Geometric Descriptions of Couplings in Fluids and Circuits

Thesis by

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to the memories of

Manuel Jacobs,

Chin Hyung Kim,

and Jerrold E. Marsden

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Abstract

Geometric mechanics is often commended for its breadth (e.g., fluids, circuits, controls) and depth (e.g., identification of stability criteria, controllability criteria, conservation laws). However, on the interface between disciplines it is commonplace for the analysis previously done on each discipline in isolation to break down. For example, when a solid is immersed in a fluid, the particle relabeling symmetry is broken because particles in the fluid behave differently from particles in the solid. This breaks conservation laws, and even changes the configuration manifolds. A second example is that of the interconnection of circuits. It has been verified that LC-circuits satisfy a variational principle. However, when two circuits are soldered together this variational principle must transform to accommodate the interconnection.

Motivated by these difficulties, this thesis analyzes the following couplings: fluid-particle, fluid-structure, and circuit-circuit. For the case of fluid-particle interactions we understand the system as a Lagrangian system evolving on a Lagrange-Poincaré bundle. We leverage this interpretation to propose a class of particle methods by "ignoring" the vertical Lagrange-Poincaré equation. In a similar vein, we can analyze fluids interacting with a rigid body. We then generalize this analysis to view fluid-structure problems as Lagrangian systems on a Lie algebroid. The simplicity of the reduction process for Lie algebroids allows us to propose a mechanism in which swimming corresponds to a limit-cycle in a reduced Lie algebroid. In the final section we change gears and understand non-energetic interconnection as Dirac structures. In particular we find that any (linear) non-energetic interconnection is equivalent to some Dirac structure. We then explore what this insight has to say about variational principles, using interconnection of LC-circuits as a guiding example.

Contents

Chapter 1

Introduction

Many systems exhibit couplings, and it is very common for these couplings to corrupt information we have about the isolated subsystems. To illustrate what we mean, consider the following examples:

1. A rigid body is well-understood as a geodesic flow on SE(3). An ideal fluid is well-understood as a geodesic flow on the set of special diffeomorphisms, $\mathcal{D}_\mu(\mathbb{R}^3)$. Can the system consisting of a rigid body immersed in an ideal fluid be understood as a geodesic flow in any sense?

2. LC circuits can be understood as Poisson systems. When we connect two LC circuits with wires, we get another circuit, and therefore another Poisson system. How does the Poisson structure of the connected circuit relate to the Poisson structure of the disconnected circuits?

3. Similarly, by attaching a wheel to a circuit through an ideal motor we also get a Poisson system. Again, how is the poisson structure of the interconnected system related to the Poisson structure of the isolated subsystems.

A pattern is emerging. We often have information about the subsystems which we would like to see expressed by the interconnected system. However, this information is not expressed in the interconnected system as a simple cartesian product of the information of the subsystems. If it were a cartesian product then the coupling is not much of a coupling (this is the defining characteristic of a decoupled system). Somehow, the information is transformed, and the question we seek to answer is "What is the transformation?". We depict this schematically in Figure 1.1

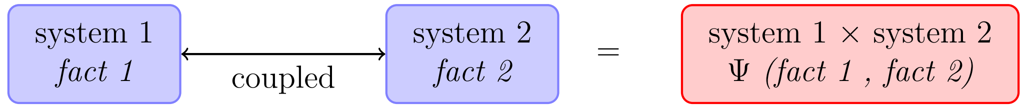

Figure 1.1 – Coupled systems tend to change what we know about systems in isolation. This thesis seeks to find a map, Ψ, which expresses this change in a variety of situations.

The use of differential geometry We have found differential geometry to be an indispensable tool in answering the questions previously posed. The examples mentioned involve couplings of distinctly different characters. However, each can be understood through the use of differential geometry. Why do I claim this? The use of differential geometry necessitates the use of coordinate-free language. If one is working on $\mathbb{R}^n$, the use of coordinate-free notation is questionable. However, if one is working on SO(3), then the use of local coordinate charts (i.e. smooth maps $\varphi : \mathrm{SO}(3) \to \mathbb{R}^3$) can trick one into making assumptions about SO(3) which only hold for $\mathbb{R}^n$. Thus, the power of coordinate-free expressions lies in their ability to communicate coordinate-free information. This is especially useful when it comes to couplings. Ball and socket interconnections of mechanical systems do not care about which coordinate system you use. Moreover, even if there exists a set of convenient coordinates for two subsystems in isolation, it is unlikely that the cartesian product of these coordinates is a convenient system for expressing a given coupling. In conclusion, when describing coupled systems, it is not uncommon for coordinate-free notation to have a distinct rhetorical advantage over coordinate based language. Of course, coordinates have their purposes (e.g. creating models which can be input into a computer). Therefore, we will use coordinates when they promote understanding, and we will avoid coordinates when they detract from understanding.

The utility of studying couplings Finally, the findings we obtain by studying couplings will have corollaries with substantial potential for applications. Our understanding of the coupling between a fluid and passive particles will have implications for our view of particle based methods for fluids. The understanding we will obtain of fluid-structure interaction will have implications for a theory of swimming as a stable limit cycle. Finally, our understanding of the interconnection of systems through Dirac structures will allow us to relate the equations of motion of the disconnected system to equations of motion for the connected system, and thus may have substantial benefits for modular modeling.

In the next three sections we will describe the main projects of this thesis.

1.1 A Fluid and its Particles

The first project of this dissertation seeks to a fluid flowing on a Riemannian manifold M. For the sake of modeling, it is necessary to obtain a finite dimensional approximation to this system. Additionally, even the most fundamental expositions in fluid dynamics view the system as a momentum equation on "volume elements" which (to 0th order approximation) may be represented as particles. Perhaps we can approximate the motion of the fluid using a finite set of particles. What are the obstacles to doing this well? The most glaring obstacle is the fact that the fluid particles are all coupled to each other, and so it is not clear how to ignore any of them. Moreover, particle methods are usually developed by taking the Euler equations, which evolve on $\mathfrak{X}_{\mathrm{div}}(M)$, and making approximations. However, the error analysis is fairly difficult because it is hard to describe the information lost during the approximation (see Figure 1.2).

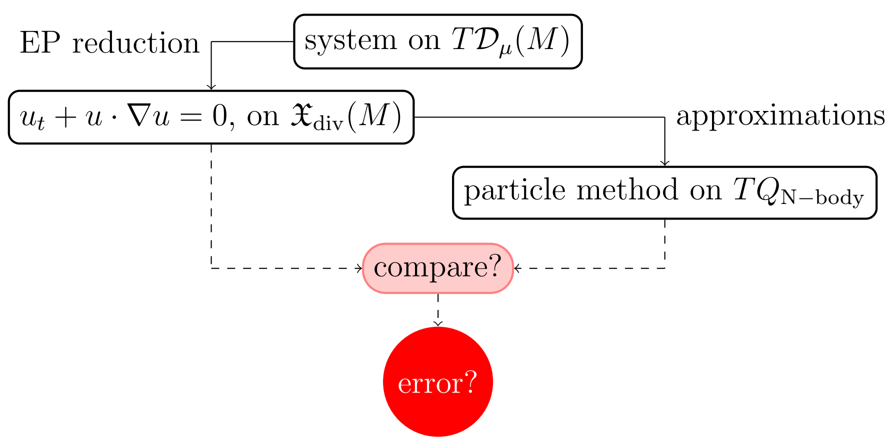

Figure 1.2 – A Signal Flow Diagram for the Error Analysis of Particle Methods?

However, we can start by choosing a good decomposition for the equations of motion. In particular, we may view a fluid as evolving on the set of volume-preserving diffeomorphisms, $\mathcal{D}_\mu(M)$. By applying Lagrange-Poincaré reduction with respect to a carefully chosen symmetry group, we can write the equations of motion on the space $TQ_{\text{part}} \oplus E$, where Q_{part} is the configuration manifold for a finite set of particle and E is some infinite dimensional vector-bundle over Q_{part}. Armed with these equations (which express the exact motion for the fluid) we can begin to surmise approximations to the equations of motion (evolving on Q_{part}) which ignore the E component. These approximations are equivalent to the creation of particle methods. However, unlike particle methods such as SPH, these are derived by approximating the Lagrange-Poincaré equations on $TQ_{\text{part}} \oplus E$ rather than the inviscid fluid equations on $\mathfrak{X}_{\text{div}}(M)$. We will find that this makes determining error bounds particularly simple (see Figure 1.3).

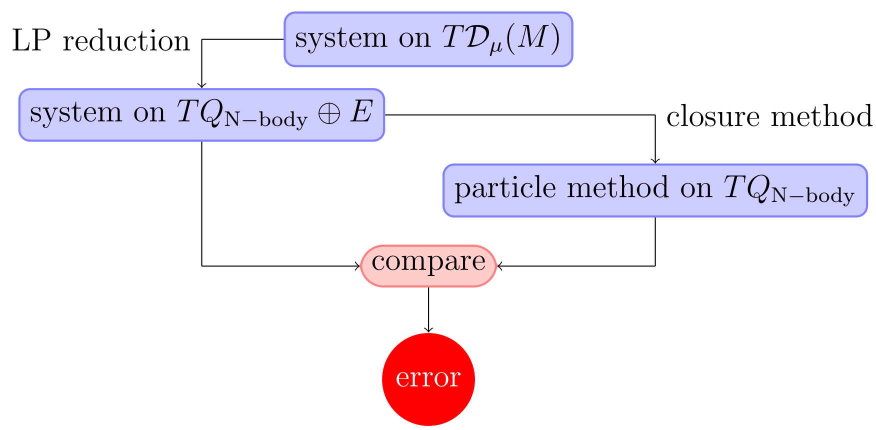

Figure 1.3 – A Signal Flow Diagram for the Error Analysis of Particle Methods

There is much room for development in this perspective on finite dimensional approximations of fluid motion. In particular, there are other symmetry groups which we could reduce which result in particles that have shape and orientation and that satisfy analogues of the circulation theorem. Additionally, there is no reason that the main ideas developed could not be applied to any continuum system with particle relabeling symmetry.

1.2 Swimming

When one watches a fish swim, something about it appears periodic. However it is not really periodic because with each flap of the fins, the fish has changed locations and so the state of the system has changed. In particular there is some translation and rotation which relates the previous flapping of the fins to the current one. Therefore, the motion is really only periodic modulo a (fixed) rotation and translation. Therefore, the motion appears to be a *relative* closed orbit in phase-space. One would conjecture that this motion is stable in some sense, which leads to the main conjecture of this project:

Is swimming a relative limit cycle?

To begin thinking about this we first need to understand the system consisting of a solid body immersed in a fluid as an unforced mechanical system. We let $)\!(\subset \mathbb{R}^3$ be a reference configuration, and we use the set

$$B := \{ b :)\!(\hookrightarrow \mathbb{R}^3 \quad | \quad \det([\frac{\partial b^i}{\partial x^j}]) = 1 \}$$

as the set of configurations for the body. For each $b \in B$ the state of the fluid may be described by a vector field, u, on the set

$$\approx\!\approx_b := \text{closure}\left(\mathbb{R}^3 \backslash b()\!() \right).$$

Therefore, given a particular $b \in B$, the state of the system may be described by the set

$$\mathcal{A}_b := \{ (\dot{b}, u) \in T_b B \times \mathfrak{X}(\approx\!\approx_b) \quad | \quad \dot{b} = u \circ b \text{ on } \partial)\!(\}$$

and the phase space for the system can be given by the vector bundle over B given by $\mathcal{A} = \cup_{b \in B} \mathcal{A}_b$. It is fairly simple to find a natural Lie bracket on sections of $\mathcal{A}$, and prove that $\mathcal{A}$ is a Lie algebroid. This allows us to use [Wei95] to derive the equations of motion for a solid body in a fluid.

How can we use this to understand swimming as a relative limit cycle? First, we must find a limit cycle in a reduced phase space. How do we do that? A quick answer is provided by the averaging theorem. The averaging theorem suggests that a sufficiently small periodic perturbation of a system on a Banach manifold with an asymptotically stable point results in a system with a stable limit cycle. Therefore, most of the work of this section will be directed towards proving the existence of an asymptotically stable point in some reduced phase space, $[\mathcal{A}]$. This would bring us well on our way to interpreting swimming as a relative limit cycle.

$$\begin{tikzcd}
\text{stable submanifold} \subset \mathcal{A} \arrow[r, "\text{periodic force}"] \arrow[d, "\text{reduction}"'] & \text{swimming?} \arrow[d, "\text{reduction}"] \\
\text{stable point} \in [\mathcal{A}] \arrow[r, "\text{periodic force}"'] & \text{stable limit cycle} \in [\mathcal{A}]
\end{tikzcd}$$

Figure 1.4 – A proposed understanding of swimming

Finding a stable point on $[\mathcal{A}]$ will require performing reduction by stages. The first reduction is with respect to the particle relabeling symmetry and the second reduction is with respect to frame-invariance (right-SE(3) symmetry). We will refer to [CMR01] when we need equations of motion, and we will use [Wei95] when performing SE(3) reduction. In any case, the left side of Figure 1.4 will be described in full. Unfortunately, the process of adding a periodic force to get a stable limit cycle remains at the conjecture level due to certain topological difficulties with infinite dimensional spaces (they are not Banach). However, any finite dimensional model of a fluid which is sufficiently well behaved (e.g. dissipates energy correctly) should exhibit these limit cycles.

1.3 Interconnections as Dirac Structures

In the final chapter of this dissertation we will be using the following concept.

Definition 1.3.1. *A linear Dirac structure D on a vector-space V is a* $\dim(V)$ *di-*

mensional subspace of $V \times V^*$ *such that*

$$\langle \beta, v\rangle + \langle \alpha, w\rangle = 0$$

for any $(v, \alpha), (w, \beta) \in D$.

The generalization of Dirac structure to manifolds roughly consists of applying this definition to each fiber of the tangeant bundle. In the last few decades Dirac structures have emerged as generalization of symplectic and Poisson structures. In particular, there is now a new formalism which one could call the Dirac formalism

formalism	*function*	*structure*	*eq. of motion*
Lagrange	L	$\delta \int (\cdot) dt = 0$	$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$
Hamilton	H	$\{\cdot, \cdot\}$	$\dot{x} = \{x, H\}$
Dirac	E	D	$(\dot{x}, dE(x)) \in D(x)$

In summary, just as Poisson and symplectic structures can be used to derive dynamics from a Hamiltonian, so can a Dirac structure be used to derive dynamics from an energy function. Additionally, one could take any power-conserving coupling (e.g. soldering the wires of two circuits, or connecting an electrical system to a mechanical one through an ideal motor). By the definition of Dirac structures, the power-conserving coupling can be written as a Dirac structure. We call the Dirac structures which express couplings "interaction Dirac structures." However, the observation that power-conserving interconnections could be expressed as Dirac structures has not been used until recently. The question we seek to answer in this final chapter is "how do we use interaction Dirac structures?"

More specifically, given mechanical systems with Dirac structures D_1 and D_2 on manifolds M_1 and M_2, how do we use a interaction Dirac structure, D_{int} on $M_1 \times M_2$?

In order to answer this question, we will define a product, $\boxtimes$, and form the Dirac structure

$$D_C := (D_1 \oplus D_2) \boxtimes D_{\text{int}},$$

which is a Dirac structure over $M_1 \times M_2$. We will find that D_C is the Dirac structure for the system which couples Dirac systems on Dirac manifolds (M_1, D_1) and (M_2, D_2) using the power-conserving interconnection given by D_{int}.

Applications for this insight can be fairly broad. However, in this thesis we have restricted our examples to the case of circuit-circuit interconnections and mechanical interconnections by non-holonomic constraints.

1.4 Conclusion

We hope that this introduction has sufficiently whet the reader's appetite. A fresh graduate student looking for a new project to embark on should find ample material here to start one. If there is a "moral" to this thesis, it is that much of the understanding known for isolated systems can carry over to coupled systems - it is just a matter of working on the right spaces and resisting the urge to use coordinates. For example, many couplings of Poisson systems result in Poisson systems. Ideal fluids and rigid bodies in free space satisfy geodesic equations. We will find rigid bodies immersed in ideal fluids satisfy geodesic equations as well. In conclusion, couplings need not be so mysterious. Couplings may destroy desirable properties of subsystems, but one should never lose hope. It is not uncommon for the beautiful aspects of the subsystems to be reincarnated as new creatures in the coupled system. Working to find these reincarnations can have significant benefits.

Reader's Guide The chapters on particle methods and swimming require some understanding of reduction by symmetry. A reader who is unfamiliar with the literature is encouraged to read chapter 2 and perhaps the first two chapters of [CMR01]. Chapters 3 through 5 are written such that they may be read in isolation from one another. Therefore, the reader is generally encouraged to start with whichever project most interests him or her.

Chapter 2

Background Material

In this section we provide some necessary background material in geometric mechanics, drawing primarily from [AM00] and [CMR01]. Two of the three projects contained in this thesis are related to fluids. Such systems evolve on spaces which are difficult to coordinatize. Therefore, there will be a bias in favor of geometrically intrinsic expressions over coordinate based ones. We begin with a brief statement on interpreting commutative diagrams in §2.1. In §2.2 we will describe how the Euler-Lagrange equations can be written in a coordinate free notation upon choosing a Covariant derivative for the tangent bundle. Additionally, there will be a heavy use of Lie groups and geodesics on Lie groups (in particular, SE(3) and SDiff(M)). In §2.3 we will review the nature of Riemannian geometry and Lagrangian mechanics on Lie groups. Finally, as Lagrange-Poincaré reduction is not thoroughly covered in most classical mechanics courses, I will provide the minimal amount of concepts needed in order to write down the Lagrange-Poincaré equation in §2.4. This chapter is intended as a reference and the reader is encouraged to skip the sections for which he or she is already familiar with. The following material is not intended as an introduction to geometric mechanics. An advanced undergraduate level introduction to geometric mechanics is [Hol11a] and [Hol11b]. A slightly more advanced introduction is [MR99]. Finally, [AM00] is largely considered the "Bible" of the subject.

2.1 Reading Commutative Diagams

Let $f : A \to A'$ and $g : B \to B'$. Let $\Psi : A \to B$ and $\Psi' : A' \to B'$. We say the diagram in figure 2.1 commutes if $g(\Psi(a)) = \Psi'(f(a)$ for any $a \in A$. Commutative

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\Psi"'] & A' \arrow[d, "\Psi'"] \\ B \arrow[r, "g"'] & B' \end{tikzcd}$$

Figure 2.1 – A commutative square

diagrams will make a few appearances in this thesis and they will generally convey the message 'Ψ *acts on* A *like* Ψ' *acts on* A'' or possibly 'f *acts on* A *like* g *acts on* B''. If g is the identity, so that $B = B'$, we get a commutative triangle (see Figure 2.2) which says that f will send elements of the equivalence classes $\Psi^{-1}(b)$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\Psi"'] & A' \arrow[dl, "\Psi'"] \\ B & \end{tikzcd}$$

Figure 2.2 – A commutative triangle

to $(\Psi')^{-1}(b)$. In summary, these diagrams convey the message that some kind of structure is preserved. For a gentle introduction to this perspective see [LS09], or for a more advanced introduction see [Mac00].

2.2 Lagrangian Mechanics

Given a Lagrangian, $L : TQ \to \mathbb{R}$, on an n-dimensional configuration manifold Q, we can express the Euler-Lagrange equations in local coordinate $(q^1, \ldots, q^n)$ by the expression:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0.$$

Implicitly we are using the Euclidean inner product on $\mathbb{R}^n$ to write the equations of motion locally. If we desire a geometrically intrinsic expression for the Euler-Lagrange equation, we need to replace $\frac{d}{dt}$ with a covariant derivative and provide intrinsic notions to replace the partial derivatives $\frac{\partial L}{\partial q^i}$ and $\frac{\partial L}{\partial \dot{q}^i}$. All of these issues are solved by choosing a covariant derivative. For a general Q there is no canonical way of making this choice. In the case that Q is a Riemannian manifold, however, there is. In this section we will provide the necessary ingredients from Riemannian geometry to write down a geometrically intrinsic expression for the Euler-Lagrange equation. We begin by defining a connection.

Definition 2.2.1 (Connection)**.** *Let Q be a manifold. A* connection *is a mapping $\nabla : \mathfrak{X}(Q) \times \mathfrak{X}(Q) \to \mathfrak{X}(G)$ such that:*

1. $\nabla_X(Y + Z) = \nabla_X(Y) + \nabla_X(Z)$.

2. $\nabla_X(fY) = X[f]Y + f \cdot \nabla_X(Y)$.

3. $\nabla_{X+Y}(Z) = \nabla_X(Z) + \nabla_Y(Z)$.

One can understand a connection as a way of differentiating vector fields. It is worth noting that the first argument of a connection need not be a vector field, but may be a single vector $v \in TQ$, while the second item need only be a vector field along a path $q_\epsilon \in Q$ tangent to v. Therefore, using ∇ we can define a covariant derivative.

Definition 2.2.2 (Covariant Derivative, Geodesic)**.** *Given a path $q(t) \in Q$ and a vector field $v(t) \in T_{q(t)}Q$ above $q(t)$ we can define the* covariant derivative

$$\frac{Dv}{Dt} := \nabla_{\dot{q}}(v).$$

Additionally, the path $q(t)$ is called a geodesic *if*

$$\frac{D\dot{q}}{Dt} := 0,$$

where we define $\dot{q} = \frac{dq}{dt}$. Lastly, the covariant derivative acts on covector fields above $\dot{q}$ by the formula

$$\langle \frac{D\alpha}{Dt}, v\rangle = \langle \alpha, \frac{Dv}{Dt}\rangle - \frac{d}{dt}\langle \alpha, v\rangle.$$

Given a Lagrangian $L : TQ \to \mathbb{R}$ we can define the Legendre transform $\frac{\partial L}{\partial \dot{q}} : TQ \to T^*Q$ by

$$\langle \frac{\partial L}{\partial \dot{q}}(q,v), \delta q \rangle := \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} (l(q, v + \epsilon \delta q)) .$$

It is notable that if $q(t)$ is a curve in Q, then both $\frac{\partial L}{\partial \dot{q}}(q,\dot{q})(t)$ and $\frac{D}{Dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$ are covector fields above $q(t)$. Finally we define the partial derivative $\frac{\partial L}{\partial q}$ by:

$$\langle \frac{\partial L}{\partial q}(q,v), \delta q \rangle := \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} (l(q_\epsilon, v_\epsilon))$$

for an arbitrary curve $(q_\epsilon, v_\epsilon) \in TQ$ such that $\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} q_\epsilon = \delta\epsilon$ and $\frac{Dv_\epsilon}{D\epsilon} = 0$.

The Euler-Lagrange equations are given by

$$\frac{D}{Dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0.$$

We can stop here, but we have not addressed the issue of how one chooses a connection. In general, one should simply do what is easiest for the circumstances at hand. However, if Q is a Riemannian manifold there is a “natural” choice [AM00, §2.7].

Theorem 2.2.1 (The Fundamental Theorem of Riemannian Geometry)**.** *If Q is a Riemannian manifold with metric $\ll,\gg$, then there exists a unique connection, ∇, on Q such that*

$$\nabla_X Y - \nabla_Y X = [X,Y]$$

and

$$X[\ll Y, Z \gg] = \ll \nabla_X Y, Z \gg + \ll Y, \nabla_X Z \gg$$

we call ∇ the Levi-Cevita connection*.*

The Levi-Cevita connection is given implicitly by *Koszul's formula*,

$$\begin{aligned} 2 \ll \nabla_X Y, Z \gg =& X[\ll Y, Z \gg] + Y[\ll X, Z \gg] - Z[\ll Y, X \gg] \\ &+ \ll [X,Y], Z \gg - \ll [X,Z], Y \gg - \ll [Y,Z], X \gg . \end{aligned}$$

In particular, if $L(q,\dot{q}) = \frac{1}{2} \ll \dot{q}, \dot{q} \gg$ then the Euler-Lagrange equations become

$$\frac{D}{Dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0,$$

and are equivalent to the geodesic equations on Q with respect to the metric field, $\ll, \gg$ [AM00, §3.7].

2.3 Lagrangian Mechanics on Lie Groups

In this section we will review the geometry and Lagrangian mechanics of Lie groups. In particular we will pay attention to the case of Riemannian metrics which are invariant with respect to group translations.

Definition 2.3.1 (Group)**.** *A* group, $(G, \circ)$, *is a pair which consists of a set G and a composition $\circ : G \times G \to G$ which satisfies the following properties:*

1. *The composition is associative (i.e., $g \circ (h \circ k) = (g \circ h) \circ k$).*
2. *There exists an identity element, $e \in G$, defined by the condition $e \circ g = g$ for every $g \in G$.*
3. *For each $g \in G$ there exists an inverse g^{-1} defined by the condition $g^{-1} \circ g = e$.*

If G is a manifold, and $\circ$, and $g \mapsto g^{-1}$ are smooth, then G is called a Lie Group.

The examples of Lie groups most useful for this thesis are the special Euclidean group, SE(3), and the special diffeomorphism group, $\mathcal{D}_\mu(M)$, for a volume manifold.[1]

In addition to the concept of a Lie Group we will also use its infinitesimal counterpart, the *Lie Algebra*.

Definition 2.3.2 (Lie algebra)**.** *A* Lie algebra *is a pair $\{\mathfrak{g}, [,]\}$ consisting of a vector space $\mathfrak{g}$ and a bracket $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which satisfies the following properties:*

1. *The bracket satisfies the Jacobi identity, $[a,[b,c]] - [b,[a,c]] + [c,[a,b]] = 0$.*

[1] This latter group is actually an infinite dimensional Lie group. The consequences of this are not investigated in this dissertation. For more information see [BK09].

2. *The bracket is anti-symmetric,* $[a, b] = 0$.

In particular, the Lie group, G, with identity, e, is equipped with a Lie algebra, $\mathfrak{g} = T_e G$. The Lie bracket is induced by the commutator of vector fields [AMR09, Chapter 5].

Deriving Lie brackets on Lie groups . Given a Lie group, G, with identity, e, one can use the following procedure to derive the bracket on the Lie algebra, $\mathfrak{g} = T_e G$.

1. For each $g \in G$ define the *AD-map* or *inner automorphism* $\mathrm{AD}_g : G \to G$ by $I_g(h) = ghg^{-1}$.

2. Define the *Ad-map*[2], $\mathrm{Ad}_g = T_e \,\mathrm{AD}$, which can be written as $\mathrm{Ad}_g(\eta) \approx g \cdot \eta \cdot g^{-1}$.

3. For each $\xi \in \mathfrak{g}$ define the ad-map[3] $\mathrm{ad}_\xi = \frac{d}{dt}\big|_{t=0} \mathrm{Ad}_g$ for a curve $g(t)$ with $\xi = \frac{dg}{dt}\big|_{t=0}$. This defines the Lie bracket $[\xi, \eta] := \mathrm{ad}_\xi \, \eta$.

Jacobi's identity follows from interpreting elements of $\mathfrak{g}$ as left invariant vector fields on G. Noting that left invariant vector fields form a Lie subalgebra of the set of all vector fields on G allows us to carry the Lie bracket on $\mathfrak{X}(G)$ to $\mathfrak{g}$ [AMR09, Chapter 5].

Invariant metrics and Euler-Poincaré reduction . Let G be a Lie group with identity, e, and a left invariant metric, $\ll, \gg$. Left invariance means

$$\ll v, w \gg = \ll g \cdot v, g \cdot w \gg$$

for any $(v, w) \in TG \oplus TG$ and $g \in G$. Equivalently, we can choose an inner product $\ll , \gg_e$ on the Lie algebra $\mathfrak{g} := T_e G$ and construct the left invariant metic $\ll v, w \gg := \ll \lambda_{\mathrm{triv}}(v), \lambda_{\mathrm{triv}}(w) \gg_e$ using the left-trivializing diffeomorphism $\lambda_{\mathrm{triv}}(g, v) = Tg^{-1} \cdot v$ which takes $TG \to \mathfrak{g}$.

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[2]Also called the "adjoint map"
[3]Called the "adjoint map" as well

If we define a Lagrangian, $L(g,\dot g) = \frac{1}{2} \ll \dot g, \dot g \gg$, then the previous section tells us that the Euler-Lagrange equations are given by

$$\frac{D}{Dt}\left(\dot g^\flat\right) = 0$$

where $\dot g^\flat$ is a curve in T^*G given by the contraction of $\dot g(t) \in TG$ with the metric tensor. We may write $\dot g(t) = g(t) \cdot \xi(t)$ for some curve $\xi(t) \in \mathfrak{g} := T_eG$. From Koszul's formula one can observe that left invariance of $\ll, \gg$ implies left invariance of the covariant derivative, so that

$$T^*g \cdot \frac{D}{Dt}\left(\dot g^\flat\right) = \frac{D}{Dt}\left(g^* \dot g^\flat\right).$$

Let η be an arbitrary element of $\mathfrak{g}$ and note that $T^*g \cdot \xi^\flat = \dot g^\flat$. Then we find along a curve $g(t)$ which satisfies the Euler-Lagrange equations

$$\begin{aligned}
0 &= \langle \frac{D\dot g^\flat}{Dt}, Tg \cdot \eta \rangle \\
&= \langle T^*g^{-1} \cdot \frac{D}{Dt}(\xi^\flat), Tg \cdot \eta \rangle \\
&= \langle \nabla_\xi(\xi^\flat), \eta \rangle \\
&= \frac{d}{dt} \ll \xi^\flat, \eta \gg - \langle \xi^\flat, \nabla_\xi \eta \rangle \\
&= \langle \frac{d\xi^\flat}{dt}, \eta \rangle - \frac{1}{2} \langle \xi^\flat, [\xi,\eta] - (\mathrm{ad}^*_\xi \eta^\flat)^\sharp - (\mathrm{ad}^*_\eta \xi^\flat)^\sharp \rangle.
\end{aligned}$$

We can simplify this further using the identities

$$\begin{aligned}
\langle \xi^\flat, [\xi,\eta] \rangle &= \langle \mathrm{ad}^*_\xi \xi^\flat, \eta \rangle \\
\langle \xi^\flat, (\mathrm{ad}^*_\xi \eta^\flat)^\sharp \rangle &= \langle \mathrm{ad}^*_\xi \eta^\flat, \xi \rangle = \langle \eta^\flat, \mathrm{ad}_\xi \xi \rangle = 0 \\
\langle \xi^\flat, (\mathrm{ad}^*_\eta \xi^\flat)^\sharp \rangle &= \langle \mathrm{ad}^*_\eta \xi^\flat, \xi \rangle = \langle \xi^\flat, [\eta,\xi] \rangle = -\langle \mathrm{ad}^*_\xi \xi^\flat, \eta \rangle
\end{aligned}$$

which imply

$$0 = \langle \frac{d\xi^\flat}{dt} - \mathrm{ad}^*_\xi \xi^\flat, \eta \rangle.$$

Letting η vary over all $\mathfrak{g}$ implies the *Euler-Poincaré equation*:

$$\frac{d\xi^\flat}{dt} = \operatorname{ad}^*_\xi \xi^\flat.$$

(see [MR99, Chap. 13] for matrix Lie groups).

Example 2.3.1 (Rigid Body on SO(3))**.** *Consider the Lie Group* SO(3)*. We can equate the Lie algebra with* $\mathbb{R}^3$ *through a map* $\vee$*. Then, both the adjoint action and coadjoint action are represented by the cross product. The Euler-Lagrange equations for the Lagrangian*

$$L(R, \dot{R}) = \frac{1}{2}\left(\vee(\mathbb{R}^{-1}\dot{R})^T \cdot \mathbb{I} \cdot \vee(\mathbb{R}^{-1}\dot{R})\right)$$

therefore reduce to the Euler-Poincaré equation

$$\dot{\Pi} = \Pi \times \Omega$$

where $\Pi = \mathbb{I} \cdot \Omega$*.*

Finally, we could have considered a right invariant metric. That is to say, a metric which satisfies the invariance property

$$\ll vg, wg \gg = \ll v, w \gg \quad , \quad \forall g \in G.$$

This would result in the (right) Euler-Poincaré equation

$$\frac{d\xi^\flat}{dt} = -\operatorname{ad}^*_\xi \xi^\flat.$$

Example 2.3.2 (Ideal Fluids, [Arn66])**.** *Let* M *be a Riemannian manifold and consider the Lie Group,* $\operatorname{SDiff}(M)$*, consisting of the volume-preserving diffeomorphisms of* M*. The Lagrangian,* $L : T\operatorname{SDiff}(M) \to \mathbb{R}$*, given by*

$$L(\varphi, \dot{\varphi}) := \frac{1}{2}\int_M \|\dot{\varphi}(x)\|^2 dx$$

is right $\mathrm{SDiff}(M)$ *invariant. The resulting (right) Euler-Poincaré equations occur on the Lie algebra of* $\mathrm{SDiff}(M)$*, which is identified with the set of divergence-free vector fields,* $\mathfrak{X}_{\mathrm{div}}(M)$*, equipped with the Lie bracket given by the commutator of vector fields. The resulting Euler-Poincaré equations are the inviscid fluid equations* $u_t + \nabla_u u = \nabla p$*. On* $\mathbb{R}^n$ *this takes the more familiar form* $u_t + u \cdot \nabla u = \nabla p$*. See [AK92, Chapter 1] for details on the Hamiltonian perspective.*

2.4 Lagrange-Poincaré Equations

In this section we state the (right) Lagrange-Poincaré equations. The material of this section is taken from [CMR01]. Let $\pi : Q \to [Q]$ be a principal bundle with structure group G induced by a right action. Given a $q \in Q$ we use the notation $[q] := \pi(q)$ to denote the orbit given by $q \cdot G$. Additionally there exists a lifted action on TQ through the tangent lift of the action of G. We denote the action of $g \in G$ on $q \in Q$ by qg and on $v \in TQ$ by vg. We denote the equivalence class of a $v \in TQ$ by $[v] = v \cdot G$. The collection of these equivalence classes is denoted by $[TQ]$. Given a Lagrangian on TQ which is right invariant with respect to the action of G there must exist a well-defined Lagrangian on $[TQ]$ given by $l([q, v]) = L(q, v)$. One would expect the Euler-Lagrange equations to have the same symmetry, since they are determined by the Lagrangian. Such a symmetry would induce consistent dynamics on $[TQ]$. To find these equations we must first understand how variations of curves in Q will lead to variation of curves in $[TQ]$. Unfortunately, the quotient $[TQ]$ is a fairly abstract space to consider writing equations of motion on. More concrete formulations of the equations can be found on the bundle $T[Q] \oplus \tilde{\mathfrak{g}}$ (to be defined), which is isomophic to $[TQ]$. In this section we will produce an isomorphism from $[TQ]$ to $T[Q] \oplus \tilde{\mathfrak{g}}$ where we will also be able to state the reduced equations of motion using suitably chosen covariant derivatives. We start by defining the adjoint bundle, $\tilde{\mathfrak{g}}$.

Definition 2.4.1 (Adjoint Bundle)**.** *Given the action of* G *on* Q *we may define an action on* $Q \times \mathfrak{g}$ *by* $(q, \xi) \mapsto (qg, \mathrm{Ad}_g^{-1}(\xi))$*. The* adjoint bundle *is the vector bundle*

$\tilde{\pi} : \tilde{\mathfrak{g}} \to [Q]$ where

$$\tilde{\mathfrak{g}} := \frac{Q \times \mathfrak{g}}{G},$$

and $\tilde{\pi}([q, \xi]) = [q]$. This bundle is also equipped with a fiber-wise Lie bracket given by

$$[[q, \xi], [q, \eta]] = [q, [\xi, \eta]] \, .$$

Definition 2.4.2 (Principal Connection)**.** *A* principal connection *is a mapping $A : TQ \to \mathfrak{g}$ which satisfies*

1. *For any $\xi \in \mathfrak{g}$, $A(\xi_Q(g)) = \xi$ where ξ_Q is the infinitesimal generator of ξ.*
2. *$A(vg) = \mathrm{Ad}_g^{-1} \cdot A(v)$ for any $v \in TQ, g \in G$.*

The definition of principal connection provided here is designed to handle symmetries with respect to right group action (see [CMR01] for the case of left actions). In particular, principal connections serve as morphisms which carry the action on TQ to the Ad-map on $\mathfrak{g}$. That is to say, A is designed to make the following diagram commute.

$$\begin{tikzcd}
TQ \arrow[r, "g"] \arrow[rr, bend left, "gh"] \arrow[d, "A"'] & TQ \arrow[r, "h"] \arrow[d, "A"] & TQ \arrow[d, "A"] \\
\mathfrak{g} \arrow[r, "g"'] \arrow[rr, bend right, "gh"'] & \mathfrak{g} \arrow[r, "h"'] & \mathfrak{g}
\end{tikzcd}$$

where we are using the right action on Q in the top row of the diagram and the right adjoint action, Ad_g^{-1}, on $\mathfrak{g}$ in the bottom row.

We define the *horizontal distribution* to be the constraint distribution $\mathbb{H} := \mathrm{kernel}(A) \subset TQ$. Additionally, the vertical distribution is defined as

$$\mathbb{V} := \{\xi_Q(q) \in TQ : q \in Q, \xi \in \mathfrak{g}\}.$$

The distributions satisfy the following properties:

- The horizontal distribution is G invariant. That is to say, $\mathbb{H} \cdot G \subset \mathbb{H}$.
- The horizontal distribution is complementary to the vertical distribution. That is to say, $\mathbb{H} \cap \mathbb{V}$ is the 0 section of TQ, and $\mathbb{H} \oplus \mathbb{V} = TQ$.
- The vertical distribution is integrable. This is because the infinitesimal generators satisfy $[\xi_Q, \eta_Q] = [\xi, \eta]_Q$.

Since $\mathbb{H}$ and $\mathbb{V}$ are complementary distributions, we may define projection onto the horizontal and vertical distributions, denoted by $\mathrm{hor} : TQ \to \mathbb{H}$ and $\mathrm{ver} : TQ \to \mathbb{V}$, respectively. Additionally, given a $\dot{x} \in T[Q]$ and a $q \in \pi^{-1}(x)$, there is a unique vector $\dot{x}_q^\uparrow \in T_q Q$ such that $\dot{x}_q^\uparrow \in \mathbb{H}$ and $T\pi(\dot{x}_q^\uparrow) = \dot{x}$. We call the mapping, $\dot{x} \mapsto \dot{x}_q^\uparrow$ the *horizontal lift* of $\dot{x}$ above q . One may hope that $\mathbb{H}$ is integrable. However, this is generally not the case. A measurement of the integrability of $\mathbb{H}$ is given by the curvature tensor.

Definition 2.4.3 (Curvature Tensor)**.** *The* curvature tensor *of a principal connection $A : TQ \to \mathfrak{g}$ is the $\mathfrak{g}$ valued two form on Q given by the expression*

$$B(\dot{q}, \delta q) = dA(\mathrm{hor}(\dot{q}), \mathrm{hor}(\delta q)).$$

Additionally, the reduced curvature tensor is $\tilde{g}$ valued two form on $[Q]$ given by

$$\tilde{B}(\dot{x}, \delta x) = [q, B(\dot{x}_q^\uparrow, \delta x_q^\uparrow)].$$

The choice of a principal connection induces a bundle map from $\tilde{A} : TQ \to \tilde{\mathfrak{g}}$ given by $\tilde{A}(q, v) = [q, A(q, v)]$. This makes $\Psi_A := T\pi \oplus \tilde{A} \circ [\tau_Q] : [TQ] \to T[Q] \oplus \tilde{\mathfrak{g}}$ into an isomorphism. A principal connection, A, also induces a covariant derivative on $\tilde{\mathfrak{g}}$, given by

$$\frac{D}{Dt}([q(t), \xi(t)]) = [q(t), [\xi, A(\dot{q})] + \dot{\xi}]$$

for a curve $[q(t), \xi(t)] \in \tilde{\mathfrak{g}}$. The covariant derivative on $\tilde{\mathfrak{g}}$ induces a covariant derivative on the dual bundle $\tilde{\mathfrak{g}}^*$, defined by the condition

$$\langle \frac{D}{Dt}[q(t), \alpha(t)], [q(t), \xi(t)]\rangle = \frac{d}{dt}\langle \alpha(t), \xi(t)\rangle - \langle \frac{D}{Dt}[q(t), \xi(t)], [q(t), \alpha(t)]\rangle$$

for a curve $[q(t), \alpha(t), v(t)] \in \tilde{\mathfrak{g}}^* \oplus \tilde{\mathfrak{g}}$. These notions are enough to state the *Lagrange-Poincaré reduction theorem* [CMR01, Theorem 3.4.1].

Theorem 2.4.1 (Lagrange-Poincaré Reduction Theorem)**.** *Let $\pi : Q \to [Q]$ be a principal bundle with structure group G. Let $L : TQ \to \mathbb{R}$ be a Lagrangian with right G-symmetry. Finally, let there exist a covariant derivative on $[Q]$. Then given a curve $q(t) \in Q$, the following are equivalent:*

1. *$q(t)$ satisfies the Euler-Lagrange equations for L.*

2. *$q(t)$ extremizes the action $\int L(q, \dot{q})dt$ with respect to variations with fixed end-points.*

3. *For $x(t) = \pi(q(t)) \in [Q], \tilde{\xi}(t) = [q(t), A(q(t), \dot{q}(t))] \in \tilde{\mathfrak{g}}$ and $l = L \circ \Psi_A : T[Q] \oplus \tilde{\mathfrak{g}} \to \mathbb{R}$ the Lagrange-Poincaré equations*

$$\begin{aligned} \frac{D}{Dt}\left(\frac{\partial l}{\partial \dot{x}}\right) - \frac{\partial l}{\partial x} &= \langle \frac{\partial l}{\partial \tilde{\xi}}, i_{\dot{x}} \cdot \tilde{B}\rangle \ \text{on } T[Q] \\ \frac{D}{Dt}\left(\frac{\partial l}{\partial \tilde{\xi}}\right) &= -\operatorname{ad}^*_{\tilde{\xi}}\left(\frac{\partial l}{\partial \tilde{\xi}}\right) \ \text{on } \tilde{\mathfrak{g}}. \end{aligned}$$

4. *$(x, \tilde{\xi})(t) = (\pi(q(t)), [q(t), A(\dot{(q, (q)})(t)) \in T[Q] \oplus \tilde{\mathfrak{g}}$ extremizes the action $\int l(x, \dot{x}, \tilde{\xi})dt$ with respect to variations $(\delta x, \delta\tilde{\xi}) \in T[Q] \oplus \tilde{\mathfrak{g}}$ where δx is a variation of x with fixed endpoints and $\delta\tilde{\xi} = \frac{D\tilde{\eta}}{Dt} - [\xi, \eta] + B(\delta x, \dot{x})$.*

The derivatives $\frac{\partial l}{\partial \tilde{\xi}}$ and $\frac{\partial l}{\partial \dot{x}}$ can be viewed as fiber derivatives, while the derivative $\frac{\partial l}{\partial x}$ should be viewed as induced by the covariant derivative on $[Q]$ as in §2.2

Example 2.4.1 (The Kaluza-Klein Formalism for Charged Particles)**.** *Consider an electron moving in $\mathbb{R}^3$. Let the configuration space be $Q = \mathbb{R}^3 \times \mathbb{S}^1$. Consider the Lagrangian*

$$L(q, \dot{q}, \theta, \dot{\theta}) = \frac{1}{2}m\|\dot{q}\|^2 + \frac{1}{2}\|\omega_1(q) \cdot \dot{q} + \dot{\theta}\|^2.$$

We observe that L has $\mathbb{S}^1$ symmetry, as it does not depend on θ. The quotient space is $[Q] = \mathbb{R}^3$ and we choose the principal connection

$$A(q, \dot{q}, \theta, \dot{\theta}) = \dot{\theta},$$

where we interpret $\dot{\theta}$ as a vector above $\theta = 0$ on the right-hand side. The curvature of A is given by $B = d\omega$. We observe that the horizontal equations are

$$m\ddot{x} = i_{\dot{x}} d\omega$$

while the vertical equations are simply $\dot{\theta}(t) = \dot{\theta}(0)$. Equating $\mathbb{R}^*$ with $\mathbb{R}$ through the standard Euclidean metric induces the Hodge star, which sends $dx \wedge dy \mapsto dz$, $dz \wedge dx \mapsto dy$, and $dy \wedge dz \mapsto dx$. Upon setting $d\omega = e(B_1 dy \wedge dz + B_2 dz \wedge dx + B_3 dy \mapsto dz)$ we find that the horizontal equations become $m\ddot{x} = e\dot{x} \times B$. This is the standard Lorentz force law for charged particles.

