QUANTUM THEORY OF COUPLED SYSTEMS

HAVING APPLICATION TO MASERS

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Abstract

Non-relativistic quantum mechanics is treated using Feynman's space-time approach and Wigner's probability density of position and momentum. The mechanics of a system is described by the probability of each path of the coordinates in space-time. The influence of one system on another is described by the probability of a force function.

It is shown that linear systems follow only classical paths. The quantum theory merely puts uncertainty into their initial conditions, just like thermal noise. The formalism apparently does not simplify intrinsically non-linear problems. However, linear systems are sometimes coupled to non-linear quantum systems which operate only on a linear portion of their characteristics (like a vacuum tube in a class A amplifier). Then the reaction on the linear systems can be reduced to an equivalent impedance and noise power spectrum, so that the linear problem can be solved by classical methods.

Solid state and beam type masers are treated as examples. Masers employ excited non-linear systems to achieve linear amplification of the electromagnetic signal. The modes of the waveguides and cavities are the linear systems. To them the amplifying systems appear as negative resistance, reactance, and noise.

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I. INTRODUCTION

We shall treat non-relativistic quantum mechanics using Feynman's space-time approach, reference (1), and Wigner's joint probability distribution of position and momentum, reference (2). The formalism is quite different from the conventional one, an important difference being that the concepts in the new formalism are more closely akin to those of classical mechanics. In specific problems we shall take advantage of this resemblance in several ways, the most important of which is the near classical behavior of linear systems. Two of the classical concepts which we shall incorporate are the direct use of probability densities instead of squaring amplitudes, and the concept of the path of a particle. Our use of the word "path" will always include time as a dimension. For instance, a particle constrained to move in a line has but one path in the usual sense of the word, but many paths in our sense, each represented by a graph of position versus time. Of course a quantum system will not have a single definite path, but rather a probability density of paths in accordance with the uncertainty principle.

With the concept of the path of a particle comes the classical machinery for studying those paths. Such quantities as resistance, impedance, driving force, and power spectrum of noise will be used. We shall find for example,

that when certain non-linear systems are weakly coupled to any linear system, the complete effect on the linear system is to introduce a positive or negative resistance plus a distribution of random forces due to the quantum uncertainties in the non-linear systems.

Another advantage of the resemblance to classical mechanics is the ease with which we can understand a problem involving classical limits. For example, an antenna may be radiating a strong signal which is received at a great distance, where it causes only a few quanta of excitation in the first wave guide of the receiver. After several stages of amplification the signal is strong again. One might ask, what noise proportional to h appears in the output as a result of the weak signal in the receiver input? This problem is classical at both ends and quantum in the middle.

The material is presented in the following order: First, the concept of probability of a path is discussed, putting classical mechanics in the notation which the quantum theory will resemble. Second, the quantum theory is derived in general. Special attention is given linear systems since their simplicity is the major reason for the formalism. Third, we digress to examine the classical limit, $h \to 0$, in the new formalism. Fourth, we return to the main discussion and, under the heading "forcing functionals of

various systems", we find the forces which certain linear and non-linear systems exert on any system to which they are coupled. The systems chosen are components in the maser example to follow; a classical driving force, classical noise, a harmonic oscillator, a harmonic oscillator at temperature θ, a resistor, a general non-linear system in the approximation of short interaction time, a molecular beam, and a lattice of spins, or three-level systems, being "pumped" between upper and lower states. The resistor is the first example of a formula (equation 51) for noise that is important for all that follows. Therefore, we digress in the resistor section to relate our formalism to conventional noise theory. Finally, we discuss the maser amplifier as a whole with particular attention to noise of quantum mechanical origin. The original maser articles are references (3), (4), and (5).

The reader, who is primarily interested in qualitative conclusions concerning maser amplifiers, may wish to skip much of the derivation beginning with the influence functional (III D 1). He may pick up necessary notation and the form of the noise forces by reading the sections on the harmonic oscillator (V C) and the resistor (V D), skipping the proof following equation 34, and the derivation leading to equation 46. Then the qualitative conclusions are found in the section on the maser amplifier system (VI).

II. PROBABILITY OF A PATH

Since we shall take advantage of the resemblance of the new quantum formalism to classical concepts, we devote this section to a brief discussion of classical mechanics using the space-time notation in which the quantum theory will appear.

A. Single System

Classically, we usually speak of the path of a particle or system as being perfectly definite; however, the exact path may be uncertain either because the initial conditions are uncertain, or because some parameters in the equation of motion are uncertain, or both. If the equation of motion is uncertain, the best we can do is to find a functional $\mathcal{P}[x(t)]$ which gives the probability of the path $x(t)$, where x represents all the coordinates needed to describe the system. That is, x stands for a point in the n dimensional space of all the variables $x^{(1)}, x^{(2)}, \cdots x^{(n)}$, if the system has n degrees of freedom.

To define $\mathcal{P}[x]$ completely, we must define the spacing between paths and a method of summing over paths. For example, suppose we are interested in some number which depends on the path $x(t)$ (any functional of $x(t)$), say the time integrated square displacement $\int_0^T x^2 dt$. Then we can calculate the expected value of this functional by ordinary probability concepts, once the spacing of paths is defined

so that we can sum over them.

$$\left\langle \int_0^T [x(t)]^2 dt \right\rangle = \sum_{\text{paths } x(t)} \mathcal{P}[x(t)] \int_0^T x^2 dt$$

We define the spacing and the method of summing paths by the path integral method of Feynman described in reference (1). We approximate the path x(t) by the values x_k of x at the discrete times $k\epsilon$. Straight lines join the points x_k. All integrals with x(t) in the integrand become Riemann sums. The sum over all paths is performed by integrating over all x_k from $-\infty$ to ∞. (An integral over x means a volume integral in the n dimensional space $x^{(1)}, x^{(2)}, \cdots x^{(n)}$.) Finally, after suitable normalisation, we take the limit as ϵ (the time spacing) tends to zero. The path integral is symbolised by

$$\int_J [\text{functional of } x(t)]\, \mathcal{D}x(t) .$$

We now have enough definitions to illustrate with an example of $\mathcal{P}[x]$. Suppose we have a harmonic oscillator satisfying

$$m(\ddot{x} + \omega_0^2 x) = f(t),$$

but the forces f(t) are uncertain. Suppose the probability of f(t) is given by

$$\mathcal{P}_F[f(t)] = \exp\left\{ \int_0^T \left[\frac{f_1(t) - f_0(t)}{\phi(t)} \right]^2 \frac{dt}{T} \right\} .$$

This simply says that the most probable force is $f_0(t)$, and that the deviation is Gaussian, with the force $f_0(t) \pm \phi(t)$ having 1/e relative probability. It is most practical to not consider normalization of these functional probabilities until the integrals are expressed in Riemann sums for the purpose of performing a path integration. Now we substitute from the equation of motion and find the probability of a path:

$$\mathcal{P}[x(t)] = \exp\left\{-\int_0^T \left[\frac{m(\ddot{x}+\omega_0^2 x)-f_0(t)}{\phi(t)}\right]^2 \frac{dt}{T}\right\}.$$

If the normalization of $\mathcal{P}_F[F(t)]$ is some constant in the variables f_j, then the normalization of $\mathcal{P}[x(t)]$ in the variables x_j is also a constant, since the transformation

$$f_j = m\left[(x_{j+1} - 2x_j + x_{j-1})\epsilon^{-2} + \omega_0^2 x_j\right]$$

is linear, and so the Jacobian is constant.

For the purpose of a path integration we express $\mathcal{P}$ in a Riemann sum. Normally, x_0 and the initial velocity v_0 are specified, which gives us $x_1 = x_0 + v_0\epsilon$. We have for $\mathcal{P}$:

$$\mathcal{P}[x_0, x_1, \cdots\cdots x_{T/\epsilon}] =$$

$$\exp\left\{-\sum_j \left[\frac{m[\epsilon^{-2}(x_{j-1} - 2x_j + x_{j+1}) + \omega_0^2 x_j] - f_0(j\epsilon)}{\phi(j\epsilon)}\right]^2 \frac{\epsilon}{T}\right\}.$$

The normalization condition can now be written

$$\iint\cdots\int_{-\infty}^{\infty} \mathcal{P}_{norm}[x_0, x_1, \cdots x_{T/\epsilon}]\, dx_1\, dx_2 \cdots dx_{T/\epsilon} = 1.$$

This is just a series of Gaussian integrations. We obtain a normalizing factor N_i, $i=2,\cdots T/\epsilon$, for each integration. We can write the expected value of any functional $\mathcal{F}$ as

$$\langle \mathcal{F}[x_0, x_1 \cdots x_{T/\epsilon}] \rangle =$$
$$\int\cdots\int_{-\infty}^{\infty} \mathcal{F}[x_0, x_1, \cdots x_{T/\epsilon}]\, \mathcal{P}[x_0 \cdots x_{T/\epsilon}]\, (N_1 dx_1)\cdots(N_{T/\epsilon} dx_{T/\epsilon}).$$

In this expression, $\mathcal{F}$ will normally be the Riemann sum of an integral. The result will normally be recognized as another Riemann sum, which should be expressed as an integral to perform the final step in path integration, namely, taking the limit $\epsilon \to 0$.

Having illustrated the concept of the expected value of a functional, we discontinue the example without calculating a specific $\langle \mathcal{F} \rangle$, because we shall employ a neater trick for such calculations later on.

Suppose we are given that a particle starts with position x_0 and momentum p_0 at $t=0$, and that the uncertain mechanics are described by $\mathcal{P}[x(t)]$. Then we can find the probability $K(x_T, p_T; x_0, p_0; T)$ that the particle has position x_T and momentum p_T at the time T, just by summing the probability of the various paths which have the specified boundary conditions. We indicate the restrictions on the paths being summed by "limits" on the path integral sign. Thus,

we write

$$K(x_T, p_T; x_0, p_0; T) = \int_{x_0, p_0}^{x_T, p_T} \mathcal{P}[x(t)]\, \mathcal{D}x(t) \,. \qquad (1)$$

Looking back at the normalization condition, we see that K differs from one only because the integrals on $x_{T/\epsilon}$ and $x_{T/\epsilon - 1}$ are not performed. Instead, we put $x_{T/\epsilon} = x_T$, $x_{T/\epsilon - 1} = x_T - \epsilon p_T/m$. If in addition to the mechanics, we have uncertainty in the initial conditions, the above expression must be multiplied by the probability $P(x_0, p_0, 0)$ of having x_0 and p_0, then summed over all x_0 and p_0. Thus the probability P of finding x_T and p_T at time T is

$$P(x_T, p_T, T) = \int\int_{-\infty}^{\infty} K(x_T, p_T; x_0, p_0)\, P(x_0, p_0, 0)\, dx_0\, dp_0 \,. \qquad (2)$$

We make a notation change to simplify a discussion to follow. Suppose the equation of motion for x has the form

$$m\ddot{x} + \frac{\partial V(x)}{\partial x} - \phi(t) = f(t) \,,$$

where the known forces are all on the left, and the uncertain terms are in f. Suppose the probability of the forces f(t) is given by $\mathcal{P}_F[x(t), f(t)]$. Here we have written x(t) in the functional dependence to include cases in which the uncertain forces depend on the coordinates of the system,

perhaps an uncertain potential. Now we can substitute from the equation of motion and write the probability of a path

$$\mathcal{P} = \mathcal{P}_F\left[x(t), \left(m\ddot{x} + \frac{\partial V}{\partial x} - \phi(t)\right)\right] = \mathcal{P}[x, M(x,\dot{x},\ddot{x})],$$

where we have written $M(x,\dot{x},\ddot{x})$ for the terms $m\ddot{x} + \frac{\partial V}{\partial x} - \phi(t)$. We continue to write $\mathcal{P}[x,M]$ instead of just $\mathcal{P}[x]$, so that when we have occasion to add another force $f'(t)$, we can write the new probability immediately, namely $\mathcal{P}[x,M-f']$. One term of M is always $m\ddot{x}$.

B. Influencing System

Suppose that the system x is coupled to another system whose variables are represented by y. Furthermore, suppose we do not care how the variables y and their momenta change. We only care to know what influence the y system has on x, that is we want an expression for $\mathcal{P}[x,M]$, with y coupled to x, in terms of $\mathcal{P}_o[x,M]$, the probability of paths if y had not been coupled. The molecules of a maser beam are examples of systems we do not measure, but which influence (drive) systems that we do measure.

If the mechanics of y are sure, and if y is massive so that we can neglect the reaction of x on the motion of y, then we can find a definite force f(t) that y exerts on x.

This case gives us

$$\mathcal{P}[x,M] = \mathcal{P}_0[x, M-f(t)] .$$

However, if the y mechanics are uncertain, we must multiply $\mathcal{P}_0$ by the probability $\mathcal{L}[f(t)]$ that the force is f, and then sum over all forces f(t). We now obtain

$$\mathcal{P}[x, M] = \int_{\infty} \mathcal{P}_0[x, M-f(t)]\, \mathcal{L}[f(t)]\, \mathcal{D}f(t) .$$

We shall call $\mathcal{L}$ the forcing functional.

If we no longer assume that y is massive, the probability of f(t) at each time t depends on x earlier, since x reacted back on the y system changing its motion. So we write $\mathcal{L}[x(t), f(t)]$ for the probability that the system y exerts the force f(t) on x, given that x(t) is the path of the system x. Now we obtain the general expression

$$\mathcal{P}[x,M] = \int_{\infty} \mathcal{P}_0[x, M - f(t)]\ \mathcal{L}[x(t), f(t)]\ \mathcal{D}f(t). \qquad (3)$$

This says that the probability of x(t), when x is coupled to y, is the probability of x(t) under the external driver f(t), times the probability that y exerts this force given the path x(t), summed over all the forces f(t).

Surprisingly enough, in the quantum theory to follow, the equations will take the form of equations 1, 2, and 3, entirely unchanged. The mechanics will be contained in

expressions for $\mathcal{P}$ and $\mathcal{J}$. The only way in which the probability concepts of quantum mechanics will differ from the classical is that the "probabilities" ($\mathcal{P}$ or $\mathcal{J}$) of some paths (x(t) or f(t)) may be negative. This gives rise to the familiar interference effects of quantum mechanics. However, these negative probabilities cannot be measured without disturbing the system. (Recall the old game of which slit did the electron pass through.) Measurable probabilities will always turn out positive.

The attitude expressed by equation 3, that we only care about y insofar as it influences x, is the main reason we shall use path integral quantum mechanics. We express the whole problem of systems x and y in path integrals. Then we dispose of y by summing the final states (a simplification mathematically), then performing all the integrations on the y variables first. There remains only a problem in x. The process of integrating Schrödinger's equation for $\psi(x,y,t)$ would carry from time 0 to T extra information about the state of the y system which would appear as extra mathematical complexity.

6. Linear Systems

It is interesting to look ahead at this point to the case of a linear system. When one expands an arbitrary initial oscillator wave function ψ in terms of energy

eigenfunctions to see how ψ changes with time, he soon notices that the probability density $\psi^*\psi$ swings back and forth at the classical frequency with shape unchanged after each period. This is because the time factors, $\exp(-iE_n t/\hbar) = \exp i(n+\frac{1}{2})\omega t$, in ψ merely change sign after the time $2\pi/\omega$, the classical period. Thus it appears that in some sense probabilities propagate along classical paths. We will give precise meaning to this sense by showing that $\mathcal{P}[x(t)] = 0$ unless $x(t)$ is a classical path, in which case $\mathcal{P} = \infty$ (like a delta function).

Let us consider equation 3 now in the important case that the linear systems are modes of the electromagnetic field. Much terminology has been invented to conveniently describe the electromagnetic fields classically. A resonant cavity is described by giving the shape of each mode, its resonant frequency, and its Q. Couplings between modes of cavities, modes of wave guides, and modes of the great outdoors have been studied and described. But when the fields are very weak, they must be treated as quantum systems even if it means abandoning the convenient classical description entirely. The quantum theory is treated in reference (6). It is not very difficult if the fields are not coupled to anything but classical driving forces. The normal modes of the system simply become driven quantum oscillators. When we

prove that linear systems follow only classical paths, this case becomes even simpler. We only need quantum mechanics to find the probability distribution (P) of initial conditions, then the classical description still holds. However, we very often have non-linear quantum systems coupled to the electromagnetic field. Spins are coupled in nuclear magnetic resonance or paramagnetic resonance experiments. Radiation from hot atoms is observed over a wide range of the spectrum. Beam type maser amplifiers and oscillators use excited molecules, and solid state masers use non-linear three-level systems in crystals. In these cases, it seems as though the convenient classical description is lost if the non-linear systems are treated by conventional quantum mechanics. For example, when the electromagnetic modes are driven by coupling to hot atoms, how does one find the driving force on a mode in terms of the amplitude for the atom to be in certain states, and the matrix elements connecting these states? We will be able to answer this and rescue the classical description of the fields by finding the forcing functional $\mathcal{H}$ of equation 3, which tells the probable forces that the non-linear systems exert on the electromagnetic field.

In reference (6), Fermi quantized the electromagnetic field using a coordinate proportional to the vector

potential A, and chosen for unit "mass"; i.e. m in the oscillator equation $m(\ddot{x}+\omega^2 x)=f(t)$ was equal to one and x was proportional to A. However, a quantity proportional to the electric field $\mathcal{E}$ or the magnetic field B would make an equally good coordinate, because the amplitudes of $\mathcal{E}$, B, and A all vary sinusoidally with time in any normal mode. The field energy (in MKS units),

$$E=\int \frac{1}{2}\left[\epsilon_0 \mathcal{E}^2+\frac{B^2}{\mu_0}\right] dV,$$

of a particular mode can be expressed in the form:

$$E=\frac{p^2}{2m}+\frac{m\omega^2 x^2}{2}, \quad p=m\dot{x},$$

where x is either $\mathcal{E}$, B, or A at some reference point in the cavity. Then E becomes the Hamiltonian. The most convenient choice of x depends on the particular problem. If the mode is driven by an oscillating electric dipole, then $\mathcal{E}$ at the location of the driver is probably the most convenient choice for x. Similarly, B is convenient for a magnetic moment. Therefore, we keep the form with x and m throughout this paper, and the conversion to electromagnetic quantities can be made with ease in any particular problem using any system of units.

III. DERIVATION

A. Wigner Density

Instead of using wave functions, we shall use the joint

probability density of position and momentum introduced by Wigner in reference (2) and defined by

$$P(x,p) = \left(\frac{1}{\pi\hbar}\right)^n \int_{-\infty}^{\infty} e^{2ipx'/\hbar}\, \psi^*(x+x')\, \psi(x-x')\, dx',$$

where px' means $p^{(1)}x^{(1)} + p^{(2)}x^{(2)} + \cdots + p^{(n)}x^{(n)}$. Nothing is lost in integration, since $\psi^*(x)\ \psi(x')$ can be recovered by inverse Fourier transformation. P is real. To see this, take the complex conjugate, change x' to $-x'$, and note that the same expression results. However, P is not everywhere positive, so it is not a true probability density in the classical sense. Nevertheless, in most important respects this density can be used like a true one. It is easily shown that integrating either x or p leads to the true density for the other one, that is,

$$\int_{-\infty}^{\infty} P(x,p)\, dp = |\psi(x)|^2$$

$$\int_{-\infty}^{\infty} P(x,p)\, dx = \frac{1}{2\pi\hbar} \left| \int_{-\infty}^{\infty} \psi(x)\, e^{-ipx/\hbar}\, dx \right|^2 .$$

Furthermore, the expected value of a function of position or momentum can be calculated in the usual classical manner

i.e. it can be readily shown that

$$\iint_{-\infty}^{\infty} f(x)\,P(x,p)\,dx\,dp = \int_{-\infty}^{\infty} \psi^*(x)\,f(x)\,\psi(x)\,dx,$$

$$\iint_{-\infty}^{\infty} g(p)\,P(x,p)\,dx\,dp = \int_{-\infty}^{\infty} \psi^*(x)\,g\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\psi(x)\,dx \quad .$$

The operator expression for the expected value of a function of x and p may be ambiguous, until a particular order of operation of the non-commuting x and p operators has been specified, or a linear combination of the ways of ordering them. However, the expected value calculated with $P(x,p)$ is unambiguous and corresponds to one of the possible operator expressions. A consequence is that "eigenproperties" of the energy eigenfunctions ψ_n are not apparent when ψ_n is converted to P_n. If the Hamiltonian has the usual form

$$H = \left(\frac{p^2}{2m} + V(x)\right), \qquad (4)$$

then,

$$\iint H(x,p)\,P_n(x,p)\,dp\,dx = E_n \ ,$$

the eigenvalue, as expected. However, E_n now appears as only an average value, not a definite value, because

$$\iint P_n(x,p)\left[H(x,p)\right]^m dp\,dx \neq (E_n)^m ,$$

in particular,

$$\iint P_n(x,p)\left[H-E_n\right]^2 dp\,dx > 0 .$$

Suppose we do not have a pure state ψ, but rather a state described by the density matrix:

$$\rho(x',x) = \sum_n w_n \psi_n^*(x')\psi_n(x), \quad \sum_n w_n = 1 . \tag{5}$$

Then we need the following generalization of P:

$$P(x,p) = \left(\frac{1}{\pi\hbar}\right)^n \int \rho(x+x', x-x') e^{2\frac{i}{\hbar}px'} dx', \tag{6}$$

the inverse of which is

$$\rho(x+x', x-x') = \int P(x,p)\, e^{-2ipx'/\hbar}\, dp . \tag{7}$$

Clearly, the remarks about correct expected values still hold.

B. Notation

Before proceeding with path integral theory of the Wigner probability function, we introduce some notation and collect some already introduced. We define sum and difference

coordinates

$$\begin{aligned} X &= x + x' & x &= \tfrac{1}{2}(X+\varkappa) \\ \varkappa &= x - x' & x' &= \tfrac{1}{2}(X-\varkappa) \end{aligned} \tag{8}$$

A similar set of script, capital and primed letters may be used for any coordinates.

Analagous to the delta function

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\varkappa s}\, ds\,,$$

we shall use a delta functional for x(t) between 0 and T,

$$\delta[x(t)] = \int\!\!\!\int \exp\left[i\int_0^T x(t)\,q(t)\,dt\right] \mathcal{D}q(t)\,, \tag{9}$$

which can be thought of as

$$\delta[x(t)] = \lim_{\epsilon \to 0} \prod_{j=0}^{T/\epsilon} \delta(x(j\epsilon))\,.$$

We use brackets to indicate a functional, parenthoses to indicate a function. The delta functional is zero unless its argument function is identically zero from time 0 to T.

