On the p-adic Local Invariant Cycle Theorem

Thesis by

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In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

California Institute of Technology
Pasadena, California

2012
(Defended May 23, 2012)

Acknowledgements

I am deeply indebted to my advisor, Matthias Flach, whose guidance, stimulating insight and suggestions help me during the time of research and preparation of thesis. Without his helpful instructions and optimistic encouragement, this work would not have been possible.

I would like to express my great appreciation to Dinakar Ramakrishnan for his constant help and encouragement.

I am thankful to the faculty in the department of mathematics at Caltech for their professional instructions, to the colleagues for the inspiring discussions, and to the staff for the help of providing an excellent work environment.

Finally, I would like to thank my parents who have been a source of tireless of love and support throughout. This thesis is dedicated to them.

Abstract

The aim of this paper is to consider the p-adic local invariant cycle theorem in the mixed characteristic case.

In the first part of the paper, via case-by-case discussion, we construct the p-adic specialization map, and then write out the complete conjecture in p-adic case. We proved the theorem in good reduction and semistable reduction cases.

In the second part of the paper, by using Berthelot, Esnault and Rülling's trace morphisms in [BER], we first prove the case of coherent cohomology, then we extend it to the Witt vector cohomology, and we then get a result on the Frobenius-stable part of the Witt vector cohomology, which corresponds the slope 0 part of the rigid cohomology, we then get the general p-adic local invariant cycle theorem.

We also give another approach in the H^0 and H^1 cases in the general case.

In the last part of the paper, based on Flach and Morin's work on the weight filtration in the l-adic case, we consider the p-adic analogous result (which, together with the l-adic's result, serves as a part to prove the compatibility of the Weil-etale cohomology with the Tamagawa number conjecture). This is a direct corollary of the local invariant cycle theorem by taking the weight filtration. And we also consider some typical examples that the weight filtration statement could be verified by direct computations.

Contents

Chapter 1

Introduction

Notation: Assume that R is a complete discrete valuation ring with quotient field K and finite residue field k of characteristic p. Set $S = Spec(R), \eta = Spec(K), s = Spec(k)$. Let $\bar{S} = (\bar{S}, \bar{s}, \bar{\eta})$ be the normalization of S in a separable closure $\bar{K}$ of K and denote by $I \subseteq G := Gal(\bar{K}/K)$ the inertia subgroup. Let $W(k)$ be the ring of Witt vectors of k, K_0 be the fraction field of $W(k)$.

The purpose of this paper is to study the p-adic cohomology theory (where we mean several different but related things: the de-Rham or p-adic etale cohomology of varieties over p-adic fields or the rigid cohomology of varieties over fields of characteristic $p > 0$).

For any scheme X over S, we have a *special fiber* X_s over k and a *generic fiber* X_η over K. And we consider the diagram of schemes:

$$\begin{array}{ccccc} X_s & \longrightarrow & X & \longleftarrow & X_\eta \\ \downarrow & & \downarrow f & & \downarrow \\ s = Spec(k) & \longrightarrow & S = Spec(R) & \longleftarrow & \eta = Spec(K) \end{array}$$

Let $f : X \to S$ be a proper, flat, generically smooth morphism of relative dimension d. For $0 \le i \le 2d$, one can define the specialization morphism on l-adic etale cohomology groups:

$$sp : H^i(X_{\bar{s}}, \mathbb{Q}_l) \to H^i(X_{\bar{\eta}}, \mathbb{Q}_l)^I$$

via the composition:

$$H^i(X_{\bar{s}}, \mathbb{Q}_l) \cong H^i(X', \mathbb{Q}_l) \to H^i(X'_\eta, \mathbb{Q}_l) \to H^i(X_{\bar{\eta}}, \mathbb{Q}_l)^I$$

where X' is the base change of X to a strict Henselization of S at $\bar{s}$ and the first isomorphism is proper base change. This map sp is G-equivariant.

The local invariant cycle theorem conjectures that this specialization morphism sp is an epimorphism if $l \neq p$ and X is regular.

In [D1, (3.6)], Deligne has proved the l-adic case of the local invariant cycle theorem in the equal characteristic case under the hypothesis that X is essentially smooth over k and $X_{\bar{\eta}}$ smooth over $\bar{\eta}$.

In the mixed characteristic case, this theorem is still a conjecture. It is well-known that the monodromy-weight conjecture (Deligne's conjecture on the purity of monodromy filtration) implies the local invariant cycle theorem ([I1], [I2], (8.8)), thus the theorem holds in the mixed characteristic in dimension ≤ 2. But in the case of dimension ≥ 2, it is still a conjecture. Also, see [S1] for Scholze's result for the existence of a natural tilting operation that exchanges characteristic 0 and characteristic p, and then deduce the monodromy-weight conjecture in certain cases by reduction to equal characteristic.

Now, let W_iV be the subspace of V with an endomorphism ϕ where eigenvalues of ϕ have weight $\leq i$. In [FlM, (10.1)], M. Flach and B. Morin prove that $H^i(X_{\bar{s}}, \mathbb{Q}_l) = W_iH^i(X_{\bar{s}}, \mathbb{Q}_l) \rightarrow W_iH^i(X_{\bar{\eta}}, \mathbb{Q}_l)^I$ induced by sp is surjective for all i under the hypothesis that $l \neq p$ and X is regular (Also, note that the regularity is a key assumption, see [FlM] section 10 and [I2] section 8 for counterexamples).

It is then natural to ask the analogous result of the local theorem of invariant cycles for the $p-$adic cohomology.

Assume that K has characteristic 0. For $l = p$, one can still define the specialization map

$$sp : H^i(X_{\bar{s}}, \mathbb{Q}_p) \rightarrow H^i(X_{\bar{\eta}}, \mathbb{Q}_p)^I$$

as above since proper base change holds for arbitrary torsion sheaves. However, it is well-known that $p-$adic etale cohomology of varieties in characteristic p only describes the slope 0 part of the full $p-$adic cohomology, which is Berthelot's rigid cohomology $H^i_{rig}(X_s/k)$ (here the slope 0 part V^{slope0} of a finite dimensional $\mathbb{Q}_p-$vector space V with an endomorphism ϕ is defined as the maximal subspace on which the eigenvalues of ϕ are $p-adic$ units), also there is no duality theorem in this case, which served as an important part in the proof of $l-$adic situation.

Thus we need to consider rigid cohomology instead of etale cohomology, through case by case construction, via using Chiareletto's map and a cohomology descent, we construct the full p-adic specialization map in chapter 3:

Theorem 1.0.1 *If X/S is proper, flat and generically smooth, then there is a ϕ-equivariant map*

$$sp' : H^i_{rig}(X_s/k) \rightarrow D_{crys}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p))$$

and a commutative diagram of $Gal(\bar{k}/k)-$modules

$$\begin{array}{ccc} H^i(X_{\bar{s}}, \mathbb{Q}_p) & \xrightarrow{sp} & H^i(X_{\bar{\eta}}, \mathbb{Q}_p)^I \\ \lambda_s \downarrow & & \lambda_\eta \downarrow \\ H^i_{rig}(X_s/k) \otimes_{K_0} \hat{K_0}^{ur} & \xrightarrow{sp' \otimes 1} & D_{crys}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p)) \otimes_{K_0} \hat{K_0}^{ur} \end{array}$$

where $\hat{K_0}^{ur} = Frac(W(\bar{k}))$ *is the p-adic completion of the maximal unramified extension of* K_0*, and the vertical maps* λ_s, λ_η *induce isomophisms:*

$$\lambda_s : H^i(X_{\bar{s}}, \mathbb{Q}_p) \cong (H^i_{rig}(X_s/k) \otimes_{K_0} \hat{K_0}^{ur})^{\phi\otimes\phi=1} = H^i_{rig}(X_s/k)^{slope0}$$

and

$$\lambda_\eta : H^i(X_{\bar{\eta}}, \mathbb{Q}_p)^I \cong (D_{crys}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p)) \otimes_{K_0} \hat{K_0}^{ur})^{\phi\otimes\phi=1} = D_{crys}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p))^{slope0}$$

Theorem 1.0.2 *Let* X/S *be proper, flat, and generically smooth, we have:*
1. If X *has good reduction, then* $sp' : H^i_{rig}(X_s/k) \to D_{crys}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p))$ *is an isomorphism.*
2. If X *has semistable reduction, then* $sp : H^i(X_{\bar{s}}, \mathbb{Q}_p) \to H^i(X_{\bar{\eta}}, K_0)^I$ *is an isomorphism, and* $sp' : H^i_{rig}(X_s/k) \to D_{crys}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p))$ *is an isomorphism,* $i = 0, 1$

In [BER], Berthelot, Esnault and Rülling constructed a series of trace morphisms between the Witt vector cohomology of the special fibers of two flat regular R-schemes X and Y of the same dimension.

By using these trace morphisms, in chapter 4, we first prove some results concerned the coherent cohomology:

Proposition 1.0.1 *Let* X/S *be regular, proper, flat and generically smooth, then the cohomology of the sequence*

$$H^i(X, O_X) \to H^i(X^{(1)}, O_{X^{(1)}}) \to H^i(X^{(2)}, O_{X^{(2)}})$$

is annihilated by a fixed integer d_g*, where* $X, X^{(1)}, X^{(2)}$ *is defined via* $X^{(2)} \to Y = X^{(1)} \times_X X^{(1)} \rightrightarrows X^{(1)} \to X$ *such that* $X^{(1)} \to X$ *is surjective ,* $X^{(2)} \to X^{(1)} \times_X X^{(1)}$ *is generically surjective, both are alterations and have semistable reductions by De Jong's theorem ([D-J]).*

Then we use reductions and want to lift the result to the Witt vector cohomology, and we have proved the following:

Theorem 1.0.3 *(p-adic local invariant cycle theorem) Let* X/S *be regular, proper, flat and generically smooth, we have:*

$$sp : H^i(X_{\bar{s}}, WO_X)_{\mathbb{Q},s} \simeq (D_{crys}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p) \otimes_{K_0} \hat{K_0}^{ur})^{slope0}$$

induced from the p-adic specialization map is an isomorphism, and we have a commutative diagram:

$$\begin{array}{ccc} H^i(X_{\bar{s}}, \mathbb{Q}_p) & \xrightarrow{sp} & H^i(X_{\bar{\eta}}, \mathbb{Q}_p)^I \\ \lambda_s \downarrow & & \lambda_\eta \downarrow \\ H^i(X_{\bar{s}}, WO_X)_{\mathbb{Q},s} & \xrightarrow{sp} & (D_{crys}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p) \otimes_{K_0} \hat{K_0}^{ur})^{slope 0} \end{array}$$

where λ_s and λ_η are isomorphisms as above, and the subscript s denotes the Frobenius stable part as in [C2].
In particular, $sp : H^i(X_{\bar{s}}, \mathbb{Q}_p) \to H^i(X_{\bar{\eta}}, K_0)^I$ is an isomorphism.

In chapter 5, we give another proof of H^0 and H^1 cases via Grothendieck's fundamental group and purity.

In chapter 6, based on Flach and Morin's work on the weight filtration in the l-adic case, we consider the p-adic analogous result (which, together with the l-adic's result, serves as a part to prove the compatibility of the Weil-etale cohomology with the Tamagawa number conjecture), i.e., to verify that the morphism $W_0H^i(X_{\bar{s}}, \mathbb{Q}_p) \xrightarrow{sp} W_0H^i(X_{\bar{\eta}}, \mathbb{Q}_p)^I$ induced by sp is an isomorphism, which is a direct corollary of the local invariant cycle theorem by taking the weight filtration. And we also consider some typical examples where this weaker statement could be verified by direct computations.

Chapter 2

Preliminaries

2.1 $\mathbf{B_{crys}}$ and $\mathbf{B_{st}}$ Conjectures

We briefly recall Fontaine's definitions of B_{crys}, D_{crys}, B_{st}, D_{st}, B_{dR} and D_{dR} here([F1], [F2]) .

Let A be a $\mathbb{Z}_p$-algebra such that $A/pA \neq 0$. Let

$$R = \underrightarrow{lim}(\cdots \xrightarrow{F} A/pA \xrightarrow{F} A/pA \xrightarrow{F} A/pA)$$

where F denotes the Frobenius of A/pA.

Then we can consider the Witt vector ring $W(R)$ and a ring morphism

$$\theta : W(R) \to \hat{A}$$

$$u = (u_0, u_1, \dots) \mapsto \lim_{m\to\infty} \sum_{i=0}^{m} p^i \widetilde{u_{im}}^{p^{m-i}}$$

where $\hat{A}$ is the p-adic completion of A, $u_n = (u_{n0}, u_{n1}, \dots)$, and $\tilde{}$ denotes a lifting from A/pA to A.

Now $Ker\theta$ is a principal ideal generated by $\xi = [\hat{p}] - p$, where $\hat{p} = (\cdots \to \sqrt[p^2]{p} \to \sqrt[p]{p} \to p)$. Also, denote $[] : R \to W(R)$ to be the multiplicative Techmuller lift.

Define

$$B_{dR}^{+} = \underleftarrow{lim} W(R)[\frac{1}{p}]/\xi^n$$

$$B_{dR} = Frac B_{dR}^{+} = B_{dR}^{+}[\frac{1}{\xi}]$$

$$A_{crys}^{0} = \{\sum_{n=0}^{N} a_n \frac{\xi^n}{n!}, N < \infty, a_n \in W(R)\}$$

$$A_{crys} = \underleftarrow{lim} A_{crys}^{0}/p^n A_{crys}^{0}$$

$$B^+_{crys} = A_{crys}[\frac{1}{p}]$$

$$B_{crys} = B^+_{crys}[\frac{1}{t}] = A_{crys}[\frac{1}{t}]$$

$$B_{st} = B_{crys}[log[\varpi]]$$

where $t = log[\epsilon] = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{([\epsilon]-1)^n}{n}$, $\epsilon \in R$ such that $\epsilon^{(0)} = 1, \epsilon^{(1)} \neq 1$ and $\varpi \in R$ such that $\varpi^{(0)} = p, v(\varpi) = 1$.

Then for any p-adic representation V, define:

$$D_{dR}(V) = (V \otimes_{\mathbb{Q}_p} B_{dR})^{G_K}$$

$$D_{st}(V) = (V \otimes_{\mathbb{Q}_p} B_{st})^{G_K}$$

$$D_{crys}(V) = (V \otimes_{\mathbb{Q}_p} B_{crys})^{G_K} = D_{st}(V)^{N=0}$$

Then we have Fontaine's C_{crys} conjecture([FM]) (proved by Fontaine and Messing via p-adic nearby cycles, Faltings via almost etale extensions, and Niziol via K-theory):

(C_{crys}): Assume X proper and smooth over S, then there exists a natural isomorphism:

$$B_{crys} \otimes_{K_0} H^i_{crys}(X_s/k) \simeq B_{crys} \otimes_{\mathbb{Q}_p} H^i(X_{\bar{\eta}}, \mathbb{Q}_p)$$

compatible with the actions of ϕ and G on both sides (here the action of $g \in G$ on LHS (resp. RHS) is $g \otimes 1$ (resp. $g \otimes g$), ϕ's action on LHS (resp. RHS) is $\phi \otimes \phi$ (resp. $\phi \otimes 1$)), as well as with the filtrations after tensoring both sides with K.

And we also have Fontaine-Jannsen's C_{st} conjecture ([T]) (proved by Kato and Tsuji via p-adic nearby cycles, Faltings via almost etale extensions, and Niziol via K-theory):

(C_{st}): Assume X proper and has semistable reduction over S, then there exists a natural isomorphism:

$$B_{st} \otimes_{K_0} H^i_{HK}(X_s/k) \simeq B_{st} \otimes_{\mathbb{Q}_p} H^i(X_{\bar{\eta}}, \mathbb{Q}_p)$$

compatible with the actions of ϕ, N and G on both sides(here the action of $g \in G$ on LHS (resp. RHS) is $g \otimes 1$ (resp. $g \otimes g$), ϕ's action on LHS (resp. RHS) is $\phi \otimes \phi$ (resp. $\phi \otimes 1$), N's action on LHS (resp. RHS) is $N \otimes 1 + 1 \otimes N$ (resp. $N \otimes 1$)), as well as with the filtrations after tensoring both sides with K.

2.2 Chiarellotto's Map

Chiarellotto's map plays an important role in our construction of p-adic specialization map, and we briefly recall Chiarellotto's construction in [C] here:

First, given a semistable scheme Y, we suppose that Y is proper and be the union of irreducible smooth components

$$Y = \cup_{i \in I} Y_i$$

Denote $Y^{(j)}$ be the space of disjoint union of the intersections of j components. These $Y^{(j)}$ are smooth by the definition of semistable schemes.

By [M3, (3.7)], we know that there exists an isomorphism on Y_{et} for each $j \geq 1$

$$Res : Gr_j^W W_n \tilde{\omega}_Y^{\bullet} \to W_n \Omega_{Y^{(j)}}^{\bullet}[-j](-j)$$

where $W_n \tilde{\omega}_Y^{\bullet}$ is a complex defined by Hyodo and Mokrane ([M3]), $W_n \Omega_{Y^{(j)}}^{\bullet}$ is the usual de Rham-Witt complex of $Y^{(j)}$(thought of as a complex in Y_{et}) and $(-j)$ is the Tate-shift related to the Frobenius structure ([I4]).

By these, we could consider the Hyodo-Steenbrink bicomplex $W_n A^{ij}(i, j \geq 0)$ of sheaves in Y_{et} given by

$$W_n A^{ij} = \frac{W_n \tilde{\omega}_Y^{i+j+1}}{P_j W_n \tilde{\omega}_Y^{i+j+1}}$$

where $P_j W_n \tilde{\omega}_Y^{\bullet}$ is the usual logarithmic filtration on $W_n \tilde{\omega}_Y^{\bullet}$, these $P_j W_n \tilde{\omega}_Y^{\bullet}$ are coherent $W_n O_{Y_{et}}$-modules and form the *weight filtration.*

For $x \in W_n A^{ij}$, the first differential $d'x \in W_n A^{i+1j}$ is $(-1)^j$ times the usual one, while $d'' : W_n A^{ij} \to W_n A^{ij+1}$ is given by the multiplication by Θ: $d''(x) = x \wedge \Theta$, where Θ is a global section on $W_n \tilde{\omega}_Y^1$ which locally coincides with dt/t.

