Symmetries in Three-Dimensional Superconformal Quantum Field Theories

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I dedicate this thesis to my family.

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Abstract

Many examples of gauge-gravity duality and quantum equivalences of different-looking three-dimensional Quantum Field Theories indicate the existence of continuous symmetries whose currents are not built from elementary, or perturbative, fields used to write down the Lagrangian. These symmetries are called hidden or nonperturbative.

We describe a method for studying continuous symmetries in a large class of three-dimensional supersymmetric gauge theories which, in particular, enables one to explore non-perturbative global symmetries and supersymmetries. As an application of the method, we prove conjectured supersymmetry enhancement in strongly coupled ABJM theory from $\mathcal{N}=6$ to $\mathcal{N}=8$ and find additional nonperturbative evidence for its duality to the $\mathcal{N}=8$ $U(N)$ SYM theory for the minimal value of the Chern-Simons coupling. Hidden supersymmetry is also shown to occur in $\mathcal{N}=4$ $d=3$ SQCD with one fundamental and one adjoint hypermultiplets. An infinite family of $\mathcal{N}=6$ $d=3$ ABJ theories is proved to have hidden $\mathcal{N}=8$ superconformal symmetry and hidden parity on the quantum level. We test several conjectural dualities between ABJ theories and theories proposed by Bagger and Lambert, and Gustavsson by comparing superconformal indices of these theories. Comparison of superconformal indices is also used to test dualities between $\mathcal{N}=2$ $d=3$ theories proposed by Aharony, the analysis of whose chiral rings teaches some general lessons about nonperturbative chiral operators of strongly coupled $3d$ supersymmetric gauge theories.

As another application of our method we consider examples of hidden global symmetries in a class of quiver three-dimensional $\mathcal{N}=4$ superconformal gauge theories. Finally, we point out to the relations between some basic propeties of superconformal $\mathcal{N}\geq 6$ theories and their symmetries.

The results presented in this thesis were obtained in a series of papers [1, 2, 3, 4, 5].

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Chapter 1

Introduction

This thesis summarizes research on continuous symmetries in three-dimensional superconformal field theories presented in a series of papers [1, 2, 3, 4, 5]. The bulk of the research was the analysis of hidden, or nonperturbative, symmetries whose conserved currents were the so-called monopole operators introduced in [6, 7]. These operators are a central concept in realization of nonperturbative symmetries. In this introductory chapter we give a short motivation for the introduction of monopole operators as essential objects responsible for the enhancement of continuous symmetries.

In three space-time dimensions a photon has just one polarizations, the same number as a scalar. It was pointed by Polyakov [8] thirty-five years ago that the action for a free photon can in fact be written as an action for a free scalar called a "dualized photon" γ. In terms of equations satisfied by the field strength $F = F_{\mu\nu} dx^\mu \wedge dx^\nu$, this just corresponds to swapping the roles of the equation of motion $d * F = 0$ and the Bianchi identity $dF = 0$. The first equation is regarded as defining a new "gauge potential" γ by means of $*F = d\gamma$, while the second equation becomes the equation of motion for the free scalar γ. The new gauge transformations are trivial. Because of the quantization of the magnetic flux, the target space for the scalar γ is a circle with radius g^2, the dimensionful parameter in the Maxwell action $S = -\frac{1}{4g^2}\int d^3x F \wedge \star F$.

The old Bianchi identity $dF = 0$ can also be seen as a conservation $\partial^\mu J_\mu = 0$ of the topological current $J_\mu = \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}F^{\nu\lambda}$. When written in terms of the "dualized photon" γ, this current $J_\mu = \frac{1}{2\pi}\partial_\mu\gamma$ is seen to generate global rotations of the target space circle: $\gamma(x) \to \gamma(x) + \alpha$. It is useful to introduce a global coordinate for the circle: $\phi(x) = e^{i\frac{\gamma(x)}{g^2}}$.

This field has charge one under the global symmetry generated by the topological current. It is interesting that in the initial formulation of the theory in terms of the gauge potential there was no gauge invariant operator charged under this symmetry. The introduction of magnetic monopole operators in the approach with the gauge fields eliminates this mismatch in the two descriptions of the same physics.

Of course, the example with the free electromagnetic field is too easy to be an interesting system with monopole operators playing any significant role. However, there are many theories with nontrivial interesting dynamics where introduction of monopole operators is well worth it. For example, consider following [9] an $\mathcal{N} = 2$ supersymmetric nonabelian gauge theory with a gauge group $U(N)$. At a general point on the Coulomb branch the gauge group is broken to its maximal torus $U(1)^N$ by the expectation value of the scalar ϕ in the adjoint representation of $U(N)$ that lives in the $\mathcal{N} = 2$ vector multiplet together with the gauge fields. Below the gauge symmetry breaking scale all massive fields are integrated out, and the remaining N photons can be dualized to N scalars γ_i because there are no fields that couple to them. They can be combined with ϕ_i into chiral fields with scalars $\Phi_i = e^{(\phi_i+i\gamma_i)/g^2}$. A natural question to ask is what these chiral operators correspond to in the ultraviolet. The answer is the monopole operators.

There is, however, a very practical (from a theoretical point of view) and not just academic reason to introduce monopole operators. This practical reason comes from dualities between quantum field theories with different Lagrangians and also from gauge-gravity duality. For example, there are two theories with different Lagrangians, which are conjectured to be equivalent quantum mechanically. However, at a first glance the global symmetries do not match. One of the actions has a symmetry group G and the other one has symmetry group $H \subset G$ which is a proper subgroup of G. In all known cases in three dimensions monopole operators come to rescue, completing H to G. Consider a specific example where G and H are global symmetries. One of the theories Intriligator and Seiberg analyzed in their paper [10] is an $\mathcal{N} = 4$ gauge theory with gauge group $U(1)^3$ and matter fields in the bifundamental representation of each of the pairs of $U(1)$ gauge factors. This theory is an example of a quiver gauge theory. In the present case the diagonal $U(1)$ gauge group decouples, and the remaining gauge group is $U(1)^3/U(1) = U(1)^2$. They argued that this theory is equivalent to another theory whose global symmetry group $G = SU(3)$. As was pointed out in [7], a Cartan subalgebra of $SU(3)$ can be easily identified in the quiver theory as symmetries produced by the two topological currents $J^i = \frac{1}{2\pi} * F^i$. The identification of the remaining six currents that, together with the two topological currents, complete the symmetry group to $SU(3)$ is more diffucult. This identification is explained in chapter 5. One thing can be seen immediately, though. The remaining six currents correspond to the roots of $SU(3)$ and thus must be charged under the two topological currents which correspond to Cartan generators. Because monopole operators are charged under topological symmetries, it is natural to expect that they help resolve the puzzle.

The gauge-gravity duality, or more precisely AdS_4/CFT_3 duality, provides examples where the continuous symmetries that are enhanced are supersymmetries and corresponding R-symmetries. The most famous example is the ABJM theory describing the low energy limit of the dynamics of a stack of $M2$-branes probing a conical singularity $\mathbb{C}^2/\mathbb{Z}_k$. It can be described as quantum gravity or string theory in $AdS_4 \times S^7$ background or as a three-dimensional Chern-Simons gauge theory with the parameter k determining the Chern-Simons level of the gauge group factors [11]. The gravity description of this system for $k = 1, 2$ has more supersymmetries that are seen in the gauge theory description. There are again some topological currents under which the missing conserved currents in the gauge theory must be charged, and so monopoles operators which by definition are charged under topological symmetries are relevant for the supersymmetry enhancement.

This thesis organized as follows. Chapter 2 reviews the definition of monopole operators and their basic properties and introduces the method to find and prove enhancement of continuos symmetries. Enhancement of supersymmetries is discussed in chapters 2 and 3. Chapter 4 is devoted to an analysis of chiral rings and dualities in a certain class of $\mathcal{N} = 2$ supersymmetirc theories. Chapter 5 discusses enhancement of global symmetries in $\mathcal{N} = 4$ supersymmetric theories of Intriligator and Seiberg and some other quiver theories. Some very general properties of $\mathcal{N} \geq 6$ superconformal theories which our approach to global symmetries helps to illuminate are considered in chapter 6.

Chapter 2

Supersymmetries Enhancement by Monopole Operators

2.1 Introduction

In this chapter we describe a method which enables one to study hidden, or accidental, continuous symmetries in strongly coupled superconformal field theories in three space-time dimensions. The existence of such hidden symmetries has been conjectured for many three-dimensional theories. We apply our method to two models. The first one is the recently proposed ABJM model [11] which has gauge group $U(N) \times U(N)$, an integral parameter k (the Chern-Simons level), and a manifest $\mathcal{N} = 6$ supersymmetry. It is believed to have hidden $\mathcal{N} = 8$ superconformal symmetry for $k = 1, 2$ [11]. The second model is the infrared limit of $\mathcal{N} = 4$ $d = 3$ super Yang-Mills theory with an adjoint and a fundamental hypermultiplets. It is believed to be dual to the ABJM model with $k = 1$, as well as to the infrared limit of $\mathcal{N} = 8$ super-Yang-Mills theory with gauge group $U(N)$, and consequently also must have hidden $\mathcal{N} = 8$ superconformal symmetry. In this chapter we demonstrate the existence of supersymmetry enhancement in all three models. We also provide some evidence in favor of the duality with $\mathcal{N} = 8$ super-Yang-Mills.

By definition, a hidden symmetry is generated by a conserved current whose existence does not follow from any symmetry of an action. A simple example of such a symmetry corresponds to a topological conserved current which exists in any three-dimensional gauge theory whose gauge group contains a $U(1)$ factor:

$$J^{\mu} = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\mathrm{Tr}\, F_{\nu\lambda}. \tag{2.1}$$

In this chapter we study more complicated hidden symmetries whose conserved currents are monopole operators, i.e., disorder operators defined by the condition that the gauge field has a Dirac monopole singularity at the insertion point. More concretely, in a $U(N)$ gauge theory the singularity corresponding to a monopole operator must have the form

$$A^{N,S}(\vec{r}) = \frac{H}{2}(\pm 1 - \cos\,\theta)d\phi \tag{2.2}$$

for the north and south charts, correspondingly. In this formula $H = diag(n_1, n_2, \ldots, n_N)$, and the integers $n_1, \ldots, n_N$ are defined up to a permutation.[1] These integers are called magnetic or GNO charges [12].

If we require the monopole operator to preserve some supersymmetry (such operators may be called BPS operators), matter fields must also be singular, in such a way that BPS equations are satisfied in the neighborhood of the insertion point.

The idea that hidden symmetry currents can arise from monopole operators is not new. Even before the discovery of the ABJM model, it has been mentioned in [7] in connection with the hidden flavor symmetries proposed by Intriligator and Seiberg [10]. More recently there have been several works which studied monopole operators in the ABJM model with the goal of showing the existence of hidden conserved currents enhancing $\mathcal{N} = 6$ supersymmetry to $\mathcal{N} = 8$ supersymmetry [13, 14, 15]; other works which studied monopole operator are [16, 17]. Our approach is similar to Benna, Klebanov and Klose (BKK) [13] in that we deform the theory in a controlled manner which makes it weakly coupled but breaks part of the conformal symmetry. The details are rather different because the deformation we use breaks a different subset of the conformal symmetry. The deformation of ABJM theory used by BKK preserved the Poincare subgroup of the conformal group as well as the $Spin(3) \times Spin(3)$ subgroup of the $Spin(6)$ R-symmetry. The conformal and dilatational symmetries were

[1] One often chooses a particular representative satisfying $n_1 \geq n_2 \geq \ldots \geq n_N$. We will not always follows this convention.

broken. The deformation we use preserves the rotational and dilatational symmetry of $\mathbb{R}^3$ and the $Spin(2) \times Spin(4)$ subgroup of the $Spin(6)$ R-symmetry. The translational and conformal symmetries are broken. This is the same deformation as that used by S. Kim to compute the superconformal index of the ABJM theory [16]. We will see that the same kind of deformation can be used to study any three-dimensional gauge theory with enough supersymmetry. One big advantage of this method is that we have control over the conformal dimensions of an important class of monopole operators. We show that for $k = 1, 2$ the ABJM theory has monopole operators which are conformal primaries of dimension 2 and transform as vectors under Lorenz transformations. Such operators must be conserved currents, which enables us to conclude that the R-symmetry and consequently supersymmetry are enhanced.

The other model we consider is an $\mathcal{N} = 4$ $d = 3$ $U(N)$ gauge theory with an adjoint and a fundamental hypermultiplet. This theory has no Chern-Simons term and is not conformal but flows to a nontrivial IR fixed point. String theory arguments show that it must be IR dual to $\mathcal{N} = 8$ super-Yang-Mills theory with gauge group $U(N)$. This implies that it must have enhanced supersymmetry in the infrared, and we show that this is indeed the case. There are several important differences compared to the case of the ABJM theory. In particular, we find that some currents predicted by the duality are realized by monopole operators with a vanishing topological charge (but nonzero GNO charges). This is a nice illustration of the importance of nontopological disorder operators in quantum field theory.

We also show that for $N > 1$ the $U(N) \times U(N)$ $k = 1$ ABJM theory as well as the IR limit of $\mathcal{N} = 4$ $U(N)$ theory with an adjoint and a fundamental hypermultiplet have a free sector (also with $\mathcal{N} = 8$ supersymmetry). This decoupled sector is not visible on the perturbative level, but its existence is predicted by the conjecture [11] that both theories are dual to the IR limit of $\mathcal{N} = 8$ $U(N)$ super-Yang-Mills theory.[2]

The organization of this chapter is as follows. In section 2 we study monopole operators in the ABJM theory. In section 3 we study monopole operators in the $\mathcal{N} = 4$ $U(N)$ gauge theory with an adjoint and a fundamental hypermultiplet. In section 4 we discuss out results; in particular we show that supersymmetry enhancement is quite delicate and does not occur in other similar gauge theories. In the appendices we provide some details of the arguments;

[2] Other tests of this conjecture have been performed in [16] and [18].

in particular, we rederive a formula for the charges of a bare monopole proposed by Gaiotto and Witten [19].

2.2 Supersymmetry Enhancement in the ABJM Model

2.2.1 Field Content, Action, and Symmetries

The ABJM model is an $\mathcal{N} = 6$ $d = 3$ Chern-Simons gauge theory with the gauge group $U(N) \times U(N)$. It is convenient to use $\mathcal{N} = 2$ superfield formalism to describe its field content and action. The $U(N) \times U(N)$ vector multiplet consists of gauge fields $A_\mu, \tilde{A}_\mu$, adjoint-valued scalars $\sigma, \tilde{\sigma}$ and adjoint-valued Dirac fermions $\lambda, \tilde{\lambda}$. Fields with a tilde take values in the Lie algebra of the second $U(N)$ factor, while fields without a tilde take values in the Lie algebra of the first $U(N)$ factor. The matter sector contains complex scalars C_I and Dirac fermions Ψ^I in the representations 4 and $\bar{4}$ of the $SU(4)_R \simeq Spin(6)_R$ R-symmetry and are in the bifundamental $(N, \bar{N})$ representation of the gauge group. Written as $C_I = (A_1, A_2, \bar{B}^{\dot{1}}, \bar{B}^{\dot{2}})$ and $\Psi^I = (-\psi_2, \psi_1, -\bar{\chi}^{\dot{2}}, \bar{\chi}^{\dot{1}})$ they can be grouped into four $\mathcal{N} = 2$ chiral multiplets

$$(A_a, \psi_a) \in (N, \bar{N}), \qquad (B_{\dot{a}}, \chi_{\dot{a}}) \in (\bar{N}, N). \tag{2.3}$$

The indices mark the representations of the fields under the group $SU(2)_A \times SU(2)_B \subset SU(4)_R$ which is manifest in the $\mathcal{N} = 2$ superfield formalism.

The Lagrangian is

$$\mathcal{L} = \mathcal{L}_{CS} + \mathcal{L}_{matter}, \tag{2.4}$$

with the Chern-Simons term

$$\mathcal{L}_{CS} = \frac{k}{4\pi} tr \left(A \wedge dA - \frac{2i}{3} A^3 + i\bar{\lambda}\lambda + 2D\sigma \right) - \frac{k}{4\pi} tr \left(\tilde{A} \wedge d\tilde{A} - \frac{2i}{3} \tilde{A}^3 + i\bar{\tilde{\lambda}}\tilde{\lambda} + 2\tilde{D}\tilde{\sigma} \right), \tag{2.5}$$

and the matter term

$$\begin{aligned}
\mathcal{L}_{matter} = tr[&-D_\mu \bar{A}^a D^\mu A_a - D_\mu \bar{B}^{\dot{a}} D^\mu B_{\dot{a}} - i\bar{\psi}^a \not{D}\psi_a - i\bar{\chi}^{\dot{a}} \not{D}\chi_{\dot{a}} \\
&(\sigma A_a - A_a\tilde{\sigma})(\bar{A}^a\sigma - \tilde{\sigma}\bar{A}^a) - (\tilde{\sigma}B_{\dot{a}} - B_{\dot{a}}\sigma)(\bar{B}^{\dot{a}}\tilde{\sigma} - \sigma\bar{B}^{\dot{a}}) + \\
&+ i\bar{\psi}^a\sigma\psi_a - i\psi_a\tilde{\sigma}\bar{\psi}^a + i\bar{A}^a\lambda\psi_a + i\bar{\psi}^a\bar{\lambda}A_a - i\psi_a\tilde{\lambda}\bar{A}^a - iA_a\bar{\tilde{\lambda}}\bar{\psi}^a \\
&- \chi_{\dot{a}}\sigma\bar{\chi}^{\dot{a}} + i\bar{\chi}^{\dot{a}}\tilde{\sigma}\chi_{\dot{a}} - i\chi_{\dot{a}}\lambda\bar{B}^{\dot{a}} - iB_{\dot{a}}\bar{\lambda}\bar{\chi}^{\dot{a}} + i\bar{B}^{\dot{a}}\tilde{\lambda}\chi_{\dot{a}} + i\bar{\chi}^{\dot{a}}\bar{\tilde{\lambda}}B_{\dot{a}}] + \mathcal{L}_{sup},
\end{aligned} \tag{2.6}$$

where $\mathcal{L}_{sup}$ contains Yukawa interaction terms and scalar potential coming from the quartic superpotential

$$W = -\frac{2\pi}{k}\epsilon^{ab}\epsilon^{\dot{a}\dot{b}}tr(A_a B_{\dot{a}} A_b B_{\dot{b}}). \tag{2.7}$$

The $\mathcal{N} = 6$ supercharges transform in the vector representation of $Spin(6)_R$ or, equivalently, rank two antisymmetric tensor representation of $SU(4)_R$ with a reality condition

$$Q_{IJ} = \frac{1}{2}\epsilon_{IJKL}\bar{Q}^{KL}, \tag{2.8}$$

where I, J, K, L are indices of the fundamental representation of $SU(4)_R$.

Apart from Noether currents corresponding to symmetries of the action the ABJM theory also has two conserved topological currents

$$J_T^\mu = \frac{1}{2\pi}\mathrm{Tr}\, F^\mu, \quad \tilde{J}_T^\mu = \frac{1}{2\pi}\mathrm{Tr}\, \tilde{F}^\mu,$$

where $F^\mu, \tilde{F}^\mu$ are Hodge-dual to $F_{\mu\nu}, \tilde{F}_{\mu\nu}$. Equations of motion of the ABJM theory imply $k\mathrm{Tr}\, F^\mu = k\mathrm{Tr}\, \tilde{F}^\mu$, i.e., the two currents may be identified. Thus the theory has a topological symmetry $U(1)_T$ (it was called $U(1)_b$ in [11]). ABJM proposed that at $k = 1, 2$ $U(1)_T \times Spin(6)_R$ is enhanced to $Spin(8)$. The adjoint of $Spin(8)$ decomposes under $U(1) \times Spin(6)$ as follows:

$$\mathbf{28} = \mathbf{15}_0 \oplus \mathbf{1}_0 \oplus \mathbf{6}_1 \oplus \mathbf{6}_{-1}.$$

Here the subscript indicates the $U(1)_T$ charge. The first two subrepresentations correspond to the $U(1) \times Spin(6)$ currents. The last two subrepresentations have nonvanishing topological charge and therefore the corresponding currents are monopole operators. Our goal is to show that such monopole operators indeed exist for $k = 1, 2$.

More precisely, we will see that for $k = 1, 2$ monopole currents have $U(1)_T$ charge $\pm 2/k$. If we want the charge to be ± 1 for both values of k, we need to change the normalization of the $U(1)_T$ current. From now on we will define the $U(1)_T$ current as

$$J_T^\mu = -\frac{k}{4\pi} \mathrm{Tr}\, F^\mu.$$

The sign is convention dependent.

2.2.2 Deformation to Weak Coupling

Since the ABJM model is strongly coupled at k of order 1, we will deform it by adding terms to the action suppressing fluctuations of all fields. The additional terms in the action are multiplied by a parameter t, so that the deformed theory becomes essentially free in the limit $t \to \infty$. In order to be able to relate the spectrum of operators in the deformed and undeformed theory, we need to have some control over the behavior of the theory as t is decreased from $+\infty$ to 0. A measure of control is achieved if the additional terms are Q-exact for some nilpotent supercharge Q. To construct a deformation with all the desired properties we follow S. Kim [16].

We pick the supercharge $Q \equiv Q_{12-}$ where "$-$" stands for the spinor index corresponding to $j_3 = -\frac{1}{2}$. Quantum numbers of supercharges $Q_\pm = Q_{12\pm}$, their Hermitean conjugates (in the radial quantization) $S^\pm$, and the fields of the theory are summarized in table 2.1.

The deformation

$$\begin{aligned} \Delta\mathcal{L}_V &= (rW^\alpha W_\alpha + r\tilde{W}^\alpha \tilde{W}_\alpha)|_{\theta^2} = \\ &= \frac{1}{2} r \left((F_\mu - D_\mu \sigma)^2 - D^2 + \lambda \sigma^\mu D_\mu \bar{\sigma} \right) + \frac{1}{2} r \left((\tilde{F}_\mu - \tilde{D}_\mu \tilde{\sigma})^2 - \tilde{D}^2 + \tilde{\lambda} \sigma^\mu D_\mu \bar{\tilde{\sigma}} \right) \end{aligned} \quad (2.9)$$

proposed in [16] suppresses fluctuations of the fields (A_μ, σ) and $(\tilde{A}_\mu, \tilde{\sigma})$. This expression is Q-exact for the following reason. Recall that $W^\alpha W_\alpha$ (and $\tilde{W}^\alpha \tilde{W}_\alpha$) is a chiral superfield

fields	h_1	h_2	h_3	j_3	ϵ
(A_1, A_2)	$(-\frac{1}{2}, -\frac{1}{2})$	$(\frac{1}{2}, -\frac{1}{2})$	$(\frac{1}{2}, -\frac{1}{2})$	0	$\frac{1}{2}$
$(B_{\dot{1}}, B_{\dot{2}})$	$(-\frac{1}{2}, -\frac{1}{2})$	$(\frac{1}{2}, -\frac{1}{2})$	$(-\frac{1}{2}, \frac{1}{2})$	0	$\frac{1}{2}$
$(\psi_{1\pm}, \psi_{2\pm})$	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, -\frac{1}{2})$	$(\frac{1}{2}, -\frac{1}{2})$	$\pm\frac{1}{2}$	1
$(\chi_{\dot{1}\pm}, \chi_{\dot{2}\pm})$	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, -\frac{1}{2})$	$(-\frac{1}{2}, \frac{1}{2})$	$\pm\frac{1}{2}$	1
$A_\mu, \tilde{A}_\mu$	0	0	0	$(1, 0, -1)$	1
$\lambda_\pm, \tilde{\lambda}_\pm$	-1	0	0	$\pm\frac{1}{2}$	$\frac{3}{2}$
$\sigma, \tilde{\sigma}$	0	0	0	0	1
$Q_\pm$	1	0	0	$\pm\frac{1}{2}$	$\frac{1}{2}$
$S^\pm$	-1	0	0	$\mp\frac{1}{2}$	$-\frac{1}{2}$

Table 2.1: Quantum numbers of fields and supercharges. The charges h_1, h_2, h_2 are weights of $Spin(6)$ R-symmetry, j_3 is the projection of spin, ϵ is the conformal dimension. Our conventions are such that spinors of $Spin(6)$ have half-integral weights.

which can be written in the form $W^\alpha W_\alpha = A(y) + \sqrt{2}\theta\Psi(y) + \theta^2 F(y)$ with $\sqrt{2}\xi F = \xi Q\Psi$ and $y^\mu \equiv x^\mu + i\theta\sigma^\mu\bar{\theta}$. In other words, the component $W^\alpha W_\alpha|_{\theta^2}$ (as well as $\tilde{W}^\alpha \tilde{W}_\alpha|_{\theta^2}$) is Q-exact, and multiplication by r does not change this fact. Of course, we lost invariance with respect to $\bar{Q}$. Note that terms $\bar{W}^\alpha \bar{W}_\alpha|_{\theta^2}$ and $\bar{\tilde{W}}^\alpha \bar{\tilde{W}}_\alpha|_{\theta^2}$ are not included in the deformation.

On the other hand, $\Delta\mathcal{L}_V$ does not suppress fluctuations of the chiral multiplets which therefore interact strongly via the quartic superpotential. It is easy to come up with a Q-exact term serving to fix this problem. For chiral multiplets whose scalars have conformal dimension 1/2 we introduce the usual kinetic term

$$\Delta\mathcal{L}_h = \mathrm{Tr}\left[\int d^2\theta d^2\bar{\theta}\bar{A}_a e^{-2V} A^a + \int d^2\theta d^2\bar{\theta}\bar{B}_{\dot{a}} e^{2V} B^{\dot{a}}\right] + \mathrm{Tr}\left[\int d^2\theta d^2\bar{\theta}\bar{A}_a e^{2\tilde{V}} A^a + \int d^2\theta d^2\bar{\theta}\bar{B}_{\dot{a}} e^{-2\tilde{V}} B^{\dot{a}}\right] \tag{2.10}$$

Strictly speaking, what is Q-exact is not this expression but another one differing by a total derivative. This makes no difference as we integrate the Lagrangian over the entire space-time to construct the action. The full Lagrangian on $\mathbb{R}^3/\{0\}$

$$\mathcal{L}_t = \mathcal{L}_0 + t\Delta\mathcal{L}_V + t\Delta\mathcal{L}_h. \tag{2.11}$$

gives rise to a Lagrangian on $S^2 \times \mathbb{R}$ which determines a deformation of the ABJM model. The deformed theory on $S^2 \times \mathbb{R}$ becomes free in the limit $t \to \infty$.

2.2.3 Monopole Operators

Using the state-operator correspondence in the undeformed ABJM theory, we replace the study of BPS monopole operators with the study of BPS states on $S^2 \times \mathbb{R}$ with a magnetic flux on S^2. Such a magnetic flux corresponds to a singular gauge field on $\mathbb{R}^3$ of the form

$$F^\mu \sim \frac{H}{2}\frac{\hat{r}^\mu}{r^2}, \quad \tilde{F}^\mu \sim \frac{\tilde{H}}{2}\frac{\hat{r}^\mu}{r^2},$$

where $\hat{r}^\mu$ is the unit vector in the radial direction and $H = (n_1, \dots, n_N)$ and $\tilde{H} = (\tilde{n}_1, \dots, \tilde{n}_N)$ are GNO charges for the two $U(N)$ factors of the gauge group. The BPS equations $F_\mu = D_\mu \sigma$, $\tilde{F}_\mu = D_\mu \tilde{\sigma}$ imply that a BPS field configuration must have singular σ and $\tilde{\sigma}$:

$$\sigma \sim \frac{H}{2r}, \quad \sigma \sim \frac{\tilde{H}}{2r}.$$

After a conformal rescaling $\sigma \to \sigma/r$, $\tilde{\sigma} \to \tilde{\sigma}/r$ needed to go from $\mathbb{R}^3$ to $S^2 \times \mathbb{R}$ this becomes a constant scalar background at $\tau = \log r = -\infty$:

$$\sigma \sim \frac{1}{2}H, \quad \sigma \sim \frac{1}{2}\tilde{H}.$$

Another way to understand these values for scalars is to note that for τ-independent fields the action for bosonic fields $A, \tilde{A}, \sigma, \tilde{\sigma}$ reduces to

$$\frac{t}{2}\int d\tau d\Omega \left(\mathrm{Tr}(F_{12} - \sigma)^2 + \mathrm{Tr}(\tilde{F}_{12} - \tilde{\sigma})^2\right)$$

where F_{12} is the magnetic field on S^2. Thus for constant magnetic fields $H/2, \tilde{H}/2$ the absolute minimum of the action is reached for $\sigma = H/2$, $\tilde{\sigma} = \tilde{H}/2$.

This a good place to discuss the difference between the deformation we use and the one used by Benna, Klebanov, and Klose [13]. An obvious difference is that our deformation is time independent if one regards the factor $\mathbb{R}$ in $S^2 \times \mathbb{R}$ as time, while the BKK deformation is time dependent and interpolates between weak coupling in the far past and strong coupling in the far future. Using a time-independent deformation has the advantage that one can compute the conformal dimensions of monopole operators. Another important difference is that the BKK deformation introduces three dynamical scalar fields in the adjoint representation (the scalar part of the $\mathcal{N}=3$ vector multiplet) which transform as a triplet of $SU(2)_R$ symmetry. This leads to a continuous degeneracy of classical vacua which are parametrized by points in a 2-sphere.[3] BKK propose to deal with this degeneracy by regarding the 2-sphere as a space of collective coordinates and quantizing it. In contrast, we introduce only one dynamical scalar in the adjoint. For a given magnetic flux there is a unique value of the scalar which minimizes the energy, and therefore a unique classical vacuum.

2.2.4 Strategy of the Computation

We would like to study the spectrum of monopole operators in the ABJM theory using the above deformation. Let us outline the idea of the computation. Since the deformation we use preserves the dilatational symmetry, it is best to think about the theory we want to study as defined on $S^2\times\mathbb{R}$ with a product metric. Then dilatational invariance becomes translational invariance of $\mathbb{R}$, which we therefore regard as Euclidean time. Local operators in a conformal theory on $\mathbb{R}^3$ are in one-to-one correspondence with states on $S^2\times\mathbb{R}$. Instead of studying monopole operators on $\mathbb{R}^3$ we will study states on $S^2\times\mathbb{R}$ with nonabelian magnetic flux (GNO charge) on S^2. The deformed theory on $S^2\times\mathbb{R}$ is not conformal, so once the deformation is turned on we can only talk about states not operators. The deformation we have constructed breaks the number of supersymmetry generators (counting the superconformal ones) from 24 down to 4. They can be assembled into a spinor representation of $SO(3)$, the rotational symmetry of S^2. The supercharge Q used to construct the deformartion is a particular component of the spinor with $j_3=-1/2$. The R-symmetry group $Spin(6)$ is broken down to $U(1)_R\times Spin(4)$, so that the supercharges have charge 1 with respect to the $U(1)_R$ subgroup and are $Spin(4)$-singlets.

We will call a state annihilated by both Q and $Q^\dagger$ a BPS state. One reason to be interested in BPS states is because their spectrum changes in a controlled manner as one varies the deformation parameter t. For example, a BPS state $|\psi\rangle$ can disappear only if it pairs up with another BPS state $|\psi'\rangle$ whose quantum numbers are related to those of $|\psi\rangle$ in a well-defined

[3]This is a 2-sphere in the space of scalars and is acted on by $SU(2)_R$. It should not be confused with the S^2 on which the theory lives.

manner. If such a BPS state $|\psi'\rangle$ is absent, the BPS state $|\psi\rangle$ is stable with respect to deformations. Using such considerations we will infer that at $t = 0$ and $k = 1, 2$ there exist scalar BPS states which transform in a particular representation of $Spin(4) \times U(1)_R \times U(1)_T$. Another reason to be interested in BPS states is that supersymmetry algebra on $S^2 \times \mathbb{R}$ implies that the energy of a BPS state is related to its $U(1)_R$ charge and spin:

$$E = h_1 + j_3,$$

where h_1 is the $U(1)_R$ charge. From this we will infer that the BPS states we will have found have energy 1 for all t, including $t = 0$.

We now recall that at $t = 0$ the theory has at least $\mathcal{N} = 6$ superconformal symmetry, and therefore the scalar BPS states must be part of some $Spin(6) \times U(1)_T$ multiplet. We will argue that this multiplet must be $\mathbf{10}_{-1}$, i.e., a 3rd-rank anti-selfdual skew-symmetric tensor with $U(1)_T$ charge -1. Acting on it with two supercharges we can get vector states with energy 2 which transform in $\mathbf{6}_{-1}$ of $Spin(6)$. By state-operator correspondence (which holds only at $t = 0$!) we will be able to conclude that the ABJM theory at $k = 1, 2$ has conserved currents realized by monopole operators which transform in $\mathbf{6}_{-1}$ of $Spin(6) \times U(1)_T$. By charge-conjugation symmetry, there are also monopole currents in $\mathbf{6}_1$. Conserved currents in any theory must fit into an adjoint of a Lie group, and it is easy to see that monopole currents together with the $U(1)_T$ current and the $Spin(6)$ currents assemble into an adjoint of $Spin(8)$. This implies that the superconformal symmetry must also be enhanced at least to $\mathcal{N} = 8$ superconformal symmetry.

2.2.5 Quantization of the Deformed ABJM Theory

In the limit $t \to \infty$ fluctuations of all fields, inlcuding A and σ, are suppressed, and each magnetic flux gives rise to a sector (summand) in the Hilbert space of the theory. If we ignore the issue of gauge invariance, each sector is a Fock space for excitations of fields coupled to a monopole background but not between themselves. (The constraints following from gauge invariance will be discussed later). The amount of magnetic flux for each mode is summarized in table 2.2.

Mode	Flux
$(A_\mu)_{ij}, \sigma_{ij}$	$n_i - n_j$
$(\tilde{a}_{ij})_\mu, \tilde{\sigma}_{ij}$	$\tilde{n}_i - \tilde{n}_j$
λ_{ij}	$n_i - n_j$
$\tilde{\lambda}_{ij}$	$\tilde{n}_i - \tilde{n}_j$
$(A_1, A_2)_{ij}$	$n_i - \tilde{n}_j$
$(\psi_1, \psi_2)_{ij}$	$n_i - \tilde{n}_j$
$(B_1, B_2)_{ij}$	$\tilde{n}_i - n_j$
$(\chi_1, \chi_2)_{ij}$	$\tilde{n}_i - n_j$

Table 2.2: Magnetic flux for gauge and matter modes in the ABJM theory. The integers $n_i, \tilde{n}_i$, $i = 1, \ldots, N$, are GNO charges of the monopole state.

The energy spectrum of a free chiral field X in a Dirac monopole background with magnetic flux q was calculated in [7]. In appendix A we summarize these results and also compute the energy spectrum of a vector multiplet. To perform the computation for the vector multiplet one needs to choose a gauge. If the GNO charges are all zero, the most convenient choice is the three-dimensional Coulomb gauge which says that the spatial part of the gauge field A is divergence free. If the GNO charges are nonzero, the vacuum value of the scalar field breaks the gauge symmetry down to a subgroup. Consider a general monopole background[4] with flux $(\{n_i\})$ where the first k_1 fluxes are equal and strictly greater in magnitude than the second group of equal fluxes and so on until the last group of k_m equal fluxes with the obvious condition $k_1 + k_2 + \cdots + k_m = N$. A choice of a classical vacuum σ_0 breaks the gauge group down to a subgroup $U(k_1) \times U(k_2) \times \cdots \times U(k_m)$ represented by block-diagonal matrices. This means that we can choose a "unitary" gauge for the quantum part $\sigma - \sigma_0$ by requiring it to be block-diagonal as well. For the residual $U(k_1) \times U(k_2) \times \cdots \times U(k_m)$ gauge symmetry we again use the three-dimensional Coulomb gauge.

The outcome of this computation is that none of the fields have zero modes, and therefore one may quantize each magnetic flux sector by defining the vacuum in this sector as the unique state annihilated by all annihilation operators. We will refer to the vacuum state as the bare monopole.

[4]For definiteness,we focus on one of the $U(N)$ factors in the $U(N) \times U(N)$ gauge group.

2.2.6 Quantum Numbers of Bare Monopoles

To compute quantum numbers of the bare monopole we follow the usual procedure. For definiteness, let us discuss the computation of energy. We regularize the vacuum energy using point splitting in the time direction and subtract a similarly regularized vacuum energy for the trivial magnetic flux sector. The difference has a well-defined limit as one removes the regulator and gives the renormalized energy of the bare monopole. The final answer for the energy is (see appendix A for details):

$$E = \sum_{i,j=1}^{N} |n_i - \tilde{n}_j| - \sum_{i<j} |n_i - n_j| - \sum_{i<j} |\tilde{n}_i - \tilde{n}_j|. \tag{2.12}$$

The first term is the contribution of chiral multiplets, the second and third terms are the contributions of the vector multiplets for first and second factors in the $U(N) \times U(N)$ gauge group respectively. The same result was obtained in [16] and [13].

It is easy to show that the energy of a bare monopole is nonnegative; it is equal to zero if and only if $n_i = \tilde{n}_i$ for all i.