Normalization is a nuisance when working with path integrals and is not an essential consideration, because a function P or ψ can always be normalized after it is propagated by path integrals and before expected values or probabilities are calculated. Therefore, we shall freely use the

same symbols $P, \rho, \mathcal{P}, \mathcal{Y}$, etc. with different normalization from one equation to the next. We shall use "limits" on the path integral sign to indicate that the paths for which the integrand is to be summed are restricted to those of certain initial and final conditions. In particular,

$$\int_{x_0, \bar{p}_0}^{x_T, \bar{p}_T} f[x(t)]\, \mathcal{D}x(t)$$

means that the functional f is summed only for those paths $x(t)$ which start at time 0 at x_0 with momentum $\bar{p}_0$ and end at time T, position x_T, and momentum $\bar{p}_T$, where $\bar{p}$ is the classical momentum of the path derived from the Lagrangian:

$$\bar{p}^{(i)} = \partial L(x, \dot{x}, t) / \partial \dot{x}^{(i)}. \tag{10}$$

We distinguish between $\bar{p}$ and the quantum p, which we do not yet know to be the same. We write without "limits"

$$\int f[x(t)]\, \mathcal{D}x(t)$$

to mean the integral over x at all times from 0 to T.

When many variables $x^{(j)}$ are involved we use the ab-

breviations

$$p\dot{x}' \text{ to mean } \sum_j p^{(j)} \dot{x}^{(j)} ,$$

$$\frac{\partial L}{\partial \dot{x}} \dot{x}' \text{ to mean } \sum_j \frac{\partial L}{\partial \dot{x}^{(j)}} \dot{x}'^{(j)} ,$$

$$f(x) \text{ to mean } f(x^{(1)}, x^{(2)}, \cdots x^{(n)}),$$

$$\int \cdots\cdots dx \text{ to mean } \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \cdots dx^{(1)} dx^{(2)} \cdots dx^{(n)},$$

and $\int_{x_0, p_0}^{x_T, p_T} \cdots \mathcal{D}x$ to mean $\int_{x_0^{(1)}, p_0^{(1)}}^{x_T^{(1)}, p_T^{(1)}} \cdots \int_{x_0^{(n)}, p_0^{(n)}}^{x_T^{(n)}, p_T^{(n)}} \cdots \prod_{j=1}^{n} \mathcal{D}x^{(j)}(t)$.

C. Probability of a Path. Single System

Now we begin a fairly lengthy derivation which leads to equation 15, the quantum mechanical probability of a path. Equations 1 and 2 are the rules for propagating $P(x,p)$, given $\mathcal{P}[x(t)]$. So equations 1, 2, and 15 constitute a theorem for the propagation of the Wigner density. The theorem of equation 15 is restricted to systems with Lagrangians such that $\dot{x}$ does not appear in any term higher than the second power in x and $\dot{x}$. This includes in the theorem all physical systems described by the ordinary Hamiltonian of equation 4, plus systems with velocity terms $\dot{x}^{(i)} \dot{x}^{(j)}$ and $\dot{x}^{(i)} x^{(j)}$.

First, we must propagate $\rho(\xi, x)$, then convert to $P(x,p)$ by equation 6. From reference (1) we obtain the propagation

rule for ψ:

$$\psi(x_T, T) = \int dx_0 \int_{x_0}^{x_T} e^{\frac{i}{\hbar} S[x]} \mathcal{D}x(t)\, \psi(x_0, 0) \quad ,$$

where S is the classical action of the path x(t). From the definition of ρ, equation 5, it follows that the rule for ρ is

$$\rho(x_T, x_T', T) = \iint \int_{x_0}^{x_T} \int_{x_0}^{x_T'} e^{\frac{i}{\hbar}(S[x'] - S[x])} \mathcal{D}x\, \mathcal{D}x' \cdot \rho(x_0, x_0', 0)\, dx_0\, dx_0' \quad . \tag{11}$$

The integrations are more easily performed in x, x' coordinates, so we first extract and simplify the exponent,

$$S[x(t)] - S[x(t)] = s[x(t), x'(t)] = \int_0^T \left[L(x, \dot{x}, t) - L(x, \dot{x}, t) \right] dt \quad , \tag{12}$$

where L is the Lagrangian of the system. Substituting sum and difference coordinates, equations 8, into 12 gives

$$s[x, x'] = \int_0^T \left[L(x - x', \dot{x} - \dot{x}', t) - L(x + x', \dot{x} + \dot{x}', t) \right] dt.$$

Expanding L in powers of x' and $\dot{x}'$, the even powers cancel and the odd powers add, leaving:

$$s = -2 \int_0^T \left\{ \frac{\partial L}{\partial \dot{x}} \dot{x}' + \frac{\partial L}{\partial x} x' + \lambda(x, x') \right\} dt,$$

where the third and higher order terms of x' have been collected into $\lambda(x,x')$. Integrating the first term by parts, we have

$$\frac{\mathcal{A}[x,x']}{2} = \int_0^T \left\{ \left[\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} \right] x' - \lambda(x,x') \right\} dt$$

$$+ \left(\frac{\partial L}{\partial \dot{x}}\right)_{t=0} x'_0 \; - \left(\frac{\partial L}{\partial \dot{x}}\right)_{t=T} x'_T \quad .$$

We let

$$M^{(j)}(x,\dot{x},\ddot{x}) = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^{(j)}} - \frac{\partial L}{\partial x^{(j)}} \quad ; \tag{13}$$

that is, $M^{(j)}=0$ are the classical equations of motion. Substituting equations 10 and 13 gives for $\mathcal{A}$:

$$\frac{\mathcal{A}[x,x']}{2} = \int_0^T \left[M(x,\dot{x},\ddot{x})x' - \lambda(x,x') \right] dt + \bar{P}_0 x'_0 - \bar{P}_T x'_T \quad . \tag{14}$$

We do not let M=0 in this expression, because paths other than the classical one may be important in quantum theory.

Now we propagate ρ in the integrand of the definition of P, equation 6. Using equations 11 and 12, we obtain

$$P(x_T, p_T, T) = \int dx'_T e^{2ip_T x'_T/\hbar} \iint dx_0\, dx'_0 \int_{x_0}^{x_T} \mathcal{D}x \cdot$$

$$\cdot \int_{x_0}^{x_T} \mathcal{D}x\, e^{i\omega[x,x']/\hbar}\, \rho(x_0, x'_0, 0) \quad .$$

Changing integration variables from x_0, x to x, x', and substituting the inverse transformation, equation 7 for $\rho(0)$, and equation 14 for ω gives

$$P(T) = \iiint dx'_T\, dx_0\, dx'_0\, dp_0\, e^{2ip_T x'_T/\hbar} \int \mathcal{D}x' \int_{x_0}^{x_T} \mathcal{D}x \cdot$$

$$\exp\left\{\frac{2i}{\hbar}\int_0^T [Mx' - \lambda(x,x')]\, dt - \frac{2i}{\hbar}\bar{p}_T x'_T + \frac{2i}{\hbar}\bar{p}_0 x'_0\right\}$$

$$P(x_0, p_0, 0)\, e^{-2ip_0 x'_0/\hbar} \quad .$$

The integrations on x'_0 and x'_T give $\delta(p_0 - \bar{p}_0)\delta(p_T - \bar{p}_T)$. We express these conditions $p_0 = \bar{p}_0$, $p_T = \bar{p}_T$, as "limits" on the x path integrals, which gives finally:

$$P(T) = \iint \left\{ \int_{x_0,p_0}^{x_T,p_T} \left[\int \exp\left\{\frac{2i}{\hbar}\int_0^T (Mx' - \lambda)\, dt\right\} \mathcal{D}x' \right] \mathcal{D}x \right\} \cdot$$

$$\cdot P(x_0, p_0, 0)\, dx_0\, dp_0 \quad .$$

Comparing this expression to equations 1 and 2, we see that it has the same form when the square bracket is identified as $\mathcal{P}$.

Thus we have our first theorem: The quantum mechanical probability of a path is given by

$$\mathcal{P}_{[x(t)]} = \int\!\!\!\int\!\!\!\int \exp\left\{\frac{2i}{\hbar}\int_0^T \left[M(x,\dot{x},\ddot{x})x' - \lambda(x,x')\right]dt\right\} \mathcal{D}x', \qquad (15)$$

where M is given by equation 13, and λ is derived from the potential V(x) by

$$\lambda(x,x') = \frac{1}{2}\left[V(x-x') - V(x+x')\right] + \frac{\partial V(x)}{\partial x}\,x' \quad . \qquad (16)$$

Note that $\mathcal{P}$ is real, because the exponent in equation 15 is an odd function of x', so the imaginary sine part of the integral vanishes.

We can immediately perform the integration of equation 15 in the case of a quadratic Lagrangian (linear system), because λ vanishes. The result of integration over the variables x'(t) is then

$$\mathcal{P}_{[x,\,M(x,\dot{x},\ddot{x})]} = \prod_{j=1}^{n} \delta\left[M^{(j)}(x,\dot{x},\ddot{x})\right], \qquad (17)$$

but $M^{(j)}=0$ are the equations of motion of the classical path

$x_c(t)$. So we can rewrite equation 17 as

$$\mathcal{P}[x(t)] = \delta[x(t) - x_c(t)] \quad . \tag{18}$$

(The switch from force to position units in the argument of δ just introduces a constant at each time, i.e. a renormalization.) We now have our second theorem as follows: For the most general linear system, derivable from a Lagrangian, P(x,p) propagates exactly like a classical probability density*, i.e. only along classical paths. It should be remembered that the Lagrangian was sufficiently general to include terms like $\dot{x}^{(i)}x^{(j)}$, $\dot{x}^{(i)}\dot{x}^{(j)}$, $x^{(k)}f(t)$, i.e. velocity coupling and external driving forces f(t).

Equation 15 for non-linear systems is not too useful. It can be used to do scattering theory by showing that the probable paths are the straight ones and those having momentum jumps equal to $h \div$ wavelengths in the Fourier spectrum of the scattering potential. However, this is an old result and no simplification is obtained.

Equations 17 and 18 are more interesting. They say

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* In reference (2) Wigner showed that for a quadratic V(x), P satisfies

$$\frac{\partial P}{\partial t} = \frac{p}{m}\frac{\partial P}{\partial x} - \frac{\partial V}{\partial x}\frac{\partial P}{\partial p} \quad ,$$

which is just the classical equation of continuity. Thus, Wigner proved the same theorem, except for velocity dependent coupling terms.

that we only need to solve classical equations for linear systems, then use the initial wave function to see what range of boundary conditions to expect. But linear systems are fairly simple in any formalism. The real usefulness comes when equation 17 is the $\mathcal{P}_o$ of equation 3, where $\mathcal{L}$ describes forces which result from coupling to non-linear systems.

D. Influencing System

Let us write down the linear case of equation 3 and discuss its usefulness further. It is

$$\mathcal{P}[x,M] = \int \delta\left[M(x,\dot{x},\ddot{x}) - f(t)\right] \mathcal{L}[x(t), f(t)] \mathcal{D} f(t). \qquad (19)$$

Recall that this reads as follows: The probability of a path vanishes unless it satisfies the classical equation of motion with an additional external driving force f(t). This equation of motion has the form:

$$M(x,\dot{x},\ddot{x}) = m\ddot{x} + \text{----} = f(t).$$

But more than one force f(t) may occur. In fact, for a given x(t), the probability of a force f(t) is $\mathcal{L}[x(t),f(t)]$. So, the probability of a path x(t) is just the probability of the force f(t) required to make that path satisfy the

above equation. This last statement is just the statement of the equation we obtain by performing the integration in equation 19, namely,

$$\mathcal{P}[x(t),n]=\mathcal{L}[x(t),\ M(x,x,x)]\ . \qquad (20)$$

The fact that $\mathcal{L}[x,f]$ depends on x means, for the examples in this paper, that part or all of the forces will be described as an impedance. That is, when x(t) with certain frequency components is substituted into $\mathcal{L}$, this makes $\mathcal{L}[x,f]$ large for those forces f which have a certain phase relation with the frequency components of x. Thus the influencing system may absorb or emit power to x (positive or negative resistance), or shift the frequencies (reactance).

D. 1. Influence Functional, Path Integral Form

In most cases we shall not find the forcing functionals by first finding the probabilities of paths of the nonlinear systems, because the path integral of equation 15 is too difficult. Instead, we shall relate the forcing functionals to more conventional quantum mechanics. Toward this end, we now return to a discussion of the density matrix and wave functions propagated by $e^{iS/\hbar}$.

In the discussion of equation 3 it was pointed out that we want to consider two systems, x and y, and that we do not measure y, but are interested in it only insofar

as it influences x. To say this mathematically, we follow a procedure due to Feynman, reference (7), which leads to equations 22 and 23. We first propagate the density matrix $\rho(x',y';x,y;t)$ from 0 to T. Then we ask the probability that y is in the mth state of some orthogonal set ϕ_m, leaving the variables x and x' alone so any simultaneous question can be asked about x. We have

$$\iint \rho(x'_\tau, y'_\tau; x_\tau, y_\tau, T)\, \phi_m(y'_\tau)\, \phi_m^*(y_\tau)\, dy_\tau\, dy'_\tau .$$

We sum this expression over all m and call the result $\rho(x'_\tau, x_\tau, T)$. In so doing, we say that we do not care what the final state of y is. We have

$$\rho(x'_\tau, x_\tau, T) = \sum_m \iint \rho(x'_\tau, y'_\tau; x_\tau, y_\tau; T)\, \phi_m(y'_\tau)\, \phi_m^*(y_\tau)\, dy_\tau\, dy'_\tau$$

$$= \iint \rho(x'_\tau, y'_\tau; x_\tau, y_\tau; T)\, \delta(y_\tau - y'_\tau)\, dy\, dy' ,$$

where in the second step we have changed the order of integration and summation and have used the closure relation,

$$\delta(y-y') = \sum_m \phi_m(y')\, \phi_m^*(y) . \qquad (21)$$

Writing out the path integrals using equation 11, we have

$$\rho(x_T', x_T, T) = \iiiint\!\!\iiint \delta(y_T - y_T') \int_{x_0}^{x_T} \mathcal{D}x \int_{x_0'}^{x_T'} \mathcal{D}x' \int_{y_0}^{y_T} \mathcal{D}y \int_{y_0'}^{y_T'} \mathcal{D}y' \cdot$$
$$\cdot \exp\left\{\frac{i}{\hbar}\left(S[x,y] - S[x',y']\right)\right\} \rho(x_0', y_0'; x_0, y_0; 0) \cdot$$
$$\cdot dx_0\, dx_0'\, dy_0\, dy_0'\, dy_T\, dy_T' \quad .$$

We specialize to the case in which the x and y systems are initially independent, i.e. $\rho(0) = \rho_x(x_0', x_0, 0)\rho_y(y_0', y_0, 0)$. We break the action into three parts, $S_x[x]$, $S_y[y]$, and $S_I[x,y]$, which are respectively the terms in x only, the terms in y only, and the interaction terms. With these substitutions we can factor out of the above integrand everything that depends on y and perform the y integrations first, calling the result $\mathcal{J}$, the influence functional:

$$\mathcal{J}[x', x] = \iiiint dy_0\, dy_0'\, dy_T\, dy_T'\, \delta(y_T - y_T') \int_{y_0}^{y_T} \mathcal{D}y \int_{y_0'}^{y_T'} \mathcal{D}y' \cdot \tag{22}$$
$$\cdot \exp\left\{\frac{i}{\hbar}\left(S_y[y] + S_I[x,y] - S_y[y'] - S_I[x',y']\right)\right\} \rho_y(y_0', y_0, 0) \, .$$

Substituting this into $\rho(T)$, we have

$$\rho(x_T', x_T, T) = \iint dx_0\, dx_0' \int_{x_0}^{x_T} \mathcal{D}x \int_{x_0'}^{x_T'} \mathcal{D}x' \, \mathcal{J}[x', x] \cdot \tag{23}$$
$$\cdot \exp\left\{\frac{i}{\hbar}\left(S_x[x] - S_x[x']\right)\right\} \rho_x(x_0', x_0, 0).$$

Note that equation 23 is, apart from the factor $\mathcal{J}$, the same as the expression for $\rho(T)$ if x were an isolated sys-

tem described by the action S_x. The influence functional is a functional which, if inserted into the path integrand for the density matrix of any x system whatsoever, describes completely the effects on x of coupling to the y system with interaction S_I.

D. 2. Forcing Functional

We now obtain a new functional $\mathscr{L}[x(t), f(t)]$ from $\mathscr{J}[X(t), x(t)]$ by a transformation process. We are not supposed to know yet that this $\mathscr{L}$ is the desired forcing functional of equations 3, 19 and 20, but we shall shortly prove that it is. The transformation is just the path integral analog of the Fourier transformation by which P(x,p) was obtained from $\rho(X, x)$, equation 6, namely,

$$\mathscr{L}[x(t), f(t)] = \int_{\mathcal{D}} \exp\left\{\frac{2i}{\hbar}\int_0^T f(t)\, x'(t)dt\right\} \mathscr{J}[x+x', x-x']\, \mathscr{D}x'. \qquad (24)$$

Note that when the integral is changed to a Riemann sum with variables x'_j, f_j, this is just a Fourier transformation of the variables x'_j into f_j. $\mathscr{J}$ may be recovered by the inverse transformation:

$$\mathscr{J}[X, x] = \int_{\mathcal{D}} \exp\left\{-\frac{2i}{\hbar}\int_0^T f(t)\, x'(t)\, dt\right\} \mathscr{L}[x, f]\, \mathscr{D}f(t). \qquad (25)$$

To prove that this $\mathcal{L}$ is indeed the forcing functional, we refer back to equation 11. There we were propagating $\rho(X, \alpha)$ from time 0 to T, preparing to find a propagation rule for P in the steps to follow that equation. Now we have the new propagation rule of equation 23 for ρ with $\mathcal{J}[X, \alpha]$ in the path integrand. To find the new propagation rule for P, the steps following equation 11 still apply except the factor $\mathcal{J}$ is included in the integrand up to equation 15, which now reads

$$\mathcal{P}[x(t)] = \int_{\alpha} \mathcal{J}[x + x', x - x'] \exp\left\{\frac{2i}{\hbar}\int_0^T [Mx' - \lambda]\, dt\right\} \mathcal{D}x'.$$

Substituting equation 25 into the above, and combining the two exponential factors gives

$$\mathcal{P}[x(t)] = \int_{\alpha} \mathcal{L}[x, f]\left[\int_{\alpha} \exp\left\{\frac{2i}{\hbar}\int_0^T \left[\left(M(x, \dot{x}, \ddot{x}) - f(t)\right)x'(t) - \lambda(x, x')\right] dt\right\} \mathcal{D}x'(t)\right] \mathcal{D}f(t) \quad .$$

Comparing to equation 15 we note that the big bracket is $\mathcal{P}_0$, the probability of a path with no coupling to y, except now the driving force f(t) appears in the parentheses with $M(x, \dot{x}, \ddot{x})$. Therefore, we identify the big bracket as $\mathcal{P}_0[x, M - f(t)]$, and equation 3 results. This identifies the new $\mathcal{L}$ of equation 24 as the old $\mathcal{L}$ of equation 3. Remembering the interpretation of equation 3 (so

that it can be reconstructed from the words to follow), we state our third theorem of quantum mechanics: Given a possible path x(t) of any system x, and given the influence functional $\mathcal{F}$, defined by equation 22 for a system y influencing x, then the functional $\mathcal{P}[x,f]$, derived from $\mathcal{F}$ by equation 24, is the probability that y exerts the force f on x.

Since the $\mathcal{P}$ functionals are real, and $\mathcal{D}$ is used in equation 3 to obtain one $\mathcal{P}$ from another, we suspect that $\mathcal{D}$ is also real. To show that this is in fact true, we first note from the definition of ρ, equation 5, that $\rho(y',y) = \rho^*(y,y')$, i.e. ρ is Hermitian. Using this fact in equation 22, it easily follows that $\mathcal{F}[x',x] = \mathcal{F}^*[x,x']$. Finally, using this second fact in equation 24 for $\mathcal{D}$, and changing the dummy variable x' to -x', it follows that $\mathcal{D} = \mathcal{D}^*$, or $\mathcal{D}$ is real.

2. 1. Many Systems

Suppose we have several systems $y_1, \cdots y_n$*, each coupled to x but not to each other, and each initially independent of x and the other y_j. Also suppose we have the influence functionals $\mathcal{F}_j[x',x]$, each calculated as though y were the only influencing system. Then we can easily show

* These subscripts have nothing to do with the time index used in Riemann sums.

that the effective influence functional $\mathcal{F}$ of all n systems is given by

$$\mathcal{F}[x',x] = \prod_{j=1}^{n} \mathcal{F}_j[x',x]. \tag{26}$$

To see this, we note that the path integrand for everything contains the n factors:

$$\exp\left\{\frac{i}{\hbar}\left(S_{y_j}[y_j] - S_{y_j}[y_j'] + S_{I_j}[x,y_j] - S_{I_j}[x',y_j']\right)\right\}\rho_j(y_{j_0}', y_{j_0}, 0).$$

Each of these factors can be taken out and integrated separately, and the integrals of these factors are just the $\mathcal{F}_j$. Restoring the factors is the same as using the single $\mathcal{F}$ of equation 26.

To compound the corresponding $\mathcal{L}_j$, we have two choices. (1) We can find the effective $\mathcal{F}$ by equation 26, then transform to a single effective $\mathcal{L}$ by equation 24. (2) We can write $\mathcal{P}[x]$ as in the two equations following equation 25, except using $\prod_j \mathcal{F}_j$, where each $\mathcal{F}_j$ is expressed as the inverse transformation of the corresponding $\mathcal{L}_j$. Then we have

$$\mathcal{P}[x,M] = \int_{x_0}^{x} \cdots \int_{x_0}^{x} \mathcal{P}_0[x, M - \textstyle\sum_j f_j(t)] \prod_j \mathcal{L}_j[x, f_j] \mathcal{D} f_j(t). \tag{27}$$

This equation corresponds to our classical concept of

probability of forces just as equation 3 does.