The double complex $(W_n A^{\bullet\bullet}, d', d'')$ is endowed with a Frobenius endomorphism Φ_n defined on each $W_n A^{ij}$ by the usual Frobenius morphism twisted by p^{-j-1}. Taking the inverse limit on $n \in \mathbb{N}$, we get a bicomplex $WA^{\bullet\bullet}$ whose associated simple complex $WA^{\bullet}$ is isomorphic to $W\omega_Y^{\bullet}$ under multiplication by Θ (see [M3, (3.17)]).

By [M3, (3.18)], $WA^{\bullet}$ admits an operator v as follows: v_n is the endomorphism induced on the simple complex by the endomorphism on $W_n A^{\bullet\bullet}$ such that $(-1)^{i+j+1} v_n$ is the natural projection

$$W_n A^{ij} \to W_n A^{i-1j+1}$$

Taking inverse limit of v_n, we then construct

$$v = \varprojlim v_n$$

which induces the monodromy operator in cohomology, and by the exactness property, one obtains an isomorphism of complexes in (Y_{et}, W):

$$Kerv = \varprojlim Kerv_n$$

Chiarellotto's paper has given an interpretation of $Kerv \otimes K_0$ in terms of rigid cohomology.

We have an explicit formula for $Kerv$ involving the usual de Rham-Witt complexes of the various intersections of the components of Y by Chiarellotto:

Proposition 2.2.1 *([C,Prop. 1.8]) Consider a proper semistable scheme Y which is the union of irreducible smooth components*

$$Y = \cup_{i \in I} Y_i$$

and denote $Y^{(j)}$ be the space of disjoint union of the intersections of j components. Then the kernel of the operator

$$v_n : W_n A^\bullet \to W_n A^\bullet$$

is the simple complex associated to the double complex

$$0 \to W_n \Omega^\bullet_{Y^{(1)}} \xrightarrow{\rho_1} W_n \Omega^\bullet_{Y^{(2)}} \xrightarrow{\rho_2} W_n \Omega^\bullet_{Y^{(3)}} \cdots$$

in Y_{et}, where $\rho_t : W_n \Omega^\bullet_{Y^{(t)}} \to W_n \Omega^\bullet_{Y^{(t+1)}}$ is defined by

$$\rho_t = (-1)^t \sum_{1 \le j \le t+1} (-1)^{j+1} \delta_j^*$$

(Here $\delta_j : Y^{(t)} \to Y^{(t-1)}$'s restriction on the components are defined to be the inclusions

$$Y_{i_1} \cap \cdots Y_{i_k} \hookrightarrow Y_{i_1} \cap \cdots \cap Y_{i_{j-1}} \cap Y_{i_{j+1}} \cap \cdots \cap Y_{i_k}$$

(see [M3] and [S2]).)

Then we need an interpretation of a complex calculating the rigid cohomology of the k-scheme Y in terms of the components of Y and their various intersections.

Let P be a formal W-scheme locally of finite type. Then its generic fiber P_{K_0} is a rigid analytic space and we could define

$$sp : P_{K_0} \to P$$

to be the specialization([B4]).

Let Y be a k-scheme of finite type, and assume that there is a closed immersion of Y into a smooth formal W-scheme P:

$$j : Y \to P$$

Define $H^*_{conv}(Y/K_0) = H^*(sp^{-1}(Y), \Omega^\bullet_{sp^{-1}(Y)})$ (here $sp^{-1}(Y) =]Y[_p$ is the tube of Y). As Y being proper, we have $H^*_{conv}(Y/K_0) = H^*_{rig}(Y/K_0)$.

Consider the restriction $sp :]Y[_p \to Y$ and $sp_*\Omega^\bullet_{sp^{-1}(Y)}$. Note that the hypercohomology of the complex $sp_*\Omega^\bullet_{sp^{-1}(Y)}$ calculates the rigid cohomology of Y ([B4]), and this is the intuition of Chiarellotto's construction.

The existence of a closed embedding of Y into a smooth formal W-scheme is only true locally. The method of dealing with the generic situation is to use the technique of "diagrams of topos". In fact, one can always find an open covering $\{T_\alpha\}$ of Y, and for each T_α, we could construct a closed embedding into a smooth formal W-scheme:

$$i_\alpha : T_\alpha \to P_\alpha$$

Thus we have the close embedding given by the diagonal map

$$i_{\alpha_0\cdots\alpha_n} : T_{\alpha_0\cdots\alpha_n} = T_{\alpha_0} \cap \cdots \cap T_{\alpha_n} \to P_{\alpha_0\cdots\alpha_n} = P_{\alpha_0} \times \cdots \times T_{\alpha_n}$$

These $\{T_{\alpha_0\cdots\alpha_n} = T_{\alpha_0} \cap \cdots \cap T_{\alpha_n}\}$ form a diagram of topos $T_\bullet$ endowed with the Zariski topology. The complex of sheaves $\{sp_*\Omega^\bullet_{]T_{\alpha_0\cdots\alpha_n}[P_{\alpha_0\cdots\alpha_n}}\}$ form a complex of sheaves $sp_*\Omega^\bullet_{sp^{-1}(Y)}$ on this diagram of topos. On the other hand, there is a natural map $\epsilon : T_\bullet \to Y_{Zar}$. The convergent cohomology of Y is defined by $R\epsilon_* sp_*(\Omega^\bullet_{]Y[_P})$. As Y is proper, these cohomology groups coincide with the rigid cohomology groups of Y ([B2]).

Now, given a k-scheme of finite type Y, and suppose that it is the union of smooth irreducible components

$$Y = \cup_{i \in I} Y_i$$

denote $Y^{(j)}$ similarly as above and $i : Y^{(j)} \to Y$ be the natural maps.

We have the following exact sequence in Y_{Zar} by [C, Prop. 2.3]

$$0 \to sp_*\Omega^\bullet_{]Y[_P} \xrightarrow{\rho_0} i_* sp_*\Omega^\bullet_{]Y^{(1)}[_P} \xrightarrow{\rho_1} i_* sp_*\Omega^\bullet_{]Y^{(2)}[_P} \cdots$$

where ρ_0 is the natural restriction and ρ_t is defined via δ_t similarly as above.

Now one can connect $Kerv$ with the rigid cohomology, which both are related to complexes defined using the smooth component of the proper semistable scheme Y:

Proposition 2.2.2 *([C, Theorem 3.6]) Consider a proper semistable scheme Y which is the union* *of irreducible smooth components*

$$Y = \cup_{i \in I} Y_i$$

Then, in $D^+(Y_{Zar})$ *there is an isomorphism*

$$sp_* \Omega^\bullet_{]Y[} \to Kerv \otimes K_0$$

In particular,

$$H^*_{rig}(Y/K_0) \simeq H^*(Y_{Zar}, Kerv) \otimes K_0 \simeq H^*(Y_{et}, Kerv) \otimes K_0$$

Now, from the exact sequence:

$$0 \to Kerv^j \to WA^\bullet \xrightarrow{v^j} WA^\bullet \to Cokerv^j \to 0$$

We then get the map, by taking inverse limit and the above proposition:

$$H^i_{rig}(X_s/k) \to H^i_{HK}(X_s/k)^{N=0}$$

as $H^i_{HK}(X_s) = H^i(X_s, WA^\bullet)$ is defined as the hypercohomology of the Hyodo-Mokrane complex $WA^\bullet$.

Remark 2.2.1 *For more details and discussions on Chiarellotto's constructions, we refer to his paper [C], Also, Chiarellotto conjectured in [C] that for each* $i \geq 0$, *the sequence*

$$H^i_{rig}(X_s/k) \to H^i_{HK}(X_s/k) \xrightarrow{N} H^i_{HK}(X_s/k)$$

is exact, while here we do not require this conjecture to be true and only use this morphism to construct our $sp' : H^i_{rig}(X_s/k) \to D_{crys}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p))$ *for general* X.

Chapter 3

Construction of p-adic Specialization Map

In this chapter, we will construct the p-adic specialization map $sp' : H^i_{rig}(X_s/k) \to D_{crys}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p))$, and we make some simplifications first.

3.1 Some Simplifications

Define a ϕ-ring R(or a ϕ-field F) to be a ring(or a field) with an endomorphism ϕ, a ϕ-module D over a ϕ-ring R(or over a ϕ-field F) is defined to be a finite free R-module(or a finite dimensional F-vector space) equipped with a semi-linear ϕ-action on D, i.e., $\phi(rx) = \phi(r)\phi(x)$ for any $r \in R, x \in D$.

Now, for any $\phi-$module D, the $Gal(\bar{k}/k)-$module $V(D) := (D \otimes_{K_0} \hat{K_0}^{ur})^{\phi\otimes\phi=1}$ can be viewed as a $\phi-$module via the action of $\phi \otimes 1$ and we defined it as D^{slope0}, where the action of $Frob_p^{-1} \in Gal(\bar{k}/k)$ coincides with that of $\phi^{-[k:\mathbb{F}_p]} \otimes 1 = 1 \otimes \phi^{-[k:\mathbb{F}_p]}$.

Thus we have:

$$(H^i_{rig}(X_s/k) \otimes_{K_0} \hat{K_0}^{ur})^{\phi\otimes\phi=1} = H^i_{rig}(X_s/k)^{slope0}$$

and

$$(D_{crys}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p)) \otimes_{K_0} \hat{K_0}^{ur})^{\phi\otimes\phi=1} = D_{crys}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p))^{slope0}$$

Remark 3.1.1 *In fact, using the Dieudonne-Manin classification([M1]), any $\hat{K_0}^{ur}[\phi]-$module is isomorphic to a sum of $E_{r,s} = \hat{K_0}^{ur}[\phi]/(\phi^r - p^s)$, where $t = \frac{s}{r} \in \mathbb{Q}$ is the slope.*

Note that $E_{r,s}$ has a basis $\{e_i\}$ extended via ϕ-action, i.e., we choose $e_1 \in E_{r,s}$, choose $e_2 = \phi(e_1), e_3 = \phi(e_2)\cdots$, and $\phi(e_r) = \phi^r(e_1) = p^s e_1$, which causes a contradiction when $s \neq 0$ by the valuation criterion, thus is shows that $E_{r,s}^{\phi=1} = 0$ when $t = \frac{s}{r} \neq 0$.

Lemma 3.1.1 *Let V be a finite dimensional $\mathbb{Q}_p$-vector space with a continuous $G_p := Gal(\bar{K}/K)$-action, and such that $D_{dR}(V)/Fil^0 D_{dR}(V) = 0$. Then we have an isomorphism*

$$V^{I_p} \simeq (D_{crys}(V) \otimes_{K_0} \hat{K}_0^{ur})^{\phi\otimes\phi=1}$$

Proof:

Since $D_{dR}(V)/Fil^0 D_{dR}(V) = 0$, and we have an isomorphism of $Frob_p-$modules with the diagonal G_p-action:

$$H^0(I_p, B^0(V)) = D_{crys}(V) \otimes_{K_0} \hat{K}_0^{ur}$$

where $B^0(V) = B_{crys} \otimes_{K_0} V$

We have

$Fil^0 D_{crys}(V) \otimes_{K_0} \hat{K}_0^{ur} = Fil^0(V \otimes_{K_0} B_{crys})^{I_p} = Fil^0(V \otimes_{K_0} B_{dR})^{I_p} \cap (V \otimes_{K_0} B_{crys})^{I_p} = (Fil^0 D_{dR}(V) \otimes_{K_0} \hat{K}_0^{ur}) \cap (V \otimes_{K_0} B_{crys})^{I_p} = (D_{dR}(V) \otimes_{K_0} \hat{K}_0^{ur}) \cap (V \otimes_{K_0} B_{crys})^{I_p} = (V \otimes_{K_0} B_{crys})^{I_p} = D_{crys}(V) \otimes_{K_0} \hat{K}_0^{ur}$.

Thus

$(B_{crys}^{\phi=1} \otimes_{K_0} V)^{I_p} = (B_{crys} \otimes_{K_0} V)^{I_p,\phi=1} = (D_{crys}(V) \otimes_{K_0} \hat{K}_0^{ur})^{\phi\otimes\phi=1} = (Fil^0 D_{crys}(V) \otimes_{K_0} \hat{K}_0^{ur})^{\phi\otimes\phi=1} = Fil^0(D_{crys}(V) \otimes_{K_0} \hat{K}_0^{ur})^{\phi\otimes\phi=1} = Fil^0(B_{crys}^{\phi=1} \otimes_{K_0} V)^{I_p}$.

Now, $(B_{crys} \otimes_{K_0} V)^{I_p,\phi=1} = (B_{crys}^{\phi=1} \otimes_{K_0} V)^{I_p} = Fil^0(B_{crys}^{\phi=1} \otimes_{K_0} V)^{I_p} = (Fil^0 B_{crys}^{\phi=1} \otimes_{K_0} V)^{I_p} = (K_0 \otimes_{K_0} V)^{I_p} = V^{I_p}$, we get $V^{I_p} \simeq (D_{crys}(V) \otimes_{K_0} \hat{K}_0^{ur})^{\phi\otimes\phi=1}$. □

Corollary 3.1.1 *Let X be a proper and generically smooth scheme over S, Then we have an isomorphism*

$$H^i(X_{\bar{\eta}}, \mathbb{Q}_p)^I \simeq (D_{crys}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p)) \otimes_{K_0} \hat{K_0}^{ur})^{\phi\otimes\phi=1}$$

Proof:

Note that $D_{dR}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p)) = H^i_{dR}(X/K)$ and $Fil^0 H^i_{dR}(X/K) = H^i_{dR}(X/K)$, we get the consequence by the above lemma. □

Lemma 3.1.2 *For a proper variety X over S, we have an isomorphism*

$$H^i(X_{\bar{s}}, \mathbb{Q}_p) \simeq (H^i_{rig}(X_s/k) \otimes_{K_0} \hat{K}_0^{ur})^{\phi\otimes\phi=1}$$

Proof:

By [BBE], we have a canonical isomorphism:

$$H^i_{rig}(X_s/k)^{[0,1)} \simeq H^i(X_s, WO_X)_{K_0}$$

where $H^i_{rig}(X_s/k)^{[0,1)}$ is the maximal subspace on which ϕ acts with slopes in [0,1), $H^i(X_s, WO_X)$ denotes the Witt vector cohomology, and the subscript K_0 means tensorization with K_0.

On the other hand, we have short exact sequence from [I4, Prop. 3.28]:

$$0 \longrightarrow \mathbb{Z}_p \longrightarrow WO_X \xrightarrow{1-\phi} WO_X \longrightarrow 0$$

Thus, by [I4, Lemma 5.3], we have:

$$H^i(X_{\bar{s}}, \mathbb{Q}_p) = H^i(X_{\bar{s}}, WO_X)^{\phi=1}_{K_0} \simeq (H^i_{rig}(X_{\bar{s}}/\bar{k})^{[0,1)})^{\phi=1} = (H^i_{rig}(X_{\bar{s}}/\bar{k}))^{\phi=1} \simeq (H^i_{rig}(X_s/k) \otimes_{K_0} \hat{K_0}^{ur})^{\phi\otimes\phi=1}$$

□

Combining the above lemmas, we get the isomorphisms

$$\lambda_s : H^i(X_{\bar{s}}, \mathbb{Q}_p) \simeq (H^i_{rig}(X_s/k) \otimes_{K_0} \hat{K_0}^{ur})^{\phi\otimes\phi=1} = H^i_{rig}(X_s/k)^{slope0}$$

and

$$\lambda_\eta : H^i(X_{\bar{\eta}}, \mathbb{Q}_p)^I \simeq (D_{crys}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p)) \otimes_{K_0} \hat{K_0}^{ur})^{\phi\otimes\phi=1} = D_{crys}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p))^{slope0}$$

It remains to construct sp' and then show the diagram commutes, however, it is complicated to construct this morphism in general, and we have to discuss case by case.

3.2 Case of Good Reduction

Note that in the good reduction case, the crystalline cohomology coincides with the rigid cohomology ([B2, Prop.1.9]):

$$H^i_{rig}(X_s/k) \simeq H^i_{crys}(X_s/k)$$

Using Fontaine's C_{crys} conjecture([FM]), we know that

$$H^i_{crys}(X_s/k) \otimes B_{crys} \simeq H^i(X_{\bar{\eta}}, \mathbb{Q}_p) \otimes B_{crys}$$

Taking the G_K-invariants, since $B_{crys}^{G_K} = K_0$, we can then define a $\phi-$equivariant isomorphism

$$sp' : H^i_{rig}(X_s/k) \to D_{crys}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p))$$

Also, note that the isomorphism $sp' \otimes 1$ is induced by taking $I-$invariants on the C_{crys} conjecture.

And we can adjust λ_s and λ_η(since all the maps in the diagram are isomorphisms) to make the diagram

$$\begin{array}{ccc} H^i(X_{\bar{s}}, \mathbb{Q}_p) & \xrightarrow{sp} & H^i(X_{\bar{\eta}}, \mathbb{Q}_p)^I \\ \lambda_s \downarrow & & \lambda_\eta \downarrow \\ (H^i_{rig}(X_s/k) \otimes_{K_0} \hat{K_0}^{ur})^{slope0} & \xrightarrow{sp' \otimes 1} & (D_{crys}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p)) \otimes_{K_0} \hat{K_0}^{ur})^{slope0} \end{array}$$

commute. This finishes the case of good reduction.

3.3 Case of Semistable Reduction

Similar as in the good reduction case, we can use Fontaine-Jannsen's C_{st} conjecture ([T]):

$$H^i_{HK}(X_s/k) \otimes B_{st} \cong H^i(X_{\bar{\eta}}, \mathbb{Q}_p) \otimes B_{st}$$

Thus we need to replace the above crystalline cohomology in good reduction case by the Hyodo-Kato cohomology first, in fact, we need to construct $\tilde{\lambda}_s : H^i(X_{\bar{s}}, \mathbb{Q}_p) \to (H^i_{HK}(X_s/k)^{N=0}) \otimes_{K_0} \hat{K_0}^{ur}$, prove the commutativity of the following diagram:

$$\begin{array}{ccc} H^i(X_{\bar{s}}, \mathbb{Q}_p) & \xrightarrow{sp} & H^i(X_{\bar{\eta}}, \mathbb{Q}_p)^I \\ \tilde{\lambda}_s \downarrow & & \lambda_\eta \downarrow \\ (H^i_{HK}(X_s/k)^{N=0}) \otimes_{K_0} \hat{K_0}^{ur} & \xrightarrow{sp'_{HK} \otimes 1} & D_{crys}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p)) \otimes_{K_0} \hat{K_0}^{ur} \end{array}$$

and also that $\tilde{\lambda}_s$ is injective.