Chapter 3

Geometric Foundations for Particle Methods

There are two major competing representations for the state of a fluid, Eulerian and Lagrangian (alternatively described as the spatial description and the material description). In the Lagrangian description, the fluid is represented by a volume-preserving diffeomorphism. This diffeomorphism evolves with time and stores the data of where fluid particles go. In contrast, the Eulerian description of fluids only keeps track of the velocity field from the reference frame of the observer, and thus particle locations are forgotten. In this chapter we will be concerned with understanding numerical methods for fluids which adopt the Lagrangian description.

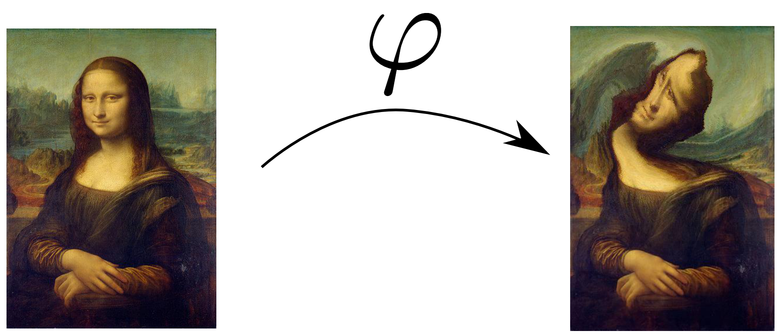

Figure 3.1 – How a volume-preserving diffeomorphism represents the state of a fluid

A Lagrangian method of particular concern is a particle method which represents the corresponding diffeomorphism by the motion of a finite number of particles[1]. Unlike Eulerian methods, such as fixed-grid finite-difference, it is not entirely clear how to estimate or even write down error bounds for particle methods over infinitesimal times. In the case of Eulerian methods with fixed grids, the accuracy is controlled

[1]This excludes meshless methods such as the Vortex Method.

by the grid spacing, which is decoupled from time. This separation of space and time allows for the error analysis of most Eulerian schemes. However in the case of particle methods, the analogue of grid spacing is particle spacing, and the spacing of the particles is not static but governed by the dynamics of the chosen method. Due to the time-dependence of the particle spacing, discovering error bounds for particle methods is particularly difficult. Upon encountering this problem, it becomes desirable to find a way to move particles around to regulate particle spacing. However, this presents one with another problem. Upon moving the particles around to a preferable arrangement, how does one choose the velocities? More specifically, given a method for estimating vector-fields from particle positions and velocities, how does one rearrange the particles and choose the velocities in such a way that the estimate is unaltered? The key to answering this is to tie everything to a reconstruction mapping, a method which estimates the fluid velocity field given particle positions and velocities.

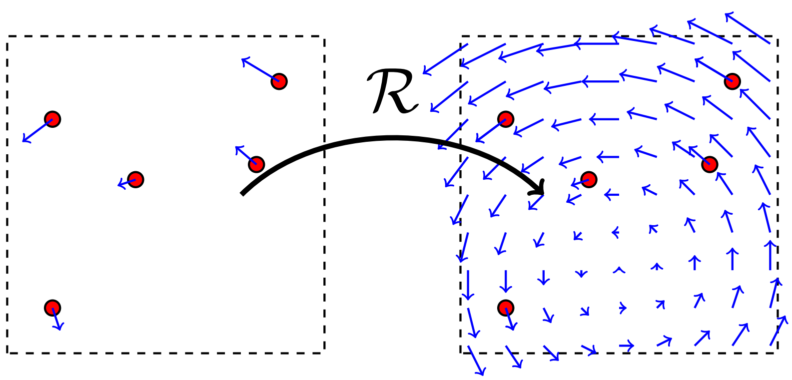

Figure 3.2 – Schematic of a Reconstruction Map

In particular, this chapter will demonstrate the following claims:

Claim 3.0.1. *Each reconstruction mapping induces a corresponding particle method.*

Claim 3.0.2. *Certain reconstruction mappings provide a means of moving the particles manually and choosing velocities so as to leave the estimated velocity field unaltered.*

Claim 3.0.3. *Claim 3.0.1 and Claim 3.0.2 can be combined to create particle methods with error bounds.*

Additionally, we will attempt to analyze the smoothed particle method (SPM). However, the correspondence between reconstruction mappings and particle methods is many-to-one. Therefore we are faced with an arbitrary choice when trying to find a reconstruction mapping corresponding to SPM.

Warning! One would naturally expect a chapter on numerical methods to include computational experiments. In order to avoid creating unmet expectations, let me warn the reader now that this will not be done in this chapter. Indeed, we will be proposing a new framework for making new numerical methods and analyzing them. However, the task of getting an appropriate configuration manifold for error analysis of particle methods is a substantial one. Clearly computation is the next step in this project, and some ideas are presented in the final section.

Problem formulation Let $(M, \ll, \gg)$ be a Riemannian manifold filled with an inviscid fluid. It is not difficult to imagine that the state of an incompressible fluid may be represented by a volume-preserving diffeomorphism (see Figure 3.1).

It was proven by V. I. Arnold that the inviscid fluid equations can be derived from the Lagrangian

$$L(\varphi, \dot{\varphi}) = \frac{1}{2} \int_M \|\dot{\varphi}(x)\| dx$$

on the set of volume-preserving diffeomorphisms of M, denoted as $\mathcal{D}_\mu(M)$. This was done through a symplectic reduction with respect to the particle relabeling symmetry of the system. The reduction yielded the traditional Euler equations for an ideal fluid on the vector space of divergence-free vector fields over M, denoted $\mathfrak{X}_{\mathrm{div}}(M)$ [Arn66, AK92]. However, we may reduce by other symmetries instead. In particular, if we let Q_{part} be the configuration manifold for n non-overlapping point particles embedded in M, then there exists a vector bundle, $\pi_E : E \to Q_{\mathrm{part}}$, as well as a symmetry reduction procedure which places the equations of motions on the vector bundle $TQ_{\mathrm{part}} \oplus E$. The significance of being able to write the equations of motion on $TQ_{\mathrm{part}} \oplus E$ is that TQ_{part} is a submanifold. Therefore, given a particle method (which is an ordinary differential equation on TQ_{part}) we may compare it with the exact equations of motion. In particular, the reduction procedure is that of Lagrange-Poincaré reduction as demonstrated in [CMR01] (see also Section §2.4) and our perspective on error is communicated by the cartoon signal flow diagram in Figure 3.3.

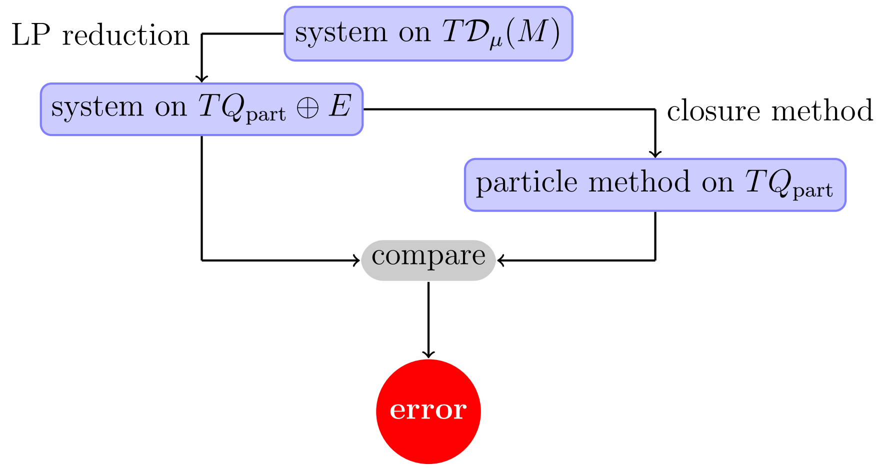

Figure 3.3 – A signal flow diagram for the error analysis of particle methods

The primary task of this paper is to specialize the equations and geometric concepts of [CMR01] to the case of reduction by a certain Lie-subgroup, $G_\odot \subset \mathcal{D}_\mu(M)$.

Previous work There are a number of preexisting mesh-free methods which have been applied to fluids. Motivations for the development of such methods include:

- They avoid complex mesh generation techniques.
- There is no mesh entanglement.
- They can handle odd-shaped and/or time-dependent domains with relative ease.
- They are easy to modify for use in multi-physics simulations.

We are interested in the class of methods known as "particle methods" which store the data of the fluid on a finite number of moving particles. The most well-established particle method today, smooth particle hydrodynamics (SPH), was introduced in [GM77] and [Luc77] for the purpose of astrophysical simulations. It was realized that the basic idea of SPH could be generalized to deal with a variety of partial differential equations (PDEs) including fluids [GM82]. However, SPH was found to have a number of problems (e.g., consistency, boundary conditions, stability), which then motivated the search for improvements and new methods. Most notably, the reproducing kernel particle method (RKPM) was developed to address the inaccuracy of SPH on the boundary of the domain [LJZ95, GL98]. Since then, a number of improvements have been made to SPH, and a variety of other methods have appeared on the scene. Finite-point methods approximate functions involved in a PDE through a set of moving collocation points [OIT96, OIZ+96]. Local Petrov-Galerkin schemes use the "local weak form" of the PDE in question via a moving set of basis functions [AS02]. A thorough survey of mesh-free methods (including a few not mentioned here) can be found in [Liu03]. All of these methods are designed to carry the velocity field data by storing it, at least implicitly, on a finite number of nodes. This is distinct from both Chorin's vortex method and vortex blob method, where each "particle" stores vorticity data (although a velocity field is implied by the vorticity equation) [Cho73].

In this chapter we will form geometric foundations for the creation and error analysis of methods in the spirit of SPH, where the particles carry velocity information. In particular we will be using much work from the field of geometric mechanics and Lie group theory. The analysis of mechanical systems on Lie groups traces back to the time of Poincaré, and the corresponding brackets are closely related to the results of Arnold, Kirillov, Kostant, and Souriau in the 1960s [MR99, chapter 10]. In particular, Arnold discovered that the Euler fluid equations are Euler-Poincaré equations corresponding to the geodesic equations on the group of special diffeomorphisms of a manifold [Arn66]. This result was then leveraged to prove local existence in time of the Euler equations by using Sobolev norms [EM70]. Simultaneously, an understanding of more general forms of reduction by symmetry were desired, and symplectic reduction by the action of a Lie group on a symplectic manifold was articulated in [MW74]. Since then, it has been observed repeatedly that a number of systems appear to be symmetry reduced systems by non-transitive group actions. In particular, a number of systems in particle physics and gauge theory had this structure (see [Ble81] and references therein). In order to understand the resulting quotient space of reduction by non transitive group actions a principal connection is chosen which places the dynamics on a vector bundle (dubbed a “Lagrange-Poincaré bundle”). This process is called Lagrange-Poincaré reduction [CMR01]. Lagrange-Poincaré reduction was explicitly used in [Kel98] and [KM00] to understand the locomotion of a vehicle in potential flow. It can be argued that Lagrange-Poincaré reduction was implicitly understood in the analysis of swimming at low Reynolds numbers [SW89], even though this work was decades before Lagrange-Poincaré reduction took on the form found in [CMR01].

Outline In §3.1 we seek to translate [CMR01] to the case of (right) subgroup reduction. That is to say, we hope to reduce a system on a Lie group G by some subgroup $G_s \subset G$ equipped with the action of right translation. We will not review Langrage-Poincaré reduction in full, but only translate the concepts necessary to write down the Lagrange-Poincaré equations. These necessary concepts are: the quotient bundle, the adjoint bundle, the principal connection, the curvature tensor, and the covariant derivatives. We then apply these constructions to fluids in §3.2 for the case of reduction by the isotropy subgroup of a finite set of points, $G_\odot \subset \mathcal{D}_\mu(M)$. We will find that the quotient manifold is isomorphic to the configuration manifold for point particles, Q_{part}, and the adjoint bundle is isomorphic to the vector bundle of vector fields which vanish at n-points, $\pi_E : E \to Q_{\text{part}}$. These identities put the Lagrange-Poincaré equations on the space $TQ_{\text{part}} \oplus E$. In §3.4 we use a closure method which effectively “ignores the vertical equation”. Additionally, a higher-order closure method is proposed which provides an extra order of accuracy in time, while sacrificing geometric properties in the process. This yields an ODE on TQ_{part} which we interpret as a particle method. We will find that smoothed particle methods fit under this construction (modulo time reparametrization), but we will not be able to perform error analysis on them. Additionally, we will find that under certain circumstances it is possible to re-arrange particles, and thus to bound the spacing between the particles. Enforcing this bound would open the door for error analysis of a new class of particle methods. In §3.5 we concoct a numerical method where error analysis is possible. We find the proposed method to be accurate to second order in space (via the bound enforced on particle spacing) and second or third order in time, depending on the closure method.

3.1 Subgroup Reduction

In this section we review Lagrange-Poincaré reduction in the context of reduction by subgroups. For brevity we state all claims and theorems while referring to [CMR01] for proofs[2].

Let G be a Lie-group and $G_s \subset G$ a Lie-subgroup. We equip G_s with the action defined by right translation on G. That is to $g_s \in G_s$ acts on G by $g_s : g \in G \mapsto gg_s \in G$ for each $g_s \in G_s$. Let $\mathfrak{g}_s$ denote the Lie algebra of G_s. The infinitesimal generator of $\xi_s \in \mathfrak{g}_s$ is the vector field on G given by the map $g \in G \mapsto g \cdot \xi \in T_g G$.

If G_s acts on a manifold X we may define the equivalence relation

$$x_1 \sim_{G_s} x_2 \iff \exists g_s \in G_s \text{ such that } x_1 = x_2 \cdot g_s$$

for $x_1, x_2 \in X$. Throughout this paper we will denote the equivalence class of $x \in X$ under a right G_s action by $[x]_s$ and the set of equivalence classes by $\frac{X}{G_s}$. For this section we define $[G] = \frac{G}{G_s}$ the set of equivalence classes with the quotient projection

$$\pi : g \in G \mapsto [g]_s \in [G].$$

Because Lie groups act freely and properly on themselves, their subgroups do as well. This ensures that $\pi : G \to [G]$ is a principal bundle (see Proposition 4.2.23 in [AM00], see also [Ebi70] for technicalities related to infinite dimensional Lie groups). Additionally, G_s is equipped with a right action on TG given by tangent lift of the

[2] In this paper we will be concerned with reduction by right group actions. The paper [CMR01] is concerned with left group actions. However, all the content remains intact upon substituting left actions with right actions, Ad_g with $\mathrm{Ad}_{g_s}^{-1}$ and ad_ξ with $-\mathrm{ad}_{\xi_s}$.

action on G. That is to say

$$g_s : v \in T_g G \mapsto \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} (g_\epsilon \cdot g_s) \in T_{g \cdot g_s} G, \forall g_s \in G_s$$

where $v = \left. \frac{dg_\epsilon}{d\epsilon} \right|_{\epsilon=0}$. We may denote the quotient projection $\tau : TG \to [TG] := \frac{TG}{G_s}$. One should note that $T\pi \neq \tau$ and $T[G] \neq [TG]$. These quotient bundles relate to Lagrangian mechanics for the following reason: If the Lagrangian, $L : TG \to \mathbb{R}$, possesses right G_s-invariance then there exists a reduced Lagrangian $l : [TG] \to \mathbb{R}$ defined by the condition $l \circ \tau = L$. One could hypothetically compute dynamics on $[TG]$. However, $[TG]$ is not a tangent bundle but a Lie-algebroid, and we can not resort to traditional Lagrangian mechanics[3]. To write the dynamics in a familiar form it is useful to find an isomorphism to the bundle $T[G] \oplus \tilde{\mathfrak{g}}_s$ where $\tilde{\mathfrak{g}}_s$ is the adjoint bundle (§2.4) and $T[G]$ is the tangent bundle of $[G]$. There are many such isomorphisms, but one of particular interest is an isomorphism induced by a principal connection (§3.1).

Principal connections for subgroups The infinitesimal action of G_s on G will not produce the entire tangent bundle, TG, but only a sub-bundle, $\mathbb{V}^s$, known as the vertical bundle. The choice of a principal connection allows one to split any vector in TG into a part produced by infinitesimal actions of G_s and a part which is not. We may refer to [CMR01] for proofs with left group actions. In this paper we are concerned with right group actions which yield a slightly different set of conventions (see for example [MMO+07] or [Ble81]). For the case of subgroups, Definition 2.4.2 for principal connections can be written in more specific terms. A *principal connection* for a subgroup $G_s \subset G$ equipped with right action on G is a $\mathfrak{g}_s$-valued one-form $A \in \bigwedge^1(G, \mathfrak{g}_s)$ such that

$$A(v \cdot g_s) = \mathrm{Ad}_{g_s}^{-1} A(v) \tag{3.1}$$

$$A(Tg \cdot \xi_s) = \xi_s \tag{3.2}$$

[3] this non-traditional type of mechanics is found in [Wei95, Mar01]

where $v \in TG$, $\xi_s \in \mathfrak{g}_s$. Similarly the *horizontal* and *vertical distributions* take a specific form in the case of subgroups.

Definition 3.1.1 (Vertical and Horizontal Distributions for subgroups)**.** *We define the* vertical distribution

$$\mathbb{V}^s := \{g \cdot \xi_s : \xi_s \in \mathfrak{g}_s, g \in G\}.$$

Additionally, a horizontal distribution *is any distribution,* $\mathbb{H}$*, such that* $\mathbb{H} \oplus \mathbb{V}^s = TG$*. We say that* $\mathbb{H}$ *is* G_s *invariant if* $\mathbb{H} \cdot G_s \subset \mathbb{H}$*.*

Proposition 3.1.1. *Given a* G_s *invariant horizontal distribution,* $\mathbb{H}$*, there exists a unique principal connection* $A \in \bigwedge^1(G, \mathfrak{g}_s)$ *such that* $\mathrm{kernel}(A) \equiv \mathbb{H}$*. Conversely, a principal connection,* $A \in \bigwedge^1(G, \mathfrak{g}_s)$*, induces a* G_s*-invariant horizontal distribution* $\mathbb{H} = \mathrm{kernel}(A)$*.*

As $\mathbb{V}^s$ is the kernel of $T\pi$ we find that $T\pi$ restricted to $\mathbb{H}$ is an isomorphism between the fiber $\mathbb{H}_g \subset T_gG$ and the fiber $T_{[g]}[G]$. Therefore, given $x \in [G]$ and $g \in \pi^{-1}(x)$, there exists a mapping called the *horizontal lift* which takes a vector $\dot{x} \in T_x[G]$ to the unique vector $\dot{x}_g^{\uparrow} \in \mathbb{H}_g$, such that $T\pi(\dot{x}_g^{\uparrow}) = \dot{x}$. Additionally, one can take a curve $x(t) \in [G]$, set $\dot{x} = \frac{dx}{dt}$, and solve the ODE on G given by:

$$\dot{g}(t) = \dot{x}_{g(t)}^{\uparrow}(t)$$

for some initial condition $g \in \pi^{-1}(x(0))$. The solution curve, denoted $x_g^{\uparrow}(t) \in G$, is called the *horizontal lift* of the curve $x(t) \in [G]$.

Proposition 3.1.2. *Given a principal connection,* $A : TG \to \mathfrak{g}_s$*, the induced horizontal lift satisfies*

$$\dot{x}_g^{\uparrow} \cdot g_s = \dot{x}_{g \cdot g_s}^{\uparrow}$$

for any vector $\dot{x} \in T[G], g \in \pi^{-1}(x), g_s \in G_s$*. Moreover, for any curve* $x(t) \in [G]$ *we have:*

$$x_{g_0}^{\uparrow}(t) \cdot g_s = x_{g_0 \cdot g_s}^{\uparrow}(t).$$

Since $\mathbb{H} \oplus \mathbb{V}^s = TG$ is a direct sum, there exist projections $\mathrm{ver} : TG \to \mathbb{V}^s$ and $\mathrm{hor} : TG \to \mathbb{H}$ such that $v = \mathrm{hor}(v) + \mathrm{ver}(v)$ for any $v \in TG$. This allows us to state the following corollary to 3.1.2:

Corollary 3.1.1. *Given a principal connection $A : TG \to \mathfrak{g}_s$ the Lie bracket of vector fields satisfies*

$$[\mathrm{ver}(X), \mathrm{hor}(Y)] = 0$$

for any vector fields $X, Y \in \mathfrak{X}(G)$.

We conclude this section with some thoughts on how infinitesimal loops in $T[G]$ should be expressed by a principal connection. Note that $\mathbb{V}^s$ is integrable by the Frobenius theorem. However, the horizontal distribution, $\mathbb{H} = \mathrm{kernel}(A)$, may not be. The curvature tensor (to be defined) is a $\mathfrak{g}_s$-valued two-form which measures the non-integrability of $\mathbb{H}$.

Moreover, the curvature tensor can also be written more specifically in the case of subgroup reduction. Given $A : TG \to \mathfrak{g}_s$, we define dA to be the unique $\mathfrak{g}_s$-valued two-form such that

$$\langle \mu, dA \rangle = d\langle \mu, A \rangle$$

for an arbitrary $\mu \in \mathfrak{g}_s^*$. The *curvature tensor* of A is the $\mathfrak{g}_s$ valued two-form defined by

$$B(v, w) = dA(v, w) + [A(v), A(w)].$$

From this, it may not be entirely clear how B measures non-integrability. The following proposition addresses this.

Proposition 3.1.3. *Given vector-fields $X, Y \in \mathfrak{X}(G)$ we have the identity*

$$B(X, Y) = A([\mathrm{hor}(X), \mathrm{hor}(Y)])$$

where $[,]$ is the Lie-bracket on vector-fields.

As a result, $\mathbb{H}$ is integrable if and only if $B = 0$.

3.2 Subgroup Reduction for Fluids

In the previous section we assembled the necessary constructions for subgroup reduction. In this section we then specialize this construction to the case of fluids. Again, $\mathcal{D}_\mu(M)$ is the infinite-dimensional Lie group of volume-preserving diffeomorphisms of a Riemannian manifold $(M, \ll, \gg)$. Let $\odot_1, \ldots, \odot_n \in M$ be a set of points. We use the shorthand $\odot = (\odot_1, \ldots, \odot_n)$ and $F(\odot) = (F(\odot_1), \ldots, F(\odot_n))$ for any map F with domain M. We can then define the isotropy subgroup,

$$G_\odot := \{\varphi \in \mathcal{D}_\mu(M) : \varphi(\odot) = \odot\}.$$

We will denote an arbitrary element of $G_\odot$ by $g_\odot$ and an arbitrary element of $\mathcal{D}_\mu(M)$ by φ. It is simple to observe that $G_\odot$ is a Lie subgroup of $\mathcal{D}_\mu(M)$. The Lie algebra of $G_\odot$ is the Lie subalgebra $\mathfrak{g}_\odot \subset \mathfrak{X}_{\mathrm{div}}(M)$ consisting of divergence-free vector fields $\xi_\odot$ on M such that $\xi_\odot(\odot) = 0$ (see Figure 3.4). Additionally, note that a tangent vector in $T_\varphi \mathcal{D}_\mu(M)$ is a mapping $\delta\varphi : m \in M \mapsto \delta\varphi(m) \in T_{\varphi(m)}M$. This allows us to understand the infinitesimal generator of $\xi \in \mathfrak{g}_\odot$ on $\mathcal{D}_\mu(M)$ as the mapping $\varphi \mapsto T\varphi \cdot \xi$, where we view ξ as mapping from $m \in M \to T_m M$ and $T\varphi$ as a mapping $T_m M \to T_{\varphi(m)} M$.

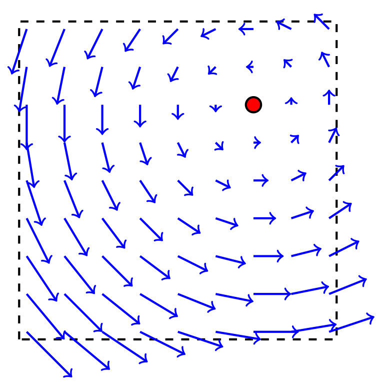

Figure 3.4 – The vector field represents an element of $\mathfrak{g}_\odot$ where $\odot$ is given by the red dot.

The goal of this section will be to perform LP-reduction on $T\mathcal{D}_\mu(M)$ by the symmetry group $G_\odot$. As in the previous section, we equip $G_\odot$ with the right action on $\mathcal{D}_\mu(M)$ and set $[\mathcal{D}_\mu(M)] := \frac{\mathcal{D}_\mu(M)}{G_\odot}$ with projection $\pi : \mathcal{D}_\mu(M) \to [\mathcal{D}_\mu(M)]$.

Proposition 3.2.1. *Let Q_{part} be the configuration manifold for n non-overlapping particles. That is to say,*

$$Q_{\text{part}} = \{(m_1, \ldots, m_n) \in M \times \cdots \times M : \quad i \neq j \implies m_i \neq m_j\}.$$

Then $[\mathcal{D}_\mu(M)] \equiv Q_{\text{part}}$.

Proof. Define the set-valued map $\Psi : Q_{\text{part}} \to \wp(\mathcal{D}_\mu(M))$,

$$\Psi(m_1, \ldots, m_n) := \{\varphi \in \mathcal{D}_\mu(M) : \varphi(\odot) = (m_1, \ldots, m_n)\}.$$

We first prove that Ψ maps to co-sets. Take an arbitrary $g \in G$. Then we find

$$\begin{aligned}
\Psi(m_1, \ldots, m_n) \cdot g &= \{\varphi \circ g : \varphi(\odot) = (m_1, \ldots, m_n)\} \\
&= \{\varphi \circ g : \varphi \circ g(\odot) = (m_1, \ldots, m_n)\} \\
&= \{\varphi : \varphi(\odot) = (m_1, \ldots, m_n)\} \\
&= \Psi(m_1, \ldots, m_n).
\end{aligned}$$

This implies that Ψ maps to co-sets of $G_\odot$, i.e., elements of $[\mathcal{D}_\mu(M)]$. Additionally we find

$$\Psi^{-1}(\varphi \cdot G_\odot) = \varphi(\odot),$$

so that Ψ is a bijection between Q_{part} and $[\mathcal{D}_\mu(M)]$. □

This proposition highlights the link that this paper seeks to make between LP-reduction and particle-based numerical methods for fluids. From this point on, we may write an element $q \in Q_{\text{part}}$ as $q = (q_1, \ldots, q_n)$, where $q_i \in M$ for $i = 1, \ldots, n$. Additionally, the projection map $\pi : \mathcal{D}_\mu(M) \to Q_{\text{part}}$ is given explicitly by

$$\pi(\varphi) \equiv \varphi(\odot) := (\varphi(\odot_1), \ldots, \varphi(\odot_n)).$$

As explained in §3.1 for arbitrary Lie subgroups, $G_\odot$ also acts on $T\mathcal{D}_\mu(M)$ by the tangent lift of the right action. We can view a vector $\delta\varphi \in T_\varphi\mathcal{D}_\mu(M)$ as a mapping

$\delta\varphi : m \in M \to \delta\varphi(m) \in T_{\varphi(m)}M$. The right-action of G on $T(\mathcal{D}_\mu(M))$ is given by composition of maps. That is, $\delta\varphi \cdot g := \delta\varphi \circ g$. This right-action on $T\mathcal{D}_\mu(M)$ allows us to define the quotient space $[T\mathcal{D}_\mu(M)] := \frac{T\mathcal{D}_\mu(M)}{G_\odot}$ with the quotient projection $\tau : T\mathcal{D}_\mu(M) \to [T\mathcal{D}_\mu(M)]$. A Lagrangian on $T\mathcal{D}_\mu(M)$ is right invariant with respect to $G_\odot$ if and only if there exists a reduced Lagrangian $l : [T\mathcal{D}_\mu(M)] \to \mathbb{R}$ defined by the property that $l \circ \tau \equiv L$.

Principal connections for fluids In this section we seek to understand principal connections of the type $A \in \bigwedge^1(\mathcal{D}_\mu(M), \mathfrak{g}_\odot)$. First, it is notable that the vertical space above $\varphi \in \mathcal{D}_\mu(M)$ is given by

$$\mathbb{V}^\odot(\varphi) \equiv \{T\varphi \cdot \xi \in T_\varphi\mathcal{D}_\mu(M) \quad | \xi \in \mathfrak{g}_\odot\}$$

and the vertical bundle is the union of these vertical spaces. We begin this section by establishing a correspondence between principal connections and reconstruction methods for vector fields.

Definition 3.2.1 (Reconstruction Method)**.** *A* reconstruction method *is a linear immersion $\mathcal{R} : TQ_{\mathrm{part}} \hookrightarrow \mathfrak{X}_{\mathrm{div}}(M)$ such that for $(q, \dot{q}) \in TQ_{\mathrm{part}}$ we have that $\mathcal{R}(\dot{q})(q) = \dot{q}$.*

A reconstruction mapping takes a set of velocities for n point particles and returns a vector field on all of M such that the data at the location of the particles matches (see Figure 3.2). Thus, $\mathcal{R}$ serves as a protocol for estimating the spatial velocity field of a fluid given only the velocity of a finite set of particles. This interpretation of $\mathcal{R}$ will allow us to do error analysis in the final sections of this paper.

Proposition 3.2.2. *Let $\mathcal{R} : TQ_{\mathrm{part}} \hookrightarrow \mathfrak{X}_{\mathrm{div}}(M)$ be a reconstruction mapping. Then $\mathcal{R}$ induces a horizontal distribution*

$$\mathbb{H} = \{\mathcal{R}(q, \dot{q}) \circ \varphi : q = \varphi(\odot)\}$$

and a principal connection

$$A(\dot{\varphi}) = T\varphi^{-1} \cdot \delta\varphi - \varphi^* \mathcal{R}(\delta\varphi(\odot)).$$

Conversely, given a principal connection, A, we can define a reconstruction mapping, $\mathcal{R}(\dot{q}) = \dot{q}_{\varphi}^{\uparrow} \circ \varphi^{-1}$.

Proof. Let $\mathbb{H} := \{\mathcal{R}(q, \dot{q}) \circ \varphi : \pi(\varphi) = q\}$. We first prove that $\mathbb{H}$ is a horizontal distribution: It is simple to observe that $\mathbb{H}$ is a distribution because $\mathcal{R}$ is linear on each tangent fiber of TQ_{part}. We see that for an arbitrary $g_\odot \in G_\odot$,

$$\begin{aligned}
\mathbb{H} \cdot g_\odot &= \{\mathcal{R}(q, \dot{q}) \circ \varphi \circ g_\odot : \pi(\varphi) = q\} \\
&= \{\mathcal{R}(\dot{q}) \circ (\varphi \circ g_\odot) : \pi(\varphi \circ g_\odot) = q\} \\
&= \mathbb{H}.
\end{aligned}$$

Therefore $\mathbb{H}$ is $G_\odot$ invariant. We need only prove that $\mathbb{H}$ is complementary to the vertical distribution $\mathbb{V}^\odot$. Let $(q, \dot{q}) \in TQ_{\text{part}}$. For an arbitrary $\varphi \in \pi^{-1}(q)$ we find $\mathcal{R}(q, \dot{q}) \circ \varphi$ is not contained in the fiber $\mathbb{V}^\odot(\varphi)$ since

$$\begin{aligned}
T\pi(\mathcal{R}(q, \dot{q}) \circ \varphi) &= (\mathcal{R}(q, \dot{q}) \circ \varphi)(\odot) \\
&= \mathcal{R}(q, \dot{q})(q) \\
&= \dot{q}.
\end{aligned}$$

Yet $T\pi(\mathbb{V}^\odot(\varphi)) = 0$. Since $\dot{q}$ is arbitrary we find $T\pi(\mathcal{R}(T_q Q_{\text{part}}) \circ \varphi) = T_q Q_{\text{part}}$ so that $\mathbb{H}(\varphi)$ is complementary to $\mathbb{V}^\odot(\varphi)$. Allowing q to vary we see that $\mathbb{H}$ and $\mathbb{V}^\odot$ are complementary distributions. Thus $\mathbb{H}$ is a horizontal distribution.

Finally, let A be the unique principal connection defined by the distribution $\mathbb{H}$ as in Proposition 3.1.1. This would mean A satisfies

$$\delta\varphi = \underbrace{T\varphi \cdot A(\delta\varphi)}_{\text{vertical part}} + \underbrace{(\delta\varphi(\odot))_{\varphi}^{\uparrow}}_{\text{horizontal part}} .$$

Upon substituting the identity $\mathcal{R}(\delta\varphi(\odot))\circ\varphi = (\delta\varphi(\odot))^{\uparrow}_{\varphi}$ we find $A(\delta\varphi)$ is given by the desired expression.

To prove the converse, let A be a principal connection with horizontal distribution $\mathbb{H}$. Let $\mathcal{R}(q,\dot{q}) := \dot{q}^{\uparrow}_{\varphi}\circ\varphi^{-1}$, where $\dot{q}^{\uparrow}_{\varphi}$ is the horizontal lift above some $\varphi\in\pi^{-1}(q)$. We must prove that $\mathcal{R}$ is a reconstruction mapping. This can be done by alternatively proving that $\mathcal{R}$ is independent of φ and is $G_{\odot}$ invariant. Let $g_{\odot}\in G_{\odot}$. We then find by Proposition 3.1.2 that:

$$
\begin{aligned}
\dot{q}^{\uparrow}_{\varphi\circ g_{\odot}}(\varphi\circ g_{\odot})^{-1} &= \dot{q}^{\uparrow}_{\varphi\circ g_{\odot}}\circ g_{\odot}^{-1}\circ\varphi^{-1} \\
&= \dot{q}^{\uparrow}_{\varphi}\circ\varphi^{-1}.
\end{aligned}
$$

By inspection, $\mathcal{R}$ is linear and injective. Lastly we note that $T\pi(\dot{q}^{\uparrow}_{\varphi}) = \dot{q}$ by the definition of the horizontal lift. Finally $\dot{q} = T\pi(\dot{q}^{\uparrow}_{\varphi}) \equiv \dot{q}^{\uparrow}_{\varphi}(\odot) = \dot{q}^{\uparrow}_{\varphi}(\varphi^{-1}(q)) = \mathcal{R}(q,\dot{q})(q)$. Thus $\mathcal{R}$ is a reconstruction mapping. $\square$

Proposition 3.2.2 allows us to replace any instance of a chosen principal connection with a reconstruction map. There are a number of constructions which are induced by the principal connection. It would be advantageous to rewrite all of them using reconstruction mappings instead. Whenever there is an opportunity to express a more obscure concept with a more intuitive one, we should take it. For example, the reduced curvature tensor will be expressed as a difference of brackets composed with the reconstruction method. We will postpone this construction for the next subsection.

For now we shall continue making the geometry more tractable. To begin, we will eliminate the use of the projection hor in the curvature tensor formula (or alternatively we will eliminate the "dA" term in the definition). However, first we must state the following lemma which will provide a Lie bracket on the tangent fibers of $\mathcal{D}_{\mu}(M)$ (as opposed to a bracket on the set of vector fields).

Lemma 3.2.1. *Let $[,]_M$ be the Lie-bracket of vector fields on M and $[,]_{\mathcal{D}_{\mu}(M)}$ be the Lie bracket of vector fields on $\mathcal{D}_{\mu}(M)$. Let $\rho_{\mathrm{triv}} : T\mathcal{D}_{\mu}(M)\to\mathfrak{X}_{\mathrm{div}}(M)$ be the right* *trivializing morphism $\delta\varphi \mapsto \delta\varphi \circ \varphi^{-1}$. Then for $X, Y \in \mathfrak{X}(\mathcal{D}_\mu(M))$ we have that*

$$[\rho_{\text{triv}} \circ X, \rho_{\text{triv}} \circ Y]_M = \rho_{\text{triv}} \circ [X, Y]_{\mathcal{D}_\mu(M)}.$$

Proof. Using the dynamic definition of Lie-derivative and evaluating at a $\varphi \in \mathcal{D}_\mu(M)$ we have that

$$[X, Y]_{\mathcal{D}_\mu(M)}(\varphi) = \partial_t \partial_s \varphi_{t,s} - \partial_s \partial_t \varphi_{t,s}$$

where $\partial_s = \frac{d}{ds}\big|_{s=0}$, $\partial_t = \frac{d}{dt}\big|_{t=0}$ and $\varphi_{s,t}$ is such that $\varphi_{0,0} = \varphi$ and $\partial_t \varphi_{t,0} = X(\varphi)$ and $\partial_s \varphi_{0,s} = Y(\varphi)$. Applying ρ_{triv}, we find

$$\begin{aligned}
\rho_{\text{triv}}([X, Y]_{\mathcal{D}_\mu(M)}(\varphi)) &= \partial_t \partial_s (\varphi_{t,s} \circ \varphi^{-1}) - \partial_s \partial_t (\varphi_{t,s} \circ \varphi^{-1}) \\
&= [\rho_{\text{triv}}(X(\varphi)), \rho_{\text{triv}}(Y(\varphi))]_M.
\end{aligned}$$

□

The take-away message from this lemma is that for each $\varphi \in \mathcal{D}_\mu(M)$

$$[X, Y]_{\mathcal{D}_\mu(M)}(\varphi) = [X(\varphi) \circ \varphi^{-1}, Y(\varphi) \circ \varphi^{-1}]_M \circ \varphi$$

for $X, Y \in \mathfrak{X}(\mathcal{D}_\mu(M))$. The right-hand side only depends on the values of X and Y at φ and provides a bracket on the vector space $T_\varphi \mathcal{D}_\mu(M)$, as opposed to the set of vector fields $\mathfrak{X}(\mathcal{D}_\mu(M))$. With Lemma 3.2.1, we are now prepared to derive a more tractable expression for the curvature tensor.

Proposition 3.2.3. *Given a principal connection $A : T\mathcal{D}_\mu(M) \to \mathfrak{g}_\odot$, the curvature tensor B is given by*

$$B(\delta\varphi, \dot\varphi) = A([\delta\varphi, \dot\varphi]) - [A(\delta\varphi), A(\dot\varphi)]$$

for $\delta\varphi, \dot\varphi \in T_\varphi \mathcal{D}_\mu(M)$.

Proof. First we use Proposition 3.1.3 to write

$$B(X, Y) = A([\mathrm{hor} \circ X, \mathrm{hor} \circ Y]_{\mathcal{D}_\mu(M)})$$

for vector fields $X, Y \in \mathfrak{X}(\mathcal{D}_\mu(M))$. By Lemma 3.2.1 we can apply B to vectors $\delta\varphi, \dot{\varphi} \in T_\varphi \mathcal{D}_\mu(M)$ using the expression

$$B(\delta\varphi, \dot{\varphi}) = A([\mathrm{hor}(\delta\varphi), \mathrm{ver}(\dot{\varphi})]).$$

Substituting $\mathrm{hor}(\delta\varphi) = \delta\varphi - \mathrm{ver}(\delta\varphi)$ we find

$$B(\delta\varphi, \dot{\varphi}) = A([\delta\varphi, \dot{\varphi}]) - A([\delta\varphi, \mathrm{ver}(\dot{\varphi})]) - A([\mathrm{ver}(\delta\varphi), \dot{\varphi}]) + A([\mathrm{ver}(\delta\varphi), \mathrm{ver}(\dot{\varphi})]).$$

Since $[\mathrm{hor}(\delta\varphi), \mathrm{ver}(\dot{\varphi})] = 0$, by Corollary 3.1.1 we see that

$$A([\delta\varphi, \mathrm{ver}(\dot{\varphi})]) = A([\mathrm{ver}(\delta\varphi), \mathrm{ver}(\dot{\varphi})])$$

and similarly

$$A([\mathrm{ver}(\delta\varphi), \dot{\varphi}]) = A([\mathrm{ver}(\delta\varphi), \mathrm{ver}(\dot{\varphi})]).$$

Finally, given $\xi, \eta \in \mathfrak{g}_\odot$ such that $\mathrm{ver}(\delta\varphi) = T\varphi \cdot \xi$ and $\mathrm{ver}(\dot{\varphi}) = T\varphi \cdot \eta$, we find

$$\begin{aligned} A([\mathrm{ver}(\delta\varphi), \mathrm{ver}(\dot{\varphi})]) &= A([T\varphi \cdot \xi, T\varphi \cdot \eta]) \\ &= A(T\varphi \cdot [\xi, \eta]) \\ &= [\xi, \eta] \\ &= [A(\mathrm{ver}(\delta\varphi)), A(\mathrm{ver}(\dot{\varphi}))] \\ &= [A(\delta\varphi), A(\dot{\varphi})]. \end{aligned}$$

Thus,

$$B(\delta\varphi, \dot{\varphi}) = A([\delta\varphi, \dot{\varphi}]) - [A(\delta\varphi), A(\dot{\varphi})].$$

□

The adjoint bundle, $\tilde{\mathfrak{g}}_\odot$ In this section we prove that the adjoint bundle $\tilde{\mathfrak{g}}_\odot$ is isomorphic to the set

$$E := \{(q, \xi_q) \in Q \times \mathfrak{X}_{\mathrm{div}}(M) \quad : \quad \xi_q(q) = 0\}$$

equipped with the bundle projection $\pi_E(q, \xi_q) = q$ and the bracket $[(q, \xi_q), (q, \eta_q)] = (q, [\xi_q, \eta_q])$.[4]

Proposition 3.2.4. *For each $q \in Q$ the map*

$$\Psi([\varphi, \xi_\odot]_\odot) = (\pi(\varphi), \varphi_* \xi_\odot)$$

is a vector bundle and bracket-preserving isomorphism from $\tilde{\mathfrak{g}}_\odot$ to E with inverse

$$\Psi^{-1}(q, \xi_q) = [\varphi, \varphi^* \xi_q]_\odot$$

for arbitrary $\varphi \in \pi^{-1}(q)$. That is, Ψ makes the following diagrams commute.

$$\begin{tikzcd} \tilde{\mathfrak{g}}_\odot \arrow[rr, "\Psi"] \arrow[dr, "\tilde{\pi}"'] & & E \arrow[dl, "\pi_E"] \\ & Q_{\mathrm{part}} & \end{tikzcd} \qquad \begin{tikzcd} \tilde{\mathfrak{g}}_\odot \oplus \tilde{\mathfrak{g}}_\odot \arrow[r, "\Psi \oplus \Psi"] \arrow[d, "{[,]}"'] & E \oplus E \arrow[d, "{[,]}"] \\ \tilde{\mathfrak{g}}_\odot \arrow[r, "\Psi"'] & E \end{tikzcd},$$

Proof. First, we check that Ψ is well defined. Let $[\varphi, \xi_\odot]_\odot \in \tilde{\mathfrak{g}}_\odot$. Then in order for Ψ to be well defined we must verify it maps the expression "$[\varphi, \xi]_\odot$" to the same place as the expression "$[\varphi \circ g_\odot, \mathrm{Ad}_{g_\odot}^{-1} \xi_\odot]_\odot$" for an arbitrary $g_\odot \in G_\odot$. We see that:

$$\begin{aligned} \Psi([\varphi \circ g_\odot, \mathrm{Ad}_{g_\odot}^{-1} \xi_\odot]_\odot) &= (\pi(\varphi \circ g_\odot), (\varphi \circ g_\odot)_*(\mathrm{Ad}_{g_\odot}^{-1}(\xi_\odot))) \\ &= (\pi(\varphi), T\varphi \cdot Tg_\odot \cdot (Tg_\odot^{-1} \cdot \xi_\odot \circ g_\odot) \circ (\varphi \circ g_\odot)^{-1}) \\ &= (\pi(\varphi), T\varphi \cdot \xi_\odot \circ \varphi^{-1}) \\ &= \Psi([\varphi, \xi_\odot]_\odot). \end{aligned}$$

[4] The notation ξ_q for a vector field such that $\xi_q(q) = 0$ is intentionally suggestive. The element ξ_q is in the Lie algebra for the isotropy group of q, just as $\xi_\odot$ has stood for a Lie algebra element of the isotropy group of $\odot$.

