PART ONE

ON THE INITIAL VALUE PROBLEMS OF RADIATION AND SCATTERING OF WATER WAVES BY IMMERSED OBSTACLES

PART TWO

GRAVITY WAVES DUE TO A POINT DISTURBANCE IN A STRATIFIED FLOW

Thesis by

Chiang Chung Mei

In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1963

ACKNOWLEDGMENT

The author owes his profoundest gratitude to Professor T. Yao-Tsu Wu who has suggested and guided the present research. He feels very fortunate indeed to have benefited immeasurably from Professor Wu's enlightening criticism, suggestions and encouragement throughout the entire course of this work. He is also much indebted to Dr. Din-Yu Hsieh for being helpful in many discussions.

Sincere thanks are due to Mrs. Barbara Hawk and Mrs. Mary Goodwin for providing expert assistance in typing the manuscript, and to Miss Cecilia Lin and Mrs. Zora Harrison for preparing the drawings.

Finally the author is grateful to the Li Foundation Inc. New York for a fellowship and to the California Institute of Technology for the Institute Scholarships during the years of his graduate study. Part of this investigation has been carried out under Contract Nonr 220(35) with the Office of Naval Research, Department of the Navy, U.S.A.

PART ONE (pp 1 - 81)

ON THE INITIAL VALUE PROBLEMS OF RADIATION AND SCATTERING OF WATER WAVES BY IMMERSED OBSTACLES

ABSTRACT

Some initial value problems are studied regarding the radiation and scattering of gravity waves by finite bodies in an infinitely deep ocean. Emphasis is placed on the case where a finite number of thin plates lie on a vertical line, for which the general solution is obtained by transforming the boundary value problem to one of the Riemann-Hilbert type. Explicit investigations are made for the large time behavior of the free surface elevation for the case of a rolling plate, and for the Cauchy-Poisson problems in the presence of a stationary plate. By taking the limit as $t \to \infty$, the steady state solution is derived for a harmonic point pressure acting on the free surface near a vertical barrier. Finally a formal asymptotic representation of the free surface elevation is given for large time when the geometry of the submerged bodies is arbitrary.

TABLE OF CONTENTS

TABLE OF CONTENTS

I. INTRODUCTION

In the recent past much has been attempted to study the problem of generation and scattering of surface waves by solid obstacles immersed in a heavy fluid. This class of problems is of basic academic interest as well as of great importance in naval hydrodynamics, for it provides the basis of predicting the ship behavior in waves. The development of a rigorous hydrodynamic theory for this problem, even in the linearized sense, has been recognized as a difficult mathematical task, because the boundary value problem involved is of a highly mixed nature in that different linear combinations of the unknown function and its normal derivative are prescribed on different parts of the boundary. For two-dimensional cylindrical bodies of general shape the analytical methods developed thus far have all been more or less approximate.

The characteristic difficulties encountered in general are fully represented in the time-harmonic steady state problems; a brief survey of the latter may therefore be of value. When there are rigid bodies oscillating with a single frequency ω in or beneath the free surface of a perfect fluid, a potential function $\Phi(x,y,t)$ exists in the flow field. Assuming the free stream velocity to be zero, we may describe the steady-state problem by a time-independent potential $\phi(x,y)$ governed by the following equations (see, e.g., reference 1, pp. 554-555).

$$\Phi(x,y,t) = \phi(x,y)e^{-j\omega t} \tag{1.1}$$

$$\nabla^2\phi(x,y) = 0 \qquad y < 0 \tag{1.2}$$

$$\frac{\partial\phi}{\partial y} - \beta\phi = 0 \qquad y = 0\ , \qquad \text{with} \qquad \beta = \omega^2/g \tag{1.3}$$

$$\phi \;,\; \nabla\phi \quad \text{bounded as} \quad x^2 + y^2 \to \infty \tag{1.4}$$

$$\frac{\partial\phi}{\partial n} \quad \text{is given on the wetted body surfaces} \tag{1.5}$$

$$\Phi = \phi e^{-i\omega t} \propto e^{\beta y}\, e^{j(\beta|x| - \omega t)} \quad \text{as} \quad |x| \to \infty \;. \tag{1.6}$$

For radiation problems ϕ is the true velocity potential, while for scattering problems it is the difference of the true velocity potential and the potential of the incident wave train. The last condition, 1.6 is the so-called radiation condition which is necessary to insure a unique solution. Furthermore, equation 1.3 implies a tacit assumption that the water surface is free of external pressure.

Consider for instance the case of a single floating cylinder. In the extreme cases of $\beta \to \infty$ and $\beta = 0$ the boundary condition on the free surface, equation 1.3, is simplified and the velocity potential can be continued analytically into the upper half plane. One then obtains a Neuman problem with $\frac{\partial\phi}{\partial n}$ given on a closed curve composed of the wetted body contour and its mirror reflection in the upper half plane. The reduced problem can be solved exactly (reference 2). However, for finite values of β the situation becomes much more complicated. Although a formal representation may be obtained for ϕ by using Green's function $G(x,y;x_o,y_o)$ satisfying

$$\nabla^2 G = \delta(x-x_o)\,\delta(y-y_o)$$

equations 1.3, 1.4 and 1.6, the boundary condition on the solid body (equation 1.5) leads to an integral equation which is usually difficult to solve. For a semi-circular cylinder and large values of β, Ursell (references 3 and 4) has treated the integral equation by an iteration procedure. The analysis seems to be too laborious, however, to be extended to cylinders of other geometry. An effective and unified perturbation method for either small or large values of β is still lacking.

The most successful method developed so far is essentially a numerical one first introduced by Ursell (reference 5) and later extended by himself and others (references 6, 7, and 8) to cylinders of various other shapes. The central idea of Ursell hinges on the superposition of a polynomial consisting of terms satisfying equations 1.2 and 1.3 and vanishing at infinity, and a singular solution satisying conditions 1.2 - 4 and 1.6. A numerical patching is done by applying the condition 1.4 at a number of chosen points on the solid boundary to determine the coefficients in the polynomial and the strength of the singularity. As an example, for a half-immersed circular cylinder of unit radius and centered at the origin, one may take the following complex potential,

$$f(z) = \phi(x,y) + i\psi(x,y) = \sum_{m=1}^{\infty} (A_m + jB_m)\left(\frac{1}{z^{2m}} + \frac{i\beta}{2m-1}\,\frac{1}{z^{2m-1}}\right)$$

$$+ \frac{\omega}{\pi\rho g}(P+jQ)\int_{0}^{\infty} \frac{dk}{k-\beta}\, e^{-ikz}$$

where the integral term above is the singular solution corresponding to an oscillating point pressure of strength $P + jQ$ applied at the origin. The path of integration in the complex k-plane, $k = k_1 + jk_2$, lies mostly on the positive real k-axis from the origin to infinity except for a small detour along a semi-circular arc beneath the pole at $k = \beta$. It is to be noted that two separate imaginary units i and j are used for the z-and k-planes respectively. Taking the imaginary part of f(z) with respect to i and noting $z = re^{i\theta}$ we have the expression for the stream function:

$$\psi(x,y) = \sum_{m=0}^{\infty} (A_m + jB_m)\left[-\frac{\sin 2m\theta}{r^{2m}} + \frac{\beta}{2m-1}\frac{\cos(2m-1)\theta}{r^{2m-1}}\right]$$

$$-\frac{\omega}{\pi\rho g}(P+jQ)\int_0^{\infty}\frac{dk}{k-\beta}e^{kr\sin\theta}\sin(kr\cos\theta)\ . \qquad (1.7)$$

To calculate the coefficients A_m, B_m, P and Q by applying the boundary condition on the body (equation 1.5), $r = 1$, $-\pi < \theta < 0$, we lack the advantage of orthogonality present in ordinary Fourier sine series and orthogonalization does not seem practical. In order to simplify the calculation, the series in equation 1.7 can be truncated to N terms, say, and the boundary values of $\psi = \int \frac{\partial\phi}{\partial r}\, d\theta$ at any N points on $r = 1$ provide 2N simultaneous equations for an equal number of unknown real coefficients. Evidently this procedure is suitable only for relatively low values of β. A survey of many other approximate methods can be found in a recent review by Kaplan and Kotik (reference 9).

An important exception is the case of a thin vertical plate because the corresponding boundary value problem can be solved exactly, whether the plate is floating or completely submerged. The scattering of steady monochromatic waves by a submerged stationary plate barrier extending vertically to the bottom of the ocean has been investigated by Dean (reference 10); the same problem for a surface-piercing plate barrier has been solved by Ursell (reference 11) who also studied the associated radiation problem for a rolling vertical plate (reference 12).

Haskind (reference 13) combined both radiation and scattering and allowed the floating plate to have swaying in addition to rolling motion. The methods adopted by these authors can be classified into two categories (cf. reference 14): In the first category one uses either Fourier transform or a Green's function technique to obtain and solve a singular integral equation, whereas in the second one introduces an auxiliary function to simplify the boundary conditions and then uses some function-theoretic argument. In the present thesis the extension of both methods will be employed.

In regard to the transient gravity wave phenomena the rigorous theory began with the classical Cauchy-Poisson problem for initial disturbances on the water surface (see e.g., reference 15, pp. 384 - 394). DePrima and Wu (reference 16) treated in great detail the transient waves due to a uniformly travelling point pressure on the water surface with surface tension. Wu (reference 17) subsequently extended the investigation to include the case where the strength of the point pressure is sinusoidal in time. In both studies the basic mathematical tool is the combination of Laplace and Fourier transforms. Apparently unaware of the more general work of Wu, Miles (reference 18) has recently analysed the case of an oscillating point pressure, disregarding the forwarding velocity and the surface tension. He used, however, a different approach of superimposing the Cauchy-Poisson results. The analysis by Kennard (reference 19) for a wave-maker in a wall seems to be the only two dimensional initial value problem treated with the presence of a solid object in water.

In the present thesis several initial value problems of the radiation and scattering of surface waves by finite objects in an infinitely deep ocean will be studied. We shall, however, restrict in most part the geometry of the solid bodies to a series of thin plates lying vertically on the negative y-axis. Except for this limitation, the investigation is formally carried out in the most general manner, i.e., arbitrary initial and boundary conditions compatible with the basic assumptions of linearization will be allowed. Thus in Chapter II we formulate and then reduce the mixed initial-boundary value problem to a pure boundary value problem of Riemann-Hilbert type, which is later solved by a function-theoretic method. The general solution is left in a form which requires no further analysis other than explicit evaluation of some integrals. In the particular case of a single floating plate in the water surface, all the relevant integrals can be explicitly evaluated. Hence in Chapter III the radiation of transient waves from a rolling surface plate is investigated in detail, ignoring the disturbances on the free surface. When the rolling motion is simple harmonic in time, the transient phenomenon is compared with the steady state diffraction of light and an interesting analogy is pointed out. Chapter IV deals with the scattering by a barrier of waves generated by a point disturbance in the immediate neighborhood of the plate. Solutions obtained are exact and explicit. Finally in Chapter V the restriction on the shape and the geometrical arrangement of solid bodies is entirely removed and the transient surface elevation is formally found for simple-harmonic disturbances. There a different method of attack using a Green's function is developed. It will be seen that in the special case where the bodies are a string of thin rigid plates lying on the vertical y-axis the transient response is essentially the same as that of a single plate or of a point pressure at the origin.

II GENERAL SOLUTION OF THE INITIAL VALUE PROBLEM FOR VERTICAL PLATES

2.1 Formulation of the problem

We assume the fluid to be perfect, free of surface tension, and the flow irrotational. The flow field can therefore be conveniently described by a complex potential $f(z,t) = \phi(x,y,t) + i\psi(x,y,t)$ analytic in $z = x + iy$ for $y < 0$, with ϕ and ψ representing the velocity potential and the stream function respectively. With the flow further assumed to be of small amplitude about an equilibrium position, the usual linearized boundary conditions on the surface read as follows (reference 1, p. 604):

$$-\frac{\partial \phi}{\partial y} + \frac{\partial \zeta}{\partial t} = 0 \quad \text{and} \quad \frac{\partial \phi}{\partial t} + g\zeta = -\frac{p}{\rho} \tag{2.1.a}$$

for $y = 0$ and $t > 0$, or, equivalently,

$$\operatorname{Im} \frac{\partial f}{\partial z} + \frac{\partial \zeta}{\partial t} = 0 \quad \text{and} \quad \operatorname{Im} i \frac{\partial f}{\partial t} + g\zeta = -\frac{p}{\rho} \tag{2.1.b}$$

where ζ denotes the displacement of the water surface measured from $y = 0$. Let the external pressure applied on the free surface be given in a general form,

$$\begin{aligned} p(x,o,t) &= 0 \qquad t \leq 0 \\ &= p_o(x,t) \quad t > 0 \; . \end{aligned} \tag{2.2}$$

We allow initial disturbances of the following type to be created on the free surface at $t = 0+$

$$\phi(x,0,0+) = \operatorname{Re} f(x,0+) = \phi_o(x) \tag{2.3.a}$$

$$\zeta(x,0+) = \zeta_o(x) \quad . \tag{2.3.b}$$

The condition 2.3.a can be interpreted as prescribing an initial impulse of total magnitude $I = -\rho\phi_o$. From equations 2.1.a it is evident that giving $\zeta(x,0+)$ is equivalent to the description of an initial value of $\frac{\partial\phi}{\partial t} + p/\rho$ on the free surface.

Let it be assumed that the obstacles, denoted by L as a whole, consist of $N+1$ separate vertical plates having no thickness and all lying on the negative y-axis, i.e.,

$$L = \sum_{n=o}^{N} L_n \quad \text{with} \quad L_n : x = 0 \ , \ -b_n \geqq y \geqq -a_n \quad \text{and}$$
$$0 = b_o < a_o < b_1 \ . \ . \ . \ . \ . \ . \ . < b_N < a_N \ . \tag{2.4}$$

The motion of L now provides a condition on its boundary:

$$\begin{aligned} \frac{\partial}{\partial x}\phi(0,y,t) &= U(y,t) \qquad t > 0 \\ &= 0 \qquad\qquad\quad t \leqq 0 \end{aligned} \qquad \text{for } y \text{ on } L \tag{2.5.a}$$

where U denotes the prescribed normal velocity of the plates, or, after integration,

$$\begin{aligned} \operatorname{Im} f(iy,t) &= \psi_L(y,t) \qquad t > 0 \ , \\ &= 0 \qquad\qquad\quad t = 0 \end{aligned} \qquad \text{for } y \text{ on } L \tag{2.5.b}$$

where

$$\psi_L(y,t) = \int_{-a_n}^{y} \frac{\partial\phi}{\partial x}(y,t)\,dy + \psi_n(t) \text{ for } y \text{ on } L_n, \ n = 0,1,2 \ . \ . \ .N \ . \tag{2.5.c}$$

Among all the integration "constants", $\psi_n(t)$, one of them can be taken arbitrary, say $\psi_o(t)$, while the rest have to be determined in the solution.

As usual square root singularities will be allowed for the fluid velocity at the submerged edges of the plates, i.e.,

$$\left|\frac{\partial f}{\partial z}\right| \sim |z - z_e|^{-\frac{1}{2}} \qquad \text{as } z \longrightarrow z_e \tag{2.6}$$

where $z_e = -ia_o, -ia_1, \ldots -ia_N; -ib_1, -ib_2, \ldots -ib_N;$ and $\partial f/\partial z$ will be bounded everywhere else. The potential itself and its first time derivative are everywhere finite for all t. In the region infinitely far away from the origin the velocity field must be vanishingly small for all finite values of t, namely,

$$\left|\frac{\partial f}{\partial z}\right| \longrightarrow 0 \qquad \text{as} \quad |z| \longrightarrow \infty \text{ for all } \quad t < \infty \;. \tag{2.7}$$

This completes the formulation.

2.2 An auxiliary function $F(z, s)$

By the method of Laplace transform, the initial-boundary value problem just formulated can be immediately reduced to a pure boundary value problem. We define the Laplace transform of a given function $M(t)$ by:

$$\widetilde{M}(s) = \int_0^\infty dt\, e^{-st} M(t)$$

then

$$M(t) = \frac{1}{2\pi j} \int_\Gamma ds\, e^{st} \widetilde{M}(s) \tag{2.8}$$

gives the inverse transform of $\widetilde{M}(s)^*$, where Γ denotes a

---

* It is to be noted that two different imaginary units are used here, i for z and j for s, where $i = \sqrt{-1}$ and $j = \sqrt{-1}$ but $ij \neq -1$.

vertical path extending from $c - j\infty$ to $c + j\infty$ with the real number c greater than the real parts of all singularities of $\widetilde{M}(s)$. Taking the transform of equations 2.1.b, one gets,

$$\mathrm{Im}_i \frac{\partial}{\partial z} \widetilde{f}(x,s) + s\widetilde{\zeta}(x,s) - \zeta(x,o) = 0$$

$$\mathrm{Im}_i i s \widetilde{f}(x,s) - \mathrm{Im}_i i f(x,o) + g\widetilde{\zeta}(x,s) = \frac{\widetilde{p}_{o}(x,s)}{\rho}$$

which can be combined to give a single boundary condition on the free surface,

$$\mathrm{Im}_i \left(\frac{\partial}{\partial z} - i \frac{s^2}{g}\right)\widetilde{f}(x,s) = \frac{s\widetilde{p}_o(x,s)}{\rho g} + \zeta(x,o) - \mathrm{Im}_i \frac{is}{g} f(x,o) . \qquad (2.9)$$

The right hand side of the preceding equation is of course known from equations 2.2 and 2.3. The boundary condition on the plates becomes

$$\mathrm{Im}_i \widetilde{f}(iy,s) = \widetilde{\psi}(o,y,s) = \widetilde{\psi}_L(y,s) \qquad (2.10.a)$$

where

$$\widetilde{\psi}_L(y,s) = \int_{-a_n}^{y} \frac{\partial\widetilde{\phi}}{\partial x}(y,s)dy + \widetilde{\psi}_n(s) \quad \text{for } y \text{ on } L_n \ n = 0, 1, \ldots N .$$

(2.10.b)

By introducing an auxiliary function $F(z,s)$ defined by

$$F(z,s) = \Phi(x,y,s) + i\Psi(x,y,s) = \left(\frac{\partial}{\partial z} - i \frac{s^2}{g}\right)\widetilde{f}(z,s) \qquad (2.11)$$

which is analytic in z for all $y < 0$, we can rewrite equation 2.9 as

$$\mathrm{Im}_i F(x,s) = \frac{s\widetilde{p}_o(x,s)}{\rho g} + \zeta_o(x) - \frac{s}{g}\phi_o(x) \equiv \Psi_o(x,s) . \qquad (2.12)$$

On the moving solid boundary L the real part of F is known, since

$$\mathrm{Re}_i F(iy) = \Phi(o, y, s) = \left(\frac{\partial}{\partial y} + \frac{s^2}{g}\right) \widetilde{\psi}_L(y, s) = \Phi_L(y, s) \quad , \quad y \text{ on } L \qquad (2.13)$$

where $\widetilde{\psi}_L$ is given by 2.10.b. The singularity condition clearly permits that,

$$|F| \sim |z - z_e|^{-\frac{1}{2}} \quad \text{as} \quad z \rightarrow z_e \; . \qquad (2.14)$$

F is bounded everywhere else, in particular

$$|F| < \infty \quad \text{as} \quad |z| \rightarrow \infty \qquad 2.15)$$

Summarizing, we note that F has its imaginary part given on the free surface, its real part on the solid surface and its singular behavior prescribed at sharp corners. It should be added here that the given functions $p_o(x,t)$, $\zeta_o(x)$ and $\phi_o(x)$ are assumed to be such as to render all the related integrals convergent. Hereafter we shall use the notations Re for Re_i and Im for Im_i to represent the real and imaginary parts respectively with reference to the unit "i".

2.3 General solution for $F(z,s)$ and $\widetilde{f}(z,s)$

We proceed to determine $F(z,s)$ and $\widetilde{f}(z,s)$. The variable s in the arguments of these functions will be omitted so long as it remains as a parameter. First of all we express F as the sum of two parts F_1 and F_2:

$$F(z) = F_1(z) + F_2(z) \qquad (2.16)$$

such that, F_1 satisfies conditions 2.12 and 2.15 in the absence of the plates. The solution F_1 is readily given by the Poisson's formula for the lower half plane $(\mathrm{Im}\, z < 0)$:

$$F_1(z) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Psi_o(x_o)dx_o}{x_o - z} + \Phi_{1\infty} \tag{2.17}$$

where $\Phi_{1\infty}$ is a real constant equal to the value of $F_1(z)$ at infinity. From this the value of F_1 on the solid plates L can be calculated and is of course continuous across the line segments L. In fact, on $x = 0$,

$$F_1(iy) = \Phi_1(o,y) + i\Psi(o,y) = \Phi_{1\infty} - \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(x_o + iy)\Psi_o(x_o)dx_o}{x_o^2 + y^2} . \tag{2.18}$$

In accordance with the conditions imposed on F, the second part F_2 must now have the following boundary value:

$$\operatorname{Im} F_2 = 0 \qquad y = 0 \tag{2.19}$$

and

$$\operatorname{Re} F_2 = \Phi_2 = \operatorname{Re}(F - F_1) = \Phi_L - \Phi_1 = \Phi_{2L} \quad x = 0, \ y \text{ on } L \tag{2.20}$$

with Φ_L and Φ_1 given by equations 2.13 and 2.18 respectively. The requirement on the behavior of F at the sharp edges (equation 2.14) and infinity (equation 2.15) apply to F_2 without change. The solution of F_2 will now be given as follows.

Because of equation 2.19 one may use Schwarz's reflection principle to continue $F_2(z)$ analytically to the upper half z-plane:

$$F_2(z) = \overline{F_2(\bar{z})} \qquad \text{or}$$

$$\Phi_2(x,y) + i\Psi_2(x,y) = \Phi_2(x,y) - i\Psi_2(x,-y) . \tag{2.21}$$

Let $\bar{L}$ be defined as the mirror image of L reflected into the x-axis and the boundary condition 2.20 is extended in accordance with equation

2.21 so that

$$\begin{aligned} &\text{Re}\, F_2 = \Phi_{2L}(y) && && L \\ \text{and,} \quad &\phantom{\text{Re}\, F_2} = \Phi_{2L}(-y) && \text{for } x = \pm 0 \quad y \text{ on} && \Sigma \\ \text{or,} \quad &\phantom{\text{Re}\, F_2} = \Phi_{2L}(-|y|) && && L + \Sigma \, . \end{aligned} \tag{2.22}$$

Under the continuation defined by equation 2.21 the condition 2.19 is satisfied identically. Hence we have arrived at a simpler boundary value problem for F_2 as specified by equation 2.22 together with equations 2.14 and 2.15, regarding the behavior at the edges and infinity.

The solution is most readily obtained by the method of singular integral equations (reference 20, section 91). The version presented below is tailored to bring the result in a form particularly suitable for the case on hand.

Let the positive direction of the contour $L + \Sigma$ be vertically upwards and the half planes $x > 0$ and $x < 0$ be designated by S_+ and S_- respectively. Since its boundary value is the same on both sides of $L + \Sigma$ the function $\Phi_2(x,y)$ must be even in x, which, by Cauchy-Riemann relations, implies that $\Psi_2(x,y)$ must be odd in x. In other words, the following relation holds:

$$F_2(z) = \overline{F_2(-\bar{z})} \qquad \text{or}$$

$$\Phi_2(x,y) + i\Psi_2(x,y) = \Phi_2(-x,y) - i\Psi_2(-x,y) \, . \tag{2.23}$$

When $-\bar{z} \to z_0$ on $L + \Sigma$ from S_+, $z \to z_0$ from S_-, therefore,

$$\overline{F_2^+(z_0)} \Leftarrow \overline{F_2(-\bar{z})} = F_2(z) \Rightarrow F_2^-(z_0) \tag{2.24}$$

where

$$F_2^{\pm}(z_o) = \lim_{x \to \pm 0} F_2(z) \qquad \text{for } z_o \quad \text{on } L + \Sigma . \qquad (2.25)$$

The boundary condition 2.22 then requires that, for $z_o = iy$ on $L + \Sigma$,

$$\operatorname{Re} F_2(z) = \tfrac{1}{2}[F_2^+(z_o) + \overline{F_2^+(z_o)}]$$
$$= \tfrac{1}{2}[F_2^+(z_o) + F_2^-(z_o)] = \Phi_{2L}(-|y|)$$

by making use of equation 2.24. This leads to a so-called Riemann-Hilbert problem with

$$F_2^+(z_o) + F_2^-(z_o) = 2\Phi_{2L}(-|y|) \qquad \text{for } z_o = iy \text{ on } L + \Sigma . \quad (2.26.a)$$

and

$$F_2^+(z_o) - F_2^-(z_o) = 2i\,\Psi_2(o,y) = 0 \quad \text{for } z_o = iy \text{ not on } L + \Sigma . \; (2.26.b)$$

The last condition follows from the fact that Ψ_2 is odd in x.

The corresponding homogeneous solution satisfying

$$F_H^+(z_o) + F_H^-(z_o) = 0 \qquad z_o \text{ on } L + \Sigma \qquad (2.27.a)$$

$$F_H^+(z_o) - F_H^-(z_o) = 0 \qquad z_o = iy \text{ not on } L + \Sigma \qquad (2.27.b)$$

assumes the following form:

$$F_H(z) = P(z)/D(z) \; , \text{ with}$$

$$P(z) = \sum_{n=o}^{N} A_{2n+1} z^{2n+1} \; , \text{ and} \qquad (2.28)$$

$$D(z) = [(z^2 + a_o^2) \prod_{n=1}^{N} (z^2 + a_n^2)(z^2 + b_n^2)]^{\frac{1}{2}}$$

where all the coefficients A_{2n+1} are real. The function D will be made single-valued by introducing $L + \Sigma$ as cuts in the z-plane and choosing the branch such that in the cut plane

$$D(z) \simeq z^{2N+1} \quad \text{as} \quad |z| \to \infty . \tag{2.29}$$

It is obvious that

$$\lim_{x \to +0} D(z) = D^{+}(z_o) = - \lim_{x \to -0} D(z) = -D^{-}(z_o) \quad \text{for } y \text{ on } L + \Sigma .$$

Furthermore, so far as only the condition 2.27.a), that is

$$\operatorname{Re} \lim_{x \to 0} P/D = 0 \qquad y \text{ on } L + \Sigma ,$$

(2.27.b) and the conditions at the edges and infinity are concerned, we may take

$$P(z) = \sum_{n=0}^{N} A_{2n+1} z^{2n+1} + i \sum_{n=0}^{N} A_{2n} z^{2n}$$

with all coefficients real. However the requirement of equation 2.19 that

$$\operatorname{Im} P/D = 0 \qquad \text{for } y = 0$$

demands the vanishing of all A_{2n}'s.

We shall now look for the inhomogeneous part of the solution $F_I(z)$ satisfying equation 2.26. To do so we let the following subsidiary function be introduced:

$$K(z) = \prod_{n=0}^{N} (z^2 + a_n^2)^{\frac{1}{2}} \prod_{n=1}^{N} (z^2 + b_n^2)^{-\frac{1}{2}} \tag{2.30}$$ *

* It is immaterial which factors are in the numerator or the denominator as long as equation 2.31 is satisfied.

which is single-valued in a plane cut at the position of the plates and their images, $L + \bar{L}$, and behaves like

$$K(z) \approx z \quad \text{as} \quad |z| \to \infty \quad . \tag{2.31}$$

Like $D(z)$, $K(z)$ has the following property:

$$\lim_{x \to +0} K(z) = K^+(z_o) = -\lim_{x \to -0} K(z) = -K^-(z_o) \quad \text{for} \quad z_o = iy \quad \text{on} \quad L + \bar{L}$$

and

$$\lim_{x \to \pm 0} K(z) = K^{\pm}(z_o) = K^+(z_o) = K^-(z_o) \quad \text{for} \quad z_o = iy \quad \text{not on} \quad L + \bar{L} \ .$$

which may be used to rewrite equation 2.26 as follows,

$$\frac{F_I^+(z_o)}{K^+(z_o)} - \frac{F_I^-(z_o)}{K^-(z_o)} = \frac{2\Phi_{zL}(-|y|)}{K^+(z_o)} \quad z_o = iy \quad \text{on} \quad L + \bar{L} \ . \tag{2.32}$$

$$= 0 \qquad\qquad z_o = iy \quad \text{not on} \quad L + \bar{L} \ .$$

Direct use may now be made of the Plemelj's formulae (reference 20, section 17) stating that if

$$\Gamma(z) = \frac{1}{2\pi i} \int_c \frac{\gamma(\xi) d\xi}{\xi - z} \tag{2.33.a}$$

in which c is an arbitrary smooth curve and $\gamma(\xi)$ is Hölder-continuous on c, then for a point z_o on c,

$$\Gamma^R(z_o) - \Gamma^L(z_o) = -\gamma(z_o) \tag{2.33.b}$$

and,

$$\Gamma^R(z_o) + \Gamma^L(z_o) = \frac{1}{\pi i} \int_c \frac{\gamma(\xi) d\xi}{\xi - z_o} \tag{2.33.c}$$

or, equivalently,

$$\left.\begin{matrix}\Gamma^R(z_o)\\ \Gamma^L(z_o)\end{matrix}\right\} = \mp \frac{1}{2}\,\gamma(z_o) + \frac{1}{2\pi i} ⨍_c \frac{\gamma(\xi)d\xi}{\xi - z_o} \tag{2.33.d}$$

The symbol $⨍$ represents the Cauchy principle value of the integral whereas $\Gamma^R(z_o)$ and $\Gamma^L(z_o)$ are the limiting values of Γ as z approaches z_o on the contour c from the right and the left respectively. From equations above it is evident that equation 2.32 is satisfied by:

$$F_I(z) = -\frac{1}{\pi i}\, K(z) \int_{L+\bar{L}} \frac{\Phi_{2L}(-|\eta|)dz'}{(z'-z)K^+(z')} \qquad z' = i\eta \ , \tag{2.34}$$

the path of integration being along $L + \bar{L}$, in the direction of increasing η. An equivalent form of F_I may be obtained by making use of the evenness of $\Phi_{2L}(-|\eta|)$ and $K^+(z')$ in η, giving

$$F_I(z) = \frac{2z}{\pi}\, K(z) \int_L \frac{\Phi_{2L}(\eta)d\eta}{(z^2+\eta^2)K^+(i\eta)} \ . \tag{2.35}$$

Combining with equation 2.28 one gets the general solution of the present Riemann-Hilbert problem as,

$$F_2(z) = F_H + F_I = \frac{P(z)}{D(z)} + \frac{2z}{\pi}\, K(z) \int_L \frac{\Phi_{2L}(\eta)\,d\eta}{(z^2+\eta^2)K^+(i\eta)} \ . \tag{2.36}$$

It is easy to check that this solution meets all the requirements, in particular, F_2 tends to a real constant, say $\phi_{2\infty}$, at infinity.