The $U(1)_R$ charge of a bare monopole is equal to its energy. This happens because the bare monopole is a BPS state. It transforms in a trivial representation of $Spin(4)_R$ and the rotational $SU(2)$ symmetry. The topological charges are $\sum_i n_i$ and $\sum_i \tilde{n}_i$; as discussed above the equations of motion imply that they are equal. The $U(1)_T$ charge is $-\frac{k}{2}\sum_i n_i$.

Gaiotto and Witten proposed in [19] a general formula for the R-charge of a bare monopole in an $\mathcal{N} = 4$ $d = 3$ gauge theory. According to this formula the R-charge receives a contribution $|q|/2$ from every (twisted) hypermultiplet which couples to magnetic flux q and a contribution $-|q|/2$ from every charged component of a vector multiplet which couples to magnetic flux q. For example, in ABJM theory for every pair of indices i, j there is a hypermultiplet and a twisted hypermultiplet which both couple to magnetic flux $n_i - \tilde{n}_j$ and four vector multiplets which couple to magnetic fluxes $\pm(n_i - n_j)$ and $\pm(\tilde{n}_i - \tilde{n}_j)$. Our computation in appendix A can be viewed as a derivation of the Gaiotto-Witten formula valid for an arbitrary three-dimensional gauge theory with at least $\mathcal{N} = 3$ supersymmetry. Another derivation can be found in [13].

2.2.7 Gauss Law Constraint

So far the value of the Chern-Simons coupling appeared irrelevant. Its significance emerges when we turn to the Gauss law constraint. The Coulomb gauge for the residual $U(k_1) \times \cdots \times U(k_m)$ symmetry does not fix the gauge symmetry completely: we still have the freedom to perform constant gauge transformations on S^2. Physical states must be annihilated by the charge corresponding to this symmetry. In the undeformed theory this charge is

$$-\frac{k}{2\pi} \int_{S^2} F_{12} + N,$$

where N is the gauge charge of the matter fields. Similarly, the second $U(N)$ factor in the gauge group is broken down to $U(\tilde{k}_1) \times \cdots \times U(\tilde{k}_{\tilde{m}})$ by the scalar background, and the charge for constant gauge transformations is

$$\frac{k}{2\pi} \int_{S^2} \tilde{F}_{12} + \tilde{N},$$

where $\tilde{N}$ is the gauge charge of the matter fields. These formulas remain true in the deformed theory if we understand N and $\tilde{N}$ to include the charges of fields in the vector multiplet, i.e., σ, $\tilde{\sigma}$, and the gauginos.[5] Thus the gauge charges have a Chern-Simons contribution and a matter contribution.

Note that in a given magnetic flux sector the Chern-Simons contribution to the charge is a c-number. Concretely, for the $U(k_i)$ factor in the residual gauge group the Chern-Simons contribution is $-kn_i$, and for the $U(\tilde{k}_i)$ factor the Chern-Simons contribution is $k\tilde{n}_i$. We may interpret this as saying that the bare monopole has $U(k_i)$ charge $-kn_i$ and $U(\tilde{k}_i)$ charge $k\tilde{n}_i$. Physical states must have vanishing gauge charge, so bare monopoles are not physical if $k \neq 0$. To construct physical states we need to act on bare monopoles by creation operators of matter fields or fields in the vector multiplet.

[5] The term $\partial_i E_i$ in the Gauss law constraint does not contribute because it is a total derivative and integrates to zero on S^2.

2.2.8 Superconformal Multiplet of the Stress-Tensor

The following table summarizes some conformal primaries of the $\mathcal{N} = 8$ superconformal rmultiplet which includes the stress-tensor.

	E	j	R (h.w.)
Φ	1	0	(1, 1, 1, -1)
Ψ_α	3/2	1/2	(1, 1, 1, 0)
$R_{\alpha\beta}$	2	1	(1, 1, 0, 0)
$\mathcal{Q}_{\alpha\beta\gamma}$	5/2	3/2	(1, 0, 0, 0)
$T_{\alpha\beta\gamma\delta}$	3	2	(0, 0, 0, 0)

Table 2.3: Some conformal primaries of the $\mathcal{N} = 8$ stress-tensor multiplet. Greek letters denote space-time spinor indices.

Operators at zero level transform as the rank-four anti-selfdual tensor of $Spin(8)_R$. With respect to its subgroup $Spin(6)_R \times U(1)_T$ this representation decomposes as $\mathbf{15}_0 \oplus \mathbf{10}_{-1} \oplus \bar{\mathbf{10}}_1$. Here $\mathbf{15}$ is the adjoint representation of $Spin(6)_R$. In the ABJM theory it is built from the fundamental fields and is given by $tr_G(C_I C_J^\dagger) - \frac{1}{4} tr_G(\sum_{I=1}^4 C_I C_I^\dagger)$.[6] Representations $\mathbf{10}$ and $\bar{\mathbf{10}}$ carry topological $U(1)_T$ charge -1 and 1, respectively, and should be realized by monopole operators.[7] Their highest weights are $(1, 1, 1)$ and $(1, 1, -1)$, respectively.

Our method is based on studying a deformation which breaks $Spin(6)_R$ symmetry down to $Spin(4) \times U(1)_R \simeq SU(2) \times SU(2) \times U(1)_R$, so we need to decompose $\mathbf{10}$ and $\bar{\mathbf{10}}$ with respect to this subgroup and identity BPS states in these representations. The decompositions look as follows:

$$\mathbf{10} = (\mathbf{2}, \mathbf{2})_0 \oplus (\mathbf{3}, \mathbf{1})_1 \oplus (\mathbf{1}, \mathbf{3})_{-1}, \quad \bar{\mathbf{10}} = (\mathbf{2}, \mathbf{2})_0 \oplus (\mathbf{3}, \mathbf{1})_{-1} \oplus (\mathbf{1}, \mathbf{3})_1.$$

Scalar BPS states have $E = h_1$ and live in representations with positive $U(1)_R$ charge,

[6] The trace is over the gauge indices.

[7] This was also mentioned in [20].

i.e., $(\mathbf{3},\mathbf{1})_1$ and $(\mathbf{1},\mathbf{3})_1$.These two representations have opposite $U(1)_T$ charge: -1 for the former one and $+1$ for the latter one. Scalar anti-BPS states[8] have $E = -h_1$ and live in representations with negative $U(1)_R$ charge, i.e., $(\mathbf{1},\mathbf{3})_{-1}$ and $(\mathbf{3},\mathbf{1})_{-1}$. They have $U(1)_T$ charges -1 and $+1$, respectively. Assuming that BPS states survive in the deformed theory (we will justify this assumption below), we expect to see them as elements of the Fock space built on a bare monopole.

The GNO charge of a monopole state with $U(1)_T$ charge 1 must either have the form $(n, 0, \ldots, 0)$ with $kn = 2$, or $(n, n, 0, \ldots, 0)$ with $kn = 1$ (for both $U(N)$ factors). Indeed, the energy of a bare monopole is a nonnegative integer. If it is nonzero, then it cannot give rise to a physical state with energy 1, because to construct such a state one needs to act on the bare monopole with creation operators, and they all have positive energy. If the energy of the bare monopole is zero, then the GNO charges for the two $U(N)$ factors must be identical. Further, to construct physical states we need to act on bare monopole states by creation operators, and it is easy to see that these must be creation operators for chiral multiplets, so that the energy does not exceed 1. Bosonic creation operators for chiral multiplets have energy $1/2$ or larger, while fermionic creation operators have energy at least 1. Hence the states we are looking for must be obtained by acting on the bare monopole by two bosonic creation operators with the lowest possible energy. Such a state can satisfy the Gauss law constraint only if the GNO charges are of the above form.

Since both k and n are integral, for $k = 2$ there is a unique possible GNO charge $(1, 0, 0, \ldots, 0)$. For $k = 1$ there are two possible GNO charges: $(2, 0, \ldots, 0)$ and $(1, 1, 0, \ldots, 0)$. For $k > 2$ there are no candidate GNO charges, and therefore no BPS scalars with $E = 1$. This agrees with the expectation that for $k > 2$ there is no supersymmetry enhancement. The difference between the $k = 1$ and $k = 2$ case arises from the fact that in the former case the theory has two copies of $\mathcal{N} = 8$ superconformal algebra as discussed below.

For $k = 2$ we have the following BPS states with $E = 1$ satisfying the Gauss law

[8] That is, states annihiated by both $\bar{Q}$ and $(\bar{Q})^\dagger$.

constraint:[9]

$$(\mathbf{3},\mathbf{1})_1 \sim \bar{A}_a^{1\tilde{1}}(j=0)\bar{A}_b^{1\tilde{1}}(j=0)|1,0,0,...,0\,\rangle, \tag{2.13}$$

$$(\mathbf{1},\mathbf{3})_1 \sim \bar{B}_{\dot{a}}^{1\tilde{1}}(j=0)\bar{B}_{\dot{b}}^{1\tilde{1}}(j=0)|-1,0,0,...,0\,\rangle. \tag{2.14}$$

Here $|n_1, n_2, \ldots, n_N\,\rangle$ denotes the bare monopole with the indicated GNO charge in one $U(N)$ subgroup and identical charge in the other $U(N)$ subgroup. The $Spin(4) \times U(1)_R \times U(1)_T$ quantum numbers of these states are exactly as predicted by enhanced supersymmetry. Similarly, the anti-BPS states are obtained by acting on bare anti-BPS monopoles with lowest-energy modes of A_a and $B_{\dot{a}}$.

For $k = 1$ we have very similar scalar BPS states with $E = 1$:

$$(\mathbf{3},\mathbf{1})_1 \sim \bar{A}_a^{1\tilde{1}}(j=0)\bar{A}_b^{1\tilde{1}}(j=0)|2,0,0,...,0\,\rangle, \tag{2.15}$$

$$(\mathbf{1},\mathbf{3})_1 \sim \bar{B}_{\dot{a}}^{1\tilde{1}}(j=0)\bar{B}_{\dot{b}}^{1\tilde{1}}(j=0)|-2,0,0,...,0\,\rangle. \tag{2.16}$$

In addition, we have the following scalar BPS states with $E = 1$ and $U(1)_T$ charge ∓ 1:

$$(\mathbf{3},\mathbf{1})_1 \sim \epsilon_{pp'}\epsilon_{\tilde{q}\tilde{q}'}\bar{A}_a^{p\tilde{q}}(j=0)\bar{A}_b^{p'\tilde{q}'}(j=0)|1,1,0,...,0\,\rangle, \tag{2.17}$$

$$(\mathbf{1},\mathbf{3})_1 \sim \epsilon_{pp'}\epsilon_{\tilde{q}\tilde{q}'}\bar{B}_{\dot{a}}^{p\tilde{q}}(j=0)\bar{B}_{\dot{b}}^{p'\tilde{q}'}(j=0)|-1,-1,0,...,0\,\rangle. \tag{2.18}$$

The indices p, p' and $\tilde{q}, \tilde{q}'$ take values in the set $\{1,2\}$. The manner in which these indices are contracted is determined uniquely by the the Gauss law constraint. Indeed, the GNO magnetic flux breaks the gauge symmetry down to $U(2) \times U(2) \times U(N-2) \times U(N-2)$. The Gauss law constraint for $k = 1$ says that the combination of oscillators acting on the bare monopole $|\pm 1, \pm 1, 0, \ldots, 0\,\rangle$ must be a singlet of the $SU(2) \times SU(2) \times U(N-2) \times U(N-2)$ subgroup and have charge ∓ 2 under the $U(1)$ subgroups of both $U(2)$ factors. The requirement of $SU(2) \times SU(2)$ invariance tells us that gauge indices must be contracted with epsilon tensors.

[9]The superscripts are gauge indices.

2.2.9 Evidence for Duality at $k = 1$

The existence of extra scalar BPS states (2.17) might seem surprising, but in fact it is implied by the conjecture that for $k = 1$ the ABJM theory is dual to the IR limit of $\mathcal{N} = 8$ $U(N)$ super-Yang-Mills theory. For $N > 1$ the latter theory decomposes into two noninteracting sectors corresponding to the decomposition of the adjoint of $U(N)$ into trace and traceless parts. The trace sector is a free $\mathcal{N} = 8$ $U(1)$ gauge theory which flows in the infrared to a free $\mathcal{N} = 8$ SCFT (a free $\mathcal{N} = 4$ hypemultiplet plus a free $\mathcal{N} = 4$ twisted hypermultiplet). The traceless part flows to an interacting $\mathcal{N} = 8$ SCFT. Thus we expect that for $N > 1$ the $k = 1$ ABJM theory has a decoupled sector which is the free $\mathcal{N} = 8$ SCFT described above, and correspondingly has two copies of $\mathcal{N} = 8$ superconformal symmetry algebra. This is the reason we see the doubling of $E = 1$ BPS scalars at $k = 1$. Note also that the extra BPS states (2.17) exist only for $N > 1$.

We can go further and directly demonstrate the presence of a free sector in the $k = 1$ ABJM theory. In a unitary conformal three-dimensional theory a free scalar must have dimension 1/2. A free $\mathcal{N} = 8$ SCFT contains eight real scalars which transform in a spinor representation of $Spin(8)$. With respect to the $Spin(4) \times U(1)_R \times U(1)_T \simeq SU(2) \times SU(2) \times U(1)_R \times U(1)_T$ subgroup they transform as

$$(\mathbf{2}, \mathbf{1})_{1/2,-1/2} \oplus (\mathbf{1}, \mathbf{2})_{1/2,1/2} \oplus (\mathbf{2}, \mathbf{1})_{-1/2,1/2} \oplus (\mathbf{1}, \mathbf{2})_{-1/2,-1/2}.$$

The first two subrepresentations are BPS, and the last two are anti-BPS. The corresponding BPS states in the deformed theory are

$$(\mathbf{2}, \mathbf{1})_{1/2,-1/2} \sim \bar{A}_a^{1\tilde{1}}(j = 0)|1, 0, \ldots, 0\rangle, \tag{2.19}$$

$$(\mathbf{1}, \mathbf{2})_{1/2,1/2} \sim \bar{B}_{\dot{a}}^{1\tilde{1}}(j = 0)| - 1, 0, \ldots, 0\rangle. \tag{2.20}$$

Similarly the anti-BPS states satisfying the Gauss law constraint can be obtained by acting on bare anti-BPS monopoles with a single creation operator for $A_a^{1\tilde{1}}$ or $B_{\dot{a}}^{1\tilde{1}}$. All these states have $E = 1/2$, and if the spectrum of BPS scalars does not change as one decreases t from $t = \infty$ to $t = 0$, then these states must correspond to free scalar fields in the undeformed theory. Acting on them with supercharges we get the free sector of the theory. Note that it is not possible to construct BPS states with $E = 1/2$ satisfying the Gauss law constraint for $k > 1$.

2.2.10 Protected States and Enhanced Supersymmetry

We have seen above that $\mathcal{N} = 8$ supersymmetry of the ABJM theory implies the existence of scalar BPS states in particular representations of $Spin(4) \times U(1)_R \times U(1)_T$, and that such states do indeed exist in the weakly coupled limit for $k = 1, 2$. In this subsection we will argue that such scalar BPS states are protected and their existence at $t = 0$ implies their existence at $t = \infty$ and vice versa. Then we will reverse the logic and show that existence of scalar BPS states in the weakly coupled limit implies that R-symmetry at $t = 0$ is enhanced from $Spin(6)$ to $Spin(8)$. This in turn implies that supersymmetry is enhanced from $\mathcal{N} = 6$ to $\mathcal{N} = 8$.

The argument that BPS states are protected is standard and based on the observation that as one varies a parameter cohomology classes appear and disappear in pairs, so that members of the pair have R-charge differing by 1 and energy and j_3 differing by 1/2. Thus the number of scalars with R-charge 1 can change as one varies t only if there exist either BPS spinors with $R = 0$, $E = 1/2$ or BPS spinors with $R = 2$, $E = 3/2$. These spinors must also transform in $(\mathbf{3}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{3})$ and have $U(1)_T$ charge ∓ 1. At $t = 0$ there can be no such states because they would violate unitarity bounds. Therefore scalar BPS states predicted by $\mathcal{N} = 8$ supersymmetry cannot disappear at $t > 0$, and this is why we expect to see them at $t = \infty$. Conversely, we can explicitly check that at $t = \infty$ there are no spinor BPS states with $R = 0$, $E = 1/2$ or $R = 2, E = 3/2$ in the sectors with $U(1)_T$ charge ± 1 (see appendix B). However, in principle, at some intermediate value of the deformation parameter t where we cannot check the spectrum of BPS states some BPS states could pair up with others to form long multiplets. Fortunately, this never happens to the scalar BPS states with minimal energy $E = 1/2$ or $E = 1$. A quick and easy way to see this is to consider the superconformal index for the theory under consideration. We will explain this argument after we introduce the superconformal index in chapter 3.

The conclusion is that the states (2.13), (2.15, (2.17) are protected and cannot disappear as one decreases t to 0.

We have established that in the undeformed ABJM theory with $k = 1, 2$ there exist scalar BPS states which transform in the following representations of $Spin(4) \times U(1)_R \times U(1)_T$:

$$(\mathbf{3}, \mathbf{1})_{1,-1} \oplus (\mathbf{1}, \mathbf{3})_{1,1}.$$

There are also anti-BPS states which are obtained from the BPS states by charge conjugation; they transform as

$$(\mathbf{3}, \mathbf{1})_{-1,1} \oplus (\mathbf{1}, \mathbf{3})_{-1,-1}.$$

At $t = 0$ these states must be part of some $Spin(6) \times U(1)_T$ multiplets. A generic state in a $Spin(6)$ multiplet is not BPS, but if it contains any BPS states at all, the highest weight state must be among them (otherwise the unitarity bound $E \geq h_1$ would be violated). Hence the $Spin(6)$ multiplet containing the $Spin(4) \times U(1)_R$ multiplet $(\mathbf{3}, \mathbf{1})_1$ must have the highest weight $(1, 1, 1)$. This is the representation $\mathbf{10}_{-1}$ of $Spin(6) \times U(1)_T$. It also contains anti-BPS states in the representation $(\mathbf{1}, \mathbf{3})_{-1,-1}$ of $Spin(4) \times U(1)_R \times U(1)_T$. By charge-conjugation symmetry, the BPS states in $(\mathbf{1}, \mathbf{3})_{1,1}$ and $(\mathbf{3}, \mathbf{1})_{-1,1}$ are parts of the $Spin(6) \times U(1)_T$ multiplet $\bar{\mathbf{10}}_1$ with highest weight $(1, 1, -1)$.

Now let us act on these scalar states with two supercharges with symmetrized spinor indices and antisymmetrized $Spin(6)$ indices. This combination of supercharges transforms as a vector with respect to rotations and as a rank-2 antisymmetric tensor with respect to $Spin(6)$. Since $\mathbf{10}$ and $\bar{\mathbf{10}}$ are self-dual and anti-self-dual components of a rank-3 antisymmetric tensor, acting on them with this combination of supercharges will produce, among other things, states which are vectors with respect to both $Spin(6)$ and the rotation group. They also have $U(1)_T$ charge -1 and $+1$, respectively and energy 2. Local operators corresponding to such states must be conserved currents, by unitarity.

To complete the argument we only need to show that the vector states in $\mathbf{6}_1$ and $\mathbf{6}_{-1}$ constructed as above are nonzero. The norm of these states is determined by $\mathcal{N} = 6$ superconformal algebra alone, thus we may use any unitary $\mathcal{N} = 6$ theory where the scalar states $\mathbf{10}$ and $\bar{\mathbf{10}}$ are present and check that the corresponding conserved currents in $\mathbf{6}$ are nonvanishing. For example, one can take a free $\mathcal{N} = 8$ superconformal theory and consider the $\mathcal{N}=8$ superconformal multiplet of the stress energy tensor. When decomposed with respect to $\mathcal{N}=6$ subalgebra it contains both dimension-1 scalars in $\mathbf{10}$ and $\bar{\mathbf{10}}$ (arising from decomposing $\mathbf{35}$ of $Spin(8)$ with respect to $Spin(6)$) and conserved currents in $\mathbf{6}$ (arising from decomposing $\mathcal{N}=8$ R-currents).

We have shown that the ABJM theory at $k=1,2$ has extra conserved currents which transform as $\mathbf{6}_1$ and $\mathbf{6}_{-1}$ with respect to $Spin(6)\times U(1)_T$. Conserved currents in any theory must fit into an adjoint representation of some Lie group. In our case the only possible choice of such a Lie group is $Spin(8)$; its adjoint decomposes with respect to $Spin(6)$ as $\mathbf{15}_0\oplus\mathbf{1}_0\oplus\mathbf{6}_1\oplus\mathbf{6}_{-1}$. This implies that supersymmetry is enhanced from $\mathcal{N}=6$ to $\mathcal{N}=8$.

2.2.11 Construction of States Corresponding to Conserved Currents

Instead of relying on group-theoretic arguments and unitarity, one might try to construct directly vector BPS states with energy 2 at $t=\infty$ and then argue that they persist all the way down to $t=0$. The first step is easily accomplished: the desired states are obtained by acting on the bare monopoles by two bosonic creation operators with spin 1 and spin 0

$$\begin{aligned}
|E=2,j=1\,\rangle_1 &= \epsilon^{\alpha\beta}\bar{A}^{1\tilde{1}}_{\alpha}(j=1)\bar{A}^{1\tilde{1}}_{\beta}(j=0)|2,0,0,...,0\,\rangle, \qquad k=1,\\
|E=2,j=1\,\rangle_1 &= \epsilon^{\alpha\beta}\bar{A}^{1\tilde{1}}_{\alpha}(j=1)\bar{A}^{1\tilde{1}}_{\beta}(j=0)|1,0,0,...,0\,\rangle, \qquad k=2,
\end{aligned} \tag{2.21}$$

or two fermionic creation operators with spin 1/2

$$\begin{aligned}
|E=2,j=1\,\rangle_2 &= \epsilon^{\alpha\beta}\chi^{1\tilde{1}}_{\alpha+}\chi^{1\tilde{1}}_{\beta+}|2,0,0,...,0\,\rangle, \qquad k=1,\\
|E=2,j=1\,\rangle_2 &= \epsilon^{\alpha\beta}\chi^{1\tilde{1}}_{\alpha+}\chi^{1\tilde{1}}_{\beta+}|1,0,0,...,0\,\rangle, \qquad k=2.
\end{aligned} \tag{2.22}$$

In the above formula the superscripts of bosonic and fermionc creation operators are the gauge indices.

As long as we consider $t = \infty$, both states states have the quantum numbers appropriate for a conserved current, and we cannot decide what linear combination of them is the correct one. Presumably, when we consider small nonzero values of $\frac{1}{t}$, this degeneracy is lifted, and the secular equation gives us a unique linear combination of states $|E = 2, j = 1\rangle_1$ and $|E = 2, j = 1\rangle_2$ which has the right quantum numbers to be a conserved current.

Unfortunately, it might happen that all vector BPS states with $E = 2$ "disappear" (i.e. become non-BPS) at $t < \infty$. This appears possible because at $t = \infty$ there are enough fermionic BPS states with $E = 5/2, R = 2$ which could pair up with vector states with $E = 2, R = 1$. It is for this reason that we had to resort to a more roundabout argument using scalar BPS states with $E = 1, R = 1$.

2.3 $\mathcal{N} = 4$ SQCD with an Adjoint Hypermultiplet.

2.3.1 Field Content and RG Flow

The second model we consider is $\mathcal{N} = 4$ $d = 3$ $U(N)$ gauge theory with the following field content: a $U(N)$ vector multiplet, a hypermultiplet in the fundamental representation of $U(N)$, and another hypermultiplet in the adjoint representation (the B-model in the terminology of [21]). We will use $\mathcal{N} = 2$ superfield formalism, so that an adjoint hypermultiplet contains two adjoint chiral superfields which we denote X and $\tilde{X}$, and a fundamental hypermultiplet contains a fundamental chiral superfield f and an antifundamental chiral superfield $\tilde{f}$. The $\mathcal{N} = 4$ vector multiplet contains an $\mathcal{N} = 2$ vector multiplet and an adjoint chiral superfield Φ. This theory is IR-dual to $\mathcal{N} = 4$ $d = 3$ $U(N)$ gauge theory with only a vector multiplet and an adjoint hypermultiplet (the A-model in the terminology of [21]). The A-model has $\mathcal{N} = 8$ supersymmetry in the UV and therefore expected to flow to an IR fixed point with $\mathcal{N} = 8$ superconformal symmetry. More precisely, for $N > 1$ the IR theory has two copies of $\mathcal{N} = 8$ superconformal symmetry. Indeed, both the vector multiplet and the adjoint hypermultiplet have a traceless part and a trace part, and the latter is decoupled at all scales. The trace part can be regarded as an abelian $\mathcal{N} = 8$ gauge theory which flows to a free $\mathcal{N} = 8$ superconformal field theory in the infrared. The traceless part is described by

$SU(N)$ gauge theory and flows to an interacting $\mathcal{N}=8$ superconformal field theory in the infrared. By duality, we expect that the B-model has the same behavior, even though in the UV there is only $\mathcal{N}=4$ supersymmetry, and the only decoupled field is the trace part of the adjoint hypermultiplet. Our goal is to verify these predictions of duality.

The B-model has $SU(2)_R \times SU(2)_N$ R-symmetry with respect to which the supercharges transform as $(\mathbf{2},\mathbf{2})$, the lowest components of the hypermultiplets as $(\mathbf{2},\mathbf{1})$, and the scalars of the vector multiplet as $(\mathbf{1},\mathbf{3})$. In the $\mathcal{N}=2$ superfield formalism only the maximal torus $U(1)_R \times U(1)_N$ of $SU(2)_R \times SU(2)_N$ is manifest. With respect to this subgroup $\mathcal{N}=2$ chiral superfields transform as follows:

$$\begin{aligned}
U(1)_R: &\quad \Phi \to \Phi(e^{-i\alpha}\theta),\\
U(1)_N: &\quad \Phi \to e^{2i\alpha}\Phi(e^{-i\alpha}\theta),\\
U(1)_R: &\quad X \to e^{i\alpha}X(e^{-i\alpha}\theta), \quad \tilde{X} \to e^{i\alpha}\tilde{X}(e^{-i\alpha}\theta),\\
U(1)_N: &\quad X \to X(e^{-i\alpha}\theta), \quad \tilde{X} \to \tilde{X}(e^{-i\alpha}\theta),\\
U(1)_R: &\quad f \to e^{i\alpha}f(e^{-i\alpha}\theta), \quad \tilde{f} \to e^{i\alpha}\tilde{f}(e^{-i\alpha}\theta),\\
U(1)_N: &\quad f \to f(e^{-i\alpha}\theta), \quad \tilde{f} \to \tilde{f}(e^{-i\alpha}\theta).
\end{aligned} \tag{2.23}$$

If we assume that $SU(2)_R \times SU(2)_N$ becomes part of $\mathcal{N}=4$ superconformal symmetry in the infrared, then the IR conformal dimensions of hypermultiplets are the same as in the UV (i.e., scalars have dimension 1/2 and spinors have dimension 1), while the IR conformal dimension of Φ is 1. This means that the kinetic term for the vector multiplet is irrelevant in the IR and may be dropped. In other words, the IR limit is the naive limit $g^2 \to \infty$. While this assumption is very natural, it is not true for all $\mathcal{N}=4$ $d=3$ gauge theories. For example, it is known to fail for the A-model. A necessary condition for the assumption to hold has been formulated by Gaiotto and Witten [19]: the R-charges of all chiral monopole operators must be positive. Here the R-charge is defined as

$$-\frac{1}{2}(h_R + h_N),$$

where h_R and h_N are $U(1)_R$ and $U(1)_N$ charges, respectively. For the A-model the condition is not satisfied since the contributions of the vector multiplet and the adjoint hypermultiplet to the energy cancel ([19], see also a discussion below). For the B-model there is also a contribution of the fundamental hypermultiplet which is strictly positive, so the Gaiotto-Witten condition is satisfied.

2.3.2 Symmetries and Their Expected Enhancement

Let us now discuss the symmetries of the B-model and their expected enhancement in the infrared. Apart from $SU(2)_R \times SU(2)_N \simeq Spin(4)$ symmetry, there is also a flavor $Sp(1) \simeq SU(2)$ symmetry acting on the adjoint hypermultuplet; we will denote it $SU(2)_X$ and its maximal torus will be denoted $U(1)_X$. $SU(2)_X$ acts on $(X, -\tilde{X})$ as a doublet, so X and $\tilde{X}$ have $U(1)_X$ charge ± 1. There are no nontrivial flavor symmetries acting on the fundamental hypermultiplet (the $U(1)$ symmetry is gauged). In addition, there is a topological symmetry $U(1)_T$ whose current is

$$J^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\rho}\mathrm{Tr}F_{\nu\rho}.$$

We expect that in the IR the R-symmetry is enhanced to $Spin(8)$. We propose that the symmetry $Spin(4) \times SU(2)_X \times U(1)_T$ visible in the UV embeds as follows into the $Spin(8)$ group. First of all, $Spin(8)$ has an obvious $Spin(4)\times Spin(4)$ subgroup. We identify the first $Spin(4)$ factor with the $Spin(4)$ R-symmetry visible in the UV. The second $Spin(4)$ factor is isomorphic to a product $SU(2) \times SU(2)$. We identify the first $SU(2)$ factor with $SU(2)_X$, and identify the maximal torus of the second $SU(2)$ factor with $U(1)_T$. In what follows we will denote the second $SU(2)$ factor by $SU(2)_T$.

To motivate this choice of embedding, consider the case when the gauge group is abelian, i.e., $N = 1$. In this case the B-model reduces to an $\mathcal{N} = 4$ SQED with a single charge-1 hypermultiplet plus a decoupled uncharged hypermultiplet. It is well-known that in the IR $\mathcal{N} = 4$ SQED with one charged hypermultiplet flows to a theory of a free twisted hypermultiplet [22]. The lowest component of the free twisted hypermultiplet is constructed as a bare monopole with $U(1)_T$ charge ± 1 [7]. The $U(1)_T$ symmetry of SQED is therefore enhanced in the IR to $SU(2)_T$, with the lowest component of the bare monopole transforming as $(\mathbf{1}, \mathbf{2}, \mathbf{2})$ of $SU(2)_R \times SU(2)_N \times SU(2)_T$. The theory of a free hypermultiplet and a free twisted hypermultiplet is well known to have $\mathcal{N} = 8$ superconformal symmetry. For example, the scalars transform as

$$(\mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{2})$$

of $SU(2)_R \times SU(2)_N \times SU(2)_X \times SU(2)_T$, which corresponds to the decomposition of the spinor of $Spin(8)$.

The adjoint of $Spin(8)$ decomposes with respect to the $SU(2)_R \times SU(2)_N \times SU(2)_X \times U(1)_T$ as follows:

$$\mathbf{28} = (\mathbf{3}, \mathbf{1}, \mathbf{1})_0 \oplus (\mathbf{1}, \mathbf{3}, \mathbf{1})_0 \oplus (\mathbf{1}, \mathbf{1}, \mathbf{3})_0 \oplus \mathbf{1}_0 \oplus \mathbf{1}_2 \oplus \mathbf{1}_{-2} \oplus (\mathbf{2}, \mathbf{2}, \mathbf{2})_1 \oplus (\mathbf{2}, \mathbf{2}, \mathbf{2})_{-1}.$$

Thus we expect to see currents in all these representations. In fact, as explained above, for $N > 1$ we expect to see a doubling of all conserved currents. For example, we expect to see not one but two R-currents which transform as an adjoint of $SU(2)_R \times SU(2)_N$ and a singlet of $SU(2)_X \times U(1)_T$. This might seem surprising: while we already got used to the idea that monopole operators may provide extra conserved currents, the extra currents we need here have vanishing topological charge! The resolution of this conundrum is rather mundane: a monopole operator may have nontrivial GNO charges but vanishing topological charge. This is a new phenomenon which is observed only for a nonabelian gauge group. We will see that all additional operators predicted by duality are monopole operators, some of which have vanishing $U(1)_T$ charge.

Given the assumption about symmetry enhancement, the group $U(1)_R \times U(1)_N \times U(1)_X \times U(1)_T$ can be identified with the maximal torus of $Spin(8)$. More precisely, our convention for the weights h_i of $Spin(8)$ is such that the precise relationship is

$$h_N = -(h_1 - h_2), \quad h_R = -(h_1 + h_2), \quad h_X = h_3 - h_4, \quad h_T = h_3 + h_4. \tag{2.24}$$

The peculiar minus signs in the first two equations arise because we define BPS operators as operators annihilated by Q rather than $\bar{Q}$, i.e., they are elements of the antichiral ring.

2.3.3 Deformation to Weak Coupling

Deformation to weak coupling is constructed along the same lines as for the ABJM model. The only difference is the presence of an adjoint chiral multiplet Φ which is part of the $\mathcal{N} = 4$ vector multiplet. As explained above, its lowest component has dimension 1, and consequently in the IR limit the usual kinetic term should be dropped. Then Φ enters the undeformed action only through the $\mathcal{N} = 4$ superpotential,

$$i\sqrt{2}\mathrm{Tr}\,(\tilde{X}[\Phi, X]) + i\sqrt{2}\mathrm{Tr}\,(\tilde{f}\Phi f). \tag{2.25}$$

Thus in the IR limit Φ is a Lagrange multiplier field whose presence enforces a quadratic constraint on the hypermultiplets. To go to weak coupling we need to suppress its fluctuations. The usual kinetic term on $\mathbb{R}^3$ is not conformally invariant, and adding it would result in an action on $S^2 \times \mathbb{R}$ which is time dependent. Instead, we may use the following Q-exact deformation which is conformally invariant:

$$\Delta\mathcal{L}_\Phi = r \int d^2\theta d^2\bar{\theta}\, \bar{\Phi} e^{-2ad(V)}\Phi. \tag{2.26}$$

Adding the term $\Delta\mathcal{L}_\Phi$ with a large coefficient suppresses fluctuations of Φ. In appendix A we show that the contribution of the field Φ to the energy of a bare monopole vanishes. Essentially this happens because the fermion contribution is the same as for the $\mathcal{N} = 2$ vector multiplet, and because we have scalars instead of vectors, the bosonic contribution increases resulting in a net zero.

There is a way to reach the same conclusion without any computations. Instead of adding the term $\Delta\mathcal{L}_\Phi$ to the action, we add a Q-exact F-term

$$\Delta\mathcal{L}_m(\Phi) = m \int d^2\theta\, \Phi^2$$

It looks like a mass term but is conformally-invariant since the conformal dimension of Φ is 1 rather than 1/2. The nice thing about this Q-exact deformation is that it leaves Φ non-dynamical. Integrating it out, we get a quartic superpotential for the hypermultiplet fields proportional to $1/m$. In the limit $m \to \infty$ the effect of this quartic superpotential disappears, and we see that the field Φ may be simply ignored for the purposes of computing the BPS spectrum on $S^2 \times \mathbb{R}$.

Either way of constructing the deformation leaves only four supercharges unbroken (out of the original sixteen, if we include superconformal generators). If we use $\Delta\mathcal{L}_\Phi$ to suppress the fluctuations of Φ, then $SU(2)_R \times SU(2)_N$ R-symmetry is broken down to its maximal torus $U(1)_R \times U(1)_N$. If we use $\Delta\mathcal{L}_m$, then $SU(2)_R \times SU(2)_N$ is broken down to the diagonal $U(1)$ subgroup of $U(1)_R \times U(1)_N$. Since we would like to keep track of both $U(1)_R$ and $U(1)_N$ charges of the states, we will assume in what follows that the former deformation is used.

The expression for the energy of a bare monopole is

$$E = \frac{1}{2}\sum_{i=1}^{N} |n_i| + \frac{1}{2}\sum_{i,j=1}^{N} |n_i - n_j| - \sum_{i<j} |n_i - n_j| = \frac{1}{2}\sum_{i=1}^{N} |n_i|. \tag{2.27}$$

The first term is the contribution of the hypermultiplet (one flavor) in the fundamental representation of the gauge group, the second term is the contribution of the adjoint hypermultiplet (one flavor) and the last one is the vector multiplet's contribution. The $U(1)_N$ charge of a bare monopole is twice the energy, while the $U(1)_R$ charge vanishes. The relationship $E = h_1 = -\frac{1}{2}(h_N + h_R)$ is satisfied, in agreement with the fact that a bare monopole is a BPS state.

Because we do not have a Chern-Simons term in this theory, the Gauss law simply says that the total charge of the excitations with respect to the unbroken gauge group is zero. In particular the bare monopole is a physical state.