Additionally, it is simple to observe that if $q = \varphi(\odot)$ then $\varphi_*\xi_\odot(q) = 0$, so that $\Psi([\varphi, \xi_\odot]_\odot) \in \hat{\mathfrak{g}}_\odot$

Second, we see that $\tilde{\pi} = \pi_E \circ \Psi$ by inspection, so that Ψ is a bundle morphism.

Third, we find

$$\begin{aligned}
[\Psi([\varphi, \xi_\odot]_\odot), \Psi([\varphi, \eta_\odot]_\odot)] &= (\pi(\varphi), [\varphi_*\xi_\odot, \varphi_*\eta_\odot]) \\
&= (\pi(\varphi), \varphi_*[\xi_\odot, \eta_\odot]) \\
&= \Psi([\varphi, [\xi_\odot, \eta_\odot]]_\odot) \\
&= \Psi([\varphi, \xi_\odot]_\odot, [\varphi, \eta_\odot]_\odot).
\end{aligned}$$

Finally, we show Ψ is invertible. We see that

$$\Psi^{-1}(\Psi([\varphi, \xi_\odot])) = \Psi^{-1}(q, \varphi_*\xi_\odot) = [\varphi, \varphi^*(\varphi_*\xi_\odot)] = [\varphi, \xi_\odot],$$

and conversely,

$$\Psi(\Psi^{-1}(q, \xi_q)) = \Psi([\varphi, \varphi^*\xi_q]) = (\pi(\varphi), \varphi_*(\varphi^*\xi_q)) = (q, \xi_q).$$

$\square$

Additionally, the parallel translation on $\tilde{\mathfrak{g}}_\odot$ given by

$$[\varphi, \xi_\odot]_\odot \mapsto [\phi, \xi_\odot]_\odot$$

induces a covariant derivative along a curve in $[\varphi_t, \xi_\odot(t)] \in \tilde{\mathfrak{g}}_\odot$ given by

$$\frac{D}{Dt}[\varphi_t, \xi_\odot(t)] = [\varphi_t, [A(\dot{\varphi}), \xi_\odot] + \dot{\xi}_\odot],$$

where $\dot{\xi}_\odot$ is the time derivative of the curve $\xi_\odot(t) \in \mathfrak{X}_{\mathrm{div}}(M)$.

Proposition 3.2.5. *The covariant derivative $\frac{D}{Dt}$ on E which satisfies the commutative diagram,*

$$
\begin{tikzcd}
\textit{curves in } \tilde{\mathfrak{g}}_{\odot} \arrow[r, "\frac{D}{Dt}"] \arrow[d, "\Psi"'] & \textit{curves in } \tilde{\mathfrak{g}}_{\odot} \arrow[d, "\Psi"] \\
\textit{curves in } E \arrow[r, "\frac{D}{Dt}"'] & \textit{curves in } E
\end{tikzcd}
$$

is given by

$$\frac{D}{Dt}(q, \xi_q) = [q, [\mathcal{R}(\dot{q}), \xi_q] + \dot{\xi}_q]$$

above a curve $q(t) \in Q_{\text{part}}$ *with* $\dot{q} = \frac{dq}{dt}$.

Proof. We need to show

$$\frac{D}{Dt}(q, \xi_q) = \Psi\left(\frac{D}{Dt}(\Psi^{-1}(q, \xi_q)\right)$$

for an arbitrary curve $(q, \xi_q)(t) \in E$. Let $\varphi_t = q_\varphi^\uparrow$ be the horizontal lift of the curve $q(t)$ with the initial condition $\varphi \in \pi^{-1}(q(0))$. Then we find

$$\begin{aligned}
\frac{D}{Dt}\left(\Psi^{-1}(q, \xi_q)\right) &= \frac{D}{Dt}([\varphi_t, \varphi_t^* \xi_q]) \\
&= [\varphi, [A(\dot{\varphi}), \varphi^* \xi_q] + \frac{d}{dt}(\varphi^* \xi_q)].
\end{aligned}$$

Noting that $\dot{\varphi}$ is horizontal, we see that

$$\frac{D}{Dt}\left(\Psi^{-1}(q, \xi_q)\right) = [\varphi, \frac{d}{dt}(\varphi_t^* \xi_q)].$$

Applying the product rule and using the dynamic definition of the Lie-derivative gives us

$$\begin{aligned}
\frac{D}{Dt}\left(\Psi^{-1}(q, \xi_q)\right) &= [\varphi, \varphi^*[\dot{\varphi}_t \circ \varphi^{-1}, \xi_q] + \varphi^*(\dot{\xi}_q)] \\
&= [\varphi, \varphi^*[r(\dot{q}), \xi_q] + \varphi^*(\xi_q)].
\end{aligned}$$

Applying the map Ψ to both sides completes the proof. $\square$

All of this exploration of the adjoint bundle, $\tilde{\mathfrak{g}}_{\odot}$, is merely prelude to an understanding of the coadjoint bundle, $\tilde{\mathfrak{g}}^*_\odot$, the dual vector bundle to $\tilde{\mathfrak{g}}_\odot$. To get a feel for the coadjoint bundle, note that the dual space to $\mathfrak{X}_{\mathrm{div}}(M)$ is identical to the space of one-forms modulo the space exact forms M (see [AK92, Arn66]). Therefore, the coadjoint bundle is the quotient space

$$\tilde{\mathfrak{g}}^*_\odot := \{(\varphi, \alpha_\odot) \cdot G_\odot : \varphi \in \mathcal{D}_\mu(M), \alpha_\odot \in \frac{\bigwedge^1(M\backslash\odot)}{d\bigwedge^0(M\backslash\odot)}\}.$$

We define the bundle

$$E^* := \{(q, \alpha_q) \in Q \times \frac{\bigwedge^1(M\backslash q)}{d\bigwedge^0(M\backslash q)}\}$$

dual to E to find that E^* is isomorphic to $\tilde{\mathfrak{g}}^*_\odot$ through the isomorphism

$$[\varphi, \alpha_\odot] \in \tilde{\mathfrak{g}}^*_\odot \mapsto (\pi(\varphi), \varphi_* \alpha_\odot) \in \hat{\mathfrak{g}}^*_\odot$$

in the same way that E is isomorphic to $\tilde{\mathfrak{g}}_\odot$. In summary, $E^* \equiv (\Psi^*)^{-1}(\tilde{\mathfrak{g}}^*_\odot)$.

Last, the covariant derivative, $\frac{D}{Dt}$, on E induces a unique covariant derivative on E^*.

Proposition 3.2.6. *Given a curve $(q, \alpha_q)(t) \in E^*$, the covariant derivative on E^* such that*

$$\frac{d}{dt}\langle (q, \alpha_q), (q, \xi_q)\rangle = \langle \frac{D}{Dt}(q, \alpha_q), (q, \xi_q)\rangle + \langle (q, \alpha_q), \frac{D}{Dt}(q, \xi_q)\rangle$$

for an arbitrary curve $(q, \xi_q)(t) \in E$ is given by the expression

$$\frac{D}{Dt}(q, \alpha_q) = (q, \dot{\alpha}_q - \mathrm{ad}^*_{\mathcal{R}(q,\dot{q})}\, \alpha_q)$$

where $\dot{\alpha}_q$ is the time-derivative of the curve $\alpha_q(t) \in \frac{\bigwedge^1(M\backslash q(t))}{d\bigwedge^0(M\backslash q(t))}$ viewed as a one-form on M modulo an exact form on M by arbitrary extension.

Proof. By the required condition and the expression for $\frac{D}{Dt}$ on E, we find:

$$\begin{aligned}
\langle \frac{D}{Dt}(q,\alpha_q),(q,\xi_q)\rangle &= \langle \alpha_q, -[\mathcal{R}(q,\dot{q}),\xi_q] - \dot{\xi}_q\rangle + \frac{d}{dt}(\langle (q,\alpha_q),(q,\xi_q)\rangle) \\
&= \langle (q,\dot{\alpha}_q - \mathrm{ad}^*_{\mathcal{R}(q,\dot{q})}\,\alpha_q),(q,\xi_q)\rangle + \langle (q,\alpha_q) - (q,\alpha_q),(q,\dot{\xi}_q)\rangle \\
&= \langle (q,\dot{\alpha}_q - \mathrm{ad}^*_{\mathcal{R}(q,\dot{q})}\,\alpha_q),(q,\xi_q)\rangle.
\end{aligned}$$

Since (q,ξ_q) is arbitrary the result follows. □

From this point on, we equate $\tilde{\mathfrak{g}}_\odot$ with E and casually pass between the notations (q,ξ_q) and $[\varphi,\xi_\odot]_\odot$ for elements of $\tilde{\mathfrak{g}}_\odot$ and E. In terms of calculations we will have a preference for E. The same statement applies to $\tilde{\mathfrak{g}}^*_\odot$ and E^* as well.

Proposition 3.2.7. *The reduced curvature tensor (expressed on E) is given by*

$$\begin{aligned}
\tilde{B}(\dot{q},\delta q) &= [\mathcal{R}(q,\dot{q}),\mathcal{R}(q,\delta q)] - \mathcal{R}(q,[\mathcal{R}(q,\dot{q}),\mathcal{R}(q,\delta q)](q)) \\
&\equiv \mathrm{ver}_q\left([\mathcal{R}(q,\dot{q}),\mathcal{R}(q,\delta q)]\right).
\end{aligned}$$

Proof. Simply use the definition and the map Ψ to find

$$\begin{aligned}
\tilde{B}(\dot{q},\delta q) &:= [\varphi, B(\dot{q}^\uparrow_\varphi,\delta q^\uparrow_\varphi)]_\odot \\
&\equiv \varphi_* B(\dot{q}^\uparrow_\varphi,\delta q^\uparrow_\varphi).
\end{aligned}$$

Since $B(X,Y) = A([X,Y]) - [A(X),A(Y)]$ and $\dot{q}^\uparrow_\varphi,\delta q^\uparrow_\varphi$ are horizontal we find.

$$\begin{aligned}
&= \varphi_* A([\dot{q}^\uparrow_\varphi,\delta q^\uparrow_\varphi]) \\
&= \varphi_*\left(T\varphi^{-1}\cdot[\dot{q}^\uparrow_\varphi,\delta q^\uparrow_\varphi] - \varphi^*(\mathcal{R}([\dot{q}^\uparrow_\varphi,\delta q^\uparrow_\varphi](q)))\right) \\
&= \varphi_*\left(T\varphi^{-1}[\mathcal{R}(q,\dot{q})\circ\varphi,\mathcal{R}(q,\delta q)\circ\varphi]\right) - \mathcal{R}(q,[\mathcal{R}(q,\dot{q}),\mathcal{R}(q,\delta q)](q)) \\
&= \varphi_*\left(T\varphi^{-1}[\mathcal{R}(q,\dot{q}),\mathcal{R}(q,\delta q)]\circ\varphi\right) - \mathcal{R}(q,[\mathcal{R}(q,\dot{q}),\mathcal{R}(q,\delta q)](q)) \\
&= [\mathcal{R}(q,\dot{q}),\mathcal{R}(q,\delta q)] - \mathcal{R}(q,[\mathcal{R}(q,\dot{q}),\mathcal{R}(q,\delta q)](q)).
\end{aligned}$$

□

3.3 Equations of Motion on $TQ_{\text{part}} \oplus E$

Consider the kinetic Lagrangian, $L : T\mathcal{D}_\mu(M) \to \mathbb{R}$, given by

$$L(\varphi, \dot{\varphi}) = \frac{1}{2} \int_M \|\dot{\varphi}\|^2 d^3x.$$

As L is $G_\odot$ invariant we may alternatively use the reduced Lagrangian on $TQ_{\text{part}} \oplus \tilde{\mathfrak{g}}_\odot$ given by

$$l(q, \dot{q}, \xi_q) = \frac{1}{2} \int_M \|\dot{q}^\uparrow_\varphi \circ \varphi^{-1} + \xi_q\|^2 d^3x, \tag{3.3}$$

where φ is an arbitrary element of $\pi^{-1}(q)$. In particular, l has the form

$$l(q, \dot{q}, \xi_q) = L_{\text{part}}(q, \dot{q}) + L_\times(q, \dot{q}, \xi_q) + l_\odot(\xi_q), \tag{3.4}$$

where

$$\begin{aligned}
L_{\text{part}}(q, \dot{q}) &:= \frac{1}{2} \int_M \|\mathcal{R}(q, \dot{q})\|^2 d^3x := \frac{1}{2} \ll \dot{q}, \dot{q} \gg_{\text{part}} := \frac{1}{2} g_{ij} \dot{q}^i \dot{q}^j, \\
L_\times(q, \dot{q}, \xi_q) &:= \int_M \langle \mathcal{R}(q, \dot{q}), \xi_q \rangle d^3x = \langle \mathbb{P}(q, \dot{q}), \xi_q \rangle \\
l_\odot(\xi_q) &:= \frac{1}{2} \int_M \|\xi_q\|^2 d^3x := \frac{1}{2} \ll \xi_q, \xi_q \gg_E .
\end{aligned}$$

We have imposed a coordinate system $(\dot{q}^1, \ldots, \dot{q}^N)$ on the finite-dimensional space Q so that L_{part} is induced by some metric tensor on Q_{part} given in coordinates by g_{ij}. It is simple to observe that L_{part} and $l_\odot$ come from inner products on the vector bundles TQ and E, respectively, which we have denoted by $\ll, \gg_{\text{part}}: TQ \oplus TQ \to \mathbb{R}$ and $\ll, \gg_E: E \oplus E \to \mathbb{R}$. Additionally $l_\times$ comes from a vector bundle morphism $\mathbb{P} : TQ \to \tilde{\mathfrak{g}}^*$ defined by:

$$\langle \mathbb{P}(q, \dot{q}), (q, \xi_q) \rangle := \int_M \langle \mathcal{R}(q, \dot{q}), \xi_q \rangle_M dx.$$

Before we derive the equations of motion, we should understand the expressions

$\frac{\partial l}{\partial q}$ and $\tilde{B}_\mu$ in more detail.

Proposition 3.3.1. *Let $q(t)$ be a curve in Q_{part} and $\dot{q}(t) = \frac{dq}{dt} \in TQ_{\text{part}}$. For each time t we can define the linear map $\Gamma_{\dot{q}(t)} : T_{q(t)}Q_{\text{part}} \to T_{q(t)}Q_{\text{part}}$ by*

$$\Gamma_{\dot{q}(t)}(\delta q) = \nabla_{\delta q}(\dot{q}).$$

Additionally, since $\mathcal{R}$ is a vector bundle map we may define the linear map on the tangent fiber above q by $\mathcal{R}_q : T_qQ_{\text{part}} \to \mathfrak{X}_{\text{div}}(M)$ and the dual map $\mathcal{R}_q^* : (\mathfrak{X}_{\text{div}}(M))^* \to T_q^*Q_{\text{part}}$. The fiber derivative of $\mathcal{R}$ above $q \in Q_{\text{part}}$ is denoted as $\mathbb{F}\mathcal{R}_q : T_qQ \to \mathfrak{X}_{\text{div}}(M)$. The dual is denoted as $\mathbb{F}\mathcal{R}_q^* : (\mathfrak{X}_{\text{div}}(M))^* \to T^*Q_{\text{part}}$. Then, the partial derivative $\frac{\partial l}{\partial q} = \frac{\partial L_{\text{part}}}{\partial q} + \frac{\partial L_\times}{\partial q} + \frac{\partial l_\odot}{\partial q}$ at the point $(q, \dot{q})$ is the sum of the terms:

$$\begin{aligned} \frac{\partial L_{\text{part}}}{\partial q} &= 0, \\ \frac{\partial L_\times}{\partial q} &= -\mathcal{R}_q^* \cdot \xi_q[\flat\left(R(q, \dot{q})\right)] + \Gamma_{\dot{q}}^* \cdot \mathbb{F}\mathcal{R}_q^* \cdot \flat(\xi_q), \\ \frac{\partial l_\odot}{\partial q} &= \mathcal{R}_q^* \cdot \mathrm{ad}_{\xi_q}^* \left(\xi_q^\flat\right), \end{aligned}$$

where $\flat$ refers to contraction with the metric tensor $\ll, \gg$.

Proof. It is easiest to consider L_{part} first. We find by our definition of the expression $\frac{\partial}{\partial q}$ given in §2.2 that

$$\langle \frac{\partial L_{\text{part}}}{\partial q}, \delta q \rangle = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} (L(q_\epsilon, \dot{q}_\epsilon))$$

for some curve $(q_\epsilon, \dot{q}_\epsilon)$, such that $\left.\frac{dq_\epsilon}{d\epsilon}\right|_{\epsilon=0} = \delta q$ and $\frac{D\dot{q}_\epsilon}{D\epsilon} = 0$. However, this parallel transport of $\dot{q}$ is precisely the type of change which does not alter the value of L_{part} which comes from the metric that induced the parallel transport. Thus $\frac{\partial L_{\text{part}}}{\partial q} = 0$. In particular, the parallel transport of $(q, \dot{q}, \xi)$ along a curve q_ϵ yields the variations $\delta\dot{q} = \nabla_{\delta q}\dot{q}$ and $\delta\xi_q = -[\mathcal{R}(q, \delta q), \xi_q]$. By applying the same idea to the other Lagrangians, we find

$$\langle \frac{\partial L_\times}{\partial q}, \delta q \rangle = \underbrace{\ll \mathbb{F}\mathcal{R}(q, \dot{q}) \cdot \nabla_\delta q\dot{q}, \xi_q \gg}_{T_1} + \underbrace{\ll \mathcal{R}(q, \dot{q}), -[\mathcal{R}(q, \delta q), \xi_q] \gg}_{T_2}.$$

We see that

$$
\begin{aligned}
T_1 &= \ll \mathbb{F}\mathcal{R}_q \cdot \Gamma_{\dot{q}}(\delta q), \xi_q \gg \\
&= \langle \xi_q^\flat, \mathbb{F}\mathcal{R}_q \cdot \Gamma_{\dot{q}}(\delta q) \rangle \\
&= \langle \Gamma_{\dot{q}}^* \cdot \mathbb{F}\mathcal{R}_q^* \xi_q^\flat, \delta q \rangle.
\end{aligned}
$$

By manipulations in the same spirit, we find

$$
T_2 = \langle R_q^* \operatorname{ad}_{\xi_q}^* \left(\xi_q^\flat \right), \delta q \rangle.
$$

Thus

$$
\frac{\partial L_\times}{\partial q} = \Gamma_{\dot{q}}^* \cdot \mathbb{F}\mathcal{R}_q^* \xi_q^\flat + R^* \operatorname{ad}_{\xi_q}^* \left(\xi_q^\flat \right).
$$

The final derivative $\frac{\partial l_\odot}{\partial q}$ is derived in the same vein. □

Proposition 3.3.2. *The force from the curvature term is*

$$
i_{\dot{q}} \tilde{B}_\mu = \mathcal{R}_q^* \left(\operatorname{ad}_{R(q,\dot{q})}^* \left[\operatorname{ver}^* \left(\xi_q^\flat + (\mathcal{R}(q,\dot{q}))^\flat \right) \right] \right).
$$

Proof. Define the momentum as $\mu = \frac{\partial l}{\partial \xi} = \xi_q^\flat + (\mathcal{R}(q,\dot{q}))^\flat$. Thus for an arbitrary δq, we find

$$
\begin{aligned}
\langle i_{\dot{q}} \tilde{B}_\mu, \delta q \rangle : &= \langle \mu, B(\dot{q}, \delta q) \rangle \\
&= \langle \xi_q^\flat + (\mathcal{R}(q,\dot{q}))^\flat, \operatorname{ver}_q ([\mathcal{R}(q,\dot{q}), \mathcal{R}(q,\delta q)]) \rangle \\
&= \langle \operatorname{ver}_q^* \left(\xi_q^\flat + (\mathcal{R}(q,\dot{q})^\flat) \right), \operatorname{ad}_{\mathcal{R}(q,\dot{q})} (\mathcal{R}(q,\delta q)) \rangle \\
&= \langle R_q^* \left(\operatorname{ad}_{\mathcal{R}(q,\dot{q})}^* \left[\operatorname{ver}_q^* \left(\xi_q^\flat + (\mathcal{R}(q,\dot{q}))^\flat \right) \right] \right), \delta q \rangle.
\end{aligned}
$$

The result follows because δq is arbitrary. □

Putting together all of the pieces, the Lagrange-Poincaré equations can then be written as

$$i_{\dot{q}}\tilde{B}_\mu = \frac{D}{Dt}\left(\dot{q}^\flat + \mathbb{P}^*(q,\xi_q)\right) - \frac{\partial l}{\partial q}, \tag{3.5}$$

$$\frac{D\xi_q^\flat}{Dt} = \mathrm{ad}^*_{\xi_q}\left(\mathbb{P}(q,\dot{q}) + \xi_q^\flat\right) - \frac{D}{Dt}\left(\mathbb{P}(q,\dot{q})\right), \tag{3.6}$$

where

$$i_{\dot{q}}\tilde{B}_\mu = \mathcal{R}_q^*\left(\mathrm{ad}^*_{R(q,\dot{q})}\left[\mathrm{ver}^*\left(\xi_q^\flat + (\mathcal{R}(q,\dot{q}))^\flat\right)\right]\right), \tag{3.7}$$

$$\frac{\partial l}{\partial q} = \mathcal{R}_q^* \cdot \mathrm{ad}^*_{\xi_q}\left(\xi_q^\flat\right) - \mathcal{R}_q^* \cdot \xi_q[\flat\left(R(q,\dot{q})\right)] + \Gamma^*_{\dot{q}} \cdot \mathbb{F}\mathcal{R}_q^* \cdot \flat(\xi_q). \tag{3.8}$$

These equations still evolve on the infinite-dimensional vector bundle $TQ_{\mathrm{part}} \oplus E$ and are by most regards "more difficult" than the inviscid Euler equations. At this point, the reader may feel as if we have taken a step backwards. Here is the upshot: Particle methods are equivalent to ODEs on TQ_{part}. Therefore, error analysis of particle methods may be performed by comparing the ODE of the particle method to equation (3.5) via the natural embedding $TQ_{\mathrm{part}} \hookrightarrow TQ_{\mathrm{part}} \oplus E$.

3.4 Particle Methods

The inner product $\ll,\gg_Q$ makes $(Q,\ll,\gg_Q)$ into a Riemannian manifold. If one desires to estimate the dynamics in Q without reference to the bundle E, then a simple estimate would be given by L_{part}.[5] Additionally, when computing dynamics, we could apply the non-holonomic constraint that the spatial velocity field is always in the range of $\mathcal{R}$, so that the E component remains 0 for all time. The solution of this constrained system would be a geodesic flow on $(Q,\ll,\gg_Q)$, and would produce a particle method for estimating fluid motion. In this section we will explore this idea in the context of the existing method known as "smoothed particle hydrodynamics", and additionally create new particle methods. Lastly, the non-holonomic constraint

[5]This would be an entropy-minimizing approximation if we view the vertical component as a random variable with mean 0.

to the range of $\mathcal{R}$ can be relaxed to allow for a first-order (in time) correction to the equations of motion. This will be explored in the final section on error analysis.

Smoothed particle hydrodynamics Smooth particle methods track the position of a finite number of particles in an open set $M \subset \mathbb{R}^3$. The motion of the particles satisfies the Euler-Lagrange equations for a Lagrangian of the form $\frac{1}{2}\dot{q}^2 - V(q)$ using the Euclidean metric on $\mathbb{R}^d$ and a potential energy of the form

$$V(q) = \sum_{i<j} W(q_i - q_j).$$

The function W is usually taken to be a spherically symmetric function of $\mathbb{R}^d$ with concentrated mass about the origin (e.g., a Gaussian blob).

The integral curves of this system are identical to those of the Lagrangian system given by

$$L_{spm}(q, \dot{q}) = \frac{1}{2}(E_0 - V(q)) \sum_i \dot{q}_i \cdot \dot{q}_i$$

where E_0 is the energy of the initial condition and $\cdot$ is a dot product on $\mathbb{R}^d$ (see "Jacobi Metric" in [AM00]).

Therefore, if there exists a principal connection, A, such that

$$\langle \dot{q}, \delta q \rangle_Q = (E_0 - V(q)) \sum_i \dot{q}_i \cdot \delta q_i, \tag{3.9}$$

then the dynamics of the exact Lagrangian (3.4) constrained to $TQ_{\text{part}} \subset TQ_{\text{part}} \oplus E$ will match the dynamics of a smoothed particle method. We can do this by setting the horizontal space above $q \in Q_{\text{part}}$ to be $\mathbb{H}_q := \text{span}\{X^q_{i,\mu}\}_{i,\mu}$, where we take the vector fields $\{X^q_{i,\mu}\}$ to be an orthonormal basis. Then the vector fields $X^q_{i,\mu}$ must satisfy (3.9), implying:

$$X^q_{i,\mu}(q_j) = \delta_{ij} \left. \frac{\partial}{\partial x^\mu} \right|_{q_i}, \tag{3.10}$$

$$\int_M \langle X^q_{i,\mu}, X^q_{j,\nu} \rangle dx = \delta_{ij}\delta_{\mu\nu} \cdot \sqrt{E_0 - V(q)}. \tag{3.11}$$

However, determining the basis $\{X^q_{i,\mu}\}$ is an underdetermined problem. We could make it well posed by defining an energy to minimize, such as

$$C\{X^q_{i,\mu}\} = \sum_{i,\mu} \int_M |\mathrm{shear}(X^q_{i,\mu})|^2 dx$$

where $\mathrm{shear}(X) = \sum_{i,j} \frac{\partial X^i}{\partial x^j}$. If M has a boundary, we would need to impose boundary conditions as well. This is admittedly a sizable problem of a scale which is impractical to solve. Additionally, this cost function is chosen in a somewhat arbitrary fashion. In summary, there does not appear to be a clear choice of the "best" principal connection for the smoothed particle method. However, there are a number of preexisting reconstruction methods. In particular, one takes the smoothing kernel of the method, $\delta_h(x)$, and defines the mapping $\mathcal{S} : TQ_{\mathrm{part}} \to \mathfrak{X}(M)$ by $\mathcal{S}(q, \dot{q})(x) = \sum_i \dot{q}_i \delta_h(x)$. However this does not produce divergence-free vector fields. Taking the divergence-free component of $\mathcal{S}$ by the Hodge decomposition would provide us with a valid reconstruction method. However, we would not have a reconstruction method that is compatible with (3.9). We suspect that the choice of "best" reconstruction method will depend upon circumstances which vary between scenarios. Therefore we end our discussion of SPM here so that we can discuss the construction of new particle methods in which the error analysis is taken into account by construction.

New particle methods In the case of smoothed particle methods it was found that the error analysis could not be completed without choosing a principal connection compatible with the dynamics. It was found that finding such a principal connection was an underdetermined problem. In this section we seek to construct a particle method by choosing this principal connection first. One advantage to building particle methods this way is that the error analysis is clear by construction. More specifically, new particle methods can be made by adding the constraint that the E component be 0. This suppresses the vertical equation (3.6) and sets $\frac{\partial l}{\partial q} = 0$. The remaining constrained horizontal equation then reads:

$$i_{\dot{q}}\tilde{B}_\mu = \frac{D}{Dt}\left(\dot{q}^\flat + \mathbb{P}^*(q,\xi_q)\right),$$
$$i_{\dot{q}}\tilde{B}_\mu = \mathcal{R}_q^*\left(\mathrm{ad}^*_{R(q,\dot{q})}\left[\mathrm{ver}^*\left(\mathcal{R}(q,\dot{q})\right)^\flat\right]\right).$$

We will call the integration of these equations *method A*. Another way to describe method A is to consider the exact equations of motion on $TQ_{\text{part}} \oplus E$ as a direct sum of two vector-field $X(q,\dot{q},\xi) \oplus Y(q,\dot{q},\xi)$. Then method A is given by the vector field $\tilde{X}$ on TQ_{part} defined by $\tilde{X}(q,\dot{q}) := X(q,\dot{q},0)$.

Method A seeks to minimize the infinitesimal time error, but neglects much the geometry of Lagrangian mechanics. Alternatively, one could construct methods based on variational principles by maximizing the Lagrangian on the submanifold of $TQ_{\text{part}} \oplus E$ where the E component is 0 by adding a non-holonomic constraint which forces the spatial velocity field to be in the range of some reconstruction mapping. This would be equivalent to executing the following sequence:

1. Choose a reconstruction mapping $\mathcal{R} : TQ_{\text{part}} \to \mathfrak{X}_{\text{div}}(M)$.

2. Compute the inner product

$$\ll \dot{q}, \delta q \gg_{\text{part}} := \int_M \ll \mathcal{R}(q,\dot{q}), \mathcal{R}(q,\delta q) \gg dx.$$

3. Set

$$L_{\text{part}}(q,\dot{q}) = \frac{1}{2} \ll \dot{q}, \dot{q} \gg_{\text{part}}.$$

4. The Euler-Lagrange equations for L_{part} are the geodesic equations on $(Q, \ll , \gg_{\text{part}})$; these equations are the numerical method induced by $\mathcal{R}$.

We will call this second approach *method B*. In the limit of infinitesimal time steps, method B will preserve energy (which happens to be equal to L_{part}). Using a variational integrator, method B can conserve energy over large timescales (see [HLW02] or [MW01]).

However, method A is more accurate than method B in infinitesimal time because it incorporates the curvature term $i_{\dot{q}}\tilde{B}_{\mu}$. This is a major difference with traditional smooth particle methods which do not exhibit these forces. However, introducing the curvature term destroys the geometric structure of solving geodesic equations. If preservation of conserved quantities such as energy and momenta are important, then method A may be a better choice.

Infinitesimal time error analysis We can perform error analysis of method A at time $t = 0$ by comparing the geodesic equations on $(Q, \ll, \gg_{\text{part}})$ to the Lagrange-Poincaré equations (3.5) and (3.6). At time $t = 0$ let $q(0) = q_0 \in Q_{\text{part}}$. if $u_0 \in \mathfrak{X}_{\text{div}}(M)$ is the exact initial condition of a fluid spatial velocity field and $\dot{q}_0 = u(q_0)$, then the vertical component is

$$\xi_q(0) = \mathcal{R}(q_0, \dot{q}_0) - u_0 \in E.$$

The vertical component ξ_q at $t = 0$ represents the error of the estimate $\mathcal{R}(q_0, \dot{q}_0)$ in reconstructing u_0. In the case where $\xi_q(0) = 0$, then $\xi_q(t)$ satisfies the equation

$$\frac{D\xi_q^\flat}{Dt} = -\frac{D}{Dt}(\mathbb{P}(q, \dot{q})) \tag{3.12}$$

at time $t = 0$.

Additionally, at $t = 0$ our estimate of the horizontal equations will be wrong because we will be missing the curvature term. In method B, we include the curvature term. For the sake of accuracy in infinitesimal time, method B deceptively seems like a good idea. However, method B would likely destroy the symplectic structure and produce a non-conservative ODE on TQ.

Remeshing One may desire to remove particles from dense areas and place them in less-dense areas in order to keep the particle spacing below some threshold. One of the difficulties with particle methods is that it is not clear how to do this. Even if one knows how they want to place the particles, it is not clear how to choose the velocity of the particles. However, the map $\mathcal{R} : TQ_{\text{part}} \to \mathfrak{X}_{\text{div}}(M)$ provides one means of doing just this. Given a configuration of particles $q \in Q_{\text{part}}$ with velocity $\dot{q}$, if we wish to move the particles to the configuration $q_{\text{new}} \in Q_{\text{part}}$ we can assign them the velocity $\dot{q}_{\text{new}} = \mathcal{R}(\dot{q})(q_{\text{new}})$.

This change from configuration q to configuration q_{new} would alter the velocity field estimate (given by $\mathcal{R}$) by the amount

$$\Delta = \mathcal{R}(q_{\text{new}}, \dot{q}_{\text{new}}) - \mathcal{R}(q, \dot{q}).$$

Due to our choice of $\dot{q}_{\text{new}}$, the vector field $\Delta \in \mathfrak{X}_{\text{div}}(M)$ is such that $\Delta(q_{\text{new}}) = 0$, so $(q_{\text{new}}, \Delta) \in E$. This is consistent with our interpretation of the vector bundle, E, as the space which stores the error of our reconstructed estimates. For certain cases Δ is zero, and we can ignore it. That is, we may move from configuration q to q_{new} without changing the estimated spatial velocity field. This would change the dynamics, but it will allow us to derive an error bound. We will see such a case in the next section.

3.5 An Example on $\mathbb{T}^2$

In this section we will carry out this procedure on $\mathbb{T}^2$ viewed as $\frac{\mathbb{C}}{2\pi\mathbb{Z}+2\pi i\mathbb{Z}}$. Assume the number of particles is $n = N^2$ for some integer N. Define the complex vector fields

$$L_k = ie^{i(k_1x+k_2y)}(k_2\frac{\partial}{\partial x} - k_1\frac{\partial}{\partial y}), \quad k_1, k_2 = 1, \ldots, N.$$

Each L_k corresponds to two real vector fields by taking the real and imaginary parts. We find that

$$[L_k, L_j] = -i \cdot (k_2 j_1 - j_2 k_1) L_{k+j}.$$

Additionally the vector fields $\{L_k\}$ form an orthogonal basis for square integrable divergence-free vector fields (see [Zei91])[6]. Now let $X_1, \ldots, X_n$ be the sequence

$$\begin{array}{llll} L_{1,0} & , L_{2,0} & , \ldots & , L_{N,0} \\ L_{1,1} & , L_{2,1} & , \ldots & , L_{N,1} \\ \vdots & , \vdots & , & , \vdots \\ L_{1,N} & , L_{2,N} & , \ldots & , L_{N,N}. \end{array}$$

We define the reconstruction mapping $\mathcal{R}(q, \dot{q}) = \sum_{j=1}^{n} c_j(q, \dot{q}) X_j$, where the coefficients $c_j(q, \dot{q}) \in \mathbb{C}$ are the solution to the inverse problem $\sum_{j=1}^{n} c_j(q, \dot{q}) X_j(q_i) = \dot{q}_i$. In matrix form, this is written as

$$\dot{q} = [w] \cdot c,$$

where $w_{ij} = X_j(q_i), c = (c_1, \ldots, c_n)$. This reconstruction mapping defines the horizontal space above $\varphi \in \pi^{-1}(q)$ to be

$$\mathbb{H}(\varphi) := \operatorname{span}(X_1 \circ \varphi, \ldots, X_n \circ \varphi)$$

and the horizontal lift to be

$$\dot{q}_{\varphi}^{\uparrow} = \sum_{j=1}^{n} c_j(q, \dot{q}) X_j \circ \varphi.$$

The kinetic energy of the particles is given by:

$$L_{\text{part}}(q, \dot{q}) = \sum_{j=1}^{n} \|X_j\|^2 \|c_j(q, \dot{q})\|^2.$$

[6] Our integrator is different from [Zei91] because we choose a different closure method. In [Zei91] the spatial velocity field was constrained to $\operatorname{span}(L_{1,1}, \ldots, L_{N,N})$ by equating L_{k_1+N,k_2} with L_{k_1,k_2} (and similarly for L_{k_1,k_2+N}). Here we constrain spatial velocity to $\operatorname{span}(L_{1,1}, \ldots, L_{N,N})$ through holonomic constraints.

The magnitudes for each X_j can be obtained from the magnitude $\|L_k\|^2 = 4\pi^2\|k\|^2$. This choice of horizontal space induces the horizontal projection

$$\mathrm{hor}(\varphi, \delta\varphi) = [\delta\varphi(\odot)]^{\uparrow}_{\varphi}$$

and the principal connection

$$A(\varphi, \delta\varphi) = (\delta\varphi - \mathrm{hor}(\varphi, \delta\varphi)) \circ \varphi^{-1}.$$

Finally the reduced curvature form is

$$\tilde{B}(\dot{q}, \delta q) = \varphi_* A\left([\dot{q}^{\uparrow}_{\varphi}, \delta q^{\uparrow}_{\varphi}]\right),$$

where we define the bracket on $T\,\mathrm{SDiff}(M)$ as the pullback of the standard Lie bracket on the Lie algebra. That is, $[\dot{\varphi}, \delta\varphi] := [\dot{\varphi}\varphi^{-1}, \delta\varphi\circ\varphi^{-1}]\circ\varphi$. Additionally, because $\mathbb{H}(\varphi)$ is spanned by the complex vector fields $L_k \circ \varphi$, it is useful to calculate the curvature in this basis. We find:

$$B(L_k \circ \varphi, L_j \circ \varphi) = \begin{cases} i(k_1 j_2 - j_1 k_2)\varphi_* A(L_{k+j} \circ \varphi) & \text{if } k_1 + j_1 > N \text{ or } k_2 + j_2 > N \\ 0 & \text{else} \end{cases}.$$

To discretize time and calculate trajectories we can invoke the framework of "discrete Lagrangian mechanics" [MW01, MV91] by choosing the discrete Lagrangian:

$$L_d(q, q^+) = L_{\mathrm{part}}\left(\frac{q + q^+}{2}, \frac{q^+ - q}{h}\right).$$

The integrator is equal to the discrete Euler-Lagrange equations

$$D_2 L_d(q^-, q) + D_1 L_d(q, q^+) = 0,$$

which are then solved with a root-finding algorithm, such as Newton's method. This produces a symplectic variational integrator with approximate conservation of energy over large times and exact conservation of Noetherian momenta. This would be method A.

To implement method B, one can approximate the curvature force, $i_{\dot{q}} B_\mu$, in discrete time by substituting $\dot{q}$ with $\frac{1}{2h}(q^+ - q^-)$ to get a covector F_d, and then solving the forced discrete Euler-Lagrange equations

$$D_2 L_d(q^-, q) + D_1(q, q^+) = F_d.$$

Infinitesimal time error analysis The complex vector fields, $\{L_k\}$, serve as a basis for square integrable vector fields on M. Given the initial condition $u_0 \in \mathfrak{X}_{\text{div}}(M)$ we set $\dot{q} = u_0(\odot)$ as the initial velocity for our particle method.[7] Using the fact that $\mathcal{R}(q, \dot{q})(\odot_i) - u_0(\odot_i) = 0$, we find that the reconstructed vector field $\mathcal{R}(q, \dot{q})$ satisfies the error bound

$$\|\mathcal{R}(q, \dot{q}) - u_0\|_\infty \leq \|\nabla u_0\|_\infty \|\Delta x\|,$$

where Δx is the largest distance between neighboring particles.

Method B: We can get an error bound which is second order in time. By using the remeshing method of §3.4 to guarantee that Δx remains below some some threshold Δx_{max}, we notice that translating q to q_{new} does not alter the reconstructed vector field. This is because we set:

$$\dot{q}_{\text{new}} := \mathcal{R}(q, \dot{q})(q_{\text{new}})$$

and then we find $\mathcal{R}(q_{\text{new}}, \dot{q}_{\text{new}})(q_{\text{new}}) = \dot{q}_{\text{new}} = \mathcal{R}(q, \dot{q})(q_{\text{new}})$. This last equation places the same constraints on the space spanned by $X_1, \ldots, X_n$ and so $\mathcal{R}(q_{\text{new}}, \dot{q}_{\text{new}}) =$

[7] This would make the estimated spatial velocity field, $\mathcal{R}(q, \dot{q})$, less then optimal (in the Euclidean 2-norm) because a better approximation would be to orthogonally project the desired initial velocity, u_0, onto the horizontal space. However, the "improved" approximation would induce an initial condition which would lead to first order in time accuracy in predicting particle velocities. It is imperative to get particle velocities correct at time $t = 0$ in order to say anything meaningful about error for particle methods over infinitesimal times.

$\mathcal{R}(q, \dot{q})$. Therefore, remeshing leaves the estimated spatial velocity unaltered and can be used at each time step to keep the particles within a distance Δx_{max} without penalty. This generalizes the idea of semi-Lagrangian methods by relieving the constraint that we drag the data carried by the particles to predetermined nodes; instead, we may drag the data to whatever nodes we consider convenient. Finally, the error of method B will come from neglecting the curvature term, $i_{\dot{q}} B_\mu$. If u is the exact solution and C_u is a bound on $\|u\|$ over the time interval $[0, \Delta t]$, then the magnitude of the force satisfies $\|i_{\dot{q}} B_\mu\| \leq C_u^2 B_{\max}$, where $B_{\max}$ is the supremum of the expression "$\|i_v B_\mu\|$" on unit vectors $v \in TQ_{\text{part}}$. If there exists a bound, $C_{\nabla u}$, on $\|\nabla u\|$ over the time interval $[0, \Delta t]$, then the bound, $B_{\max}$, on the missing curvature term produces the second-order error bound

$$\|\mathcal{R}(q, \dot{q}) - u\|_\infty \leq (C_{\nabla u} \Delta x_{max}) \cdot \left(C_u^2 B_{\max}\right) \Delta t^2$$

for the reconstructed velocity field at time Δt.

Method A: If the vertical component, $\xi_q = u - \mathcal{R}(q, \dot{q}) = 0$ at time $t = 0$, then we can implement method A by including the curvature term, $i_{\dot{q}} B_\mu(\dot{q}, 0)$, to get an extra order of accuracy. If we estimate the vertical component to be 0 for all time (as a closure method), then by equation (3.12) the error of this estimate would satisfy the bound $\|\Delta \xi\| < C_\xi \|\dot{q}_0\| \Delta t$ for some constant C_ξ. The important thing to note is that this error bound is first order in time. The error $\Delta \xi$ would introduce an error of size $\|i_{\dot{q}} B_{\Delta \xi}(\dot{q}, 0)\| \sim O(\Delta t)$ in our estimate of $\ddot{q}$. This would produce the error bound

$$\|\mathcal{R}(q, \dot{q}) - u\|_\infty \leq (C_{\nabla u} \Delta x_{max}) \cdot \left(C_u^2 C_\xi B_{\max}\right) \Delta t^3.$$

This makes method A third-order accurate in time.

A finite time error bound A second advantage to the geometric framework is that we can consider finite-time error bounds. That is to say, we can construct conservative bounds on the error over times of order 1. We are able to consider this possibility because we know exactly what we are missing: the dynamics on E. We know that if $\xi = 0$ at time $t = 0$, then the exact equations of motion would satisfy $\dot{\xi} = \mathcal{R}(q,\dot{q}) \cdot \nabla (\mathcal{R}(q,\dot{q})) - \nabla p$. However, in the context of methods A and B we are consistently neglecting ξ and effectively setting it to 0 for all time. Thus, ξ stores the error of our reconstructed vector field. Forming the quantity $\delta e = \|\mathcal{R}(q,\dot{q}) \cdot \nabla (\mathcal{R}(q,\dot{q})) - \nabla p$ at each time step, we may construct an error bound at time T given by

$$e_{\max} = \int_0^T \delta e dt.$$

Assuming e can be calculated or approximated within some tolerance, we can use it as a stopping criterion. Such stopping criteria are important when accuracy is desired over long times.

3.6 Simple Extensions and Future Work

So far, we have illustrated how one can take the horizontal Lagrange-Poincaré equation to derive an ODE on TQ_{part}, which can be used as a candidate particle method. Naturally, one is reluctant to simply drop the vertical equations. From simply judging by the dimensions, the vertical equations contain “more of the action”. In this section, we present some possible ideas for further investigation into producing higher-accuracy meshless methods.

