Asymptotic Properties of Orthogonal and Extremal Polynomials

Thesis by

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"Success is sweet: the sweeter if long delayed
and attained through manifold struggles and defeats."

– A. Branson Alcott

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Abstract

The content of this thesis includes results describing the asymptotic behavior of extremal polynomials in a variety of settings. Special attention will be paid to the orthonormal and monic orthogonal polynomials. Given a finite measure μ with compact and infinite support in the complex plane, let $\{p_n(z;\mu)\}_{n=0}^{\infty}$ be the associated orthonormal polynomials. We will study the asymptotic behavior of these polynomials. More precisely, we will consider so-called ratio asymptotics:

$$\lim_{n\to\infty} \frac{p_{n-1}(z;\mu)}{p_n(z;\mu)},$$

and Szegő asymptotics:

$$\lim_{n\to\infty} \frac{p_n(z;\mu)}{\varphi(z)^n},$$

for an appropriate analytic function φ. If the measure μ is supported on the unit circle $\partial\mathbb{D} = \{z : |z| = 1\}$ or a compact subset of the real line, then both of these properties are well understood in terms of the coefficients appearing in the recurrence relation for the polynomials $\{p_n(z;\mu)\}_{n\geq 0}$. We will work in settings where no such recurrence relation exists and prove analogous results.

We will prove Szegő asymptotics when the measure is supported on an analytic region and is of a certain very general form, of which area measure is a special case. For this class of measures, we will prove analogs of several theorems from orthogonal polynomials on the unit circle such as describing when the probability measures $\{|p_n(z;\mu)|^2 d\mu(z)\}_{n\geq 0}$ converge weakly to the equilibrium measure for the support of μ and positivity of the *Christoffel function* $\lambda_\infty(z;\mu)$ at all points inside the region.

We will prove ratio asymptotics in a variety of settings including the closed unit disk $\overline{\mathbb{D}} = \{z : |z| \leq 1\}$ and a region with sufficiently smooth boundary whose complement is simply connected in the extended plane. The key tool for our ratio asymptotic results will be a nonlinear formula involving the polynomials $\{p_n(z;\mu)\}_{n\geq 0}$ that was recently introduced by Saff.

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Contents

Index of Notation

a_n, b_n	n^{th} Jacobi parameters
α_n	n^{th} Verblunsky coefficient
$\overline{\mathbb{C}}$	$\mathbb{C} \cup \{\infty\}$, the one point compactification of the complex plane
$c_t(\gamma)$	the t^{th} moment of the measure γ
$\text{cap}(X)$	the logarithmic capacity of the set X
$\text{ch}(\mu)$	the convex hull of the support of the measure μ
$\mathbb{D}$	the unit disk $\{z : \lvert z\rvert < 1\} \subseteq \mathbb{C}$
$\mathbb{D}_r$	the disk $\{z : \lvert z\rvert < r\} \subseteq \mathbb{C}$
∂X	the boundary of the set X
$\overline{X}$	the closure of the set X
F_n	the Faber polynomial of degree n
$g_\Omega(z, \infty)$	the Green function for the region Ω with pole at infinity
κ_n	the leading coefficient of the orthonormal polynomial
$\kappa_n(\mu, q)$	the leading coefficient of $p_n(z; \mu, q)$
$K_n(z, w; \mu)$	the $L^2(\mu)$ reproducing kernel for polynomials of degree at most n
$\lambda_n(z; \mu, q)$	the n^{th} $L^q(\mu)$ Christoffel function
$\lambda_\infty(z; \mu, q)$	the limiting $L^q(\mu)$ Christoffel function
$\hat{\mu}$	balayage of the measure μ
μ^n	the Bernstein-Szegő approximating measure for OPUC
μ_x	the Uvarov transform of the measure μ with added pure point at x
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
ν^x	the Christoffel transform of a measure with corresponding zero at x
ν_n	the normalized zero counting measure for $P_n(z; \mu, 2)$
$P_n(z; \mu, q)$	a polynomial of degree n having minimal $L^q(\mu)$-norm
$p_n(z; \mu, q)$	the polynomial $P_n(z; \mu, q)$ divided by its $L^q(\mu)$-norm
$\text{Pch}(\mu)$	the polynomial convex hull of the support of the measure μ
ϕ	the inverse to the map ψ
ψ	a conformal map initially defined on $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$
ρ_n	the Geronimus-type approximating measure for OPRL
$S(z)$	the Szegő function
$\text{supp}(\mu)$	the support of the measure μ
U^μ	the potential funtion of the measure μ
ω_K	the equilibrium measure for a compact set K

Introduction

This text is devoted to the general theory of orthogonal polynomials in one complex variable. We begin our study by fixing some notation and basic terminology. Let μ be a finite measure with compact and infinite support $\text{supp}(\mu)$ in the complex plane $\mathbb{C}$. By performing Gram-Schmidt orthogonalization on the sequence $\{1, z, z^2, z^3, \ldots\}$ in the space $L^2(\mu)$, one obtains a sequence of orthonormal polynomials $\{p_n(z;\mu)\}_{n\geq 0}$, which satisfy

$$\int_{\text{supp}(\mu)} p_n(z;\mu)\overline{p_m(z;\mu)}d\mu(z) = \delta_{nm}, \tag{0.0.1}$$

and are normalized so that $p_n(z;\mu)$ has positive leading coefficient κ_n. The polynomial $p_n(z;\mu)\kappa_n^{-1}$ is a monic polynomial of degree n, which we refer to as the monic orthogonal polynomial of degree n and denote by $P_n(z;\mu)$. This polynomial satisfies an extremal property, namely

$$\|P_n\|_{L^2(\mu)} = \inf\{\|Q\|_{L^2(\mu)} : Q = z^n + \text{ lower order terms}\}. \tag{0.0.2}$$

We will find this extremal property very useful in our investigation.

We will be interested in describing the asymptotic behavior of the polynomials $\{p_n(z;\mu)\}_{n\geq 0}$ and $\{P_n(z;\mu)\}_{n\geq 0}$ as n tends to infinity. One usually studies the asymptotics of $p_n(z;\mu)$ in one of three ways; the first is *root asymptotics*:

$$\lim_{n\to\infty} |p_n(z;\mu)|^{1/n};$$

the second is *ratio asymptotics*:

$$\lim_{n\to\infty} \frac{p_n(z;\mu)}{p_{n-1}(z;\mu)};$$

and the third is *Szegő asymptotics*:

$$\lim_{n\to\infty} \frac{p_n(z;\mu)}{\varphi(z)^n} \qquad , \qquad \varphi(z) \text{ analytic on } \mathbb{C}\setminus \text{ch}(\mu),$$

where $\text{ch}(\mu)$ denotes the convex hull of the support of the measure μ. It is easy to see that the existence of the limit for Szegő asymptotics implies the existence of the limit for ratio asymptotics, which in turn implies the existence of the limit for root asymptotics, and in general none of the converse statements hold. Perhaps the most useful tools for studying the root asymptotic behavior of the orthonormal polynomials have their foundations in potential theory, which we will discuss in Section 1.1. The book [65] by Stahl and Totik contains many deep results on root asymptotics that are proved using potential theoretic techniques. Our main focus in Chapter 3 will be on ratio and Szegő asymptotics.

In Chapter 3, we will present recent results, most of which directly address one of the following questions:

- If κ_n is the leading coefficient of $p_n(z;\mu)$, what is the asymptotic behavior of κ_n as $n \to \infty$?
- Is there an analytic function $\phi(z)$ so that the sequence of functions

$$\left\{\frac{P_n(z;\mu)}{\phi(z)^n}\right\}_{n\in\mathbb{N}}$$

  approaches a limit as $n \to \infty$ and what is the maximal domain of convergence?
- What are the weak limits of the sequence of probability measures $\{|p_n(z;\mu)|^2 d\mu(z)\}_{n\geq 0}$?
- What is the asymptotic behavior of the sum

$$\sum_{j=0}^{n} |p_j(z;\mu)|^2$$

  as $n \to \infty$, and how does the limit depend on z?
- What is the asymptotic behavior of the ratio $p_n(z;\mu)/p_{n-1}(z;\mu)$ as $n \to \infty$?
- What properties of the measure μ, if changed, do not change the answer to some of the above questions?

The two most heavily studied classes of orthogonal polynomials are those whose measure of orthogonality is supported on a compact subset of the real line and those whose measure of orthogonality is supported in the unit circle $\{z : |z| = 1\}$. These two "classical" settings are commonly referred to as orthogonal polynomials on the real line (OPRL) and orthogonal polynomials on the unit circle (OPUC) respectively. In both cases, one of the fundamental properties of the orthonormal polynomials is that they satisfy a finite term recurrence relation, i.e., the polynomial $p_n(z;\mu)$ can be expressed in a concise way in terms of $p_{n-1}(z;\mu)$ and $p_{n-2}(z;\mu)$. This recurrence relation has been an invaluable tool in the study of OPRL and OPUC, and the current state of knowledge in both settings is very advanced. Indeed, the answers to many of the aforementioned questions are already known for OPRL and OPUC. Much of this thesis is devoted to taking known theorems from the settings of OPRL and OPUC and adapting them to more general situations where the orthonormal polynomials do not satisfy a finite term recurrence relation.

Aside from examining the asymptotics for orthonormal and monic orthogonal polynomials, we can consider a related problem of examining the monic polynomials that minimize the expression

$$\left(\int_{\mathbb{C}}\left|z^n+\sum_{j=0}^{n-1}a_jz^j\right|^q d\mu(z)\right)^{1/q}, \tag{0.0.3}$$

where $q\in(0,\infty)$. A short compactness argument shows that a minimizer exists and if $q>1$, the strict convexity of the norm implies that the minimizer is unique. If $q\in(0,1]$, the minimizer need not be unique (see Proposition 3.2.1 below). We will denote a minimizer of (0.0.3) by $P_n(z;\mu,q)$ and denote the integral in (0.0.3) by $\|P_n(\cdot;\mu,q)\|_{L^q(\mu)}$ and refer to it as the L^q-*norm* of $P_n(\cdot;\mu,q)$ (this is an abuse of terminology since it is technically not a norm when $q\in(0,1)$, but we give it this name for convenience). If we divide $P_n(z;\mu,q)$ by $\|P_n(\cdot;\mu,q)\|_{L^q(\mu)}$, we get a degree n polynomial having $L^q(\mu)$-norm equal to 1, which we denote by $p_n(z;\mu,q)$. For many of the questions that we mentioned above concerning the polynomials $p_n(z;\mu)=p_n(z;\mu,2)$ and $P_n(z;\mu)=P_n(z;\mu,2)$, we will investigate analogous questions for the polynomials $p_n(z;\mu,q)$ and $P_n(z;\mu,q)$ for all $q\in(0,\infty)$.

Before we dive into the more technical matters, let us briefly summarize the main idea of the results that follow. Recall that the polynomials $\{P_n(z;\mu)\}_{n\geq0}$ satisfy the extremal property (0.0.2). This would lead one to believe that the polynomial $P_n(z;\mu)$ is smallest where the measure μ has greatest density and is largest where the measure has smallest density (to the extent this is possible). One would therefore expect that under reasonable hypotheses, the measures $\{|p_n(z;\mu)|^2d\mu(z)\}_{n\geq0}$ converge to a measure that is somehow as uniformly distributed as possible. The extremal characterization of the equilibrium measure given in Section 1.1 will show that the equilibrium measure for the support of μ is a likely candidate. In the settings of OPUC and OPRL, a precise description of the measures for which this convergence property holds is well-known. We will show that this convergence property often holds in more general settings. Although not every measure has this convergence property, the intuition is helpful and also leads one to guess that the orthonormal polynomial p_n resembles the function $e^{ng_{\overline{\mathbb{C}}\setminus\mathrm{supp}(\mu)}(z;\infty)}$ in some ways (here $g_{\overline{\mathbb{C}}\setminus\mathrm{supp}(\mu)}(z;\infty)$ is the Green function, which we define in Section 1.1). The main idea of our results is to make this resemblance as precise as possible under the weakest possible assumptions on the measure μ.

Throughout this paper, $\overline{\mathbb{C}}$ will denote the extended complex plane $\mathbb{C}\cup\{\infty\}$. We will often omit one or more parameters in our notation for $p_n(z;\mu,q)$ if we feel there is no possibility for confusion. In particular, $p_n(z;\mu)$ will always mean $p_n(z;\mu,2)$ and similarly for $P_n(z;\mu)$.

Chapter 1

Tools and Methods

"All men by nature desire knowledge."

– Aristotle

This chapter is meant to be an introduction to many of the ideas and tools we will employ in Chapter 3 to prove our new results. The material we present here will be relevant both for understanding existing results in the literature and for proving new theorems. We assume the reader has at least some familiarity with the concepts of analysis presented in standard texts such as [46, 47, 51] and we will not review this material here. Much of the notation that we introduce in this chapter will be retained for the remainder of this work.

We start this chapter by surveying some potential theoretic techniques in Section 1.1. Such methods are very useful for studying root asymptotic behavior of orthonormal polynomials by considering the corresponding zero counting measure. Potential Theory will also naturally lead us to define objects such as the equilibrium measure and the Green function, which will be important in our later analysis. In Section 1.2 we will discuss some important results about conformal maps that we will make extensive use of in Chapter 3. We then turn our attention to Chebyshev and Faber polynomials in Section 1.3. Chebyshev and Faber polynomials are sequences of polynomials that can be canonically defined for any simply connected and bounded domain. In Section 1.4 we define the Szegő function, which is an analytic and nonvanishing function whose boundary values are determined by the absolutely continuous part of a measure. Our discussion of the Szegő function in this chapter will provide the framework we need to understand Szegő's Theorem on the unit circle and some of its generalizations, which we will discuss in Section 2.3. Finally, we conclude this chapter with a complete proof of the Keldysh Lemma, which provides us with a useful criterion for convergence in Hardy space. Further details on all of these topics may be found in the references provided.

1.1 Potential Theory

We begin our discussion with an introduction to potential theory. The tools we discuss here have been used extensively to study orthogonal polynomials (see for example [16, 49, 58, 65, 72, 73, 74]), especially in deriving root asymptotics and exploring the consequences of the notion of *regularity* (see (1.1.6) below). Throughout this thesis, we will often make reference to objects such as the equilibrium measure, logarithmic potential, and Green function of a compact set; all of which will be discussed in this section. We refer the reader to the books [14, 45, 50] for additional background in potential theory.

Given a finite measure γ of compact support, we can define its *logarithmic potential*

$$U^{\gamma}(z) := \int_{\mathbb{C}} \log \frac{1}{|z-w|} d\gamma(w),$$

though for some values of z, the integral may be $+\infty$. We define the *equilibrium measure* of a compact set K as the unique probability measure ω_K satisfying

$$\int_K \int_K \log \frac{1}{|z-w|} d\omega_K(z) d\omega_K(w) = \inf \left\{ \int_K \int_K \log \frac{1}{|z-w|} d\gamma(z) d\gamma(w) : \gamma(K) = 1 = \gamma(\mathbb{C}) \right\}$$

provided the right-hand side is finite. In this case we call the left-hand side the *logarithmic energy* of ω_K and denote it by $E(\omega_K)$. It is always true that the support of the equilibrium measure ω_K is contained in the boundary of K (see Theorem 3.7.6 in [45]). If a compact set K admits an equilibrium measure ω_K, we define the *logarithmic capacity* of K to be $e^{-E(\omega_K)}$ and denote it by $\mathrm{cap}(K)$. If K does not admit an equilibrium measure (for example, if K is a single point), then we define the capacity of K to be 0. A straightforward argument using the minimizing property of the equilibrium measure shows that the support of the equilibrium measure of a compact set K is always contained in the boundary of the set. It is also customary to say that a property holds *quasi-everywhere* if the set of points where the property fails has logarithmic capacity zero.

The physical intuition behind the equilibrium distribution is quite simple. In two dimensions, the electrostatic interaction is given by a logarithmic repulsion. Therefore, if a large collection of charged particles is confined to a perfect conductor in the shape of a particular compact set $K \subseteq \mathbb{C}$, then the equilibrium measure describes the distribution of charge over the boundary of the set K when the system has reached equilibrium. While this reasoning is not mathematically rigorous, it is often extremely useful for developing intuition and understanding deeper mathematical results.

Armed with the notions of equilibrium measure and capacity, we can define the Green function with pole at infinity of a compact set K (of positive capacity) as

$$g_{\overline{\mathbb{C}} \setminus K}(z;\infty) := -U^{\omega_K}(z) - \log(\mathrm{cap}(K)). \tag{1.1.1}$$

It follows from Theorem 4.4.4 in [45] that the Green function is conformally invariant, i.e., if K_1 and K_2 are simply connected compact sets in the plane and $\mathcal{F}$ is the conformal map that sends the complement of K_1 to the complement of K_2 mapping ∞ to itself and having positive derivative there, then

$$g_{\overline{\mathbb{C}}\setminus K_1}(z;\infty) = g_{\overline{\mathbb{C}}\setminus K_2}(\mathcal{F}(z);\infty). \tag{1.1.2}$$

Example. Let $K = \overline{\mathbb{D}} = \{z : |z| \leq 1\}$ and let us consider the quantities just defined. From the uniqueness property of the equilibrium measure, we know that $\omega_{\overline{\mathbb{D}}}$ must be rotation invariant. Since $\omega_{\overline{\mathbb{D}}}$ must be supported in $\partial\mathbb{D} = \{z : |z| = 1\}$, we must have

$$d\omega_{\overline{\mathbb{D}}}(z) = \frac{d|z|}{2\pi}.$$

We then calculate

$$E(\omega_{\overline{\mathbb{D}}}) = \int_{\overline{\mathbb{D}}}\int_{\overline{\mathbb{D}}} \log\frac{1}{|z-w|}d\omega_{\overline{\mathbb{D}}}(z)d\omega_{\overline{\mathbb{D}}}(w) = \int_0^{2\pi}\int_0^{2\pi} \log\frac{1}{|e^{it}-e^{is}|}\frac{d|s|}{2\pi}\frac{d|t|}{2\pi} = 0$$

by Example 0.5.7 in [50]. Therefore, $\mathrm{cap}(\overline{\mathbb{D}}) = 1$. Using this calculation, from (1.1.1) we calculate

$$g_{\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}}(z;\infty) = \log|z|.$$

It follows from (1.1.2) that if K is any compact set of positive capacity such that $\overline{\mathbb{C}} \setminus K$ is simply connected in $\overline{\mathbb{C}}$, then

$$g_{\overline{\mathbb{C}}\setminus K}(z;\infty) = \log|\phi(z)|, \tag{1.1.3}$$

where ϕ is any conformal map from $\overline{\mathbb{C}} \setminus K$ to $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ satisfying $\phi(\infty) = \infty$.

In the above example, we heavily relied on the fact that the equilibrium measure of a compact set is always supported on the boundary of the set. If we consider normalized arc-length measure on the circle centered at zero and of radius 1/2, then this measure has the same potential as $\omega_{\overline{\mathbb{D}}}$ at all points outside of $\overline{\mathbb{D}}$. Indeed, to any measure ν supported in $\overline{\mathbb{D}}$ we can associate a measure supported on $\partial\mathbb{D}$ that has the same potential on $\{z : |z| > 1\}$. We call this measure the *balayage* of ν and it is the content of our next theorem. Before we state this theorem, we must introduce additional terminology and notation.

Definition. Given a domain G and a bounded function $f \in C(\partial G)$, the *Dirichlet problem* for f on G is to find a function f_0 that is continuous on $\overline{G}$, equal to f on ∂G, and harmonic on G. If G is unbounded, we also require that f_0 is continuous at infinity.

It is not true that the Dirichlet problem admits a solution on every domain. For example, $\mathbb{D}\setminus\{0\}$ does not admit a solution to the Dirichlet problem for certain continuous functions f, the difficulty being that $f(0)$ cannot be chosen arbitrarily (see page 269 in [8]). If a domain does admit a solution to the Dirichlet problem for every bounded and continuous f, we say that the domain is a *Dirichlet Region* (following terminology from [8]). Several conditions that are intimately connected to solvability of the Dirichlet problem are discussed in Section III.6 of [14] and Section I.4 in [50]. One such condition involves the Green function and can be described as follows. Let Ω be the unbounded component of $\overline{\mathbb{C}}\setminus\overline{G}$. For a point $z\in\partial\Omega$, consider the condition

$$\lim_{\substack{\zeta\to z\\ \zeta\in\Omega}} g_{\overline{\mathbb{C}}\setminus\overline{G}}(\zeta;\infty)=0. \tag{1.1.4}$$

Any point $z\in\partial\Omega$ for which (1.1.4) holds is called a *regular* point for G (regularity will usually mean something different throughout this text (see (1.1.6) below); this abuse of terminology – while unfortunate – is in agreement with standard terminology in the literature). Now we can state our theorem asserting the existence of the balayage measure.

Theorem 1.1.1. *Let G be a bounded domain whose closure is simply connected. Assume furthermore that ∂G is a Jordan curve and G is a Dirichlet Region. Let ν be a probability measure with support in $\overline{G}$. There exists a probability measure $\hat{\nu}$ supported on ∂G so that*

1. *if h is continuous on $\overline{G}$ and harmonic on G then $\int h d\nu=\int h d\hat{\nu}$,*

2. *if $z\notin\overline{G}$ or $z\in\partial G$ and z is regular for G then $U^{\nu}(z)=U^{\hat{\nu}}(z)$.*

Remark 1. We will discuss uniqueness of the balayage measure in Section 1.2.

Remark 2. One can assert the existence of a balayage measure under more general hypotheses than those given in Theorem 1.1.1. See Section II.4 in [50] for details.

Proof. We will only prove (1) since most of (2) can be recovered from (1) because $\log|t-z|$ is a bounded harmonic function of $t\in G$ when $z\notin\overline{G}$. Also observe that it suffices to show that (1) is true when $\nu=\delta_t$ for some $t\in\overline{G}$ for then the general result follows by setting

$$\hat{\nu}=\int\hat{\delta}_t d\nu(t).$$

By this we mean that for every $f\in C(\partial G)$ we set

$$\int_{\partial G} f(z)d\hat{\nu}(z)=\int_G\int_{\partial G} f(z)d\hat{\delta}_t(z)d\nu(t).$$

Therefore, we may assume $\nu=\delta_t$. If $t\in\partial G$ then $\nu=\hat{\nu}$ so we may also assume $t\notin\partial G$.

Let f be a bounded continuous function on ∂G. Since G is a Dirichlet Region, we may extend f to a harmonic function f_0 that is harmonic on G and continuous on $\overline{G}$. The maximum principal easily implies that if f is positive, then $f_0(t)$ is also positive. Therefore, the map $f \to f_0(t)$ is a positive linear functional on the set of continuous functions on ∂G. By the Riesz-Markov Theorem, there exists a measure $\hat{\delta}_t$ supported on ∂G with the desired properties. □

The measure $\hat{\delta}_t$ from the above proof is often called the *harmonic measure* at t. Harmonic measure can also be defined in terms of the hitting distribution of a Brownian motion. More specifically, if $E \subseteq \partial G$ is a Borel measurable set, then $\hat{\delta}_t(E)$ is the probability that a two-dimensional Brownian motion that begins at t leaves G for the first time by passing through a point of E. This realization of harmonic measure is a beautiful result due to Kakutani and is discussed in detail in Appendix F in [14]. Furthermore, it makes sense to discuss the Dirichlet problem in unbounded domains in the extended plane $\overline{\mathbb{C}}$. In this sense, the equilibrium measure for the boundary of a set K is the harmonic measure at infinity (see Theorem 4.3.14 in [45]). If ν is a measure that can be written as $\nu_1 + \nu_2$ where $\mathrm{supp}(\nu_1) \subseteq \overline{G}$ and $\mathrm{supp}(\nu_2) \subseteq \mathbb{C} \setminus G$, then we define the balayage of ν onto $\mathbb{C} \setminus G$ as $\nu_2 + \hat{\nu}_1$, where $\hat{\nu}_1$ is the balayage of ν_1 onto ∂G.

In the study of orthogonal polynomials, potential theoretic methods play an important role for the following reason: outside the convex hull of K, we know p_n does not vanish and so we can write

$$\log\left(|p_n(z;\mu)|^{1/n}\right) = \frac{1}{n}\log(\kappa_n) - U^{\nu_n}(z)$$

(recall κ_n is the leading coefficient of $p_n(z;\mu)$), where ν_n is the normalized zero counting measure for the polynomial $p_n(z;\mu)$, i.e., it is a point measure of total mass 1 that assigns weight n^{-1} to each zero of $p_n(z;\mu)$ (where we repeat each zero a number of times equal to its multiplicity as a zero of $p_n(z;\mu)$). It is then clear that the asymptotics of $|p_n(z;\mu)|^{1/n}$ can be deduced from the asymptotics of the leading coefficient κ_n and the asymptotic behavior of the measures $\{\nu_n\}_{n\in\mathbb{N}}$. We refer the reader to references such as [49, 65, 74] for examples of results obtained in this way. A typical result of this nature is contained in Theorem 1.1.4 in [65] and reads as follows:

Theorem 1.1.2 (Stahl & Totik, [65] pg. 4)**.** *Let μ be a finite measure with compact support K. Then locally uniformly in $\mathbb{C} \setminus \mathrm{ch}(\mu)$ one has*

$$\liminf_{n\to\infty} |p_n(z;\mu)|^{1/n} \geq e^{g_{\mathbb{C}\setminus K}(z;\infty)}. \tag{1.1.5}$$

In fact, in [65] Stahl and Totik prove much more. They introduce the notion of *regularity* by saying that a measure μ is a regular measure if

$$\lim_{n\to\infty} \kappa_n^{-1/n} = \mathrm{cap}(\mathrm{supp}(\mu)). \tag{1.1.6}$$

They then use potential theoretic methods to prove that regularity is equivalent to equality in the limit (1.1.5) (with the lim inf replaced by a true limit; see Theorem 3.1.1 in [65]).

Another elegant application of potential theoretic techniques was used to prove the following:

Theorem 1.1.3 (Saff & Totik, [49])**.** *Let Ω_μ be the unbounded component of $\mathbb{C} \setminus \mathrm{supp}(\mu)$ and let $\gamma \subseteq \Omega_\mu$ be a Jordan curve. Let $N_n(\gamma)$ denote the number of zeros of $p_n(z;\mu)$ inside γ. If the interior of γ contains infinitely many points of $\mathrm{supp}(\mu)$, then $\lim_{n\to\infty} N_n(\gamma) = \infty$. If γ contains exactly $k \in \mathbb{N} \cup \{0\}$ points of $\mathrm{supp}(\mu)$, then $\liminf_{n\to\infty} N_n(\gamma) \geq k$.*

Remark. Although not explicitly stated in [49], the same proof shows that we can make the same conclusion about the polynomials $\{p_n(z;\mu,q)\}_{n\geq 0}$.

In [49], explicit examples are provided to show that in Theorem 1.1.3, one cannot replace $\gamma \subseteq \Omega_\mu$ with $\gamma \subseteq \mathbb{C} \setminus \mathrm{supp}(\mu)$ and still arrive at the same conclusion.

1.2 Conformal Maps

The discussion of conformal maps usually begins with the most relevant result: the Riemann Mapping Theorem, which establishes the existence of a conformal map from a simply connected region $G \subsetneq \mathbb{C}$ to the unit disk. Furthermore, such a map is unique if we further specify the preimage of 0 and the argument of the derivative at this point.

Given a compact set $K \subseteq \mathbb{C}$, let Ω denote the unbounded component of $\overline{\mathbb{C}} \setminus K$. The boundary of Ω will also be called the *outer boundary* of K. Let us assume that Ω is simply connected in $\overline{\mathbb{C}}$ and K is not equal to a single point. In this case, the Riemann Mapping Theorem implies that there is a unique conformal map ϕ that maps Ω to $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ and satisfies $\phi(\infty) = \infty$ and $\phi'(\infty) > 0$. Let us denote the inverse to ϕ by ψ.

One is often interested in studying properties of conformal maps such as the existence of an analytic continuation to a larger domain or continuity properties of the map ϕ as a function of the domain Ω (we will make this precise shortly). The first relevant result is the following theorem due to Carathéodory:

Theorem 1.2.1. *Let φ be a conformal map from the unit disk $\mathbb{D}$ onto a Jordan domain G. Then φ has a continuous extension to $\overline{\mathbb{D}}$ and the extension is an injective mapping from $\overline{\mathbb{D}}$ to $\overline{G}$.*

It is also true that φ^{-1} can be extended to a homeomorphism of $\overline{G}$ onto $\overline{\mathbb{D}}$ (see page 12 in [13]). For a proof of Theorem 1.2.1, see page 13 in [14]. The utility of this result rests in the mild hypothesis it requires, namely only that of being a Jordan domain, meaning its boundary is a Jordan curve. Theorem 1.2.1 clearly applies also for mappings of regions in the extended plane, so it applies to the map ϕ in the setting considered above provided the outer boundary of K is a Jordan curve. This is very useful because it gives us a way to understand the equilibrium measure for the compact set K. This is the content of the following result:

Theorem 1.2.2 (Totik, [71]). *Let K be a compact set whose boundary is a Jordan curve and let ϕ be the conformal map as described above. Then for any Borel measurable set E, it holds that*

$$\omega_K(E) = \frac{|\phi(E)|}{2\pi},$$

where $|\cdot|$ denotes arc-length measure on the unit circle $\partial\mathbb{D}$.

Theorem 1.2.2 tells us that the equilibrium measure for the compact set K is the pull-back of arc-length measure on the unit circle under the conformal map ϕ. Indeed, the injectivity of ϕ on $\partial\Omega$ allows us to pull back any measure on $\partial\mathbb{D}$ to $\partial\Omega$. Similarly, if one has a measure μ defined on $\partial\Omega$, one can push it forward to a measure $\phi_*\mu$ on $\partial\mathbb{D}$ using the map ϕ. This is done by setting

$$(\phi_*\mu)(A) = \mu(\psi(A))$$

for all measurable sets $A \subseteq \partial\mathbb{D}$. We can integrate with respect to this measure by using the formula

$$\int_{\partial\mathbb{D}} f(w) d(\phi_*\mu)(w) = \int_{\partial\Omega} f(\phi(z)) d\mu(z)$$

for every $f \in C(\partial\mathbb{D})$. We can similarly take a measure ν on $\partial\mathbb{D}$ and push it to $\partial\Omega$ using ψ and obtain the measure $\psi_*\nu$. From our definitions, it is clear that $\phi_*(\psi_*\nu) = \nu$ and $\psi_*(\phi_*\mu) = \mu$. Theorem 1.2.2 can then be recast in this notation to read

$$\omega_K = \psi_*\omega_{\overline{\mathbb{D}}}.$$

Furthermore, we can now determine the uniqueness of the balayage measure in Theorem 1.1.1. Assume the hypotheses of that theorem and let φ be any conformal map from G to $\mathbb{D}$. Since ∂G was assumed to be a Jordan curve, we may extend φ continuously and injectively to all of $\overline{G}$. Since ν is given and we know $\hat{\nu}$ exists, we have

$$\int_{\overline{G}} \varphi(z)^k d\nu(z) = \int_{\partial G} \varphi(z)^k d\hat{\nu}(z)$$

for every $k \in \mathbb{N}_0$. Pushing the measure forward to $\partial\mathbb{D}$ using φ, we get

$$\int_{\overline{G}} \varphi(z)^k d\nu(z) = \int_{\partial\mathbb{D}} z^k d(\varphi_*\hat{\nu})(z).$$

In other words, the moments of $\varphi_*\hat{\nu}$ are uniquely determined, and since the moments of a measure on $\partial\mathbb{D}$ determine the measure, we have established uniqueness of $\hat{\nu}$.

Just as Theorem 1.2.2 establishes a relationship between exterior conformal maps and equilibrium measures, there exists a similar relationship between arbitrary conformal maps and harmonic measures. We first need the following definition:

Definitiion. Let $G \subsetneq \mathbb{C}$ be a simply connected domain and fix $z_0 \in G$. The conformal map $\varphi : G \to \mathbb{D}$ is called the *canonical conformal map at* z_0 if $\varphi(z_0) = 0$ and $\varphi'(z_0) > 0$.

Given a Dirichlet Region G and a point $z_0 \in G$, let $\omega_{\overline{G},z_0}$ denote the harmonic measure for the boundary of G at the point z_0. Our result is the following:

Theorem 1.2.3. *Let $G \subsetneq \mathbb{C}$ be a Jordan domain and a Dirichlet Region and fix any $z_0 \in G$. Assume further that ∂G is rectifiable and that $\omega_{\overline{G},z_0}$ is mutually absolutely continuous with arc-length measure on ∂G and has a continuous density function. If φ is the canonical conformal map at z_0 then*

$$d\omega_{\overline{G},z_0}(z) = |\varphi'(z)| \frac{d|z|}{2\pi}, \tag{1.2.1}$$

where $d|z|$ is arc-length measure on ∂G.

Proof. Let χ denote the inverse map to φ. Theorem 1.2.1 implies χ can be extended to $\partial\mathbb{D}$ continuously. Let $\omega'_{\overline{G},z_0}$ be the derivative of $\omega_{\overline{G},z_0}$ with respect to arc-length measure on ∂G. If f is a continuous function on $\overline{\mathbb{D}}$ that is harmonic on $\mathbb{D}$ then we get

$$\int_{\partial\mathbb{D}} f(\chi(z)) \frac{d|z|}{2\pi} = f(\chi(0)) = f(z_0) = \int_{\partial G} f(w) d\omega_{\overline{G},z_0}(w) = \int_{\partial G} f(w) \omega'_{\overline{G},z_0}(w) d|w|. \tag{1.2.2}$$

Taking the far right-hand side of (1.2.2) and setting $w = \chi(x)$, we get

$$\int_{\partial\mathbb{D}} f(\chi(x)) \omega'_{\overline{G},z_0}(\chi(x)) |\chi'(x)| d|x|.$$

Therefore,

$$\omega'_{\overline{G},z_0}(\chi(x)) |\chi'(x)| = \frac{1}{2\pi}$$

for almost every $x \in \partial\mathbb{D}$ and hence for all such x by continuity. This is easily seen to be equivalent to (1.2.1). □

We know now that under very mild hypotheses we can extend the conformal map ϕ to the boundary of Ω. Under stronger hypotheses, we can make a more powerful conclusion. The notions of analytic Jordan curve and analytic region will be very important for us throughout this thesis, so we take the time here to define them rigorously.

Definitiion. We define an *analytic Jordan curve* to be the image of $\partial\mathbb{D}$ under a map that is injective and analytic in a neighborhood of $\partial\mathbb{D}$ (see page 42 in [14]). If Γ is an analytic Jordan curve and G is the bounded component of $\mathbb{C} \setminus \Gamma$, then we say G is an *analytic region.*

It follows from a simple argument using the reflection principle that if Γ is an analytic Jordan curve, then ψ can be univalently (that is, injectively and analytically) continued to be analytic on the exterior of a disk of radius $\tilde{\rho} < 1$.

Theorem 1.2.3 establishes a connection between canonical conformal maps and harmonic measures. The intuition derived from the Brownian Motion definition of harmonic measure suggests that similar domains should have similar harmonic measures at a given point and hence similar canonical conformal maps at that point. In other words, one expects the conformal maps to exhibit some kind of continuity as a function of the domain.

To make this idea precise, we first need to define an appropriate notion of convergence of sets. All of our notation will be consistent with the notation in [13]. Let $\{D_n\}_{n\in\mathbb{N}}$ be a sequence of simply connected domains, all of which contain 0 and none of which are equal to $\mathbb{C}$. Let f_n be the conformal map from $\mathbb{D}$ to D_n satisfying $f_n(0) = 0$ and $f_n'(0) > 0$. If 0 is an interior point of the intersection of the domains D_n then we define the *kernel* D of $\{D_n\}_{n\in\mathbb{N}}$ to be the largest open subset with the property that every compact subset of D is contained in all but finitely many of the domains D_n. If 0 is a boundary point of the intersection of the domains D_n then we define the kernel to be $\{0\}$. The sequence $\{D_n\}_{n\in\mathbb{N}}$ is said to converge to its kernel D if every subsequence has the same kernel.