2.3.4 Spectrum of Protected Scalars

As in the case of the ABJM theory, it is more useful to focus on scalar BPS states with energy 1 than on vector BPS states with energy 2. The lowest component of the superconformal multiplet of the stress-tensor is a dimension-1 scalar in the $Spin(8)$ representation $\bar{\mathbf{35}}$ which has highest weight $(1, 1, 1, -1)$ (4th rank anti-self-dual tensor). With respect to the manifest $SU(2)_R \times SU(2)_N \times SU(2)_X \times U(1)_T$ symmetry it decomposes as follows:

$$\bar{\mathbf{35}} = (\mathbf{3}, \mathbf{1}, \mathbf{3})_0 \oplus (\mathbf{1}, \mathbf{3}, \mathbf{1})_0 \oplus \mathbf{1}_0 \oplus (\mathbf{1}, \mathbf{3}, \mathbf{1})_2 \oplus (\mathbf{1}, \mathbf{3}, \mathbf{1})_{-2} \oplus (\mathbf{2}, \mathbf{2}, \mathbf{2})_1 \oplus (\mathbf{2}, \mathbf{2}, \mathbf{2})_{-1}.$$

From the point of view of the $\mathcal{N}=4$ superconformal algebra these scalars are not part of the stress-tensor supermultiplet. Some of them can be thought of as lowest components of the $\mathcal{N}=4$ supermultiplets containing the $SU(2)_X$ and $U(1)$ currents. We recall that in an $\mathcal{N}=4$ superconformal theory there are two kinds of supermultiplets containing conserved currents. The lowest component of either multiplet is a dimension-1 scalar either in $(\mathbf{3},\mathbf{1})$ or $(\mathbf{1},\mathbf{3})$ of $SU(2)_R \times SU(2)_N$. Currents corresponding to the flavor symmetries of hypermultiplets sit in the former kind of a supermultiplet, while topological currents arising from $\mathcal{N}=4$ vector multiplets sit in the latter kind of a supermultiplet.

As discussed above, for $N>1$ we expect a doubling of the stress-tensor multiplet and therefore two copies of $\bar{\mathbf{35}}$. Let us begin by constructing scalars in $\bar{\mathbf{35}}$ which exist for all N, and then show that for $N>1$ one can construct another copy of the same representation which we will call $\bar{\mathbf{35}}'$.

The construction of $\bar{\mathbf{35}}$ valid for all N is suggested by the abelian case $N=1$. First of all, we can construct quadratic combinations of the scalar which is the lowest component of the decoupled hypermultiplet $(\mathrm{Tr}\, X, \mathrm{Tr}\, \tilde{X})$. This scalar is an ordinary operator, not a monopole operator. This gives us a representation $(\mathbf{3},\mathbf{1},\mathbf{3})_0$ of $SU(2)_R \times SU(2)_N \times SU(2)_X \times U(1)_T$. Second, the trace part of the scalars in the $\mathcal{N}=4$ vector multiplet gives us a representation $(\mathbf{1},\mathbf{3},\mathbf{1})_0$.

For the remaining representations we construct only the BPS or anti-BPS states. The trivial representation $\mathbf{1}_0$ is neither BPS nor anti-BPS, so we do not consider it. The representation $(\mathbf{1},\mathbf{3},\mathbf{1})_2$ contains a BPS scalar with $h_T=-h_N=2, h_R=h_X=0$ and an anti-BPS scalar with $h_T=h_N=2, h_R=h_X=0$. In the deformed theory the corresponding states are bare monopoles

$$(1,-1,1,1) = |2,0,0,\ldots,0\,\rangle_+, \quad (-1,1,1,1) = |2,0,0,\ldots,0\,\rangle_-.$$

Here the numbers in parentheses are the weights of $Spin(8)$, and subscripts $\pm$ indicate whether the state is BPS or anti-BPS. Similarly, the representation $(\mathbf{1},\mathbf{3},\mathbf{1})_{-2}$ contains a BPS scalar with $h_T=-h_N=-2, h_R=h_X=0$ and an anti-BPS scalar with $h_T=h_N=$

$-2, h_R = h_X = 0$. The corresponding states are also bare monopoles

$$(1,-1,-1,-1) = |-2,0,0,\ldots,0\,\rangle_+, \quad (-1,1,-1,-1) = |-2,0,0,\ldots,0\,\rangle_-.$$

The representation $(\mathbf{2},\mathbf{2},\mathbf{2})_1$ contains two BPS scalars with $h_N = h_R = -1$ and two anti-BPS scalars with $h_N = h_R = 1$. Both BPS scalars and anti-BPS scalars transform as $\mathbf{2}_1$ of $SU(2)_X \times U(1)_T$. The corresponding states are obtained by acting on bare monopoles with GNO charge $|1,0,\ldots,0\,\rangle$ with $\mathrm{Tr}X^\dagger$, $\mathrm{Tr}\,\tilde{X}^\dagger$ (for BPS states) and by $\mathrm{Tr}\,X, \mathrm{Tr}\,\tilde{X}$ (for anti-BPS states):

$$(1,0,1,0) = \mathrm{Tr}\,X^\dagger|1,0,0,\ldots,0\,\rangle_+, \qquad (1,0,0,1) = \mathrm{Tr}\,\tilde{X}^\dagger|1,0,0,\ldots,0\,\rangle_+, \tag{2.28}$$

$$(-1,0,1,0) = \mathrm{Tr}\,\tilde{X}|1,0,0,\ldots,0\,\rangle_-, \qquad (-1,0,0,1) = \mathrm{Tr}\,X|1,0,0,\ldots,0\,\rangle_-. \tag{2.29}$$

Similarly, BPS and anti-BPS states in $(\mathbf{2},\mathbf{2},\mathbf{2})_1$ transform in $\mathbf{2}_{-1}$ of $SU(2)_X \times U(1)_T$ and are represented by

$$(1,0,0,-1) = \mathrm{Tr}\,X^\dagger|-1,0,0,\ldots,0\,\rangle_+, \qquad (1,0,-1,0) = \mathrm{Tr}\,\tilde{X}^\dagger|-1,0,0,\ldots,0\,\rangle_+, \tag{2.30}$$

$$(-1,0,0,-1) = \mathrm{Tr}\,\tilde{X}|-1,0,0,\ldots,0\,\rangle_-, \qquad (-1,0,-1,0) = \mathrm{Tr}\,X|-1,0,0,\ldots,0\,\rangle_-. \tag{2.31}$$

We can now see how a decoupled free $\mathcal{N}=8$ CFT arises for all N. It is obvious that for all N there is a free hypermultiplet $(\mathrm{Tr}\,X, \mathrm{Tr}\,\tilde{X})$. It follows from the formula for the energy of a monopole operator that the bare monopole with GNO charge $|\pm 1,0,\ldots,0\,\rangle_+$ is a BPS scalar of dimension 1/2. By unitarity, the corresponding local operators must be complex free fields with $U(N)$ charge ± 1. Such fields are lowest components of a free twisted hypermultiplet, which together with the free hypermultiplet forms a free $\mathcal{N}=8$ SCFT. Note that the BPS and anti-BPS states in the representation $\bar{\mathbf{35}}$ constructed above all lie in this free sector of the theory.

Now let us construct BPS and anti-BPS scalars with $E=1$ which exist only for $N>1$. The representation $(\mathbf{3},\mathbf{1},\mathbf{3})'_0$ is essentially the lowest component of the $SU(2)_X$ current multiplet. More precisely, it is constructed by taking various gauge-invariant quadratic expressions built out of the traceless parts of X and $\tilde{X}$. If we denote these traceless parts by x and $\tilde{x}$, the operators are

$$\mathrm{Tr}\, x^2, \quad \mathrm{Tr}\, \tilde{x}^2, \quad \mathrm{Tr}\, x\tilde{x}, \quad \mathrm{Tr}\,(x^\dagger)^2, \quad \mathrm{Tr}\,(\tilde{x}^\dagger)^2, \quad \mathrm{Tr}\, x^\dagger\tilde{x}^\dagger, \quad \mathrm{Tr}\, x^\dagger\tilde{x}, \quad \mathrm{Tr}\, x\tilde{x}^\dagger, \quad \mathrm{Tr}\,(xx^\dagger - \tilde{x}\tilde{x}^\dagger).$$

Out of these nine states the first three are anti-BPS, the next three are BPS, and the last three are neither. The corresponding operators are ordinary operators, not monopole operators.

Representations with a nonzero topological charge correspond to monopole operators, so for these representations we only construct BPS and anti-BPS states. The representation $(\mathbf{1}, \mathbf{3}, \mathbf{1})'_2$ contains a BPS scalar with $h_T = -h_N = 2, h_R = h_X = 0$ and an anti-BPS scalar with $h_T = h_N = 2, h_R = h_X = 0$. In the deformed theory the corresponding states are bare monopoles

$$(1, -1, 1, 1) = |1, 1, 0, \ldots, 0\,\rangle_+, \quad (-1, 1, 1, 1) = |1, 1, 0, \ldots, 0\,\rangle_-.$$

Note that these states exist only for $N > 1$ so presumably they do not belong to the free sector of the theory. Similarly, the representation $(\mathbf{1}, \mathbf{3}, \mathbf{1})'_{-2}$ contains a BPS scalar with $h_T = -h_N = -2, h_R = h_X = 0$ and an anti-BPS scalar with $h_T = h_N = -2, h_R = h_X = 0$. The corresponding states are also bare monopoles

$$(1, -1, -1, -1) = |-1, -1, 0, \ldots, 0\,\rangle_+, \quad (-1, 1, -1, -1) = |-1, -1, 0, \ldots, 0\,\rangle_-.$$

The representation $(\mathbf{2}, \mathbf{2}, \mathbf{2})'_1$ contains two BPS scalars with $h_N = h_R = -1$ and two anti-BPS scalars with $h_N = h_R = 1$. Both BPS scalars and anti-BPS scalars transform as $\mathbf{2}_1$ of $SU(2)_X \times U(1)_T$. The corresponding states are obtained by acting on bare monopoles with GNO charge $|1, 0, \ldots, 0\,\rangle$ with $X^{11\dagger}$, $\tilde{X}^{11\dagger}$ (for BPS states) and by $X^{11}, \tilde{X}^{11}$ (for anti-BPS states):

$$(1, 0, 1, 0) = X^{11\dagger}|1, 0, 0, \ldots, 0\,\rangle_+, \qquad (1, 0, 0, 1) = \tilde{X}^{11\dagger}|1, 0, 0, \ldots, 0\,\rangle_+, \tag{2.32}$$

$$(-1, 0, 1, 0) = \tilde{X}^{11}|1, 0, 0, \ldots, 0\,\rangle_-, \qquad (-1, 0, 0, 1) = X^{11}|1, 0, 0, \ldots, 0\,\rangle_-. \tag{2.33}$$

The point is that a monopole background with a GNO charge of the form $|n_1, 0, 0, \ldots, 0\,\rangle$

breaks the gauge symmetry down to $U(1) \times U(N-1)$, and X^{11} and $\tilde{X}^{11}$ are invariant with respect to the residual gauge symmetry. Note that X^{11} by itself is not gauge invariant, so the operators thus constructed cannot be viewed as products of free fields (corresponding to the bare monopole states $|\pm 1, 0, 0, \ldots, 0\rangle$) and some other gauge invariant operators.

Similarly, BPS and anti-BPS states in $(\mathbf{2}, \mathbf{2}, \mathbf{2})'_{-1}$ transform in $\mathbf{2}_{-1}$ of $SU(2)_X \times U(1)_T$ and are represented by

$$(1, 0, 0, -1) = X^{11\dagger}| -1, 0, 0, \ldots, 0\rangle_+, \qquad (1, 0, -1, 0) = \tilde{X}^{11\dagger}| -1, 0, 0, \ldots, 0\rangle_+, \quad (2.34)$$

$$(-1, 0, 0, -1) = \tilde{X}^{11}| -1, 0, 0, \ldots, 0\rangle_-, \quad (-1, 0, -1, 0) = X^{11}| -1, 0, 0, \ldots, 0\rangle_-. \quad (2.35)$$

The most interesting representation inside $\mathbf{35}'$ is $(\mathbf{1}, \mathbf{3}, \mathbf{1})'_0$. It contains a BPS state with $h_N = -2, h_R = h_X = h_T = 0$, an anti-BPS state with $h_N = 2, h_R = h_X = h_T = 0$ and a state which neither BPS nor anti-BPS and has $h_N = h_R = h_X = h_T = 0$. It turns out that we can construct BPS and anti-BPS states as bare monopole operators with zero topological charge but nonzero GNO charge, namely

$$(1, -1, 0, 0) = |1, -1, 0, ..., 0\rangle_+, \quad (-1, 1, 0, 0) = |1, -1, 0, ..., 0\rangle_-.$$

2.3.5 Symmetry Enhancement

So far we have confirmed that scalar states with $E = 1$ predicted by the hypothesis of hidden $\mathcal{N} = 8$ supersymmetry are indeed present. We can do better: we can argue that the spectrum of BPS and anti-BPS scalars in the theory at $t = \infty$ is such that the theory at $t = 0$ must have enhanced $Spin(8)$ R-symmetry and therefore enhanced supersymmetry.

The argument proceeds along the same lines as for the ABJM theory. We have seen that all weights of $(\mathbf{1}, \mathbf{3}, \mathbf{1})_{\pm 2}$ which are BPS states are realized by monopole operators of conformal dimension 1. Hence the whole representation must be present in the theory at $t = 0$. The commutator of two $\mathcal{N} = 4$ supercharges contains a piece which is symmetric in the spinor indices and antisymmetric in the $Spin(4)_R$ indices. This piece is a vector in the adjoint of $SU(2)_R \times SU(2)_N$, so letting it act on a scalar in $(\mathbf{1}, \mathbf{3}, \mathbf{1})_{\pm 2}$ of $SU(2)_R \times SU(2)_N \times SU(2)_X \times U(1)_T$ we get, among other things, a vector in $(\mathbf{1}, \mathbf{1}, \mathbf{1})_{\pm 2}$ which has dimension 2. One can check that this vector has nonzero norm by considering the theory of a free twisted hypermultiplet. By unitarity, the corresponding vector operators are conserved currents, which combine with $U(1)_T$ current into an $SU(2)_T$ current multiplet. Thus $U(1)_T$ is enhanced to $SU(2)_T$.

Next consider the representations $(\mathbf{2},\mathbf{2},\mathbf{2})_{\pm 1}$. All its BPS weights are realized by monopole operators of conformal dimension 1, so the whole representation must be present at $t = 0$. Further, since $U(1)_T$ is enhanced to $SU(2)_T$, these two representations assemble into $(\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2})$ of $SU(2)_R \times SU(2)_N \times SU(2)_X \times SU(2)_T$. Acting on it with the same combination of supercharges as above, we can get a vector of conformal dimension 2 which transforms as $(\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2})$. The corresponding operator must be a conserved current. Together with $SU(2)_R \times SU(2)_N \times SU(2)_X \times SU(2)_T$ currents they assemble into an adjoint of $Spin(8)$. Thus the theory at $t = 0$ has hidden $Spin(8)$ R-symmetry and consequently hidden $\mathcal{N} = 8$ supersymmetry.

For $N > 1$ we have an additional set of scalars of conformal dimension 1 which leads to another copy of $Spin(8)$ R-symmetry. So all in all the theory at $t = 0$ has two copies of $\mathcal{N} = 8$ superconformal symmetry in agreement with the predictions of duality.

2.4 Discussion

2.4.1 Gauge Group $SU(N)$

One may study other models in a similar way. For example one may take the model considered in the previous section but with gauge group $SU(N)$ instead of $U(N)$. This results in a very different spectrum of protected scalars and no supersymmetry enhancement. The manifest symmetry in this case is $SU(2)_R \times SU(2)_N \times SU(2)_X \times U(1)_F$, where $U(1)_F$ is the flavor symmetry of the fundamental hypermultiplet. The adjoint scalars X and $\tilde{X}$ are now traceless, so there are no decoupled hypermultiplets in the theory. In addition, since the gauge group is $SU(N)$, the GNO charge must satisfy $\sum_i n_i = 0$. Hence the bare monopole operator $|\pm 1, 0, \ldots, 0\rangle$ is no longer allowed, and there are no decoupled twisted hypermultiplets.

The only scalar (anti-)BPS monopole states with $E = 1$ are

$$[0, -2, 0, 0] = |1, -1, 0, \ldots, 0\,\rangle_+, \quad [0, 2, 0, 0] = |1, -1, 0, \ldots, 0\,\rangle_-,$$

where the numbers in brackets denote charges with respect to $U(1)_R \times U(1)_N \times U(1)_X \times U(1)_F$. They are obviously part of a representation $(\mathbf{1}, \mathbf{3}, \mathbf{1})_0$ of $SU(2)_R \times SU(2)_N \times SU(2)_X \times U(1)_F$. Such a scalar is the lowest component of a supermultiplet which contains a conserved $U(1)$ current. Hence there is a hidden $U(1)$ symmetry in this model whose current is a monopole operator, but there is no enhanced supersymmetry.

2.4.2 Adding More Flavors

Another obvious modification of the model is to add more hypermultiplets in the fundamental representation. The Gaiotto-Witten condition is still satisfied, so it is reasonable to assume that $SU(2)_R \times SU(2)_N$ multiplet of currents becomes part of the stress-tensor supermultiplet in the IR. Fundamental hypermultiplets make a positive contribution to the R-charge of BPS monopole operators, so if we are looking for states with $E = 1$, their number is decreased compared to the case $N_f = 1$. In fact, for $N_f > 2$ the energy of a monopole operator is strictly greater than 1, so there are no enhanced symmetries at all. For $N_f = 2$ the only way to get scalars with $E = 1$ is to consider a bare monopole operator with a GNO charge $|\pm 1, 0, \ldots, 0\,\rangle$. Such scalar BPS states have $h_N = -2$, $h_R = h_X = 0$, $h_T = \pm 1$, so they indicate the presence of protected scalars in the undeformed theory which have $E = 1$ and transform in the representations $(\mathbf{1}, \mathbf{3}, \mathbf{1})_{\pm 1}$ of $SU(2)_R \times SU(2)_N \times SU(2)_X \times U(1)_T$. Such scalars are lowest components of a supermultiplet which includes a conserved current. Since the $U(1)_T$ charge of these conserved currents is ± 1, we conclude that $U(1)_T$ symmetry is enhanced to $SU(2)_T$.

In the case $N = 1$ this result is well known and follows from the usual three-dimensional mirror symmetry. Indeed, for $N = 1$ the model reduces to $\mathcal{N} = 4$ SQED with two charged flavors and a decoupled hypermultiplet (the adjoint of $U(1)$). Apart from this decoupled hypermultiplet, the theory is self-mirror, and the $SU(2)$ flavor symmetry acting on the charged hypermultiplets is mapped by the mirror duality to the $SU(2)_T$ symmetry. For

$N > 1$ the model we are considering is not self-mirror, even if we drop the trace part of the adjoint hypermultiplet. Nevertheless, the symmetry enhancement occurs just like in the abelian case.

One can also understand these results from the standpoint of string theory. One can realize $\mathcal{N} = 4$ $U(N)$ gauge theory with one adjoint and N_f fundamental hypermultiplets via a system of N D2-branes and N_f D6-branes in Type IIA string theory. The infrared description of this system is provided by N M2-branes in a multi-Taub-NUT space with N_f centers. In the extreme infrared limit one can replace multi-Taub-NUT space with an orbifold $\mathbb{C}^2/\mathbb{Z}_{N_f}$. For $N_f > 1$ orbifolding breaks $\mathcal{N} = 8$ supersymmetry down to $\mathcal{N} = 4$, so we do not expect to have enhanced SUSY in the infrared. In addition, for $N_f > 2$ orbifolding breaks the $Spin(4)$ symmetry acting on $\mathbb{C}^2$ down to $SU(2)_N \times U(1)_T$, while for $N_f = 2$ it does not break it at all. Thus for $N_f = 2$ we expect that $U(1)_T$ is enhanced to $SU(2)_T$.

2.4.3 Concluding Remarks

We have studied in detail supersymmetry enhancement in the $U(N)$ ABJM model and $\mathcal{N} = 4$ SQCD with adjoint and fundamental matter. We found that supersymmetry enhancement is rather delicate: in the ABJM model it occurs only for Chern-Simons level 1 or 2, while in $\mathcal{N} = 4$ SQCD it occurs only if $N_f = 1$ and the gauge group is $U(N)$ rather than $SU(N)$. We also showed that the latter model has a decoupled free sector with $\mathcal{N} = 8$ supersymmetry.

The same method can be used to study enhancement of global symmetries in other $\mathcal{N} = 4$ supersymmetric gauge theories. Some examples of global symmetry enhancement have already been discussed along similar lines by Gaiotto and Witten [19]; We extend this discussion to other models in cfhapter 5.

2.5 Appendix A. Quantization in a Monopole Background

In this appendix we compute the spectrum of fluctuations and the energy of the ground state in the presence of a background magnetic flux in the theory deformed to weak coupling

$(t = \infty)$.

Energy spectrum

The contribution of a hypermultiplet has been computed in [7], so we will focus on the vector multiplet. We will follow the approach of S. Kim [16]. Let a_μ and ρ denote deviations of A_μ and σ from the background values. The quadratic part of the Lagrangian for a_μ and ρ (in the Euclidean signature) is

$$\left|\vec{D} \times \vec{a} - \vec{D}\rho - i[\sigma, a]\right|^2 = \sum_{i,j} \left|\vec{D}_{ij} \times \vec{a}_{ij} - \vec{D}_{ij}\rho_{ij} - iq_{ij}\vec{a}_{ij}\right|^2 . \tag{2.36}$$

Here $\vec{D}_{ij} = \vec{\partial} - iq_{ij}\vec{\mathcal{A}}$, $\vec{\mathcal{A}}$ is the vector potential of a Dirac monopole with unit magnetic charge, and $q_{ij} = n_i - n_j$.

The analysis is easier to carry through if we expand the fluctuations in terms of vector monopole harmonics [23], [16]. Let q be the magnetic charge of a monopole.[10] The values of spin j start with the minimal value $j_{min} = \frac{q}{2} - 1$ if this is nonnegative and from $j_{min} = \frac{q}{2}$ otherwise.

For $j \geq \frac{q}{2} + 1$ there are three kinds of vector monopole harmonics which were denoted in [23] as $\vec{C}^\lambda_{qjm}$ (with $\lambda = +1, 0, -1$). For the value of spin $j = q/2$ the harmonic $\vec{C}^{-1}_{qjm}$ is absent, while for $j = q/2 - 1$, both $\vec{C}^{-1}_{qjm}$ and $\vec{C}^0_{qjm}$ are absent. We expand the fluctuations of fields around their background values as

$$\vec{a} = \sum_{j,m} \sum_{\lambda=0,\pm1} a^\lambda_{jm} \vec{C}^\lambda_{qjm} \ , \quad \rho = \sum_{j,m} \alpha_{jm} \frac{Y_{qjm}}{r}, \tag{2.37}$$

where Y_{qjm} are monopole spherical harmonics [23], [24]. Substituting these expressions into the action (2.36) and using some properties of the vector monopole harmonics written down in [23] and [16] we obtain the action for the modes a^λ_{jm} and α_{jm}.

Recall that we are interested in only those components that are coupled to the monopole background and in their counterparts in the trivial background. For the latter we use the usual scalar and vector harmonics and have the action

[10] In this subsection we consider the case $q \geq 0$. The energy, of course, depends only on $|q|$.

(i)

$$S=\int d^3x\,r|\vec{\partial}\times\vec{a}-\vec{\partial}\sigma|^2=\int d\tau|\dot{\alpha}_{00}-\alpha_{00}|^2$$
$$+\sum_{\substack{j=1\\ m=-j,\dots,j}}^{\infty}\int d\tau[|\alpha_{jm}-\dot{\alpha}_{jm}+is_j(a^{(-)}_{jm}-a^{(+)}_{jm})|^2+|s_j(a^{(0)}_{jm}+i\alpha_{jm})-\dot{a}^{(+)}_{jm}|^2$$
$$+|s_j(a^{(0)}_{jm}-i\alpha_{jm})-\dot{a}^{(-)}_{jm}|^2], \tag{2.38}$$

where $\tau=\log r$, $s_j\equiv\sqrt{\frac{j(j+1)}{2}}$ and the Coulomb gauge condition is $s_j(a^{(-)}_{jm}+a^{(+)}_{jm})=0$.

For the former case we work in the unitary gauge which puts the relevant σs to zero, so the action is

$$S=\int d^3x\,r|\vec{D}\times\vec{a}-iq\vec{a}/r|^2=S_0+\sum_{\substack{j=j_0+2\\ m=-j,\dots,j}}^{\infty}\int d\tau\left[|s^+_j a^{(+)}_{jm}-s^-_j a^{(-)}_{jm}+qa^{(0)}|^2\right.$$
$$\left.+|\dot{a}^{(+)}_{jm}+qa^{(+)}_{jm}-s^+_j a^{(0)}_{jm}|^2+|\dot{a}^{(-)}_{jm}-qa^{(-)}_{jm}-s^-_j a^{(0)}_{jm}|^2\right], \tag{2.39}$$

where $s^+_j\equiv\sqrt{\frac{\mathcal{J}^2+q/2}{2}}$, $s^-_j\equiv\sqrt{\frac{\mathcal{J}^2-q/2}{2}}$ with $\mathcal{J}^2\equiv j(j+1)-q^2/4$. In the above formula we decomposed the action into two pieces: S_0 which depends on the modes corresponding to the two lowest values of spin j_0 which in turn depends on q, and the piece which depends on other modes. The reason for this distinction is that there are (potentially) fewer vector harmonics for the two lowest spins than for higher spins, so we need to treat them separately.[11]

[11] Indeed, if $q/2-1\geq 0$ then $j_0=q/2-1$ and for this spin there is only the mode $\vec{C}^{+1}$. For $j=j_0+1=q/2$ there are modes $\vec{C}^{+1}$ and $\vec{C}^{0}$, and for higher spins all three modes $\vec{C}^{+1}$, $\vec{C}^{0}$ and $\vec{C}^{-1}$ are present. If $q\geq 1$ then $j_0=q/2$ and this spin has two modes $\vec{C}^{+1}$ and $\vec{C}^{0}$ while $j=j_0+1$ and all higher spins have three modes for each. See [23].

(ii)

$$q=1 \Rightarrow j_0=q/2=\frac{1}{2}, \quad S_0=\int d\tau[|\dot{a}^{(+)}_{j_0m}+qa^{(+)}_{j_0m}/2-s^+a^{(0)}_{j_0m}|^2$$
$$+|s^+a^{(+)}_{j_0m}+qa^{(0)}_{j_0m}/2|^2]+$$
$$\sum_{m=-j,\dots,j}\int d\tau[|s^+_j a^{(+)}_{jm}-s^-_j a^{(-)}_{jm}+qa^{(0)}|^2$$
$$+|\dot{a}^{(+)}_{jm}+qa^{(+)}_{jm}-s^+_j a^{(0)}_{jm}|^2+|\dot{a}^{(-)}_{jm}-qa^{(-)}_{jm}-s^-_j a^{(0)}_{jm}|^2|_{j=j_1=j_0+1=3/2},$$
$$s^+=\sqrt{q/2}, \tag{2.40}$$

(iii)

$$q/2\geq 1 \Rightarrow j_0=q/2-1, \quad S_0=\int d\tau|\dot{a}^{(+)}_{j_0m}+qa^{(+)}_{j_0m}/2|^2+$$
$$\int d\tau[|\dot{a}^{(+)}_{q,m}+qa^{(+)}_{q,m}/2-s^+a^{(0)}_{q,m}|^2+|s^+a^{(+)}_{q,m}+qa^{(0)}_{q,m}/2|^2],$$
$$s^+=\sqrt{q/2}. \tag{2.41}$$

These systems are coupled harmonic oscillators with normal frequencies

(i)

$$\omega^{(1)}_j=j, \quad \omega^{(2)}_j=j+1 \quad \text{for} \quad j\geq 1,$$
$$\omega_{j_0}=j_0+1=1 \quad \text{for} \quad j=0, \tag{2.42}$$

(ii)

$$\omega^{(1)}_j=j, \quad \omega^{(2)}_j=j+1 \quad \text{for} \quad j\geq j_0+1,$$
$$\omega_{j_0}=j_0+1 \quad \text{for} \quad j=j_0=q/2, \tag{2.43}$$

Field	Energy spectrum	Spin	Degeneracy
ψ	$-\lvert q\rvert/2-p, \quad \mp\lvert q\rvert/2, \quad \lvert q\rvert/2+p$	$j=\lvert E\rvert-1/2$	$2j+1=2\lvert E\rvert$

Table 2.4: Spectrum of Dirac fermions in a monopole background [7]. p is an arbitrary natural number.

(iii)

$$\begin{aligned}\omega_j^{(1)}=j,\quad \omega_j^{(2)}=j+1 \quad \text{for} \quad j\geq j_0+2,\\ \omega_{j_0}=j_0+1 \quad \text{for} \quad j=j_0=q/2-1,\\ \omega_{j=j_0+1}=j+1=j_0+2 \quad \text{for} \quad j=j_0+1.\end{aligned} \tag{2.44}$$

The presence of only one frequency for the lower spin reflects the fact that there is only one complex degrees of freedom (for fixed m) for each of these values of j in contrast to two complex degrees of freedom for higher j.

Next we consider the kinetic term for fermions in the vector multiplet. The only difference between fermions in the vector multiplet and fermions in the hypermultiplet is an extra factor $r=\exp\tau$ in the action for the latter. It has been shown in [16] that the additional factor of r shifts all energies by 1/2, so we can use the results of [7] where the spectrum for the hypermultiplet has been computed (table 2.1).

The energies are $E(j)=j+\frac{1}{2}$ in terms of angular momentum values, which gives us $E=j+\frac{1}{2}+\frac{1}{2}=j+1$ and also from shifts of negative frequencies $-E=-j-\frac{1}{2}+\frac{1}{2}=-j$. Thus we get $E^{(1)}(j)=j$, $E^{(2)}(j)=j+1$ except for lowest $j=j_0=|q|/2-\frac{1}{2}$: the lowest j corresponds to the case when there is no negative-energy mode and $E(j_0)=j_0+1=|q|/2+1/2$.

2.6 Appendix B. Casimir Energies

Contribution of the fields to the vacuum energy are summarized below.

(i) $q=0$

Bosons:

$$E_b(0) = e^{-\beta} + \sum_{j=1}^{\infty}(2j+1)[je^{-\beta j} + (j+1)e^{-\beta(j+1)}]. \tag{2.45}$$

Fermions:

$$E_f(0) = -\sum_{j=\frac{1}{2}}^{\infty}(2j+1)[je^{-\beta j} + (j+1)e^{-\beta(j+1)}]. \tag{2.46}$$

(ii) $|q|/2 = 1/2$

Bosons:

$$E_b(q/2 = \frac{1}{2}) = 3e^{-\frac{3}{2}\beta} + \sum_{j=\frac{3}{2}}^{\infty}(2j+1)[je^{-\beta j} + (j+1)e^{-\beta(j+1)}]. \tag{2.47}$$

Fermions:

$$E_f(q/2 = \frac{1}{2}) = -e^{-\beta} - \sum_{j=1}^{\infty}(2j+1)[je^{-\beta j} + (j+1)e^{-\beta(j+1)}]. \tag{2.48}$$

(iii) $|q|/2 \geq 1$

Bosons:

$$\begin{aligned} E_b(q) &= |q/2|(|q|-1)e^{-\beta|q|/2} + (|q|+1)(|q|/2+1)e^{-\beta(|q|/2+1)} \\ &+ \sum_{j=|q|/2+1}^{\infty}(2j+1)[je^{-\beta j} + (j+1)e^{-\beta(j+1)}]. \end{aligned} \tag{2.49}$$

Fermions:

$$E_f(q) = -|q/2|(|q|+1)e^{-\beta(|q|+1)/2} - \sum_{j=|q|/2+1/2}^{\infty}(2j+1)[je^{-\beta j} + (j+1)e^{-\beta(j+1)}]. \tag{2.50}$$

The contribution of the vector multiplet to the the energy of the bare Dirac monopole of charge q is then given by

$$E(q) = E_b(q) + E_f(q) - E_b(0) - E_f(0) = -|q|. \tag{2.51}$$

Let us now specialize to the case of the ABJM theory. First of all he have abelian vector multiplets $(\vec{a}_{ij}, \sigma_{ij})$ interacting with Dirac monopoles of charges $q_{ij} = n_i - n_j$ and their tilded copies. Their contribution to the vacuum energy is

$$E_v = -\sum_{i<j} |n_i - n_j| - \sum_{i<j} |\tilde{n}_i - \tilde{n}_j|. \tag{2.52}$$

The contribution of a (twisted) hypermultiplet in the Dirac monopole background of charge q is $E(q) = |q|/2$ [7]. In the ABJM model for each pair of indices i, j we have two hypermultiplets (one of them twisted) coupling to the Dirac monopole of charge $n_i - \tilde{n}_j$, so the total vacuum energy is[12]

$$E_{tot} = \sum_{i,j} |n_i - \tilde{n}_j| - \sum_{i<j} |n_i - n_j| - \sum_{i<j} |\tilde{n}_i - \tilde{n}_j|. \tag{2.53}$$

[12]The expression below was also obtained in [16] as an expression for $E_{tot} + j_3$. Since bare monopoles are spherically symmetric, our result agrees with [16].

Chapter 3

Dualities in Three-Dimensional SCFTs

3.1 Introduction and Summary

Over the last few years several new classes of $\mathcal{N} = 8$ $d = 3$ superconformal field theories have been discovered [25, 26, 11]. Until then, it had been widely assumed that the only such theories are infrared limits of $\mathcal{N} = 8$ super-Yang-Mills theories and therefore are infinitely strongly coupled. The newly discovered theories are not of this type. Rather they are Chern-Simons-matter theories which are superconformal already on the classical level. First of all, there are BLG theories [25, 26] which have gauge group $SU(2) \times SU(2)$ [27] and an arbitrary Chern-Simons coupling. $\mathcal{N} = 8$ supersymmetry in these theories is visible on the classical level. Then there are $\mathcal{N} = 8$ ABJM theories [11] which have gauge group $U(N) \times U(N)$ and have Chern-Simons coupling $k = 1$ or $k = 2$. These theories have $\mathcal{N} = 6$ supersymmetry on the classical level, and $\mathcal{N} = 8$ supersymmetry arises as a quantum effect. $\mathcal{N} = 8$ ABJM theories are strongly coupled, but they have a a weakly coupled AdS-dual description in the large-N limit [11] and describe the physics of M2-branes.

In this chapter we exhibit another class of $\mathcal{N} = 8$ $d = 3$ superconformal Chern-Simons-matter theories. The theories themselves are not new: they are a special class of ABJ theories describing fractional M2-branes [28]. The gauge group of ABJ theories is $U(M) \times U(N)$ with Chern-Simons couplings k and $-k$ for the two factors. These theories have $\mathcal{N} = 6$ superconformal symmetry on the classical level for all values of M, N, and k. We will show that for $M = N + 1$ and $k = \pm 2$ they have hidden $\mathcal{N} = 8$ supersymmetry on the quantum level. The same kind of arguments were used by us in [1] to show that ABJM theories with gauge group $U(N)_k \times U(N)_{-k}$ and $k = 1, 2$ have hidden $\mathcal{N} = 8$ supersymmetry.

At first sight it might seem unlikely that ABJ theories may have $\mathcal{N} = 8$ supersymmetry for $N \neq M$. These theories are not parity invariant on the classical level, while all hitherto known $\mathcal{N} = 8$ $d = 3$ theories are parity invariant. On the other hand, we know of no reason why $\mathcal{N} = 8$ supersymmetry should imply parity invariance. We will see that $U(N + 1)_2 \times U(N)_{-2}$ theories do have hidden parity-invariance on the quantum level. The definition of the parity transformation involves a nontrivial duality on one of the gauge group factors.

ABJ theories with $M = N+1$ and $k = 2$ have the same moduli space as $U(N)_2 \times U(N)_{-2}$ ABJM theories. Nevertheless we show that at least for $N = 1$ and $N = 2$ (and presumably for higher N) these two $\mathcal{N} = 8$ theories are not isomorphic. We do this by comparing superconformal indices [29] of both theories. The indices are computed using the localization method of [16].

The existence of two nonisomorphic $\mathcal{N} = 8$ superconformal field theories with the moduli space $(\mathbb{R}^8/\mathbb{Z}_2)^N/S_N$ is unsurprising from the point of view of M-theory. Such theories should describe N M2-branes on an orbifold $\mathbb{R}^8/\mathbb{Z}_2$, and it is well-known that there are exactly two such orbifolds differing by G-flux taking values in $H^4(\mathbb{RP}^7, \mathbb{Z}) = \mathbb{Z}_2$ [30].

The interpretation of Bagger-Lambert-Gustavsson theories in terms of M2-branes is unclear in general. However, for low values of k it has been proposed that BLG theories describe systems of two M2-branes on $\mathbb{R}^8$ or $\mathbb{R}^8/\mathbb{Z}_2$ [31, 32, 33]. Such systems of M2-branes are also described by ABJM and ABJ theories [11, 28]. Thus we may reinterpret these proposals in field-theoretic terms as isomorphisms between certain BLG theories and ABJM or ABJ theories. We test these proposals by computing the superconformal indices of BLG theories and comparing them with those of ABJM and ABJ theories. Based on this comparison, we propose that the following $\mathcal{N} = 8$ theories are isomorphic on the quantum level:

- $U(2)_1 \times U(2)_{-1}$ ABJM theory and $(SU(2)_1 \times SU(2)_{-1})/\mathbb{Z}_2$ BLG theory
- $U(2)_2 \times U(2)_{-2}$ ABJM theory and $SU(2)_2 \times SU(2)_{-2}$ BLG theory
- $U(3)_2 \times U(2)_{-2}$ ABJ theory and $(SU(2)_4 \times SU(2)_{-4})/\mathbb{Z}_2$ BLG theory

The first two of these isomorphisms have been discussed in [33].