Higher-order isotropy groups The analysis presented in this chapter addresses methods in which the particles carry 0th-order data about the spatial velocity field, $u \in \mathfrak{X}_{\text{div}}(M)$, of the fluid. That is, the *only* datum a particle carries is the velocity of the fluid at a single point. The particles do not carry 1st order data, i.e., data obtained from ∇u, such as the vorticity at a point or the local stretching. This is unfortunate considering how important vorticity is for turbulence modeling [Cho94]. To address this, one could consider the isotropy group

$$G_{\odot}^{(1)} := \{\phi \in \mathcal{D}_\mu(M) : \phi(\odot) = \odot, \quad T_{\odot}\phi = 0\}.$$

This is the set of special diffeomorphisms which are equal to the identity on $\odot$ to 1st order. We can calculate the quotient $\mathcal{D}_\mu(M)/G^{(1)}_\odot$ to be the frame bundle of Q_{part}. The fibers of the frame bundle contain the vorticity and stretch data around each particle. For example, if M is a flat Riemannian manifold then the quotient space is the set of particle configurations with a $(1,1)$ tensor attached to them. The anti-symmetric part of the time derivative of the $(1,1)$ is the vorticity, and the symmetric part is the stretch. It would be interesting to see if and how vortex methods can be approached in some limiting case of this perspective.

More generally one can consider the kth-order isotropy group

$$G^{(k)}_\odot := \{\phi \in \mathcal{D}_\mu(M) : \phi(\odot) = \odot, \quad T^i_\odot \phi = 0, i = 1, \ldots, k\}.$$

The quotient space $\mathcal{D}_\mu(M)/G^{(k)}_\odot$ is the kth-order frame bundle (i.e., the frame of the frame of the frame ... bundle). If M is a flat Riemannian manifold, then the quotient space is the set of particle configurations, each carrying tensors of rank $(1,1), (1,2), \ldots, (1,k)$ above them. Thus, we can create new methods which carry higher-order data above u using roughly the same constructions presented in this chapter.

Navier-Stokes and complex fluids The Navier-Stokes equations can be viewed as a dissipative version of the Euler equations (see §1.12 of [AK92]). Moreover, there are a number of fluids on slightly more complex spaces where Euler-Poincaré reduction has been performed, such as in magnetohydrodynamics and liquid crystals [Hol02, GBR09]. The particle relabeling symmetry of these systems makes the procedure presented in this chapter applicable to them as well. In the case of complex fluids, the unreduced configuration is a semidirect product with the special diffeomorphism group as the first component. Particle methods for complex fluids would attach data to the particles in addition to the instantaneous velocity.

Practical considerations for implementations When obtaining a method in any context, there are a number of things to keep in mind. Besides accuracy, we have not addressed some of the basic issues that one comes across when evaluating the performance of a new method. In particular, consistency (i.e., convergence to the exact solutions as the number of particles goes to infinity) is something which needs to be investigated. This will likely depend on the choice of principal connection used to estimate the spatial velocity fields. For example, consistency of the SPH method relies heavily on the smoothness of Gaussian kernel functions. One would expect the smoothness of the image of the reconstruction mapping to play a similar role.

Additionally, the practical performance of a method depends heavily on the coupling between the particles. In the example provided in §3.5 all of the particles were coupled to each other; for high-accuracy computation, this scales very badly. Principal connections which yield a large amount of coupling should be avoided when one is planning on using a large number of particles. However, this constraint will likely have some trade-off in accuracy. This also should be investigated.

Finally, the boundary conditions have not been sufficiently addressed in this chapter. It is certainly possible to satisfy the boundary conditions by construction simply by requiring that the range of the reconstruction mapping satisfy them.

3.7 Conclusion

In this chapter we have demonstrated that error analysis for particle methods is possible. In particular, it is possible to create remeshing procedures which do not sacrifice accuracy, and to define the error in a rigorous manner. The key insight is that a large family of particle methods can be obtained by taking the horizontal component of the Lagrange-Poincaré equations. Additionally, these particle methods can be modified to include the case of Navier-Stokes fluids and complex fluids, since one can apply Lagrange-Poincaré reduction to these systems as well. In summary, we have a new playground for creating new particle-based methods for fluids which can be both easily generalized and rigorously analyzed.

Chapter 4

Interpreting Swimming as a Limit Cycle

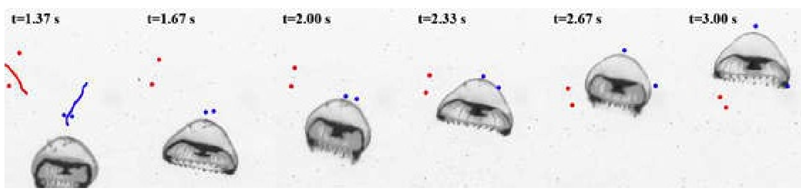

Figure 4.1 – The periodic motion of a jellyfish. Every other snapshot appears to be identical modulo a rigid transformation. Photo taken from [KCDC11], courtesy of Kakani Katija Young.

It has long been suspected that swimming via undulatory motion has a passive component to it [AS05, LBLT03a]. This is of interest to control theorists, roboticists, and biomechanicians because passivity would reduce demands on active controllers and provide robustness to a variety of perturbations. In particular, we pose the conjecture, "*Is swimming is a limit cycle?*". Upon first listen, this statement may sound like a plausible hypothesis. A basic example used in introductory control courses is that of the damped harmonic oscillator with external forcing u,

$$\ddot{x} = -kx - \nu\dot{x} + u.$$

We think of a signal, $u(t)$, as an input and the state, $x(t)$, as an output. The step response is characterized by a transient oscillatory phase which settles to steady-state behavior at a fixed value of x. More importantly, if one inputs a sinusoidal signal, the output is also sinusoidal and of the same period. The goal of this chapter is to interpret the coherent motion of an undulating body immersed in a Navier-Stokes fluid in a similar manner. For a shape-changing body in a viscous fluid, the contraction of muscles will induce a change in shape and also move the surrounding fluid. After a while, dissipation will bring the body and the fluid to rest at a new location in space, perhaps with a different shape as well. We could view this as the step response of the system with respect to muscle contraction. Building upon the analogy with the damped harmonic oscillator, one could hope that time-periodic muscle contractions could produce limit cycles of the same period in an appropriate phase space.

Moreover, we know that a dissipative Hamiltonian system with an asymptotically stable equilibrium will produce a limit cycle when a sufficiently small periodic force is applied (see "The Averaging Theorem" in [GH83]). A viscous fluid is a dissipative system with an asymptotically stable fixed point (i.e., still motion). An undulating body may exert a periodic force. It may appear that we need only write down the Hamiltonian, the viscous friction, an oscillating shape potential, and then QED. Right? Wrong! As we begin to probe the idea, we come across ambiguities:

(Q1) What is the configuration manifold?

(Q2) If a body moves through each cycle of undulation to a new location, then it is not returning to its previous position. Therefore, the state of the system is not cyclic unless the animal produces 0 *net motion. Does this argument negate the hypothesis that swimming is a limit cycle?*

(Q3) Conversely, if we had a limit cycle, then the system returns to the same state it began in. Would this imply no motion is produced?

The answers we have provided are:

(A1) The configuration manifold is a Lie groupoid. The particle relabeling symmetry of the fluid allows us to describe the system on the the corresponding Lie algebroid, $(\mathcal{A}, \rho, [,])$.

(A2) No. The hypothesis is not negated. The Lie algebroid, $\mathcal{A}$*, exhibits an* $\mathrm{SE}(3)$ *invariance. Upon reducing by this invariance we can view swimming as a limit cycle in a reduced algebroid,* $[\mathcal{A}]$.

(A3) If $[\gamma](t)$ *is a closed orbit in* $[\mathcal{A}]$ *then it must be reduced from a path,* $\gamma(t)$*, in* $\mathcal{A}$ *such that* $\gamma(T) = z \cdot \gamma(0)$ *for some* $z \in \mathrm{SE}(3)$*.*

These answers may be "Greek" upon first reading. We will spend the remainder of the paper explaining them. Additionally, while we are not able to definitively conclude that periodic forces on the shape of a body limits to swimming, the contraction of phase space for finite dimensional dissipative Hamiltonian systems is strongly suggestive that such limiting behavior is likely.

4.1 Background

This section is not intended as a comprehensive overview. We seek to merge a number of seemingly disparate subjects in this paper, first and foremost motivated by observations in biology and numerical studies.

Biological, computational, and experimental evidence The passive component to swimming is under increasingly intensive investigation. For example, fish have been observed passively harvesting kinetic energy from the surrounding fluid vorticity by decreasing their muscle activity when trailing a bluff body [LBLT03a, LBLT03b]. Additionally, numerical experiments involving rigid bodies with oscillating forces suggest that uniform motion (i.e., flapping flight) is an attracting state for certain combinations of frequencies and Reynolds numbers [AS05]. Experiments with model "bugs" suggest that the vortices shed from periodically forced bodies have the apparent result of stabilizing a top-heavy body despite steady-state analysis which would suggest instability [LRW+12]. This all suggests further investigation into the role of non-stationary flows as means of achieving stable motion in viscous fluids.

Swimming as a gauge theory In this paper we desire to understand how changes in the shape of a body immersed in a fluid can alter its position and orientation in space. It is fairly common to view the system as an application of gauge theory, where the gauge symmetry is the particle relabeling symmetry of the fluid. This was first investigated in detail for the case of propulsion in Stokes fluids in [SW89]. This perspective was later expanded to the case of potential flow in [Kel98] and[KM00]. Swimming in potential flow for the case of articulated bodies was understood as a Lagrange-Poincaré reduced system (with 0-momenta) in [KMRMH05]. The framework of [KMRMH05] was used for a numerical investigation of motion planning in [MRR06]. Finally, an understanding of shape-changing bodies immersed in ideal fluid with nonzero circulation was proposed in [Rad03], where a monoid called the *fluid-body group* was introduced in an attempt to mimic the Lie group centric theory of [Arn66]. The framework of [KMRMH05] was extended to handle interactions with point vortices in [VKM09] where the use of Lagrange-Poincaré reduction as in [CMR01] was applied to understand the non-zero momentum. All gauge-theoretic approaches to understanding swimming have focused on extreme Reynolds regimes. The analysis presented in this chapter will consider the intermediate Reynolds regime as well.

Groupoids and reduction As demonstrated by the papers mentioned in the previous paragraph, fluid structure interaction problems can be viewed as Lagrangian systems with a gauge symmetry. Therefore, the equations of motion may be described by reduced versions of the Euler-Lagrange equations known as the Lagrange-Poincare equations [CMR01]. Such systems evolve on vector bundles known as Lagrange-Poincaré bundles, for which the dual bundle is a Poisson manifold. If the quotient space is non-trivial, the Poisson structure is partially symplectic and partially Lie-Poisson, as predicted by the Lie-Weinstein theorem [Wei78]. Inspired by [Ves88], it was observed in [Wei95] that Lie algebroids contained the necessary structure to produce a generalized form of Euler-Lagrange equations which satisfy a generalized form of Hamilton's principle. Later, a Lagrangian formalism for degenerate Lagrangian on Lie algebroids was developed in [Mar01] providing intrinsic formulations of the equations of motion. The Euler-Lagrange equations of Lie algebroids satisfy a variational principal that is isomorphic to the one satisfied by the Lagrange-Poincaré equations. Thus both systems exhibit the same dynamics modulo an isomorphism which is determined by an arbitrarily chosen principal connection. The equivalence of the Lagrange-Poincaré equations and the Euler-Lagrange equations on algebroids was implicitly provided in [Wei95] for the case of non-degenerate Lagrangians. The equivalence for degenerate Lagrangians was provided explicitly in [dLMM05]. Additionally, the reduction theory provided in [Wei95] was verified for the degenerate Lagrangian case in [CnNdCS07]. Due to the equivalence of the Lagrange-Poincaré formalism of [CMR01] and Lie algebroid formalism of [Wei95], we will be invoking each depending on convenience.

4.2 Organization of the chapter

We desire to understand the system consisting of a shape-changing body immersed in a viscous fluid as a dissipative Lagrangian system on a Lie algebroid. As motivation, and a sanity check, we will derive the known equations for a rigid body in a fluid using Lagrange-Poincaré reduction in §4.3. We do this for both the case of ideal fluids and viscous fluids, each of which correspond to a different configuration manifold. In §4.4 we review Lagrangian mechanics on Lie groupoids and algebroids as performed in [Wei95] and [Mar01]. We then discuss the relationship with Lagrange-Poincaré reduction, which will allow us to characterize fluid-structure problems as Lagrangian systems on Lie algebroids. Finally in §4.5 we derive a set of asymptotically stable states corresponding to immersions of the body in stagnant fluid. We find the collection of these stable states to be an embedding of SE(3). Upon performing an SE(3) reduction we project this stable manifold to a single asymptotically stable equilibrium, and project the base manifold of the algebroid to the shape space of the body. Adding an oscillating potential energy on the shape space with an isolated minima suggests the existence of a limit cycle in the reduced algebroid. Such a limit cycle would correspond to a time T-map in the unreduced algebroid with a constant translation and rotation.

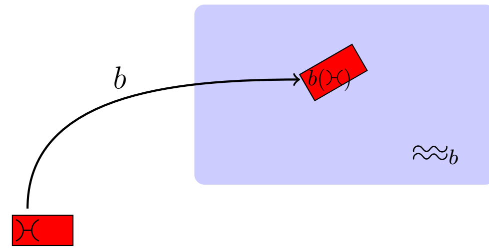

Figure 4.2 – embedding of a rigid body

4.3 A Rigid Body in a Fluid

We ultimately desire to study shape-changing bodies immersed in fluids. However, to motivate future concepts and keep our feet on the ground, we begin with rigid bodies. The configuration manifold is introduced and related to the matrix group SE(3) in §4.3. We then describe the rigid body in free space as a Lagrangian system on the matrix group SE(3) in §4.3. In §4.3 we study rigid bodies immersed in ideal fluids. The fluid component is represented with a volume-preserving diffeomorphism which respects the movement of the body through the ambient space, $\mathbb{R}^3$. Additionally, the Lagrangian exhibits particle relabeling symmetry. Therefore, it is possible to apply Lagrange-Poincaré reduction as done in [CMR01]. By performing this reduction we obtain the well-established equations of motion for a rigid body immersed in an ideal fluid as would be derived from Newton's laws and deduced by a standard reference in continuum mechanics through the stress tensor [Tru91]. Finally, in §4.3 we extend our theory to viscous fluids by adding a viscous dissipation force and a no-slip condition on the boundary.

The set of rigid embeddings Consider a rigid body given by a closed 3-submanifold with boundary, $\text{♓} \subset \mathbb{R}^3$, which we represent by the zodiac Pisces. The configuration manifold, B_{rb}, for a rigid body is the set of rigid embeddings of ♓ into $\mathbb{R}^3$. Since SE(3) is naturally identified as the set of rigid diffeomorphisms of $\mathbb{R}^3$, we observe that for any two rigid embeddings $b_1, b_2 \in B_{\text{rb}}$ there exists a unique $z \in \text{SE}(3)$ such that $b_2 = z \cdot b_1$. This gives us a transitive action of SE(3) on B_{rb}. Given a curve $b_\epsilon \in B_{\text{rb}}$, we see that $\frac{db_\epsilon}{d\epsilon} = (\omega, v) \cdot b$ for a unique $(\omega, v) \in \mathfrak{se}(3)$. We call (ω, v) the spatial velocity of the body, and we may casually equate it with $\dot{b}b^{-1}$. More formally, we define the right trivializing map, $\rho_{\mathrm{triv}} : TB_{\mathrm{rb}} \to \mathfrak{se}(3)$, which outputs the spatial velocity corresponding to a vector in B_{rb}. Similarly, for each $\dot{b} \in TB_{\mathrm{rb}}$ there exists a body velocity $(\Omega, V) \in \mathfrak{se}(3)$ given by the condition $\dot{b} = b \cdot (\Omega, V)_{)\!-\!(}$. This defines the left trivializing map, $\lambda_{\mathrm{triv}} : TB_{\mathrm{rb}} \to \mathfrak{se}(3)$.

Proposition 4.3.1. *The map, λ_{triv}, is left invariant with respect to the lifted action of* $\mathrm{SE}(3)$ *on TB_{rb}. That is to say*

$$\lambda_{\mathrm{triv}}(z \cdot \dot{b}) = \lambda_{\mathrm{triv}}(\dot{b}) \quad , \quad \forall z \in \mathrm{SE}(3).$$

Proof. Consider the defining condition of λ_{triv} acting on $z \cdot \dot{b}$ where $\dot{b} \in T_b B_{\mathrm{rb}}$. This gives us,

$$z \cdot b \left(\lambda_{\mathrm{triv}}(z \cdot \dot{b})_{)\!-\!(}(x) \right) = z\dot{b}(x) = z \cdot b \left(\lambda_{\mathrm{triv}}(\dot{b})_{)\!-\!(}(x) \right), \quad \forall x \in \;)\!-\!(.$$

Left multiplication by $b^{-1} \circ z^{-1}$ gives us the result. $\square$

Finally, it is common to choose a reference configuration, $b_0 \in B_{\mathrm{rb}}$, and describe the dynamics relative to b_0. If we do this, then we can effectively view the system as evolving on $\mathrm{SE}(3)$ because each element of B_{rb} can be reached by multiplying b_0 by an element of $\mathrm{SE}(3)$. Therefore, choosing the initial configuration is equivalent to first choosing a reference configuration and then viewing the rigid body system as evolving on $\mathrm{SE}(3)$([Hol11b, Chapter 1–6]). This is undoubtedly an easier route to take when studying free rigid body dynamics, or even the case of rigid bodies immersed in fluids. However, in later chapters we will consider non-rigid embeddings where the equivalence with $\mathrm{SE}(3)$ breaks down. Therefore, to ease the transition to shape-changing bodies we will pay the initial cost of working with B_{rb} to analyze rigid bodies, rather than use $\mathrm{SE}(3)$.

Rigid bodies In this paragraph we will use the description of the rigid body Lagrangian (in body coordinates) as described in ([Hol11b, Chapter 6]). In order to write down the kinetic energy Lagrangian it is useful to introduce the hat-map, $\wedge : \mathbb{R}^3 \to \mathfrak{so}(3)$ given by

$$\wedge(\Omega) := \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix} \equiv \hat{\Omega}.$$

In particular the hat-map has the convenient property that $\hat{\Omega} \cdot x = \Omega \times x$ where $\times$ is the cross product on $\mathbb{R}^3$. We overload the symbol, $\wedge$, by defining a second hat-map, $\wedge : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathfrak{se}(3)$, given by

$$\wedge(\Omega, x) = \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 & x_1 \\ \Omega_3 & 0 & -\Omega_1 & x_2 \\ -\Omega_2 & \Omega_1 & 0 & x_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \hat{\Omega} & x \\ 0 & 1 \end{bmatrix}.$$

We denote the inverse of both hat-maps by $\vee$. Given a body with density $\mu : \asymp \to \mathbb{R}^+$we can define the mass

$$M = \int_{\asymp} \mu(x) d^3x$$

and the rotational inertia tensor

$$(\mathbb{I}_{\text{rot}})_{ij} = \begin{cases} -\int_{\asymp} \mu(x) x^i x^j d^3x, & i \neq j \\ \int_{\asymp} \mu(x) \left(\|x\|^2 - (x^i)^2\right) d^3x, & i = j \end{cases}.$$

Finally, we define the inertia tensor $\mathbb{I}$ in block diagonal form as an operator on $\mathbb{R}^3 \times \mathbb{R}^3$ by

$$\mathbb{I} = \begin{bmatrix} \mathbb{I}_{\text{rot}} & 0 \\ 0 & M \cdot I \end{bmatrix}$$

where 0 is the 3×3 null-matrix and I is the 3×3 identity on $\mathbb{R}^3$.

Using the hat-map, we define the left invariant metric on SE(3) given by

$$\ll \dot{z}, \delta z \gg_{\mathrm{rb}} := \operatorname{trace}\left(\vee\left[z^{-1} \cdot \dot{z}\right] \cdot \mathbb{I} \cdot \vee\left[z^{-1} \cdot \delta z\right]\right).$$

The kinetic energy Lagrangian, $L_{\mathrm{rb}} : T\,\mathrm{SE}(3) \to \mathbb{R}$, for the rigid body is given by

$$L_{\mathrm{rb}}(z, \dot{z}) = \frac{1}{2} \ll \dot{z}, \dot{z} \gg_{\mathrm{rb}}.$$

Note that L_{rb} is left invariant with respect to SE(3) because it is built from composition with λ_{triv}, which is SE(3) invariant by Proposition 4.3.1. That is to say, $L_{\mathrm{rb}}(\tilde{z} \cdot \dot{z}) = L_{\mathrm{rb}}(\dot{z})$. This will allows us to derive the equations of motion as an instance of the Euler-Poincaré equations.

Proposition 4.3.2. *The rigid-body equations in free space,*

$$\dot{\Pi} := \Pi \times \Omega, \quad \dot{P} = P \times \Omega$$

where $\Pi = \mathbb{I} \cdot \Omega, P = M \cdot V, (\Omega, V) = \wedge(z^{-1}\dot{z})$, *are equivalent to the Euler-Lagrange equations of* L_{rb}.

Proof. We observe that L_{rb} is left invariant with respect to the action of SE(3), and that the quotient space $\mathrm{SE}(3)/\mathrm{SE}(3) = \bullet$. We can thus define a reduced Lagrangian, $[L_{\mathrm{rb}}] : \mathfrak{se}(3) \to \mathbb{R}$, by the condition $[L_{\mathrm{rb}}](\lambda_{\mathrm{triv}}(\dot{b})) = L_{\mathrm{rb}}(\dot{b})$. The dynamics must satisfy the Euler-Poincaré equation

$$\frac{d}{dt}\left(\frac{\partial [L_{\mathrm{rb}}]}{\partial \xi}\right) = \mathrm{ad}^*_{\xi}\left(\frac{\partial [L_{\mathrm{rb}}]}{\partial \xi}\right)$$

paired with the reconstruction formula $\xi = \lambda_{\mathrm{triv}}(\dot{b})$ ([MR99, Chapter 13]). To do this we must derive the bracket on the Lie algebra $\mathfrak{se}(3)$ following the process provided in §2.3.

If we represent SE(3) using pairs, (R, x), of rotation matrices and position vectors, and the action, $(R, x) \cdot (Q, y) = (RQ, Ry + x)$, then we find that the inner-automorphism is

$$\mathrm{AD}_{R,x}(Q,y) = (RQR^T, -RQR^T x + Ry).$$

If we substitute (R,x) with a curve (R_s, x_s) such that $(R_0, x_0) = (I, 0)$ is the identity in SE(3) and $\frac{d}{ds}\big|_{s=0}(R_s, x_s) = (\hat{\Omega}, v) \in \mathfrak{se}(3)$, and similarly substitute (Q,y) with a curve (Q_t, y_t) with $\frac{d}{dt}\big|_{t=0}(Q_t, y_t) = (\hat{\Gamma}, w) \in \mathfrak{se}(3)$, then we find the Lie-bracket on $\mathfrak{se}(3)$ is given by

$$\begin{aligned}
[(\Omega, V), (\Gamma, W)] &:= \left.\frac{d}{ds}\right|_{s=0} \left.\frac{d}{dt}\right|_{t=0} \left(RQR^T, -RQR^T x + Ry\right) \\
&= \left.\frac{d}{ds}\right|_{s=0} \left(R\hat{\Gamma}R^T, -R\hat{\Gamma}R^T x + Rw\right)
\end{aligned}$$

(and through liberal use of the product rule from Calculus I)

$$\begin{aligned}
&= \left(\hat{\Omega}\cdot\hat{\Gamma} - \hat{\Gamma}\cdot\hat{\Omega}, \hat{\Omega}\cdot W - \hat{\Gamma}\cdot V\right) \\
&= \left(\hat{\Omega}\cdot\Gamma, \hat{\Omega}\cdot W + \hat{V}\cdot\Gamma\right).
\end{aligned}$$

We see that, given $(\Pi, P) \in \mathfrak{se}(3)^*$, the coadjoint action is given by

$$\begin{aligned}
\langle \mathrm{ad}^*_{(\Omega,V)}(\Pi, P), (\Gamma, W)\rangle &= \langle (\Pi, P), (\hat{\Omega}\cdot\Gamma, \hat{\Omega}\cdot W + \hat{V}\cdot\Gamma)\rangle \\
&= \langle \Pi, \hat{\Omega}\cdot\Gamma\rangle + \langle P, \hat{\Omega}\cdot W + \hat{V}\cdot\Gamma\rangle \\
&= \langle \hat{\Omega}^T\cdot\Pi, \Gamma\rangle + \langle \hat{\Omega}^T\cdot P, W\rangle + \langle \hat{V}^T P, \Gamma\rangle \\
&= \langle \Pi\times\Omega + P\times V, \Gamma\rangle + \langle P\times\Omega, W\rangle.
\end{aligned}$$

Therefore, if we set $\Pi := \frac{\partial[L_{\mathrm{rb}}]}{\partial\Omega} = \mathbb{I}_{\mathrm{rot}}\cdot\Omega$ and $P := \frac{\partial[L_{\mathrm{rb}}]}{\partial V} = MV$, the Euler-Poincaré equations may be written as

$$(\dot{\Pi}, \dot{P}) = \mathrm{ad}^*_{(\Omega,V)}(\Pi, P) = (\Pi\times\Omega, P\times\Omega).$$

By the Euler-Poincaré theorem ([MR99, Theorem 13.5.3]), the above equation paired with the reconstruction formulas are equivalent to the Euler-Lagrange equations for

L_{rb}. □

Ideal fluids For each rigid embedding, $b : \mathcal{H} \hookrightarrow \mathbb{R}^3$, we desire to express the region which will be occupied by the fluid. Appropriately, we use the zodiac of Aquarius, $\approx$, to do this. We define the set-valued function $\approx : B_{\mathrm{rb}} \to \wp(\mathbb{R}^3)$ by

$$\approx(b) \equiv \approx_b := \text{closure}\left(\mathbb{R}^3 \backslash b(\mathcal{H})\right).$$

The fluid at time t is described by a volume-preserving diffeomorphism,

$$\varphi_t \in \mathcal{D}_\mu\left(\approx_{b_0}, \approx_{b_t}\right),$$

which approaches a rigid transformation at infinity. The map φ_t can be thought of as representing the motion of a fluid by taking a particle position at time 0 and outputting the particle position at time t. The construction so far allows us to define the configuration manifold

$$\begin{aligned} G_{b_0} = \{(b, \varphi)| \quad & b \in \mathrm{SE}(3), \\ & \varphi \in \mathcal{D}_\mu\left(\approx_{b_0}, \approx_b\right) \\ & \lim_{\|x\| \to \infty} \|\varphi(x) - z' \cdot x\| = 0 \text{ for some } z' \in \mathrm{SE}(3)\}. \end{aligned}$$

For convenience we replace $b \in B_{\mathrm{rb}}$ with an element $z \in \mathrm{SE}(3)$ defined by the condition $z \cdot b_0 = b$. Thus elements of G_{b_0} may equivalently be represented as pairs (z, φ) such that $z \in \mathrm{SE}(3)$ and $\varphi \in \mathcal{D}_\mu(\approx_{b_0}, \approx_{z \cdot b_0})$. This set is identical to the *fluid body group* defined in [Rad03][1]. Additionally, this means any vector in B_{rb} may be written as $\dot{z} \cdot b_0$ for some $\dot{z} \in T\,\mathrm{SE}(3)$. Thus, the left trivializing map may be written as $\lambda_{\mathrm{triv}}(\dot{b}) = z^{-1}\dot{z}$ and the right trivializing map may be written as $\rho_{\mathrm{triv}}(\dot{b}) = \dot{z}z^{-1}$.

[1] In [Rad03], the tangent space at the element, $(e, I) \in G_{b_0}$, was treated as a generalization of a Lie-algebra. The equations of motion were generalizations of the Euler-Poicare equations (using the $+$ bracket on the rigid body component, and the $-$ bracket on the fluid). The link with Lagrange-Poincaré reduction was obscured by this generalization.

We denote the group of gauge symmetries by

$$G_{b_0}^{b_0} := \{\phi \in \mathcal{D}_\mu(\approx\!\approx_{b_0}) | \quad \lim_{\|x\|\to\infty} \|\phi(x) - z' \cdot x\| = 0 \text{ for some } z' \in \mathrm{SE}(3)\}$$

which represents the set of particle relabeling symmetries through the right action of $G_{b_0}^{b_0}$ on G_{b_0} given by $(z, \varphi) \cdot \phi := (z, \varphi \circ \phi)$. The Lie algebra of $G_{b_0}^{b_0}$ is the vector space

$$\begin{aligned} \mathfrak{g}_{b_0}^{b_0} := \{X \in \mathfrak{X}_{\mathrm{div}}\left(\mathbb{R}^3 \backslash \approx\!\approx(b_0)\right) : X|_{\partial\{b_0()\!-\!()\}} \in \mathfrak{X}(\partial\{b_0()\!-\!()\}), \\ \lim_{\|x\|\to\infty} (X(x) - \xi(x)) = 0 \text{ for some } \xi \in \mathfrak{se}(3)\}. \end{aligned}$$

The boundary condition on $\partial\{b_0()\!-\!()\}$ ensures that we only consider flows which do not penetrate the body.

Next we define a principal connection which we will use later to perform Langrange-Poincaré reduction.

Proposition 4.3.3. *Consider the map* $A : TG_{b_0} \to \mathfrak{g}_{b_0}^{b_0}$ *defined by*

$$A(z, \varphi, \dot{z}, \dot{\varphi}) = T\varphi^{-1} \cdot \dot{\varphi} - T\varphi^{-1}(\dot{z}z^{-1}) \circ \varphi.$$

The map, A, is a principal connection for the Lie-algebra $\mathfrak{g}_{b_0}^{b_0}$ whose curvature tensor is 0.

Proof. First we check that A truly maps to $\mathfrak{g}_{b_0}^{b_0}$ as we claim. It should be clear that $T\varphi^{-1} \cdot \dot{\varphi}$ and $T\varphi^{-1} \cdot (\dot{z}z^{-1}) \circ \varphi$ are both vector fields on $\approx\!\approx_{b_0}$. We must prove that A maps to a vector field corresponding to an element of $\mathfrak{g}_{b_0}^{b_0}$. In particular, we must check that A satisfies the correct boundary condition. Let $x \in \partial)\!-\!($. Since φ maps the boundary of $\approx\!\approx_{b_0}$ to the boundary of $\approx\!\approx_{zb_0}$ for all time, it must be the case that $\dot{\varphi}(b_0(x)) - \dot{z}(x) \in T_{zb_0(x)}\partial\approx\!\approx_{zb_0}$. Therefore:

$$A(z, \varphi, \dot{z}, \dot{\varphi})(z(x)) = T\varphi^{-1}(\dot{\varphi}(x) - \dot{z}(x)) \in \partial\approx\!\approx_{b_0}$$

for all $x \in \partial)\!-\!($. Thus A maps to $\mathfrak{g}_{b_0}^{b_0}$.

We must check that A is equivariant. Let $\phi \in G_{b^0}^{b_0}$. Then

$$\begin{aligned} A(z, \varphi \circ \phi, \dot{z}, \dot{\varphi} \circ \phi) &= \phi^{-1}\varphi^{-1} \circ \dot{\varphi}\varphi - \phi^*\varphi^* \left(\dot{z}z^{-1}\right)_{\approx\approx_b} \\ &= \phi^*(\varphi^{-1} \circ \dot{\varphi} - \varphi^* \left(\dot{z}z^{-1}\right)_{\approx\approx_b}) \\ &= \mathrm{Ad}_{\phi}^{-1} \cdot A(b, \dot{b}, \varphi, \dot{\varphi}). \end{aligned}$$

Additionally, the infinitesimal generator of $\xi \in \mathfrak{g}_{b_0}^{b_0}$ on G_{b_0} is the vector field which maps as $(z, \varphi) \mapsto (z, \varphi, 0, \varphi \cdot \xi)$. Thus

$$A(\xi_{G_{b_0}}) = A(b, \varphi, 0, \varphi \cdot \xi) = \varphi^{-1} \cdot \varphi \cdot \xi = \xi$$

so that A is a valid principal connection. We can observe by inspection that the kernel of A is given by the infinitesimal generators of $\mathfrak{se}(3)$, which is an integrable distribution. Thus the curvature is $B^A = 0$. □

Finally, we define the fluid Lagrangian,

$$L_{\mathrm{f}}(z, \varphi, \dot{z}, \dot{\varphi}) := \frac{1}{2} \int_{\approx\approx_{b_0}} \|\dot{\varphi}(x)\|^2 d^3x.$$

This Lagrangian is degenerate with respect to motion of the body, however we are actually going to use the total kinetic energy Lagrangian,

$$L(z, \varphi, \dot{z}, \dot{\varphi}) = L_{\mathrm{rb}}(z, \dot{z}) + L_{\mathrm{f}}(z, \varphi, \dot{z}, \dot{\varphi}),$$

to derive the equations of motion. L is non-degenerate.

Theorem 4.3.1. *The Lagrangian, L, is right invariant with respect to the action of $G_{b_0}^{b_0}$ on G_{b_0}. The resulting Lagrange-Poincaré equations on $T[G_{b_0}] \oplus \tilde{\mathfrak{g}}_{b_0}^{b_0}$ are equivalent* *to*

$$\frac{du}{dt} + u \cdot \nabla u = \nabla p, \tag{4.1}$$

$$\operatorname{div}(u) = 0, \tag{4.2}$$

$$\vee(b^{-1}\dot{b}) = (\Omega, V), \tag{4.3}$$

$$\Pi = \mathbb{I}_{\text{rot}} \cdot \Omega, P = MV, \tag{4.4}$$

$$\dot{\Pi} = \Pi \times \Omega + \tau, \tag{4.5}$$

$$\dot{P} = P \times \Omega + F, \tag{4.6}$$

$$\tau = \int_{\partial \between} (x \wedge b^*(\hat{n}))p(b(x))dA \tag{4.7}$$

$$F = \int_{\partial \between} p(b(x))b^*(\hat{n})dA \tag{4.8}$$

where u is vector field on $\approx\!\approx_b$ such that $u(b(x)) = \dot{b}(x)$ for $x \in \partial \between$.

In order to prove Theorem 4.3.1 we will be using the following lemmas:

Lemma 4.3.1. $\mathrm{SE}(3) \equiv \frac{G_{b_0}}{G_{b_0}^{b_0}} := [G_{b_0}]$.

Proof. Let $z \in \mathrm{SE}(3)$; we can equate it with the equivalence class

$$C_z = \{(z, \varphi) : \varphi \in \mathcal{D}_\mu(\approx\!\approx_{b_0}, \approx\!\approx_{zb_0})\}$$

and we can do the converse as well, since for any two elements $(z, \varphi_1), (z, \varphi_2) \in C_z$ there exists a map $\phi \in G_{b_0}^{b_0}$ such that $\varphi_2 = \varphi_1 \circ \phi$. $\square$

Lemma 4.3.2. *The adjoint bundle $\tilde{\mathfrak{g}}_{b_0}^{b_0}$ is equivalently described by the set*

$$\begin{aligned} \{(z, \xi) : \quad & z \in \mathrm{SE}(3), \xi \in \mathfrak{X}_{\text{div}}(\approx\!\approx_b) \\ & \xi|_{\partial b(\between)} \in \mathfrak{X}(\partial\{Z \cdot b_0(\between)\}) \\ & \lim_{\|x\| \to \infty} \|\xi(x) - \eta(x)\| = 0 \text{ for some } \eta \in \mathfrak{se}(3)\} \end{aligned}$$

with the projection $\tilde{\pi}(z, \xi) = z$ and the bracket $[(z, \xi), (z, \eta)] = (z, [\xi, \eta])$.

Proof. Let E be the proposed bundle. We prove the equivalence by showing that the map $\Psi : \tilde{\mathfrak{g}}_{b_0}^{b_0} \to E$ is an adjoint-bundle isomorphism. Writing elements of G_{b_0} as pairs (z, φ) with $z \in \mathrm{SE}(3)$, we define

$$\Psi([(z, \varphi), \xi_{b_0}]) := (z, \varphi_* \xi_{b_0}).$$

We first prove Ψ is well defined, in that it is independent of which element we choose from the equivalence class $[(z, \varphi), \xi_{b_0}]$ (which we choose to represent the right-hand side). By letting ϕ be an arbitrary element of $G_{b_0}^{b_0}$, we observe:

$$\begin{aligned}
\Psi([(z, \varphi) \circ \phi, \mathrm{Ad}_{\phi}^{-1} \xi_{b_0}]) &= \Psi([(z, \varphi \circ \phi), \phi^* \xi_{b_0}]) \\
&= (z, (\varphi \circ \phi)_* (\phi^* \xi_{b_0})) \\
&= (z, \varphi_* \phi_* \phi^* \xi_{b_0}) \\
&= (z, \varphi_* \xi_{b_0}) \\
&= \Psi([(z, \varphi), \xi_{b_0}]).
\end{aligned}$$

Second, it can be observed that Ψ is invertible with inverse $\Psi^{-1}(z, \xi) = [(z, \varphi), \varphi^* \xi]$, where φ is an arbitrary element of $\mathcal{D}_\mu(\approx\!\approx_{b_0}, \approx\!\approx_{zb_0})$.

Third, we must prove Ψ actually maps to E. Since φ is a volume-preserving map which sends from $\approx\!\approx_{b_0}$ to $\approx\!\approx_{Z \cdot b_0}$, and ξ_{b_0} is a divergence-free vector field on $\approx\!\approx_{b_0}$ tangent to the boundary, we see that $\varphi_* \xi_{b_0}$ is a divergence-free vector field on $\approx\!\approx_{zb_0}$ tangent to the boundary. Additionally, $\varphi_* \xi_{b_0}$ limits to an element of $\mathfrak{se}(3)$ at infinity since both φ and ξ_{b_0} do. Thus $\Psi([(z, \varphi), \xi_{b_0}])$ maps bijectively to E.

By observation, we see that the map, Ψ, sends the projection, $\tilde{\pi}$, to the projection, $(z, u) \mapsto z$, which makes Ψ a vector bundle morphism. Lastly, we prove Ψ preserves the Lie bracket.

$$\begin{aligned}\Psi([[(z,\varphi),\xi_{b_0}],[(z,\varphi),\eta_{b_0}]]) &= \Psi([(z,\varphi),[\xi_{b_0},\eta_{b_0}]]) \\ &= (z,\varphi_*[\xi_{b_0},\eta_{b_0}]) \\ &= (z,[\varphi_*\xi_{b_0},\varphi_*\eta_{b_0}]) \\ &= [(z,\varphi_*\xi_{b_0}),(z,\varphi_*\eta_{b_0})] \\ &= [\Psi([(z,\varphi),\xi_{b_0}]),\Psi([(z,\varphi),\eta_{b_0}])]\end{aligned}$$

□

Lemma 4.3.3. *The covariant derivative along a curve, $(z,\xi)(t) \in \tilde{\mathfrak{g}}_{b_0}^{b_0}$, with respect to the principal connection, A, is*

$$\frac{D}{Dt}(z,\xi) := (z,\dot{\xi}+[\dot{z}z^{-1},\xi]).$$

Proof. We use the map, Ψ, from the previous proof, and let $[(z,\varphi),\xi_{b_0}] = \Psi^{-1}(z,u)$. Thus $\xi_{b_0} = \varphi^* u$. By taking the time derivative, we find that

$$\dot{\xi}_{b_0} = \varphi^*[u,\dot{\varphi}\varphi^{-1}] + \varphi^*\dot{u}.$$

Additionally,

$$\begin{aligned}\Psi(\frac{D}{Dt}[(z,\varphi),\xi_{b_0}]) &= \Psi([(z,\varphi),[\varphi^{-1}\dot{\varphi}-\varphi^*(\dot{z}z^{-1}),\xi_{b_0}]+\dot{\xi}_{b_0}]) \\ &= (z,[\dot{\varphi}\varphi^{-1}-\dot{z}z^{-1},\xi_{b_0}]+\varphi_*\dot{\xi}_{b_0}) \\ &= (z,[\varphi^{-1}\dot{\varphi},u]-[\dot{z}z^{-1},u]+[u,\dot{\varphi}\varphi^{-1}]+\dot{u}) \\ &= (Z,\dot{u}-[\dot{z}z^{-1},u]).\end{aligned}$$

□

Corollary 4.3.1. *The covariant derivative, $\frac{D}{Dt}$, along a curve $(z,\alpha)(t) \in (\tilde{\mathfrak{g}}_{b_0}^{b_0})^*$ is given by*

$$\frac{D}{Dt}(z,\alpha) = (z,\dot{\alpha}+\mathrm{ad}^*_{\dot{z}z^{-1}}(\alpha)).$$

Proof. We apply the definition of the covariant derivative on the coadjoint bundle by choosing an arbitrary curve $\xi(t) \in \tilde{\mathfrak{g}}_{b_0}^{b_0}$ such that $\tilde{\pi}(\xi) = \tilde{\pi}\alpha$ for each t. Then:

$$\langle \frac{D\alpha}{Dt}, \xi \rangle + \langle \alpha, \frac{D\xi}{Dt} \rangle = \frac{d}{dt}\langle \alpha, \xi \rangle = \langle \dot{\alpha}, \xi \rangle + \langle \alpha, \dot{\xi} \rangle.$$

Therefore

$$\begin{aligned} \langle \frac{D\alpha}{Dt}, \xi \rangle &= \langle \dot{\alpha}, \xi \rangle + \langle \alpha, \dot{\xi} \rangle - \langle \alpha, \dot{\xi} - [\dot{z}z^{-1}, \xi] \rangle \\ &= \langle \dot{\alpha} + \operatorname{ad}^*_{\dot{z}z^{-1}} \alpha, \xi \rangle. \end{aligned}$$

□

By using the above lemmas we are better prepared for a proof of Theorem 4.3.1.

Proof of Theorem 4.3.1. We observe that $L(z, \varphi, \dot{z}, \dot{\varphi}) = L_{\mathrm{rb}}(z, \dot{z}) + L_{\mathrm{f}}(z, \varphi, \dot{z}, \dot{\varphi})$ is right invariant with respect to $G_{b_0}^{b_0}$ since L_{f} is the kinetic energy Lagrangian for an ideal fluid and thus exhibits particle relabeling symmetry. This symmetry allows us to invoke the Lagrange-Poincaré equations (see §2.4). We call the reduced Lagrangian $l : T[G_{b_0}^{b_0}] \oplus \tilde{\mathfrak{g}}_{b_0}^{b_0} \to \mathbb{R}$.

We first derive the vertical Lagrange-Poincaré equations. Let $(z, \dot{z}, u)(t)$ be a curve in $T[G_{b_0}] \oplus \tilde{\mathfrak{g}}_{b_0}^{b_0}$. Note that the vertical component of $(z, \dot{z}, u) \in T[G_{b_0}] \oplus \tilde{\mathfrak{g}}_{b_0}^{b_0}$ is given by $\xi = u - \dot{z}z^{-1}$. We find

$$\frac{\partial l}{\partial \xi} = u^\flat$$

where $u^\flat$ is the curve in $(\tilde{\mathfrak{g}}_{b_0}^{b_0})^*$ defined by the condition $\langle u^\flat, v \rangle := \frac{1}{2}\int_{\approx_b} u(x) \cdot v(x) d^3x$. The vertical equations imply

$$\dot{u}^\flat + \operatorname{ad}^*_{\dot{z}z^{-1}} u^\flat = -\operatorname{ad}^*_{u - \dot{z}z^{-1}} \left(u^\flat \right).$$

On $\mathbb{R}^3$ the coadjoint action on $u^\flat$ is given by $\operatorname{ad}_v u^\flat = v \cdot \nabla u - \nabla p$, where ∇p is a Lagrangian parameter such that the output is a divergence-free vector field ([AK92, §1.7]). Upon removing the $\flat$s and canceling the terms involving $\dot{z}z^{-1}$, the previous line becomes

$$\dot{u} - u \cdot \nabla u = \nabla p.$$

Since $(z, u) \in \tilde{\mathfrak{g}}_{b_0}^{b_0}$ we automatically have the condition that $u \in \mathfrak{X}_{\mathrm{div}}(\approx\!\approx_{zb_0})$, so that $\mathrm{div}(u) = 0$ and u is a vector field on the bulk region occupied by the fluid. Therefore *the vertical Lagrange-Poincaré equations are identical to* (4.1) *and* (4.2).