The transformed complex potential, $\tilde{f}(z)$, is obtained simply by integrating equation 2.11,

$$\tilde{f}(z) = e^{i\alpha z} \int_{-i\infty}^{z} dp\, e^{-i\alpha p} F(p)$$

$$= e^{i\alpha z} \int_{-i\infty}^{z} dp\, e^{-i\alpha p} [F_1(p) + F_2(p)] \qquad (2.37)$$

with

$$\alpha = s^2/g \ . \qquad (2.38)$$

A term proportional to $e^{i\alpha z}$ could have been added in equation 2.37 since it satisfies the relation

$$\left(\frac{\partial}{\partial z} - i\alpha\right) e^{i\alpha z} = 0 \ ,$$

but is discarded since its derivative with respect to z does not vanish at infinity.

2.4 Determination of the constants

It remains to determine the $N + 1$ coefficients, A_{2n+1}, in the polynomial $P(z)$ and N integration constants, $\tilde{\psi}_n$ (cf. equations 2.10.b and 2.28). For this purpose $2N + 1$ relations are needed, of which $N + 1$ are provided by applying the boundary condition 2.10 on each of the $N + 1$ plates. In the expression 2.37 for $\tilde{f}$, we let the path of integration lie slightly to the right of the negative y-axis, i.e., $p = +0 + i\eta$, then on any plate,

$$\operatorname{Im} \tilde{f}(iy) = \tilde{\psi}(o, y) = e^{-\alpha y} \int_{-\infty}^{y} d\eta\, e^{\alpha\eta} \operatorname{Re} F^{+}(i\eta) \ .$$

If C_n denotes the complement of L on the portion of y-axis between $-\infty$ and the upper limit of integration, y, i.e., if $-b_n \geqq y \geqq -a_n$, then,

$$C_n = \sum_{m=n}^{N} (-a_m > \eta > -b_{m+1}) \quad \text{with } b_{N+1} = \infty \tag{2.39}$$

and if the definition of Φ_L given by equation 2.13 is extended for all $y \leq 0$, then from the boundary condition that $\mathrm{Re}\, F^+(i\eta) = \Phi^+(\eta) = \Phi_L(\eta)$ for η on L, one gets

$$\tilde{\psi}(o,y) = e^{-\alpha y} \int_{-\infty}^{y} d\eta\, e^{\alpha\eta} \Phi_L(\eta) + e^{-\alpha y} \int_{C_n} d\eta\, e^{\alpha\eta} [\Phi^+(\eta) - \Phi_L(\eta)] \ .$$

Since by equation 2.13 and partial integration

$$e^{-\alpha y} \int_{-\infty}^{y} d\eta\, e^{\alpha\eta} \Phi_L(\eta) = e^{-\alpha y} \int_{-\infty}^{y} d\eta\, e^{\alpha\eta} \left(\alpha + \frac{\partial}{\partial\eta}\right) \tilde{\psi}_L(\eta) = \tilde{\psi}_L(y) \ ,$$

which reduces the previous equation to

$$\tilde{\psi}(o,y) = \tilde{\psi}_L(y) + e^{-\alpha y} \int_{C_n} d\eta\, e^{\alpha\eta} [\Phi(\eta) - \Phi_L(\eta)] \ .$$

Thus the boundary condition 2.10 is satisfied on every plate if

$$\int_{C_n} d\eta\, e^{\alpha\eta} [\Phi(\eta) - \Phi_L(\eta)] = 0 \ , \qquad n = 0, 1, 2, \ldots N \ . \tag{2.40}$$

This provides N + 1 linear algebraic equations. Since in each relation above the path of integration does not involve either side of the plates, the choice made at the beginning of this section to integrate along the right side of y-axis is clearly immaterial.

In addition to the ones just given, N more conditions are needed. We shall now make the physical assumption that the circulation around any closed contour, which may enclose an arbitrary number of the submerged plates, vanishes. Since,

$$\text{circulation} = \oint \vec{q} \cdot \vec{ds} = \mathrm{Re} \oint \frac{df}{dz}\, dz \ ,$$

our assumption implies that $\mathrm{Re}\, f(z)$ must be a single-valued function of position (x, y), which in turn implies that

$$\mathrm{Re}\ \{e^{i\alpha z} \oint dp e^{-i\alpha p} F(p)\} = 0$$

in view of equation 2.37. In particular, if the boundary contour of a submerged plate, L_n, is chosen to be the closed path of the above integral, it follows that

$$\int_{-a_n}^{-b} dy \quad e^{\alpha y}\ \Psi^{+}(y) = 0 \qquad n = 1, 2, 3, \ldots N \tag{2.41}$$

where use has been made of the oddness of $\Psi(x, y)$ in x (cf. equation 2.23). This provides N conditions which, together with equation 2.40, form a set of $2N + 1$ simultaneous equations for an equal number of unknowns $\{A_{2n+1}\}$ and $\{\tilde{\psi}_n\}$. The solution of these equations gives in principle the complete formal solution of the problem.

The preceding assumption implies that the effect of viscosity is completely ignored. Let us however, examine the actual situation of the flow near the sharp edge of a plate. When the plate first starts to move transversely in a given direction, relative to the wall a particle travelling nearby is accelerated and experiences a decreasing pressure on the upstream side, and is not seriously affected by viscosity since the boundary layer is very thin at the early instants of the motion. As soon as it passes over the sharp edge it is decelerated and enters an increasing pressure field and a boundary layer which has grown thicker. Due to the additional loss of kinetic energy in the form of viscous dissipation, the particle may now find itself unable to overcome the adverse pressure gradient and be driven back. Thus a reverse flow is created and a vortex formed on the downstream side of the edge. As the plate oscillates back and forth, vortices may be continuously generated in this way on both sides of the edge. Moreover, under the combined influences of the plate and the mutual induction between the vortices, these vortices move about and gradually disperse in a very complicated manner. Hence even in the region at some distance away from the plate, our assumption of zero circulation along a closed path is not strictly valid. However, it is known experimentally that such a separation phenomenon does not occur for motions of sufficiently small amplitudes and high frequencies. It is hoped that under a wider range of circumstances our solution will also be valid with errors of only secondary importance.

2.5 Alternative representation of $\tilde{f}$

The solution in the form of equation 2.37 is not always convenient for the general purpose of later applications and an alternative expression will now be derived. By Poisson's formula for a half plane:

$$F_2(z) = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{\Psi_2^+(\eta)\, d\eta}{z - i\eta} + \Phi_{2\infty} \qquad x > 0$$

and

$$F_2(z) = -\frac{i}{\pi} \int_{-\infty}^{\infty} \frac{\Psi_2^-(\eta)\, d\eta}{z - i\eta} + \Phi_{2\infty} \qquad x < 0$$

where

$$\Psi_2^{\pm}(y) = \lim_{x \to \pm 0} \operatorname{Im} F_2(z) \quad . \tag{2.42}$$

Since from equation 2.23, $\Psi_2^+ = -\Psi_2^-$, one may write

$$F_2(z) = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{\Psi_2^+(\eta)d\eta}{z-i\eta} + \Phi_{2\infty} = -\frac{i}{\pi} \int_{-\infty}^{\infty} \frac{\Psi_2^-(\eta)d\eta}{z-i\eta} + \Phi_{2\infty}$$

for all z. Furthermore F_2 suffers a jump only across the branch cuts $L + \bar{L}$ and not the remaining part of the y-axis, therefore,

$$\Psi_2^+ = \Psi_2^- = 0 \quad , \qquad \text{for } y \text{ not on } L + \bar{L} \ ,$$

$$\Psi_2^+ = -\Psi_2^- = \frac{1}{2}(\Psi_2^+ - \Psi_2^-) = \frac{1}{2}[\Psi_2]_-^+ \quad \text{for } y \text{ on } L + \bar{L}$$

and

$$F_2(z) = \Phi_{2\infty} + \frac{i}{2\pi} \int_{L+\bar{L}} \frac{[\Psi_2]_-^+ d\eta}{z-i\eta} \quad .$$

It follows from equation 2.21 that $\Psi_2^{\pm}(y) = -\Psi_2^{\pm}(-y)$, hence,

$$F_2(z) = \Phi_{2\infty} + \frac{i}{2\pi} \int_L d\eta [\Psi_2]_-^+ \left(\frac{1}{z-i\eta} - \frac{1}{z+i\eta}\right).$$

Combining with equation 2.17 one obtains

$$\begin{aligned} F(z) &= F_1(z) + F_2(z) \\ &= \Phi_\infty + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\Psi_o(x_o)dx_o}{z-x_o} + \frac{i}{2\pi} \int_L d\eta [\Psi_2]_-^+ \left(\frac{1}{z-i\eta} - \frac{1}{z+i\eta}\right) \end{aligned} \tag{2.43}$$

in which the quantity $[\Psi_2]_-^+$ can be replaced by $[\Psi]_-^+$ since

$$[\Psi_2]_-^+ = [\Psi]_-^+ - [\Psi_1]_-^+ = [\Psi]_-^+ \tag{2.44}$$

as a consequence of the continuity of Ψ_1. Substituting into equation 2.37 one has

$$\widetilde{f}(z) = \frac{i\Phi_\infty}{\alpha} + \frac{1}{\pi}\int_{-\infty}^{\infty} dx_o \Psi_o(x_o) \int_{-i\infty}^{z} dp \frac{e^{i\alpha(z-p)}}{p-x_o}$$

$$+ \frac{i}{2\pi}\int_{i\infty}^{z} dp e^{i\alpha(z-p)} \int_L d\eta\, [\Psi(\eta)]_-^+ \left(\frac{1}{p-i\eta} - \frac{1}{p+i\eta}\right)$$

$$= \frac{i\Phi_\infty}{\alpha} + \frac{1}{\pi}\int_{-\infty}^{\infty} dx_o \Psi_o(x_o) \int_{-i\infty}^{z} dp \frac{e^{i\alpha(z-p)}}{p-x_o}$$

$$+ \frac{i}{2\pi}\int_{-i\infty}^{z} dp e^{i\alpha(z-p)} \int_L d\eta \left(\frac{1}{p+i\eta} - \frac{1}{p-i\eta}\right)\left(\alpha + \frac{\partial}{\partial\eta}\right)[\widetilde{\phi}]_-^+$$

(2.45)

after using equation 2.11. The imaginary constant $i\phi_\infty/\alpha$, like the arbitrary constant $\widetilde{\psi}_o$ in equation 2.10, is of no significance whatsoever and will be ignored hereafter. By partial integration,

$$\int_{-i\infty}^{z} dp e^{i\alpha(z-p)} \int_L d\eta \; \frac{1}{p+i\eta} - \frac{1}{p-i\eta} \; \frac{\partial}{\partial\eta} \; [\widetilde{\phi}]_-^+$$

$$= - \int_{-i\infty}^{z} dp e^{i\alpha(z-p)} \int_L d\eta [\widetilde{\phi}]_-^+ \frac{\partial}{\partial\eta}\left(\frac{1}{p+i\eta} - \frac{1}{p-i\eta}\right)$$

$$= - \int_{-i\infty}^{z} dp e^{i\alpha(z-p)} \int_L d\eta [\widetilde{\phi}]_-^+ \frac{\partial}{\partial p}\left(\frac{1}{p+i\eta} + \frac{1}{p-i\eta}\right)$$

$$= - i \int_L d\eta [\widetilde{\phi}]_-^+ \left\{\frac{1}{z+i\eta} + \frac{1}{z-i\eta} + i\alpha \int_{-i\infty}^{z} dp e^{i\alpha(z-p)} \cdot \left(\frac{1}{p+i\eta} + \frac{1}{p-i\eta}\right)\right\}$$

where use has been made of the fact that $[\widetilde{\phi}]_-^+ = 0$ at the edges of the plates. Thus equation 2.45 may be expressed as

$$\tilde{f}(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} dx_o \Psi_o(x_o) \int_{-i\infty}^{z} \frac{dp}{p-x_o} e^{i\alpha(z-p)}$$

$$+ \frac{1}{2\pi}\int_L d\eta [\tilde{\phi}]_-^+ \left\{ i\alpha \int_{-i\infty}^{z} dp \left(\frac{1}{p+i\eta} - \frac{1}{p-i\eta} \right) e^{i\alpha(z-p)} \right.$$

$$\left. + i\alpha \int_{-i\infty}^{z} dp \left(\frac{1}{p+i\eta} + \frac{1}{p-i\eta} \right) e^{i\alpha(z-p)} + \frac{1}{z+i\eta} + \frac{1}{z-i\eta} \right\}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} dx_o \Psi_o(x_o) \int_{-i\infty}^{z} \frac{dp}{p-x_o} e^{i\alpha(z-p)} + \frac{1}{2\pi}\frac{\partial}{\partial z}\int_L d\eta [\tilde{\phi}]_-^+ \left\{ \log \frac{z-i\eta}{z+i\eta} \right.$$

$$\left. +2 \int_{-i\infty}^{z} \frac{dp}{p+i\eta} e^{i\alpha(z-p)} \right\} .$$

The transformation from p to k by the relations

$$\alpha(p-z) = k(z-x_o) \qquad \text{and} \qquad \alpha(p-z) = k(z+i\eta)$$

for the first and the second integrals respectively finally gives the following result

$$\tilde{f}(z) = \frac{-1}{\pi}\int_{-\infty}^{\infty} dx_o \Psi_o(x_o) \int_0^{\infty} \frac{dk}{k+\alpha} e^{-ik(z-x_o)}$$

$$+ \frac{1}{2\pi}\frac{\partial}{\partial z}\int_L d\eta [\tilde{\phi}]_-^+ \left\{ \log \frac{z-i\eta}{z+i\eta} - 2\int_0^{\infty} \frac{dk}{k+\alpha} e^{-ik(z+i\eta)} \right\} .$$

(2.46)

The velocity potential is the real part of the preceding formula,

$$\tilde{\phi}(x,y) = -\frac{1}{\pi}\int_{-\infty}^{\infty} dx_o \Psi_o(x_o) \int_o^{\infty} \frac{dk}{k+\alpha}\; e^{ky} \cos k(x-x_o)$$

$$-\frac{1}{2\pi}\frac{\partial}{\partial x}\int_L d\eta [\tilde{\phi}]_-^+ \left\{ \log \frac{R^*}{R} + 2\int_o^{\infty} \frac{dk}{k+\alpha}\; e^{k(y+\eta)} \cos kx \right\}$$

(2.47)

with

$$R^2 = x^2+(y-\eta)^2 \qquad \text{and} \qquad R^{*2} = x^2+(y+\eta)^2 \tag{2.48}$$

It may be remarked that the formal representation of $\tilde{\phi}(x,y)$ given in equation 2.47 can also be obtained by the Green's function method (cf. Chapter V) or the method of Fourier transform.

In using the result just obtained the knowledge of $[\tilde{\phi}]_-^+$ on the plates is needed. It is thus desirable to have a general expression for $[\tilde{\phi}]_-^+$ in terms of more explicit result already deduced for F. Referring to figure 1 and recalling equation 2.37 and that $\Psi_2^+ = -\Psi_2^-$ on L, we get for $-b_n \geqq y \geqq -a_n$ $n = 0, 1, 2, \ldots N$

$$[\tilde{\phi}(y)]_-^+ = [\tilde{f}(iy)]_-^+ = e^{-\alpha y}\int_C dp e^{-i\alpha p}[F_1(p) + F_2(p)]$$

$$= -e^{-\alpha y}\left\{ \int_{-a_n}^{y} d\eta e^{\alpha\eta}\Psi_2^+(\eta) + \int_y^{-a_n} d\eta e^{\alpha\eta}\Psi_2^-(\eta) \right\}$$

$$= -2e^{-\alpha y}\int_{-a_n}^{y} d\eta e^{\alpha\eta}\Psi_2^+(\eta) \tag{2.49}$$

III. RADIATION OF TRANSIENT GRAVITY WAVES BY A VERTICAL PLATE ROLLING IN THE WATER SURFACE; LARGE TIME ASYMPTOTIC BEHAVIOR

Because of the linearity of the problem the effects of disturbances of various origin may be treated individually and later superposed in a proper manner whenever the combined influence is desired. In this chapter we shall study in greater detail a special case of the general result obtained in the previous chapter, namely the class of problems in which all surface disturbances vanish at all time

$$p_o(x,t) = \phi_o(x) = \zeta_o(x) = 0 \quad ; \tag{3.1}$$

the only sources of energy are the plates whose forced motion induces in water gravity waves propagating away from the plates. We further restrict the subsequent discussion to the case of a single surface-piercing plate.

The steady state problem of a vertical plate rolling simple-harmonically in the surface of an infinitely deep ocean has been solved by Ursell (reference 12) and investigated further by Haskind (reference 13). We shall, however, be interested in the transient waves created by a surface piercing plate rolling either (i) impulsively or (ii) simple-harmonically. In both cases the motion of the plate is assumed to start at $t = 0_+$ in a fluid initially at rest. Let the axis of rolling be at a depth c below the mean water surface, the boundary condition on the plate L can then be expressed as,

$$\begin{aligned} \frac{\partial\phi}{\partial x}(o,y,t) = \frac{\partial\psi}{\partial y}(o,y,t) &= 0 \qquad t \leqq 0 \\ &= \eta(t)(c+y), \quad t > 0 \end{aligned} \qquad 0 \geqq y \geqq -a \tag{3.2}$$

where,

$$\tau(t) = \theta_o \delta(t-o+) \quad \text{in Case (i)} \tag{3.2.i}$$

and,

$$\tau(t) = -j\theta_o \omega e^{-j\omega t} \quad \text{in Case (ii)} . \tag{3.2.ii}$$

Integrating the Laplace transform of the above condition and omitting an arbitrary function of s which bears no effect on the velocity field, one has,

$$\widetilde{\psi}(y,s) = \widetilde{\psi}_L(y) = T(s)(cy + \tfrac{1}{2}y^2) \ , \tag{3.3}$$

with $\quad T(s) = \theta_o \quad$ in Case (i) (3.3.i)

and $\quad = -j\omega\theta_o/(s+j\omega) \quad$ in Case (ii) (3.3.ii)

One may now use these boundary values to calculate explicitly the auxiliary function $F(z,s)$. An immediate simplification due to equation 3.1 is that $\tilde{w}_o(x) = 0$ from equation 2.12 and hence $F_1(z) = \Phi_{1\infty}$ from equation 2.17. As has been observed earlier (see the remark after equation 2.46), a real constant in F results in $\widetilde{f}$ an imaginary constant of no importance, we may therefore ignore F_1 altogether and obtain

$$F(z) = F_2(z) \ . \tag{3.4}$$

Equations 3.4 and 2.13 together give

$$\Phi_L(y,s) = \Phi_{2L}(y,s) = \left(\frac{\partial}{\partial y} + \alpha\right)\widetilde{\psi}_L = T\left[c + (1+2c)y + \alpha y^2/2\right] \ . \tag{3.5}$$

Since there is only one surface-piercing plate, equations 2.28 and 2.30 are much simplified, i.e.,

$$P(z)/D(z) = A_1 z(z^2+a^2)^{-\frac{1}{2}} \tag{3.6.a}$$

and

$$K(z) = (z^2+a^2)^{\frac{1}{2}} \ . \tag{3.6.b}$$

Subetituting into equation 2.36 and carrying out the elementary integrations, one gets,

$$F(z) = \frac{TAz}{\sqrt{z^2+a^2}} + \frac{1}{\pi}\, T(1+\alpha c) z \log \frac{\sqrt{z^2+a^2}-a}{\sqrt{z^2+a^2}+a} + \frac{1}{2}\, T\alpha(z\sqrt{z^2+a^2}-z^2) + Tc \tag{3.7}$$

in which a slight change of notation has been made to replace $A_1(s)$ by $T(s)A(s)$. The constant term Tc may be discarded in the sequel again for the reason just mentioned. In order to find A we first observe that due to the absence of totally submerged plates, equations 2.41 are not needed at all and A can be found from equation 2.40 alone. It follows from equation 3.7 that on C_o, i.e., $-\infty < y \leq -a$, $x = 0$

$$\Phi(o,y) = \frac{-TAy}{\sqrt{y^2-a^2}} + T[c+(1+\alpha c)y+\alpha y^2/2] + T[\frac{\alpha y}{2}\sqrt{y^2-a^2} - \frac{2}{\pi}(1+\alpha c)y \cos^{-1} a/y] \ .$$

Making use of this result in equation 2.40 we obtain a single equation for A:

$$A\int_{-\infty}^{-a} \frac{dy\, y e^{\alpha y}}{\sqrt{y^2-a^2}} = \int_{-\infty}^{-a} dy\, e^{\alpha y} \left[\frac{\alpha}{2}\, y\sqrt{y^2-a^2} - \frac{2}{\pi}(1+\alpha c) y \cos^{-1} a/y\right] \ .$$

The integrals involved above can all be evaluated in terms of the modified Bessel functions $K_n(\alpha a)$ and the Struve functions $L_n(\alpha a)$; the result is as follows (see Appendix A.1).

$$A(s) = A(\alpha a) = \frac{a}{K_1(\alpha a)} \left\{ \frac{1}{2} K_2(\alpha a) + \frac{1+\frac{c}{a}(\alpha a)}{\alpha a} \left[K_o(\alpha a) L_1(\alpha a) \right.\right.$$

$$\left.\left. + L_o(\alpha a) K_1(\alpha a) - \frac{1}{\alpha a} \right] \right\} . \qquad (3.8)$$

Another preliminary step for using equation 2.47 is the calculation of the difference $[\widetilde{\phi}(y)]^+_-$ for $0 \geqq y \geqq -a$. From equations 2.49 and 3.7 one has,

$$[\widetilde{\phi}(y)]^+_- = -2T e^{-\alpha y} \int_{-a}^{y} d\eta\, e^{\alpha\eta} G(\alpha, \eta) \quad \text{for } 0 \geqq y \geqq -a \qquad (3.9.a)$$

where,

$$G(\alpha, y) = \Psi^+(y)/T = -\Psi^-(y)/T$$

$$= \frac{Ay}{\sqrt{a^2-y^2}} + \frac{1}{\pi}(1+\alpha c) y \log \frac{a-\sqrt{a^2-y^2}}{a+\sqrt{a^2-y^2}} + \frac{\alpha}{2} y \sqrt{a^2-y^2} \qquad (3.9.b)$$

by taking the imaginary part of equation 3.7. Hence the function $\widetilde{\phi}$ assumes the following form

$$\widetilde{\phi}(x, y, x) = \frac{T(s)}{\pi} \frac{\partial}{\partial x} \int_{-a}^{o} d\eta\, e^{-\alpha n} \int_{-a}^{\eta} d\sigma\, e^{\alpha\sigma} G(\alpha, \sigma) \left\{ \log \frac{R^*}{R} \right.$$

$$\left. + 2 \int_o^{\infty} \frac{dk}{k+\alpha} e^{k(y+\eta)} \cos kx \right\} \qquad (3.10)$$

and its inversion gives the velocity potential,

$$\phi(x, y, t) = \frac{1}{2\pi j} \int_{\Gamma} ds\, e^{st} \widetilde{\phi}(x, y, s) \quad . \qquad (3.11)$$

This is an integral representation of the solution; it does not seem feasible to perform the inversion explicitly in closed form. One important and interesting information which can be derived from this solution is the asymptotic behavior of the transient waves long after the motion is begun. This will be investigated in the remainder of this section.

Upon inspecting equation 3.10 together with equations 3.3, ii, 3.8, 3.9, and 3.10 we observe the presence of the following singularities of $\tilde{\phi}(x,y,s)$ in the complex s-plane:

$$\text{a logarithmic branch point at } s_o = 0$$

$$\text{simple poles at} \begin{cases} s_1 = j\sqrt{gk} & \text{Cases (i) and (ii)} \\ s_2 = -j\sqrt{gk} & \\ s_3 = -j\omega & \text{Case (ii) only} \end{cases} \quad (3.12)$$

The branch point at s_o arises from the modified Bessel functions $K_n(\alpha a)$. We now make use of the following known theorem: If s_i is the singular point of $\tilde{\phi}(s)$ having the largest real part and if $\tilde{\phi}(s)$ can be expanded near s_i in the form

$$\tilde{\phi}(s) = \sum_{n=o} a_n(s-s_i)^{n-1} + \log(s-s_i) \sum_{n=o} b_n(s-s_i)^n$$

or, (3.13.a)

$$\tilde{\phi}(s) = \sum_{n=o} a_n(s-s_i)^{n-1} + (s-s_i)^{\beta-1} \sum_{n=o} b_n(s-s_i)^n, \quad 0 < \beta < 1$$

then the asymptotic expression of $\phi(t)$ for large t is

$$\phi(t) \simeq e^{s_i t} \left\{ a_o + \sum_{n=o}^{N} (-)^{n+1} b_n n! \, t^{-n-1} \right\}$$

or (3.13.b)

$$\phi(t) \simeq e^{s_i t} \left\{ a_o + \frac{1}{\pi} \sin\beta\pi \sum_{n=o}^{N} (-)^n b_n \Gamma(\beta+n) t^{-\beta-n} \right\}$$

respectively. If there exists several singular points and m of them have a common real part, $\mathrm{Re}_j s$, which is larger than those of all the rest, then a term as given by equation 3.13.b should be included for each of the m singular points. The detailed proof of the statement above is available elsewhere (see, e.g., reference 21) and will be omitted here. It should be noted however that the contribution due to the simple pole term alone, i.e., $a_o e^{s_i t}$ is valid for all t as it is obviously the exact result of Cauchy's integral theorem after properly closing the contour Γ on the left.

Since all four singular points s_o, s_1, s_2, s_3 (see equation 3.12) have zero real part, they must be all taken into account in the calculation of the asymptotic behavior of the solution for large t. It is straightforward to show from equation 3.10 that the expansion of $\tilde{\phi}(x,y,s)$ for small s (near $s_o = 0$) is

$$\tilde{\phi}(x,y,s) \sim B + B' s^4 \log s + \ldots$$

where B and B' are independent of s. Our theorem then gives for fixed x,y,

$$\phi_o(x,y,t) = 0(t^{-5}) \qquad (3.14)$$

and the corresponding surface height is

$$\zeta_o(x,t) = -\frac{1}{g}\frac{\partial}{\partial t}\phi_o(x,o^-,t) = 0(t^{-6}) \quad . \tag{3.15}$$

As will be seen later, these results are very insignificant compared with the simple pole terms at s_1, s_2, s_3.

For these simple poles we shall consider Cases (i) and (ii) separately for large t.

3.1 Case (i): impulsive rolling

Writing out explicitly, we have by Theorem 3.13

$$\phi(x,y,t) \cong \phi_1 + \phi_2 + 0(t^{-5}) \tag{3.16.a}$$

where

$$\begin{Bmatrix}\phi_1\\ \phi_2\end{Bmatrix} = \mp\frac{\theta_o}{\pi j}\int_o^\infty dk\sqrt{gk}\,e^{ky}\sin kx\,e^{\pm j\sqrt{gk}\,t}\left[\int_{-a}^{o}d\eta\,e^{2k\eta}\int_{-a}^{\eta}d\sigma e^{-k\sigma}G(ke^{\pm j\pi},\sigma)\right] \tag{3.16}$$

are the exact contributions from the residues at the poles s_1 and s_2 respectively as has been remarked before. In the foregoing expression the bracket can be evaluated in terms of Bessel and Struve functions. Leaving the details in Appendix A.2 we merely quote the following result:

$$\int_{-a}^{o}d\eta e^{2k\eta}\int_{-a}^{\eta}d\sigma\,e^{-k\sigma}G(ke^{\pm j\pi},\sigma) = \frac{j\pi a}{2k^2}\begin{Bmatrix}-V(ka)\\ \bar{V}(ka)\end{Bmatrix} \tag{3.17.a}$$

with

$$\begin{Bmatrix}V(X)\\ \bar{V}(X)\end{Bmatrix} = \frac{\frac{1}{2} - \dfrac{1-\frac{c}{a}X}{X}\left[I_1(X)+L_1(X)\right]}{\pi I_1(X) \mp jK_1} \tag{3.17.b}$$

Thus it follows after substitution and differentiation that the surface height is

$$\zeta(x,t) = -\frac{1}{g}\frac{\partial}{\partial t}\ \phi(x,o^-,t)$$

$$= \frac{g\theta_o a}{4}\int_o^{\infty}\frac{dk}{k}\left\{ V(ka)\left[e^{-j(kx-\sqrt{gk}\,t)}-e^{j(kx+\sqrt{gk}\,t)}\right]\right.$$

$$\left. + \bar{V}(ka)\left[e^{j(kx-\sqrt{gk}\,t)}-e^{-j(kx+\sqrt{gk}\,t)}\right]\right\} \qquad (3.18)$$

We are now in a position to explore the dispersion phenomenon of waves along fixed rays $\xi = x/t =$ constant for large t. The well-known method of stationary phase will be used. For simplicity all the following calculations will be made for positive x only. Consider first the exponential functions,

$$\exp\left[\pm j(kx-\sqrt{gk}\,t)\right] = \exp\left[\pm jt(k\xi-\sqrt{gk})\right] \equiv \exp jtu$$

where

$$u(k) = \pm(k\xi-\sqrt{gk}\,)\ , \qquad \xi = x/t\ .$$

Then

$$u'(k) = \pm\left(\xi-\frac{1}{2}\sqrt{\frac{g}{k}}\right) \qquad \text{and} \qquad u''(k) = \pm\sqrt{g}\ /\ 4k^{3/2}\ .$$

At the stationary point k_o, $u'(k_o) = 0$, so that $\sqrt{gk_o} = \frac{g}{2\xi}$, $k_o = g/4\xi^2$ and $u'(k_o) = \pm 2\xi^2/g$. Let $\sqrt{gk}$ be defined such that it is real and positive for positive real k and is one-valued in the complex k-plane cut along the negative real axis. Then it is evident that k_o lies in the range of integration $(0, \infty)$ since $\xi = x/t > 0$. Similar analysis for the other two exponentials in equation 3.18

$$\exp\left[\pm j(kx + \sqrt{gk}\,t)\right]$$

shows that there is no stationary point on the line $0 \leq k < \infty$. According to the theory of the stationary phase method, the first member of each bracket in equation 3.18 plays a more dominant role than the second since they make the major contribution to the following asymptotic result:

$$\zeta(x,t) \simeq 2\theta_o a \sqrt{\frac{\pi g}{t}}\ \ \xi\, \mathrm{Re}_j \left\{ V\left(\frac{ga}{4\xi^2}\right) e^{j\Theta} \right\} + O(t^{-1}) \qquad (3.19.a)$$

with

$$\Theta = \frac{gt}{2\xi} - \frac{gx}{4\xi^2} - \frac{\pi}{4} = \frac{gt^2}{4x} - \frac{\pi}{4} \quad . \qquad (3.19.b)$$

Clearly the waves are dispersive in nature, i.e., along any ray $\xi = x/t =$ constant one observes a train of simple harmonic waves with a speed 2ξ and an amplitude decaying with t like $t^{-\frac{1}{2}}$.