We provide further evidence for the first of these dualities by showing that on the quantum level $(SU(2)_1 \times SU(2)_{-1})/\mathbb{Z}_2$ BLG theory has a free sector realized by monopole operators with minimal GNO charge. This sector has $\mathcal{N} = 8$ supersymmetry and can be thought of as a free $\mathcal{N} = 4$ hypermultiplet plus a free $\mathcal{N} = 4$ twisted hypermultiplet. Thus this BLG theory has not one but two copies of $\mathcal{N} = 8$ supersymmetry algebra, one acting on the free sector and one acting on the remainder. This quantum doubling of the $\mathcal{N} = 8$ supercurrent multiplet is required by duality, because $U(2)_1 \times U(2)_{-1}$ theory also has such a doubling on the quantum level, as well as a free sector [1]. All these peculiar properties stem from the fact that the theory of N M2-branes in flat space must have a free $\mathcal{N} = 8$ sector describing the center-of-mass motion. In the "traditional" approach to the theory of N M2-branes via the $U(N)$ $\mathcal{N} = 8$ super-Yang-Mills theory, this decomposition is apparent on the classical level (one can decompose all fields into trace and traceless parts which then do not interact, with the trace part being free). In the ABJM description of the same system this decomposition arises only on the quantum level [1]. For $N = 2$ we also have a BLG description of the same system, and the existence of a free sector is again a quantum effect.

Superconformal index provides a simple tool for distinguishing $\mathcal{N} = 8$ theories which have the same moduli space. We can apply this method to other BLG theories which do not have an obvious interpretation in terms of M2-branes. For example, as noted in [33], $SU(2)_k \times SU(2)_{-k}$ and $(SU(2)_{2k} \times SU(2)_{-2k})/\mathbb{Z}_2$ BLG theories have the same moduli space for all k and one may wonder if they are in fact isomorphic. We compare the indices of these theories for $k = 1, 2$ and show that they are different. We also find that for $k = 1$ both BLG theories have an extra copy of the $\mathcal{N} = 8$ supercurrent multiplet realized by monopole operators. This indicates that each of these theories decomposes as a product two $\mathcal{N} = 8$ SCFTs which do not interact with each other. For higher k there is only one copy of the $\mathcal{N} = 8$ supercurrent multiplet.

3.2 The Moduli Space

Consider the family of $\mathcal{N}=6$ Chern-Simons-matter theories constructed by Aharony, Bergman and Jafferis [28]. The gauge group of such a theory is $U(M)\times U(N)$, with Chern-Simons couplings k and $-k$. If we regard it as an $\mathcal{N}=2$ $d=3$ theory, then the matter consists of two chiral multiplets A_a, $a=1,2$ in the representation $(M,\bar{N})$ and two chiral multiplets $B_{\dot{a}}, \dot{a}=1,2$ in the representation $(\bar{M},N)$. The theory has a quartic superpotential

$$W=\frac{2\pi}{k}\epsilon^{ab}\epsilon^{\dot{a}\dot{b}}\mathrm{Tr}A_aB_{\dot{a}}A_bB_{\dot{b}}$$

which preserves $SU(2)\times SU(2)$ symmetry as well as $U(1)_R$ R-symmetry. The chiral fields A_a and $B_{\dot{a}}$ transform as $(\mathbf{2},\mathbf{1})_1$ and $(\mathbf{1},\mathbf{2})_1$ respectively. It was shown in [28] that the Lagrangian of such a theory has $Spin(6)$ symmetry which contains $Spin(4)=SU(2)\times SU(2)$ and $U(1)_R$ as subgroups. This implies that the action has $\mathcal{N}=6$ superconformal symmetry, and the supercharges transform as a $\mathbf{6}$ of $Spin(6)$ R-symmetry.

We wish to explore the possibility that on the quantum level some of these theories have $\mathcal{N}=8$ supersymmetry. A necessary condition for this is that at a generic point in the moduli space of vacua the theory has $\mathcal{N}=8$ supersymmetry. The moduli space can be parameterized by the expectation values of the fields A_a and $B_{\dot{a}}$. Let us assume $M\geq N$ for definiteness. The superpotential is such that the expectation values can be brought to the diagonal form [28]:

$$\langle A_{a}{}^{i}_{j}\rangle=a_{ja}\delta^i_j,\quad \langle B_{\dot{a}}{}^{j}_{i}\rangle=b^j_{\dot{a}}\delta^i_j\quad i=1,\ldots,M,\quad j=1,\ldots,N.$$

Thus the classical moduli space is parameterized by $2N$ complex numbers a_{ja} and $2N$ complex numbers $b^j_{\dot{a}}$ which together parameterize $\mathbb{C}^{4N}$. Unbroken gauge symmetry includes a $U(M-N)$ factor which acts trivially on the moduli, as well as a discrete subgroup of $U(N)$. The low-energy effective action for the $U(M-N)$ gauge field is the Chern-Simons action at level $k'=k-\mathrm{sign}(k)(M-N)$. Thus along the moduli space the theory factorizes into a free theory describing the moduli and the topological $U(M-N)$ Chern-Simons theory at level k'. Note that for $M-N>|k|$ the sign of k' is different from that of k. This has been interpreted in [28] as a signal that for $M - N > |k|$ supersymmetry is spontaneously broken on the quantum level, and that the classical moduli space is lifted. Therefore from now on we will assume $M - N \leq |k|$.

The putative $\mathcal{N} = 8$ supersymmetry algebra must act trivially on the topological sector, so we need to analyze for which M, N, and k the free theory of the moduli has $\mathcal{N} = 8$ supersymmetry. This theory is a supersymmetric sigma-model whose target space is the quotient of $\mathbb{C}^N$ by the discrete subgroup of $U(N)$ which preserves the diagonal form of the matrices A_a and B_b. This discrete subgroup is a semidirect product of the permutation group S_N and the $\mathbb{Z}_k^N$ subgroup of the maximal torus of $U(N)$ [28]. Thus the target space is $(\mathbb{C}^4/\mathbb{Z}_k)^N/S_N$. The action of $\mathbb{Z}_k$ on $\mathbb{C}^4$ is given by

$$z_i \mapsto \eta z_i, \quad i = 1, \ldots, 4, \quad \eta^k = 1.$$

Here $z_{1,2}$ are identified with a_{ia}, $a = 1, 2$, while $z_{3,4}$ are identified with $b^j_{\dot{a}}$, $\dot{a} = 1, 2$.

Free $\mathcal{N} = 2$ sigma-model with target $\mathbb{C}^4 \simeq \mathbb{R}^8$ has $\mathcal{N} = 8$ supersymmetry and $Spin(8)$ R-symmetry. Supercharges transform as $\mathbf{8}_c$ of $Spin(8)$, while the moduli parameterizing $\mathbb{R}^8$ transform as $\mathbf{8}_v$. The above $\mathbb{Z}_k$ action on $\mathbf{8}_v$ factors through the $Spin(8)$ action on the same space, and for $|k| > 2$ its commutant with $\mathbb{Z}_k$ is $U(4)$. $\mathbb{Z}_k$ itself can be identified with the $\mathbb{Z}_k$ subgroup of the $U(1)$ subgroup of $U(4)$ consisting of scalar matrices. Under the $U(4)$ subgroup $\mathbf{8}_c$ decomposes as $\mathbf{6}_0 + \mathbf{1}_2 + \mathbf{1}_{-2}$, and therefore for $|k| > 2$ only $\mathbf{6}_0$ is $\mathbb{Z}_k$-invariant. Thus for $|k| > 2$ the moduli theory has only $\mathcal{N} = 6$ supersymmetry.

For $|k| = 1, 2$ the $\mathbb{Z}_k$ subgroup acts trivially on $\mathbf{8}_c$, and therefore these two cases are the only ones for which the theory of moduli has $\mathcal{N} = 8$ supersymmetry. In view of the above, for $|k| = 1$ we may assume that $M - N \leq 1$ while for $|k| = 2$ we may assume $M - N \leq 2$.

For $N = M$ and $|k| = 1, 2$ it has been argued in [11] that the full theory has $\mathcal{N} = 8$ supersymmetry on the quantum level. The hidden symmetry currents are realized by monopole operators. This proposal has been proved using controlled deformation to weak coupling [1]; for other approaches see [14, 13, 15].

It remains to consider the case $0 < M - N \leq |k|$ for $|k| = 1, 2$. Some of these theories are dual to the $\mathcal{N} = 8$ ABJM theories with $N = M$ and $k = 1, 2$. Indeed, it has been argued in [28] that for $M - N \leq |k|$ the theory with gauge group $U(M)_k \times U(N)_{-k}$ is dual to the theory with gauge group $U(2N - M + |k|)_{-k} \times U(N)_k$. We will call it the ABJ duality.[1] It maps $M - N$ to $|k| - (M - N)$ and k to $-k$. Hence the ABJ theory with gauge group $U(N+1)_1 \times U(N)_{-1}$ is dual to the ABJ theory with gauge group $U(N)_{-1} \times U(N)_1$. Similarly, the ABJ theory with gauge group $U(N+2)_2 \times U(N)_{-2}$ is dual to the ABJ theory with gauge group $U(N)_{-2} \times U(N)_2$.

The only remaining case is the ABJ theory with gauge group $U(N+1)_2 \times U(N)_{-2}$ and its parity reversal. Each theory in this family is self-dual under the ABJ duality combined with parity. Put differently, the combination of naive parity and ABJ duality is a symmetry for all N, i.e., while these theories are not parity invariant on the classical level, they have hidden parity on the quantum level. In the remainder of this paper we will argue that this family of theories in fact has hidden $\mathcal{N} = 8$ supersymmetry and is not isomorphic to any other known family of $\mathcal{N} = 8$ $d = 3$ SCFTs. We will also present evidence that certain BLG theories with $k = 1, 2$ are isomorphic to $\mathcal{N} = 8$ ABJ and ABJM theories for $N = 1, 2$.

3.3 Monopole Operators and Hidden $\mathcal{N} = 8$ Supersymmetry

In this section we will show that the ABJ theory with gauge group $U(N+1)_2 \times U(N)_{-2}$ has hidden $\mathcal{N} = 8$ supersymmetry. We will follow the method described in the first chapter. The main step is to demonstrate the presence of protected scalars with scaling dimension $\Delta = 1$ which live in the representation $\mathbf{10}_{-1}$ of the manifest symmetry group $Spin(6) \times U(1)_T$. Here $U(1)_T$ is the topological symmetry of the ABJ theory whose current

$$J^\mu = -\frac{k}{16\pi}\epsilon^{\mu\nu\rho}\left(\mathrm{Tr}F_{\nu\rho} + \mathrm{Tr}\tilde{F}_{\nu\rho}\right).$$

is conserved off-shell. Once the existence of these scalars is established, acting on them with two manifest supercharges produces conserved currents with $\Delta = 2$ transforming in

[1] Alternatively, the ABJ duality follows from the $\mathcal{N} = 3$ version of the Giveon-Kutasov duality applied to the $U(M)$ factor [34]. One can also verify that the S^3 partition functions of the dual ABJ theories agree [34].

the representation $\mathbf{6}_{-1}$ of $Spin(6) \times U(1)_T$. Since conserved currents in any field theory form a Lie algebra, these currents together with their Hermitian-conjugate currents, $Spin(6)$ currents and the $U(1)_T$ current must combine into an adjoint of some Lie algebra containing $Spin(6) \times U(1)_T$ Lie algebra as a subalgebra. The unique possibility for such a Lie algebra is $Spin(8)$, which implies that the theory has $\mathcal{N} = 8$ supersymmetry.

The existence of $\Delta = 1$ scalars transforming in $\mathbf{10}_{-1}$ is established using a controlled deformation of the theory compactified on S^2 to weak coupling. This deformation preserves $Spin(4) \times U(1)_R$ subgroup of $Spin(6)$ as well as $U(1)_T$. Decomposing $\mathbf{10}_{-1}$ with respect to this subgroup, we find that it contains BPS scalars in $(\mathbf{3}, \mathbf{1})_{1,-1}$ of $Spin(4) \times U(1)_R \times U(1)_T$ and anti-BPS scalars in $(\mathbf{1}, \mathbf{3})_{-1,-1}$. Such BPS scalars cannot disappear as one changes the coupling (see appendix A for a detailed argument), so it is sufficient to demonstrate the presence of BPS scalars at extremely weak coupling. Note that the scaling dimension Δ of an operator is now reinterpreted as the energy of a state on S^2.

The BPS scalars we are looking for have nonzero $U(1)_T$ charge and therefore are monopole operators [7]. At weak coupling monopole operators in ABJ theories are labeled by GNO "charges" $(m_1, \ldots, m_M)$ and $(\tilde{m}_1, \ldots, \tilde{m}_N)$. GNO charges label spherically symmetric magnetic fields on S^2 and are defined up to the action of the Weyl group of $U(M) \times U(N)$ [12]. They do not correspond to conserved currents and can be defined only at weak coupling. Their sum however is related to the $U(1)_T$ charge:

$$Q_T = -\frac{k}{4}\left(\sum m_i + \sum \tilde{m}_i\right).$$

Equations of motion of the ABJ theory imply that $\sum m_i = \sum \tilde{m}_i$, so Q_T is integral for even k but may be half-integral for odd k. We are interested in the case $Q_T = -1$, $k = 2$, which implies

$$\sum m_i = \sum \tilde{m}_i = 1.$$

Consider a bare BPS monopole, i.e., the vacuum state, with GNO charges $m_1 = \tilde{m}_1 = 1$ and all other GNO charges vanishing. This state has $\Delta = 0$ but because of Chern-Simons terms it is not gauge-invariant (does not satisfy the Gauss law constraint). One can construct a gauge invariant state by acting on the bare BPS monopole with two creation operators corresponding to the fields $\bar{A}_a^{1\tilde{1}}$, $a = 1, 2$. These states are completely analogous to the BPS scalars for the $U(N)_2 \times U(N)_{-2}$ ABJM theory constructed in [1] (see equation (13) in that paper). The resulting multiplet of states transforms as $(\mathbf{3}, \mathbf{1})_{1,-1}$ of $Spin(4) \times U(1)_R \times U(1)_T$. It also has $\Delta = 1$ and zero spin, since the creation operators for the field $\bar{A}$ with lowest energy have $\Delta = 1/2$ and zero spin.

Similarly, by starting from an anti-BPS bare monopole with the same GNO charges and acting on it with two creation operators belonging to the fields $B_a^{1\tilde{1}}$ we obtain anti-BPS scalars which transform in $(\mathbf{1}, \mathbf{3})_{-1,-1}$. One can also check that no other GNO charges give rise to BPS scalars with $\Delta = 1$. In view of the above discussion this implies that the $U(N+1)_2 \times U(N)_{-2}$ ABJ theory has hidden $\mathcal{N} = 8$ supersymmetry.

3.4 Superconformal Index and Comparison with Other $\mathcal{N} = 8$ Theories

One may question if $U(N+1)_2 \times U(N)_{-2}$ ABJ theories are genuinely distinct from other known $\mathcal{N} = 8$ $d = 3$ theories. The moduli space of such a theory is $(\mathbb{C}^4/\mathbb{Z}_2)^N/S_N$, which is exactly the same as the moduli space of the $U(N)_2 \times U(N)_{-2}$ ABJ theory. They differ in that along the moduli space the former theory has an extra topological sector described by $U(1)$ Chern-Simons theory at level 1. The latter theory is not quite trivial [35], but it is very close to being trivial; for example, it does not admit any nontrivial local or loop observables. In any case, one could conjecture that even at the origin of the moduli space the two $\mathcal{N} = 8$ $d = 3$ theories differ only by this decoupled topological sector. Some evidence in support of this conjecture is that BPS scalars in the two theories are in one-to-one correspondence, as we have seen in the previous section.

Fortunately, in the last few years there has been substantial progress in understanding superconformal $d = 3$ gauge theories which allows us to compute many quantities exactly. One such quantity is the partition function on S^3 [36]; another one is the superconformal index on $S^2 \times S^1$ [29, 16]. The superconformal index receives contribution from BPS scalars as well as from other protected states with nonzero spin. In what follows we will compute the index for several low values of N and verify that it is different for the two families of $\mathcal{N}=8$ theories. The perturbative contribution to the superconformal index for ABJM theories has been computed in [37]; the contributions of sectors with a nontrivial GNO charge has been determined in [16]. We will follow the approach of [16].

Bagger and Lambert [25] and Gustavsson [26] constructed another infinite family of $\mathcal{N}=8$ $d=3$ superconformal Chern-Simons-matter theories with gauge group $SU(2)\times SU(2)$ and matter in the bifundamental representation. More precisely, as emphasized in [11, 33], there are two versions of BLG theories which have gauge groups $SU(2)_k\times SU(2)_{-k}$ or $(SU(2)_k\times SU(2)_{-k})/\mathbb{Z}_2$ where k is an arbitrary natural number. The moduli space is $(\mathbb{C}^4\times\mathbb{C}^4)/\mathbf{D}_{2k}$ and $(\mathbb{C}^4\times\mathbb{C}^4)/\mathbf{D}_k$ respectively, where $\mathbf{D}_k$ is the dihedral group of order $2k$ [31, 32, 33]. For large enough k the moduli space is different from the moduli space of ABJ theories and so BLG theories cannot be isomorphic to any of them. However, for low values of k there are some coincidences between moduli spaces which suggest that perhaps some of BLG theories are isomorphic to ABJ theories.

One such case is $k=1$ and $G=(SU(2)\times SU(2))/\mathbb{Z}_2$. The moduli space is $(\mathbb{C}^4\times\mathbb{C}^4)/\mathbb{Z}_2$ where $\mathbb{Z}_2$ exchanges the two $\mathbb{C}^4$ factors. It is natural to conjecture that this theory is isomorphic to $U(2)_1\times U(2)_{-1}$ ABJM theory. A derivation of this equivalence was proposed in [33]. Another special case is $k=2$ and $G=SU(2)\times SU(2)$. In that case the moduli space is isomorphic to $(\mathbb{C}^4/\mathbb{Z}_2\times\mathbb{C}^4/\mathbb{Z}_2)/\mathbb{Z}_2$, where the first two $\mathbb{Z}_2$ factors reflect the coordinates on the two copies of $\mathbb{C}^4$, while the third one exchanges them [31, 32, 33]. This is the same moduli space as that of $U(2)_2\times U(2)_{-2}$ ABJM theory and $U(3)_2\times U(2)_{-2}$ ABJ theory. It was conjectured in [33] that this BLG theory is isomorphic to the $U(2)_2\times U(2)_{-2}$ ABJM theory. Finally, one can take $k=4$ and $G=(SU(2)\times SU(2))/\mathbb{Z}_2$. The moduli space is the same as in the previous case, so one could conjecture that this BLG theory is isomorphic to either the $U(2)_2\times U(2)_{-2}$ ABJM theory or the $U(3)_2\times U(2)_{-2}$ ABJ theory.

Below we will first of all compute the superconformal index for the $U(N)_2\times U(N)_{-2}$ ABJM theories and $U(N+1)_2\times U(N)_{-2}$ ABJ theories for $N=1,2$ and verify that although these theories have the same moduli space, they have different superconformal indices and therefore are not isomorphic. We will also compute the index for the special BLG theories with low values of k discussed above and test the proposed dualities with the ABJM and

ABJ theories. We will see that certain BLG theories have an additional copy of the $\mathcal{N} = 8$ supercurrent multiplet which is realized by monopole operators. In some cases this is predicted by dualities.

3.4.1 $\mathcal{N} = 8$ ABJM vs. $\mathcal{N} = 8$ ABJ Theories

The superconformal index for a supersymmetric gauge theory on $S^2 \times \mathbb{R}$ is defined as

$$\mathcal{I}(x, z_i) = Tr[(-1)^F x^{E+j_3} \prod_i z_i^{F_i}], \tag{3.1}$$

where F is the fermion number, E is the energy, j_3 is the third component of spin and F_i are flavor symmetry charges. The index receives contributions only from states satisfying $\{Q, Q^\dagger\} = E - r - j_3 = 0$, where Q is one of the 32 supercharges and r is a $U(1)$ R-charge. For details the reader is referred to [29, 16].

The localization method [16] enables one to express the index in a simple form[2]

$$\mathcal{I}(x, z_i) = \sum_{\{n_i\}} \int [da]_{\{n_i\}} x^{E_0(n_i)} e^{S^0_{CS}(n_i, a_i)} exp(\sum_{m=1}^{\infty} f(x^m, z_i^m, ma_i)), \tag{3.2}$$

where the sum is over GNO charges, the integral whose measure depends on GNO charges is over a maximal torus of the gauge group, $E_0(n_i)$ is the energy of a bare monopole with GNO charges $\{n_i\}$, $S^0_{CS}(n_i)$ is effectively the weight of the bare monopole with respect to the gauge group and the function f depends on the content of vector multiplets and hypermultiplets. For details see [16].

We computed the indices for the $U(2)_2 \times U(1)_{-2}$ and $U(1)_2 \times U(1)_{-2}$ theories up to the sixth order in x and found the following pattern. In each topological sector the indices agree at the leading order in x as a consequence of the identical spectra of BPS scalars of the lowest dimension. However, next-to-leading terms are different which signals nonequivalence

[2] The formula is written for the case of zero anomalous dimensions of all fields which is true for all theories with at least $\mathcal{N} = 3$ supersymmetry.

of these theories. We summarize our results in tables 3.1 and 3.2 in appendix B.

It is possible to single out contributions from different topological sectors by treating topological $U(1)_T$ symmetry as a flavor symmetry and introducing a new variable z into the index. The result is a double expansion in x and z with powers of z multiplying contributions of the appropriate topological charge. Alternatively, one can restrict summation over all GNO charges to those giving the desired topological charge. We used the second type of calculation.

We also compared the indices for the ABJ theory $U(3)_2 \times U(2)_{-2}$ and the ABJM theory $U(2)_2 \times U(2)_{-2}$ up to the fourth order in x. The contributions from different GNO sectors are summarized in tables 3.3 and 3.4 in Appendix B. Note that we count the contributions from the topological sectors $T \geq 1$ twice because there is an identical contribution from the sectors with opposite topological charges. Starting at order x^3 the indices disagree, which means that these two $\mathcal{N} = 8$ theories, despite having the same moduli space, are not equivalent.

3.4.2 Comparison with BLG Theories

There are two BLG theories which have the same moduli space as $U(2)_2 \times U(2)_{-2}$ ABJM and $U(3)_2 \times U(2)_{-2}$ ABJ theories. They have gauge groups $SU(2)_2 \times SU(2)_{-2}$ and $(SU(2)_4 \times SU(2)_{-4})/\mathbb{Z}_2$. It is natural to conjecture that these four theories are pairwise isomorphic. Indeed, the moduli space is $(\mathbb{C}^4/\mathbb{Z}_2 \times \mathbb{C}^4/\mathbb{Z}_2)/\mathbb{Z}_2$ in all four cases, suggesting that all these theories describe two M2-branes on an $\mathbb{R}^8/\mathbb{Z}_2$ orbifold. It is well-known that there are two distinct $\mathbb{R}^8/\mathbb{Z}_2$ orbifolds in M-theory [30], which means that there should be only two nonisomorphic $\mathcal{N} = 8$ theories with this moduli space.

Comparison of the indices of the $U(2)_2 \times U(2)_{-2}$ ABJM theory and the $SU(2)_2 \times SU(2)_{-2}$ BLG theory (see table 3.5) reveals their agreement up to the fourth order in x. Thus we conjecture that the two theories are equivalent.

This conjecture can be checked further by comparing contributions to the indices from individual topological sectors on the ABJM side and sectors parametrized by the corresponding $U(1)$ charge on the BLG side. Recall that the topological charge Q_T on the ABJM side is a charge of a $U(1)$ subgroup of the $Spin(8)$ R-symmetry group. The commutant of this subgroup is $Spin(6)$ R-symmetry visible already on the classical level. Furthermore, the supercharge used in the deformation and the definition of the index is charged under a $U(1)$ subgroup of this $Spin(6)$. On the BLG side, the whole $Spin(8)$ R-symmetry is visible on the classical level. Recall that one can think of the BLG theory as a $\mathcal{N} = 2$ field theory with gauge group $SU(2) \times SU(2)$ and four chiral multiplets in the bifundamental representation. In this description, there is a manifest $SU(4) = Spin(6)$ symmetry under which the four chiral superfields transform as $\mathbf{4}$. The commutant of this $Spin(6)$ symmetry is $U(1)_R$ symmetry with respect to which all four chiral superfields have charge 1/2 and the supercharge has charge 1. The topological charge Q_T on the ABJM side corresponds to the charge of a $U(1)$ subgroup of $Spin(6)$ which we denote as $U(1)_t$.[3] Thus we should compare the ABJM index in a particular topological sector with the BLG index in a sector with a particular $U(1)_t$ charge. The four chiral fields of the BLG theory decompose as $\mathbf{4} = \mathbf{2}_1 + \mathbf{2}'_{-1}$ under $U(1)_t \times Spin(4)$. To keep track of $U(1)_t$ charges we introduce a new variable z in accordance with (4.1). To the fourth order in x only the $(|0\,\rangle|0\,\rangle, |1\,\rangle|1\,\rangle, |2\,\rangle|2\,\rangle)$ GNO charges contribute. The two-variable index is

$$\begin{aligned}&\mathcal{I}_{BLG,k=2}(x,z) = 1 + 4x + 21x^2 + 32x^3 + 53x^4 + z^2(3x + 16x^2 + 36x^3 + 48x^4)+\\&z^4(11x^2 + 36x^3 + 54x^4) + z^6(22x^3 + 64x^4) + 45x^4z^8 + z^{-2}(3x + 16x^2 + 36x^3 + 48x^4)+\\&z^{-4}(11x^2 + 36x^3 + 54x^4) + z^{-6}(22x^3 + 64x^4) + 45x^4z^{-8} + O(x^5).\end{aligned} \tag{3.3}$$

This is in a complete agreement with the index for the $U(2)_2 \times U(2)_{-2}$ ABJM theory.

Similarly, we can compute the two-variable index for the $(SU(2)_4 \times SU(2)_{-4})/\mathbb{Z}_2$ BLG theory. The difference compared to the $SU(2) \times SU(2)$ case is that the GNO charges are allowed to be half-integral, but their difference is required to be integral. The contributions of individual GNO charges are summarized in table 3.6. We see that the total index agrees with that of the $U(3)_2 \times U(2)_{-2}$ ABJ theory at least up to the fourth order in x. The

[3] We now adopt the notation $T \equiv \sum_i m_i$ for the topological charge and normalize the $U(1)_t$ charge of fundamental scalars of the BLG theories to ± 1 for notational convenience. The $U(1)_R$ charges are not shown in what follows.

two-variable index for this BLG theory is given by

$$\mathcal{I}'_{BLG}(x,z) = 1 + 4x + 21x^2 + 36x^3 + 39x^4 + z^2(3x + 16x^2 + 39x^3 + 40x^4)+$$
$$z^4(11x^2 + 36x^3 + 56x^4) + z^6(22x^3 + 64x^4) + 45z^8x^4 + z^{-2}(3x + 16x^2 + 39x^3 + 40x^4)+$$
$$z^{-4}(11x^2 + 36x^3 + 56x^4) + z^{-6}(22x^3 + 64x^4) + 45z^{-8}x^4 + O(x^5) \tag{3.4}$$

and agrees with the two-variable index of the $U(3)_2 \times U(2)_{-2}$ ABJ theory.

Lambert and Papageorgakis [33] argued that the $(SU(2)_1 \times SU(2)_{-1})/\mathbb{Z}_2$ BLG theory is isomorphic to the $U(2)_1 \times U(2)_{-1}$ ABJM theory. We can test this proposal in the same way by comparing the two-variable superconformal indices of the two theories. We find that they agree up to at least the fourth order in x. The contributions from different GNO charges are written down in tables 3.7 and 3.8. They happen to match in each GNO sector separately. For a fixed topological charge on the ABJM side and the corresponding value of the $U(1)_t$ charge on the BLG side which manifests itself in the index as a power of z, the contribution to the index comes from a sum over different GNO charges, and the two sums happen to coincide term by term. For example, in the topological sector $T = 1$ on the ABJM side the contribution from the GNO charge $|n, 1-n\,\rangle|n, 1-n\,\rangle$ equals the contribution from the GNO charge $|n-1/2\,\rangle|n-1/2\,\rangle$ with the first power of z on the BLG side.

The index makes apparent a peculiar feature of these two theories: they have twice the number of BPS scalars needed to enhance supersymmetry from $\mathcal{N} = 6$ to $\mathcal{N} = 8$. The first set of scalars has vanishing GNO charge. The corresponding contribution to the index is $\Delta\mathcal{I} = 4x + 3xz^2 + 3xz^{-2}$. It represents the decomposition $\mathbf{10} = \mathbf{4}_0 + \mathbf{3}_2 + \mathbf{3}_{-2}$ under $U(1)_t \times Spin(4) \subset Spin(6)$. The corresponding operators are gauge-invariant bilinear combinations of four chiral superfields present in the BLG model. The second set of ten BPS scalars comes from the GNO charge $|1\,\rangle|1\,\rangle$ and makes an identical contribution to the index. Ten BPS states are obtained by acting with ten scalar bilinears on the bare monopole to form gauge-invariant states $Q^{(i}Q^{j)}|1\,\rangle|1\,\rangle$. Here Q^i is an off-diagonal component of the ith complex scalar, $i = 1, \ldots, 4$. Among these ten states there are representations $(\mathbf{3},\mathbf{1})_{1,-1} + (\mathbf{1},\mathbf{3})_{1,1}$ of $Spin(4) \times U(1)_R \times U(1)_t$ with the normalization of the $U(1)_t$ charge as in section 1. Together with their Hermitian-conjugates, these BPS scalars lead to supersymmetry enhancement as in [1].

The existence of two copies of the $\mathcal{N}=8$ supersymmetry algebra for the $U(2)_1 \times U(2)_{-1}$ ABJM theory was noted in [1]. It was shown there that the extra copy arises because the theory has a free sector with $\mathcal{N}=8$ supersymmetry realized by monopole operators. The same is true about the $(SU(2)_1 \times SU(2)_{-1})/\mathbb{Z}_2$ BLG theory, giving further support for the duality. The sector with the GNO charge $|1/2\rangle|1/2\rangle$ contains four gauge-invariant BPS scalars $Q^i|1/2\rangle|1/2\rangle$ with energy $\Delta = 1/2$ whose contribution to the index is $\Delta\mathcal{I}' = 2x^{1/2}z + 2x^{-1/2}z$. This expression corresponds to the decomposition $\mathbf{4} = \mathbf{2}_1 + \mathbf{2'}_{-1}$ under $U(1)_t \times Spin(4) \subset Spin(6)$. By virtue of state-operator correspondence these states correspond to four free fields with conformal dimension $\Delta = 1/2$. Their bilinear combinations give rise to ten BPS scalars with GNO charge $|1\rangle|1\rangle$ discussed above. This is in a complete agreement with the structure of the $U(2)_1 \times U(2)_{-1}$ ABJM theory explored in [1].

We can also use superconformal index to test whether certain BLG theories with identical moduli spaces are isomorphic on the quantum level. It has been noted in [33] that the moduli spaces of $SU(2)_k \times SU(2)_{-k}$ and $(SU(2)_{2k} \times SU(2)_{-2k})/\mathbb{Z}_2$ BLG theories are the same (they are both given by $(\mathbb{C}^4 \times \mathbb{C}^4)/\mathbf{D}_{2k}$. We have seen above that for $k=2$ these two theories are not isomorphic. We also computed the index for $k=1$ and found that the indices disagree already at the second order in x (tables 3.9 and 3.10), so the theories are not equivalent. Examining BPS scalars, we find that neither of these theories has a free sector, but they both have two copies of the $\mathcal{N}=8$ supercurrent multiplet. One copy is visible on the classical level, while the BPS scalars of the other copy carry GNO charges, so it is intrinsically quantum mechanical in origin. The presence of the second copy of $\mathcal{N}=8$ superalgebra indicates that on the quantum level both of these theories decompose into two $\mathcal{N}=8$ SCFTs which do not interact with each other. This phenomenon does not occur for higher k.

3.5 Appendix A. Protected BPS States

Our method of detecting hidden supersymmetry is based on deforming the theory to weak coupling and analyzing the spectrum of BPS scalars. In general, a local operator (or the corresponding state in the radial quantization) which lives in a short representation of the superconformal algebra can pair up with another short multiplet to form a long multiplet; quantum number of a long multiplet can change continuously as one deforms the coupling. We would like to show that this cannot happen for the cases of interest to us.

The kind of short multiplet we are interested in has a BPS scalar among its primaries. In the radial quantization such a state has energy Δ equal to its $U(1)_R$ charge r. To form a long multiplet there must be a short multiplet containing a spinor with energy $\Delta' = \Delta \pm 1/2$ and R-charge $r = r' \pm 1$. The option with $\Delta' = \Delta + 1/2$ and $r' = r + 1$ is ruled out by unitarity constraints [29]. These constraints also specify the short multiplet with the spinor. This is a so-called regular short multiplet [29] with a scalar $\Delta'' = \Delta - 1$, $r'' = r - 2$ as the superconformal primary state satisfying $\Delta'' = r'' + 1$. The zero-norm state is also a scalar, appears on the second level, and has the quantum numbers of a BPS scalar $\Delta = r$. The spinor itself is on the first level.

We conclude that a necessary condition for a BPS scalar with quantum numbers $\Delta = r$ to pair up into a long multiplet and flow away is the existence of a regular short multiplet with quantum numbers $\Delta'' = \Delta - 1$ and $r'' = r - 2$.

In the particular case of a $U(N+1)_k \times U(N)_{-k}$ ABJ theory and $\Delta = 1$ such "regular short multiplets" do not exist at the value of the deformation parameter $t = \infty$ because $\Delta'' = 0$ and all physical states have $\Delta \geq 1$.

However, as we mentioned earlier, the spectrum of BPS states at intermediate values of t (between $t = 0$ and $t = \infty$) is unknown and the pairing could occurr in principle. Indeed, there are examples of theories in which BPS scalar with high conformal dimension dissappear on the way from $t = \infty$ to $t = 0$. Fortunately, this never happens for scalar BPS states with the energy $E = 1/2$ or $E = 1$.

The argument uses the invariance of the superconformal index under the deformation. The Taylor expansion of the index around $x = 0$ has the form $\mathcal{I}(x) = 1 + \alpha x^{1/2} + \beta x + \mathcal{O}(x^{3/2})$. A BPS state with quantum numbers E and j_3 gives contribution $\pm x^{E+j_3}$ where the sign depends on whether the state is bosonic or fermionic. Because the state is BPS $E = r + j_3$ and because of the unitarity constraints [38], $r > 0$ and $j_3 \geq 0$. Note that although only part of the full superconformal group is preserved at $t = \infty$ this is the part that gives the constraints. Hence, BPS scalars and only BPS scalars contribute to $x^{1/2}$ and x^1 terms in the

Taylor expansion of the index. It follows that the spectra of BPS scalars at $t = 0$ and $t = \infty$ are identical.

3.6 Appendix B: Superconformal Indices for $\mathcal{N} = 8$ ABJM, ABJ, and BLG Theories

GNO charges	Index contribution
$T = 0$	$1 + 4x + 2x^2 + 15x^4 - 16x^5 + 11x^6$
$\lvert 0, 0\rangle\lvert 0\rangle$ $\lvert 1, -1\rangle\lvert 0\rangle$	$1 + 4x + 2x^2 + 15x^4 - 16x^5 + 2x^6$ $9x^6$
$T = 1$	$3x + x^2 - 4x^3 + 20x^4 - 32x^5 + 24x^6$
$\lvert 1, 0\rangle\lvert 1\rangle$	$3x + x^2 - 4x^3 + 20x^4 - 32x^5 + 24x^6$
$T = 2$	$5x^2 + 4x^3 - 5x^4 + 4x^5 - 4x^6$
$\lvert 2, 0\rangle\lvert 2\rangle$	$5x^2 + 4x^3 - 5x^4 + 4x^5 - 4x^6$
$T = 3$	$7x^3 + 4x^4 + x^6$
$\lvert 3, 0\rangle\lvert 3\rangle$	$7x^3 + 4x^4 + x^6$
$T = 4$	$9x^4 + 4x^5$
$\lvert 4, 0\rangle\lvert 4\rangle$	$9x^4 + 4x^5$
$T = 5$	$11x^5 + 4x^6$
$\lvert 5, 0\rangle\lvert 5\rangle$	$11x^5 + 4x^6$
$T = 6$	$13x^6$
$\lvert 6, 0\rangle\lvert 6\rangle$	$13x^6$
total	$1 + 10x + 14x^2 + 14x^3 + 71x^4 - 42x^5 + 39x^6$

Table 3.1: $U(2)_2 \times U(1)_{-2}$. T stands for the topological charge.