Recall that the horizontal equation is given by

$$\frac{D}{Dt}\left(\frac{\partial l}{\partial \dot{z}}\right) - \frac{\partial l}{\partial z} = i_{\dot{z}} \tilde{B}_{\mu}$$

where $\tilde{B}_{\mu}(\dot{z}, \delta) = \langle \frac{\partial l}{\partial \xi}, \tilde{B}(\dot{z}, \delta z) \rangle$. However, we can drop this curvature force because $B = 0 \implies \tilde{B}_{\mu} = 0$. Now let $(\Omega, V) := z^{-1}\dot{z} \equiv \lambda_{\mathrm{triv}}(\dot{b})$. This allows us to derive the generalized momenta

$$(\Pi, P) := (\mathbb{I}_{\mathrm{rot}} \cdot \Omega, MV) \equiv T^* z \cdot \frac{\partial L_{\mathrm{rb}}}{\partial \dot{z}}.$$

As in the derivation of the free rigid body equations, multiplying $\frac{D}{Dt}\left(\frac{\partial l}{\partial \dot{z}}\right)$ by $T^* z$ yields $(\dot{\Pi} - \Pi \times \Omega, \dot{P} - P \times \Omega)$.

Because L_{rb} comes from a left-invariant metric on SE(3), we see that $\frac{\partial L_{\mathrm{rb}}}{\partial z} = 0$. Therefore, the partial derivative, $\frac{\partial l}{\partial z}$, is given by

$$\langle \frac{\partial l}{\partial z}, \delta z \rangle = \langle \frac{\partial l_{\mathrm{f}}}{\partial z}, \delta z \rangle.$$

To compute $\frac{\partial l_{\mathrm{f}}}{\partial z}$, let (z_ϵ, u_ϵ) be a curve in $\tilde{\mathfrak{g}}_{b_0}^{b_0}$ such that $\frac{D}{D\epsilon}(z_\epsilon, u_\epsilon) = 0$ and $\frac{d}{d\epsilon}\big|_{\epsilon=0}(z_\epsilon) = \delta z$. This implies that $\delta u = \frac{d}{d\epsilon} u = \mathrm{ad}_{\delta z \cdot z^{-1}}(u)$. We can neglect variations of $\dot{z}$ because l_{f} is not sensitive to them to first order. Thus we find

$$\begin{aligned}
\langle \frac{\partial l}{\partial z}, \delta z \rangle &= \langle \frac{\partial l_{\mathrm{f}}}{\partial u}, \mathrm{ad}_{\delta z \cdot z^{-1}}(u) \rangle \\
&= -\langle \mathrm{ad}_u^* \frac{\partial l_{\mathrm{f}}}{\partial u}, \delta z \cdot z^{-1} \rangle \\
&= -\int_{\approx\approx_{zb_0}} (\ll u \cdot \nabla u - \nabla p) \cdot \delta z \cdot z^{-1} \gg d^3 x.
\end{aligned}$$

We may write $\delta z \cdot z^{-1}(x) = \omega \times x + v$ for $\omega, v \in \mathbb{R}^3$. Then we find

$$
\begin{aligned}
\int_{\approx\!\approx_{zb_0}} (u \cdot \nabla u) \cdot \delta z \cdot z^{-1} d^3 &= \int_{\approx\!\approx_b} u^j \frac{\partial u^i}{\partial x^j} (\epsilon_{ikl} \omega_k x_l + v_i) d^3 x \\
&= \oint_{\partial \approx\!\approx_{zb_0}} \left(u^j u^i (\epsilon_{ikl} \omega_k x_l + v_i) \right) n^j da \\
&\qquad - \int_{\approx\!\approx_{zb_0}} \frac{\partial u^j}{\partial x^j} u^i (\epsilon_{ijk} \omega_j x_k + v^i) + u^j u^i \epsilon_{ijk} \omega_j \delta_{jk} d^3 x.
\end{aligned}
$$

The first integral vanishes since the unit normal, $\hat{n} = (n^1, n^2, n^3)$, is orthogonal to u. Therefore:

$$
= \int_{\approx\!\approx_{zb_0}} \frac{\partial u^j}{\partial x^j} u^i (\epsilon_{ijk} \omega_j x_k + v^i) + u^j u^i \epsilon_{ijk} \omega_j \delta_{jk} d^3 x.
$$

The first summand vanishes from the divergence condition, $\sum_j \frac{\partial u^j}{\partial x^j} = 0$. Thus:

$$
\begin{aligned}
&= \int_{\approx\!\approx_{zb_0}} u^j u^i \epsilon_{ijk} \omega_j \delta_{jk} d^3 x \\
&= \int_{\approx\!\approx_{zb_0}} u^j u^i \epsilon_{ijj} \omega d^3 x \\
&= 0.
\end{aligned}
$$

This implies:

$$\begin{aligned}
\langle \frac{\partial l}{\partial b}, \delta z \rangle &= \int_{\approx\approx_{zb_0}} \nabla p \cdot (\delta z \cdot z^{-1}) d^3 x \\
&= \oint_{\partial \approx\approx_{zb_0}} (p \cdot (\omega \wedge x + v)) \cdot \hat{n} da - \int_{\approx\approx_{zb_0}} p \frac{\partial}{\partial x^i} (\epsilon_{ijk} \omega_j x_k + v^i) d^3 x \\
&= \oint_{\partial \approx\approx_{zb_0}} (p \cdot (\omega \wedge x + v)) \cdot \hat{n} da - \int_{\approx\approx_{zb_0}} p \epsilon_{ijk} \omega_j \delta_{ik} d^3 x \\
&= \oint_{\partial \approx\approx_{zb_0}} (p \cdot (\omega \wedge x + v)) \cdot \hat{n} da - \int_{\approx\approx_{zb_0}} p \epsilon_{ijj} \omega_j d^3 x \\
&= \oint_{\partial \approx\approx_{zb_0}} (p \cdot (\omega \wedge x + v)) \cdot \hat{n} da.
\end{aligned}$$

Multiplying by $T^* z$ gives us the force on the body:

$$\begin{aligned}
\langle T^* z \cdot \frac{\partial l}{\partial z}, (\tilde{\Omega}, \tilde{V}) \rangle &= \oint_{\partial)(} \Big(p(zb_0(x)) \cdot (\tilde{\Omega} \wedge zb_0(x) + \tilde{V}) \Big) \cdot \hat{n} da \\
&= - \oint_{\partial)(} \Big(p(zb_0(x)) \cdot (n(zb_0(x)) \wedge zb_0(x)) \cdot \tilde{\Omega}) + p \hat{n} \cdot \tilde{V}) \Big) \cdot \hat{n} da.
\end{aligned}$$

Therefore, the horizontal equations (multiplied by $T^* z$) imply

$$(\dot{\Pi} - \Pi \times \Omega, P - P \times \Omega) - (\tau, F) = 0,$$

where

$$\begin{aligned}
\tau &= - \oint_{\partial)(} (p(zb_0(x)) \cdot (n(zb_0(x)) \times zb_0(x))) \, da \\
F &= - \oint_{\partial)(} p \hat{n} da.
\end{aligned}$$

$\square$

Having recovered the standard equations for a rigid body in an ideal fluid, the choice of the kinetic energy Lagrangian should seem especially reasonable. Additionally, the proof allows us to state the following (perhaps more significant) corollary:

Corollary 4.3.2 (Corollary to Theorem 4.3.1)**.** *The equations of motion for a rigid body in an ideal fluid are geodesic equations on G_{b_0} with respect to the metric*

$$\ll (\dot{b}_1, \dot{\varphi}_1), (\dot{b}_2, \dot{\varphi}_2) \gg = \ll \dot{b}_1, \dot{b}_2 \gg_{\text{rb}} + \ll \dot{\varphi}_1, \dot{\varphi}_2 \gg_{\text{f}} .$$

Viscous fluids The case of viscous fluids is similar to that of inviscid fluids. In fact, the only differences are the addition of a dissipative force from the viscosity and a "no-slip" boundary condition. We can now consider the configuration manifold

$$\begin{aligned} G_{b_0} := \{(b, \varphi) : &\varphi \in \mathcal{D}_\mu(\approx\approx_{b_0}, \approx\approx_b), \\ &\varphi(x) = b(x) \text{ for all } x \in \partial)(\} \end{aligned} \tag{4.9}$$

and the symmetry group

$$G_{b_0}^{b_0} := \{\phi \in \mathcal{D}_\mu(\approx\approx_{b_0}) : \phi(b_0(x)) = b_0(x) \text{ for all } x \in \partial)(\}. \tag{4.10}$$

The boundary constraint ensures the no-slip condition, which is the appropriate condition for a Navier-Stokes fluid. The Lagrangian is the same one used in the ideal fluids case; however, this time we include a dissipative force $F_\nu : TG_{b_0} \to T^*G_{b_0}$ given by

$$\langle F_\nu(\dot{b}, \dot{\varphi}), (\delta b, \delta\varphi)\rangle = -\nu \int_{\approx\approx_b} \operatorname{trace}\left([\nabla(\dot{\varphi} \circ \varphi^{-1})]^T [\nabla(\delta\varphi \circ \varphi^{-1})]\right) d^3x,$$

where $\nu > 0$. It is clear that F_ν is symmetric with respect to particle relabeling by $G_{b_0}^{b_0}$ since it only depends on the velocity field $\dot{\varphi} \circ \varphi^{-1}$. In particular, if $b = b_0$ and $\varphi = Id$, then $\dot{\varphi}$ is given by a vector field, $u \in \mathfrak{X}_{\text{div}}(\approx\approx_{b_0})$. This allows us to say:

$$\begin{aligned} \langle F_\nu(\dot{b}, u), (\delta b, v)\rangle &= -\nu \int_{\approx\approx_b} \operatorname{trace}\left([\nabla u]^T [\nabla v]\right) d^3x \\ &= -\nu \int_{\approx\approx_b} (\partial_i v^j)(\partial_i u^j) d^3x \end{aligned}$$

(using the orthogonality of v to the unit normal of the boundary, we may add a 0 valued boundary integral)

$$= \nu \int_{\partial \approx\!\approx_b} (v^j \partial_i u^j) n^i dA - \nu \int_{\approx\!\approx_b} (\partial_i v^j)(\partial_i u^j) d^3x$$

(and by reversing integration by parts)

$$\begin{aligned} &= \nu \int_{\approx\!\approx_b} v_j \partial_i \partial_i u_j d^3x \\ &= \ll \nu \Delta u, v \gg_{\mathrm{f}} . \end{aligned}$$

Thus we observe that F_ν reduces to the viscous force for a Navier-Stokes fluid. Through the particle relabling symmetry of F_ν we may define the reduced force $f_\nu : \mathfrak{X}_{\mathrm{div}}(\approx\!\approx_{b_0}) \to (\mathfrak{X}_{\mathrm{div}}(\approx\!\approx_{b_0}))^*$ by

$$\langle f_\nu(u), v \rangle = \ll \Delta u, v \gg_{\mathrm{f}} .$$

This gives us the following theorem regarding rigid bodies in viscous fluids.

Theorem 4.3.2. *The forced Euler-Lagrange equations for the kinetic energy Lagrangian L on the manifold G_{b_0} with the force F_ν are identical to the forced Lagrange-Poincaré equations*

$$\begin{aligned} \frac{D}{Dt}\left(\frac{\partial l}{\partial \dot{b}}\right) - \frac{\partial l}{\partial b} &= \langle \frac{\partial l}{\partial \xi}, i_{\dot{b}} \tilde{B} \rangle, \\ \frac{D}{Dt}\left(\frac{\partial l}{\partial \xi}\right) &= -\operatorname{ad}^*_\xi \left(\frac{\partial l}{\partial \xi}\right) + f_\nu(\xi), \end{aligned}$$

which can be explicitly written as

$$\frac{du}{dt} + u \cdot \nabla u = \nabla p + \nu \Delta u, \tag{4.11}$$

$$\operatorname{div}(u) = 0, \tag{4.12}$$

$$\vee(b^{-1}\dot{b}) = (\Omega, V), \tag{4.13}$$

$$\Pi = \mathbb{I}_{\mathrm{rot}} \cdot \Omega, P = MV, \tag{4.14}$$

$$\dot{\Pi} = \Pi \times \Omega + \tau, \tag{4.15}$$

$$\dot{P} = P \times \Omega + F, \tag{4.16}$$

$$\tau = \int_{\partial)(} (x \wedge b^*(\hat{n}))p(b(x)) + \nu b^*[\operatorname{sym}(\nabla u)] \cdot (b^*(\hat{n}) \wedge x) dA \tag{4.17}$$

$$F = \int_{\partial)(} p(b(x)) b^*(\hat{n}) dA \tag{4.18}$$

where $\operatorname{sym}(\nabla u)$ *is the* $1-1$ *symmetric tensor field on* $\approx\!\approx_b$ *given by* $(\nabla u) + (\nabla u)^T$*.*

The proof of the theorem is virtually identical to that of 4.3.1, except we have included a dissipation force and added a no-slip condition. The constraint force which enforces the no-slip condition on the boundary is where the term involving $\operatorname{sym}(\nabla u)$ arises.

Additionally we can observe that f_ν makes $\langle f_\nu(u), u \rangle$ into a positive semi-definite symmetric form on the argument u. This makes f_ν a dissipative force, and will allow us to use the energy as a Lyapunov function when doing proofs of stability. By observation, the energy is minimized when the velocity of the fluid is 0 and $\dot{b} = 0$. In particular, due to the no-slip condition, when $u = 0$, it is implied that $\dot{b} = 0$. This will imply that stagnant fluid with a rigid body is an asymptotically stable state for the system, assuming the fluid velocity at infinity is 0. That is to say, solutions which start infinitesimally close to $(b_0, Id, 0, 0) \in TG_{b_0}$ will tend towards some (b_1, Id) where b_1 is close to b_0. Note that the set of these equilibria forms an attractor which is essentially an embedding of SE(3) into G_{b_0}. Thus if we perform an SE(3) reduction, we can reduce this attractor to a single stable equilibrium point in the reduced space. This is precisely what we will do in the final section.

At this point, our sanity check is complete and it appears safe to consider more general fluid structure interaction problems as Lagrange-Poincaré reduced systems.

From rigid to non-rigid bodies In the previous sections we let B_{rb} be the set of rigid embeddings of $)\!\!-\!\!(\hookrightarrow \mathbb{R}^3$. However, we ultimately desire to understand shape-changing bodies. Thus we define the manifold, B, consisting of volume-preserving embeddings from $)\!\!-\!\!(\hookrightarrow \mathbb{R}^3$. Such objects have the potential to change the shape of the body. Of course we are free to consider various submanifolds of B, such as the

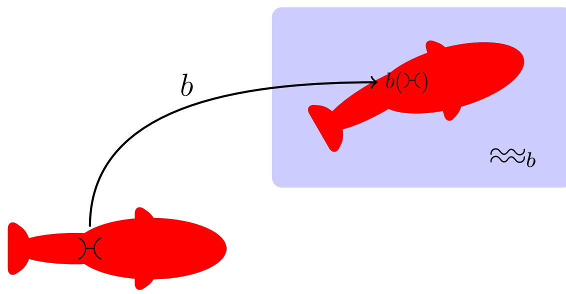

Figure 4.3 – embedding of a fish

case of rigid bodies connected by a ball-socket joint. Additionally, we assume that the body has a mass density $\mu(x)$ so that the kinetic energy of the body is

$$L_{\mathrm{b}}(b, \dot{b}) = \frac{1}{2} \int_{)\!-\!(} \mu(x) \|\dot{b}(x)\|^2 dx.$$

In order to study interaction with Navier-Stokes fluids, we define the set G_{b_0} as in equation (4.9) and symmetry group $G_{b_0}^{b_0}$ as in equation (4.10), except now we allow b_0 to be a volume-preserving embedding, rather than just a rigid one. This will be the setup used in the remainder of the chapter. We will not be deriving equations of motion explicitly, but we will assume that a principal connection has been chosen so that the Lagrange-Poincaré equations can be invoked as the equations of motion when necessary. In particular, the following theorem is merely an instance of the Lagrange-Poincaré reduction theorem (see Theorem 2.4.1).

Theorem 4.3.3. *Let B be the set of volume-preserving embeddings of $)\!\!-\!\!(\hookrightarrow \mathbb{R}^3$ and* *let*

$$G_{b_0} = \{(b, \varphi) : \quad b \in B, \varphi \in \mathcal{D}_\mu(\approx\!\approx_{b_0}, \approx\!\approx_b), \varphi(b_0(x)) = b(x) \text{ for } x \in \partial)\!(\}.$$

Additionally, let $G_{b_0}^{b_0} = \mathcal{D}_\mu(\approx\!\approx_{b_0})$. *Let* $L : TG_{b_0} \to \mathbb{R}$ *be*

$$L(b, \varphi, \dot{b}, \dot{\varphi}) = L_{\mathrm{b}}(b, \dot{b}) + L_{\mathrm{f}}(b, \varphi, \dot{b}, \dot{\varphi}).$$

Then L *is* $G_{b_0}^{b_0}$ *invariant and has a reduced Lagrangian,* l*, defined on the Lagrange-Poincaré bundle,* $TB \oplus \tilde{\mathfrak{g}}_{b_0}^{b_0}$*, through an arbitrarily chosen principal connection,* $A : TG_{b_0} \to \mathfrak{g}_{b_0}^{b_0}$*. Moreover, the Euler-Lagrange equations of* L *are equivalent to the Lagrange-Poincaré equations of* l*.*

4.4 Groupoids and algebroids

In the case of Lagrange-Poincaré reduction we had to choose a reference configuration, which meant we committed to a specific $b_0 \in B$. In the final section we are going to perform an SE(3) reduction, which acts on b_0 as well as G_{b_0}. In order to reduce clutter, it is helpful to embrace an alternative theory of mechanics which incorporates b_0 into the state of the system. For example, we can consider the manifold G given by

$$G := \{(b_1, \varphi, b_0) : b_0, b_1 \in B, \varphi \in \mathcal{D}_\mu(\approx\!\approx_{b_0}, \approx\!\approx_{b_1})\}.$$

We can then define a left SE(3) action G by $z \cdot (b_1, \varphi, b_2) = (z \cdot b_1, z \cdot \varphi \cdot z^{-1}, z \cdot b_2)$. This augmented manifold, G, happens to be a special case of a groupoid which contains the fluid body group, G_{b_0}, as a submanifold (it is a source fiber). Additionally, the SE(3) action will be shown to be a groupoid morphism.

Here we provide definitions for Lie groupoids and Lie algebroids and discuss the link with Lagrange-Poincaré bundles and Lagrange-Poincaré reduction.

Definition 4.4.1 (Groupoid)**.** *Let* G *and* B *be sets and let* $\mathrm{src} : G \to B$, $\mathrm{tar} : G \to B$

be submersions. Define the set

$$G_2 = \{(g_2, g_1) \in G \times G : \mathrm{tar}(g_1) = \mathrm{src}(g_2)\}.$$

Given a partial composition, $\circ : G_2 \to G$*, and a map,* $i : B \hookrightarrow G$*, such that:*

1. $g_3 \circ (g_2 \circ g_1) = (g_3 \circ g_2) \circ g_1$,
2. *for each* $b \in B$ *there exists an identity element* $i(b) \in G$ *such that* $i(b) \circ g = g$ *or* $g \circ i(b) = g$ *whenever these expressions are defined,*
3. *for each* $g \in G$ *there exists an inverse* $g^{-1} \in G$ *such that* $g^{-1}g = i(\mathrm{src}(g))$ *and* $gg^{-1} = i(\mathrm{tar}(g))$,

the collection $\{G, \mathrm{src}, \mathrm{tar}, i, \circ\}$ *is called a* groupoid*. If* $\mathrm{src}, \mathrm{tar}, i, \circ$*, and* $g \mapsto g^{-1}$ *are smooth maps, we call* $\{G, \mathrm{src}, \mathrm{tar}, i, \circ\}$ *a* Lie groupoid*.*

Given a groupoid $\{G, \mathrm{src}, \mathrm{tar}, i, \circ\}$ we have the following labels.

src	source map
tar	target map
i	unit embedding
B	the base

.

Example 4.4.1 (Groups)**.** *A group, G, is a groupoid where B is the singleton set,* $\bullet$, $\mathrm{src}(g) = \mathrm{tar}(g) = \bullet$*, and* $i(\bullet) = e$ *is the identity element of G. If G is a Lie group then it is also a Lie groupoid.*

Example 4.4.2 (Pair groupoid)**.** *Given a manifold M we define the pair groupoid to be the set* $M \times M$ *with* $\mathrm{src}(m_1, m_0) = m_0$ *and* $\mathrm{tar}(m_1, m_0) = m_1$*. The unit embedding is the map* $i(m) = (m, m)$ *and the composition is the map* $(m_2, m_1) \cdot (m_1, m_0) = (m_2, m_0)$*. Inversion is mapping* $(m_1, m_0) \mapsto (m_0, m_1)$*. By inspection, all of these notions are smooth, and so the pair groupoid for a smooth manifold is a Lie groupoid.*

Example 4.4.3 (Swimming)**.** *Let B be the set of volume-preserving embeddings, $)\!(\hookrightarrow \mathbb{R}^3$. Consider the set:*

$$
\begin{aligned}
G = \{(b_1, \varphi, b_0) : & b_1, b_0 \in B, \varphi \in \mathcal{D}_\mu(\approx\!\approx_{b_0}, \approx\!\approx_{b_1}), \\
& \varphi(b_0(x)) = b_1(x) \textit{ for all } x \in \partial)\!(\}
\end{aligned}
$$

equipped with the source target and unit maps

$$
\begin{aligned}
\mathrm{src}(b_1, \varphi, b_0) &= b_0 \\
\mathrm{tar}(b_1, \varphi, b_1) &= b_1 \\
i(b) &= (b, Id, b)
\end{aligned}
$$

and the composition

$$
(b_2, \varphi_2, b_1) \circ (b_1, \varphi_1, b_0) = (b_2, \varphi_2 \circ \varphi_1, b_0).
$$

The collection $\{G, \mathrm{src}, \mathrm{tar}, i, \circ\}$ is actually an infinite dimensional Lie Groupoid over B. However, we are going to act as if G is a legitimate Lie groupoid. The issues with this assumption are not clear, but such assumptions go back to the initial observations on the group theoretic structure of hydrodynamics [Arn66].

Remark 4.4.1. *By allowing the base manifold, B, to include non-rigid embeddings, we are effectively permitting the body to change its shape. An embedding roughly consists of a position, orientation, and a shape. Thus the quotient space, $[B] = B/\,\mathrm{SE}(3)$, is refered to as the* shape space. *We will be performing an* $\mathrm{SE}(3)$ *reduction later to understand how loops in shape space can lead to* $\mathrm{SE}(3)$ *motion in a fluid.*

Lie algebroids The tangent space to a Lie group at the identity is known as a Lie algebra. Additionally, the group structure is encoded into the Lie algebra by an anti-symmetric bracket which satisfies Jacobi's identity. Similar concepts exist for Lie groupoids. However, a Lie groupoid has a distinct identity element for every single element of the base. As a result, the analogue of a Lie algebra becomes a vector bundle over the base manifold (rather than a vector space), and the groupoid structure is encoded with a Lie bracket on sections of this vector bundle.

Definition 4.4.2. *A* Lie algebroid *is a vector bundle, $\tau : E \to B$, equipped with a Lie bracket, $[,]$, on $\Gamma(E)$*[2]*, and a map $\rho : E \to TB$ which is also a Lie algebra homomorphism from $\Gamma(E)$ to $\mathfrak{X}(B)$.*

We call the mapping ρ the "anchor". That ρ is a Lie algebra homomorphism implies

$$[X, fY] = f[X, Y] + \rho(X)[f]Y$$

for each $X, Y \in \Gamma(E)$ and $f \in C^\infty(B)$.

Example 4.4.4 (Lie algebras)**.** *A Lie algebra $\mathfrak{g}$ is a Lie algebroid with $B = \bullet$. This forces the choice $\tau(\xi) = \bullet$ and $\rho(\xi) = 0$. The bracket is inherited from the Lie bracket of the Lie algebra.*

Example 4.4.5 (Tangent bundles)**.** *A tangent bundle $\tau_M : TM \to M$ is a Lie algebroid with base M. In this case, $\tau = \tau_M$ and ρ is the identity on the set of vector fields. The bracket is the Lie bracket of vector fields on M.*

Example 4.4.6 (The Lie algebroid of a Lie groupoid)**.** *The most important Lie algebroid for us is the Lie algebroid of a Lie groupoid. Given a Lie groupoid $\{G, \mathrm{src}, \mathrm{tar}, i, \circ\}$ we set $\mathcal{A} = \mathrm{kernel}(T\,\mathrm{src})$. Let $\tau_G : TG \to G$ be the tangent bundle projection so that we may define the projection*

$$\tau = \mathrm{tar} \circ \tau_G|_{\mathcal{A}} : \mathcal{A} \to B.$$

The anchor is $\rho = T\,\mathrm{tar}|_{\mathcal{A}}$ and the bracket is given by extending sections of $\mathcal{A}$ to vector fields on G and taking the standard Lie bracket of vector fields.

In the case of the pair groupoid, $M \times M$, the corresponding Lie algebroid is TM. In the case of a Lie Group, the corresponding Lie algebroid is the Lie algebra of the group.

[2]The set $\Gamma(E)$ refers to the set of sections of E

Example 4.4.7 (Swimming)**.** *Let B and G be as in Example 4.4.3. The Lie algebroid for G is the set*

$$\{(b,u) : b \in B, u \in \mathfrak{X}_{\mathrm{div}}(\approx\!\approx_b)\}$$

with the projection $\tau(b,u) = b$, the anchor $\rho(b,u) = u \circ b$, and the (fiberwise) bracket $[(b,u),(b,v)] = (b,[u,v])$.

Mechanics on Lie algebroids In this section we review the contents of [Wei95]. A Lagrangian on a Lie algebroid, $\mathcal{A}$, is a real valued function $L : \mathcal{A} \to \mathbb{R}$. The *Legendre tranformation* is the mapping $\mathbb{F}L : \mathcal{A} \to \mathcal{A}^*$ defined by

$$\langle \mathbb{F}L(\xi), \eta \rangle := \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} (L(\xi + \epsilon\eta))$$

for any $(\eta, \xi) \in \mathcal{A} \oplus \mathcal{A}$. We say that L is weakly/strongly non-degenerate if the bilinear mapping $\langle \mathbb{F}L(\xi), \eta \rangle$ is weakly/strongly non-degenerate. In general, weak non-degeneracy implies strong non-degeneracy. In the case that $\mathcal{A}$ is finite dimensional, weak and strong non-degeneracy are the same thing.

We can construct a Poisson form on the dual algebroid, $\mathcal{A}^*$ by describing how it operates on functions which are affine on each fiber. Note that a function $f \in C^\infty(B)$ can be lifted to a function on $\mathcal{A}$ by the map $f \to f \circ \tau = \tau^* f$. The set of function $\{\tau^* f : f \in C^\infty(B)\}$ is the set of real valued function on E that are constant on each fiber. Additionally a section $\xi \in \Gamma(\mathcal{A})$ can be viewed as a function on $\mathcal{A}^*$ via the canonical pairing. The set of sections of $\mathcal{A}$ are linear on each fiber. We define a Poisson structure on $\mathcal{A}^*$ by the relations

$$\{\xi, \eta\} = [\xi, \eta], \quad \{\xi, \tau^* f\} = \rho(\xi)[f], \quad \{\tau^* g, \tau^* f\} = 0$$

for sections $\xi, \eta \in \Gamma(\mathcal{A})$ and functions $f, g \in C^\infty(B)$.[3] If $\mathbb{F}L$ is weakly non-degenerate we can pull-back the Poisson form on $\mathcal{A}^*$ to get a Poisson form on $\mathcal{A}$, which we denote $\{\cdot,\cdot\}_L$.

[3] This is a modest generalization of the commutation relations described in [Mar67] for the case of cotangent bundles.

We define the generalized energy $E : \mathcal{A} \to \mathbb{R}$ by

$$E(\xi) = \langle \mathbb{F}L(\xi), \xi \rangle - L(\xi)$$

and define the equations of motion for an unforced Lagrangian system on $\mathcal{A}$ by

$$\dot{x} = \{x, E\}_L \tag{4.19}$$

for arbitrary coordinate functions $x \in C^\infty(\mathcal{A})$.

In our case, we are concerned with dissipative systems with periodic forces. We define a force to be a vector bundle map $f : \mathcal{A} \to \mathcal{A}^*$. Given a force, f, we defined the forced Euler-Lagrange equations to be

$$\dot{x} = \{x, E\}_L + i_f \Lambda_L[x] \tag{4.20}$$

where Λ_L is the Poisson tensor of $\{,\}_L$.

So far we have shown that it is hypothetically possible to do everything on algebroids instead of Lagrange-Poincaré bundles. The primary use of Lie algebroids in this paper will be the economy of language they provide with respect to reduction by symmetry. We should be able to use either framework for the following reason. The Euler-Lagrange equations on $\mathcal{A}$ are derived from the usual Euler-Lagrange equations on TG_{b_0} via right trivialization (see Corollary 4.6 of [Wei95]). The Lagrange-Poincaré equations are also derived from the Euler-Lagrange equations on TG_{b_0}, but are written on the space $TB \oplus \tilde{\mathfrak{g}}_{b_0}^{b_0}$ instead of $\mathcal{A}$ through the isomorphism Ψ_A^{-1}. In particular we will be making use of the following proposition.

Proposition 4.4.1 ([Wei95])**.** *Let $A : TG_{b_0} \to \mathfrak{g}_{b_0}^{b_0}$ be a principal connection. Let $\mathcal{A}$ be the Lie algebroid $TG_{b_0}/G_{b_0}^{b_0}$ with base $B = G_{b_0}/G_{b_0}^{b_0}$. Let L be the kinetic energy Lagrangian and $f : \mathcal{A} \to \mathcal{A}^*$ be a force. Then the generalized Euler-Lagrange equation* (4.19) *on $\mathcal{A}$ must map to the Lagrange-Poincaré equations on $TB \oplus \tilde{\mathfrak{g}}_{b_0}^{b_0}$ through the isomorphism Ψ_A.*

A thorough proof of Proposition 4.4.1 can also be found in [dLMM05, §9]. It is equivalent to [Wei95, Corollary 4.6], although the isomorphism Ψ_A is not invoked there.

Due to the relationship between the Lagrange-Poincaré equations and the Euler-Lagrange equations on Lie algebroids, we should be able to use either framework. In particular, when referencing equations of motions we will have a tendency to use the Lagrange-Poincaré equations. However, when performing reduction we will use the reduction theory developed on Lie algebroids. Reduction theory on Lie algebroids is less tedious than on Lagrange-Poincaré bundles because there is one less structure to worry about (i.e. the connection-forms). We state the reduction theorems here, the first being [Wei95, Theorem 4.5].

Theorem 4.4.1. *Let $\square : \mathcal{A} \to [\mathcal{A}]$ be a Lie algebroid morphism which is an isomorphism on each fiber. Let $[L]$ be a Lagrangian on $[\mathcal{A}]$ and let $L = [L] \circ \square$. Then solutions to the Euler-Lagrange equations on $\mathcal{A}$ for the Lagrangian L project via $\square$ to the solutions of Euler-Lagrange equations on $[\mathcal{A}]$ for the Lagrangian $[L]$.*

Corollary 4.4.1. *Let $\square : \mathcal{A} \to [\mathcal{A}], [L], L$ be as in Theorem 4.4.1. Let $[f] : [\mathcal{A}] \to [\mathcal{A}]^*$ be an external force, and let $f : \mathcal{A} \to \mathcal{A}^*$ be defined by the condition $\langle f(\xi), \eta \rangle = \langle [f](\square(\xi), \square(\eta) \rangle$ (that is to say $f = \square^*[f]$). Then the forced Euler-Lagrange equations on $\mathcal{A}$ with force f project via $\square$ to the forced Euler-Lagrange equations on $[\mathcal{A}]$ with force $[f]$.*

4.5 Swimming

Let B be the set of volume-preserving embeddings (possibly non-rigid) from a closed subset $\rtimes \subset \mathbb{R}^3$ into $\mathbb{R}^3$. At this point it should be clear that fluid-structure dynamics in a Navier-stokes fluid in $\mathbb{R}^3$ with vanishing motion at infinity can be described as a dissipative (traditional) Lagrangian systems on the source fibers, $\{G_{b_0} : b_0 \in B\}$, of the Lie groupoid

$$\begin{aligned} G := \{(b_1, \varphi, b_0) : & b_0, b_1 \in B, \\ & \varphi(b_0(x)) = b_1(x), \forall x \in \partial)\!(, \\ & \lim_{\|y\| \to \infty} \varphi(y) - y = 0\} \end{aligned}$$

with $\operatorname{src}(b_1, \varphi, b_0) = b_0, \operatorname{tar}(b_1, \varphi, b_0) = b_1$ and $i(b) = (b, Id, b)$ and the standard kinetic energy Lagrangian

$$K_G(b_1, \varphi, b_0, \dot{b_1}, \dot{\varphi}, 0) = \int_{)\!(} \mu(x) \|\dot{b_1}(x)\|^2 d^3x + \int_{\approx\!\approx_{b_0}} \|\dot{\varphi}(y)\|^2 d^3y,$$

where $\mu :)\!(\to \mathbb{R}^+$ represents the mass distribution in $)\!($. By the particle relabeling symmetry of the system (i.e., reduction by symmetry groups $\{G_{b_0}^{b_0} := \operatorname{src}^{-1}(b_0) \cap \operatorname{tar}^{-1}(b_0) : b_0 \in B\}$) we can describe the system as a dissipative Lagrangian system on the Lie algebroid

$$\begin{aligned} \mathcal{A} := \{(b, \dot{b}, u) : & u \in \mathfrak{X}_{\mathrm{div}}(\approx\!\approx_b) \\ & (b, \dot{b}) \in TB, \\ & u(b(x)) = \dot{b}(x) \text{ for } x \in \partial)\!(, \\ & \lim_{\|y\| \to \infty} u(y) = 0\} \end{aligned}$$

with projection $\tau(b, \dot{b}, u) = b$, anchor $\rho(b, \dot{b}, u) = (b, \dot{b})$, bracket $[(b, \dot{b}, u), (b, \delta b, v)] = (b, [u, v] \circ b, [u, v])$, and kinetic energy Lagrangian

$$\begin{aligned} K(b, \dot{b}, u) &= \frac{1}{2} \int_{)\!(} \mu(x) \|\dot{b}(x)\|^2 d^3x + \frac{1}{2} \int_{\approx\!\approx_b} \|u\|^2 d^3x \\ &= \frac{1}{2} \left(\ll \dot{b}, \dot{b} \gg_{\mathrm{b}} + \ll u, u \gg_{\mathrm{f}} \right). \end{aligned}$$

We would like to amend this Lagrangian with a potential energy which corresponds to the shape of the body. Animals, and even some robots, change their shape by contracting muscles. The contraction of these muscles involves stiffening various biological tissues/synthetic materials/springs. Therefore it could be reasonable to describe the potential energy in the muscles as a real-valued function on the shape space. An animal or robot could change its shape by varying this potential and then shifting the shape which minimizes potential energy. In the next proposition, we seek to understand what happens if we do not vary this potential. As before, stagnate fluid is a stable state for the system with a fixed shape.

Proposition 4.5.1. *Let* $)\!\!-\!\!(\subset \mathbb{R}^3$ *be a closed subset and* B *be the set of volume-preserving embeddings* $)\!\!-\!\!(\hookrightarrow \mathbb{R}^3$*. Define the left* $\mathrm{SE}(3)$ *action on* B *by* $(z \cdot b)(x) = z \cdot (b(x))$ *for* $z \in \mathrm{SE}(3)$ *and* $b \in B$*. Then we define the shape space to be* $[B] = B/\mathrm{SE}(3)$*. Let* $V : [B] \to \mathbb{R}$ *be a potential energy with isolated minimum* $[s] \in [B]$*. Let* $f_s : T[B] \to T^*[B]$ *be such that* $\langle f_s(\cdot), \cdot\rangle$ *is a positive definite form on shape space. Then the set of points* $(b, 0, 0) \in \mathcal{A}$ *for* $b \in [s]$ *forms a stable manifold for the Lagrangian system with Lagrangian* $L = K - V$ *and dissipative forces* f_ν *and* f_s*.*

Proof. Consider the generalized energy

$$E(b, \dot{b}, u) = \langle \frac{\partial L}{\partial \dot{b}}, \dot{b}\rangle + \langle \frac{\partial L}{\partial u}, u\rangle - L(b, \dot{b}, u).$$

We see that the time derivative is

$$\begin{aligned}\dot{E} =& \langle \frac{D}{Dt}\left(\frac{\partial L}{\partial \dot{b}}\right), \dot{b}\rangle + \langle \frac{\partial L}{\partial \dot{b}}, \frac{D\dot{b}}{Dt}\rangle + \langle \frac{D}{Dt}\left(\frac{\partial L}{\partial u}\right), u\rangle + \langle \frac{\partial L}{\partial u}, \frac{Du}{Dt}\rangle \\ &- \langle \frac{\partial L}{\partial b}, \dot{b}\rangle - \langle \frac{\partial L}{\partial \dot{b}}, \frac{D\dot{b}}{Dt}\rangle - \langle \frac{\partial L}{\partial u}, \frac{Du}{Dt}\rangle\end{aligned}$$

and upon invoking the forced Lagrange-Poincaré equations with the force f_ν on the fluid, and f_s on the shape we find

$$\begin{aligned}&= \langle -\operatorname{ad}^*_u\left(\frac{\partial L}{\partial u}\right) + f_\nu(u), u\rangle + \langle f_s(T\square(b)), T\square(b)\rangle \\ &= \langle \frac{\partial L}{\partial u}, -\operatorname{ad}_u(u)\rangle + \langle f_\nu(u), u\rangle + \langle f_s(T\square(b)), T\square(b)\rangle \\ &= \langle f_\nu(u), u\rangle + \langle f_s(T\square(b)), T\square(b)\rangle.\end{aligned}$$

Using the energy as a Lyapunov function we see that the system evolves towards the trajectories in the kernel of $f_\nu + \square^*(f_s)$. This only includes velocity fields which are linear in space, i.e., $u(x) = Ax + b$. However, the only such velocity field which goes to 0 at infinity is $u = 0$. Applying the same process to b tells us that any $b \in [s]$ would minimize E since $[s]$ is an isolated minima. Therefore, the set $\{(b, 0, 0) \in G : b \in [s]\}$ forms a stable manifold for the system. $\square$

Frame invariance In this paragraph we prove a number of things in regards to frame indifference. Objects such as the kinetic energy, the viscous force, and the configuration manifold itself should alter their behavior when we rotate our heads. In particular, we desire to study SE(3) invariance. Consider the SE(3) action on G given by

$$z \cdot (b_1, \varphi, b_0) := (z \cdot b_1, z \cdot \varphi \cdot z^{-1}, z \cdot b_0)$$

for $z \in \mathrm{SE}(3)$. It can be quickly verified that for $z_1, z_2 \in \mathrm{SE}(3)$ and any $g \in G$ we have

$$z_2 \cdot z_1 \cdot g = z_2 \cdot (z_1 \cdot g) = (z_2 \cdot z_1) \cdot g,$$

so that this indeed is a group action. Additionally, SE(3) acts by Lie groupoid morphisms. That is, $\mathrm{src}(z \cdot g) = z \cdot \mathrm{src}(g), \mathrm{tar}(z \cdot g) = z \cdot \mathrm{tar}(g), i(z \cdot b) = z \cdot i(b)$, and $z \cdot (g_1 \cdot g_2) = (z \cdot g_1) \cdot g_2$. This is equivalent to stating that the following diagrams commute.

$$\begin{tikzcd} G \arrow[r, "z"] \arrow[d, "\mathrm{src}"'] & G \arrow[d, "\mathrm{src}"] \\ B \arrow[r, "z"'] & B \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "z"] \arrow[d, "\mathrm{tar}"'] & G \arrow[d, "\mathrm{tar}"] \\ B \arrow[r, "z"'] & B \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "z"] & G \\ B \arrow[r, "z"'] \arrow[u, "i"] & B \arrow[u, "i"] \end{tikzcd}$$

In other words, the action preserves the structure of the groupoid. Therefore, we may multiply the groupoid by all of SE(3) to get the reduced groupoid $[G] = G/\mathrm{SE}(3)$ over the shape space $[B] = B/\mathrm{SE}(3)$. We can define a source map $[\mathrm{src}] : [G] \to [B]$ and target map $[\mathrm{tar}] : [G] \to [B]$ by the relations $[\mathrm{tar}]([g]) = [\mathrm{tar}(g)]$ and $[\mathrm{src}]([g]) = [\mathrm{src}(g)]$. These definitions are valid because of the commutativity relations. We also get a unit map $[i] : [B] \hookrightarrow [G]$ by $[i]([b]) = [i(b)]$ and the composition

$$[(g_2, \psi, g_1)] \cdot [(g_1, \varphi, g_0)] = [(g_2, \psi \circ \varphi, g_0)]. \tag{4.21}$$

It may not be clear that the last item is well defined, so we will verify it.

Proposition 4.5.2. *The composition described in* (4.21) *is a well-defined composition on* $[G]$ *using the source and target maps* $[\mathrm{src}], [\mathrm{tar}]$.

Proof. Let $(b_2, \psi, b_1), (b_1', \varphi, b_0) \in G$ be such that $[\mathrm{src}]([(b_2, \psi, b_1)]) = [\mathrm{tar}]([(b_1', \varphi, b_0)])$. By the commutation relations, this means $[b_1] = [b_1']$. Therefore, there exists a $z \in \mathrm{SE}(3)$ such that $b_1 = z \cdot b_1'$. We can then write the composition as

$$\begin{aligned}
[(b_2, \psi, b_1)] \cdot [(b_1', \varphi, b_0)] &\equiv [(b_2, \psi, b_1)] \cdot [(b_1, z \cdot \varphi \cdot z^{-1}, z \cdot b_0)] \\
&= [(b_2, \psi \circ z \cdot \varphi \cdot z^{-1}, z \cdot b_0)] \\
&= [(b_2, \psi, b_1) \circ (b_1, z \cdot \varphi \cdot z^{-1}, z \cdot b_0)] \\
&= [(b_2, \psi, b_1)] \cdot [(b_1, z \cdot \varphi \cdot z^{-1}, z \cdot b_0)].
\end{aligned}$$

$\square$

This makes the collection $\{[G], [\mathrm{src}], [\mathrm{tar}], [i], \cdot\}$ into a Lie groupoid. We can derive the corresponding Lie algebroid, $[\mathcal{A}]$, as in Example 4.4.6. Or we can find the natural action of SE(3) on $\mathcal{A}$ and derive the quotient space with respect to this action. The action of SE(3) on $\mathcal{A}$ is given by $z \cdot (b, u) = (z \cdot b, z \cdot u \circ z^{-1}) \equiv (zb, z_*(u))$ for $z \in \mathrm{SE}(3)$ and $(b, u) \in \mathcal{A}$. We observe that each $z \in \mathrm{SE}(3)$ acts by Lie algebroid morphisms. That is to say, for each $z \in \mathrm{SE}(3)$, the following diagrams commute.