3.2 Case (ii): simple-harmonic rolling

Neglecting again the contribution from the logarithmic branch point s_o, we obtain from the three poles, s_1, s_2 and s_3, (cf. equations 3.3.ii, 3.10, 3.12 and 3.13)

$$\phi(x,y,t) = \phi_1 + \phi_2 + \phi_3 + O(t^{-5}) \qquad (3.20.a)$$

in which

$$\left\{ \begin{matrix} \phi_1 \\ \phi_2 \end{matrix} \right\} = \pm \frac{1}{\pi}\, j\theta_o \omega g\, \frac{\partial}{\partial x} \int_0^\infty \frac{e^{ky \pm j\sqrt{gk}\,t} \cos kx}{\sqrt{gk}\ (\omega \pm \sqrt{gk})} \int_{-a}^{0} d\eta$$

$$e^{2k\eta} \int_{-a}^{\eta} d\sigma\, e^{-k\sigma} G(ke^{\pm j\pi}, \sigma) \qquad (3.20.b)$$

and, with the new notation

$$\beta = \omega^2/g \quad , \tag{3.20.c}$$

$$\phi_3 = -\frac{1}{\pi} j\theta_o \omega e^{-j\omega t} \frac{\partial}{\partial x} \int_{-a}^{o} d\eta e^{\beta\eta} \int_{-a}^{\eta} d\sigma e^{-\beta\sigma} G(\beta e^{-j\pi}, \sigma) \cdot$$

$$\left\{ \log \frac{R^*}{R} + 2 \int_o^{\infty} \frac{dk}{k-\beta} e^{k(y+\eta)} \cos kx \right\} . \tag{3.20.d}$$

The k-integrands of ϕ_2 and ϕ_3 are both singular at $\sqrt{gk} = \omega$ or $k = \beta$, but actually when combined together the singularity is removed as can be seen from the above equations and

$$\frac{1}{\beta - k} = \frac{g}{2\omega} \left(\frac{1}{\omega - \sqrt{gk}} + \frac{1}{\omega + \sqrt{gk}} \right) .$$

Thus,

$$\phi_2 + \phi_3 = \frac{1}{\pi} j\theta_o \omega g \frac{\partial}{\partial x} \int_o^{\infty} dk \frac{e^{ky} \cos kx}{\omega - \sqrt{gk}} \left\{ \frac{e^{-j\omega t}}{\omega} \int_{-a}^{o} d\eta e^{(k+\beta)\eta} \int_{-a}^{\eta} d\sigma \right.$$

$$\left. e^{-\beta\sigma} G(\beta e^{-j\pi}, \sigma) - \frac{e^{-j\sqrt{gk}t}}{\sqrt{gk}} \int_{-a}^{o} d\eta e^{2k\eta} \int_{-a}^{\eta} d\sigma e^{-k\sigma} G(k e^{-j\pi}, \sigma) \right\} + \ldots$$

in which we have omitted the integral with the regular integrand. As $\sqrt{gk} \to \omega$ or $k \to \beta$, the curly bracket tends to zero and therefore cancels the simple pole in front. This enaües us to interpret the k-integrals of ϕ_2 and ϕ_3 separately by their Cauchy principal values.

By using the closed contour shown in figure 2 one can rewrite the following principal-valued integral

$$\int_o^{\infty} \frac{dk}{k-\beta} e^{ky} \cos kx = -\pi e^{\beta y} \sin \beta |x| + \int_o^{\infty} \frac{dk\, e^{-k|x|}}{k^2 + \beta^2} (k \cos ky + \beta \sin ky) \tag{3.21}$$

The integral on the right hand side and the term $\log R^*/R$ in 3.20.d have only local effect near the plate; their contributions diminish as x^{-1} with increasing $|x|$. Hence,

$$\phi_3 = 2j\theta_o\omega\beta(\mathrm{sgn}x)e^{\beta y - j\omega t}\cos\beta x \int_{-a}^{o} d\eta e^{2\beta\eta}\int_{-a}^{\eta} d\sigma e^{-\beta\sigma}G(\beta e^{-j\pi},\sigma)$$

$$+ O(|x|^{-2}) .$$

Making use of equation 3.17 one gets,

$$\phi_3(x,y,t) = -\frac{1}{\beta}\pi\omega\theta_o a\bar{V}(\beta a)(\mathrm{sgn}x)e^{\beta y - j\omega t}\cos\beta x + O(|x|^{-2}) \qquad (3.22)$$

and the corresponding surface elevation is

$$\zeta_3(x,t) = -\frac{1}{Z}\frac{\partial}{\partial t}\phi_3(x,o^-,t)$$

$$= -j\pi\theta_o a\bar{V}(\beta a)(\mathrm{sgn}x)e^{-j\omega t}\cos\beta x + O(|x|^{-2}) . \qquad (3.23)$$

Equations 3.20.b can also be simplified by applying equation 3.17,

$$\begin{Bmatrix}\phi_1\\ \phi_2\end{Bmatrix} = -\frac{1}{Z}\theta_o a\omega g\int_o^\infty dk\,\frac{\sin kx\, e^{ky \pm j\sqrt{gk}\,t}}{k\sqrt{gk}(\omega \mp \sqrt{gk})}\begin{Bmatrix}V(ka)\\ \bar{V}(ka)\end{Bmatrix} . \qquad (3.24)$$

The corresponding surface heights have the following expressions:

$$\begin{Bmatrix}\zeta_1\\ \zeta_2\end{Bmatrix} = -\frac{1}{g}\frac{\partial}{\partial t}\begin{Bmatrix}\phi_1\\ \phi_2\end{Bmatrix}_{y=o} = \pm\frac{1}{Z}j\theta_o a\omega\int_o^\infty\frac{dk\,\sin kx}{k(\omega \mp \sqrt{gk})}e^{\pm j\sqrt{gk}\,t}\begin{Bmatrix}V(ka)\\ \bar{V}(ka)\end{Bmatrix} .$$

(3.25)

Only the second integrals in equations 3.24 and 3.25 are to be interpreted by their principal values.

In view of the fact that ζ_3 is of order $O(1)$ as far as t is concerned, we shall study ζ_1 and ζ_2 up to the same order of magnitude only. For convenience, only $x > 0$ will be considered in all the calculations and results that follow. By the method of stationary phase one may again show that $\zeta_1 = O(t^{-\frac{1}{2}})$ for large t and x/t = constant. To treat the principal-valued integral for ζ_2 we first break up the path of integration into three parts, i.e.,

$$\zeta_2(x,t) = \zeta_{21} + \zeta_{22} + \zeta_{23}$$

$$= -\frac{1}{4}\theta_o a\omega \left\{ \int_0^{\beta-\delta} + \int_{\beta+\delta}^{\infty} + \int_{\beta-\delta}^{\beta+\delta} \right\} \frac{dk\overline{V}(ka)}{k(\omega-\sqrt{gk})} (e^{jkx} - e^{-jkx}) e^{-j\sqrt{gk}t} \tag{3.26}$$

where δ is a small positive quantity. ζ_{21} and ζ_{22} can again be shown to be of the order $O(t^{-\frac{1}{2}})$. Now consider,

$$\zeta_{23}(x,t) = -\frac{1}{4}\theta_o a\omega \int_{\beta-\delta}^{\beta+\delta} \frac{dk\,\overline{V}(ka)}{k(\omega-\sqrt{gk})} (e^{jkx} - e^{-jkx}) e^{-j\sqrt{gk}t}$$

$$= -\frac{1}{4}\theta_o a\omega (E_+ - E_-) \; . \tag{3.27}$$

Let $k = \beta(1+\epsilon)$, then

$$\sqrt{gk} = \omega(1+\epsilon)^{\frac{1}{2}} = \omega\left[1 + \frac{\epsilon^2}{2} - \frac{\epsilon^2}{8} + O(\epsilon^3)\right]$$

and

$$\exp\left[j(\pm kx - \sqrt{gk}\,t)\right] = \exp\left\{j\left[(\pm\beta x - \omega t) + t(\pm\beta\xi - \frac{\omega}{2})\epsilon + \frac{\omega t}{8}\epsilon^2 + O(\epsilon^3)\right]\right\} .$$

Thus the integrals in ζ_{23} becomes,

$$E_{\pm} = \int_{\beta-\delta}^{\beta+\delta} \frac{dk\,\overline{V}(ka)}{k(\omega-\sqrt{gk})} e^{\pm jkx - j\sqrt{gk}\,t}$$

$$= -\frac{2}{\omega} e^{j(\pm\beta x-\omega t)} \int_{-\delta/\beta}^{\delta/\beta} \frac{d\epsilon}{\epsilon} \overline{V}(\beta a)[1+O(\epsilon)]\exp\left\{jt\left[\left(\pm\beta\xi - \frac{\omega}{2}\right)\epsilon + \frac{\omega\epsilon^2}{8} + O(\epsilon^2)\right]\right\}.$$

Neglecting the order terms for the moment, we obtain after the change of variable $n = \sqrt{\frac{\omega t}{8}}\ \epsilon$, that

$$E_{\pm} \cong + \frac{4}{\omega} j e^{j(\pm\beta x-\omega t)}\overline{V}(\beta a) \int_{0}^{\frac{\delta}{\beta}\sqrt{\frac{\omega t}{8}}} \frac{dn}{n} e^{jn^2} \sin 2q_{\pm} n \tag{3.28.a}$$

$$q_{\pm} = \left(\frac{\omega}{2} \mp \beta\xi\right)\sqrt{\frac{2t}{\omega}} \quad . \tag{3.28.b}$$

Since $\frac{\delta}{\beta}\sqrt{\frac{\omega t}{8}} \to \infty$ with t, the integral in equation 3.28.a can be approximated by means of the following asymptotic relation (see reference 16 p. 21 for proof):

$$\int_{0}^{\frac{\delta}{\beta}\sqrt{\frac{\omega t}{8}}} \frac{dn}{n}\ e^{jn^2} \sin 2qn = \frac{\pi}{\sqrt{2}}\ e^{j\frac{\pi}{4}}\left[C\left(\sqrt{\frac{2}{\pi}}\ q\right) - jS\left(\sqrt{\frac{2}{\pi}}\ q\right)\right] + O(t^{-\frac{1}{2}}) \tag{3.29.a}$$

where

$$C(u) = \int_{0}^{u} \cos\frac{\pi m^2}{2}\ dm \quad \text{and} \quad S(u) = \int_{0}^{u} \sin\frac{\pi m^2}{2}\ dm \tag{3.29.b}$$

are the Fresnel integrals. Combining equation 3.27, 3.28 and 3.29 we obtain for large t

$$\zeta_{23} \cong j\ \frac{\pi}{\sqrt{2}}\ \theta_o a\overline{V}(\beta a) e^{j\frac{\pi}{4}}\left\{ e^{j(\beta x-\omega t)}\left[C\left(\sqrt{\frac{2}{\pi}}\ q_+\right) - jS\left(\sqrt{\frac{2}{\pi}}\ q_+\right)\right]\right.$$

$$\left. - e^{-j(\beta x+\omega t)}\left[C\left(\sqrt{\frac{2}{\pi}}\ q_-\right) - jS\left(\sqrt{\frac{2}{\pi}}\ q_-\right)\right]\right\}. \tag{3.30}$$

The error term that we have neglected is proportional to

$$\sim \int_{-\delta/\beta}^{\delta/\beta} d\epsilon \, \exp\left\{ jt \left[\left(\pm\beta\xi - \frac{\omega}{2} \right)\epsilon + \frac{\omega\epsilon^2}{8} \right] \right\}$$

$$\sim \frac{1}{\sqrt{t}} \int_0^{\frac{\delta}{\beta}\sqrt{\frac{\omega t}{8}}} dn \, e^{jn^2} \cos 2q_{\pm} n \; .$$

It can also be shown that the integral above is of order $0(1)$ (see reference 16 p.21), i.e.,

$$\int_0^{\frac{\delta}{\beta}\sqrt{\frac{\omega t}{8}}} dn \, e^{jn^2} \cos 2qn = \frac{\sqrt{\pi}}{2} \, e^{-j\left(q^2 - \frac{\pi}{4}\right)} + O(t^{-\frac{1}{2}}) \quad .$$

The information derived thus far can now be collected to give,

$$\begin{aligned} \zeta(x,t) &= \zeta_{2s} + \zeta_3 + O(t^{-\frac{1}{2}}) \\ &= -j\pi\theta_o a\overline{V}(\beta a) \left\{ \frac{1}{2} + \frac{1}{\sqrt{2}} e^{j\frac{\pi}{4}} \left[C\left(\sqrt{\frac{2}{\pi}}\, q_+\right) - jS\left(\sqrt{\frac{2}{\pi}}\, q_+\right) \right] \right\} e^{j(\beta x - \omega t)} \\ &\quad -j\pi\theta_o a\overline{V}(\beta a) \left\{ \frac{1}{2} - \frac{1}{\sqrt{2}} e^{j\frac{\pi}{4}} \left[C\left(\sqrt{\frac{2}{\pi}}\, q_-\right) - jS\left(\sqrt{\frac{2}{\pi}}\, q_-\right) \right] \right\} e^{-j(\beta x + \omega t)} \\ &\quad + O(t^{-\frac{1}{2}}) \end{aligned} \tag{3.30}$$

for t large. In the region where the above asymptotic result is valid, the second term represents a train of left-going waves the amplitude of which is, however, vanishingly small. This can be seen from the fact q_- is positive and large for all positive x and large t and that we have the following asymptotic expansions:

$$\left.\begin{matrix} C(u) \\ S(u) \end{matrix}\right\} = \frac{1}{2}\, \mathrm{sgn}\, u + 0(u^{-1}) \quad . \tag{3.31}$$

As a consequence, the bracket in the second term of equation 3.30 becomes

$$\frac{1}{2} - \frac{1}{\sqrt{2}} e^{j\frac{\pi}{4}}(1-j)\frac{1}{2}\,\mathrm{sgn}\,q_- + O(t^{-\frac{1}{2}}) = O(t^{-\frac{1}{2}})$$

and hence the leftward waves do not actually exist up to the first order. The expression for the surface height is finally reduced to a simpler form

$$\zeta(x,t) = -j\pi\theta_o a\bar{V}(\beta a)\left\{\frac{1}{2} + \frac{1}{\sqrt{2}} e^{j\frac{\pi}{4}}\left[C\left(\sqrt{\frac{2}{\pi}}\,q_+\right) - jS\left(\sqrt{\frac{2}{\pi}}\,q_+\right)\right]\right\} e^{j(\beta x-\omega t)}$$

$$+ O(t^{-\frac{1}{2}}, x^{-2}) \qquad (3.32)$$

In a region far away from the wave front, $x/t = g/2\omega$ = group velocity, the quantity $\frac{2}{\pi}|q_+| = 2\left|\frac{\omega t}{2} - \beta x\right| \sqrt{\pi\omega t} >> 1$, consequently equation 3.31 may be applied to approximate equation 3.32, giving

$$\zeta(x,t) = -j\pi\theta_o a\bar{V}(\beta a)\,H\left(\frac{\omega t}{2} - \beta x\right) e^{j(\beta x-\omega t)} + O(t^{-\frac{1}{2}}, |q_+|^{-1}, x^{-2}) \;. \qquad (3.33)$$

If it is further required that $\frac{1}{2}\,\omega t >> \beta x$, one observes a train of steady waves propagating to the right with the complex amplitude

$$A_S = -j\pi\theta_o a\bar{V}(\beta a) \qquad (3.34)$$

which is a result already obtained by Ursell (reference 12). On the other hand if $\frac{1}{2}\,\omega t << \beta x$, the surface height

$$\zeta = O(t^{-\frac{1}{2}}, \; |q_+|^{-1}, \; x^{-2}) \qquad (3.35)$$

will be small. Clearly the interior of the parabolic region:

$$\frac{2}{\pi}\,|q_+| = O(1) \qquad \text{or} \qquad |x - gt/2\omega| = O(t^{\frac{1}{2}}) \qquad (3.36)$$

defines a sort of transition zone relating the steady state and the transient state. Near the wave front, $\frac{2}{\pi}|q_+| << 1$. Hence, by using the properties of the Fresnel integrals,

$$C(u) = u + O(u^5)$$
$$S(u) = O(u^3) \qquad (3.37)$$

it is easy to show that near the wave front,

$$\zeta(x,t) = -\frac{1}{2} j\pi\theta_o a\bar{V}(\beta a)e^{j(\beta x-\omega t)} + O(t^{-\frac{1}{2}}, |q_+|, x^{-2}) \qquad (3.38)$$

namely the amplitude is only one half that of the steady waves.

As for the situation throughout the transition region we must study the curly bracket in equation 3.32 which is the amplitude ratio of the transient waves, A_t, to the steady state waves, A_s.:

$$A_t/A_s = \frac{1}{2} + \frac{1}{\sqrt{2}} e^{j\frac{\pi}{4}} \left[C\left(\sqrt{\frac{2}{\pi}}\, q_+\right) - jS\left(\sqrt{\frac{2}{\pi}}\, q_+\right)\right]$$
$$= \frac{1}{2}\left[1+(1+j)(C-jS)\right]$$
$$= \frac{1}{2} j\left\{\left[\left(\frac{1}{2}+C\right) - \left(\frac{1}{2}+S\right)\right] - j\left[\left(\frac{1}{2}+C\right) + \left(\frac{1}{2}+S\right)\right]\right\}$$

where the argument of C and S is understood to be

$$Q = \sqrt{\frac{2}{\pi}}\, q_+ = \frac{2}{\sqrt{\pi\omega t}}\left(\frac{\omega t}{2} - \beta x\right). \qquad (3.39)$$

Thus,

$$I(Q) = |A_t/A_s|^2 = \frac{1}{2}\left[\left(\frac{1}{2}+C\right)^2 + \left(\frac{1}{2}+S\right)^2\right]. \qquad (3.40)$$

This quantity is familiar in the Fresnel theory of optical diffraction (see reference 22 section 8.7, in particular p. 433). For the case of a half plane screen in the field of monochromatic light source, equation 3.40 represents the normalized intensity of illumination I(Q) at a point of observation where Q is a measure of the distance between the observer and the edge of the geometrical shadow. In the illuminated region $Q > 0$, whereas in the geometrical shadow $Q < 0$. The variation of $\sqrt{I(Q)}$ is shown graphically in figure 3. It is interesting to view the pheonmenon of the radiation of transient gravity waves in comparison with optical diffraction. Since in the former Q is a measure of the distance from the wave front ($Q = 0$ or $\beta x = \omega t/2$), the physical correspondence between the two cases is clear, i.e., the relatively undisturbed state to the geometrical shadow, the wave front to the boundary of the shadow and the highly disturbed state to the illuminated region. As the wave front approaches a stationary observer, Q increases from negative values; the quantity $|A_t/A_s|$, grows monotonically to $\frac{1}{2}$ at the front ($Q = 0$), rises to a peak shortly after the wave front passes, then oscillates with diminishing amplitude and eventually settles to the value 1 in the steady state.

Comparing the present example with the case of an oscillating point pressure investigated by Wu (reference 17) and later by Miles (reference 18) we may note that the transient responses are quite the same except that in the latter case the steady state wave amplitude A_s should be replaced by

$$+jP\omega^2/\rho g^2 \quad .$$

IV. SCATTERING OF SURFACE WAVES BY A STATIONARY VERTICAL BARRIER

In many scattering problems it is customary to assume that there is a steady train of simple harmonic waves, generated at infinity from a permanent energy source, incident upon obstacles situated in the finite part of space. Such steady state problems for vertical plate barriers in a deep ocean have received a rather comprehensive treatment from Dean (reference 10), Ursell (reference 11), Levine and Rodemich (reference 14), etc. In this chapter the scattering of surface waves will be investigated with a view to reveal the effects of disturbances not necessarily simple harmonic in time and originated within a finite past and a finite region from the obstacle. Specifically three kinds of surface disturbances will be dealt with: an initial impulse, an initial displacement and an oscillating pressure.

4.1 The fundamental solution

We consider only one surface-piercing vertical plate of finite depth a, i.e.,

$$L \; : \; x = 0, \qquad -a \leq y \leq 0,$$

which is held fixed in the presence of external disturbances. The linearity of the problem enables one to treat any general disturbance as a superposition of concentrated ones. It is therefore useful to construct a fundamental solution defined as the solution to the problem formulated in section 2.1 with the following special initial and boundary values

$$\psi(o,y,t) = \psi_L(y,t) = 0 \quad , \qquad t > 0 \quad y \text{ on } L \tag{4.1.a}$$

$$\phi(x,o-,o) = \phi_o(x) = Q_1\,\delta(x-x_1)/\rho \tag{4.1.b}$$

$$\zeta(x,o) = \zeta_o(x) = Q_2\,\delta(x-x_2) \tag{4.1.c}$$

and

$$p_o(x,t) = Q_3\,\tau(t)\delta(x-x_3) \qquad t > 0 \tag{4.1.d}$$

where the constants Q_1, Q_2 and Q_3 denote the total strength of the concentrated impulse, displacement and pressure on the free surface respectively.

Referring to equations 2.12 and 2.13 we now have the following boundary values for the auxiliary function F:

$$\operatorname{Im} F(z) = \sum_{m=1}^{3} T_m(s)\delta(x-x_m) = \Psi_o \qquad \text{on} \qquad y = 0 \tag{4.2.a}$$

where

$$T_1(s) = Q_1\,s/\rho g \tag{4.2.b}$$

$$T_2(s) = Q_2 \tag{4.2.c}$$

$$T_3(s) = Q_3\,s\tilde{\tau}(s)/\rho g \tag{4.2.d}$$

and,

$$\operatorname{Re} F(iy) = \Phi(o,y) = \Phi_L(y) = 0 \qquad \text{for} \quad y \text{ on } L\ . \tag{4.2.e}$$

It is obvious that we may first take

$$\operatorname{Im} F(z) = T_m\delta(x-x_m) = \Psi_o \qquad \text{on } y = 0 \tag{4.2.f}$$

instead of equation 4.2.a as our boundary value on the free surface and sum over m in the final step. Substituting 4.2.f into 2.17 we get,

$$F_1(z) = \frac{T_m}{\pi} \frac{1}{z - x_m} + \Phi_{1\infty} \tag{4.3}$$

and,

$$\Phi_1(o,y) = \Phi_{1\infty} - \frac{T_m}{\pi} \frac{x_m}{y^2 + x_m^2} \quad \text{for } y \text{ on } L .$$

Hence by equation 2.20, for y on L,

$$\Phi_2(o,y) = \Phi_L - \Phi_1 = \Phi_{2L} = -\Phi_{1\infty} + \frac{T_m}{\pi} \frac{x_m}{y^2 + x_m^2} . \tag{4.4}$$

Noting that the results in equation 3.6 is still valid in the present case, we may put 4.3 in equation 2.36 to obtain

$$F_2(z) = \frac{T_m}{\pi a} \frac{Az}{\sqrt{z^2 + a^2}} + \frac{T_m}{\pi} \frac{1}{z^2 - x_m^2} \left(\frac{z\sqrt{z^2+a^2}\,\mathrm{sgn}\, x_m}{\sqrt{x_m^2 + a^2}} - x_m \right) - \Phi_{1\infty} .$$

(4.5)

It then follows by combining equations 4.3 and 4.5 that,

$$F(z) = F_1 + F_2$$

$$= \frac{T_m}{\pi a} \frac{Az}{\sqrt{z^2 + a^2}} + \frac{T_m}{\pi} \frac{z}{z^2 - x_m^2} \left[1 + \frac{(\mathrm{sgn}\, x_m)\sqrt{z^2+a^2}}{\sqrt{x_m^2 + a^2}} \right] . \tag{4.6}$$

The coefficient A can be calculated from equation 2.40, and the fact that $\Phi_L(y) = 0$,

$$\int_{-\infty}^{-a} d\eta\, e^{\alpha\eta} \Phi(o,\eta) = -\frac{T_m}{\pi a} \int_{-\infty}^{-a} d\eta\, e^{\alpha\eta} \left[\frac{A\eta}{\sqrt{\eta^2 - a^2}} + \frac{a\eta\sqrt{\eta^2 - a^2}\,\mathrm{sgn}\, x_m}{(\eta^2 + x_m^2)\sqrt{x_m^2 + a^2}} \right] = 0$$

or

$$A = \frac{\operatorname{sgn} x_m}{K_1(\alpha a)} \frac{1}{\sqrt{x_m^2 + a^2}} \int_{-\infty}^{-a} d\eta e^{\alpha\eta} \frac{\eta\sqrt{\eta^2 - a^2}}{\eta^2 + x_m^2} \tag{4.7}$$

whereas the function $\Psi_2^+(y)$ needed in computing the difference $[\tilde{\phi}]_-^+$ of equation 2.49 is simply

$$\Psi_2^+(y) = \frac{T_m}{\pi a} \frac{Ay}{\sqrt{a^2 - y^2}} - \frac{(\operatorname{sgn} x_m) a y \sqrt{a^2 - y^2}}{x_m^2 + a^2 \;\; (y^2 + x_m^2)} \qquad -a \leq y \leq 0 \;. \tag{4.8}$$

Finally the fundamental solution including only one kind of surface disturbance is

$$\phi_m(x, x_m, y, t) = \frac{1}{2\pi j} \int_\Gamma ds e^{st} T_m(s) \left\{ -\frac{1}{\pi} \int_0^\infty \frac{dk}{k+\alpha} e^{ky} \cos k(x - x_m) \right.$$

$$\left. - \frac{1}{2\pi} \frac{\partial}{\partial x} \int_{-a}^{0} d\eta [\tilde{\phi}]_-^+ \left[\log \frac{R^*}{R} + 2 \int_0^\infty \frac{dk}{k+\alpha} e^{k(y+\eta)} \cos kx \right] \right\} \tag{4.9}$$

with $[\tilde{\phi}]_-^+$ given by equations 2.49 and 4.8.

The solution for general disturbances is formally obtained by superposition:

$$\phi(x, y, t) = \sum_{m=1}^{3} \int_{-\infty}^{\infty} dx_m Q_m(x_m) \phi_m^0(x, x_m, y, t) \tag{4.10}$$

where $Q_m(x)$ is the strength per unit length of the free surface and ϕ_m^0 is given by normalizing equation 4.9 for unit strength.

In order to obtain the solution in an explicit form it is however desirable to confine ourselves to a rather special case when the concentrated disturbance is confined to the immediate neighborhood of the plate, say $x_m = 0_+$. Thus from equations 4.7 and 4.8,

$$\Psi_2^+(y) = \frac{T_m}{\pi a}\left[\frac{Ay}{\sqrt{a^2-y^2}} - \frac{\sqrt{a^2-y^2}}{y}\right] \tag{4.11}$$

and

$$A(\alpha a) = \frac{\pi/2}{K_1(\alpha a)}\left\{1-\alpha a[K_1(\alpha a)L_{-2}(\alpha a) + K_2(\alpha a)L_{-1}(\alpha a)]\right\} . \tag{4.12}$$

The detailed integration for the result above is performed in Appendix A. 3. In the following sections we proceed to study this particular case for the three types of disturbances separately.