The result we need is the following (see Theorem 3.1 in [13]):

Carathéodory Convergence Theorem. *With the above notation, $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly on compact subsets of $\mathbb{D}$ to a function f if and only if $\{D_n\}_{n\in\mathbb{N}}$ converges to its kernel D. In the case of convergence there are two cases. If $D = \{0\}$, then $f = 0$. If $D \neq \{0\}$ then D is a simply connected domain, f maps $\mathbb{D}$ to D conformally with $f(0) = 0$ and $f'(0) > 0$, and $\{f_n^{-1}\}_{n\in\mathbb{N}}$ converges to f^{-1} uniformly on each compact subset of D.*

We will not prove the Carathéodory Convergence Theorem here and we refer the reader to a complete and straightforward proof presented in [13]. One important consequence of this result is that if the domains D_n are nested with the closure of their union compact, then the conformal maps converge to the conformal map of their union. Under some additional (but mild) assumptions on the domains $\{D_n\}_{n\in\mathbb{N}}$, Snipes and Ward in [64] are able to prove convergence of the maps $\{f_n\}_{n\in\mathbb{N}}$ uniformly on $\overline{\mathbb{D}}$.

1.3 Chebyshev and Faber Polynomials

In this section, we will explore two special sequences of polynomials. This first sequence we will study is the sequence of Chebyshev polynomials and the second is the sequence of Faber polynomials.

Each of these sequences is useful in applications for a particular reason. The Chebyshev polynomials are defined by the fact that they have minimal supremum norm and so are useful as trial functions when solving an extremal problem. The Faber polynomials are polynomial approximations to a particular conformal map and hence serve as an analog for the monomials $\{z^n\}_{n\geq 0}$ for regions other than the unit disk. We begin with a formal definition.

Definition. Let $K \subseteq \mathbb{C}$ be a compact and infinite set. The n^{th} *Chebyshev polynomial* is the unique monic polynomial T_n of degree n satisfying

$$\|T_n\|_{L^\infty(K)} = \min\left\{\|Q\|_{L^\infty(K)} : Q = z^n + \text{ lower order terms}\right\}.$$

Basic compactness results imply that the minimum in the definition of T_n is actually attained and a simple argument using the triangle inequality implies the minimizer is unique (though we must assume K is infinite for this to be true). Computing the Chebyshev polynomial explicitly for an arbitrary compact set K is in general a very difficult problem. However, it is known that $\lim_{n\to\infty} \|T_n\|_{L^\infty(K)}^{1/n} = \text{cap}(K)$ (see Theorem III.3.1 in [50]). Let us consider some examples.

Example. If $K = \overline{\mathbb{D}}$, then $T_n(z) = z^n$. This is an easy consequence of the uniqueness of the degree n Chebyshev polynomial.

Example. If $K = [-1, 1]$, then the Chebyshev polynomials are given by the formula

$$T_n(x) = \frac{1}{2^n}\left[\left(x + \sqrt{x^2-1}\right)^n + \left(x - \sqrt{x^2-1}\right)^n\right], \tag{1.3.1}$$

which is in fact a polynomial. It is often the case that if one refers to the Chebyshev polynomials without reference to a compact set, then it is understood to be this particular sequence of Chebyshev polynomials.

Now we turn our attention to Faber polynomials. Let K be a compact and simply connected set that is not equal to a single point and let $\Omega = \overline{\mathbb{C}} \setminus K$. As in Section 1.1, ϕ will denote the conformal mapping from the region Ω to the complement of the closed unit disk in the extended plane. The injectivity of ϕ implies that its Laurent expansion around infinity can be written as

$$\phi(z) = \xi_{-1}z + \xi_0 + \frac{\xi_1}{z} + \frac{\xi_2}{z^2} + \cdots.$$

We then define the degree n Faber polynomial as the polynomial part of ϕ^n and denote it by $F_n(z)$. More precisely, we can write

$$\phi(z)^n = F_n(z) + \ell_n(z),$$

where $F_n(z)$ is a polynomial and $\ell_n(\infty) = 0$. The leading coefficient of F_n is ξ_{-1}^n. It will be important for us that $\xi_{-1} = \text{cap}(K)^{-1}$.

One can also use alternative methods and formulas to characterize the Faber polynomials (see [35]). For example, let $\psi : \overline{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \Omega$ be the conformal map that is inverse to ϕ. One may also define the Faber polynomials using a generating function and the formula

$$\frac{\psi'(w)}{\psi(w) - z} = \sum_{n=0}^{\infty} \frac{F_n(z)}{w^{n+1}}. \tag{1.3.2}$$

If we let

$$\Gamma_r = \{\psi(z) : |z| = r\} \tag{1.3.3}$$

(for appropriate $r > 0$), then if z lies interior to Γ_r, a simple application of the Cauchy Integral Formula yields

$$F_n(z) = \frac{1}{2\pi i} \oint_{\{|t|=r\}} \frac{t^n \psi'(t)}{\psi(t) - z} dt. \tag{1.3.4}$$

If z lies exterior to Γ_r, then (1.3.4) and the Residue Theorem imply

$$F_n(z) = \phi(z)^n + \frac{1}{2\pi i} \oint_{\{|t|=r\}} \frac{t^n \psi'(t)}{\psi(t) - z} dt. \tag{1.3.5}$$

Equation (1.3.5) easily implies that if $\{z : |z| = r\}$ is inside the domain of ψ then all of the accumulation points of zeros of $\{F_n\}_{n\in\mathbb{N}}$ are inside Γ_r. Furthermore, as mentioned in Section 1.2, if $\partial\Omega$ is an analytic Jordan curve, then it is well-known that the map ψ can be univalently extended to the exterior of a disk with radius $\tilde{\rho}$ smaller than 1. In this case, formula (1.3.5) remains true if we take as our contour $\{|z| = r\}$ for any $r > \tilde{\rho}$. It easily follows from this that if $\partial\Omega$ is an analytic Jordan curve, then in the region $\{z : |z| > \rho > \tilde{\rho}\}$, one has

$$F_n(\psi(z)) = z^n + O(\rho^n), \tag{1.3.6}$$

where the implied constant is independent of z for z in this region. It is in this sense that the sequence $\{F_n\}_{n\geq 0}$ is a suitable analog of $\{z^n\}_{n\geq 0}$ for regions other than the unit disk.

Equation (1.3.6) impies that in the closed region $\overline{\Omega}$, the functions $F_n(z) - \phi(z)^n$ converge uniformly to 0. This is often a convenient property to exploit in applications and occurs under even more general circumstances. In [17], Geronimus lists several conditions on the boundary of Ω that imply this convergence property holds – analyticity of the boundary being one of them (see also [42]). One meaningful consequence of this property is the conclusion that $\|F_n\|_{L^\infty(K)} \to 1$ as $n \to \infty$.

Let us consider some examples:

Example. Let $K = \overline{\mathbb{D}}$. In this case $\phi(z) = z$ and so $F_n(z) = z^n$ for every $n \in \mathbb{N}_0$.

Example. Let $K = [-2, 2]$ so that $\phi(z) = \frac{1}{2}(z + \sqrt{z^2 - 4})$. It is an easy exercise using the binomial theorem to show that

$$\left(\frac{z + \sqrt{z^2 - 4}}{2}\right)^n + \left(\frac{z - \sqrt{z^2 - 4}}{2}\right)^n \tag{1.3.7}$$

is a polynomial in z, while the Laurent expansion of $\frac{1}{2}(z - \sqrt{z^2 - 4})$ contains only negative powers of z. Therefore, (1.3.7) is the expression for the Faber polynomial F_n corresponding to $[-2, 2]$. However, we recall from (1.3.1) that (1.3.7) is also equal to $2^n T_n(z/2)$ where T_n is the degree n Chebyshev polynomial for $[-1, 1]$. Therefore – just as in the case of the unit disk – the Faber polynomials and Chebyshev polynomials are the same for the interval $[-2, 2]$.

Example. This example is based on Example 3.8 in [35]. Fix $m \in \mathbb{N}$ and let $K = \{z : |z^m - 1| \leq 1\}$ (a *lemniscate*). In this case $\phi(z)^m = z^m - 1$ so if $j \in \mathbb{N}_0$ and $\ell \in \{0, \ldots, m - 1\}$, then

$$F_{mj+\ell}(z) = \sum_{k=0}^{j} (-1)^k \binom{j + \ell/m}{k} z^{s(j-k)+\ell},$$

where $\binom{a}{b}$ stands for the generalized binomial coefficient $\Gamma(a + 1)\Gamma(b + 1)^{-1}\Gamma(a - b + 1)^{-1}$. This immediately implies $F_{km}(z) = (z^m - 1)^k$ for all $k \in \mathbb{N}$.

Further examples of Faber polynomials with explicit formulas can be found in [25].

By using the generating function (1.3.2), one can define *generalized Faber polynomials*, which we denote by $F_n(z; g)$, where $g : \Omega \to \mathbb{C}$ is an analytic function satisfying $g(\infty) > 0$. In this case, one has

$$\frac{g(\psi(w))\psi'(w)}{\psi(w) - z} = \sum_{n=0}^{\infty} \frac{F_n(z; g)}{w^{n+1}}$$

so that $F_n(z) = F_n(z; 1)$. We will not use generalized Faber polynomials to prove any of our new results, but they are an essential ingredient in many proofs in [37, 68]. Further properties of Faber polynomials are discussed in [9, 35].

1.4 The Szegő Function

To prove many of our results, we will need to treat certain equalities as two simultaneous inequalities and prove each inequality separately. We will often use the extremal property to derive an upper bound on $\|P_n(z; \mu, q)\|_{L^q(\mu)}$. To obtain a lower bound, we will often use inequalities concerning the measure or subharmonicity of certain functions. The key will be to use well-known results from the theory of H^q spaces to realize the absolutely continuous part of a measure μ as the boundary values of the absolute value of an analytic function, often called the *Szegő function*, which was introduced in [69].

The Szegő function will take on different forms in different settings. One common form will be the function that we denote $S(z)$, which is defined on $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ and is useful in the context of OPUC. However, in order to define $S(z)$, we need to make some assumptions about the measure in question. As just indicated, the function $S(z)$ will depend only on the absolutely continuous part of the measure μ, so we will require the measure μ to have an absolutely continuous component, but we must assume even more. For a measure μ defined on the unit circle, we can analogously define a measure on $[0, 2\pi)$ (which we also call μ) and write

$$d\mu(\theta) = w(\theta)\frac{d\theta}{2\pi} + d\mu_s(\theta), \tag{1.4.1}$$

where $d\mu_s$ is singular with respect to the linear Lebesgue measure on $[0, 2\pi)$ and $w(\theta)$ is the Radon-Nikodym derivative of μ, which we may also denote by $\mu'(\theta)$. To define the Szegő function, we require

$$\int_0^{2\pi} \log(w(\theta))\frac{d\theta}{2\pi} > -\infty. \tag{1.4.2}$$

We refer to (1.4.2) as the *Szegő condition* and any measure for which (1.4.2) holds will be called a *Szegő measure* on the unit circle. For reasons discussed in [22], if (1.4.2) holds we will also say μ has finite entropy. Since $\log(x) < x$ for all $x > 0$, the finiteness of μ implies that the integral in (1.4.2) cannot diverge to $+\infty$, so (1.4.2) is equivalent to saying $\log(w(\theta)) \in L^1(\frac{d\theta}{2\pi})$.

Before we define the Szegő function, we must define some notation that we will retain throughout this work. For any $q \in (0, \infty)$, we say that a function f is in the Hardy space $H^q(\mathbb{D})$ if it is analytic in the unit disk and

$$\lim_{r \to 1^-} \left(\int_0^{2\pi} |f(re^{i\theta})|^q \frac{d\theta}{2\pi} \right)^{1/q} < \infty. \tag{1.4.3}$$

We call the limit in (1.4.3) the H^q-norm of f and denote it by $\|f\|_{H^q(\mathbb{D})}$. We say that a function g is in the Hardy space $H^q(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}})$ if $f(z) = g(1/z) \in H^q(\mathbb{D})$ (as in [19]) and define

$$\|g(z)\|_{H^q(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}})} = \|g(1/z)\|_{H^q(\mathbb{D})} = \lim_{r \to 1^+} \left(\int_0^{2\pi} |g(re^{i\theta})|^q \frac{d\theta}{2\pi} \right)^{1/q}.$$

We refer the reader to the references [12, 47] for more information on the Hardy spaces H^q.

Now, for a Szegő measure μ on $\partial\mathbb{D}$, we define the function $S(z)$ as follows:

$$S(z) = \exp\left(-\frac{1}{4\pi}\int_0^{2\pi} \log(\mu'(\theta))\frac{e^{i\theta}+z}{e^{i\theta}-z}d\theta\right), \qquad |z|>1, \tag{1.4.4}$$

which is analytic in the region $\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}$. If we want to make the dependence on μ explicit, we will write $S(z;\mu)$. The importance for us comes from the following result, which is contained in Theorem 2.4.1 in [56]:

Theorem 1.4.1 ([56], pg. 144)**.** *Let $S(z)$ be defined as in (1.4.4). Then*

1. *$S(z)$ is analytic and nonvanishing in $\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}$.*
2. *$S(z)$ lies in the Hardy Space $H^2(\overline{\mathbb{C}}\setminus\overline{\mathbb{D}})$.*
3. *$\lim_{r\to 1^+} S(re^{i\theta}) = S(e^{i\theta})$ exists for Lebesgue almost every $\theta\in[0,2\pi]$ and*

$$|S(e^{i\theta})|^2 = \mu'(\theta). \tag{1.4.5}$$

Proof. The theorem is proven using a short argument from [56], which we reproduce here. The property (1) is two statements. The analyticity is obvious while the fact that it is nonvanishing in $\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}$ follows from the fact that it is the exponential of a function with finite real part for each $z\notin\overline{\mathbb{D}}$.

We will prove properties (2) and (3) at the same time. If $\log(\mu'(\theta))$ is bounded, then

$$|S(re^{i\theta})|^2 = \exp\left(\int_0^{2\pi}\log(\mu'(t))\frac{r^2-1}{|e^{it}-re^{i\theta}|^2}\frac{dt}{2\pi}\right).$$

We recognize this as the exponential of the Poisson extension of $\log(\mu')$ to $\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}$. Therefore, $\lim_{r\to 1^+}|S(re^{i\theta})|^2 = \mu'(\theta)$ and since $\log(\mu')$ is bounded, Dominated Convergence shows that $S\in H^2$. If $\log(\mu')$ is not bounded, then a simple double approximation argument proves the general result whenever μ is a Szegő measure. □

When we consider the polynomials $P_n(z;\mu,q)$ in Section 3.2, it will be convenient to generalize our definition of the Szegő function by defining

$$S(z;q) = \exp\left(-\frac{1}{2q\pi}\int_0^{2\pi}\log(\mu'(\theta))\frac{e^{i\theta}+z}{e^{i\theta}-z}d\theta\right), \qquad |z|>1. \tag{1.4.6}$$

In this case, the proof of Theorem 1.4.1 shows that $S(\cdot;q)\in H^q(\overline{\mathbb{C}}\setminus\overline{\mathbb{D}})$ and $|S(e^{i\theta})|^q = \mu'(\theta)$.

The Szegő function can also appear in the context of measures supported on curves other than the unit circle. If G is an analytic region and μ is supported on ∂G, we let $w(z) = d\mu(z)/d|z|$ be the Radon-Nikodym derivative of μ with respect to arc-length measure on ∂G. Let $\varphi : \overline{G}\to\overline{\mathbb{D}}$ be any conformal bijection and let its inverse be denoted by χ. If

$$\frac{1}{2\pi i}\int_0^{2\pi}\log(w(\chi(e^{i\theta})))\frac{d\theta}{2\pi}>-\infty,$$

then for any $q\in(0,\infty)$ we define (as in [34])

$$\Delta_q(z)=\exp\left(\frac{1}{2q\pi i}\oint_{\partial G}\log(w(\zeta))\frac{1+\overline{\varphi(\zeta)}\varphi(z)}{\varphi(\zeta)-\varphi(z)}\varphi'(\zeta)d\zeta\right),\qquad z\in G. \tag{1.4.7}$$

The argument in the proof of Theorem 1.4.1 can be easily adapted to this setting to show that $\Delta_q(z)\in H^q(G)$ (see Chapter 10 in [12] for a discussion of H^q spaces for regions other than the unit disk) and $|\Delta_q(z)|^q=w(z)$ for almost every $z\in\partial G$ with respect to arc-length measure. In the special case $G=\mathbb{D}$ and $q=2$, the function $\Delta_2(z)$ is the function $D(z)$ used throughout [56, 57].

1.5 The Keldysh Lemma

Recall that the polynomial $P_n(z;\mu,q)$ is defined in terms of an extremal property for a particular integral. Consequently, it is often the case that the asymptotic behavior of $P_n(z;\mu,q)$ must be determined by the behavior of the corresponding integral. The situation is not dire since convergence in the space $H^q(\mathbb{D})$ is determined by the behavior of an integral and is stronger than pointwise convergence in $\mathbb{D}$. Therefore, if one wants pointwise asymptotics of $P_n(z;\mu,q)$, it is useful to have a general convergence criterion that can be applied to functions in $H^q(\mathbb{D})$.

Fortunately, such a result exists, and is known as the *Keldysh Lemma* (see [19]). This result tells us that a sequence $\{f_n\}_{n\in\mathbb{N}}$ converges in $H^q(\mathbb{D})$ whenever the norms $\{\|f_n\|_{H^q}\}_{n\in\mathbb{N}}$ and the values at zero $\{f_n(0)\}_{n\in\mathbb{N}}$ converge to the same limit. In the case $q=2$, this is a triviality because of the isometry between $H^2(\mathbb{D})$ and $\ell^2(\mathbb{N}_0)$ realized by the Taylor coefficients. It is not as easy to see that the result holds for all q, but it does and we will prove it in this section. The Keldysh Lemma was applied to obtain orthogonal polynomial asymptotics in [19] and we will use it in several places in this thesis, including in Chapter 3. Additionally, Theorem 3.3.2, which is the key to many of our relative and ratio asymptotic results in Chapter 3, will be reminiscent of the Keldysh Lemma.

We begin with a statement of the result.

Keldysh Lemma. *Let $q\in(0,\infty)$ be fixed and suppose $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of functions in $H^q(\mathbb{D})$ with boundary values $f_n(e^{i\theta})$. If $f_n(0)\to1$ and $\|f_n\|_{H^q}\to1$ as $n\to\infty$, then $f_n\to1$ in H^q as $n\to\infty$.*

The remainder of this section will be devoted to the proof of this result. We begin with a discussion of the simplest case, namely $q=2$. This case is especially simple because $g\in H^2$ if and only if the Taylor coefficients of g around 0 are in $\ell^2(\mathbb{N}_0)$ (see Theorem 17.12 in [47]) and in fact the

H^2 norm of g is the same as the ℓ^2 norm of its Taylor coefficients. Let us write

$$f_n(z) = \sum_{j=0}^{\infty} a_j^{(n)} z^j.$$

We are assuming $a_0^{(n)} \to 1$ and $\sum_{j=0}^{\infty} |a_j^{(n)}|^2 \to 1$. Therefore, $\sum_{j=1}^{\infty} |a_j^{(n)}|^2 \to 0$, which is equivalent to saying the H^2 norm of $f_n - 1$ converges to 0, which concludes the proof in the case $q = 2$.

The proof for arbitrary $q \in (0, \infty)$ comes in two steps. The first step is to show that $f_n \to 1$ uniformly on compact subsets of $\mathbb{D}$ and the second step is to show that this and the convergence of norms (to the same limit) implies convergence in H^q. We begin with a lemma showing that convergence in H^q implies uniform convergence on compact subsets (note that the following lemma does not take the q^{th} power of $\|f\|_{H^q}$ as is done in [19]).

Lemma 1.5.1 (Kaliaguine, [19])**.** *Let $K \subseteq \mathbb{D}$ be a compact set. There exists a constant C_K such that for all $f \in H^q(\mathbb{D})$ one has*

$$\sup_K |f(z)| \leq C_K \|f\|_{H^q}.$$

Proof. If $q \geq 1$, we may apply Jensen's inequality. Indeed, if $t < 1$ and $K \subseteq \{z : |z| < t\}$, then uniformly for $z \in K$ we calculate

$$|f(z)|^q \leq \left| \frac{1}{2\pi i} \oint_{\{|z|=t\}} \frac{f(\zeta)}{\zeta - z} d\zeta \right|^q \leq C_K^q \int_0^{2\pi} |f(te^{i\theta})|^q \frac{d\theta}{2\pi}.$$

Since this last integral is increasing in t (see Theorem 17.8 in [47]), we may further bound this by $\|f\|_{H^q}^q$ as desired.

For arbitrary $q \in (0, \infty)$, we define $f_r(z) = f(rz)$ for $r \in (0, 1)$ and let g_r be the harmonic extension of $|f_r|^q$ to $\mathbb{D}$, that is,

$$g_r(z) = \int_0^{2\pi} |f_r(e^{i\theta})|^q \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \frac{d\theta}{2\pi}.$$

We clearly have

$$\sup_K |g_r(z)| \leq C_K^q \|f_r\|_{H^q}^q \tag{1.5.1}$$

by the uniform boundedness of the Poisson Kernel on compact subsets of the disk. Let us now write $f_r(z) = B_r(z) h_r(z)$ where B_r is a Blaschke product and h_r is nonvanishing. Then

$$g_r(z) = \int_0^{2\pi} |f_r(e^{i\theta})|^q \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \frac{d\theta}{2\pi} \geq \left| \int_0^{2\pi} h_r(e^{i\theta})^q \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \frac{d\theta}{2\pi} \right| = |h_r(z)|^q$$

since h_r^q is analytic in $\mathbb{D}$. Now, $h_r(z) B_r(z) = f(rz)$ so $|h_r(z)| \geq |f(rz)|$ and hence $g_r(z) \geq |f(rz)|^q$.

Combining this with (1.5.1) and noting that C_K is independent of r, we get

$$|f(rz)|^q \le C_K^q \|f_r\|_{H^q}^q$$

for each $r < 1$. Sending r to 1 gives the desired result. □

Lemma 1.5.1 tells us that the sequence $\{f_n\}_{n\in\mathbb{N}}$ as in the statement of the Keldysh Lemma is uniformly bounded on compact subsets of $\mathbb{D}$ and hence forms a normal family. Let f be a limit – which exists by Montel's Theorem – and let $\mathcal{N} \subseteq \mathbb{N}$ be a subsequence so that f_n converges to f uniformly on compact sets as $n \to \infty$ through $\mathcal{N}$. It is clear that $f(0) = 1$. Since the convergence is uniform on compact subsets, then for every $r < 1$ we calculate

$$\int_0^{2\pi} |f(re^{i\theta})|^q \frac{d\theta}{2\pi} = \lim_{\substack{n\to\infty \\ n\in\mathcal{N}}} \int_0^{2\pi} |f_n(re^{i\theta})|^q \frac{d\theta}{2\pi} \le \lim_{\substack{n\to\infty \\ n\in\mathcal{N}}} \|f_n\|_{H^q}^q = 1$$

and so $f \in H^q$ and $\|f\|_{H^q} = 1$. This and the fact that $f(0) = 1$ easily imply that $f \equiv 1$. Furthermore, since the same reasoning can be applied to any subsequence of $\{f_n\}_{n\in\mathbb{N}}$, we conclude $f_n \to 1$ uniformly on compact subsets of $\mathbb{D}$.

The next lemma shows that we can deduce the desired convergence in $L^q(\partial\mathbb{D})$.

Lemma 1.5.2 (Duren, [12] pg. 21)**.** *Suppose $\{g_n\}_{n\in\mathbb{N}}$ is a sequence in $L^q(\partial\mathbb{D})$ and $g \in L^q(\partial\mathbb{D})$ is such that g_n converges to g almost everywhere and $\|g_n\|_{L^q} \to \|g\|_{L^q}$. It follows that $g_n \to g$ in L^q.*

We include Duren's proof so as to provide here a complete proof of the Keldysh Lemma.

Proof. For a measurable set $E \subseteq \partial\mathbb{D}$, let $J_n(E) = \int_E |g_n(e^{i\theta})|^q \frac{d\theta}{2\pi}$ and $J(E) = \int_E |g(e^{i\theta})|^q \frac{d\theta}{2\pi}$. Let $Y = \partial\mathbb{D} \setminus E$. Then

$$J(E) \le \liminf_{n\to\infty} J_n(E) \le \limsup_{n\to\infty} J_n(E) \le \lim_{n\to\infty} J_n(\partial\mathbb{D}) - \liminf_{n\to\infty} J_n(Y) \le J(\partial\mathbb{D}) - J(Y) = J(E),$$

which shows $J_n(E) \to J(E)$ for every E.

Given $\epsilon > 0$, choose $E \subseteq \partial\mathbb{D}$ so that $J(Y) < \epsilon$. Choose $\delta > 0$ so that $J(Q) < \epsilon$ for every $Q \subseteq E$ satisfying $|Q| < \delta$. By Egorov's Theorem, there exists a set $Q \subseteq E$ of measure less than δ such that $g_n \to g$ uniformly on $E' = E \setminus Q$. Therefore,

$$\begin{aligned}\int_{\partial\mathbb{D}} |g_n - g|^q \frac{d|z|}{2\pi} &= \int_Y |g_n - g|^q \frac{d|z|}{2\pi} + \int_Q |g_n - g|^q \frac{d|z|}{2\pi} + \int_{E'} |g_n - g|^q \frac{d|z|}{2\pi} \\ &\le 2^q(J_n(Y) + J(Y) + J_n(Q) + J(Q)) + \int_{E'} |g_n - g|^q \frac{d|z|}{2\pi}.\end{aligned}$$

The last integral tends to 0 as $n \to \infty$ by the uniform convergence on E'. Also, we know $J_n(Y) \to J(Y)$ and $J_n(Q) \to J(Q)$ as $n \to \infty$ and $J(Y)$ and $J(Q)$ are both less than ϵ. Since $\epsilon > 0$ was arbitrary, we get the desired result. ☐

We now conclude the proof of the Keldysh Lemma with the following special case of a theorem from [3].

Theorem 1.5.3 ([3]). *Suppose $q \in (0,\infty)$ and $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of functions in $H^q(\mathbb{D})$ satisfying $\lim_{n\to\infty} \|f_n\|_{H^q} = 1$ and for all $z \in \mathbb{D}$ we know $\lim_{n\to\infty} f_n(z) = 1$ where the convergence is uniform on compact subsets of $\mathbb{D}$. Then $f_n \to 1$ in $H^q(\mathbb{D})$.*

Proof. Our proof is based on ideas presented in [3]. If $q \in (1,\infty)$, the result is easily obtained by noting that

$$\lim_{n\to\infty} \left\| \frac{f_n+1}{2} \right\|_{H^q} = 1$$

and so the uniform convexity of the norm implies

$$\lim_{n\to\infty} \left\| \frac{f_n-1}{2} \right\|_{H^q} = 0$$

as desired.

If $q \in (0,1]$, then we again divide the proof into two cases. First, we suppose $f_n(z) \neq 0$ for any $n \in \mathbb{N}$ and any $z \in \mathbb{D}$. Let s be any real number in $(0,q)$ so that $s^{-1} \in \mathbb{N}$. In this case, we may define $f_n^{s/2}$, which is an analytic function. The Hölder inequality shows that $f_n^{s/2} \in H^2(\mathbb{D})$ and since $f_n^{s/2}(0) \to 1$, a simple calculation shows $\|f_n^{s/2}\|_{H^2} \to 1$. By our previous discussion, we know that $f_n^{s/2}$ converges to 1 in H^2. By Theorem 3.12 in [47], we know that there is a subsequence $\mathcal{N} \subseteq \mathbb{N}$ so that $f_n^{s/2}(e^{i\theta})$ converges to 1 almost everywhere as $n \to \infty$ through $\mathcal{N}$, which in turn implies $f_n(e^{i\theta})$ converges to 1 almost everywhere as $n \to \infty$ through $\mathcal{N}$ (since $2s^{-1} \in \mathbb{N}$). Lemma 1.5.2 then implies $f_n \to 1$ in $L^q(\partial\mathbb{D})$ and hence in $H^q(\mathbb{D})$ (see Theorem 17.11 in [47]) as $n \to \infty$ through $\mathcal{N}$. Since this reasoning can be applied to any subsequence of the functions $\{f_n\}_{n\in\mathbb{N}}$, we conclude $f_n \to 1$ in $H^q(\mathbb{D})$ as claimed.

The final step is to consider the case when $0 < q \leq 1$ and f_n is allowed to have zeros in $\mathbb{D}$. In this case, we write $f_n = B_n h_n$ where B_n is a Blaschke product and h_n is nonvanishing and satisfies $\|h_n\|_{H^q} = \|f_n\|_{H^q}$. By Lemma 1.5.1, the families $\{h_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ are normal. Therefore, our above arguments imply there exists a subsequence $\mathcal{N} \in \mathbb{N}$ so that $\lim_{n\in\mathcal{N}} h_n = h$ and $\lim_{n\in\mathcal{N}} B_n = B$ uniformly on compact subsets of $\mathbb{D}$ with $B, h \in H^q$, $Bh = 1$, and $\|h\|_{H^q} \leq 1$. Clearly $\|B\|_{L^\infty} \leq 1$, so

$$\|h\|_{H^q} \leq 1 = \|Bh\|_{H^q} \leq \|h\|_{H^q},$$

so $\|h\|_{H^q} = 1$. However, $|h(0)| \geq 1$ so we must have $h \equiv 1 \equiv B$. Our previous considerations imply that $h_n \to 1$ in H^q. Furthermore, we have shown $B_n \to 1$ uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$ through $\mathcal{N}$. Combining this with the fact that $\|B_n\|_{L^\infty(\mathbb{D})} \leq 1$ shows that $\|B_n\|_{H^2} \to 1$ as

$n \to \infty$ through $\mathcal{N}$ so our previous considerations imply we have convergence in H^2. Now we apply Theorem 3.12 in [47] again to conclude there is a subsequence $\mathcal{N}' \subseteq \mathcal{N}$ so that $h_n(e^{i\theta}) \to 1$ and $B_n(e^{i\theta}) \to 1$ almost everywhere as $n \to \infty$ through $\mathcal{N}'$. This means $f_n(e^{i\theta}) \to 1$ almost everywhere as $n \to \infty$ through $\mathcal{N}'$, which by Lemma 1.5.2 implies we have H^q convergence as $n \to \infty$ through $\mathcal{N}'$. Since the same reasoning can be applied to any subsequence of the functions $\{f_n\}_{n\in\mathbb{N}}$, we get the desired conclusion. $\square$

The Keldysh Lemma tells us that if we have a sequence of functions that converges at a particular point and the norms of those functions converge in the space $L^q(\partial\mathbb{D}, \frac{d\theta}{2\pi})$ to the same limit, then in fact we have convergence in the space H^q. A variant of this theme will be prominent when we discuss our new results concerning ratio and relative asymptotics in Sections 3.3.2, 3.3.3, and 3.3.4. Furthermore, the map $z \to z^{-1}$ induces a bijection between $H^q(\mathbb{D})$ and $H^q(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}})$ in such a way that we can easily adapt the results of this section to the setting of $H^q(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}})$. Namely, if a sequence $\{f_n\}_{n\in\mathbb{N}}$ in $H^q(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}})$ satisfies $f_n(\infty) \to 1$ and $\|f_n\|_{H^q(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}})} \to 1$ as $n \to \infty$, then $f_n \to 1$ as $n \to \infty$ in $H^q(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}})$.

Chapter 2

Overview of Polynomial Asymptotics

"Those who cannot remember the past are condemned to repeat it."

– George Santayana

In this chapter, we will motivate and introduce the problems that we will tackle in Chapter 3. Many of these problems are motivated by well-known theorems in the subject of OPUC and OPRL that we wish to adapt to more general settings. A prominent theme in this chapter will be to precisely state these known results so that we may draw meaningful comparisons to them in later sections.

We begin our exposition in Section 2.1 with a brief summary of some basic properties of OPUC and OPRL. In Section 2.2, we will discuss regularity and some associated ideas from potential theory. In Section 2.3, we will state Szegő's Theorem on the unit circle and explore some consequences and generalizations of this important result. We then turn to Bergman polynomials in Section 2.4, which are polynomials that are orthogonal with respect to area measure on a domain. Our discussion of Bergman polynomials will include one of the most important applications of Bergman polynomials, which is the *Bergman Kernel method* for numerically approximating conformal maps. We will discuss this method in Section 2.6 and state some of the key results. In Section 2.5 we define Christoffel functions and the associated minimization problem and describe the relationship between the solution to this problem and orthogonal polynomials. Our focus in Section 2.7 will be on ratio asymptotic results while we will discuss a related phenomenon in Section 2.8 when we consider relative asymptotics. Finally, in Section 2.9 we will examine the problem of finding the weak limit points of the measures $\{|p_n(z;\mu,2)|^2 d\mu(z)\}_{n\geq 0}$, which we call the *weak asymptotic measures*.

Throughout this chapter, we will write $p_n(z;\mu)$ for $p_n(z;\mu,2)$.

2.1 OPUC and OPRL: A Brief Introduction

Any discussion of the history of orthogonal polynomials must begin with the classical settings of the unit circle and the real line. Our exposition here will be very brief and we refer the reader to the references [38, 56, 57, 63, 69] for extensive background and additional results on this topic. Of central importance to the theory of both OPUC and OPRL is the existence of a finite term recurrence relation satisfied by the orthonormal polynomials. If μ is is a probability measure on a compact subset of the real line, then there are two bounded sequence $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ such that

$$xp_n(x;\mu) = a_{n+1}p_{n+1}(x;\mu) + b_{n+1}p_n(x;\mu) + a_np_{n-1}(x;\mu). \tag{2.1.1}$$

For each $n \in \mathbb{N}$ it is true that $a_n > 0$ and $b_n \in \mathbb{R}$. These sequences are often called the recursion coefficients or *Jacobi parameters* for the measure μ. If μ is a probability measure on the unit circle, then there is a sequence of complex numbers $\{\alpha_n\}_{n\geq 0}$ such that

$$p_{n+1}(z;\mu) = \frac{1}{\sqrt{1-|\alpha_n|^2}}\left(zp_n(z;\mu) - \overline{\alpha}_n p_n^*(z;\mu)\right), \tag{2.1.2}$$

where $p_n^*(z;\mu) = z^n\overline{p_n(\bar{z}^{-1};\mu)}$. For each $n \in \mathbb{N}_0$ it is true that $\alpha_n \in \mathbb{D}$. The numbers $\{\alpha_n\}_{n\geq 0}$ have many names in the literature, including recursion coefficients, *Verblunsky coefficients*, and *Schur parameters*. Furthermore, there is an inverse Szegő recursion:

$$zP_n(z;\mu) = \frac{P_{n+1}(z;\mu) + \overline{\alpha}_n P_{n+1}^*(z;\mu)}{1-|\alpha_n|^2}. \tag{2.1.3}$$

Since $\overline{\alpha}_n = -P_{n+1}(0;\mu)$, (2.1.3) allows us to recover the Verblunsky coefficients $\alpha_0,\ldots,\alpha_n$ if we are given $P_{n+1}(\cdot;\mu)$.

The existence of the recurrence relation (2.1.1) enables one to prove Favard's Theorem, which asserts that there is a bijection between pairs of bounded sequences $\{a_n,b_n\}_{n\in\mathbb{N}} \subseteq (0,\infty)^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ and compactly supported probability measures on the real line with infinite support. Similarly, the relation (2.1.2) allows one to establish Verblunsky's Theorem, which states that there is a bijection between $\mathbb{D}^{\mathbb{N}}$ and probability measures on the unit circle with infinite support. A key step in one approach to proving both Favard's Theorem and Verblunsky's Theorem is to use the recursion coefficients to construct a particular operator, which is self-adjoint when we are given $\{a_n,b_n\}_{n\in\mathbb{N}}$ and unitary when we are given $\{\alpha_n\}_{n\geq 0}$. These operators are respectively called Jacobi matrices and CMV matrices, and the measure corresponding to the recursion coefficients is realized as the spectral measure of this operator. Since the correspondence between measures and sequences is established through the use of the recursion coefficients, it should not be surprising then that there are many results relating properties of the measure to properties of the corresponding sequence(s) (see for example [5, 7, 10, 22, 23, 32, 55, 56, 57, 61, 63, 66, 75] among many others).