We observe a close relation between simple probability theory and the theory expressed in equations 26 and 27. In the simple theory, suppose we want the probability density of the sum of several random variables, given the densities of the individual variables. We first find the Fourier transformations of the individual densities. Each of these functions is called the characteristic function of the random variable it describes. We multiply characteristic functions in terms of the same transformation variable, and the result is the characteristic function of the sum. The inverse transformation gives the probability density of the sum. In equation 27 we are interested in the sum of the forces, $f = \sum_i f_i$. In order to find a single probability density $\mathcal{P}[x, f]$ for the sum, instead of integrating over each individual force (f_i) as is equation 27, we just showed that we do the operation analogous to the simple theory. We transform the random variables $f_i(t)$ to $x'(t)$, obtaining the functionals $\mathcal{J}_i[x+x', x-x']$. Then we take the product of these (equation 26) and perform the inverse transformation obtaining $\mathcal{P}[x,f]$. Thus, we add meaning to the functional $\mathcal{J}[x+x', x-x']$, which originally was just a quantity in a mathematical manipulation. It is the characteristic functional of the random force function f(t).

D. 4. Influence Functional, Amplitude Form

When we find forcing functionals for various systems, we shall usually start with equation 22 for $\mathcal{J}$. However, we also want an expression for $\mathcal{J}$ which does not require that the y system be described in coordinate representation. We want to be able to find $\mathcal{J}$ when y is described in a general representation by the amplitude a_n to be in the nth state of an orthogonal set. Toward this end we substitute into equation 22 the closure relation, equation 21, and the definition of ρ, equation 5, using the same orthogonal set ϕ_n in each case. Now the path integrals factor, and we have,

$$\mathcal{J} = \sum_{ij} w_i \left\{ \int dy_T \, \phi_i^*(y_T) \left[\int dy_0 \int_{y_0}^{y_T} \exp\left\{ \frac{i}{\hbar} \left(S_y[y] + S_I[x,y] \right) \right\} \mathcal{D}y \; \phi_i(y_0) \right] \right\} \cdot$$
$$\cdot \left\{ y \to y', x \to x', \text{complex conjugate} \right\}.$$

The expression in the square bracket is the same as the wave function $\psi_i[y_T, T, x(t)]$ at time T, having started from ϕ_i at time 0, and having progressed under the perturbing influence of $S_I[x,y]$. For the limited purpose of understanding ψ_i, the function x(t) in S_I can simply be regarded as the time dependence of a term in the action of an isolated system, for example, a time dependent potential. Expanding ψ_i, in

the orthogonal set, we have

$$\psi_i[y_\tau, \tau, x(t)] = \sum_j a_{ij}[\tau, x(t)] \quad \phi_j(y_\tau)$$

In principle, we can find the $a_{ij}[\tau,x]$ by solving Schrödinger's equation in the ϕ_n representation where, instead of $S[x,y]$ we have the corresponding Hamiltonian term $h[x(t), y]$. For example, if $S_I = -\int_0^\tau V[x(t), y(t)]dt$, then the Hamiltonian term is just the time dependent potential $h = V[x(t), y]$. Substituting the expansion of ψ_i for the bracket in $\mathcal{J}$, we have

$$\mathcal{J} = \sum_{ijkr} w_i \left\{ \int dy_\tau \phi_j^*(y_\tau) a_{ik}[\tau, x(t)] \phi_k(y_\tau) \right\} \cdot \cdot \left\{ y \to y', x \to x', k \to r, \text{ C.C.} \right\} .$$

But since the ϕ_n are orthogonal, we obtain finally

$$\mathcal{J}[x',x] = \sum_{ij} w_i a_{ij}[\tau, x(t)] \; a_{ij}^*[\tau, x'(t)] . \qquad (28)$$

Equation 28 does not make it particularly easy to find the forcing functional, because the a_{ij} must be found for arbitrary $x(t)$, since $\mathcal{J}$ goes into the path integrand of equation 24.

Equation 28 can be used for spin systems that are not conveniently described by the path integral theory with which we started.

In the section after next (V) we calculate specific

examples of forcing functionals using the formulae developed here, but first, we digress to discuss the classical limit of the quantum formalism we have developed.

IV. CLASSICAL LIMIT

Since our formalism is so close to classical concepts, it is a convenient one to use in a problem in which we pass to the classical limit. An example is an amplifier in which the incoming signal causes only a few quanta of excitation (in the first wave guide of a maser, for instance), and the outgoing signal is a few milliwatts. We discuss the transition to classical physics in three steps: first, the ability to make measurements without appreciably disturbing the system, second, positive probabilities, and third, the classical equation of motion.

A. Classical Measurements and Probability

Suppose as before the y system influences the x system as described by $\mathcal{Y}$, but now y is being used to measure x. Equation 3 then shows the criterion for making the measurement with an inappreciable disturbance. $\mathcal{Y}$ gives a certain range of forcing functions f(t) which may occur with appreciable probability. This could be expressed as a range of frequency and amplitude. When x is so large that $M(x,\dot{x},\ddot{x})$

and $M(x,\dot{x},\ddot{x}) - f(t)$ give solutions $x(t)$ which are inappreciably different for f's in this range, then we may replace $M_x - f$ by M_x in equation 3. $\mathscr{P}$ factors out of the integrand, and we have $\mathscr{P}[x,M] = \text{constant} \cdot \mathscr{P}_o[x,M]$. Thus, after suitable normalization, $\mathscr{P}$ is the same as $\mathscr{P}_o$, the probability with no coupling.

Similarly, if x is measuring y, and y is so large that the reaction on x may be neglected, we need only $\mathscr{P}[0,f(t)]$, which can be treated as a probability distribution of driving forces f(t), which are independent of the response.

We now consider the disappearance of negative probabilities in the classical limit. If we average the probability of a path x(t), or that of a force f(t), over enough paths we are sure to get a positive number, since the total probabilities in the theory are normalizable to one. In any probability theory, some detail is lost if the probability density of the random variable is averaged over neighboring values of the random variable, but such averaging is inconsequential if the range averaged is an inappreciable fraction of the total range of the random variable, or of the range of accuracy of any measuring apparatus. If the probable functions x(t), f(t) vary with time over a large enough range of x and f, the bundles of

paths over which we must average to obtain a positive probability distribution are small compared to the whole range. Then we can perform the averaging and say that our probabilities are classical.

3. Classical Equation of Motion

In order for a system to be classical in all respects, the paths which have any appreciable probability by equation 15 must obey the classical equation of motion. For linear systems, this is always true as we showed. Although we shall not treat non-linear systems quantum mechanically by actually performing the path integration of equation 15, we shall use 15 to examine the classical limit. For simplicity let us consider a one dimensional potential problem. The argument is easily generalised. Equation 15 becomes

$$\mathcal{K}[x] = \int_{x'} \exp\left\{ \frac{i}{\hbar} \phi[x,x'] \right\} \mathcal{D}x'(t), \qquad (29)$$

where

$$\phi[x,x'] = \frac{2}{\hbar} \int_0^T \left[m\ddot{x}x' + \frac{V(x+x') - V(x-x')}{2} \right] dt,$$

and V is the potential.

As $\hbar \to 0$ in the classical limit, the phase ϕ fluctuates more and more rapidly for small changes in the path x'. This cancels the contributions of paths $x'(t)$ to the integral except in the neighborhood of a path x' where the phase is stationary. To find the path for stationary phase, we take the variation of ϕ with respect to $x'(t)$.

$$\frac{\hbar}{2}\delta\phi = \frac{\hbar}{2}\left\{\phi[x, x'+\delta x'] - \phi[x, x']\right\}$$

$$= \int_0^T \left[m\ddot{x} + \tfrac{1}{2}\left(V'(x+x') + V'(x-x')\right)\right]\delta x'\, dt$$

$$+ \int_0^T \tfrac{1}{4}\left[V''(x+x') - V''(x-x')\right]\left[\delta x'(t)\right]^2 dt + \cdots .$$

The phase is stationary when

$$m\ddot{x} + \tfrac{1}{2}\left[V'(x+x') + V'(x-x')\right] = 0 \qquad (30)$$

identically from 0 to T. The amount of probability we get from paths near the stationary one depends on the size of the bundle of paths for which the phase changes only slightly. The bundle is largest, giving the largest probability, if the second order term in the variation also vanishes, that is

$$V''(x+x') - V''(x-x') = 0. \qquad (31)$$

An obvious solution of equations 30 and 31 is $x' = 0$ and $m\ddot{x} + V'(x) = 0$; that is $x = x_c(t)$, a solution of the classical equation . For this solution, $\phi = 0$, so that we have

positive probability.

In general there are other solutions to equations 30 and 31. This means that there are paths x (t) that do not satisfy the classical equation of motion, but which have comparable probability for very small h. For example, consider the potential

$$V = \frac{x^6}{6} + x^3 .$$

Equation 30 becomes

$$m\ddot{x} + \left(x^5 + 10x^3x'^2 + 5xx'^4 + 3x^2 + 3x'^2\right) = 0 .$$

Equation 31 is satisfied by either

$$x' = 0, \text{ or } x' = \pm\sqrt{-\frac{3}{10x} - x^2} .$$

The first solution of 31, $x' = 0$, gives the classical equation of motion when substituted into the differential equation. However, the second solution gives a different equation of motion when substituted into the differential equation. This queer equation of motion is satisfied by non-classical paths $x_q(t)$. Since higher order variations of ϕ do not vanish for the classical paths, we are forced to accept the fact that the paths $x_c(t)$ and $x_q(t)$ have comparable prob-

ability in the limit as $h \to 0$.

The solution to this apparent dilemma lies in the fact that probabilities of the non-classical paths are not always positive. Any classical apparatus measuring the path $x(t)$ does not find $x(t)$ exactly, but rather observes the coordinates to be in some small range near $x(t)$. Therefore, we really want the average probability of $x(t)$ in a small range $\delta x(t)$. We shall show that the average probability in a bundle of paths near $x_c(t)$ vanishes as $h \to 0$, because the probabilities in the bundle oscillate more and more rapidly between positive and negative values from one path to the next as h becomes smaller. In contrast to this behavior, the probabilities are stationary and positive in the neighborhood of $x_c(t)$.

To prove the above statements about small bundles, we refer to equation 29. Within any bundle, $\mathcal{P}[x]$ clearly oscillates rapidly to zero for sufficiently small h, unless the phase ϕ is stationary for variations δx. Now we have the added condition for the classical limit, that ϕ be stationary with respect to independent variations in x and x'. We readily obtain the new set of conditions:

$$\text{(a)} \quad x(t) = \tfrac{1}{2}\left[x_{c1}(t) + x_{c2}(t)\right],$$

$$\text{(b)} \quad x'(t) = \tfrac{1}{2}\left[x_{c1}(t) - x_{c2}(t)\right],$$

$$\text{(c)} \quad V''(x_{c1}) = V''(x_{c2}),$$

$$\text{(d)} \quad x'_0 = 0, \quad x'_T = 0, \quad \dot{x}'_0 = 0, \quad \dot{x}'_T = 0,$$

where x_{c1} and x_{c2} are two solutions of the classical equations of motion. Conditions (d) require that x' start at zero with no velocity, which, taken with (b), requires that x_{c1} and x_{c2} start at the same position and velocity. This makes x_{c1} and x_{c2} the same function, which in turn satisfies (c), makes $x' = 0$, and makes $x(t)$ a solution of the classical equation of motion. The paths $x_Q(t)$ no longer appear.

V. FORCING FUNCTIONALS OF VARIOUS SYSTEMS

Using as examples the circuit elements of beam and solid state maser amplifier systems, we return now to the basic equations for the influence functional, 22 and 26, and to the transformation equation for $\mathcal{J}$, 24. We calculate the forcing functionals, and interpret them in terms of classical concepts. The circuit elements are a classical driver, a noisy driver, a harmonic oscillator with arbitrary initial conditions, a harmonic oscillator initially at temperature θ, a resistor, a general non-linear system initially in an energy eigenstate and treated in the approximation of short interaction time, a molecular beam (the negative resistance element of a beam-type maser), and a three-level system being "pumped" between upper and lower states (the negative resistor of a solid state maser).

A. Classical force

Suppose we have a classical force $f_c(t)$ acting on the x system. This case is so degenerate that we are tempted to write $\mathcal{L}[x,f] = \delta[f(t)-f_c(t)]$ immediately, based on our interpretation of $\mathcal{L}$ as the probability of force. However, we must prove this mathematically. The Lagrangian term corresponding to the external force is $xf_c(t)$, and the action is $S_I = \int_o^T x(t)f_c(t)dt$. Therefore, we have for the influence functional:

$$\mathcal{J}[x,\bar{x}] = \exp\left\{\frac{i}{\hbar}\left(S_I[\bar{x}]-S_I[x]\right)\right\} = \exp\left\{\frac{i}{\hbar}\int_o^T(\bar{x}-x)f_c(t)\,dt\right\}$$

$$\mathcal{J}[x,\bar{x}] = \exp\left\{-\frac{2i}{\hbar}\int_o^T x'(t)f_c(t)\,dt\right\}. \qquad (32)$$

Substituting this into equation 24 we have

$$\mathcal{L}[x,f] = \iint_{\mathcal{U}} \exp\int_o^T \frac{2i}{\hbar}x'(t)\left[f(t)-f_c(t)\right]dt \;\; \mathcal{D}x'(t)$$

$$= \delta\left[f(t)-f_c(t)\right],$$

the expected result.

B. Classical Noise

Suppose we have classical driving forces $f_c(t)$ which are not certain, but are distributed by a probability den-

sity $\mathcal{J}'[f_c(t)]$. Again this case is so degenerate that we are tempted to write immediately $\mathcal{J}[x,f]=\mathcal{J}'[f]$, but we must carry out the mathematical proof.

First, we must find the effective influence functional when the true one is not known, but can be any one of many functionals $\mathcal{J}_j$ with a probability w_j of being the jth. We show that the effective $\mathcal{J}$ is

$$\mathcal{J}=\sum_i w_i \mathcal{J}_i \quad . \tag{33}$$

Each functional $\mathcal{J}_i$, gives a density matrix $\rho_i(x'_T, x_T, T)$ by the propagation rule, equation 23. In order to calculate correct expected values with ρ , it is evident from classical probability theory that we must use as the effective density matrix

$$\rho(x'_T, x_T, T) = \sum_j w_j \rho_j(x'_T, x_T, T).$$

But this verifies equation 33, because this is just the expression that will result from substituting equation 33 into 23 and calculating each ρ_j .

In the present case, the functional $\mathcal{J}_j$ is that given by equation 32 for the force $f_{cj}(t)$, namely

$$\mathcal{J}_j = \exp \int_0^T -\frac{2i}{\hbar} x'(t) f_{cj}(t)\, dt \quad .$$

Substituting this into equation 33, and passing from the

discrete w_j to the continous $\mathcal{L}'$, we have

$$\mathcal{J}(x, x) = \int\!\!\int \mathcal{L}'[f_c(t)] \exp\left\{-\int_0^T \frac{i}{\hbar} x' f_c(t)\, dt\right\} \mathcal{D} f_c(t).$$

Transforming to $\mathcal{L}$ (equation 24), we have

$$\mathcal{L} = \int\!\!\int\!\!\int \mathcal{L}'[f_c(t)] \exp\left\{\frac{2i}{\hbar}\int_0^T x'[f(t) - f_c(t)]\, dt\right\} \mathcal{D} f_c \mathcal{D} x'$$

$$= \int\!\!\int \mathcal{L}'[f_c]\, \delta[f - f_c]\, \mathcal{D} f_c(t)$$

$$\mathcal{L}[x(t), f(t)] = \mathcal{L}'[f(t)] \quad ,$$

the expected result.

C. Harmonic Oscillator

1. Arbitrary Initial Conditions

We shall find the forcing functional for a harmonic oscillator (y system) linearly coupled to x through the term $\gamma(t)\, xy$ in the Lagrangian. Since we know that linear systems propagate only along classical paths, it is interesting to construct $\mathcal{L}$ from classical concepts before discussing a mathematical proof. The y and interaction terms of the Lagrangian are

$$L_y + L_I = \gamma(t) x y + \frac{m}{2}\left(\dot{y}^2 - \omega^2 y^2\right) \quad .$$

The classical equations are

$$M(x, \dot{x}, \ddot{x}) = \gamma(t) y$$

$$m(\ddot{y} + \omega^2 y) = \gamma(t) x .$$

The solution to the second equation is

$$y_c(t) = \frac{1}{m\omega} \int_0^t \sin \omega(t-s)\, \gamma(s)\, x(s)\, ds + y_0 \cos \omega t + \frac{q_0}{m\omega} \sin \omega t ,$$

where $q = m\dot{y}$. The probability of a y path, therefore, is $\delta[y - y_c]$ times the probability $P_y(y_0, q_0, 0)$ of the initial conditions y_0, q_0 in y_c, summed over y_0, q_0. That is,

$$\mathcal{P}_y[y(t)] = \iint \delta[y - y_c]\, P_y(y_0, q_0, 0)\, dy_0\, dq_0 .$$

But the force on x is $f = \gamma y$, therefore

$$\mathcal{P}[x, f] = \mathcal{P}_y[f(t)/\gamma(t)]$$
$$= \iint \delta\left[\frac{f(t)}{\gamma(t)} - \frac{1}{m\omega} \int_0^t \sin \omega(t-s)\, \gamma(s)\, x(s)\, ds - y_0 \cos \omega t - \frac{q_0}{m\omega} \sin \omega t\right] P_y(y_0, q_0, 0)\, dy_0\, dq_0 .$$

We change the integration variables and multiply the argument of the delta functional by $\gamma(t)$ changing $\mathcal{P}$ by a con-

stant factor. Then we have finally

$$\mathcal{L}[x,\gamma] = \iint \delta\Big[\gamma(t) - \frac{\gamma(t)}{m\omega}\int_0^t \sin\omega(t-s)\,\gamma(s)\,x(s)\,ds - a_c\,\gamma(t)\cos\omega t - a_s\,\gamma(t)\sin\omega t\Big] P_\gamma(a_c, m\omega a_s, 0)\,da_c da_s . \tag{34}$$

This expression, constructed from the classical interpretation of $\mathcal{L}$, is proved most easily by first performing the inverse transformation of equation 25. The path integration is trivial because we have a delta functional in the integrand. We obtain

$$\mathcal{J}[x,\chi] = \iint \exp\left\{\frac{-2i}{\hbar}\int_0^T \chi'(t)\left[\frac{\gamma(t)}{m\omega}\int_0^T \sin\omega(t-s)\gamma(s)x(s)\,ds - a_c\gamma(t)\cos\omega t - a_s\gamma(t)\sin\omega t\right]dt\right\}\cdot \tag{35}$$
$$\cdot P_\gamma(a_c, m\omega a_s, 0)\,da_s da_c .$$

We shall obtain our quantum mechanical proof of equation 34 by first proving 35; for evidently 34 follows from 35 by the transformation of equation 24. The quantum expression for $\mathcal{J}$ is given in general by equation 22, which in this case reads

$$\mathcal{J}[x,\chi] = \iiiint\int_{y_0}^{y_T}\mathcal{D}y\int_{Y_0}^{Y_T}\mathcal{D}Y\;\delta(y_T - Y_T)\cdot$$
$$\cdot\exp\int_0^T \frac{i}{\hbar}\left\{\gamma(t)xy + \frac{m}{2}(\dot y^2 - \omega^2 y^2) - \gamma(t)XY - \frac{m}{2}(\dot Y^2 - \omega^2 Y^2)\right\}dt$$
$$\cdot P_\gamma(y_0, q_0, 0)\,e^{-2iq_0\chi_0/\hbar}\,dy_0\,dY_0\,dy_T\,dY_T\,dq_0 ,$$

where the inverse of Wigner's Fourier transform, equation 7, has been substituted for $\rho_y(0)$. The path integrals here have been carried out by Feynman, and the results appear in section 13 of reference (1). However, the integrations are more conveniently performed by another method also due to Feynman, reference (7). We show only the main steps of this method since it resembles closely the original derivation of the quantum mechanical probability of a path. We perform all the y integrations in y, y' coordinates. The integrand becomes, after integrating by parts in the time integral,

$$\delta(y'_\tau)\exp\left\{-\frac{2i}{\hbar}\left(m y_\tau \dot{y}'_\tau - m y_0 \dot{y}'_0 + q_0 y'_0\right)/\hbar\right\}\cdot$$

$$\exp\left\{\int_0^\tau -\frac{2i}{\hbar}\left[m(\ddot{y}' + \omega^2 y') - \gamma(t)x'\right] y\, dt\right\}\cdot$$

$$\exp\left\{\int_0^\tau -\frac{2i}{\hbar}\,\gamma(t)\,y'(t)\,x\,dt\right\} P_y(y_0, q_0, 0) \quad .$$

The path integral on y gives a delta functional of y' paths that satisfy the classical equation of motion. The y_τ integration gives $\delta(\dot{y}'_\tau)$. These deltas together with $\delta(y'_\tau)$ leave only the single y' path

$$y'_c(t) = -\frac{1}{m\omega}\int_t^\tau \sin\omega(t-s)\;\gamma(s)\,x'(s)\,ds \quad .$$

This gives y'_{c0}, $\dot{y}'_{c0}$, and $y'_c(t)$ to be substituted into the

remaining integral for $\mathcal{J}$, namely:

$$\mathcal{J} = \iint \exp\left\{\frac{2i}{\hbar}\left[m y_o \dot{y}'_o - q_o y'_o + \right.\right.$$
$$\left.\left. - \int_o^T \gamma(t) y'_c(t) x(t)\, dt\right]\right\} P_y(y_o, q_o, 0) \ .$$

This substitution and a little rearranging give equation 35 to complete the proof.

Equation 34 says that the oscillator only exerts forces of the form

$$f(t) = f_1(t) + f_2(t), \text{ where}$$

$$f_1(t) = a_c \gamma(t) \cos \omega t + a_s \gamma(t) \sin \omega t, \tag{36}$$
$$f_2(t) = \frac{\gamma(t)}{m\omega} \int_o^t \sin \omega(t-s)\, \gamma(s)\, x(s)\, ds.$$

The force f_1 represents the ringing of the initial conditions of y. It is a transient if there is resistance in the system. The distribution of f_1, described in equation 34 by $P_y(a_c, m\omega a_s, 0)$, always includes noise due to the quantum uncertainty, and may include other excitation, either known or uncertain. In some problems $P_y(a_c, m\omega a_s, 0)$ has the form of a function of $a^2 = a_c^2 + a_s^2$, say $P'(a^2)$. In this case the phase of f_1 is completely random (i.e. a uniform probability distribution from 0 to 2π), and the an-

plitude a is distributed by $aP'(a^2)$. We can write in this case

$$\mathcal{Y} = \int_0^{2\pi}\int_0^{\infty} \delta[f - f_2 - a\gamma(t)\cos(\omega t + \beta)P'(a^2)\,a\,da\,d\beta \qquad (37).$$

It is convenient to Fourier transform $f_2(t)$ for the case γ = constant, which mates the integral a convolution. Then we obtain*

$$F_2(\nu) = G_+(\nu)X_+(\nu) \; . \qquad (38)$$

Here capitals stand for the Fourier transforms, i.e.

$$F(\nu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-i\nu t}\,dt,$$

and the subscript + for the positive one-sided transform

$$X_+(\nu) = \frac{1}{2\pi}\int_0^{\infty} x(t)e^{-i\nu t}\,dt \; .$$

Also, G_+ is the one-sided transform of

$$g(t) = \frac{2\pi\gamma^2}{m\omega}\sin\omega t \; ,$$

---

* The symbol ν, like ω, stands for 2π times the frequency.

and is given by

$$G_+ = \frac{\nu^2}{m\omega}\left[-\pi i\,\delta(\nu-\omega) + \pi i\,\delta(\nu+\omega) + \frac{\omega}{\omega^2-\nu^2}\right].$$

If the system has been coupled forever, so that the integral in $f_2(t)$ has $-\infty$ as its lower limit, then $X(\nu)$ replaces $X_+(\nu)$.