4.2 Cauchy-Poisson problem for an initial impulse

Purely transient waves will arise when the water surface is acted upon by an impulsive pressure for a short duration. In the present case we let the initial displacement and the surface pressure vanish identically, thus we have

$$T_1 = -Q_1 s/\rho g \ , \quad T_2 = T_3 = 0 \ . \tag{4.13}$$

From equation 4.9 the velocity potential is therefore,

$$\begin{aligned} \frac{\rho g}{Q_1}\,\phi(x,y,t) = &+ \frac{1}{2\pi j}\int_\Gamma ds\, e^{st}\,\frac{s}{\pi}\int_0^\infty dk\,\frac{e^{ky}\cos kx}{k+\alpha} \\ &+ \frac{1}{2\pi j}\, ds\, e^{st}\,\frac{s}{2\pi^2}\int_{-a}^{0} d\eta[\tilde{\phi}(\eta)]_-^+\,\frac{\partial}{\partial x}\log\frac{R^*}{R} \\ &+ \frac{1}{2\pi j}\int_\Gamma ds\, e^{st}\,\frac{s}{\pi^2}\int_{-a}^{0} d\eta\,[\tilde{\phi}(\eta)]_-^+\int_0^\infty \frac{dk}{k+\alpha}\, e^{k(y+\eta)}k\sin kx \ . \end{aligned} \tag{4.14}$$

As was mentioned earlier the first term on the right side of the preceding expression represents the potential without the barrier, of which the inversion can be performed to give the classical Cauchy-Poisson result (reference 15):

$$\phi_I(x,y,t) = \frac{Q_1}{\pi\rho} \int_0^\infty dk\, e^{ky} \cos kx \cos\sqrt{gk}\,t \qquad (4.15)$$

The corresponding surface elevation is

$$\zeta_I(w,t) = -\frac{1}{g}\frac{\partial}{\partial t}\,\phi_I(x,0-,t) = \frac{Q_1}{\pi\rho g}\int_0^\infty dk\sqrt{gk}\cos kx \sin\sqrt{gk}\,t\ . \qquad (4.16)$$

For large t and fixed $\xi = x/t$, one finds

$$\zeta_I(x,t) = \frac{Q_1}{4\rho}\sqrt{\frac{g}{\pi t}}\,|\xi|^{-5/2}\sin\Theta + O(\frac{1}{t}) \qquad (4.17)$$

where Θ is given by equation 3.19.b.

The second integral in equation 4.14 represents a local effect which is significant only in a small neighborhood of the origin and dies out like $1/x^2$ for all t; it will therefore be ignored.

The asymptotic expression valid for large t can be obtained for the third term in equation 4.14 by considering the three singular points having the largest real part, i.e., $s = 0$ and $\pm j\sqrt{gk}$. It can again be shown that at $s = 0$ there is only a logarithmic branch point whose contribution to the surface elevation is of a negligibly small magnitude: $O(t^{-7})$. As for the two poles at $s = j\sqrt{gk}$ and $-j\sqrt{gk}$, we can follow the procedure used in section 3.1 by first rewriting the third term as

$$\phi_m(x,y,t) = \frac{2Q_1}{\pi^2 a\rho g}\,\frac{1}{2\pi j}\int_\Gamma ds\, e^{st}s \int_0^\infty dk\, k\,\frac{\sin kx\, e^{ky}}{k+\alpha}\int_{-a}^{0} d\eta\, e^{(k-\alpha)\eta} \int_{-a}^{\eta} d\sigma\, e^{\alpha\sigma} G(\alpha,\sigma) \qquad (4.18.a)$$

where

$$G(\alpha,\sigma) = \frac{A(\alpha)\sigma}{\sqrt{a^2-\sigma^2}} - \frac{\sqrt{a^2-\sigma^2}}{\sigma} \quad . \tag{4.18.b}$$

The corresponding surface elevation is

$$\zeta_{III}(x,t) = -\frac{1}{g}\frac{\partial}{\partial t}\Phi_{III}(x,o-,t)$$

$$= \frac{-2Q_1}{\pi^2 a\rho g}\frac{1}{2\pi j}\int_\Gamma ds\, e^{st} s^2 \int_o^\infty dk\, \frac{k \sin kx}{s^2+gk}\int_{-a}^{o} d\eta\, e^{(k-\alpha)\eta}\int_{-a}^{\eta} d\sigma\, e^{\alpha\sigma} G(\alpha,\sigma)\,. \tag{4.19}$$

Applying the asymptotic theorem of section 3 (equation 3.13.a and b) to the two poles at $s = \pm j\sqrt{gk}$ and integrating by parts, we get,

$$\zeta_{III}(x,t) = \zeta_+ + \zeta_- + O(t^{-7}) \tag{4.20.a}$$

in which

$$\zeta_{\pm} = \pm\frac{jQ_1}{\pi^2 a\rho g}\int_o^\infty dk\,\sqrt{gk}\, e^{\pm j\sqrt{gk}\,t} \sin kx \int_{-a}^{o} d\eta \sin \quad k\eta G(ke^{\pm j\pi},\eta) \tag{4.20.b}$$

where the order term refers to the neglected part from $s = 0$. The last integral is evaluated in Appendix A4 from which the following result is quoted:

$$\int_{-a}^{o} d\eta \sinh k\eta \left[\frac{A(ke^{\pm j\pi})\eta}{\sqrt{a^2-\eta^2}} - \frac{\sqrt{a^2-\eta^2}}{\eta}\right] = \pm j\,\frac{\pi^2 a}{4}\left\{\begin{matrix} V(ka) \\ \bar{V}(ka)\end{matrix}\right\} \tag{4.21.a}$$

where,

$$\left\{\begin{matrix} V(ka) \\ \bar{V}(ka)\end{matrix}\right\} = \frac{I_1(ka)+L_{-1}(ka)}{\pi I_1(ka) \mp jK_1(ka)} \tag{4.21.b}$$

Putting this into the expression for $\zeta_{\pm}$ and applying the method of stationary phase as in sectinn 3.1, we can then combine with ζ_I to obtain,

$$\zeta(x,t) = \frac{Q_1}{4\rho}\sqrt{\frac{g}{\pi t}}\ |\xi|^{-5/2}\left\{\sin\Theta + \frac{\pi}{2}(\operatorname{sgn} x)\ \mathrm{Im}_j\left[V\left(\frac{ga}{4\xi^2}\right)e^{j\Theta}\right]\right\} + O(t^{-1}). \quad (4.22)$$

From the following properties of Bessel and Struve functions (references 23 and 24),

$$I(X),\ L_{-1}(X) \to \frac{e^{X}}{2\pi X} \quad \text{and} \quad K_1(X) \to \frac{\pi}{2X}\, e^{-X} \quad \text{as} \quad X \to \infty$$

and

$$I_1(X) \to \frac{1}{2}X,\ \ L_{-1}(X) \to \frac{1}{\pi} \quad \text{and} \quad K_1(X) \to \frac{1}{X} \quad \text{as} \quad X \to 0,$$

one can show that

$$V(X) \to 0 \quad \text{as} \quad X \to 0 \quad \text{and}$$

$$\to 2/\pi \quad \text{as} \quad X \to \infty\ . \quad (4.23)$$

Thus, when there is no barrier, i.e., $a = 0$,

$$\zeta(x,t) = \frac{Q_1}{4\rho}\sqrt{\frac{g}{\pi}}\ \frac{t^2}{|x|^{5/2}}\ \sin\Theta + O(t^{-1}) \quad (4.24)$$

and when the barrier extends down to the bottom of the ocean, i.e., $a \to \infty$,

$$\zeta(x,t) = \frac{Q_1}{4\rho}\sqrt{\frac{g}{\pi}}\ \frac{t^2}{|x|^{5/2}}\ (1+\operatorname{sgn} x)\sin\Theta + O(t^{-1})\ , \quad (4.25)$$

indicating no transmission to the left and perfect reflection to the right. Comparing the two limiting cases one finds that in the latter the amplitude of waves traveling to the right is twice in magnitude as in the former. This agrees with the simple fact that mathematically the cliff is equivalent to an additional image disturbance of the same strength applied on the free surface at $x = 0-$ and $t = 0$. For an observer not too close to the origin, the two concentrated disturbances may of course be replaced by a single one with twice the strength.

4.3 Cauchy-Poisson problem for an initial displacement

In this case $T_1 = T_3 = 0$ and $T_2 = Q_2$. An analysis similar to the one given in the foregoing section leads to the following result for large t and constant $\xi = x/t$:

$$\zeta(x,t) \simeq \frac{Q_2}{2}\sqrt{\frac{g}{\pi t}}\ |\xi|^{-3/2}\left\{\cos\Theta + \frac{\pi}{2}(\operatorname{sgn} x)\mathrm{Re}_j\left[V\left(\frac{ga}{4\xi^2}\right)e^{j\Theta}\right]\right\} + O(t^{-1}) \tag{4.26}$$

where V is given by equation 4.21.b.

When no barrier exists, $a = 0$,

$$\zeta(x,t) = \frac{Q_2}{2}\sqrt{\frac{g}{\pi t}}\ |\xi|^{-3/2}\cos\Theta + O(t^{-1}) \tag{4.27}$$

and when the barrier forms a cliff, $a = \infty$.

$$\zeta(x,t) = \frac{Q_2}{2}\sqrt{\frac{g}{\pi t}}\ |\xi|^{-3/2}(1+\operatorname{sign} x)\cos\Theta + O(t^{-1}) \ . \tag{4.28}$$

4.4 An oscillating point pressure

We suppose that starting from $t = 0_+$, a concentrated pressure having a harmonic time dependence is acting on the free surface at $x = 0_+$. Allowing no initial disturbance, we have for equation 4.1.d that

$$\tau(t) = e^{-j\omega t}$$

and from equation 4.2 that

$$T_1 = T_2 = 0 \ , \quad T_3 = Q_3 \, s/\rho g(s+j\omega) \ . \tag{4.29}$$

Equation 4.9 then becomes

$$\frac{\rho g}{Q_3} \phi(x,y,t) = -\frac{1}{\pi}\frac{1}{2\pi j}\int_\Gamma ds \frac{se^{st}}{s+j\omega}\int_0^\infty \frac{dk}{k+\alpha} e^{ky}\cos kx$$

$$-\frac{1}{\pi^2 a}\frac{1}{2\pi j}\int_\Gamma ds \frac{se^{st}}{s+j\omega}\int_{-a}^{0} d\eta e^{-\alpha\eta}\int_{-a}^{\eta} d\sigma e^{\alpha\sigma} G(\alpha,\sigma)\frac{\partial}{\partial x}$$

$$\left\{\log\frac{R^*}{R} + 2\int_0^\infty \frac{dk}{k+\alpha} e^{k(y+\eta)}\cos kx\right\} \tag{4.30}$$

with G defined by equation 4.18.b.

The asymptotic behavior of the solution for large t can be carried out as in section 3.2. This will be omitted here however, and only the steady state limit will be derived. Adopting the procedure of De Prima and Wu (reference 16), we shall make use of a theorem by Tauber. Let

$$M(x,t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} e^{\lambda t} N(\lambda,x)\frac{d\lambda}{\lambda} \tag{4.31.a}$$

and let $N(x,\lambda)$ be analytic in λ and regular in the half plane $Re_j\lambda \geq \lambda_o \geq 0$ and $c > \lambda_o$, then

$$\lim_{t\to+\infty} M(x,t) = \lim_{\lambda\to 0+} N(\lambda) \tag{4.31.b}$$

if and only if

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t t\frac{\partial M}{\partial t}\, d\tau = 0 \ . \tag{4.31.c}$$

Supposing that the qualifying condition 4.31.c can always be verified a posteriori, we now introduce in equation 4.30 a new variable $\lambda = s+j\omega$

so that the theorem may be directly applied. It is apparent that the k-integrals deserve special attention because of the appearance of a singularity in the integrands as $\lambda \to 0+$,

$$\lim_{\lambda \to 0+} (k+s^2/g) = \lim_{\lambda \to 0+} [k+(\lambda-j\omega)^2/g]$$

$$= \lim_{\lambda \to 0+} [k-(\frac{\omega^2}{g} + 2j\frac{\omega\lambda}{g})] .$$

Thus the k-integrals are Cauchy-singular and their limits as $\lambda \to 0+$ can be obtained by Plemelj's formula (equation 2.33) or by deforming the path of k to by-pass the pole at ω^2/g. The complex number $\omega^2/g + 2j\omega\lambda/g$ approaches the original path of integration from above, the new path is therefore taken to circumvent the point $k = \omega^2/g$ from below, as shown in figure 44. Replacing α by $\beta e^{-j\pi} = \frac{\omega^2}{g} e^{-j\pi}$ everywhere else we arrive at the steady state limit:

$$\phi_s(x,y,t) = \frac{j\omega Q_3}{\pi\rho g} e^{-j\omega t} \int \frac{dk}{k-\beta} e^{ky} \cos kx$$

$$+ \frac{j\omega Q_3}{\pi^2 \rho g a} e^{-j\omega t} \int_{-a}^{0} d\eta\, e^{\beta\eta} \int_{-a}^{\eta} d\sigma\, e^{-\beta\sigma} G(\beta e^{-j\pi}\sigma) \frac{\partial}{\partial x}$$

$$\left\{ \log \frac{R^*}{R} + 2 \int \frac{dk}{k-\beta} e^{k(y+\eta)} \cos kx \right\} . \qquad (4.32)$$

The curly bracket in the preceding formula may be recognized as the Green's function in the theory of steady, time-harmonic gravity waves in deep water. The correct indentation of the contour as obtained above can also be arrived at by various other means, for which a brief discussion is available in reference 16. The present method using Tauberian theorem relates Rayleigh's approach of employing a fictitious damping factor and Peters' approach of an initial value problem (reference 25); it has the same mathematical simplicity but not the artificiality of the first, and seems to be more direct than the second.

To be sure that equation 4.32 is the valid steady state limit, one has of course to check that the necessary and sufficient condition is satisfied, i.e.,

$$\lim \frac{1}{t} \int_0^t \tau \frac{\partial}{\partial t} \left[\phi(x, y, \tau) e^{j\omega t} \right] d\tau = 0 . \tag{4.33}$$

In doing so it suffices to know the asymptotic order of magnitude φ for large t and fixed x and y. For, by taking any large T the left hand side of equation 4.33 may be written as

$$\lim \frac{1}{t} \left\{ \int_0^T + \int_T^t \right\} \tau \frac{\partial}{\partial \tau} \left[\phi(x, y, \tau) e^{j\omega t} \right] d\tau = 0 .$$

The first integral above is of order $O(t^{-1})$ due to the finiteness of ϕ whereas the second can also be proved to approach zero after substituting the approximate order of magnitude of ϕ into the integrand. We shall, however, omit the details which are fully illustrated in reference 16.

Since for large $|x|$,

$$\frac{1}{\pi} \int \frac{dk}{k-\beta} e^{ky} \cos kx = j e^{j\beta|x| + \beta y} + O(|x|^{-1})$$

as can be shown readily, we have, from equation 4.32

$$\phi_s(x,y,t) = -\frac{\omega Q_3}{\rho g} e^{\beta y} e^{j(\beta|x|-\omega t)} \Big\{ 1$$

$$+ \frac{j\beta}{a} (\mathrm{sgn}\, x) \frac{2}{\pi} \int_{-a}^{0} d\eta\, e^{2\beta\eta} \int_{-a}^{\eta} d\sigma\, e^{-\beta\sigma} G(\beta e^{-j\pi}, \sigma) \Big\} + O(|x|^{-1})$$

$$= -\frac{\omega Q_3}{\rho g} e^{\beta y} e^{j(\beta|x|-\omega t)} \Big\{ 1 + \frac{\pi}{2} (\mathrm{sgn}\, x) \overline{V}(\beta a) \Big\} + O(|x|^{-1}) \quad (4.44)$$

where the function $\overline{V}(\beta a)$ is given by equation 4.21.b and the term $O(|x|^{-1})$ denotes the local effect diminishing as $1/|x|$ for large $|x|$. In getting the present expression use has been made of the analysis in section 4.1. The surface elevation far away from the plate is

$$\zeta_s(x,t) = -\frac{1}{g} \frac{\partial}{\partial t} \phi_s(x, o-, t)$$

$$= \frac{j\omega^2 Q_3}{\rho g^2} e^{j(\beta|x|-\omega t)} \Big\{ 1 + \frac{\pi}{2} (\mathrm{sgn}\, x) \overline{V}(\beta a) \Big\} + O(|x|^{-1}) \quad . \quad (4.45)$$

Applying equation 4.23, one obtains for the limiting case of $a = 0$ that

$$\zeta_s(x,t) \cong \frac{j\omega^2 Q_3}{\rho g^2} e^{j(\beta|x|-\omega t)} \quad , \quad |x| \to \infty \qquad (4.46)$$

while in the other extreme case of a vertical cliff, $a \to \infty$,

$$\zeta_s(x,t) \cong \frac{j\omega^2 Q_3}{\rho g^2} e^{j(\beta|x|-\omega t)} (1 + \mathrm{sgn}\, x) \quad , \quad |x| \to \infty \quad . \qquad (4.47)$$

The transmission coefficient can be defined as

$$C_T = 1 - \frac{\pi}{2} \overline{V}(\beta a) = |C_T| e^{j\theta_T} \qquad (4.48)$$

and the reflection coefficient as

$$C_R = 1 + \frac{\pi}{2} \overline{V}(\beta a) = |C_R| e^{j\theta_R} \quad . \qquad (4.49)$$

The quantities $|C_T|$, θ_T, $|C_R|$ and θ_R are plotted as a function of βa, in figure 5.

V. RADIATION AND SCATTERING OF TRANSIENT GRAVITY WAVES DUE TO DISTURBANCES AND SOLID BODIES OF ARBITRARY GEOMETRY

In Chapter III we have derived an asymptotic formula, equation 3.32, which describes the transient gravity waves due to the rolling oscillation of a vertical plate. The analysis performed there can be extended formally to arbitrary sinusoidal disturbances distributed over a finite part of the ocean. We shall consider in this section the effect of oscillatory motion of distributed pressure on the water surface and of floating and submerged bodies of any geometrical arrangement. The result will be formal since it contains a quantity which depends on the complete solution of the problem.

Let the fluid field be described by a velocity potential $\phi(\vec{r},t)$ where $\vec{r}$ denotes the vector (x,y), then ϕ is governed by the following conditions:

$$\nabla^2 \phi = 0 \qquad y < 0 \tag{5.1}$$

$$\left(\frac{\partial^2}{\partial t^2} + g\frac{\partial}{\partial y}\right)\phi = \frac{+j}{\rho}\,\omega p_o(x) H(t) e^{-j\omega t} \qquad \text{on } S_f \tag{5.2}$$

$$\frac{\partial \phi}{\partial y} = V(\vec{r})\, e^{-j\omega t} H(t) \qquad \text{on } S_b \tag{5.3}$$

$$\phi(x,o,o) \text{ and } \frac{\partial \phi}{\partial t}(x,o,o) \text{ given for } |x| > X = 0$$

$$\text{for } |x| > 0 \tag{5.4}$$

$$\text{and } \phi \to 0 \text{ sufficiently fast as } |r| \to \infty \text{ for finite } t\,. \tag{5.5}$$

In the above, the water surface has been designated by S_f and the solid boundaries of floating and submerged bodies by S_b. The phrase "sufficiently fast" will be made more specific as the situation arises.

We now define a Green's function $G(\vec{r}, t \mid \vec{r}_o, t_o)$ by the following requirements:*

$$\nabla^2 G = \delta(\vec{r} - \vec{r}_o)\, \delta(t-t_o) \tag{5.6}$$

$$\left(\frac{\partial^2}{\partial t^2} + g \frac{\partial}{\partial y}\right) G = 0 \qquad y = 0 \tag{5.7}$$

$$G = \frac{\partial G}{\partial t} = 0 \qquad t < t_o \qquad y = 0 \tag{5.8}$$

and

$$G \rightarrow 0 \text{ sufficiently fast as } |r| \rightarrow \infty \text{ for finite } t\ . \tag{5.9}$$

Using the reciprocity theorm which is proved in Appendix B

$$G(\vec{r}, t \mid \vec{r}_o, t) = G(\vec{r}_o, -t_o \mid \vec{r}, -t) \tag{5.10}$$

we can show for the same Green's function that,

$$\nabla^2_o G(\vec{r}, t \mid \vec{r}_o, t_o) = \delta(\vec{r} - \vec{r}_o)\, \delta(t - t_o) \qquad y_o < 0 \tag{5.11}$$

$$\left(\frac{\partial^2}{\partial t_o^{\ 2}} + g \frac{\partial}{\partial y_o}\right) G = 0 \qquad y_o = 0 \tag{5.12}$$

$$G = \frac{\partial}{\partial t_o} G = 0 \qquad t_o > t \qquad y_o = 0 \tag{5.13}$$

and,

$$G \longrightarrow 0 \text{ sufficiently fast as } |r| \rightarrow \infty \text{ for finite } t_o\ . \tag{5.14}$$

---

* This Green's function is different from the one used by Finkelstein in proving a uniqueness theorem for a situation identical to the present. See reference 26.

where $\nabla_o^2 = \frac{\partial^2}{\partial x_o^2} + \frac{\partial^2}{\partial y_o^2}$ in equation 5.11.

From equations 5.1 and 5.11 and the second Green's theorem it follows that,

$$\phi(\vec{r},t) = \int_0^{t_+} dt_o \int_D dx_o dy_o [\phi(\vec{r}_o,t_o) \nabla_o^2 G - G \nabla_o^2 \phi(\vec{r}_o,t_o)]$$

$$= \int_0^{t_+} dt_o \int_{S_f+S_b+S_\infty} dS_o \left[\phi(\vec{r}_o,t_o) \frac{\partial G}{\partial n_o} - G \frac{\partial \phi}{\partial n_o} (\vec{r}_o,t_o) \right]$$

in which D is the domain bounded by S_f, S_b and a great semi-circular arc S_∞. The direction of the contour will be taken as positive, when the enclosed region is to the left, and the direction of the normal $(\vec{n}_o)$ is positive if it points outward. We now take conditions 5.5 and 5.14 to mean that $\phi(r_o,t_o)$ and $G(r,t \mid r_o,t_o)$ vanishes as $|\vec{r}_o| \to \infty$ so fast that the line integral along S_∞ also vanishes as the radius of the arc grows indefinitely. Applying the boundary conditions 5.2 and 5.12 we have,

$$\phi(r,t) = \int_0^{t_+} dt_o \int_{S_b} dS_o \left[\phi \frac{\partial G}{\partial n_o} - G \frac{\partial \phi}{\partial n_o} \right] + \frac{j\omega}{\rho g} \int_0^{t_+} dt_o e^{-j\omega t_o} \int_{-\infty}^{\infty} dx_o p_o(x_o) G$$

$$+ \frac{1}{g} \int_0^{t_+} dt_o \int_{-\infty}^{\infty} dx_o \left[\phi \frac{\partial^2 G}{\partial t_o^2} - G \frac{\partial^2 \phi}{\partial t^2} \right] .$$

The third integral can be integrated with respect to t_o

$$\int_0^{t_+} dt_o \int_{-\infty}^{\infty} dx_o \left[\phi \frac{\partial^2 G}{\partial t_o^2} - G \frac{\partial^2 \phi}{\partial t_o^2} \right] = \int_{-\infty}^{\infty} dx_o \left[\phi \frac{\partial G}{\partial t_o} - G \frac{\partial \phi}{\partial t_o} \right]_{t_o=0}^{t_o=t_+}$$

$$= - \int_{-\infty}^{\infty} dx_o \left[\phi \frac{\partial G}{\partial t_o} - G \frac{\partial \phi}{\partial t_o} \right]_{t_o=0}$$

on account of equation 5.13. Hence

$$\phi(r,t) = \int_0^{t_+} dt_o \int_{S_b} dS_o \left[\phi \frac{\partial G}{\partial n_o} - G \frac{\partial \phi}{\partial n_o} \right] + \frac{j\omega}{\rho g} \int_0^{t_+} dt_o e^{-j\omega t_o} \int_{-\infty}^{\infty} dx_o p_o(x_o) G$$

$$- \frac{1}{g} \int_{-\infty}^{\infty} dx_o \left[\phi \frac{\partial G}{\partial t_o} - G \frac{\partial \phi}{\partial t_o} \right]_{t_o = 0} . \qquad (5.15)$$

If G is found, the second and third integrals then involve known boundary and initial values (cf. equation 5.4), while in the first integral $\frac{\partial \phi}{\partial n_o}$ on S_b is given by the boundary condition 5.3. Thus equation 5.15 provides the solution for the velocity potential once the value of ϕ on S_b can be calculated. Obviously this formula is valid also in three space dimensions with the line integrals replaced by surface integrals.

The construction of the Green's function defined by equations 5.6 to 5.9 can be performed by the customary integral transform method, as shown in Appendix C; the result is:

$$G(\vec{r},t|\vec{r}_o,t_o) = \frac{1}{2\pi j} \int_\Gamma ds\, e^{s(t-t_o)} \tilde{G}(\vec{r}, \vec{r}_o, s) \qquad (5.16)$$

where,

$$\tilde{G}(\vec{r}, \vec{r}_o, s) = -\left[\frac{1}{2\pi} \log R^*/R + 2 \int_0^\infty dk \cos k(x-x_o) \frac{e^{k(y+y_o)}}{k+s^2/g} \right] \qquad (5.17)$$

is the Laplace transform of G for $t_o = 0$,

$$R = [(x-x_o)^2+(y-y_o)^2]^{\frac{1}{2}} \qquad R^* = [(x-x_o)^2+(y+y_o)^2]^{\frac{1}{2}} \qquad (5.18)$$

and Γ is a vertical path in the s-plane to the right of all singularities of the function $\tilde{G}(r, r_o, s)$.

Now let the Laplace transform of equation 5.15 be taken. Since the dependence of G on t and t_o is in the form of their difference $t - t_o$ only, the convolution theorem may be used to yield,

$$\tilde{\phi}(\vec{r}, s) = \int_{S_b} dS_o \left[\tilde{\phi}(\vec{r}_o, s) \frac{\partial \tilde{G}}{\partial n_o} - \frac{V(\vec{r}_o)}{s+\omega j} \tilde{G} \right] + \frac{j\omega}{\rho g} \frac{1}{s+j\omega} \int_{-\infty}^{\infty} dx_o p_o(x) \tilde{G}$$

$$+ \frac{1}{g} \int_{-\infty}^{\infty} dx_o \left[s\phi_o(x_o) + N_o(x_o) \right] \tilde{G} \qquad (5.19)$$

with,

$$\tilde{\phi}(\vec{r}, s) = \int_o^{\infty} dt\, e^{-st} \phi(\vec{r}, t) \quad , \quad \tilde{\phi}(\vec{r}_o, s) = \int_o^{\infty} dt_o e^{-st_o} \phi(\vec{r}_o, t_o) \qquad (5.20)$$

$$\phi_o(x_o) = \phi(x_o, o, o) \quad , \quad N_o(x_o) = \frac{\partial}{\partial t_o} \phi(x_o, o, t_o) \Big|_{t_o = 0} \quad . \qquad (5.21)$$

In getting the third integral to the present form we have made use of the fact that

$$\widetilde{\frac{\partial G}{\partial t_o}} \Big|_{t_o = 0} = - \widetilde{\frac{\partial G}{\partial t}} \Big|_{t_o = 0} = -s\tilde{G}$$

on account of equation 5.16. Equation 5.19 and

$$\phi(\vec{r}, t) = \frac{1}{2\pi j} \int_{\Gamma} ds\, e^{st} \tilde{\phi}(\vec{r}, s) = \phi_b + \phi_p + \phi_i$$

where ϕ_b, ϕ_p and ϕ_i correspond respectively to the three integrals in equation 5.19, form a more convenient basis of asymptotic analysis than equation 5.15 as will be seen presently.

Let us consider ϕ_p and ϕ_i first. For $y_o = 0$, $R = R^*$, hence it follows from equation 5.17 that,

$$\tilde{G}(\vec{r}, x_o, s) = -\frac{1}{\pi}\int_o^\infty \frac{dk}{k+s^2/g} \; e^{ky}\cos k(x-x_o)$$

and from equation 5.19 that

$$\phi_p+\phi_1 = -\frac{j\omega}{\pi\rho g}\int_{-\infty}^{\infty} dx_o p_o(x_o)\left\{\frac{1}{2\pi j}\int_\Gamma ds \frac{e^{st}}{s+j\omega}\int_o^\infty dk \frac{e^{ky}}{k+s^2/g}\cos k(x-x_o)\right\}$$

$$-\frac{1}{\pi g}\int_{-\infty}^{\infty} dx_o \phi_o(x_o)\left\{\frac{1}{2\pi j}\int_\Gamma ds\, s e^{st}\int_o^\infty dk \frac{e^{ky}}{k+s^2/g}\cos k(x-x_o)\right\}$$

$$-\frac{1}{\pi g}\int_{-\infty}^{\infty} dx_o N_o(x_o)\left\{\frac{1}{2\pi j}\int_\Gamma ds\, e^{st}\int_o^\infty dk \frac{e^{ky}}{k+s^2/g}\cos k(x-x_o)\right\} \quad .(5.22)$$

The curly brackets in the three terms above can be identified with known results, i.e., the first with the periodic point pressure by Wu (1957) and the remaining two with the classical Cauchy-Poisson problem for a concentrated initial impulse and ϕ_t on the surface S_f. By either quoting from the literature or repeating the procedure demonstrated in Chapter III one gets the time derivative of the first bracket in equation 5.22 the following asymptotic expression for large t.

$$-\frac{1}{g}\frac{\partial}{\partial t}\left\{\frac{1}{2\pi j}\int_\Gamma ds \frac{e^{st}}{s+j\omega}\int_o^\infty \frac{dk}{k+s^2/g} e^{k(y+y_o)}\cos k(x-x_o)\right\}_{y=0}$$

(5.23.a)

$$\simeq -\frac{\pi\omega}{g}\left\{\frac{1}{2}+\frac{1}{\sqrt{2}}e^{j\frac{\pi}{4}}\left[C\left(\sqrt{\frac{2}{\pi}}\, q\right)-jS\left(\sqrt{\frac{2}{\pi}}\, q\right)\right]\right\}e^{\beta y_o}e^{j[\beta(x-x_o)-\omega t]} + O[t^{-\frac{1}{2}}]$$

$$\text{where } \sqrt{\frac{2}{\pi}}\, q = \frac{2}{\sqrt{\pi\omega t}}\left[\frac{\omega}{2}t-\beta(x-x_o)\right] . \quad (5.23.b)$$

For later application y_o is kept arbitrary without being equated to zero. Since the contribution to ζ_1 from the initial values ϕ_o and N_o is purely transient in nature it will be ignored on account of its relative order of magnitude. Consequently from equation 5.22

$$\zeta_p+\zeta_i = -\frac{1}{g}\frac{\partial}{\partial t}(\phi_p+\phi_i)_{y=0} \approx \frac{j\omega^2}{\rho g^2}\int_{-\infty}^{\infty} dx_o\, e^{-j\beta x_o} p_o(x_o)\left\{\frac{1}{2}+\frac{1}{\sqrt{2}}e^{j\frac{\pi}{4}}[C-jS]\right\}\cdot$$

$$e^{j[\beta(x-x_o)-\omega t]} + O(t^{-\frac{1}{2}}) \quad . \qquad (5.24)$$

It should be noted that the effect of the pressure distribution and initial disturbances on the water surface is not entirely represented by the terms ϕ_p and ϕ_i; it is also contained implicitly in the value of $\phi(\vec{r}_o, t)$ on the solid boundaries.