In the classical settings of OPUC and OPRL, the orthonormal polynomials also enable one to study the corresponding measure by providing a sequence of measures that approximate the measure of orthogonality and are defined in terms of the orthonormal polynomials. In the case of OPUC, the approximating measures – which we denote by μ^n – are defined by

$$d\mu^n(e^{i\theta}) = \frac{1}{|p_{n+1}(e^{i\theta};\mu)|^2}\frac{d\theta}{2\pi}. \tag{2.1.4}$$

These measures are called the Bernstein-Szegő Approximating measures and are in fact probability measures on the unit circle. Theorem 1.7.8 in [56] tells us that the measures $\{\mu^n\}_{n\in\mathbb{N}}$ converge weakly to the measure μ as $n \to \infty$. More precisely, if Q is a polynomial of degree at most $n+1$ then

$$\int_{\partial\mathbb{D}} Q(z)d\mu^n(z) = \int_{\partial\mathbb{D}} Q(z)d\mu(z).$$

Furthermore, it is trivial to see that $p_{n+1}(z;\mu^n) = p_{n+1}(z;\mu)$ and so by (2.1.3), we conclude that the Verblunsky coefficients $\alpha_0,\ldots,\alpha_n$ are the same for μ and μ^n and so in fact $p_m(z;\mu^n) = p_m(z;\mu)$ for all $m \le n+1$.

In the case of OPRL, there is more than one sequence of approximating measures, some of which are discussed in [26, 28]. We will be interested in the measures defined by

$$d\rho_n(x) = \frac{1}{\pi(a_{n+1}^2 p_{n+1}^2(x;\mu) + p_n(x;\mu)^2)}dx, \tag{2.1.5}$$

which are analyzed in [59] where it is shown that the measures $\{\rho_n\}_{n\in\mathbb{N}}$ converge weakly to the measure μ as $n\to\infty$. It is well-known that the zeros of $p_n(x;\mu)$ and $p_{n+1}(x;\mu)$ strictly interlace for OPRL (see Section 1.2.5 in [56]) so the denominator on the right-hand side of (2.1.5) is never zero. As discussed in Section 2 in [59], it holds that

$$\int_{\mathbb{R}} x^\ell d\rho_n(x) = \int_{\mathbb{R}} x^\ell d\mu(x), \qquad \ell = 0,1,\ldots,2n. \tag{2.1.6}$$

Notice that for $\ell > 2n$, the left-hand side of (2.1.6) is infinite. However, we can still construct the orthonormal polynomials $p_m(z;\rho_n)$ for $m \le n$ and we easily see that $p_m(z;\rho_n) = p_m(z;\mu)$ for all $m \le n$.

We will continue to discuss OPRL and OPUC throughout this chapter and our exposition will often make reference to the recurrence relations (2.1.1) and (2.1.2). In the Introduction, we mentioned some questions that will guide our investigation. In the remainder of this chapter, we will discuss some of these problems in more detail and state explicitly much of what is known about these problems in the context of OPUC and OPRL.

2.2 Regularity

Recall that in (1.1.6), we said that a measure is called regular if the leading coefficients of its corresponding orthonormal polynomials satisfy

$$\lim_{n\to\infty} \kappa_n^{-1/n} = \mathrm{cap}(\mathrm{supp}(\mu)).$$

For a measure μ with compact support $\mathrm{supp}(\mu)$, we recall the notation $\mathrm{ch}(\mu)$ for the convex hull of the support of μ and we will let $\mathrm{Pch}(\mu)$ denote the polynomial convex hull of the support of μ, where the polynomial convex hull of a set X is defined as in [65] by

$$\mathrm{Pch}(X) = \bigcap_{\text{polynomials } p\neq 0} \left\{z : |p(z)| \leq \|p\|_{L^\infty(X)}\right\}.$$

It is not difficult to see that if Ω is the unbounded component of $\overline{\mathbb{C}}\setminus\mathrm{supp}(\mu)$ then $\mathrm{Pch}(\mu) = \overline{\mathbb{C}}\setminus\Omega$ (see [65]). If K is a compact set, then a measure μ is said to be *regular on* K if μ is regular, $\mathrm{supp}(\mu) \subseteq K$, and the boundary of $\mathrm{Pch}(K)$ is contained in $\mathrm{supp}(\mu)$. Generally speaking, a measure is regular if it is sufficiently dense near the support of the equilibrium measure for $\mathrm{supp}(\mu)$. There are many ways to understand what is meant by "sufficiently dense" and we will discuss some of them in this section.

As a motivating example, consider the case when μ is area measure on the unit disk. Let $\{Q_n\}_{n\in\mathbb{N}}$ be a sequence of polynomials satisfying $\deg(Q_n) \leq n$. Suppose also that $\liminf_{n\to\infty} \|Q_n\|_{L^\infty(\mathbb{D})}^{1/n} = \alpha > 1$. In this case, for any $\Lambda \in (1, \alpha)$ it is true that for every sufficiently large n there is a $z_n \in \partial\mathbb{D}$ so that $|Q_n(z_n)| = \|Q_n\|_{L^\infty(\mathbb{D})} > \Lambda^n$. Bernstein's Theorem (see Theorem 2.2.5 in [56]) tells us that $|Q_n(z)| \geq \Lambda^n/2$ for all $z \in \{w : |w - z_n| \leq (100n)^{-1}, |w| \leq 1\}$. Therefore, $\liminf \|Q_n\|_{L^2(\mu)}^{1/n} \geq \Lambda$ and since this is true for every $\Lambda \in (1, \alpha)$, we get $\liminf \|Q_n\|_{L^2(\mu)}^{1/n} = \alpha$. One might guess that this phenomenon always occurs, namely that if a sequence of polynomials has L^∞-norm growing exponentially quickly, then the $L^2(\mu)$-norms must grow exponentially quickly and at the same rate. This is in fact not always the case, but when it is, we call the measure regular. The result is the following:

Theorem 2.2.1 (Stahl & Totik, [65] pg. 66). *A measure μ is regular if and only if for every sequence of nonzero polynomials $\{Q_n\}_{n\in\mathbb{N}}$ satisfying $\deg(Q_n) \leq n$ we have*

$$\limsup_{n\to\infty} \left(\frac{|Q_n(z)|}{\|Q_n\|_{L^2(\mu)}}\right)^{1/n} \leq e^{g_\Omega(z;\infty)}$$

locally uniformly for $z \in \mathbb{C}$*.*

Remark. Our use of the phrase "locally uniformly" is the same as in [65], i.e., for every $z \in \mathbb{C}$ and sequence $\{z_n\}_{n\in\mathbb{N}}$ converging to z, we have

$$\limsup_{n\to\infty} \left(\frac{|Q_n(z_n)|}{\|Q_n\|_{L^2(\mu)}} \right)^{1/n} \leq e^{g_\Omega(z;\infty)}.$$

Now we will consider some conditions on a measure that imply regularity. In keeping with the above observations, all of these conditions will imply some amount of density near the outer boundary of the measure's support. First we need to define the notion of a carrier of a measure.

Definition. A carrier of a measure μ is a measurable set X such that $\mu(\mathbb{C} \setminus X) = 0$.

We call a measure μ an *Ullman measure* if for every carrier X of μ it is true that

$$\mathrm{cap}(X \cap \mathrm{supp}(\mu)) = \mathrm{cap}(\mathrm{supp}(\mu)).$$

Now we can state our first regularity criterion.

Proposition 2.2.2 (Stahl & Totik, [65] pg. 102)**.** *If* μ *is an Ullman measure, then* μ *is regular.*

A condition equivalent to being an Ullman measure is known as *Widom's criterion* and is satisfied if for any carrier C of μ, there exists a sequence of compact sets $\{X_n\}_{n\in\mathbb{N}}$ (possibly depending on C) such that $\omega_{X_n}(C) \to 1$ and $\mathrm{cap}(X_n) \to \mathrm{cap}(\mathrm{supp}(\mu))$ as $n \to \infty$.

Proposition 2.2.3. *Let* μ *be a measure on the unit disk of the form* $w(z)dA(z)$ *where* $dA(z)$ *is area measure and* $w \in L^1(dA)$ *satisfies* $w(z) > 0$ *whenever* $|z| \in (1-\epsilon, 1)$ *for some* $\epsilon > 0$*. Then* μ *satisfies Widom's criterion.*

Proof. Fix a representative of the equivalence class $w \in L^1(dA)$ and also denote it by w. Let X be a carrier of μ. It must be the case that X has full measure in the annulus $\{z : 1-\epsilon < |z| < 1\}$ since otherwise X would not be a carrier of μ. Therefore, without loss of generality we may assume w is the characteristic function of this annulus.

Let $C_r = \{z : |z| = r\}$ and suppose for contradiction that there is some $r_1 \in (1-\epsilon, 1)$ and $\epsilon_1 > 0$ so that for almost every $r \in (r_1, 1)$ one has $|X \cap C_r|_r < 1 - \epsilon_1$ (here $|\cdot|_r$ denotes normalized arc-length measure on C_r). This clearly contradicts the fact that X is a carrier of μ, so we can find a sequence of circles $\{C_{r_n}\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} r_n = 1$ and $|X \cap C_{r_n}|_{r_n} \to 1$ as $n \to \infty$, which means μ satisfies Widom's criterion. $\square$

There are various other criteria that imply regularity and some of them are listed in Section 4.2 in [65]. One of the weakest such conditions is known as condition Λ^* and is satisfied precisely when there exists a constant $L > 0$ such that

$$\lim_{r\to 0} \operatorname{cap}\left(\left\{z : \mu\left(\{w : |w - z| < r\}\right) \geq r^L\right\}\right) = \operatorname{cap}(\operatorname{supp}(\mu)).$$

The relevant result is the following:

Theorem 2.2.4 (Stahl & Totik, [65] pg. 109)**.** *If every point in the boundary of* $\operatorname{Pch}(\mu)$ *is a regular point with respect to the Dirichlet problem in* $\overline{\mathbb{C}} \setminus \operatorname{Pch}(\mu)$*, then criterion* Λ^* *implies* μ *is regular.*

The condition of regularity has many important consequences. As mentioned in Section 1.1, it is a necessary and sufficient condition for root asymptotics of the orthonormal polynomials. It also has important implications for the location of the zeros of the orthonormal polynomials. If ν_n denotes the normalized zero counting measure for the polynomials $p_n(z;\mu)$ (that is, the measure with a point mass of weight n^{-1} at each zero of $p_n(z;\mu)$) and we let $\hat{\nu}_n$ denote its balayage onto the unbounded component of $\overline{\mathbb{C}} \setminus \operatorname{Pch}(\mu)$, then the regularity of μ implies that the measures $\hat{\nu}_n$ converge weakly to the equilibrium measure for $\operatorname{supp}(\mu)$ as $n \to \infty$ (see Theorem 3.6.1 in [65]). If $\operatorname{Pch}(\mu)$ has empty interior, then the weak convergence of ν_n to the equilibrium measure for $\operatorname{supp}(\mu)$ is in fact equivalent to regularity provided every carrier of μ has capacity bounded away from 0 (see Theorem 3.1.4 in [65]). In Section 3.3 we will observe some further consequences of regularity.

2.3 Szegő's Theorem

One of the motivations for the present body of work is to explore generalizations of Szegő's Theorem on the unit circle. This is a very profound result, several proofs of which are presented in Chapter 2 of [56]. The result can be stated as follows:

Szegő's Theorem. *If* μ *is a probability measure on the unit circle, then*

$$\lim_{n\to\infty} \kappa_n^{-1} = \exp\left(\frac{1}{4\pi}\int_0^{2\pi} \log(\mu'(\theta))d\theta\right) = S(\infty;\mu).$$

Perhaps one of the deepest consequences of this result is given as Theorem 2.7.14 in [56], where a large list of quantities related to orthogonal polynomials are (perhaps surprisingly) shown to be equal. This theorem relates the limiting behavior of the monic orthogonal polynomial norms to several other properties of the measure such as its entropy (see Section 2.3 in [56]) or the behavior of the associated Christoffel functions (see Section 2.5 below). This list of equivalences invites one to think about generalizations of Szegő's Theorem to settings where recursion coefficients do not exist.

One of the earliest generalizations was achieved in [17], where Geronimus generalized Szegő's Theorem to the case of a sufficiently smooth Jordan curve Γ. His result can be stated as follows:

Theorem 2.3.1 (Geronimus, [17]). *Let μ be a finite measure on an analytic Jordan curve Γ where* $\mathrm{cap}(\Gamma) = 1$. *If $\phi_*\mu$ is a Szegő measure on the unit circle and $q \in (0, \infty)$, then*

$$\lim_{n\to\infty} \|P_n(\cdot;\mu,q)\|^q_{L^q(\mu)} = \exp\left(\int_0^{2\pi} \log((\phi_*\mu)'(\theta))\frac{d\theta}{2\pi}\right). \tag{2.3.1}$$

Before we proceed with the proof of Theorem 2.3.1, we need to make an observation. The upper bound will be obtained using the extremal property, while for the lower bound we will invoke subharmonicity of a particular integrand. This is simple enough when $q \geq 1$ because every $H^1(\overline{\mathbb{C}}\setminus\overline{\mathbb{D}})$ function is the Poisson integral of its boundary values (see Theorem 17.11 in [47]). However, some care is required when $0 < q < 1$. We simply note here that Theorem 17.11(c) in [47] combined with a well-known L^q inequality (see page 74 in [47]) imply that if $f \in H^q(\overline{\mathbb{C}}\setminus\overline{\mathbb{D}})$, then

$$\int_0^{2\pi} |f(e^{i\theta})|^q \frac{d\theta}{2\pi} \geq |f(\infty)|^q. \tag{2.3.2}$$

Now we are ready to prove Theorem 2.3.1. The proof we present here is essentially the same as the proof from [17]. We present it here in english for the reader's convenience.

Proof. We begin with the proof of the lower bound. By definition of $\phi_*\mu$, we have

$$\begin{aligned}
\|P_n(\cdot;\mu,q)\|^q_{L^q(\mu)} &= \int_\Gamma |P_n(z;\mu,q)|^q d\mu(z) = \int_{\partial\mathbb{D}} |P_n(\psi(w);\mu,q)|^q d(\phi_*\mu)(w) \\
&= \int_{\partial\mathbb{D}} \left|\frac{P_n(\psi(w);\mu,q)}{w^n}\right|^q d(\phi_*\mu)(w) \\
&\geq \int_{\partial\mathbb{D}} \left|\frac{P_n(\psi(w);\mu,q)S(w;\phi_*\mu,q)}{w^n}\right|^q \frac{d|w|}{2\pi}.
\end{aligned}$$

Now, notice that the integrand in this last expression is analytic in $\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}$ and hence we are integrating a subharmonic function around a circle. Therefore, (2.3.2) implies that we can bound this integral from below by the value of the integrand at infinity. Since Γ has capacity 1, $P_n(\psi(w);\mu,q)$ grows like w^n at ∞, so we end up with

$$\|P_n(\cdot;\mu,q)\|^q_{L^q(\mu)} \geq |S(\infty;\phi_*\mu,q)|^q = \exp\left(\int_0^{2\pi} \log((\phi_*\mu)'(\theta))\frac{d\theta}{2\pi}\right),$$

which is the desired lower bound.

For the upper bound, we need to first recall a generalization of Szegő's Theorem. For any measure ν, define the quantity

$$\lambda_n(z;\nu,q) = \inf\left\{\int_{\mathbb{C}} |Q(w)|^q d\nu(w) : \deg(Q) \leq n\,,\ Q(z) = 1\right\} \tag{2.3.3}$$

and we also define $\lambda_\infty(z;\mu,q) = \lim_{n\to\infty}\lambda_n(z;\mu,q)$ (the limit clearly exists as the limit of a nonincreasing sequence of positive real numbers). We will discuss this object in greater detail later (see Section 2.5). Its importance for us in this section is derived from Theorem 2.5.4 in [56], which tells us the following:

Theorem 2.3.2 ([56]). *If ν is supported on the unit circle, then*

$$\lambda_\infty(0;\nu,q) = S(\infty;\nu,q)^q. \tag{2.3.4}$$

We note here that the right-hand side of (2.3.4) is independent of q. The proof of Theorem 2.3.2 is in fact quite elementary and requires only two facts. The first is that for the case $q=2$, the polynomial $P_n^*(z;\mu,2) = z^n\overline{P_n(\bar{z}^{-1},\mu,2)}$ is the unique minimizer of $\lambda_n(0;\nu,q)$. The second ingredient is the fact that $P_n^*(z;\mu,2)$ has all of its zeros outside $\overline{\mathbb{D}}$, so $(P_n^*(z;\mu,2))^{2/q}$ is an analytic function in a neighborhood of $\overline{\mathbb{D}}$ and hence can be uniformly approximated by polynomials. These two realizations make proving Theorem 2.3.2 quite simple, though Theorem 2.5.4 in [56] is, in fact, much more general because it applies to $\lambda_\infty(z;\nu,q)$ for any $z\in\mathbb{D}$.

Returning to the proof of the upper bound, recall that since Γ is an analytic Jordan curve, we know $F_n(z)-\phi(z)^n\to 0$ uniformly for z in the closure of the unbounded component of $\overline{\mathbb{C}}\setminus\Gamma$. To see this, we use (1.3.5) to get

$$|F_n(z)-\phi(z)^n| \leq \frac{1}{2\pi}\int_{\{|z|=r\}}\left|\frac{t^n\psi'(t)}{\psi(t)-z}\right| d|t|.$$

Analyticity of Γ implies we may take $r<1$ in this integral and see that for z in the desired set we in fact have convergence to 0 at an exponential rate.

As is often the case, we exploit the extremal property of $P_n(\cdot;\mu,q)$ to derive an upper bound on its norm. We know that F_n is a monic polynomial, so for any $m\in\mathbb{N}$ and any choice of constants $\gamma_1,\ldots,\gamma_m\in\mathbb{C}$ we have

$$\begin{aligned}
\|P_n(\cdot;\mu,q)\|_{L^q(\mu)}^q &\leq \int_\Gamma \left|F_n(z)+\sum_{j=1}^m \gamma_j F_{n-j}(z)\right|^q d\mu(z)\\
&= \int_{\partial\mathbb{D}} \left|F_n(\psi(w))+\sum_{j=1}^m \gamma_j F_{n-j}(\psi(w))\right|^q d(\phi_*\mu)(w)\\
&= \int_{\partial\mathbb{D}} \left|w^n+\sum_{j=1}^m \gamma_j w^{n-j}\right|^q d(\phi_*\mu)(w)+o(1)\\
&= \int_{\partial\mathbb{D}} \left|1+\sum_{j=1}^m \overline{\gamma}_j w^{j}\right|^q d(\phi_*\mu)(w)+o(1)\\
&= \lambda_m(0;\phi_*\mu,q)+o(1)
\end{aligned}$$

if we choose the constants $\gamma_1, \ldots, \gamma_m$ correctly. Therefore,

$$\limsup_{n\to\infty} \|P_n(\cdot;\mu,q)\|_{L^q(\mu)}^q \leq \lambda_m(0;\phi_*\mu,q)$$

and since $m \in \mathbb{N}$ was arbitrary, the desired conclusion follows from Theorem 2.3.2. □

Remark. The same proof works if Γ is any curve for which $F_n - \phi^n$ tends to zero uniformly on the curve Γ and outside it. This is how Geronimus stated his result in [17], where he provides several smoothness conditions on Γ that imply this convergence property holds.

Notice that in deriving the lower bound in Theorem 2.3.1, we only used the absolutely continuous part of the measure μ (with respect to arc-length measure). Since this lower bound matches the upper bound, we have proven the following:

Corollary 2.3.3. *Under the assumptions of Theorem 2.3.1, if μ_{sing} is the component of μ that is singular with respect to arc-length measure, then one has*

$$\lim_{n\to\infty} \|p_n(\cdot;\mu,q)\|_{L^q(\mu_{\text{sing}})} = 0.$$

Furthermore, in the proof of Theorem 2.3.1, we showed that

$$\lim_{n\to\infty} \left\| \frac{p_n(\psi(z);\mu,q)S(z;\phi_*\mu,q)}{z^n} \right\|_{H^q(\overline{\mathbb{C}}\setminus\overline{\mathbb{D}})} = 1, \qquad \lim_{n\to\infty} \frac{p_n(\psi(z);\mu,q)S(z;\phi_*\mu,q)}{z^n}\bigg|_{z=\infty} = 1.$$

Therefore, the Keldysh Lemma (see Section 1.5 above) proves the following:

Theorem 2.3.4. *If μ is as in Theorem 2.3.1, then*

$$\lim_{n\to\infty} \frac{p_n(\psi(z);\mu,q)}{z^n} = \frac{1}{S(z;\phi_*\mu,q)}, \qquad |z| > 1$$

and the convergence is uniform on compact subsets of $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$.

Theorem 2.3.4 gives a very clear picture of the behavior of the orthonormal polynomials when the measure μ has sufficiently nice properties. The leading order behavior of the polynomials is $S(z)\phi(z)^n$ for an explicitly computable function S that is independent of the singular component of the measure μ. This is precisely the kind of statement that we will try to make about orthonormal polynomials whose measure of orthogonality has a more general support.

2.4 Bergman Polynomials

One of the main themes of this thesis is to study orthogonal polynomials in the most general possible setting, in particular in settings where the orthonormal polynomials do not satisfy a finite term recurrence relation. The simplest and most natural such setting is when the measure is area measure on a region of the plane. The corresponding orthonormal polynomials are called *Bergman polynomials*. The earliest substantial results concerning Bergman polynomials were obtained by Carleman in [6]. His main result can be stated as follows:

Theorem 2.4.1 (Carleman, [6])**.** *Let G be a bounded region in the plane whose boundary is an analytic Jordan curve. Let μ be area measure on G and suppose the conformal map ψ has a univalent extension to the exterior of the circle of radius $\tilde{\rho} < 1$. Then uniformly for $z \in \overline{\Omega}$ we have*

$$p_n(z;\mu) = \sqrt{\frac{n+1}{\pi}}\phi'(z)\phi(z)^n\left(1 + O(\sqrt{n}\rho_1^n)\right), \tag{2.4.1}$$

where $\rho_1 \in (\tilde{\rho}, 1)$ and the implied constant in the error term depends on ρ_1.

Since Carleman's paper [6], there have been many generalizations that have extended Carleman's result by either relaxing the smoothness conditions on the boundary of the region G (see [67]) or allowing for more general measures (see [68]). A brief description of many such generalizations can be found in the introduction to [68]. One of the strongest such results is the following, which is Theorem 3.1 in [68]:

Theorem 2.4.2 (Suetin, [68])**.** *Let G be a bounded region in the plane whose boundary is an analytic Jordan curve having capacity 1. Let μ be the measure on $\overline{G}$ defined by $h(z)dA(z)$ where $dA(z)$ is area measure on G and $h(z)$ is Lipschitz continuous of order $\alpha < 1$ on $\overline{G}$. There is a function $g(z)$, which is analytic in $\overline{\mathbb{C}} \setminus \overline{G}$, so that if $z \notin \overline{G}$ then*

$$p_n(z;\mu) = \sqrt{\frac{n+1}{\pi}}g(z)\phi(z)^n\left(1 + O\left(\left(\frac{\log(n)}{n}\right)^{\alpha/2}\right)\right),$$

where the implied constant may be chosen uniformly on compact subsets of $\overline{\mathbb{C}} \setminus \overline{G}$.

Due to the importance of Theorem 2.4.2, we will now sketch its proof (from [68]).

Sketch of proof of Theorem 2.4.2. Define the function

$$D(w) = \exp\left(-\frac{1}{4\pi}\int_0^{2\pi}\log(h(\psi(e^{i\theta})))\frac{e^{i\theta}+w}{e^{i\theta}-w}d\theta\right), \qquad |w| > 1$$

and define $g(z) = \phi'(z)D(\phi(z))^{-1}$ for $z \in \mathbb{C} \setminus \overline{G}$. Let $\{q_n\}_{n\in\mathbb{N}}$ be a sequence of numbers that monotonically increases to 1 so that $\{z : |z| = q_1\}$ is in the domain of ψ. Let X_n be the image of $\{z : |z| = q_n\}$ under the map ψ and let G_n be the region bounded by X_n. If we define

$$g(z, q_n) = \frac{\phi'(z)}{D\left(\frac{\phi(z)}{q_n}\right)},$$

then a key estimate is given by Lemma 3.1 in [68], which states

$$A(q_n) := \max_{z\in\overline{G}\setminus G_n} \frac{h(z)|g(z,q_n)|^2}{|\phi'(z)|^2} \le 1 + O[(1-q_n)^\alpha],$$
$$B(q_n) := \min_{z\in\overline{G}\setminus G_n} \frac{h(z)|g(z,q_n)|^2}{|\phi'(z)|^2} \ge 1 - O[(1-q_n)^\alpha].$$

In other words, the Lipschitz continuity of the weight function h enables is to use the same Szegő function D to approximate the weight up to an acceptable error.

A second key step is an analysis of the generalized Faber polynomial $B_n(z, q_m)$, which is the polynomial part of $g(z, q_m)\phi(z)^n$. It is a simple estimate to conclude that there is a constant $C_1 > 0$ so that

$$|B_n(z, q_m)| \le C_1 q_m^n, \qquad z \in \overline{G}_m.$$

A more subtle estimate comes from equation (3.10) in [68], which tells us that there is a constant $C_2 > 0$ so that

$$|B_n(z, q_m) - g(z, q_m)\phi(z)^n| \le \frac{C_2 q_m^n \log(n)}{n^\alpha}, \qquad |z| \ge q_m.$$

With this estimate, it is a short argument using basic harmonic analysis techniques to show that

$$\int_{G\setminus G_m} \left|\frac{B_n(z,q_m)}{g(z,q_m)}\right|^2 |\phi'(z)|^2 dA(z) \le \frac{\pi}{n+1}\left(1 + O\left(\frac{\log(n)^2}{n^{2\alpha}}\right)\right)$$

(see Lemma 3.2 in [68]).

Now we have the necessary estimates to finish the proof relatively easily. Let $g_0 = g(\infty)$ so that by the extremal property, we conclude

$$\int_G h(z)|P_n(z;\mu)|^2 dA(z) \le \int_G h(z)\left|\frac{B_n(z,q_n)}{g_0}\right|^2 dA(z).$$

Therefore,

$$\begin{aligned} g_0^2\kappa_n^{-2} &\le \int_{G_n} h(z)|B_n(z,q_n)|^2 dA(z) + A(q_n)\int_{G\setminus G_n}\left|\frac{B_n(z,q_n)}{g(z,q_n)}\right|^2 |\phi'(z)|^2 dA(z) \\ &\le C_1 q_n^{2n} + (1 + O[(1-q_n)^\alpha])\,\frac{\pi}{n+1}\left[1 + O\left(\frac{\log(n)^2}{n^{2\alpha}}\right)\right]. \end{aligned}$$

Meanwhile

$$1 = \int_G h(z)|p_n(z;\mu)|^2 d\mu(z) \ge B(q_n)\int_{G\setminus G_n}\left|\frac{p_n(z;\mu)}{g(z,q_n)}\right|^2 dA(z).$$

Now we make the change of variables $z = \psi(w)$ and switch to polar coordinates ($w = re^{i\theta}$). For each fixed r, the angular integral is the integral of the absolute value of an H^2 function around its boundary circle, so we bound this integral from below by the value at infinity as in (2.3.2). We then integrate in the variable r and use our previous estimates on $B(q_n)$ to get

$$1 \geq (1 - O[(1-q_n)^{\alpha}])\,\frac{\pi}{n+1}(1-q_n^{2n+2})\kappa_n^2 g_0^{-2}.$$

Using these estimates, the desired conclusion follows by setting $q_n = 1 - n^{-1}\log(n)$. □

Theorem 2.4.2 is in many ways still the strongest result of its kind. We will generalize it in the sense that we will consider a more general class of measures, but we will pay the price of replacing the error term in the expression for $p_n(z;\mu)$ by simply $o(1)$. However, notice that even with this slightly weaker conclusion we still have enough information to determine the asymptotic behavior of the leading coefficient of $p_n(z;\mu)$ as $n \to \infty$ as well as the ratio and root asymptotic behavior of the orthonormal polynomials.

2.5 Christoffel Functions

In this section we will consider an important minimization problem. While the nature of this problem may at first appear unremarkable, its solution in terms of the orthonormal polynomials is of substantial importance because it provides a very useful tool for proving universality estimates (see for example [27]) and a shape reconstruction algorithm (see Section 5 in [18]). In this section we will explicitly state the problem at hand and explain its connection to orthogonal polynomials. We will then state some important results on this subject in the setting of OPUC. We conclude with a description of a generalized version of the problem.

The Christoffel function is defined as the limit of a sequence of functions that are defined in terms of a very natural minimization problem. For each $n \in \mathbb{N}$, define

$$\lambda_n(z;\mu) = \inf\left\{\int_{\mathbb{C}} |Q(w)|^2 d\mu(w) : \deg(Q) \leq n\,,\; Q(z) = 1\right\}. \tag{2.5.1}$$

A simple compactness argument shows that in fact the infimum is a minimum and a convexity argument shows that there is a unique minimizer. It is clear that the sequence $\{\lambda_n(z;\mu)\}_{n\geq 0}$ is non-decreasing in n and so the sequence has a limit, which we denote by $\lambda_\infty(z;\mu)$. Our definitions obviously imply that

$$\lambda_\infty(z;\mu) = \inf\left\{\int_{\mathbb{C}} |Q(w)|^2 d\mu(w) : Q \text{ a polynomial}\,,\; Q(z) = 1\right\}. \tag{2.5.2}$$

The definition of the function λ_∞ makes it a natural object to consider. Its relationship to orthogonal polynomials comes from our next theorem, but before we state it we must define some additional notation. For every $n \in \mathbb{N}$, define

$$K_n(z,w;\mu) = \sum_{j=0}^{n} p_j(z;\mu)\overline{p_j(w;\mu)}. \tag{2.5.3}$$

Notice that $K_n(z,w)$ is a polynomial in z of degree n and $K_n(z,w;\mu) = \overline{K_n(w,z;\mu)}$. Its most important property is the *reproducing property*, which is described in the following:

Proposition 2.5.1. *If Q is a polynomial of degree at most n, then*

$$\int_{\mathbb{C}} Q(w)K_n(z,w;\mu)d\mu(w) = Q(z). \tag{2.5.4}$$

Proof. Since $\deg(Q) \leq n$, we write $Q(w) = \sum_{j=0}^{n} d_j p_j(w;\mu)$ for some complex numbers $d_0, \ldots, d_n$. We then use (0.0.1) to calculate

$$\int_{\mathbb{C}} Q(w)K_n(z,w;\mu)d\mu(w) = \sum_{j=0}^{n}\sum_{k=0}^{n}\int_{\mathbb{C}} d_j p_j(w;\mu)p_k(z;\mu)\overline{p_k(w;\mu)}d\mu(w) = \sum_{j=0}^{n} d_j p_j(z;\mu) = Q(z)$$

as desired. □

We can now state the relevant result.

Theorem 2.5.2 ([63] pg. 124)**.** *The unique minimizer of the right-hand side of (2.5.1) is*

$$\frac{K_n(w,z;\mu)}{K_n(z,z;\mu)}$$

and

$$\lambda_n(z;\mu) = \frac{1}{K_n(z,z;\mu)}. \tag{2.5.5}$$

Proof. The key to the proof is the reproducing property of the kernel $K_n(z,w;\mu)$. Indeed, if $\deg(Q) \leq n$ and $Q(z) = 1$, then

$$1 = \int_{\mathbb{C}} Q(w)K_n(z,w;\mu)d\mu(w).$$

Applying the Schwarz inequality yields

$$1 \leq \|Q\|_{L^2(\mu)}^2 \|K_n(z,\cdot;\mu)\|_{L^2(\mu)}^2.$$

However,

$$\|K_n(z,\cdot;\mu)\|_{L^2(\mu)}^2 = \int_{\mathbb{C}} K_n(z,w;\mu)K_n(w,z;\mu)d\mu(w) = K_n(z,z)$$

since $K_n(w,z;\mu)$ is a polynomial in w of degree n. Therefore, $\|Q\|_{L^2(\mu)}^2 \geq K_n(z,z;\mu)^{-1}$. It is easily checked that we have equality if $Q(w) = K_n(w,z;\mu)K_n(z,z;\mu)^{-1}$, which is what we wanted to show. $\square$

Theorem 2.5.2 tells us that we can deduce the behavior of the Christoffel function in any case where we have sufficiently detailed information about the orthonormal polynomials. Conversely, if we know something about the behavior of the Christoffel function, than we know something about the behavior of the orthonormal polynomials. For example, if $\lambda_\infty(z_0;\mu) > 0$, then we know that the sequence $\{p_n(z_0;\mu)\}_{n\geq 0}$ lies in $\ell^2(\mathbb{N}_0)$, implying the orthonormal polynomials tend to zero very rapidly at z_0. Therefore, studying the properties of the Christoffel function is a natural problem associated to the general study of orthogonal polynomials and we will study it more in Chapter 3.

In the case of OPUC, much is known about the Christoffel function. For example, the following theorem comes from Section 2.2 in [56].

Theorem 2.5.3 ([56])**.** *Let μ be a probability measure on the unit circle. The following hold*

1. *If $z \in \partial\mathbb{D}$ then $\lambda_\infty(z;\mu) = \mu(\{z\})$.*
2. *If $|z| > 1$ then $\lambda_\infty(z;\mu) = 0$.*
3. *Either $\lambda_\infty(z;\mu)$ is identically zero on $\mathbb{D}$ or it is never zero on $\mathbb{D}$. The latter case holds if and only if μ is a Szegő measure on $\partial\mathbb{D}$, in which case λ_∞ is given by*

$$\lambda_\infty(z;\mu) = \exp\left(\int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\log(\mu'(\theta))\frac{d\theta}{2\pi}\right).$$

Items (1) and (2) continue to hold in much more general settings and with essentially the same proof. As the statement suggests, the usual proof of (3) uses the Szegő function, so generalizing it to measures with more general support is not immediate. We will prove a result analogous to item (3) in Section 3.2.3. The main step in the proof will be to derive some amount of uniformity in the harmonic measures for a sequence of nested domains whose union carries the measure μ.

In fact we will consider a more general problem that admits the same solution. We will consider a more generalized Christoffel function by defining

$$\lambda_n(z;\mu,q) = \inf\left\{\int_{\mathbb{C}} |Q(w)|^q d\mu(w) : \deg(Q) \leq n\,,\ Q(z) = 1\right\}. \tag{2.5.6}$$

and letting $\lambda_\infty(z;\mu,q) = \lim_{n\to\infty} \lambda_n(z;\mu,q)$ as in (2.3.3). We will see that obtaining a result comparable to Theorem 2.5.3 for this more general function is no more difficult than considering only $\lambda_\infty(z;\mu) = \lambda_\infty(z;\mu,2)$, although when $q \neq 2$, there is no obvious connection to orthonormal polynomials. Indeed, the full version of Theorem 2.5.4 in [56] (which was cited earlier in Theorem 2.3.2), tells us that if μ is supported on $\partial\mathbb{D}$ and $z \in \mathbb{D}$, then $\lambda_\infty(z;\mu,q)$ is independent of q.

2.6 The Bergman Kernel Method

In this section we will take a more detailed look at one of the many applications of Bergman polynomials. This application is a tool for numerically estimating a conformal map from a given region to the unit disk and is called the *Bergman Kernel Method*. Recall (1.1.3), which tells us that one can express the Green function for a region in terms of the corresponding conformal map of the exterior domain. Therefore, if one wants to numerically estimate the Green function – which is often the case in applications – one can approach the problem by estimating the conformal map and the Bergman kernel method gives an explicit algorithm for doing so. Much of the material we present here can be found in [30].

Let G be a bounded simply connected domain whose boundary is a Jordan curve. If we let $dA(z)$ denote two dimensional Lebesgue measure, then we may define the Hilbert space

$$L_a^2(G) = \left\{ f : f \text{ analytic in } G\,,\ \int_G |f(z)|^2 dA(z) < \infty \right\},$$

which is known to be a reproducing kernel Hilbert space, and so has a reproducing kernel $K(z,w)$. For any fixed $z_0 \in G$, we will consider the minimization problem given by

$$\tilde{\lambda}(z_0) = \inf \left\{ \int_G |f'(z)|^2 dA(z) : f \in L_a^2(G),\ f(z_0) = 0,\ f'(z_0) = 1 \right\}.$$

If we let χ be the inverse to the canonical conformal map at z_0, then for any $f \in L_a^2(G)$ we have

$$\int_G |f'(z)|^2 dA(z) = \int_{\mathbb{D}} |f'(\chi(w))|^2 |\chi'(w)|^2 dA(w) = \int_{\mathbb{D}} |(f \circ \chi)'(w))|^2 dA(w) \geq \pi |(f \circ \chi)'(0))|^2,$$

with equality in this last inequality if and only if $(f \circ \chi)'(z)$ is constant on $\mathbb{D}$. It follows that if f is an extremizer of $\tilde{\lambda}(z_0)$ then f is a conformal map from G to some disk of radius $r_0 > 0$ satisfying $f(z_0) = 0$ and $f'(z_0) = 1$. Let us denote this conformal map by φ_0.