Using Lorenzon's result in [Lo], we can prove the following:

Lemma 3.3.1 *For a proper variety X over S, we have an isomorphism*

$$\tilde{\lambda}_s : H^i(X_{\bar{s}}, \mathbb{Q}_p) \cong (H^i_{HK}(X_s/k)^{N=0} \otimes_{K_0} \hat{K}_0^{ur})^{\phi \otimes \phi = 1}$$

Proof:

Using [Lo, (3.4.6)], we have a canonical isomorphism

$$H^i(X_s, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong (H^i_{HK}(X_s/k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)^{slope0}$$

So, similarly as in lemma 2.1.3, using the exact sequence (2.14) and theorem 3.4 in [Lo], we have

$$H^i(X_{\bar{s}}, \mathbb{Q}_p) \cong (H^i_{HK}(X_s/k) \otimes_{K_0} \hat{K}_0^{ur})^{\phi \otimes \phi = 1} = (H^i_{HK}(X_s/k)^{N=0} \otimes_{K_0} \hat{K}_0^{ur})^{\phi \otimes \phi = 1}$$

where the latter equality follows from: let $x \in (H^i_{HK}(X_s/k) \otimes_{K_0} \hat{K}_0^{ur})^{\phi \otimes \phi = 1}$, then $x \in (H^i_{HK}(X_s/k) \otimes_{K_0}$

$K')^{\phi\otimes\phi=1}$ for a finite extension K' of K_0 in K^{ur}, thus $\phi^a(Nx) = (1/p^a)\phi(x)$, where p^a is the number of elements in the residue field of K', since ϕ^a is induced by a $W(k)$-endomorphism of $H^i_{HK}(X_s/W(k))$, it then follows that $Nx = 0$. (see [B1], page 674). □

Thus, we have constructed

$$\tilde{\lambda}_s : H^i(X_{\bar{s}}, \mathbb{Q}_p) \cong (H^i_{HK}(X_s/k)^{N=0} \otimes_{K_0} \hat{K}_0^{ur})^{\phi\otimes\phi=1} \cong (H^i_{HK}(X_s/k)^{N=0})^{slope0}$$

Also, by the C_{st} conjecture proved in [T], we know that $H^i_{HK}(X_s/k) \otimes B_{st} \cong H^i(X_{\bar{\eta}}, \mathbb{Q}_p) \otimes B_{st}$. Note that $B_{st}^{N=0,\phi=1} \cap Fil^0 B_{dR} = K_0$ and $B_{st}^G = K_0$, we have the canonical isomorphisms:

$$H^m_{et}(X_{\bar{K}}, \mathbb{Q}_p) \cong (B_{st} \otimes_{K_0} H^m_{HK}(X))^{N=0,\phi=1} \cap Fil^0(B_{dR} \otimes_{K_0} H^m_{HK}(X))$$

$$\beta : H^m_{et}(X_{\bar{K}}, \mathbb{Q}_p)^I \cong (\hat{K}_0^{ur} \otimes_{K_0} H^m_{HK}(X))^{N=0,\phi\otimes\phi=1} = (\hat{K}_0^{ur} \otimes_{K_0} H^m_{HK}(X))^{\phi\otimes\phi=1}$$

and

$$H^m_{HK}(X) \cong (B_{st} \otimes_{K_0} H^m_{et}(X_{\bar{K}}, \mathbb{Q}_p))^G = D_{st}(H^m_{et}(X_{\bar{K}}, \mathbb{Q}_p))$$

So we indeed have an isomorphism of $\phi-$modules

$$H^i_{HK}(X_s/k)^{N=0} \to D_{st}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p))^{N=0} = D_{crys}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p))$$

and we then define this isomorphism to be our required sp'_{HK}.

Now, we will use the morphism $H^i_{rig}(X/k) \to H^i_{HK}(X_s/k)$ constructed by Chiarellotto in [C] to construct our diagram for the rigid cohomology as in the conjecture.

By looking at the proof of lemma 3.1.2 and lemma 3.3.1, note that λ_s and $\tilde{\lambda}_s$ are isomorphisms such that map $H^i(X_{\bar{s}}, \mathbb{Q}_p)$ to the slope 0 part of $H^i_{rig}(X_s/k) \otimes_{K_0} \hat{K_0}^{ur}$ and $(H^i_{HK}(X_s/k)^{N=0}) \otimes_{K_0} \hat{K_0}^{ur}$, also, Chiarellotto's morphism induces an isomorphism on these two slope 0 part, we could then assume that $\tilde{\lambda}_s$ factor through λ_s via Chiarellotto's morphism, thus we deduce our diagram from the above diagram of Hyodo-Kato cohomology:

$$\begin{array}{ccc} H^i(X_{\bar{s}}, \mathbb{Q}_p) & \xrightarrow{sp} & H^i(X_{\bar{\eta}}, \mathbb{Q}_p)^I \\ {\scriptstyle\lambda_s}\downarrow & & {\scriptstyle\lambda_\eta}\downarrow \\ H^i_{rig}(X_s/k) \otimes_{K_0} \hat{K_0}^{ur} & \xrightarrow{sp'\otimes 1} & D_{crys}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p)) \otimes_{K_0} \hat{K_0}^{ur} \end{array}$$

Note that, as sp'_{HK} is an isomorphism, we then have sp is also an isomorphism in the semistable reduction case, however, sp' is not an isomorphism in general. But for the H^1 case, we can prove it is indeed an isomorphism:

Proposition 3.3.1 *Using the above notations, if X has semistable reduction, then $H^1_{rig}(X/k) \cong H^1_{HK}(X_s/k)^{N=0}$, in particular, $sp' : H^1_{rig}(X_s/k) \to D_{crys}(H^1(X_{\bar{\eta}}, \mathbb{Q}_p))$ is an isomorphism.*

Proof:

Note that in [M3, (5.4.3)], by a computation of low degrees of his spectral sequence and also uses the Neron model in [SGA 7, IX, (12.3.6)], Mokrane has proved the local invariant cycle theorem in H^1 case, i.e., exactness of

$$0 \to H^1(X_s, Kerv) \to H^1_{HK}(X_s/k) \xrightarrow{N} H^1_{HK}(X_s/k)$$

Combined with [C, (3.8)], which said that $H^*_{rig}(X_s/k) \cong H^*(X_s, Kerv) \otimes k$.

We then get Chiarellotto's conjecture $H^1_{rig}(X/k) \cong H^1_{HK}(X_s/k)^{N=0}$. □

Similarly as in the good reduction case, we can adjust λ_s, $\tilde{\lambda}_s$ and λ_η(since all the maps in the diagram are isomorphism when restricted to the slope 0 part) to make the diagrams

$$\begin{array}{ccc} H^i(X_{\bar{s}}, \mathbb{Q}_p) & \xrightarrow{sp} & H^i(X_{\bar{\eta}}, \mathbb{Q}_p)^I \\ \tilde{\lambda}_s \downarrow & & \lambda_\eta \downarrow \\ (H^i_{HK}(X_s/k)^{N=0}) \otimes_{K_0} \hat{K_0}^{ur} & \xrightarrow{sp'_{HK}\otimes 1} & D_{crys}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p)) \otimes_{K_0} \hat{K_0}^{ur} \end{array}$$

and

$$\begin{array}{ccc} H^i(X_{\bar{s}}, \mathbb{Q}_p) & \xrightarrow{sp} & H^i(X_{\bar{\eta}}, \mathbb{Q}_p)^I \\ \lambda_s \downarrow & & \lambda_\eta \downarrow \\ (H^i_{rig}(X_s/k) \otimes_{K_0} \hat{K_0}^{ur})^{slope0} & \xrightarrow{sp'\otimes 1} & (D_{crys}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p)) \otimes_{K_0} \hat{K_0}^{ur})^{slope0} \end{array}$$

commute, and then we finish the case of semistable reduction.

Remark 3.3.1 *As we shown above, sp' is not an isomorphism in general, while it is an isomorphism when restricted to the slope 0-part. In fact, it is also an isomorphism when restricted to the bigger part slope $[0, 1)$, which both describe the Witt vector cohomology (see [Lo]).*

Remark 3.3.2 *We can also deduce the commutativity of the diagram in semistable reduction case from the good reduction case via the compatibility of the isomorphism of C_{crys} and C_{st} with the specialization map.*

Let $\{Y_i\}_{i \in I}$ be the family of irreducible components of X_s, $Y_J = \cap_{i \in J} Y_i$ for a subset $J \subseteq I$, and let Y^j be the disjoint union of all Y_j with $card(J) = j + 1, j \geq 0$. We consider the weight spectral sequences

$$E_1^{i,j} = H^i(Y^j, \mathbb{Q}_p) \Longrightarrow H^{i+j}(X_{\bar{s}}, \mathbb{Q}_p)$$

and

$$E_2^{i,j} = \bigoplus_{k=0}^{\infty} H_{crys}^{j-2k}(Y^{i+2k}/W)(-k) \Longrightarrow H_{HK}^{i+j}(X_{\bar{s}}/W)$$

Now let $u^{log}_{X_s/W_n}$ *denote the canonical morphism of topoi* $(\widetilde{X_s/W_n})_{HK} \to (\widetilde{X_s})_{et}$ *(recall that* $H^i_{HK}(X_s/W) = \lim H^i_{HK}(X_s/W_n)$*, where* $W_n = W/p^nW$*).*

Note that the eigenvalues of ϕ^a *(where* p^a *is the number of elements in* K*) on* $H^{j-2k}_{crys}(Y^{i+2k}/W) \otimes K_0$ *are Weil numbers of weight* $j-2k$*, thus the canonical morphism of spectral sequence* $E_1 \to E_2$ *induced by* $u^{log}_{X_s/W_n}$*, gives rise to an isomorphism* $\gamma : H^i(X_{\bar{s}}, \mathbb{Q}_p) \cong H^i_{HK}(X)^{slope0} = (K_0^{ur} \otimes_{K_0} H^m_{HK}(X))^{\phi\otimes\phi=1}$*.*

In fact, by the compatibility of the isomorphism of C_{st} *with the specialization morphism, we have* $\gamma = \beta \circ sp$ *(see [B1], page 675).*

3.4 General Case

In general, when X is proper, flat and generically smooth over S, we can also use Chiarellotto's morphism $c : H^i_{rig}(X'/k) \to H^i_{HK}(X'_s/k)^{N=0}$ defined on semistable scheme X' to construct our $sp' : H^i_{rig}(X_s/k) \to D_{crys}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p))$.

Consider the diagram

$$\begin{array}{ccccc}
X_s^{(2)} & \longrightarrow & X^{(2)} & \longleftarrow & X_{\bar{\eta}}^{(2)} \\
\downarrow & & \downarrow & & \downarrow \\
X_s^{(1)} \times_{X_s} X_s^{(1)} & \longrightarrow & X^{(1)} \times_X X^{(1)} & \longleftarrow & X_{\bar{\eta}}^{(1)} \times_{X_{\bar{\eta}}} X_{\bar{\eta}}^{(1)} \\
\downdownarrows & & \downdownarrows & & \downdownarrows \\
X_s^{(1)} & \longrightarrow & X^{(1)} & \longleftarrow & X_{\bar{\eta}}^{(1)} \\
\downarrow & & \downarrow & & \downarrow \\
X_s & \longrightarrow & X & \longleftarrow & X_{\bar{\eta}}
\end{array}$$

where $X^{(1)} \to X$ is surjective , $X^{(2)} \to X^{(1)} \times_X X^{(1)}$ is generically surjective (i.e., surjective on the generic fibre, while not necessarily surjective on all generic points), both are alterations and have semistable reductions by De Jong's theorem ([D-J]).

From the above diagrams, we then deduce a commutative diagram for the cohomology (for convenience, we omit the subscript when writing the etale cohomology).

$$\begin{array}{ccccc}
H^i_{rig}(X_s^{(2)}/k) & \xrightarrow{c} & H^i_{HK}(X_s^{(2)}/k)^{N=0} & \xrightarrow{\simeq} & D_{st}(H^i(X_{\bar\eta}^{(2)},\mathbb{Q}_p))^{N=0} \\
\uparrow & & & & \uparrow \\
H^i_{rig}(X_s^{(1)}\times_{X_s}X_s^{(1)}/k) & & & & D_{st}(H^i(X_{\bar\eta}^{(1)}\times_{X_{\bar\eta}}X_{\bar\eta}^{(1)},\mathbb{Q}_p))^{N=0} \\
\upuparrows & & & & \upuparrows \\
H^i_{rig}(X_s^{(1)}/k) & \xrightarrow{c} & H^i_{HK}(X_s^{(1)}/k)^{N=0} & \xrightarrow{\simeq} & D_{st}(H^i(X_{\bar\eta}^{(1)},\mathbb{Q}_p))^{N=0} \\
\uparrow & & & & \uparrow \\
H^i_{rig}(X_s/k) & & & & D_{st}(H^i(X_{\bar\eta},\mathbb{Q}_p))^{N=0}
\end{array}$$

where the first arrow in each row is Chiraletto's morphism, and the second one is the isomorphism in the semistable reduction case as we proved above.

We then have the following result:

Proposition 3.4.1 *With the above notations, the equalizer of*

$$H^i(X_{\bar\eta}^{(1)},\mathbb{Q}_p) \rightrightarrows H^i(X_{\bar\eta}^{(1)}\times_{X_{\bar\eta}}X_{\bar\eta}^{(1)},\mathbb{Q}_p)$$

is $H^i(X_{\bar\eta},\mathbb{Q}_p)$ *and the equalizer of*

$$D_{st}(H^i(X_{\bar\eta}^{(1)},\mathbb{Q}_p))^{N=0} \rightrightarrows D_{st}(H^i(X_{\bar\eta}^{(1)}\times_{X_{\bar\eta}}X_{\bar\eta}^{(1)},\mathbb{Q}_p))^{N=0}$$

is $D_{st}(H^i(X_{\bar\eta},\mathbb{Q}_p))^{N=0}$.

Before the proof of this proposition, we first prove a lemma for the existence of a trace map:

Lemma 3.4.1 *Given* $\pi : X \to Y$ *proper,* $\eta : Y \to SpecK, \xi = \eta\circ\pi : X \to SpecK$ *proper, smooth of dimension* d*, we have a trace map* $tr : R\pi_*\mathbb{Q}_l \to \mathbb{Q}_l$.

Proof:

We have the adjunction: $R\pi_!\pi^!\mathbb{Q}_l \to \mathbb{Q}_l$ in the derived category of l-adic sheaves.

Since $\mathbb{Q}_l = \xi^!\mathbb{Q}_l(-d)[-2d] = \pi^!\eta^!\mathbb{Q}_l(-d)[-2d] = \pi^!\mathbb{Q}_l$, and $R\pi_! = R\pi_*$ as π is proper, we then get the trace map. □

Proof of Proposition 3.4.1:

For the fiber product projection $X_{\bar\eta}^{(1)}\times_{X_{\bar\eta}}X_{\bar\eta}^{(1)} \rightrightarrows X_{\bar\eta}^{(1)} \to X_{\bar\eta}$ and the associated p-adic etale cohomology $H^i(X_{\bar\eta},\mathbb{Q}_p) \to H^i(X_{\bar\eta}^{(1)},\mathbb{Q}_p) \rightrightarrows H^i(X_{\bar\eta}^{(1)}\times_{X_{\bar\eta}}X_{\bar\eta}^{(1)},\mathbb{Q}_p)$, note that it suffices to show that

$$H^i(X_{\bar\eta},\mathbb{Q}_p) \hookrightarrow H^i(X_{\bar\eta}^{(1)},\mathbb{Q}_p) \rightrightarrows H^i(X_{\bar\eta}^{(1)}\times_{X_{\bar\eta}}X_{\bar\eta}^{(1)},\mathbb{Q}_p)$$

is exact.

Assume $\pi : X_{\bar\eta}^{(1)} \to X_{\bar\eta}$ is a surjection of genereic degree d.

By lemma 3.4.1, from $X_{\bar\eta}^{(1)} \times_{X_{\bar\eta}} X_{\bar\eta}^{(1)} \overset{\pi_2}{\to} X_{\bar\eta}$ and $X_{\bar\eta}^{(1)} \overset{\pi}{\to} X_{\bar\eta}$, we have $\mathbb{Q}_p \overset{\curvearrowleft}{\to} R\pi_*\mathbb{Q}_p \rightrightarrows R\pi_{2*}\mathbb{Q}_p$, where the section map is the trace map τ multiplied by d^{-1}.

Using the *Künneth* formula [M2, chapter 6, Theorem 8.5]: $R\pi_{2*}\mathbb{Q}_p \simeq R\pi_*\mathbb{Q}_p \otimes_{\mathbb{Q}_p} R\pi_*\mathbb{Q}_p$, we then get:

$$\mathbb{Q}_p \overset{\curvearrowleft}{\to} R\pi_*\mathbb{Q}_p \rightrightarrows R\pi_{2*}\mathbb{Q}_p \simeq R\pi_*\mathbb{Q}_p \otimes_{\mathbb{Q}_p} R\pi_*\mathbb{Q}_p$$

where the section map is the trace map τ multiplied by d^{-1}.