We now treat ϕ_b formally as if $\tilde{\phi}(\vec{r}, s)$ were known on S_b. The argument following will be mainly based on the Tauberian theorem (cf. equation 4.31) and Theorem 3.13 with its inverse which is assumed to be true without proof. Since from physical ground one expects $\phi(\vec{r}, t)$ to settle down finally to a steady state of periodic oscillation at the frequency ω, $\tilde{\phi}(\vec{r}, s)$ must possess a simple pole at $s = -j\omega$. Thus one may write

$$\tilde{\phi}(\vec{r}, s) = \frac{\Phi(\vec{r}, s)}{s+j\omega} = \frac{\Phi(\vec{r})}{s+j\omega} + \tilde{Q}(\vec{r}, s) \qquad (5.25)$$

where

$$\Phi(\vec{r}) = \lim_{s+j\omega \to 0+} \Phi(\vec{r}, s) \quad \text{and} \quad \tilde{Q}(\vec{r}, s) = \frac{\Phi(\vec{r}, s) - \Phi(\vec{r})}{s+j\omega} \qquad (5.26)$$

It is clear from Tauberian theorem that $\Phi(\vec{r})$ must be the time independent part of the steady state solution, i.e., $\phi_s = \Phi \exp(-j\omega t)$, and that $\tilde{Q}(\vec{r}, s)$ contains only purely transient waves. At large t the slowest rate at which $Q(\vec{r}, t)$ dies out may be assumed to be algebraic, i.e., $Q(\vec{r}, t) \sim O(t^{-\alpha})$, $\alpha > 0$. Allowing this we see that $\tilde{Q}(\vec{r}, s)$ can have singularities s_i of the following kind:

(i) They lie on the imaginary axis, i.e., $s = j\xi$, ξ real.

(ii) For any $\xi_i = 0$, $a_o = 0$ in the expansion of the form of equation 3.13.a.

Substituting 5.25 into 5.19 we have

$$\phi_b = \frac{1}{2\pi j}\int_\Gamma \frac{ds\, e^{st}}{s+j\omega}\int_{S_b} dS_o \left[\Phi(\vec{r}_o)\frac{\partial\tilde{G}}{\partial n_o} - \tilde{G}\,V(\vec{r}_o)\right]$$

$$+\frac{1}{2\pi j}\int_\Gamma ds\, e^{st}\int_{S_b} dS_b\, \tilde{Q}(\vec{r}_o, s)\frac{\partial\tilde{G}}{\partial n_o} \quad . \qquad (5.27)$$

In view of Theorem 3.13 and the singular behavior of $\tilde{Q}$ and $\tilde{G}$ one may expect that the second integral above vanishes for large t at least as fast as $O(t^{-\beta})$, $0 < \beta < 1$, for all x. As for the first integral, we shall only be interested in approximating the corresponding surface height. First, the function $\tilde{G}$ in equation 5.27 will be replaced by its explicit form (equation 5.17) in which the term $\log R/R^*$ is to be neglected as it vanishes on the free surface $(y = 0)$, while the remaining term is handled simply by applying the result of equation 5.23, i.e.,

$$\zeta_b = -\frac{1}{g}\frac{\partial\phi_b}{\partial t}\bigg|_{y=0} \cong \frac{\omega}{g}e^{-j\omega t}\int_{S_b} dS_o\left[\Phi(\vec{r}_o)\frac{\partial K}{\partial n_o} - V(\vec{r}_o)K\right] + O(t^{-\frac{1}{2}}, x^{-2}) \qquad (5.28)$$

with

$$K(x-x_o, y_o) = \left\{\frac{1}{2} + \frac{1}{\sqrt{2}}e^{j\frac{\pi}{4}}\left[C\left(\sqrt{\tfrac{2}{\pi}}\,q\right) - jS\left(\sqrt{\tfrac{2}{\pi}}\,q\right)\right]\right\}e^{\beta y_o}e^{j\beta(x-x_o)} . \qquad (5.29)$$

Summarizing equations 5.24 and 5.28 we have finally

$$\zeta(x,t) = \zeta_b + \zeta_p + \zeta_i \simeq \frac{\omega}{g} e^{-j\omega t} \int_{S_b} dS_o \left\{ \Phi(\vec{r}_o) \frac{\partial K}{\partial n_o} - V(\vec{r}_o) K \right\}$$

$$+ \frac{j\omega^2}{\rho g^2} e^{-j\omega t} \int_{-\infty}^{\infty} dx_o p_o(x_o)(K)_{y_o=0} + O(t^{-\frac{1}{2}}) \quad . \tag{5.30}$$

For $\sqrt{\frac{2}{\pi}}\, q = \dfrac{\omega t - 2\beta(x-x_o)}{\sqrt{\pi\omega t}} \gg 1$ the steady state prevails; from equation 3.31 we have the following formula

$$\zeta(x,t) \simeq \frac{\omega}{g} e^{-j\omega t} \int_{S_b} dS_o \left\{ \Phi(\vec{r}_o) \frac{\partial}{\partial n_o} [e^{+j\beta(x-x_o)} e^{\beta y_o}] - V(\vec{r}_o) e^{+j\beta(x-x_o)} e^{\beta y_o} \right\}$$

$$+ \frac{j\omega^2}{\rho g^2} e^{-j\omega t} \int_{-\infty}^{\infty} dS_o\, p_o(x_o) e^{j\beta(x-x_o)} \tag{5.31}$$

which may also be obtained from a steady state formulation (cf. reference 27).

As a special case we consider the situation in Chapter III where there are a series of thin plates denoted by L, lying vertically on the negative y-axis. Noting that

$$\frac{\partial}{\partial n_o} = \frac{\partial}{\partial x_o}, \quad x_o = 0^- \qquad \text{for } y_o \text{ on } L$$
$$= -\frac{\partial}{\partial x_o}, \quad x_o = 0^+$$

and

$$\frac{\partial}{\partial x_o} K(x-x_o, y_o, t) \Big|_{x_o=0} = -\frac{\partial}{\partial x} K(x, y_o, t)$$

we have, from equation 5.30, that

$$\zeta(x,t) \cong \frac{\omega}{g} e^{-j\omega t} \int_L dy_o \, [\Phi(y_o)]_-^+ \frac{\partial}{\partial x} K(x,y_o,t)$$

$$+ \frac{j\omega^2}{\rho g^2} e^{j(\beta x-\omega t)} \int_{-\infty}^{\infty} dx_o p_o(x_o) K(x-x_o,o,t)$$

$$= \frac{j\beta\omega}{g} e^{j(\beta x-\omega t)} \left\{ \int_L dy_o [\Phi(y_o)]_-^+ e^{\beta y_o} \right\} \left\{ \frac{1}{2} + \frac{1}{\sqrt{2}} e^{j\frac{\pi}{4}} [C-jS] \right\}$$

$$+ O(t^{-\frac{1}{2}}) + \frac{j\omega^2}{\rho g^2} e^{j(\beta x-\omega t)} \int_{-\infty}^{\infty} dx_o p_o(x_o) K(x-x_o,0,t) \qquad (5.32.a)$$

in which,

$$C = C(Q) \quad , \quad S = S(Q)$$

and

$$Q = \frac{2}{\sqrt{\pi\omega t}} \left(\frac{\omega t}{2} - \beta x \right) \qquad (5.32.b)$$

If the external pressure $p_o(x)$ is identically zero on the free surface, we have

$$\zeta(x,t) \cong A_s \left\{ \frac{1}{2} + \frac{1}{\sqrt{2}} e^{j\frac{\pi}{4}} [C-jS] \right\} e^{j(\beta x-\omega t)} \qquad (5.33)$$

with

$$A_s = \frac{j\beta\omega}{g} \int_L dy_o [\Phi(y_o)]_-^+ e^{\beta y_o} \qquad (5.34)$$

which is equal to the steady state wave amplitude as $x \to \infty$, as can be derived from equation 5.31. Recalling the result of Wu (reference 17) and Miles (reference 18), and that of Chapter III equation 3.32, we may conclude that equation 5.33 represents the transient wave response as long as the sources of disturbance are located on the y-axis.

REFERENCES

1. J. V. Wehausen and E. V. Laitone: Surface Waves, an article in Handbuch der Physik, Band IX, ed. by S. Flugge, Springer-Verlag, Berlin, 446-778 (1960).

2. L. Landweber and M. Macagno: Added mass of two-dimensional forms oscillating in a free surface. J. Ship Res. 1, 20-30 Nov. (1957).

3. F. Ursell: Short surface waves due to an oscillating immersed body. Proc. Roy. Soc. Lond., Series A, 220, 90-103 (1953).

4. F. Ursell: The transmission of surface waves under surface obstacles. Proc. Cambr. Phil. Soc. 57, 638-668, (1961).

5. F. Ursell: On the heaving motion of a circular cylinder on the surface of a fluid. Quart. J. Mech. Appl. Math. 2, 218-231 (1949).

6. F. Ursell: On the rolling motion of cylinders in the surface of a fluid. Quart. J. Mech. Appl. Math. 2, 335-353 (1949).

7. F. Tasai: On the damping force and added mass of ship's heaving and pitching. Rep. of Res. Inst. for Appl. Mech., Kyushu Univ., Fukuoka, Japan, VII, 131-152 (1959).

8. W. R. Porter: Pressure distributions, added masses, and damping coefficients for cylinders oscillating in a free surface. Inst. of Engrg. Res., University of California, Berkeley, Series No. 82, Issue No. 16, July (1960).

9. P. Kaplan and J. Kotik: Report on a seminar on the hydrodynamic theory associated with ship motion in waves. Technical Research Group, 2, Aerial Way, Syosset, New York, TRG-140-FR (1962).

10. W. R. Dean: On the reflection of surface waves by a submerged plane barrier. Proc. Cambr. Phil. Soc. 41, 231-238 (1945).

11. F. Ursell: The effect of fixed vertical barrier on surface waves in deep water. Proc. Cambr. Phil. Soc. 43, 374-382 (1947).

12. F. Ursell: On the waves due to the rolling of a ship. Quart. J. Mech. Appl. Math. 1, 246-252 (1948).

13. M. D. Haskind (Khaskind): Radiation and diffraction of surface waves by a flat plate floating vertically. J. Appl. Math. Mech. (PMM) 23, 770-783 (1959).

14. H. Levine and E. Rodemich: Scattering of surface waves on an ideal fluid. Rep. No. 78, Appl. Math. Statis. Lab., Standord Univ. (1958).

15. H. Lamb: Hydrodynamics. 6th ed. Dover Publications, New York (1945).

16. C. R. De Prima and T. Y. Wu: On the theory of surface waves in water generated by moving disturbances. Engrg. Div. California Inst. of Tech., Pasadena, Calif. Rep. 21-23 (1957).

17. T. Y. Wu: Water waves generated by the translatory and oscillatory surface disturbance. Engrg. Div. California Inst. of Tech., Pasadena, Calif., Rep. 85-3 (1957).

18. J. W. Miles: Transient gravity wave response to an oscillating pressure. J. Fluid Mech. 13, 145-150 (1962).

19. E. H. Kennard: Generation of surface waves by a moving partition. Quart. Appl. Math., 7, 303-312 (1949).

20. N. I. Muskhelishvilli: Singular Integral Equations. Noordhoff, Groningen, Holland (1953).

21. H. S. Carslaw and J. C. Jaeger: Operational Methods in Applied Mathematics. Oxford University Press, London, 2nd ed. (1953).

22. M. Born and E. Wolf: Principles of Optics. Pergamon Press, London (1959)

23. A. Erdelyi (ed.): Higher Transcendental Functions, Vol. 2, McGraw-Hill, New York (1953).

24. N. W. McLachlan and A. L. Meyer: Integrals involving Bessel and Struve functions. Phil. Mag., 7th series, 21, 437-447 (1936).

25. A. S. Peters. A new treatment of the ship wave problem. Comm. Pure Appl. Math., 2. 123-148 (1949).

26. A. B. Finkelstein: The initial value problem for transient water waves. Comm. Pure Appl. Math., 10, 511-522 (1957).

27. F. Ursell: Water waves generated by oscillating bodies. Quart. J. Mech. Appl. Math. 7, 427-437 (1954).

APPENDICES

Appendix A. Evaluation of integrals.

A.a. Let

$$N = \int_{-\infty}^{-a} dy\, e^{\alpha y} \left[\frac{\alpha}{a}\, y\sqrt{y^2-a^2} - \frac{2}{\pi}(1+\alpha c)\, y \cos^{-1}\frac{a}{y} \right]$$

$$= a^2 \int_{-\infty}^{-1} dy\, e^{vy} \left[\frac{v}{2}\, y\sqrt{y^2-1} - \frac{2}{\pi}\left(1 + V\frac{c}{a}\right) y \cos^{-1}\frac{1}{y} \right] \qquad (A.1)$$

with $v = \alpha a$. The first integral is elementary

$$N_1 = \int_{-\infty}^{-1} dy\, e^{vy} y\sqrt{y^2-1} = \frac{d}{dv}\int_1^{\infty} dy\, e^{-vy}\sqrt{y^2-1} = -\frac{1}{v} K_2(v) \quad . \quad (A.2)$$

Integrating the second one in equation A.1 by parts, we get

$$N_2 = \int_{-\infty}^{-1} dy\, e^{vy} y \cos^{-1}\frac{1}{y} = \frac{d}{dv}\int_1^{\infty} dy\, e^{-vy} \cos^{-1}\frac{1}{y} = -\frac{d}{dv}\frac{1}{v}\int_1^{\infty} \cos^{-1}\frac{1}{y}\, d(e^{-vy})$$

$$= \frac{d}{dv}\frac{1}{v}\int_1^{\infty} dy\, \frac{e^{-vy}}{y\sqrt{y^2-1}} \quad . \qquad (A.3)$$

Set $N_3 = \int_1^{\infty} dy\, \dfrac{e^{-vy}}{y\sqrt{y^2-1}}$, then $\dfrac{dN_3}{dv} = -\int_1^{\infty} dy\, \dfrac{e^{-vy}}{\sqrt{y^2-1}} = -K_0(v)$.

Hence,

$$N_3 = C - \int_0^{v} K_0(\sigma)\, d\sigma \quad .$$

The constant of integration, C, is easily seen to be $\pi/2$ by noting that

$$N_3(v=0) = \int_1^{\infty} \frac{dy}{y\sqrt{y^2-1}} = \cos^{-1}\frac{1}{y}\Big|_1^{\infty} = \pi/2 \quad .$$

We may now make use of the following known formulas (p. 439, reference 25):

$$\int^{z} z^n K_n(z)dz = 2^{n-1}\sqrt{\pi}\,\Gamma(n+\tfrac{1}{2})z[K_n(z)L_{n-1}(z)-L_n(z)K_{n-1}(z)] \tag{A.4}$$

$$= 2^{n-1}\sqrt{\pi}\,\Gamma(n+\tfrac{1}{2})z[K_n(z)L_{n+1}(z)+L_n(z)K_{n+1}(z)] + \frac{z^{n+1}K_n(z)}{2n+1} \tag{A.5}$$

where,

$$L_n(z) = \sum_{m=0}^{\infty} \frac{(z/2)^{n+2m+1}}{\Gamma(m+\frac{3}{2})\,\Gamma(n+m+\frac{3}{2})} \tag{A.6}$$

is a Struve function of imaginary argument. Upon substitution we get,

$$N_2 = -\frac{1}{v^2}\left[\frac{\pi}{2} - \int_0^v K_0(\sigma)d\sigma\right] - \frac{1}{v}K_0(v)$$

$$= \frac{\pi}{2}\left\{\frac{1}{v}\,[K_0(v)L_1(v)+L_0(v)K_1(v)] - \frac{1}{v^2}\right\}. \tag{A.7}$$

Finally, equations A.1 - 3 and A.7 together give,

$$N = a^2\left[\frac{v}{2}N_1 - \frac{2}{\pi}\left(1+v\frac{c}{a}\right)N_2\right]$$

$$= -a^2\left\{\frac{1}{2}K_2(v) + \frac{1+v\frac{c}{a}}{v}\left[K_0(v)L_1(v)+L_0(v)K_1(v) - \frac{1}{v}\right]\right\} \tag{A.8}$$

which leads to equation 3.8.

A.b. Integrating by parts, one obtains,

$$N = \int_{-a}^{0} d\eta\, e^{2k\eta}\int_{-a}^{\eta} d\sigma\, e^{-k\sigma}G(ke^{\pm j\pi},\sigma) = -\frac{1}{k}\int_{-a}^{0} d\eta \sinh k\eta\, G(ke^{\pm j\pi},\sigma). \tag{A.9}$$

Using the expression for G as given by equation 3.9.b the function N assumes the following form:

$$N = -\frac{a^2}{k}\int_{-1}^{0} dy \sinh uy\left(\frac{\frac{1}{a}Ay}{\sqrt{1-y^2}} + \frac{1-u\frac{c}{a}}{\pi}\, y \log\frac{1-\sqrt{1-y^2}}{1+\sqrt{1-y^2}} - \frac{u}{2}\, y\sqrt{1-y^2}\right) \tag{A.10}$$

where $u = ka$. The first and the third term can be integrated immediately as follows,

$$N_1 = \int_{-1}^{0} dy\, \frac{y \sinh uy}{\sqrt{1-y^2}} = \frac{d}{du}\int_{0}^{1} dy\, \frac{\cosh uy}{\sqrt{1-y^2}} = \frac{\pi}{2}\frac{d}{du} I_o(u) = \frac{\pi}{2} I_1(u)\ , \tag{A.11}$$

$$N_3 = \int_{-1}^{0} dy\, y\sqrt{1-y^2}\, \sinh uy = \frac{d}{du}\int_{0}^{1} dy\sqrt{1-y^2}\, \cosh uy = \frac{\pi}{2}\frac{d}{du}\frac{I_1(u)}{u} = \frac{\pi}{2}\frac{I_2(u)}{u}\ . \tag{A.12}$$

The second term in A.10 is,

$$N_2 = \int_{-1}^{0} dy\, y \log\frac{1-\sqrt{1-y^2}}{1+\sqrt{1-y^2}}\, \sinh uy = \frac{d}{du}\frac{1}{u}\int_{0}^{1} \log\frac{1-\sqrt{1-y^2}}{1+\sqrt{1-y^2}}\, d(\sinh uy)$$

$$= -2\frac{d}{du}\frac{1}{u}\int_{0}^{1} dy\, \frac{\sinh uy}{y\sqrt{1-y^2}}\ .$$

Let $N_4 = \int_{0}^{1} dy\, \frac{\sinh uy}{y\sqrt{1-y^2}}$ then $\frac{dN_4}{du} = \int_{0}^{1} dy\, \frac{\cosh uy}{\sqrt{1-y^2}} = \frac{\pi}{2} I_o(u)$

and,

$$N_4(u) = C + \frac{\pi}{2}\int_{0}^{u} d\sigma\, I_o(\sigma)\ .$$

Since $N_4(u = 0) = 0$, we get $C = 0$ also. Making use of the known result (p. 438, reference 25):

$$\int_{0}^{z} dz\, z^n I_n(z) = 2^{n-1}\sqrt{\pi}\ \Gamma(n+\tfrac{1}{2})\, z\,[I_n(z) L_{n-1}(z) - L_n(z) I_{n-1}(z)] \tag{A.13}$$

$$= 2^{n-1}\sqrt{\pi}\,\Gamma(n+\tfrac{1}{2})[I_n(z)L_{n+1}(z)-L_n(z)I_{n+1}(z)]\ \frac{z^{n+1}I_n(z)}{2n+1} \qquad (A.14)$$

we further have

$$N_2 = -2\ \frac{d}{du}\ \frac{N_4}{u} = -\pi\ \frac{d}{du}\ \frac{1}{u}\int_0^u d\sigma\, I_0(\sigma)$$

$$= -\pi\left[\frac{1}{u}\ I_0(u) - \frac{1}{u^2}\int_0^u d\sigma\, I_0(\sigma)\right]$$

$$= \frac{\pi^2}{2u}\ [I_0(u)L_1(u) - L_0(u)I_1(u)] \quad . \qquad (A.15)$$

Combining the expression above and that of A (equation 3.8), and employing the following identities (p. 80, reference 24):

$$L_n(-u) = (-1)^{n+1}\ L_n(u) \qquad (A.16)$$

$$K_n(ue^{\pm j\pi}) = \mp j[\pi I_n(u) \pm (-1)^n j\,K_n(u)] \qquad (A.17)$$

and

$$I_n(u)K_{n+1}(u) + I_{n+1}(u)K_n(u) = 1/u \qquad (A.18)$$

we obtain the simple formula,

$$N = \mp\ \frac{j\pi a}{2k^2}\left\{\frac{1}{2} - \frac{1-kc}{u}\ [I_1(u)+L_1(u)]\right\}\Big/[\pi I_1(u) \mp jK_1(u)] \qquad (A.19)$$

which leads to equation 3.17.

A.c Let $v = \alpha a$ and,

$$N = \int_{-\infty}^{-a} dy\, e^{\alpha y}\sqrt{y^2-a^2}/y = a\int_{-\infty}^{-1} dy\, e^{vy}\sqrt{y^2-1}\,/y \quad . \qquad (A.20)$$

Since,

$$\frac{1}{a}\ \frac{dN}{dv} = \int_{-\infty}^{-1} dy\, e^{vy}\sqrt{y^2-1} = \frac{1}{v}\ K_{-1}(v)$$

it follows from equation A.4 that,

$$N/a = C + \int_0^v d\sigma \, \frac{1}{\sigma} \, K_{-1}(\sigma)$$

$$= C - \frac{\pi}{2} v \left[K_{-1}(v) L_{-2}(v) + L_{-1}(v) K_{-2}(v) \right] \, .$$

It is evident from the defining equation that $N(v = \infty) = 0$. Thus by using the formula (p. 439, reference 25) below,

$$\int_0^\infty \frac{1}{\sigma} \, K_{-1}(\sigma) d\sigma = - \pi/2 \qquad \text{(A.21)}$$

and the asymptotic expressions of K_n and L_n for a large argument, we can determine the constant of integration C to be $\pi/2$. The identity that

$$K_{-n}(z) = K_n(z)$$

enables us to write,

$$N = \frac{\pi}{2} \, a \left\{ 1 - v\left[K_1(v) L_{-2}(v) - L_{-1}(v) K_2(v)\right] \right\} \qquad \text{(A.22)}$$

A.d Let

$$N = \int_{-a}^{0} dy \sinh uy \left[\frac{A(ue^{\pm j\pi})y}{\sqrt{a^2 - y^2}} - \frac{\sqrt{a^2 - y^2}}{y} \right]$$

$$= a \int_0^1 dy \, \sinh uy \left(\frac{Ay}{\sqrt{1 - y^2}} - \frac{\sqrt{1 - y^2}}{y} \right) \qquad \text{(A.23)}$$

with $u = ka$. The first integral in the preceding equation is,

$$N_1 = \int_0^1 dy \, \frac{y \sinh uy}{\sqrt{1 - y^2}} = \frac{d}{du} \int_0^1 \frac{\cosh uy}{\sqrt{1 - y^2}}$$

$$= \frac{\pi}{2} \frac{d}{du} \, I_0(u) = \frac{\pi}{2} \, I_1(ka)$$

whereas the second is

$$N_2 = \int_0^1 dy\ \frac{1}{y}\sqrt{1-y^2}\ \sinh uy\ .$$

Since,

$$\frac{dN_2}{du} = \int_0^1 dy\sqrt{1-y^2}\ \cosh uy = \frac{\pi}{2}\ \frac{1}{u}\ I_1(u)$$

it follows from the identity $I_n(z) = I_{-n}(z)$,

$$N_2 = \frac{\pi}{2}\int_0^u d\sigma\ \frac{1}{\sigma}\ I_{-1}(\sigma)$$

$$= -\frac{\pi}{4}\ u\ [I_1(u)L_{-2}(u) - L_{-1}(u)I_2(u)]$$

where use has been made of formula A.13 and the integration constant is fixed by the special choice of the lower limit, 0. Adding up, we have,

$$N = a(N_1 - N_2)$$

$$= \frac{\pi^2}{4}\ a\left\{\frac{2}{\pi}\ A(ue^{\pm j\pi})I_1(u) + u[I_1(u)L_{-2}(u) - L_{-1}(u)I_2(u)]\right\}\ .$$

Substituting into the foregoing equation the expression for A as given by equation 4.12 and applying formulas A.16-18, we get,

$$N = \pm\ \frac{j\pi^2 a}{4}\ \frac{I_1(u) + L_{-1}(u)}{\pi I_1(u) \mp jK_1(u)}\ . \qquad (A.24)$$

Appendix B. A Reciprocity Relation:

The effect at a point $\vec{r}$ and time t caused by an impulsive disturbance at $\vec{r}_o$ and an earlier time $t_o < t$ is equal to the effect at $\vec{r}_o$ and $-t_o$ caused by an equal impulse at $\vec{r}$, and $-t$. That is, if,

$$\nabla^2 G(\vec{r}, t \mid \vec{r}_o, t_o) = \delta(\vec{r} - \vec{r}_o)\, \delta(t - t_o) \tag{B.1}$$

$$G = G_t = 0 \quad \text{for} \quad y = 0, \; t < t_o \tag{B.2}$$

$$G_{tt} + gG_y = 0 \qquad \text{for} \qquad y = 0, \quad \text{all } t > 0 \tag{B.3}$$

then

$$G(\vec{r}, t \mid \vec{r}_o t_o) = G(\vec{r}_o, -t_o \mid \vec{r}, -t) \; . \tag{B.4}$$

PROOF:

By definition

$$\nabla^2 G(\vec{r}, t \mid \vec{r}_o t_o) = \delta(\vec{r} - \vec{r}_o)\, \delta(t - t_o) \tag{B.5}$$

$$\nabla^2 G(\vec{r}, -t \mid \vec{r}_1, -t_1) = \delta(\vec{r} - \vec{r}_1)\, \delta(t - t_1) \; . \tag{B.6}$$

Multiplying equation B.5 by $G(\vec{r}, -t \mid \vec{r}_1, -t_1)$ and equation B. 6 by $G(\vec{r}, t \mid \vec{r}_o, t_o)$ and integrating with respect to space over the volume bounded by the free surface S_f and a great sphere S_∞, and with respect to time from $-\infty$ to $t' > t_o, t_1$, we have,

$$\int_{-\infty}^{t'} dt \int_V dV \,\{G(\vec{r}, t \mid \vec{r}_o, t_o)\nabla^2 G(\vec{r}, -t \mid \vec{r}_1, -t_1) - G(\vec{r}, -t \mid \vec{r}_1, -t_1)\nabla^2 G(\vec{r}, t \mid \vec{r}_o, t_o)\}$$

$$= G(\vec{r}_1, t_1 \mid \vec{r}_o, t_o) - G(\vec{r}_o, -t_o \mid r_1, -t_1) \tag{B.7}$$

from equations B.5 and B.6. Applying to the left hand side successively the second Green's formula and the boundary condition on the free surface equation B.3 and integrating with respect to t, we get,

$$\text{L.H.S.} = \int_{-\infty}^{t'} dt \int_{S_f+S_\infty} dS\{G(\vec{r},t|\vec{r}_o,t_o)\frac{\partial}{\partial n} G(\vec{r},-t|\vec{r}_1,-t_1)$$

$$-G(\vec{r}_1,-t|r_1,-t_1) \frac{\partial}{\partial n} G(\vec{r},t|\vec{r}_o,t_o)\}$$

$$=-\frac{1}{g} \int_{-\infty}^{t'} dt \int_{S_f} dS\{G(\vec{r},t|\vec{r}_o,t_o) \frac{\partial^2}{\partial t^2} G(\vec{r},-t|\vec{r}_1,-t_1)$$

$$-G(\vec{r},-t|\vec{r}_1,-t_1) \frac{\partial^2}{\partial t^2} G(\vec{r},t|\vec{r}_o,t_o)\}$$

$$= -\frac{1}{g} \int_{S_f} \{G(\vec{r},t|\vec{r}_o,t_o) \frac{\partial}{\partial t} G(\vec{r},-t|\vec{r}_1,-t_1)$$

$$-G(\vec{r},t|\vec{r}_1,-t_1) \frac{\partial G}{\partial t} (\vec{r},t|\vec{r}_o,t_o)\}_{t=-\infty}^{t=t'} .$$

In dropping the surface integral over S_∞ as the radius of S_∞ grows indefinitely large, we have assumed that $|G| \rightarrow 0$ sufficiently fast as $|r| \rightarrow 0$. On account of the initial conditions, at the lower limit we have

$$G(x,-\infty|x_o,t_o) = G_t(x,-\infty|x_o,t_o) = 0$$

whereas at $t = t'$, $-t = -t' < -t_1$, since $t' > t_1$,

$$G(x_1,-t'|x_1,-t_1) = G_t(x,-t'|x_1,-t_1) = 0$$

also. Thus the left hand side of equation B.7 vanishes and the reciprocity relation B.4 is proved.