Now consider the problem of finding an extremizer for the function $\tilde{\lambda}_n(z_0)$, which is defined similarly to $\tilde{\lambda}(z_0)$, except now we also insist that f is a polynomial of degree at most n. In this case, a similar analysis to the one in Section 2.5 shows that if π_n is an extremizer of $\tilde{\lambda}_n(z_0)$, then

$$\pi_n'(z) = \frac{K_{n-1}(z, z_0; A_G)}{K_{n-1}(z_0, z_0; A_G)}, \qquad \pi_n(z) = \frac{1}{K_{n-1}(z_0, z_0; A_G)} \int_{z_0}^{z} K_{n-1}(\zeta, z_0; A_G) d\zeta,$$

where A_G denotes area measure on the region G. The polynomials $\{\pi_n\}_{n\in\mathbb{N}}$ are known as the *Bieberbach polynomials* for the region G and the point z_0. Given the nature of the extremal problems defining $\tilde{\lambda}$ and $\tilde{\lambda}_n$, it is natural to wonder if the Bieberbach polynomials converge to the conformal map φ_0. It is known that this convergence does hold when ∂G is a Jordan curve, and one can estimate the rate of convergence given various smoothness conditions on ∂G (see page 76 in [68]).

Indeed, it is true that

$$\varphi_0'(z) = \frac{K(z,z_0)}{K(z_0,z_0)}, \qquad \varphi_0(z) = \frac{1}{K(z_0,z_0)} \int_{z_0}^{z} K(\zeta,z_0)d\zeta$$

The Bieberbach polynomials are an extremal sequence of polynomial approximations to the map φ_0 in that

$$\|\varphi_0' - \pi_n'\|_{L_a^2(G)} = \inf\{\|\varphi_0' - Q'\|_{L_a^2(G)} : \deg(Q) \le n\,,\ Q(z_0) = 0\,,\ Q'(z_0) = 1\},$$

(see Section 1 in [30]). Note that the Bieberbach polynomials are not intrinsic to the region G, but are determined by the region and a fixed point $z_0 \in G$.

The Bieberbach polynomials provide an explicit application of the Bergman polynomials to an important problem in approximation theory, namely that of numerically approximating the conformal map of an arbitrary simply connected Jordan domain to the unit disk. One can then ask how good the approximation is and where the sequence of approximants convergences. This problem has a long and ongoing history (see Chapter 5 in [68]) with many deep results. One very general result is the following theorem, which is given as a remark following Theorem 5.2 in [68].

Theorem 2.6.1 (Suetin, [68])**.** *Let Γ be the boundary of G and suppose Γ is p times differentiable and the p^{th} derivative is Lipschitz continuous of order $\alpha \in (0,1)$. If $p+\alpha > 7/4$, then there is a constant $C > 0$ so that*

$$|\varphi_0(z) - \pi_n(z)| \le \frac{C\log(n)}{n^{p+\alpha}}$$

for all $z \in \overline{G}$.

We conclude that if the boundary of the region in question is sufficiently smooth, then we have convergence of the polynomials π_n to the conformal map φ_0 in the closed region $\overline{G}$ and a reasonable estimate on the rate of convergence.

2.7 Ratio Asymptotics

Ratio asymptotics for orthonormal polynomials on the unit circle or an interval have been studied extensively (see for example [1, 2, 11, 38, 43, 44, 55, 56, 57, 70]). In particular, in [55], Simon gives necessary and sufficient conditions for the existence of the limit

$$\lim_{n\to\infty} \frac{p_n(z;\mu)}{p_{n-1}(z;\mu)}$$

when the measure of orthogonality is (compactly) supported on the real line. The criteria he provides are in terms of the recursion coefficients for the polynomials $\{p_n(z;\mu)\}_{n\ge 0}$. Similarly, in [56, 57]

Simon shows that if the measure μ is supported on the unit circle, then the ratio $p_n p_{n-1}^{-1}$ tends to z outside $\mathbb{D} = \{z : |z| < 1\}$ if and only if the recursion coefficients decay to 0.

For measures with more general support, the problem of ratio asymptotics is more difficult and there are fewer results in the literature. The Szegő asymptotic results discussed in Section 2.4 clearly imply ratio asymptotic results, but substantial results devoted specifically to ratio asymptotics have arisen only recently. A recent paper of Saff (see [48]) develops some interesting and powerful techniques that allow him to prove the existence of ratio asymptotics along a subsequence for a large class of measures with very general compact support. His approach is relatively straightforward and is based on showing that $\{zp_{n-1}(z;\mu)p_n(z;\mu)^{-1}\}_{n\geq 0}$ is a normal family on the domain $\{z : |z| > R\}$ for some sufficiently large R. In Section 3.3, we will present many new results concerning ratio asymptotics when the measure of orthogonality has very general support. Special attention will be paid to the closed unit disk $\overline{\mathbb{D}}$ and to the lemniscate $E_m = \{z : |z^m - 1| \leq 1\}$ for technical reasons that will be explained later.

When μ is supported on the unit circle or real line, there are detailed results regarding ratio asymptotic behavior. The following two theorems are simple consequences of the proofs presented in [55] and Section 1.7 in [56] respectively.

Theorem 2.7.1. *Let μ be a probability measure supported on a compact subset of the real line and let $\{a_n, b_n\}_{n\in\mathbb{N}}$ be the recursion coefficients from the three-term recurrence relation for the orthonormal polynomials $\{p_n(x;\mu)\}_{n\geq 0}$. Let $\mathcal{N} \subseteq \mathbb{N}$ be a subsequence so that for every $m \in \mathbb{Z}$, one has*

$$\lim_{\substack{n\to\infty\\ n\in\mathcal{N}}} a_{n+m} = 1, \qquad \lim_{\substack{n\to\infty\\ n\in\mathcal{N}}} b_{n+m} = 0.$$

For any $x \notin \operatorname{supp}(\mu)$ it is true that

$$\lim_{\substack{n\to\infty\\ n\in\mathcal{N}}} \frac{p_n(x;\mu)}{p_{n-1}(x;\mu)} = \frac{x + \sqrt{x^2 - 4}}{2}. \tag{2.7.1}$$

Theorem 2.7.2. *Let μ be a probability measure supported on the unit circle with Verblunsky coefficients $\{\alpha_n\}_{n\geq 0}$. Let $\mathcal{N} \subseteq \mathbb{N}$ be a subsequence so that*

$$\lim_{\substack{n\to\infty\\ n\in\mathcal{N}}} \alpha_{n-1} = 0.$$

For any z satisfying $|z| > 1$ we have

$$\lim_{\substack{n\to\infty\\ n\in\mathcal{N}}} \frac{p_n(z;\mu)}{p_{n-1}(z;\mu)} = z.$$

In both of Theorems 2.7.1 and 2.7.2, the ratio p_n/p_{n-1} converges to the conformal map ϕ. A similar result was obtained in [2]. Although the proof of Theorem 2.7.2 is very simple, we will present here the proof of Theorem 2.7.1 since it is not explicitly stated in this way in [55]. However, we use essentially the same argument as in the second proof of Theorem 2.1 in [55].

Proof of Theorem 2.7.1. First note that the theorem is known to be true in the case when $a_n \equiv 1$, and $b_n \equiv 0$ (the so-called free case) due to the results in [55]. Let J be the self-adjoint matrix defined by

$$J = \begin{pmatrix} b_1 & a_1 & & & \\ a_1 & b_2 & a_2 & & \\ & a_2 & b_3 & a_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

(where all entries away from the three main diagonals are 0) and let $J^{(n)}$ be the upper left $n \times n$ block of J. It is well-known that $P_n(x;\mu) = \det(x - J^{(n)})$. Since $a_n \to 1$ as $n \to \infty$ through $\mathcal{N}$, it suffices to prove the ratio asymptotics for the monic orthogonal polynomials. By Cramer's rule

$$\frac{P_{n-1}(x;\mu)}{P_n(x;\mu)} = (x - J^{(n)})^{-1}_{nn} = (x - \tilde{J}^{(n)})^{-1}_{11},$$

where $\tilde{J}^{(n)}_{ij} = J^{(n)}_{n+1-i,n+1-j}$. Our hypotheses clearly imply that $\tilde{J}^{(n)}$ converges strongly to the free Jacobi matrix J_0 as $n \to \infty$ through $\mathcal{N}$, and hence $(x - \tilde{J}^{(n)})^{-1}$ converges strongly to the resolvent of the free Jacobi matrix for any x satisfying $\liminf_{n\to\infty, n\in\mathcal{N}} \mathrm{dist}(x, \mathrm{spectrum}(J^{(n)})) > 0$. It is well-known that this property holds for any x not in the spectrum of J (as mentioned in [55]) and so for any such x we have

$$\lim_{\substack{n\to\infty \\ n\in\mathcal{N}}} \frac{P_{n-1}(x;\mu)}{P_n(x;\mu)} = (x - J_0)^{-1}_{11}.$$

The result now follows from formula (2.25) in [55]. □

By using the argument in the proof of Theorem 2.9.4 (see below), Theorem 2.7.1 implies that if μ has compact support in $\mathbb{R}$, is regular, and has essential support equal to $[-2, 2]$ then we have ratio asymptotics along a sequence of asymptotic density 1. We will discuss generalizations of Theorem 2.7.1 to the setting when the measure μ is supported on the closed unit disk in Section 3.3.2.

2.8 Relative Asymptotics

Let μ and ν be two measures with compact (though not necessarily identical) support. One expects that if the measures μ and ν are very similar, then this similarity would manifest itself in some way in the polynomials $p_n(z;\mu)$ and $p_n(z;\nu)$. This reasoning inspires the study of the quantity

$$\lim_{n\to\infty} \frac{p_n(z;\mu)}{p_n(z;\nu)}, \tag{2.8.1}$$

a quantity referred to as *relative asymptotics*.

Relative asymptotics have been studied in a variety of contexts. A typical relative asymptotic result is Theorem 2.3.4. In that theorem, one starts with the equilibrium measure on an analytic Jordan curve and then perturbs it by adding a weight function satisfying (1.4.2) and a singular component. Theorem 2.3.4 exhibits two properties that are typical of relative asymptotic results. First, it shows that a particular perturbation – in this case the singular component – does not affect the leading order behavior of the orthonormal polynomials. It also provides an explicit formula for the limit (2.8.1) in terms of the relationship between μ and ν. These are features of relative asymptotic results that we will also try to capture in Section 3.3.

Additional relative asymptotic results on the unit circle can be found in [2, 33]. In a more general setting, some recent results of Saff have appeared in [48]. We will use techniques similar to those of Saff later in this text and much of our work is motivated by the conjecture in [48], so we mention some of his results here. His main result on relative asymptotics is phrased in terms of varying weights, that is, he considers a sequence of measures $\{\mu_n\}_{n\in\mathbb{N}}$ where

$$d\mu_n(z) = w_n(z)dA(z) \tag{2.8.2}$$

and $dA(z)$ is area measure on the unit disk. His main result is the following theorem, which in a qualitative sense tells us that if w_n has some reasonable constraints on it, then the corresponding orthonormal polynomials cannot differ too greatly from the orthonormal polynomials for area measure.

Theorem 2.8.1 (Saff, [48])**.** *Let μ_n be defined by (2.8.2) and let $\kappa_{n,n}$ be the leading coefficient of $p_n(z;\mu_n)$. If there exist positive constants M_1 and M_2 so that*

$$\kappa_{n,n} \leq M_1\sqrt{n} \qquad \textit{and} \qquad w_n(z) \leq M_2,$$

then for any closed set $E \subseteq \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ there exist positive constants c_1 and c_2 so that

$$c_1 \leq \left|\frac{p_n(z;\mu_n)}{\sqrt{n}z^n}\right| \leq c_2, \qquad z \in E.$$

We will consider relative asymptotics in several different settings. For example, we will consider the *Uvarov Transform* of a measure, which is obtained by adding a pure point to the measure:

$$\mu_x = \mu + t\delta_x, \qquad t > 0. \tag{2.8.3}$$

A straightforward computation reveals (see Proposition 8 in [15]):

$$P_n(z;\mu_x) = P_n(z;\mu) - \frac{tP_n(x;\mu)}{1+tK_{n-1}(x,x;\mu)}K_{n-1}(z,x;\mu) \tag{2.8.4}$$

(recall the definition of $K_n(z,w;\mu)$ in (2.5.3)). While the extremal property suggests that the polynomials $\{P_n(z;\mu_x)\}_{n\in\mathbb{N}}$ will have a zero very close to x when n is large, Theorem 2 in [49] tells us that this is often not the case and provides examples where the point x is not a limit point of the zeros of $\{P_n(\cdot;\mu_x)\}_{n\in\mathbb{N}}$.

We will also consider the *Christoffel Transform*, which is obtained by multiplying the measure by the square modulus of a monomial:

$$d\nu^x(z) = |z-x|^2 d\mu(z). \tag{2.8.5}$$

A straightforward computation reveals (see Proposition 3 in [15]):

$$P_n(z;\nu^x) = \frac{1}{z-x}\left(P_{n+1}(z;\mu) - \frac{P_{n+1}(x;\mu)}{K_n(x,x;\mu)}K_n(z,x;\mu)\right). \tag{2.8.6}$$

The Christoffel Transform has a second interpretation that makes it a natural object to study. Recall that the monic polynomial $P_n(z;\mu)$ satisfies the extremal property of having the smallest $L^2(\mu)$ norm of any monic polynomial of degree n. The polynomial $P_{n-1}(z;\nu^x)$ is such that $(z-x)P_{n-1}(z;\nu^x)$ has the smallest $L^2(\mu)$ norm of any monic polynomial of degree n *with a zero at* x. Therefore, a relative asymptotic result in this case will give us some idea about how changing the extremal problem in this way manifests itself in the corresponding extremal polynomials. It may be helpful to keep the following example in mind:

Example. Consider the case when μ is area measure on the unit disk and we take the Christoffel Transform ν^1. In this case then, we have (by the example in Section 4.6 in [68])

$$p_n(z;\mu) = \sqrt{\frac{n+1}{\pi}}\,z^n,$$
$$p_n(z;\nu^1) = \frac{2}{\sqrt{\pi(n+1)(n+2)(n+3)}}\sum_{k=0}^{n}(k+1)z^k(1+z+z^2+\cdots+z^{n-k}).$$

Clearly all of the zeros of $p_n(z;\mu)$ are located at 0. Figure (2.1) shows a Mathematica plot of the zeros of $p_{17}(\cdot;\nu^1)$. We see that the zeros of $p_n(z;\nu^1)$ are not near 0 and further analysis shows that the zeros of $p_n(\cdot;\nu^1)$ migrate to the unit circle as n tends to infinity. However, in spite of this apparent lack of similarity, a simple calculation (see the example in Section 3.3.4) shows that $(z-1)p_n(z;\nu^1)p_{n+1}(z;\mu)^{-1}$ tends to 1 outside $\overline{\mathbb{D}}$ as $n\to\infty$. Indeed this is a special case of a more general result that we will prove in Section 3.3.4.

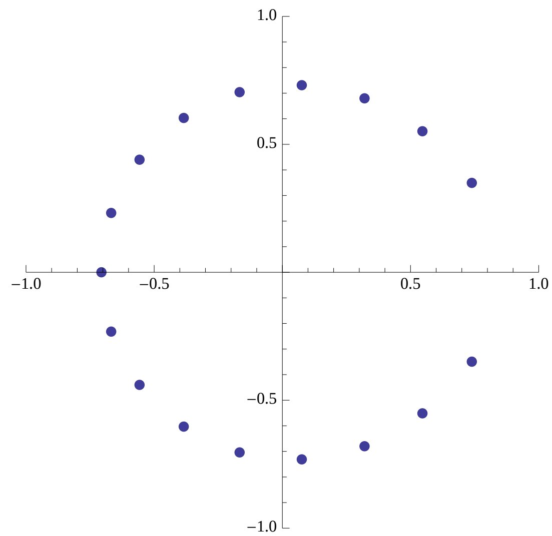

Figure 2.1: A Mathematica plot of the zeros of $p_{17}(\cdot;\nu^1)$.

The above example shows that it is interesting to consider not only the value of the limit in (2.8.1), but also to specify those values of z for which the limit exists.

2.9 Weak Asymptotic Measures

As mentioned in the Introduction, one expects the orthonormal polynomials to behave in such a way that the resulting measures $\{|p_n(z;\mu)|^2 d\mu(z)\}_{n\geq 0}$ are asymptotically as equidistributed as possible over the support of μ. This intuition does not always lead to the right conclusions, but it is helpful in understanding the behavior of the polynomials to some extent. In fact one can make a precise study of the collection of weak limits of $\{|p_n(z;\mu)|^2 d\mu(z)\}_{n\geq 0}$ and when the measure of orthogonality is on the unit circle, we have the following two theorems:

Theorem 2.9.1 ([57] pg. 523-524)**.** *There is an explicit collection of measures* $\{\nu^{\{k\}}_{a,b,\lambda}\}$ *indexed by parameters* $k\in\mathbb{N}$, $0<a\leq b\leq 1$, *and* $\lambda\in\partial\mathbb{D}$ *so that if there is a unique weak limit point of the probability measures* $\{|p_n(z;\mu)|^2 d\mu(z)\}_{n\geq 0}$, *then this weak limit must be from the collection* $\{\nu^{\{k\}}_{a,b,\lambda}\}\cup\{\frac{d\theta}{2\pi}\}$. *Furthermore, the weak limit is exactly* $\nu^{\{k\}}_{a,b,\lambda}$ *if and only if there exists an* $r\in\{0,1,\ldots,2k-1\}$ *so that as* $n\to\infty$

1. $\alpha_{2nk+r+j}\to 0 \quad$ *if* $\quad j\neq 1,2,\ldots,k-1,k+1,\ldots,2k-1,$
2. $|\alpha_{2nk+2}|\to a,$
3. $|\alpha_{2nk+r+k}|\to b,$
4. $\alpha_{2nk+r+k}/\alpha_{2nk+r}\to\lambda b/a.$

The weak limit is $\frac{d\theta}{2\pi}$ *if and only if for every* $k \in \mathbb{N}$, $\alpha_n \alpha_{n+k} \to 0$ *as* $n \to \infty$.

Theorem 2.9.2 ([56] pg. 408)**.** *There is a probability measure* μ *on the unit circle such that the collection of weak limit points of* $\{|p_n(z;\mu)|^2 d\mu(z)\}_{n\geq 0}$ *is every probability measure on the unit circle.*

The proofs of Theorems 2.9.1 and 2.9.2 can be found in [56, 57] and we will not include them here. However, we will state and prove Theorem 2.9.4 below, which provides a set of sufficient conditions for the equilibrium measure on $[-2, 2]$ to be among the weak asymptotic measures for OPRL. Before we can state and prove that result, we need to introduce a technical lemma. We recall that for a set of natural numbers $\mathcal{N} \subseteq \mathbb{N}$, we define its *asymptotic density* as

$$\lim_{n\to\infty} \frac{\#\{\mathcal{N} \cap \{1, 2, \ldots, n\}\}}{n}$$

provided this limit exists.

Lemma 2.9.3. *Let* $\mathcal{N} \subseteq \mathbb{N}$ *be a subsequence with asymptotic density* 1*. There exists a subsequence* $\mathcal{N}_1 \subseteq \mathcal{N}$ *also of asymptotic density* 1 *so that if* $\ell \in \mathbb{Z}$ *is fixed then every sufficiently large* $m \in \mathcal{N}_1$ *can be written as* $q + \ell$ *for some* $q \in \mathcal{N}$.

Remark. An equivalent condition on $\mathcal{N}_1$ in the statement of the lemma is that if $\ell \in \mathbb{Z}$ is fixed then for all sufficiently large $m \in \mathcal{N}_1$, the set $\{m - |\ell|, m - |\ell| + 1, \ldots, m + |\ell|\}$ is contained in $\mathcal{N}$.

Proof. The idea is to think of the set $\mathbb{N} \setminus \mathcal{N}$ as being gaps in the set $\mathcal{N}$ and then to widen the gaps in smart way. More precisely, let $\mathcal{M} = \mathbb{N} \setminus \mathcal{N}$ and let $[n] = \{1, 2, \ldots, n\}$. If $k \in \mathbb{N}$ is fixed, then by definition of asymptotic density, one has

$$\lim_{n\to\infty} \frac{k|\mathcal{M} \cap [n]|}{n} = 0,$$

where $|X|$ denotes the cardinality of the set X. Therefore, by a standard argument, we can find a sequence of natural numbers $\{k_n\}_{n=1}^{\infty}$, which is non-decreasing and is unbounded so that

$$\lim_{n\to\infty} \frac{k_n|\mathcal{M} \cap [n]|}{n} = 0.$$

For every $m \in \mathcal{M}$, let $U_m = \{m - (k_m - 1), \ldots, m, \ldots, m + k_m - 1\}$ and define

$$\widetilde{\mathcal{M}} = \bigcup_{m\in\mathcal{M}} U_m.$$

Then

$$\limsup_{n\to\infty} \frac{|\widetilde{\mathcal{M}} \cap [n]|}{n} \leq \limsup_{n\to\infty} \frac{2k_n|\mathcal{M} \cap [n]|}{n} = 0,$$

so $\mathbb{N} \setminus \widetilde{\mathcal{M}}$ has density 1. Define $\mathcal{N}_1 = \mathbb{N} \setminus \widetilde{\mathcal{M}}$and let $\ell \in \mathbb{Z}$ be fixed. Clearly $\widetilde{\mathcal{M}}$ is divided into blocks so that the first and last $|\ell|$ elements of any sufficiently large block are not in $\mathcal{M}$. In other words, if we shift every block of $\mathcal{N}_1$ to the left or right by $|\ell|$, all but finitely many blocks land in $\mathcal{N}$, which is the desired conclusion. $\square$

The relevant theorem is the following theorem that is proven using an argument from [55]:

Theorem 2.9.4. *Let μ be a measure supported on some compact subset of the real line. Assume further that μ is regular and has essential support equal to $[-2, 2]$. There exists a subsequence $\mathcal{N} \subseteq \mathbb{N}$ of asymptotic density 1 so that*

$$w\text{-}\lim_{\substack{n\to\infty \\ n\in\mathcal{N}}} |p_n(x; \mu)|^2 d\mu(x) = d\omega_{[-2,2]}(x). \tag{2.9.1}$$

Proof. Let $\{a_n, b_n\}_{n\in\mathbb{N}}$ be the recursion coefficients for the orthonormal polynomials (see Section 2.1). First note that Theorem 2 in [55] combined with formula (5.23) in [55] imply the result holds with $\mathcal{N} = \mathbb{N}$ when the orthonormal polynomials have recursion coefficients given by $a_n = 1$ and $b_n = 0$ for all $n \in \mathbb{N}$. Let $t = (t_0, t_1, t_2, \ldots)$ be a walk on the lattice $\{0, 1, 2, \ldots\}$ so that $|t_j - t_{j+1}| \leq 1$. Define

$$W(t) = \prod_{j=0}^{\ell-1} w(t_j, t_{j+1}),$$

where

$$w(t_j, t_{j+1}) = \begin{cases} b_{k+1}, & \text{if } t_{j+1} = t_j = k, \\ a_{k+1}, & \text{if } t_{j+1} = t_j + 1 = k + 1, \\ a_k, & \text{if } t_{j+1} = t_j - 1 = k - 1. \end{cases}$$

Let $Q_{n,m,j}$ be the set of all paths of length j with $t_0 = n$ and $t_j = m$. We claim

$$\int x^\ell |p_n(x; \mu)|^2 d\mu(x) = \sum_{t\in Q_{n,n,\ell}} W(t).$$

To see this, recall (2.1.1):

$$xp_n(x; \mu) = a_{n+1}p_{n+1}(x; \mu) + b_{n+1}p_n(x; \mu) + a_n p_{n-1}(x; \mu).$$

By induction then, we easily recover

$$x^\ell p_n(z; \mu) = \sum_m \left(\sum_{t\in Q_{n,m,\ell}} W(t) \right) p_m(x; \mu),$$

from which the claim follows since

$$\int x^\ell |p_n(x;\mu)|^2 d\mu(x) = \langle p_n(x;\mu), x^\ell p_n(x;\mu)\rangle_\mu.$$

With this claim in hand, we need to show that $W(t)$ converges for every $t \in Q_{n,n,\ell}$ as $n \to \infty$ through some subsequence of asymptotic density 1. For this, we recall Theorem 1 in [61], which tells us that there is a subsequence $\mathcal{N}' \subseteq \mathbb{N}$ of asymptotic density 1 so that

$$\lim_{\substack{n\to\infty \\ n\in\mathcal{N}'}} a_n = 1, \qquad \lim_{\substack{n\to\infty \\ n\in\mathcal{N}'}} b_n = 0.$$

Lemma 2.9.3 then implies that we can find a subsequence $\mathcal{N} \subseteq \mathcal{N}'$ also of asymptotic density 1 so that for every $m \in \mathbb{Z}$ fixed, we have

$$\lim_{\substack{n\to\infty \\ n\in\mathcal{N}}} a_{n+m} = 1, \qquad \lim_{\substack{n\to\infty \\ n\in\mathcal{N}}} b_{n+m} = 0$$

(this can be interpreted in terms of right limits; see Section 7.1 in [63]). For each $t \in Q_{n,n,\ell}$, the quantity $W(t)$ is defined in terms of a set of recursion coefficients with indices having a bounded distance from n, which implies that

$$\lim_{\substack{n\to\infty \\ n\in\mathcal{N}}} \int x^\ell |p_n(x;\mu)|^2 d\mu(x) = \lim_{\substack{n\to\infty \\ n\in\mathcal{N}}} \sum_{t\in Q_{n,n,\ell}} W(t)$$

exists for each $\ell \in \mathbb{N}$ and the limit is the same as when $a_n = 1$ and $b_n = 0$ for all $n \in \mathbb{N}$. Since the moments of a compactly supported measure on the real line determine the measure uniquely, this implies the desired conclusion. □

Because of Theorems 2.9.1 and 2.9.2, we will consider the problem of characterizing the weak asymptotic measures on the unit circle solved. We have necessary and sufficient conditions for a unique weak limit to exist, when a unique weak limit does exist then we know what the possible limiting measures are, and we know that without these conditions the collection of weak limit points need not have any restrictions at all. Therefore, we will turn our attention to a related problem, namely understanding the weak limit points of the probability measures

$$d\mu_n(z) = \frac{1}{n+1}\sum_{j=0}^{n} |p_j(z;\mu)|^2 d\mu(z).$$

In the context of this problem, we require substantially weaker hypotheses to reach a conclusion. In fact we can consider measures with arbitrary compact and infinite support. The following result is Proposition 2.3 in [62].

Theorem 2.9.5 (Simon, [62])**.** *Let $N(\mu) = \sup\{|z| : z \in \mathrm{supp}(\mu)\}$ and let ν_n be the normalized zero counting measure for the polynomial $P_n(z;\mu)$. For any $k \in \mathbb{N}$, we have*

$$\left|\int_{\mathbb{C}} z^k d\mu_n(z) - \int_{\mathbb{C}} z^k d\nu_{n+1}(z)\right| \leq \frac{2kN(\mu)^k}{n+1}. \tag{2.9.2}$$

We will now sketch Simon's elegant proof of this result. Later, we will present an alternate proof in the setting of OPUC and OPRL (see Section 3.1 below).

Sketch of Proof. Let M_z be the multiplication by z operator on the space $L^2(\mu)$ and let Q_n be the orthogonal projection onto the polynomials of degree at most n. Then it is a straightforward calculation (given as Proposition 2.2 in [62]) to show that

$$\frac{1}{n+1}\mathrm{Tr}((Q_nM_zQ_n)^k) = \int z^k d\nu_{n+1}(z), \qquad \mathrm{Tr}(Q_nM_z^kQ_n) = \int z^k K_n(z,z;\mu)d\mu(z).$$

Next, one shows that $Q_nM_z^kQ_n - (Q_nM_zQ_n)^k$ is an operator of rank at most k and norm at most $2N(\mu)^k$. This is very easy to show because the operator in question is zero on $\mathrm{ran}(1-Q_n)$ and $\mathrm{ran}(Q_{n-k})$, which implies it has rank at most k. The statement about the norms follows from $\|M_z\| = N(\mu)$. This then implies

$$\left|\mathrm{Tr}((Q_nM_zQ_n)^k) - \mathrm{Tr}(Q_nM_z^kQ_n)\right| \leq 2kN(\mu)^k,$$

which is the desired conclusion.

□

Theorem 2.9.5 is especially helpful because one is often able to prove something about the asymptotics of the measures $\{\nu_n\}_{n\in\mathbb{N}}$ under very weak hypotheses (indeed just regularity, as mentioned in Section 2.2). This allows us to make conclusions about the measures μ_n under similarly weak hypotheses, while making conclusions about the measures $\{|p_n(z;\mu)|^2d\mu(z)\}_{n\in\mathbb{N}}$ requires stronger hypotheses. In Section 3.1, we will provide a new proof of Theorem 2.9.5 in the case when the measure μ has compact support on the real line or the unit circle.

From our discussion so far, we see that the measures $\{|p_n(z;\mu)|^2d\mu(z)\}_{n\geq 0}$ are well-studied and well-understood in the literature when the measure μ is supported on the unit circle or a compact subset of the real line. However, it is also reasonable to consider the measures $\{|p_n(z;\mu,q)|^qd\mu(z)\}_{n\geq 0}$ for any $q \in (0,\infty)$ and measures μ with arbitrary support in the plane. None of the results from this section extend easily to this more general case due to the lack of a recurrence relation satisfied by the polynomials $p_n(z;\mu,q)$. We will consider this problem again in Sections 3.2.2 and 3.3.2.2.

Chapter 3

Results

"The greatest results in life are usually attained by simple means and the exercise of ordinary qualities. These may for the most part be summed in these two: common-sense and perseverance."

– Owen Feltham

In this, the final chapter of this thesis, we will present new results. The results we present here can be found in [52, 54, 53], though in many instances our presentation or exposition differs from that given in the journal articles. Many of these results are generalizations of theorems from Chapter 2 and we will repeatedly mention the similarities between our results and those mentioned above.

We will begin in Section 3.1, where we present a new proof of Theorem 2.9.5 when μ is a probability measure supported on the unit circle or a compact subset of the real line (see also [52]). In Section 3.2, we will present several new results concerning the polynomials $P_n(z;\mu,q)$ for arbitrary $q>0$ when the measure of orthogonality has a certain form and is supported on an analytic region (see also [54]). These results include the asymptotic behavior of the $L^q(\mu)$-norm of these polynomials, strong asymptotic behavior outside the region of orthogonality, a description of the weak asymptotic measures, and the behavior of the Christoffel functions. Finally, in Section 3.3, we will present new results on relative and ratio asymptotics for orthonormal polynomials (see also [53]). In Section 3.3, we make no assumptions about the support of the measure of orthogonality except that it is infinite and compact.

Throughout this chapter, we will often abbreviate our notation for clarity. We will often denote $P_n(z;\mu,q)$ by $P_n(\mu;q)$ or just $P_n(\mu)$ if there is no possibility for confusion. We will also use notation from Chapters 1 and 2 throughout this chapter.

3.1 Weak Convergence of CD Kernels: A New Approach on the Circle and Real Line

In this section we will provide a new proof of Theorem 2.9.5 when μ is a probability measure supported on the unit circle or a compact subset of the real line. We recall that Theorem 2.9.5 is originally due to Simon and can be found in [62]. If the measure is supported on a compact subset of the real line, Totik has a different proof, which can also be found in [62]. The key idea in our proof will be to look at Prüfer phases of the appropriate ratio of the orthonormal polynomials.

If μ is supported on $\partial\mathbb{D}$, we define $\eta_n(\theta) : [0, 2\pi] \to \mathbb{R}$ to be a continuous function so that

$$e^{i\eta_n(\theta)} = \frac{p_{n+1}(e^{i\theta};\mu)}{p^*_{n+1}(e^{i\theta};\mu)}, \tag{3.1.1}$$

where $p^*_{n+1}(z) = z^{n+1}\overline{p_{n+1}(\bar{z}^{-1})}$ (so that the right-hand side of (3.1.1) is a Blaschke product). If μ is supported on $\mathbb{R}$ (we always assume compact support), then we may define $\theta_n(x) : \mathbb{R} \to (-\pi/2, \infty)$ to be a continuous function so that

$$\tan(\theta_n(x)) = \frac{a_n p_n(x;\mu)}{p_{n-1}(;\mu)}, \tag{3.1.2}$$

(see Proposition 6.1 in [23]) where a_n is as in (2.1.1). In our proofs, we will use the functions η_n and θ_n (and their derivatives) to obtain measures that approximate the measure μ in a sense suitable for our purposes. More precisely, we will use the measures μ^n and ρ_n defined in (2.1.4) and (2.1.5) respectively.

In the next section we discuss some properties of the measure ρ_n that will be relevant to our proof in Section 3.1.3. In Section 3.1.2, we provide our new proof of Theorem 2.9.5 when μ is supported on the unit circle. In Section 3.1.3 we consider μ supported on the real line and prove Theorem 2.9.5 with the right-hand side of (2.9.2) replaced by $O(n^{-1})$.

Throughout this section, μ will always be a probability measure.

3.1.1 Gaussian Quadrature

Before we move on to the proof of our results in this section, let us take a moment to better understand the measures defined in (2.1.5). Recall the measure ρ_n is defined by

$$d\rho_n(x) = \frac{1}{\pi(a_{n+1}^2 p_{n+1}^2(x;\mu) + p_n(x;\mu)^2)}dx.$$

To understand the origins of this measure, we must introduce the notion of *Gaussian Quadrature* for orthogonal polynomials on the real line. Let us fix some $N \in \mathbb{N}$. The orthogonality relation for

OPRL

$$\int_{\mathbb{R}} p_n(x;\mu)p_m(x;\mu)d\mu(x) = \delta_{mn}, \qquad m,n \le N$$

determines the first $2N$ moments of the measure μ and knowing these first $2N$ moments uniquely determines the orthonormal polynomials $\{p_n\}_{n=0}^N$. Therefore, any measure having the same first $N+1$ orthonormal polynomials has the same first $2N$ moments. In fact, this reasoning applies to any measure with at least $N+1$ points in its support, since for such measures it still makes sense to talk about orthonormal polynomials up to degree N.

Let $\{e_j\}_{j=0}^\infty$ be the standard basis for $\ell^2(\mathbb{N}\cup\{0\})$. An example of a measure with $N+1$ points in its support is the spectral measure of the upper $(N+1)\times(N+1)$ block of the Jacobi matrix corresponding to the measure μ (call this block $J^{(N)}$) and the vector e_0. Since the orthonormal polynomials are determined by the recursion coefficients, it follows that this spectral measure has the same first $2N$ moments as the measure μ.

Notice that the matrix $J_b^{(N)}$ defined by

$$J_b^{(N)} = J^{(n)} + b\langle\cdot, e_N\rangle e_N$$

has the same entries as $J^{(N)}$ except in the bottom right-hand entry, where we have added b. Since this coefficient is only used to determine p_{N+1}, we see that the spectral measure of $J_b^{(N)}$ and e_0 (call it $\rho_{N,b}$) has the same first $N+1$ orthonormal polynomials as μ and hence the same first $2N$ moments as μ. To summarize, we have

$$\int_{\mathbb{R}} x^k d\mu(x) = \int_{\mathbb{R}} x^k d\rho_{N,b}(x), \qquad k = 0,1,\ldots,2N. \tag{3.1.3}$$

The measure $\rho_{N,0}$ is often called the degree $N+1$ quadrature measure for the measure μ. The following facts can all be found in [60]:

- The measure $\rho_{N,b}$ is a pure point measure supported on $N+1$ distinct points. The measure $\rho_{N,0}$ is supported on the zeros of the polynomial $p_{N+1}(\cdot;\mu)$.
- The supports of $\rho_{N,b}$ and $\rho_{N,b'}$ strictly interlace if $b \neq b'$. Additionally, if $b > b'$ then the largest value of a point in the support of $\rho_{N,b}$ is larger than the largest value of a point in the support of $\rho_{N,b'}$.
- If $x \in \mathrm{supp}(\rho_{N,b})$ then $\rho_{N,b}(\{x\}) = K_N(x,x;\mu)^{-1}$.

