p-3}{2}} (T_o g(x))$ (C F) as $n \to \infty$. Since $\hat{T}_1 g_n(x) \to T_2^{-1} V(x)$ $(C C^*)$ as $n \to \infty$ we have $\beta_p s^{\frac{p-3}{2}} (T_o g(x)) = T_2^{-1} V(x)$ is in $(C C^*)$. Thus by the definition of W we have $g(x) \in W$ and by the definition of $\hat{T}_1$ we have $\hat{T}_1 g(x) = T_2^{-1} V(x)$ or $\hat{T} g(x) = V(x)$. This proves the theorem.

5.4. A function $U(x) \in C C^*$ which is the image under T of a function $g(x)$ in C will be called a generalized wave function for the boundary value $g(x)$.

Theorem 12. If $U(x)$ is a generalized wave function for the boundary value $g(x)$ then $U_t(x,0)$ exists and

i) $U(x,0) = 0$

ii) $U_t(x,0) = g(x)$.

Proof. Suppose $p = 2m+4$ is even. Let $V(x) = T_2^{-1} U(x)$. Then

$$V(x) = \beta_p \, s^{m+1/2} \left\{ \bar{g}(x,\sqrt{t})\, t^{m+1} \right\}$$

and

$$h^{m+1} V(x) = \beta_p \, h^{1/2} \left\{ \bar{g}(x,\sqrt{t})\, t^{m+1} \right\}$$

or

$$\int_o^t \frac{(t^2-u^2)^m}{\Gamma(m+1)} u V(x,u^2)\, du = \frac{\beta_p}{\Gamma(1/2)} \int_o^t \frac{\bar{g}(x,u)}{(t^2-u^2)^{1/2}} u^{2m+3}\, du.$$

Thus

$$\int_o^t (t^2-u^2)^m u U(x,u)\, du = \frac{\beta_p \Gamma(m+1)}{\Gamma(1/2)} \int_o^t \frac{g(x)+o(1)}{(t^2-u^2)^{1/2}} u^{2m+3}\, du$$

$$= \frac{\beta_p \quad (m+1)\, B(m+1.\, 1/2)}{\Gamma(1/2)} t^{2m+3} (g(x)+o(1))$$

as $t \to 0$. If we let $\dfrac{\beta_p \Gamma(m+1)\, B(m+1, 1/2)}{\Gamma(1/2)} = \bar{\beta}_p$ we have

$$\frac{\int_o^t (t^2-u^2)^m u U(x,u)\, du}{t^{2m+3}} = \bar{\beta}_p (g(x)+o(1)).$$

The right hand side of the above equation tends to $\bar{\beta}_p g(x)$ as $t \to 0$ and thus the left hand side must also. The integral in the numerator on the left is at least $m+1$ times continuously differentiable with respect to t even if $U(x)$ is only continuous. Upon applying L'Hospital's rule $m+1$ times we find

$$\frac{U(x,t)}{t} \to g(x)$$

as $t \to 0$. Thus $U(x,t) \to 0$ as $t \to 0$ and since $\{U(x,t)\}$ is in $C\,C^*$ we have

$$U(x,0) = 0$$

and also

$$\frac{U(x,t)}{t} \to g(x) = U_t(x,0)$$

as $t \to 0$.

When $p = 2m+3$ is odd the proof is similar but less difficult.

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