Appendix C : A Green's Function

To construct the Green's function satisfying equations B.6-9 we start by letting $t_o = 0+$ and $x_o = o+$ and return in the final result to $t - t_o$ and $x - x_o$ respectively. Taking the Laplace transform with respect to t and the Fourier transform with respect to x,

$$\hat{\tilde{G}}(k,y,s) = \int_{-\infty}^{\infty} dx e^{-ikx} \int_0^{\infty} dt\, e^{st} G(x,y,t)$$

we get, from equations B.6-9

$$\left(\frac{d^2}{dy^2} - k^2\right)\hat{\tilde{G}} = \delta(y-y_o) \qquad y < 0$$

$$\frac{s^2}{g}\hat{\tilde{G}} + \frac{d}{dy}\hat{\tilde{G}} = 0 \qquad y = 0$$

and

$$|\hat{\tilde{G}}| \longrightarrow 0 \qquad y \longrightarrow -\infty \quad .$$

It is easy to show that the solution for $\tilde{G}$ is

$$\hat{\tilde{G}} = -\frac{1}{2}\frac{1}{|k|}\left[e^{|k|(y_<-y_>)} - e^{|k|(y_<+y_>)}\right] - \frac{e^{|k|(y_<+y_>)}}{|k| + s^2/g}$$

where $\begin{matrix} y_> = y \\ y_< = y_o \end{matrix}$ if $y > y_o$, and $\begin{matrix} y_> = y_o \\ y_< = y \end{matrix}$ if $y < y_o$.

Since $\hat{\tilde{G}}$ is even in k its inverse Fourier transform is

$$\tilde{G}(x,y,s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx}\hat{\tilde{G}}(|k|) = \frac{1}{\pi}\int_0^{\infty} dk \cos kx\, \hat{\tilde{G}}(k) \quad .$$

Use of the following formula

$$\int_0^\infty dk \cos kx \, \frac{1}{k} \left(e^{-ky_1} - e^{-ky_2}\right) = \log\left(\frac{x^2+y_2^{\,2}}{x^2+y_1^{\,2}}\right)^{1/2}, \quad y_1, y_2 > 0$$

finally leads to

$$\tilde{G} = -\frac{1}{2\pi}\left[\log\frac{R^*}{R} + 2\int_0^\infty dk \cos kx \, \frac{e^{k(y+y_0)}}{k+s^2/g}\right]$$

where $R^2 = x^2+(y-y_0)^2$ and $R^{*2} = x^2+(y+y_0)^2$.

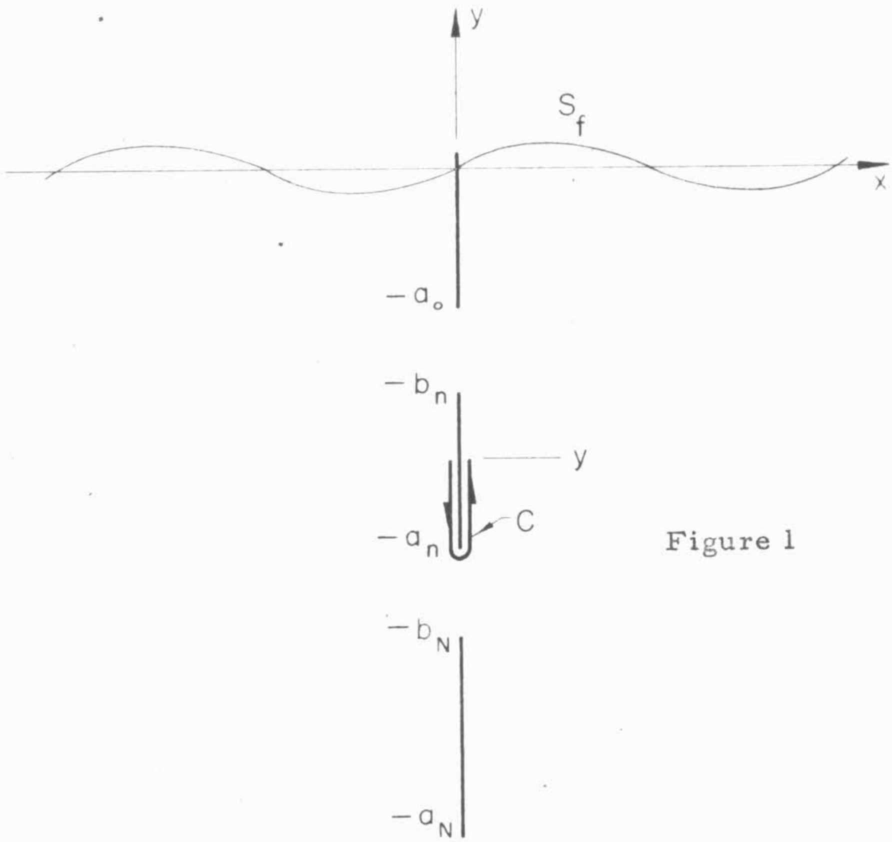

Figure 1

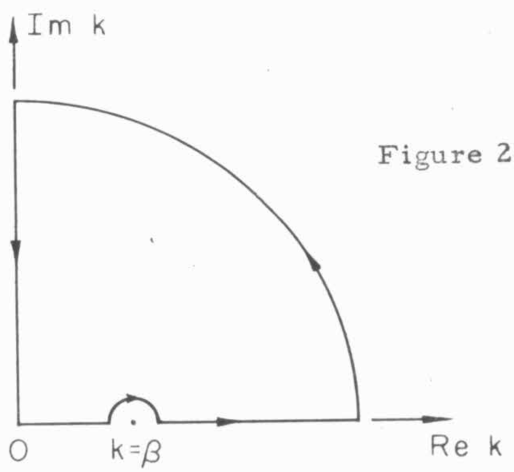

Figure 2

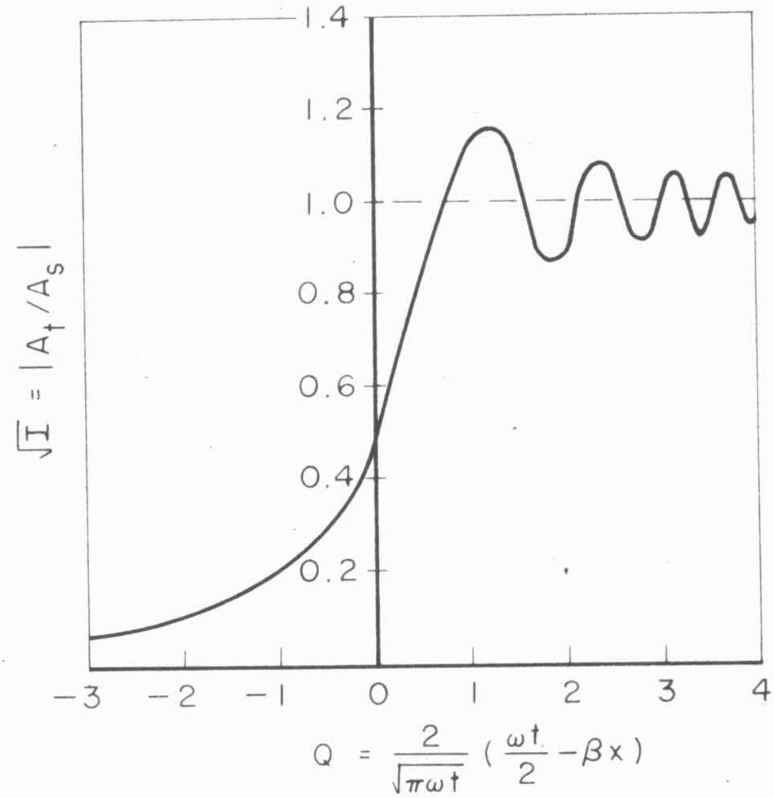

Figure 3 (cf. equation 3.40)

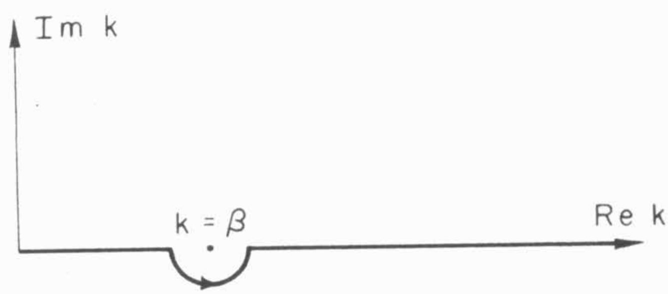

Figure 4

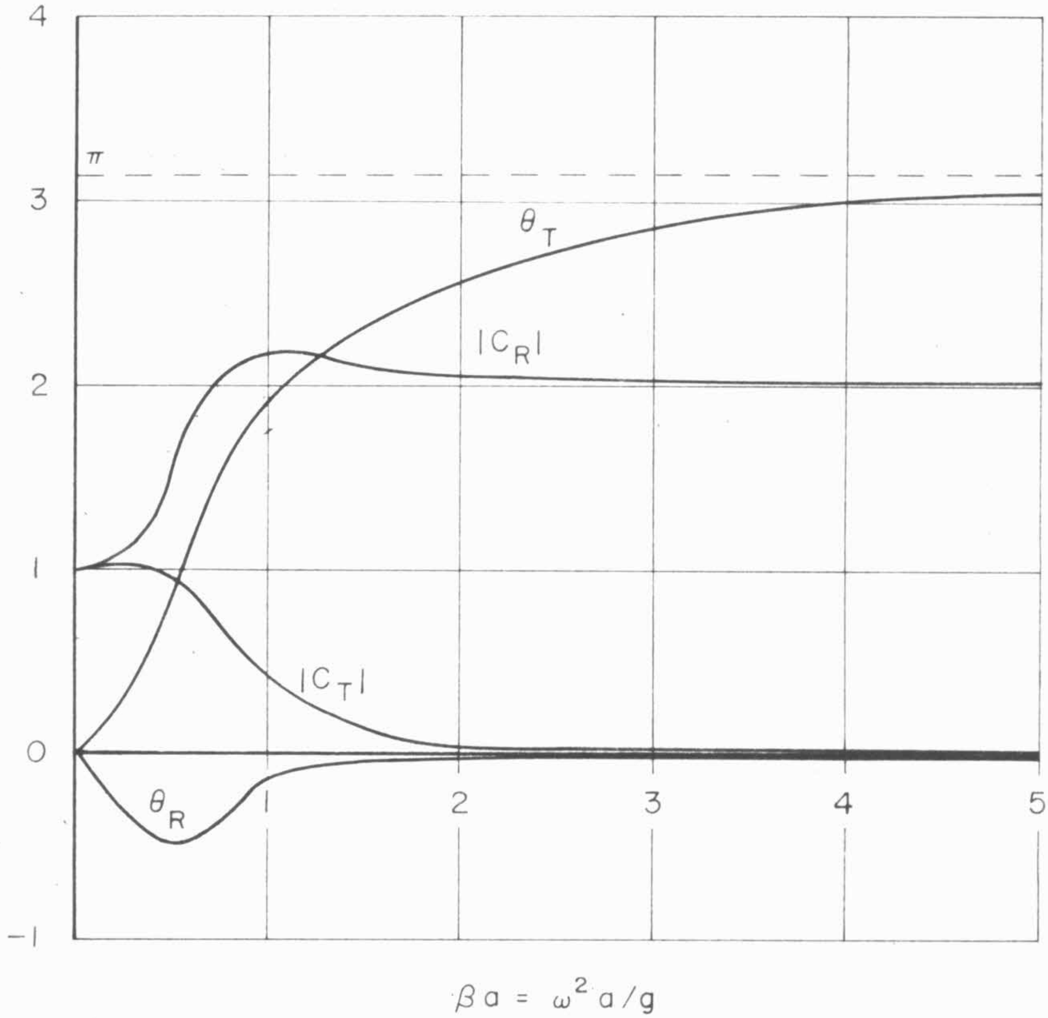

Figure 5 (cf. equations 4.48 and 4.49)

PART TWO (pp 82 - 128)

GRAVITY WAVES DUE TO A POINT DISTURBANCE IN A STRATIFIED FLOW

ABSTRACT

The subject of gravity waves in the two dimensional flow of a vertically stratified fluid is investigated with regard to the dynamic effects of a submerged singularity. Love's linearized equations are adopted as the basis for the theory. Two specific cases are treated according as the parameter N^2 being a constant or a function of depth, where

$$N^2 = \frac{g}{\rho_0} \frac{d\rho_0}{dy}$$

characterizes the density variation in the fluid. The first example of constant N^2 is physically a hypothetical case but can be given an exact mathematical analysis; it is found that in a deep ocean with such a density variation the internal waves are local in nature, i.e., their amplitudes diminish to zero as the distance from the singularity becomes very large. In the second example an asymptotic theory for small Froude number, $U^2/gL \ll 1$, is developed when $N^2(y)$ assumes the profile roughly resembling the actual situation in an ocean where a pronounced maximum called a seasonal thermocline occurs. Internal waves are now progagated to the downstream infinity in a manner analogous to the channel propagation of sound in an inhomogenous medium.

TABLE OF CONTENTS

I. INTRODUCTION

When a layer of lighter fluid is superposed on a heavier one it is well known that waves may occur not only at the upper free surface but at the interface as well. In an ocean the difference in salinity and temperature due, for example, to the melting of ice, frequently gives rise to two such distinct layers. For small density change the wave motion may sometimes be quite pronounced at the interface while the free surface remains relatively calm (reference 7, pp 521-523, Vol. II). This has been attributed as the cause of the so-called dead water phenomenon in which a ship may find herself unable to maintain a normal speed because more power is spent with the interfacial wave motion. Various physical situations also arise in the atmosphere for which the density gradient in air is largely responsible. An example is provided by the lee waves behind a mountain which is evidenced by the experience of glider pilots and the appearance of certain cloud patterns.

Much theoretical work has been done in the past on stratified fluids composed of a number of homogeneous layers of different densities. However, gradual density variation is the case more often found in reality, and therefore seems to deserve further exploration.

We shall begin with a brief discussion on the parameter which characterizes a heavy stratified fluid, such as an ocean. Clearly, if the fluid at rest is stably stratified, the density must increase with increasing depth, since otherwise a fluid parcel slightly displaced upward would experience a buoyancy tending to push it up further. Using a coordinate system in which the positive y axis points downward, we have the following stability criterion:

$$\frac{\delta \rho_0}{dy} \begin{matrix} > \\ = \\ < \end{matrix} 0 \qquad \begin{matrix} \text{stable} \\ \text{neutrally stable} \\ \text{unstable} \end{matrix} \tag{1.1}$$

In practice it is more convenient to use the so-called stability frequency defined by

$$N^2 = \frac{g}{\rho_0} \frac{\delta \rho_0}{dy} \tag{1.2}$$

The precise meaning of $\delta\rho_0$ is made clear by actually computing it as follows (reference 1, p. 196, Vol. 1). Let p, s, and T be the static pressure, salinity and temperature respectively at the level y, and p + dp, s + ds and T + dT the corresponding quantities at a slightly lower level y + dy. When a fluid parcel is displaced from y + dy to y , it will subject to a new pressure p and adjusts its temperature by an amount $-d\theta$ due to an adiabatic expansion, whereas the salinity remains unaffected. Thus the density excess of the parcel relative to its surrounding is

$$\delta\rho_0 = \rho_0(p, s+ds, T+dT-d\theta) - \rho_0(p, s, T) = \frac{\partial\rho}{\partial s} ds + \frac{\partial\rho}{\partial T}(dT - d\theta)$$

Hence,

$$N^2(y) = \frac{g}{\rho_0}\left(\frac{\partial\rho}{\partial s}\frac{ds}{dy} + \frac{\partial\rho}{\partial T}\frac{dT}{dy} - \frac{\partial\rho}{\partial T}\frac{d\theta}{dy} \right) \tag{1.3}$$

The first two terms in the bracket above are due respectively to the variable salinity and temperature of the <u>undisturbed</u> fluid; they may be jointly referred to as

$$\frac{g}{\rho_0}\frac{d\rho_0}{dy} \left(= \frac{g}{\rho_0}\left[\frac{\partial\rho}{\partial s}\frac{ds}{dy} + \frac{\partial\rho}{\partial T}\frac{dT}{dy} \right] \right) \tag{1.4}$$

The third term, being the consequence of adiabatic expansion when the fluid is slightly displaced, can be expressed in terms of the sound speed c:

$$\frac{1}{\rho_0}\frac{\Delta\rho_0}{dy}\bigg|_{ADIA} = \frac{1}{\rho_0}\frac{\partial\rho_0}{\partial T}\frac{d\theta}{dy} = \frac{g}{c^2}$$

Thus equation 1.3 may be written in the following form

$$N^2(y) = g\left(\frac{1}{\rho_0}\frac{d\rho_0}{dy} - \frac{g}{c^2}\right) \qquad (1.5)$$

which is called the Väisälä frequency and can be obtained from measurements.

In an actual ocean it is observed that $N^2(y)$ usually is pretty small in the top layer, rises sharply to a peak at the depth of about 30m, then gradually decreases and rises again slightly to another relatively low peak at a few hundred meters below, and finally diminishes to zero at great depths. The first and the second peaks are called respectively the seasonal and the permanent thermoclines. A typical profile is shown in figure 1, (taken from reference 2). As for the relative magnitude of the two terms in equation 1.5, it is known that $(g/c)^2$ is almost a constant (about 0.44×10^{-4} sec.$^{-2}$) and is comparable with respect to the first term $\frac{g}{\rho_0}\frac{d\rho_0}{dy}$ at large depths.

Simple harmonic waves in stratified fluids have been studied quite extensively, for example, by Yih (reference 3) and Eckart (reference 4) with emphasis on the dispersion relations. In particular, Eckart has investigated the effect of a thermocline. Since the compressiblity of the fluid is included, he was able to deal with waves of both acoustic and gravity types. For oceans a mathematical model using an incompressible fluid i. e.,

$$N^2 = \frac{1}{\rho_0}\frac{d\rho_0}{dy} \tag{1.6}$$

is sufficient to bring forth the essential features of internal waves of the gravity type, and this was first adopted by Love, (reference 5, pp 378-380). Based on Love's equations Yanowitch (reference 6) has investigated the dispersion relation for a stratified fluid with a piece-wise smooth density function, $\rho_0(y)$. In the present thesis we shall follow the approach of Love to neglect the effect of compressibility.

Since the influence of a finite body in a uniform stream is perhaps of some fundamental interest, we shall formulate the problem of a submerged doublet. Most of the discussion, however, will be made with respect to an associated Green's function which is not a physical solution by itself but nevertheless contains the essential features of the problem, and from which various flow quantities for the case of a doublet can be easily obtained by simple differentiation. In Chapter III we study the simplest case of a constant N^2 and obtain the exact solution for an infinitely deep ocean. The internal waves will be found to have only the local effect near the disturbance while the surface waves may or may not appear according as

$$U^2 \left(2g\rho_0 \Big/ \frac{d\rho_0}{dy} \right)^{-1} \lessgtr 1 \tag{1.7}$$

a situation reminiscent of the open channel flow of a homogeneous fluid. Internal waves of the type

$$A(\kappa, y) \begin{matrix} \cos \kappa x \\ \sin \kappa x \end{matrix} \tag{1.8}$$

may occur for the case of constant N^2 only when the depth of the fluid is finite. The corresponding solution can be worked out in an analogous manner as that in Chapter III but is not given here. An approximate solution for constant N^2 and large $\frac{gH}{U^2}$ has been studied by Long (reference 7) when the disturbance is caused by the unevenness of the bottom of a channel with a finite depth. He found that many modes of waves of the above-mentioned type (cf. equation 1. 3) can be excited, the number of which depends on the magnitude of the parameter gH/U^2. In Chapter IV we assume the function $N^2(y)$ to decrease monotonically with depth and thus to resemble to some extent the shape of a thermocline. An asymptotic theory for small Froude number, or large gL/U^2, will be developed using Langer's well-known theory of a second order differential equation with a large parameter. The internal waves generated will appear as a large number of discrete modes each confined inside a channel, similar to the high frequency sound propagation in an atmosphere with a temperature inversion.

II. FORMULATION AND THE GENERAL SOLUTION FOR A SUBMERGED DOUBLET

Consider the steady, two-dimensional flow of an inviscid and incompressible fluid bounded above by a free surface, and having a density which varies with depth. Let

$$
\begin{aligned}
(\bar{u}, \bar{v}) &= \text{velocity components} \\
\bar{p} &= \text{pressure} \\
\bar{\rho} &= \text{density, and} \\
(\bar{x}, \bar{y}) &= \text{spatial coordinates}
\end{aligned}
\tag{2.1}
$$

with positive $\bar{v}$ and $\bar{y}$ pointing vertically downwards. In an otherwise uniform flow with a streaming velocity U, let there be a fluid doublet of strength m immersed at a depth h and oriented in the negative x-direction. The continuity equation then requires that

$$(\bar{\rho}\bar{u})_{\bar{x}} + (\bar{\rho}\bar{v})_{\bar{y}} = -m\,\delta'(\bar{x})\,\delta(\bar{y}-\bar{h}) \tag{2.2}$$

and by the principle of momentum conservation,

$$\bar{\rho}(\bar{u}\bar{u}_{\bar{x}} + \bar{v}\bar{u}_{\bar{y}}) + \bar{p}_{\bar{x}} = 0 \tag{2.3}$$

and,

$$\bar{\rho}(\bar{u}\bar{v}_{\bar{x}} + \bar{v}\bar{v}_{\bar{y}}) + \bar{p}_{y} - g\bar{\rho} = 0 \tag{2.4}$$

For an incompressible fluid, on assumes the following "equation of state":

$$\frac{D\bar{\rho}}{Dt} = \bar{u}\bar{\rho}_{\bar{x}} + \bar{v}\bar{\rho}_{\bar{y}} = 0 \tag{2.5}$$

which states the constancy of density along a streamline in steady flows. This reduces equation 2.2 to a simpler form:

$$\bar{u}_{\bar{x}} + \bar{v}_{\bar{y}} = -[m/\bar{\rho}(\bar{h})]\, \delta'(\bar{x})\, \delta(\bar{y}-\bar{h}) \tag{2.6}$$

Let the free surface of the fluid be described by

$$F(\bar{x}, \bar{y}) = \bar{y} - \bar{\zeta}(\bar{x}) = 0, \tag{2.7}$$

then two conditions hold for a fluid particle lying in the surface. Kinematically the particle never leaves the surface F, i.e.,

$$\frac{DF}{Dt} = \bar{u} F_{\bar{x}} + \bar{v} F_{\bar{y}} = 0 \qquad \text{on } F(x, y) = 0 , \tag{2.8}$$

and dynamically the surface pressure remains unchanged throughout the course of the particle motion, i.e.,

$$\frac{D\bar{p}}{Dt} = \bar{u}\, \bar{p}_{\bar{x}} + \bar{v}\, \bar{p}_{\bar{y}} = 0 \qquad \text{on } F(x, y) = 0 . \tag{2.9}$$

Let each of the flow quantities be expressed as the sum of the equilibrium and the perturbation values:

$$(\bar{u}, \bar{v}) = \{ U + u(\bar{x}, \bar{y}),\ v(\bar{x}, \bar{y}) \}$$

$$\bar{p} = p_0(\bar{y}) + p(\bar{x}, \bar{y}) \tag{2.10}$$

and

$$\bar{\rho} = \rho_0(\bar{y}) + \rho(\bar{x}, \bar{y})$$

in which the static fluid density ρ_0 is a given function of $\bar{y}$. Assuming the disturbance caused by the submerged singularity is so weak that the perturbation quantities are much smaller than the corresponding equilibrium values, we may ignore the second order terms when equations 2.10 are substituted into equations 2.3-6. Thus, in terms of dimensionless variables:

$$(x, y, h, \zeta) = (\bar{x}, \bar{y}, \bar{h}, \bar{\zeta})/L \tag{2.11}$$

with $L = \left(\frac{1}{\rho_0} \frac{d\rho_0}{d\bar{y}} \right)^{-1}_{\bar{h}_0}$ where $\bar{h}_0$ is some depth at which $d\rho_0/dy \neq 0$, we have the following linearized equations valid in the region $y > 0$;

$$p_0' - gL\rho_0 = 0 \tag{2.12}$$

$$\rho_0 U u_x + p_x = 0 \tag{2.13}$$

$$\rho_0 U v_x + p_y - gL\rho = 0 \tag{2.14}$$

$$U\rho_x + \rho_0' v = 0 \tag{2.15}$$

and,

$$u_x + v_y = -[m/L^2\rho_0(h)]\,\delta'(x)\,\delta(y-h) \tag{2.16}$$

As usual, the dash is referred to the ordinary differentiations.

On the free surface the boundary conditions 2.8 and 2.9 are linearized to give:

$$v = U\zeta_x \tag{2.17}$$

for $y = 0$.

$$Up_x + v p_0' = 0 \tag{2.18}$$

Furthermore, the flow field should be unperturbed at both far upstream and a great depth below the free surface, i.e.,

$$(u, v, p, \rho) \to 0 \qquad x \to -\infty \tag{2.19}$$

$$y \to \infty \tag{2.20}$$

The simultaneous equations 2.13 - 16 can be combined to give a single equation for the vertical component of the velocity, v,

$$v_{xx} + v_{yy} + n^2(v_y + \lambda^2 v) = -M\,\delta'(x)\,[\delta'(y-h) + n^2\delta(y-h)], \quad y > 0 \tag{2.21}$$

where

$$\lambda^2 = gL/U^2 = \text{Froude number}^{-1} \tag{2.22}$$

$$n^2(y) = \rho_0'(y)/\rho_0(y) > 0 \tag{2.23}$$

and

$$M = m/L^2\rho_0(h) \tag{2.24}$$

The positive-definiteness of the function $n^2(y)$ implies that the equilibrium density increases with increasing depth, as required for the stability of the stratified fluid. The boundary condition of **v** on the free surface is obtained from equations 2.12, 2.13, 2.16 and 2.18

$$v_y + \lambda^2 v = 0 \qquad\qquad y = 0 \ . \tag{2.25}$$

As for the behavior at infinity, equations 2.19 and 2.20 require that

$$v \to 0 \quad \text{as} \quad \begin{array}{ll} x \to -\infty & (2.26) \\ y \to \infty & (2.27) \end{array}$$

Once $v(x,y)$ is solved from equations 2.21, 2.25-27 the other flow quantieies u, p, ρ and ζ can be obtained from equations 2.16 2.13, 2.15 and 2.17 respectively, giving

$$u(x,y) = -\int_{-\infty}^{x} d\xi \; v_y(\xi,y) - M\,\delta(x)\,\delta(y-h) \tag{2.28}$$

$$p(x,y) = -U\rho_0(y)\,u = U\rho_0(y)\int_{-\infty}^{0} d\xi \; v_y(\xi,y) + \frac{mU}{L^2}\,\delta(x)\,\delta(y-h) \tag{2.29}$$

$$\rho(x,y) = -\frac{1}{U}\rho_0'(y)\int_{-\infty}^{x} d\xi \; v(\xi,y) \tag{2.30}$$

and,

$$\zeta(x) = \frac{1}{U}\int_{-\infty}^{x} d\xi \; v(\xi,0) \tag{2.31}$$

In the preceding formulas the lower limits of integration have been chosen in accordance with the fact that all the perturbation quantities vanish as $x \to -\infty$. Any stream line within the fluid is given implicitly by the following integral equation:

$$y(x) = y_\infty + \int_{-\infty}^{x} d\xi \; v(\xi,y)/[U + u(\xi,y)] \tag{2.32}$$

which can be approximated for points far away from the singularity at (0, h) as,

$$y(x) \cong y_\infty + \frac{1}{U}\int_{-\infty}^{x} d\xi \, v(\xi, y_\infty) \tag{2.33}$$

with

$$y_\infty = y(-\infty) \; . \tag{2.34}$$

The fact that

$$M\,\delta'(x)\left[\,\delta'(y-h) + n^2(h)\,\delta(y-h)\,\right] = -\mathcal{L}\,\delta(x)\,\delta(y-h) \tag{2.35.a}$$

where

$$\mathcal{L} = M\,\frac{\partial}{\partial x}\left[\,\frac{\partial}{\partial h} - n^2(h)\,\right], \tag{2.35.b}$$

suggests the introduction of a new function $G(x,y)$ satisfying

$$v(x,y) = -\mathcal{L}\,G(x,y)\;, \tag{2.36}$$

for, upon substitution, it is easy to see that $G(x,y)$ is governed by the following equations:

$$G_{xx} + G_{yy} + n^2(G_y + \lambda^2 G) = -\,\delta(x)\,\delta(y-h) \qquad y > 0 \tag{2.37}$$

$$G_y + \lambda^2 G = 0 \qquad y = 0 \tag{2.38}$$

$$G \to 0 \quad \text{as} \quad x \to -\infty \tag{2.39}$$

$$\qquad\qquad\quad y \to \infty \;. \tag{2.40}$$

These equations obviously identify G as a Green's function.

It may also be observed that in terms of another function $v^s(x,y)$* such that,

$$v = \frac{\partial}{\partial x}\,v^s \tag{2.41.a}$$

---

* Physically v^s corresponds to the solution for a source immersed at the point $(0,h)$ emitting fluid of the same density as the surrounding.

or, equivalently,

$$V^s = \left[\frac{\partial}{\partial h} - n^2(h) \right] G \tag{2.42}$$

equations 2.28-31 and 2.33 all become simpler:

$$u(x,y) = - V^s_y(x,y) - M\,\delta(x)\,\delta(y-h) \tag{2.43}$$

$$p(x,y) = U\rho_0(y)\,V^s_y(x,y) + \frac{mU}{L^2}\,\delta(x)\,\delta(y-h) \tag{2.44}$$

$$\rho(x,y) = -\frac{1}{U}\,\rho_0'(y)\,V^s(x,y) \tag{2.45}$$

$$\zeta(x) = \frac{1}{U}\,V^s(x,0) \tag{2.46}$$

and

$$y(x) \cong y_\infty + \frac{1}{U}\,V^s(x, y_\infty) \tag{2.47}$$

Summarizing, we note that the whole problem hinges on the solution of the Green's function G from equations 2.37 - 40 ; the flow quantities are then given by equations 2.36 and 2.43 - 47 . Most of the important features in the solution will evidently be revealed by the behavior of the Green's function also.