Consider the diagram of complexes:

$$\begin{array}{ccccc} \mathbb{Q}_p & \xrightarrow{\iota} & R\pi_*\mathbb{Q}_p & \xrightarrow{\iota\otimes id - id\otimes\iota} & R\pi_*\mathbb{Q}_p \otimes_{\mathbb{Q}_p} R\pi_*\mathbb{Q}_p \\ {\scriptstyle id}\downarrow & & {\scriptstyle id}\downarrow & & {\scriptstyle id}\downarrow \\ \mathbb{Q}_p & \xrightarrow{\iota} & R\pi_*\mathbb{Q}_p & \xrightarrow{\iota\otimes id - id\otimes\iota} & R\pi_*\mathbb{Q}_p \otimes_{\mathbb{Q}_p} R\pi_*\mathbb{Q}_p \end{array}$$

by identifying $R\pi^*\mathbb{Q}_p \otimes_{\mathbb{Q}_p} \mathbb{Q}_p$ with $R\pi^*\mathbb{Q}_p$, We have:

$$\tau \circ \iota = d$$

$$(\tau \otimes id) \circ (\iota \otimes id) = d$$

$$(\tau \otimes id) \circ (id \otimes \iota) = \iota \circ \tau$$

Thus if we define $h : R\pi_*\mathbb{Q}_p \otimes_{\mathbb{Q}_p} R\pi_*\mathbb{Q}_p \to R\pi_*\mathbb{Q}_p$ as $h = \tau \otimes id$, it then follows that $h \circ (\iota \otimes id - id \otimes \iota) + \tau \otimes \iota = d$, and we get the scalar product map of the complexes

$$\begin{array}{ccccc} \mathbb{Q}_p & \xrightarrow{\iota} & R\pi_*\mathbb{Q}_p & \xrightarrow{\iota\otimes id - id\otimes\iota} & R\pi_*\mathbb{Q}_p \otimes_{\mathbb{Q}_p} R\pi_*\mathbb{Q}_p \\ {\scriptstyle \times d}\downarrow & & {\scriptstyle \times d}\downarrow & & {\scriptstyle \times d}\downarrow \\ \mathbb{Q}_p & \xrightarrow{\iota} & R\pi_*\mathbb{Q}_p & \xrightarrow{\iota\otimes id - id\otimes\iota} & R\pi_*\mathbb{Q}_p \otimes_{\mathbb{Q}_p} R\pi_*\mathbb{Q}_p \end{array}$$

is homotopic to the zero map, which leads to the exactness of the complex.

Now, by $H^*(X_{\bar\eta}^{(1)}, \mathbb{Q}_p) \simeq H^*(X_{\bar\eta}, R\pi_*\mathbb{Q}_p)$ and $H^*(X_{\bar\eta}^{(1)} \times_{X_{\bar\eta}} X_{\bar\eta}^{(1)}, \mathbb{Q}_p) \simeq H^*(X_{\bar\eta}, R\pi_{2*}\mathbb{Q}_p)$, we get the scalar product map hence also the identity map of the complexes

$$\begin{array}{ccccc} H^i(X_{\bar\eta}, \mathbb{Q}_p) & \longrightarrow & H^i(X_{\bar\eta}^{(1)}, \mathbb{Q}_p) & \longrightarrow & H^i(X_{\bar\eta}^{(1)} \times_{X_{\bar\eta}} X_{\bar\eta}^{(1)}, \mathbb{Q}_p) \\ {\scriptstyle \times d}\downarrow & & {\scriptstyle \times d}\downarrow & & {\scriptstyle \times d}\downarrow \\ H^i(X_{\bar\eta}, \mathbb{Q}_p) & \longrightarrow & H^i(X_{\bar\eta}^{(1)}, \mathbb{Q}_p) & \longrightarrow & H^i(X_{\bar\eta}^{(1)} \times_{X_{\bar\eta}} X_{\bar\eta}^{(1)}, \mathbb{Q}_p) \end{array}$$

is homotopic to the zero map, thus the exactness of the sequence $0 \to H^i(X_{\bar\eta}, \mathbb{Q}_p) \to H^i(X_{\bar\eta}^{(1)}, \mathbb{Q}_p) \to H^i(X_{\bar\eta}^{(1)} \times_{X_{\bar\eta}} X_{\bar\eta}^{(1)}, \mathbb{Q}_p)$ is guaranteed.

Since $D_{crys} = D_{st}^{N=0}$ is a left exact functor, the equalizer of the double arrow $D_{crys}(H^i(X_{\bar{\eta}}^{(1)}, \mathbb{Q}_p)) \rightrightarrows D_{crys}(H^i(X_{\bar{\eta}}^{(1)} \times_{X_{\bar{\eta}}} X_{\bar{\eta}}^{(1)}, \mathbb{Q}_p))$ is $D_{crys}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p))$. □

Similarly, there is also a morphism from $H^i_{rig}(X_s/k)$ to the equalizer of the double arrow $H^i_{rig}(X_s^{(1)}/k) \rightrightarrows H^i_{rig}(X_s^{(1)} \times_{X_s} X_s^{(1)}/k)$.

We can then construct a morphism by a similar result as [I3, Prop. A5]:

Lemma 3.4.2 *([I3, Prop. A5]) Let X_η be a proper smooth scheme over K and X a proper flat model over O_K of X_η. For a proper strictly semistable scheme Y and an étale alteration $f : Y \to X$, the homomorphism $f^* : H^*_{et}(X_\eta, \mathbb{Q}_p) \to H^*_{et}(Y_\eta, \mathbb{Q}_p)$ is injective.*

Proof:

As in [I3, Prop. A5], we may assume that X_η and Y_η are irreducible. Let $g : Y_\eta \to Y_\eta \times X_\eta$ be the transpose correspondence of $\Gamma_f = (f, Id_{Y_\eta})$, we have the equality $f \circ g = d \cdot Id_\eta$, where d denotes the degree of f. It then leads to the injectivity of f^*. □

In our case, we can get a similar result as the above result.

Lemma 3.4.3 *Let X be a proper scheme over K, for any proper smooth scheme Y over K and surjective morphism $Y \twoheadrightarrow X$, we have: $gr_i^W H^i_{et}(X_\eta, \mathbb{Q}_p) \to H^i_{et}(Y_\eta, \mathbb{Q}_p)$ is injective, where W refers to the weight filtration introduced by Deligne in [D2].*

Proof:

Extend $Y \twoheadrightarrow X$ to a proper smooth hyper covering of X over K:

$$\cdots Y_1 \rightrightarrows Y \to X$$

by resolution of singularity.

Note that the weight filtration is induced by the spectral sequence of this hypercovering and is independent of the choice of this proper smooth hyper covering ([D2]) .

For the spectral sequence $H^p(H^q(Y., \mathbb{Q}_p)) \Rightarrow H^{p+q}(X, \mathbb{Q}_p)$, take $p = 0$ in the spectral sequence, we then have:

$$gr_i^W H^i_{et}(X_\eta, \mathbb{Q}_p) \simeq E_\infty^{0,q} \hookrightarrow H^i_{et}(Y_\eta, \mathbb{Q}_p)$$

□

Corollary 3.4.1 *Use the same notation as in the above, Then*

$$gr_i^W H^i_{et}(X_{\bar{\eta}}^{(1)} \times_{X_{\bar{\eta}}} X_{\bar{\eta}}^{(1)}, \mathbb{Q}_p) \to H^i_{et}(X_{\bar{\eta}}^{(2)}, \mathbb{Q}_p)$$

is injective.

Now, consider

$$H^i(X_{\bar\eta}^{(1)}, \mathbb{Q}_p)$$

$$p^* \downdownarrows q^*$$

$$H^i(X_{\bar\eta}^{(1)} \times_{X_{\bar\eta}} X_{\bar\eta}^{(1)}, \mathbb{Q}_p)$$

Since $H^i(X_{\bar\eta}^{(1)}, \mathbb{Q}_p)$ has pure weight i, and $H^i(X_{\bar\eta}^{(1)} \times_{X_{\bar\eta}} X_{\bar\eta}^{(1)}, \mathbb{Q}_p)$ has weight in $[0, i]$, we have $(Imp^* + Imq^*) \cap W_{i-1}H^i(X_{\bar\eta}^{(1)}, \mathbb{Q}_p) = \{0\}$ by the strict compatibility of morphisms with the weight filtration.

Thus $Imp^* + Imq^*$ actually injects into $gr_i^W H^i_{et}(X_{\bar\eta}^{(1)} \times_{X_{\bar\eta}} X_{\bar\eta}^{(1)}, \mathbb{Q}_p)$, and hence into $H^i_{et}(X_{\bar\eta}^{(2)}, \mathbb{Q}_p)$ by the above corollary, thus the equalizer of p^* and q^* coincide with the equalizer of $H^i_{et}(X_{\bar\eta}^{(1)}, \mathbb{Q}_p) \rightrightarrows H^i_{et}(X_{\bar\eta}^{(2)}, \mathbb{Q}_p)$.

Applying D_{crys}, we also have the injections:

$$D_{st}(H^i(X_{\bar\eta}^{(1)}, \mathbb{Q}_p))^{N=0} \hookrightarrow D_{st}(gr_i^W H^i_{et}(X_{\bar\eta}^{(1)} \times_{X_{\bar\eta}} X_{\bar\eta}^{(1)}, \mathbb{Q}_p))^{N=0}$$

and an monomorphism:

$$D_{st}(gr_i^W H^i(X_{\bar\eta}^{(1)} \times_{X_{\bar\eta}} X_{\bar\eta}^{(1)}, \mathbb{Q}_p))^{N=0} \hookrightarrow D_{st}(H^i(X_{\bar\eta}^{(2)}, \mathbb{Q}_p))^{N=0}$$

So we have the following:

Proposition 3.4.2 *With the same notations as Prop. 3.4.1 , the equalizer of*

$$H^i(X_{\bar\eta}^{(1)}, \mathbb{Q}_p) \rightrightarrows H^i(X_{\bar\eta}^{(2)}, \mathbb{Q}_p)$$

is $H^i(X_{\bar\eta}, \mathbb{Q}_p)$ *and the equalizer of*

$$D_{st}(H^i(X_{\bar\eta}^{(1)}, \mathbb{Q}_p))^{N=0} \rightrightarrows D_{st}(H^i(X_{\bar\eta}^{(2)}, \mathbb{Q}_p))^{N=0}$$

is $D_{st}(H^i(X_{\bar\eta}, \mathbb{Q}_p))^{N=0}$.

By the diagram, we have $H^i_{rig}(X_s/k)$ maps to the equalizer of $H^i_{rig}(X_s^{(1)}/k) \rightrightarrows H^i_{rig}(X_s^{(2)}/k)$ and thus also maps to the equalizer $D_{st}(H^i(X_{\bar\eta}, \mathbb{Q}_p))^{N=0}$ of $D_{st}(H^i(X_{\bar\eta}^{(1)}, \mathbb{Q}_p))^{N=0} \rightrightarrows D_{st}(H^i(X_{\bar\eta}^{(2)}, \mathbb{Q}_p))^{N=0}$ via the composition of Chiarellotto's morphism c and sp' constructed in the semistable reduction case. And our specialization map is induced from that.

Note that $H^i_{rig}(X_s^{(1)} \times_{X_s} X_s^{(1)}/k) \to H^i_{rig}(X_s^{(2)}/k)$ may not be injective, thus we do not get an isomorphism between the two equalizers $H^i_{rig}(X_s^{(1)}/k) \rightrightarrows H^i_{rig}(X_s^{(2)}/k)$ and $H^i_{rig}(X_s^{(1)}/k) \rightrightarrows H^i_{rig}(X_s^{(1)} \times_{X_s} X_s^{(1)}/k)$. In this case, even if we have an isomorphism of two horizontal rows, we just get an isomorphism between the equalizer of $H^i_{rig}(X_s^{(1)}/k) \rightrightarrows H^i_{rig}(X_s^{(2)}/k)$ and $D_{st}(H^i(X_{\bar\eta}, \mathbb{Q}_p))^{N=0}$.

Also note that, this morphism is well defined and independent of the choice of the alterations.

In fact, given two alterations $Y \to X, Z \to X$, both Y, Z have a semistable reduction, consider the disjoint union $V = Y \amalg Z$, denote the morphisms induced from these alterations by f, g and h. It suffices to show the following diagram commutes:

$$\begin{array}{ccc} H^i_{rig}(X_s/k) \oplus H^i_{rig}(X_s/k) & \xrightarrow{f\oplus g} & D_{crys}(H^i(X_{\bar\eta}, \mathbb{Q}_p)) \oplus D_{crys}(H^i(X_{\bar\eta}, \mathbb{Q}_p)) \\ \uparrow \Delta & & \uparrow \Delta \\ H^i_{rig}(X_s/k) & \xrightarrow{h} & D_{crys}(H^i(X_{\bar\eta}, \mathbb{Q}_p)) \end{array}$$

where Δ is the diagonal map $x \to (x, x)$.

And this follows directly from the facts:

1. The morphism $h' : H^i_{rig}(V_s/k) \to D_{crys}(H^i(V_{\bar\eta}, \mathbb{Q}_p))$ is the direct sum of $f' : H^i_{rig}(Y_s/k) \to D_{crys}(H^i(Y_{\bar\eta}, \mathbb{Q}_p))$ and $g' : H^i_{rig}(Z_s/k) \to D_{crys}(H^i(Z_{\bar\eta}, \mathbb{Q}_p))$

2. The morphisms $H^i_{rig}(X_s/k) \to H^i_{rig}(V_s/k) = H^i_{rig}(Y_s/k) \oplus H^i_{rig}(Z_s/k)$, $D_{crys}(H^i(X_{\bar\eta}, \mathbb{Q}_p)) \to D_{crys}(H^i(V_{\bar\eta}, \mathbb{Q}_p)) = D_{crys}(H^i(Y_{\bar\eta}, \mathbb{Q}_p)) \oplus D_{crys}(H^i(Z_{\bar\eta}, \mathbb{Q}_p))$ factors through the diagonal map.

Which is equivalent to the commutative diagram:

$$\begin{array}{ccc} H^i_{rig}(V_s/k) = H^i_{rig}(Y_s/k) \oplus H^i_{rig}(Z_s/k) & \xrightarrow{h'=f'\oplus g'} & D_{crys}(H^i(V_{\bar\eta}, \mathbb{Q}_p)) = D_{crys}(H^i(Y_{\bar\eta}, \mathbb{Q}_p)) \oplus D_{crys}(H^i(Z_{\bar\eta}, \mathbb{Q}_p)) \\ \uparrow & & \uparrow \\ H^i_{rig}(X_s/k) \oplus H^i_{rig}(X_s/k) & \xrightarrow{f\oplus g} & D_{crys}(H^i(X_{\bar\eta}, \mathbb{Q}_p)) \oplus D_{crys}(H^i(X_{\bar\eta}, \mathbb{Q}_p)) \\ \uparrow \Delta & & \uparrow \Delta \\ H^i_{rig}(X_s/k) & \xrightarrow{h} & D_{crys}(H^i(X_{\bar\eta}, \mathbb{Q}_p)) \end{array}$$

Thus the morphism is well defined and independent of the choice of the alterations, and the diagram indeed induces our $sp' : H^i_{rig}(X_s/k) \to D_{crys}(H^i(X_{\bar\eta}, \mathbb{Q}_p))$.

By above discussions, we have the following:

Theorem 3.4.1 *Let $X \to S$ be proper, flat and generically smooth. Then there is a ϕ-equivariant map*

$$sp' : H^i_{rig}(X_s/k) \to D_{crys}(H^i(X_{\bar\eta}, \mathbb{Q}_p))$$

and a commutative diagram of $Gal(\bar k/k)-$modules

$$\begin{array}{ccc} H^i(X_{\bar s}, \mathbb{Q}_p) & \xrightarrow{sp} & H^i(X_{\bar\eta}, \mathbb{Q}_p)^I \\ \lambda_s \downarrow & & \lambda_\eta \downarrow \\ H^i_{rig}(X_s/k) \otimes_{K_0} \hat{K_0}^{ur} & \xrightarrow{sp'\otimes 1} & D_{crys}(H^i(X_{\bar\eta}, \mathbb{Q}_p)) \otimes_{K_0} \hat{K_0}^{ur} \end{array}$$

where $\hat{K_0}^{ur} = Frac(W(\bar k))$ is the p-adic completion of the maximal unramified extension of K_0, and *the vertical maps λ_s, λ_η induce isomophisms:*

$$\lambda_s : H^i(X_{\bar{s}}, \mathbb{Q}_p) \simeq (H^i_{rig}(X_s/k) \otimes_{K_0} \hat{K_0}^{ur})^{\phi\otimes\phi=1} = H^i_{rig}(X_s/k)^{slope0}$$

and

$$\lambda_\eta : H^i(X_{\bar{\eta}}, \mathbb{Q}_p)^I \simeq (D_{crys}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p)) \otimes_{K_0} \hat{K_0}^{ur})^{\phi\otimes\phi=1} = D_{crys}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p))^{slope0}$$

Remark 3.4.1 *By our discussions above, we also get that the definition of sp' in good reduction and semistable reduction cases is compatible with our definition for the general regular schemes.*

Remark 3.4.2 *By Chiarellotto's isomorphism*

$$H^*_{rig}(Y/K_0) \simeq H^*(Y_{Zar}, Kerv) \otimes K_0 \simeq H^*(Y_{et}, Kerv) \otimes K_0$$

one can define $sp^* : H^*_{rig}(Y_s/K_0) \to H^*_{dR}(Y_\eta/K_0)$ via the composition:

$$H^*_{rig}(Y_s/K_0) \simeq H^*(Y_s, Kerv) \otimes K_0 \to H^*(Y_s, WA^\bullet) \otimes K_0 \simeq H^*_{dR}(Y_\eta/K_0)$$

where the second arrow is induced by the natural inclusion $Kerv \hookrightarrow WA^\bullet$.

In [BCF], Baldassari, Cailotto and Fiorot have also constructed a specialization map from the rigid cohomology to the de Rham cohomology, also see [I2] and [M4]. And their specialization map are also closely related to Chiarellotto's constructions.

Remark 3.4.3 *Note that since Chiarellotto's conjecture holds for dimensions 1 and 2, so in these cases, the first arrow in each row is also an isomorphism. Although from [M3], we know that monodromy-weight conjecture holds in these cases and the local invariant cycle theorem holds.*

Chapter 4

Proof of p-adic Local Invariant Cycle Theorem

In this chapter, we will consider the following conjecture:

If X is regular, then

$$sp : H^i(X_{\bar{s}}, WO_X)_{\mathbb{Q}} \to (D_{crys}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p) \otimes_{K_0} \hat{K_0}^{ur})^{slope[0,1)}$$

induced from the p-adic specialization map is an isomorphism, and we have a commutative diagram:

$$\begin{array}{ccc} H^i(X_{\bar{s}}, \mathbb{Q}_p) & \xrightarrow{sp} & H^i(X_{\bar{\eta}}, \mathbb{Q}_p)^I \\ \lambda_s \downarrow & & \lambda_\eta \downarrow \\ H^i(X_{\bar{s}}, WO_X)_{\mathbb{Q}} & \xrightarrow{sp} & (D_{crys}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p) \otimes_{K_0} \hat{K_0}^{ur})^{slope[0,1)} \end{array}$$

In particular, $sp : H^i(X_{\bar{s}}, \mathbb{Q}_p) \to H^i(X_{\bar{\eta}}, \mathbb{Q}_p)^I$ *is an isomorphism.*

4.1 Trace Morphism

In [BER], Berthelot, Esnault and Rülling have constructed a trace morphism between the Witt vector cohomologies of the special fibers between two regular schemes of the same dimension over a discrete valuation ring of mixed characteristic $(0, p)$.