Topological charge	Index contribution
$T = 0$	$1 + 4x + x^2 + 4x^3 + 7x^4 - 12x^5 + 26x^6$
$T = 1$	$3x + 4x^2 + 8x^4 - 4x^5 + 8x^6$
$T = 2$	$5x^2 + 4x^3 = 8x^5 - 4x^6$
$T = 3$	$7x^3 + 4x^4 + 8x^6$
$T = 4$	$9x^4 + 4x^5$
$T = 5$	$11x^5 + 4x^6$
$T = 6$	$13x^6$
total	$1 + 10x + 19x^2 + 26x^3 + 49x^4 + 26x^5 + 92x^6$

Table 3.2: $U(1)_2 \times U(1)_{-2}$

GNO charges	Index contribution
$T=0$	$1+4x+21x^2+36x^3+39x^4$
$\vert 0,0,0\rangle\vert 0,0\rangle$	$1+4x+12x^2+12x^3+5x^4$
$\vert 1,0,-1\rangle\vert 1,-1\rangle$	$9x^2+24x^3+10x^4$
$\vert 2,0,-2\rangle\vert 2,-2\rangle$	$25x^4$
$\vert 1,0,-1\rangle\vert 0,0\rangle$	$-x^4$
$T=1$	$3x+16x^2+39x^3+40x^4$
$\vert 1,0,0\rangle\vert 1,0\rangle$	$3x+16x^2+24x^3+8x^4$
$\vert 2,0,-1\rangle\vert 2,-1\rangle$	$15x^3+32x^4$
$T=2$	$11x^2+36x^3+56x^4$
$\vert 1,1,0\rangle\vert 1,1\rangle$	$6x^2+12x^3+9x^4$
$\vert 2,0,0\rangle\vert 2,0\rangle$	$5x^2+24x^3+26x^4$
$\vert 3,0,-1\rangle\vert 3,-1\rangle$	$21x^4$
$T=3$	$22x^3+64x^4$
$\vert 2,1,0\rangle\vert 2,1\rangle$	$15x^3+32x^4$
$\vert 3,0,0\rangle\vert 3,0\rangle$	$7x^3+32x^2$
$T=4$	$45x^4$
$\vert 2,2,0\rangle\vert 2,2\rangle$	$15x^4$
$\vert 3,1,0\rangle\vert 3,1\rangle$	$21x^4$
$\vert 4,0,0\rangle\vert 4,0\rangle$	$9x^4$
total	$1+10x+75x^2+230x^3+445x^4$

Table 3.3: $U(3)_2 \times U(2)_{-2}$. T stands for the topological charge.

GNO charges	Index contribution
$T=0$	$1+4x+21x^2+32x^3+53x^4$
$\|0,0\rangle\|0,0\rangle$	$1+4x+12x^2+8x^3+12x^4$
$\|1,-1\rangle\|1,-1\rangle$	$9x^2+24x^3+16x^4$
$\|2,-2\rangle\|2,-2\rangle$	$25x^4$
$T=1$	$3x+16x^2+36x^3+48x^4$
$\|1,0\rangle\|1,0\rangle$	$3x+16x^2+21x^3+16x^4$
$\|2,-1\rangle\|2,-1\rangle$	$15x^3+32x^4$
$T=2$	$11x^2+36x^3+54x^4$
$\|1,1\rangle\|1,1\rangle$	$6x^2+12x^3+12x^4$
$\|2,0\rangle\|2,0\rangle$	$5x^2+24x^3+21x^4$
$\|3,-1\rangle\|3,-1\rangle$	$21x^4$
$T=3$	$22x^3+64x^4$
$\|2,1\rangle\|2,1\rangle$	$15x^3+32x^4$
$\|3,0\rangle\|3,0\rangle$	$7x^3+32x^4$
$T=4$	$45x^4$
$\|2,2\rangle\|2,2\rangle$	$15x^4$
$\|3,1\rangle\|3,1\rangle$	$21x^4$
$\|4,0\rangle\|4,0\rangle$	$9x^4$
total	$1+10x+75x^2+220x^3+475x^4$

Table 3.4: $U(2)_2 \times U(2)_{-2}$. T stands for the topological charge.

GNO charges	Index contribution
$\|0\rangle\|0\rangle$	$1+10x+40x^2+76x^3+114x^4$
$\|1\rangle\|1\rangle$	$35x^2+144x^3+196x^4$
$\|2\rangle\|2\rangle$	$165x^4$
total	$1+10x+75x^2+220x^3+475x^4$

Table 3.5: $SU(2)_2 \times SU(2)_{-2}$

GNO charges	Index contribution
$\|0\rangle\|0\rangle$	$1+4x+12x^2+8x^3+12x^4+$ $z^2(3x+8x^2+12x^3+8x^4)+z^{-2}(3x+8x^2+12x^3+8x^4)+$ $z^4(6x^2+12x^3+12x^4)+z^{-4}(6x^2+12x^3+12x^4)+$ $z^6(10x^3+16x^4)+z^{-6}(10x^3+16x^4)+15z^8x^4+15z^{-8}x^4$
$\|1/2\rangle\|1/2\rangle$	$9x^2+28x^3+2x^4+$ $z^2(8x^2+27x^3+8x^4)+z^{-2}(8x^2+27x^3+8x^4)+$ $z^4(5x^2+24x^3+23x^4)+z^{-4}(5x^2+24x^3+23x^4)+$ $z^6(12x^3+32x^4)+z^{-6}(12x^3+32x^4)+21z^8x^4+21z^{-8}x^4$
$\|1\rangle\|1\rangle$	$25x^4+24z^2x^4+24z^{-2}x^4+24z^4x^4+24z^{-4}x^4+16z^6x^4+$ $16z^{-6}x^4+9z^8x^4+9z^{-8}x^4$

Table 3.6: $(SU(2)_4 \times SU(2)_{-4})/\mathbb{Z}_2$

GNO charges	Index contribution
$\|0\rangle\|0\rangle$	$1+4x+12x^2+8x^3+12x^4+$ $z^2(3x+8x^2+12x^3+8x^4)+z^{-2}(3x+8x^2+12x^3+8x^4)+$ $z^4(6x^2+12x^3+12x^4)+z^{-4}(6x^2+12x^3+12x^4)+$ $z^6(10x^3+16x^4)+z^{-6}(10x^3+16x^4)+15z^4x^4+15z^{-8}x^4$
$\|1/2\rangle\|1/2\rangle$	$2z(x^{\frac{1}{2}}+6x^{\frac{3}{2}}+10x^{\frac{5}{2}}+7x^{\frac{7}{2}})+2z^{-1}(x^{\frac{1}{2}}+6x^{\frac{3}{2}}+10x^{\frac{5}{2}}+7x^{\frac{7}{2}})+$ $2z^3(3x^{\frac{3}{2}}+10x^{\frac{5}{2}}+9x^{\frac{7}{2}})+2z^{-3}(3x^{\frac{3}{2}}+10x^{\frac{5}{2}}+9x^{\frac{7}{2}})+$ $2z^5(6x^{\frac{5}{2}}+14x^{\frac{7}{2}})+2z^{-5}(6x^{\frac{5}{2}}+14x^{\frac{7}{2}})$
$\|1\rangle\|0\rangle$	$-x^4$
$\|0\rangle\|1\rangle$	$-x^4$
$\|1\rangle\|1\rangle$	$4x+16x^2+16x^3+33x^4+$ $z^2(3x+16x^2+19x^3+24x^4)+z^{-2}(3x+16x^2+19x^3+24x^4)+$ $z^4(8x^2+24x^3+16x^4)+z^{-4}(8x^2+24x^3+16x^4)+$ $z^6(15x^3+32x^4)+z^{-6}(15x^3+32x^4)+24z^8x^4+24z^{-8}x^4$
$\|3/2\rangle\|3/2\rangle$	$2z(3x^{\frac{3}{2}}+10x^{\frac{5}{2}}+8x^{\frac{7}{2}})+2z^{-1}(3x^{\frac{3}{2}}+10x^{\frac{5}{2}}+8x^{\frac{7}{2}})+$ $2z^3(2x^{\frac{3}{2}}+10x^{\frac{5}{2}}+10x^{\frac{7}{2}})+2z^{-3}(2x^{\frac{3}{2}}+10x^{\frac{5}{2}}+10x^{\frac{7}{2}})+$ $2z^5(5x^{\frac{5}{2}}+14x^{\frac{7}{2}})+2z^{-5}(5x^{\frac{5}{2}}+14x^{\frac{7}{2}})+18z^7x^{\frac{7}{2}}+18z^{-7}x^{\frac{7}{2}}$
$\|2\rangle\|2\rangle$	$9x^2+24x^3+16x^4+$ $z^2(8x^2+24x^3+16x^4)+z^{-2}(8x^2+24x^3+16x^4)+$ $z^4(5x^2+24x^3+21x^4)+z^{-4}(5x^2+24x^3+21x^4)+$ $z^6(12x^3+32x^4)+z^{-6}(12x^3+32x^4)+21z^8x^4+21z^{-8}x^4$
$\|5/2\rangle\|5/2\rangle$	$2z(6x^{\frac{5}{2}}+14x^{\frac{7}{2}})+2z^{-1}(6x^{\frac{5}{2}}+14x^{\frac{7}{2}})+$ $2z^3(5x^{\frac{5}{2}}+14x^{\frac{7}{2}})+2z^{-3}(5x^{\frac{5}{2}}+14x^{\frac{7}{2}})+$ $2z^5(3x^{\frac{5}{2}}+14x^{\frac{7}{2}})+2z^{-5}(3x^{\frac{5}{2}}+14x^{\frac{7}{2}})+14z^7x^{\frac{7}{2}}+14z^{-7}x^{\frac{7}{2}}$
$\|3\rangle\|3\rangle$	$16x^3+32x^4+z^2(15x^3+32x^4)+z^{-2}(15x^3+32x^4)+$ $z^4(12x^3+32x^4)+z^{-4}(12x^3+32x^4)+z^6(7x^3+32x^4)+$ $z^{-6}(7x^3+32x^4)+16z^8x^4+16z^{-4}x^4$
$\|7/2\rangle\|7/2\rangle$	$x^{\frac{7}{2}}(20z+20z^{-1}+18z^3+18z^{-3}+14z^5+14z^{-5}+8z^7+8z^{-7})$
$\|4\rangle\|4\rangle$	$x^4(25+24z^2+24z^{-2}+21z^4+21z^{-4}+16z^6+16z^{-6}+9z^8+9z^{-8})$

Table 3.7: $(SU(2)_1 \times SU(2)_{-1})/\mathbb{Z}_2$

GNO charges	Index contribution	GNO charges	Index contribution
$T=0$		$T=5$	
$\|0,0\rangle\|0,0\rangle$	$1+4x+12x^2+8x^3$	$\|3,2\rangle\|3,2\rangle$	$2(6x^{\frac{5}{2}}+14x^{\frac{7}{2}})$
$\|1,-1\rangle\|1,-1\rangle$	$4x+16x^2+16x^3$	$\|4,1\rangle\|4,1\rangle$	$2(5x^{\frac{5}{2}}+14x^{\frac{7}{2}})$
$\|1,-1\rangle\|0,0\rangle$	$-x^4$	$\|5,0\rangle\|5,0\rangle$	$2(3x^{\frac{5}{2}}+14x^{\frac{7}{/}2})$
$\|0,0\rangle\|1,-1\rangle$	$-x^4$	$\|6,-1\rangle\|6,-1\rangle$	$14x^{\frac{7}{2}}$
$\|2,-2\rangle\|2,-2\rangle$	$9x^2+24x^3+16x^4$		
$\|3,-3\rangle\|3,-3\rangle$	$16x^3+32x^4$		
$\|4,-4\rangle\|4,-4\rangle$	$25x^4$		
$T=1$		$T=6$	
$\|1,0\rangle\|1,0\rangle$	$2(x^{\frac{1}{2}}+6x^{\frac{3}{2}}+10x^{\frac{5}{2}}+7x^{\frac{7}{2}})$	$\|3,3\rangle\|3,3\rangle$	$10x^3+16x^4$
$\|2,-1\rangle\|2,-1\rangle$	$2(3x^{\frac{3}{2}}+10x^{\frac{5}{2}}+8x^{\frac{7}{2}})$	$\|4,2\rangle\|4,2\rangle$	$15x^3+32x^4$
$\|3,-2\rangle\|3,-2\rangle$	$2(6x^{\frac{5}{2}}+14x^{\frac{7}{2}})$	$\|5,1\rangle\|5,1\rangle$	$12x^3+32x^4$
$\|4,-3\rangle\|4,-3\rangle$	$20x^{\frac{7}{2}}$	$\|6,0\rangle\|6,0\rangle$	$7x^3+32x^4$
		$\|7,-1\rangle\|7,-1\rangle$	$16x^4$
$T=2$		$T=7$	
$\|1,1\rangle\|1,1\rangle$	$3x+8x^2+12x^3+8x^4$	$\|4,3\rangle\|4,3\rangle$	$20x^{\frac{7}{2}}$
$\|2,0\rangle\|2,0\rangle$	$3x+16x^2+19x^3+24x^4$	$\|5,2\rangle\|5,2\rangle$	$18x^{\frac{7}{2}}$
$\|3,-1\rangle\|3,-1\rangle$	$8x^2+24x^3+16x^4$	$\|6,1\rangle\|6,1\rangle$	$14x^{\frac{7}{2}}$
$\|4,-2\rangle\|4,-2\rangle$	$15x^3+32x^4$	$\|7,0\rangle\|7,0\rangle$	$8x^{\frac{7}{2}}$
$\|5,-3\rangle\|5,-3\rangle$	$24x^4$		
$T=3$		$T=8$	
$\|2,1\rangle\|2,1\rangle$	$2(3x^{\frac{3}{2}}+10x^{\frac{5}{2}}+9x^{\frac{7}{2}})$	$\|4,4\rangle\|4,4\rangle$	$15x^4$
$\|3,0\rangle\|3,0\rangle$	$2(2x^{\frac{3}{2}}+10x^{\frac{5}{2}}+10x^{\frac{7}{2}})$	$\|5,3\rangle\|5,3\rangle$	$24x^4$
$\|4,-1\rangle\|4,-1\rangle$	$2(5x^{\frac{5}{2}}+14x^{\frac{7}{2}})$	$\|6,2\rangle\|6,2\rangle$	$21x^4$
$\|5,-2\rangle\|5,-2\rangle$	$18x^{\frac{7}{2}}$	$\|7,1\rangle\|7,1\rangle$	$16x^4$
		$\|8,0\rangle\|8,0\rangle$	$9x^4$
$T=4$			
$\|2,2\rangle\|2,2\rangle$	$6x^2+12x^3+12x^4$		

$\|3,1\rangle\|3,1\rangle$	$8x^2+24x^3+16x^4$		
$\|4,0\rangle\|4,0\rangle$	$5x^2+24x^3+21x^4$		
$\|5,-1\rangle\|5,-1\rangle$	$12x^3+32x^4$		
$\|6,-2\rangle\|6,-2\rangle$	$21x^4$		

Table 3.8: $U(2)_1 \times U(2)_{-1}$. T stands for the topological charge.

GNO charges	Index contribution
$\|0\rangle\|0\rangle$	$1+4x+12x^2+z^2(3x+8x^2)+z^{-2}(3x+8x^2)$
$\|1\rangle\|1\rangle$	$4x+16x^2+z^2(3x+16x^2)+z^{-2}(3x+16x^2)$
$\|2\rangle\|2\rangle$	$9x^2+8z^2x^2+8z^{-2}x^2$

Table 3.9: $SU(2)_1 \times SU(2)_{-1}$

GNO charges	Index contribution
$\|0\rangle\|0\rangle$	$1+4x+12x^2+z^2(3x+8x^2)+z^{-2}(3x+8x^2)$
$\|1/2\rangle\|1/2\rangle$	$4x+17x^2+z^2(3x+16x^2)+z^{-2}(3x+16x^2)$
$\|1\rangle\|1\rangle$	$9x^2+8z^2x^2+8z^{-2}x^2$

Table 3.10: $(SU(2)_2 \times SU(2)_{-2})/\mathbb{Z}_2$

Chapter 4

Monopoles and Aharony Duality

4.1 Introduction

An important class of dualities of four-dimensional gauge theories are Seiberg dualities which relate minimally supersymmetric $\mathcal{N}=1$ SQCD theories with gauge group $SU(N_c)$ and N_f flavors of quarks and antiquarks to $SU(N_f-N_c)$ gauge theories with N_f flavors of quarks and antiquarks as well as a singlet field coupled through a superpotential. This duality has a generalization to symplectic and to special orthogonal groups.

More than a decade ago Aharony proposed a three dimensional analog of Seiberg duality. It is a duality between the infrared limits of $\mathcal{N}=2$ gauge theories with fundamental matter and unitary or symplectic gauge groups. Namely, an $\mathcal{N}=2$ supersymmetric theory with gauge group $U(N_c)$ with N_f chiral fundamental multiplets and N_f chiral antifundamental muliplets is conjectured to be dual to an $\mathcal{N}=2$ theory with gauge group $U(N_f-N_c)$, N_f chiral fundamentals, N_f chiral antifundamentals together with additional gauge singlet chiral fields and a superpotential. For the symplectic gauge groups the duality relates $USp(2N_c)$ gauge theory with $2N_f$ fundamental chiral fields to $USp(2N_f-2N_c-2)$ gauge theory with $2N_f$ fundamental chiral fields together with a number of gauge singlets and a superpotential.

Another class of three-dimensional dualities for $\mathcal{N}=2$ and $\mathcal{N}=3$ theories with Chern-Simons terms was introduced by Giveon and Kutasov [40]. It was noticed by these authors that these dualities could be obtained from the Aharony dualities by integrating out some matter fields (see also [41] and [42]). Recently, it was shown [34] that $\mathcal{N}=6$ dualities proposed by Aharony, Bergman, and Jafferis [28] are descendant from Aharony dualities.

The fact that Aharony-type dualities generate a large class of dualities in three dimensions makes their verification and further understanding an important task.

Such a verification was recently performed by Willett and Yaakov [41] who showed that partition functions on S^3 agree for theories which are related by Aharony duality.

In the present chapter we verify that the superconformal indices of theories related by Aharony duality agree to a high order in the Taylor expansion for several low values of N_c and N_f. This is of interest because agreement of indices is a check independent of agreement of partition functions on S^3. We also discuss the role played in the duality by monopole operators. In particular, we discuss the matching of chiral rings in dual theories taking account of monopole operators.

4.2 Index for $\mathcal{N} = 2$ Theories

The superconformal index of an $\mathcal{N} = 2$ superconformal theory on $S^2 \times \mathbb{R}$ is defined by the expression

$$\mathcal{I}(x, z_i) = Tr[(-1)^F x^{E+j_3} \prod_i z_i^{F_i}], \tag{4.1}$$

where F is the fermion number, E is the energy, j_3 is the third component of spin and F_i are charges of abelian flavor symmetries. As usual, contributions to the index come from states with $\{Q, Q^\dagger\} = E - r - j_3 = 0$ [29],[16]. r is the R-charge and Q has spin $-1/2$.

An important feature of $\mathcal{N} = 2$ superconformal theories in three dimensions is that the conformal dimensions of fields are not canonical in general and generically are irrational. The formula for the superconformal index of a theory with canonical conformal dimensions Δ_Φ of chiral superfields Φ from the UV Lagrangian was obtained by Kim [16] and recently generalized to any conformal dimensions by Imamura and Yokoyama, [43]

$$\mathcal{I}(x_2, z_i) = \sum_{\{n\}} \int [da]_{\{n\}} x^{E_0(\{n\})} e^{iS^0_{CS}(\{n\},a)} \prod_i z^{F_i^0} exp(\sum_{m=1}^{\infty} f(x^m, z_i^m, a^m)). \tag{4.2}$$

The sum $\sum_{\{n\}}$ is over all GNO charges [12] $\{n\} = (n_1, ..., n_c)$ with $n_i \equiv w_i(H)$ where w_i are the weights of the fundamental representation and Hs are all element of a Cartan subalgebra defining a Dirac monopole. The integral whose measure depends on GNO charges is over a maximal torus of the gauge group, $E_0(\{n\})$ is the energy of a bare monopole with GNO charges $\{n\}$ and $F_i^0 = -\sum_\Phi \sum_{\rho\in R_\Phi} |\rho(H/2)|F_{i\Phi}$ is its global charge under a global symmetry $U(1)_i$, the sum being over all gauge weights of all chiral fields with $F_{i\Phi}$ being their $U(1)_{F_i}$ charges. $S^0_{CS}(\{n\}, a)$ is effectively the weight of the bare monopole with respect to the gauge group and a is in a Cartan subalgebra.The function $f = f_{ch} + f_v$ depends on the content of vector multiplets and hypermultiplets.

$$f_{ch} = \frac{1}{1-x^2}\sum_\Phi\sum_\rho x^{|\rho(H)|}(x^{\Delta_\Phi}e^{i\rho(a)}\prod_i z^{F_i} - x^{2-\Delta_\Phi}e^{-i\rho(a)}\prod_i z^{-F_i})$$
$$f_v = -\sum_\alpha x^{|\alpha(H)|}e^{i\alpha(a)} \tag{4.3}$$

The first sum in the expression for f_{ch} is over all chiral multiplets Φ. The second sum is over weights ρ of the representations of the gauge group in which the chiral fields Φ live. The contribution of the vector multiplet f_v contains a sum over all roots α and does not contain any anomalous dimensions because it is assumed that the superconformal R-current at the IR fixed point is a linear combination of a UV R-current and some global $U(1)$ symmetry current visible classically (in the UV). This guarantees that the vector multiplet retains its classical dimension. In general, the superconformal R-current can mix with accidental symmetry currents. In such a case the above formula for the index is not correct. We assume, following Gaiotto and Witten [19], that this manifests itself in violation of unitarity bounds on conformal dimensions of chiral operators including monopole operators, and thus, in principle we know when the formula for the index is correct. The closed-form expression for the index is not known for nonabelian gauge theories[1]See paper [44] for the abelian case. but a finite number of terms in its Taylor expansion around point $x = 0$ can be computed on the computer.

The fact that conformal dimensions Δ_Φ are not known does not pose a problem if the goal is to perform a check of duality. As usual, the index can be computed as a path integral

[1],

with (twisted) periodic boundary conditions along the time line $\mathbb{R}$. That is, it is a path integral on $S^2 \times S^1$. There are many ways to put the theory on $S^2 \times S^1$ parametrized by the choice of the R-current [43]. For the present theories any R-current is a linear combination $J_R = J_R^{UV} + \alpha J_A$ of the UV R-current J_R and the global current J_A generating the $U(1)_A$ symmetry. For a special choice of the current, that is, for a special value of parameter α which determines anomalous dimensions of fields, the theory on $S^2 \times S^1$ is superconformal.[2] In this case the quantity computed by the path integral is the index in the sense of definition (4.1) with the trace over the Hilbert space of states living on S^2. For other values of the parameter, it does not have this interpretation, but it is nevertheless a quantity characterizing the theory which is independent of the description of the theory, that is, independent of a duality frame. Thus the "indices" of dual theories must coincide as functions of the parameter α. So we can introduce a new variable $y \equiv x^\alpha$ following [43] and compare the indices as functions of two variables x and y.

4.3 Aharony Duality for Unitary Groups

The duality relates two theories which we will call electric and magnetic. The electric theory is the $\mathcal{N} = 2$ supersymmetric gauge theory with gauge group $U(N_c)$ with N_f flavors of fundamental chiral fileds Q_i and N_f flavors of antifundamental chiral fields $\tilde{Q}^{\tilde{\imath}}$. The global symmetry group is $SU(N_f) \times \widetilde{SU(N_f)} \times U(1)_A \times U(1)_T \times U(1)_R$. The first two factors are flavor symmetries, the third factor is a rotation of both Q_i and $\tilde{Q}^i$ by the same phase, $U(1)_R$ is the microscopic R-symmetry and $U(1)_T$ is the topological symmetry with the current $J^\mu = -\frac{1}{4\pi}\epsilon^{\mu\nu\rho}\mathrm{Tr}F_{\nu\rho}$ under which no elementary field is charged. We summarize the action of the global symmetry group in table 4.1.

Fields	$U(1)_R$	$U(1)_A$	$SU(N_f)$	$\widetilde{SU(N_f)}$	$U(1)_T$
Q	1/2	1	$\mathbf{N_f}$	$\mathbf{1}$	0
$\tilde{Q}$	1/2	1	$\mathbf{1}$	$\mathbf{N_f}$	0
$M_i^{\tilde{\jmath}}$	1	2	$\mathbf{N_f}$	$\mathbf{N_f}$	0

[2]This special value of the the parameter $\alpha = \Delta - 1/2$ is determined by the extremization of the absolute value of the partition function of the theory put on S^3 with respect to Δ [45].

$v_\pm$	$N_f - N_c + 1$	$-N_f$	$\mathbf{1}$	$\mathbf{1}$	± 1

Table 4.1: Global charges of fields of the electric theory.

Here $M_i^{\tilde{j}} \equiv Q_i \tilde{Q}^{\tilde{j}}$ is the meson field and $v_\pm$ are monopole fields. In the ultraviolet theory the monopole operators are defined as disorder operators in the path integral [7] with topological charges ± 1. On the Coulomb branch below the Higgs scale with all charged fields integrated out they appear in the path integral as $\prod_{i=1}^{N_c} e^{\frac{\sigma_i \pm i\gamma_i}{N_c}}$ where σ_i are real scalars from the vector multiplets of the broken gauge group $U(N_c) \to \prod_{i=1}^{N_c} U(1)_i$ and γ_i are dualized photons. More precisely, in the UV description the correlation functions of monopole operators with fundamental fields are defined by performing the path integral over fields configuration having a Dirac monopole type singularity for gauge fields,

$$A^{N,S} = \frac{H}{2r}(\pm 1 - \cos\theta) d\phi, \tag{4.4}$$

together with the corresponding singularity $\sigma = -\frac{H}{2r}$ for the real scalar σ in the vector multiplet at the insertion point to make the operator chiral. The GNO charges of monopole operators $v_\pm$ are $(\pm 1, \underbrace{0, \cdots, 0}_{N_c - 1})$.

On the magnetic side is the $\mathcal{N} = 2$ supersymmetric gauge theory with gauge group $U(N_f - N_c)$ with N_f flavors of fundamental chiral fileds q^i and N_f flavors of anti-fundamental chiral fields $\tilde{q}_{\tilde{i}}$. In addition, there are two gauge-singlet chiral fields $v_\pm$ which correspond to the monopole operators of the electric theory and a gauge-singlet chiral field $M_i^{\tilde{j}}$ which is a counterpart to the meson $Q_i \tilde{Q}^{\tilde{j}}$. The theory has a superpotential $W = M_i^{\tilde{j}} q^i q_{\tilde{j}} + v_+ V_- + v_- V_+$ where $V_\pm$ are monopole chiral operators with GNO charges $(\pm 1, \underbrace{0, ..., 0}_{N_f - N_c - 1})$. The representations of the fields under the action of the global symmetry group $SU(N_f) \times \widetilde{SU(N_f)} \times U(1)_A \times U(1)_T \times U(1)_R$ are written in Table 2.

Fields	$U(1)_R$	$U(1)_A$	$SU(N_f)$	$\widetilde{SU(N_f)}$	$U(1)_T$
q	1/2	1	$\mathbf{N_f}$	$\mathbf{1}$	0
$\tilde{q}$	1/2	1	$\mathbf{1}$	$\mathbf{N_f}$	0

$M_i^{\tilde{j}}$	1	2	$\mathbf{N_f}$	$\mathbf{N_f}$	0
$v_\pm$	$N_f - N_c + 1$	$-N_f$	$\mathbf{1}$	$\mathbf{1}$	± 1
$V_\pm$	$N_c - N_f + 1$	N_f	$\mathbf{1}$	$\mathbf{1}$	± 1

Table 4.2: Global charges of fields of the magnetic theory.

Note that some of the "elementary fields", $v_\pm$, are now charged under the topological symmetry. This is compatible with the invariance of the superpotential. The only information about the superpotential in the formula for the index (4.2) is the constraints on the superconformal IR R-charges of fields it provides.

We computed indices for several dual pairs of theories.

4.3.1 Indices for Dual Pairs of Theories with Unitary Gauge Groups

We use the notation $U(N_c)_{N_f}$ to denote the electric theory with gauge group $U(N_c)$ and N_f pairs of fundamental and antifundamental chiral fieds. The magnetic theory with gauge group $U(N_c)$ and N_f pairs of fundamentals and antifundamentals and additional singlets is denoted by $U(N_c)_{N_f} + M_i^{\tilde{j}} + v_\pm$.

(i) Electric theory: $U(2)_2$. Magnetic theory: $U(0) + M_i^{\tilde{j}} + v_\pm$.

In this case there is no vector mutiplet and no superpotential in the magnetic theory. The chiral fields 2×2 matrix $M_i^{\tilde{j}}$ and two $SU(2)_f \times \widetilde{SU(2)}_f$ flavor singlets v_+ and v_- are free. The conformal dimension $\Delta \equiv \Delta(Q) = \Delta(\tilde{Q})$ was computed in [41] to be 1/4. This is one of the rare cases when the conformal dimension is rational. The conformal dimensions of the fields of the magnetic theory are easy to find using the duality dictionary. The conformal dimension $\Delta(M) = 2\Delta$ of the $M_i^{\tilde{j}}$ is twice the conformal dimension of Q because these fields correspond to the meson of the electric theory. The conformal dimensions of singlet fields $v_\pm$ are equal to the conformal dimensions of bare monopole fields $(\pm 1, 0)$ on the electric side: $\Delta(v_\pm) = 1/2$. Of course, this is obvious because all chiral fields of the magnetic theory are free and thus have conformal dimension one-half. To the second order in x the index of the magentic theory is

$$\mathcal{I}_B = 1 + 6x^{1/2} + 21x + 50x^{3/2} + 90x^2 + \mathcal{O}(x^{5/2}). \tag{4.5}$$

The first term is the contribution of the vacuum and the second term comes from the six free chiral fileds. The contribution to the index on the A-side comes from sectors with different GNO charges. It is summarized in table 4.3.

GNO charges	Index contribution
$(0,0)$	$1 + 4x^{1/2} + 10x + 20x^{3/2} + 27x^2$
$(1,-1)$	x
$(2,-2)$	x^2
$(1,0)$	$x^{1/2} + 4x + 9x^{3/2} + 16x^2$
$(2,-1)$	$x^{3/2}$
$(2,0)$	$x + 4x^{3/2} + 9x^2$
$(3,-1)$	x^2
$(3,0)$	$x^{3/2} + 4x^2$
$(4,0)$	x^2

Table 4.3: Contribution to the index from different GNO sectors in $U(2)_2$ theory.

Summation of these contributions over the topological charges (the contribution from the negative topological charges are the same as from the positive ones) reproduces the answer on the magnetic side, which consitutes a nontrivial check of the duality.

In general, we do not expect the GNO charges within a sector with a fixed $U(1)_T$ charge to mark sectors in the Hilbert space of the theory because they do not arise from any conserved currents. Rather, it is an artifact of the weakly coupled description of the theory. We saw it in chapter 3 where the indices of dual theories were in agreements within a given topological sector only after summation over all GNO charges and there was no mapping of GNO charges between dual theories. This was also noticed in [46].

However, in certain situations GNO charges may acquire invariant meaning if they correlate with other quantum numbers. This is the present case. For each value of the $U(1)_T$ charge and the $U(1)_A$ charge the GNO charge of a bare monopole is determined uniquely. We list the global charges of some of the low-energy bare monopoles in Table 4.4.

Bare monopole	Conformal dimension	Topological charge	$U(1)_A$-charge
$(1,0)$	1/2	1	-2
$(-1,0)$	1/2	-1	-2
$(1,-1)$	1	0	-4
$(2,-2)$	2	0	-8
$(2,-1)$	3/2	1	-6
$(1,1)$	3	2	-4
$(2,0)$	1	2	-4
$(3,-1)$	2	2	-8
$(3,0)$	3/2	3	-6
$(4,0)$	2	4	-8

Table 4.4: Quantum numbers of bare monopole operators in $U(2)_2$ theory.

The duality relates monopole operators of the electric theory to (composite) chiral fields of the magnetic theory. Using matching of quantum numbers it is easy to establish a dictionary for this correspondance. For some of the low-dimension operators it is

Chiral operator and OPE
$v_+ \equiv T_{(1,0)}$
$v_- \equiv T_{(-1,0)}$
$T_{(1,-1)} \sim v_+ v_-$
$T_{(2,-2)} \sim v_+^2 v_-^2$
$T_{(2,-1)} \sim v_+^2 v_-$
$T_{(1,1)} \sim M^2 v_+^3 v_-$
$T_{(2,0)} \sim v_+^2$

$T_{(3,-1)} \sim v_+^3 v_-$
$T_{(3,0)} \sim v_+^3$
$T_{(4,0)} \sim v_+^4$

Table 4.5: Mapping of chiral operators under duality. M^2 is the $SU(2) \times \widetilde{SU(2)}$ flavor singlet quadratic in meson fields.

(ii) Electric theory is $U(2)_3$, magnetic theory is $U(1)_3 + M_i^{\tilde{j}} + v_+ + v_-$.

In this case the conformal dimensions of all fields are irrational and $\Delta \equiv \Delta(Q) = \Delta(\tilde{Q}) \approx 0.3417$.[3] We introduce additional variable $y \equiv x^{2\Delta - 1}$ and expand the indices of both theories in powers of x. The contribution from different topological and GNO sectors are given in tables 4.9 and 4.10 in appendix A. We find a perfect agreement for each value of the topological charge up to the third power in x.

(iii) Electric theory: $U(2)_4$, magnetic theory: $U(2)_4 + M_i^{\tilde{j}} + v_\pm$. The conformal dimension of Q is $\Delta \approx 0.3852$. Naively, the magnetic theory contains more degrees of freedom than the electric theory by weak-coupling counting. Nevertheless, they flow to the same infrared fixed point. The indices agree in each topological sectors of both theories up to at least the third power in x (tables 4.11 and 4.12 in appendix A).

(iv) As our last check of the duality for unitary groups we chose the following pair. Electric theory is $U(3)_4$, and magnetic theory is $U(1)_4 + M_i^{\tilde{j}} + v_+ + v_-$ ($\Delta \approx 0.3058$). We found agreement of indices for each topological sector up to the fourth power in x (tables 4.13 and 4.14).

[3] We took the approximate values of conformal dimensions from [41].

4.4 Chiral Ring

4.4.1 Examples

(i) There are two ways to look at the table 4.5. One way is to view it as a correspondence between operators on different sides of duality. Another way is a relation in the chiral ring of the electric theory if we regard $v_\pm$ as chiral monopole operators with GNO charges $(\pm 1, 0)$. In particular, we see that the chiral ring is generated by 6 generators: chiral fields $M_i^{\tilde{j}}$ and two chiral monopole operators $v_\pm$.

(ii) The situation is more involved for greater number of flavors and larger gauge groups. As the next simplest case we consider the chiral rings of the dual pair of theories: the electric theory $U(2)_3$ and the magnetic theory $U(1)_3 + M_i^{\tilde{j}} + v_+ + v_-$.

First we look at the magnetic side. The generators of the chiral ring include eleven operators: mesons $M_i^{\tilde{j}} \equiv Q_i \tilde{Q}^{\tilde{j}}$ and $v_\pm$. Other candidates for generators are monopole operators. The monopole operators $V_\pm$ having GNO charges (± 1) are dismissed right away because they are Q-exact due to the presence of the superpotential $v_+ V_- + v_- V_+$. There remain monopole operators with higher values of GNO charge. However, they are also Q-exact because they are just powers of $V_\pm$. Namely, $V_{n>0} = V_+^n$ and $V_{n<0} = V_-^{-n}$. This can be seen from the fact that all global charges agree and contribution of these operators to the index cancels. This does not consitute a proof. Nevertheless, it appears to be very natural. Thus we assume that the eleven chiral operators are all generators of the chiral ring. We provide an additional argument in favor of this conclusion later.

On the electric side of duality there are chiral operators: $M_i^{\tilde{j}}$ and $v_\pm$ where the last two are now monopole operators. We should address the question of whether some of the monopole operators are in fact not generated by $M_i^{\tilde{j}}$ and $v_\pm$. For example, are there any monopole operators whose quantum numbers are such that no monomial in the generators $M_i^{\tilde{j}}$ and $v_\pm$ can reproduce them? Naively, such a monopole operator does exist. In fact, there are many of them and they all are generated in terms of quantum numbers by the operator corresponding to the bare monopole state $|1, 1\,\rangle$ which has GNO charge $(1, 1)$. To understand the origin of this phenomenon one should recall the framework in which the monopole operators are treated. We will discuss the general case $U(N_c)_{N_f}$ and use the duality conjecture to recover some information about monopole operators in the next few paragraphs and then return to the special case $U(2)_3$ to illustrtate the general conclusions that we make.