Now, let us integrate both sides of (3.1.3) along $\mathbb{R}$ with respect to the measure $\frac{db}{\pi(1+b^2)}$, which is a probability measure. Obviously the left-hand side is unchanged since it is independent of b. On the right-hand side, we get the k^{th} moment of a weighted average of the spectral measures of the matrices $J_b^{(N)}$. This weighted average is precisely the measure ρ_N. For a proof of this fact, see Theorem 2.1 in [59]. It follows that

$$\int_{\mathbb{R}} x^k d\mu(x) = \int_{\mathbb{R}} x^k d\rho_N(x), \qquad k = 0, 1, \ldots, 2N,$$

which proves (2.1.6).

3.1.2 The Unit Circle Case

Our goal in this section is to provide a new proof of Theorem 2.9.5 when μ is supported on the unit circle. We begin our proof by noting that the theorem is equivalent to the statement that the moments of the signed measures $d\hat{\nu}_{n+1} - d\mu_n$ converge to 0 at a certain rate where $\hat{\nu}_n$ is the balayage of the measure ν_n onto $\partial\mathbb{D}$. Let $\{z_j^{(n)}\}_{j=1}^n$ be the (not necessarily distinct) zeros of $p_n(z;\mu)$. It is easy to check that (see equation (8.2.8) in [56])

$$d\hat{\nu}_{n+1} = \frac{1}{n+1} \sum_{j=1}^{n+1} \frac{1 - |z_j^{(n+1)}|^2}{|e^{i\theta} - z_j^{(n+1)}|^2} \frac{d\theta}{2\pi}.$$

If we define $\eta_n : [0, 2\pi] \to \mathbb{R}$ as in (3.1.1) above, then equation (6.10) in [66] implies that

$$\frac{d}{d\theta}\eta_n(\theta) = \sum_{j=1}^{n+1} \frac{1 - |z_j^{(n+1)}|^2}{|e^{i\theta} - z_j^{(n+1)}|^2}.$$

Furthermore, equation (10.8) in [23] tells us that

$$\frac{d}{d\theta}\eta_n(\theta) = \frac{K_n(e^{i\theta}, e^{i\theta}; \mu)}{|p_{n+1}(e^{i\theta})|^2}$$

so we conclude that

$$d\hat{\nu}_{n+1} = \frac{K_n(e^{i\theta}, e^{i\theta}; \mu)}{n+1} d\mu^n(\theta).$$

Therefore, if $k \in \mathbb{N}$, we can write

$$\int_{\overline{\mathbb{D}}} z^k \, d\nu_{n+1}(z) - \int_{\partial\mathbb{D}} z^k d\mu_n(z) = \frac{1}{n+1} \sum_{j=0}^{n} \left[\langle p_j(z;\mu), z^k p_j(z;\mu) \rangle_{\mu^n} - \langle p_j(z;\mu), z^k p_j(z;\mu) \rangle_{\mu} \right].$$

Since μ and μ^n have the same first n moments, at most k of these summands are nonzero and each nonzero summand has absolute value at most 2. We have therefore proven

$$\left| \int_{\overline{\mathbb{D}}} z^k \, d\mu_n(z) - \int_{\partial\mathbb{D}} z^k \, d\nu_{n+1}(z) \right| \leq \frac{2k}{n+1},$$

exactly as in Theorem 2.9.5.

Example. Let μ be the normalized arc-length measure on the unit circle. In this case we have $p_n(z;\mu) = z^n$ for all n and $\mu_n = \mu$ for all n. The measures $\{\nu_n\}_{n\in\mathbb{N}}$ are all simply the point mass at 0 with weight 1. This example illustrates the fact that, in general, the measures μ_n and ν_n need not resemble each other as measures on $\overline{\mathbb{D}}$, so it really is important that we consider the balayage.

3.1.3 The Real Line Case

Our goal in this section is to provide a new proof of Theorem 2.9.5 when μ is supported on a compact subset of the real line and with the right-hand side of (2.9.2) replaced by $O(n^{-1})$ where the implied constant depends on k. There is a proof of this result due to Totik, also appearing in [62], but with the right-hand side of (2.9.2) replaced by $o(1)$ (though it can be modified to produce the same $O(n^{-1})$ discrepancy estimate for the moments as in (2.9.2) above). Totik's proof uses Gaussian quadratures and the monotonicity (in n) of the sequence $K_n(x,x;\mu)$ to establish the weak convergence result for all polynomials that are positive on the convex hull of the support of μ. The proof we present here will be analogous to the proof in Section 3.1.2 and will rely on the sequence of approximating measures ρ_n (see (2.1.5) above). We will make use of formula (3.1.4) below, which relates a set of perturbed zero-counting measures to a set of perturbed quadrature measures. Combining this with an interlacing property will allow us to derive the $O(n^{-1})$ estimate in (2.9.5).

Our computation will be a bit longer than in the unit circle case partly because in Section 3.1.2, the most difficult calculation was already done for us in [23] and partly because the high moments of the measure ρ_n defined in equation (2.1.5) are infinite, so we need to use a cutoff function.

Let us assume μ has support contained in $[-M,M]$ and define

$$\tau(x) = \chi_{[-M-1,M+1]}(x).$$

Let $\nu_{n,b}$ be the measure placing weight n^{-1} on each point in the support of $\rho_{n,b}$ (so that $\nu_{n,0} = \nu_n$). It follows from formula (6.16) in [60] that

$$\frac{1}{n+1}d\rho_{n,b} = \frac{1}{K_n(x,x;\mu)}d\nu_{n+1,b}. \tag{3.1.4}$$

Therefore for any fixed $k \in \mathbb{N}$, we have

$$\int_{\mathbb{R}} x^k\tau(x)d\nu_{n+1,b} = \frac{1}{n+1}\int_{\mathbb{R}} x^k\tau(x)K_n(x,x;\mu)d\rho_{n,b}. \tag{3.1.5}$$

After taking a suitable average (in b), the expression on the left-hand side of (3.1.5) approximates the k^{th} moment of ν_{n+1} as $n \to \infty$ while the right-hand side approximates the k^{th} moment of μ_n as $n \to \infty$. Indeed, our first step is to integrate the left-hand side of (3.1.5) from $-\infty$ to ∞ with respect to $\frac{db}{\pi(1+b^2)}$. Notice that for any value of b, at most one point in the support of $\rho_{n+1,b}$ lies outside $[-M-1, M+1]$ because of the interlacing property. Therefore, we have

$$\int_{-\infty}^{\infty}\int_{\mathbb{R}} x^k \tau(x) d\nu_{n+1,b}(x) \frac{db}{\pi(1+b^2)} = \int_{\mathbb{R}} x^k \tau(x) d\nu_{n+1,0}(x) + O(n^{-1}) \tag{3.1.6}$$

as $n \to \infty$.

If we integrate the right-hand side of (3.1.5) in the same way, this becomes

$$\frac{1}{n+1}\int_{\mathbb{R}} x^k \tau(x) K_n(x,x;\mu)\, d\rho_n(x) \tag{3.1.7}$$

by Theorem 2.1 in [59]. Notice that this integral would be infinite without the cutoff function τ. As an aside, we note that by Proposition 6.1 in [23], (3.1.7) is just

$$\frac{1}{n+1}\int_{\mathbb{R}} x^k \tau(x) \frac{1}{\pi}\frac{d\theta_{n+1}(x)}{dx}\, dx,$$

which is why we call this the analog of the proof in Section 3.1.2. Notice that for any fixed $m \leq n$ we have

$$\int_{\mathbb{R}} x^k \tau(x) |p_m(x;\mu)|^2 \, d\rho_n(x) \leq (M+1)^k.$$

This follows from the fact that p_m is also the degree m orthonormal polynomial for the measure $d\rho_n$ by (2.1.6). Therefore, we can rewrite (3.1.7) as

$$\frac{1}{n+1}\int_{\mathbb{R}} x^k \tau(x) K_{n-k}(x,x;\mu)\, d\rho_n(x) + O(n^{-1})$$

as $n \to \infty$. We can rewrite this again as

$$\frac{1}{n+1}\int_{\mathbb{R}} x^k K_{n-k}(x,x;\mu)\, d\rho_n(x) - \frac{1}{n+1}\int_{|x|>M+1} x^k K_{n-k}(x,x;\mu)\, d\rho_n(x) + O(n^{-1}) \tag{3.1.8}$$

as $n \to \infty$. Notice that $x^k K_{n-k}(x,x;\mu)$ is a polynomial of degree $2n-k$ while the denominator of the weight defining the measure ρ_n is a polynomial of degree $2n+2$. Therefore, both integrals in (3.1.8) are finite. The first term in (3.1.8) is equal to

$$\frac{1}{n+1}\int_{\mathbb{R}} x^k K_n(x,x;\mu) d\mu(x) + O(n^{-1}) = \int_{\mathbb{R}} x^k \, d\mu_n(x) + O(n^{-1})$$

as $n \to \infty$ again by (2.1.6). We will be finished if we can show that the second term in (3.1.8) tends to zero like $O(n^{-1})$ as $n \to \infty$ and for this it suffices to put a uniform bound on

$$\left| \int_{|x|>M+1} x^k K_{n-k}(x,x;\mu)\, d\rho_n(x) \right|. \tag{3.1.9}$$

To do this, we rewrite the integral in (3.1.9) as

$$\int_{-\infty}^{\infty}\int_{|x|>M+1} x^k K_{n-k}(x,x;\mu)\,d\rho_{n,b}(x)\,\frac{db}{\pi(1+b^2)}.$$

Recall that for each fixed b, at most one point in the support of $\rho_{n,b}$ has absolute value larger than $M+1$. Let us denote this point (if it exists) by $x_{n+1,b}$. Therefore, the above integral is just

$$\int_A x_{n+1,b}^k \frac{K_{n-k}(x_{n+1,b},x_{n+1,b};\mu)}{K_n(x_{n+1,b},x_{n+1,b};\mu)}\frac{db}{\pi(1+b^2)}, \tag{3.1.10}$$

where we used (3.1.4) and the integral is taken over some set $A\subseteq\mathbb{R}$ such that $x_{n+1,b}$ exists if and only if $b\in A$. Using the Christoffel Variational Principal (Theorem 9.2 in [60]), it is easily seen that

$$\frac{K_{n-k}(x_{n+1,b},x_{n+1,b};\mu)}{K_n(x_{n+1,b},x_{n+1,b};\mu)}\leq\left(\frac{M}{|x_{n+1,b}|}\right)^{2k}.$$

Therefore, we can bound (3.1.10) from above in absolute value by

$$\int_A \frac{M^{2k}}{|x_{n+1,b}|^k}\frac{db}{\pi(1+b^2)},$$

which is uniform in n since $|x_{n+1,b}|>M+1$. This completes the proof.

Although the proof we just presented is analogous to the proof in Section 3.1.2, an alternate (though not dissimilar) proof follows from (3.1.5) evaluated at the value $b=0$. In this case, we have

$$\int_{\mathbb{R}} x^k d\nu_{n+1}=\frac{1}{n+1}\int_{\mathbb{R}} x^k K_n(x,x;\mu)d\rho_{n,0} \tag{3.1.11}$$

because the cutoff function plays no role here. As mentioned in Section 3.1.1, the polynomial $p_m(x;\mu)$ is also the degree m orthonormal polynomial for $\rho_{n,0}$ if $m\leq n$. Therefore, by the reasoning of the above proof, we can rewrite the right-hand side of (3.1.11) as

$$\frac{1}{n+1}\int_{\mathbb{R}} x^k K_{n-k}(x,x;\mu)d\rho_{n,0}+\epsilon_n,\qquad |\epsilon_n|\leq\frac{kM^k}{n+1}.$$

The expression $x^kK_{n-k}(x,x;\mu)$ is a polynomial of degree $2n-k$, and since $\rho_{n,0}$ has the same first $2n$ moments as μ, we can rewrite this again as

$$\frac{1}{n+1}\int_{\mathbb{R}} x^k K_{n-k}(x,x;\mu)d\mu(x)+\epsilon_n=\frac{1}{n+1}\int_{\mathbb{R}} x^k K_n(x,x;\mu)d\mu(x)+\tilde{\epsilon}_n,$$

where $|\tilde{\epsilon}_n|\leq\frac{2kM^k}{n+1}$, exactly as in (2.9.2). This is our desired conclusion.

3.2 L^p-Extremal Polynomials on Analytic Regions

In this section, we will consider the behavior of the polynomials $\{P_n(z;\mu,q)\}_{n\geq 0}$ for arbitrary $q>0$ when the measure μ is of a certain form. We will be interested in describing the behavior of the polynomials $P_n(z;\mu,q)$ in several ways. The first step will be to determine the asymptotic behavior of the leading coefficient of $p_n(z;\mu,q)$ (which we denote by $\kappa_n(\mu,q)$) as n tends to infinity. Then using the Keldysh Lemma (see Section 1.5) and related ideas, we will deduce the Szegő asymptotics of the polynomials $\{P_n(z;\mu,q)\}_{n\geq 0}$ as $n\to\infty$. We will also discuss the behavior of the Christoffel functions $\lambda_\infty(z;\mu,q)$ (see Section 2.5) and the weak asymptotics of the measures $\{|p_n(z;\mu,q)|^q d\mu(z)\}_{n\geq 0}$. If the measure μ is a Szegő measure on the unit circle, then a concise description of all of these objects can be found in [56, 57], so our investigation here can be interpreted as a generalization of those critical results.

Before we begin our investigation, a word is required concerning the case $q\leq 1$, where the polynomial $P_n(z;\mu,q)$ is not uniquely defined. On page 84 in [65], it is stated that one does not have uniqueness of the L^q-extremal polynomial when $0<q<1$ and the following proposition extends this to include the case $q=1$:

Proposition 3.2.1. *If μ is a finite measure supported on $[-2,-1]\cup[1,2]$ and $\mu(A)=\mu(-A)$ for all measurable sets A, then one does not have uniqueness of the L^1-extremal polynomial $P_n(\mu,1)$ for every odd n.*

Proof. Suppose for contradiction that $P_{2n+1}(\mu,1)$ can be uniquely defined. By the symmetry of the measure, we must have that $P_{2n+1}(0;\mu,1)=0$. We may then write $P_{2n+1}(z;\mu,1)=zQ_n(z)$, for some polynomial Q_n of degree $2n$ and satisfying $Q_n(x)=Q_n(-x)$ for all $x\in\mathbb{R}$. For $a\in(-1,1)$, define

$$P_{2n+1}^{(a)}(z)=(z-a)Q_n(z)$$

so that $P_{2n+1}^{(0)}(z)=P_{2n+1}(z;\mu,1)$. We then have

$$\begin{aligned}\frac{\partial}{\partial a}\|P_{2n+1}^{(a)}\|_{L^1(\mu)}&=\frac{\partial}{\partial a}\left(\int_{-2}^{-1}(a-z)|Q_n(z)|d\mu(z)+\int_1^2(z-a)|Q_n(z)|d\mu(z)\right)\\&=\int_{-2}^{-1}|Q_n(z)|d\mu(z)-\int_1^2|Q_n(z)|d\mu(z)=0,\end{aligned}$$

which contradicts our uniqueness assumption. □

Remark. If in Proposition 3.2.1 we also assume μ has no pure points then an alternative proof can be found by appealing to Theorem 2.1 in [41].

Therefore, in the case $q\leq 1$, we will always let $P_n(z;\mu,q)$ denote *some* monic polynomial with minimal $L^q(\mu)$-norm (we use the word "norm" here loosely as it is not technically a norm when

$q < 1$).

We will consider measures whose support is contained in some compact and simply connected set $\overline{G}$ along with finitely many points not in $\overline{G}$. We will also assume that G is a region with analytic boundary and that the logarithmic capacity (see Section 1.1) of G is equal to 1. The analyticity of ∂G allows us to univalently extend the conformal map ψ to the exterior of a disk of radius $\tilde{\rho} < 1$. Therefore, if a measure μ is carried by the set $\{z : |z| > \tilde{\rho}\}$, then we may define the measure $\psi_*\mu$ as in Section 1.2. Similarly, we may define the measure $\phi_*\mu$ when μ is carried by the exterior of $\Gamma_{\tilde{\rho}} := \{\psi(z) : |z| = \tilde{\rho}\}$. Henceforth, we will always assume that ρ is some fixed number in the interval $(\tilde{\rho}, 1)$.

Throughout this section, for a measure γ (on any set), we denote

$$c_t(\gamma) = \int_{\mathbb{C}} |z|^t \, d\gamma(z), \tag{3.2.1}$$

where we do *not* insist $t \in \mathbb{N}$. We will see that these "moments" provide the appropriate rate of decay of the norms of the extremal polynomials. One of our main results is the following:

Theorem 3.2.2. *Consider the measure* $\tilde{\mu}(re^{i\theta}) = h(re^{i\theta})\,(\nu(\theta) \otimes \tau(r)) + \sigma_2(re^{i\theta})$ *where*

1. $h(z)$ *is a continuous function on* $\overline{\mathbb{D}}$ *that is nonvanishing in a neighborhood of* $\partial\mathbb{D}$*,*
2. σ_2 *is a measure carried by* $\{z : \rho \le |z| \le 1\}$ *that satisfies* $\lim_{t\to\infty} c_t(\sigma_2)c_t(\tau)^{-1} = 0$*,*
3. ν *is a measure on the unit circle such that* $\nu'(\theta) > 0$ *Lebesgue almost everywhere,*
4. τ *is a measure on* $[\rho, 1]$ *such that* $1 \in \mathrm{supp}(\tau)$*.*

Let μ *be the measure on* $\mathbb{C}$ *be given by*

$$\mu = \psi_*\tilde{\mu} + \sigma_1 + \sum_{j=1}^{m} d_j \delta_{z_j} + \sum_{j=1}^{\ell} \beta_j \delta_{\zeta_j},$$

where $\mathrm{supp}(\sigma_1) \subseteq G$*,* $d_j > 0$*,* $\beta_j > 0$*,* $z_j \notin \overline{G}$ *for all* $j \in \{1, \ldots, m\}$*, and* $\zeta_j \in \partial G$ *for all* $j \in \{1, \ldots, \ell\}$*. Then*

$$\lim_{n\to\infty} \frac{\|P_n(z;\mu,q)\|_{L^q(\mu)}^q}{c_{qn}(\tau)} = \exp\left(\int_0^{2\pi} \log\left(h(e^{i\theta})\nu'(\theta)\right) \frac{d\theta}{2\pi}\right) \prod_{j=1}^{m} |\phi(z_j)|^q. \tag{3.2.2}$$

If the logarithmic capacity of G is equal to $\gamma > 0$, then Theorem 3.2.2 still allows us to deduce the asymptotic behavior of the extremal monic polynomial norms. Observe that the region $\gamma^{-1}G = \{x : \gamma x \in G\}$ has logarithmic capacity 1 (see Theorem 5.1.2 in [45]). If $M_{\gamma^{-1}} : \mathbb{C} \to \mathbb{C}$ is given by $M_{\gamma^{-1}}(z) = \gamma^{-1}z$, then we define the measure $(M_{\gamma^{-1}})_*\mu$ as usual. Notice that for any monic polynomial Q of degree n, the polynomial $\gamma^{-n}Q(\gamma z)$ is also monic and

$$\gamma^{-n}\left\|Q(z)\right\|_{L^q(\mu)} = \left\|\gamma^{-n}Q(\gamma z)\right\|_{L^q((M_{\gamma^{-1}})_*\mu)}. \tag{3.2.3}$$

Taking the infimum of both sides of (3.2.3) over all monic polynomials Q of degree n and invoking Theorem 3.2.2, we conclude that if $\mathrm{cap}(G) = \gamma$, then

$$\lim_{n\to\infty}\frac{\|P_n(z;\mu,q)\|^q_{L^q(\mu)}}{c_{qn}(\tau)\gamma^n} = \exp\left(\int_0^{2\pi}\log\left(h(e^{i\theta})\nu'(\theta)\right)\frac{d\theta}{2\pi}\right)\prod_{j=1}^m|\phi(z_j)|^q.$$

Therefore, we suffer no loss of generality by only considering regions G with capacity 1.

During the preparation of [54], we discovered the recent announcement of Baratchart and Saff, which is described in [4]. They consider measures on the unit disk that in may ways resemble the measures we consider in Theorem 3.2.2. They obtain a similar description of the asymptotic behavior of the monic orthogonal polynomial norms, though Theorem 3.2.2 seems to be more general.

The factor of $\prod_{j=1}^m|\phi(z_j)|^q$ in (3.2.2) is exactly what one would expect given the results of [19, 20, 21, 22, 37, 24, 40]. We will call a measure μ as in the statement of Theorem 3.2.2 a *push-forward of a product measure*. Let us consider some examples of measures to which we can apply Theorem 3.2.2.

Example. If we set $q=2$, $d\nu=\frac{d\theta}{2\pi}$, $d\tau=2rdr$, $\sigma_1=\sigma_2=0$, and $\ell=m=0$, then we are dealing with measures of the form $h(z)d^2z$ for a function h continuous and nonvanishing on $\overline{\mathbb{D}}$. Such measures with an added Hölder continuity assumption on h were considered by Suetin in [68]. Theorem 3.2.2 recovers the leading term in the conclusion of Theorem 2.4.2.

Example. If we set $G=\mathbb{D}$, $\tau=\delta_1$, and $h=1$, then we recover a result similar to that of [37] (when $q=2$) that allows for a singular component of the measure on $\partial\mathbb{D}$, but only finitely many pure points outside $\overline{\mathbb{D}}$. If we further set $\sigma_1=\sigma_2=0$, then we can recover the result from Theorem 2.2 in [19] (for any G with analytic boundary).

Example. Let us set $G=\mathbb{D}$ and $\tau=\frac{6}{\pi^2}\sum_{j=1}^\infty j^{-2}\delta_{1-2^{-j}}$, $d\nu=\frac{d\theta}{2\pi}$, $h=1$, $\ell=m=0$, and $\sigma_2=\sigma_1=0$. If s is a sufficiently large power of 2, then

$$c_s(\tau)=\frac{6}{\pi^2}\sum_{j=1}^\infty j^{-2}(1-2^{-j})^s\geq\frac{6}{\pi^2\log_2(s)^2}\left(1-\frac{1}{s}\right)^s\geq\frac{C}{\log_2(s)^2}$$

for some constant $C>0$. Theorem 3.2.2 implies that in this example, the extremal polynomial norms do *not* decay like $O(n^{-1})$ as $n\to\infty$.

Example. Let us set $G=\mathbb{D}$, $\tau=(1-r)dr$, $d\nu=d\nu_{\mathrm{ac}}$, $\ell=m=0$, and $\sigma_2=\sigma_1=0$. In this case, we have $d\mu(z)=w(z)d^2z$, where the weight w vanishes on the boundary. Theorem 3.2.2 still applies to this measure, and we will see below that we can still derive Szegő asymptotics for the extremal polynomials outside $\overline{\mathbb{D}}$.

Theorem 3.2.2 provides the asymptotic behavior of the norms of the $L^q(\mu)$-extremal polynomials for every $q \in (0, \infty)$. We can also deduce the behavior of the extremal polynomials outside the compact set $\overline{G}$, i.e., we can prove Szegő asymptotics. If μ is of the form considered in Theorem 3.2.2 with ν a Szegő measure on $\partial\mathbb{D}$, then we can prove the following:

1. there are polynomials $\{y_n\}_{n\in\mathbb{N}}$ (depending on $P_n(\mu, q)$ and q) of degree m and a function $S = S_q$ analytic and nonvanishing in $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ and positive at ∞ so that
$$\lim_{n\to\infty} \frac{P_n(\psi(z); \mu, q)S(z)}{y_n(\psi(z))z^{n-m}S(\infty)} = 1$$
uniformly on compact subsets of $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$,

2. the probability measures $|p_n(z; \mu, q)|^q d\mu(z)$ converge weakly to the equilibrium measure for $\overline{G}$ as $n \to \infty$,

3. for any $z \in G$, we have $\sum_{n=0}^{\infty} |p_n(z; \mu, 2)|^2 < \infty$.

Item (3) follows from an argument based on Christoffel functions and the associated minimization problem. We will discuss this in more detail in Section 3.2.3 and for all values of $q > 0$. The function S in item (1) will be of the form given in (1.4.6). We will see that the polynomial y_n in item (1) has a single zero near each z_i for $i \in \{1, \ldots, m\}$ and shares all of its zeros with $P_n(z; \mu, q)$.

The remainder of the section is organized as follows. In Section 3.2.1, we prove Theorem 3.2.2. One key step will be to use Faber polynomials and look at weak limits of the measures $\left\{\frac{|F_n(z)|^q d\mu}{c_{qn}(\tau)}\right\}_{n\in\mathbb{N}}$. In Section 3.2.2 we will discuss Szegő asymptotics of the extremal polynomials for measures of the form considered in Theorem 3.2.2. In Section 3.2.3 we will discuss Christoffel functions and their behavior on the set $\overline{G}$, especially inside the region G. A major theme throughout will be the many similarities with the theory of orthogonal polynomials on the unit circle. Many of our results produce interesting corollaries and we will point these out as we go.

Throughout this section, we will let Γ_r be the contour given by $\{\psi(z) : |z| = r\}$ for $r > \tilde{\rho}$, $\mathcal{G}_r$ will denote the region bounded by Γ_r, and G_r will denote $\overline{G} \setminus \mathcal{G}_r$.

3.2.1 Push-Forward of Product Measures on the Disk

In this section, we will derive norm asymptotics for the extremal polynomials corresponding to measures of the form considered in Theorem 3.2.2. We will use Faber polynomials in conjunction with the extremal property to eventually derive an upper bound in the proof of Theorem 3.2.2 and we will use subharmonicity of appropriate functions to derive a lower bound. For the remainder of this section, we will let $q > 0$ be fixed but arbitrary, and we will denote $P_n(z; \mu, q)$ by $P_n(\mu)$ and $\|P_n(\mu)\|_{L^q(\mu)}$ by $\|P_n(\mu)\|_\mu$ when there is no possibility for confusion. We begin with the following crude estimate, which applies even when ν is not a Szegő measure:

Proposition 3.2.3. *If μ is as in Theorem 3.2.2 then μ is regular.*

Proof. We will in fact show that μ satisfies Widom's criterion (see Section 2.2) from which regularity immediately follows by Theorem 4.1.6 in [65].

For each $r \in (\rho, 1]$, the equilibrium measure of the curve Γ_r is absolutely continuous with respect to arc-length measure with continuous derivative bounded above and below by positive constants (see Theorem II.4.7 in [14]; the constants are allowed to depend on r). Let C be a carrier of μ. Since $\nu'(\theta) > 0$ Lebesgue almost everywhere, we conclude that

$$\lambda_r(C \cap \Gamma_r) = \ell(\Gamma_r)$$

for τ almost every $r \in (\rho, 1]$ where λ_r is arc-length measure on Γ_r and $\ell(\Gamma_r)$ is the length of the curve Γ_r. It follows that there is a sequence $r_n \to 1$ such that $\omega_{\Gamma_{r_n}}(C) = 1$ while clearly $\text{cap}(\Gamma_{r_n}) \to 1$. This shows μ satisfies Widom's criterion. ☐

We will now begin developing the ideas necessary to prove the more refined estimate of $\|P_n(\mu)\|_\mu^q$ given in Theorem 3.2.2. Let $\{F_n\}_{n\in\mathbb{N}}$ be the sequence of Faber polynomials corresponding to the region G. Since we are assuming $\text{cap}(\overline{G}) = 1$, we recover from our earlier discussion of Faber polynomials (see Section 1.3) the following two facts:

1. $F_n(z)$ is a monic polynomial of degree n,

2. for $\rho < |z| \leq 1$ we have

$$F_n(\psi(z)) = z^n + O(\rho_0^n), \tag{3.2.4}$$

   where $\rho_0 \in (\tilde{\rho}, \rho)$ and the implied constant is uniformly bounded from above in the annulus considered.

We will henceforth assume that some value of $\rho_0 \in (\tilde{\rho}, \rho)$ has been fixed so that (2) holds.

We begin with a lemma that immediately highlights the importance of these important polynomials to our results.

Lemma 3.2.4. *Let $\mathcal{N} \subseteq \mathbb{N}$ be a subsequence such that*

$$\underset{\substack{n\to\infty \\ n\in\mathcal{N}}}{\text{w-lim}} \frac{|F_n(z)|^q d\mu(z)}{a_n} = d\gamma$$

where γ *is a measure on* ∂G *and* $\{a_n\}_{n\in\mathbb{N}}$ *is a sequence of positive real numbers satisfying* $\lim_{n\to\infty} a_n a_{n+1}^{-1} = 1$*. Then for any fixed* $k \in \mathbb{N}$*, we have*

$$\underset{\substack{n\to\infty \\ n\in\mathcal{N}}}{\text{w-lim}} \frac{|F_{n-k}(z)|^q d\mu}{a_n} = d\gamma.$$

Proof. Recall our notation $G_\rho = \{\psi(z) : \rho \leq |z| \leq 1\}$. It is clear from our observations above (specifically (3.2.4)) that all weak limits in question are measures on ∂G and that F_n has no zeros in G_ρ for all sufficiently large n. Now, let f be a continuous function on G_ρ. We have

$$\begin{aligned}
\int_{G_\rho} f(z)\frac{|F_n(z)|^q}{a_n}d\mu(z) - \int_{G_\rho} f(z)\frac{|F_{n-k}(z)|^q}{a_n}d\mu(z) = \\
&= \int_{G_\rho} f(z)\left(1 - \frac{|F_{n-k}(z)|^q}{|F_n(z)|^q}\right)\frac{|F_n(z)|^q}{a_n}d\mu(z) \\
&= \int_{G_\rho} f(z)\left(1 - \frac{|\phi(z)|^{q(n-k)} + O(\rho_0^n)}{|\phi(z)|^{qn} + O(\rho_0^n)}\right)\frac{|F_n(z)|^q}{a_n}d\mu(z) \\
&\to \int_{\partial G} f(z)\left(1 - |\phi(z)|^{-qk}\right)d\gamma(z) \\
&= 0
\end{aligned}$$

since $|\phi(z)| = 1$ when $z \in \partial G$. ☐

Our next lemma will identify some ideal choices for the sequence $\{a_n\}_{n\in\mathbb{N}}$ of Lemma 3.2.4.

Lemma 3.2.5. *Let* γ *be a probability measure on the unit interval* $[0, 1]$ *and let* $c_t = c_t(\gamma)$ *be defined as in (3.2.1). The following are equivalent:*

1. $1 \in \text{supp}(\gamma)$,
2. $\lim_{t\to\infty} c_t^{1/t} = 1$,
3. $\lim_{n\to\infty} c_{q(n+1)}c_{qn}^{-1} = 1$.

Proof. It is obvious that (1)⇒(2) and (3)⇒(1) so we need only prove that (2)⇒(3). To this end, we have

$$\frac{c_{qn+q}}{c_{qn}} = 1 + \frac{\int_0^1 r^{qn}(r^q - 1)d\gamma(r)}{\int_0^1 r^{qn}d\gamma(r)}.$$

If $\lim_{t\to\infty} c_t^{1/t} = 1$, then the measures $\frac{r^{qn}d\gamma(r)}{\int_0^1 r^{qn}d\gamma(r)}$ converge weakly to the point mass at 1 as $n \to \infty$, which implies the desired conclusion. ☐

Now we can prove the following lemma, which will be of critical importance in our proof of Theorem 3.2.2.

Lemma 3.2.6. *Let κ be a measure on $\overline{G}$ and γ a measure on $\partial\mathbb{D}$ and let $\mathcal{N} \subseteq \mathbb{N}$ be a subsequence such that*

$$\underset{\substack{n\to\infty \\ n\in\mathcal{N}}}{\text{w-lim}} \frac{|F_n(z)|^q}{a_n} d\kappa = d(\psi_*\gamma),$$

where $\{a_n\}_{n\in\mathbb{N}}$ is as in Lemma 3.2.4. Then

$$\limsup_{\substack{n\to\infty \\ n\in\mathcal{N}}} \frac{\|P_n(\kappa)\|_\kappa^q}{a_n} \leq \exp\left(\int_0^{2\pi} \log(\gamma'(\theta))\frac{d\theta}{2\pi}\right).$$

Proof. By the extremal property, we have $\|P_n(\kappa)\|_\kappa^q \leq \|F_{n-k}(z)P_k(\psi_*\gamma)\|_\kappa^q$. By Lemma 3.2.4, we can write

$$\int_{\overline{G}} \frac{|P_k(z;\psi_*\gamma)|^q|F_{n-k}(z)|^q}{a_n} d\kappa(z) \to \int_{\partial G} |P_k(z;\psi_*\gamma)|^q d(\psi_*\gamma)$$

as $n \to \infty$ through $\mathcal{N}$. Therefore

$$\limsup_{\substack{n\to\infty \\ n\in\mathcal{N}}} a_n^{-1}\|P_n(\kappa)\|_\kappa^q \leq \|P_k(\psi_*\gamma)\|_{\psi_*\gamma}^q$$

for every $k > 0$. Since k here is arbitrary, we can take the infimum over all k, which is no larger than the limit as k tends to infinity. The result now follows from Theorem 2.3.1. □

The following calculation will be useful also.

Proposition 3.2.7. *If $x \notin G$ and $r \in [\rho, 1]$, then*

$$\int_0^{2\pi} \log|\psi(re^{i\theta}) - x|^q \frac{d\theta}{2\pi} = \log|\phi(x)|^q.$$

Proof. First, consider the case when $x \notin \overline{G}$. It is clear that $\text{cap}(\overline{\mathcal{G}}_r) = r$. Define $\psi_r(z) = \psi(rz)$ on $\{z : |z| > \tilde{\rho}r^{-1}\}$. Then we calculate

$$\begin{aligned}
\log|\phi(x)|^q &= \int_0^{2\pi} \log|e^{i\theta} - \phi(x)r^{-1}|^q \frac{d\theta}{2\pi} + q\log(r) \\
&= -qU^{\omega_{\overline{\mathbb{D}}}}\left(\frac{\phi(x)}{r}\right) + q\log(r) \\
&= qg_{\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}}(\phi(x)r^{-1}, \infty) + q\log(r) \\
&= qg_{\overline{\mathbb{C}}\setminus\overline{\mathcal{G}}_r}(x, \infty) + q\log(\text{cap}(\overline{\mathcal{G}}_r)) \\
&= \int_{\Gamma_r} \log|y - x|^q d\omega_{\overline{\mathcal{G}}_r}(y) \\
&= \int_0^{2\pi} \log|\psi_r(e^{i\theta}) - x|^q \frac{d\theta}{2\pi}.
\end{aligned}$$

The first line follows from Example 0.5.7 in [50]. The second line is just the definition of the logarithmic potential. The third line then follows from (1.1.1) above and the fact that $\overline{\mathbb{D}}$ has logarithmic capacity 1. The fourth line then follows from the conformal invariance of the Green's function (see Section 1.1). The fifth line follows as the third did from the first. Finally, the last line follows from the definition of equilibrium measure as given in Theorem 3.1 in [71].

The case $x \in \partial G$ follows by dominated convergence as in Example 0.5.7 in [50]. □

Now we are ready to prove Theorem 3.2.2.

Proof of Theorem 3.2.2. For now, let us assume that $\ell = m = 0$ in our definition of μ. We will appeal to Lemma 3.2.6 to prove our upper bound. As mentioned in the proof of Lemma 3.2.4, all weak limits of the measures considered there are supported on ∂G, so it suffices to consider functions that are continuous in a neighborhood of ∂G. For any $k \in \mathbb{N}_0$, we have

$$\lim_{n\to\infty}\int_{G_\rho}\frac{\phi(z)^k|F_n(z)|^q}{c_{qn}(\tau)}d\mu(z) = \lim_{n\to\infty}\frac{\int_\rho^1\int_0^{2\pi} r^{k+qn}e^{ik\theta}h(re^{i\theta})d\nu(\theta)d\tau(r)}{c_{qn}(\tau)} + \lim_{n\to\infty}\frac{\int_{\overline{\mathbb{D}}} z^k|z^n|^q d\sigma_2}{c_{qn}(\tau)}$$
$$= \int_0^{2\pi} e^{ik\theta}h(e^{i\theta})d\nu(\theta) = \int_{\partial G}\phi(z)^k d(\psi_*(h\nu)).$$

It follows that the measures $\frac{|F_n(z)|^2}{c_{qn}(\tau)}d\mu$ converge weakly to $d(\psi_*(h\nu))$ as measures on $\overline{G}$. The upper bound in this case now follows from Lemma 3.2.6.