To make our formalism exactly that of AC theory, we eliminate the use of negative frequencies. We write for $f(t)$ (or similarly for $x(t)$ or $y(t)$) the inverse transform

$$f(t) = \int_{-\infty}^{\infty} F(\nu)e^{i\nu t}\,d\nu = \int_0^{\infty}\left(F(\nu)e^{i\nu t} + F(-\nu)e^{-i\nu t}\right)d\nu.$$

But since f(or x, or y) is real, $F(-\nu) = F^*(\nu)$. Now we write $\hat{F}(\nu) = 2F(\nu)$, throwing away terms that are either zero or negligible for positive frequencies, and we have

$$f(t) = \int_0^{\infty}\frac{1}{2}\left[\hat{F}(\nu)e^{i\nu t} + \hat{F}^*(\nu)e^{-i\nu t}\right]d\nu$$

$$= \int_0^{\infty}\left\{\mathrm{Re}\,\hat{F}(\nu)\cos\nu t - \mathrm{Im}\,\hat{F}(\nu)\sin\nu t\right\}d\nu.$$

Thus, $\hat{F}$ is the familiar phasor of AC theory. Its real part is the amplitude with cosine phase, and its imaginary part

the amplitude with minus sine phase. When using phasors, we can discard terms in G, or other quantities derived from a convolution, that are negligible for positive frequency. However, we omit the circumflex on such quantities, because it indicates a factor of two, and quantities derived from a convolution always multiply a phasor which already has this factor.

We can now rewrite equation 38 as

$$\hat{F}_2(\nu) = G(\nu)\hat{X}(\nu) = -\left(\frac{i}{\nu}G(\nu)\right)\left(i\nu\hat{X}(\nu)\right) = -Z(\nu)\hat{\dot{x}}(\nu) \ ,$$

where

$$Z(\nu) = \frac{i}{\nu}G(\nu) = \frac{\gamma^2}{m\omega^2}\left[\pi\delta(\nu-\omega) - \frac{i\omega}{\nu^2-\omega^2}\right] \ .$$

Since $i\nu X(\nu)$ is the transform of $\dot{x}(t)$, we interpret Z as the impedance which x sees as a result of coupling to y.

We can also relate G to the complex admittance $A(\nu)$ that the force γx sees in the y circuit. The response velocity is $\dot{y} = F/\gamma$, and we can write

$$\hat{\dot{Y}}(\nu) = i\nu\hat{Y}(\nu) = i\nu\frac{F(\nu)}{\gamma} = i\nu\frac{G\hat{x}}{\gamma} \ .$$

But admittance is defined by

$$\hat{\dot{x}}(\nu) = -A(\nu)\ \widehat{(\text{force})} = -A(\nu)\gamma\,\hat{x}(\nu).$$

We compare and identify

$$A(\nu) = -\frac{i\nu\, G(\nu)}{\gamma^2}.$$

We substitute $G(\nu)$ into the above and separate the conductance, $g = \mathrm{Re}A$, and the susceptance, $h = \mathrm{Im}A$, obtaining,

$$g(\nu) = \frac{\pi\nu}{m\omega}\,\delta(\nu-\omega),$$

$$h(\nu) = \frac{1}{m}\,\frac{\nu}{\nu^2-\omega^2}.$$

This compares to the susceptance of an LC circuit in AC theory,

$$h(\nu) = \frac{1}{L}\,\frac{\nu}{\nu^2-\omega^2},$$

which can easily be changed to the familiar expression for the reactance,

$$\frac{1}{h} = \nu L - \frac{1}{\nu C},$$

by the substitution $\omega^2 = (LC)^{-1}$.

C. 2. Oscillator Initially at Temperature θ

We shall now find the function $P_y(a_c, m\omega a_s, 0)$ of equa-

tion 34, which describes the noise from a harmonic oscillator which was initially at temperature θ. The initial density matrix that we want is

$$\rho(Y_o, y_o) = \sum_n e^{-E_n/k\theta} \phi_n^*(Y_o)\,\phi_n(y_o)\,.$$

We can find this sum in closed form because the propagator for the wave function of a harmonic oscillator has this form and is given in reference (1). The propagator is

$$\begin{aligned} K(x_T, x_o, T) &= \int_{x_o}^{x_T} e^{iS[x]/\hbar}\,\mathcal{D}x(t) \\ &= \sum_n e^{-iE_n t/\hbar}\,\phi_n(x_T)\,\phi_n^*(x_o) \\ &= e^{iS[x_c(t)]/\hbar} \\ &= \exp\left\{\frac{im\omega}{2\hbar \sin\omega T}\left[\cos\omega T(x_T^2 + x_o^2) - 2x_o x_T\right]\right\}. \end{aligned}$$

Comparison of these expressions for K and ρ shows that

$$\begin{aligned} \rho(Y_o, y_o) &= K(y_o, Y_o, \hbar/ik\theta) \\ &= \exp\left\{\frac{-m\omega}{2\hbar \sinh \hbar\omega/k\theta}\left[\cosh\frac{\hbar\omega}{k\theta}\,(y_o^2 + Y_o^2) + y_o^2 - Y_o^2\right]\right\}. \end{aligned}$$

The transformation (equation 6) to P(y, q, 0) is a simple Gaussian integration. The result is

$$P(y, q, 0, \theta) = \exp\left\{-\frac{1}{\alpha(\theta,\omega)}\left(\frac{m\omega^2 y^2}{2} + \frac{q^2}{2m}\right)\right\}, \qquad (39)$$

where α is just the mean energy of a harmonic oscillator given by

$$\alpha(\theta,\omega)=\hbar\omega\left(\frac{1}{2}+\frac{1}{e^{\hbar\omega/k\theta}-1}\right). \tag{40}$$

Note that $\alpha\to\theta k$ as $\theta\to\infty$, $\alpha\to\frac{\hbar\omega}{2}$ as $\theta\to 0$.

Note in equation 39 that the quantity in parentheses is just the potential plus kinetic energy. Finally, the noise distribution for equation 34 is given by

$$P(a_c, m\omega a_s, 0, \theta)=\exp\left\{-\frac{m\omega^2}{2\alpha(\theta,\omega)}(a_c^2+a_s^2)\right\}. \tag{41}$$

Thus the noise phase is completely random, and the noise amplitude is $\gamma(t)a$, where a has the probability density $aP'(a^2)$ of equation 37, which in this case is

$$aP'(a^2)=a\exp\left\{-\frac{m\omega^2a^2}{2\alpha(\theta,\omega)}\right\}.$$

D. Resistor (Continuum of Oscillators) at Temperature θ

If an excited system is coupled to a relatively cold resonant system, the excited system at first loses energy as it would to a lossy circuit element or resistor, but soon the energy comes back like the beating of two normal

modes of weakly coupled oscillators. To represent a resistance, we need a series of coupled systems so vast that the energy is hopelessly sidetracked for the time intervals involved in the problem. That is, we need an infinite number of systems weakly coupled to x, and coupled any way among themselves. This is illustrated in figure 1 with couplings γ' to x(t).

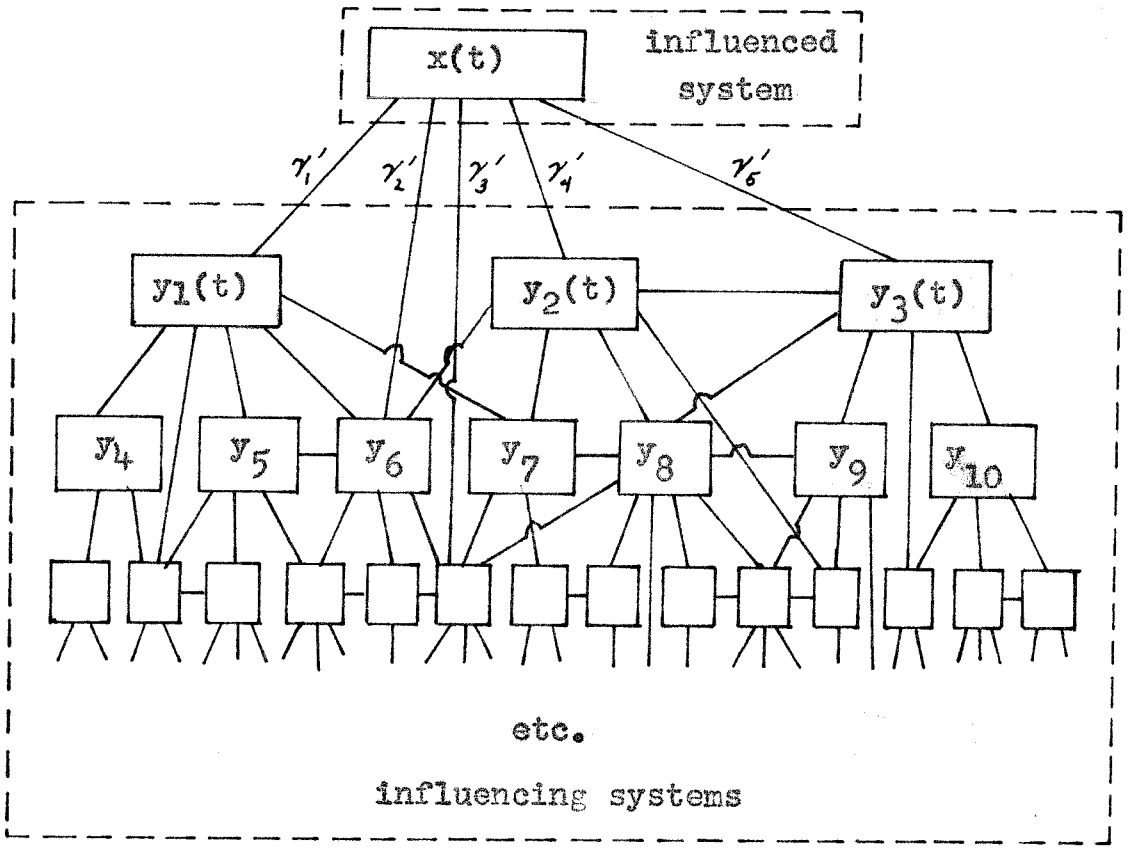

Figure 1

As long as there are enough y systems to make a con-

tinuum of resonances, the x system cannot distinguish between linear and non-linear y systems. Therefore, we take the y systems to be harmonic oscillators, in which case they can be represented by normal modes with new coupling constants γ to x(t). That is, y becomes an infinite number of systems, each weakly coupled to x, but not to one another, as in figure 2.

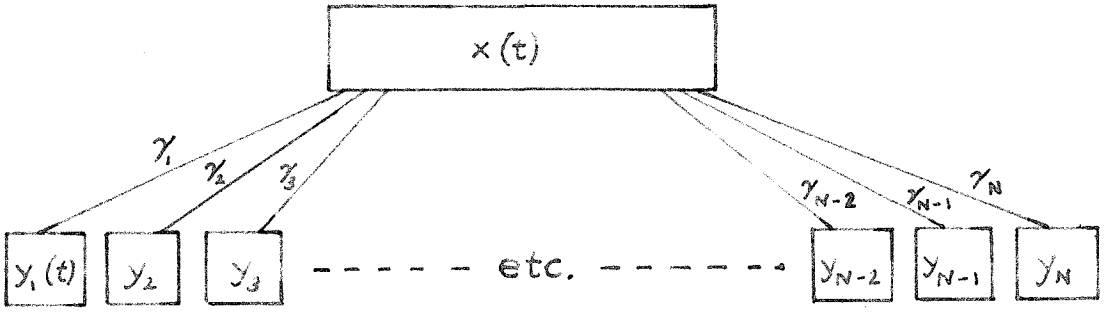

Figure 2

For a good representation of a resistor, there should be so many modes that we can regard them as a continuum. We take $D(\omega)\, d\omega$ to be the number of oscillators resonant between ω and $\omega+d\omega$, each with constant coupling $\gamma(\omega)$.

D.1. Derivation of Impedance and Noise

We can most easily pass from one oscillator to a continuum by using the functional $\mathcal{F}$ instead of $\mathcal{L}$, since $\mathcal{F}$ for many systems in parallel is just the product of indi-

vidual $\mathcal{F}$'s, equation 26. Since the individual $\mathcal{F}$'s have exponential form, the product will be just an integration in the exponent.

We first find $\mathcal{F}_1$ for one oscillator at temperature θ. We have only to substitute equation 41 into 35 and perform the Gaussian integrations. The result is

$$\mathcal{F}_1 = \exp\left\{ \iint_{s<t}^{\infty}_{-\infty} \frac{-2\gamma^2}{\hbar m\omega}\left[i x'(t)x(s)\sin\omega(t-s) + \frac{2\alpha(\theta,\omega)}{\hbar\omega} x'(t)\,x'(s)\cos\omega(t-s)\right] dt\,ds \right\}, \tag{42}$$

where we have used infinite time limits assuming the resistor is coupled forever. The complete $\mathcal{F}$ is formed by multiplying the exponent of equation 42 by $D(\omega)$ and integrating over all ω.

$$\mathcal{F} = \exp\left\{ \int_0^{\infty} \iint_{s<t}^{\infty}_{-\infty} -\frac{2\Gamma(\omega)}{\hbar\omega}\left[i x'(t)x(s)\sin\omega(t-s) + \frac{2\alpha(\theta,\omega)}{\hbar\omega} x'(t)\,x'(s)\cos\omega(t-s)\right] dt\,ds\,d\omega \right\}, \tag{43}$$

where $\Gamma(\omega) = D(\omega)\,\gamma^2(\omega)/m(\omega)$.

We now wish to transform back to $\mathcal{L}$. This is difficult to do with equations 43 and 24 as they stand, because the

factors $\sin \omega(t-s)$ and $\cos \omega(t-s)$ couple x' at one time to x and x' at other times. Hence, the path integrand does not factor into many identical integrals. However, we can sum paths of x' easily by expressing x' as a Fourier series,

$$x'(t) = X'^{(c)}(\nu_1)\cos \nu_1 t + X'^{(c)}(\nu_2)\cos \nu_2 t + \cdots$$

$$- X'^{(s)}(\nu_1)\sin \nu_1 t - X'^{(s)}(\nu_2)\sin \nu_2 t + \cdots ,$$

and then integrating over the amplitudes $X'^{(c)}$ and $X'^{(s)}$ of each frequency. We use only positive frequencies, so $X'^{(c)}$ and $X'^{(s)}$ are just the real and imaginary parts of our phasor $\hat{X}'(\nu)$. In the limit as the frequency spacing vanishes this becomes a path integral in frequency space over the paths $X'^{(c)}(\nu)$ and $X'^{(s)}(\nu)$. In other words, we make a transformation from the variables $x'(t_1)$, $x'(t_2)$, $\cdots$ at each time, to the variables $X'^{(c)}(\nu_1)$, $X'^{(s)}(\nu_1)$, $X'^{(c)}(\nu_2)$, $\cdots$, at each frequency. The transformation is linear, so the Jacobian is constant, and we ignore it like any other normalization; that is,

$$\mathcal{D}x'(t) = dx'_1 dx'_2 \cdots dx'_j \cdots = \left| \frac{\partial x'_j}{\partial X'^{(c,s)}_k} \right| dX'^{(c)}_1 dX'^{(s)}_1 dX'^{(c)}_2 \cdots$$

$$= \text{constant} \cdot \mathcal{D}X'^{(c)} \mathcal{D}X'^{(s)}(\nu) .$$

We can put the path integrand in terms of $X'^{(c)}$ and $X'^{(s)}$ by

first rewriting equation 43 in the form:

$$\mathcal{J} = \exp\left\{\iint_{-\infty}^{\infty} -\frac{e}{\hbar}\left[ix'(t)x(s)u_s(t-s) + x'(t)x'(s)u_c(t-s)\right]dt\,ds\right\}, \tag{44}$$

where

$$u_s(t) = 1(t)\int_0^{\infty}\frac{\Gamma(\omega)}{\omega}\sin\omega t\,d\omega\,,$$

$$u_c(t) = 1(t)\int_0^{\infty} 2\frac{\Gamma(\omega)\alpha(\theta,\omega)}{\hbar\omega^2}\cos\omega t\,d\omega\,, \tag{45}$$

and $$1(t) = \begin{cases} 0, & t<0 \\ 1, & t>0 \end{cases}.$$

We can now express equation 44 in terms of Fourier transforms by using the well known theorem for the integral of the product of two transforms, and the convolution theorem. Equation 44 becomes

$$\mathcal{J} = \exp\left\{-\frac{2\pi}{\hbar}\int_{-\infty}^{\infty}\left[i\left(X'^*(\nu)X(\nu)U_s(\nu) + X'X^*U_s^*\right) + X'X'^*\left(U_c + U_c^*\right)\right]d\nu\right\}.$$

The functional used to transform this expression to $\mathcal{J}$ is given by

$$\exp\int_{-\infty}^{\infty}\frac{2i}{\hbar}f(t)x'(t)dt = \exp\left\{\frac{2\pi i}{\hbar}\int_{-\infty}^{\infty}\left[F^*(\nu)X'(\nu) + FX'^*\right]d\nu\right\}.$$

We change to only positive frequencies in these two expressions, substituting $\hat{F}, \hat{X}$, and $\hat{X}' = X'^{(c)} + iX'^{(s)}$. Then we multiply them and perform the path integrations on $X'^{(c)}$ and $X'^{(s)}$ to find the forcing functional. The path integrals are

$$\mathcal{J} = \int\int\int \exp \int_0^\infty \frac{2\pi}{\hbar} \Big\{ -i \big[X'^{(c)} \operatorname{Re} \hat{F} + X'^{(s)} \operatorname{Im} \hat{F} \big]$$

$$+ i \big[X'^{(c)} \operatorname{Re} U_s \hat{X} + X'^{(s)} \operatorname{Im} U_s \hat{X} \big]$$

$$+ \big[X'^{(c)2} + X'^{(s)2} \big] \operatorname{Re} U_c \Big\} \, d\nu \, \mathcal{D}X'^{(c)} \mathcal{D}X'^{(s)} .$$

When we express the exponent as a Riemann sum with frequency spacing ϵ, the whole expression factors into an infinite number of independent identical integrals. Since they are identical, we need do only one, namely,

$$\int_{-\infty}^{\infty} \exp \left\{ -\frac{2\pi\epsilon}{\hbar} \left[X'^{(c)2} \operatorname{Re} U_c + iX'^{(c)} \left(\operatorname{Re} U_s \hat{X} - \operatorname{Re} \hat{F} \right) \right] \right\} dX'^{(c)} \cdot$$

$$\cdot \int_{-\infty}^{\infty} \exp \left\{ -\frac{2\pi\epsilon}{\hbar} \left[X'^{(s)2} \operatorname{Re} U_c + iX'^{(s)} \left(\operatorname{Im} U_s \hat{X} - \operatorname{Im} \hat{F} \right) \right] \right\} dX'^{(s)} .$$

This is a simple Gaussian integration. The result is

$$\exp \left\{ -\frac{\pi\epsilon \left| \hat{F} - U_s(\nu) \hat{X}(\nu) \right|^2}{2\hbar \operatorname{Re} U_c} \right\} .$$

When all such factors are put together, the sum in the exponent is clearly the Riemann sum of an integral, so we have

$$\mathcal{P}[x,f] = \exp\left\{-\frac{\pi}{2\hbar}\int_0^\infty \frac{|\hat{F}(\nu) - U_s(\nu)\hat{x}(\nu)|^2}{\mathrm{Re}\ U_c(\nu)}\,d\nu\right\}. \qquad (46)$$

Now we calculate U_c and U_s using equations 45 and obtain

$$U_s = -\frac{i}{2}\frac{\Gamma(\nu)}{\nu} + \frac{1}{2\pi}\int_0^\infty \frac{\Gamma(\omega)}{\nu^2-\omega^2}\,d\omega$$

and

$$\mathrm{Re}\ U_c = \frac{\alpha(\theta,\nu)\,\Gamma(\nu)}{\hbar\nu^2}.$$

Substitution in equation 46 gives finally

$$\mathcal{P}[x,f] = \exp\left\{-\int_0^\infty \frac{|\hat{F}(\nu) + i\nu Z(\nu)\hat{x}(\nu)|^2}{2W(\nu)}\,d\nu\right\}, \qquad (47)$$

where $Z(\nu) = i\,U_s(\nu)/\nu$ and $W(\nu) = \frac{\hbar}{\pi}\,\mathrm{Re}\ U_c(\nu)$,

or $$Z(\nu) = \frac{\Gamma(\nu)}{2\nu^2} + \frac{i}{2\pi\nu}\int_0^\infty \frac{\Gamma(\omega)\,d\omega}{\nu^2-\omega^2}, \qquad (48)$$

and $$W(\nu) = \Gamma(\nu)\,\alpha(\theta,\nu)/\pi\nu^2. \qquad (49)$$

To interpret equation 47, we note that the most prob-

able force is

$$\hat{F}(\nu) = -Z(\nu)\,(i\nu\hat{x}(\nu)).$$

Since $i\nu\hat{x}(\nu)$ is the phasor of $\dot{x}(t)$, we identify $Z(\nu)$ as the impedance in the x system caused by coupling to the y systems. The resistive and reactive parts are respectively

$$R = \frac{\Gamma(\nu)}{2\nu^2}, \qquad S = \frac{1}{2\pi\nu}\int_0^\infty \frac{\Gamma(\omega)}{\nu^2-\omega^2}\,d\omega .$$

Note that the reactance is inductive (positive) if the y systems have mostly low frequencies, and capacitive (negative) if they have mostly high frequencies.