4.4.2 General Discussion

The definition of monopole operators as a certain class of disordered operators is only constructive in weakly interacting theories. When the theory of interest is not weakly coupled yet supersymmetric one can proceed in two steps to make use of these operators. First, the theory is put on $S^2 \times R$, that is, radially quantized. Second, a supersymmetic deformation to a weak coupling is performed. In the first step monopole operators become states in the radially quantized picture as all local operators do. In the second step the supersymmetry guarantees that some information about the original theory is preserved in the deformed theory which describes dynamics of free fields quanta in the classical monopole backgrounds parametrized by GNO charges. The Fock vacua in every GNO sector of this theory are the bare monopoles. The index formula (4.2) computes the index (4.1) of this free theory which by the supersymmetry of the deformation is the index of the original radially quantized theory. This is an example of the preserved information. Another example is the spectrum of chiral scalars which are bottom components of different current multiplets [1, 2].

Unfortunately, the chiral ring as a vector space is not part of the structure of the original theory preserved by the deformation. We show it in the next subsection. Two things can happen. First, a state corresponding to a nontrivial element of the chiral ring of the original theory may become Q-exact when the deformation is switched on if there are states with appropriate quantum numbers to pair up with it. Then the energy of this long multiplet may be changed in the deformed theory so that no traces of the original state are seen in the deformed theory. Even if the energy is not changed, we do not pay attention to long multiplets in the deformed theory because they will remain long when the deformation is switched on and what happens to them is anyone's guess. The $U(2)_2$ theory provides an example: in the deformed electric theory there is no state corresponding to chiral operator $v_+v_-M_i^{\tilde{j}}$. There is a manifestation of this in the index: there is no contribution with the quantum numbers of $v_+v_-M_i^{\tilde{j}}$ (see tables 4.3 and 4.4). In the magnetic theory this happens because the contribution of $v_+v_-M_i^{\tilde{j}}$ is canceled by the contribution of the BPS spinor $\Psi^{\dagger\tilde{j}}_{i}$, the conjugate of the superpartner of $M_i^{\tilde{j}}$. Second, there may appear accidental Q-cohomology classes in the deformed theory by essentially the opposite process. In fact, as explained below, these two processes become more likely with the increase of the energy of states and rank of the gauge group.

Yet, some low-energy states are, in fact, protected. These are states corresponding to operators $M_{\tilde{j}}^i$ and $v_\pm$ that are naturally expected to be the complete set of generators of the chiral ring. Of course, the presence of meson operators in the chiral ring of the electric theory is obvious, and, due to the duality, the presence of monopole operators $v_\pm$ is guaranteed. From the point of view of the electric theory their presence is ensured as they are the lowest energy states in the sector with topological charge one and they cannot pair up with fermions of higher energy. More precisely, for a BPS scalar to become a part of a longer multiplet there must be a fermion available with appropriate quantum numbers. In particular, by unitarity, its energy must be less than that of the scalar.

There remains a possibility that some other monopole operators can complete the set of generators of the chiral ring. Below we argue that the assumption that this does not happen is consistent with the information preserved along the deformation.

4.4.3 Scalar BPS States in the Deformed Theory

The Hilbert space of the deformed theory is the direct sum of Fock spaces whose vacua are bare monopole states with different GNO charges. All these vacua are BPS scalars. Other BPS scalar states are obtained by acting on the bare monopoles with the creation operators corresponding to the fields of the theory. It is not a problem to obtain scalar states in this way but the BPS condition is quite restrictive. Consider a bare monopole state $|n_1, ..., n_{N_c}\rangle$ with GNO charges $(n_1, ..., n_{N_c})$. A matter field creation operator φ_i with gauge index i interacts with n_i units of magnetic charge. As a result [7, 6], it obtaines "anomalous" spin with minimal value $j_0 = \frac{|n_i|}{2}$ if φ was a scalar field and $j_0 = \frac{|n_i|-1}{2}$ if it was a spinor. The energy of this mode is also changed compared to the case when the mode does not interact with magnetic flux. We list the different modes and their energies when they are coupled to

n units of magnetix flux in table 4.6 below.

fields	$U(1)_{R*}$	Spin	Energy
$Q^\dagger$	$\frac{1+\alpha}{2}$	$j_0 = \frac{\lvert n\rvert}{2}$	$j + \frac{1+\alpha}{2} = r + j$
$\tilde{Q}^\dagger$	$\frac{1+\alpha}{2}$	$j_0 = \frac{\lvert n\rvert}{2}$	$j + \frac{1+\alpha}{2} = r + j$
ψ_Q	$\frac{1-\alpha}{2}$	$j_0 = \frac{\lvert n\rvert+1}{2}$	$j + \frac{1-\alpha}{2} = r + j$
$\psi_{\tilde{Q}}$	$\frac{1-\alpha}{2}$	$j_0 = \frac{\lvert n\rvert+1}{2}$	$j + \frac{1-\alpha}{2} = r + j$
$\psi_Q^\dagger$	$-\frac{1-\alpha}{2}$	$j_0 = \frac{\lvert n\rvert-1}{2}$	$j + \frac{1+\alpha}{2} > r + j$
$\psi_{\tilde{Q}}^\dagger$	$-\frac{1-\alpha}{2}$	$j_0 = \frac{\lvert n\rvert-1}{2}$	$j + \frac{1+\alpha}{2} > r + j$
$a^{(1)}$	0	$j_0 = \frac{\lvert n\rvert}{2} + 1$	$j = r + j$
$a^{(2)}$	0	$j_0 = \frac{\lvert n\rvert}{2} + 1$	$j + 1 > r + j$
λ	-1	$j_0 = \frac{\lvert n\rvert+1}{2}$	$j > r + j$
$\lambda^\dagger$	1	$j_0 = \frac{\lvert n\rvert-1}{2}$	$j + 1 = r + j$

Table 4.6: Quantum numbers of fields and supercharges.

Here $U(1)_{R*}$ is the IR superconformal R-symmetry and r is its charge. The last four modes come from the vector mutiplet. A scalar state is BPS iff its quantum numbers satisfy the relation $E = r$. This requirement can be met only if the modes that excite the bare monopole are modes of scalar fields Q_i or $\tilde{Q}^{\tilde{j}}$ that do not interact with magnetic flux or modes of gluino $\lambda_{ij}^\dagger$ that interact with one unit of the magnetic flux $|n_i - n_j| = 1$. Here i and j are gauge indices. In the first case this means that among GNO charges $(n_1, ..., n_{N_c})$ at least one must be zero $n_i = 0$. In the second case the difference of at least two GNO charges must be one. Moreover, one must use at least two gaugino modes to guarantee gauge invariance. For example, the gauge invariant state built on $|2, 1\rangle$ is $\lambda_{12}^\dagger \lambda_{21}^\dagger |2, 1\rangle$.

So, a scalar BPS state in the deformed theory is either a bare monopole with arbitrary GNO charges or a bare monopole excited with free squark modes and/or gluino modes interacting with one unit of magnetic flux. All gauge indices of the squark and gluino modes must be contracted in a gauge-invariant way. Here gauge invariance is with respect to the unbroken by the fluxes subgroup $U(N_1) \times \cdots \times U(N_k) \subset U(N_c)$. Note that the number of squark modes Q must be equal the number of squark modes $\tilde{Q}$ for the state to be gauge invariant.

Now we can look for counterparts of the chiral ring operators in the deformed theory. For a monomial in the monopole operators $v_{\pm}$, comparison of quantum numbers gives $v_+^n v_-^m \to |n, -m, \underbrace{0, ..., 0}_{N_c-2}\rangle$ where the ket-vector is the bare monopole state with GNO charges $(n, -m, \underbrace{0, ..., 0}_{N_c-2})$. Multiplying this operator by a meson field $M_i^{\tilde{j}}$ naturally corresponds to $Q_i \tilde{Q}^{\tilde{j}} |n, -m, \underbrace{0, ..., 0}_{N_c-2}\rangle$ where the gauge indices of scalar modes of squarks run over $N_c - 2$ values corresponding to the unbroken gauge group $U(N_c - 2) \subset U(N_c)$ and are contracted properly to form a gauge singlet. There must be no gauge indices corresponding to the commutant of the $U(N_c - 2)$ in $U(N_c)$ because such modes interact with the monopole background and as a result their energy is increased [7], [6] which makes it impossible to build BPS scalars with them. Multiplying by more powers of mesons corresponds to putting more squarks modes on the bare monopole. If $N_c \leq 2$, then there is no state in the deformed theory corresponding to the operator $v_+^n v_-^m M_i^{\tilde{j}}$.

The next question is whether there is a scalar BPS monopole operator with such quantum numbers that it cannot be generated by mesons $M_{\tilde{j}}^{\tilde{j}}$ and monopole operators $v_{\pm}$, which then is a new generator of the chiral ring. As usual, the direct analysis of the original theory which is strongly coupled is out of reach, so one can try looking at the deformed dual theories.

If in the deformed electric theory there is a BPS state with quantum numbers which cannot be reproduced by a monomial in $M_{\tilde{j}}^i$ and $v_{\pm}$, then, by the above argument, this state must be a bare monopole excited with free squark modes and/or gluino modes. The free squark modes correspond to (a product of) meson operators, so we can strip the state of them. This new state corresponds to a BPS monopole operator which is still not generated by the mesons and monopole operators $v_{\pm}$. Now we make use of the conjectured duality. In the dual magnetic theory this operator correspond to a (dressed) monopole state. If it contains free squark modes, we repeat the procedure again to obtain a monopole state which is either bare or excited with only gluino modes. Then we again look at the electric side, and so on. This process reduces energy, so it must stop at some step. It stops only if a monopole operator not generated from the mesons and $v_{\pm}$ corresponds to states in the deformed theories which are both either bare monopoles or bare monopoles excited with gluino modes. However, bare monopoles or bare monopoles excited with only gluino modes on the different sides of the duality can never have the same $U(1)_A$ charge. Indeed, the $U(1)_A$ charges of such monopoles on the electric side $A = -N_f \sum_{i=1}^{N_c} |n_i|$ are always negative while the charges of monopoles on the magnetic side $A = N_f \sum_{j=1}^{N-f-N_c-1} |n_j|$ are always positive. Thus, they never match. So the assumption that mesons and minimal monopole operators $v_\pm$ exhaust the generators of the chiral ring is consistent with the information preserved by the deformation. Moreover, the chiral ring is freely generated by them as long the IR superconformal R-current is not accidental. On the magnetic side this is obvious in view of the absence of a superpotential monomial depending on $v_\pm$ and $M_i^{\tilde{j}}$ simultaneously. On the electric side this can be proved not using the duality conjecture: matching quantum numbers of any relation between them lead to negative energies of either the mesons or $v_\pm$.[4]

The conclusion is that in all Aharony-type theories with arbitrary N_f and N_c[5] the deformation does not preserve the chiral ring as a vector space. Indeed, if the chiral ring is generated not by only mesons and minimal monopole operators $v_\pm$ then by the above reasoning there cannot be one-to-one correspondance between BPS scalar states in the deformed theories and chiral operators in the original one. If, on the other hand, the entire chiral ring is generated by mesons and $v_\pm$, then it is not preserved by the deformations either, because there are many BPS monopoles in the deformed theories (for $N_c > 2$) whose quantum numbers forbid them to correspond to monomials in mesons and $v_\pm$. Thus the spectra of scalar BPS states in the original and deformed theories are not the same.

4.4.4 Illustration of the General Conclusions

Returning to the state $|1,1\rangle$ in the $U(2)_3$ theory, the most natural explanation of its appearence in view of the duality is that it is accidental in the deformed theory. When the interactions are switched on it gets paired up with a fermion and is not present in the original theory as a nontrivial element of the chiral ring. In other words, it is zero in the chiral ring. Indeed, its contribution to the index is x^3y^{-3} while there are fermionic monopole operators with GNO charge $(2,0)$ which contribute $-18x^3y^{-3}$ to the index. Among these

[4]See appendix B.
[5]As long as N_f is big enough compared to N_c for an accidental R-charge not to appear.

fermionic operators there are those with quantum numbers necessary for $|1,1\,\rangle$ to become their superdescendant once the interactions are turned back on.

This example illustrates a general fact about monopole operators in the $\mathcal{N}=2$ SQCD theories. Assumption of the completeness of the chiral ring freely generated by meson operators M together with minimal monopole operators $v_\pm$ leads to the conclusion that all bare monopole states in the deformed theory with GNO charges different from $(n,-m,0,...,0)$ are accidental BPS states. We provide evidence in favor of this statement in appendix C.

As an additional example we consider the bare monopole state $|1,1,0\,\rangle$ in $U(3)_4$ theory from example (**iv**). Its contribution to the index is $x^8y^{-4}z^{-8}$ where the power of z indicates the $U(1)_A$ charge. In the same topological sector monopole states with GNO charge $(2,0,0)$ contribute $x^4y^{-4}z^{-8}-32x^8y^{-4}z^{-8}+\mathcal{O}(x^{10})$. The 32 fermionic states have the form $\bar\psi^iQ_j|2,0,0\,\rangle$ and $\bar{\tilde\psi}_{\tilde\imath}\tilde Q^{\tilde\jmath}|2,0,0\,\rangle$ where (i,j) are indices of flavor group $SU(4)$, $(\tilde\imath,\tilde\jmath)$ are indices of flavor group $\widetilde{SU(4)}$ and gauge indices corresponding to the unbroken $U(2)$ are contracted properly and not shown. In terms of the representation of the flavor group $SU(4)\times\widetilde{SU(4)}$ the 32 fermions are $(\bar{\mathbf{4}},\mathbf{1})\times(\mathbf{4},\mathbf{1})+(\mathbf{1},\bar{\mathbf{4}})\times(\mathbf{1},\mathbf{4})$. There are two flavor singlets among them, one of which can pair up with the bare monopole $|1,1,0\,\rangle$.

Another conclusion is that not all elements of the chiral ring are present in the deformed theory. For instance, in the example (**i**) there is no state in the deformed theory corresponding to operator $v_+v_-M^i_{\tilde\jmath}$. This is possible because this state does not make a distinguished contribution to the index. Indeed, the term $20x^{3/2}$ originates from states with $U(1)_A$ charge 6 instead of -2 which would be if contribution of the operator $v_+v_-M^i_{\tilde\jmath}$ was not canceled by a potential fermionic superpartner.[6]

Finally, an important conclusion is that GNO charges do not parametrize sectors in the Hilbert space as charges of global symmetries do. They are just labels of operators or states in the radially quantized picture. This follows from the fact that the $\mathcal{N}=2$ $U(N)$ SQCD with N_f flavors of quarks and N_f flavors of antiquarks contain bare monopole operators that must be superdescendants of fermions that have different GNO charges,[7] because these fermions have lower conformal dimension and the bare monopoles are the lowest conformal dimension

[6]This is seen when the additional parameter z corresponding to the $U(1)_A$ symmetry introduced into the index.

[7]Assuming validity of Aharony duality which is now well tested.

operators with given GNO charges. The same conclusion can be reached if one notes that all monopole operators on the B-side are superdescendants, so the monopole operators of the electric theory which are nontrivial elements of the chiral ring do not correspond to any monopole operators of the magnetic theory. Rather, they correspond to operators which are generated by elementary fields.

4.5 Aharony Duality for Symplectic Groups

The duality for symplectic groups is quite similar to the case of unitary groups. The electric theory is a $USp(2N_c)$ $\mathcal{N}=2$ gauge theory with $2N_f$ chiral multiplets in the fundamental representation. The composite chiral gauge invariant fields include the meson $M_{ij} \equiv Q_i Q_j$ and the monopole field Y. Their quantum numbers are displayed in table 4.7.

Fields	$U(1)_R$	$U(1)_A$	$SU(2N_f)$
Q	$1/2$	1	$\mathbf{2N_f}$
M	1	2	$\mathbf{N_f(2N_f-1)}$
Y	$2(N_f-N_c)$	$-2N_f$	$\mathbf{1}$

Table 4.7: Global charges of fields of the electric theory.

The dual theory is an $USp(2(N_f-N_c-1))$ $\mathcal{N}=2$ gauge theory with $2N_f$ chiral multiplets q_i in the fundamental representation together with singlet chiral fields M_{ij} and Y which correspond to the composite chiral fields on the electric theory. There is a superpotential $W = M_{ij} q_i q_j + Y\tilde{Y}$ where $\tilde{Y}$ is the monopole field in the magnetic theory. The global charges of all fields are written in table 4.8

Fields	$U(1)_R$	$U(1)_A$	$SU(2N_f)$
q	$1/2$	1	$\mathbf{2N_f}$
M	1	2	$\mathbf{N_f(2N_f-1)}$
Y	$2(N_f-N_c)$	$-2N_f$	$\mathbf{1}$
$\tilde{Y}$	$-2(N_f-N_c-1)$	$2N_f$	$\mathbf{1}$

Table 4.8: Global charges of fields of the magnetic theory.

Unlike the previously discussed theories with unitary gauge groups, the gauge groups in the present case are simple which means there is no topological current. The monopole operator Y does not carry any quantum numbers in addition to the perturbative ones.[8] The GNO charges are merely labels distinguishing different operators. When comparing the indices of dual theories we must sum over GNO charges. The GNO charges of Y are $(1, \underbrace{0, ..., 0}_{N_c-1})$ and those of $\tilde{Y}$ are $(1, \underbrace{0, ..., 0}_{N_f-N_c-2})$. We compared the indices for the following three dual pairs of theories and found complete agreement in the lower orders in x. Similarly to the case of unitary gauge groups the subscript of the gauge group stands for N_f.

(i) Electric theory: $USp(2)_3$, magnetic theory: $USp(2)_3 + M + Y$. The index is

$$\mathcal{I} = 1 + 15xy - 36x^2 + 105x^2y^2 + x^2y^{-6} + 21x^3y^{-1} - 384x^3y + 490x^3y^3 + x^3y^{-9} + \cdots , \tag{4.6}$$

where $y \equiv x^{2\Delta-1}$. The contributions from different GNO sectors are summarized in tables 4.14 and 4.15 in appendix A.

(ii) Electric theory is $USp(4)_5$, magnetic theory is $USp(4)_5 + M + Y$.

The index is

$$\mathcal{I} = 1 + 45xy - 100x^2 + xy^{-5} + 1035x^2y^2 + 45x^2y^{-4} + x^2y^{-10} + 55x^3y^{-1} - 4400x^3y + 16005x^3y^3 - 99x^3y^{-5} + 825x^3y^{-3} + 45x^3y^{-9} + x^3y^{-10} + \cdots . \tag{4.7}$$

The contribution from different GNO sectors are written down in Tables 4.15 and 4.16 in appendix A.

Arguments analogous to those for unitary gauge groups make plausible the assumption that the chiral rings of symplectic theories of Aharony types are freely generated by meson

[8] By perturbative quantum numbers we mean Noether charges associated with symmetries of the UV Lagrangian.

operators M and operators Y which are monopole operators of minimal GNO charges for electric theories and fundamental fields for magnetic theories. Analogously to the case of unitary gauge groups all bare monopoles with GNO charges different from $(n, 0, ..., 0)$ are accidental BPS states in the deformed theory. The states $|n, 0, ..., 0\rangle$ correspond to the chiral operator Y^n.

For instance, in the second example (**ii**) some bare monopole states on the electrical side are not generated by only Y and the mesons. There is $|1, 1\rangle$ among the states, whose energy makes it impossible for it to correspond to any monomial in Y and Ms. Therefore, one expects that it gets paired up with a fermion on the way from the weak coupling to the original theory and is not present as a nontrivial element of the chiral ring in the original theory. Indeed, there is an indication of that in the index. The contribution of $|1, 1\rangle$ is x^4y^{-10} is canceled by the contribution $-100x^4y^{-10}$ of fermionic excited monopole with GNO charge $(2, 0)$. In other words, the index suggests that it pairs up with a fermion with GNO charge $(2, 0)$ and appropriate $U(1)_A$ charge.

In the example (**i**) all bare monopole operators of the electric theory are nontrivial elements of the chiral ring and generated by the minimal bare monopole operator Y: if $T_{n>0}$ is the bare monopole operator with GNO charge $n > 0$, then $T_{n>0} = Y^n$.[9]

4.6 Appendix A. Contribution to Indices from Different GNO Sectors

GNO charges	Index contribution
$(0, 0)$	$1 + 9xy - 18x^2 + 45x^2y^2 + 9x^3y^{-1} - 144x^3y + 164x^3y^3$
$(1, -1)$	xy^{-3}
$(2, -2)$	x^2y^{-6}
$(3, -3)$	x^3y^{-9}
$(1, 0)$	$x^{1/2}y^{-3/2} + 9x^{3/2}y^{-1/2} - 17x^{5/2}y^{-3/2} + 36x^{5/2}y^{1/2}$
$(2, -1)$	$x^{3/2}y^{-9/2}$

[9]The Weyl group of $USp(2) = SU(2)$ identifies GNO charges n and $-n$, and we choose representatives $n > 0$.

$(3,-2)$	$x^{5/2}y^{-15/2}$
$(1,1)$	x^3y^{-3}
$(2,0)$	$xy^{-3}+9x^2y^{-2}-18x^3y^{-3}+36x^3y^{-1}$
$(3,-1)$	x^2y^{-6}
$(4,-2)$	x^3y^{-9}
$(3,0)$	$x^{3/2}y^{-9/2}+9x^{5/2}y^{-7/2}$
$(4,-1)$	$x^{5/2}y^{-15/2}$
$(4,0)$	$x^2y^{-6}+9x^3y^{-5}$
$(5,-1)$	x^3y^{-9}
$(5,0)$	$x^{5/2}y^{-15/2}$
$(6,0)$	x^3y^{-9}

Table 4.9: Contribution to the index from different GNO sectors in $U(2)_3$ theory.

GNO charge	Top. charge	Index contribution
0	0	$1+xy^{-3}+9xy+x^2y^{-6}-20x^2+45x^2y^2+x^3y^{-9}-2x^3y^{-3}+27x^3y^{-1}-162x^3y^{-1}+166x^3y^3$
	1	$x^{1/2}y^{-3/2}+x^{3/2}y^{-9/2}+9x^{3/2}y^{-1/2}-x^{3/2}y^{3/2}+x^{5/2}y^{-15/2}-19x^{5/2}y^{-3/2}+45x^{5/2}y^{1/2}-9x^{5/2}y^{5/2}$
	2	$xy^{-3}+x^2y^{-6}+9x^2y^{-2}-x^2+x^3y^{-9}-19x^3y^{-3}+45x^3y^{-1}-9x^3y^1$
	3	$x^{3/2}y^{-9/2}+x^{5/2}y^{-15/2}+9x^{5/2}y^{-7/2}-x^{5/2}y^{-3/2}$
	4	$x^2y^{-6}+x^3y^{-9}+9x^3y^{-5}-x^3y^{-3}$
	5	$x^{5/2}y^{-15/2}$
	6	x^3y^{-9}
1	0	$x^2+x^3y^{-3}-9x^3y^{-1}+9x^3y-x^3y^3$
	1	$x^{5/2}y^{-3/2}$
	2	x^3y^{-3}
-1	0	$x^2+x^3y^{-3}-9x^3y^{-1}+9x^3y-x^3y^3$

	1	$x^{3/2}y^{3/2} + x^{5/2}y^{-3/2} - 9x^{5/2}y^{1/2} + 9x^{5/2}y^{5/2}$
	2	$x^2 + x^3y^{-3} - 9x^3y^{-1} + 9x^3y - x^3y^3$
	3	$x^{5/2}y^{-3/2}$
	4	x^3y^{-3}
-2	2	x^3y^3

Table 4.10: Contribution to the index from different GNO sectors in $U(1)_3+M+v_\pm$ theory. GNO charge coincides with the topological charge for the bare monopole, but different for excited states due to the fact that fields $v_\pm$ carry topological charge.

GNO charges	Index contribution
$(0,0)$	$1 + 16xy - 32x^2 + 136x^2y^2 + 16x^3y^{-1} - 480x^3y + 800x^3y^3$
$(1,-1)$	x^2y^{-4}
$(2,-2)$	x^4y^{-8}
$(1,0)$	$xy^{-2} + 16x^2y^{-1} - 31x^3y^{-2} + 100x^3$
$(2,-1)$	x^3y^{-6}
$(2,0)$	$x^2y^{-4} + 16x^3y^{-3}$
$(3,0)$	x^3y^{-6}

Table 4.11: Contribution to the index from different GNO sectors in $U(2)_4$ theory.

GNO charge	Top. charge	Index contribution
$(0,0)$	0	$1 + 16xy + x^2y^{-4} - 34x^2 + 136x^2y^2 + x^4y^{-2}+$ $16x^3y^{-1} + -512x^3y + 816x^3y^3 + 16x^3y^5$
	1	$xy^{-2} - xy^2 + 16x^2y^{-1} - 16x^2y^3 + x^3y^{-6}-$ $33x^3y^{-2} + 136x^3 + 32x^3y^2 - 136x^3y^4$
	2	$x^2y^{-4} - x^2 + 16x^3y^{-3} - 16x^3y^2$

	3	$x^3y^{-12} - x^3y^{-2}$
$(1,-1)$	0	$x^2y^4 - 16x^3y^3 + 16x^3y^5$
	1	$x^3y^2 - x^3y^6$
$(1,0)$	0	$x^2 - x^2y^4 + 16x^3y - 16x^3y^5$
	1	$x^3y^{-2} - x^3y^2$
$(0,-1)$	0	$x^2 - x^2y^4 + 16x^3y - 16x^3y^5$
	1	$xy^2 + 16x^2y^3 - 36x^3 + x^3y^{-2} - 33x^3y^2 +$ $136x^3y^4 + x^3y^6$
	2	$x^2 - x^2y^4 + 16x^3y - 16x^3y^5$
	3	$x^3y^{-2} - x^3y^2$
$(0,-2)$	1	$x^3y^2 - x^3y^6$
	2	$x^2y^4 + 16x^3y^5$
	3	$x^3y^2 - x^3y^6$
$(0,-3)$	3	x^3y^6
$(1,-2)$	1	x^3y^6

Table 4.12: Contribution to the index from different GNO sectors in $U(2)_4 + M + v_\pm$ theory.

GNO charges	Index contribution
$(0,0,0)$	$1 + 16xy + 136x^2y^2 + 816x^3y^3 - 32x^4 + 3875x^4y^4$
$(1,0,-1)$	x^4y^{-4}
$(1,0,0)$	$x^2y^{-2} + 16x^3y^{-1} + 136x^4$
$(2,0,0)$	x^4y^{-4}

Table 4.13: Contribution to the index from different GNO sectors in $U(3)_4$ theory.

GNO charge	Top. charge	Index contribution

0	0	$1 + 16xy + 136x^2y^2 + 816x^3y^3 - 34x^4 + x^4y^{-4} + 3877x^4y^4$
	1	$x^2y^{-2} - x^2y^2 + 16x^3y^{-1} - 16x^3y^3 + 136x^4 - 136x^4y^4$
	2	$-x^4 + x^4y^{-4}$
1	0	$x^4 - x^4y^4$
-1	0	$x^4 - x^4y^4$
	1	$x^2y^2 + 16x^3y^3 + 136x^4y^4$
	2	$x^4 - x^4y^4 + 16x^3y - 16x^3y^5$
-2	2	x^4y^4

Table 4.14: Contribution to the index from different GNO sectors in $U(1)_4 + M + v_\pm$ theory.

GNO charge	Index contribution
0	$1 + 15xy - 36x^2 + 105x^2y^2 + 21x^3y^{-1} - 384x^3y + 490x^3y^3$
1	xy^{-3}
2	x^2y^{-6}
3	x^3y^{-9}

Table 4.15: Contribution to the index from different GNO sectors in $USp(2)_3$ theory.

GNO charge	Index contribution
0	$1 + xy^{-3} + 15xy - xy^3 + x^2y^{-6} - 37x^2 + 120x^2y^2 - 15x^2y^4 + x^3y^{-9} - x^3y^{-3} + 36x^3y^{-1} - 504x^3y + 715x^3y^3 - 120x^3y^5$
1	$xy^3 + x^2 - 15x^2y^2 + 15x^2y^4 - x^2y^6 + x^3y^{-3} - 15x^3y^{-1} + 120x^3y - 226x^3y^3 + 135x^3y^5 - 15x^3y^7$
2	$x^2y^6 + x^3y^3 - 15x^3y^5 + 15x^3y^7 - x^3y^9$

3	x^3y^9

Table 4.16: Contribution to the index from different GNO sectors in $USp(2)_3 + 15M + Y$ theory.

GNO charge	Index contribution
$(0,0)$	$1 + 45xy - 100x^2 + 1035x^2y^2 + 55x^3y^{-1} - 4400x^3y + 16005x^3y^3$
$(1,0)$	$xy^{-5} + 45x^2y^{-4} - 99x^3y^{-5} + 825x^3y^{-3}$
$(2,0)$	$x^2y^{-10} + 45x^3y^{-9}$
$(3,0)$	x^3y^{-10}

Table 4.17: Contribution to the index from different GNO sectors in $USp(4)_5$ theory.

GNO charge	Index contribution
$(0,0)$	$1xy^{-5} + 45xy - xy^5 - 101x^2 + x^2y^{-10} + 45x^2y^{-4} + 1035x^2y^2 - 45x^2y^6 + x^3(y^{-10} + 45y^{-9} - 100y^{-5} + 825y^{-3} + 55y^{-1} - 4445y + 16215y^3 + 99y^5 - 1035y^7)$
$(1,0)$	$xy^5 + x^2(1 + 45y^6 - y^{10}) + x^3(y^{-5} + 45y - 210y^3 - 100y^5 + 1035y^7 - 45y^{11})$
$(2,0)$	$x^2y^{10} + x^3(y^5 + 45y^{11} - y^{15})$
$(3,0)$	x^3y^{15}

Table 4.18: Contribution to the index from different GNO sectors in $USp(4)_5 + 45M + Y$ theory.

4.7 Appendix B. Relations Between Generators of the Chiral Ring

The fact that chiral operators $M_i^{\tilde{j}}$ and $v_\pm$ are free generators of the chiral ring is obvious in the magnetic theory since there is no superpotential including both mesons and operators $v_\pm$. In this appendix we prove this fact from the electric theory point of view not using duality.

If there is a relation between generators $M_i^{\tilde{j}}$ and $v_\pm$ of the chiral ring, then there exist a monomial in these fields with zero topological, $U(1)_A$ charges and conformal dimension. This monomial has the expression $(v_- v_+)^n M^m = 1$ where n and m are integral numbers, not necessarily positive. The condition of zero $U(1)$ charge is $-2N_f n + 2m = 0$. The equality to zero of the conformal dimension is equivalent to the condition of zero R-charge, which due to the condition on the $U(1)_A$-charge is just equality to zero of the UV R-charge $N_c = N_f + 1$. This gives the conformal dimension of operators $v_\pm$: $\Delta(v_\pm) = -\frac{N_f}{2}\Delta(M)$. Thus either the mesons or the minimal monopole operators $v_\pm$ have negative conformal dimension which violates unitarity. We conclude that there is no realtion between these operators.

4.8 Appendix C. Consistency of the Chiral Ring

The purpose of this appendix is to show that for every bare monopole state in the deformed theory with GNO charges different from $(n, -m, 0, ..., 0)$ there exists a possibility to become a part of a long supermultiplet and, correspondingly, become a Q-exact operator in the original theory.

It was motivated in the main text that the only nonaccidental BPS bare monopole states in the deformed theory are those with GNO charges $(n, -m, 0, ..., 0)$ with nonnegative integral n and m. A bare monopole with any other GNO charges must correspond to a Q-exact operator in the original theory. There are two scenarios how this can happen. The simplest one is that in the deformed theory for each bare monopole with GNO charges different from $(n, -m, 0, ..., 0)$ there is a fermionic spinor state with quantum numbers appropriate for a Q-ascendant of the bare monopole. This cannot happen for bare monopoles $|n, -m, 0, ..., 0\rangle$

because this is the lowest energy state in the sector with topological $U(1)_T$ charge $t = n - m$ and $U(1)_A$ charge $A = -N_f(|n| + |m|)$. In the second scenario there is not an appropriate fermionic superpartner for each bare monopole. But two conditions must be satisfied in order for the bare monopole to become Q-exact in the original theory. First, it cannot give a distinguished contribution to the index which unambiguously could be deciphered as that of a scalar BPS state. Second, there must be a mechanism explaining pairings of bare monopole states when the interactions are switched on. Below it is shown that the first scenario is not realized, but both conditions for the realization of the second scenario are met at least for several low values of N_c and N_f.

Consider some bare monopole state with GNO charges $(n_1, n_2, ..., n_{N_c})$. The potential superpatner must be a state of the form $\Psi|n, -m, 0, ..., 0\,\rangle$ where Ψ is some monomial in the matter and gauge modes with spin one half. Moreover, Ψ is the $SU(N_f) \times \widetilde{SU(N_f)}$ flavor singlet because all bare monopoles are flavor singlets. To get an idea how to build such a state in general, let us consider an example from the theory $U(3)_4$ in addition to those already discussed in the main text.

Bare monopole $|2, 1, 0\,\rangle$. Its topological charge is $t = 3$, $U(1)_A$-charge is $A = -12$ and the UV R-charge is $h = 2$. The potential superpartner must be of the form $\Psi|3, 0, 0\,\rangle$, or $\Psi'|4, -1, 0\,\rangle$ or $\Psi''|5, -2, 0\,\rangle$, etc. The monomial Ψ in the matter modes must be a singlet with respect to the flavor symmetry group $SU(4) \times \widetilde{SU(4)}$ because all bare monopoles are, so it is natural to look for an elementary monomial which is a singlet. These are $w \equiv \psi^i Q_i$ and $\tilde{w} \equiv \psi_{\tilde{i}} Q^{\tilde{i}}$. They have energy $3/2$ and spin $1/2$ as long as we take the lowest spin components of the matter scalars and the fermions. Moreover, these modes must not interact with nonzero magnetic charges. An obvious candidate for the fermionic state is $w_{+1/2}|3, 0, 0\,\rangle$ where the bare monopole $|3, 0, 0\,\rangle$ is chosen to have the same $U(1)_A$-charge and the UV R-charge as the state $|1, 1, 1\,\rangle$.

An important restriction on building a fermionic Q-ascendant of a bare monopole is that it must be a BPS state with $J_3 = +1/2$. Indeed, the energy of this state is lower by one half, the R-charge is lower by one and J_3 is higher by one-half. Thus $E - r - J_3 = E_0 - r_0 = 0$ where E_0 and r_0 are the energy and the R-charge of the bare monopole. As follows from the table with quantum numbers of the modes of all fields, all modes used to build a BPS spinor with $J_3 = +1/2$ must be scalars with the exception only one which must have $J_3 = j = 1/2$. Moreover, Ψ must be a flavor singlet. This means that we can use either $w_{+1/2}$ or $\tilde{w}_{+1/2}$ only once while the other modes must factorize into "flavor baryons" and gauge-invariant scalar gluinos. Using this it is easy to show that many bare monopoles do not have appropriate fermionic states. Examples for the theory $U(3)_4$ are bare monopoles $|2, 2, -1\,\rangle$ and $|3, 2, -2\,\rangle$.

The second scenario implies the two requirements whose satisfaction we show now.

(a) No distinguished contribution to the index.

It was mentioned above that this requirement is not met for bare monopoles $|n, -m, 0, ..., 0\,\rangle$ which sets them aside and guarantees their existence as BPS scalars in the original theory (put on $S^2 \times \mathbb{R}$).

For all other monopoles their contribution to the index, in principle, can be canceled by certain fermionic modes. For a bare monopole with topological $U(1)_T$ charge t and $U(1)_A$ charge A one available fermionic state is $w_{N/2}|n, -m, 0, ..., 0\,\rangle$ where $n = \frac{t-A/N_f}{2}$, $m = -\frac{t+A/N_f}{2}$ and the mode $w_{N/2}$ is $\psi^i Q_i(s)$ where the Q-mode has spin s determined from the requirement that the energy difference between the original bare monopole and $|n, -m, 0, ..., 0\,\rangle$ is equal $2s + 2$. Other fermionic states are obtained from different bare monopoles containing zero GNO charges. We have been unable to show that for each bare monopole not of the type $|n, -m, 0, ..., 0\,\rangle$ the contribution to the index is canceled by a fermionic state in general. With the increase in N_c the numbers of unwanted bare monopoles grow, but the number of compensating fermions grows as well, so it is not implausible that all contributions can be cancelled. We verified this for a number of low-energy monopoles for several low values of N_f and N_c.

One should note that all these modes w and $\tilde{w}$ have even contributions to the value of $E + j_3$. So, for them to be useful, the energy difference between the bare monopoles must be even. This is always the case because, having equal $U(1)_A$ charges, their energy difference is determined by the difference in contributions coming from the vector multiplet $\delta E = \sum_{i<i}(|n_i - n_j| - |m_i - m_j|)$ which is always even for $\sum_i n_i = \sum_i m_i = t$. Moreover, for a given values of $U(1)_A$- and topological $U(1)_T$-charges the bare monopole $|n, -m, 0, ..., 0\,\rangle$ has the lowest energy, which makes such states distinct. Both statements are easy to prove

by going from the initial bare monopole to the $|n, -m, 0, ..., 0\rangle$ using a number of steps at each of which one of the GNO charges n_i is increased by one while another n_j is decreased by one without changing the $U(1)_A$. There is a sequence of such steps when the value of the expression $\sum_{i<j} |n_i - n_j|$ is increased by two at each step until the GNO charge $(n, -m, 0, ..., 0)$ is reached.[10]

(b) The pairing.