Let us proceed to derive an integral representation of G. Applying the Fourier transform defined by the following pair of formulas,

$$f(k,y) = \int_{-\infty}^{\infty} dx\, e^{-i\lambda k x}\, G(x,y) \tag{2.48.a}$$

and

$$G(x,y) = \frac{\lambda}{2\pi}\int_{-\infty}^{\infty} dk\, e^{i\lambda k x}\, f(k,y) \tag{2.48.b}$$

we obtain from equations 2.37 - 40 that,

$$f''(y) + n^2(y)\,f'(y) + \lambda^2[n^2(y) - k^2]\,f(y) = -\delta(y-h), \qquad y > 0 \tag{2.49}$$

$$f' + \lambda^2 f = 0, \qquad y = 0 \tag{2.50}$$

$$f \to 0 \qquad y \to \infty \tag{2.51}$$

The solution for $f(y)$ is known to be of the following form:

$$f(y) = f_1(y_>)\, f_2(y_<)\, / \, W(h) \tag{2.52.a}$$

where

$$y_< = h, \quad y_> = y \qquad \text{if} \quad y > h$$

$$y_< = y, \quad y_> = h \qquad \text{if} \quad y < h \tag{2.52.b}$$

and

$$W(y) = f_1(y)\, f_2'(y) - f_1'(y)\, f_2(y) \tag{2.52.c}$$

is the Wronskian of the functions f_1 and f_2 which are two linearly independent solutions of the corresponding homogeneous equation

$$f'' + n^2 f' + \lambda^2 (n^2 - k^2) f = 0 \tag{2.53}$$

and are such that,

$$f_1 \to 0 \qquad \text{as} \qquad y \to \infty \tag{2.54}$$

$$f_2' + \lambda^2 f_2 = 0 \qquad \text{for} \qquad y = 0 \tag{2.55}$$

Finally the inversion formula gives the Green's function G.

Explicit examples will be worked out in the subsequent chapters.

III. THE CASE OF CONSTANT $n^2(y)$

Let us consider the special case of a deep ocean with the simple property:

$$n^2(y) = \rho_0'/\rho_0 = n^2 = \text{constant} \tag{3.1}$$

which represents a fluid with an equilibrium density increasing exponentially with depth, i.e.,

$$\rho_0(y) = \rho_0(0)\, e^{n^2 y} \tag{3.2}$$

This is of course a highly idealized model of which the discrepancies are perhaps comparable with those of an isothermal atmosphere of infinite height in the theory of atmospheric waves, where ρ_0 decreases exponentially with the height above the ground. In both cases the solution should not be expected to agree with physical reality at the levels where the distances from the water surface or ground are great.

Since the coefficients involved in the governing equations are now constants, an exact solution is possible. It is easy to show that the following functions, f_1 and f_2, are the two linearly independent solutions of equation 2.53, satisfying the boundary conditions 2.54 and 2.55 respectively:

$$f_1 = e^{-(A-B)y} \tag{3.3.a}$$

$$f_2 = (\lambda^2 - A - B)\, e^{-(A-B)y} - (\lambda^2 - A + B)\, e^{-(A+B)y} \tag{3.3.b}$$

in which,

$$A = n^2/2 , \qquad B = i\lambda\sqrt{\beta^2 - k^2} , \qquad \beta^2 = n^2 - n^4/4\lambda^2 \tag{3.3.c}$$

and the square root $\sqrt{\beta^2 - k^2}$ will be defined to behave like

$$\sqrt{\beta^2 - k^2} \longrightarrow i|k| \qquad \text{as} \qquad |k| \to \infty \tag{3.3.d}$$

along the real axis of the k-plane cut along two vertical lines: $(\beta, \beta + i\infty)$ and $(-\beta, -\beta - i\infty)$. The solution for f follows immediately from equation 2.52 . In order to have a real Green's function, we shall take,

$$f(k, y) = Re\, f_1(k, y_>) f_2(k, y_<)/W(k, h) \tag{3.4}$$

which may be substituted into equation 2.48.b to give,

$$G(x, y) = \frac{\lambda}{2\pi} Re \int_{-\infty}^{\infty} dk\, e^{i\lambda k x} f_1(k, y_>) f_2(k, y_<)/W(k, h)$$

$$= \frac{1}{2\pi} e^{-A(y-h)} Re\, \frac{i}{2} \int_{-\infty}^{\infty} dk \frac{e^{i\lambda k x}}{\sqrt{\beta^2 - k^2}} \left\{ e^{i\lambda\sqrt{\beta^2-k^2}(y_> - y_<)} - e^{i\lambda\sqrt{\beta^2-k^2}(y_> + y_<)} \right\}$$

$$- \frac{1}{2\pi} e^{-A(y-h)} Re \int_{-\infty}^{\infty} dk \frac{e^{i\lambda[kx + \sqrt{\beta^2-k^2}(y+h)]}}{i\sqrt{\beta^2 - k^2} + \alpha} \tag{3.5.a}$$

with

$$\alpha = \lambda - A/\lambda = \lambda - n^2/2\lambda \tag{3.5.b}$$

In obtaining the first equality of equation 3.5.a use has been made of the evenness of $f(k, y)$ in k. It may be remarked that according to our nomenclature (cf. equation 2.52.b) $y_> - y_< = |y - h|$ and $y_> + y_< = y + h$, and both quantities are positive.

The integral,

$$\mathcal{J} = \int_{-\infty}^{\infty} dk \frac{e^{i(kX + \sqrt{\beta^2-k^2}\,Y)}}{\sqrt{\beta^2 - k^2}} \tag{3.6.a}$$

can be transformed to a well-known representation of the Hankel function of the first kind by introducing,

$$k = \beta\cos\theta, \quad X = \bar{R}\cos\phi \qquad Y = \bar{R}\sin\phi$$

The result is (see, e.g., reference 8, pp. 823-824),

$$\mathcal{J} = \int_{-\frac{\pi}{2}+i\infty}^{\frac{\pi}{2}-i\infty} d\theta \, e^{i\beta \bar{R}\cos\theta} = \pi H_0^{(1)}(\beta \bar{R}) . \qquad (3.6.b)$$

Equation 3.5.a may now be rewritten in the following form:

$$G(x,y) = \frac{1}{4} e^{-A(y-h)} \, Re \, i\left[H_0^{(1)}(\lambda\beta R) - H_0^{(1)}(\lambda\beta R^*)\right]$$
$$- \frac{1}{2\pi} e^{-A(y-h)} \, Re \int_{-\infty}^{\infty} dk \, \frac{e^{i\lambda[kx + \sqrt{\beta^2-k^2}(y+h)]}}{i\sqrt{\beta^2-k^2} + \alpha} \qquad (3.7.a)$$

where

$$R = [x^2 + (y-h)^2]^{1/2} , \quad R^* = [x^2 + (y+h)^2]^{1/2} . \qquad (3.7.b)$$

When $\beta^2 > 0$, or β = real and positive, we may use the fact that

$$H_0^{(1)}(\lambda\beta R) = J_0(\lambda\beta R) + i\, Y_0(\lambda\beta R)$$

and hence the **integrated term in equation 3.6.a may be written**

$$Re \, i\left[H_0^{(1)}(\lambda\beta R) - H_0^{(1)}(\lambda\beta R^*)\right] = Y_0(\lambda\beta R^*) - Y_0(\lambda\beta R) . \qquad (3.8.a)$$

On the other hand, when $\beta^2 < 0$, or $\beta = ib$ = imaginary, we have instead,

$$Re \, i\left[H_0^{(1)}(\lambda\beta R) - H_0^{(1)}(\lambda\beta R^*)\right] = \frac{2}{\pi}\left[K_0(\lambda b R) - K_0(\lambda b R^*)\right] \qquad (3.8.b)$$

by noting that $i H_0^{(1)}(i\lambda bR) = \frac{2}{\pi} K_0(\lambda bR)$. These various representations will be useful in later discussions.

To proceed the examination of the obtained Green's function we first observe that the integrand of the remaining integral in equation 3.7.a possesses simple poles on the real k-axis if simple zeroes can

be found from the following equation:

$$i\sqrt{\beta^2 - k^2} = -\alpha = -(\lambda - n^2/2\lambda) \tag{3.9}$$

The right hand side of the preceding equation is always real. In the case when $\beta^2 = n^2(1-n^2/4\lambda^2) > 0$ the left hand side is real only when k is real and $|k| > \beta$, and is then equal to $-|\sqrt{k^2-\beta^2}|$ on our chosen branch. Consequently real roots will exist only if $\alpha > 0$, or $1 > n^2/2\lambda^2$. If $\beta^2 = -b^2 < 0$, $i\sqrt{\beta^2-k^2} = |\sqrt{k^2+b^2}|$ for all real k, hence the same conditions holds for the existence of real roots. It is easy to show that when they exist, the real zeroes are located at the points $k = \pm\lambda$. We may summarize with the statement as follows:

$$\text{Simple real zeroes of eq. 3.9} \begin{matrix} \text{exist at } k = \pm\lambda \\ \text{do not exist} \end{matrix} \quad \text{if } \alpha \begin{matrix} > \\ < \end{matrix} 0 \text{ or } 1 \begin{matrix} > \\ < \end{matrix} \frac{n^2}{2\lambda^2} \tag{3.10}$$

When $\alpha = 0$, the roots at $k = \pm\beta$ are only two branch points.

When its integrand has simple poles on the real k-axis, the integral in equation 3.7.a is not well defined as it stands. However, we can define it by fixing the path of integration such that the singularities are avoided. At each pole this can be achieved by deforming the real k-axis either above or below the pole; the correct choice is dictated by the requirement of equation 2.39 that $G(x,y)$ should die out as $x \to -\infty$. Let us decide to adopt the deformed path C which circumvents both poles at $k = \lambda$ and $-\lambda$ from below along two infinitesimal semi-circles S_+ and S_- respectively, as shown in figure 2.a, and then study the behavior the resulting integral as $|x| \to \infty$, i.e.,

$$\mathcal{I} = \int_C dk \, \frac{e^{i(kX + \sqrt{\beta^2 - k^2}\, Y)}}{i\sqrt{\beta^2 - k^2} + \alpha} , \qquad (X, Y) = \lambda(x, y+h) \tag{3.11.a}$$

The above integral can be transformed successively as follows:

$$\mathcal{I} = \left\{ P.V.\int_{-\infty}^{\infty} + \int_{S_+} + \int_{S_-} \right\} dk \, \frac{e^{i(kX + \sqrt{\beta^2 - k^2}\, Y)}}{i\sqrt{\beta^2 - k^2} + \alpha} \tag{3.11.b}$$

$$= P.V.\int_{-\infty}^{\infty} dk \, \frac{e^{i(k|X| + \sqrt{\beta^2 - k^2}\, Y)}}{i\sqrt{\beta^2 - k^2} + \alpha} + \left\{ \int_{S_+} + \int_{S_-} \right\} dk \, \frac{e^{i(kX + \sqrt{\beta^2 - k^2}\, Y)}}{i\sqrt{\beta^2 - k^2} + \alpha} \tag{3.11.c}$$

$$= \left\{ \int_{C^*} + \int_{S_+^*} + \int_{S_-^*} \right\} dk \, \frac{e^{i(k|X| + \sqrt{\beta^2 - k^2}\, Y)}}{i\sqrt{\beta^2 - k^2} + \alpha} + \left\{ \int_{S_+} + \int_{S_-} \right\} dk \, \frac{e^{i(kX + \sqrt{\beta^2 - k^2}\, Y)}}{i\sqrt{\beta^2 - k^2} + \alpha} \tag{3.11.d}$$

$$= \int_{C^*} dk \, \frac{e^{i(k|X| + \sqrt{\beta^2 - k^2}\, Y)}}{i\sqrt{\beta^2 - k^2} + \alpha} - H(\alpha)\, H(x)\, \frac{4\pi}{\lambda} \sqrt{\lambda^2 - \beta^2}\, \sin \lambda^2 x \, \exp[-\lambda\sqrt{\lambda^2 - \beta^2}\,(y+h)] \tag{3.11.e}$$

$$= \int_{\Gamma} dk \, \frac{e^{i(k|X| + \sqrt{\beta^2 - k^2}\, Y)}}{i\sqrt{\beta^2 - k^2} + \alpha} - H(\alpha)\, H(x)\, \frac{4\pi}{\lambda} \sqrt{\lambda^2 - \beta^2}\, \sin \lambda^2 x \, \exp[-\lambda\sqrt{\lambda^2 - \beta^2}\,(y+h)] . \tag{3.11.f}$$

The procedure involved above will now be explained step by step with reference to figures 2.a and 2.b. From the construction of the contour C, equation 3.11.b follows at once from 3.11.a where the symbol P.V. refers to the Cauchy principal value. In the first term of equation 3.11.b

3.11.b the "spectrum function"

$$\frac{e^{i\sqrt{\beta^2-k^2}\,Y}}{i\sqrt{\beta^2-k^2}+\alpha}$$

is even in k which is purely real within the range of integration, hence x may be replaced by $|x|$, as expressed by equation 3.11.c. The broken path may then be considered as the sum of a continuous contour C^*, which is the mirror reflection of C, and of two small semi-circles S_+^* and S_-^* spanning the gaps at $k = \lambda$ and $-\lambda$ respectively from above in the counter-clockwise direction (cf. figure 2.b), thus equation 3.11.d is resulted. To see how to arrive at equation 3.11.e, we note that when $x > 0$, S_+^* and S_+ together form a closed circuit and the sum of their corresponding integrals can be evaluated by using the residue theorem; the same is true for the pair S_-^* and S_-. When $x < 0$, $|x| = -x$; by changing the variable from k to $-k$ the integral along S_+^* is easily seen to cancel exactly the one along S_-. In a similar manner the integrals along S_-^* and S_+ cancel also for $x < 0$. Thus equation 3.11.e follows. A Heaviside function $H(\alpha)$ is used as a multiplier to the residue term, on account of equation 3.10. Finally the contour C^* can be replaced by the contour Γ along the sides of the branch cut extending from β to $\beta + i\infty$ (cf. figure 2.b), without altering the value of the corresponding integral. This leads to the last equation 3.11.f. It may be added that

$$\lambda^2-\beta^2 = \lambda^2\left[1-\frac{n^2}{\lambda^2}\left(1-\frac{n^2}{4\lambda^2}\right)\right] = \lambda^2\left(1-\frac{n^2}{2\lambda^2}\right)^2 = \lambda^2\alpha^2 \geq 0 \qquad (3.12)$$

It will be shown in Appendix A that the branch cut integral in 3.11.f dies out as $|x| \to \infty$. Upon substituting the above result into equation 3.7.a and making use of the asymptotic properties of the Hankel functions, one readily sees that the "radiation condition" 2.39 is indeed satisfied with the contour C chosen by us. The solution for the Green's function is therefore:

$$G(x,y) = -\frac{1}{4} e^{-A(y-h)} \operatorname{Re}\, i\left[H_0^{(1)}(\lambda\beta R) - H_0^{(1)}(\lambda\beta R^*)\right]$$

$$-\frac{1}{2\pi} e^{-A(y-h)} \operatorname{Re}\int_C dk\, \frac{e^{i\lambda[kx+\sqrt{\beta^2-k^2}(y+h)]}}{i\sqrt{\beta^2-k^2}+\alpha} \qquad (3.13.a)$$

$$= -\frac{1}{4} e^{-A(y-h)} \operatorname{Re}\, i\left[H_0^{(1)}(\lambda\beta R) - H_0^{(1)}(\lambda\beta R^*)\right]$$

$$+ H(\alpha) H(x) \frac{1}{\lambda}\sqrt{\lambda^2-\beta^2}\, \sin\lambda^2 x\, \exp\left[-A(y-h) - \lambda\sqrt{\lambda^2-\beta^2}(y+h)\right]$$

$$-\frac{1}{2\pi} e^{-A(y-h)} \operatorname{Re}\, P.V.\int_{-\infty}^{\infty} dk\, \frac{e^{i\lambda[kx+\sqrt{\beta^2-k^2}(y+h)]}}{i\sqrt{\beta^2-\lambda^2}+\alpha} \qquad (3.13.b)$$

$$= H(\alpha) H(x) \frac{2}{\lambda}\sqrt{\lambda^2-\beta^2}\, \sin\lambda^2 x\, \exp\left[-A(y-h) - \lambda\sqrt{\lambda^2-\beta^2}(y+h)\right]$$

$$-\frac{1}{4} e^{-A(y-h)} \operatorname{Re}\, i\left[H_0^{(1)}(\lambda\beta R) - H_0^{(1)}(\lambda\beta R^*)\right]$$

$$-\frac{1}{2\pi} e^{-A(y-h)} \operatorname{Re}\int_\Gamma dk\, \frac{e^{i\lambda[k|x|+\sqrt{\beta^2-k^2}(y+h)]}}{i\sqrt{\beta^2-k^2}+\alpha} \qquad (3.13.c)$$

The preceding formula will be studied in greater detail later. At this stage we may recall a few facts from the theory of gravity waves in homogeneous fluids. In the case when there is a disturbance near the origin, in an otherwise uniform flow, simple poles on the real k-axis correspond to waves that propagate downstream to infinity. If the depth of the fluid is finite, say H, then those far-going waves will or will not be present according as

$$1 > U^2/gH \qquad \text{or} \qquad 1 < U^2/gH \tag{3.14}$$

respectively. The value $\sqrt{gH}$ is called the critical speed of the open channel flow. In stratified fluid the simple poles also give rise to the downstream-going waves as represented by the first term in equation 3.13.b. Comparing the two criteria 3.10 and 3.14, and noting by definition that

$$n^2/2\lambda^2 = U^2/gH_e \qquad H_e = 2L\rho_0/\rho_0' \tag{3.15}$$

we may conclude that in the present example, the increasing fluid density produces an effect equivalent to an effective depth H_e of an open channel. The quantity $\sqrt{gH_e}$ may therefore be considered as an equivalent critical speed.

Before going into the discussion of the general case, we cite the following limiting cases which may serve as useful checks.

<u>(i) $\alpha = 0(n^2 = 2\lambda^2,\ \beta = \lambda)$:</u>

The integral in equation 3.7.a can be expressed as a Hankel function (cf. equations 3.6.a and b) which in turn leads to a Weber function (cf. equation 3.8.a). Thus we have

$$G(x,y) = -\frac{1}{4} e^{-\lambda^2(y-h)} [\, Y_0(\lambda^2 R) + Y_0(\lambda^2 R^*) \,] \tag{3.16}$$

and from equation 2.36, the vertical component of the velocity

$$V(x,y) = -M \frac{\partial}{\partial x}\left(\frac{\partial}{\partial h} - 2\lambda^2\right) G(x,y)$$

$$= \lambda^2 M \left[\frac{\partial}{\partial h} - 2\lambda^2\right] \times e^{-\lambda^2(y-h)} \left\{ \frac{1}{R} Y_1(\lambda^2 R) + \frac{1}{R^*} Y_1(\lambda^2 R^*) \right\} \tag{3.17}$$

No far-going surface waves appear in this case. Since

$$Y_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\pi}{4} - \frac{\nu\pi}{2}\right), \tag{3.18}$$

the internal waves die out with $|x|$ at an equal rate in both upstream and downstream directions like $|x|^{-\frac{1}{2}}$; they are strong above and diminish exponentially below the level of the doublet $(y = h)$. However, unlike a surface wave which decays monotonically with depth, the present solution is oscillatory in y because of the behavior of the Weber functions.

(ii) $n^2 = 0$ ($\beta^2 = A = 0$, $\alpha = \lambda > 0$, $\mathcal{L} = \frac{\partial}{\partial h}$) - - - homogeneous fluid:

Using the formula

$$Y_0(\lambda\beta R) \cong \frac{2}{\pi} \log(\lambda\beta R), \quad \beta \to 0 \tag{3.19}$$

we obtain from 3.13.a and b two equivalent expressions as follows

$$G(x,y) = \frac{1}{2\pi} \log R^*/R + \frac{Re}{2\pi} \int_C dk \, \frac{e^{\lambda[ikx - |k|(y+h)]}}{|k| - \lambda}$$

$$= \frac{1}{2\pi} \log R^*/R + \frac{1}{\pi} P.V. \int_0^\infty dk \, \frac{\cos kx \, e^{-k(y+h)}}{k - \lambda^2}$$

$$+ \sin \lambda^2 x \, e^{-\lambda^2 (y+h)} \; .$$

Since, $\frac{\partial}{\partial h} R = - \frac{\partial}{\partial y} R$

we have

$$V(x,y) = -M \frac{\partial^2}{\partial x \partial y} \Big[\frac{1}{2\pi} \log RR^* + \frac{1}{\pi} P.V. \int_0^\infty dk \, \frac{\cos kx \, e^{-k(y+h)}}{k - \lambda^2}$$

$$+ \sin \lambda^2 x \, e^{-\lambda^2 (y+h)} \Big] \qquad (3.20)$$

in which the bracket may be recognized as the velocity potential of a submerged source in a uniform stream of a homogeneous fluid (reference 9, p. 489).

We now return to equation 3.13.c. Consider first the case $\beta^2 = n^2 \left(1 - \frac{n^2}{4\lambda^2}\right) > 0$. **The Hankel functions can be replaced by the Weber functions (cf. equation 3.8.a) and the two branch points at** $k = \pm \beta$ **are located on the real k-axis. For large** $|x|$ **and a fixed** y,

$$R^* = [x^2 + (y+h)^2]^{1/2} = [x^2 + (y-h)^2 + 4hy]^{1/2}$$

$$\cong R(1 + 2yh/R^2),$$

thus one may use the asymptotic property of the Weber function (cf. equation 3.18) to obtain

$$\frac{1}{4}[Y_0(\lambda\beta R^*) - Y_0(\lambda\beta R)] \cong \frac{1}{4}[Y_0(\lambda\beta R + 2\lambda\beta h \frac{y}{R}) - Y_0(\lambda\beta R)]$$

$$\cong -\frac{1}{4}(2\lambda\beta h)\frac{y}{R} Y_1(\lambda\beta R)$$

$$\cong -\sqrt{\frac{\lambda\beta}{2\pi}} \frac{hy}{|x|^{3/2}} \sin(\lambda\beta|x| - \frac{3\pi}{4}) \tag{3.21}$$

The integral along the branch cuts is evaluated asymptotically for large $|x|$ and a fixed y in Appendix A with the following result:

$$\int_\Gamma dk \frac{e^{i(k|X| + \sqrt{\beta^2-k^2}\,Y)}}{i\sqrt{\beta^2-k^2}+\alpha} \cong i\sqrt{2\pi\beta} \frac{Y - \frac{1}{\alpha}}{\alpha|X|^{3/2}} e^{i(\beta|X| - \frac{3\pi}{4})} \tag{3.22}$$

Combining equations 3.21 and 3.22 with 3.13.c, we have

$$G(x,y) \cong e^{-\frac{n^2}{2}(y-h)} \{ H(\alpha) H(x) \frac{2}{\lambda}\sqrt{\lambda^2-\beta^2} \sin\lambda^2 x \; e^{-\lambda\sqrt{\lambda^2-\beta^2}(y+h)}$$

$$+ [\frac{1}{\lambda\alpha}(y + h - \frac{1}{\lambda\alpha}) - yh]\sqrt{\frac{\lambda\beta}{\alpha\pi}} |x|^{-3/2} \sin(\lambda\beta|x| - \frac{3\pi}{4}) \} \tag{3.23}$$

The first term in the above expression represents obviously a surface wave of wavelengh $2\pi/\lambda^2$, which exists only downstream of the disturbance and persists at $x \rightarrow \infty$. The second term represents an internal wave which propagates with equal strength both ahead and behind the disturbance, and with the wavelength $2\pi/\lambda\beta$ which is longer than that of the surface wave since $\lambda > \beta$ (cf. equation 3.12). Although they die out like $|x|^{-3/2}$ in the horizontal direction, the internal waves do not diminish with depth as fast as the surface waves; in fact the former starts decaying exponentially only below the depth of the disturbance $(y = h)$.

In the other case of $\beta^2 = -b^2 < 0$, the surface wave is the only significant term for large $|x|$:

$$G(x,y) \cong H(\alpha) H(x) \frac{2}{\lambda} \sqrt{\lambda^2 + b^2} \sin \lambda^2 x \exp\left[-\frac{n^2}{2}(y-h) - \lambda\sqrt{\lambda^2+b^2}(y+h)\right] \qquad (3.24)$$

The remaining terms in equation 3.13.c are exponentially small for large $|x|$ as can be seen by first using equation 3.8.b and the following asymptotic property of the modified Bessel function:

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z}, \qquad z \rightarrow \infty$$

and by carrying out the same analysis as before for the integral around the branch cut, which is now entirely on the imaginary k-axis. The details are omitted.

IV. ASYMPTOTIC SOLUTION FOR LARGE λ^2: CHANNEL PROPAGATION OF INTERNAL GRAVITY WAVES

In this chapter, we shall study a case where the density stratification is such that $n^2(y)$ decreases monotonically with depth, i. e.,

$$n'(y) < 0 , \quad \text{for} \quad y \geqq 0 \tag{4.1}$$

as depicted by figure 3, which partly fits the shape of a thermocline. From the end of Chapter II, the solution of our problem is seen to hinge on the explicit solution of a second order ordinary differential equation. This in general is not feasible unless the expression for $n^2(y)$ is extremely simple. However, for very large values of the parameter $\lambda^2 = gL/U^2$, the results of Langer's asymptotic theory (reference 10, p. 91 ff) can be used even when $n^2(y)$ is prescribed in as general a way as the present*. We shall therefore limit our investigations to the case where $\lambda^2 \gg 1$. Furthermore, only the Green's function will be discussed in detail as it reveals to a sufficiently full extent the essential features of the problem, and from it the physical quantities u, v, p and ρ can be easily computed in a straightforward manner.

The transformation,

$$f(y) = e^{-\frac{1}{2}\int^y n^2 dy} F(y) \tag{4.2}$$

---

* For a similar application of Langer's theory to the acoustics of inhomogeneous media, see, e. g., reference 11 and 12.

brings equation 2.53 to a standard form:

$$F'' + \{\lambda^2[n^2(y) - k^2] - (nn' + n^4/4)\} F = 0 \tag{4.3}$$

which belongs to the general class treated by Langer. To obtain asymptotic solutions uniformly valid even near the turning point $y = Y$ at which

$$n^2(Y) = k^2, \qquad |k| < n(O), \tag{4.4}$$

Langer introduces the following further transformation:

$$F(y) = (\phi')^{-1/2}\, \xi(y) \tag{4.5}$$

with
$$\phi \phi'^2 = n^2(y) - k^2. \tag{4.6}$$

Equation 4.3 can then be written as

$$\frac{d^2\xi}{d\phi^2} + \lambda^2 \phi \xi = \Omega \xi \tag{4.7.a}$$

where,

$$\Omega = \frac{1}{2}\frac{\phi'''}{(\phi')^2} - \frac{3}{4}\frac{(\phi'')^2}{(\phi')^4} - (\phi')^{-2}(nn' + n^4/4) \tag{4.7.b}$$

Since the function Ω is bounded if

$$\phi'(y) \neq 0 \qquad \text{for} \quad y \geqq 0, \tag{4.8}$$

and if $\phi(y)$ is three times differentiable, for large λ^2 equation 4.7.a becomes approximately,

$$\frac{d^2\xi}{d\phi^2} + \lambda^2 \phi \xi = 0. \tag{4.9}$$

The solutions of the preceding equations are the Airy functions, the properties of which are briefly listed in Appendix B. Thus one may expect that

$$\left\{ \begin{matrix} F_a \\ F_b \end{matrix} \right\} \cong (\phi')^{-1/2} \left\{ \begin{matrix} Ai(-\lambda^{2/3}\phi) \\ Bi(-\lambda^{2/3}\phi) \end{matrix} \right\} \tag{4.10}$$

are two linearly independent solutions of equation 4. 3 in the asymptotic sense; this is indeed true and is justified rigorously in the reference cited. The omitted terms are of the order $0(\lambda^{-1})$ relative to those given above.

The function $\phi(y)$ can be solved from equation 4. 6 with the following result:

for $y < Y$,
$$\frac{2}{3}\phi^{3/2} = \int_y^Y \sqrt{n^2(\sigma) - k^2}\, d\sigma , \quad \phi > 0 \tag{4.11.a}$$

for $y > Y$,
$$\frac{2}{3}(-\phi)^{3/2} = \int_Y^y \sqrt{k^2 - n^2(\sigma)}\, d\sigma , \quad \phi < 0 \tag{4.11.b}$$

in which the positive values of all fractional powers are to be taken.

The condition that ϕ' should be non-sero may now be examined. The only place where ϕ' may vanish is at the turning point, as is evident from equation 4. 6. In the neighborhood of such a point we have, from, for example, equation 4. 11. b

$$\frac{2}{3}(-\phi)^{3/2} \cong \int_Y^y \left\{ -[n^2(Y)]'(\sigma - Y) - \frac{1}{2}[n^2(Y)]''(\sigma - Y)^2 - \cdots \right\}^{1/2} d\sigma$$

$$\cong \left\{ -[n^2(Y)]' \right\}^{1/2} \frac{2}{3}(y - Y)^{3/2} + O[(y - Y)^2]$$

Consequently for ϕ' to be non-zero it is necessary to have $[n^2(y)]' \neq 0$ which is guaranteed by our monotonicity assumption (cf. equation 4. 1).