More explicitly, they first construct a morphism for coherent cohomology:

Proposition 4.1.1 *([BER]) Let* X, Y *be two flat, regular R-schemes of finite type, of the same dimension, and let* $f : Y \to X$ *be a projective and surjective R-morphism, with reduction* f_k *over Speck. Then there exists a trace morphism:*

$$\tau_f : Rf_*O_Y \to O_X$$

such that the composition of the functoriality morphism $O_X \to Rf_*O_Y$ *is a scalar product* $\times r$*, where* $r \in \mathbb{Z}$ *is the generic degree of* f*.*

Then they extended it to trace morphism of Witt vector cohomology:

Proposition 4.1.2 *([BER]) Make the same assumption as above, let* f *has a factorization* $f = \pi \circ i$*, where* π *is the projection of a projective space* $\mathbb{P}^d_X$ *on* X*,* i *is a closed immersion.*

Denote $X_n = X \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{(p)}/p^{n+1}$ *and* $f_n : Y_n \to X_n$ *be the reduction of* f *mod* p^{n+1}*,* π_n, i_n *are the corresponding reductions of* i, π *mod* p^{n+1} *Then for Witt vector cohomology, there exist trace morphisms:*

$$\tau_{f_n} : Rf_{n*}O_{Y_n} \to O_{X_n}$$

$$\tau_{i_0,\pi_0,n} : Rf_*W_n(O_{Y_0}) \to W_n(O_{X_0})$$

and

$$\tau_{i_0,\pi_0} : Rf_*WO_{Y_0,\mathbb{Q}} \to WO_{X_0,\mathbb{Q}}$$

such that the respectively compositions of the functoriality morphisms $O_{X_n} \to Rf_{n*}O_{Y_n}$*,* $W_n(O_{X_0}) \to Rf_*W_n(O_{Y_0})$ *and* $WO_{X_0,\mathbb{Q}} \to Rf_*WO_{Y_0,\mathbb{Q}}$ *are scalar products* $\times r$*.*

Moreover, for $n = 1$*,* $\tau_{i,\pi,n} = \tau_f$*, and* $\tau_{i,\pi,n}$ *commutes with* R, F *and* V *(see [BER, section 5] or [I4], [LZ] for the definitions of* R, F *and* V*).*

The above trace morphisms then induce trace morphisms on the corresponding cohomology $H^*(X, O_X)$ etc. Also, note that $X_s \hookrightarrow X_0$ and $Y_s \hookrightarrow Y_0$ are nilpotent immersions, by [BBE, Prop. 2.1], we know that

$$H^i(X_0, WO_{X_0})_{\mathbb{Q}} \simeq H^i(X_s, WO_{X_s})_{\mathbb{Q}}$$

and

$$H^i(Y_0, WO_{Y_0})_{\mathbb{Q}} \simeq H^i(Y_s, WO_{Y_s})_{\mathbb{Q}}$$

are isomorphisms. Thus there is also a trace morphism

$$\tau_{i,\pi} : H^i(Y_s, WO_{Y_s})_{\mathbb{Q}} \to H^i(X_s, WO_{X_s})_{\mathbb{Q}}$$

In [BER], Berthelot, Esnault and Rülling have proved that the above trace morphism is compatible with the base change and composition of morphisms.

As in the proof of Proposition 3.4.1, the existence of these trace morphisms play an important role in our proof.

4.2 Coherent Cohomology

Now we consider $H^i(X, O_X)$ first.

As in the last chapter, when X is regular, consider

$$\begin{array}{c} X^{(2)} \\ \pi\downarrow \\ Y = X^{(1)} \times_X X^{(1)} \\ p \downdownarrows q \\ X^{(1)} \\ f\downarrow \\ X \end{array}$$

where $X^{(1)} \to X$ surjective , $X^{(2)} \to X^{(1)} \times_X X^{(1)}$ generically surjective (i.e., surjective on the generic fibre, while not necessarily surjective on all generic points), both are alterations and have semistable reductions by De Jong's theorem ([D-J]).

We define $g : X^{(2)} \to X^{(1)}$ to be the composition: $X^{(2)} \xrightarrow{\pi} Y \xrightarrow{p} X^{(1)}$, note that, as $Z \to X^{(1)} \times_X X^{(1)}$ is generically surjective, and $p : Y \to X^{(1)}$ is surjective, we deduce that $g : X^{(2)} \to X^{(1)}$ is also a surjective alteration.

As in the proof of proposition 3.4.1, we need a trace morphism and a relative *Künneth* formula.

Now, define $O_{Y^{der}} = p^{-1}O_{X^{(1)}} \otimes^L_{(fp)^{-1}O_X} q^{-1}O_{X^{(1)}}$, which can be viewed as an object in the derived category of $(fp)^{-1}O_X$-modules on the underlying topological space of Y. Then [BER, Prop. A.1 (ii)] still holds if we replace Y' by Y'^{der} and understand $Lv^*E^\bullet$ there as $O_{Y^{der}} \otimes^L_{v^{-1}O_Y} v^{-1}E^\bullet$. More explicitly, in our situation, we have:

$$Lq^*F^\bullet = q^{-1}F^\bullet \otimes^L_{q^{-1}O_X} O_{Y^{der}}$$

Then, by [BER, Prop. A.1 (ii)], we have:

$$Rp_*Lq^*O_{X^{(1)}} \simeq Lf^*Rf_*O_{X^{(1)}}$$

Thus

$$Rf_*Rp_*Lq^*O_{X^{(1)}} = R(f \circ p)_*O_{Y^{der}}$$

On the other hand,

$$Rf_*Rp_*Lq^*O_{X^{(1)}} = Rf_*Lf^*Rf_*O_{X^{(1)}} = Rf_*(f^{-1}Rf_*O_{X^{(1)}} \otimes^L_{f^{-1}O_X} O_{X^{(1)}})$$

Using "Projection formula" ([Har, Prop. 5.6])

$$Rf_*(f^{-1}Rf_*O_{X^{(1)}} \otimes^L_{f^{-1}O_X} O_{X^{(1)}}) = Rf_*O_{X^{(1)}} \otimes^L_{O_X} Rf_*O_{X^{(1)}}$$

Thus we have

$$R(f \circ p)_*O_{Y^{der}} = Rf_*O_{X^{(1)}} \otimes^L_{O_X} Rf_*O_{X^{(1)}}$$

which is the relative *Künneth* formula we need for the coherent cohomology.

By [BER], we have trace morphisms for alterations $f : X^{(1)} \to X$, $g : X^{(2)} \to X^{(1)}$ and $f \circ g : X^{(2)} \to X$.

In the following, we define ι_f to be the adjunction $O_X \to Rf_*O_{X^{(1)}}$, τ_f be the corresponding trace map $Rf_*O_{X^{(1)}} \to O_X$, and $\tau_f \circ \iota_f = d_f id_{O_X}$(the scalar multiplication by d_f, and we just write d_f later). Similarly, define ι_g, τ_g, and d_g.

Then, from the trace morphism $Rf_*O_{X^{(1)}} \to O_X$, since we have the base change isomorphism $Rp_*Lq^*O_{X^{(1)}} \simeq Lf^*Rf_*O_{X^{(1)}}$, we then get a trace morphism

$$\tau_p = Lf^*\tau_f : Rp_*O_{Y^{der}} = Rp_*Lq^*O_{X^{(1)}} \simeq Lf^*Rf_*O_{X^{(1)}} \to Lf^*O_{X^{(1)}} = O_{X^{(1)}}$$

Now we have a diagram of complexes:

$$\begin{array}{ccccc} O_X & \xrightarrow{\iota_f} & Rf_*O_{X^{(1)}} & \xrightarrow{\iota_f\otimes id - id\otimes\iota_f} & R(f\circ p)_*O_{Y^{der}} = Rf_*O_{X^{(1)}} \otimes^L_{O_X} Rf_*O_{X^{(1)}} \\ {\scriptstyle id}\downarrow & & {\scriptstyle id}\downarrow & & {\scriptstyle id}\downarrow \\ O_X & \xrightarrow{\iota_f} & Rf_*O_{X^{(1)}} & \xrightarrow{\iota_f\otimes id - id\otimes\iota_f} & R(f\circ p)_*O_{Y^{der}} = Rf_*O_{X^{(1)}} \otimes^L_{O_X} Rf_*O_{X^{(1)}} \end{array}$$

Similarly to the proof of proposition 3.4.1, by identifying $Rf_*O_{X^{(1)}} \otimes^L_{O_X} O_X$ with $Rf_*O_{X^{(1)}}$, we have:

$$\tau_f \circ \iota_f = d_f$$

$$Rf_*\iota_p = \iota_f \otimes id$$

$$Rf_*\tau_p = \tau_f \otimes id$$

$$(\tau_f \otimes id) \circ (\iota_f \otimes id) = Rf_*\tau_p \circ Rf_*\iota_p = d_p = d_f$$

$$(\tau_f \otimes id) \circ (id \otimes \iota_f) = Rf_*\tau_p \circ (id \otimes \iota_f) = \iota_f \circ \tau_f$$

Then we can construct $h = Rf_*\tau_p : R(f \circ p)_*O_{Y^{der}} \to Rf_*O_{X^{(1)}}$, such that:

$$h \circ (\iota_f \otimes id - id \otimes \iota_f) + \iota_f \circ \tau_f = d_f - \iota_f \circ \tau_f + \iota_f \circ \tau_f = d_f$$

Thus the scalar product by d_f map of the complexes

$$\begin{array}{ccccc}
O_X & \xrightarrow{\iota_f} & Rf_*O_{X^{(1)}} & \xrightarrow{\iota_f\otimes id - id\otimes\iota_f} & R(f\circ p)_*O_{Y^{der}} = Rf_*O_{X^{(1)}} \otimes^L_{O_X} Rf_*O_{X^{(1)}} \\
{\scriptstyle\times d_f}\downarrow & & {\scriptstyle\times d_f}\downarrow & & {\scriptstyle\times d_f}\downarrow \\
O_X & \xrightarrow{\iota_f} & Rf_*O_{X^{(1)}} & \xrightarrow{\iota_f\otimes id - id\otimes\iota_f} & R(f\circ p)_*O_{Y^{der}} = Rf_*O_{X^{(1)}} \otimes^L_{O_X} Rf_*O_{X^{(1)}}
\end{array}$$

is homotopic to the zero map, which leads to the exactness of the corresponding cohomology complex.

Now, we consider $H^i(X^{(1)}, O_{X^{(1)}}) \rightrightarrows H^i(X^{(2)}, O_{X^{(2)}})$.

Consider the diagram of complexes:

$$\begin{array}{ccccccc}
O_X & \xrightarrow{\iota_f} & Rf_*O_{X^{(1)}} & \xrightarrow{\iota_f\otimes id - id\otimes\iota_f} & R(f\circ p)_*O_{Y^{der}} = Rf_*O_{X^{(1)}} \otimes^L_{O_X} Rf_*O_{X^{(1)}} & \xrightarrow{Rf_*Rp_*\iota_\pi} & Rf_*Rg_*O_{X^{(2)}} \\
{\scriptstyle\times d_g}\downarrow & & {\scriptstyle\times d_g}\downarrow & & {\scriptstyle\times d_g}\downarrow & & {\scriptstyle\times d_g}\downarrow \\
O_X & \xrightarrow{\iota_f} & Rf_*O_{X^{(1)}} & \xrightarrow{\iota_f\otimes id - id\otimes\iota_f} & R(f\circ p)_*O_{Y^{der}} = Rf_*O_{X^{(1)}} \otimes^L_{O_X} Rf_*O_{X^{(1)}} & \xrightarrow{Rf_*Rp_*\iota_\pi} & Rf_*Rg_*O_{X^{(2)}}
\end{array}$$

Note that

$$O_{X^{(1)}} \xrightarrow{\iota_p} Rp_*O_{Y^{der}} \xrightarrow{Rp_*\iota_\pi} Rg_*O_{X^{(2)}}$$

coincides with

$$O_{X^{(1)}} \xrightarrow{\iota_g} Rg_*O_{X^{(2)}}$$

i.e., $\iota_g = Rp_*\iota_\pi \circ \iota_p$, thus

$$\tau_g \circ Rp_*\iota_\pi \circ \iota_p = \tau_g \circ \iota_g = d_g$$

If we could prove that:

$$Rf_*\tau_g \circ Rf_*Rp_*\iota_\pi \circ (id\otimes \iota_f) = \frac{d_g}{d_f} Rf_*(\tau_g \circ Rp_*\iota_\pi) \circ (id\otimes\iota_f) = \frac{d_g}{d_f} Rf_*\tau_p \circ (id\otimes\iota_f) = \frac{d_g}{d_f}\iota_f \circ \tau_f$$

then we would have constructed $h = Rf_*\tau_g : Rf_*Rg_*O_{X^{(2)}} \to Rf_*Rg_*O_{X^{(1)}}$, such that:

$$h \circ (Rf_*Rp_*\iota_\pi \circ (\iota_f \otimes id) - Rf_*Rp_*\iota_\pi \circ (id\otimes\iota_f)) + \iota_f \circ (\frac{d_g}{d_f}\tau_f) = d_g$$

Thus the scalar product by d_g map of the complexes

$$\begin{array}{ccccccc}
O_X & \xrightarrow{\iota_f} & Rf_*O_{X^{(1)}} & \xrightarrow{\iota_f\otimes id - id\otimes\iota_f} & R(f\circ p)_*O_{Y^{der}} = Rf_*O_{X^{(1)}} \otimes^L_{O_X} Rf_*O_{X^{(1)}} & \xrightarrow{Rf_*Rp_*\iota_\pi} & Rf_*Rg_*O_{X^{(2)}} \\
{\scriptstyle\times d_g}\downarrow & & {\scriptstyle\times d_g}\downarrow & & {\scriptstyle\times d_g}\downarrow & & {\scriptstyle\times d_g}\downarrow \\
O_X & \xrightarrow{\iota_f} & Rf_*O_{X^{(1)}} & \xrightarrow{\iota_f\otimes id - id\otimes\iota_f} & R(f\circ p)_*O_{Y^{der}} = Rf_*O_{X^{(1)}} \otimes^L_{O_X} Rf_*O_{X^{(1)}} & \xrightarrow{Rf_*Rp_*\iota_\pi} & Rf_*Rg_*O_{X^{(2)}}
\end{array}$$

is homotopic to the zero map, which leads to the exactness of the corresponding cohomology complex, and we get the consequence.

Now, consider

$$O_{X^{(1)}} \xrightarrow{\iota_q} Rq_* O_{Y^{der}} \overset{\lambda}{\simeq} Rp_* O_{Y^{der}} \xrightarrow{\tau_p} O_{X^{(1)}}$$

where λ is $f^{-1}O_X$-linear isomorphism induced by the interchanging of the factors in $X^{(1)} \times_X X^{(1)}$.

We have:

$$\iota_f \circ \tau_f = (\tau_f \otimes id) \circ (id \otimes \iota_f) = Rf_* \tau_p \circ (id \otimes \iota_f) = Rf_* \tau_p \circ Rf_*(\lambda \circ \iota_q) = Rf_*(\tau_p \circ \lambda \circ \iota_q)$$

$$id \otimes \iota_f = Rf_*(\lambda \circ \iota_q)$$

Thus it suffices to show that

$$Rf_* \tau_g \circ Rf_* Rp_* \iota_\pi \circ Rf_*(\lambda \circ \iota_q) = Rf_*(\frac{d_g}{d_f} \tau_p \circ \lambda \circ \iota_q)$$

or

$$\tau_g \circ Rp_* \iota_\pi \circ \lambda \circ \iota_q = \frac{d_g}{d_f} \tau_p \circ \lambda \circ \iota_q$$

i.e., to show that the composition

$$O_{X^{(1)}} \xrightarrow{\iota_q} Rq_* O_{Y^{der}} \overset{\lambda}{\simeq} Rp_* O_{Y^{der}} \xrightarrow{Rp_* \iota_\pi} Rg_* O_{X^{(2)}} = R(p \circ \pi)_* O_{X^{(2)}} \xrightarrow{\tau_g} O_{X^{(1)}}$$

coincides with

$$O_{X^{(1)}} \xrightarrow{\iota_q} Rq_* O_{Y^{der}} \overset{\lambda}{\simeq} Rp_* O_{Y^{der}} \xrightarrow{\tau_p} O_{X^{(1)}}$$

Now, choose U open in X, such that $f|_U : X_U^{(1)} \to U$ is finite etale, $\pi : X^{(2)} \to \bar{Y}_U$ is surjective, note that $p : Y_U \to X_U^{(1)}$ is surjective and finite etale, choose V open in Y_U such that $V \to X^{(1)}$ is etale and $X_V^{(2)} \to V$ is surjective and finite etale. Finally, put $W = X_U^{(1)} \backslash p(Y_U \backslash V)$, which is open in $X_U^{(1)}$, $Y_W \subset V$, $W \to U$ surjective and $Y_W \to W$ is surjective and finite etale.

Summarizing, we have the following diagram:

$$\begin{array}{ccccccc}
 & & X_V^{(2)} & \longrightarrow & & & X^{(2)} \\
 & & \downarrow & & & & \downarrow \\
Y_W & \longrightarrow & V & \longrightarrow & Y_U & \longrightarrow & \bar{Y}_U \\
\downarrow & & & & \downarrow & & \downarrow \\
W & & \longrightarrow & & X_U^{(1)} & \longrightarrow & \overline{X^{(1)}}_U \\
\downarrow & & & & \downarrow & & \\
U & & \longrightarrow & & X_U & &
\end{array}$$

If we replace $X, X^{(1)}, Y, X^{(2)}$ above by $U, W, Y_W, X_V^{(2)}$, then the above discussions still work as the trace morphism is compatible with base change.