The second condition necessary for Q-exactness of a scalar monopole operator is existence of a long multiplet near some value t_0 of the deformation parameter t whose energy changes along the deformation so that at the point t_0 it breaks into short multiplets providing the bare BPS monopole with a Q-superpartner. There can be such multiplets in principle. An example of this is a long multiplet whose lower component is a spinor with energy satisfying the unitary inequality $E_s > r_s + j_s + 1 = r + 3/2$. On the first level there is a scalar with energy $E_b > r_b + 1$ and a vector. On the second level there is a spinor. At some point t_0 along the defrmation it may happen that $E_s = r_s + 3/2$. In this situation the scalar from the first level with energy $E_b = r_b + 1$ and spinor from the second level with energy $E = r + 1/2$ become part of a separate short multiplet. This short multiplet has a zero-norm scalar state on the second level with energy $E = r$. Thus, the initial BPS scalar can take this place as the parameter of the deformation is varied further.

The same mechanism can also govern the fate of the monopole operators defined in the asymptotically free UV theory along the RG flow. First, the free UV theory is put on $S^2 \times \mathbb{R}$. Perturbation by the relevant operator of the theory on $\mathbb{R}^3$ that switches on the gauge interaction corresponds to turning on a time-dependent perturbation in the radially quantized picture in the far past. The nonunitary evolution leads to the radially quantized IR fixed point of the theory on $\mathbb{R}^3$ in the far future. Although this perturbation breaks time-translation invariance, the supersymmetry is preserved and states on the sphere S^2 are combined into supermultiplets. Initially, in the far past, the monopole operators live in short BPS multiplet, but when the interaction is switched on they can pair up with appropraite

[10] The energy of a bare monopole $|n_1, n_2, ..., n_{N_c}\rangle$ is given by the expression $E = -\sum_{i<j} |n_i - n_j| + N_f(1 - \Delta)\sum_{i=1}^{N_c} |n_i|$. The $U(1)_A$ charge is $A = -N_f \sum_{i=1}^{N_c} |n_i|$.

fermions into long multiplets. This can explain why most of the monopole operators may be absent in the chiral ring of the IR superconformal fixed point. Checking that the pairings actually occur is out of reach, but these pairings are possible in principle. The analysis of potential superpartners performed above for the deformed theory did not depend on any assumptions about values of anomalous dimensions. Thus, it is applicable to the case of canonical dimensions of all fields, and because the two analysises are identical, the picture is consistent.

Chapter 5

Enhancement of Global Symmetries

5.1 Introduction

In this chapter we continue the investigation of hidden symmetries in supersymmetric gauge theories in three dimensions using the method developed in [19],[1],[16] and described in the previous chapters. The emphasis is now placed upon global symmetries in the infrared limit of $\mathcal{N}=4$ theories whose currents do not lie in the same $\mathcal{N}=4$ supermultiplet as the stress-tensor. The lowest components of the global symmetries multiplets are scalars with conformal dimension 1 and in the adjoint representation of the corresponding group of global symmetries.

The $\mathcal{N}=4$ theories have microscopic $SO(4)_R \simeq SU(2)_R \times SU(2)_N$ R-symmetry. Because the R-symmetry group of a superconformal $\mathcal{N}=4$ theory to which the microscopic theory flows in the IR is $SO(4)$, one is tempting to assume the equality between the microscopic and the superconformal R-groups. This is known to be a wrong assumption in general, as there may appear accidental symmetries in the infrared whose currents are not conserved along the full RG flow from the ultraviolet. Luckily, in a large class of models the IR superconformal R-symmetry group is the microscopic one. Although, given a UV theory, it is not known how to prove this statement, there is a necessary condition for $R_{UV}=R_{IR}$ to hold which is easy to check: if, with respect to any subgroup $U(1) \subset SO(4)^{UV}_R$ there is a chiral operator with nonpositive R-charge, the IR R-symmetry is not the microscopic one. This condition is a simple consequence of unitarity in the IR.[1] Moreover, in all known

[1] See [19] for discussion of this point and some examples.

cases where the infrared R-symmetry is not the UV R-symmetry, this manifests itself by appearence of chiral operators with R-charges violating unitarity. Thus it seems reasonable to assume the condition to be suffucient as well.

5.2 Models

We consider three-dimensional $\mathcal{N}=4$ supersymmetric gauge theories with a compact group G. The fields form two $\mathcal{N}=4$ multiplets: vector multiplet $\mathcal{V}$ consisting of a vector $\mathcal{N}=2$ V and a chiral $\mathcal{N}=2$ Φ multiplets in the adjoint representation of the gauge group G, and a hypermultiplet $\mathcal{H}$ consisting of two chiral $\mathcal{N}=2$ multiplets Q and $\tilde{Q}$ in fundamental and antifundamental representations of G, respectively. Each chiral multiplet contains a complex scalar and a complex spinor (two Majorana spinors). The vector multiplet has a gauge field, a real scalar (dimensional reduction from a gauge field in 4d) and a complex spinor.

5.3 A Brief Review of Monopole Operators

Here we recap the basic facts about monopole operators in three dimensional gauge theories [7].

By definition, a hidden symmetry is generated by a conserved current whose existence does not follow from any symmetry of an action. The simplest example of such a symmetry corresponds to a topological conserved current which exists in any three-dimensional gauge theory whose gauge group contains a $U(1)$ factor:

$$J^{\mu} = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\mathrm{Tr}\, F_{\nu\lambda}. \tag{5.1}$$

There may be more complicated hidden symmetries whose conserved currents are monopole operators, i.e., disorder operators defined by the condition that the gauge field has a Dirac monopole singularity at the insertion point. More concretely, in a $U(N)$ gauge theory the singularity corresponding to a monopole operator must have the form

$$A^{N,S}(\vec{r}) = \frac{H}{2}(\pm 1 - \cos\,\theta)d\phi \tag{5.2}$$

for the north and south charts, correspondingly. In this formula $H = diag(n_1, n_2, \ldots, n_N)$ and integers $n_1 \geq n_2 \geq \ldots \geq n_N$ are called magnetic charges (or GNO charges [12]). If we require the monopole operator to preserve some supersymmetry (such operators may be called BPS operators), matter fields must also be singular, in such a way that BPS equations are satisfied in the neighborhood of the insertion point.

In special curcimstances it is possible to determine the spectrum of chiral monopole operators with low values of conformal dimension. Namely, if we have a superconformal theory we can implement the radial quantization to obtain a supersymmetric theory on $\mathbb{R}\times S^2$ whose states are in one-to-one correspondence with local operators of the original theory on $\mathbb{R}^3$. The quantum numbers match on both sides of the correspondence with energies of the states being equal to conformal dimensions of the corresponding operators. For $\mathcal{N} = 4$ gauge theories with vanishing anomalous dimensions of operators, it is possible to continuously deform the theory on the sphere in a controlled supersymmetric way to a free theory of fields in a fixed spherically symmetric background determined by the Dirac monopole singularity at the insertion point [16]. It is then possible to find the spectrum of chiral monopole operators with lowest values of conformal dimensions [1]. In the absence of Chern-Simons couplings the lowest energy states in the radial quantization are bare monopoles, that is, "vacuum" states in sectors determined by magnetic charges which are not excited with fields modes.

5.4 Review of the Method

In their paper [10] Intriligator and Seiberg suggested a dual description of the infrared limit of a class of three-dimensional $\mathcal{N} = 4$ quiver gauge theories. A particular prediction of this correspondence was presence of hidden global symmetries in the quiver theories. It was realized long ago that conserved currents that span the cartan subalgebra are simply the topological currents of each of the U(1) factors of the compact gauge group (see [7]). The realization of the rest of the global currents was claimed to be by means of monopole operators [7]. This claim was verified in paper [19] whose authors showed that all the neccesary currents are monopole operators by finding all chiral scalars of conformal dimension 1 and making sure that their topological charges are exactly the ones appropriate for the roots of the global symmetry groups. This means that the conserved currents produced from them taken together with the topological currents form the required Lie algebra.

Moreover, authors of [19] showed that any quiver whose nodes are (1) unitary groups, (2) balanced nodes is necessary one of the ADE quivers. A node corresponding to gauge group $U(n_c)$ and n_f fundamental hypermultiplets[2] is called balanced if $n_f = 2n_c$. They also conjectured that a quiver, each node of which is good ($n_f \geq 2n_c$), is good in whole, that is, all chiral monopole operators have $E \geq 1$ and the corresponding theory flows to the standard IR limit (the R-symmetry is the microscopic one) with monopole symmetries being products of ADE groups and $U(1)$.

In this chapter we provide examples of such quiver theories and find their monopole symmetries.

In order to establish notations and illustrate the procedure we review the cases of D_4 and D_5 quivers that will also serve as starting points for deformed quivers with new global symmeties considered later in the paper.

5.4.1 D_4 and D_5 Quivers

Let us start with the smallest quiver of D-type corresponding to the extended Dynkin diagram D_4. This diagram represents the Lie algebra $so(8)$. It translates to a quantum field theory as follows.

The central node with index 2 denotes the gauge group $U(2)$, the other four nodes are U(1) gauge groups and the edges are bifundamental hypermultiplets. Because there are no fundamental hypers there is a decoupled U(1) gauge subgroup which is manifested in the invariance of the energy of bare chiral monopoles [19],[1],[16]

[2]Here $U(n_c)$ is treated in isolation from the other nodes in the sense that all bifundamentals and fundamentals themselves are included in n_f. For example, a bifundamental hyper of $U(n_c) \times U(N)$ is considered as $n_f = N$ fundamental hypers.

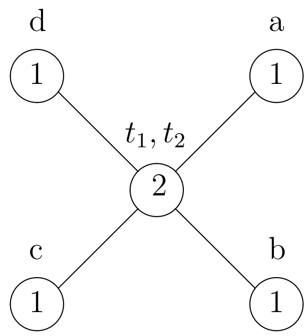

Figure 5.1: D_4 quiver. Letters stand for magnetic fluxes.

$$E = -|t_1 - t_2| + \frac{1}{2}(|t_1 - b| + |t_2 - b| + |t_1 - c| + |t_2 - c| + \\ |t_1 - d| + |t_2 - d| + |t_1 - a| + |t_2 - a|) \tag{5.3}$$

under shifts by equal fluxes $\{t_1, t_2; b, c, d, a\} \to \{t_1 + m, t_2 + m; b + m, c + m, d + m, a + m\}$. To deal with this redundacy we fix "the gauge" by setting $a = 0$. It is important that no nonzero flux distributions give nonpositive energy (after gauge fixing only the zero fluxes give zero energy). This means that the microscopic R-symmetry can be the R-symmetry that enters the superconformal algebra in the infrared. Note that energy positivity for bare monopoles is nontrivial in this case because the vector multiplet gives negative contribution.

A calculation gives 24 bare monopole scalars with energy 1 corresponding to different magnetic (and topological!) charges (see appendix A). In the basis (h_1, h_2, h_3, h_4) where $\{t_1 + t_2 = h_2 - h_4, b = h_3 + h_4, c = h_3 - h_4, d = h_1 - h_2\}$ it is obvious that the scalars together with 4 nontopological chiral scalars $tr\phi$ (they are superpartners of four topological currents and are lowest components from the chiral multiplets $tr\Phi$) are in adjoint representation of $so(8)$. This leads to 28 conserved currents forming the Lie algebra $so(8)$.

A similar analysis can be performed for the quiver diagram D_5. It is shown in appendix A that this leads to a global symmetry group $SO(10)$ with its currents being monopole operators.

Note that we did not prove the absence of nonzero fluxes leading to nonpositive energies.

The necessary condition $n_f \geq 2n_c - 1$ for each node is obviously satisfied. Moreover the stronger condition $n_f = 2n_c$ holds. The condition $n_f \geq 2n_c - 1$ is necessary because if it is not satisfied one gets a bare monopole with nonpositive energy which is magnetically charged under the corresponding gauge subgroup and magnetically neutral under all the rest factors in the full gauge group. Authors of [19] showed that D_n models have no bare monopoles with nonpositive energy.

5.4.2 $E_{6,7,8}$-type Quivers

The gauge group and the field content can be read off from figures B1, B2, and B3 in appendix B.

When we run the procedure from the previous subsection for E_6 we find lowest energy states being 72 bare monopole scalars of energy $E = 1$ with magnetic charges just right to form an adjoint representation after completing them with 6 nonmagnetical chiral scalars: traces $tr\phi$ of the chiral scalars which are the lowest components of the $\mathcal{N} = 2$ chiral multiplets Φ. Acting twice with supercharges on the 72 scalars we get conserved currents which correspond to roots of the global group E_6. The six independent topological charges correspond to the cartan operators.

For the E_7 quiver theory we find 126 bare monopoles with energy $E = 1$ and appropriate topological charges[3] leading to the existence of E_7 group of symmetries realized by monopole operators.

The situation for the E_8 quiver theory is similar: 240 bare monopoles with energy $E = 1$ give rise to the E_8 symmetry of the theory.

5.5 Engineering Nonlinear Quivers

It turns out to be difficult if possible at all to construct quiver theories with hidden non-ADE-type groups of global symmetries realized by monopole operators. More precisely, it is possible to build $Sp(N)$ with the symmetry currents lying in a free sector of the IR theory.

[3]For all theories considered in this section to each set of topological charges there corresponds a unique set of magnetic charges.

Examples of such theories of linear-quiver type were given in [19]. In the next section we provide some examples of nonlinear-quiver theories with free symplectic symmetry group. However, it is very easy to build a large class of theories whose symmetry groups contain nonfree factors of A-D-E type.

Indeed, given an A-D-E-type theory consider connecting arbitrary theory to some nodes of the original quiver. This means that we take two theories A and B that contain gauge subgroups G_A and G_B, correspondingly, as factors and add a hypermultiplet in a (nontrivial) representation of $G_A \times G_B$. If the two original theories had monopole symmetries S_A and S_B, then the engineered theory will have at least a subgroup of $S_A \times S_B$ corresponding to zero fluxes for $G_A \times G_B$. Note that the engineered theory $A \times B$ is always good if A and B are good theories separately. This is because the expression for the energy of bare monopoles in the engineered theory is a sum of those in the original theories and a positive contribution from the new hypermultiplet. Similarly, ugly theories produce an ugly or a good theory. Ugly means that the minimal nonzero energy of chiral operators is 1/2, that is, they are free.

Let us consider several examples illustrating this construction taking the D_n quiver theory as an original theory.

Example 1.

Consider the D_4 quiver as an G_A theory and an $U(1)$ theory with two fundamental hypermultiplets as an G_B theory and modify them to the $D_4 \times U(1)$ gauge theory as in figure 5.2. This new theory has $SU(4) \times SU(2) \times U(1)$ as its symmetry group in the monopole sector. The first factor is inhereted from the original D_4 theory (it was possible for the new $U(1)$ factor to have such magnetic flux so that not to excite[4] any new edge compared with the fluxes distribution producing the $SU(4)$ symmetry subgroup in D_4 theory) while the $SU(2)$ corresponds to putting one unit (and minus the unit) of flux for the $U(1)$ gauge factor while leaving all the rest fluxes zero. The last $U(1)$ factor is just one of the five topological charges under which none of the bare monopoles carries a charge. The whole symmetry group is nonfree.[5]

[4] An excited edge corresponds to a hypermultiplet that gives a nontrivial contribution to the energy of a bare monopole.

[5] Quivers of this type appeared in [49].

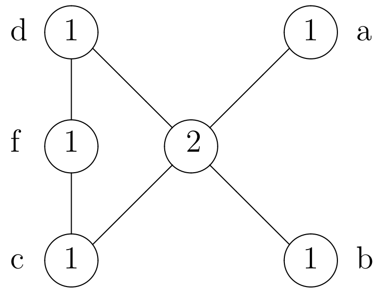

Figure 5.2: *Example 1* quiver.

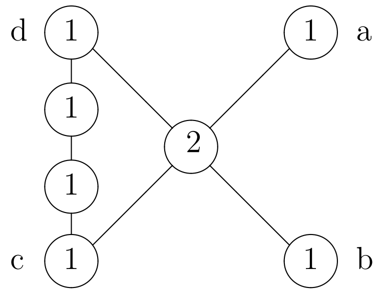

Figure 5.3: *Example 2* quiver.

Example 2.

We start with a D_4 quiver again and add two more $U(1)$ factor and bifundamental hypermultiplets as in figure 5.3. This gives an $SU(4) \times SU(3) \times U(1)$ nonfree monopole symmetry.

Example 3.

This time we take the D_5 quiver and add one $U(1)$ factor and bifundamental hypers as figure 5.4. The resulting monopole symmetry group is nonfree $SO(8) \times SU(2) \times U(1)$.

Example 4.

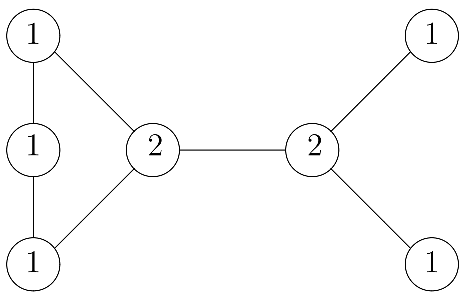

Figure 5.4: *Example 3* quiver.

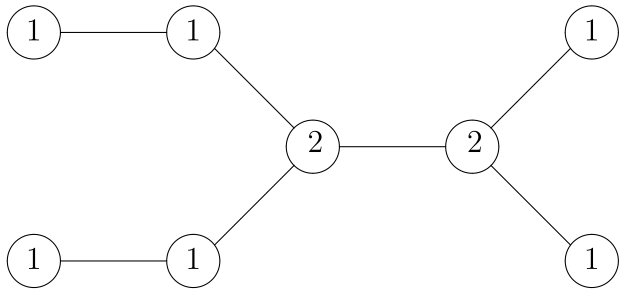

Figure 5.5: *Example 4* quiver.

Take the D_5 quiver and add two $U(1)$ factors as in Fig.5.5. This time the whole $SO(10)$ group is preserved and there appears an additional factor $SU(2)_{free} \times SU(2)_{free}$ in the monopole symmetry group $SO(10) \times SU(2)_{free} \times SU(2)_{free}$. This happens because any original distribution of fluxes can be embedded in the new quiver without changing the net energy by simply putting fluxes on the new factors so that no new edge is excited, that is, no new hypermultiplet contributes to the energy. Moreover, now we can put fluxes on the two new $U(1)$ gauge group factors and set the rest fluxes to zero. This produces the $SU(2)_{free} \times SU(2)_{free}$ free factor. The subscript "free" is used to stress that the currents of the corresponding group (or, equivalently, $E = 1$ scalars) are built from free fields. A natural guess then is that the infrared limit of this theory is that of $X \times X \times D_5$ with X being the theory of a free twisted hypermultiplet. The bare monopoles with $E = 1/2$ correspond to the lowest component scalars in the free twisted hypermultiplets. This also is in accord with the argument in favor of a similar factorization on page 24 of [19].

5.6 Quiver Theories with Nonunitary Gauge Groups

So far we have considered only theories whose gauge groups are products of unitary groups. Let us analize nonsimply laced gauge groups $SO(5)$ and G_2.

The formula for energies of bare monopoles is trivially generalized to arbitrary gauge groups G.

$$E(H) = -\frac{1}{2}\sum_r |r(H)| + \frac{1}{2}\sum_w |w(H)| = -\sum_{r^+} |r^+(H)| + \frac{1}{2}\sum_w |w(H)| \tag{5.4}$$

In this formula H is the cartan generator containing magnetic charges, that is $e^{iH\alpha}$ is the image of $e^{i\alpha}$ under an embedding $U(1) \hookrightarrow G$ that defines magnetic (GNO) charges; r stands for roots, r^+ for positive roots and w for weights of representations of all hypermultiplets. Magnetic weights, or cartan generators H, can take any values satifying the Dirac quantization condition

$$w(H) \in Z \quad \text{for all weights } w \text{ of all representations present.} \tag{5.5}$$

5.6.1 G_2 Case

G_2 is a rank two simple Lie algebra of dimension 14. The root space is a two dimensional vector space $\mathbb{R}^2$, in which positive simple roots can be taken as

$$\alpha = (1, 0), \quad \beta = (-\frac{3}{2}, \frac{\sqrt{3}}{2}). \tag{5.6}$$

Other positive roots are

$$r_3 = \alpha + \beta, \quad r_4 = 2\alpha + \beta, \quad r_5 = 3\alpha + \beta, \quad r_6 = 3\alpha + 2\beta. \tag{5.7}$$

The cartan algebra is a two-dimensional vector space dual to $\mathbb{R}^2$ which can be identified with it by means of the standard metric. Let us chose a basis $\{H_1, H_2\}$ in it dual to $\{\alpha, \beta\}$ and write an arbitrary cartan as $H = n_1 H_1 + n_2 H_2$ where (n_1, n_2) are a priori real numbers.

$(m_1m_2m_3m_4, n_1n_2n_3n_4, p_1p_2)$		
(-1000,-1000,00)	(-2000,0000,00)	(1 -100,0000,00)
(-1000,1000,00)	(2000,0000,00)	
(1000,-1000,00)	(0000,-2000,00)	(0000,1 -100,00)
(1000,1000,00)	(0000,2000,00)	

Table 5.1: *Example 5.* Bare monopole states with energy $E = 1$.

$(m_1m_2m_3m_4, n_1n_2n_3n_4, p_1p_2)$
(-1000,0000,00)
(1000,0000,00)
(0000,-1000,00)
(0000,1000,00)

Table 5.2: *Example 5.* Bare monopole states with energy $E = 1/2$.

Then the contribution of the vector multiplet to the energy evaluated on such a cartan is

$$\begin{aligned} E_v(H) &= -(|\alpha(H)| + |\beta(H)| + |r_3(H)| + |r_4(H)| + |r_5(H)| + |r_6(H)|) = \\ &(|n_1| + |n_2| + |2n_1 + n_2| + |n_1 + n_2| + |3n_1 + n_2| + |3n_1 + 2n_2|). \end{aligned} \tag{5.8}$$

Comparing this expression with (5.4) and (5.5) we conclude that n_1 and n_2 must be integral. We consider three examples with a fundamental hypermultiplets. G_2 has two fundamental representations with highest weights $2\alpha + \beta$ and $3\alpha + 2\beta$. We focus on the former. It has dimension 7 and weights

$$\begin{aligned} &w_1 = \alpha, \quad w_2 = 2\alpha + \beta, \quad w_3 = \alpha + \beta, \\ &w_4 = -w_1, \quad w_5 = -w_2, \quad w_6 = -w_6, \quad w_7 = 0. \end{aligned} \tag{5.9}$$

Example 5

This is a theory with gauge group $G_2 \times U(4)^2$ corresponding to the quiver with a bifundamental hyper of $G_2 \times U(4)$ for both $U(4)$ factors.[6] The symmetry group is $Sp(2)_{free} \times U(1)^2$. The corresponding monopole scalars with energy $E = 1$ are in the table 5.1.

In addition there are four monopole operators with energy $E = 1/2$ (table 5.2).

[6]Note that $n_f = 2n_c - 1$ for both $U(n_c)$ factors.

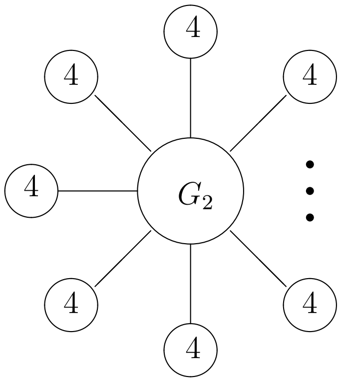

Figure 5.6: *Example 5* quiver. All lines stand for representations $\mathbf{7} \times \mathbf{4}$ of $G_2 \times U(4)$.

On $\mathbb{R}^3$ they correspond to free chiral operators. The eight states in the first two columns of table 5.1 and four states in table 5.2 are naturally reproduced in the theory X^2 which is the product of two free twisted hypermultiplets. In fact, by the argument of [19] on page 24 the whole theory is equivalent to a product $X \times X \times \mathcal{H}$ where $\mathcal{H}$ is a good theory with gauge group $G_2 \times U(3)^2$ and bifundamental hypermultiplets for the two pairs $G_2 \times U(3)$. The two topologically neutral states in the third column of table 5.1 correspond to two scalars $tr\Phi$ of the two gauge $U(3)$. They are the lowest components of the multiplets containing topological currents for the two $U(3)$ gauge factors. If we add a fundamental hypermultiplet for each $U(4)$ we obtain the nonfree symmetry group $SU(2)_{nonfree} \times SU(2)_{nonfree}$. One can generalize this example by considering the gauge group $G_2 \times U(4)^N$ with a bifundamental hypermutiplet for each pair $G_2 \times U(4)$ for arbitrary natural number N. This gives the monopole symmetry group $Sp(N)_{free} \times U(1)^N$.

Example 6

As a gauge group we take $G_2 \times U(4)$ with a bifundamental hypermultiplet and two fundamentals of G_2 needed to exclude chiral monopole operators with nonpositive energies. We use gauge groups with $U(1)$ factors to get nonabelian monopole symmetries. Each $U(1)$ factor provides a conserved topological current whose charge serves as a Cartan generator. Because magnetic charges are not produced from any conserved currents simple gauge groups can only produce abelian monopole symmetries. The global symmetry group[7] is $SU(2)_{free} \times U(1)$. The $SU(2)_{free}$ factor is free in the sense that the currents are build from a doublet of free fields which are bare monopole operators with energy $E = 1/2$. By the conformal algebra such chiral operators are free fields. These fields carry magnetic charges only with respect to the $U(4)$ factor. Alternatively, this theory can be described as the infrared limit of $X \times \mathcal{H}$ where X is the theory of a free twisted hypermultiplet and $\mathcal{H}$ is the original theory with $U(4)$ gauge group replaced by $U(3)$. Adding a fundamental hyper of $U(4)$ eliminates the free doublet and reduces the symmetry group to $SU(2)$ which is now nonfree. This quiver describes a "good theory" in the terminology of [19].

We could also take one fundamental of G_2. This theory has the same symmetry group as the above.

5.6.2 $SO(5)$ Case

The rank of the group is two and the dimension is 10. The roots are

$$\begin{aligned} &\alpha_1 = (1,0), \quad \alpha_2 = (0,1), \quad \alpha_3 = (1,1), \quad \alpha_4 = (1,-1), \\ &\alpha_5 = -\alpha_1, \quad \alpha_6 = -\alpha_2, \quad \alpha_7 = -\alpha_3, \quad \alpha_4 = -\alpha_4, \end{aligned} \tag{5.10}$$

where α_2 and α_4 are positive simple roots. The basis of cartans $\{H_1, H_2\}$ are chosen to be dual to $\{\alpha_1, \alpha_2\}$. On a cartan $H = n_1 H_1 + n_2 H_2$ the contribution of the vector multiplet to the energy of bare monopoles is

$$E_v = -(|n_1| + |n_2| + |n_1 - n_2| + |n_1 + n_2|), \tag{5.11}$$

so the magnetic charges n_1 and n_2 are at least integer.

[7] In all cases we mention a (global) symmetry group we refer to the part of the symmetry group generated by monopole operators.

The weights of a fundamental representation **5** are

$$w_1 = \alpha_1, \quad w_2 = \alpha_2, \quad w_3 = -\alpha_1, \quad w_4 = -\alpha_2, \quad w_2 = 0. \tag{5.12}$$

Example 7

The gauge group is $SO(5) \times U(3)$ with one hypermultiplet in the bifundamental representation and two hypermultiplets in representation $\mathbf{5} \times \mathbf{1}$. The symmetry group is $SU(2)_{free} \times U(1)^3$. The two bare monopoles in the adjoint of $SU(2)$ have magnetic charges

$$(0,0;-2,0,0), \quad (0,0;2,0,0). \tag{5.13}$$

The three bare monopoles corresponding to the three $U(1)^3$ currents are

$$(-1,0;1,-1,0), \quad (0,0;1,-1,0), \quad (1,0;1,-1,0). \tag{5.14}$$

We see that although the three $U(1)^3$ currents do not carry topological charge, they are magnetically charged with respect to both gauge subgroups. This $U(1)^3$ symmetry is nonfree.

It is possible to give a discription of this theory in which the free part and the interacting part of the IR theory are factorized already in the UV Lagrangian [19]. This theory is a product $X \times \mathcal{H}$ of the free twisted hypermultiplet X and theory $\mathcal{H}$ which is obtained from the original one by replacing $U(3)$ gauge factors with $U(2)$ factors. In this description one of the $U(1)$ currents is the topological current of $U(2)$ while the other two are magnetically charged with respect to $SO(5)$ currents.

Example 8

The gauge group is $SO(5) \times U(3)^N$ with a bifundamental of $SO(5) \times U(3)$ for each $U(3)$. The symmetry group is $Sp(N)_{free} \times U(1)^N_{nonfree}$.[8] There are bare monopoles with energy $E = 1/2$ with nozero magnetic charges for only one of the $U(3)$ factors and bare monopoles with energy $E = 1$ with nonzero magnetic charges for any two of the $U(3)$ factors. As in previous examples we can give a description of this theory where the factorization of the free

[8]For $N = 2$ there is an additional $U(1)^2$ symmetry with currents magnetically charged under $SO(5)$ but topologically neutral.

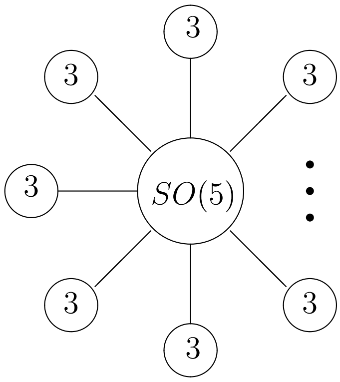

Figure 5.7: *Example 8* quiver.

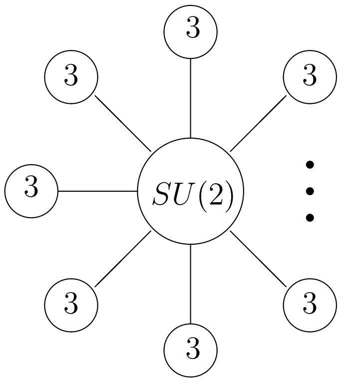

Figure 5.8: *Example 8* quiver. All lines stand for representations $\mathbf{5} \times \mathbf{3}$ of $SU(2) \times U(3)$.

sector is manifest already in the UV. This theory is $X^N \times \mathcal{H}$ where $\mathcal{H}$ is obtained from the original theory by replacing all $U(3)$ gauge factors by $U(2)$ factors.

In this example we meet a certain universality. Because for all $E = 1/2$ and $E = 1$ bare monopoles $SO(5)$ magnetic charges are zero we can reproduce all the scalars by putting any other group in the center of the quiver instead of $SO(5)$ as long as it has a five-dimensional representation and the resulting theory does not have any bare monopoles with nonpositive energy. For example, we can take $SU(2) \times U(3)^N$ with hypers in representations $\mathbf{5} \times \mathbf{3}$ for each $U(3)$.

If we take all hypermultiplets in representation $\mathbf{6} \times \mathbf{3}$ of $SU(2) \times U(3)$ the monopole symmetry becomes nonfree $SU(2) \times SU(2)$.

Example 9

The gauge group is $SO(5) \times U(2)$ but the theory is not of a quiver type because we take a

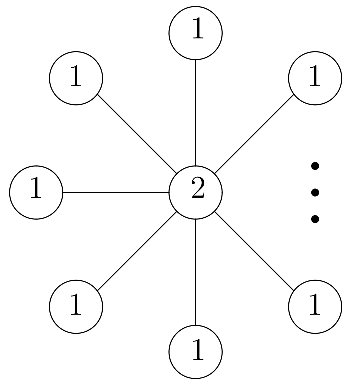

Figure 5.9: *Example 11* quiver.

hypermultiplet in the representation **10**$\otimes$**4** and two hypers in the fundamental representation of $U(2)$. The symmetry group is $SU(2)_{nonfree}$. Again, the magnetic charges of $SO(5)$ are zero.[9]

5.6.3 Unitary Quivers

Example 10

Guided by the principle to take $n_f \geq 2n_c - 1$ for unitary quivers we can build the quiver theory depicted in figure B4 in appendix B. It turns out to have a nonfree $SO(14) \times U(1)$ monopole symmetry. The $U(1)$ factor is just one of the eight topological charges (one topological charge corresponds to the decoupled $U(1)_{diag}$ which gives eight topological charges) under which no $E = 1$ bare monopoles are charged.

Example 11

Consider a quiver with gauge group $U(2) \times U(1)^N$ with bifundamental hypermultiplets for each subgroup $U(2) \times U(1)$ as in figure 5.9. This theory has a nonfree symmetry $SU(2)^N$ except in the case $N = 4$ which corresponds to quiver D_4 and enhanced symmetry $SO(8)$. The enhancement happens because for this particular value of N the central node can have nonzero magnetic flux without spoiling condition $E = 1$.

[9]**10** is the adjoint representation of $SO(5)$ and **4** is the adjoint of $U(2)$.

tbcd	tbcd	tbcd	tbcd	tbcd	tbcd
0001 0100	0010 1100	1000 1101	1001 1110	1010 1111	1011 2111

Table 5.3: Positive topological charges of D_4 theory.

$h_1h_2h_3h_4$	$h_1h_2h_3h_4$	$h_1h_2h_3h_4$	$h_1h_2h_3h_4$	$h_1h_2h_3h_4$	$h_1h_2h_3h_4$
1-100 100-1	001-1 0101	01-10 1001	0011 0110	10-10 1010	010-1 1100

Table 5.4: Positive topological charges of D_4 in new basis.

5.7 Appendix A

5.7.1 D-type Quivers

For D_4-quiver theory with bare monopole energy

$$E = -|t_1 - t_2| + \frac{1}{2}(|t_1 - b| + |t_2 - b| + |t_1 - c| + \\ |t_2 - c| + |t_1 - d| + |t_2 - d| + |t_1 - a| + |t_2 - a|) \tag{5.15}$$

fixing the shift symmetry by setting $s = 0$ we get 24 $E = 1$ scalars. They are devided in two equal parts. One is obtained from the other by flipping signs of magnetic charges. This is obvious because the expression for the energy is invariant under flipping the signs as well as our "gauge fixing" condition. The topological charges for one of the parts are given in table 5.3.

For a new basis (h_1, h_2, h_3, h_4) where

$$c = h_3 - h_4, \quad b = h_3 + h_4, \quad t = h_2 - h_4, \quad d = h_1 - h_2 \tag{5.16}$$

they are in table 5.4.

This is a set of positive roots of $SO(8)$ [47].

For D_5-type quiver theory we have the following tables.

t_1t_2bcd	t_1t_2bcd	t_1t_2bcd	t_1t_2bcd	t_1t_2bcd
00001	01010	11001	00100	11110
00010	01011	11010	10100	11111
01000	10000	11011	11100	12111
01001	11000	12011	11101	22111

Table 5.5: Positive topological charges of D_5.

$h_1h_2h_3h_4h_5$	$h_1h_2h_3h_4h_5$	$h_1h_2h_3h_4h_5$	$h_1h_2h_3h_4h_5$	$h_1h_2h_3h_4h_5$
0001 -1	00101	0100 -1	1 -1000	10001
00011	00110	01001	10-100	10010
001 -10	01 -100	01010	100 -10	10100
0010 -1	010 -10	01100	1000 -1	11000

Table 5.6: Positive topological charges of D_5 in the new basis.

In table 5.5 $t_1 \equiv x_1 + x_2$, $t_2 \equiv z_1 + z_2$.

Using relations

$$t_1 = h_2 - h_3, \quad t_2 = h_3 - h_4, \quad b = h_1 - h_2, \quad c = h_4 + h_5, \quad d = h_4 - h_5 \tag{5.17}$$

Table 5.5 becomes table 5.6 which is obviously the table of positive roots of $SO(10)$.

5.7.2 E_6 Quiver

After we set flux for the node X to zero the energy for bare monopoles is given by the expression

$$\begin{aligned} E = &-(|s_1 - s_2| + |l_1 - l_2| + |l_1 - l_3| + |l_2 - l_3| + |m_1 - m_2| + |p_1 - p_2|) + \frac{1}{2}(|s_1| + |s_2| + |m_1 - n| \\ &+ |m_2 - n| + |p_1 - q| + |p_2 - q| + |l_1 - s_1| + |l_1 - s_2| + |l_2 - s_1| + |l_2 - s_2| \\ &+ |l_3 - s_1| + |l_3 - s_2| + |l_1 - m_1| + |l_1 - m_2| + |l_2 - m_1| + |l_2 - m_2| + |l_3 - m_1| + |l_3 - m_2| \\ &+ |l_1 - p_1| + |l_1 - p_2| + |l_2 - p_1| + |l_2 - p_2| + |l_3 - p_1| + |l_3 - p_2|) \end{aligned} \tag{5.18}$$

Denote the six topological charges by

$k_1k_2k_3k_4k_5k_6$					
000001	001100	011001	012111	111100	112211
000010	001101	011100	012211	111101	122101
000100	001110	011101	100000	111110	122111
000110	001111	011110	110000	111111	122211
001000	010000	011111	111000	112101	123211
001001	011000	012101	111001	112111	123212

Table 5.7: Positive topological charges of E_6.

$$\begin{aligned} &k_1 = n, \quad k_2 = m_1 + m_2, \quad k_3 = l_1 + l_2 + l_3, \\ &k_4 = p_1 + p_2, \quad k_5 = q, \quad k_6 = s_1 + s_2. \end{aligned} \tag{5.19}$$

The 36 topological scalars with positive values of topological charges reproduce 36 positive roots of E_6 (see [48]). The remaining 36 scalars have opposite topological charges appropriate for 36 negative roots.