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "z"] \arrow[d, "\tau"'] & \mathcal{A} \arrow[d, "\tau"] \\
B \arrow[r, "z"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A} \arrow[r, "z"] \arrow[d, "\rho"'] & \mathcal{A} \arrow[d, "\rho"] \\
TB \arrow[r, "Tz"'] & TB
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma(\mathcal{A} \oplus \mathcal{A}) \arrow[r, "z"] \arrow[d, "{[,]}"'] & \Gamma(\mathcal{A} \oplus \mathcal{A}) \arrow[d, "{[,]}"] \\
\Gamma(\mathcal{A}) \arrow[r, "z"'] & \Gamma(\mathcal{A})
\end{tikzcd}$$

By using this, we can multiply $\mathcal{A}$ by all of SE(3) to get the reduced algebroid $[\mathcal{A}]$ where each element is an orbit of SE(3). That is, $[\mathcal{A}]$ consists of elements of $\mathcal{A}$ modulo the equivalence

$$(b,u) \sim_{\mathrm{SE}(3)} (z\cdot b, z_*(u)), \quad \forall z \in \mathrm{SE}(3).$$

We overload the symbol $\Box$ as both quotient projections $\Box : \mathcal{A} \to [\mathcal{A}]$ and $\Box : B \to [B]$. Because of the commutations we can define a projection, $[\tau] : [\mathcal{A}] \to [B]$, and an anchor $[\rho] : [\mathcal{A}] \to T[B]$ by the condition

$$[\tau]([(b,u)]) = \Box(\tau(b,u)), \quad [\rho]([(b,u)]) = T\Box \cdot \rho(b,u).$$

Finally we may define a bracket on $\Gamma([\mathcal{A}] \oplus [\mathcal{A}])$ given by

$$[\![[\xi], [\eta]]\!] = \Box\left([\xi,\eta]\right)$$

for sections $\xi, \eta \in \Gamma(\mathcal{A})$. This choice of $[\tau]$, $[\rho]$, and $[\![,]\!]$ make the following diagrams commute

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\Box"] \arrow[d, "\tau"'] & {[\mathcal{A}]} \arrow[d, "{[\tau]}"] \\ B \arrow[r, "\Box"'] & {[B]} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A} \arrow[r, "\Box"] \arrow[d, "\rho"'] & {[\mathcal{A}]} \arrow[d, "{[\rho]}"] \\ TB \arrow[r, "T\Box"'] & T[B] \end{tikzcd} \qquad \begin{tikzcd} \Gamma(\mathcal{A}\oplus\mathcal{A}) \arrow[r, "\Box"] \arrow[d, "{[,]}"'] & \Gamma([\mathcal{A}]\oplus[\mathcal{A}]) \arrow[d, "{[\![,]\!]}"] \\ \Gamma(\mathcal{A}) \arrow[r, "\Box"'] & \Gamma([\mathcal{A}]) \end{tikzcd}$$

and proves that $[\mathcal{A}]$ is a Lie algebroid over $[B]$ with projection $[\tau]$, anchor $[\rho]$, and bracket $[\![,]\!]$. Note that $[B]$ is the shape space.

Notice that $\Box : \mathcal{A} \to [\mathcal{A}]$ is an isomorphism on each fiber since $\Box(b,u) = \Box(b,v)$ if and only if there is a $z \in \mathrm{SE}(3)$ such that $u = z_* v$. This will eventually allow us to use Corollary 4.4.1 to reduce the system by SE(3). Hopefully the next proposition comes as no surprise.

Proposition 4.5.3. *The kinetic energy, $K : \mathcal{A} \to \mathbb{R}$, and the viscous force, $f_\nu : \mathcal{A} \to \mathcal{A}^*$, are both* SE(3) *invariant.*

Proof. We first prove the statement for K. Let $z \in \mathrm{SE}(3)$. We see

$$K(z\cdot b, z\cdot \dot{b}, z_*(u)) = \int_{)(} f(x)\|z\dot{b}\|^2 d^3x + \int_{\approx\approx_{z\cdot b}} \|z_*(u)\| d^3x.$$

Because $\dot{b}(x)$ has the same magnitude as $z \cdot \dot{b}(x)$ we find

$$= \int_{)(} f(x)\|\dot{b}\|^2 d^3x + \int_{\approx_{z\cdot b}} \|z_* u\| d^3x.$$

By applying the change of variables, $x \to zx$, to the second integral implies

$$= \int_{)(} f(x)\|\dot{b}\|^2 d^3x + \int_{\approx_b} \|u\| d^3x = K(b, \dot{b}, u).$$

Similarly for f_ν we find:

$$\langle f_\nu(z \cdot b, z \cdot \dot{b}, z_* u), (z \cdot b, z \cdot \delta b, z_* v)\rangle = \int_{\approx_{z\cdot b}} \mathrm{trace}(\nabla(z_* u)^T \nabla(z_* u)) d^3x,$$

by change of variables $x \to zx$ we find

$$\begin{aligned} &= \int_{\approx_b} \mathrm{trace}(\nabla(u)^T \nabla(u)) d^3x \\ &= \langle f_\nu(b, \dot{b}, u), (b, \delta b, v)\rangle. \end{aligned}$$

In other words, $f_\nu \circ z = T^* z f_\nu$. $\square$

Corollary 4.5.1. *There exists a reduced kinetic energy $[K] : [\mathcal{A}] \to \mathbb{R}$ defined by the condition*

$$[K]([(b, \dot{b}, u)]) = K(b, \dot{b}, u)$$

for all $(b, \dot{b}, u) \in \mathcal{A}$. There also exists a reduced viscous force $[f_\nu] : [\mathcal{A}] \to [\mathcal{A}]^*$ defined by

$$\langle [f_\nu]([(b, \dot{b}, u)]), [(b, \delta b, v)]\rangle = \langle f_\nu(b, \dot{b}, u), (b, \delta b, v)\rangle.$$

This gives us the following corollary to Proposition (4.5.1)

Corollary 4.5.2. *Let $[V] : [B] \to \mathbb{R}$ be a shape potential with an isolated minimum $[b^\infty] \in [B]$. Then the point $([b^\infty], 0, 0) \in [\mathcal{A}]$ is a stable point for the Lagrangian system with Lagrangian, $[L] = [K] - [V]$, viscous force $[f_\nu] : [\mathcal{A}] \to [\mathcal{A}]^*$, and strong dissipative friction force $f_b : T[B] \to T^*[B]$.*

Proof. By the SE(3) invariance of the system and 4.4.1, a trajectory in $[\mathcal{A}]$ which serves as a solution to the Euler-Lagrange equations of $[L]$ can be lifted to a solution to the Lagrangian $L = [L] \circ \square$ on $\mathcal{A}$. By Proposition 4.5.1, the entire equivalence class $([b^\infty], 0, 0)$ is stable in $\mathcal{A}$. Additionally, since f_b is a strong dissipative force on the shape, it will drive the shape $[b] \in [B]$ to a minima of $[V]$, of which there is only one ([MR99, Chapter 7.8]). Applying $\square$ to this maps to the single point $([b^\infty], 0, 0) \in [\mathcal{A}]$. $\square$

Limit cycles and swimming In this paragraph we formalize the conjecture that swimming can be expressed as a limit cycle. Unfortunately we stop short of a proof. However we should state that we are motivated by the following proposition which formalizes the cartoon in Figure 4.4.

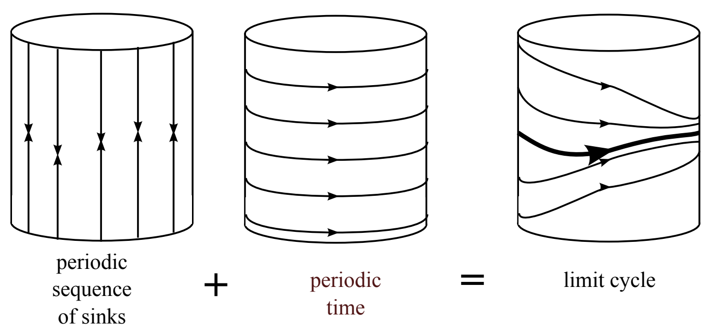

Figure 4.4 – Each cylinder represents the augmented phase space of a time-periodic system. On the first cylinder an asymptotically stable point is perturbed by a time-periodic perturbation, leading to a periodic sequence of stable points. In the second cylinder we depict the vector field which corresponds to progression of time. The third cylinder depicts the vector field obtained by summing the first two vector fields. This third cylinder is the phase portrait of the system and exhibits a limit cycle highlighted in bold.

Proposition 4.5.4. *Let $X \in \mathfrak{X}(\mathbb{R}^n)$ and $Y_\theta \in \mathfrak{X}(\mathbb{R}^n)$ be a time-periodic vector field parametrized by $\theta \in S^1$. If x_0 is a hyperbolic stable point of X, then there exists an $\bar{\epsilon} > 0$ such that the time-periodic ODE given by $\dot{x}(t) = X(x(t)) + \epsilon Y_t(x(t))$ exhibits a stable limit cycle for any $\epsilon < \bar{\epsilon}$.*

The following proof is adapted from the proof of [GH83, Theorem 4.1.1].

Proof. Let P_ϵ be the flow for the vector field $X + \epsilon Y_t$ from time 0 to 2π. For $\epsilon = 0$ we see that x_0 is a stable point of P_0. Additionally, because x_0 is a *hyperbolic* stable point, the local Lyapunov exponent of P_0 at x_0 are strictly negative. Because the Lyapunov exponents depend continuously on $x \in \mathbb{R}^n$ there must exist a neighborhood U of x_0 such that the local Lyapunov exponents P_0 are strictly negative at each $x \in U$ and $P_0(U) \subset U$. Additionally, because P_ϵ converges to P_0 as ϵ approaches 0 there exists a $\bar{\epsilon} > 0$ such that the Lyapunov exponents of $T_x P_\epsilon$ are strictly negative for all $x \in U$. This means that P_ϵ is a contraction mapping on U. The local compactness of $\mathbb{R}^n$ allows us to invoke the Banach fixed-point theorem and obtain a stable fixed point, $x_\epsilon \in U \subset \mathbb{R}^n$, of the map P_ϵ for each $\epsilon < \bar{\epsilon}$. The fixed points, x_ϵ, correspond to a stable limit cycles for each $\epsilon < \bar{\epsilon}$. $\square$

Because all finite dimensional manifolds are locally homeomorphic to $\mathbb{R}^n$ we can generalize the above theorem to periodically perturbed systems on manifolds. We could then specialize to the case of dissipative Lagrangian systems on manifolds with periodic forces. One would hope we could apply Proposition 4.5.4 to our Lagrangian system on $[\mathcal{A}]$. Unfortunately, our proof does not generalize to the infinite dimensional manifolds because it depends on a spectral gap in the differential of the flow map. In finite dimensions, hyperbolicity of a fixed point implies that the Lyapunov exponent closest to zero is still a finite distance away from zero. However, for infinite dimensions there are infinitely many eigenvalues to pay attention to, and so it is conceivable that the spectrum gets arbitrarily close to zero. Additionally, the final step of the proof involved the use of local compactness. However, the space $[\mathcal{A}]$ is not locally compact because $\mathcal{D}_\mu(M)$ is not locally compact [Les67]. Therefore, we cannot be confident that these limit cycles exist always exist for arbitrary periodic forces on the shape and arbitrary initial conditions. Proposition 4.5.4 suggests that any finite dimensional model of swimming will exhibit the desired limit cycles, but the full infinite dimensional system is difficult to address. To avoid a discussion beyond our expertise, we state the remainder of our paper with the following assumption, which would be a corollary if not for these difficulties.

Assumption 4.5.1. *Consider the Lagrangian system on* $[\mathcal{A}]$ *with time-periodic potential energy* $V : [B] \times \mathbb{S}^1 \to \mathbb{R}$, *Lagrangian*

$$L([b,u];\theta) = [K]([b,u]) - V([b];\theta),$$

viscous friction force $[f_\nu] : [\mathcal{A}] \to [\mathcal{A}]^*$, *and a strong dissipation force on the shape space* $f_s : T[B] \to T^*B$. *We assume there exists a set of "reasonable" initial conditions such that the system exhibits a limit cycle.*

If one accepts the above assumption then we have managed to express swimming as a limit cycle. A cartoon of this perspective is sketched in Figure 4.5.

$$\begin{tikzcd}
\text{asy. stable manifold in } \mathcal{A} \arrow[r, "\text{periodic force}", "\text{on shape}"'] \arrow[d, "\text{SE(3) reduction}"'] & \text{swimming in } \mathcal{A} \arrow[d, "\text{SE(3) reduction}"] \\
\text{asy. stable point in } [\mathcal{A}] \arrow[r, "\text{periodic force}", "\text{on shape}"'] & \text{limit cycle in } [\mathcal{A}]
\end{tikzcd}$$

Figure 4.5 – A proposed relationship between swimming and limit cycles in $[\mathcal{A}]$

Formally, if $[\gamma](t)$ is the limit cycle with period T in $[\mathcal{A}]$, then it must integrate to an element $[g] = [(b_1, \varphi, b_0)] \in [G]$ via the reconstruction formula $\frac{d[g]}{dt} = [g] \cdot [\gamma](t)$. Considering $[\gamma]$ is a closed curve we see that $[\mathrm{src}]([g]) = [b_1] = [b_0] = [\mathrm{tar}]([g])$. However, the motion achieved after one cycle is encoded in $[g]$. In particular, it is encoded as a conjugacy class of SE(3). If we define the left action of SE(3) on itself by the inner-automorphism, AD, we define the conjugacy class of a $z \in \mathrm{SE}(3)$ by $[z] = \mathrm{AD}_{\mathrm{SE}(3)}(z) = \{wzw^{-1} : w \in \mathrm{SE}(3)\}$. Consistent with previous notation, we define the set of conjugacy classes of SE(3) by [SE(3)]. Conjugacy classes of SE(3) correspond to rigid motions, modulo a choice of coordinate system. Therefore elements of [SE(3)] represent relative motion rather than absolution motion.

Lemma 4.5.1. *There exists a map* $[\mathrm{motion}] : [G] \to [\mathrm{SE}(3)]$ *defined by the condition that* $[\mathrm{motion}]([g])$ *produces the unique conjugacy class* $[z]$ *such that for any* $g \in [g]$

there is a unique $z' \in [z]$ *such that.*

$$\mathrm{tar}(g) = z' \,\mathrm{src}(g) \forall g \in [g].$$

Proof. We need only show [motion] is a well-defined map to a unique element of [SE(3)]. Let $g_1, g_2 \in [g]$ and let $w \in \mathrm{SE}(3)$ be the unique element such that $g_2 = wg_1$. Let $z_1, z_2 \in \mathrm{SE}(3)$ be the unique elements such that $\mathrm{tar}(g_1) = z_1 \,\mathrm{src}(g_1)$ and $\mathrm{tar}(g_2) = z_2 \,\mathrm{src}(g_2)$. We observe that

$$w \,\mathrm{tar}(g_1) = \mathrm{tar}(g_2) = z_2 \,\mathrm{src}(g_2) = z_2 w \,\mathrm{src}(g_1) = z_2 w z_1^{-1} \,\mathrm{tar}(g_1).$$

The action of an element of SE(3) on B determines that element uniquely, so that we may drop the term $\mathrm{tar}(g_1)$ from both sides. This gives us $z_2 = w^{-1} z_1 w$. Therefore z_2 and z_1 are in the same conjugacy class. □

This produces the following theorem.

Theorem 4.5.1. *If we take Assumption 4.5.1 to be true, then the corresponding limit cycle ,* $[\gamma](t) \in [\mathcal{A}]$*, with period* T *leads to a motion of the body represented by an element of* $[\mathrm{motion}]([g](T))[\mathrm{SE}(3)]$ *where* $[g](t)$ *is defined by the reconstruction formula* $\frac{d[g]}{dt} = [g](t) \cdot [\gamma](t)$*.*

Theorem 4.5.1 would mean that the swimming body settles to a motion in which each period is just a constant rotation and translation relative to the state of the previous period.

4.6 Conclusion

In this investigation we had hoped to discover how to express swimming in a Navier-Stokes fluid as a limit cycle. Unfortunately we have stopped just short of this. However, we managed to accomplish a number of useful things on the way:

1. We derived the equations of motion for a rigid body in an ideal fluid as an instance of the Lagrange-Poincaré equations.

2. We presented the Lie algebroid ($\mathcal{A}$), Lagrangian, and forces necessary to study fluid structure interaction with Navier-Stokes fluids.

3. We showed that the set of states corresponding to stagnant Navier-Stokes fluid is an embedding of SE(3) into $\mathcal{A}$.

4. We investigated the SE(3) invariance of the system and the resulting SE(3) reduction. This sent the stable manifold corresponding to stagnant fluid to a single asymptotically stable point in the reduced Lie algebroid, $[\mathcal{A}]$.

5. We conjectured how these observations could lead to a limit cycle based on a theorem known for asymptotically stable points in finite dimensional systems.

6. Assuming the conjecture, we showed how one expresses the relative motion as a conjugacy class of SE(3).

The findings open a number of routes for future studies. In particular we mention the following:

- A better understanding of phase space contraction on infinite dimensional manifolds could help determine criteria for when limit cycles are possible.

- Variational integrators have been constructed for fluids based on analogies between $\mathcal{D}_\mu(M)$ and certain finite dimensional Lie groups [GMP+11]. It is conceivable that one could carry out a similar project for fluid structure interaction by using analogies with finite dimensional groupoids.

- Control theorists are commonly interested in path planning. To use the results given here, one would likely desire upper bounds on convergence times.

- Additionally, methods for computing limit cycles could be developed. The limit cycle should satisfy a variational principle on $\mathcal{A}$ with periodic boundary conditions. This frames the problem of finding the limit cycle as a minimization problem in the space of loops.

Chapter 5

Couplings with Interaction Dirac Structures

This final chapter concerns work done with Professor Hiroaki Yoshimura and is significantly more general that the previous two chapters. The goal is to understand a family on nonenergetic couplings which have sufficient structure for us to say something meaningful. A large class of physical and engineering problems can be described in terms of Lagrangian and Hamiltonian systems. However, analysis becomes difficult when these systems are large and heterogeneous. For example, systems which can involve a mixture of mechanical and electrical components with flexible and rigid parts and magnetic couplings (see, for instance, [Yos95, Blo03, ABM06]). To handle these complex situations, methods of breaking the problem into simpler sub-problems have been devised. The final step in obtaining the dynamics of the connected systems is what we call *interconnection*.

5.1 Background

Early work. Early work on interconnection was developed by Kron in his book, "Diakoptics" [Kro63]. The word "diakoptics" denotes a procedure where one *tears* a dynamical system into well-understood subsystems. Each tear is associated with a constraint on the interface between the two systems. The original system is restored by *interconnecting* the subsystems through these constraints. This theory was further developed to handle power-conserving interconnections through bond graph theory [Pay61]. Additionally, there exist specific procedures to handle the interconnection of electrical networks through (nonenergic) multiports (see [Bra71, WC77]). In the case of electrical networks, Kirchhoff's current law provides the interconnection constraint. In mechanics, it was shown in [Yos95] that kinematic constraints due to mechanical joints, nonholonomic constraints, and force equilibrium conditions in d'Alembert's principle lead to the proper constraints.

Dirac Structures and Interconnection. It has gradually been revealed that Dirac structures provide a natural geometric framework for describing interconnections between "easy-to-analyze" subsystems. Dirac structures generalize Poisson and symplectic structures from maps to relations between cotangent and tangent bundles. This generalization transforms Hamiltonian and Lagrangian systems from ODEs to DAEs, in which case we call the resulting system an *implicit Lagrangian* or *implicit Hamiltonian system*. In particular [vdSM95] demonstrated that certain interconnections could be described by Dirac structures associated to Poisson structures and that nonholonomic systems and L-C circuits could be represented by implicit Hamiltonian systems. On the Lagrangian side, [YM06b] showed that nonholonomic mechanical systems and L-C circuits (as degenerate Lagrangian systems) could be formulated as implicit Lagrangian systems associated with Dirac structures induced from Kirchhoff's current law. Finally, [YM06c] demonstrated how the implicit Euler-Lagrange equations for unconstrained systems could be derived from the Hamilton-Pontryagin principle and how constrained implicit Lagrangian systems with forces could be formulated in the context of the Lagrange-d'Alembert-Pontryagin principle.

Port Systems. In the realm of control theory, implicit port-controlled Hamiltonian (IPCH) systems (systems with external control inputs) were developed by [vdSM95] (see also [BC97], [Bla00] and [VdS96]) and much effort is devoted to understanding passivity based control for interconnected IPCH systems ([OPNSR98]). The equivalence between controlled Lagrangian (CL) systems and controlled Hamiltonian (CH) systems was shown by [CBL+02] for non-degenerate Lagrangians. For the case in which the Lagrangian is degenerate, an implicit Lagrangian analogue of IPCH systems, namely *implicit port-controlled Lagrangian (IPCL) systems* for electrical circuits were constructed by [YM06a] and [YM07a], where it was shown that L-C transmission lines can be represented in the context of the IPLC system by employing induced Dirac structures.

Composition of Dirac Structures. A product dubbed *composition* was developed in [CvDSBn07] for the purpose of connecting IPCH systems. This provided a new tool for the passive control of IPCH systems. In particular, it was shown that the feedback interconnection of a "plant" port-Hamiltonian system with a "controller" port-Hamiltonian system could be represented by the composition of the plant Dirac structure with the controller Dirac structure. Finally, it was shown in [JY11] that the composition operator can be written using the Dirac tensor product, and "forgetting" the shared variables.

Goals and Main Contributions. We are concerned with the following problem. Consider two implicit Hamiltonian or Lagrangian systems whose equations of motion are given by Dirac structures, D_1 and D_2 on manifolds M_1 and M_2, respectively. A non-energetic interconnection between these systems can be represented by a Dirac structure, D_{int} on the manifold $M_1 \times M_2$. It is observed that the connected system is also an implicit Lagrangian/Hamiltonian system, whose Lagrangian/Hamiltonian is the sum of the Lagrangians/Hamiltonians of the separate systems. However, it is not well known how the Dirac structure of the connected system relates to the old ones. In this paper we propose a way to alter D_1 and D_2 to yield the Dirac structure of the connected system using only D_{int}.

Outline In §5.2, we review the use of Dirac structures in Lagrangian mechanics following [YM06b, YM06c]. In §5.3, we show how to take a *direct sum* of Dirac structures D_1 and D_2 to yield a single Dirac structure $D_1 \oplus D_2$ on $M_1 \times M_2$. We then show how a non-energetic interconnection can be represented by a Dirac structure (usually labeled D_{int} in this paper). Finally, we show how one could obtain the Dirac structure of the interconnected system by using the *Dirac tensor product*, $\boxtimes$. In particular, the Dirac structure of the connected system is $D = (D_1 \oplus D_2) \boxtimes D_{\mathrm{int}}$. In §5.4 we explore how this procedure alters the variational structure of implicit Lagrangian systems. In §5.5, we apply the theory to an LCR circuit, a nonholonomic system, and a simple mass-spring system. In §5.6, we summarize our results and mention some future work.

5.2 Review of Dirac Structures in Mechanics

Linear Dirac Structures. First, we recall the definition of a linear Dirac structure, namely, a *Dirac structure on a vector space* V; we assume that V is finite dimensional for simplicity (see, [CW88]). Let V^* be the dual space of V, and $\langle \cdot\,, \cdot \rangle$ be the natural pairing between V^* and V. Define the symmetric pairing $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ on $V \oplus V^*$ by

$$\langle\!\langle\, (v, \alpha), (\bar{v}, \bar{\alpha})\, \rangle\!\rangle = \langle \alpha, \bar{v} \rangle + \langle \bar{\alpha}, v \rangle,$$

for any $(v, \alpha), (\bar{v}, \bar{\alpha}) \in V \oplus V^*$.

A Dirac structure on V is a subspace $D \subset V \oplus V^*$ such that $D = D^\perp$, where $D^\perp$ is the orthogonal complement of D relative to the pairing $\langle\!\langle \cdot, \cdot \rangle\!\rangle$.

Dirac Structures on Manifolds. Let M be a smooth manifold and let $TM \oplus T^*M$ denote the Whitney sum bundle over M, namely, the bundle over the base M and with fiber over $x \in M$ equal to $T_xM \times T_x^*M$. A subbundle $D \subset TM \oplus T^*M$ is called an *almost Dirac structure* on M, when $D(x)$ is a Dirac structure on the vector space T_xM at each $x \in M$. We may derive an almost Dirac structure from a two-form $\Omega \in \bigwedge^2(M)$ and a regular distribution Δ_M on M as follows: For each $x \in M$, set

$$\begin{aligned} D(x) = \{(v, \alpha) \in T_xM \times T_x^*M \ \mid v \in \Delta_M(x) \text{ and} \\ \langle \alpha, w \rangle = \Omega_x(v, w) \ \text{ for all } \ w \in \Delta_M(x)\}; \end{aligned} \tag{5.1}$$

we call the pair (M, D) a *Dirac manifold*.

Integrablity. We call D an *integrable Dirac structure* if the integrability condition

$$\langle \pounds_{X_1}\alpha_2, X_3\rangle + \langle \pounds_{X_2}\alpha_3, X_1\rangle + \langle \pounds_{X_3}\alpha_1, X_2\rangle = 0 \tag{5.2}$$

is satisfied for all pairs of vector fields and one-forms (X_1, α_1), (X_2, α_2), (X_3, α_3) that take values in D, where $\pounds_X$ denotes the Lie derivative along the vector field X on M.

Remark. Let $\Gamma(TM \oplus T^*M)$ be a space of local sections of $TM \oplus T^*M$, which is endowed with the skew-symmetric bracket $[\ ,\ ] : \Gamma(TM \oplus T^*M) \times \Gamma(TM \oplus T^*M) \to \Gamma(TM \oplus T^*M)$ defined by

$$\begin{aligned}
[(X_1, \alpha_1), (X_2, \alpha_2)] &:= \left([X_1, X_2], \pounds_{X_1}\alpha_2 - \pounds_{X_2}\alpha_1 + \mathbf{d}\langle \alpha_2, X_1\rangle\right) \\
&= \left([X_1, X_2], \mathbf{i}_{X_1}\mathbf{d}\alpha_2 - \mathbf{i}_{X_2}\mathbf{d}\alpha_1 + \mathbf{d}\langle \alpha_1, X_2\rangle\right).
\end{aligned}$$

This bracket is given in [Cou90] and does not necessarily satisfy the Jacobi identity. It was shown by [Dor93] that the integrability condition of the Dirac structure $D \subset TM \oplus T^*M$ given in equation (5.2) can be expressed as

$$[\Gamma(D), \Gamma(D)] \subset \Gamma(D),$$

which is the closedness condition of the Courant bracket (see [DVDS98] and [JR08]).

Pull-backs and Push-forwards. Given a covector $\alpha \in T_x^*M$ and a map $\varphi : N \to M$ we can think of α as a real valued function and form the composition $\alpha \circ T_y\varphi$ for any $y \in N$ such that $\varphi(y) = x$. Then we see that $\alpha \circ T_y\varphi$ is a covector above $y \in N$. We use this observation to define the natural notions of push-forward and pull-back of Dirac structures.

Definition 5.2.1. *Let (M, D) be a Dirac manifold and $\varphi : N \to M$ a smooth injective map. We define the pull-back of D by φ as*

$$\varphi^*D := \{(T\varphi(v), \alpha) \in TN \oplus T^*N : (v, \alpha \circ T_x\varphi) \in D, x = \pi_M(\alpha)\}.$$

Additionally, if $\psi : M \to N$ is smooth and surjective, we define the push-forward of D by ψ as

$$\psi_* D := \{(v, \alpha \circ T_x \psi) \in TN \oplus T^*N : (T\psi(v), \alpha) \in D, x = \pi_M(\alpha)\}.$$

Note that the push-forward and pull-back of a Dirac structure is itself a Dirac structure ([BR03] and [YM07b]).

Notions from Lagrangian Mechanics A Lagrangian is a real valued function on TQ. In this paper we denote a generic Lagrangian by $L \in C^\infty(TQ)$. We define the *Legendre transformation* as the mapping $\mathbb{F}L : TQ \to T^*Q$ given by the condition

$$\langle \mathbb{F}L(q,v), w\rangle := \frac{d}{d\epsilon}\left(L(q, v + \epsilon w)\right)$$

for arbitrary $w \in TQ$. We define the *Pontryagin bundle* as the Whitney sum:

$$TQ \oplus T^*Q := \{(v,p) \in TQ \times T^*Q : \tau_Q(v) = \pi_Q(p)\},$$

which is locally coordinatized by the chart (q, v, p). Finally, we define the *generalized energy*, $E_L : TQ \oplus T^*Q \to \mathbb{R}$, to be the function

$$E_L(q, v, p) := \langle p, v\rangle - L(q, v).$$

When $\mathbb{F}L$ is invertible we say that L is *non-degenerate* and we define the *Hamiltonian* as the function on T^*Q given by:

$$H(q,p) := E_L(q, \mathbb{F}L^{-1}(q,p)).$$

However, when $\mathbb{F}L$ is *not* invertible we say that L is *degenerate*. If L is degenerate and/or we impose a velocity constraint distribution $\Delta_Q \subset TQ$, only the subset $\mathbb{F}L(\Delta_Q) \subset T^*Q$ is physically meaningful as the set of possible momenta for the system. Thus we define the *primary constraint manifold*

$$P := \mathbb{F}L(\Delta_Q).$$

In the following sections we will define systems which are well defined even when P is a strict subset of T^*Q with embedding $i_P : P \hookrightarrow T^*Q$. For the remainder of this section we will be use the following notation

Q	a configuration manifold
Δ_Q	a regular distribution on TQ
L	a generic Lagrangian in $C^\infty(TQ)$, possibly non-degenerate
$\mathbb{F}L$	the Legendre transformation of L
E_L	the generalized energy of L
P	the primary constraint manifold $\mathbb{F}L(\Delta_Q)$

Induced Dirac Structures. Define the lifted distribution on T^*Q by

$$\Delta_{T^*Q} = (T\pi_Q)^{-1}(\Delta_Q) \subset TT^*Q,$$

where $\pi_Q : T^*Q \to Q$ is the cotangent projection. Let Ω be the canonical two-form on T^*Q. Define a Dirac structure D_Δ on T^*Q whose fiber is given for each $(q,p) \in T^*Q$ by

$$\begin{aligned} D(q,p) = \{(v,\alpha) \in T_{(q,p)}T^*Q \times T^*_{(q,p)}T^*Q \mid v \in \Delta_{T^*Q}(q,p) \text{ and} \\ \langle \alpha, w\rangle = \Omega_{(q,p)}(v,w) \ \text{ for all } \ w \in \Delta_{T^*Q}(q,p)\}. \end{aligned} \tag{5.3}$$

Let us call this Dirac structure an *induced Dirac structure*. This is an instance of equation (5.1). Alternatively, the induced Dirac structure can be restated by using the bundle map $\Omega^\flat : TT^*Q \to T^*T^*Q$ as follows:

$$\begin{aligned} D(q,p) = \{(v,\alpha) \in T_{(q,p)}T^*Q \times T^*_{(q,p)}T^*Q \mid v \in \Delta_{T^*Q}(q,p) \text{ and} \\ \alpha - \Omega^\flat(q,p)\cdot v \in \Delta^\circ_{T^*Q}(q,p)\}, \end{aligned}$$

where $\Delta^{\circ}_{T^*Q}$ is the annihilator of Δ_{T^*Q}. Finally, if one desires to be terse, we could use the includsion map $i_{\Delta_{T^*Q}} : \Delta_{T^*Q} \hookrightarrow TT^*Q$ to write the induced Dirac structure as

$$D_{\Delta_Q} := i^*_{\Delta_{T^*Q}} \left(\text{graph}(\Omega^\flat)\right).$$

Remark. If there exists no constraint, then $\Delta_Q = TQ$, and the Dirac structure D_{TQ} corresponds to the graph of the bundle map $\Omega^\flat : TT^*Q \to T^*T^*Q$. That is:

$$D_{TQ} := \{(v, \Omega^\flat(v)) \in TT^*Q \oplus T^*T^*Q : \forall v \in TT^*Q\}.$$

Local Expressions. Let V be a model space for Q and $U_q \subset Q$ an open subset. A coordinate chart for Q is a smooth bijective map $q : U_q \to U \subset V$. Since $TU \equiv U \times V$, then a chart on TQ is a bijective mapping from $(q, v) : TU_q \to U \times V$ such that $\tau_U \circ (q, v) = q$. Similar constructions provide charts for T^*Q, TT^*Q, T^*T^*Q given by:

$$\begin{aligned}
(q, p) &: T^*U_q \to U \times V^* \\
(q, p, \dot{q}, \dot{p}) &: TT^*U_q \to U \times V^* \times V \times V^* \\
(q, p, \alpha, w) &: T^*T^*U_q \to U \times V^* \times V^* \times V.
\end{aligned}$$

Using $\pi_Q : T^*Q \to Q$, locally denoted by $(q, p) \mapsto q$, and $T\pi_Q : (q, p, \dot{q}, \dot{p}) \mapsto (q, \dot{q})$, it follows that

$$\Delta_{T^*Q} = \{(q, p, \dot{q}, \dot{p}) \mid q \in U, \dot{q} \in \Delta(q)\}$$

and the annihilator of Δ_{T^*Q} is locally represented as

$$\Delta^{\circ}_{T^*Q} = \{(q, p, \alpha, 0) \mid q \in U,\ \alpha \in \Delta^{\circ}(q)\}.$$

Since we have the local formula $\Omega^\flat(q, p) \cdot (q, p, \dot{q}, \dot{p}) = (q, p, -\dot{p}, \dot{q})$, the condition $(q, p, \alpha, w) - \Omega^\flat(q, p) \cdot (q, p, \dot{q}, \dot{p}) \in \Delta^{\circ}_{T^*Q}$ reads $\alpha + \dot{p} \in \Delta^{\circ}(q)$ and $w - \dot{q} = 0$. Thus, the induced Dirac structure is locally represented by

$$D(q,p) = \{((\dot q,\dot p),(\alpha,w)) \mid \dot q \in \Delta(q),\ w=\dot q,\quad \alpha+\dot p \in \Delta^\circ(q)\}, \tag{5.4}$$

where $\Delta^\circ(q) \subset T_q^*Q$ is the annihilator of $\Delta(q) \subset T_qQ$.

Implicit Lagrangian Systems. Here we recall the definition of implicit Lagrangian systems (sometimes called Lagrange-Dirac dynamical systems) following [YM06b] and [YM06c]. A *partial vector field* on T^*Q is locally given by writing $\dot q = dq/dt$ and $\dot p = dp/dt$ as functions of (q,v,p). Formally, a partial vector field is a mapping $X : TQ \oplus T^*Q \to TT^*Q$ such that $\tau_{T^*Q} \circ X = \mathrm{pr}_{T^*Q}$.

An *implicit Lagrangian system* is a triple (E_L, D, X), where $X : TQ \oplus T^*Q \to TT^*Q$ is a *partial vector field* which satisfies the constraint:

$$(X(q,v,p), \mathbf{d}E_L(q,v,p)|_{T_{(q,p)}P}) \in D(q,p), \tag{5.5}$$

for any $(q,p) = \mathbb{F}L(q,v)$ with $v \in \Delta_Q$.

The reader may be disturbed that dE is a covector field on $TQ \oplus T^*Q$. However, the restriction $\mathbf{d}E_L(q,v,p)|_{T_{(q,p)}P} \cong (-\partial L/\partial q, v)$ may be regarded as a linear function on $T_{(q,p)}P$ when $(q,p) = (q, \partial L/\partial v) \in P = \mathbb{F}L(\Delta_Q)$. We can embed this covector on P to one on T^*Q using the cotangent lift of the embedding $i_P : P \hookrightarrow T^*Q$.

Local Expressions. It follows from equation (5.4) that the implicit dynamical system $(X(q,v,p), \mathbf{d}E_L|_{TP}(q,v,p)) \in D(q,p)$ is locally given by

$$p = \frac{\partial L}{\partial v}, \quad \dot q = v \in \Delta_Q(q), \quad \dot p - \frac{\partial L}{\partial q} \in \Delta_Q^\circ(q).$$

Remark. The partial vector field of an implicit Lagrangian system is uniquely given on the graph of the Legendre transformation. Equation (5.5) does not constrain the partial vector field outside of the graph of the Legendre transformation, and so the partial vector field of an implicit Lagrangian system is generally *not* uniquely defined on all of $TQ \oplus T^*Q$.

For the case in which no kinematic constraint is imposed, i.e., $\Delta_Q = TQ$, we can develop the standard implicit Lagrangian system, which is expressed in local coordinates as

$$p = \frac{\partial L}{\partial v}, \quad \dot{q} = v, \quad \dot{p} = \frac{\partial L}{\partial q},$$

which we shall call the *implicit Euler–Lagrange equation*. Note that the implicit Euler–Lagrange equation contains the Euler–Lagrange equation $\dot{p} = \partial L/\partial q$, the Legendre transformation $p = \partial L/\partial v$, and the second-order condition $\dot{q} = v$. In summary, the implicit Euler–Langrange equation provides an ODE on $TQ \oplus T^*Q$ which handles the degeneracy of L, while the original Euler–Lagrange equation is an ODE on TQ.

The Hamilton-Pontryagin Principle. In the absence of constraints an integral curve, $q(t) \in Q$, of the Euler-Lagrange equation is known to satisfy Hamilton's principle:

$$\delta \int_{t_1}^{t_2} L(q(t), \dot{q}(t)) dt = 0$$

for arbitrary variations $\delta q(t) \in TQ$ with fixed end points. However, in the case of a degenerate Lagrangian, L, with the constraint, Δ_Q, certain variations will induce no variation in the generalized momenta, $\frac{\partial L}{\partial v}$, and variations outside of Δ_Q are not physically meaningful. To deal with these degenerate Lagrangians and velocity constraints we prefer to express variational principles on $TQ \oplus T^*Q$. The natural choice is the Hamilton-Pontryagin principle (or HP-principle), and is given by the stationary condition on the space of curves $(q(t), v(t), p(t))$, $t \in [t_1, t_2]$ in $TQ \oplus T^*Q$ by:

$$\delta \int_{t_1}^{t_2} L(q(t), v(t)) + \langle p(t), \dot{q}(t) - v(t) \rangle \; dt = 0$$

for variations $\delta q(t) \in \Delta_Q$ with fixed end points and arbitrary fiberwise variations $\delta p(t) \in TT^*_{q(t)}Q$ and $\delta v(t) \in TT_{q(t)}Q$. The HP-principle can be restated as

$$\delta \int_{t_1}^{t_2} \langle p(t), \dot{q}(t) \rangle - E_L(q(t), v(t), p(t))\, dt = 0.$$

The HP-principle was shown to be equivalent to the implicit Euler–Lagrange equations [YM06c].

Example: Harmonic Oscillators. Here we derive an implicit Lagrangian system associated to a linear harmonic oscillator. In this case, the configuration space is $Q = \mathbb{R}$ where $q \in Q$ represents the position of a particle on the real line. The Lagrangian is given by $L(q, v) = v^2/2 - q^2/2$ and the generalized energy is $E_L(q, v, p) = pv - v^2/2 + q^2/2$. Recall that the canonical Dirac structure on T^*Q is given by $D = \text{graph}(\Omega^\flat)$.

A partial vector field $X(q, v, p) = (q, p, \dot{q}, \dot{p})$ for this Lagrange-Dirac system satisfies

$$(X(q, v, p), dE_L(q, v, p)|_{T_{(q,p)}P}) \in D(q, p),$$

where $p = v \in P = \mathbb{F}L(TQ)$. Equating $(\dot{q}, \dot{p})$ with $X(q, v, p)$ we find $dE_L(q, v, p)|_{T_{(q,p)}P} = \Omega^\flat(q, p) \cdot (\dot{q}, \dot{p})$. In local coordinates we may write $dE_L(q, v, p)|_{T_{(q,p)}P} = vdp + qdq$ and $\Omega^\flat(q, p)(\dot{q}, \dot{p}) = -\dot{p}dq + \dot{q}dp$. Thus, the dynamics are given by the equations:

$$\dot{q} = v, \quad \dot{p} = -q, \quad p = v.$$

Implicit Lagrangian Systems with External Forces. In this section we discuss how external forces alter the dynamics of implicit Lagrangian systems. This will be useful later in describing the interaction between connected systems through interaction forces. However, since the differential of the generalized energy lives in T^*T^*Q and forces live in T^*Q, we need to define the horizontal lift.

Definition 5.2.2. *Given a covector $F \in T^*Q$ we define the horizontal lift above*

$p \in T^*Q$ to be the covector $\tilde{F} \in T^*T^*Q$ such that

$$\langle \tilde{F}, x \rangle = \langle F, T\pi_Q(x) \rangle$$

for any $x \in TT_p^*Q$.

Locally the horizontal lift of a covector $(q, F) \in T_q^*Q$ above $p \in T^*Q$ is given by $(q, p, F, 0) \in T^*T^*Q$.

A *force* is a mapping $F : TQ \oplus T^*Q \to T^*Q$. Given a Lagrangian $L : TQ \to \mathbb{R}$, an *implicit Lagrangian system with an external force field* is defined by a *quadruple* (E_L, D, X, F) such that for each $(q, v) \in \Delta_Q$ and $(q, p) \in P = \mathbb{F}L(q, v)$ we have:

$$(X(q, v, p), (\mathbf{d}E_L - \tilde{F})(q, v, p)|_{T_{(q,p)}P}) \in D(q, p). \tag{5.6}$$

It follows that the local Lagrange-Dirac system in equation (5.6) may be given by

$$\dot{q} = v \in \Delta_Q(q), \quad \dot{p} - \frac{\partial L}{\partial q} - F \in \Delta_Q^\circ(q), \quad p = \frac{\partial L}{\partial v}. \tag{5.7}$$

A curve $(q(t), v(t), p(t))$, $t_1 \leq t \leq t_2$ in $TQ \oplus T^*Q$ which satisfies (5.6) is called a *solution curve* of (E_L, D, X, F).

The Lagrange-d'Alembert-Pontryagin principle. In this paragraph we provide a variational principle for forced implicit Lagrangian systems. Consider a mechanical system with kinematic constraints given by a regular distribution Δ_Q on Q. The motion of the mechanical system $q : [t_1, t_2] \to Q$ is said to be constrained if $\dot{q}(t) \in \Delta_Q(q(t))$ for all t, $t_1 \leq t \leq t_2$. Let L be a Lagrangian on TQ and let $F : TQ \oplus T^*Q \to T^*Q$ be an external force field. The Lagrange-d'Alembert-Pontryagin principle (LAP principle) for a curve $(q(t), v(t), p(t))$, $t_1 \leq t \leq t_2$, in

$TQ \oplus T^*Q$ with the constraint $v(t) \in \Delta_Q(q(t))$ is given by

$$\begin{aligned}
&\delta \int_{t_1}^{t_2} \langle p(t), \dot{q}(t) \rangle - E_L(q(t), v(t), p(t))\, dt + \int_{t_1}^{t_2} \langle F(q(t), v(t), p(t)), \delta q(t) \rangle \ dt \\
&= \delta \int_{t_1}^{t_2} L(q(t), v(t)) + \langle p(t), \dot{q}(t) - v(t) \rangle \ dt + \int_{t_1}^{t_2} \langle F(q(t), v(t), p(t)), \delta q(t) \rangle \ dt \\
&= 0
\end{aligned}$$

for variations $\delta q(t) \in \Delta(q(t))$ with fixed endpoints and arbitrary variation of v and p.

Proposition 5.2.1. *A curve in $TQ \oplus T^*Q$ satisfies the LAP principle if and only if it satisfies the equations of motion* (5.7).

Proof. Taking an appropriate variation of $q(t)$ with fixed end points yields:

$$\int_{t_1}^{t_2} \left\langle \frac{\partial L}{\partial q} - \dot{p} + F, \delta q \right\rangle + \left\langle \frac{\partial L}{\partial v} - p, \delta v \right\rangle + \langle \delta p, \dot{q} - v \rangle \, dt = 0.$$

This is satisfied for all variations $\delta q(t) \in \Delta_Q(q(t))$ and arbitrary variations $\delta v(t)$ and $\delta p(t)$, and with the constraint $v(t) \in \Delta_Q(q(t))$ if and only if (5.7) is satisfied. $\square$

Coordinate Representation. Let $\dim(Q) = n$ so that we may choose $\mathbb{R}^n$ as a model space and we have local coordinates q^i for $i = 1, \ldots, n$ on an open set $U \subset \mathbb{R}^n$. Additionally, TQ is locally given by local coordinates (q^i, v^i) on $U \times \mathbb{R}^n$. Similarly T^*Q may be locally coordinatized by charts (q^i, p_i) to $U \times \mathbb{R}^n$. The constraint set Δ_Q defines a subspace on each fiber of TQ, which can locally be expressed as a subset of $\mathbb{R}^n$. If the dimension of $\Delta_Q(q)$ is $n-m$, then we can choose a basis $e_{m+1}(q), e_{m+2}(q), \ldots, e_n(q)$ of $\Delta(q)$. Recall that the constraint sets can be also represented by the annihilator $\Delta^\circ(q)$, which is spanned by m one-forms $\omega^1, \omega^2, \ldots, \omega^m$ on Q. It follows that equation (5.7) can be represented, in coordinates, by employing the Lagrange multipliers μ_a, $a =$

$1, ..., m$, as follows:

$$\begin{pmatrix} \dot{q}^i \\ \dot{p}_i \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{\partial L}{\partial q^i} - F_i \\ v^i \end{pmatrix} + \begin{pmatrix} 0 \\ \mu_a \, \omega_i^a \end{pmatrix},$$

$$p_i = \frac{\partial L}{\partial v^i},$$

$$0 = \omega_i^a \, v^i,$$

where we employ the local expression $\omega^a = \omega_i^a \, dq^i$.