If we add finitely many pure points outside G, we get the desired upper bound by placing a single zero at each z_i and ζ_i. More precisely, if we define the polynomials $y_\infty(z)$ and $\Upsilon_\infty(z)$ by

$$y_\infty(z) = \prod_{j=1}^m (z - z_j) \qquad , \qquad \Upsilon_\infty(z) = \prod_{j=1}^m (z - \zeta_j), \tag{3.2.5}$$

then we have

$$\|P_n(\mu)\|_\mu^q \le \|y_\infty\Upsilon_\infty P_{n-m-\ell}(|y_\infty(z)\Upsilon_\infty(z)|^q\mu)\|_\mu^q = \|P_{n-m-\ell}(|y_\infty(z)\Upsilon_\infty(z)|^q\mu)\|^q_{|y_\infty(z)\Upsilon_\infty(z)|^q\mu}$$

and then proceed as in the case when $\ell = m = 0$ and apply Proposition 3.2.7.

For the lower bound we will use an argument inspired by the proof of Theorem 2.4.2. Recall that Theorem 1.1.3 (and the remark following it) implies that for each z_i, we can choose a sequence $\{w_{i,n}\}_{n\in\mathbb{N}}$ so that $P_n(w_{i,n};\mu) = 0$ and $\lim_{n\to\infty} w_{i,n} = z_i$ (we will establish later that such a sequence has a unique tail, but we do not need this now). Define

$$y_n(z) = \prod_{j=1}^m (z - w_{j,n}) \tag{3.2.6}$$

(so that $y_n(z) \to y_\infty(z)$ pointwise). We now can calculate

$$\|P_n(z;\mu)\|_\mu^q \geq \int_\rho^1 \int_0^{2\pi} \left|\frac{P_n(\psi(re^{i\theta}))}{y_n(\psi(re^{i\theta}))}\right|^q \prod_{j=1}^m |\psi(re^{i\theta}) - w_{j,n}|^q h(re^{i\theta}) d\nu_{ac}(\theta) d\tau(r) \tag{3.2.7}$$

For $|z| > 1$ and $r \in [\rho, 1]$, define the functions

$$S_{r,n}(z) = \exp\left(-\frac{1}{2q\pi}\int_0^{2\pi} \log\left(\prod_{j=1}^m |\psi(re^{i\theta}) - w_{j,n}|^q h(re^{i\theta})\nu'(\theta)\right) \frac{e^{i\theta}+z}{e^{i\theta}-z} d\theta\right).$$

By our discussion in Section 1.4, we can rewrite (3.2.7) as

$$\|P_n(z;\mu)\|_\mu^q \geq \int_\rho^1 \int_0^{2\pi} \left|\frac{P_n(\psi(re^{i\theta}))}{e^{i(n-m)\theta} y_n(\psi(re^{i\theta}))}\right|^q |S_{r,n}(e^{i\theta})|^q \frac{d\theta}{2\pi} d\tau(r)$$

(notice that we arbitrarily added a factor of $e^{-i(n-m)\theta}$ to the integrand, which is acceptable since it is inside the absolute value bars). For each fixed r, we invoke the subharmonicity of the integrand (or equation (2.3.2)) to obtain

$$\|P_n(z;\mu)\|_\mu^q \geq \int_\rho^1 r^{qn-qm} S_{r,n}(\infty)^q d\tau(r). \tag{3.2.8}$$

Since $w_{j,n}$ converges to z_j as $n \to \infty$ for each j (by construction), we find that

$$\liminf_{n\to\infty} \frac{\|P_n(z;\mu)\|_\mu^q}{c_{qn}(\tau)} \geq \exp\left(\int_0^{2\pi} \log\left(h(e^{i\theta})\nu'(\theta)\right) \frac{d\theta}{2\pi}\right) \prod_{j=1}^m |\phi(z_j)|^q$$

by Proposition 3.2.7. This is the desired lower bound.

□

The proof of Theorem 3.2.2 produces several interesting corollaries. The first of these shows that certain parts of the measure μ contribute only negligibly to the norm of the extremal polynomial. The following corollary is reminiscent of Theorem 2.4.1(vii) in [56].

Corollary 3.2.8. *If μ is as in Theorem 3.2.2 with ν a Szegő measure on $\partial\mathbb{D}$, then*

$$\lim_{n\to\infty}\left(\int_{\overline{\mathbb{D}}} |p_n(\psi(re^{i\theta});\mu)|^q\, h(re^{i\theta}) d\nu_{sing}(\theta) d\tau(r) + \int_{\overline{G}} |p_n(z;\mu)|^q d\sigma_1(z) + \right.$$
$$\left. + \int_{\overline{\mathbb{D}}} |p_n(\psi(re^{i\theta});\mu)|^q\, d\sigma_2(re^{i\theta}) + \sum_{j=1}^m b_j |p_n(z_j;\mu)|^q + \sum_{j=1}^\ell \beta_j |p_n(\zeta_j;\mu)|^q\right) = 0.$$

Proof. Let us write $\mu = \mu^0 + \mu^1$ where $\mu^0 = \psi_*(h(\nu \otimes \tau)) + \sum_{j=1}^m d_j \delta_{z_j}$. Then

$$\frac{\|P_n(\mu)\|_\mu^q}{c_{qn}(\tau)} = \frac{\|P_n(\mu)\|_{\mu^0}^q}{c_{qn}(\tau)} + \frac{\|P_n(\mu)\|_{\mu^1}^q}{c_{qn}(\tau)}. \tag{3.2.9}$$

The proof of Theorem 3.2.2 shows that the left-hand side of (3.2.9) and the first term on the right-hand side of (3.2.9) both converge to the right-hand side of (3.2.2). This shows that everything except μ^0 contributes only negligibly to the norm of $p_n(z;\mu)$. To show that the pure points outside $\overline{G}$ contribute only negligibly to the norm, we keep our definition of $w_{1,n}$ from the proof of Theorem 3.2.2 and we write $\mu^0 = \mu_1^0 + w_1\delta_{z_1}$. We can now calculate

$$\begin{aligned}
1 &\geq \frac{\int_{\mathbb{C}} \left|\frac{P_n(z;\mu)}{z-w_{1,n}}\right|^q |z-w_{1,n}|^q d\mu_1^0}{\|P_n(\mu)\|_\mu^q} + d_1|p_n(z_1)|^q \\
&\geq \frac{\|P_{n-1}(|z-w_{1,n}|^q\mu_1^0)\|_{|z-w_{1,n}|^q\mu_1^0}^q}{\|P_n(\mu)\|_\mu^q} + d_1|p_n(z_1)|^q \\
&= \frac{c_{qn}(\tau)\exp\left(\int_0^{2\pi}\log\left(h(e^{i\theta})\nu'(\theta)|\psi(e^{i\theta})-w_{1,n}|^q\right)\frac{d\theta}{2\pi}\right)\prod_{j=2}^m|\phi(z_j)|^q}{c_{qn}(\tau)\exp\left(\int_0^{2\pi}\log\left(h(e^{i\theta})\nu'(\theta)\right)\frac{d\theta}{2\pi}\right)\prod_{j=1}^m|\phi(z_j)|^q} + \\
&\qquad + d_1|p_n(z_1)|^q + o(1) \\
&= 1 + o(1) + d_1|p_n(z_1)|^q,
\end{aligned}$$

which implies the desired conclusion for z_1. An identical proof works for each z_j for $j = 2, 3, \ldots, m$. □

Remark. As a consequence of Corollary 3.2.8, we see that if $K \subseteq G$ is compact and μ is of the form considered in Theorem 3.2.2 with ν a Szegő measure on $\partial\mathbb{D}$, then

$$\int_K |p_n(z;\mu,q)|^q d\mu(z) \to 0$$

as $n \to \infty$.

An additional consequence of Theorem 3.2.2 is the following corollary, which is a refinement of Theorem 1.1.3.

Corollary 3.2.9. *Let μ be as in Theorem 3.2.2 with ν a Szegő measure on $\partial\mathbb{D}$. There exists a $\delta > 0$ and $N \in \mathbb{N}$ such that for all $n \geq N$, the polynomial $P_n(\mu)$ has a single zero in $\{u : |u - z_i| < \delta\}$ for each $i \leq m$. If we denote this zero by $w_{i,n}$, then there is an $a > 0$ so that $|w_{i,n} - z_i| \leq e^{-an}$ for all large n.*

Proof. Theorem 1.1.3 (and the remark following it) establishes the existence of at least one zero of $P_n(\mu)$ in $\{u : |u - z_i| < \delta\}$ for all i and all large n. Now, fix $\epsilon > 0$ (but small) and let $\{w_1, \ldots, w_{t(n)}\}$

denote the collection of zeros of $P_n(\mu)$ outside $\Gamma_{1+\epsilon}$.

Define for $|z|>1$ the functions

$$S_{r,n}(z)=\exp\left(-\frac{1}{2q\pi}\int_0^{2\pi}\log\left(\prod_{j=1}^{t(n)}|\psi(re^{i\theta})-w_j|^q h(re^{i\theta})\nu'(\theta)\right)\frac{e^{i\theta}+z}{e^{i\theta}-z}d\theta\right).$$

As in the proof of Theorem 3.2.2, we calculate

$$\begin{aligned}\frac{\|P_n(z;\mu)\|_\mu^q}{c_{qn}(\tau)}&\geq\frac{\int_\rho^1 r^{qn-qt(n)}S_{r,n}(\infty)^q d\tau(r)}{c_{qn}(\tau)}\geq\frac{\int_\rho^1 r^{qn-qt(n)}S_{r,n}(\infty)^q d\tau(r)}{c_{qn-qt(n)}(\tau)}\\&=\frac{\int_\rho^1 r^{qn-qt(n)}\exp\left(\frac{1}{2\pi}\int_0^{2\pi}\log\left(h(re^{i\theta})\nu'(\theta)\right)d\theta\right)d\tau(r)}{c_{qn-qt(n)}(\tau)}\prod_{j=1}^{t(n)}|\phi(w_j)|^q,\end{aligned}\tag{3.2.10}$$

where we used Proposition 3.2.7. From this expression, it follows that $n-t(n)$ tends to infinity as $n\to\infty$, for if it did not, then since $|\phi(w_j)|>1+\epsilon$ for every $j\leq t(n)$, we would have $\|P_n(z;\mu)\|_\mu^{1/n}>1+\epsilon$ for all n in some subsequence $\mathcal{N}\subseteq\mathbb{N}$, which violates the fact that $\text{cap}(\overline{G})=1$ and μ is regular (see Theorem III.3.1 in [50]).

Since $n-t(n)\to\infty$, the first factor in (3.2.10) converges to $\exp\left(\frac{1}{2\pi}\int_0^{2\pi}\log\left(h(e^{i\theta})\nu'(\theta)\right)d\theta\right)$ as $n\to\infty$ while the left-hand side has limit given by the right-hand side of (3.2.2). If for each $i\in\{1,\ldots,m\}$ we pick a sequence $\{w_{i,n}\}_{n\in\mathbb{N}}$ as in the proof of Theorem 3.2.2, then the corresponding factor in the product (3.2.10) converges to $|\phi(z_i)|^q$ as $n\to\infty$. Therefore, it must be that

$$\limsup_{n\to\infty}\prod_{j=1,w_j\neq w_{i,n}}^{t(n)}|\phi(w_j)|^q\leq 1.$$

However, each factor in this product is larger than $(1+\epsilon)^q$. We conclude that $t(n)=m$ for all sufficiently large n. This implies $P_n(\mu)$ has a single zero near each z_j for $j=1,\ldots,m$ when n is sufficiently large.

The proof of the exponential attraction now proceeds exactly as in the last portion of the proof of Theorem 8.1.11 in [56] and we provide it here for completeness. For each $i\in\{1,\ldots,m\}$, let $\{w_{i,n}\}_{n\in\mathbb{N}}$ be as above. We have just shown that this sequence has a unique tail. Suppose for contradiction that there exists $i\in\{1,\ldots,m\}$ so that for all $a>0$ it is true that $|w_{i,n}-z_i|>e^{-an}$ for infinitely many values of $n\in\mathbb{N}$. Then we can find a subsequence $n(j)$ so that $|w_{i,n(j)}-z_i|^{1/n(j)}\to 1$ as $j\to\infty$. The above proof shows that all accumulation points of the zeros of the extremal polynomials are in $\text{Pch}(\mu)$. Therefore, Corollary 1.1.5 in [65] (whose proof only depends on the extremal property and not orthogonality) implies that

$$\liminf_{j\to\infty}|p_{n(j)}(z;\mu,q)|^{1/n(j)}\geq\exp\left(-U^{\omega_{\overline{G}}}(z)\right)\tag{3.2.11}$$

for all z in some punctured neighborhood of z_i, and in particular, uniformly on some small circle $\mathcal{C}_i$ centered at z_i. It is then clear that we retain the uniformity on $\mathcal{C}_i$ in (3.2.11) if we replace $p_{n(j)}(z;\mu,q)$ by $p_{n(j)}(z;\mu,q)/(z-w_{i,n(j)})$. Our above analysis implies that for all sufficiently large $n\in\mathbb{N}$, the polynomial $p_{n(j)}(z;\mu,q)/(z-w_{i,n(j)})$ has no zeros in a neighborhood of z_i, and all other zeros tend to $\overline{G}\cup\{z_\ell\}_{\ell\neq i}$. Therefore, $p_{n(j)}(z;\mu,q)/(z-w_{i,n(j)})$ is free of zeros inside $\mathcal{C}_i$, and so the minimum principle implies

$$\liminf_{j\to\infty}\left|\frac{p_{n(j)}(z;\mu,q)}{z-w_{i,n(j)}}\right|^{1/n(j)}\geq\exp\left(-U^{\omega_{\overline{G}}}(z)\right) \tag{3.2.12}$$

uniformly for all z in some open set containing z_i. By our definition of the subsequence $\{n(j)\}_{j\in\mathbb{N}}$, we conclude that

$$\liminf_{j\to\infty}|p_{n(j)}(z_i;\mu,q)|^{1/n(j)}\geq\exp\left(-U^{\omega_{\overline{G}}}(z_i)\right)>1,$$

which is clearly impossible by the normalization of p_n. This contradiction gives us the desired conclusion. □

Remark 1. We can actually quantify the parameters δ and a in the statement of Corollary 3.2.9. The proof of Corollary 3.2.9 shows that we may take δ to be any positive real number so that $\{w:|w-z_i|\leq\delta\}\cap\text{supp}(\mu)=\{z_i\}$ for each $i\leq m$. Furthermore, equation (3.2.12) shows that

$$\limsup_{n\to\infty}|z_i-w_{i,n}|^{1/n}\leq\exp\left(U^{\omega_{\overline{G}}}(z_i)\right)$$

for all $i\leq m$.

Remark 2. Corollary 3.2.9 tells us that the polynomial $P_n(\mu,q)$ has a single zero extremely close to z_i for each $i\in\{1,\ldots,m\}$ and the remaining $n-m$ zeros are placed so as to minimize the $L^q(\mu)$ norm with respect to a varying – yet converging – weight. It would be interesting to look at the measure μ on $\mathbb{D}\cup\{z_1,\ldots,z_m\}$ given by $d\mu=dA(z)+\sum_{j=1}^m\delta_{z_j}$ (where $dA(z)$ refers to area measure on the unit disk) and see if the results from [36] continue to hold in this case, where the polynomial weight would be $y_\infty(z)$ (see (3.2.5) above).

The upper bound in the proof of Theorem 3.2.2 came from Lemma 3.2.6, which applies to arbitrary finite measures (not just product measures). We can also state the lower bound used in the proof of Theorem 3.2.2 in a more general form.

Proposition 3.2.10. *Let $\tilde{\mu}$ be a measure on $\overline{G}$ so that $\tilde{\mu}\geq\mu$ and μ is the push-forward (via ψ) of the measure $w(re^{i\theta})\frac{d\theta}{2\pi}d\tau(r)$ where $1\in\text{supp}(\tau)$ and $w\in L^1(\frac{d\theta}{2\pi}\otimes d\tau(r))$. Then*

$$\|P_n(\tilde{\mu})\|_{\tilde{\mu}}^q\geq\int_0^1 r^{nq}\exp\left(\int_0^{2\pi}\log(w(re^{i\theta}))\frac{d\theta}{2\pi}\right)d\tau(r).$$

Remark. The statement here is very general because we do not insist on any continuity of w.

Proof. By the inequality of the measures and the extremal property, we have

$$\|P_n(\tilde{\mu})\|_{\tilde{\mu}}^q \geq \|P_n(\tilde{\mu})\|_{\mu}^q \geq \|P_n(\mu)\|_{\mu}^q,$$

so it suffices to put the desired bound on $\|P_n(\mu,q)\|_{L^q(\mu)}^q$. Let $X \subseteq [0,1]$ be the collection of all r so that $w(re^{i\theta})\frac{d\theta}{2\pi}$ is a Szegő measure on $\partial\mathbb{D}$. The proposition is trivial unless $\tau(X) > 0$. Therefore, we assume this is the case, and for $r \in X$ we define

$$S_r(z) = \exp\left(-\frac{1}{2q\pi}\int_0^{2\pi} \log\left(w(re^{i\theta})\right)\frac{e^{i\theta}+z}{e^{i\theta}-z}d\theta\right), \qquad |z| > 1$$

and write

$$\|P_n(\mu)\|_{\mu}^q \geq \int_X r^{nq}|S_r(\infty)|^q d\tau(r)$$

as in (3.2.8). This is the desired lower bound. □

We conclude this section with an example showing how one can apply Lemma 3.2.6 to a region without analytic boundary.

Example. Let G be the region $\left\{z : |z^3 - 1| < 1\right\}$ and assume $q > 1$. Notice that G has capacity 1 since $\phi(z)^3 = z^3 - 1$ (see Example 3.8 in [35]). Therefore, when n is a multiple of 3 we have $F_{3m}(z) = (z^3-1)^m$.

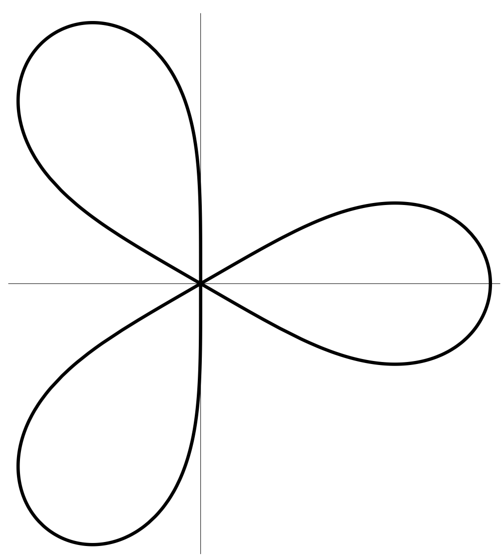

Figure 3.1: The region G of the example.

Let τ be a probability measure on $(0,1)$ with $1 \in \operatorname{supp}(\tau)$. The region G can be decomposed into level sets Ξ_r where

$$\Xi_r = \left\{z : |z^3 - 1| = r\right\}$$

and r runs from 0 to 1. Let ν_r be arc-length measure on each component of Ξ_r and let $h(z)$ be a function that is continuous on $\overline{G}$ and is invariant under rotations by $\frac{2\pi}{3}$ so that $\phi_*(h\nu_1)$ has $\mathbb{Z}_3$ symmetry as a measure on $\partial\mathbb{D}$ as in Example 1.6.14 in [56]. Let us define μ by

$$\int_{\overline{G}} f(z)d\mu(z) = \int_0^1 \int_{\Xi_r} f(z)h(z)d\nu_r(z)d\tau(r).$$

Consider the measure $h\nu_1$ on ∂G. If $m \in \mathbb{N}$ is fixed, then by the extremal property we have that for any choice of complex numbers $a_0, \ldots, a_{m-1}$ and $a_m = 1$

$$\|P_{3n}(h\nu_1, q)\|^q_{L^q(h\nu_1)} \leq \left\|\sum_{j=0}^m a_j F_{3(j+n-m)}(z)\right\|^q_{L^q(h\nu_1)} = \left\|\sum_{j=0}^m a_j \phi(z)^{3(j+n-m)}\right\|^q_{L^q(h\nu_1)}.$$

Following the proof of the upper bound in Theorem 7.1 in [17] we get

$$\|P_{3n}(h\nu_1, q)\|^q_{L^q(h\nu_1)} \leq \int_0^{2\pi} \left|1 + \sum_{k=1}^m \gamma_k e^{3ki\theta}\right|^q d\phi_*(h\nu_1) \tag{3.2.13}$$

for any $m \leq n$ and any choice of constants $\gamma_1, \ldots, \gamma_m$. The assumed $\mathbb{Z}_3$ symmetry of the measure implies that $P_{3m}(z; \phi_*(h\nu_1), q) = R_m(z^3)$ for some monic polynomial R_m of degree m (this follows from the uniqueness of the extremal polynomial in the case $q > 1$; see Example 1.6.14 in [56]). Therefore, we can choose $\gamma_1, \ldots, \gamma_m$ appropriately so that the right-hand side of (3.2.13) is equal to $\|P_{3m}(\phi_*(h\nu_1), q)\|^q_{\phi_*(h\nu_1)}$. The reasoning of Lemma 3.2.6 then implies

$$\limsup_{n\to\infty} \|P_{3n}(h\nu_1, q)\|^q_{L^q(h\nu_1)} \leq \exp\left(\int_0^{2\pi} \log\left(\phi_*(h\nu_1)'(\theta)\right) \frac{d\theta}{2\pi}\right).$$

Now, as in Lemma 3.2.6, we calculate (for $f \in C(\overline{G})$)

$$\begin{aligned} c_{qm}(\tau)^{-1} \int_{\overline{G}} f(z)|F_{3m}(z)|^q d\mu(z) &= c_{qm}(\tau)^{-1} \int_0^1 \left(\int_{\Xi_r} f(z)h(z)d\nu_r(z)\right) r^{qm} d\tau(r) \\ &\to \int_{\Xi_1} f(z)h(z)d\nu_1(z) \end{aligned}$$

as $m \to \infty$. Therefore, the measures $\frac{|F_{3m}|^q}{c_{qm}(\tau)} d\mu$ converge weakly to $hd\nu_1$ and the reasoning of Lemma 3.2.6 implies

$$\limsup_{n\to\infty} \frac{\|P_{3n}(z; \mu, q)\|^q_{L^q(\mu)}}{c_{qn}(\tau)} \leq \exp\left(\int_0^{2\pi} \log(\phi_*(h\nu_1)'(\theta)) \frac{d\theta}{2\pi}\right).$$

□

In the next section, we explore more detailed asymptotic properties of the polynomials $P_n(z;\mu,q)$ and $p_n(z;\mu,q)$.

3.2.2 Szegő Asymptotics for Extremal Polynomials

The main idea of Theorem 3.2.2 is that the asymptotic behavior of the extremal polynomial norms is comparable to the behavior of the L^q norms $\{\|\phi(z)^n\|_{L^q(\mu)}\}_{n\in\mathbb{N}}$. It should not be surprising then that in some cases we can make a stronger statement about the extremal polynomials' resemblance to $\phi(z)^n$ in certain regions of the plane, which is the essence of what we call Szegő asymptotics. Theorems 3.2.11 and 3.2.12 will provide us with detailed information about the behavior of $P_n(z;\mu,q)$ outside of $\overline{G}$ and near the boundary of G. In the next section we will see how $P_n(z;\mu,2)$ behaves inside G (see Corollary 3.2.18).

In the previous section we established that the polynomial $P_n(\mu,q)$ has a single zero near each pure point of μ outside of $\overline{G}$ (for large n) and asymptotically, all other zeros tend to $\overline{G}$. If we label the zero of $P_n(\mu,q)$ near z_j as $w_{j,n,q}$, let us define

$$y_n(z;q) = \prod_{j=1}^{m}(z - w_{j,n,q}),$$

which can be uniquely defined for all sufficiently large n by Corollary 3.2.9. It will be convenient for us to define

$$\Lambda_n(z;\mu,q) = \frac{P_n(z;\mu,q)}{y_n(z;q)} \tag{3.2.14}$$

for all sufficiently large n. We also recall the definition

$$S_{r,n}(z;q) = \exp\left(-\frac{1}{2q\pi}\int_0^{2\pi}\log\left(h(re^{i\theta})\nu'(\theta)|y_n(\psi(re^{i\theta});q)|^q\right)\frac{e^{i\theta}+z}{e^{i\theta}-z}d\theta\right) \tag{3.2.15}$$

for $r\in[\rho,1]$ and $z\in\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}$. We begin by considering the behavior of $P_n(z;\mu,q)$ when $z\notin\overline{G}$ and any $q>0$. We will prove a result reminiscent of the convergence result in Theorem 2.4.1(iv) in [56] and the corollary in [37].

Theorem 3.2.11. *Let $S_{r,n}(z;q)$ be defined as in (3.2.15). If μ is a measure as in Theorem 3.2.2 with ν a Szegő measure on $\partial\mathbb{D}$ and $q>0$, then*

$$\frac{\Lambda_n(\psi(z);\mu,q)S_{1,\infty}(z;q)}{z^{n-m}S_{1,\infty}(\infty;q)} \to 1$$

as $n\to\infty$ uniformly on compact subsets of $\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}$.

Proof. Let $q>0$ be fixed throughout this proof and denote $S_{r,n}(z;q)$ by $S_{r,n}(z)$ and $\Lambda_n(z;\mu,q)$ by

$\Lambda_n(z;\mu)$.

We showed in Section 3.2.1 (or see equation (2.3.2)) that if $r \in [\rho, 1]$, then

$$r^{q(n-m)} S_{r,n}(\infty)^q \leq \int_0^{2\pi} |\Lambda_n(\psi(re^{i\theta});\mu) S_{r,n}(e^{i\theta})|^q \frac{d\theta}{2\pi} \tag{3.2.16}$$

for all $r \in [\rho, 1]$. Let us fix some $t < 1$. If we divide both sides of (3.2.16) by $c_{qn}(\tau)$ and then integrate in the variable r from t to 1 with respect to τ, then both sides converge to $S_{1,\infty}(\infty)^q$ as $n \to \infty$ (by Theorem 3.2.2). Therefore, (3.2.16) is optimal in that we cannot multiply the right-hand side by a factor smaller than 1 and have the inequality remain valid for all $r \in [t, 1]$ when n is sufficiently large. It follows that for any $\epsilon > 0$, there exists a sequence $\{r_n\}_{n=1}^{\infty}$ converging to 1 from below as $n \to \infty$ so that

$$r_n^{q(n-m)} S_{r_n,n}(\infty)^q \geq (1-\epsilon) \int_0^{2\pi} |\Lambda_n(\psi(r_n e^{i\theta});\mu) S_{r_n,n}(e^{i\theta})|^q \frac{d\theta}{2\pi}. \tag{3.2.17}$$

By a standard argument, we can choose our sequence $\{r_n\}_{n=1}^{\infty}$ converging to 1 from below so that (3.2.17) remains true for some sequence ϵ_n tending monotonically to 0 from above. Let $a_n := \|\Lambda_n(\psi(r_n e^{i\theta});\mu) S_{r_n,n}(e^{i\theta})\|_{L^q(\frac{d\theta}{2\pi})}$. By using (3.2.16) and (3.2.17) we see that

$$1 - \epsilon_n \leq \left| \lim_{z\to\infty} \frac{\Lambda_n(\psi(r_n z);\mu) S_{r_n,n}(z)}{a_n z^{n-m}} \right|^q \leq 1. \tag{3.2.18}$$

Let

$$f_n(z) = \frac{\Lambda_n(\psi(r_n z);\mu) S_{r_n,n}(z)}{a_n z^{n-m}}.$$

Clearly, $\|f_n\|_{H^q(\overline{\mathbb{C}}\setminus\overline{\mathbb{D}})} = 1$ for all n and equation (3.2.18) shows that $\lim_{n\to\infty} f_n(\infty) = 1$. Therefore, the Keldysh Lemma (see Section 1.5) implies $\{f_n\}_{n\in\mathbb{N}}$ converges to 1 in $H^q(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}})$ as $n \to \infty$ so f_n converges to 1 uniformly on compact subsets of $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$. Equations (3.2.16) and (3.2.17) show that $a_n = (1 + \delta_n) r_n^{(n-m)} S_{r_n,n}(\infty)$ with $\delta_n \to 0$ as $n \to \infty$. Therefore, if $|z| > 1$, we have

$$\frac{\Lambda_n(\psi(r_n z)) S_{r_n,n}(z)}{(r_n z)^{(n-m)} S_{r_n,n}(\infty)} \to 1 \tag{3.2.19}$$

and the convergence is uniform on compact subsets of $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$. By plugging in $z = w/r_n$ into (3.2.19) and using the uniformity of convergence on compact subsets we recover

$$\frac{\Lambda_n(\psi(w)) S_{r_n,n}(w/r_n)}{w^{n-m} S_{r_n,n}(\infty)} \to 1, \tag{3.2.20}$$

and the convergence is uniform on compact subsets. Equation (3.2.20) is sufficient to guarantee that the sequence $\{\Lambda_n(\psi(w)) w^{-(n-m)}\}_{n\in\mathbb{N}}$ is a normal family on $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$. Dominated Convergence easily implies that

$$\frac{S_{r_n,n}(w/r_n)}{S_{r_n,n}(\infty)} \to \frac{S_{1,\infty}(w)}{S_{1,\infty}(\infty)}, \qquad |w| > 1$$

pointwise as $n \to \infty$, so we must have the desired uniform convergence on compact subsets. □

Remark. Notice that Theorem 3.2.11 tells us that the leading-order Szegő asymptotic behavior of the polynomials $P_n(\mu, q)$ is independent of τ.

Now that we have some information about the behavior of $P_n(z;\mu,q)$ outside $\overline{G}$, we will consider what happens close to the boundary of G. Our next result is motivated in part by Theorem 9.3.1 in [57]. As in Theorem 3.2.11, we will consider all $q > 0$.

Theorem 3.2.12. *If μ is as in Theorem 3.2.2, $q > 0$, and ν is a Szegő measure on $\partial\mathbb{D}$, then*

$$\operatorname*{w-lim}_{n\to\infty} |p_n(z;\mu,q)|^q d\mu(z) = d\omega_{\overline{G}}(z)$$

as measures on $\mathbb{C}$.

Proof. Let $q > 0$ be fixed and denote by p_n the polynomial $p_n(z;\mu,q)$. Corollary 3.2.8 and the remark following it imply that any weak limit of the measures $\{|p_n|^q d\mu\}_{n\in\mathbb{N}}$ must be a measure on ∂G and that we may, without loss of generality, assume that $\sigma_1 = \sigma_2 = 0$, $\ell = 0$, and ν is purely absolutely continuous with respect to Lebesgue measure. Let us recall the definition of $S_{r,n}(z) = S_{r,n}(z;q)$ from (3.2.15) for $r \in [\rho, 1]$ and $z \in \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$.

We showed in Theorem 3.2.2 that

$$\int_\rho^1 \int_0^{2\pi} \frac{\left|e^{-i(n-m)\theta}\Lambda_n(\psi(re^{i\theta}))\right|^q |S_{r,n}(e^{i\theta})|^q}{c_{qn}(\tau) S_{1,n}(\infty)^q} \frac{d\theta}{2\pi} d\tau(r) \to 1 \tag{3.2.21}$$

as $n \to \infty$.

For fixed $n \in \mathbb{N}$ and $r \in [\rho, 1]$, let $\{u_1, \ldots, u_{\eta_n(r)}\}$ be the zeros of $\Lambda_n(\psi(rz))$ ($\eta_n \in \mathbb{N}_0$) lying outside of $\overline{\mathbb{D}}$, each listed a number of times equal to its multiplicity as a zero. We may then define the Blaschke product

$$B_{r,n}(z) = \prod_{j=1}^{\eta_n(r)} \frac{z - u_j}{z\bar{u}_j - 1} \cdot \frac{\bar{u}_j}{|u_j|}.$$

With this notation, we may define $J_{r,n}(z)$ so that

$$z^{-(n-m)}\Lambda_n(\psi(rz))S_{r,n}(z) = B_{r,n}(z)J_{r,n}(z). \tag{3.2.22}$$

From (3.2.22), we know that $J_{r,n}(z)$ is analytic and nonvanishing in $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$, so we may write

$$J_{r,n}(z)^{q/2} = J_{r,n}(\infty)^{q/2} + g_{r,n}(z) = \left(\frac{r^{n-m} S_{r,n}(\infty)}{B_{r,n}(\infty)}\right)^{q/2} + g_{r,n}(z), \tag{3.2.23}$$

where $g_{r,n}(z)$ is in $H^2(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}})$ and is orthogonal to the constant functions in $H^2(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}})$ (that is, $g_{r,n}(z) \in H^2_0(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}})$ in the notation of [12]). Notice that

$$|e^{-i(n-m)\theta}\Lambda_n(\psi(re^{i\theta}))S_{r,n}(e^{i\theta})|^q = |J_{r,n}(e^{i\theta})^{q/2}|^2.$$

If we plug (3.2.23) into (3.2.21), we get

$$\int_\rho^1 \frac{r^{qn-qm}S_{r,n}(\infty)^q}{c_{qn}(\tau)S_{1,n}(\infty)^q B_{r,n}(\infty)^q} + \frac{\|g_{r,n}\|^2_{H^2}}{c_{qn}(\tau)S_{1,n}(\infty)^q} d\tau(r) \to 1 \tag{3.2.24}$$

as $n \to \infty$. However, $B_{r,n}(\infty)^{-q} > 1$ and the second term is always nonnegative, so we conclude that the first term in (3.2.24) has integral tending to 1 as $n \to \infty$, and hence

$$\int_\rho^1 \frac{\|g_{r,n}\|^2_{H^2}}{c_{qn}(\tau)S_{1,n}(\infty)^q} d\tau(r) \to 0 \tag{3.2.25}$$

as $n \to \infty$.

Now fix $k \in \mathbb{N}$. We have

$$\int_{G_\rho} \phi(z)^k |p_n(z)|^q d\mu(z) \tag{3.2.26}$$

$$\begin{aligned}
&= \int_\rho^1 \int_0^{2\pi} \frac{r^k e^{ik\theta}|e^{-i(n-m)\theta}\Lambda_n(\psi(re^{i\theta}))|^q |S_{r,n}(e^{i\theta})|^q}{c_{qn}(\tau)S_{1,n}(\infty)^q} \frac{d\theta}{2\pi} d\tau(r) + o(1) \\
&= \int_\rho^1 \int_0^{2\pi} \frac{r^k e^{ik\theta}|r^{q(n-m)/2}S_{r,n}(\infty)^{q/2}B_{r,n}(\infty)^{-q/2} + g_{r,n}(e^{i\theta})|^2}{c_{qn}(\tau)S_{1,n}(\infty)^q} \frac{d\theta}{2\pi} d\tau(r) + o(1) \\
&= \int_\rho^1 \int_0^{2\pi} \frac{r^{k+q(n-m)}e^{ik\theta}S_{r,n}(\infty)^q}{c_{qn}(\tau)S_{1,n}(\infty)^q B_{r,n}(\infty)^q} \frac{d\theta}{2\pi} d\tau(r) + \int_\rho^1 \int_0^{2\pi} \frac{r^k e^{ik\theta}|g_{r,n}(e^{i\theta})|^2}{c_{qn}(\tau)S_{1,n}(\infty)^q} \frac{d\theta}{2\pi} d\tau(r) \qquad (3.2.27) \\
&\quad + \int_\rho^1 \int_0^{2\pi} \frac{r^{k+q(n-m)/2}e^{ik\theta}S_{r,n}(\infty)^{q/2} \cdot 2\,\mathrm{Re}[g_{r,n}(e^{i\theta})]}{c_{qn}(\tau)S_{1,n}(\infty)^q B_{r,n}(\infty)^{q/2}} \frac{d\theta}{2\pi} d\tau(r) + o(1)
\end{aligned}$$

as $n \to \infty$. If we send n to infinity, the first term in (3.2.27) converges to 0 since $k \in \mathbb{N}$. The second term in (3.2.27) can be bounded from above in absolute value by

$$\int_\rho^1 \frac{\|g_{r,n}\|^2_{H^2}}{c_{qn}(\tau)S_{1,n}(\infty)^q} d\tau(r), \tag{3.2.28}$$

which tends to 0 by (3.2.25). By applying the Schwarz inequality, the third term in (3.2.27) can be bounded from above in absolute value by

$$\left(\int_\rho^1 \frac{r^{2k+q(n-m)}S_{r,n}(\infty)^q}{c_{qn}(\tau)S_{1,n}(\infty)^q B_{r,n}(\infty)^q} d\tau(r)\right)^{1/2} \left(\int_\rho^1 \frac{4|\int_0^{2\pi} e^{ik\theta}\,\mathrm{Re}[g_{r,n}(e^{i\theta})]\frac{d\theta}{2\pi}|^2}{c_{qn}(\tau)S_{1,n}(\infty)^q} d\tau(r)\right)^{1/2}.$$

The first factor tends to 1 as $n \to \infty$ (as in (3.2.24)). After applying Jensen's inequality to the second factor, we can bound it from above by twice the square root of (3.2.28). Therefore, the integral (3.2.26) tends to 0 as $n \to \infty$.