In view of the asymptotic properties of the Airy functions the solution f_1 which is to vanish as $y \to \infty$ should be of the following form,

$$f_1 \cong e^{-\frac{1}{2}\int^y n^2 dy} (\phi')^{-1/2} Ai(-\lambda^{2/3}\phi) \tag{4.12}$$

To determine f_2 we first rewrite the free surface boundary condition 2. 55 in terms of $\xi(y)$,

$$\begin{aligned} f' + \lambda^2 f &= 0 \\ &= e^{-\frac{1}{2}\int^y n^2 dy} [F' - (\lambda^2 + n^2/2) F] \\ &= e^{-\frac{1}{2}\int^y n^2 dy} (\phi')^{-1/2} \{ \xi' + \lambda^2 \xi + [n^2 - \phi''/2\phi'] \xi \}, \quad \text{for } y = 0 \end{aligned}$$

Since the square bracket in the last expression is always bounded, we obtain for large λ^2, that

$$\xi' + \lambda^2 \xi \cong 0 \qquad \text{for } y = 0, \tag{4.13}$$

in agreement with the present degree of approximation. The asymptotic expression of f_2 should therefore be

$$f_2 \cong e^{-\frac{1}{2}\int^y n^2 dy} (\phi')^{-1/2} \{ Q\, Ai(-\lambda^{2/3}\phi) - P\, Bi(-\lambda^{2/3}\phi) \} \tag{4.14. a}$$

where

$$\begin{Bmatrix} P \\ Q \end{Bmatrix} = \left(\frac{\partial}{\partial y} + \lambda^2\right) \begin{Bmatrix} Ai(-\lambda^{2/3}\phi) \\ Bi(-\lambda^{2/3}\phi) \end{Bmatrix}_{y=0} \tag{4.14. b}$$

With the aid of identity B. 1 in Appendix B the Wronskian as defined by equation 2. 52. c can be easily calculated; the result

$$W(k, y) \cong \frac{1}{\pi} \lambda^{2/3} e^{-\int^{y} n^2 dy} P \tag{4.15}$$

Equations 4. 12, 4. 14. a and 4. 15 may now be substituted into 2. 52. a for the Creen's function:

$$G(x, y) = \frac{\lambda}{2\pi} \int_{-\infty}^{\infty} dk\, e^{i\lambda k x} f_1(k, y_>) f_2(k, y_<)/W(k, h)$$

$$\cong -\frac{1}{2} \lambda^{1/3} e^{-\frac{1}{2}\int_h^y n^2 dy} \int_{-\infty}^{\infty} dk\, e^{i\lambda k x} Ai(-\lambda^{2/3}\phi_>) \{$$

$$Q\, Ai(-\lambda^{2/3}\phi_<) - P\, Bi(-\lambda^{2/3}\phi_<)\} (\phi_>' \phi_<')^{-1/2} P^{-1} \tag{4.16. a}$$

in which,

$$\begin{matrix} \phi_> \\ \phi_< \end{matrix} \equiv \begin{matrix} \phi(y_>) \\ \phi(y_<) \end{matrix} \tag{4.16. b}$$

The integrand of equation 4. 16. a is a single-valued analytic function of k except at the possible poles in the complex k-plane. At the first glance the points $k = \pm n(y_i)$ where $y_i = 0, y_<$, and $y_>$ may appear to be branch points; a more careful study shows, however, that they are not. The details of the argument is given in Appendix C.

As is usual in many wave problems, if the integrand of the Fourier integral representation of the solution has simple poles on the real k-axis, then these poles contribute most significantly to the far field. An examination of equation 4. 16. a shows that such poles exist at $k = \pm |k_m|$, m = 1, 2, 3... which are the real roots of the following equation:

$$P = \lambda^2 \{ Ai(-\lambda^{2/3}\phi_m) - \lambda^{-4/3} \phi_m' \dot{A}i(-\lambda^{2/3}\phi_m) \} = 0 \qquad (4.17.a)$$

where

$$\frac{2}{3}(\phi_m)^{3/2} = \int_0^{Y_m} \sqrt{n^2(\sigma) - k_m^2} \quad d\sigma \qquad (4.17.b)$$

and

$$\dot{A}i(z) = \frac{d}{dz} Ai(z) \qquad (4.17.c)$$

More detailed study of the foregoing equation will be postponed until later. Following exactly the same procedure which led to equation 3.13 we can show that, in order not to have waves at upstream infinity, the integral representation of $G(x, y)$ must be defined on a path C constructed by deforming the real k-axis below all the simple poles at $k = \pm|k_m|$, as shown in figure 4. Then the contour integral along C may be transformed to an integral along a great circular arc D in the upper half k-plane, plus the sum of residues from the simple poles enclosed within D + C. By using the asymptotic formulas of the Airy functions (cf. Appendix B) the integral along D can be shown to vanish as the radius of the arc becomes infinite. Since the contribution from the possible poles which do not lie on the real k-axis are exponentially small as $|x| \to \infty$, only the residues from the real poles are retained to give the following result,

$$G(x,y) \cong -2\pi\lambda^{1/3} e^{-\frac{1}{2}\int_h^y n^2 dy} H(x) \sum_{m=1} \sin(\lambda|k_m|x) Q_m \, Ai[-\lambda^{2/3}\phi_m(y_>)] \cdot$$

$$Ai[-\lambda^{2/3}\phi_m(y_<)] \left[\phi_m'(y_>) \, \phi_m'(y_<) \right]^{-1/2} \left(\frac{\partial P}{\partial k} \right)^{-1}_{k=|k_m|}$$

$$= -2\pi\lambda^{1/3} e^{-\frac{1}{2}\int_h^y n^2 dy} H(x) \sum_{m=1} \sin(\lambda|k_m|x) Q_m \, Ai[-\lambda^{2/3}\phi_m(y)] \cdot$$

$$Ai[-\lambda^{2/3}\phi_m(h)] \left[\phi_m'(y) \, \phi_m'(h) \right]^{-1/2} \left(\frac{\partial P}{\partial k} \right)^{-1}_{k=|k_m|} \qquad (4.18)$$

in which $\phi_m(y)$ is obtained by replacing k by k_m and Y by Y_m in equations 4.11. a and b, whereas Q_m denotes $(Q)_{\phi=\phi_m}$.

Some general observations may be made with regard to the dependence on y of the Green's function just derived. In view of the definition of ϕ and the properties of Ai($-z$) the following two features are evident:

a). $\phi_m(y)$ always decreases with increasing y and

b). when $Y_m \gtrless y$, $\phi_m(y) \gtrless 0$, $Ai[-\lambda^{2/3}\phi_m(y)]$ is $\begin{matrix}\text{oscillatory} \\ \text{monotonic}\end{matrix}$ in y.

It follows from equation 4.18 that for increasing depth and for a fixed x > 0, the m-th mode, which is represented by the m-th term in the series, oscillates when $y < Y_m$, and decays exponentially when $y > Y_m$. Hence the quantity Y_m plays the role of the depth of a channel within which the m-th mode is effectively trapped.

We now return to equation 4.17.a for the approximate location of the real poles. Since both Ai(-z) and $\dot{A}$i(-z) have zeroes only when z is real and positive, and since λ is large, we may use the asymptotic formulas B.4 of Appendix B to get,

$$P = 0 = \frac{\lambda^{4/3}}{\sqrt{\pi}\,\phi_m^{1/4}} \left\{ \cos\left(s_m - \frac{\pi}{4}\right) - \lambda^{-1}\sqrt{n^2(0) - k_m^2}\,\cos\left(s_m - \frac{3\pi}{4}\right) \right\} \left[1 + O(\lambda^{-1}) \right] \quad (4.19.a)$$

with

$$s_m = \frac{2}{3}\lambda\phi_m^{3/2} \quad (4.19.b)$$

The value of s_m must then fall into the following range,

$$\left(m + \frac{1}{4}\right)\pi > s_m > \left(m - \frac{1}{4}\right)\pi \quad (4.20)$$

Since $\sqrt{n^2(0) - k_m^2}$ is bounded for real k_m ($|k_m| < n(0)$) and λ is large, s_m is closer to the m-th zero of Ai than to that of $\dot{A}$i; in fact we may say that,

$$s_m = \left(m - \frac{1}{4}\right)\pi\left[1 + |O(\lambda^{-1})|\right] \quad (4.21)$$

Due to the monotonicity of n(y), $|k_m|$ decreases while both Y_m and $\phi_m(0)$ increase with increasing m. It follows from equation 4.17.b that if

$$\int_0^{\infty} n(y)\,dy < N$$

where N is some finite number, there is an upper limit M for the integer m beyond which no real roots can be found for k_m. However, if the integral above is infinite, such a limit does not exist and k = 0 is the accumulation point of the entire sequence $\{|k_m|\}$ as $m \to \infty$.

As is clearly exhibited in equation 4.13, each $|k_m|$ corresponds to an eigenmode and is inversely proportional to the associated wavelength. Thus while the spectrum of the wavelength is discrete and has a lower bound in both cases, it also has an upper bound in the first, if $|k_M| \neq 0$, but not in the second. The following table summarizes the situation just described.

$\int_0^\infty n(y)\,dy$	No. of eigenmodes	Max. eigen-wavelength
finite	finite (= M)	finite, if $|k_M| \neq 0$
		infinite, if $|k_M| = 0$ i.e., $\int_0^\infty n(y)\,dy = \frac{2}{3}[\phi_m]^{3/2}$
infinite	infinite	infinite

In equation 4.18 the unspecified upper limit of the summation, which is also the number of possible normal modes, is given by the second column of the table above. When the integral $\int_0^\infty n(y)\,dy$ is finite we may calculate roughly the value of M, as follows. Clearly M must be the largest integer such that

$$\frac{2}{3}(\phi_m)^{3/2} \leqq \int_0^\infty n(y)\,dy$$

Using equation 4.21, we have

$$M \leqq \frac{1}{4} + \frac{\lambda}{\pi}\int_0^\infty n(y)\,dy \qquad (4.22)$$

which indicates that the larger the parameter λ, the more modes will be excited.

We shall investigate the lower modes in greater detail. For the first few values of m, $\phi_m(0)$ is small for large λ. It follows from equation 4.17.b that $Y_m \cong 0$. We further recall that in our $n'(0) < 0$ model hence the following approximation is valid:

$$n^2(y) \cong n_o^2 - 2n_o|n_o'|y \qquad (4.23.a)$$

where

$$n_o = n(0) \quad , \qquad n_o' = \left[\frac{d}{dy}n(y)\right]_{y=0} \qquad (4.23.b)$$

By definition,

$$n^2(Y_m) = k_m^2 \cong n_o^2 - 2n_o|n_o'|Y_m$$

therefore we have,

$$Y_m \cong (n_o^2 - k_m^2)(2n_o|n_o'|)^{-1} .$$

From equations 4.17.b and 4.21 we get,

$$\frac{2}{3}[\phi_m(0)]^{3/2} \cong (m - \tfrac{1}{4})\frac{\pi}{\lambda} \cong (2n_o|n_o'|)^{1/2}\int_0^{Y_m} d\sigma\sqrt{Y_m - \sigma}$$

$$= \frac{2}{3}(2n_o|n_o'|)^{1/2}\, Y_m^{3/2}$$

$$= \frac{2}{3}(n_o^2 - k_m)^{3/2}(2n_o|n_o'|)^{-1} ,$$

which can be solved for the location of the real poles:

$$k_m = \pm |k_m| \cong \pm \left\{ n_o - \left(\frac{n_o'^2}{2n_o}\right)^{1/3} \phi_m(0) \right\}$$

$$= \pm \left\{ n_o - \frac{1}{2}(n_o)^{-1/3} \left[\frac{3\pi}{\lambda}\left(m - \frac{1}{4}\right) |n_o'| \right]^{2/3} \right\} \qquad (4.24)$$

Let a_m denote the wavelength of the m-th mode in horizontal propagation, i.e.,

$$2\pi / a_m = \lambda |k_m| \qquad a_m = 2\pi / \lambda |k_m| \qquad (4.25)$$

then,

$$a_m \cong \frac{2\pi}{\lambda n_o} \left\{ 1 + \frac{1}{2} \left[\frac{3\pi}{\lambda}\left(m - \frac{1}{4}\right) \frac{|n_o'|}{n_o^2} \right]^{2/3} \right\} \qquad (4.26)$$

Thus the wavelengths become longer for higher modes. The channel depth for the m-th mode can also be calculated to give

$$Y_m \cong \left[\frac{3\pi}{2\lambda}\left(m - \frac{1}{4}\right) \right]^{2/3} \left(2 n_o / n_o'\right)^{-1/3} \qquad (4.27)$$

REFERENCES

1. A. Defant: Physical Oceanography, vols. 1 and 2, Pergamon Press, New York (1961).

2. C. Eckart: Internal waves in the ocean, Physics of Fluids, 4, 791-799, (1961).

3. C. S. Yih: Gravity waves in a stratified fluid. J. Fluid Mech. 8, 481-508, (1960).

4. C. Eckart: Hydrodynamics of Oceans and Atmospheres. Pergamon Press, New York (1960).

5. H. Lamb: Hydrodynamics. 6th ed. Dover Publications, New York (1945).

6. M. Yanowitch: Gravity waves in a heterogeneous incompressible fluid. Comm. Pure Appl. Math. 15, 45-61 (1962).

7. R. R. Long: Some aspects of the flow of stratified fluids, I. a theoretical investigation. Tellus, 5, 42-58 (1953).

8. P. M. Morse and H. Feshbach: Methods of Theoretical Physis. vol. 1. McGraw-Hill, New York (1953).

9. J. V. Wehausen and E. V. Laitone: Surface Waves, an article in Handbuch der Physik, band IX, ed. by S. Flugge, Springer-Verlag, Berlin, 446-778 (1960).

10. A. Erdelyi: Asymptotic Expansions. Dover Publications, (1956).

11. N. A. Haskel: Asymptotic approximation for the normal modes in sound channel wave propagation. J. Appl. Phys. 22, 157-168 (1951).

12. B. D. Seckler and J. B. Keller: Asymptotic theory of diffraction in inhomogeneous media. J. Acous. Soc. Am. 31, 206-216 (1959).

13. F. W. J. Olver: The asymptotic expansion of Bessel functions of large order. Philo. Trans. Roy. Soc. Lond. Series A, 247 328-368 (1954). See, in particular, pp 364-5.

APPENDICES

Appendix A.

We shall evaluate asymptotically for large $|X|$ the following integral (cf. figure 2.b):

$$\mathcal{I} = \int_{\Gamma} dk \, \frac{e^{i(k|X| + \sqrt{\beta^2 - k^2}\, Y)}}{i\sqrt{\beta^2 - k^2} + \alpha}$$

By the change of variable $k = \beta + i\xi$ one gets.

$$\mathcal{I} = i\, e^{i\beta|X|} \left\{ \int_{\infty L}^{0} + \int_{0}^{\infty R} \right\} d\xi \, \frac{e^{-|X|\xi} \, e^{iY\sqrt{-2i\beta\xi + \xi^2}}}{i\sqrt{-2i\beta\xi + \xi^2} + \alpha}$$

where L and R refer to the left and the right sides of the cut respectively. The above integrals are of the Laplace type,

$$\int_0^{\infty} d\xi \, e^{-|X|\xi} g(\xi)$$

for which the asymptotic expansion for large $|X|$ can be obtained by first expanding $g(\xi)$ about $\xi = 0$ and then integrating the first few terms in the series. This amounts to saying that the short stretches near the branch point give the most significant contribution to the contour integral. Since for $\xi \cong 0$

$$\arg \sqrt{-2i\beta\xi + \xi^2} \cong 3\pi/4$$

on the right side of the cut, and

$$\arg\sqrt{-2i\beta\xi+\xi^2} \cong -\pi/4$$

on the left, we have

$$\mathcal{I} \cong \frac{i}{\alpha} e^{i\beta|X|}\Big\{\int_0^\infty d\xi\, e^{-|X|\xi}\big[1+i\sqrt{2\beta\xi}\,\gamma e^{i\frac{3\pi}{4}}+\cdots\big]\big[1-\frac{i}{\alpha}\sqrt{2\beta\xi}\,e^{i\frac{3\pi}{4}}+\cdots\big]$$

$$-\int_0^\infty d\xi\, e^{-|X|\xi}\big[1+i\sqrt{2\beta\xi}\,\gamma e^{-i\frac{\pi}{4}}+\cdots\big]\big[1-\frac{i}{\alpha}\sqrt{2\beta\xi}\,e^{-i\frac{\pi}{4}}+\cdots\big]$$

$$\cong \frac{i}{\alpha} e^{i\beta|X|}\, 2i e^{i\frac{3\pi}{4}}\sqrt{2\beta}\,(\gamma-\frac{1}{\alpha})\int_0^\infty d\xi\, e^{-|X|\xi}\sqrt{\xi}$$

$$= \frac{i}{\alpha}\sqrt{2\pi\beta}\,(\gamma-\frac{1}{\alpha})\,|X|^{-3/2}\, e^{i(\beta|X|-\frac{3\pi}{4})}.$$

Appendix B.

Some important properties of the Airy functions (reference 12):

$$Ai(z)\,\dot{B}i(z) - \dot{A}i(z)\,Bi(z) = 1/\pi \qquad \text{(B.1)}$$

Let

$$s = \tfrac{2}{3} z^{3/2} \qquad \text{(B.2)}$$

then for large $|z|$:

$$Ai(z) \sim \tfrac{1}{2}\pi^{-1/2} z^{-1/4} e^{-s} \qquad \dot{A}i(z) \sim -\tfrac{1}{2}\pi^{-1/2} z^{1/4} e^{-s} \qquad |\arg z| < \pi \qquad \text{(B.3)}$$

$$Ai(-z) \sim \pi^{-1/2} z^{-1/4} \cos(s - \tfrac{\pi}{4}) \qquad Ai(-z) \sim \pi^{-1/2} z^{1/4} \cos(s - \tfrac{3\pi}{4}) \qquad |\arg z| < \tfrac{2\pi}{3} \qquad \text{(B.4)}$$

$$Bi(z) \sim \pi^{-1/2} z^{-1/4} e^{s} \qquad \dot{B}i(z) \sim \pi^{-1/2} z^{1/4} e^{s} \qquad |\arg z| < \tfrac{\pi}{3} \qquad \text{(B.5)}$$

$$Bi(-z) \sim \pi^{-1/2} z^{-1/4} \cos(s + \tfrac{\pi}{4}) \qquad \dot{B}i(-z) \sim \pi^{-1/2} z^{1/4} \cos(s - \tfrac{\pi}{4}) \qquad |\arg z| < \tfrac{2\pi}{3} \qquad \text{(B.6)}$$

$$\left.\begin{aligned} Bi(z e^{\pm i\frac{\pi}{3}}) &\sim \left(\tfrac{2}{\pi}\right)^{1/2} e^{\pm i\frac{\pi}{6}} z^{-1/4} \cos\left(s - \tfrac{\pi}{4} \mp \tfrac{i}{2}\log 2\right) \\ \dot{B}i(z e^{\pm i\frac{\pi}{3}}) &\sim \left(\tfrac{2}{\pi}\right)^{1/2} e^{\mp i\frac{\pi}{6}} z^{1/4} \cos\left(s + \tfrac{\pi}{4} \mp \tfrac{i}{2}\log 2\right) \end{aligned}\right\} |\arg z| < \tfrac{2}{3}\pi \qquad \text{(B.7)}$$

Appendix C.

We shall show that the points $k = \pm n(y_i)$, where $y_i = 0$, $y_<$ and $y_>$, are not branch points of the integrand in equation 4. 16. a

Consider the behavior of ϕ near the point $k = n(y_i)$ with

$$\frac{2}{3}[\phi(y,k)]^{3/2} = \int_y^Y \sqrt{n^2(\sigma) - k^2}\, d\sigma \qquad \text{(C. 1)}$$

When $k \cong n(y_i)$ and $Y \cong y_i$, the following approximation can be made:

$$n(\sigma) \cong n_i + n_i'(\sigma - y_i) \qquad \text{(C. 2)}$$

$$k = n(Y) \cong n_i + n_i'(Y - y_i) \qquad \text{(C. 3)}$$

in which

$$n_i = n(y_i) \;, \qquad n_i' = n'(y)\big|_{y=y_i} \qquad \text{(C. 4)}$$

Substituting C. 2 into the C. 1, we get,

$$\frac{2}{3}(\phi)^{3/2} \cong \sqrt{2n_i} \int_{y_i}^Y \sqrt{(n_i - k) + n_i'(\sigma - y_i) + \cdots}\; d\sigma$$

$$\cong \frac{2}{3n_i'} \sqrt{2n_i}\, \Big[n_i - k + n_i'(\sigma - y_i)^{3/2} \Big]_{y_i}^{Y} \qquad \text{(C. 5)}$$

From equation C. 3 it follows that

$$n_i - k \cong -n_i'(Y - y_i) \;.$$

Thus equation C. 5 gives

$$\frac{2}{3}(\phi)^{3/2} \cong -\frac{2}{3n_i'}\sqrt{2n_i}\;(n_i - k)^{3/2}$$

or

$$\phi \cong \left(\frac{\sqrt{2n_i}}{|n_i'|}\right)^{2/3}(n_i - k)$$

Hence $k = n(y_i)$ is not a branch point for ϕ . Similarly we can show that $(\phi')^{1/2}$ is analytic in k also. Since both $Ai(-\lambda^{2/3}\phi)$ and $B_i(-\lambda^{2/3}\phi)$ are entire functions of ϕ , it follows that the integrand in equation 4.16.a is single-valued at $k = n(y_i)$. The proof for $k = -n(y_i)$ is analogous.

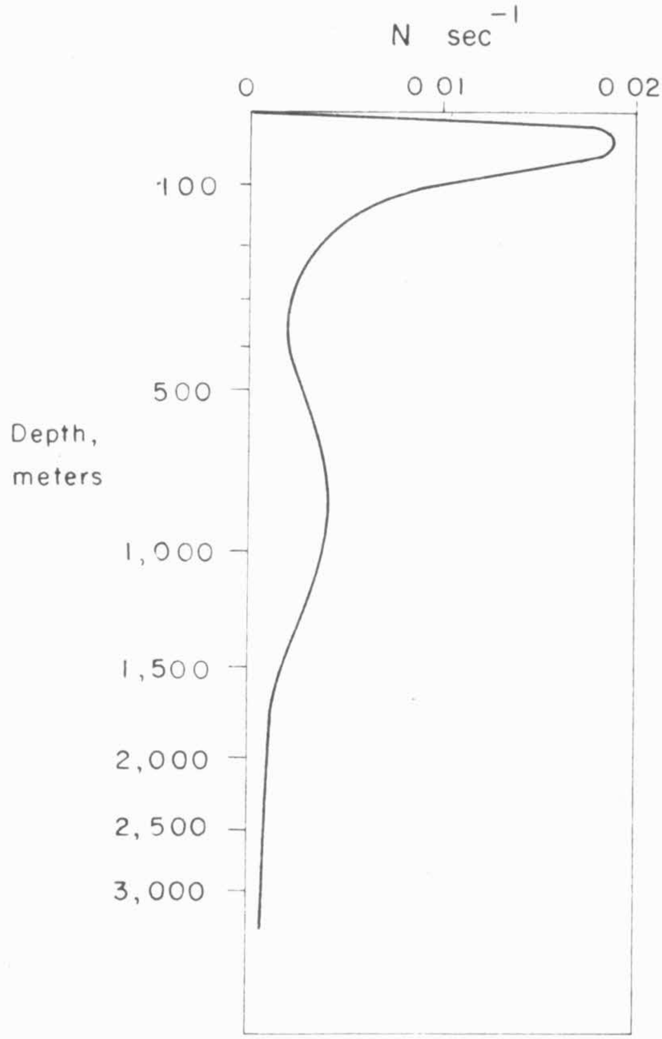

Figure 1. Typical variation of stability frequency in an ocean

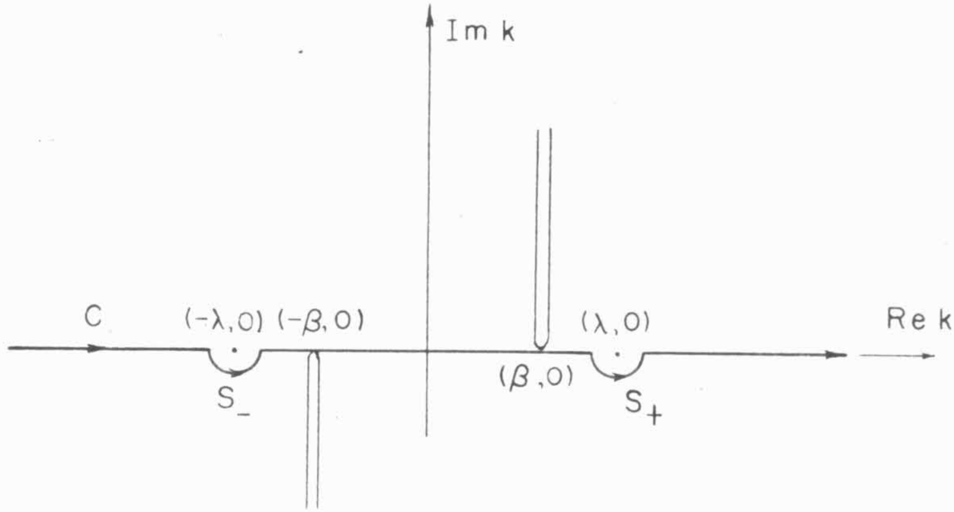

Figure 2.a

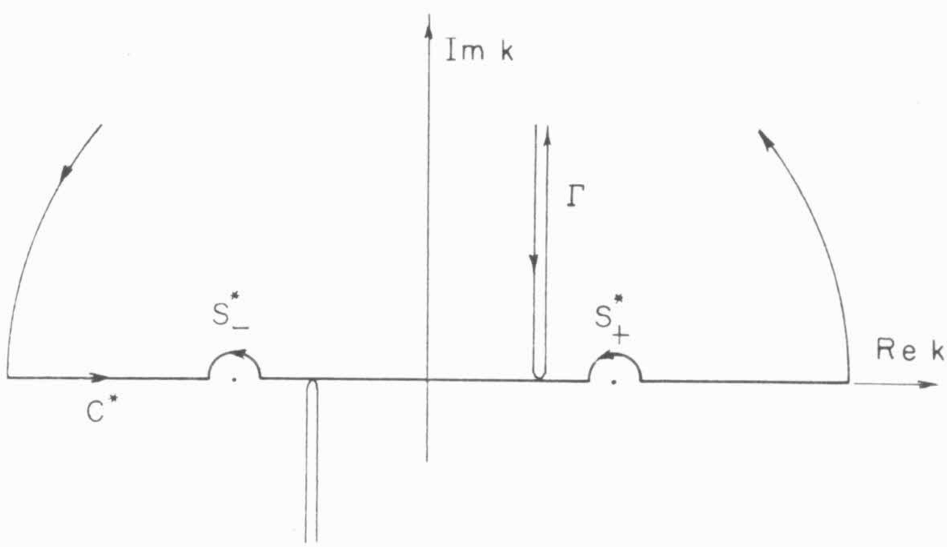

Figure 2.b

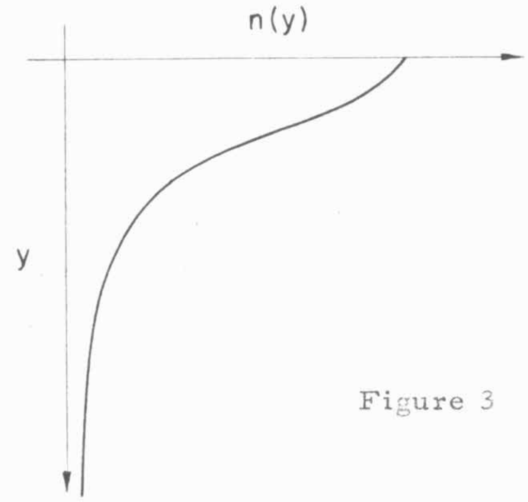

Figure 3

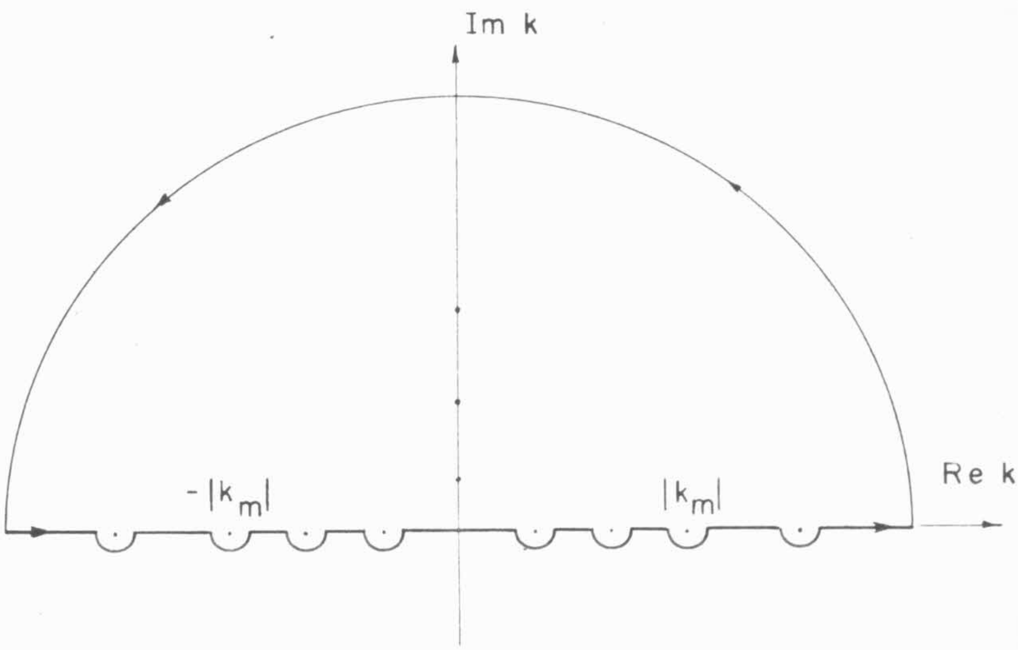

Figure 4