5.7.3 E_7 Quiver

For the E_7 quiver theory we get 126 bare monopoles with energy $E = 1$. Expressing the topological charges k_i through magnetic charges

$$\begin{aligned} &k_1 = d_1 + d_2, \quad k_2 = f_1 + f_2 + f_3, \quad k_3 = g_1 + g_2 + g_3 + g_4, \\ &k_4 = h_1 + h_2 + h_3, \quad k_5 = x_1 + x_2, \quad k_6 = b, \quad k_7 = c_1 + c_2, \end{aligned} \tag{5.20}$$

we obtain exactly 126 roots of the Lie algebra E_7 as can be checked by comparing the spectrum of topological charges Table 5.8 with 63 positive roots of E_7 written down in [48].

k_i	k_i	k_i	k_i	k_i	k_i
0000001	0011100	0111110	1110001	1122111	1232112
0000010	0011101	0111111	1111000	1122211	1232211
0000100	0011110	0121001	1111001	1221001	1232212
0000110	0011111	0121101	1111100	1221101	1233211
0001000	0100000	0121111	1111101	1221111	1233212
0001100	0110000	0122101	1111110	1222101	1243212
0001110	0110001	0122111	1111111	1222111	1343212
0010000	0111000	0122211	1121001	1222211	2343212
0010001	0111001	1000000	1121101	1232101	
0011000	0111100	1100000	1121111	1232102	
0011001	0111101	1110000	1122101	1232111	

Table 5.8: Positive topological charges of E_7.

k_i	k_i	k_i	k_i	k_i	k_i
00000001	00111111	01222211	11222101	12322102	13432212
00000010	01000000	10000000	11222111	12322111	13433212
00000100	01100000	11000000	11222211	12322112	13433212
00000110	01100001	11100000	12210001	12322211	13443212
00001000	01110000	11100001	12211001	12322212	13543212
00001100	01111000	11110000	12211101	12332101	13543213
00001110	01111001	11110001	12211111	12332102	23432102
00010000	01111100	11111000	12221001	12332111	23432112
00011000	01111101	11111001	12221101	12332112	23432212
00011100	01111110	11111100	12221111	12332211	23433212
00011110	01111111	11111101	12222101	12332212	23443212
00100000	01210001	11111110	12222111	12333211	23543212
00100001	01211001	11111111	12222211	12333212	23543213
00110000	01211101	11210001	12321001	12432102	24543212
00110001	01211111	11211001	12321002	12432112	24543213
00111000	01221001	11211101	12321101	12432212	24643213
00111001	01221101	11211111	12321102	12433212	24653213
00111100	01221111	11221001	12321111	12443212	24654213
00111101	01222101	11221101	12321112	13432102	24654313
00111110	01222111	11221111	12322101	13432112	24654323

Table 5.9: Positive topological charges of E_8.

5.7.4 E_8 Quiver

Finally, for the E_8 quiver theory the spectrum of topological charges

$$\begin{aligned}
&k_1 = x_1 + x_2, \quad k_2 = c_1 + c_2 + c_3 + c_4, \quad k_3 = b_1 + b_2 + b_3 + b_4 + b_5 + b_6, \\
&k_4 = a_1 + a_2 + a_3 + a_4 + a_5, \quad k_5 = g_1 + g_2 + g_3 + g_4, \quad k_6 = f_1 + f_2 + f_3, \\
&k_7 = d_1 + d_2, \quad k_8 = h_1 + h_2 + h_3
\end{aligned} \tag{5.21}$$

on the energy level $E = 1$ (table 5.9) coincides with 240 roots of E_8 the positive part of which can be compared with [48].

5.8 Appendix B

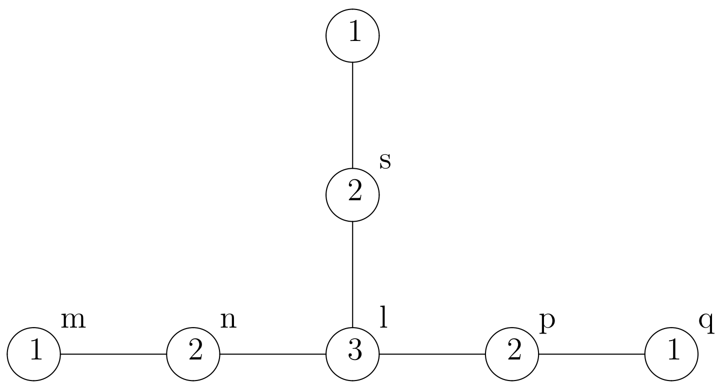

Figure B1. E_6 quiver.

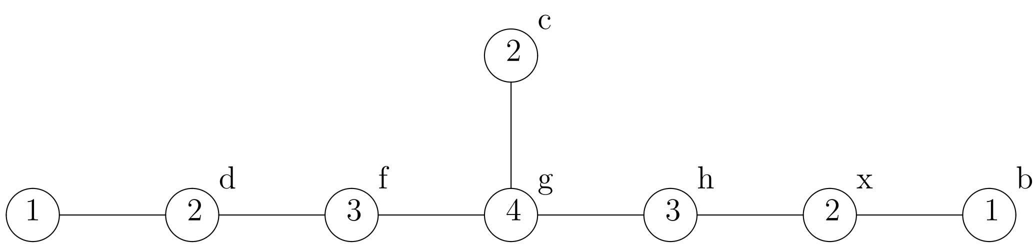

Figure B2. E_7 quiver.

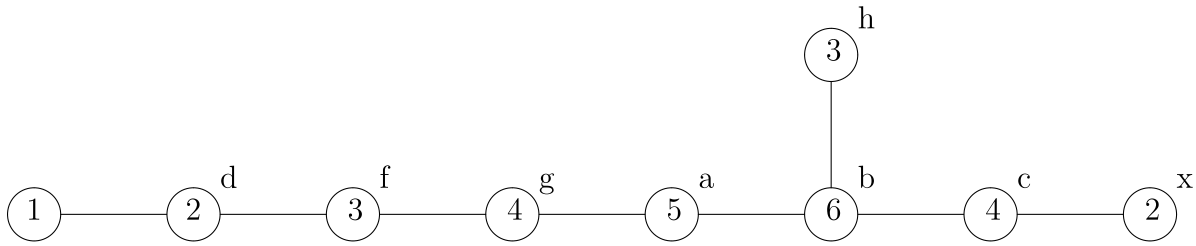

Figure B3. E_8 quiver.

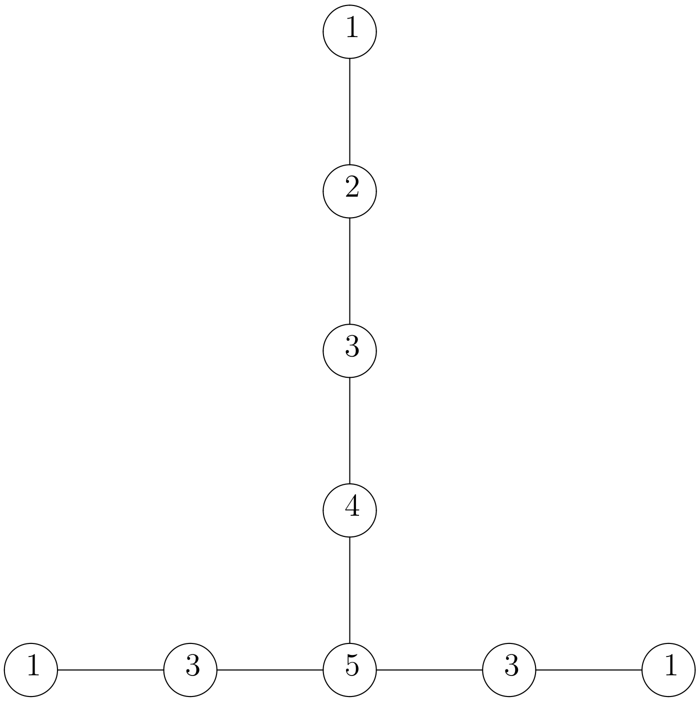

Figure B4.

Chapter 6

Structure of the Stress-Tensor Supermultiplet in $\mathcal{N} \geq 6$ SCFTs

6.1 Introduction

Conformal quantum field theories are not only an interesting special case of general quantum field theories. They are essential part of the definition of any quantum field theory as a relevant perturbation of an ultraviolet fixed point, that is, a conformal field theory. Superconformal field theories are needed to define supersymmetric quantum field theories. Beside this, the infrared limit of a quantum field theory is controlled by another conformal field theory which is called the infrared fixed point.

The simplest cases of (super)conformal field theories are free field theories without dimensionful parameters. These are always explicitly described in terms of Lagrangians. Only a small part of interacting superconformal field theories are known to have a Lagrangian description, and sometimes even in these cases not the entire superconformal structure is seen in the Lagrangian. The now classical example is the ABJM theory [11] in three dimenions with Chern-Simons levels $|k| = 1, 2$. In this particular case only the $\mathcal{N} = 6$ part of the entire $\mathcal{N} = 8$ superconformal structure is seen in the Lagrangian.

Another, more general way to define a superconformal field theory is as the infrared fixed point of some supersymmetric QFT which is usually a perturbation of an UV free fixed point by a relevant operators. In this case we know, for example, the symmetries[1] of the IR

[1]Except the so-called accidental symmetries whose currents are not conserved along the entire RG flow but only in the infrared.

superconformal field point but not its Lagrangian. In fact there is no reason to believe that such a Lagrangian should exist in general.

This makes it clear that classification of superconformal quantum field theories is not reduced to the classification of the superconformal Lagrangians. To find general properties of superconformal theories more abstract approach not relying on the possibility of a Lagrangian description should be employed. In this chapter we make use of only the most fundamental characteristics of $\mathcal{N} \geq 6$ superconformal field theories in three dimensions to find and prove some of their properies. These characteristics are: existence of the superconformal algebra, unitarity and the existence of the stress-tensor.

In the recent years many $\mathcal{N} = 6$ and $\mathcal{N} = 8$ superconformal quantum field theories were found in three space-time dimenions [25, 14, 11, 28]. All these theories have an interesting property: they contain a global $U(1)$ symmetry. In the first part of this chapter we explain this "empirical" fact as stemming only from the properties of $\mathcal{N} = 6$ superconformal algebra, unitarity and the existence and uniquess of the stress-tensor. As a result, every "irreducible" [2] $\mathcal{N} = 6$ superconformal quantum field theory contains a single conserved global current. This immediately implies that any $\mathcal{N} = 8$ superconformal theory has no global symmetries. In the second part of the chapter we make use of this result to explain another "empirical" fact – the fact than no purely $\mathcal{N} = 7$ superconformal field theories have been found so far. We show that every $\mathcal{N} = 7$ superconformal field theory has actually a larger, $\mathcal{N} = 8$, supersymmetry.

6.2 $\mathcal{N} = 6$ Superconformal Field Theories

We start by reviewing the structures of the stress-tensor and global conserved currents multiplets in $\mathcal{N} \geq 4$ three-dimensional superconformal field theories. In three-dimensional $\mathcal{N}$ superconformal field theory the stress-tensor is a primary field with conformal dimension three which is spin-two tensor $T_{(\alpha\beta\gamma\delta)}$ with respect to the rotation group $SO(3)$[3] of $\mathbb{R}^3$ and a singlet of the $SO(\mathcal{N})_R$ $R-$ symmetry and any global symmetry groups. This tensor belongs to a supermultiplet $\mathcal{T}$ which can be decomposed with respect to the bosonic subgroup

[2] That is, possessing a unique stress-tensor.
[3] We work with the Euclidean version of the theory.

$SO(2) \times SO(3) \times SO(\mathcal{N})_R$ of the superconformal group as $\mathcal{T} = \oplus_{n,j,R}\mathcal{T}_{n,j,R}$ where n denotes the so-called level and runs from zero to infinity. As n increases by one the conformal dimension of the representations living on the same level increases by one-half. The only operators from the superconformal algebra that raise or lower the level are supercharges Q and their conjugates (in the radially quantized picture) superconformal charges S as they are the only operators that do not commute with the dilatation operator. Indices j and R label the spins and $SO(\mathcal{N})_R$ representations. The lowest level $n = 0$ contains a single representation of $SO(3) \times SO(\mathcal{N})_R$ with spin j_0, with the highest weight $(r_1, ..., r_{[\mathcal{N}]/2})$ of $SO(\mathcal{N})_R$[4] and the conformal dimension ϵ_0 subject to a certain inequality stemming from the requirements of unitarity [38].

The lowest component of the stress-tensor multiplet is an $SO(3)$ scalar and absolutely antisymmetric rank-four $SO(\mathcal{N})_R$ tensor[5] with conformal dimension $\epsilon_0 = r_1 = 1$. On the second level of the stress-tensor multiplet are the R-currents, on the third level are the supercurrents and on the fourth level is the stress-tensor itself.

Similarly, a conserved global current, if it exists, belongs to a supermultiplet. The lowest component of this supermultiplet is an $SO(3)$ scalar which is the rank-two antisymmetric tensor of $SO(\mathcal{N})_R$ with conformal dimension $\epsilon_0 = r_1 = 1$ [19].

Unless $\mathcal{N} = 6$ the stress-tensor and a global current multiplets are two distinct multiplets. When $\mathcal{N} = 6$ a rank-two antisymmetric tensor is equivalent to a rank-four antisymmetric tensor. This means that the stress-tensor multiplet may contain a conserved global $U(1)$-current on the second level. Below we argue that this is indeed the case: every $\mathcal{N} = 6$ superconformal field theory has a global $U(1)$ symmetry whose current lives on the second level of the stress-tensor multiplet together with the R-currents.

First of all we note that all known $\mathcal{N} = 6$ superconformal field theories possess a global $U(1)$ symmetry. In the case of ABJM theories with gauge groups $SU(N) \times SU(N)$ the $U(1)$ symmetry is the barion number symmetry, while in the case of the gauge group $U(N) \times U(N)$ it is the symmetry generated by the topological current $J = \star\frac{1}{4\pi}(tr(F) - tr(\tilde{F}))$.

Now consider a simple case of the ABJM theory [11] with gauge group $U(1)_k \times U(1)_{-k}$

[4] We use the convention $r_1 \geq r_2 \geq ... \geq r_{[\mathcal{N}/2]}$

[5] In the case of $\mathcal{N} = 8$ the antisymmetric rank-four tensor is decomposed into a self-dual and anti-selfdual parts. Choosing either of them is a matter of convention.

where the subscipts stand for the Chern-Simons levels. In addition to the two $\mathcal{N}=2$ vector multiplets there are matter fields: two hypermultiplets in the bifundamental representation of the gauge group $U(1)\times U(1)$. In terms of the chiral multiplets these are (A_1, B_1) and (A_2, B_2). The theory contains a quartic superpotential and the Chern-Simons kinetic term for the gauge fields with Chern-Simos level k for the first $U(1)$ and $-k$ for the second $U(1)$. This theory has $\mathcal{N}=6$ supersymmetry unless $k=1,2$[6]. Moreover, as we mentioned, there is a $U(1)$ global symmetry. The rank-two antisymmetric tensor of $SO(6)_R$ is equivalent to a rank-four antisymmetric tensor. This is the representation $\mathbf{15}$ of $SO(6)_R$. Because there is a $U(1)$ global symmetry we must have another $\mathbf{15}$ with conformal dimension one in addition to that corresponding to the stress-tensor. However, we only see one copy of $\mathbf{15}$ with conformal dimension one:[7] the binomials in the matter scalars $C_I C_J^\dagger$ where $C_I=(A_1, A_2, B_1^\dagger, B_2^\dagger)$ is in the spinor representation $\mathbf{4}$ of $SO(6)_R$. There are no candidates for another copy of $\mathbf{15}$.

Actually, it is easy to prove that there is no other $\mathbf{15}$. Let us consider only chiral scalars, those scalars which are annihilated by a complex supercharge Q corresponding to an $SO(2)$ subgroup of $SO(6)$. The representation $\mathbf{15}$ is decomposed as $\mathbf{15}=\mathbf{6}_0+\mathbf{4}_1+\mathbf{4}_{-1}+\mathbf{1}_0$ under $SO(4)\times SO(2)\subset SO(6)_R$. The representation $\mathbf{4}_1$ consists of four chiral scalars. Now we can compute the superconformal index for these theories on $S^2\times\mathbb{R}$ [29], that is, in the radially quantized picture

$$\mathcal{I}(x) = Tr[(-1)^F x^{\epsilon+j_3}] \tag{6.1}$$

using the localization technique [16]. In the above expression F is the fermion number.

In the Taylor expansion of the index around $x=0$ the coefficient in front of the first power of x counts the number of chiral scalars with conformal dimension one. No other states can contribute to this coefficient because of the unitarity constraints. If the coefficient is four, there are only 4 chiral scalars and, correspondingly, only one representation $\mathbf{15}$ of $SO(6)_R$. If the coefficient is eight, there are two copies of $\mathbf{15}$. Of course, the coefficient does not depend on the particlular choice of an ABJM theory. We computed the index for the

[6] If $k=1$ or $k=2$ the supersymmetry is enhanced to $\mathcal{N}=8$.

[7] Due to a large amount of supersymmetry ($\mathcal{N}>2$) all fields have their UV conformal dimensions.

$U(N)_k \times U(N)_{-k}$ theory for several low values of $k > 2$ and N and found

$$\mathcal{I}(x) = 1 + 4x + \mathcal{O}(x^2). \tag{6.2}$$

The conclusion is that there is only one representation **15** of $SO(6)_R$ which is an $SO(3)$ singlet with conformal dimension $\epsilon_0 = 1$. This means that both the R-currents and the $U(1)$ symmetry current are obtained from the same fifteen scalars by acting on them with a bilinear combination of supercharges $Q^{[i}_{(\alpha}Q^{j]}_{\beta)}$. Here latin indices are fundamental indices of $SO(6)_R$ while the greek indices are spinor indices corresponding to space-time rotations. The group theory indeed allows that because $\mathbf{15} \times \mathbf{15} = \mathbf{1} + \mathbf{15} + ...$ Note that the group theory argument alone is insufficient because the norm of the $SO(6)_R$ singlet could turn out to be zero, and this state then would be absent in the Hilbert space of the radially quantized theory or as an operator on $\mathbb{R}^3$. Our example shows that this is not the case. The $SO(6)_R$ singlet current does exist and is conserved by virtue of its quantum numbers.

The conclusion about the existence of a global $U(1)$ symmetry is in fact true for any $\mathcal{N} = 6$ superconformal quantum field theory. Indeed, the norm of this $SO(6)_R$ singlet current which is obtained from $Q^{[i}_{(\alpha}Q^{j]}_{\beta)}|\mathbf{15}\,\rangle$ where **15** are the scalars on the zero level of the stress-tensor supermultiplet is determined by only the superconformal $\mathcal{N} = 6$ algebra. So, in any $\mathcal{N} = 6$ superconformal theory the $U(1)$ current has a nonzero norm. Furhermore, if there is more than one global symmetry in the theory, then there is more than one set of associated scalars in the representation **15** of $SO(6)_R$. Each such set generates the whole stress-tensor supermultiplet (to which the global current belongs). In particular, there is more than one stress-tensor. Such a theory is reducible, i.e., decomposes into a direct sum of several superconformal field theories. Thus the presence of more than one $U(1)$ global symmetry is an indicator of reducibility of $\mathcal{N} = 6$ SCFT. There are, in fact, examples of reducible superconformal quantum field theories both in four dimensions [50] and in three dimensions [1, 2] where reducibility is not obvious in the (UV) Lagrangian description.

We come to the main conclusion of this section: every irreducible $\mathcal{N} = 6$ superconformal theory has a single global $U(1)$ symmetry with its current appearing on the second level of the stress-tensor supermultiplet together with the $SO(6)_R$ currents.

6.3 $\mathcal{N}=7$ Superconformal Field Theories

Now that the existence of a global $U(1)$ symmetry for any $\mathcal{N}=6$ superconformal field theory is established there arises a natural question: how the global $U(1)$ symmetry fits into the cases of $\mathcal{N}=7$ and $\mathcal{N}=8$ superconformal symmetries?[8] For the case of $\mathcal{N}=8$ the answer is obvious: just as in the case of ABJM theories [11] the global $U(1)$ symmetry becomes the commutant of $SO(6)_R$ in the full R-symmetry group $SO(8)_R$. Because an irreducible $\mathcal{N}=8$ superconformal theory is an irreducible $\mathcal{N}=6$ superconformal theory with a global (with respect to the $\mathcal{N}=6$ superconformal structure) $U(1)$ symmetry which corresponds to the commutant of $SO(6)_R$ in $SO(8)_R$, there is no room for any global symmetries. Thus any $\mathcal{N}=8$ SCFT has no global symmetries. In this section we explore the way in which the structure of $\mathcal{N}=7$ theories is affected. We find that there are no purely $\mathcal{N}=7$ superconformal field theory: every $\mathcal{N}=7$ superconformal theory is in fact $\mathcal{N}=8$ supersymmetric.

Because an $\mathcal{N}=7$ superconformal field theory is a particular case of $\mathcal{N}=6$ superconformal field theories there is a $U(1)$ symmetry which is global as long as the $\mathcal{N}=6$ subgroup of the $\mathcal{N}=7$ supergroup is considered. There are two options for the $U(1)$ group to fit into the $SO(7)_R$ R-symmetry group. First is that the $U(1)$ does not commute with the $SO(7)_R$. This immediately implies that the full R-symmetry group is $SO(8)$ and so the theory is $\mathcal{N}=8$ supersymmetric.

The second option is that the $U(1)$ commutes with the $SO(7)_R$. Let us check if this option is self-consistent. The lowest components (operators on $\mathbb{R}^3$) of the stress-tensor supermultiplet are scalars with the conformal dimension one and form a fourth-rank antisymmetric tensor with respect to $SO(7)$. This tensor is equivalent to a third-rank antisymmetric tensor. It is easy to see how this representation decomposes with respect to $SO(6)_R \subset SO(7)_R$: $\mathbf{35} = \mathbf{15} + \mathbf{10} + \bar{\mathbf{10}}$. This is exactly how the lowest component of the stress-tensor supermultiplet of an $\mathcal{N}=8$ theory decomposes under $SO(6)_R \subset SO(8)_R$ [1].

Here we remind the reader that the lowest component of the stress-tensor supermultiplet of an $\mathcal{N}=8$ superconformal theory is a rank-four selfdual antisymmetric tensor $\mathbf{35}$.[9] Under the reduction $SO(8)_R \to SO(7)_R$ it becomes a single irreducible representation $\mathbf{35}$ of $SO(7)$.

[8] I thank Anton Kapustin for asking me this natural question.
[9] Or an antiselfdual tensor, this depends on the convention.

If the global symmetry $U(1)$ commutes with $SO(7)_R$ then all three irreps $\mathbf{15}+\mathbf{10}+\bar{\mathbf{10}}$ of $SO(6)$ have zero $U(1)$ charges. In the case of $\mathcal{N}=8$ it was $\mathbf{15}+\mathbf{10}_1+\bar{\mathbf{10}}_{-1}$ instead. Forgetting for a moment about the $U(1)$ charges consider the action of the supercharges bilinears $Q^{[i}_{(\alpha}Q^{j]}_{\beta)}$ on the scalars in the representation $\mathbf{10}+\bar{\mathbf{10}}$ where i,j are fundamental indices of $SO(6)_R$ and greek indices are rotation spinor indices. In [1] it was shown that the result of this operation is the set of conserved currents in the representation $\mathbf{6}$ of $SO(6)_R$ needed to enhance $SO(6)_R\times U(1)$ to $SO(8)_R$. Namely, the currents are in the representation $\mathbf{15}+\mathbf{6}+\mathbf{6}$ of $SO(6)_R$, where the representation $\mathbf{15}$ comes form $Q^{[i}_{(\alpha}Q^{j]}_{\beta)}|\mathbf{15}\,\rangle$. For $\mathcal{N}=7$ the conserved currents in the representation $\mathbf{6}$ of $SO(6)_R\subset SO(7)_R$ are the currents which enlarge $SO(6)_R$ to $SO(7)_R$: $\mathbf{21}=\mathbf{15}+\mathbf{6}$. However there are twice as many of them as needed for $SO(7)_R$. Thus there are conserved currents (with nonzero norm) in addition to those in $SO(7)_R$ which do not commute with the $SO(7)_R$. This means that the supersymmetry is enhanced to $\mathcal{N}=8$ which contradicts the assumption that the theory has only $\mathcal{N}=7$ superconformal symmetry.

This proves the claim that every $\mathcal{N}=7$ superconformal theory is, in fact, $\mathcal{N}=8$ supersymmetric.

Chapter 7

Summary

Chapter 2 of this thesis is based on the paper [1] written in collaboration with Anton Kapustin. In this chapter we describe a method to study hidden symmetries in a large class of strongly coupled supersymmetric gauge theories in three dimensions. We applied this method to the ABJM theory and to the infrared limit of $\mathcal{N} = 4$ SQCD with adjoint and fundamental matter. We showed that the $U(N)$ ABJM model with Chern-Simons level $k = 1$ or $k = 2$ has hidden $\mathcal{N} = 8$ supersymmetry. Hidden supersymmetry is also shown to occur in $\mathcal{N} = 4$ $d = 3$ SQCD with one fundamental and one adjoint hypermultiplet. The latter theory, as well as the $U(N)$ ABJM theory at $k = 1$, are shown to have a decoupled free sector. This provides evidence that both models are dual to the infrared limit of $\mathcal{N} = 8$ $U(N)$ super-Yang-Mills theory.

Chapter 3 is based on the paper [2] coauthored by Anton Kapustin. In this chapter we show that an infinite family of $\mathcal{N} = 6$ $d = 3$ superconformal Chern-Simons-matter theories has hidden $\mathcal{N} = 8$ superconformal symmetry and hidden parity on the quantum level. This family of theories is different from the one found by Aharony, Bergman, Jafferis, and Maldacena, as well as from the theories constructed by Bagger and Lambert, and Gustavsson. We also tested several conjectural dualities between BLG theories and ABJ theories by comparing superconformal indices of these theories.

Chapter 4 is based on paper [4]. In this chapter we tested dualities between three dimensional N = 2 gauge theories proposed by Aharony in [39] by comparing superconformal indices of dual theories. We also extended the discussion of chiral rings matching to include monopole operators.

Chapter 5 is based on paper [3]. In this chapter we considered examples of global symmetry enhancement by monopole operators in three dimensional $\mathcal{N} = 4$ gauge theories. These examples include unitary overbalanced quivers, quivers with non-simply laced gauge groups and nonlinear quivers.

Chapter 6 is based on the paper [5]. In this chapter, based on the structure of the three-dimensional superconformal algebra, we showed that every irreducible $\mathcal{N} = 6$ three-dimensional superconformal theory containes exactly one conserved $U(1)$-symmetry current in the stress-tensor supermultiplet and that superconformal symmetry of every $\mathcal{N} = 7$ superconformal theory is in fact enhanced to $\mathcal{N} = 8$. Moreover, an irreducible $\mathcal{N} = 8$ superconformal theory does not have any global symmetries. The first observation explains why all known examples of $\mathcal{N} = 6$ superconformal theories have a global abelian symmetry.

Bibliography

[1] D. Bashkirov and A. Kapustin, "Supersymmetry enhancement by monopole operators," *JHEP* **05** (2011) 015 [arXiv:1007.4861 [hep-th]].

[2] D. Bashkirov and A. Kapustin, "Dualities between $\mathcal{N} = 8$ superconformal field theories in three dimensions," *JHEP* **05** (2011) 074 [arXiv:1103.4861 [hep-th]].

[3] D. Bashkirov, "Examples of global symmetry enhancement by monopole operators," [arXiv:1009.3047[hep-th]].

[4] D. Bashkirov, "Aharony duality and monopole operators in three dimensions," [arXiv:1106.4110[hep-th]].

[5] D. Bashkirov, "A Note on $\mathcal{N} \geq 6$ Superconformal Quantum Field Theories in three Dimensions," [arXiv:1108.4081[hep-th]].

[6] V. Borokhov, A. Kapustin and X. k. Wu, "Topological Disorder Operators in Three-Dimensional Conformal Field Theory," *JHEP* **0211**, 049 (2002) [arXiv:hep-th/0206054].

[7] V. Borokhov, A. Kapustin and X. k. Wu, "Monopole operators and mirror symmetry in three dimensions," *JHEP* **0212**, 044 (2002) [arXiv:hep-th/0207074].

[8] A. M. Polyakov, "Quark Confinement and the Topology of Gauge Groups, *Nucl. Phys. B* **120**, (1977) 429.

[9] O. Aharony, A. Hanany, K. Intriligator, N. Seiberg, M. J. Strassler, "Aspects of N = 2 Supersymmetric Gauge Theories in Three Dimensions," *Nucl. Phys. B* **499**, (1997), 67 [arXiv:9703110[hep-th]].

[10] K. A. Intriligator and N. Seiberg, "Mirror symmetry in three dimensional gauge theories," *Phys. Lett. B* **387**, 513 (1996) [arXiv:hep-th/9607207].

[11] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, "$N = 6$ superconformal Chern-Simons-matter theories, M2-branes and their gravity duals," *JHEP* **0810**, 091 (2008) [arXiv:0806.1218 [hep-th]].

[12] P. Goddard, J. Nuyts and D. I. Olive, "Gauge Theories And Magnetic Charge," *Nucl. Phys. B* **125**, 1 (1977).

[13] M. K. Benna, I. R. Klebanov and T. Klose, "Charges of Monopole Operators in Chern-Simons Yang-Mills Theory," *JHEP* **1001**, 110 (2010) [arXiv:0906.3008 [hep-th]].

[14] A. Gustavsson and S. J. Rey, "Enhanced $N = 8$ Supersymmetry of ABJM Theory on R(8) and R(8)/Z(2)," arXiv:0906.3568 [hep-th].

[15] O. K. Kwon, P. Oh and J. Sohn, "Notes on Supersymmetry Enhancement of ABJM Theory," *JHEP* **0908**, 093 (2009) [arXiv:0906.4333 [hep-th]].

[16] S. Kim, "The complete superconformal index for $N = 6$ Chern-Simons theory," *Nucl. Phys. B* **821**, 241 (2009) [arXiv:0903.4172 [hep-th]].

[17] D. Berenstein and J. Park, "The BPS spectrum of monopole operators in ABJM: towards a field theory description of the giant torus," *JHEP* **1006**, 073 (2010) [arXiv:0906.3817 [hep-th]].

[18] A. Kapustin, B. Willett and I. Yaakov, "Nonperturbative Tests of Three-Dimensional Dualities," [arXiv:1003.5694 [hep-th]].

[19] D. Gaiotto and E. Witten, "S-Duality of Boundary Conditions In $N = 4$ Super Yang-Mills Theory," [arXiv:0807.3720 [hep-th]].

[20] I. R. Klebanov and G. Torri, "M2-branes and AdS/CFT," *Int. J. Mod. Phys. A* **25**, 332 (2010) [arXiv:0909.1580 [hep-th]].

[21] J. de Boer, K. Hori, H. Ooguri, Y. Oz and Z. Yin, "Mirror symmetry in three-dimensional gauge theories, SL(2,Z) and D-brane moduli spaces," *Nucl. Phys. B* **493**, 148 (1997) [arXiv:hep-th/9612131].

[22] N. Seiberg and E. Witten, "Gauge dynamics and compactification to three dimensions," [arXiv:hep-th/9607163].

[23] E. J. Weinberg, "Monopole vector spherical harmonics," *Phys. Rev. D* **49**, 1086 (1994) [arXiv:hep-th/9308054].

[24] T. T. Wu and C. N. Yang, *Nucl. Phys. B* **107**, 365 (1976).

[25] J. Bagger and N. Lambert, "Gauge Symmetry and Supersymmetry of Multiple M2-Branes," *Phys. Rev. D* **77**, 065008 (2008) [arXiv:0711.0955 [hep-th]].

[26] A. Gustavsson, "Algebraic structures on parallel M2-branes," *Nucl. Phys. B* **811**, 66 (2009) [arXiv:0709.1260 [hep-th]].

[27] M. Van Raamsdonk, "Comments on the Bagger-Lambert theory and multiple M2-branes," *JHEP* **0805**, 105 (2008) [arXiv:0803.3803 [hep-th]].

[28] O. Aharony, O. Bergman and D. L. Jafferis, "Fractional M2-branes," *JHEP* **0811**, 043 (2008) [arXiv:0807.4924 [hep-th]].

[29] J. Bhattacharya, S. Bhattacharya, S. Minwalla, S. Raju, "Indices for Superconformal Field Theories in 3, 5 and 6 Dimensions," *JHEP***0802**, 064 (2008) [arXiv:hep-th/0801.1435].

[30] S. Sethi, "A relation between $N = 8$ gauge theories in three dimensions," *JHEP* **9811**, 003 (1998) [arXiv:hep-th/9809162].

[31] N. Lambert and D. Tong, "Membranes on an Orbifold," *Phys. Rev. Lett.* **101**, 041602 (2008) [arXiv:0804.1114 [hep-th]].

[32] J. Distler, S. Mukhi, C. Papageorgakis and M. Van Raamsdonk, "M2-branes on M-folds," *JHEP* **0805**, 038 (2008) [arXiv:0804.1256 [hep-th]].

[33] N. Lambert and C. Papageorgakis, "Relating $U(N)xU(N)$ to $SU(N)xSU(N)$ Chern-Simons Membrane theories," *JHEP* **1004**, 104 (2010) [arXiv:1001.4779 [hep-th]].

[34] A. Kapustin, B. Willett and I. Yaakov, "Tests of Seiberg-like Duality in Three Dimensions," arXiv:1012.4021 [hep-th].

[35] E. Witten, "SL(2,Z) action on three-dimensional conformal field theories with Abelian symmetry," arXiv:hep-th/0307041.

[36] A. Kapustin, B. Willett and I. Yaakov, "Exact Results for Wilson Loops in Superconformal Chern-Simons Theories with Matter," *JHEP* **1003**, 089 (2010) [arXiv:0909.4559 [hep-th]].

[37] J. Bhattacharya and S. Minwalla, "Superconformal Indices for $\mathcal{N} = 6$ Chern Simons Theories," *JHEP* **0901**, 014 (2009) [arXiv:0806.3251 [hep-th]].

[38] S. Minwalla, "Restrictions Imposed by Superconformal Invariance on Quantum Field Theories," *Adv. Theor. Math. Phys.* **2**, 781-846 (1998) [arXiv: 9712074[hep-th]].

[39] O. Aharony, "IR duality in $d = 3$ $N = 2$ Supersymmetric $USp(2N_c)$ and $U(N_c)$ Gauge theories," *JHEP* **0811**, 043 (2008) [arXiv:9703215 [hep-th]].

[40] A. Giveon and D. Kutasov, "Seiberg Duality in Chern-Simons Theory," *Nucl. Phys. B* **812** 1 (2009) [arxiv:0808.0360[hep-th]].

[41] B. Willett and I. Yaakov, "$N = 2$ Dualities and Z Extremization in Three Dimensions," [arXiv:1104.0487[hep-th]].

[42] A. Kapustin, "Seiberg-like duality in three dimensions for orthogonal gauge groups," [arXiv:1104.0466]

[43] Y. Imamura and S. Yokoyama, "Index for three dimensional superconformal field theories with general R-charge assignments," [arXiv:1101.0557[hep-th]].

[44] C. Krattenthaler, V. P. Spiridonov and G. S. Vartanov, "Superconformal Indices of Three-Dimensional Theories related by Mirror Symmetry,"[arXiv:1103.4075[hep-th]].

[45] D. L. Jafferis, "The exact Superconformal R-Symmetry extremizes Z," [arXiv:1012.3210[hep-th]].

[46] Daniel Jafferis and X. Yin, "A Duality Appetizer," [arXiv:1103.5700[hep-th]].

[47] B. Simon: *Representation of Finite and Compact Groups*, *American Mathematical Society*, 1996.

[48] J. F. Cornwell: *Group Theory in Physics*, volume 2, Appendix F, *Academic Press*, 1984.

[49] U. Lindstrom, M. Rocek, R. von Unge, "Hyperkahler quotients and algebraic curves," *JHEP* 0001 (2000) 022 [arXiv:hep-th/9908082].

[50] D. Gaiotto, N. Seiberg, Y. Tachikawa, "Comments on scaling limits of 4d $N = 2$ theories," *JHEP* **1101**, 078 (2011) [arXiv: 1011.4568[hep-th]].