The function $W(\nu)$ describes the noise. The larger W is in a given frequency range, the more $\hat{F}$ can deviate from $-i\nu Z\hat{x}$ in that range with appreciable probability. In the next subsection we shall relate W to conventional noise theory. Let us define a noise force phasor as the amount by which $\hat{F}$ deviates from $-i\nu Z\hat{x}$.

$$\hat{F}_N = \hat{F} + i\nu Z\hat{x} \tag{50}$$

Then we can substitute this in equation 47 and write the probability of a noise force as

$$\mathcal{P}_N[\hat{F}_N(\nu)] = \exp\left\{-\int_0^\infty \frac{|\hat{F}_N(\nu)|^2}{2W(\nu)}\,d\nu\right\}. \tag{51}$$

Normally, the resistance is obtained by measurement rather than by calculation from the couplings described by $\Gamma(\omega)$. For example, one would measure the Q of a lossy maser cavity directly, rather than calculate all the couplings to the electronic modes in the metal. So we wish to express the noise function $W(\nu)$ directly in terms of resistance. Comparing equations 48 and 49 we have

$$W(\nu) = \frac{2}{\pi} R(\theta, \nu)\, \alpha(\theta, \nu) \ . \qquad (52)$$

If the resistance does not vanish at absolute zero temperature, then neither does the noise, since $\alpha \to h\nu/2$. In other words, if couplings do not disappear as $\theta \to 0$, then neither does the noise. This statement, combined with similar statements about the circuit elements we have not yet treated, constitutes one of the more important results of this paper. The nature of zero temperature noise is discussed fully in the section on the maser amplifier system (VI). Briefly, we can understand this noise in terms of the theorem that linear systems propagate only along classical paths. The ground state Wigner density describes initial conditions with Gaussian uncertainty, qualitatively like any other kind of Gaussian uncertainty. Those initial conditions propagate like classical oscillations among the many coupled oscillators of figure 1.

As a result, the x system feels a noisy force qualitatively like the noise of higher temperature.

Equations 48 through 52 describe rigorously a resistor consisting of oscillators. In fact they describe a real resistor with the same density of resonances $\Gamma(\omega)$. One might suspect that linear and non-linear resonances contribute different proportions of resistance and noise. We remove this doubt when we treat non-linear systems. We shall show that each resonance contributes a noise force and a sure force. The proportion of these is independent of the other resonances which determine the linearity of the system. We shall derive these forces after we digress to interpret the noise function $W(\nu)$.

D.2. Connection to Conventional Noise Theory

We shall prove that $W(\nu)$ is the power spectrum of the noise force, reference (8). That is, we show that

$$W(\omega) = \lim_{T\to\infty} \left\langle \frac{\pi}{T} \left| \hat{F}_N(\omega) \right|^2 \right\rangle , \qquad (53)$$

where T is the length of time that the force acts, and the expected value is taken with the probability density of equation 51.* The limiting process means that we take a discrete frequency spacing $\epsilon/2\pi = 1/T$, and say that the force acts for only one period T. With discrete frequencies equation 51 becomes

* Equation 53 is identical with equation 131 of reference (8) except for a numerical factor resulting from different conventions.

after normalization

$$\mathcal{P}_N = \prod_j \left\{ \exp\left[-\frac{|\hat{F}_N(j\epsilon)|^2 \epsilon}{2W(j\epsilon)} \right] \frac{\epsilon}{2\pi W(j\epsilon)} \right\} .$$

We can immediately integrate all the F's except $F(\omega)$, because they do not appear in equation 53. This leaves for the expected value in equation 53 the integral:

$$\left\langle \frac{\epsilon}{2} |\hat{F}_N(\omega)|^2 \right\rangle = \iint_{-\infty}^{\infty} \frac{\epsilon}{2}(u^2+v^2) \exp\left\{ -\frac{u^2+v^2}{2W(\omega)}\epsilon \right\} \frac{\epsilon\, du\, dv}{2\pi W(\omega)}$$

$$= W(\omega) .$$

We have let u and v be the real and imaginary parts of $\hat{F}_N(\omega)$ under the integral signs. In the answer, $W(\omega)$, the ϵ factors have cancelled, so the limiting process in equation 53 is trivial. We have proved equation 53 and identified $W(\nu)$ as the power spectrum. We could also easily show that

$$\lim_{T\to\infty} \left\langle \frac{\pi}{T} F_N^*(\omega_1) F_N(\omega_2) \right\rangle = 0 , \quad \omega_1 \neq \omega_2 ,$$

$$\lim_{T\to\infty} \left\langle \frac{\pi}{T} F_N(\omega_1) F_N(\omega_2) \right\rangle = 0 , \quad \text{all } \omega_1, \omega_2 . \tag{54}$$

The way we have defined $W(\nu)$, the numerical factors are so adjusted that the expected total noise power delivered to the x system is given by

$$\left\langle \text{power} \right\rangle = \int_0^\infty W(\nu)\, g(\nu)\, d\nu ,$$

where $g(\nu)$ is the conductance of the x system.* In an amplifier problem it is important to have noise expressed as a frequency spectrum, because that part which is not in the band width of the signal can be filtered out in a later stage of amplification.

Another interesting way to relate $W(\nu)$ to conventional noise theory is through the autocorrelation function $R(\tau)$, defined by

$$R(t-s) = \left\langle f_N(t)\, f_N(s) \right\rangle . \qquad (55)$$

$R(0)$ is just the mean square noise force at a single time. When equation 55 is expressed in terms of phasors, we ob-

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* This is easily proved from equation 53 and

$$\left\langle \text{power} \right\rangle = \lim_{T\to\infty} \left\langle \frac{1}{T} \int_{-T/2}^{T/2} f(t)\, \dot{x}(t)\, dt \right\rangle .$$

First express the integral in terms of Fourier transforms, then put $\hat{x}(\nu) = A(\nu)\, \hat{F}(\nu)$, where A is the admittance, of which g is the real part.

tain

$$R(t-s)=\frac{1}{4}\left\{\left\langle\iint_{J J_0}^{\infty}\hat{F}_N(\nu)\hat{F}_N(\eta)e^{i(\nu t+\eta s)}d\eta d\nu\right\rangle+\left\langle\iint_{J J}\hat{F}_N(\nu)F_N^{*}(\eta)e^{i(\nu t-\eta s)}d\eta d\nu\right\rangle\right.$$

$$\left.+\text{complex conjugates}\right\}\ .$$

The calculation of the expected values in the above expression is practically the same as the calculations leading to equations 53 and 54. Therefore, we give only the answers. Since it is no added difficulty, we give the expected values of the most general functionals of second order in F. They are

$$\left\langle\iint_{J J_0}^{\infty}\hat{F}_N(\nu)\hat{F}_N^{*}(\eta)\phi(\nu,\eta)d\nu d\eta\right\rangle=\int_0^{\infty}W(\nu)\phi(\nu,\nu)d\nu \tag{56}$$

and

$$\left\langle\iint_{J J_0}^{\infty}\hat{F}_N(\nu)\hat{F}_N(\eta)\phi(\nu,\eta)d\nu d\eta\right\rangle=0\ .$$

Now we obtain R from the special case $\phi(\nu,\eta)=\exp i(\nu t\pm\eta s)$. We have

$$R(\tau)=\int_0^{\infty}\cos\nu\tau\, W(\nu)d\nu=\frac{1}{2}\int_{-\infty}^{\infty}e^{i\nu\tau}W(|\nu|)d\nu. \tag{57}$$

We have written the second form for comparison to those conventional treatments that use negative frequencies.*

* Our equations 55 and 57 correspond to equations 117 and 136 of reference (6).

Equation 57 says that the autocorrelation function is the transform of the power spectrum. This is a well known fact of conventional noise theory.

The distribution of the noise force f_N at any time, not asking about any other time, is

$$P_N(f_N) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-f_N^2/2\sigma^2},$$

where (58)

$$\sigma^2 = \int_0^\infty W(\nu)\,d\nu,$$

if σ^2 converges.

We could derive this from equation 51 however, we can prove equation 58 more elegantly by using the central limit theorem. This theorem tells us that the distribution of the sum of n independent random variables approaches a Gaussian distribution as $n \to \infty$, regardless of the distribution of the separate variables. We have not just n, but a continuum of independent normal modes. We are asking the distribution of the sum of the forces due to each, so we are assured that this distribution is Gaussian. Knowing this, we need just two parameters to prove equation 58, the mean and the variance. The noise has mean zero, and the variance is given by equation 57:

$$\sigma^2 = R(0) = \left\langle f_N^2 \right\rangle = \int_0^\infty W(\nu)\,d\nu.$$

Our description of noise by path integral theory is interesting in its own right, apart from quantum mechanics. With the probability of a path, such as $\mathcal{P}_N[\hat{F}_N(\omega)]$ in equation 51, we can answer the most general questions about the noise, within our ability to perform the integrations. For instance, we can calculate higher order correlation functions such as

$$\langle f(t_1)\, f(t_2) \cdots f(t_n) \rangle .$$

E. Non-Linear System

In this section we derive the forces of a general non-linear y system which is initially in an energy eigenstate. We restrict ourselves to the case in which there is little transition probability for y during the interaction time of the x and y systems. That is, we develop a perturbation series in powers of the coupling. Also we assume that the coupling of x and y is linear and given by $S_I[x,y]=\int_0^T \gamma(t)x(t)y(t)\,dt$. We discuss this case in order to prepare for a treatment of the molecular beam of a beam-type maser. When the signal in the maser cavity is so weak that the present treatment with quantized fields must be used, the condition that the molecules have little transition probability during their passage through the cavity is more than adequately satisfied.

First, we shall find the influence functional using its path integral definition, equation 22. We begin now a rather lengthy derivation which ends in equation 52, an expression for the influence functional, valid to order γ^2.

Since we are assuming little transition probability, the action $S_I[x,y]$ is small, and we expand $\exp\{S_I[x,y] - S_I[x',y']\}$ of equation 22 in a power series. The nth term of this series gives the nth term $\mathcal{J}^{(n)}$ in the expansion of $\mathcal{J}$. Hence, we have instead of equation 22

$$\mathcal{J}^{(n)} = \iiiint dy_0\, dy_0'\, dy_T\, dy_T' \int_{y_0}^{y_T} \mathcal{D}y \int_{y_0'}^{y_T'} \mathcal{D}y'\, \delta(y_T - y_T') \cdot$$
$$\cdot \frac{1}{n!}\left(\frac{i}{\hbar}\right)^n \left(S_I[x,y] - S_I[x',y']\right)^n \cdot$$
$$\cdot \exp\left\{\frac{i}{\hbar}\left(S_y[y] - S_y[y']\right)\right\} \rho(y_0', y_0, 0) \quad .$$

It is readily shown that

$$\frac{1}{n!}\left(S_I[x,y] - S_I[x',y']\right)^n = \frac{1}{n!}\left[\int_0^T \gamma(t)\left(xy - x'y'\right) dt\right]^n$$
$$= \int_{t_1 < t_2 \cdots < t_n} \cdots \int_0^T \prod_{r=1}^{n} \gamma(t_r)\left[x(t_r)y(t_r) - x'(t_r)y'(t_r)\right] dt_r \quad .$$

We could continue to find an expression for the nth term $\mathcal{J}^{(n)}$, but a rather cumbersome set of sums and products results, where even the dummy indices have indices. Instead,

we find $\mathcal{J}^{(2)}$ as an example. In this paper we need only $\mathcal{J}^{(0)}$, $\mathcal{J}^{(1)}$, and $\mathcal{J}^{(2)}$. Substituting the last expression into $\mathcal{J}^{(n)}$, and setting $n=2$, gives

$$\mathcal{J}^{(2)} = \int\int\int\int dy_0\, dy_0'\, dy_T\, dy_T' \int_{y_0}^{y_T}\int_{y_0'}^{y_T'} \mathcal{D}y\, \mathcal{D}y'\, \delta(y_T - y_T') \cdot$$

$$\cdot\left(-\frac{1}{\hbar^2}\right)\iint_{s<t} \gamma(t)\gamma(s)\left[x(t)y(t) - x'(t)y'(t)\right]_{t\to s} dt\, ds \cdot$$

$$\cdot \exp i\left(S_y[y] - S_y[y']\right) \phi_m^*(y_0')\, \phi_m(y_0) \quad ,$$

where ϕ_m is the initial energy eigenfunction, and we have put $\rho(0) = \phi_m^*(y_0')\, \phi_m(y_0)$. Now we perform the path integration over y at all times except 0, s, t, and T. For these integrations everything is constant except the exponential, but its integration just leads to the kernals for propagating the wave functions if there were no coupling. We express these kernals as infinite sums:

$$\int_{y_1}^{y_2}\int_{y_1'}^{y_2'} \exp\left\{\frac{i}{\hbar}\left(S_y[y] - S_y[y']\right)\right\} \mathcal{D}y\, \mathcal{D}y' =$$

$$\left[\sum_j e^{-iE_j(t_2 - t_1)}\, \phi_j(y_2)\, \phi_j^*(y_1)\right]_{y\to y'}, \text{ complex conj.},$$

where we have used t_1 to t_2 to represent any of the three intervals between times 0, s, t, and T. Substituting the

above into $\mathcal{F}^{(2)}$, we obtain

$$\mathcal{F}^{(2)}(x',x) = -\frac{1}{\hbar^2}\int\int\int\int\int\int\int\int dy_o\, dy_o'\, dy_s\, dy_s'\, dy_t\, dy_t'\, dy_T\, dy_T'$$

$$\delta(y_T - y_T')\iint_{s<t} dt\, ds\left[x(t)y_t - x'(t)y_t'\right]\left[x(s)y_s - x'(s)y_s'\right]\gamma(t)\gamma(s)\cdot$$

$$\cdot\left\{\sum_i e^{-iE_i(T-t)}\phi_i(y_T)\phi_i^*(y_t)\right\}\left\{\sum_j e^{iE_j(T-t)}\phi_j^*(y_T')\phi_j(y_t')\right\}\cdot$$

$$\cdot\left\{\sum_k e^{-iE_k(t-s)}\phi_k(y_t)\phi_k^*(y_s)\right\}\left\{\sum_\ell e^{iE_\ell(t-s)}\phi_\ell^*(y_t')\phi_\ell(y_s')\right\}\cdot$$

$$\cdot\left\{\sum_q e^{-iE_q s}\phi_q(y_s)\phi_q^*(y_o)\right\}\left\{\sum_r e^{iE_r s}\phi_r^*(y_s')\phi_r(y_o')\right\}\cdot$$

$$\cdot\phi_m^*(y_o')\phi_m(y_o) \quad .$$

Next we perform the integrations indicated in the first line of this expression. We obtain Kronecker deltas and matrix elements,

$$Y_{ij} = \int \phi_i^*(y)\, y\, \phi_j(y)\, dy, \qquad \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \quad ,$$

from integrating the orthonormal eigenfunctions. The result is

$$\mathcal{F}^{(2)}[x',x] = \iint_{s<t} dt\, ds \sum_{ijk\ell qr} \gamma(t)\,\gamma(s)\left(-\frac{1}{\hbar^2}\right)$$

$$\cdot\left\{x(t)x(s)\delta_{ij}Y_{ik}\delta_{j\ell}Y_{kq}\delta_{r\ell}\delta_{qm}\delta_{rm}\right.$$

$$-x(t)x'(s)\delta_{ij}Y_{ik}\delta_{j\ell}\delta_{kq}Y_{r\ell}\delta_{qm}\delta_{rm}$$

$$-x'(t)x(s)\delta_{ij}\delta_{ik}Y_{\ell j}Y_{kq}\delta_{r\ell}\delta_{qm}\delta_{rm}$$

$$\left.+x'(t)x'(s)\delta_{ij}\delta_{ik}Y_{\ell j}\delta_{kq}Y_{r\ell}\delta_{qm}\delta_{rm}\right\}\cdot$$

$$\cdot\exp i\left[\omega_{ij}T + \omega_{ik}t + \omega_{\ell j}t + \omega_{kq}s + \omega_{r\ell}s\right] ,$$

where $\omega_{ij} = (E_i - E_j)/\hbar$. Summing the Kronecker deltas gives

$$\mathcal{J}_2[x',x] = \iint_{s<t} \gamma(t)\gamma(s) \sum_j |Y_{mj}|^2 \{ x(t)x(s)e^{-i\omega_{jm}(t-s)}$$

$$- x(t)x'(s)e^{i\omega_{jm}(t-s)} - x'(t)x(s)e^{-i\omega_{jm}(t-s)}$$

$$+ x'(t)x'(s)e^{i\omega_{jm}(t-s)} \} \, dt\,ds \left(-\frac{1}{\hbar^2}\right) .$$

After changing from $\mathcal{J}^{(2)}[x',x]$ to $\mathcal{J}^{(2)}[X,x]$, we finally obtain

$$\mathcal{J}^{(2)}[X,x] = -\frac{4}{\hbar^2}|Y_{mj}|^2 \iint_{s<t} \gamma(t)\gamma(s)x'(t)[x'(s)\cos\omega_{jm}(t-s)$$

$$+ ix(s)\sin\omega_{jm}(t-s)]\,dt\,ds .$$

It is easy to show by a similar derivation that

$$\mathcal{J}^{(1)}[X,x] = -2\frac{i}{\hbar}Y_{mm}\int_0^T \gamma(t)x'(t)\,dt ,$$

and that $\mathcal{J}^{(0)} = 1$ (of course, since the expression for $\mathcal{J}^{(0)}$ has no coupling to x, and is just the probability that if the y system starts in state m, it ends in some final state).

Let us examine the two approximations:

$$\mathcal{J} \approx 1 + \mathcal{J}^{(1)} + \mathcal{J}^{(2)}, \quad \text{and}$$

$$\mathcal{J} \approx \exp\left\{\mathcal{J}^{(1)} + \mathcal{J}^{(2)} - \frac{1}{2}\mathcal{J}^{(1)2}\right\} .$$

They have the same expansion to second order, but it turns out that only the latter is good enough to be useable. A first comment on this point is that we can transform the second expression to the forcing functional and obtain physically reasonable forces. The first expression, however, transforms to nonsense, since the integral over x at the ith time has the form

$$\int_{-\infty}^{\infty} (\text{Quadratic polynomial in } x_1 \cdots x_{T/\epsilon}) e^{2\pi i f_i x_i \epsilon/\hbar} dx_i,$$

which gives derivatives of delta functions. Let us further discuss the exponential form of perturbation series, supposing for simplicity that we are only interested in first order effects given by $\exp(\mathcal{J}^{(1)})$. Then we shall look at $\mathcal{J}^{(2)}$ to see the corrections to the first order effects. Comparing the form $\exp(\mathcal{J}^{(1)})$ to the classical force influence functional, equation 32, we see that it corresponds to a force $Y_{mm}\gamma(t)$. Such a force is easy to understand. If the y potential is lopsided, we can think of y in two parts, an average position Y_{mm}, and motion $\eta(t)$ about the average. The coupling term in the Lagrangian becomes $\gamma xy = \gamma(t) x Y_{mm} + \gamma'(t) x\eta$. The first of these terms contributes the same force $Y_{mm}\gamma(t)$ to the classical equation of motion. Now the second order

term of $\exp(\mathcal{J}^{(1)})$ is

$$\frac{1}{2}\mathcal{J}^{(1)2} = \frac{1}{2}\left(-2\frac{i}{\hbar}\right)^2\left(\int_0^T \gamma(t)\,x'(t)\,dt\right)^2$$

$$= -\frac{4}{\hbar^2}|Y_{mm}|^2 \iint_{s<t} \gamma(t)\gamma(s)x'(t)x'(s)\,dt\,ds,$$

but this is just the $j = m$ term of $\mathcal{J}^{(2)}$. So we see that as far as the $Y_{mm}\gamma(t)$ force is concerned, $\exp(\mathcal{J}^{(1)})$, is correct to second order, and in fact, it is correct to all orders. This idea continues to higher orders in the exponential form of perturbation series. The $j \neq m$ terms of $\mathcal{J}^{(2)}$ introduce new physical forces that come into play for stronger coupling or longer coupling time, namely, transitions to $j \neq m$ states caused by the oscillating η part of $y = Y_{mm} + \eta$. The exponential form of the second order perturbation series gives, correct to all orders, the terms corresponding to the first such forces to appear after coupling y to x. However, the second order neglects to say that as time goes on, the physical forces change in a way described by still higher terms in the perturbation series. These higher order terms say that the phase relation between x and y changes in a way not like linear systems (linear systems have second order exponents), because the frequency of y de-

ponds upon its amplitude.

Now we assemble our approximation to the influence functional in exponential form:

$$\mathcal{F}[x,\alpha] = \exp\left\{-\frac{2i}{\hbar}\gamma_{mm}\int_0^T \gamma(t)x'(t)\,dt\right.$$

$$-\sum_{j\neq m}\frac{4}{\hbar^2}\iint_{s<t}|\gamma_{mj}|^2\gamma(t)\gamma(s)x'(t)\big[x'(s)\cos\omega_{jm}(t-s) \tag{59}$$

$$\left.+i\,x(s)\sin\omega_{jm}(t-s)\big]\,dt\,ds\right\}.$$

To find the forces we need not transform this expression to $\mathcal{G}$, because we can identify it in terms of systems already familiar to us. Recall that the influence functional for many systems coupled to x but not to one another is just the product of the individual $\mathcal{F}$'s, (equation 26). So we can factor equation 59 and say that coupling our single system to x, is the same, to order γ^2, as coupling the many systems described by

$$\mathcal{F}_j = \begin{cases} \exp\left\{-\frac{4}{\hbar^2}|\gamma_{mj}|^2\iint_{s<t}\gamma(t)\gamma(s)x'(t)\big[x(s)\cos\omega_{jm}(t-s)\right. \\ \qquad\left.+i\,x(s)\sin\omega_{jm}(t-s)\big]\,dt\,ds\right\}, & j\neq m \\ \exp\left\{-\frac{2i}{\hbar}\gamma_{mm}\int_0^T\gamma(t)x'(t)\,dt\right\}, & j=m\,. \end{cases}$$

As we have already seen, $\mathcal{F}_m$ represents the classical driving force $Y_{mm}\gamma(t)$. If we compare $\mathcal{F}_j$, $j \neq m$, to equation 42, we find that the other equivalent systems are just harmonic oscillators initially in their ground states. Their masses are $\hbar/2\omega|Y_{mj}|^2$ *, and their resonant frequencies are $\omega_{jm}/2\pi$. However, ω_{jm} is negative if the initial state m is a higher energy level than the state j. This is an essential difference from the harmonic oscillator case where ω is always positive. We now examine the harmonic oscillator force, equation 36, and find the significance of negative ω. (The formal derivation of equation 36 from 59 would not depend upon this sign.) The force $f_1(t)$ gives the noise due to the quantum uncertainty in the initial conditions of the y system. To find the distribution of the coefficients a_s and a_c we put $\theta = 0$, $\alpha = \hbar\omega/2$, and $m\omega = \hbar/2|Y_{mj}|^2$ into the expression for thermal noise of an oscillator, equation 41. This gives for the present non-linear case, the probability density of a_s and a_c:

* Consider this perturbation theory applied to an oscillator initially in its ground state. Then we can equate this constant to m and obtain the correct matrix element

$$Y_{01} = \sqrt{\frac{\hbar}{2m\omega}} .$$

$$P\left(a_c, \frac{\hbar a_s}{2|Y_{mj}|^2}, 0, 0\right) = \exp\left[-\frac{a_c^2 + a_s^2}{2|Y_{mj}|^2}\right] .$$

Thus we have noise at each of the frequencies ω_{jm}, such that the noise coefficients a_s and a_c are independently distributed, each with a Gaussian distribution of mean zero and variance $|Y_{mj}|^2$.