Example: Harmonic Oscillators with Damping. As before, let $Q = \mathbb{R}$, $L = v^2/2 - q^2/2$, $E_L = pv - v^2/2 + q^2/2$, and $D = \mathrm{graph}(\Omega^\flat)$. Now consider the force-field $F : TQ \oplus T^*Q \to T^*Q$ defined by $F(q, v, p) = -(rv)dq$, where $r > 0$ is a positive damping coefficient. Then, $\tilde{F}(q, v, p) = (q, p, rv, 0)$. The formulas in equation (5.7) give us the equations:

$$\dot{q} = v, \quad \dot{p} + q + rv = 0$$

with the Legendre transformation $p = v$.

5.3 Interconnection of Dirac Structures

The interconnection of physical systems is executed in a variety of ways. Examples are massless hinges, soldering of wires, conversion of current into torque by a motor, interaction potentials, etc. Many of these interconnections are expressed by Dirac structures. For example, in the case of interconnection of two mechanical systems by a massless ball-socket joint we could consider the velocity constraint of the form

$$\begin{aligned} \Delta_{\text{ball-socket}} :=& \{(v_1, v_2) \in TQ_1 \times TQ_2 \,| \\ & \text{velocity of hinge on system 1} = \text{ velocity of hinge on system 2}\}. \end{aligned}$$

Then the annihilator, $\Delta^{\circ}_{\text{ball-socket}}$, contains the possible constraint forces required to obey the ball-socket constraint. Finally, the direct sum $\Delta_{\text{ball-socket}} \oplus \Delta^{\circ}_{\text{ball-socket}}$ is a Dirac structure. Additionally, if we consider an ideal motor, then the relationship between the current/voltage through the terminals and the torque/angular velocity of the shaft is given by the graph of a two form, and is also a Dirac structure. In the following section we will explore in greater detail how interconnections may be expressed as Dirac structures, which we will call *interaction Dirac structures.* In this section we hope to convey how nonenergetic constraints between systems are naturally expressed as interaction Dirac structures. In particular, we will present a tensor product of Dirac structures, $\boxtimes$, such that the Dirac structure of an interconnected Lagrangian system is given by:

$$\underbrace{D_C}_{\text{connected system}} = \overbrace{(D_1 \oplus \cdots \oplus D_n)}^{\text{non-interacting subsystems}} \underbrace{\boxtimes}_{\text{tensor product}} \overbrace{D_{\text{int}}}^{\text{interaction}}$$

where $D_1, \ldots, D_n$ are Dirac structures for disconnected subsystems. We refer to the transition from the disconnected Dirac structures $D_1, \ldots, D_n$ to the connected one, D_C, by the phrase *interconnection of Dirac structures.*

Interaction Dirac Structures. Here we will introduce a special interaction Dirac structure called a *interaction constraint Dirac structure.*

Definition 5.3.1. *Consider a regular distribution $\Sigma_Q \subset TQ$ and define the lifted distribution on T^*Q by*

$$\Sigma_{\text{int}} = (T\pi_Q)^{-1}(\Sigma_Q) \subset TT^*Q.$$

Let $\Sigma^{\circ}_{\text{int}}$ be the annihilator of Σ_{int}. Then, an interaction constraint Dirac structure *on T^*Q is defined by, for each $(q,p) \in T^*Q$,*

$$D_{\text{int}} = \Sigma_{\text{int}} \times \Sigma^{\circ}_{\text{int}}. \tag{5.8}$$

Alternatively, we could have written the above Dirac structure as

$$D_{\text{int}} = (\pi_Q)^* \left(\Sigma_Q \oplus \Sigma_Q^\circ\right)$$

where we view $\Sigma_Q \oplus \Sigma_Q^\circ$ as a Dirac structure on Q. In any case, the above Dirac structure typically appears in mechanics as Newton's third law of action and reaction, as shown in the next example.

Remark. We can consider a more general class of interactions, which may be defined by a two-form Ω_{int} and a distribution Σ_{int} on T^*Q. In theory any Dirac structure could be used to interconnect systems. Analysis of such a system may involve *Lagrangian reduction theory* [CMR01], and will be explored in another paper. In this paper we will only consider constraint interaction Dirac structures. However we will include the example of a charged particle in a magnetic field and an ideal motor to demonstrate the flexibility of this framework.

Example: A Particle Moving Through a Magnetic Field. Consider an electron moving through a vacuum. Then the equations of motion are $\ddot{x} = 0, \ddot{y} = 0, \ddot{z} = 0$. We could think of this system as a set of 3 decoupled systems with constant dynamics. Now let $\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$ be a magnetic field (so $\text{div}(\mathbf{B}) = 0$) and let B be a closed two-form on $Q = \mathbb{R}^3$ defined by

$$\mathbf{i}_{\mathbf{B}}(dx \wedge dy \wedge dz) = B,$$

so that

$$B = B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy.$$

Using B, one can define a closed two-form Ω_{int} on $T^*Q = \mathbb{R}^3 \times \mathbb{R}^3$ by

$$\Omega_{\text{int}} = -\frac{e}{c}\pi_Q^* B.$$

The force on a charged particle moving through the magnetic field B is given the *Lorentz force*, $\alpha = -(e/c)\,\mathbf{i_v}B$. This force couples the dynamics of the x, y, and z coordinates. If we desire to express this coupling in the form of an interaction Dirac structure, one could define the Dirac structure $D_{\text{magneticfield}}$ on T^*Q by

$$D_{\text{magnetic field}} = \operatorname{graph} \Omega^\flat_{\text{int}}.$$

Example: An ideal motor. The form of the Dirac structure given in the previous paragraph also describes the structure of an ideal motor. In this case the configuration manifold is $\mathbb{R}\times S^1$, where the first component is the electric flux through the terminals and the second component is the angle of the shaft represented as an element of the unit circle. When a current I passes through the terminal it experts a torque $\tau = J\cdot I$ for some constant J. Geometrically, given coordinates (q,θ) on $\mathbb{R}\times S^1$, we can express the relationship between current and torque with the two-form $B = Jdq\wedge d\theta$ so that $\tau = B(I,\cdot)$. Finally, this can all be expressed with the Dirac structure

$$\begin{aligned}
D_{\text{motor}} &= \operatorname{graph}(B) \\
&= \{((I,\omega),(\alpha,\tau)) \in T(\mathbb{R}\times S^1)\oplus T^*(\mathbb{R}\times S^1) : \\
&\qquad \alpha = -J\cdot\omega, \quad \tau = J\cdot I\}
\end{aligned}$$

where I and α are the current and voltage on the terminals of the motor, and ω and τ are the angular velocity and torque on the shaft. Given a circuit and a mechanical system connected by an ideal motor, the above interaction Dirac structure would characterize the interconnection between systems.

The Direct Sum of Dirac Structures. So far we have shown how to express interconnections as interaction Dirac structures. We intend to use these interaction Dirac structures to connect systems on separate Dirac manifolds M_1 and M_2. However, before we can connect Dirac systems on separate manifolds, we should formalize the notion of a "direct sum" of systems on separate spaces.

Definition 5.3.2. *Let (D_1, M_1) and (D_2, M_2) be Dirac manifolds. Let $\mathrm{pr}_{M_i} : M_1 \times M_2 \to M_i$, $i = 1, 2$, be the natural Cartesian projection. We define the direct sum of D_1 and D_2 by:*

$$\begin{aligned} D_1 \oplus D_2(m) = \{((v_1, v_2), T_m^* \mathrm{pr}_{M_1}(\alpha_1) + T_m^* \mathrm{pr}_{M_2}(\alpha_2)) \in T_m(M_1 \times M_2) \times T_m^*(M_1 \times M_2)| \\ (v_1, \alpha_1) \in D_1(m_1), \quad (v_2, \alpha_2) \in D_2(m_2)\} \end{aligned}$$

for each $m = (m_1, m_2) \in M_1 \times M_2$.

Proposition 5.3.1. *If $D_1 \in \mathrm{Dir}(M_1), D_2 \in \mathrm{Dir}(M_2)$, then $D_1 \oplus D_2 \in \mathrm{Dir}(M_1 \times M_2)$.*

To prove Proposition 5.3.1 the following lemma will be useful (see [YM06b] or [Cou90]).

Lemma 5.3.1. *A subbundle $D \subset TM \oplus T^*M$ is maximally isotropic with respect to $\langle\langle \cdot, \cdot \rangle\rangle$ if and only if $\langle\langle (v, \alpha), (v, \alpha) \rangle\rangle = 0, \forall (v, \alpha) \in D$.*

Proof of Proposition 5.3.1. Let $(v, \alpha) \in D_1 \oplus D_2(m)$. Notice that $\dim(D_1 \oplus D_2(m)) = \dim(D_1(m_1)) + \dim(D_2(m_2)) = \dim(M_1 \times M_2)$ for each $m \in M$. By definition, $v = (v_1, v_2) \in T_m(M_1 \times M_2)$ and $\alpha = T_m^* \mathrm{pr}_{M_1}(\alpha_1) + T_m^* \mathrm{pr}_{M_2}(\alpha_2)$ for some $\alpha_1 \in T_{m_1}^* M_1$ and $\alpha_2 \in T_{m_2}^* M_2$ such that $(v_1, \alpha_1) \in D_1(m_1)$ and $(v_2, \alpha_2) \in D_2(m_2)$. Then, one has, for $(v, \alpha) \in D_1 \oplus D_2(m)$ at $m \in M_1 \times M_2$,

$$\begin{aligned} \langle\langle (v, \alpha), (v, \alpha) \rangle\rangle &= 2\langle \alpha, v \rangle \\ &= 2\langle T_m^* \mathrm{pr}_{M_1}(\alpha_1) + T_m^* \mathrm{pr}_{M_2}(\alpha_2), v \rangle \\ &= 2\langle \alpha_1, T_m \mathrm{pr}_{M_1}(v) \rangle + 2\langle \alpha_2, T_m \mathrm{pr}_{M_2}(v) \rangle \\ &= 2\langle \alpha_1, v_1 \rangle + 2\langle \alpha_2, v_2 \rangle \\ &= 0, \end{aligned}$$

since $(v_1, \alpha_1) \in D_1(m_1)$ and $(v_2, \alpha_2) \in D_2(m_2)$. Thus, noting that m is arbitrary, one can prove that $D_1 \oplus D_2$ is maximally isotropic by Lemma 5.3.1. $\square$

The Direct Sum of Induced Dirac Structures. The direct sum of Dirac structure will allow us to express a "Cartesian product" of distinct non-interacting implicit Lagrangian systems. In particular, let Q_1 and Q_2 be distinct configuration spaces. Let $\Delta_{Q_1} \subset TQ_1$ and $\Delta_{Q_2} \subset TQ_2$ be smooth constraint distributions which induce the Dirac structures $D_1 \in \mathrm{Dir}(T^*Q_1)$ and $D_2 \in \mathrm{Dir}(T^*Q_2)$ respectively.

Let $Q = Q_1 \times Q_2$ be an *extended configuration manifold*, and we may identify $TQ = T(Q_1 \times Q_2)$ with $TQ_1 \times TQ_2$, and $T^*Q = T^*(Q_1 \times Q_2)$ with $T^*Q_1 \times T^*Q_2$. Define the induced distribution $\mathfrak{d}_{T^*Q} = (T\pi_Q)^{-1}(\mathfrak{d}_Q)$ on T^*Q from $\mathfrak{d}_Q = \Delta_{Q_1} \times \Delta_{Q_2} \subset TQ$.

Proposition 5.3.2. *Let Ω_i be the canonical symplectic structures on T^*Q_i and D_i the Dirac structures on T^*Q_i induced from $\Delta_{Q_i} \subset TQ_i$ for $i = 1, 2$. Then, $D_1 \oplus D_2$ is equal to the Dirac structure D on T^*Q induced from $\mathfrak{d}_Q$, which is given by:*

$$D(q,p) = \{ (w, \alpha) \in T_{(q,p)}T^*Q \times T^*_{(q,p)}T^*Q \mid w \in \mathfrak{d}_{T^*Q}(q,p) \text{ and } \alpha - \Omega^\flat(q,p) \cdot w \in \mathfrak{d}^\circ_{T^*Q}(q,p) \}, \tag{5.9}$$

for each $(q,p) \in T^*Q$, where $\Omega^\flat : TT^*Q \to T^*T^*Q$ is the bundle map associated with $\Omega = \Omega_1 \oplus \Omega_2$.

Proof. Since $D_1 \oplus D_2$ and D are distributions of identical rank, it suffices to prove that $D_1 \oplus D_2 \subset D$. Let us choose $(q,p) \in T^*Q$ and $(w, \alpha) \in D_1 \oplus D_2(q,p)$. Then, we may decompose (w, α) as

$$\begin{aligned} w &= (w_1, w_2) \\ \alpha &= T^*_{(q,p)} \operatorname{pr}_{T^*Q_1}(\alpha_1) + T^*_{(q,p)} \operatorname{pr}_{T^*Q_2}(\alpha_2) \end{aligned}$$

such that $(w_1, \alpha_1) \in D_1(q_1, p_1)$ and $(w_2, \alpha_2) \in D_2(q_2, p_2)$. Then it follows that $\alpha_1 - \Omega_1^\flat(q_1, p_1) \cdot w_1 \in \Delta^\circ_{T^*Q_1}(q_1, p_1)$ and $\alpha_2 - \Omega_2^\flat(q_2, p_2) \cdot w_2 \in \Delta^\circ_{T^*Q_2}(q_2, p_2)$, where $w_1 \in \Delta_{T^*Q_1}(q_1, p_1)$ and $w_2 \in \Delta_{T^*Q_2}(q_2, p_2)$. Noting that $\Omega^\flat(q,p) = \Omega_1^\flat(q_1, p_1) \oplus \Omega_2^\flat(q_2, p_2)$ and $w = (w_1, w_2) \in \Delta_{T^*Q_1}(q_1, p_1) \times \Delta_{T^*Q_2}(q_1, p_2)$, we conclude that $\alpha - \Omega^\flat(q,p) \cdot w \in \Delta^\circ_{T^*Q_1}(q_1, p_1) \times \Delta^\circ_{T^*Q_2}(q_2, p_2)$. Additionally, $T\pi_Q(\Delta_{T^*Q_1} \times \Delta_{T^*Q_2}) = \Delta_{Q_1} \times \Delta_{Q_2} = \mathfrak{d}_Q \subset TQ$. The space $\mathfrak{d}_{T^*Q} = \Delta_{T^*Q_1} \times \Delta_{T^*Q_2}$ is annihilated by $\mathfrak{d}^\circ_{T^*Q} = \Delta^\circ_{T^*Q_1} \times \Delta^\circ_{T^*Q_2}$.

Putting these all together gives us $w \in \mathfrak{d}_{T^*Q}(q,p)$ and $\alpha - \Omega^\flat(q,p) \cdot w \in \mathfrak{d}^\circ_{T^*Q}(q,p)$, for each $(q,p) \in T^*Q$. Thus $(w, \alpha) \in D$. Since (w, α) was chosen arbitrarily, $D_1 \oplus D_2 \subset D$. $\square$

It is notable that the direct sum of Dirac structures does not express interactions between separate systems. The interaction is expressed using an interaction Dirac structure but we have not yet shown how to use the interaction Dirac structure to do anything useful. We do this in the next section.

Tensor Product of Dirac Structures. Now we show how to derive the Dirac structure of the interconnected Dirac system using D_1, D_2 and an interaction Dirac structure D_{int}. In order to do this we need an important mathematical ingredient called the *Dirac tensor product.*

Definition 5.3.3. *Let $D_a, D_b \in \mathrm{Dir}(M)$. Let $d : M \hookrightarrow M \times M$ be the diagonal embedding in $M \times M$. The* Dirac tensor product *of D_a and D_b is defined as*

$$D_a \boxtimes D_b := d^*(D_a \oplus D_b) = \frac{(D_a \oplus D_b \cap K^\perp) + K}{K}, \tag{5.10}$$

where

$$K = \{(0,0)\} \oplus \{(\beta, -\beta)\} \subset T(M \times M) \oplus T^*(M \times M)$$

and its orthogonal complement $K^\perp \subset T(M \times M) \oplus T^*(M \times M)$ is given by

$$K^\perp = \{(v,v)\} \oplus T^*(M \times M).$$

Theorem 5.3.1. *Under the assumption that $D_a \oplus D_b \cap K^\perp$ has locally constant rank at each $x \in M$, $D_a \boxtimes D_b$ is a Dirac structure on M.*

Remark. In previous publications we called the tensor product of Dirac structures the *bowtie product* and used the symbol "$\bowtie$" [YJM10, JYM10] with the definition:

$$D_a \bowtie D_b = \{(v,\alpha) \in TM \oplus T^*M \mid \exists \beta \in T^*M \\ \text{such that } (v, \alpha+\beta) \in D_a, (v, -\beta) \in D_b\}. \tag{5.11}$$

Later, it was revealed that this construction was equivalent to the tensor product of Dirac structures, $\boxtimes$, introduced by [Gua07] in the context of generalized complex geometry[1].

Properties of the Dirac Tensor Product. In this section we will prove that the tensor product of Dirac structures is associative and commutative, we will use a special restricted two-form Ω_{Δ_M} induced from a Dirac structure D on M with $\Delta_M = \mathrm{pr}_{TM}(D) \subset TM$, where $\mathrm{pr}_{TM} : TM \oplus T^*M \to TM$; $(v,\alpha) \mapsto v$ and we assume that Δ_M is smooth.

Lemma 5.3.2. *On each fiber of $T_xM \times T_x^*M$ at $x \in M$, there exists a bilinear anti-symmetric map $\Omega_{\Delta_M}(x) : \Delta_M(x) \times \Delta_M(x) \to \mathbb{R}$ defined as*

$$\Omega_{\Delta_M}(x)(v_1, v_2) = \langle \alpha_1, v_2 \rangle \text{ such that } (v_1, \alpha_1) \in D(x). \tag{5.12}$$

This restricted two-form was initially introduced by [CW88] for linear Dirac structures (see also [Cou90] and [DW04]). We can easily generalize it to the case of general manifolds since Ω_{Δ_M} may be defined fiber-wise.

Given a Dirac structure $D \in \mathrm{Dir}(M)$, it follows from equation (5.1) that, for each $x \in M$, $D(x)$ may be given by

$$D(x) = \{(v,\alpha) \in T_xM \times T_x^*M \mid v \in \Delta_M(x), \text{ and} \\ \alpha(w) = \Omega_{\Delta_M}(x)(v,w) \text{ for all } w \in \Delta_M(x)\},$$

[1]We appreciate Henrique Bursztyn for pointing out this fact in Iberoamerican Meeting on Geometry, Mechanics and Control in honor of Hernán Cendra at Centro Atómico Bariloche, January 13, 2011.

which may be also stated by

$$D(x) = \{(v, \alpha) \in T_xM \times T_x^*M \mid v \in \Delta_M(x), \text{ and } \alpha - \Omega^\flat(x) \cdot v \in \Delta_M^\circ(x)\},$$

where $\Omega^\flat(x) : T_xM \to T_x^*M$ is the skew-symmetric bundle map that is an extension of the skew-symmetric map $\Omega^\flat_{\Delta_M}(x) : \Delta_M(x) \subset T_xM \to \Delta_M^*(x) = T_x^*M/\Delta_M^\circ(x) \subset T_x^*M$, which is defined by $\langle \Omega^\flat_{\Delta_M}(x)(v_x), w_x \rangle = \Omega_{\Delta_M}(v_x, w_x)$ on $\Delta_M(x)$.

Proposition 5.3.3. *Let D_a and $D_b \in \mathrm{Dir}(M)$. Let $\Delta_a = \mathrm{pr}_{TM}(D_a)$ and $\Delta_b = \mathrm{pr}_{TM}(D_b)$. Let Ω_a and Ω_b be the Dirac two-forms for D_a and D_b, respectively. If $\Delta_a \cap \Delta_b$ has locally constant rank, then $D_a \boxtimes D_b$ is a Dirac structure with the smooth distribution $\mathrm{pr}_{TM}(D_a \boxtimes D_b) = \Delta_a \cap \Delta_b$ and with the Dirac two-form $(\Omega_a + \Omega_b)|_{\Delta_a \cap \Delta_b}$.*

Proof. Let $(v, \alpha) \in D_a \boxtimes D_b(x)$ for $x \in M$. By definition of the Dirac tensor product in (5.11), there exists $\beta \in T_x^*M$ such that $(v, \alpha + \beta) \in D_a(x), (v, -\beta) \in D_b(x)$. Hence, one has

$$\Omega_a^\flat(x) \cdot v - \alpha - \beta \in \Delta_a^\circ(x) \text{ and } \Omega_b^\flat(x) \cdot v + \beta \in \Delta_b^\circ(x), \text{ for each } x \in M,$$

where $v \in \Delta_a(x)$ and $v \in \Delta_b(x)$. This means $(\Omega_a^\flat + \Omega_b^\flat)(x) \cdot v - \alpha \in \Delta_a^\circ(x) + \Delta_b^\circ(x)$ and $v \in \Delta_a \cap \Delta_b(x)$. But $\Delta_a^\circ(x) + \Delta_b^\circ(x) = (\Delta_a \cap \Delta_b)^\circ(x)$. Therefore, upon setting $\Omega_c = \Omega_a + \Omega_b$ and $\Delta_c = \Delta_a \cap \Delta_b$, we can write $\Omega_c^\flat(x) \cdot v - \alpha \in \Delta_c^\circ(x)$ and $v \in \Delta_c(x)$; namely, $(v, \alpha) \in D_c(x)$, where D_c is a Dirac structure with Δ_c and Ω_c. Then, it follows that $D_a \boxtimes D_b \subset D_c$. Equality follows from the fact that both $D_a \boxtimes D_b(x)$ and $D_c(x)$ are subspaces of $T_xM \times T_x^*M$ with the same dimension. □

Corollary 5.3.1. *If $\Omega_b = 0$, then it follows that $D_b = \Delta_b \oplus \Delta_b^\circ$, and also that $D_c = D_a \boxtimes D_b$ is induced from $\Delta_a \cap \Delta_b$ and $\Omega_a|_{\Delta_a \cap \Delta_b}$.*

Proposition 5.3.4. *Let $D_a, D_b, D_c \in \mathrm{Dir}(M)$ with smooth distributions $\Delta_a = \mathrm{pr}_{TM}(D_a)$, $\Delta_b = \mathrm{pr}_{TM}(D_b)$, and $\Delta_c = \mathrm{pr}_{TM}(D_c)$. Assume that $\Delta_a \cap \Delta_b$, $\Delta_b \cap \Delta_c$, and $\Delta_c \cap \Delta_a$ have locally constant ranks. Then the Dirac tensor product $\boxtimes$ is associative and com-*

mutative; namely we have

$$(D_a \boxtimes D_b) \boxtimes D_c = D_a \boxtimes (D_b \boxtimes D_c)$$

and

$$D_a \boxtimes D_b = D_b \boxtimes D_a.$$

Proof. First we prove commutativity. Recall that any Dirac structure may be constructed by its associated constraint distribution $\Delta = \mathrm{pr}_{TM}(D)$ and the Dirac two-form Ω_Δ. Let Ω_a, Ω_b, and Ω_c be the Dirac two-forms corresponding to D_a, D_b, and D_c, respectively. Then we find by Proposition 5.3.3 that $D_a \boxtimes D_b$ is defined by the smooth distribution $\Delta_{ab} = \Delta_a \cap \Delta_b$ and the Dirac two-form $\Omega_{\Delta_{ab}} = (\Omega_{\Delta_a} + \Omega_{\Delta_b})|_{\Delta_{ab}}$. By commutativity of $+$ and $\cap$, we find the same distribution, and from the two-form for $D_b \boxtimes D_a$, we have $D_a \boxtimes D_b = D_b \boxtimes D_a$.

Next, we prove associativity. Let $\Delta_{(ab)c} = \mathrm{pr}_{TM}((D_a \boxtimes D_b) \boxtimes D_c)$ and $\Delta_{a(bc)} = \mathrm{pr}_{TM}(D_a \boxtimes (D_b \boxtimes D_c))$ and it follows

$$\Delta_{(ab)c} = (\Delta_a \cap \Delta_b) \cap \Delta_c = \Delta_a \cap (\Delta_b \cap \Delta_c) = \Delta_{a(bc)}.$$

If $\Omega_{\Delta_{(ab)c}}$ and $\Omega_{\Delta_{a(bc)}}$ are, respectively, the Dirac two-forms for $(D_a \boxtimes D_b) \boxtimes D_c$ and $D_a \boxtimes (D_b \boxtimes D_c)$, we find

$$\begin{aligned}
\Omega_{\Delta_{(ab)c}} &= [(\Omega_{\Delta_a} + \Omega_{\Delta_b})|_{\Delta_{ab}} + \Omega_{\Delta_c}]|_{\Delta_{(ab)c}} \\
&= (\Omega_{\Delta_a} + \Omega_{\Delta_b} + \Omega_{\Delta_c})|_{\Delta_{(ab)c}} \\
&= (\Omega_{\Delta_a} + \Omega_{\Delta_b} + \Omega_{\Delta_c})|_{\Delta_{a(bc)}} \\
&= \Omega_{\Delta_{a(bc)}}.
\end{aligned}$$

Thus, we obtain

$$(D_a \boxtimes D_b) \boxtimes D_c = D_a \boxtimes (D_b \boxtimes D_c).$$

□

Remark. We have shown that $\boxtimes$ acts on pairs of Dirac structures with clean intersections[2] to give a new Dirac structure, and also that that it is an associative and commutative product. It is easy to verify that the Dirac structure $D_e = TM \oplus \{0\}$ satisfies the property of the identity element as $D_e \boxtimes D = D \boxtimes D_e = D$ for every $D \in \mathrm{Dir}(M)$. However this does *not* make the pair $(\mathrm{Dir}(M), \boxtimes)$ into a commutative category because $\boxtimes$ is not defined on all pairs of Dirac structures. This is similar to the difficulty of defining a symplectic category [Wei09].

The previous Propositions justify the following definition for the "interconnection" of Dirac structures.

Definition 5.3.4. *Let (D_1, M_1) and (D_2, M_2) be Dirac manifolds and let $D_{\mathrm{int}} \in \mathrm{Dir}(M_1 \times M_2)$ be such that D_{int} and $D_1 \oplus D_2$ have a clean intersection. Then we call the Dirac structure*

$$D_C := (D_1 \oplus D_2) \boxtimes D_{\mathrm{int}}$$

the interconnection *of D_1 and D_2 through D_{int}.*

Interconnections by Constraints. Let Q_1 and Q_2 be distinct configuration manifolds and let $D_1 \in \mathrm{Dir}(T^*Q_1)$ and $D_2 \in \mathrm{Dir}(T^*Q_2)$ be Dirac structures induced from distributions $\Delta_{Q_1} \subset TQ_1$ and $\Delta_{Q_2} \subset TQ_2$. Let D_{int} be a Dirac structure described by a distribution $\Sigma_Q \subset TQ$, as in Definition 5.3.1. Then it is clear that $D_1 \oplus D_2$ and D_{int} intersect cleanly if and only if $\Delta_{Q_1} \oplus \Delta_{Q_2}$ and Σ_Q intersect cleanly. If we have clean intersections then the interconnection of D_1 and D_2 through D_{int} is given locally by

$$\begin{aligned} D(q,p) = \{\, (w, \alpha) \in T_{(q,p)}T^*Q \times T^*_{(q,p)}T^*Q \ | \\ w \in \Delta_{T^*Q}(q,p) \ \text{ and } \ \alpha - \Omega^\flat(q,p) \cdot w \in \Delta^\circ_{T^*Q}(q,p) \,\}, \end{aligned}$$

where $\Omega = \Omega_1 \oplus \Omega_2$, $\Delta_{T^*Q} = T\pi_Q^{-1}((\Delta_{Q_1} \oplus \Delta_{Q_2}) \cap \Sigma_Q)$.

[2] A clean intersection of two sub bundles of a vector bundle means that the intersection is a sub bundle of constant rank on each component of the base.

Interconnection of $n > 2$ Dirac Structures. It is simple to generalize the preceding constructions to the interconnection of $n > 2$ distinct Dirac structures $D_1, D_2, \ldots, D_n$ on distinct manifolds $M_1, M_2, \ldots, M_n$. Recall that the direct sum $\oplus$ is associative so that we may define the iterated sum

$$\bigoplus_{i=1}^{n} D_i = D_1 \oplus D_2 \oplus \cdots \oplus D_n.$$

By choosing an appropriate interaction Dirac structure

$$D_{\mathrm{int}} \in \mathrm{Dir}(M),$$

where $M = M_1 \times \cdots \times M_n$, and $\mathrm{rank}(\mathrm{pr}_{TM}(\oplus D_i) \cap \mathrm{pr}_{TM}(D_{\mathrm{int}}))$ is constant on each component of M, we can define the interconnection of $D_1, \ldots, D_n$ through D_{int} by the Dirac structure

$$D = \left(\bigoplus_{i=1}^{n} D_i \right) \boxtimes D_{\mathrm{int}}.$$

Composition as an Interconnection of Dirac Structures. The notion of *composition* of Dirac structures was introduced in [CvDSBn07] in the context of port-Hamiltonian systems, where the composition was constructed on vector spaces. In this section we site the results of [JY11] to clarify the link between composition and interconnection via $\boxtimes$.

Let V_1, V_2, and V_s be vector spaces. Let D_1 be a linear Dirac structure on $V_1 \oplus V_s$ and D_2 be a linear Dirac structure on $V_s \oplus V_2$. The *composition* of D_1 and D_2 is given by

$$\begin{aligned} D_1 || D_2 = \{ &(v_1, v_2, \alpha_1, \alpha_2) \in (V_1 \times V_2) \oplus (V_1^* \times V_2^*) \mid \\ &\exists (v_s, \alpha_s) \in V_s \oplus V_s^*, \text{ such that } (v_1, v_s, \alpha_1, \alpha_s) \in D_1, (-v_s, v_2, \alpha_s, \alpha_s) \in D_2 \}, \end{aligned}$$

where V_1^*, V_2^*, and V_s^* denote the dual space of V_1, V_2 and V_s. It was also shown that the set $D_1 || D_2$ is itself a Dirac structure on $V_1 \times V_2$. Moreover, given many shared variables, the operation of composition is associative. However the type of interaction given by composition of Dirac structures is specifically the interaction between systems which have shared variables. This is expressed in the next theorem which links the notion of composition of Dirac structures with the notion of interconnection of Dirac structures.

Proposition 5.3.5. *Set* $V = V_1 \times V_s \times V_s \times V_2$ *and* $\bar{V} = V_1 \times V_2$*. Let* $\Psi : V \to \bar{V}$ *be the projection* $(v_1, v_s, v_s', v_2) \mapsto (v_1, v_2)$*. Let* $\Sigma_{\text{int}} = \{(v_1, v_s, -v_s, v_2) \in V\}$ *and let* $D_{\text{int}} = \Sigma_{\text{int}} \oplus \Sigma_{\text{int}}^{\circ}$*. For linear Dirac structures* D_1 *on* $V_1 \times V_s$ *and* D_2 *on* $V_s \times V_2$*, it follows that*

$$D_1 || D_2 = \Psi_*(D_1 \oplus D_2) \boxtimes D_{\text{int}}.$$

Proof. First, set $D = (D_1 \oplus D_2) \boxtimes D_{\text{int}}$ and observe that $\Sigma_{\text{int}}^{\circ} = \{(0, \alpha_s, \alpha_s, 0) \in V^*\}$. We also observe

$$\Psi_* D = \{(\Psi(v_1, v_s, v_s', v_2), \alpha_1, \alpha_2) \mid (v_1, v_s, v_s', v_2, \Psi^*(\alpha_1, \alpha_2)) \in D\}$$

by definition of the push-forward map. Using the facts that $\Psi(v_1, v_s, v_s', v_2) = (v_1, v_2)$ and $\Psi^*(\alpha_1, \alpha_2) = (\alpha_1, 0, 0, \alpha_2) \in V^*$,

$$\Psi_* D = \{(v_1, v_2, \alpha_1, \alpha_2) \mid \exists v_s, v_s' \in V_s \text{ such that } (v_1, v_s, v_s', v_2, \alpha_1, 0, 0, \alpha_2) \in D\}.$$

Since $D = (D_1 \oplus D_2) \boxtimes D_{\text{int}}$, it follows that

$$\begin{aligned} \Psi_* D = \{&(v_1, v_2, \alpha_1, \alpha_2) \mid \exists v_s, v_s' \in V_s \text{ and } \exists \beta \in V^* \text{ such that} \\ &(v_1, v_s, v_s', v_2, \alpha_1 + \beta_1, \beta_s, \beta_s', \alpha_2 + \beta_2) \in D_1 \oplus D_2, \\ &(v_1, v_s, v_s', v_2, -\beta_1, -\beta_s, -\beta_s', -\beta_2) \in D_{\text{int}}\}. \end{aligned}$$

Utilizing the fact that $(v_1, v_s, v_s', v_2, -\beta_1, -\beta_s, -\beta_s', -\beta_2) \in D_{\text{int}}$ if and only if $v_s = -v_s'$ and $\beta_s = \beta_s', \beta_1 = 0, \beta_2 = 0$, we may restate the above as

$$\begin{aligned} \Psi_* D = \{&(v_1, v_2, \alpha_1, \alpha_2) \mid \exists v_s \in V_s, \alpha_s \in V_s^* \text{ such that} \\ &(v_1, v_s, -v_s, v_2, \alpha_1, \alpha_s, \alpha_s, \alpha_2) \in D_1 \oplus D_2\}. \end{aligned}$$

Finally, we have

$$\begin{aligned}\Psi_* D = \{(v_1, v_2, \alpha_1, \alpha_2) \mid \exists v_s \in V_s, \alpha_s \in V_s^* \text{ such that} \\ (v_1, v_s, \alpha_1, \alpha_s) \in D_1, \quad (-v_s, v_2, \alpha_s, \alpha_2) \in D_2\}.\end{aligned}$$

This is nothing but $D_1 || D_2$. □

5.4 Interconnection of Implicit Lagrangian Systems

The process of interconnection of Dirac structures allows us to couple the dynamics of implicit Lagrangian systems using interaction Dirac structures. Specifically, given a pair of implicit Lagrangian systems, (X_1, D_1, E_{L_1}) and (X_2, D_2, E_{L_2}), and an interaction Dirac structure, D_{int}, we derive the system (X_C, D_C, E_L) where $L = L_1 + L_2$, $D_C = (D_1 \oplus D_2) \boxtimes D_{\mathrm{int}}$, and X_C is a partial vector-field which satisfies the implicit Euler-Lagrange equations with respect to the Dirac structure D_C and energy E_L. We call the process of transition from (X_1, D_1, E_{L_1}) and (X_2, D_2, E_{L_2}) to (X_C, D_C, E_L) *interconnection of implicit Lagrangian systems.*

Distinct implicit Lagrangian Systems. In this section, we shall consider the interconnection of n distinct implicit Lagrangian systems. For the remainder of this section we have the following setup:

Q_i	a configuration manifold
Δ_{Q_i}	a regular distribution on TQ_i
D_i	the Dirac structure induced from the distribution Δ_{Q_i}
L_i	generic Lagrangian in $C^\infty(TQ_i)$, possibly non-degenerate
$\mathbb{F}L_i$	the Legendre transformation of L_i
E_{L_i}	the generalized energy of L_i
P_i	the primary constraint manifold $\mathbb{F}L_i(\Delta_{Q_i})$
X_i	a partial vector field for the implicit Lagrangian system (X_i, D_i, E_{L_i})
for $i = 1, \ldots, n$	

For the n distinct implicit Lagrangian systems, $(E_{L_i}, D_i, X_i), i = 1, \ldots, n$, one has the conditions

$$(X_i(q_i, v_i, p_i), \mathbf{d}E_{L_i}(q_i, v_i, p_i)|_{TP_i}) \in D_i(q_i, p_i)$$

for each $(q_i, v_i) \in \Delta_{Q_i}$ with $(q_i, p_i) = \mathbb{F}L_i(q_i, v_i)$.

Finally to save space, set

$$\begin{aligned} Q &= Q_1 \times \cdots \times Q_n \\ \mathfrak{d}_Q &= \Delta_{Q_1} \times \cdots \times \Delta_{Q_n} \subset TQ \\ L &= L_1 + \cdots + L_n \\ D &= (D_1 \oplus \cdots \oplus D_n) \boxtimes D_{\text{int}} \end{aligned}$$

and let $E_L : TQ \oplus T^*Q \to \mathbb{R}$ be the generalized energy associated to L.

Interconnection through an interaction Dirac Structure Let $\Sigma_Q \subset TQ$ be a smooth distribution such that $\Delta_Q := \mathfrak{d}_Q \cap \Sigma_Q$ is a distribution of locally constant rank.[3]. Define the interaction Dirac structure by $D_{\text{int}} = \Sigma_{\text{int}} \oplus \Sigma_{\text{int}}^\circ$ as in (5.8), where $\Sigma_{\text{int}} = (T\pi_Q)^{-1}(\Sigma_Q)$. Let $q = (q_1, ..., q_n) \in Q$, $(q, v) = (q_1, ..., q_n, v_1, ..., v_n) \in TQ$, and $(q, p) = (q_1, ..., q_n, p_1, ..., p_n) \in T^*Q$.

[3]Locally constant in this case means constant on each component of Q.

The dynamics of the interconnected implicit Lagrangian system are given by:

$$\dot{q} = v \in \Delta_Q(q), \quad \dot{p} - \frac{\partial L}{\partial q} \in \Delta_Q^{\circ}(q), \quad p = \frac{\partial L}{\partial v}. \tag{5.13}$$

However, this local expression can be split between the subsystems by introducing interaction force fields $F_i : TQ \oplus T^*Q \to T^*Q_i$ to yield equations

$$\dot{q}_i = v \in \Delta_{Q_i}(q_i), \quad \dot{p}_i - \frac{\partial L_i}{\partial q_i} + F_i(q, v, p) \in \Delta_{Q_i}^{\circ}(q_i), \quad p_i = \frac{\partial L_i}{\partial v_i}, i = 1, \ldots, n \tag{5.14}$$

along with the condition $\dot{q} \in \Sigma_Q$. Note that the domain of each F_i is $TQ \oplus T^*Q$, and involves all the subsystems thus coupling the equations. We state this formally in the following theorem.

Theorem 5.4.1. *Denote the natural projections*

$$\begin{aligned} \mathrm{pr}_{Q_i} &: TQ_i \oplus T^*Q_i \to Q_i \\ \rho_{TQ_i \oplus T^*Q_i} &: TQ \oplus T^*Q \to TQ_i \oplus T^*Q_i; (q, v, p) \mapsto (q_i, v_i, p_i) \\ \rho_{T^*Q_i} &: T^*Q \to T^*Q_i \end{aligned}$$

where we identify TQ with $TQ_1 \times \cdots \times TQ_n$ and T^*Q with $T^*Q_1 \times \cdots \times T^*Q_n$.

Then given a curve $(q(t), v(t), p(t))$ in $TQ \oplus T^*Q$, the following statements are equivalent:

(i) The curve $(q(t), v(t), p(t))$ satisfies

$$\Big((\dot{q}(t), \dot{p}(t)), dE_L(q(t), v(t), p(t))|_{T_{(q(t),p(t))}P} \Big) \in D(q(t), p(t)),$$

where $(q(t), p(t)) = \mathbb{F}L(q(t), v(t)) \in P$.

(ii) There exists a force field $F : TQ \oplus T^*Q \to \Sigma_Q^{\circ} \subset T^*Q$ such that the curves

$(q_i(t), v_i(t), p_i(t))$ satisfy

$$((\dot{q}_i(t), \dot{p}_i(t)), (dE_{L_i} - \tilde{F}_i)(q(t), v(t), p(t))|_{TP_i}) \in D_i(q_i(t), p_i(t)),$$

where $(q_i(t), p_i(t)) = \mathbb{F}L_i(q_i(t), v_i(t)) \in P_i, \dot{q}(t) = (\dot{q}_1(t), ..., \dot{q}_n(t)) \in \Sigma_Q(q(t))$ and $F = (F_1, \ldots, F_n)$.

Proof. Fix t and recall that $D = (D_1 \oplus \cdots \oplus D_n) \boxtimes D_{\text{int}}$, and the condition

$$((\dot{q}(t), \dot{p}(t)), \mathbf{d}E(q(t), v(t), p(t))|_{T_{(q(t),p(t))}P}) \in D(q(t), p(t)), \text{ for all } t_1 \leq t \leq t_2,$$

using the definition of $\boxtimes(\equiv\bowtie)$ given in (5.11), the above equation implies the existence of a covector

$$(F, w) = (F_1, \ldots, F_n, w_1, \ldots, w_n) \in T^*_{(q,p)}T^*Q,$$

such that

$$\left(\dot{q}, \dot{p}, -\frac{\partial L}{\partial q} - F, v + w\right) \in D_1(q_1, p_1) \oplus \cdots \oplus D_n(q_n, p_n), \tag{5.15}$$

and

$$(\dot{q}, \dot{p}, F, -w) \in D_{\text{int}}(q, p). \tag{5.16}$$

It follows from condition (5.15) that

$$\left(\dot{q}_i, \dot{p}_i, -\frac{\partial L_i}{\partial q_i} - F_i, v_i + w_i\right) \in D_i(q_i, p_i), \quad i = 1, ..., n,$$

and also from condition (5.16) that $\dot{q} \in \Sigma_Q(q), w = 0$, and $F \in \Sigma_Q^\circ(q)$, where we note $\partial L/\partial q_i = \partial L_i/\partial q_i$. Allowing t to vary, it follows that the curves $(q_i(t), v_i(t), p_i(t))$, $t_1 \leq t \leq t_2$ satisfy the conditions

$$\left((\dot{q}_i(t), \dot{p}_i(t)), (dE_i - F_i)(q_i(t), v_i(t), p_i(t))\big|_{T_{(q_i(t),p_i(t))}P_i}\right) \in D_i(q_i(t), p_i(t)), \quad i = 1, ..., n,$$

together with $(q_i(t), p_i(t)) = \mathbb{F}L_i(q_i(t), v_i(t)) \in P_i$. These conditions implicitly define the force field $F : TQ \oplus T^*Q \to T^*Q$ on the graph of the $\mathbb{F}L$. One can easily check the converse by reversing the order of the arguments given. $\square$

Variational Principles for Interconnected Systems. Because D is equivalent to a Dirac structure induced from the constraint Δ_Q, the dynamics must satisfy the LAP principle

$$\begin{aligned} &\delta \int_{t_1}^{t_2} \langle p(t), \dot{q}(t)\rangle - E_L(q(t), v(t), p(t))\, dt \\ &= \delta \int_{t_1}^{t_2} L(q(t), v(t)) + \langle p(t), \dot{q}(t) - v(t)\rangle \; dt = 0, \end{aligned} \tag{5.17}$$

for a curve $(q(t), v(t), p(t))$, $t_1 \le t \le t_2$ in $TQ \oplus T^*Q$ with variations $\delta q(t) \in \Delta_Q(q(t)) \subset T_{q(t)}Q$ with fixed endpoints, unconstrained variations $\delta v(t)$ and $\delta p(t)$, and $\dot{q}(t) \in \Delta_Q(q(t)) \subset T_{q(t)}Q$. In this section we provide an equivalent variational principles where each subsystem is given its own variational equation.