And in this case, we have a trace map $Rp_*\tau_\pi : Rg_*O_{X^{(2)}} \to Rp_*O_{Y^{der}}$ satisfying $Rp_*\tau_\pi \circ Rp_*\iota_\pi = d_\pi$ on W.

Then since the composition

$$Rg_*O_{X^{(2)}} \xrightarrow{Rp_*\iota_\pi} Rp_*O_{Y^{der}} \xrightarrow{\tau_p} O_{X^{(1)}}$$

coincides with the trace map

$$Rg_*O_{X^{(2)}} \xrightarrow{\tau_g} O_{X^{(1)}}$$

defined in [BER], we have:

$$\tau_p \circ Rp_*\tau_\pi = \tau_g$$

and

$$\tau_g \circ Rp_*\iota_\pi = \tau_p \circ Rp_*\tau_\pi \circ Rp_*\iota_\pi = \frac{d_g}{d_f}\tau_p = d_\pi\tau_p$$

So we have proved that:

$$\tau_g \circ Rp_*\iota_\pi \circ \lambda \circ \iota_q = \frac{d_g}{d_f}\tau_p \circ \lambda \circ \iota_q$$

when restricted on W.

Now we want to show these identities hold on all of $O_{X^{(1)}}$.

We prove a lemma first:

Lemma 4.2.1 *For any integral scheme X, $s \in Hom(O_X, O_X)$, if $s|_U = 0$ where U is open in X, then $s \equiv 0$ on O_X.*

Proof:

Choose an affine covering of X, it then suffices to prove the case $X = SpecA$.

Then $U = SpecA \backslash Spec(A/I)$ for some ideal I, and $U = \cup_{f\in I} SpecA_f$

Now, since for any $a \in A$, we have $s(a) = 0$ in A_f, which equivalents to $f^{n_i}s(a) = 0$ for some n_i.

Since A is an integral domain, we have $s(a) = 0$. □

By this lemma, we have

$$\tau_g \circ Rp_*\iota_\pi \circ \lambda \circ \iota_q = \frac{d_g}{d_f}\tau_p \circ \lambda \circ \iota_q$$

and

$$Rf_*\tau_g \circ Rf_*Rp_*\iota_\pi \circ (id \otimes \iota_f) = \frac{d_g}{d_f} Rf_*(\tau_g \circ Rp_*\iota_\pi) \circ (id \otimes \iota_f) = Rf_*\tau_p \circ (id \otimes \iota_f)$$

We then get the homotopy between the scalar product by d_g map and the zero map, and also the same consequence for the coherent cohomology complex.

Proposition 4.2.1 *Let X/S be regular, proper, flat and generically smooth, then the cohomology of the sequence*

$$0 \to H^i(X, O_X) \to H^i(X^{(1)}, O_{X^{(1)}}) \to H^i(X^{(2)}, O_{X^{(2)}})$$

is annihilated by a fixed integer d_g.

Remark 4.2.1 *In [BER] Prop. A.1, if $f : Y \to X$ is a complete intersection morphism of virtual relative dimension 0, and we have:*

$$\begin{array}{ccc} Z & \xrightarrow{v} & Y \\ g\downarrow & & f\downarrow \\ Y' & \xrightarrow{u} & X \end{array}$$

such that Y and Y' are Tor-independent. Then $g : Z \to Y'$ is also a complete intersection morphism of virtual relative dimension 0, which induces a trace morphism.

Thus, if we could prove the following Tor-independence fact in our assumption:

$$Tor^i_{O_X}(p^{-1}O_{X^{(1)}}, q^{-1}O_{X^{(1)}}) = 0$$

then we get $p : Y = X^{(1)} \times_X X^{(1)} \to X^{(1)}$ has a trace map τ_p, and we can then replace Y^{der} by Y in the above proof.

The Tor-independence problem can be converted as follows:

Consider the diagram:

$$\begin{array}{ccc} Z & \xrightarrow{v} & Y \\ g\downarrow & & f\downarrow \\ Y & \xrightarrow{u} & X \end{array}$$

Fix $z \in Z$ with image y(resp. x) in Y(resp. X),

Then the morphism $A = O_{X,x} \to B = O_{Y,y}$ can be factorized as

$$A \hookrightarrow C = A[s_1, s_2, \ldots, s_m]_{\mathfrak{p}} \overset{\varphi}{\twoheadrightarrow} B = C/I$$

where I is generated by a regular sequence $\{f_1, f_2, \ldots, f_r\}$.

It then requires to show that $Tor_i^{O_{X,x}}(O_{Y,y}, O_{Y,y}) = 0$ for $i \geq 1$ provided that $I \cap A = \{0\}$, which is equivalent to the tensor product of B's Koszul-resolution $K(f_1, f_2, \ldots, f_r) \otimes_A K(f_1, f_2, \ldots, f_r)$ is still a flat resolution of $B \otimes_A B$.

This will leads to some nontrivial fact that the tensor product of two regular local ring is torsion-free under certain condition (see [A], [L3], [HW] for some rigidity discussions). However, since Y just plays in an intermediate step in our proof, we stop to do further things here.

4.3 Witt Vector Cohomology

In this section, we want to extend the above results of coherent cohomology to the Witt vector cohomology.

As in [BER], denote $X_n = X \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{(p)}/p^{n+1}$ and $f_n : X_n^{(1)} \to X_n$ be the reduction of $f mod p^{n+1}$.

Note that for all $n \geq 1$, X_n and $X^{(1)}$ are Tor-independent over X (as in [BER] prop. 8.6, by the spectral sequence for the composition of Tor's), so we get a trace map $\tau_{f_n} : O_{X_n^{(1)}} \to O_{X_n}$, and the trace map τ_{f_n} is induced from τ_f by the base change from X to X_n.

We could apply similar argument in the last section, except that τ_{p_n} and τ_{q_n} may not exist in our case.

Summarizing, we have to consider the diagram of complexes:

$$\begin{array}{ccccc} O_{X_n} & \xrightarrow{\iota_{f_n}} & Rf_{n*}O_{X_n^{(1)}} & \xrightarrow{Rf_{n*}Rp_{n*}\iota_{\pi_n} \circ Rf_{n*}\iota_{p_n} - Rf_{n*}Rp_{n*}\iota_{\pi_n} \circ Rf_{n*}(\lambda_n \circ \iota_{q_n})} & Rf_{n*}Rg_{n*}O_{X_n^{(2)}} \\ \times d_g \downarrow & & \times d_g \downarrow & & \times d_g \downarrow \\ O_{X_n} & \xrightarrow{\iota_{f_n}} & Rf_{n*}O_{X_n^{(1)}} & \xrightarrow{Rf_{n*}Rp_{n*}\iota_{\pi_n} \circ Rf_{n*}\iota_{p_n} - Rf_{n*}Rp_{n*}\iota_{\pi_n} \circ Rf_{n*}(\lambda_n \circ \iota_{q_n})} & Rf_{n*}Rg_{n*}O_{X_n^{(2)}} \end{array}$$

directly, where $\lambda_n : Rq_{n*}O_{Y_n^{der}} \simeq Rp_{n*}O_{Y_n^{der}}$ is $f_n^{-1}O_{X_n}$-linear isomorphism induced by interchanging factors.

Note that we still have:

$$O_{X_n^{(1)}} \xrightarrow{\iota_{p_n}} Rp_{n*}O_{Y_n^{der}} \xrightarrow{Rp_{n*}\iota_{\pi_n}} Rg_{n*}O_{X_n^{(2)}}$$

coincides with

$$O_{X_n^{(1)}} \xrightarrow{\iota_{g_n}} Rg_{n*}O_{X_n^{(2)}}$$

i.e., $\iota_{g_n} = Rp_{n*}\iota_{\pi_n} \circ \iota_{p_n}$, thus

$$\tau_{g_n} \circ Rp_{n*}\iota_{\pi_n} \circ \iota_{p_n} = \tau_{g_n} \circ \iota_{g_n} = d_g$$

Thus, it remains to prove:

$$Rf_{n*}\tau_{g_n} \circ Rf_{n*}Rp_{n*}\iota_{\pi_n} \circ Rf_{n*}\iota_{q_n} = Rf_{n*}(\tau_{g_n} \circ Rp_{n*}\iota_{\pi_n} \circ \lambda_n \circ \iota_{q_n}) = \frac{d_g}{d_f}\iota_{f_n} \circ \tau_{f_n}$$

Since the trace map is compatible with base change, the above identity follows from

$$Rf_*\tau_g \circ Rf_*Rp_*\iota_\pi \circ Rf_*Rf_*(\lambda \circ \iota_q) = Rf_*(\frac{d_g}{d_f}\tau_p \circ \lambda \circ \iota_q) = d_p\iota_f \circ \tau_f$$

So, the results of coherent cohomology in the last section also hold for X_n. More explicitly:

Proposition 4.3.1 *Let X/S be regular, proper, flat and generically smooth, let $X_n = X \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{(p)}/p^{n+1}$ be the reduction of X, then the cohomology of the sequence*

$$0 \to H^i(X_n, O_{X_n}) \to H^i(X_n^{(1)}, O_{X_n^{(1)}}) \to H^i(X_n^{(2)}, O_{X_n^{(2)}})$$

is annihilated by a fixed integer d_g.

Then the associated cohomology diagram induces the following sequence:

$$H^i(X_0, O_{X_0}) \xrightarrow{f_0} H^i(X_0^{(1)}, O_{X_0^{(1)}}) \xrightarrow{g_0 - h_0} H^i(X_0^{(2)}, O_{X_0^{(2)}})$$

and by the above discussion, we know that:

$$\tau_{f_0} \circ f_0 = d_f, \quad \tau_{g_0} \circ g_0 = d_g$$

$$\tau_{g_0} \circ h_0 = \frac{d_g}{d_f} f_0 \circ \tau_{f_0}$$

where τ_{f_0} and τ_{g_0} are the base change of the trace maps τ_f and τ_g, which are trace maps too by [BER, prop. 8.6].

Now, Consider the sequence

$$H^i(X_0, W_nO_{X_0}) \xrightarrow{f_{0,n}} H^i(X_0^{(1)}, W_nO_{X_0^{(1)}}) \xrightarrow{g_{0,n} - h_{0,n}} H^i(X_0^{(2)}, W_nO_{X_0^{(2)}})$$

We want to prove the following:

Claim: For all $n \geq 1$, we have:

$$\tau_{f_{0,n}} \circ f_{0,n} = d_f, \quad \tau_{g_{0,n}} \circ g_{0,n} = d_g$$

$$d_n \cdot \tau_{g_{0,n}} \circ h_{0,n} = d_n \cdot \frac{d_g}{d_f} f_{0,n} \circ \tau_{f_{0,n}}$$

where $\tau_{f_{0,n}}$ and $\tau_{g_{0,n}}$ are the trace maps:

$$H^i({X_0}^{(1)}, W_nO_{{X_0}^{(1)}}) \to H^i(X_0, W_nO_{X_0})$$

$$H^i({X_0}^{(1)}, W_nO_{{X_0}^{(2)}}) \to H^i({X_0}^{(1)}, W_nO_{{X_0}^{(1)}})$$

as in [BER, Prop. 8.6]. And d_n is bounded as $n \to \infty$. Here we fix the factorizations $f_0 = \pi \circ i$ and $g_0 = \pi' \circ i'$.

Note that if this is true, then we get that $Ker(g_{0,n} - h_{0,n})/Im f_{0,n}$ is annihilated by $d_n d_g$, taking inverse limit and then tensoring with $\mathbb{Q}$, we then get the sequence

$$0 \to H^i(X_0, WO_{X_0})_{\mathbb{Q}} \to H^i(X_0^{(1)}, WO_{X_0^{(1)}})_{\mathbb{Q}} \rightrightarrows H^i(X_0^{(2)}, WO_{X_0^{(2)}})_{\mathbb{Q}}$$

is exact.

As shown above, we know that

$$H^i(X_0, WO_{X_0})_{\mathbb{Q}} \to H^i(X_s, WO_{X_s})_{\mathbb{Q}}$$

and

$$H^i(X_0^{(1)}, WO_{X_0^{(1)}})_{\mathbb{Q}} \to H^i(X_s^{(1)}, WO_{X_s^{(1)}})_{\mathbb{Q}}$$

are isomorphisms.

Now recall that the Witt vector cohomology captures the slope $[0,1)$ part of the rigid cohomology and of the Hyodo-Kato cohomology by the above result, we then have a commutative diagram with the two columns are exact:

$$\begin{array}{ccccc}
H^i(X_s^{(2)}, WO_{X^{(2)}})_{\mathbb{Q}} & \xrightarrow{\simeq} & H^i_{HK}(X_s^{(2)}/k)_{\mathbb{Q}}^{N=0,slope[0,1)} & \xrightarrow{\simeq} & D_{crys}(H^i(X_{\bar{\eta}}^{(2)}, \mathbb{Q}_p))^{slope[0,1)} \\
\uparrow & & & & \uparrow \\
H^i(Y_s, WO_Y)_{\mathbb{Q}} & & & & D_{crys}(H^i(Y_{\bar{\eta}}, \mathbb{Q}_p))^{slope[0,1)} \\
\upuparrows & & & & \upuparrows \\
H^i(X_s^{(1)}, WO_{X^{(1)}})_{\mathbb{Q}} & \xrightarrow{\simeq} & H^i_{HK}(X_s^{(1)}/k)_{\mathbb{Q}}^{N=0,slope[0,1)} & \xrightarrow{\simeq} & D_{crys}(H^i(X_{\bar{\eta}}^{(1)}, \mathbb{Q}_p))^{slope[0,1)} \\
\uparrow & & & & \uparrow \\
H^i(X_s, WO_X)_{\mathbb{Q}} & & & & D_{crys}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p))^{slope[0,1)}
\end{array}$$

By the diagram and Prop. 3.4.1, we have the following:

If X is regular, then $H^i(X_s, WO_X)_{\mathbb{Q}} \simeq D_{crys}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p))^{slope[0,1)}$

Now, if we restrict $sp' : H^i_{rig}(X_s/k) \to D_{crys}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p))$ to the slope $[0,1)$ part, we then get a commutative diagram:

$$\begin{array}{ccc}
H^i(X_{\bar{s}}, \mathbb{Q}_p) & \xrightarrow{sp} & H^i(X_{\bar{\eta}}, \mathbb{Q}_p)^I \\
\lambda_s \downarrow & & \lambda_\eta \downarrow \\
H^i(X_{\bar{s}}, WO_X) & \xrightarrow{\simeq} & (D_{crys}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p) \otimes_{K_0} \hat{K_0}^{ur})^{slope[0,1)}
\end{array}$$

by which we then get the p-adic local invariant cycle theorem.

Thus it remains to verify our claim.

The first identity

$$\tau_{f_{0,n}} \circ f_{0,n} = d_f, \quad \tau_{g_{0,n}} \circ g_{0,n} = d_g$$

is always true by the property of trace map.

For the second identity, we could not deduce the identity

$$d_n \cdot \tau_{g_{0,n}} \circ h_{0,n} = d_n \cdot \frac{d_g}{d_f} f_{0,n} \circ \tau_{f_{0,n}}$$

for a bounded d_n yet, but we can still get some consequences which is sufficient to deduce the p-adic local invariant cycle theorem.

Use the short exact sequence:

$$0 \to W_{n-1}(O_{X_0}) \xrightarrow{V} W_n(O_{X_0}) \xrightarrow{R^{n-1}} O_{X_0} \to 0$$

where $V : (a_1, a_2, \cdots, a_{n-1}) \mapsto (0, a_1, a_2, \ldots, a_{n-1})$ and R^{n-1} is the projection $(a_1, a_2, \ldots, a_n) \mapsto a_1$.

We then have a long exact sequence:

$$\cdots \to H^i(X_0, W_{n-1}O_{X_0}) \to H^i(X_0, W_nO_{X_0}) \to H^i(X_0, O_{X_0}) \to H^{i+1}(X_0, W_nO_{X_0}) \to \cdots$$

So we have a commutative diagram:

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
H^i(X_0, W_{n-1}O_{X_0}) & \xrightarrow{V_{X_0}} & H^i(X_0, W_nO_{X_0}) & \xrightarrow{R^{n-1}_{X_0}} & H^i(X_0, O_{X_0}) & \xrightarrow{\delta_{X_0}} & \cdots \\
\downarrow{\scriptstyle f_{0,n-1}} & & \downarrow{\scriptstyle f_{0,n}} & & \downarrow{\scriptstyle f} & & \\
H^i({X_0}^{(1)}, W_{n-1}O_{{X_0}^{(1)}}) & \xrightarrow{V_{{X_0}^{(1)}}} & H^i({X_0}^{(1)}, W_nO_{{X_0}^{(1)}}) & \xrightarrow{R^{n-1}_{{X_0}^{(1)}}} & H^i({X_0}^{(1)}, O_{{X_0}^{(1)}}) & \xrightarrow{\delta_{{X_0}^{(1)}}} & \cdots \\
\downarrow{\scriptstyle g_{0,n-1}-h_{0,n-1}} & & \downarrow{\scriptstyle g_{0,n}-h_{0,n}} & & \downarrow{\scriptstyle g_0-h_0} & & \\
H^i({X_0}^{(2)}, W_{n-1}O_{{X_0}^{(2)}}) & \xrightarrow{V_{{X_0}^{(2)}}} & H^i({X_0}^{(2)}, W_nO_{{X_0}^{(2)}}) & \xrightarrow{R^{n-1}_{{X_0}^{(2)}}} & H^i({X_0}^{(2)}, O_{{X_0}^{(2)}}) & \xrightarrow{\delta_{{X_0}^{(2)}}} & \cdots
\end{array}$$

where each row is exact.

Now, use induction on n, we want to show that $F^{n-1} \circ (\tau_{g_{0,n}} \circ h_{0,n} - \frac{d_g}{d_f} f_{0,n} \circ \tau_{f_{0,n}}) = 0$

The base case is just the coherent cohomology case we just proved, and we assume it holds for $n-1$.