When we change the sign of ω in $f_1(t)$ (equation 36 again) we get a sign change in the term $a_s \gamma(t) \sin \omega t$. This makes no difference, because the above distribution of a_s, is an even function. The term $f_2(t)$ gives the noise-free force. This term changes sign when ω does. As it stands in equation 36, f_2 represents an initial power absorption. If we put in a function $x(s)$ nearly resonant at ω, we get out a force in quadrature with x for a short time (for all time if x is exactly resonant and γ is constant). This force has the sign that results in power absorption from the x system. So the negative value of ω_{jm} when $E_m > E_j$ represents power emission to the x system, as must be the case if the y system is initially in the higher state of a pair of states resonant with x. Now we can summarize the forces of the non-linear system as follows: In our perturbation approximation the coupling of x and y by

the Lagrangian term $\gamma(t)\,xy$ causes three kinds of forces on x: (1) a classical driving force $Y_{mm}\,\gamma(t)$; (2) for every energy level j with non-zero matrix element Y_{mj} to the initial state m, a noisy force the same as f_1 of a harmonic oscillator initially in its ground state, namely,

$$f_{1j}(t) = a\,\gamma(t)\cos(\omega_{jm}t + \beta),$$

where the phase is completely random, and a has the probability density

$$P_j(a) = a\exp\left\{-\frac{a^2}{2|Y_{mj}|^2}\right\}$$

(We changed from a_s, a_c to the "polar" description a, β.); (3) for the same energy levels j, the noise-free forces

$$f_{2j}(t) = \frac{2|Y_{mj}|^2}{\hbar}\gamma(t)\int_0^T \sin\omega_{jm}(t-s)\,\gamma(s)x(s)\,ds,$$

which for $E_j > E_m$ are the same as f_2 of harmonic oscillators and absorb power from the x system, but for $E_j < E_m$ these forces emit power to x.

The nature of the approximation to order γ^2 is clear. The harmonic oscillator initially in its ground state

can go on absorbing resonant power forever, but the non-linear system emits after reaching the highest state of a particular resonance. Thus, the approximation that they are equivalent must correspond to little probability of finding the y system in this highest state. Likewise, the approximation that they are equivalent except for the sign of the f_2 part must correspond to little probability of finding the y system in the lowest state.

F. Molecular Beam

We shall consider a beam that consists entirely of identical molecules initially in the same energy eigenstate. In order to generalize to a beam that consists of molecules that are not identical and are not in the same initial state, one can simply regard the real beam as a superposition of the idealized beams. Then the $\mathcal{F}$ or $\mathcal{G}$ functionals for these beams can be compounded by the usual rules.

Each molecule of a real beam arrives at any reference point at random time. We could set up our problem with random arrival, but it is simpler and sufficient for our purposes to consider equally spaced molecules traveling at constant velocity. We are primarily interested in molecular beams as used in masers. These typically have 10^{14} molecules arriving per second, of which 2×10^{10} are coupled to the electromagnetic mode at

the same time. By considering uniform spacing, we miss fluctuations of $\sqrt{2 \times 10^{10}} = 1.4 \times 10^{5}$, less than 1 part in 10^{5}. If one wishes to correct for velocity distribution, he may consider the real beam as a superposition of idealized beams of uniform velocity.

In reference (3), Gordon, Zieger, and Townes describe the beams used in maser amplifiers and oscillators. Briefly, a nozzle and focuser are used to direct molecules into a resonant cavity. The molecules focused are the ones in the upper state of a pair of states resonant at the cavity frequency. Thus, there is net emission of radiation in the cavity, which can be used to amplify an input signal. Reference (3) also gives conventional maser theory with classical fields.

We represent a molecular beam as a group of identical non-linear systems each having a coupling function $\gamma(t-t_i)$. The ith molecule passes some reference point at a time t_i, different for each molecule. We take the t_i to be equally spaced at intervals n^{-1}, that is, there are n molecules per second arriving in the beam, and we assume that this goes on forever. We shall sum the influence of all the particles approximately by integrating over t_i. The function $\gamma(t)$ is, of course, non-zero only for a finite interval near $t = 0$.

When the molecules are coupled to oscillators that

are so weak that this quantum theory must be used, one may be assured that the perturbation expression of equation 59 is valid for any ordinary beam experiment in the RF frequency range (since spontaneous emission is slow below optical frequencies, so is emission due to only a few quanta.)

We start with equation 59, assuming that we are only interested in a resonance between two particular energy levels, the initial state n, and the final n. That is, of all the equivalent oscillators, we are interested only in the one that is nearly resonant with the RF mode being used in the maser or other molecular beam devices. Therefore, we ignore all but the nth term in the sums of equation 59, and we ignore the Y_{nn} term, since it is usually zero, and if not, it gives rise to the force $\gamma(t)\, Y_{nn}$ on x, the frequencies of which are far below RF.

Summing what remains of the exponent in equation 59 over all the molecules gives

$$\mathcal{J}[X,x] = \exp\left\{-\frac{4Y^2 n}{\hbar^2}\iint_{s<t}^{\infty}_{-\infty} r(t-s)\,x'(t)\cdot \right.$$
$$\left. \left[x'(s)\cos\omega(t-s) + ix(s)\sin\omega(t-s)\right]dt\,ds\right\}, \qquad (60)$$

where $Y = |Y_{nm}|$, $\omega = \omega_{nm}$, and

$$r(t-s) = \int_{-\infty}^{\infty} \gamma(t-t_i)\,\gamma(s-t_i)\,dt_i \quad . \qquad (61)$$

*

Equation 60 already has the form of equation 44, the resistor problem we have already solved, except now instead of u_s and u_c we have

$$r_s(t) = \frac{2Y^2 n}{\hbar}\, 1(t)\, r(t) \sin \omega t$$

$$r_c(t) = \frac{2Y^2 n}{\hbar}\, 1(t)\, r(t) \cos \omega t \quad . \qquad (62)$$

The answer appears in the form of equation 47, where now the impedance and the power spectrum of the noise are given by the transforms of equation 62:

$$Z(\nu) = \frac{i R_s(\nu)}{\nu} \quad , \qquad W(\nu) = \frac{\hbar}{\pi} \operatorname{Re} R_c(\nu) \quad . \qquad (63)$$

It is interesting to work out $R_s(\nu)$ and $R_c(\nu)$ for a simple coupling function. We take the Gaussian func-

---

* To show that γ is really a function of $t-s$, substitute $t_i = u + s$. Then equation 61 becomes

$$r = \int_{-\infty}^{\infty} \gamma\left[(t-s)-u\right] \gamma(-u)\, du \, .$$

tion

$$\gamma(t) = \frac{\gamma_o}{\sqrt{\pi}} e^{-(t/\tau)^2} ,$$

which is normalised to $\gamma_o \tau$. Performing the integration of equation 61 gives

$$r(t) = \frac{\tau \gamma_o^2}{\sqrt{2\pi}} e^{-t^2/2\tau^2} .$$

We substitute into equation 62, Fourier transform to find R_s and R_c, then substitute into equation 63. The result is

$$Z(\nu) = sg(\omega) \frac{n(\gamma \tau \gamma_o)^2}{4\pi\hbar\nu} \exp\left(-\frac{\tau^2}{2}(\nu - |\omega|)^2\right) \cdot$$

$$\cdot \left[1 + i\, \mathrm{erf}\left(\frac{-\tau}{\sqrt{2}}(\nu - |\omega|)\right)\right] \qquad \text{and} \qquad (64)$$

$$W(\nu) = \frac{n(\gamma \tau \gamma_o)^2}{4\pi^2} \exp\left(-\frac{\tau^2}{2}(\nu - |\omega|)^2\right) ,$$

where

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_o^x e^{-s^2} ds , \qquad sg(x) = \begin{cases} +1 , & x > 0 \\ -1 , & x < 0 \end{cases} .$$

Note that when the beam is initially in the lower state of the resonant pair, ω is positive, and the resistance term of Z is positive. However, if the beam is initially in the higher state, ω and the resistance are negative. It is this negative resistance of a beam that causes amplification in the resonant cavity of a beam-type maser. There is also a reactive term in Z which pulls the resonance of the whole system toward that of the molecules. Both the impedance and the noise are effective in a Gaussian width proportional to τ^{-1}.

It is possible to take account of the random arrival of molecules. To do this we would first find $\mathcal{J}$ for any set of arrival times, t_i, then average it over the distribution of these quantities. Equation 33 assures us that the effective influence functional is the quantity $\langle \mathcal{J} \rangle$ which results.

G. Solid State Maser

The solid state maser amplifier is a device which achieves amplification by stimulated emission from spin systems in a crystal lattice. To obtain emission between two energy levels, the population of the higher must exceed that of the lower. At least three energy levels are necessary so that transitions between levels not resonant with the signal can be used to prepare the populations

of the two levels that are resonant with the signal. Bloembergen, reference (4), originally proposed the solid state maser, and Scovil, Feher, and Seidel, reference (5), announced its successful operation.

We treat only the most idealized solid state maser in this paper. With our formalism we are equipped to treat many of the complicating effects of real masers; however, the treatment is lengthy and properly belongs in a paper devoted exclusively to the solid state maser. The idealizations are pointed out as they arise.

We treat the solid state maser as a collection of N three-level systems, each coupled to the field oscillators, but not to each other. Actually, the systems are spins in a crystal lattice. The energy levels are unequally spaced (figure 3). The middle level may be closer to either of the other two.

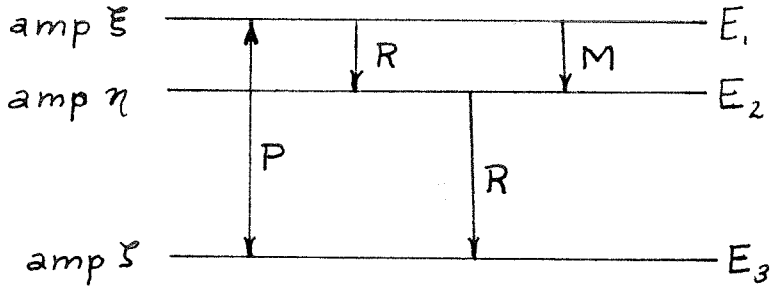

Figure 3

We let the letters ξ, η, and ζ stand for the amplitude to be in the states of energy E_1, E_2, and E_3 respectively, where $E_1 > E_2 > E_3$. A strong classical RF field is pumping levels 1 and 3 to maintain about equal population of these two states. Maser action takes place between another pair, say 1 and 2, and relaxation effects from the crystal lattice deplete the population of state 2, so that the maser action can continue in the steady state. The arrows P, R, and M of figure 3 indicate pumping, relaxation, and maser action respectively.

When there is no input signal to the maser, the probabilities of a system being in each of the three states reach the equilibrium values $|\xi_0|^2$, $|\eta_0|^2$, and $|\zeta_0|^2$. When we introduce a signal so weak that the present treatment with quantized fields is necessary, we may be assured that the signal does not upset these equilibrium values appreciably.

We could treat general equilibrium populations. However, for our purposes of illustration, we take an idealized maser described by $|\xi_0|^2 = |\zeta_0|^2 = \frac{1}{2}$, $|\eta_0|^2 = 0$; that is, we assume that the pump completely saturates the $1 \longleftrightarrow 3$ transition, and that the $2 \longrightarrow 3$ relaxation is so effective that we can neglect η_0.

Also we assume that there is no relaxation from 1 to 2; that is, only maser action connects levels 1 and 2. Physically, it is easy to see what this second idealization neglects. We can think of the relaxation mechanisms of the crystal lattice as a resistor coupled to each of the spins. The spins in turn are coupled to the field oscillator of the amplifier. Therefore, the oscillator receives thermal noise from the resistor indirectly through the spins. This noise is filtered by the spins acting as a band pass filter in bands near ω_{12}, ω_{13}, and ω_{23}. Only ω_{12} degrades the signal, but in principle we could choose resistors such that the density of modes near ω_{12} vanishes. Our idealization assumes such a resistor.

We would like to solve the problem for a fairly general pumping potential $V_2(t)$, so that we can see the effect of a driver that is noisy and slightly off frequency. However, the only two cases we can solve in terms of simple functions are

$$V_2(t) = v \cos\left[(\omega_{13} + 2\delta)(t - t_0)\right] \tag{65}$$

and

$$V_2(t) = v(t) \cos\left[\omega_{13}(t - t_0)\right]. \tag{66}$$

Fortunately, these two functions are qualitatively dif-

ferent cases, and together they give a fair picture of the situation. The first function allows the driver to be off frequency. We shall see that this splits the band of negative resistance into two bands. The average of the center frequencies of the two bands is $\omega_{12} + \delta$, where we see from equation 65 that δ is half the amount by which the driver is off resonance. The second function describes a driver that is exactly in phase with ω_{13} at all times, but the strength of the driver can vary in any way, (except not so violently that the equilibrium populations are changed). This provides us a way to look at the effect of a noisy driver, although the noise is restricted to a special type.

6. 1. Driver Off Resonance

We begin now a fairly lengthy derivation using the driving function of equation 65. We shall show that the three-level systems look to the field oscillators like two equivalent systems, one described by impedance $Z'(\nu)$ and noise $W'(\nu)$, and the other by Z'' and W''. The answers appear in equations 74 through 78.

We shall attack the problem by using a certain iteration process to solve the coupled differential equations for ξ, η, and ζ. We shall start the process at time 0 when the signal is turned on with initial con-

ditions ξ_0, η_0, and ζ_0. We calculate the negative resistance based on a single iteration. In general, this resistance is valid only for a brief time interval after $t = 0$, during which the first iteration is valid for ξ, η, and ζ. However, we use the equilibrium conditions as the initial conditions in the calculation, and since the signal is not strong enough to upset equilibrium, we know that the negative resistance found in this way is in fact the equilibrium resistance. By not iterating further, we avoid the terms of ξ, η, and ζ, that describe an effect that does not interest us. These terms describe the manner in which the amplification would decrease if the signal should become so large that it induces enough transitions to upset the equilibrium populations.

We shall find the influence functional starting with the general expression in amplitude form, equation 28. The a's in that expression are just the amplitudes to be in the various states; so the a's become ξ, η, and ζ, in the present notation. As discussed in the remarks preceeding equation 28, we solve for ξ, η, and ζ, just as though the spins (three-level systems) were not coupled to other quantum systems, but are driven by time dependent terms in the Hamiltonian. These terms are really the coupling terms to other systems, but while solving for ξ,

η , and ζ, we regard the coordinates of the other systems as just an arbitrary time dependence of the Hamiltonian terms.

The coupled differential equations for the amplitudes in matrix form are

$$\frac{d}{dt}\begin{pmatrix}\xi\\ \eta\\ \zeta\end{pmatrix} = -\frac{i}{\hbar}\begin{pmatrix} 0 & V_{12}(t)e^{i\omega_{12}t} & V_{13}(t)e^{i\omega_{13}t} \\ V_{12}^{*}(t)e^{-i\omega_{12}t} & 0 & V_{23}(t)e^{i\omega_{23}t} \\ V_{13}^{*}(t)e^{-i\omega_{13}t} & V_{23}^{*}(t)e^{-i\omega_{23}t} & 0 \end{pmatrix}\begin{pmatrix}\xi\\ \eta\\ \zeta\end{pmatrix},$$

where the V_{ij} are the matrix elements of the perturbing potential, and we have set the diagonal terms equal to zero since we are concerned with transitions, and not with perturbations in the energy levels. Furthermore, we throw away the parts of the V's that are not nearly in resonance with the exponential function multiplying them, since these parts average rapidly to zero. We let V_1 stand for the part of V_{23} resonant near ω_{23}, and similarly, V_2 and V_3 stand for part of V_{13} and V_{12} respectively. V_2 is the classical pumping potential, a notation already used in equations 65 and 66. V_1 is the coupling to the relaxation mechanisms, and V_3 is the linear coupling to the cavity mode being amplified, say $V_3 = \mu x(t)$. We conserve subscripts by letting $\omega_{12} = \omega_3$, $\omega_{13} = \omega_2$, and $\omega_{23} = \omega_1$. We now rewrite the differential equations:

$$\dot{\xi} = -\frac{i}{\hbar}\left(\quad 0 \quad + V_3 e^{i\omega_3 t}\eta + V_2 e^{i\omega_2 t}\zeta \right),$$

$$\dot{\eta} = -\frac{i}{\hbar}\left(V_3^* e^{-i\omega_3 t}\xi + \quad 0 \quad + V_1 e^{i\omega_1 t}\zeta \right), \qquad (67)$$

$$\dot{\zeta} = -\frac{i}{\hbar}\left(V_2^* e^{-i\omega_2 t}\xi + V_1^* e^{-i\omega_1 t}\eta + \quad 0 \quad \right).$$

Since V_1 and V_3 are arbitrary functions of time, we must solve these equations in powers of V_1 and V_3 by iteration. In the zero order approximation we neglect V_1 and V_3 altogether, obtaining*

$$\dot{\xi}^{(0)} = -\frac{i}{\hbar} V_2(t) e^{i\omega_2 t}\zeta^{(0)},$$

$$\dot{\zeta}^{(0)} = -\frac{i}{\hbar} V_2^*(t) e^{-i\omega_2 t}\xi^{(0)}. \qquad (68)$$

In the first order of approximation, we find $\eta^{(1)}$, by integrating the second of equations 67 using the solutions

---

* These equations are the same as the perturbation equations of spin one half in a magnetic field. We could therefore study these equations geometrically in terms of vectors that represent the direction of the precessing spin. This has been done by Feynman, Vernon, and Hellwarth in reference (9). The geometry presented there makes it evident why simply solutions to these equations can be found only for the drivers given by equations 65 and 66.

of equation 68. We obtain

$$\eta^{(1)}(t) = \int_0^t -\frac{i}{\hbar}\left[V_3^*(s)e^{-i\omega_3 s}\xi^{(0)}(s) + V_1(s)e^{i\omega_1 s}\zeta^{(0)}(s)\right]ds. \quad (69)$$

Then we solve for $\xi^{(1)}$ and $\zeta^{(1)}$ using the first and third of equations 67 and $\eta^{(1)}$. We rewrite these equations as

$$\begin{aligned} &(a)\quad \dot{\xi}^{(1)} = -\frac{i}{\hbar}V_2(t)e^{i\omega_2 t}\zeta^{(1)} + f_1(t), \\ &(b)\quad \dot{\zeta}^{(1)} = -\frac{i}{\hbar}V_2^*(t)e^{-i\omega_2 t}\xi^{(1)} + f_2(t), \end{aligned} \quad (70)$$

$$\begin{aligned} \text{where}\quad f_1(t) &= -\frac{i}{\hbar}V_3(t)e^{i\omega_3 t}\eta^{(1)}(t) \\ f_2(t) &= -\frac{i}{\hbar}V_1^*(t)e^{-i\omega_1 t}\eta^{(1)}(t). \end{aligned} \quad (71)$$

Note that on the right side of equation 70 we have $\xi^{(1)}$ and $\zeta^{(1)}$, not $\xi^{(0)}$ and $\zeta^{(0)}$. We cannot iterate in a term with the strong driver $V_2(t)$. Besides there is no necessity, since equations 70 can be solved for any function $V_2(t)$ such that equations 68 can be solved.*

First we solve equations 70 (of which 68 is the special case $f_1 = f_2 = 0$), with $V_2(t)$ given by equation 65. Substitution of equation 65 into equation 70 gives

* Equations 68 are just the reduced equations of 70. Any linear differential equation can be solved by the method of variation of parameters if the solution to the reduced equation is given.

(a) $\dot{\xi} = -i\Omega \exp i\left[-2\delta t + (\omega_2 + 2\delta)t_0\right]\zeta + f_1(t)$

(b) $\dot{\zeta} = -i\Omega \exp i\left[2\delta t - (\omega_2 + 2\delta)t_0\right]\xi + f_2(t)$,

where $\Omega = \frac{\nu}{2\hbar}$. Here we have expressed the cosine of equation 65 as a sum of exponentials and discarded the terms with $\pm 2i\omega_2 t$ in the exponent. We differentiate equation a, then substitute $\dot{\zeta}$ from b, and ζ from a as it stands. This gives an equation in ξ only which reads:

$$\ddot{\xi} + 2i\delta\dot{\xi} + \Omega^2\xi = i\Omega e^{-i\omega_2 t_0} f_2 + 2i\delta f_1 + \dot{f}_1 .$$

Substituting $u = \xi e^{i\delta t}$, we get the equation of a driven harmonic oscillator:

$$\ddot{u} + \Delta^2 u = g(t) \text{ , where } \Delta^2 = \delta^2 + \Omega^2 \text{ ,* and}$$
$$g(t) = e^{i\delta t}\left\{i\Omega \exp\left[-i\omega_2 t_0 - 2i\delta(t - t_0)\right] f_2 + 2i\delta f_1 + \dot{f}_1\right\} .$$

The solution to this is

$$u = u_0 \cos\Delta t + \frac{\dot{u}_0}{\Delta}\sin\Delta t + \frac{1}{\Delta}\int_0^t g(s)\sin\Delta(t-s)\,ds .$$

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* Our vector Δ is just the length of the vector $\vec{\Omega}$ in reference (9). Our δ and Ω are just the vertical and horizontal components of $\vec{\Omega}$.