We conclude that if γ is a weak limit point of the measures $\{|p_n(\mu)|^q d\mu\}_{n\in\mathbb{N}}$, then for every $k \in \mathbb{N}$ we have

$$\int_{\partial G} \phi(z)^k d\gamma = 0.$$

This implies that γ is induced (via ψ) by a measure κ on $\partial\mathbb{D}$ with no nontrivial moments, i.e., $d\kappa = \frac{d\theta}{2\pi}$ and it follows that γ is the equilibrium measure for $\overline{G}$ (see Theorem 3.1 in [71]). □

Theorem 3.2.12 yields the following corollary, which can be interpreted in terms of the Christoffel functions discussed in Section 2.5 (see (2.5.5)).

Corollary 3.2.13. *Under the hypotheses of Theorem 3.2.12, we have*

$$\operatorname*{w-lim}_{n\to\infty} \frac{K_n(z,z)}{n+1} d\mu(z) = d\omega_{\overline{G}}$$

as measures on $\mathbb{C}$.

Remark. Since μ is regular, one can use a polynomial approximation argument, Corollary 3.2.8, and Theorem 2.9.5 to arrive at a different proof of Corollary 3.2.13. Theorem 3.2.12 is of course much stronger.

In the next section, we will consider the behavior of the Christoffel functions on $\overline{G}$.

3.2.3 Christoffel Functions

In this section we will turn our attention to the Christoffel function $\lambda_\infty(z;\mu,q) = \lim_{n\to\infty} \lambda_n(z;\mu,q)$ where $\lambda_n(z;\mu,q)$ is defined as defined in (2.5.6). The behavior of $\lambda_\infty(z;\mu,q)$ is particularly easy to describe when $z \in \partial G$.

Proposition 3.2.14. *If μ is any measure with support in $\overline{G}$ and G has analytic boundary, then $\lambda_\infty(x;\mu,q) = \mu(\{x\})$ for all $x \in \partial G$ and all $q > 0$.*

Remark. For Proposition 3.2.14, we do not need to assume $\operatorname{cap}(\overline{G}) = 1$.

Proof. Fix $x \in \partial G$. It is obvious that $\lambda_n(x;\mu,q) \geq \mu(\{x\})$ for every $n \in \mathbb{N}$, so it remains to show the reverse inequality holds in the limit. Since ∂G is analytic, we can define a conformal map $\varphi : G \to \mathbb{D}$ satisfying $\varphi(x) = 1$. By a well-known argument, this map φ has an analytic continuation to some open set $U \supseteq \overline{G}$. Define

$$f_n(z) := 3^{-n}(\varphi(z)+2)^n, \qquad z \in U$$

so that $f_n(x) = 1 = \|f_n\|_{L^\infty(\overline{G})}$. By Theorem 2.5.7 in [51] there exists a sequence of polynomials $\{W_n\}_{n\in\mathbb{N}}$ so that $\|W_n - f_n\|_{L^\infty(\overline{G})} < n^{-1}$ (we do not assume W_n has degree n). It follows that for each $n \in \mathbb{N}$ there is a constant $a_n = 1 + o(1)$ (as $n \to \infty$) so that $a_n W_n(x) = 1$. Then (with $E_n = W_n - f_n$)

$$\lambda_{\deg(W_n)}(x; \mu, q) \leq \int_{\overline{G}} |a_n W_n(z)|^q d\mu(z) = (1 + o(1)) \int_{\overline{G}} |f_n(z) + E_n|^q d\mu(z) \to \mu(\{x\})$$

as $n \to \infty$ by Dominated Convergence. □

Remark. For results producing more precise asymptotics of $\lambda_n(z; \mu, 2)$ for $z \in \partial G$ under stronger hypotheses on μ, see [27, 71].

Now let us focus on $x \in G$. For measures supported on the unit circle, recall the discussion in Section 2.5, where we said that if ν is a Szegő measure, then $\lambda_\infty(z; \nu, q) > 0$ for all $z \in \mathbb{D}$ and $q \in (0, \infty)$. We will prove an analog for the kinds of measures considered in Theorem 3.2.2. Before we can do this, we need to define some auxiliary notation. For x interior to Γ_1, define

$$\xi(x) = \frac{1}{2} \left(1 + \inf\{r : x \in \mathcal{G}_r \, , \, r \geq \rho\}\right).$$

For each $r \in [\xi(x), 1]$, let $\varphi_{r,x}$ be the canonical conformal map at x from $\mathcal{G}_r$ to $\mathbb{D}$ (see Section 1.2). Denote the inverse to $\varphi_{r,x}$ by $\chi_{r,x}$. The following lemma will be useful:

Lemma 3.2.15. *With the above notation, it holds that $\varphi_{r,x}$ converges to $\varphi_{1,x}$ uniformly on some open set containing $\overline{G}$ as $r \to 1$ and there is an $s \in (\xi(x), 1)$ and positive constants λ_1 and λ_2 such that*

$$\lambda_1 < |\varphi'_{r,x}(z)| < \lambda_2$$

for all $r \in [s, 1]$ and $z \in \overline{G}$.

Remark. The proof of the lemma will actually show that when r is sufficiently close to 1, $\varphi_{r,x}$ is defined on all of $\overline{G}$ so the statement of the lemma makes sense.

Proof. By the Carathéodory Convergence Theorem (see Section 1.2), the maps $\varphi_{r,x}$ converge to $\varphi_{1,x}$ uniformly on compact subsets of G as $r \to 1^-$. Since G has analytic boundary, a simple argument shows that each $\varphi_{r,x}$ can be univalently continued outside of $\overline{G}$ when r is sufficiently close to 1 and in fact all such $\varphi_{r,x}$ have a common domain of holomorphy containing $\overline{G}$. A normal families argument then implies $\varphi_{r,x}$ converges to $\varphi_{1,x}$ uniformly on some open set containing $\overline{G}$ as $r \to 1$. We can then use the Cauchy integral formula to conclude that $\varphi'_{r,x}$ converges to $\varphi'_{1,x}$ on a smaller open set containing $\overline{G}$. This means that when r is sufficiently close to 1, we have $\|\varphi'_{r,x}\|_{L^\infty(\Gamma_1)} \leq 2\|\varphi'_{1,x}\|_{L^\infty(\Gamma_1)}$. The same arguments can be applied to $\{\chi_{r,x}\}_{r \in [\xi(x),1]}$, which proves the claim. □

As a final preparatory step, we will need the following lemma, which is a slight refinement of Lemma 1.1 in [19] (see also Lemma 1.5.1).

Lemma 3.2.16. *If $q \in (0,\infty)$ and $w \in \mathcal{G}_r$, then there is a constant β_w so that for every $f \in H^q(\mathcal{G}_r)$,*

$$|f(w)|^q \leq \beta_w \int_{\Gamma_r} |f(z)|^q d|z|.$$

Furthermore, the constant β_w may be taken uniform for all r sufficiently close to 1 (but perhaps depending on w).

Proof. The inequality follows from Lemma 1.1 in [19] and the equivalence of the spaces $E^q(\mathcal{G}_r)$ and $H^q(\mathcal{G}_r)$ (see Chapter 10 in [12]), so we need only focus on the uniformity. If $q \geq 1$, then this is a simple consequence of Jensen's inequality and the fact that H^1 functions are the Cauchy integral of their boundary values (see Theorem 10.4 in [12]), so we need only focus on the case $0 < q < 1$. To this end, let g be the function harmonic in $\mathcal{G}_r$ satisfying $g(\psi(re^{i\theta})) = |f(\psi(re^{i\theta}))|^q$ almost everywhere on Γ_r. Let $\omega_{r,w}$ be the harmonic measure for the region $\mathcal{G}_r$ and the point w. Then by the subharmonicity of f, we have

$$|f(w)|^q \leq g(w) = \int_{\Gamma_r} g(z) d\omega_{r,w}(z) \leq \left\| \frac{d\omega_{r,w}}{d|z|} \right\|_{L^\infty(\Gamma_r)} \int_{\Gamma_r} g(z) d|z| \leq \left\| \varphi_{r,w}' \right\|_{L^\infty(\Gamma_r)} \int_{\Gamma_r} |f(z)|^q d|z|,$$

where we used Theorem 1.2.3. We can now apply Lemma 3.2.15 with $x = w$ to provide uniformity in the constant β_w. ☐

Now we are ready to prove the main theorem of this section.

Theorem 3.2.17. *If μ and G are as in Theorem 3.2.2 with ν a Szegő measure on $\partial\mathbb{D}$, then $\lambda_\infty(z;\mu,q) > 0$ for all $z \in G$ and $q \in (0,\infty)$.*

Proof. Since h is bounded from below and $\lambda_n(z;\mu,q)$ increases as we increase μ, we may assume that $\mu = \nu_{ac} \otimes \tau$. In the region G_ρ we may write (for f continuous)

$$\int_{G_\rho} f(z) d\mu(z) = \int_\rho^1 \int_{\Gamma_t} f(z)\tilde{w}(z) d|z|, d\tau(t) \tag{3.2.29}$$

where $\tilde{w}$ is a weight on G_ρ. In fact, we can write explicitly

$$\tilde{w}(z) = \frac{1}{2\pi} \cdot \nu' \left(\frac{\phi(z)}{|\phi(z)|} \right) \frac{|\phi'(z)|}{|\phi(z)|} \tag{3.2.30}$$

(we identify $\nu'(e^{i\theta})$ and $\nu'(\theta)$). As in (1.4.7), define $\Delta_{r,q}(z)$ by

$$\Delta_{r,q}(z) = \exp\left(\frac{1}{2q\pi i} \oint_{\Gamma_r} \log\left(\tilde{w}(\zeta)\right) \frac{1 + \overline{\varphi_r(\zeta)}\varphi_r(z)}{\varphi_r(\zeta) - \varphi_r(z)} \varphi_r'(\zeta) d\zeta \right) \tag{3.2.31}$$

for each $r \in [\rho, 1]$ so that $\Delta_{r,q} \in H^q(\mathcal{G}_r)$ and $|\Delta_{r,q}(\zeta))|^q = \tilde{w}(\zeta)$ for almost every $\zeta \in \Gamma_r$ ((3.2.30) implies the integral in (3.2.31) converges).

Now fix $y \in G$ and let $Q(z)$ be any polynomial so that $Q(y) = 1$ (we make no assumptions about the degree of Q). Let $s \in (\rho, 1)$ be so that y is interior to Γ_s and so the constant β_y of Lemma 3.2.16 may be chosen independently of $t \in [s, 1]$. We calculate

$$\|Q\|^q_{L^q(\mu)} \geq \int_s^1 \int_{\Gamma_t} |Q(z)\Delta_{t,q}(z)|^q \, d|z| d\tau(t) \geq \beta_y^{-1} \int_s^1 |\Delta_{t,q}(y)|^q d\tau(t) \tag{3.2.32}$$

by Lemma 3.2.16. The function $\Delta_{t,q}(y)$ is expressed as an exponential so the fact that ν is a Szegő measure on $\partial\mathbb{D}$ implies $\Delta_{t,q}(y)$ is never equal to 0 for any t. Therefore, $|\Delta_{t,q}(y)|^q$ is not the zero function and so the integral on the far right of (3.2.32) is not equal to zero. We have therefore obtained a lower bound for the far left-hand side of (3.2.32) that is independent of the degree of Q. Taking the infimum over all such Q proves the theorem. □

If we combine Theorem 2.5.2 and Theorem 3.2.17 for the case $q = 2$ we arrive at a proof of the following corollary:

Corollary 3.2.18. *If $\tilde{\mu} \geq \mu$ and μ is as in Theorem 3.2.2 with ν a Szegő measure on $\partial\mathbb{D}$, then*

$$\sum_{n=0}^{\infty} |p_n(z; \tilde{\mu}, 2)|^2 < \infty$$

for all $z \in G$.

Now that we have some understanding of $\lambda_\infty(x; \mu, q)$ for all $x \in G$ when μ is of the form considered in Theorem 3.2.2, we want to try to calculate it exactly. Our next result will show that one can reduce the problem to considering only measures on $G = \mathbb{D}$ and only the point $x = 0$. Indeed, take any $x_0 \in G$ and let $\varphi : G \to \mathbb{D}$ be the canonical conformal map at x_0 (see Section 1.2). By the injectivity of φ on $\overline{G}$ (we used Carathéodory's Theorem here; see Section 1.2), we can push any measure μ on $\overline{G}$ forward via φ to get a measure $\varphi_*\mu$ on $\overline{\mathbb{D}}$ as in Section 1.2. With this notation, we can prove the following result:

Proposition 3.2.19. *With x_0, μ and φ as above, we have $\lambda_\infty(x_0; \mu, q) = \lambda_\infty(0; \varphi_*\mu, q)$ for all $q \in (0, \infty)$.*

Remark. We do not exclude the possibility that $G = \mathbb{D}$ and φ is an automorphism of the disk.

Remark. If $\tau \neq \delta_1$, the resulting measure $\varphi_*\mu$ may not be of the form considered in Theorem 3.2.2.

Proof. Fix $q \in (0, \infty)$. Given $\epsilon > 0$, let T be a polynomial so that $\|T\|^q_{L^q(\varphi_*\mu)} < \lambda_\infty(0; \varphi_*\mu, q) + \epsilon$ and $T(0) = 1$. Then $\tilde{Q} := T \circ \varphi$ is a function on $\overline{G}$ satisfying $\|\tilde{Q}\|^q_{L^q(\mu)} = \|T\|^q_{L^q(\varphi_*\mu)}$ and $\tilde{Q}(x_0) = 1$.

Now let Q be a polynomial satisfying $\||Q|^q - |\tilde{Q}|^q\|_{L^\infty(\overline{G})} < \epsilon$ and $Q(x_0) = 1$ (such a Q exists by the same reasoning as in the proof of Proposition 3.2.14). It follows at once that $\lambda_\infty(x_0; \mu, q) \leq \lambda_\infty(0; \varphi_*\mu, q) + 2\epsilon$ and one direction of the inequality follows by sending $\epsilon \to 0$. The reverse inequality follows by an argument symmetric to the one just given. □

Remark. If we set $\tau = \delta_1$, Proposition 3.2.19 can be used to provide a new proof of Proposition 2.2.2 in [56] and a new proof of Theorem 2.5.4 in [56].

Proposition 3.2.19 allows us to calculate $\lambda_\infty(x; \mu, q)$ by considering only measures on $\overline{\mathbb{D}}$ and only the point 0. If μ happens to be supported on ∂G, then $\varphi_*\mu$ is supported on $\partial\mathbb{D}$ so that $\lambda_\infty(0; \varphi_*\mu, q)$ is in fact independent of q (see Theorem 2.5.4 in [56]) so the same must be true of $\lambda_\infty(x; \mu, q)$. However, the following example shows that the value of $\lambda_\infty(0; \mu, q)$ is in general not as easily calculated when $\text{supp}(\mu) \not\subseteq \partial G$.

Example. Let us consider the special case of Corollary 3.2.18 where $G = \mathbb{D}$, $h = 1$, and $z = 0$. Let us further assume τ and ν are both probability measures. Fix any $N \in \mathbb{N}$ and let $Q_N(z)$ be a polynomial of degree at most N satisfying $Q_N(0) = 1$. Then for any $r < 1$ we have

$$\int_0^{2\pi} |Q_N(re^{i\theta})|^2 d\nu(\theta) \geq \lambda_N(0; \nu, 2)$$

because $Q_N(rz)$ is still a polynomial of degree N in z that is equal to 1 at 0. Integrating both sides in the variable r with respect to τ from 0 to 1, we obtain $\lambda_N(0; \mu, 2) \geq \lambda_N(0; \nu, 2)$. Sending $N \to \infty$ we obtain $\lambda_\infty(0; \mu, 2) \geq \lambda_\infty(0; \nu, 2) > 0$ (see equation (2.2.3) in [56]).

However, if $0 \in \text{supp}(\tau)$ then the reverse inequality is false unless $d\nu = \frac{d\theta}{2\pi}$ (we still assume ν is a Szegő measure on $\partial\mathbb{D}$), i.e., it is true that $\lambda_\infty(0; \mu, 2) > \lambda_\infty(0; \nu, 2)$. To see this, recall Proposition 2.16.2 in [63], which tells us that $Q_{n,z}(w) := K_n(w, z; \mu)K_n(z, z; \mu)^{-1}$ satisfies $Q_{n,z}(z) = 1$ and $\|Q_{n,z}\|_\mu^2 = \lambda_n(z; \mu, 2)$. If $G = \mathbb{D}$ and $q = 2$, then by appealing to Theorem 2.5.4 in [56] and our above arguments, one can conclude that $\{Q_{n,0}(w)\}_{n\in\mathbb{N}}$ is uniformly bounded on $\{u : |u| \leq r_1\}$ for any $r_1 < 1$. By Montel's Theorem this is a normal family so we may take $n \to \infty$ through some subsequence $\mathcal{N} \subseteq \mathbb{N}$ so that $\{Q_{n,0}(w)\}_{n\in\mathcal{N}}$ converges uniformly to a function $Q_{\infty,0}(w)$, which is analytic in $\{z : |z| < r_1\}$ and $Q_{\infty,0}(0) = 1$. By continuity and the fact that if $d\nu \neq \frac{d\theta}{2\pi}$ then $\lambda_\infty(0; \nu, 2) < 1$, it must be that

$$\int_0^{2\pi} |Q_{\infty,0}(re^{i\theta})|^2 d\nu(\theta) > \frac{1 + \lambda_\infty(0; \nu, 2)}{2}$$

for all r sufficiently small (say $r < r_0$). By Dominated Convergence, the same must be true for all

$Q_{n,0}(z)$ for n sufficiently large and $n \in \mathcal{N}$. We conclude that for sufficiently large $n \in \mathcal{N}$, we have

$$\begin{aligned}
\lambda_n(0;\mu,2) = \|Q_{n,0}(z)\|_\mu^2 &= \int_0^{r_0}\int_0^{2\pi} |Q_{n,0}(re^{i\theta})|^2 d\nu(\theta) d\tau(r) + \int_{r_0}^1 \int_0^{2\pi} |Q_{n,0}(re^{i\theta})|^2 d\nu(\theta) d\tau(r) \\
&> \frac{1+\lambda_\infty(0;\nu,2)}{2}\tau([0,r_0]) + \lambda_\infty(0;\nu,2)\tau((r_0,1]) \\
&= \frac{1-\lambda_\infty(0;\nu,2)}{2}\tau([0,r_0]) + \lambda_\infty(0;\nu,2).
\end{aligned}$$

Since $\lambda_n(0;\mu,2)$ is decreasing in n, $\tau([0,r_0]) > 0$ and $\lambda_\infty(0;\nu,2) < 1$, the desired conclusion follows.

□

3.3 New Results on Ratio and Relative Asymptotics

This section will be devoted to the new results on ratio and relative asymptotics for orthonormal polynomials presented in [53]. Recall that regularity is a necessary and sufficient condition for the existence of root asymptotics (see Theorem 3.1.1 in [65]). Although ratio asymptotics need not hold for regular measures (see the example in Section 3.3.2.1), we can say something about the asymptotic behavior of p_n/p_{n-1} when μ is regular. We will prove that if the measure μ is regular on $\overline{\mathbb{D}}$, then the ratio $zp_{n-1}(z;\mu)/p_n(z;\mu)$ converges to 1 uniformly on compact subsets of $\{z : |z| > 1\}$ as n tends to infinity through a subsequence of asymptotic density 1. This can be thought of as a unit disk analog of Theorem 2.7.1. We also show that if the measure μ is regular on the lemniscate $E_m = \{z : |z^m - 1| \le 1\}$, then the ratio $(z^m-1)p_{n-m}(z;\mu)/p_n(z;\mu)$ converges to 1 uniformly on compact subsets of $\overline{\mathbb{C}} \setminus \mathrm{ch}(\mu)$ as n tends to infinity through a subsequence of asymptotic density 1. The advantage of working on a lemniscate is that there is a monic polynomial whose L^∞-norm is 1 on the lemniscate, while for more general supports this is not necessarily the case. If this is not the case, then we cannot obtain convergence of p_n/p_{n-1} by our methods, but we can describe the behavior of p_n/p_{n-k_n} for a possibly unbounded sequence $\{k_n\}$ (see Section 3.3.3 for details).

The strength of our results is rooted in the weak assumptions we place on the measure μ in order to arrive at a ratio asymptotic result. Many ratio asymptotic results arise as a consequence of Szegő asymptotics (see Theorems 2.4.2 and 3.2.2), which is a stronger conclusion than ratio asymptotics and hence requires stronger hypotheses on the measure. In [48], Saff places bounds on $|p_n/p_{n-1}|$ for arbitrary compactly supported measures using methods similar to ours. The results in [11] concern orthogonal polynomials on the real line and are in the same spirit as our Theorem 3.3.2, though Theorem 3.3.2 is much more general.

In addition to studying ratio asymptotics for consecutive orthonormal polynomials, we will also consider ratios of orthonormal polynomials corresponding to different but related measures. In particular, we will study the *Uvarov Transform* and the *Christoffel Transform*, both of which were introduced in Section 2.8. In both cases, we show that the asymptotic behavior of the orthonormal polynomials outside of $\mathrm{ch}(\mu)$ is unchanged provided the pure point (for the Uvarov Transform) or the zero of the monomial (for the Christoffel Transform) satisfies the condition

$$\lim_{n\to\infty} \frac{|p_n(x;\mu)|^2}{\sum_{j=0}^{n-1}|p_j(x;\mu)|^2} = 0 \tag{3.3.1}$$

(see (3.3.15) in Section 3.3.4). The condition of regularity is equivalent to

$$\limsup_{n\to\infty} |p_n(z;\mu)|^{1/n} = 1$$

for every z in the outer boundary of the support of μ, except perhaps on a set of capacity 0 (see Theorem 3.1.1 in [65]). Therefore, condition (3.3.1) – when applied to a point x in the outer boundary of $\mathrm{supp}(\mu)$ – qualitatively tells us that x is not a point at which $|p_n(x;\mu)|$ grows exponentially (see also Theorem 1.3 in [5]).

After proving the key fact about ratios of polynomials in the next section, we will apply it in the case when the orthonormal polynomials correspond to a measure supported on the closed unit disk in Section 3.3.2. We also include a brief digression where we show that if μ is any regular measure on $\overline{\mathbb{D}}$, then there is a subsequence $\mathcal{N} \subseteq \mathbb{N}$ of asymptotic density 1 so that the probability measures $\{|p_n(z;\mu)|^2 d\mu(z)\}_{n\geq 0}$ converge weakly to normalized arc-length measure on the unit circle as $n \to \infty$ through $\mathcal{N}$, which is a unit disk analog of Theorem 2.9.4. In Section 3.3.3, we will apply the results of Section 3.3.1 to orthonormal polynomials whose measure of orthogonality has a more general support. The main theorem in Section 3.3.3 is analogous to results in Section 3.3.2, but requires a small sacrifice in the strength of the conclusion due to the added generality. Finally, in Section 3.3.4, we will apply the results of Section 3.3.1 to prove our stability results concerning orthonormal polynomials when the measure is perturbed in specific ways. The foundation for all that follows is Theorem 3.3.2 in the next section.

Throughout this section, $p_n(z;\mu)$ will be used to denote $p_n(z;\mu,2)$ and $P_n(z;\mu)$ will denote $P_n(z;\mu,2)$.

3.3.1 The Key Fact

We begin by recalling a formula originally due to Saff (see [48]) that will be essential to the proof of our key result. Let Q be a polynomial of degree at most n and let $z \in \mathbb{C}$ be fixed. The orthogonality relation implies

$$\int_{\mathbb{C}} \frac{Q(z)p_n(w;\mu) - Q(w)p_n(z;\mu)}{z-w}\overline{p_n(w;\mu)}\,d\mu(w) = 0.$$

By rearranging this equality, we recover

$$\frac{Q(z)}{p_n(z;\mu)} = \frac{\int_{\mathbb{C}} \frac{\overline{p_n(w;\mu)}Q(w)}{z-w} d\mu(w)}{\int_{\mathbb{C}} \frac{|p_n(w;\mu)|^2}{z-w} d\mu(w)}, \tag{3.3.2}$$

whenever both denominators in (3.3.2) are nonzero.

At first glance, the utility of (3.3.2) is not obvious, though some applications are discussed in [48]. We will apply this formula in cases where $Q(z) = Q_n(z)$ is also n-dependent. The key to our calculations will be to write the numerator on the right-hand side of (3.3.2) as a perturbation of the denominator and – under suitable hypotheses – show that the perturbation tends to zero as $n \to \infty$ while the denominator does not. In order to do so, we will require that the left-hand side of (3.3.2) tends to 1 at infinity as $n \to \infty$ and also that $Q_n(z)$ has $L^2(\mu)$-norm tending to 1 as $n \to \infty$. Obviously $Q_n(z) = p_n(z;\mu)$ satisfies these conditions, but we will show that for any sequence $\{Q_n\}_{n\in\mathbb{N}}$ of polynomials satisfying these conditions, the left-hand side of (3.3.2) tends to 1 as $n \to \infty$ when $|z|$ is sufficiently large.

Before we prove our main result of this section, we make the following simple calculation:

Lemma 3.3.1. *Let μ be a measure with compact support $\mathrm{supp}(\mu) \subseteq \mathbb{C}$ and suppose z satisfies $z \notin \mathrm{ch}(\mu)$. There is a constant $A_z > 0$ so that*

$$\left| \int_{\mathbb{C}} \frac{|p_n(w;\mu)|^2}{z-w} d\mu(w) \right| > A_z$$

for every $n \in \mathbb{N}$. Furthermore, the constant A_z may be bounded uniformly from below on any compact subset of $\mathbb{C} \setminus \mathrm{ch}(\mu)$.

Proof. Since $z \notin \mathrm{ch}(\mu)$, we can find a $\theta \in \mathbb{R}$ so that $\min_{w\in \mathrm{ch}(\mu)} \mathrm{Re}[e^{i\theta}z - e^{i\theta}w] = \mathrm{dist}(z, \mathrm{ch}(\mu))$. Therefore

$$\mathrm{Re}\left[e^{-i\theta} \int_{\mathbb{C}} \frac{|p_n(w;\mu)|^2}{z-w} d\mu(w) \right] = \int_{\mathbb{C}} \frac{|p_n(w;\mu)|^2}{|z-w|^2} \mathrm{Re}[e^{-i\theta}\bar{z} - e^{-i\theta}\overline{w}] d\mu(w) \geq \frac{\mathrm{dist}(z, \mathrm{ch}(\mu))}{\sup_{w\in\mathrm{supp}(\mu)} |z-w|^2}$$

as desired. The uniformity in A_z is now obvious. □

Remark. Lemma 3.3.1 assures us that the denominator on the right-hand side in (3.3.2) is nonzero for appropriate z.

Now we can prove the critical result. The following theorem will be used heavily for the applications in the remainder of this section. It tells us that the behavior of the orthonormal polynomials when $|z|$ is large is determined only by its normalization and its leading coefficient.

Theorem 3.3.2. *Suppose μ is a (finite) and compactly supported measure on $\mathbb{C}$. For each $n \in \mathbb{N}$, choose a polynomial Q_n of degree exactly n and leading coefficient τ_n so that the following properties* *hold:*

1. $\lim_{n\to\infty} \|Q_n\|_{L^2(\mu)} = 1$,
2. $\lim_{n\to\infty} \tau_n/\kappa_n = 1$.

Then

$$\lim_{n\to\infty} \frac{Q_n(z)}{p_n(z;\mu)} = 1 \tag{3.3.3}$$

for all $z \notin \mathrm{ch}(\mu)$*. Furthermore, the convergence is uniform on compact subsets of* $\overline{\mathbb{C}} \setminus \mathrm{ch}(\mu)$*.*

Remark 1. The proof will show that we get the same conclusion if we only define Q_n for n in some subsequence and then send $n \to \infty$ through that subsequence.

Remark 2. By evaluating $Q_n(\cdot)/p_n(\cdot;\mu)$ at infinity, we see that the second condition in Theorem 3.3.2 is necessary for (3.3.3) to hold. Additionally, since $\kappa_n^{-1} = \|P_n(\cdot;\mu)\|_{L^2(\mu)}$, the second condition and the extremal property imply

$$\liminf_{n\to\infty} \|Q_n\|_{L^2(\mu)} \geq \liminf_{n\to\infty} \tau_n \|P_n(\cdot;\mu)\|_{L^2(\mu)} = 1,$$

so the first condition of Theorem 3.3.2 is really a statement about the $\limsup$.

Remark 3. We will show by means of an example in Section 3.3.4.2 that we cannot extend the conclusion of Theorem 3.3.2 to include the boundary of $\mathrm{Pch}(\mu)$. However, we will be able to say something about what happens at points z that are outside $\mathrm{Pch}(\mu)$, but inside the convex hull of the support of μ (see the end of Section 3.3.2).

Proof. Fix $z \notin \mathrm{ch}(\mu)$. By (3.3.2), we have

$$\begin{aligned}\frac{Q_n(z)}{p_n(z;\mu)} &= \frac{\int_{\mathbb{C}} \frac{\overline{p_n(w;\mu)}Q_n(w)}{z-w} d\mu(w)}{\int_{\mathbb{C}} \frac{|p_n(w;\mu)|^2}{z-w} d\mu(w)} \\ &= \frac{\int_{\mathbb{C}} \frac{|p_n(w;\mu)|^2}{z-w} d\mu(w) + \int_{\mathbb{C}} \frac{\overline{p_n(w;\mu)}(Q_n(w)-p_n(w;\mu))}{z-w} d\mu(w)}{\int_{\mathbb{C}} \frac{|p_n(w;\mu)|^2}{z-w} d\mu(w)}.\end{aligned} \tag{3.3.4}$$

By Lemma 3.3.1, the denominator and the matching term in the numerator in (3.3.4) stay away from 0, so we need only show the second term in the numerator goes to 0 as $n \to \infty$. For this, we apply the Schwarz inequality to see that

$$\left| \int_{\mathbb{C}} \frac{\overline{p_n(w;\mu)}(Q_n(w) - p_n(w;\mu))}{z-w} d\mu(w) \right|^2 \leq \frac{\|Q_n(w) - p_n(w;\mu)\|^2_{L^2(\mu)}}{\inf_{w\in \mathrm{ch}(\mu)} |z-w|^2}. \tag{3.3.5}$$

The norm can be expanded as

$$\|p_n(\cdot;\mu)\|^2_{L^2(\mu)} + \|Q_n\|^2_{L^2(\mu)} - 2\,\mathrm{Re}\left[\langle Q_n(w), p_n(w;\mu)\rangle_\mu\right].$$

Our first hypothesis on Q_n implies that the sum of the first two terms tends to 2 as $n \to \infty$. By the orthogonality relation, we may replace $Q_n(w)$ in the inner product by $\tau_n \|P_n(\cdot;\mu)\|_{L^2(\mu)} p_n(w;\mu)$. We now apply the second hypothesis on Q_n and arrive at (3.3.3).

To prove the statement concerning uniformity, notice that Lemma 3.3.1 proves that convergence holds uniformly on compact subsets of $\mathbb{C} \setminus \mathrm{ch}(\mu)$ so by the maximum modulus principle, we get uniformity on any closed set in $\overline{\mathbb{C}} \setminus \mathrm{ch}(\mu)$, even those that include infinity. $\square$

Before we turn our attention to applications of Theorem 3.3.2, we conclude this section by exploring the behavior of the ratio (3.3.2) when z is inside the convex hull of the support of μ but outside the support of μ. The calculations in the proof of Proposition 3.3.2 imply that the second term in the numerator on the right-hand side of (3.3.4) still tends to 0 in this case, so we can obtain the same conclusion as Theorem 3.3.2 (without the uniformity) if we can show that the denominator on the right-hand side of (3.3.4) stays away from zero, perhaps on some subsequence.

It is clear that if $z \notin \mathrm{supp}(\mu)$ then the sequence

$$\left\{\int_{\mathbb{C}} \frac{|p_n(w;\mu)|^2}{z-w}\,d\mu(w)\right\}_{n\geq 0}$$

is bounded uniformly on compact subsets of $\overline{\mathbb{C}} \setminus \mathrm{supp}(\mu)$, so Montel's Theorem implies that some subsequence converges uniformly on compact subsets to an analytic function $h(z)$. It is possible that the limiting function $h(z)$ vanishes at a point inside the convex hull of the support of the measure. For example, let μ be a measure supported on $[-2,-1] \cup [1,2]$ satisfying $\mu(A) = \mu(-A)$ for all measurable sets A. Since the measure is symmetric about zero, so are the orthonormal polynomials so we conclude

$$\int_{\mathrm{supp}(\mu)} \frac{|p_n(w;\mu)|^2}{w}\,d\mu(w) = 0,$$

i.e., the limiting function $h(z)$ satisfies $h(0) = 0$. However, this example tells us how we can look for the zeros of $h(z)$; the relevant fact being that in this example any weak limit of the measures $\{|p_n(z;\mu)|^2 d\mu(z)\}_{n\geq 0}$ is an even measure.

In a general setting, suppose σ is a weak limit point of the measures $\{|p_n(z;\mu)|^2 d\mu(z)\}_{n\geq 0}$ with corresponding subsequence $\mathcal{N}_\sigma$. Let $B_\sigma(z)$ denote the Borel Transform of the measure σ with domain given by $\mathbb{C} \setminus \mathrm{supp}(\mu)$. If $B_\sigma(z) \neq 0$, then

$$\liminf_{\substack{n\to\infty \\ n\in\mathcal{N}_\sigma}} \left|\int_{\mathbb{C}} \frac{|p_n(w;\mu)|^2}{z-w}\,d\mu(w)\right| > 0, \tag{3.3.6}$$

which is exactly what we need to carry out the proof of Theorem 3.3.2. We have therefore proven the following result:

Proposition 3.3.3. *Suppose a sequence of polynomials* $\{Q_n\}_{n\geq 0}$ *is defined so that the hypotheses of Proposition 3.3.2 are satisfied. Let* σ *be a weak limit point of the measures* $\{|p_n(z;\mu)|^2 d\mu(z)\}_{n\geq 0}$ *with corresponding subsequence* $\mathcal{N}_\sigma$*. The conclusion (3.3.3) holds for all* z *outside the zero set of* B_σ *as* n *tends to infinity through* $\mathcal{N}_\sigma$*.*

Example. Proposition 2.3 in [62] tells us that if μ is a regular measure and $\mathrm{cap}(\mathrm{supp}(\mu)) > 0$, then for any function f that is analytic in a neighborhood of $\mathrm{Pch}(\mu)$ we have

$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{j=0}^{n}\int_{\mathbb{C}} f(w)|p_j(w;\mu)|^2 d\mu(w) = \int_{\mathbb{C}} f(w)d\omega_\mu(w),$$

where ω_μ is the equilibrium measure for the support of μ (we used Theorem 3.6.1 in [65] here). Therefore, if

$$\int_{\mathbb{C}} \frac{1}{z-w} d\omega_\mu(w) \neq 0, \tag{3.3.7}$$

then we can find a subsequence $\mathcal{N}_z \subseteq \mathbb{N}$ of positive density such that

$$\inf_{n\in\mathcal{N}_z}\left\{\left|\int_{\mathbb{C}} \frac{|p_n(w;\mu)|^2}{z-w} d\mu(w)\right|\right\} > 0.$$

We conclude that if μ is regular, $z \notin \mathrm{Pch}(\mu)$, (3.3.7) holds, $\mathrm{cap}(\mathrm{supp}(\mu)) > 0$, and the conditions of Theorem 3.3.2 are satisfied, then we can establish convergence as in (3.3.3) along a subsequence of positive density.

The next several sections are devoted to applications of Theorem 3.3.2.