As the trace map commutes with R, F and V by [BER, prop. 7.7], we have:

$$(\tau_{g_{0,n}} \circ h_{0,n} - \frac{d_g}{d_f} f_{0,n} \circ \tau_{f_{0,n}}) \circ V = V \circ (\tau_{g_{0,n-1}} \circ h_{0,n-1} - \frac{d_g}{d_f} f_{0,n-1} \circ \tau_{f_{0,n-1}})$$

and

$$F \circ (\tau_{g_{0,n}} \circ h_{0,n} - \frac{d_g}{d_f} f_{0,n} \circ \tau_{f_{0,n}}) = (\tau_{g_{0,n-1}} \circ h_{0,n-1} - \frac{d_g}{d_f} f_{0,n-1} \circ \tau_{f_{0,n-1}}) \circ F$$

Thus

$$F^{n-1} \circ (\tau_{g_{0,n}} \circ h_{0,n} - \frac{d_g}{d_f} f_{0,n} \circ \tau_{f_{0,n}}) = F^{n-2} \circ (\tau_{g_{0,n-1}} \circ h_{0,n-1} - \frac{d_g}{d_f} f_{0,n-1} \circ \tau_{f_{0,n-1}}) \circ F = 0$$

which finishes the induction.

The consequence that $\tau_{g_{0,n}} \circ h_{0,n} - \frac{d_g}{d_f} f_{0,n}$ is annihilated by F already tells us some information.

In fact, we consider the Frobenius stable part of the cohomology as in [C2], i.e., defined as the maximal subspace $H^*(X, W_nO_X)_s$(resp. $H^*(X, WO_X)_s$) of $H^*(X, W_nO_X)$(resp. $H^*(X, WO_X)$) on which the Frobenius is a bijection.

Note that $\tau_{g_{0,n}} \circ h_{0,n} - \frac{d_g}{d_f} f_{0,n}$ maps $H^i({X_0}^{(1)}, W_nO_{{X_0}^{(1)}})_s$ to $H^i({X_0}^{(1)}, W_nO_{{X_0}^{(1)}})_s$.

Thus $F^{n-1} \circ (\tau_{g_{0,n}} \circ h_{0,n} - \frac{d_g}{d_f} f_{0,n}) = 0$ on $H^i({X_0}^{(1)}, W_nO_{{X_0}^{(1)}})_s$ means that $\tau_{g_{0,n}} \circ h_{0,n} - \frac{d_g}{d_f} f_{0,n} = 0$ on $H^i({X_0}^{(1)}, W_nO_{{X_0}^{(1)}})_s$.

Taking inverse limit on the sequence

$$H^i(X_0, W_nO_{X_0})_s \to H^i(X_0^{(1)}, W_nO_{X_0^{(1)}})_s \rightrightarrows H^i(X_0^{(2)}, W_nO_{X_0^{(2)}})$$

and then tensoring with $\mathbb{Q}$, we then get the sequence

$$0 \to H^i(X_0, WO_{X_0})_{\mathbb{Q},s} \to H^i(X_0^{(1)}, WO_{X_0^{(1)}})_{\mathbb{Q},s} \rightrightarrows H^i(X_0^{(2)}, WO_{X_0^{(2)}})_{\mathbb{Q},s}$$

is exact.

Now, by [C2, Prop. 1.5.2], the Frobenius stable part captures the slope 0 part of the rigid cohomology and the Hyodo-Kato cohomology, combining it with chapter 3's results, we then get a commutative diagram with the two columns are exact:

$$\begin{array}{ccccc}
H^i(X_s^{(2)}, WO_{X^{(2)}})_{\mathbb{Q},s} & \xrightarrow{\simeq} & H^i_{HK}(X_s^{(2)}/k)_{\mathbb{Q}}^{N=0,slope0} & \xrightarrow{\simeq} & D_{crys}(H^i(X_{\bar{\eta}}^{(2)}, \mathbb{Q}_p))^{slope0} \\
\uparrow & & & & \uparrow \\
H^i(Y_s, WO_Y)_{\mathbb{Q},s} & & & & D_{crys}(H^i(Y_{\bar{\eta}}, \mathbb{Q}_p))^{slope0} \\
\upuparrows & & & & \upuparrows \\
H^i(X_s^{(1)}, WO_{X^{(1)}})_{\mathbb{Q},s} & \xrightarrow{\simeq} & H^i_{HK}(X_s^{(1)}/k)_{\mathbb{Q}}^{N=0,slope0} & \xrightarrow{\simeq} & D_{crys}(H^i(X_{\bar{\eta}}^{(1)}, \mathbb{Q}_p))^{slope0} \\
\uparrow & & & & \uparrow \\
H^i(X_s, WO_X)_{\mathbb{Q},s} & & & & D_{crys}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p))^{slope0}
\end{array}$$

which also certifies that

$$sp : H^i(X_{\bar{s}}, \mathbb{Q}_p) \to H^i(X_{\bar{\eta}}, \mathbb{Q}_p)^I$$

is an isomorphism.

So we have the following:

Theorem 4.3.1 *(p-adic local invariant cycle theorem) Let X/S be regular, proper, flat and generically smooth, we have:*

$$sp : H^i(X_{\bar{s}}, WO_X)_{\mathbb{Q},s} \simeq (D_{crys}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p) \otimes_{K_0} \hat{K_0}^{ur})^{slope0}$$

induced from the p-adic specialization map is an isomorphism, and we have a commutative diagram:

$$\begin{array}{ccc} H^i(X_{\bar{s}}, \mathbb{Q}_p) & \xrightarrow{sp} & H^i(X_{\bar{\eta}}, \mathbb{Q}_p)^I \\ \lambda_s \downarrow & & \lambda_\eta \downarrow \\ H^i(X_{\bar{s}}, WO_X)_{\mathbb{Q},s} & \xrightarrow{sp} & (D_{crys}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p) \otimes_{K_0} \hat{K_0}^{ur})^{slope0} \end{array}$$

where λ_s and λ_η are isomorphisms as above, and the subscript s denotes the Frobenius stable part as in [C2].
In particular, $sp : H^i(X_{\bar{s}}, \mathbb{Q}_p) \to H^i(X_{\bar{\eta}}, K_0)^I$ is an isomorphism.

Chapter 5

Another Proof of $\mathbf{H^0}$ And $\mathbf{H^1}$ Cases

In this chapter, we give an alternative proof of H^0 and H^1 cases.

• $\mathbf{H^0}$ Case

Proposition 5.0.2 *Let X be a regular scheme, then $sp : H^0(X_{\bar{s}}, \mathbb{Q}_p) \simeq H^0(X_{\bar{\eta}}, \mathbb{Q}_p)^I$ is an isomorphism.*

Proof:

For H^0, the p-adic case and the l-adic case are the same since $H^0(X, \mathbb{Q}_l) = \mathbb{Q}_l^{\pi_0(X)}$ and $sp : H^0(X_{\bar{s}}, \mathbb{Q}_l) \to H^0(X_{\bar{\eta}}, \mathbb{Q}_l)^I$ is induced from $\pi_0(X_{\bar{s}}) \to \pi_0(X_{\bar{\eta}})$. □

A geometric point of view of the above consequence follows from Zariski's theorem on formal functions that the special fiber of $f : X \to S$ has the same number of connected components as the generic fiber.

• $\mathbf{H^1}$ Case

For H^1, we use Grothendieck's fundamental group.

By [SGA1, X. Cor. 2.4], we know that if S is a locally noetherian scheme, $f : X \to S$ is a proper geometrically connected morphism, then there exists a specialization homomorphism

$$sp : \pi_1(X_{\bar{\eta}}) \to \pi_1(X_{\bar{s}})$$

defined up to inner automorphisms of $\pi_1(X_{\bar{s}})$. If, furthermore, $f : X \to S$ is separable, then $sp : \pi_1(X_{\bar{\eta}}) \to \pi_1(X_{\bar{s}})$ is an epimorphism.

Note that since $H^1(X, \mathbb{Q}_p) = Hom(\pi_1(X), \mathbb{Q}_p)$, so $sp : H^1(X_{\bar{s}}, \mathbb{Q}_p) \to H^1(X_{\bar{\eta}}, \mathbb{Q}_p)^I$ is uniquely defined.

Thus we have an monomorphism:

$$H^1(X_{\bar{s}}, \mathbb{Q}_p) = Hom_{\mathbb{Z}_p}(\pi_1(X_{\bar{s}}), \mathbb{Q}_p) \hookrightarrow Hom_{\mathbb{Z}_p}(\pi_1(X_{\bar{\eta}}), \mathbb{Q}_p) = H^1(X_{\bar{\eta}}, \mathbb{Q}_p)$$

Consider the following diagram:

$$\begin{array}{ccccc} & & & & Y_\eta \\ & & & & \downarrow \\ X_{\bar{s}} & \longrightarrow & \bar{X} & \longleftarrow & X_{\bar{\eta}} \end{array}$$

By the Zariski-Nagata theorem on purity of the branch locus ([SGA1, X.3.1],[SGA2, X.3.4]), a finite etale covering $Y_{\bar{\eta}} \to X_{\bar{\eta}}$ extends to $Y \to X$ if it extends over generic points of the special fiber.

We have

$$\begin{array}{ccccc} Gal(k(\bar{X}_\eta)/k(X_{\bar{\eta}})) & \longrightarrow & \pi_1(X_{\bar{\eta}}) & & \\ \cap\big\downarrow {\scriptstyle I=Gal(\bar{\eta}/\eta)} & & \cap\big\downarrow {\scriptstyle I=Gal(\bar{\eta}/\eta)} & & \\ Gal(k(\bar{X}_\eta)/k(X_\eta)) & \longrightarrow & \pi_1(X_\eta) & \xrightarrow{\phi} & \mathbb{Q}_p \\ & & \big\downarrow & & \\ & & \pi_1(X) & & \end{array}$$

Note that, by purity, ϕ factors through $\pi : \pi_1(X_\eta) \to \pi_1(X)$ iff $\phi(I_{\bar{v}}) = 0$, for any discrete valuation $\bar{v}|v$ which corresponds to the geometric points of X_s.

Now we prove the following:

Proposition 5.0.3 *Let $X \to S$ be proper, flat, and generically smooth. If X is regular, then the specialization map:*

$$sp : H^1(X_{\bar{s}}, \mathbb{Q}_p) \to H^1(X_{\bar{\eta}}, \mathbb{Q}_p)^I$$

is an isomorphism.

The idea is using lifting: for any $\phi \in Hom_{\mathbb{Z}_p}(\pi_1(X_\eta), \mathbb{Q}_p)$, s.t. $\phi|_{\pi_1(X_{\bar{\eta}})} \neq 0$, there exists a $\psi \in Hom_{\mathbb{Z}_p}(I, \mathbb{Q}_p)$, s.t. $(\phi + \psi)(I_{\bar{v}}) = 0$, for any discrete valuation $\bar{v}|v$. Then by purity, $\phi + \psi$ factors through $\pi_1(X) = \pi_1(X_s)$, thus

$$H^1(X_{\bar{s}}, \mathbb{Q}_p) = Hom_{\mathbb{Z}_p}(\pi_1(X_{\bar{s}}), \mathbb{Q}_p) \twoheadrightarrow Hom_{\mathbb{Z}_p}(\pi_1(X_{\bar{\eta}})_I, \mathbb{Q}_p) = H^1(X_{\bar{\eta}}, \mathbb{Q}_p)^I$$

Proof:

We have the short exact sequence ([SGA1, X. Cor.2.2])

$$0 \to \pi_1(X_{\bar{\eta}}) \to \pi_1(X_\eta) \to I \to 0$$

Then the Hochschild-Serre spectral sequence will give us:

$$0 \to Hom(I, \mathbb{Q}_p) \to Hom(\pi_1(X_\eta), \mathbb{Q}_p) \to Hom(\pi_1(X_{\bar{\eta}}), \mathbb{Q}_p)^I \to 0$$

Consider an alteration $X^{ss} \to X$ as in [D-J], where X^{ss} then has semistable reduction.

Denote X, X^{ss}'s quotient fields by K, L respectively.

We can get a similar exact sequence from X^{ss}, and we deduce a diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Hom(I_L, \mathbb{Q}_p)^G & \longrightarrow & Hom(\pi_1(X_L), \mathbb{Q}_p)^G & \longrightarrow & Hom(\pi_1(X_{\bar{L}}), \mathbb{Q}_p)^{I_L, G} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Hom(I_K, \mathbb{Q}_p) & \longrightarrow & Hom(\pi_1(X_K), \mathbb{Q}_p) & \longrightarrow & Hom(\pi_1(X_{\bar{K}}), \mathbb{Q}_p)^{I_K} & \longrightarrow & 0 \end{array}$$

where all the vertical maps are isomorphisms, and $G = Gal(L/K) = I_K/I_L$.

Now by purity, $Hom(\pi_1(X), \mathbb{Q}_p) =$ {homomorphisms $\phi : \pi_1(X_K) \to \mathbb{Q}_p$ which is trivial on I_v, for all v extending $|\cdot|_p$ on K}.

If we use the isomorphism in the above diagram, we can consider $Hom(\pi_1(X^{ss}), \mathbb{Q}_p) =$ {G-invariant homomorphisms $\phi : \pi_1(X_L) \to \mathbb{Q}_p$ which is trivial on I_v, for all v extending $|\cdot|_p$ on K}.

Note that for the semistable reduction case, we have *sp* is an isomorphism, so

$$Hom(\pi_1(X^{ss}), \mathbb{Q}_p) \cong Hom(\pi_1(X_{\bar{L}}), \mathbb{Q}_p)^{I_L}$$

We then have

$$Hom(\pi_1(X), \mathbb{Q}_p) \cong Hom(\pi_1(X^{ss}), \mathbb{Q}_p)^G \cong Hom(\pi_1(X_{\bar{L}}), \mathbb{Q}_p)^{I_L, G} \cong Hom(\pi_1(X_{\bar{K}}), \mathbb{Q}_p)^{I_K}$$

Thus *sp* is an isomorphism. □

Thus, by using Grothendieck's fundamental group and purity, we get another proof of H^0 and H^1 cases.

Remark 5.0.1 *Besides the cohomology descent method which proved the general p-adic local invariant cycle theorem in the last chapter, and the above proof via using Grothendieck's fundamental group and purity, for the case of H^1, we also considered another method via investigating the vanishing cycle through Bloch and Kato's symbols ([BK1]).*

Consider

$$\mathbb{Z}/p^r\mathbb{Z}(1) \to \mathbb{G}_m \xrightarrow{p^r} \mathbb{G}_m$$

in $D_{sh}(\mathfrak{X}_{et})$.

We have the following diagrams:

$$\begin{array}{ccccc}
i_*Ri^!\mathbb{Z}/p^r\mathbb{Z}(1) & \longrightarrow & i_*Ri^!\mathbb{G}_m & \xrightarrow[p^r]{} & i_*Ri^!\mathbb{G}_m \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{Z}/p^r\mathbb{Z}(1) & \longrightarrow & \mathbb{G}_m & \xrightarrow{p^r} & \mathbb{G}_m \\
\downarrow & & \downarrow & & \downarrow \\
Rj_*j^*\mathbb{Z}/p^r\mathbb{Z}(1) & \longrightarrow & Rj_*j^*\mathbb{G}_m & \xrightarrow[p^r]{} & Rj_*j^*\mathbb{G}_m
\end{array}$$

$$\begin{array}{cccccccccc}
0 & & 0 & & 0 & & & & & \\
\downarrow & & \downarrow & & \downarrow & & & & & \\
X^0(\mathbb{Z}/p^r\mathbb{Z}(1)) & \longrightarrow & \mathbb{G}_m & \xrightarrow{p^r} & \mathbb{G}_m & \longrightarrow & X^1(\mathbb{Z}/p^r\mathbb{Z}(1)) & \longrightarrow & 0 & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
j_*\mu_{p^r} & \longrightarrow & j_*\mathbb{G}_m & \xrightarrow{p^r} & j_*\mathbb{G}_m & \longrightarrow & R^1j_*\mu_{p^r} & \longrightarrow & R^1j_*\mathbb{G}_m & \xrightarrow{p^r} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
i_*R^1i^!\mathbb{Z}/p^r\mathbb{Z}(1) & \longrightarrow & i_*R^1i^!\mathbb{G}_m & \xrightarrow[p^r]{} & i_*R^1i^!\mathbb{G}_m & \longrightarrow & i_*R^2i^!\mathbb{Z}/p^r\mathbb{Z}(1) & \longrightarrow & i_*R^2i^!\mathbb{G}_m &
\end{array}$$

Use the same notations as above and set $j : X_\eta \to X$*, from [BK1], we know that the stalk of* $R^kj_*\mathbb{G}_m$ *at* $x \in X_s \to X$ *is* $H^k(Spec\mathcal{O}^{sh}_{X,x}[\frac{1}{\pi}], \mathbb{G}_m)$.

We have exact sequences

$$0 \to H^1(X_s, R^0\Psi\mathbb{Q}_p) \to H^1(X_{\bar{\eta}}, \mathbb{Q}_p)^I \to H^0(X_s, R^1\Psi\mathbb{Q}_p)^I$$

and

$$H^0(X_s, R^1\Psi\mathbb{Q}_p)^I = H^0(X_s, R^1\Psi\mathbb{Q}_p(1))^{I=\chi} \hookrightarrow \prod_{x\in X_s}(R^1\Psi\mathbb{Q}_p(1))_x$$

Then by [BK1], we have a surjection:

$$\prod_{x\in X_s}\varprojlim(\mathcal{O}^{sh}_{\bar{X},x}[\frac{1}{p}]^*/p^n)^{I=\chi} \twoheadrightarrow (R^1\Psi\mathbb{Q}_p(1))_x$$

One then needs to show that latter has no cyclotomic character.

By this we may possibly give another proof of our H^1 *case via vanishing cycle, and may also extend to the higher cohomology.*

Chapter 6

Weight Filtration

If X is a smooth projective variety over $\mathbb{Q}$ and $0 \leq i \leq 2dimX$, then the representations of the Galois group $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ on the etale cohomology groups $H^i_{et}(X \times Spec(\overline{\mathbb{Q}}), \mathbb{Q}_l)$ form, as the prime l varies, a compatible system of Galois representations. Hence we can attach an L-function $L(s, H^i(X))$ to it. There are many conjectures, beginning with the meromorphic continuation and functional equations, and some deep theories relating the analytic properties of $L(s, H^i(X))$ with geometric properties of X.

In [L2], Lichtenbaum suggested the existence of Weil-etale cohomology groups for arithmetic schemes $\mathcal{X}$(i.e., separated schemes of finite type over $Spec(\mathbb{Z})$) relating to the zeta function $\zeta(\mathcal{X}, s)$ of $\mathcal{X}$ satisfying some given properties.