We substitute the expressions for η and g into the above and obtain the answer:

$$\xi(t) = \xi_0 e^{-i\delta t}\cos\Delta t + i\sin\Delta t\left[\frac{\delta}{\Delta}\xi_0 + \frac{\Omega}{\Delta}e^{-i(\omega_2+2\delta)t_0}\zeta_0\right]e^{-i\delta t}$$

$$+\int_0^t e^{-i\delta(t-s)}\left[i\frac{\Omega}{\Delta}\exp i\left(-\omega_2 t_0 - 2\delta(t-t_0)\right)f_2(s)\sin\Delta(t-s)\right. \qquad (72)$$

$$\left. + i\frac{\delta}{\Delta}f_1(s)\sin\Delta(t-s) + f_1(s)\cos\Delta(t-s)\right]ds \ ,$$

where the term in g containing $\dot{f}$ has been integrated by parts. The solution for ζ is obtained by making the following changes in equation 72: $\xi \to \zeta$, $\omega_2 \to -\omega_2$, $\delta \to -\delta$, and $f_1 \leftrightarrow f_2$. Equation 72 and its counterpart for ζ give us $\xi^{(0)}$ and $\zeta^{(0)}$ when we set $f_1 = f_2 = 0$. Substitution of $\xi^{(0)}$ and $\zeta^{(0)}$ into equation 69 gives us $\eta^{(1)}$. Finally, substituting $\eta^{(1)}$ into equation 71, and 71 into 72, gives us $\xi^{(1)}$ and $\zeta^{(1)}$ as functionals of V_1 and V_3. We do not show here the lengthy expressions which result for $\xi^{(1)}$, $\eta^{(1)}$, and $\zeta^{(1)}$, since they are obtained by simple substitution and do not have any evident physical significance.

We find the influence functional by substituting into the general expression for $\mathcal{F}$, equation 26. The quantity $\sum_j a_{ij}a_{ij}^*$ on the right side of equation 26, becomes in the present case:

$$\xi^{(1)}[T, V_1(t), V_3(t)]\,\xi^{(1)*}[T, V_1'(t), V_3'(t)]$$

$$+ \eta^{(1)}\eta^{(1)*} + \zeta^{(1)}\zeta^{(1)*} \ .$$

Again, we do not write out the lengthy expression which results from substitution of $\xi^{(i)}$, $\eta^{(i)}$, and $\zeta^{(i)}$, into this. We must explain the notation V_1, V_1' and V_3, V_3' which appears in the above expression. In equation 28 we have indicated that the amplitudes are functionals of the coordinates x and x' of the influenced system. In the present notation, the coordinates of coupled systems are contained in the coupling potentials V_1 and V_3. We have not put in the coordinates explicitly, but have used V_1 and V_3 to indicate the potentials in terms of unprimed coordinates, and V_1' and V_3' to indicate primed coordinates. V_3 depends on the coordinate x of the cavity mode, so we can write $V_3 = V_3(x)$, $V_3' = V_3(x')$. We can associate coordinates $z^{(j)}$ with the relaxation mechanisms of the crystal lattice, i.e. the vibrational modes. V_1 depends on these, so we can write $V_1 = V_1(z)$, $V_1' = V_1(z')$.

The sum over j in equation 28 has been accomplished. The sum over i is an averaging over unsure quantities. The initial phase $\omega_i t_o$ is certainly not known, so we lose all terms containing t_o, since t_o only appears in imaginary exponentials that average to zero. Thus, we have for $\mathcal{F}_1$, the influence functional of a single spin system:

$$\mathcal{F}_1 = 1 + \left(\xi^{(i)}[T, V_1, V_3]\,\xi^{(i)*}[T, V_1', V_3'] - \xi_o \xi_o^*\right)$$

$$+ \left(\eta^{(i)}\eta^{(i)*} - \eta_o \eta_o^*\right) + \left(\zeta^{(i)}\zeta^{(i)*} - \zeta_o \zeta_o^*\right) \qquad (73)$$

less terms with t_o.

Here we have added one to the right side, and then subtracted it in the form $1 = \xi_0\xi_0^* + \eta_0\eta_0^* + \zeta_0\zeta_0^*$. We have in each pair of parentheses just the terms involving V_1 and V_3, which would be zero without any coupling to the x and z systems. The parentheses are second order in the couplings, so we call them $\mathcal{J}_1^{(2)}$, as we did in the molecular beam case. Also, as in that case, we put equation 73 into exponential form, that is,

$$\mathcal{J}_1[z', z; x', x] = \exp\left\{\mathcal{J}_1^{(2)}[z', z; x', x]\right\} .$$

To obtain $\mathcal{J}$ for all N spin systems, we take the product of $\mathcal{J}_1$'s, or the sum of $\mathcal{J}_1^{(2)}$'s. Recall that we only have to make some simple substitutions to obtain a lengthy explicit expression for $\mathcal{J}_1^{(2)}$. If we had this expression written out, we would observe that each term in $\mathcal{J}_1^{(2)}$ contains either $\xi_0\xi_0^*$, $\zeta_0\zeta_0^*$, $\xi_0\zeta_0^*$, or $\zeta_0\xi_0^*$. Summing $\mathcal{J}_1^{(2)}$ we have $\xi_0\xi_0^* \to N/2$, $\zeta_0\zeta_0^* \to N/2$, $\xi_0\zeta_0^* \to 0$, $\zeta_0\xi_0^* \to 0$. The first two of these are just the equilibrium populations; the second two sum to zero because they contain all different phases. Furthermore, in summing $\mathcal{J}_1^{(2)}$ for the N systems, we must average ω_z and v over a small range of values. The resonant frequency ω_z varies a little from one location to another in the crystal lattice, and the coupling v to the driver varies a little with the location of the spin systems in the driver mode. Averaging ω_z and v is the same as averaging Δ and δ,

which we derived from ω_2 and v. From equation 73 we now obtain

$$\mathcal{J}^{(2)} = \Big\langle \Big[\big(\xi^{(1)}[T, V_1, V_3]\, \xi^{(1)*}[T, V_1', V_3'] - \xi_0 \xi_0^* \big)$$

$$+ \big(\eta^{(1)} \eta^{(1)*} - 0 \big) + \big(\zeta^{(1)} \zeta^{(1)*} - \zeta_0 \zeta_0^* \big)$$

less terms with expi (frequency$\cdot 2\pi t_0$),

less terms with $\xi_0 \zeta_0^*$ and $\zeta_0 \xi_0^* \Big]_{\xi_0 \xi_0^* = \zeta_0 \zeta_0^* = N/2} \Big\rangle_{\Delta, \delta}$,

where the symbol $\langle \rangle_{\Delta,\delta}$ indicates the average over Δ and δ. In this expression we have eliminated most of the terms in the lengthy expansion of equation 73. The remainder of them combine nicely, because of our simplified equilibrium populations. The result is

$$\mathcal{J}[z', z; x', x] = \exp \mathcal{J}^{(2)}$$

$$= \exp\Big[-\frac{N}{2\hbar^2} \iint_{s<t}^{T} \Big\{ \big(V_1^*(t) - V_1'^*(t) \big) V_1(s) \cdot$$

$$\cdot \Big\langle e^{-i(\omega_1 + \delta)(t-s)} \Big[\cos \Delta(t-s) + i \frac{\delta}{\Delta} \sin \Delta(t-s) \Big] \Big\rangle_{\Delta, \delta}$$

$$- \big(V_1(t) - V_1'(t) \big) V_1'^*(s) \Big\langle e^{i(\omega_1 + \delta)(t-s)} \cdot$$

$$\cdot \Big[\cos \Delta(t-s) - i \frac{\delta}{\Delta} \sin \Delta(t-s) \Big] \Big\rangle_{\Delta, \delta} \Big\}\, dt\, ds \Big] \cdot$$

$$\cdot \exp\Big[V_1 \to V_3^*,\ V_1' \to V_3'^*,\ \omega_1 \to -\omega_3,\ \delta \to -\delta \Big].$$

This expression describes the influence of the spins on both the relaxation mechanisms, $z^{(j)}$, and the field oscillator x. Since we do not measure the $z^{(j)}$, we wish to eliminate them from the problem and find an influence functional for x only. Toward this end let us now recall the way in which the influence functional was introduced. Suppose we have the path integrand for propagating the density matrix of x and z as though the spins were not present. This integrand is

$$I_0[z',z;\ x',\ x] = \exp\left\{\frac{i}{\hbar}\left(S[x,z] - S[x',z']\right)\right\} .$$

$\mathcal{J}$ has the property that

$$I[z',z;\ x',\ x] = \mathcal{J}[z',z;\ x',\ x]\ I_0[z',z;\ x',x]$$

is the new integrand when the spins are introduced. We shall assume that, apart from the spins, x and z are independent. Thus, we are neglecting dielectric coupling between the lattice modes and the electromagnetic mode. Now we can write $S[x,z]=S_x[x]+S_z[z]$. We observe that the expression for $\mathcal{J}$ factors into $\mathcal{J} = \mathcal{J}_z[z',z]\mathcal{J}_x[x',x]$ since the first exponential factor contains V_1 and V_1' only, and the second V_3 and V_3' only. Now we can write

$$I = \left\{\mathcal{J}_z e^{i(S_z[z]-S_z[z'])}\right\}\left\{\mathcal{J}_x e^{i(S_x[x]-S_x[x'])}\right\},$$

where the first factor is a functional of z only, and the second of x only. Since we are not asking any questions about z, we integrate over all z variables. There remains a problem in x only. The z integrations involve only the first of the above factors, and the result is merely a constant. The second factor remains in tact and expresses the complete problem of x influenced by the pumped spin systems. In this second factor $\mathcal{F}_x$ plays the role of the desired influence functional. We substitute $V_3(t)=\mu x(t)$ into $\mathcal{F}_x$, drop the subscript, and change to sum and difference coordinates. A little rearranging gives the influence functional in its final form:

$$\mathcal{F}[x,x] = \exp\left\{-\frac{2}{\hbar}\iint_{s<t}^{T}\left[ix'(t)x(s)h_s'(t-s)+\right.\right.$$
$$\left.\left.+x'(t)x'(s)h_c'(t-s)\right]dtds\right\}\cdot\exp\{h'\to h''\}, \qquad (74)$$

where

$$h_s'(t)=\frac{N\mu^2}{2\hbar}1(t)\left\langle-\left(1-\frac{\delta}{\Delta}\right)\sin(\omega+\Delta+\delta)t\right\rangle_{\delta,\Delta},$$
$$h_c'(t)=\frac{N\mu^2}{2\hbar}1(t)\left\langle+\left(1-\frac{\delta}{\Delta}\right)\cos(\omega+\Delta+\delta)t\right\rangle_{\delta,\Delta}, \qquad (75)$$
$$h_s''(t)=\frac{N\mu^2}{2\hbar}1(t)\left\langle-\left(1+\frac{\delta}{\Delta}\right)\sin(\omega-\Delta+\delta)t\right\rangle_{\delta,\Delta},$$
$$h_c''(t)=\frac{N\mu^2}{2\hbar}1(t)\left\langle+\left(1+\frac{\delta}{\Delta}\right)\cos(\omega-\Delta+\delta)t\right\rangle_{\delta,\Delta}.$$

We collect now the definitions of the quantities used in equation 75:

$$\Delta = \sqrt{\delta^2 + \left(\frac{v}{2\hbar}\right)^2} \tag{76}$$

$V_3(t) = \mu x(t)$

N = number of three level systems,

where v and δ describe the driver (equation 65), and V_3 is the matrix element of maser action.

Note that in deriving equation 74 we never had to know $V_1(z)$ or $S_z[z]$. This was because $\mathcal{J}[z', z; x', x]$ factored into $\mathcal{J}_x \mathcal{J}_z$. This would not occur in approximations of higher order, because the effect of the signal on the equilibrium populations depends on the relaxation rate. Nor would it occur if we allowed relaxation from levels 1 to 2. To see this, recall that V_3 is the coupling potential between these levels. We would have to take $V_3 = \mu x + V_2(z)$ instead of just $V_3 = \mu x$. Then terms in the exponent of $\mathcal{J}[z', z; x', x]$, which are quadratic in V_3 and V_3', would have cross products such as $\mu x' V_2(z)$; and we could not factor $\mathcal{J}$. This problem can still be solved by assuming z to be a resistor at temperature θ.

The influence functional of equation 74 factors into two exponentials, so we can think of the influence on the x system as that of two equivalent systems, one for each

factor. We extend the time limit on the integral in equation 74 to $\pm\infty$, since we argued that the functional is really good for all time, since the signal does not upset equilibrium appreciably. Each factor of $\mathcal{F}$ has the familiar form of a problem already solved, equation 44, where u is replaced by h' for one of the equivalent systems, and by h'' for the other. Therefore, we can jump to the answer, equation 47. The field oscillator sees two negative resistors, one with impedance $Z'(\nu)$ and noise $W'(\nu)$, and the other with $Z''(\nu)$ and $W''(\nu)$, where these quantities are derived from the Fourier transformations of equation 75 as follows:

$$Z'(\nu) = i\frac{H_s'(\nu)}{\nu}\ , \qquad W'(\nu) = \frac{\hbar}{\pi}\,\mathrm{Re}\,H_c'(\nu)\ ,$$
$$Z''(\nu) = i\frac{H_s''(\nu)}{\nu}\ , \qquad W''(\nu) = \frac{\hbar}{\pi}\,\mathrm{Re}\,H_c''(\nu)\ . \tag{77}$$

In order to obtain the resistance and noise from equations 75 and 77, two integrations must be performed; the integration over the probability density of δ and Δ, implied by the average sign in equation 75, and the Fourier transformation integral to obtain 77. We can reverse the order of these integrations and obtain

$$R'(\nu) = -\frac{\pi N \mu^2}{2\hbar\nu}\left\langle\left(1-\frac{\delta}{\Delta}\right)\delta(\nu-\omega_3-\Delta-\delta)\right\rangle_{\delta,\Delta},$$

$$W'(\nu) = \frac{N\mu^2}{2}\left\langle\left(1-\frac{\delta}{\Delta}\right)\delta(\nu-\omega_3-\Delta-\delta)\right\rangle_{\delta,\Delta}, \quad (78)$$

For R", W", change $\Delta \longrightarrow -\Delta$.

The resistance is negative, and amplification is obtained. Of course there is also a reactive part of Z' and Z", which pulls the resonance of the field oscillator toward $\omega_3 + \langle\Delta + \delta\rangle$ or $\omega_3 - \langle\Delta - \delta\rangle$. Note in equations 78 that there is no coefficient out front describing the strength of the pump. This is because we have assumed that the pumping is so strong that it saturates the $1 \longleftrightarrow 3$ transition. The pump strength does appear in Δ .

Let us call τ the time interval equal to the reciprocal of the band width over which the resistance has appreciable negative values. In the molecular beam case τ was proportional to the time each molecule spent in the cavity. In the present case, τ is closely related to the time that the N spin systems maintain about the same phase relation for that part of their oscillations near $\omega_3 + \langle\delta \pm \Delta\rangle$.

A convenient diagram for thinking of the location of

the negative resistance bands is shown below in figure 4.

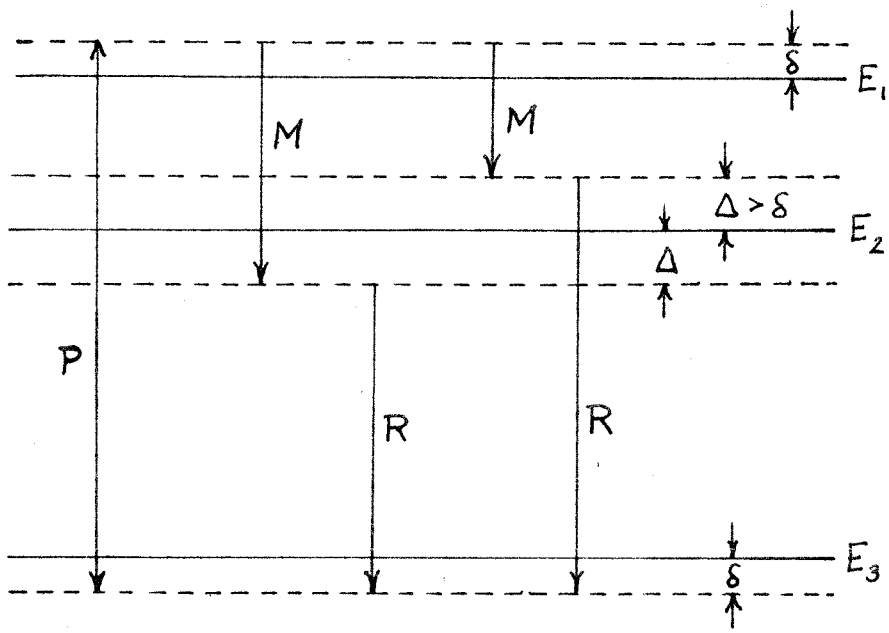

Figure 4

If we coupled to the spins a system resonant near ω_1, it would see bands of positive resistance, because the relative populations of states 2 and 3 are not inverted. Figure 4 is also valid for these bands of positive resistance.

G.2. Noisy Driver

Having solved the problem with the driver of equation 55, we start over now using the driver of equation 66. We restrict v(t) to real functions that are slowly varying compared to $e^{i\omega_1 t}$. This is consistent with our previous discussion of equation 66. The derivation of this case is the same as before except for a new differential equation.

The answer will appear in equations 82 through 89.

Substitution of equation 66 into equation 70 gives us

$$\dot{\xi} = -\frac{i}{2\hbar} V(t) e^{i\omega_2 t_0} \zeta + f_1(t)$$

$$\dot{\zeta} = -\frac{i}{2\hbar} V(t) e^{-i\omega_2 t_0} \xi + f_2(t) \quad .$$

These equations may be written in matrix form:

$$\dot{u} = -\frac{i}{2\hbar} V(t) \beta u + f(t) \quad ,$$

where

$$u = \begin{pmatrix} \xi \\ \zeta \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & e^{i\omega_2 t_0} \\ e^{-i\omega_2 t_0} & 0 \end{pmatrix}, \text{ and } f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} .$$

The solution to this equation is

$$u(t) = \exp\left\{-\frac{i\beta}{2\hbar}\int_0^t v(s)ds\right\} u_0 + \int_0^t \exp\left\{-\frac{i\beta}{2\hbar}\int_s^t v(s)ds\right\} f(s)ds.$$

Since $\beta^2 = 1$, we can separate terms of the form $e^{a\beta}$ into $\cos a + \beta \sin a$. When we do this in the above equation and perform the matrix multiplication,

we obtain*

$$\begin{pmatrix}\xi(t)\\ \eta(t)\end{pmatrix}=\begin{pmatrix}\xi_0\\ \eta_0\end{pmatrix}\cos\phi(t)-i\begin{pmatrix}\xi_0 e^{i\omega_2 t_0}\\ \eta_0 e^{-i\omega_2 t_0}\end{pmatrix}\sin\phi(t)$$
$$+\int_0^t \cos[\phi(t)-\phi(s)]\begin{pmatrix}f_1(s)\\ f_2(s)\end{pmatrix}ds \qquad (79)$$
$$-i\int_0^t \sin[\phi(t)-\phi(s)]\begin{pmatrix}e^{i\omega_2 t_0} f_2(s)\\ e^{-i\omega_2 t_0} f_1(s)\end{pmatrix}ds \quad ,$$

where

$$\phi(t)=\frac{1}{2\hbar}\int_0^t v(s)\,ds \quad .$$

---

* One might ask: Why can this matrix method be applied to solve the problem when there is an arbitrary function $v(t)$ in the expression for $V_2(t)$, but not when $V_2(t)$ is an arbitrary function? The mathematical answer lies in commutation difficulties. For arbitrary $V_2(t)$ we would have in the exponent

$$\int^t V_2(s)\begin{pmatrix}0 & e^{-i\omega_2(s-t_0)}\\ e^{i\omega_2(s-t_0)} & 0\end{pmatrix}ds$$

The matrix is now not a constant, and the value at one time does not commute with the value at another time.

It is also interesting to consider this equation in connection with reference (9). Our ϕ is the angle θ of that paper by which the vector $\vec{r}$ has been driven.

The steps we take with these equations are the same as the steps we took with equations 72 and its counterpart for ζ. We iterate using equations 69, 71, and 79. Then we find the influence functional by substitution of $\xi^{(1)}$, $\eta^{(1)}$, and $\zeta^{(1)}$ into equation 73. The result is:

$$\mathcal{J}[z',z;x',x;v(t)] = \exp\left(\mathcal{J}^{(2)}\right)$$

$$= \exp\left\{-\frac{N}{2\hbar^2}\iint_{s<t} j[t,s,v(t)]\cdot\left[(V_1^*(t)-V_1'^*(t))\cdot\right.\right. \tag{80}$$

$$\left.\left.\cdot V_1(s)e^{-i\omega_1(t-s)} - (V_1(t)-V_1'(t))V_1'^*(s)e^{i\omega_1(t-s)}\right]dtds\right\}\cdot$$

$$\cdot \exp\left\{V_1^* \to V_3,\ V_1'^* \to V_3',\ \omega_1 \to -\omega_3\right\},$$

where

$$j[t,s,v] = \left\langle \cos[\phi(t)-\phi(s)] \right\rangle_{\text{strength of } v(t).} \tag{81}$$

We emphasized in equation 81 that the average is an average over the strength of v(t), not its functional form, that is, if we write $v(t) = au(t)$, the average is taken over a small range of a. That is because each spin system sees the same time dependence of the driver, but different strength, depending on its location in the driver mode. If the driver is noisy, there is also an average to be taken over the functional form of v(t), but this is an average of the type

$$\mathcal{J}[z',z;x',x] = \left\langle \mathcal{J}[z',z;x',x;v(t)] \right\rangle_{\text{functions } v(t)}$$

(see equation 33), not an average in the exponent (equation 26).*

As before we dispose of the relaxation mechanisms (V_1 and V_1'), set $V_3(t)=\mu x(t)$, and obtain the influence functional for the field oscillator x. It is

$$\mathcal{J}[X,\varkappa,v(t)] = \exp\left\{-\frac{2}{\hbar}\iint_{s<t}\left[ix'(t)x(s)J_s'(t,s) + x'(t)x'(s)J_c(t,s)\right]dtds\right\} \cdot \exp\left\{J' \to J''\right\}, \tag{82}$$

where

$$\begin{aligned}
J_s'[t,s,v(t)] &= -\frac{N\mu^2}{2\hbar}1(t-s)\left\langle \sin\left[\omega_3(t-s)+(\phi(t)-\phi(s))\right]\right\rangle_{\substack{\text{strength}\\ \text{or } v}} \\
J_c'[t,s,v(t)] &= \frac{N\mu^2}{2\hbar}1(t-s)\left\langle \cos\left[\omega_3(t-s)+(\phi(t)-\phi(s))\right]\right\rangle_{sv} \\
J_s''[t,s,v(t)] &= \frac{-N\mu^2}{2\hbar}1(t-s)\left\langle \sin\left[\omega_3(t-s)-(\phi(t)-\phi(s))\right]\right\rangle_{sv} \\
J_c''[t,s,v(t)] &= \frac{N\mu^2}{2\hbar}1(t-s)\left\langle \cos\left[\omega_3(t-s)-(\phi(t)-\phi(s))\right]\right\rangle_{sv}.
\end{aligned} \tag{83}$$

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* Since we constructed the exponential form of $\mathcal{J}$ from its first order expansion, one might say that we do not know $\mathcal{J}$ accurately enough to distinguish between averaging $\mathcal{J}$ and averaging the exponent of $\mathcal{J}$. The two averages differ in second order. However, in the section on a single non-linear system (V. E.) we argued that the exponential form was really better than its formal derivation guaranteed. We have confidence that this is true in any problem using non-linear systems which, so to speak, operate on a linear portion of their characteristics, simply because the exponential form is exact for a system that is exactly linear.