3.3.2 Application: Measures Supported on the Unit Disk

Let us recall Theorems 2.7.1 and 2.9.4, which assert ratio asymptotics and weak convergence of the measures $\{|p_n(z;\mu)|^2 d\mu(z)\}_{n\in\mathbb{N}}$ along a subsequence of asymptotic density 1. The proofs of both of these results depend heavily on the existence of a recurrence relation satisfied by the orthonormal polynomials. Our main goal in this section is to prove analogs of (2.7.1) and (2.9.1) for measures on the closed unit disk, a setting in which the orthonormal polynomials do not, in general, satisfy a finite term recurrence relation.

3.3.2.1 Ratio Asymptotics on the Disk

We begin with a result that is related to the conjecture in [48]. There, it is conjectured that for a measure μ of a certain form on $\overline{\mathbb{D}}$, one has $p_n(z;\mu)/(zp_{n-1}(z;\mu)) \to 1$ for all z in $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$. As a corollary, one then concludes that $\kappa_n\kappa_{n-1}^{-1} \to 1$ as $n \to \infty$ (recall κ_n is the leading coefficient of $p_n(\cdot;\mu)$). We will show that in fact the corollary implies the conjecture. More precisely, we will show that we need only verify the ratio asymptotic behavior at infinity to deduce it for all of $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$. This can be viewed as a unit disk analog of Theorem 1.7.4 in [56].

Theorem 3.3.4. *Let μ be a measure on $\overline{\mathbb{D}}$ and $\mathcal{N} \subseteq \mathbb{N}$ a subsequence so that*

$$\lim_{\substack{n\to\infty \\ n\in\mathcal{N}}} \kappa_n\kappa_{n-1}^{-1} = 1. \tag{3.3.8}$$

Then

$$\lim_{\substack{n\to\infty \\ n\in\mathcal{N}}} \frac{zp_{n-1}(z;\mu)}{p_n(z;\mu)} = 1$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$.

Remark. The condition (3.3.8) does *not* imply $\partial\mathbb{D} \subseteq \operatorname{supp}(\mu)$. Indeed there are examples of measures whose essential support is exactly two points and (3.3.8) holds with $\mathcal{N} = 2\mathbb{N}+1$ (see Example 1.6.14 in [56]).

Proof. We will apply Theorem 3.3.2 with $Q_n = zp_{n-1}(z;\mu)$. We need only verify the first condition in Theorem 3.3.2; the other condition is immediate from our hypotheses. The upper bound

$$\limsup_{n\to\infty, n\in\mathcal{N}} \|Q_n\|_{L^2(\mu)} \leq 1$$

is obvious while the lower bound follows from Remark 2 following Theorem 3.3.2. □

From Theorem 3.3.4, we deduce the following corollary, which is an analog of (2.7.1) for regular measures on the unit disk. By appealing to Proposition 2.2.3, it also tells us that if the conjecture in [48] is false, then it can only fail along a sparse subsequence.

Corollary 3.3.5. *Let μ be a regular measure on $\overline{\mathbb{D}}$. There exists a subsequence $\mathcal{N} \subseteq \mathbb{N}$ of asymptotic density* 1 *so that*

$$\lim_{\substack{n\to\infty \\ n\in\mathcal{N}}} \frac{zp_{n-1}(z;\mu)}{p_n(z;\mu)} = 1 \tag{3.3.9}$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$.

Remark 1. We will generalize this result in the example in Section 3.3.3.

Remark 2. In Proposition 3.4 in [48], Saff verifies boundedness of the ratio (3.3.9) under related hypotheses.

Proof. To apply Theorem 3.3.4, we need to verify that $\kappa_n\kappa_{n-1}^{-1} \to 1$ along some subsequence of asymptotic density 1. If we define $\gamma_n = \kappa_n\kappa_{n-1}^{-1}$, then each $\gamma_n \geq 1$. Regularity implies $\left(\prod_{j=1}^{n}\gamma_j\right)^{1/n} \to 1$ so γ_n tends to 1 along a subsequence of asymptotic density 1 as desired. □

Corollary 3.3.5 cannot be improved to give us convergence as n tends to infinity through all of $\mathbb{N}$ as the following example shows.

Example. Let μ be a probability measure supported on the unit circle and let $\{\alpha_n(\mu)\}_{n\geq 0}$ be the corresponding sequence of Verblunsky coefficients. The recurrence relation (2.1.2) easily implies

$$\kappa_n^{-2} = \prod_{j=0}^{n-1}(1-|\alpha_j|^2) \tag{3.3.10}$$

(see formula (1.5.12) in [56]). Let us define the measure μ by defining

$$\alpha_n(\mu) = \begin{cases} \frac{1}{2}, & \text{if } n = 2^j \text{ for some } j \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

One can easily check that this measure is regular. However

$$\left.\frac{zp_{2^j}(z;\mu)}{p_{2^j+1}(z;\mu)}\right|_{z=\infty} = \frac{\sqrt{3}}{2},$$

so we can only apply Corollary 3.3.5 to the subsequence $\mathcal{N} = \mathbb{N} \setminus \{2^j + 1 : j \in \mathbb{N}\}$.

Now let us turn our attention to measures supported on the unit circle $\partial\mathbb{D}$. In this case, since the polynomials do satisfy a recurrence relation we can actually strengthen the conclusion of Theorem 3.3.2.

Theorem 3.3.6. *Let μ be a probability measure supported on the unit circle and let Q_n be as in Theorem 3.3.2. Then*

$$\frac{Q_n(\cdot)}{p_n(\cdot;\mu)} \to 1$$

in $L^2(\partial\mathbb{D}, \frac{d\theta}{2\pi})$.

Proof. We use the Bernstein-Szegő Approximation Theorem (Theorem 1.7.8 in [56]) to calculate

$$\int_0^{2\pi}\left|\frac{Q_n(e^{i\theta})}{p_n(e^{i\theta};\mu)}\right|^2\frac{d\theta}{2\pi} = \int_{\partial\mathbb{D}}|Q_n(z)|^2\,d\mu(z) \to 1$$

by hypothesis. Theorem 3.3.2 establishes uniform convergence on compact subsets of $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ and we have just established convergence of norms. The result now follows from Theorem 1.5.3. $\square$

Example. Let μ be a probability measure supported on the unit circle and suppose μ is *normal* in the sense defined in [32]. This means that

$$\lim_{n\to\infty} \left\| \frac{p_n'(z;\mu)}{n} \right\|_{L^2(\mu)} = 1.$$

This clearly implies $\|zp_n'(z;\mu)n^{-1}\|_{L^2(\mu)} \to 1$ while it is also clear that the leading coefficient of $zp_n'(z;\mu)n^{-1}$ is κ_n. Therefore, Theorem 3.3.6 implies

$$\lim_{n\to\infty} \left\| \frac{zp_n'(z;\mu)}{np_n(z;\mu)} - 1 \right\|_{H^2(\overline{\mathbb{C}}\setminus\overline{\mathbb{D}})} = 1$$

when μ is normal.

3.3.2.2 Weak Asymptotic Measures on the Disk

Consider now the unit disk analog of (2.9.1). Lemma 2.9.3 tells us that if μ is a regular measure on $\overline{\mathbb{D}}$, then there exists a subsequence $\mathcal{M} \subseteq \mathbb{N}$ of asymptotic density 1 so that for every $m \in \mathbb{Z}$ we have

$$\lim_{\substack{n\to\infty \\ n\in\mathcal{M}}} \kappa_{n+m}\kappa_n^{-1} = 1.$$

To see this, we let $\mathcal{N}$ be the subsequence as in Corollary 3.3.5 and let $\mathcal{M}$ be the subsequence of $\mathcal{N}$ constructed by Lemma 2.9.3. Then if $m > 0$, we have

$$\frac{\kappa_{n+m}}{\kappa_n} = \frac{\kappa_{n+m}}{\kappa_{n+m-1}} \cdot \frac{\kappa_{n+m-1}}{\kappa_{n+m-2}} \cdot \dots \cdot \frac{\kappa_{n+1}}{\kappa_n}.$$

Since $\{n, n+1, \dots, n+m\} \subseteq \mathcal{N}$ whenever $n \in \mathcal{M}$ (for large n), we see that all of the ratios in the above equality tend to 1 as $n \to \infty$ through $\mathcal{M}$. A similar argument works if $m < 0$.

This observation will allow us to make further conclusions about regular measures supported on $\overline{\mathbb{D}}$. More specifically, we will address possible weak limits of the sequence of probability measures $\{|p_n(z;\mu)|^2 d\mu(z)\}_{n\in\mathbb{N}}$. Recall our discussion in Section 2.9, where we saw that without any restrictions, the set of weak limit points can be hard to control. Also recall Theorem 2.9.5 and Section 3.1 where we learned that if μ is supported on $\partial\mathbb{D}$ and is regular, then

$$\frac{1}{n+1}\sum_{j=0}^{n} |p_j(z;\mu)|^2 d\mu(z) \to \frac{d\theta}{2\pi}$$

weakly as $n \to \infty$. This suggests convergence along a sequence of density 1 and we will show this is the case. In fact, we will show that if μ is any regular measure on $\overline{\mathbb{D}}$, then there is a subsequence

$\mathcal{N} \subseteq \mathbb{N}$ of asymptotic density 1 so that

$$\operatorname*{w-lim}_{\substack{n\to\infty \\ n\in\mathcal{N}}} |p_n(z;\mu)|^2 d\mu(z) = \frac{d\theta}{2\pi}.$$

The first step is to show that the weak limits we are interested in are measures on $\partial\mathbb{D}$. This is the content of the following lemma.

Lemma 3.3.7. *Let μ be a measure on $\overline{\mathbb{D}}$, $m \in \mathbb{N}$ fixed, and $\mathcal{N} \subseteq \mathbb{N}$ a subsequence so that*

$$\lim_{\substack{n\to\infty \\ n\in\mathcal{N}}} \kappa_{n+m}\kappa_n^{-1} = 1.$$

If $K \subseteq \mathbb{D}$ is a compact set, then

$$\lim_{\substack{n\to\infty \\ n\in\mathcal{N}}} \int_K |p_n(z;\mu)|^2 d\mu(z) = 0.$$

Proof. Let $K \subseteq \mathbb{D}$ be a fixed compact set and assume $K \subseteq \{z : |z| < R < 1\}$. For contradiction, let us suppose that there is a subsequence $\mathcal{N}_1 \subseteq \mathcal{N}$ and $\beta > 0$ such that

$$\int_K |P_n(z;\mu)|^2 d\mu \geq \beta \|P_n(\mu)\|^2_{L^2(\mu)}$$

for all $n \in \mathcal{N}_1$. Then for these n, we have

$$\int_{\overline{\mathbb{D}}\setminus K} |P_n(z;\mu)|^2 d\mu \leq (1-\beta)\|P_n(\mu)\|^2_{L^2(\mu)}.$$

We then use the extremal property to calculate

$$\begin{aligned}
\|P_{n+m}(\mu)\|^2_{L^2(\mu)} &\leq \int_K |z^m P_n(z;\mu)|^2 d\mu + \int_{\overline{\mathbb{D}}\setminus K} |z^m P_n(z;\mu)|^2 d\mu \\
&\leq R^{2m} \int_K |P_n(z;\mu)|^2 d\mu + \int_{\overline{\mathbb{D}}\setminus K} |P_n(z;\mu)|^2 d\mu \\
&= R^{2m} \int_K |P_n(z;\mu)|^2 d\mu + R^{2m} \int_{\overline{\mathbb{D}}\setminus K} |P_n(z;\mu)|^2 d\mu + (1-R^{2m}) \int_{\overline{\mathbb{D}}\setminus K} |P_n(z;\mu)|^2 d\mu \\
&\leq R^{2m} \|P_n(\mu)\|^2_{L^2(\mu)} + (1-R^{2m})(1-\beta)\|P_n(\mu)\|^2_{L^2(\mu)} \\
&= \left(1 - \beta(1-R^{2m})\right) \|P_n(\mu)\|^2_{L^2(\mu)},
\end{aligned}$$

which contradicts our hypothesis when $n \in \mathcal{N}$ is sufficiently large. □

Now we can prove an analog of (2.9.1) for regular measures on the closed unit disk.

Theorem 3.3.8. *Let μ be a regular measure on $\overline{\mathbb{D}}$. There is a subsequence $\mathcal{N} \subseteq \mathbb{N}$ of asymptotic* *density* 1 *so that*

$$\operatorname*{w-lim}_{\substack{n\to\infty\\ n\in\mathcal{N}}} |p_n(z;\mu)|^2 d\mu(z) = \frac{d\theta}{2\pi}.$$

Proof. As mentioned above, we may begin with a subsequence $\mathcal{N} \subseteq \mathbb{N}$ of asymptotic density 1 so that for every $m \in \mathbb{N}$ we have

$$\lim_{\substack{n\to\infty\\ n\in\mathcal{N}}} \kappa_{n+m}\kappa_n^{-1} = 1.$$

It then follows from Lemma 3.3.7 that if $K \subseteq \mathbb{D}$ is compact, we have

$$\lim_{\substack{n\to\infty\\ n\in\mathcal{N}}} \int_K |p_n(z;\mu)|^2 d\mu(z) = 0.$$

We conclude that any weak limit of the measures $\{|p_n(z;\mu)|^2 d\mu(z)\}_{n\in\mathcal{N}}$ is supported on $\partial\mathbb{D}$.

Let σ be such a weak limit point and $\mathcal{N}_\sigma \subseteq \mathcal{N}$ the corresponding subsequence. Then for every fixed $k \in \mathbb{N}$ we have (by the extremal property)

$$\kappa_n^2\kappa_{n+k}^{-2} \leq \int_{\overline{\mathbb{D}}} |P_k(z;\sigma)p_n(z;\mu)|^2 d\mu(z). \tag{3.3.11}$$

As $n \to \infty$ through $\mathcal{N}_\sigma$, the left-hand side of (3.3.11) tends to 1 while the right-hand side tends to $\|P_k(\cdot;\sigma)\|^2_{L^2(\sigma)}$. However, clearly $\|P_k(\cdot;\sigma)\|^2_{L^2(\sigma)} \leq \|z^k\|^2_{L^2(\sigma)} = 1$, so we must have $\|P_k(\cdot;\sigma)\|^2_{L^2(\sigma)} = 1$, which implies the Verblunsky coefficients for the measure σ satisfy

$$\alpha_j(\sigma) = 0, \qquad j = 0, 1, 2, \ldots, k-1.$$

Since $k \in \mathbb{N}$ was arbitrary, this implies σ is normalized arc-length measure on $\partial\mathbb{D}$ as desired. □

3.3.3 Application: Measures Supported on Regions

If a measure μ is supported on an arbitrary bounded region G, we cannot prove a result quite as precise as Theorem 2.7.1 or Corollary 3.3.5 using our methods. The main difficulty is that the conformal maps sending the exterior of $\overline{\mathbb{D}}$ to the exterior of $\overline{\mathbb{D}}$ or the complement of $[-2,2]$ have finite Laurent expansions, which simplifies matters computationally. To make up for this, we will approximate the exterior conformal map with polynomials. The price we will pay is that we will reach a conclusion about p_n/p_{n-k_n} for a possibly unbounded sequence $\{k_n\}$ (but see the example below).

Our proof in this setting will require use of Faber polynomials. Given a bounded region $G \subseteq \mathbb{C}$ whose boundary is a Jordan curve, let Ω be the unbounded component of $\overline{\mathbb{C}} \setminus \overline{G}$, which is simply connected in the extended complex plane. Let ϕ denote the conformal map sending Ω to $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ satisfying $\phi(\infty) = \infty$ and $\phi'(\infty) > 0$. There are three conditions given in [17] that guarantee the uniform convergence of $F_n - \phi^n$ to 0 on $\overline{\Omega}$ as $n \to \infty$. Whenever this convergence property holds (for example if G satisfies any of the three conditions in [17]), we will say G is of class $\mathcal{K}$ and write $G \in \mathcal{K}$.

Our result is the following:

Theorem 3.3.9. *Let μ be a measure on the closure of a bounded region $G \in \mathcal{K}$ with logarithmic capacity 1. Let $\mathcal{N}, \mathcal{M} \subseteq \mathbb{N}$ be infinite subsequences so that for each $j \in \mathcal{M}$, $\kappa_n \kappa_{n-j}^{-1} \to 1$ as $n \to \infty$ through $\mathcal{N}$. Then there exists a non-decreasing and unbounded sequence $\{k_n\}_{n\in\mathcal{N}}$ of elements of $\mathcal{M}$ such that*

$$\lim_{\substack{n\to\infty \\ n\in\mathcal{N}}} \frac{\phi^{k_n}(z) p_{n-k_n}(z;\mu)}{p_n(z;\mu)} = 1 \tag{3.3.12}$$

for all $z \notin \mathrm{ch}(\mu)$. Furthermore, the convergence is uniform on compact subsets of $\overline{\mathbb{C}} \setminus \mathrm{ch}(\mu)$.

Proof. We will apply Theorem 3.3.2 with $Q_n(z) = F_{k_n}(z) p_{n-k_n}(z;\mu)$ for some appropriate $k_n \in \mathcal{M}$. First note that our hypotheses imply that if the sequence $\{k_n\}_{n\in\mathcal{N}}$ grows slowly enough, then $\kappa_n \kappa_{n-k_n}^{-1}$ tends to 1 as $n \to \infty$ through $\mathcal{N}$. Therefore, the second condition of Theorem 3.3.2 is satisfied by Q_n. Remark 2 following Theorem 3.3.2 puts a lower bound on the lim inf of the $L^2(\mu)$-norm of Q_n. To put an upper bound on the lim sup, we see

$$\int_{\overline{G}} |F_{k_n}(z) p_{n-k_n}(z;\mu)|^2 d\mu(z) \leq \|F_{k_n}\|^2_{L^\infty(\overline{G})}$$

for every $n \in \mathbb{N}$. Therefore, $\|Q_n\|_{L^2(\mu)} \leq 1 + \epsilon_n$ where $\epsilon_n \geq 0$ tends to 0 as $n \to \infty$ through $\mathcal{N}$ provided $\{k_n\}_{n\in\mathcal{N}}$ is unbounded (this is because $G \in \mathcal{K}$ and $|\phi(w)| = 1$ for all $w \in \partial\Omega$). By invoking Theorem 3.3.2, we conclude that

$$\lim_{\substack{n\to\infty \\ n\in\mathcal{N}}} \frac{F_{k_n}(z) p_{n-k_n}(z;\mu)}{p_n(z;\mu)} = 1 \tag{3.3.13}$$

for all $z \notin \mathrm{ch}(\mu)$, and the convergence is uniform on compact subsets of $\overline{\mathbb{C}} \setminus \mathrm{ch}(\mu)$. Since $F_n - \phi^n$ tends to 0 on $\overline{\Omega}$ as $n \to \infty$, (3.3.13) implies (3.3.12). □

Although Theorem 3.3.9 is an analog of Theorem 3.3.4 for more general supports, proving an analog of Corollary 3.3.5 or Theorem 2.7.1 is more challenging. The difficulty lies in the fact that it is possible to have $\|P_n(\cdot;\mu)\|_{L^2(\mu)} > \|P_{n-1}(\mu)\|_{L^2(\mu)}$ when the support of the measure is not the closed unit disk. The following example shows that we can strengthen the conclusion of Theorem 3.3.9 to more closely resemble that of Theorem 2.7.1 if some power of the conformal map ϕ is a monic polynomial.

Example. Consider the set $E_m := \{z : |z^m - 1| \leq 1\}$ (pictured below for $m = 3$). In this case, $F_m(z) = z^m - 1$ (see example 3.8 in [35]) so that if μ is a measure supported on E_m, we can write

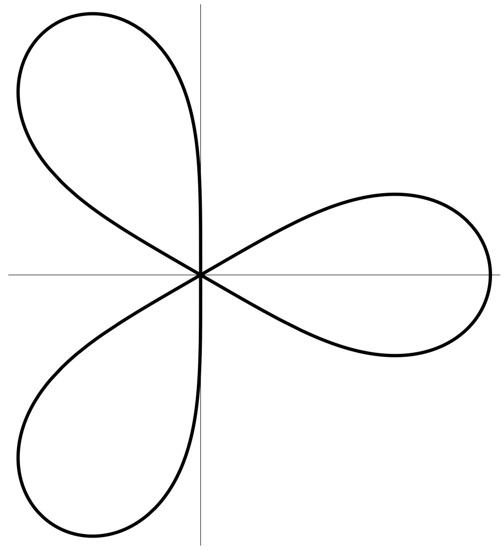

Figure 3.2: The boundary of the set E_3.

$\|P_{n+m}(\cdot;\mu)\|_{L^2(\mu)} \leq \|P_n(\cdot;\mu)\|_{L^2(\mu)}$ for all $n \in \mathbb{N}$. If μ is regular, then we have

$$1 = \lim_{n\to\infty} \left(\kappa_n\kappa_{n+1}\cdots\kappa_{n+m-1}\right)^{1/n} = \lim_{n\to\infty}\left(\kappa_1\cdots\kappa_m\prod_{j=1}^{n-1}\kappa_{j+m}\kappa_j^{-1}\right)^{1/n}.$$

We can now apply the same reasoning as in the proof of Corollary 3.3.5 to conclude that there is a subsequence $\mathcal{N} \subseteq \mathbb{N}$ of asymptotic density 1 so that $\lim_{n\to\infty, n\in\mathcal{N}} \kappa_n\kappa_{n-m}^{-1} = 1$. Furthermore, $\|F_m(z)\|_{L^\infty(E_m)} = 1$ so the proof of Theorem 3.3.9 shows that in fact we have

$$\lim_{\substack{n\to\infty\\ n\in\mathcal{N}}} \frac{F_m(z)p_{n-m}(z;\mu)}{p_n(z;\mu)} = \lim_{\substack{n\to\infty\\ n\in\mathcal{N}}} \frac{(z^m-1)p_{n-m}(z;\mu)}{p_n(z;\mu)} = 1$$

for all $z \notin \mathrm{ch}(\mu)$. Notice that if we set $m = 1$ we recover Corollary 3.3.5. The same calculation applies in any situation where some power of the conformal map ϕ is a monic polynomial.

3.3.4 Application: Stability Under Perturbation

3.3.4.1 The Uvarov Transform

Another application of Theorem 3.3.2 is to show that the behavior of the polynomials $\{p_n(z;\mu)\}_{n\geq 0}$ is stable under certain perturbations of the measure. In the following example, we consider the *Uvarov Transform* of a measure (see [15]), meaning we add a single point mass to the measure μ.

Example. Let μ be a measure with compact support and $x \in \mathbb{C}$. We will show that for any $t > 0$

we have

$$\lim_{n\to\infty} \frac{p_n(z;\mu+t\delta_x)}{p_n(z;\mu)} = 1 \tag{3.3.14}$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus \mathrm{ch}(\mu)$ if and only if

$$\lim_{n\to\infty} \frac{|p_n(x;\mu)|^2}{K_{n-1}(x,x;\mu)} = 0, \tag{3.3.15}$$

where $K_n(y,z;\mu)$ is defined as in (2.5.3). We will apply Theorem 3.3.2 with $Q_n = p_n(z;\mu+t\delta_x)$. The proof of Theorem 10.13.3 in [57] applies in this setting also to show that

$$\begin{aligned} \frac{\|P_n(\cdot;\mu+t\delta_x)\|^2_{L^2(\mu+t\delta_x)}}{\|P_n(\cdot;\mu)\|^2_{L^2(\mu)}} &= \frac{1+tK_n(x,x;\mu)}{1+tK_{n-1}(x,x;\mu)} \\ &= 1 + \frac{|p_n(x;\mu)|^2}{K_{n-1}(x,x;\mu)} \cdot \frac{t}{t+K_{n-1}(x,x;\mu)^{-1}}. \end{aligned} \tag{3.3.16}$$

Notice that,

$$\lim_{n\to\infty} \frac{t}{t+K_{n-1}(x,x;\mu)^{-1}}$$

always exists and lies in the interval $(0,1]$. Therefore, if we assume (3.3.15) holds then (3.3.16) verifies the second condition in Theorem 3.3.2 for Q_n. To verify the first condition, write $Q_n = \tau_n P_n(\cdot;\mu+t\delta_x)$ and notice

$$\|P_n(\cdot;\mu+t\delta_x)\|^2_{L^2(\mu+t\delta_x)} \geq \|P_n(\cdot;\mu)\|^2_{L^2(\mu)} + \|P_n(\cdot;\mu+t\delta_x)\|^2_{L^2(t\delta_x)}.$$

Dividing through by $\|P_n(\cdot;\mu+t\delta_x)\|^2_{L^2(\mu+t\delta_x)}$ and using our above calculations, we get $\|Q_n\|_{L^2(t\delta_x)} \to 0$ as $n\to\infty$, which verifies the first condition in Theorem 3.3.2 and hence proves (3.3.14).

If (3.3.15) does not hold, then (3.3.16) shows that we do not even get the the desired convergence at infinity so we cannot possibly have (3.3.14).

Remark. The Uvarov Transform on the unit circle was studied extensively by Wong in [75].

As an aside, we note here that the condition (3.3.15) is well studied in the context of OPUC and OPRL (see Theorem 10.13.5 in [57] and also [5]). In the context of OPRL, when $x \in \mathrm{supp}(\mu)$ it is equivalent to the *Nevai condition*, which was introduced in [38]. It holds at x precisely when

$$\operatorname*{w-lim}_{n\to\infty} \frac{K_n(x,y;\mu)^2}{K_n(y,y;\mu)} d\mu(y) = \delta_x.$$

In [5], it is conjectured that the Nevai condition holds for μ almost every x. There is an extensive literature on the Nevai condition and related phenomena. Further results can be found in [5, 29, 38, 39] and references therein.

The calculations in the above example prove our next result. It shows that if a measure is perturbed in a way that does not affect the asymptotic behavior of the monic orthogonal polynomial norms, then it also does not affect the asymptotic behavior of the orthonormal polynomials outside $\mathrm{ch}(\mu)$.

Corollary 3.3.10. *Let μ_1 and μ_2 be two measures with compact support such that*

$$\lim_{n\to\infty} \frac{\|P_n(\cdot;\mu_1)\|_{L^2(\mu_1)}}{\|P_n(\cdot;\mu_1+\mu_2)\|_{L^2(\mu_1+\mu_2)}} = 1.$$

Then

$$\lim_{n\to\infty} \frac{p_n(z;\mu_1+\mu_2)}{p_n(z;\mu_1)} = 1$$

for all $z \notin \mathrm{ch}(\mu_1)$.

3.3.4.2 The Christoffel Transform

A second kind of perturbation we will consider is the *Christoffel Transform* of a measure (see [15]), where we multiply the measure by the square modulus of a monomial; that is, we define

$$d\nu^x(z) = |z-x|^2 d\mu(z). \tag{3.3.17}$$

The location of the point x will not be arbitrary; indeed we will have to place a hypothesis on the point x as in (3.3.15). We will see later (Corollary 3.3.13 below) that this forces x to lie in the convex hull of the support of μ.

For every $n \in \mathbb{N}$, we recall the definition of $K_n(y,z;\mu)$ given in (2.5.3). A very simple calculation provides us with the following formula (see Proposition 3 in [15]):

$$P_n(z;\nu^x) = \frac{1}{z-x}\left(P_{n+1}(z;\mu) - \frac{P_{n+1}(x;\mu)}{K_n(x,x;\mu)}K_n(z,x;\mu)\right). \tag{3.3.18}$$

We can now prove the following result:

Theorem 3.3.11. *Let μ be a measure with compact support and let ν^x and μ be related by (3.3.17) where x satisfies (3.3.15). Then*

$$\lim_{n\to\infty} \frac{(z-x)p_{n-1}(z;\nu^x)}{p_n(z;\mu)} = 1 \tag{3.3.19}$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus \mathrm{ch}(\mu)$.

Proof. We wish to apply Theorem 3.3.2 with $Q_n(z) = (z-x)p_{n-1}(z;\nu^x)$. First notice that

$$\|Q_n\|^2_{L^2(\mu)} = \frac{\|(\cdot - x)P_{n-1}(\cdot;\nu^x)\|^2_{L^2(\mu)}}{\|P_{n-1}(\cdot;\nu^x)\|^2_{L^2(\nu^x)}} = 1$$

by definition, which verifies the first condition of Theorem 3.3.2. By formula (3.3.18), we calculate

$$\|P_{n-1}(\cdot;\nu^x)\|^2_{L^2(\nu^x)} = \|(\cdot - x)P_{n-1}(\cdot;\nu^x)\|^2_{L^2(\mu)} = \|P_n(\cdot;\mu)\|^2_{L^2(\mu)} + \frac{|P_n(x;\mu)|^2}{K_{n-1}(x,x;\mu)}.$$

The leading coefficient τ_n of Q_n is just $\|P_{n-1}(\cdot;\nu^x)\|^{-1}_{L^2(\nu^x)}$ so we have

$$\tau_n = \|P_n(\cdot;\mu)\|^{-1}_{L^2(\mu)}(1 + o(1))$$

as $n \to \infty$ by our assumption (3.3.15). This verifies the second condition of Theorem 3.3.2 and hence the desired conclusion follows. □

Remark. By Theorem 3.3.6, if the measure μ in Theorem 3.3.11 is supported on the unit circle, then in fact we get H^2 convergence in (3.3.19).

The hypotheses of Theorem 3.3.11 are presented in terms of the measure μ, but we can also state a condition on ν^x that implies relative asymptotics. We will follow the notation and terminology from [31], where μ is called the *Geronimus Transform* of the measure ν^x. As in [31], let us define

$$q_n(x) = \int_{\mathbb{C}} \frac{\overline{p_n(z;\nu^x)}}{x - z} d\nu^x(z), \qquad \epsilon_n(x) = \nu^x(\mathbb{C}) - \sum_{j=0}^{n} |q_j(x)|^2. \tag{3.3.20}$$

We saw in the proof of Theorem 3.3.11 that $\|(z-x)p_{n-1}(z;\nu^x)\|_{L^2(\mu)} = 1$, so we need only verify the condition on the leading coefficients to apply Theorem 3.3.2. To this end, we apply Corollary 2 in [31], which tells us that

$$\frac{\|P_n(z;\mu)\|^2_{L^2(\mu)}}{\|P_{n-1}(z;\nu^x)\|^2_{L^2(\nu^x)}} = \frac{\epsilon_{n-1}(x)}{\epsilon_{n-2}(x)}.$$

Therefore, we see that (3.3.19) holds uniformly on compact subsets of $\overline{\mathbb{C}} \setminus \mathrm{ch}(\mu)$ provided x satisfies.

$$\lim_{n\to\infty} \frac{|q_{n-1}(x)|^2}{\epsilon_{n-2}(x)} = 0. \tag{3.3.21}$$

Combining Theorem 3.3.11 with the example in Section 3.3.4.1, we deduce the following corollary:

Corollary 3.3.12. *Let μ be a measure with compact support, $x \in \mathbb{C}$, and $t > 0$. If x satisfies (3.3.15), then*

$$\lim_{n\to\infty} \frac{(z-x)p_{n-1}(z;\nu^x)}{p_n(z;\mu + t\delta_x)} = 1$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus \mathrm{ch}(\mu)$.

The following example illustrates Theorem 3.3.11 and shows that in general we cannot hope to extend the results of Theorem 3.3.2 to the boundary of $\mathrm{Pch}(\mu)$.

Example. Let us reconsider the example from Section 2.8. Let μ be two-dimensional area measure on the unit disk $\mathbb{D}$ so that $p_n(z;\mu) = \sqrt{\frac{n+1}{\pi}} z^n$. It is easily seen that in this case, the point 1 satisfies (3.3.15) so we will consider the Christoffel Transform given by ν^1. We recall that by the example in Section IV.6 in [68] (or equation (3.3.18) above), we know that

$$p_n(z;\nu^1) = \frac{2}{\sqrt{\pi(n+1)(n+2)(n+3)}} \sum_{k=0}^{n} (k+1) z^k (1 + z + z^2 + \cdots + z^{n-k}).$$

We then see that

$$\begin{aligned} &\frac{(z-1)p_n(z;\nu^1)}{p_{n+1}(z;\mu)} = \\ &= \frac{2(z-1)}{z^{n+1}(n+2)\sqrt{(n+1)(n+3)}} \sum_{k=0}^{n} (k+1) z^k (1 + z + z^2 + \cdots + z^{n-k}) \\ &= \frac{2}{(n+2)\sqrt{(n+1)(n+3)}} \left(\frac{(n+1)(n+2)}{2} - \frac{n+1}{z} - \cdots - \frac{2}{z^n} - \frac{1}{z^{n+1}} \right), \end{aligned}$$

which clearly tends to 1 as $n \to \infty$ if $|z| > 1$, in accordance with Theorem 3.3.11.

It is clear that

$$\left. \frac{(z-1)p_{n-1}(z;\nu^1)}{p_n(z;\mu)} \right|_{z=1} = 0,$$

so we cannot in general hope to extend the conclusion of Theorem 3.3.2 to include convergence on the boundary of $\mathrm{Pch}(\mu)$. However, in this example all of the zeros of $p_n(z;\mu)$ are contained in $\mathbb{D}$ so $(z-1)p_{n-1}(z;\nu^1)p_n(z;\mu)^{-1}$ is a function in $H^\infty(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}})$ and as such

$$\int_0^{2\pi} \frac{(e^{i\theta}-1)p_{n-1}(e^{i\theta};\nu^1)}{p_n(e^{i\theta};\mu)} \frac{d\theta}{2\pi} = \left. \frac{(z-1)p_{n-1}(z;\nu^1)}{p_n(z;\mu)} \right|_{z=\infty} = \frac{\kappa_{n-1}(\nu^1)}{\kappa_n(\mu)} \to 1$$

as $n \to \infty$, which suggests we do have convergence to 1 almost everywhere on $\partial\mathbb{D}$ in this example. A short calculation reveals that this is the case.

Theorem 3.3.11 also yields the following (see also Theorem 1.3 in [5]):

Corollary 3.3.13. *If* $x \notin \mathrm{ch}(\mu)$ *then (3.3.15) fails along every subsequence.*

Proof. Since all zeros of $p_n(\cdot;\mu)$ are contained in $\mathrm{ch}(\mu)$, we have

$$\left. \frac{(z-x)p_{n-1}(z;\nu^x)}{p_n(z;\mu)} \right|_{z=x} = 0$$

for every $n \in \mathbb{N}$, which means (3.3.15) cannot possibly hold along any subsequence for otherwise, by Theorem 3.3.11 (applied along the subsequence) this expression would have to converge to 1. $\square$

It is interesting to observe that we can derive a different proof of Theorem 3.3.2 (without the uniformity) if we assume Corollary 3.3.13. To see this, notice that (3.3.15) failing along every subsequence is equivalent to the statement that

$$\limsup_{n\to\infty} \frac{K_{n-1}(x,x;\mu)}{|p_n(x;\mu)|^2} < \infty. \tag{3.3.22}$$

Therefore, if we have $\{Q_n\}_{n\in\mathbb{N}}$ as in the statement of Theorem 3.3.2, then we can write each Q_n as

$$Q_n(z) = \sum_{j=0}^{n} \lambda_j^{(n)} p_j(z;\mu)$$

for appropriate $\lambda_j^{(n)} \in \mathbb{C}$, $j \in \{0,1,\ldots,n\}$. The hypothesis, $\tau_n\kappa_n^{-1} \to 1$ implies $\lambda_n^{(n)} \to 1$. Since $\|Q_n\|^2_{L^2(\mu)} = \sum_{j=0}^{n} |\lambda_j^{(n)}|^2$, the hypothesis $\|Q_n\|_{L^2(\mu)} \to 1$ implies $\sum_{j=0}^{n-1} |\lambda_j^{(n)}|^2 \to 0$. Therefore,

$$\frac{Q_n(z)}{p_n(z;\mu)} = \lambda_n^{(n)} + \frac{\sum_{j=0}^{n-1} \lambda_j^{(n)} p_j(z;\mu)}{p_n(z;\mu)}.$$

We have already seen that $\lambda_n^{(n)} \to 1$, while the remaining term can be bounded by the Cauchy-Schwarz inequality:

$$\left|\frac{\sum_{j=0}^{n-1} \lambda_j^{(n)} p_j(z;\mu)}{p_n(z;\mu)}\right|^2 \leq \left(\sum_{j=0}^{n-1} |\lambda_j^{(n)}|^2\right)\left(\frac{K_{n-1}(z,z;\mu)}{|p_n(z;\mu)|^2}\right),$$

which tends to 0 as $n \to \infty$ since we are assuming (3.3.22). Therefore, $Q_n p_n^{-1}$ tends to 1 outside of $\mathrm{ch}(\mu)$ as in (3.3.3).

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"I never see what has been done; I only see what remains to be done."

– Buddha