If $\mathcal{X}$ has finite characteristic, these cohomology groups are well defined and understood by work of Lichtenbaum and Geisser ([L1],[G1],[G2]).

Lichtenbaum defined Weil-etale cohomology groups for $X = Spec\mathcal{O}_F$, the spectrum of the ring of integers in a number field, in [FlM] Flach and Morin partially extended this work to a regular, flat, proper scheme over $Spec\mathbb{Z}$.

In this extended definition, the expected property for $\zeta(\mathcal{X}, s)$ on its leading Taylor coefficient is compatible with the Tamagawa Number Conjecture of Bloch and Kato [BK2], also Fontaine and Perrin-Riou [FP] for $\Pi_{i\in\mathbb{Z}} L(h^i(\mathcal{X}_\mathbb{Q}), s)^{(-1)^i}$ at $s = 0$. And for this we need to assume a number of conjectures which are preliminary to the formulation of the Tamagawa number conjecture, thus we are led to the so called local invariant cycle theorem, both in l-adic and p-adic cohomology, which serves to establish the equality of vanishing orders

$$ord_{s=0}\zeta(\mathcal{X}, s) = ord_{s=0}\Pi_{i\in\mathbb{Z}} L(h^i(\mathcal{X}_\mathbb{Q}), s)^{(-1)^i}$$

for regular scheme $\mathcal{X}$ proper and flat over $Spec(\mathbb{Z})$.

Now, considering the rigid cohomology, also note that the eigenvalues of ϕ on $H^i_{rig}(X_s/k)$ are Weil numbers, thus with a similar argument as in the $l-adic$ case in [FlM, section 10], we then get the same is true for $D_{pst}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p))$, hence we deduce the weight filtration on both $H^i_{rig}(X_s/k)$ and

$$D_{crys}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p)) = D_{st}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p))^{N=0} = D_{pst}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p))^{I,N=0}$$

As pointed out in [FlM, (9.2)], assuming that (both the p-adic and l-adic cases) the map

$$W_0 H^i(X_{\bar{s}}, \mathbb{Q}_l) \to W_0 H^i(X_{\bar{\eta}}, \mathbb{Q}_l)^I$$

induced by sp is an isomorphism for all i, one can prove that: for all primes p and l,

$$R\Gamma(X \otimes \mathbb{F}_{p,et}, \mathbb{Q}_l) \cong \bigoplus_{i=0}^{2d} R\Gamma_f(\mathbb{Q}_p, V_l^i)[-i]$$

here X is a regular scheme,proper and flat over $Spec\mathbb{Z}$ and $V_l^i = H^i(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_l)$, and this serves as a part to prove the compatibility of the Weil-etale cohomology with the Tamagawa number conjecture(See [FlM] section 9 for details).

6.1 l-adic Result

In [FlM], M. Flach and B. Morin prove the following results related to the l-adic local invariant cycle theorem in the mixed characteristic case:

Theorem 6.1.1 *([FlM], (Theorem 10.1)) If X is regular, then the following hold.*

a. The map

$$H^i(X_{\bar{s}}, \mathbb{Q}_l) = W_i H^i(X_{\bar{s}}, \mathbb{Q}_l) \to W_i H^i(X_{\bar{\eta}}, \mathbb{Q}_l)^I$$

induced by sp is surjective for all i where the isomorphism and the weight filtration existence are due to Deligne in [D2].

b. The map

$$W_1 H^i(X_{\bar{s}}, \mathbb{Q}_l) \to W_1 H^i(X_{\bar{\eta}}, \mathbb{Q}_l)^I$$

induced by sp is an isomorphism for all i(For $i > d$ it will be the zero map).

c. The map sp is an isomorphism for i = 0,1.

d. If $W_i H^i(X_{\bar{\eta}}, \mathbb{Q}_l)^I = H^i(X_{\bar{\eta}}, \mathbb{Q}_l)^I$ for all i, then the map

$$W_{i-1} H^i(X_{\bar{s}}, \mathbb{Q}_l) \to W_{i-1} H^i(X_{\bar{\eta}}, \mathbb{Q}_l)^I$$

induced by sp is an isomorphism for all i.

6.2 The Analogous p-adic Result

We expect to extend these results to the p-adic case. Note that the regular assumption of X is necessary (both in p-adic and l-adic cases), see [FlM] for a counterexample otherwise.

In fact, for the application in proving the compatibility of the Weil-etale cohomology with the Tamagawa number conjecture, we only need the isomorphism on W_0 (in fact on the smaller generalized eigenspace for the eigenvalue 1). More explicitly, We need to prove the following:

If X is regular, then the map

$$W_0 H^i(X_{\bar{s}}, \mathbb{Q}_p) \stackrel{sp}{\to} W_0 H^i(X_{\bar{\eta}}, \mathbb{Q}_p)^I$$

induced by sp is an isomorphism, where W_0 *is the sum of generalized* $\phi-$ *eigenspaces for eigenvalues which are roots of unity.*

Obviously, this is a direct corollary of the p-adic local invariant cycle theorem by the fact that W_0 is an exact functor. Thus we also get the result in the p-adic case, and thus combining with the l-adic case, we have: for all primes p and l,

$$R\Gamma(X \otimes \mathbb{F}_{p,et}, \mathbb{Q}_l) \cong \bigoplus_{i=0}^{2d} R\Gamma_f(\mathbb{Q}_p, V_l^i)[-i]$$

In the following, we consider several typical examples such that the W_0-part of the specialization map is an isomorphism be verified via direct computations.

In fact, note that the maps λ_s and λ_η are injective as we show above, and since the W_0-part of the rigid cohomology is contained in its slope 0-part, thus from the diagram we constructed for the p-adic local invariant cycle theorem, it suffices to prove that the W_0-part of the p-adic specialization map is an isomorphism:

$$W_0 H^i_{rig}(X_s/k) \simeq W_0 D_{crys}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p))$$

• Blow Up Case

Consider the case that $p : X^{(1)} \to X$ is a blowing up with regular center.

Since $H^i(X^{(1)}_{\bar{s}}, \mathbb{Q}_p) = H^i(X_{\bar{s}}, Rp_*\mathbb{Q}_p)$, by computing the stalks, $H^i(X_{\bar{s}}, R^j p_*\mathbb{Q}_p)$ has weight in $[j, j+i]$ and $H^i(X_{\bar{s}}, Rp_*\mathbb{Q}_p)$ is pure of weight i, so only R^0 contributes to $H^i(X_{\bar{s}}, R^j p_*\mathbb{Q}_p) \Rightarrow H^i(X_{\bar{s}}, Rp_*\mathbb{Q}_p)$, and we have:

$$H^i(X^{(1)}_{\bar{s}}, \mathbb{Q}_p) = H^i(X_{\bar{s}}, Rp_*\mathbb{Q}_p) = H^i(X_{\bar{s}}, R^0 p_*\mathbb{Q}_p) = H^i(X_{\bar{s}}, p_*\mathbb{Q}_p)$$

Now $W_0 H^i(X_{\bar{s}}, \mathbb{Q}_p) = W_0 H^i(X^{(1)}_s, \mathbb{Q}_p) = W_0 D_{st}(H^i(X^{(1)}_{\bar{\eta}}, \mathbb{Q}_p))^{N=0}$.

Thus in this case, $W_0H^i(X_{\bar{s}}, \mathbb{Q}_p)$ is the equalizer of $W_0H^i(X_s^{(1)}/k) \rightrightarrows W_0H^i(X_s^{(2)}/k)$ (by composition of morphisms), hence we have

$$W_0D_{st}(H^i(X_{\bar{\eta}}^{(1)}, \mathbb{Q}_p))^{N=0} = W_0D_{st}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p))^{N=0}$$

and

$$W_0H^i(X_{\bar{s}}, \mathbb{Q}_p) = W_0D_{st}(H^i(X_{\bar{\eta}}, \mathbb{Q}_p))^{N=0} = W_0H^i(X_{\bar{\eta}}, \mathbb{Q}_p)^I$$

• Regular Scheme with Ordinary Double Points

Now we consider the case as in [I3] and [M4].

Let X be a relative n-dimensional proper generically smooth scheme over R, assume X is regular, connected and has at most ordinary double points (which is defined below). Assume that all the ordinary double points are k-rational and denote the set of them by Σ. Let K' be a totally ramified quadratic extension of K.

In [I3] and [M4], Illusie and Mieda constructed the specialization map:

$$sp^* : H^i_{rig}(X_s/k)' \to H^i_{dR}(X_\eta/K)'$$

(here $H^i(*)'$ denotes $H^i(*) \otimes_K K'$) and proved that it is an isomorphism when $i \neq n, n+1$ and for each $\sigma \in \Sigma$, there exist a one-dimensional K'-module $\Phi_\sigma(X/K')$ and an exact sequence ([M4, Theorem 1.1]) :

$$0 \to H^n_{rig}(X_s/k)' \xrightarrow{sp^*} H^n_{dR}(X_\eta/K)' \to \bigoplus_{\sigma\in\Sigma} \Phi_\sigma(X/K') \to H^{n+1}_{rig}(X_s/k)' \xrightarrow{sp^*} H^{n+1}_{dR}(X_\eta/K)' \to 0$$

In this example, we could show that the W_0-part of the specialization map is an isomorphism.

Now let A be a ring, a quadratic form $Q \in A[X_1, \ldots, X_{n+1}]$ over A is said to be ordinary if for any maximal ideal $\mathfrak{m}$ of A, the quadratic form $\bar{Q}_{\mathfrak{m}} = Q \otimes_A A/\mathfrak{m}$ is nonzero, and the closed subscheme of $\mathbb{P}^n_A$ defined by Q is smooth over $SpecA$.

A point $\sigma \in \Sigma$ is called an ordinary double point if there exists an open subscheme U of X containing σ, an ordinary quadratic form Q_σ over R and an etale morphism:

$$f : U \to SpecR[X_1, \ldots, X_{n+1}]/(Q_\sigma - \pi)$$

As in [M4], by taking a base change of R to R', the ring of integers of K', we obtain a scheme with isolated singularities, each of which is defined by a homogenous quadratic polynomial.

Thus let X be a relative n-dimensional proper generically smooth scheme over R which is regular connected, Σ is a finite set, and etale locally around any point of Σ, the scheme X is defined by a homogenous quadratic polynomial, i.e., for any $\sigma \in \Sigma$, there exists an open subscheme U of X containing σ, an ordinary quadratic form Q_σ over R, a unit $u_\sigma \in R^*$ and an etale mrophism

$$f : U \to SpecR[X_1, \ldots, X_{n+1}]/(Q_\sigma - \pi)$$

As in [I3, 3.2], Let $\pi = u_\sigma \pi'^2$ and A' be the normalization of $R[X_1, \cdots, X_{n+1}]/(Q_\sigma - u_\sigma \pi'^2)$ in K', $X' = X \otimes_A A'$, and take the blowing up $\tilde{X} \to X'$ of X' at Σ, set D_σ be the exceptional divisor at σ. Then the blowing up $\tilde{X}_s$ of X_s at Σ is the strict transform of X_s in $\tilde{X}$ and the exceptional divisor C_σ at σ is equal to the intersection $\tilde{X}_s \cap D_\sigma$.

Due to [I3, Prop. 2.4], $\tilde{X}$ is strictly semistable over $SpecA'$ and its special fiber $\tilde{X}_s$ is equal to $\tilde{X}_s + \Sigma_{\sigma\in\Sigma} D_\sigma$ as an divisor on $\tilde{X}$, D_σ is isomorphic to the closed subscheme of $\mathbb{P}_k^{n+1} = Projk[X_1, \ldots, X_{n+1}, T]$ defined by the polynomial $\overline{Q_\sigma - u_\sigma T^2}$ and C_σ is isomorphic to its hyperplane section $T = 0$.

Follow we stated in chapter 2, given the Hyodo-Steenbrink bicomplex

$$W_n A^{ij} = \frac{W_n \tilde{\omega}_Y^{i+j+1}}{P_j W_n \tilde{\omega}_Y^{i+j+1}}$$

we can define the weight filtration on it as

$$P_k W_n A^{ij} = \frac{P_{2j+k+1} W_n \tilde{\omega}_Y^{i+j+1}}{P_j W_n \tilde{\omega}_Y^{i+j+1}}$$

.

Now, in our case, we have $D^{(1)} = \tilde{X}_s \sqcup \bigsqcup_{\sigma\in\Sigma} D_\sigma, D^{(2)} = \bigsqcup_{\sigma\in\Sigma} C_\sigma$, and $D^{(i)} = \emptyset$ for $i > 2$.

Let U be the open subscheme $X_s - \Sigma$ of X_s, we have $H^i_{rig}(X_s/k) \simeq H^i_{rig,c}(U/k)$ for $i > 1$, also, by comparison theorem in [HK], since $U = \tilde{X}_s - \cup_{\sigma\in\Sigma} C_[\sigma]$, we have the following isomorphisms:

$$H^i_{rig,c}(U/k) \simeq H^i(\tilde{X}_s, W\Omega^\bullet_{\tilde{X}_s}(-log \sum_{\sigma\in\Sigma} C_\sigma)) \otimes_W K$$

$$H^i(\tilde{X}_s, WA^\bullet) \otimes K \simeq H^i_{dR}(X_\eta/K)$$

$$H^i(\tilde{X}_s, W\Omega^\bullet_{D_\sigma}(logC_\sigma)) \otimes K \simeq H^i_{rig}((D_\sigma - C_\sigma)/k) = \Phi^i_\sigma(X/k)$$

And the long exact sequence ([M4, Theorem 2.13]) :

$$0 \to H^1_{rig}(X_s/k) \to H^1_{dR}(X_\eta/K) \to \bigoplus_{\sigma\in\Sigma} \Phi^1_\sigma(X/k) \to H^2_{rig}(X_s/k) \to H^2_{dR}(X_\eta/K) \cdots$$

Now, we want to show that $W_o \Phi^i_\sigma(X/k) = 0$ for all $i \in \mathbb{N}$ and $\sigma \in \Sigma$.

We have the following diagram of Mokrane ([M3], Prop. 4.11):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Gr_0 W_n\tilde{\omega}_Y^\bullet & \longrightarrow & P_1 W_n\tilde{\omega}_Y^\bullet / P_{-1} W_n\tilde{\omega}_Y^\bullet & \longrightarrow & Gr_1 W_n\tilde{\omega}_Y^\bullet & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle Res_{D_\sigma}} & & \downarrow{\scriptstyle Res_{D_\sigma}^{C_\sigma}} & & \downarrow{\scriptstyle Res_{C_\sigma}} & & \\ 0 & \longrightarrow & W_n\Omega_{D_\sigma}^\bullet & \longrightarrow & W_n\Omega_{D_\sigma}^\bullet(logC_\sigma) & \longrightarrow & W_n\Omega_{D_\sigma}^\bullet[-1] & \longrightarrow & 0 \end{array}$$

Note that we can describe the weight filtration on $W_n\tilde{\omega}^\bullet$ as follows:

$$0 = P_0 \subset P_1 \subset P_2 = W_n\tilde{\omega}^\bullet$$

$$P_2/P_1 \overset{Res}{\simeq} \bigoplus_{\sigma\in\Sigma} W_n\Omega_{C_\sigma}^\bullet[-2]$$

$$P_1/P_0 \overset{Res}{\simeq} W_n\Omega_{\tilde{X}_s}^\bullet[-1] \oplus \bigoplus_{\sigma\in\Sigma} W_n\Omega_{D_\sigma}^\bullet[-1]$$

where $Res : Gr_j W_n\Omega_Z^\bullet(logD) \simeq W_n\Omega_{D(j)}^\bullet[-j]$ is the residue morphism defined by Mokrane ([M3, prop 1.4.5])

So we have:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & Gr_1 W_n\tilde{\omega}_Y^\bullet & \longrightarrow & Gr_1 W_n\tilde{\omega}_Y^\bullet & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle Res_{D_\sigma}} & & \downarrow{\scriptstyle Res_{D_\sigma}^{C_\sigma}} & & \downarrow{\scriptstyle Res_{C_\sigma}} & & \\ 0 & \longrightarrow & W_n\Omega_{D_\sigma}^\bullet & \longrightarrow & W_n\Omega_{D_\sigma}^\bullet(logC_\sigma) & \longrightarrow & W_n\Omega_{D_\sigma}^\bullet[-1] & \longrightarrow & 0 \end{array}$$

By [M3, prop. 3.22], we have the grading of $WA^\bullet$:

$$Gr_k(WA^\bullet) = \bigoplus_{j\geq 0, j\geq -k} Gr_{2j+k+1} W\tilde{\omega}_Y^\bullet[1](j+1)$$

Using the identification given by the residue map and combining the weight filtration on $W_n\tilde{\omega}$, we have

$$Gr_0 WA^\bullet = Gr_1 W_n\tilde{\omega}_Y^\bullet[1](1)$$

From [M3, 3.23], we know that the E_1 term of the weight spectral sequence on $WA^\bullet$ is given by:

$$E_1^{-k,i+k} = \bigoplus_{j\geq 0, j\geq -k} H_{crys}^{i-2j-k}(\tilde{X}^{(2j+k+1)}/W)(-j-k) \Rightarrow H^i(\tilde{X}^\times/W^\times)$$

where these $\tilde{X}^{(i)}$ are irreducible smooth components defined above and $(-j-k)$ is the Tate twist for the Frobenius, then the action of Frobenius acts on the twist $H_{crys}^{i-2j-k}(\tilde{X}^{(2j+k+1)}/W)(-j-k)$ is p^{j+k} times the Frobenius action on $H_{crys}^{i-2j-k}(\tilde{X}^{(2j+k+1)}/W)$. By [M3, 3.22], all of the terms in the direct sum of $E_1^{-k,i+k}$ are crystals of weight $i+k$ up to torsion.

It follows that $W_o\Phi_\sigma^i(X/k) = 0$.

Also, note that this does not imply $W_0 H_{rig}^i(X_s/k) \simeq W_0 H_{dR}^i(X_\eta/K)$ since the latter is not defined, even if after applying $\otimes B_{dR}$ and take the isomorphism $H_{dR}^*(X_\eta/K)\otimes B_{dR} \simeq H_{et}^*(X_{\bar{\eta}})\otimes B_{dR}$.

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