Equations 83 and their counterparts, equation 75, for the other driver are the same in the special case v=constant, $\delta = 0$.

The functions in equation 83 are generally not functions of (t-s), so we do not have the familiar form of equation 44 that gives us the impedance and noise immediately. In fact we do not have the convolution form in equation 82 that makes the use of Fourier transforms so convenient. The trouble is that

$$\phi(t) - \phi(s) = \int_s^t \frac{v(\tau)}{2\hbar}\, d\tau \qquad (84)$$

is in general different for different time intervals of the same duration (t-s), because v(t) is an arbitrary function.

One can study the effects of a noisy driver given the probability distribution $\mathscr{P}_v[v(t)]$ of driving functions v(t). The effective influence functional is the average influence functional given by

$$\mathcal{J}[x, \varkappa] = \int\!\!\int \mathscr{P}_v[v(t)] \;\; \mathcal{J}[x, \varkappa, v(t)] \;\; \mathcal{D}v(t), \qquad (85)$$

where the $\mathcal{J}$ in the intogrand is given by equation 82. The transformation to $\mathcal{J}[x, f]$ (equation 24) gives the negative resistance and noise. The path integration in equation 85

is very difficult to carry out because v(t) enters into $\mathcal{J}$ in a complicated way.

In a case with sufficiently small coupling to the spins, we can average the exponent of $\mathcal{J}$ instead of $\mathcal{J}$, making errors of order $(N\mu^2)^2$. This amounts to replacing the functionals of equation 83 by their averages, namely,

$$j_S^I(t-s) = \int\!\!\!\int \mathcal{P}_v[v(t)]\, j_S'[t,s,v(t)]\, \mathcal{D}v, \quad (86)$$

and similarly j_S^{II}, j_C^I, and j_C^{II}. We know that those averages are functions of t-s if the maser is operating in the steady state, simply because there is no preferred time for v(t) to be large or small on the average. The average functions can only depend on the interval t-s.

The path integrals of equation 86 can be performed if the functional $\mathcal{P}_v[v(t)]$ is Gaussian. Let us take a noisy driver described by

$$\mathcal{P}_v[v(t)] = \exp\left\{-\int_{-\infty}^{\infty} \frac{|\hat{V}(\nu) - v\delta(\nu)|^2}{2W_v(\nu)}\, d\nu\right\} . \quad (87)$$

This says that the mean function v(t) is just the constant v, and that the noise is Gaussian as described by the power spectrum $W_v(\nu)$. (Recall that v(t) is not the whole driving function, but just a modulation of cos $\omega_2 t$.) The integration of equation 86, averaging j with the probability of equation 87 and the functionals of equation 83, is a

lengthy but straightforward application of the path integral methods we have been using. We give only the answers, which are

$$\left.\begin{aligned} j_s'(t) &= -\frac{N\mu^2}{2\hbar}\,1(t)\left\langle \sin\left(\omega_3 t + \frac{\nu t}{2\hbar}\right)\cdot\right. \\ j_c'(t) &= +\frac{N\mu^2}{2\hbar}\,1(t)\left\langle \cos\left(\omega_3 t + \frac{\nu t}{2\hbar}\right)\cdot\right. \\ j_s''(t) &= -\frac{N\mu^2}{2\hbar}\,1(t)\left\langle \sin\left(\omega_3 t - \frac{\nu t}{2\hbar}\right)\cdot\right. \\ j_c''(t) &= -\frac{N\mu^2}{2\hbar}\,1(t)\left\langle \cos\left(\omega_3 t - \frac{\nu t}{2\hbar}\right)\cdot\right. \end{aligned}\right\} \tag{88}$$

$$\left.\cdot \exp\left[-\int_0^\infty \frac{W_v(\beta)}{(2\hbar\beta)^2}(1-\cos\beta t\,)\,d\beta\right]\right\rangle_{\substack{\text{strength} \\ \text{of } v(t)}}.$$

When we take $j_s'(t-s)$, etc. from equation 88, and substitute them into $\mathcal{J}[x,\alpha]$, equation 82, we have the familiar convolution form again. This gives us immediately the impedance and noise of two equivalent resistors in terms of the Fourier transforms of the j's. We have

$$\begin{aligned} Z'(\nu) &\approx i\,\frac{J_s'(\nu)}{\nu}, & W'(\nu) &\approx \frac{\hbar}{\pi}\,\mathrm{Re}\;J_c'(\nu), \\ Z''(\nu) &\approx i\,\frac{J_s''(\nu)}{\nu}, & W''(\nu) &\approx \frac{\hbar}{\pi}\,\mathrm{Re}\;J_c''(\nu), \end{aligned} \tag{89}$$

valid to first order in $N\mu^2$.

Let us consider the qualitative behavior of equation 88. When the power spectrum W_v vanishes, equation 88 just reduces to the case of the noise free driver, equation 75 with $\delta = 0$ and j instead of h. Equation 89 becomes 77. The exponential factor in equation 88 forms an envelope for the sines and cosines. Large W_v makes the envelope small for $t > 0$, and especially for large t. This increases the range of frequencies in the transforms of equation 89. Thus, noise in the driver increases the band widths of the amplifier, as we expect. Recall that the average over the strength of v(t) results from the various spins having different locations in the driver mode. Again we can show from the envelope of the sines and cosines in equation 88 that a wide band width results from a wide range of strengths.

We have illustrated the applicability of our quantum theory to the solid state maser by finding impedance and noise in simplified cases. We discontinue the example here leaving the following loose ends for future study: (1) spin-spin interaction, (2) general equilibrium populations, (3) complete treatment of the noisy driver, (4) noise from relaxation mechanisms resonant near ω_3 (signal frequency).

VI. MASER AMPLIFIER SYSTEM

We consider in this section all the coupled systems of

a transmitter and maser receiver beginning with the antenna that transmits the signal, and ending with the stage of amplification at which we can say that the signal is classical. We especially consider the noise of quantum origin which appears as a result of the signal being very weak in the input to the receiver, say an energy equivalent to a few quanta in the input waveguide. We shall also mention briefly the case in which the maser is listening to a noise source. The application of masers to radio astronomy is such a case.

In the transmitter-receiver system the antenna acts as a classical driver of the modes in the space between the antenna and the receiver. Up to the point in the receiver where we can say that the signal is classical, all the coupled systems are linear electromagnetic modes, with the exception of the molecular beams or three-level systems used in amplification. The latter have been eliminated and their effects reduced to an equivalent impedance and noise. In fact, we can say that even these amplifying elements are linear in the same sense that a vacuum tube is linear when used in a class A amplifier. That is, we are operating on a linear portion of the characteristics.

We focus our attention on our second theorem of quantum mechanics, namely, that the linear systems propagate in time exactly like classical linear systems. Therefore, we can

calculate the signal at any point in the coupled systems exactly as though the world were classical. For each of the amplifying elements, we just insert the proper impedance into the classical equations.

Having thus shown how to calculate the signal, we have only to consider noise. Our rule for propagating linear systems introduces no uncertainty. The only uncertainty arises in the initial conditions. When we found the Wigner density for an oscillator at temperature θ, equation 39, we saw that the ground state, $\theta = 0$, uncertainty is qualitatively like any other thermal noise. Zero temperature just corresponds to the least value of the mean energy α. We can think of a group of linear systems as though the world were once classical, but entropy has increased until we can no longer find a temperature bath cold enough to rid a system of its last bit of thermal noise, which corresponds to a mean energy $\frac{1}{2}\hbar\omega$.* We showed that as long as a non-linear

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* It is interesting to speculate about what would have happened if quantum mechanics had been discovered through the rule for propagating the Wigner density, equations 1, 2, and 15. There would have been no reason in the beginning to suspect an uncertainty principle, for we can take the density $P(x,p) = \delta(x-x_0)\ \delta(p-p_0)$ and propagate it like any other density. If we take this density as the initial density of a harmonic oscillator problem, we find that no uncertainty develops as time passes. However, if we take a non-linear problem, the density in general spreads via a variety of paths. It is possible to invent a non-linear potential so that the above density returns periodically by means of interference effects among the spreading paths. The Wigner densities which violate the uncertainty principle still may be transformed to a density matrix, which in turn may be expanded in the form $\Sigma_n w_n \psi_n^*(x')\ \psi_n(x)$, but now the w_n are not all positive. For example, if the ψ_n are eigenfunctions of an even potential function, then the values $w_n = (-1)^n$, $n = 0, 1, \cdots$, give us $P = \delta(p)\ \delta(x)$.

system is nearly in an energy eigenstate, it is equivalent to a set of harmonic oscillators as far as the noise force is concerned. So to this extent, we can extend the preceding remarks about noise and initial conditions to nonlinear systems.

We have shown that the quantum theory of linear systems is a noise theory, the noise always arising from uncertain initial conditions, unless there is a classical source such as a noisy driver. In any physical system, there is at least a little resistance, so that any system in the steady state loses the memory of its initial conditions. This does not imply that a system which we call S loses its noise by weak coupling to a resistor, for the resistor itself consists of a vast number of systems at some temperature. Even if this temperature is zero, the noise of ground state oscillations in the resistor systems sifts back through the weak coupling and into S . The only difference between the noise of an isolated oscillator at temperature θ, and the noise of an oscillator S in thermal equilibrium with a coupled resistor at temperature θ, is that the former rings exactly at the resonance frequency with unknown amplitude and phase, while the latter has a band width. We can think of the resistor and S in terms of normal modes. Each mode has noise at one precise frequency, but the coordinate of S consists of a linear combination of the coordinates of many modes, the important ones resonating in a certain band width.

As in any amplifier system, we are most concerned with the noise at the point where the signal is weakest. This is the receiver input, that is the system or systems through which the signal passes just before it reaches the first maser cavity mode. Normally the input to a maser is a waveguide in which the modes are dense enough to be considered as a continuum. Suppose they are in thermal equilibrium at temperature θ. There are $2\omega^2/\pi c^3$ modes per unit frequency per unit volume, where $\omega/2\pi$ is the frequency. Equation 39 shows us that the expected noise in each mode is just an amount such that its energy calculated classically is $\alpha(\theta)$. When we multiply the density of modes by the expression for α in equation 40, we find that the

$$\left.\begin{array}{l}\text{noise energy density}\\ \text{per unit frequency}\end{array}\right\} = \frac{2\hbar\omega^3}{\pi c^3}\left(\frac{1}{2}+\frac{1}{e^{\hbar\omega/k\theta}-1}\right). \quad (90)$$

The signal requires a certain band width. If the energy density of the signal at the weakest point is less than the band width times the noise energy density of equation 90, then the signal is obscured by noise. If $\hbar\omega > k\theta$, this noise is primarily of quantum origin.

We may have more complicated cases than the above, and we may wish to ask more detailed questions. For example, we may have a linear system in which we compare the noise

to a signal consisting of pulses. We might ask, what is the probability that a pulse is lost, or that a spurious pulse appears? The system may be coupled to various other systems including resistors at different temperatures. In this case we must know the couplings. For each system treated in this paper, we have found an impedance Z, and a noise power spectrum $W(\nu)$ in terms of these couplings. When the Z's and W's are calculated, we can simply regard the linear system under consideration as a classical system. Then classical theory is available to calculate the signal from the Z's and to answer detailed noise questions using the Z's and W's.

As an example, let us consider the coordinate x of a maser cavity mode. As a result of the input couplings, x sees a certain impedance and noise, say Z_1 and W_1. If the cavity is the first stage of amplification, then the input looks like a resistor consisting of the continuum of normal modes of the input wave guide coupled to the modes of free space. Similarly, the output can be represented by Z_0 and W_0. Whether the amplifying element is a molecular beam, or a lattice of spins, we have derived its impedance and noise, Z_A and W_A. Coupling to cavity walls gives us Z_C and W_C, which can be related to the cavity's resonant frequency and Q. The signal in the input system

exerts on x the driving force $f_s(t)$ that we can calculate classically. Now we can write the expression for the signal in the cavity. It is the usual expression one obtains by transforming the linear equations of motion, namely

$$\hat{X}_s(\nu) = \frac{\hat{F}_s(\nu)}{m(\omega^2 - \nu^2) + i\nu(Z_i(\nu) + Z_o(\nu) + Z_A + Z_c)} \quad . \tag{91}$$

The power spectrum W_x of the noise in x may be calculated from the total power spectrum of noisy driving forces, $W_f = W_i + W_o + W_A + W_c$, by the equation of classical noise theory*,

$$W_x(\nu) = \left| \frac{1}{m(\omega^2 - \nu^2) + i\nu(Z_i + Z_o + Z_A + Z_c)} \right|^2 W_f(\nu) \quad .$$

Then X_s and W_x can be compared to determine whether the signal is obscured by noise. The spectrum of the signal that corresponds to the noise spectrum is $\pi|\hat{X}_s(\nu)|^2/T$, where T is the duration of the signal. The part of $W_x(\nu)$ that lies outside the band width of the signal can be filtered out in a later stage.

In equation 91, ReZ_A is a negative resistance. If this cancels the positive resistance, $Re(Z_i + Z_o + Z_c)$, then there is a frequency ν that makes the denominator vanish.

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* Equation 132 of reference (8).

The system then breaks into oscillation at this frequency.

Part or all of W_A, the noise from the amplifying element, is what conventional quantum mechanics terms spontaneous emission from the excited systems. The other W's are thermal noise of the coupled systems, where we include in the meaning of the word "thermal" the ground state excitation of quantum systems.

When we come to the place in the amplifier where we can say that the signal is classical, we have no changes whatsoever to make in our way of thinking, and hardly any in the mathematics. We simply omit W's that are negligible, and in high temperature ($k\Theta \gg \hbar\omega$) apparatus we let $\alpha = k\Theta$.

In some experiments one looks at radiation from a hot noise source, a light bulb or a radio star. One can regard these as hot resistors. Equations 48 and 49 give the impedance and noise of the resistor. Our formalism can decide what part of the noise is representative of the source, and what part arises in the couplings and amplifier.

Another kind of problem we might consider in our formalism is a collection of systems being pumped by a hot source. Then we look at the radiation of lower frequency emitted because the pumped systems relax through states between those being pumped. The systems could be molecules of gas being radiated from a star. Formally, this is like

a solid state maser, except that the pump is pure noise, an aspect not satisfactorily treated in this paper. Instead of amplifying a signal, the pumped system amplifies thermal noise in the low frequency field oscillators. If the oscillators are at zero temperature, this is the same as spontaneous emission.

VII. SUMMARY

We have formulated quantum mechanics in terms of the concept of the probability of a path, i.e. the probability that the coordinates $x^{(j)}$ of a system describe the space-time paths $x^{(j)}(t)$. Equation 15 with definitions in equations 13 and 16 gives as our first theorem, the quantum mechanical probability of a path. Taken with equations 1 and 2, this theorem constitutes a propagation rule for $P(x,p)$, the Wigner density of position x and momentum p. Equation 1 says that the probability K of x_T and p_T at time T, given the initial conditions x_0 and p_0 at 0, is just the sum of the probabilities of paths having these boundary conditions. Then equation 2 says that if x_0 and p_0 are also uncertain, we must multiply K by the probability $P(x_0,p_0,0)$, then integrate over x_0 and p_0 to find the probability $P(x_T,p_T,T)$.

Except for a side discussion of the classical limit $\hbar \to 0$, we only used equation 15 in the case of linear systems. In equations 17 and 18 we found that for linear systems the probability of a path is $\delta[x(t)-x_c(t)]$, where x_c is a solution of the classical equations of motion. Thus as our second theorem we find that linear systems follow only classical paths; the quantum theory merely describes uncertainty in their initial conditions. In equations 39 and 40 we found the uncertainty in the initial conditions of a harmonic oscillator at temperature Θ. We found that x and p have independent Gaussian distributions with a variance such that the mean energy calculated classically is just $\alpha(\Theta,\omega)$, equation 40. At high temperature α approaches $k\Theta$, but at zero temperature $\alpha = \frac{1}{2}\hbar\omega$. Thus for linear systems quantum uncertainty is qualitatively just thermal noise.

We considered two coupled systems, x and y, in the case that we follow the motion of x only. Our only concern with y was to find the forces it exerts on x. We derived the influence functional, $\mathcal{F}[x',x]$ which contains this information about y. We expressed $\mathcal{F}$ in path integral form, equation 22, and in terms of amplitudes (conventional quantum theory), equation 28. We obtained a functional

$\mathcal{G}[x,f]$ from $\mathcal{J}$ by the transformation of equation 24. As our third theorem of quantum mechanics, we showed that $\mathcal{G}$ is the probability that y exerts the force f on x, given the path x(t). To define the meaning of the probability of a force as used in this theorem, we have equation 3, which says that the probability of a path x(t) with y coupled is the same as the probability with y replaced by a driving force f(t), times the probability $\mathcal{G}$ that y exerts f(t), summed over all forces f(t). We found that the influence functional is the analog of the characteristic function of simple probability theory. Thus when we have several systems influencing x, but not each other, we can multiply their influence functionals (equation 26) and obtain a single effective functional. The forcing functional, $\mathcal{G}$, obtained from the product is the probability of the total force acting on x. If the correct functional $\mathcal{J}$ is not known, but may be one of many functionals with certain probabilities, then we showed that the effective $\mathcal{J}$ is $\langle\mathcal{J}\rangle$, equation 33. Similarly, the effective $\mathcal{G}$ is $\langle\mathcal{G}\rangle$.

In the important special case that x is a linear system, we have equation 20 for the probability of a path under the influence of y. This equation says that the

probability of x(t) is just the probability of the forces f(t) that make x(t) a solution of the classical equations of motion.

We formally derived forcing functionals for two trivial y systems, a classical driver exerting the force $f_c(t)$, and a noisy classical driver with probability $\mathcal{Y}'[f_c]$ that the force is f_c. We found as expected that $\mathcal{Y}=\delta[f-f_c]$ in the former case, and $\mathcal{Y}[x,f]=\mathcal{Y}'[f]$ in the latter.

We derived the forces of two simple y systems, a harmonic oscillator, and, in perturbation approximation, a general system initially in its mth energy eigenstate. For the oscillator we found just the forces, equation 36, that one would find from a classical oscillator with uncertain initial conditions described by $P(y_0,q_0,0)$, where $q=m\dot{y}$. By perturbation theory we found the forces that any y system linear or non-linear, exerts for a short time after it is coupled to x. We found a pair of forces, one of which is uncertain, for each state j of the y system which has a non-vanishing matrix element Y_{mj} to the initial state. These forces are summarized on page 81. Each resonance (pair of levels mj) has the property that the proportion of known and uncertain forces is determined by the couplings and Y_{jm}, not by the energy or the linearity of the y system.

We derived the forcing functionals of three complex systems in the approximation of linear operation: (1) a resistor at temperature θ represented by a continuum of oscillators, (2) a molecular beam in the approximation of equal spacing of the molecules, and (3) a lattice of three-level systems as used in a solid state maser, but with several simplifying assumptions. In each case the forcing functional had the form of equation 47, where we have changed from functions of time $f(t)$ and $x(t)$ to phasors $\hat{F}(\nu)$ and $\hat{X}(\nu)$. The phasor was defined so that

$$f(t) = \int_0^\infty [\mathrm{Re}\ \hat{F}(\nu)\cos\nu t - \mathrm{Im}\ \hat{F}(\nu)\sin\nu t]\, d\nu.$$

In equation 47, the form of $\mathcal{G}$, we were able to identify $Z(\nu)$ and $W(\nu)$ as the ordinary classical impedance and the power spectrum of noise force, which the x system sees as a result of coupling to each of the complex y systems. The power spectrum is defined classically by equation 53 or by equations 55 and 57. Hence, the important quantities to derive for each influencing system are $Z(\nu)$ and $W(\nu)$.

In the resistor case we derived the impedance and noise given by equations 43 and 49. We represented the resistor by many coupled oscillators. Clearly this was valid as far as the resistance, Re Z, is concerned, for we can always conceive of a set of oscillators with the same density of resonances that a real resistor has. The noise is also

the same as a real resistor. The perturbation theory of non-linear systems showed this, because each resonance between a pair of states causes forces with certain proportions of resistance and noise independent of the linearity of the system as determined by the spacings and matrix elements of other states. Also in thermal equilibrium, each resonant pair has relative populations $e^{-\hbar\omega/k\theta}$ independent of linearity.

For the molecular beam in the approximation of uniform spacing, velocity, and initial state of the molecules, we derived Z and W as given by equations 63. For the case of a Gaussian coupling function $\gamma(t)$, we get Z and W of equation 64. We found that Z and W have a band width proportional to τ^{-1}, where τ is the coupling time.

For the solid state maser we made several simplifying assumptions for brevity. The loose ends are summarized on page 114. We assumed two simplified driving functions. The noise free driver, equation 65, led to two equivalent negative resistors with Z and W as given by equations 77 with definitions in equations 75 and 76. The driver in exact resonance, equation 66, led to two resistors described by equations 89, with definitions in equations 87 and 88.

Finally we discussed a maser amplifier system. We had already seen that the coupled linear systems, i.e. the

electromagnetic modes, could be solved classically provided we add a little thermal noise of quantum origin. After finding the effective impedance and noise of the complex systems which couple to the field in maser amplifiers, we can also treat the maser classically. One finds the signal by inserting the proper impedance into the classical equations of motion. One can compare the signal to the noise at any point in a system by using the noise spectra $W(\nu)$ in the classical way.

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