Definition 5.4.1. *Let $(q(t), v(t), p(t)) \in TQ \oplus T^*Q$ be a curve and let $(q_i(t), v_i(t), p_i(t)) = \rho_{TQ_i \oplus T^*Q_i}(q(t), v(t), p(t))$. The* interconnection of the Lagrange-d'Alembert-Pontryagin principle *through D_{int} is:*

$$\begin{aligned} &\delta \int_{t_1}^{t_2} \langle p_i(t), \dot{q}_i(t)\rangle - E_{L_i}(q_i(t), v_i(t), p_i(t))\, dt + \int_{t_1}^{t_2} \langle F_i(q(t), v(t), p(t)), \delta q_i(t)\rangle\, dt \\ &= \delta \int_{t_1}^{t_2} L_i(q_i(t), v_i(t)) + \langle p_i(t), \dot{q}_i(t) - v_i(t)\rangle \; dt + \int_{t_1}^{t_2} \langle F_i(q(t), v(t), p(t)), \delta q_i(t)\rangle\, dt \\ &= 0, \end{aligned} \tag{5.18}$$

for variations $\delta q_i(t) \in \Delta_{Q_i}(q_i(t))$ with fixed end points, arbitrary variations $\delta v_i, \delta p_i$, and with $\dot{q}_i(t) \in \Delta_{Q_i}(q_i(t))$, and the condition

$$(\dot{q}_1, ..., \dot{q}_n) \in \Sigma_Q(q_1, ..., q_n) \text{ and } F_1 \oplus \cdots \oplus F_n \in \Sigma_Q^\circ(q_1, ..., q_n). \tag{5.19}$$

Proposition 5.4.1. *The interconnection of the LAP principle through Σ_Q given in*

(5.18) *and* (5.19) *for curves* $(q_i(t), v_i(t), p_i(t))$, $t_1 \leq t \leq t_2$ *in* $TQ_i \oplus T^*Q_i, i = 1, \ldots, n$ *is equivalent to the LAP principle in* (5.17).

Proof. It follows from (5.18) that

$$\dot{q}_i = v_i \in \Delta_{Q_i}(q_i), \quad \dot{p}_i - \frac{\partial L_i}{\partial q_i} - F_i \in \Delta^{\circ}_{Q_i}(q_i), \quad p_i = \frac{\partial L_i}{\partial v_i}, \quad i = 1, ..., n.$$

Recall that the distribution $\mathfrak{d}_Q(q_1, ..., q_n) = \Delta_{Q_1}(q_1) \times \cdots \times \Delta_{Q_n}(q_n) \subset TQ$ has the annihilator $\mathfrak{d}^{\circ}_Q(q_1, ..., q_n) = \Delta^{\circ}_{Q_1}(q_1) \times \cdots \times \Delta^{\circ}_{Q_n}(q_n)$, and impose the additional constraints

$$(\dot{q}_1, ..., \dot{q}_n) \in \Sigma_Q(q_1, ..., q_n) \quad \text{and} \quad F_1(q_1, v_1, p_1) \oplus \cdots \oplus F_n(q_n, v_n, p_n) \in \Sigma^{\circ}_Q(q_1, ..., q_n)$$

to arrive at the equations

$$\begin{aligned} (\dot{q}_1, ..., \dot{q}_n) &= (v_1, ..., v_n) \in \Delta_Q(q_1, ..., q_n), \\ \left(\dot{p}_1 - \frac{\partial L_1}{\partial q_1}, ..., \dot{p}_n - \frac{\partial L_n}{\partial q_n}\right) &\in \Delta^{\circ}_Q(q_1, ..., q_n) \\ (p_1, ..., p_2) &= \left(\frac{\partial L_1}{\partial v_1}, ..., \frac{\partial L_2}{\partial v_2}\right) \end{aligned}$$

where $\Delta_Q(q_1, ..., q_n) = \mathfrak{d}_Q(q_1, ..., q_n) \cap \Sigma_Q(q_1, ..., q_n) \subset TQ$ is the final distribution and its annihilator is given by

$$\Delta^{\circ}_Q(q_1, ..., q_n) = \mathfrak{d}^{\circ}_Q(q_1, ..., q_n) + \Sigma^{\circ}_Q(q_1, ..., q_n).$$

Reflecting upon the last group of equations we find that we have arrived at the Lagrange-d'Alembert-Pontryagin equations (5.13), which can also be derived from the Lagrange-d'Alembert-Pontryagin principle in (5.17). The converse is proven by reversing the above arguments to prove the existence of the coupling forces $F_1, \ldots, F_n$. $\square$

Thus, we obtain the following theorem:

Theorem 5.4.2. *Let $(q, v, p)(t)$ be a curve in $TQ \oplus T^*Q$ on the time-interval $[t_1, t_2]$ and set $(q_i, v_i, p_i)(t) = \rho_{TQ_i \oplus T^*Q_i}(q(t), v(t), p(t))$. Then the following statements are equivalent:*

(i) *The curve $(q, v, p)(t)$, t_1 satisfies*

$$((\dot{q}, \dot{p})(t), dE_L(q, v, p)(t)|_{TP}) \in D(q(t), p(t)),$$

where $(q(t), p(t)) = (q(t), (\partial L/\partial v)(t))$.

(ii) *There exists a force, F, such that the curves $(q_i, v_i, p_i)(t)$ satisfy*

$$((\dot{q}_i(t), \dot{p}_i(t)), (dE_{L_i} - \rho_{T^*Q_i} \cdot F(q(t), v(t), p(t)))|_{T_{(q_i(t), p_i(t))} P_i}) \in D_i(q_i(t), p_i(t)),$$

where $(q_i, p_i = (q_i, \partial L_i/\partial v_i)$, and

$$\dot{q}(t) \in \Sigma_Q(q(t)) \text{ and } F(q, v, p)(t) \in \Sigma_Q^{\circ}(q(t)).$$

(iii) *The curve $(q, v, p)(t)$, $t_1 \leq t \leq t_2$ satisfies the Hamilton–Pontryagin principle:*

$$\delta \int_{t_1}^{t_2} L(q(t), v(t)) + \langle p(t), \dot{q}(t) - v(t) \rangle dt = 0$$

with respect to chosen variations $\delta q(t) \in \Delta_Q(q(t))$ with fixed end points, $\delta v, \delta p$ arbitrary, and the constraint $\dot{q}(t) \in \Delta_Q(q(t))$.

(iv) *There exists a force, F, such that the curves $(q_i, v_i, p_i)(t)$, $t_1 \leq t \leq t_2$ satisfy the Lagrange–d'Alembert–Pontryagin principles:*

$$\delta \int_0^t L_i(q_i(t), v_i(t)) + \langle p_i(t), \dot{q}_i(t) - v_i(t) \rangle dt + \int_{t_1}^{t_2} \langle F_i(t), \delta q \rangle dt = 0,$$

for chosen variations $\delta q_i(t) \in \Delta_{Q_i}(q_i(t))$ with fixed end points, arbitrary $\delta v_i(t), \delta p_i(t)$, *and the constraints* $\dot{q}_i(t) \in \Delta_{Q_i}(q_i(t))$ *and*

$$\dot{q}(t) \in \Sigma_Q(q(t)) \;\; \textit{and} \;\; F(q(t), v(t), p(t)) \in \Sigma_Q^\circ(q(t)).$$

5.5 Examples

In this section we provide specific examples of interconnection of implicit Lagrangian systems. We have chosen simple scenarios to illustrate the essential ideas concretely. However, tearing and interconnection extend to more complicated systems. Additionally, for the first couple of examples we invoke some ideas from port-systems theory to allow for a comparison between existing theories of interconnection and the theory presented here. In the final example we do not include any port-variables. In fact one advantage of the theory presented in this paper is that port-variables are not needed.

(I) A Mass-Spring Mechanical System.

Consider a mass-spring system as in Figure 5.1. Let m_i and k_i be the i-th mass and spring for $i = 1, 2, 3$.

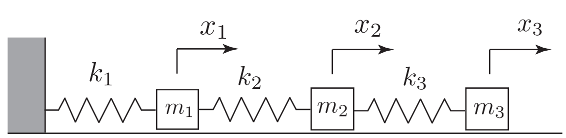

Figure 5.1 – A mass-spring system

Tearing and Interconnection. Inspired by the concept of *tearing and interconnection* developed in [Kro63], the mechanical system can be torn apart into two subsystems, as in Figure 5.2, each of which can be regarded as a subsystem the interconnected system. The procedure of tearing inevitably yields *interactive boundaries*, i.e., boundaries where dynamics or forces may be controlled externally[4]. Upon tearing, the connected system is described by obeying the following condition at the

[4]Interactive boundaries are called "ports" in circuit theory (see, for instance, [CDK87]).

interactive boundaries:

$$f_2 + \bar{f}_2 = 0, \qquad \dot{x}_2 = \dot{\bar{x}}_2. \tag{5.20}$$

Where f_2 and $\bar{f}_2$ are forces on $\dot{x}_2$ and $\dot{\bar{x}}_2$, respectively. We call (5.20) the *continuity condition*. Without the continuity condition, there would exist no interaction between the disconnected subsystems. In other words, *the original mechanical system can be*

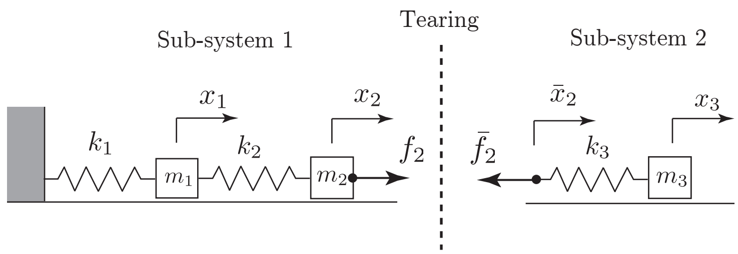

Figure 5.2 – Torn-apart systems

recovered by interconnecting the subsystems with the continuity conditions.

The continuity conditions in (5.20) imply the continuity of power flow; namely, the *power invariance* holds as

$$\mathcal{P}_2 + \bar{\mathcal{P}}_2 = 0,$$

where $\mathcal{P}_2 = \langle f_2, v_2 \rangle$ and $\bar{\mathcal{P}}_2 = \langle \bar{f}_2, \bar{v}_2 \rangle$. Needless to say, equation (5.20) can be understood as the defining condition for an interaction Dirac structure.

Subsystems. Let us consider how dynamics of the disconnected subsystems can be formulated as forced implicit Lagrangian systems.

The configuration space of subsystem 1 may be given by $Q_1 = \mathbb{R} \times \mathbb{R}$ with local coordinates (x_1, x_2), while the configuration space of the subsystem 2 is $Q_2 = \mathbb{R} \times \mathbb{R}$ with local coordinates $(\bar{x}_2, x_3)$. We can invoke the canonical Dirac structures $D_1 \in \mathrm{Dir}(T^*Q_1)$ and $D_2 \in \mathrm{Dir}(T^*Q_2)$ in this example. For Subsystem 1, the Lagrangian $L_1 : TQ_1 \to \mathbb{R}$ is given by

$$L_1(x_1, x_2, v_1, v_2) = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 - \frac{1}{2}k_1 x_1^2 - \frac{1}{2}k_2(x_2 - x_1)^2,$$

while the Lagrangian $L_2 : TQ_2 \to \mathbb{R}$ for Subsystem 2 is given by

$$L_2(\bar{x}_2, x_3, \bar{v}_2, v_3) = \frac{1}{2} m_3 v_3^2 - \frac{1}{2} k_3 (x_3 - \bar{x}_2)^2.$$

Then, we can define the generalized energy E_1 on $TQ_1 \oplus T^*Q_1$ by $E_1(x_1, x_2, v_1, v_2, p_1, p_2) = p_1 v_1 + p_2 v_2 - L_1(x_1, x_2, v_1, v_2)$, and also define the generalized energy E_2 on $TQ_2 \oplus T^*Q_2$ by $E_2(\bar{x}_2, x_3, \bar{v}_2, v_3, \bar{p}_2, p_3) = \bar{p}_2 \bar{v}_2 + p_3 v_3 - L_2(\bar{x}_2, x_3, \bar{v}_2, v_3)$. Although the original system has no external force, each disconnected subsystem has an interconnection constraint force at the interactive boundary. When viewing each system separately, the constraint force acts as an external force on each subsystem. Again, this is because *tearing always yields constraint forces at the boundaries associated with the disconnected subsystems*, as shown in Figure 5.2

Further, let $X_1 : TQ_1 \oplus T^*Q_1 \to TT^*Q_1$ be the partial vector field, which is defined at points $(x_1, x_2, v_1, v_2, p_1 = m_1 v_1, p_2 = m_2 v_2) \in TQ_1 \oplus P_1$ as $X_1(x_1, x_2, v_1, v_2, p_1, p_2) = (x_1, x_2, p_1, p_2, \dot{x}_1, \dot{x}_2, \dot{p}_1, \dot{p}_2)$, where $P_1 = \mathbb{F}L(TQ_1)$. Similarly, let $X_2 : TQ_2 \oplus T^*Q_2 \to TT^*Q_2$ be the partial vector field, which is defined at each point $(\bar{x}_2, x_3, \bar{v}_2, v_3, \bar{p}_2 = 0, p_3 = m_3 v_3)$ by $X_2(\bar{x}_2, x_3, \bar{v}_2, v_3, \bar{p}_2, p_3) = (\bar{x}_2, x_3, \bar{p}_2, p_3, \dot{\bar{x}}_2, \dot{x}_3, \dot{\bar{p}}_2, \dot{p}_3) \in TQ_2 \oplus P_2$, where $P_2 = \mathbb{F}L(TQ_2)$ and we impose the *consistency condition* $\dot{\bar{p}}_2 = 0$.

Lagrange-Dirac System 1: We can formulate dynamics of System 1 in the context of the forced implicit Lagrangian system (E_{L_1}, D_1, X_1, F_1) as

$$(X_1, \mathbf{d}E_{L_1}|_{TP_1} - \tilde{F}_1) \in D_1.$$

The above equation may be given in coordinates by

$$\dot{x}_1 = v_1, \quad \dot{x}_2 = v_2, \quad \dot{p}_1 = -k_1 x_1 - k_2(x_1 - x_2), \quad \dot{p}_2 = k_2(x_1 - x_2) + f_2,$$

and with $p_1 = m_1 v_1$ and $p_2 = m_2 v_2$.

Lagrange-Dirac System 2: Similarly, we can also formulate dynamics of System 2 in the context of the Lagrange-Dirac dynamical system (E_{L_2}, D_2, X_2, F_2) as

$$(X_2, \mathbf{d}E_2|_{TP_2} - \tilde{F}_2) \in D_2,$$

which may be given in coordinates by

$$\dot{\bar{x}}_2 = \bar{v}_2, \quad \dot{x}_3 = v_3, \quad \dot{\bar{p}}_2 = k_3(x_3 - \bar{x}_2) + \bar{f}_2, \quad \dot{p}_3 = -k_3(x_3 - \bar{x}_2),$$

together with $\bar{p}_2 = 0$ and $p_3 = m_3 v_3$ as well as $\dot{\bar{p}}_2 = 0$.

In the next paragraph we will interconnect these separate systems with a Dirac structure.

Interconnection of Distinct Dirac Structures. Let $Q = Q_1 \times Q_2 = \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ be an *extended configuration space* with local coordinates $x = (x_1, x_2, \bar{x}_2, x_3)$. Recall that the direct sum of the Dirac structures is given by $D_1 \oplus D_2$ on T^*Q. The constraint distribution due to the interconnection is given by

$$\Sigma_Q(x) = \{v \in T_xQ \mid \langle \omega_Q(x), v\rangle = 0\},$$

where $\omega_Q = dx_2 - d\bar{x}_2$ is a one-form on Q. On the other hand, the annihilator $\Sigma_Q^\circ \subset T^*Q$ is defined by

$$\Sigma_Q^\circ(x) = \{f = (f_1, f_2, \bar{f}_2, f_3) \in T_x^*Q \mid \ \langle f, v\rangle = 0 \text{ and } v \in \Sigma_Q(x)\}.$$

It follows from this codistribution that $f_2 = -\bar{f}_2$, $f_1 = 0$, and $f_3 = 0$. Hence, we obtain the conditions for the interconnection given by (5.20); namely, $f_2 + \bar{f}_2 = 0$ and $v_2 = \bar{v}_2$. Let $\Sigma_{\rm int} = (T\pi_Q)^{-1}(\Sigma_Q) \subset TT^*Q$ and let $D_{\rm int}$ be defined as in (5.8). Finally we derive the interconnected Dirac structure D on T^*Q given by

$$D = (D_1 \oplus D_2) \boxtimes D_{\rm int}.$$

Interconnection. Now, let us see how the forced implicit Lagrangian systems, namely (E_{L_1}, D_1, X_1, F_1) and (E_{L_2}, D_2, X_2, F_2), can be interconnected through D_{int} to yield a single implicit Lagrangian system. Define the Lagrangian $L : TQ \to \mathbb{R}$ for the interconnected system by $L = L_1 + L_2$, and hence the generalized energy is given by $E_L = E_{L_1} + E_{L_2} : TQ \oplus T^*Q \to \mathbb{R}$. Let $\Delta_Q = (TQ_1 \times TQ_2) \cap \Sigma_{\text{int}}$. Set a partial vector field by $X = X_1 \oplus X_2 : TQ \oplus T^*Q \to TT^*Q$, which is defined at points in $\Delta_Q \oplus P$ such that $P = \mathbb{F}L(\Delta_Q)$.

Finally, the *interconnected system* is given by (E_L, D, X), where X satisfies

$$(X(q, v, p), \mathbf{d}E_L(q, v, p)|_{TP}) \in D(q, p)$$

for each $(q, p) = \mathbb{F}L(q, v)$ with $(q, v) \in \Delta_Q$.

The Lagrange-d'Alembert-Pontryagin Principle. Additionally the interconnected system is known to satisfy the Lagrange-d'Alembert-Pontryagin principle:

$$\begin{aligned} \delta \int_a^b L_1(x_1, x_2, v_1, v_2) + p_1(\dot{x}_1 - v_1) + p_2(\dot{x}_2 - v_2) \\ + L_2(\bar{x}_2, x_3, v_3) + \bar{p}_2(\dot{\bar{x}}_2 - \bar{v}_2) + p_3(\dot{x}_3 - v_3)\, dt = 0, \end{aligned}$$

for all $\delta x_2 = \delta \bar{x}_2$, for all δv and δp, and with $v_2 = \bar{v}_2$.

Dynamics for the Interconnected System in Coordinates. Finally, we can obtain the coordinate expressions for the interconnected dynamical system as

$$\left(\begin{array}{cccc|cccc} 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&-1\\ \hline 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0 \end{array}\right)\left(\begin{array}{c} \dot{x}_1\\ \dot{x}_2\\ \dot{\bar{x}}_2\\ \dot{x}_3\\ \hline \dot{p}_1\\ \dot{p}_2\\ \dot{\bar{p}}_2\\ \dot{p}_3 \end{array}\right)=\left(\begin{array}{c} k_1x_1-k_2(x_2-x_1)\\ k_2x_2\\ -k_3(x_3-\bar{x}_2)\\ k_3(x_3-\bar{x}_2)\\ \hline v_1\\ v_2\\ \bar{v}_2\\ v_3 \end{array}\right)+\left(\begin{array}{c} 0\\ -1\\ 1\\ 0\\ \hline 0\\ 0\\ 0\\ 0 \end{array}\right)f_2,$$

together with the Legendre transformation $p_1 = m_1v_1, p_2 = m_2v_2, \bar{p}_2 = 0, p_3 = m_3v_3$, the interconnection constraint $v_2 = \bar{v}_2$, as well as the consistency condition $\dot{\bar{p}}_2 = 0$.

(II) Electric Circuits

Consider the electric circuit depicted in Figure 5.3, where R denotes a resistor, L an inductor, and C a capacitor.

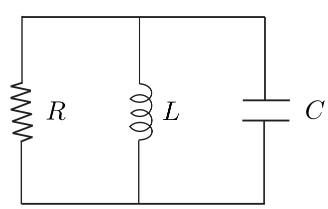

Figure 5.3 – R-L-C circuit

As in Figure 5.4, we decompose the circuit into two disconnected subsystems, "Circuit 1" and "Circuit 2". Let S_1 and S_2 denote external ports resulting from the tear. In order to reconstruct the original circuit in Figure 5.3, the external ports may be connected by equating currents across each.

Circuit 1: The configuration manifold for circuit 1 is denoted by $Q_1 = \mathbb{R}^3$ with local coordinates $q_1 = (q_R, q_L, q_{S_1})$, where q_R, q_L, and q_{S_1} are the charges associated to the

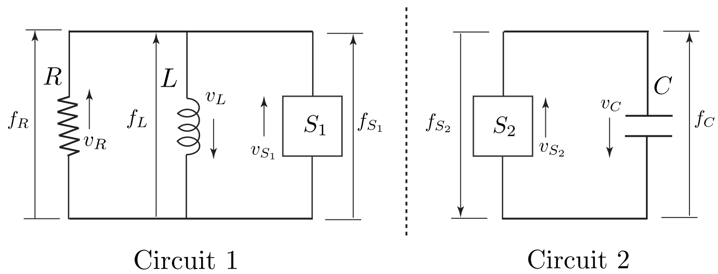

Figure 5.4 – Disconnected circuits

resistor R, inductor L, and port S_1. Kirchhoff's circuit law is enforced by applying a constraint distribution $\Delta_{Q_1} \subset TQ_1$ known as *the KCL distribution* to Circuit 1. The distribution, Δ_Q, is defined for each $q_1 = (q_R, q_L, q_{S_1}) \in Q_1$ by the subspace:

$$\Delta_{Q_1}(q_1) = \{v_1 = (v_R, v_L, v_{S_1}) \in T_{q_1}Q_1 \mid v_R - v_L - v_{S_1} = 0\},$$

where $v_1 = (v_R, v_L, v_{S_1})$ denotes the current vector at each q_1, while the KVL constraint is given by its annihilator $\Delta^\circ_{Q_1}$. Then, we can naturally define the induced Dirac structure D_1 on T^*Q_1 from Δ_{Q_1} as before.

The Lagrangian for Circuit 1, namely, $\mathcal{L}_1$ on TQ_1, is given by

$$\mathcal{L}_1(q_1, v_1) = \frac{1}{2}L_1 v_L^2,$$

which is degenerate. Define the generalized energy $E_{\mathcal{L}_1}$ on $TQ_1 \oplus T^*Q_1$ by $E_{\mathcal{L}_1}(q_1, v_1, p_1) = \langle p_1, v_1\rangle - \mathcal{L}_1(q_1, v_1)$ on $TQ_1 \oplus T^*Q_1$. Circuit 1 also has the external force field due to the resistor $F_{R,1} : TQ_1 \oplus T^*Q_1 \to \mathbb{R}$ as

$$F_R(q_1, v_1, p_1) = (-Rv_R)dq_R,$$

as well as a force on the variable S_2 denoted by $F_{1,\text{port}} : TQ \oplus T^*Q \to T^*Q_1$. Now, we can set up the implicit Lagrangian system $(E_{\mathcal{L}_1}, D_1, X_1, F_{R,1} + F_{1,\text{port}})$ where X_1 is a partial vector field which satisfies

$$(X_1(q_1, v_1, p_1), (\mathbf{d}E_{\mathcal{L}_1} - \tilde{F}_1 - \tilde{F}_{1,\text{port}})(q_1, v_1, p_1)|_{TP_1}) \in D_1(q_1, p_1),$$

where $(q_1, p_1) = \mathbb{F}\mathcal{L}_1(q_1, v_2)$ and $(q_1, v_1) \in \Delta_Q(q_1)$.

Circuit 2: The configuration manifold for Circuit 2 is $Q_2 = \mathbb{R}^2$ with local coordinates $q_2 = (q_{S_2}, q_C)$, where q_{S_2} is the charge through the port S_2 and q_C is the charge stored in the capacitor. The KCL distribution is given for each q_2 by

$$\Delta_2(q_2) = \{v_2 = (v_{S_2}, v_C) \in T_{q_2}Q_2 \mid v_C - v_{S_2} = 0\},$$

and the KVL space is given by the annihilator $\Delta_2^\circ(q_2)$. This gives us the Dirac structure D_2 on T^*Q_2. Set the Lagrangian $\mathcal{L}_2 : TQ_2 \to \mathbb{R}$ for Circuit 2 to be

$$\mathcal{L}_2 = \frac{1}{2C} q_C^2,$$

and so the generalized energy $E_{\mathcal{L}_2}(q_2, v_2, p_2) = \langle p_2, v_2 \rangle - \mathcal{L}_2(q_2, v_2)$. Circuit 2 has the external force field due to the port $F_{2,\mathrm{port}}$. Then, we can formulate the Lagrange-Dirac dynamical system $(E_{\mathcal{L}_2}, D_2, X_2, F_{2,\mathrm{port}})$ where X_2 is a partial vector field which satisfies

$$(X_2(q_2, v_2, p_2), (\mathbf{d}E_{\mathcal{L}_2} - \tilde{F}_{2,\mathrm{port}})(q_2, v_2, p_2)|_{TP_2}) \in D_2(q_2, p_2),$$

on point $(q_2, p_2) = \mathbb{F}\mathcal{L}_2(q_2, v_2)$ when $(q_2, v_2) \in \Delta_{Q_2}(q_2)$.

The Interaction Dirac Structure. Set $Q = Q_1 \times Q_2$ and set

$$\Sigma_Q = \{v = (v_1, v_2) \in TQ \mid v_{S_1} = v_{S_2}\}.$$

By the tangent lift $\Sigma_{\mathrm{int}} = T\pi_Q^{-1}(\Sigma_Q)$, we can define the interaction Dirac structure,

$$D_{\mathrm{int}} = \Sigma_{\mathrm{int}} \oplus \Sigma_{\mathrm{int}}^\circ,$$

which is denoted, in local coordinates, by

$$D_{\mathrm{int}} = \{(\dot{q}_1, \dot{q}_2, \dot{p}_1, \dot{p}_2, \alpha_1, \alpha_2, w_1, w_2) \in TT^*Q \oplus T^*T^*Q \mid \\ \dot{q}_{S_1} = \dot{q}_{S_2} = 0,\ w_1 = 0,\ w_2 = 0,\ (\alpha_1, \alpha_2) \in \mathrm{span}(dq_{S_1} - dq_{S_2})\}.$$

The Interconnected Circuit. Now, we can develop the interconnected Dirac structure

$$D = (D_1 \oplus D_2) \boxtimes D_{\mathrm{int}},$$

as the Dirac structure induced from the constraint space

$$\Delta_Q = (\Delta_1 \times \Delta_2) \cap \Sigma_{\mathrm{int}},$$

which is given, in coordinates $(q_1, q_2) = (q_R, q_L, q_{S_1}, q_{S_2}, q_C)$, by

$$\Delta(q_1, q_2) = \{(v_1, v_2) = (v_R, v_L, v_{S_1}, v_{S_2}, v_C) \mid v_R - v_L - v_C = 0,\ v_{S_1} = v_{S_2},\ v_{S_1} = v_C\}.$$

Set the Lagrangian for the interconnected system as $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ and the external force field $F = F_R$. Set also $E_{\mathcal{L}} = E_{\mathcal{L}_1} + E_{\mathcal{L}_2}$. The interconnected Lagrange-Dirac dynamical system is given by the quadruple $(E_{\mathcal{L}}, D, X, F)$, which satisfies

$$(X(q, v, p), (dE_{\mathcal{L}} - \tilde{F})(q, v, p)|_{TP}) \in D(q, p),$$

for each $(q, p) = \mathbb{F}L(q, v)$.

(III) A Ball Rolling on Rotating Tables

Consider the mechanical system depicted in Figure 5.5, where there are two rotating tables and a ball is rolling on one of the tables without slipping. We assume the system is conservative and the gears are linked by a no-slip constraint. Finally, we assume the external torque is constant. Let I_1 and I_2 be moments of inertia for the tables. We will now decompose the system into distinct three subsystems; (1) a rotating (small) table, (2) a rotating (large) table, and (3) a rolling ball.

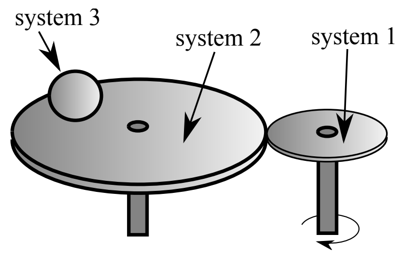

Figure 5.5 – A rolling ball on rotating tables without slipping

System 1. The configuration manifold for System 1 is the circle, $Q_1 = S^1$. The Lagrangian is the rotational kinetic energy of the system given by

$$L_1(s_1, \dot{s}_1) = \frac{I_1}{2}\dot{s}_1^2.$$

We employ the canonical Dirac structure on T^*Q_1 given by:

$$D_1 = \{(\dot{s}_1, \dot{p}_{s_1}, \alpha_{s_1}, w_{s_1}) \mid \dot{s}_1 = w_{s_1}, \dot{p}_{s_1} + \alpha_{s_1} = 0\}.$$

System 2. The configuration manifold for System 2 is also the circle, $Q_2 = S^1$ and the Lagrangian is again the rotational kinetic energy

$$L_2(s_2, \dot{s}_2) = \frac{I_2}{2}\dot{s}_2^2.$$

Again, we have the canonical Dirac structure

$$D_2 = \{(\dot{s}_2, \dot{p}_{s_2}, \alpha_{s_2}, w_{s_2}) \mid \dot{s}_2 = w_{s_2}, \dot{p}_{s_2} + \alpha_{s_2} = 0\}.$$

System 3. System 3 is a rolling sphere of uniform density and radius 1. The sphere moves in space by changing its position and orientation relative to a reference configuration. The configuration manifold is given by the special Euclidean group $Q_3 = \mathrm{SE}(3)$, which we parameterize as (R, u) where $R \in \mathrm{SO}(3), u \in \mathbb{R}^3$. Following [MR99], let β be the set of points of the sphere in the reference configuration. For configuration $(R, u) \in Q_3$, a point $x \in \beta$ is transformed into $\mathbb{R}^3$ by the action $(R, u) \cdot$

$x = (R \cdot x) + u$. The Lagrangian is given by the kinetic energy as

$$L_3(R, u, \dot{R}, \dot{u}) = \int_\beta \frac{\rho}{2} \|\dot{R}x + \dot{u}\|^2 dx,$$

where $\|\dot{R}x + \dot{u}\|^2 = x^T \dot{R}^T \dot{R}x + 2x^T \dot{R}^T \dot{u} + \dot{u}^2$. We use body coordinates such that the center of the sphere in the reference configuration is at the origin so that $\int_\beta x dx = 0$. Substituting these relations, the above Lagrangian is

$$L_3 = \int_\beta \frac{\rho}{2} \left(x^T \dot{R}^T \dot{R}x + \dot{u}^2 \right) dx.$$

Setting $m_3 = \int_\beta \rho dx = \frac{4}{3}\pi\rho$ and noting that $\int_\beta x_i x_j dx = 0$ when $i \neq j$, one finally obtains

$$L_3 = \frac{m_3}{2} \left(\mathrm{tr}(\dot{R}^T \dot{R}) + \dot{u}^2 \right).$$

Since the motion along the z-direction is constrained so that the ball does not leave the plane of table 2, we have the (holonomic) constraint

$$\Delta_{Q_3} = \{(\dot{R}, \dot{u}) \mid \dot{u}_3 = 0\}.$$

This yields the induced Dirac structure

$$\begin{aligned} D_3 = \{ &(\delta R, \delta u, \delta p_R, \delta p_u, \alpha_R, \alpha_u, w_R, w_u) \in TT^*Q_3 \oplus T^*T^*Q_3 \mid \\ &\delta u_3 = 0, \delta u = w_u, \delta R = w_R,\ \delta p_R + \alpha_R = 0,\ \delta p_u + \alpha_u = \lambda dz \text{ for some } \lambda \in \mathbb{R} \}. \end{aligned}$$

Interaction Dirac Structure. Let $Q = Q_1 \times Q_2 \times Q_3$. In order to interconnect the three subsystems, we need to impose the constraints due to the no-slip conditions. By left trivialization we interpret TS^1 as $S^1 \times \mathbb{R}$. The interconnection constraint between System 1 and System 2 is given by

$$\Sigma_{Q,1} = \{(\dot{s}_1, \dot{s}_2, \dot{R}, \dot{u}) \in TQ \mid \dot{s}_1 + \dot{s}_2 = 0\}$$

and with its annihilator

$$\Sigma^{\circ}_{Q,1} = \mathrm{span}(\omega_1)$$

where $\omega_1 = ds_1 - ds_2$. This constraint ensures that the gears rotate (without slipping) at the same speed in opposite directions.

Next, we consider the interconnection constraint between Systems 2 and 3. Note that the velocity of a point located at the bottom of the sphere is given by

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \dot{R}R^T \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + \dot{u}.$$

Note also that a point rotating on the gear of System 2 with the axle taken to be the origin has velocity

$$\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} 0 & -\dot{s}_2 \\ \dot{s}_2 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}.$$

So the no-slip condition between System 2 and 3 is given by

$$\Sigma_{Q,2} = \{(\dot{s}_1, \dot{s}_2, \dot{R}, \dot{u}) \in TQ \mid \ \mathbf{i} \cdot (-\dot{R}R^T \cdot \mathbf{k} + \dot{u}) = -\dot{s}_2 \cdot u_2, \ \mathbf{j} \cdot (-\dot{R}R^T \cdot \mathbf{k} + \dot{w}) = \dot{s}_2 \cdot u_1\},$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the basis on $\mathbb{R}^3$. Set the interconnection constraint distribution

$$\Sigma_Q = \Sigma_{Q,1} \cap \Sigma_{Q,2},$$

together with its annihilator $\Sigma^{\circ}_Q = \Sigma^{\circ}_{Q,1} + \Sigma^{\circ}_{Q,2}$. Then, one can define $\Sigma_{\mathrm{int}} = (T\pi_Q)^{-1}(\Sigma_Q)$ and with its annihilator $\Sigma^{\circ}_{\mathrm{int}}$. The interaction Dirac structure is given by

$$D_{\mathrm{int}} = \Sigma_{\mathrm{int}} \oplus \Sigma^{\circ}_{\mathrm{int}}.$$

The Interconnected Lagrange-Dirac System. The Dirac structure for the interconnected system is given by

$$D = (D_1 \oplus D_2 \oplus D_3) \boxtimes D_{\text{int}}.$$

Note that D is defined by the canonical two-form on T^*Q and the distribution

$$\Delta_Q = (TQ_1 \oplus TQ_2 \oplus \Delta_3) \cap \Sigma_{\text{int}},$$

and also that the annihilator is given by

$$\Delta_Q^\circ = \Delta_{Q_3}^\circ + \Sigma_{\text{int}}^\circ.$$

Letting $L = L_1 + L_2 + L_3$, the dynamics of the interconnected Lagrange-Dirac system is given by (E_L, D, X), which satisfies

$$(X(q, v, p), dE_L(q, v, p)|_{TP}) \in D(q, p),$$

where $X : TT^*Q \oplus T^*T^*Q \to TT^*Q$ is a partial vector field, defined at points $(q, v, p) \in TQ \oplus T^*Q$ such that $(q, p) = \mathbb{F}L(q, v)$ with $(q, v) \in \Delta_Q$, and $E_L : TQ \oplus T^*Q \to \mathbb{R}$ is the generalized energy of L.

5.6 Conclusions and Future Work

We hope to have shown how interaction Dirac structures can be used in a variety of systems. Specifically, we used the tensor product of Dirac structures, $\boxtimes$, to define the notion of *interconnection of Dirac structures* and derive interconnected implicit Lagrangian systems. This process can be repeated n-fold due to the associativity of $\boxtimes$. This enables us to understand large heterogenous systems by decomposing them and keeping track of the relevant interaction Dirac structures. We also clarified how the LAP principle of an interconnected system can be decomposed into variational equations on separate subsystems. Lastly, we demonstrated our theory with some examples. The result is a geometrically intrinsic framework for analyzing large heterogenous systems through tearing and interconnection.

We hope that the framework provided here can be explored further and we are specifically interested in the following areas for future work:

- The use of more general interaction Dirac structures such as those associated with gyrators, motors, magnetic couplings, and so on (in this paper, we mostly studied interaction Dirac structures of the form $\Sigma_{\text{int}} \oplus \Sigma_{\text{int}}^{\circ}$). For examples of these more general interconnections, see, for instance, [WC77, Yos95].

- Reduction and symmetry for interconnected Lagrange-Dirac systems ([YM07b], [YM09]). In particular, the use of the curvature tensor of a principal connection in reduction theory is related to the interaction Dirac structures mentioned in the last bullet [CMR01]. We conjecture that such interaction Dirac structures can be derived from reducing a Lagrangian we would call the *interaction Lagrangian*.

- Applications to complicated systems such as guiding central motion problems, multibody systems, fluid-structure interactions, passivity controlled interconnected systems, etc. (See, for example, [Lit83, Fea87, Yos95, VdS96] and [OvdSME02].)

- The integrability condition for the Dirac tensor product. As to the integrability condition for Dirac structures, see [Dor93] and [DVDS98]. For the link with composition of Dirac structures as well as with *symplectic categories*, see [Wei09].

- Discrete-time versions of interconnection and $\boxtimes$. By discretizing the Hamilton-Pontryagin principle one arrives at a discrete mechanical version of Dirac structures (see [BRM09] and [LO11]). A discrete-time version of $\boxtimes$ could allow for notions of interconnection of variational integrators.

- Finally, introducing noise and stochasticity into mechanical systems is at the forefront of research in geometric mechanics. There are two major camps on this issue. That of [CJL08] allows noise to enter the system by making the the Hamiltonian or Lagrangian a random variable. The other camp allows stochasticity to enter by generalizing ideas from quantization of Feynman's path integral, and extremizing the expected value of a "random path". This was successfully carried out in the case of Navier-Stokes fluids in [NYZ81]. In any case, the symplectic structures involved in these formulations could be replaced with Dirac structures, thus providing an understanding of how to add noise to interconnected systems.

Chapter 6

Conclusion

Throughout this thesis we have repeatedly demonstrated that information about isolated subsystems can often be used when describing the coupled system. That is to say, couplings do not destroy all our knowledge. However they often modify and obscure it. This suggests the diagram in Figure 6.1 as a schematic of our experience.

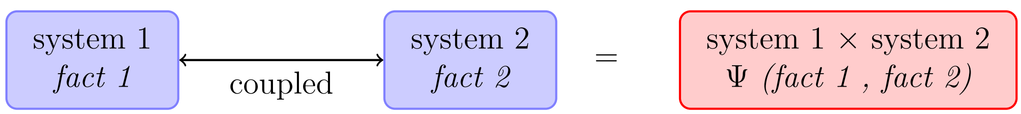

Figure 6.1 – Couplings seen in this thesis

We could claim that Figure 6.1 depicts many of the major findings within this thesis. In Figure 6.2 through 6.4 I provide an example of a finding from each of the three major chapters which fit into the framework of Figure 6.1.

In words, we have described a number of systems which can be viewed as coupled systems whose subsystems are well-understood. We then found that our knowledge of the subsystems was fairly useful in describing properties of the fully coupled system. Additionally, we found that understanding the coupled system in this way resulted in significant payoffs. In chapter 3 we noticed that the horizontal LP equations for an inviscid fluid are nearly equivalent to the equations of an N-body problem. We were able to use this insight to create new particle methods which were prone to deeper analysis than previous methods. In chapter 4 we used our understanding of fluid structure interaction as a system on a Lie algebroid to interpret swimming as a relative limit cycle. Finally, in chapter 5 our use of the tensor product of Dirac structures allowed us to view equations of motion for an interconnected system as forced versions of the equations of motion of the decoupled system.

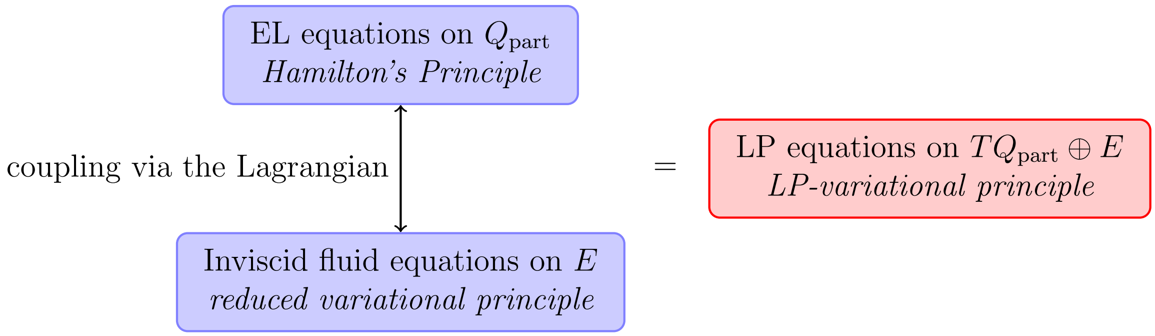

Figure 6.2 – A coupling described in chapter 3

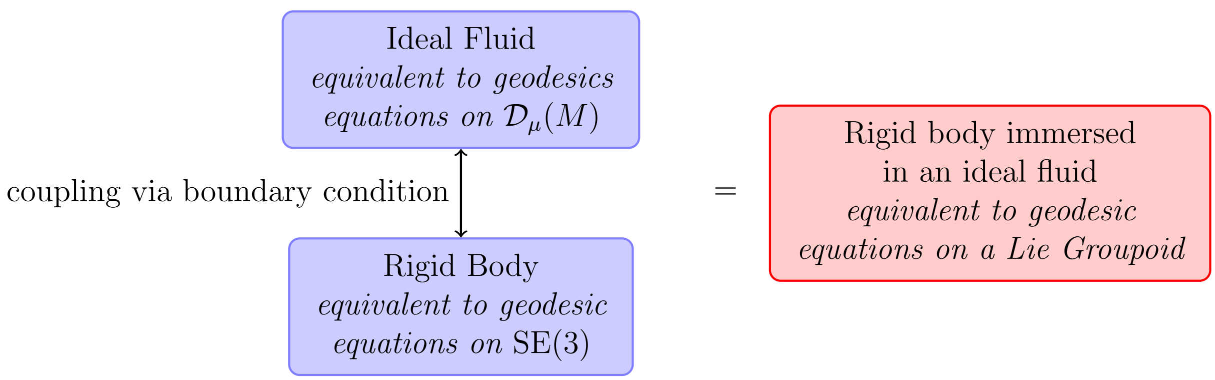

Figure 6.3 – A coupling described in chapter 4

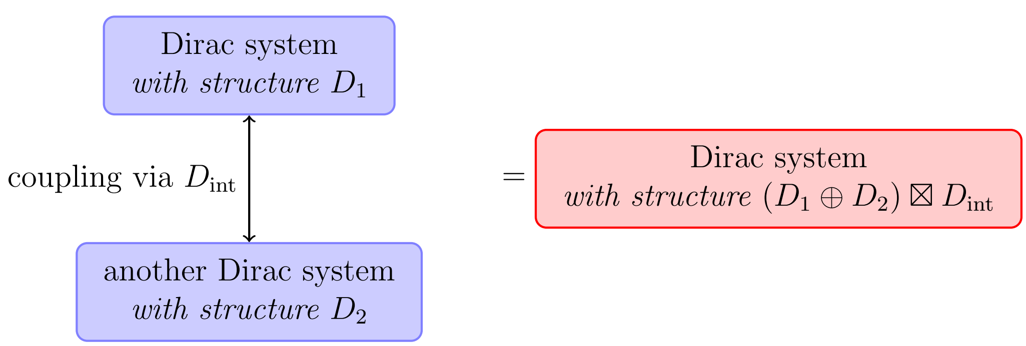

Figure 6.4 – A coupling described in chapter 5

At this point we hope the last paragraph of the introduction to this thesis appears to make a sensible claim. For the convenience of the reader we will repeat some of it here. *Couplings may destroy desirable properties of subsystems, but one should never lose hope. It is not uncommon for the beautiful aspects of the subsystems to be reincarnated as new creatures in the coupled system. Working to find these reincarnations can have significant benefits.*

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