DIRECT PAIR PRODUCTION BY ELECTRONS

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ABSTRACT

The total cross section for direct pair production by electrons in the field of a heavy nucleus is calculated, using the Born approximation for the nuclear Coulomb field. The calculation improves on previous calculations of direct pair production in four respects: (1) all of the first-order Feynman diagrams for the process are included, (2) the exchange effect is included, (3) the effect of screening is treated more accurately using the Thomas-Fermi screening function, (4) the integration of the differential cross section is done with increased accuracy. These improvements in the calculation result in a theoretical accuracy of 10-15% for low Z elements, and of the order of 25% for high Z elements, for which the neglected Coulomb corrections are more important.

The present results are compared with previous theories and with experiments. It is found that there is reasonable agreement with experiment for energies up to about 1 Bev, but that for higher energies the experimental situation is too uncertain to allow a definite conclusion.

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I. INTRODUCTION

Direct pair production is the creation of an electron-positron pair by a charged particle as it passes through the electric field of a nucleus. When the incident particle is an electron, this process is also called the trident process. The purpose of this thesis is to calculate, using the methods of quantum electrodynamics, the total cross section for direct pair production by high energy electrons, taking into account effects which have been neglected in previous theoretical discussions of this process.

The original theoretical work on direct pair production was carried out by a number of authors: notably Nishina, Tomonaga, and Kobayasi (1), who used the Weizsacker-Williams method in which the incident charged particle is replaced by an equivalent photon spectrum, which is then combined with the known cross section for ordinary pair production; Racah (2), who used the positron hole theory; and Bhabha (3), who considered both the Weizsacker-Williams method and the positron hole theory.

Each of these calculations resulted in a series expansion for the total cross section in decreasing powers of $\ln(E_1/m)$, where E_1 is the primary energy and m is the electron mass. The leading term, at high energies, of this expansion was the same in all the calculations, and was, for the unscreened case, $\sigma = z^2(e^2/\hbar c)^2(e^2/mc^2)^2(28/27\pi)\ln^3(E_1/m)$.

Although these various calculations obtained the same leading term, they disagreed as to the remaining terms, and this resulted in large numerical disagreements. In particular, over a wide range of energies the results of Bhabha and Racah differed by over a factor of three. This disagreement was partially resolved by Block, King and Wada (4), who showed that it was due to the neglect by Bhabha of certain terms of lower order in $\ln(E_1/m)$ during the integration of the differential cross section. (Specifically, the neglected terms were of order $\ln^2(E_1/m)$ and $\ln(E_1/m)$ compared to the leading term of order $\ln^3(E_1/m)$.) Block et al. modified the Bhabha cross section to include these terms, and obtained a result which was in much better agreement with that of Racah. However, the various theories still disagreed by factors ranging from about 25% at 100 Bev to about 50% at 100 Mev.

More recently, the problem has been discussed by Murota, Ueda, and Tanaka (5), using the methods of covariant perturbation theory. These authors were primarily interested not in improving the accuracy of the calculation, but rather in understanding, in terms of Feynman diagrams, the approximations made by the previous authors. The most significant result obtained by Murota, et al. was that all of the previous authors had made approximations which were equivalent to neglecting a certain group of Feynman diagrams. Murota, et al. estimated that the leading contribution from these omitted diagrams was of the same order as the terms omitted by Bhabha but included by Racah and Block et al. As a by-product of their work, Murota et al. rederived the Bhabha formula for the cross section.

The major defects in these previous theoretical calculations of the direct pair production cross section may be summarized as follows:

(1) In the integration of the differential cross section numerous approximations have been made. These approximations, besides reducing the accuracy of the final result, have resulted in the introduction of various arbitrary parameters, of order of magnitude one, into the final expression for the total cross section. The uncertainty of the values of these parameters results in an uncertainty in the total cross section of order $\ln^2(E_1/m)$, compared to the leading term of order $\ln^3(E_1/m)$.

(2) For the case of an electron as the incident particle, all of the previous calculations neglect the exchange effect. In terms of Feynman diagrams, this is equivalent to neglecting diagrams a', b', c', and d' shown in Figure 1.

(3) All of the previous calculations also neglect diagrams c and d of Figure 1. Murota *et al.* (5) estimate that this produces an error in the total cross section of order $\ln^2(E_1/m)$. However, we will show later that because of cancellations that occur between various terms, the actual error is of order $\ln(E_1/m)$.

(4) The screening of the Coulomb field of the nucleus by the atomic electrons has been treated only under the approximation of "complete screening" (3), which is only valid for $E_1/mc^2 >> 137/z^{1/3}$ where z is the nuclear charge.

Murota *et al.* (5) concluded on the basis of defects (1) and (3), that for direct pair production by electrons the existing theoretical calculations were unreliable for energies less than 10 Bev.

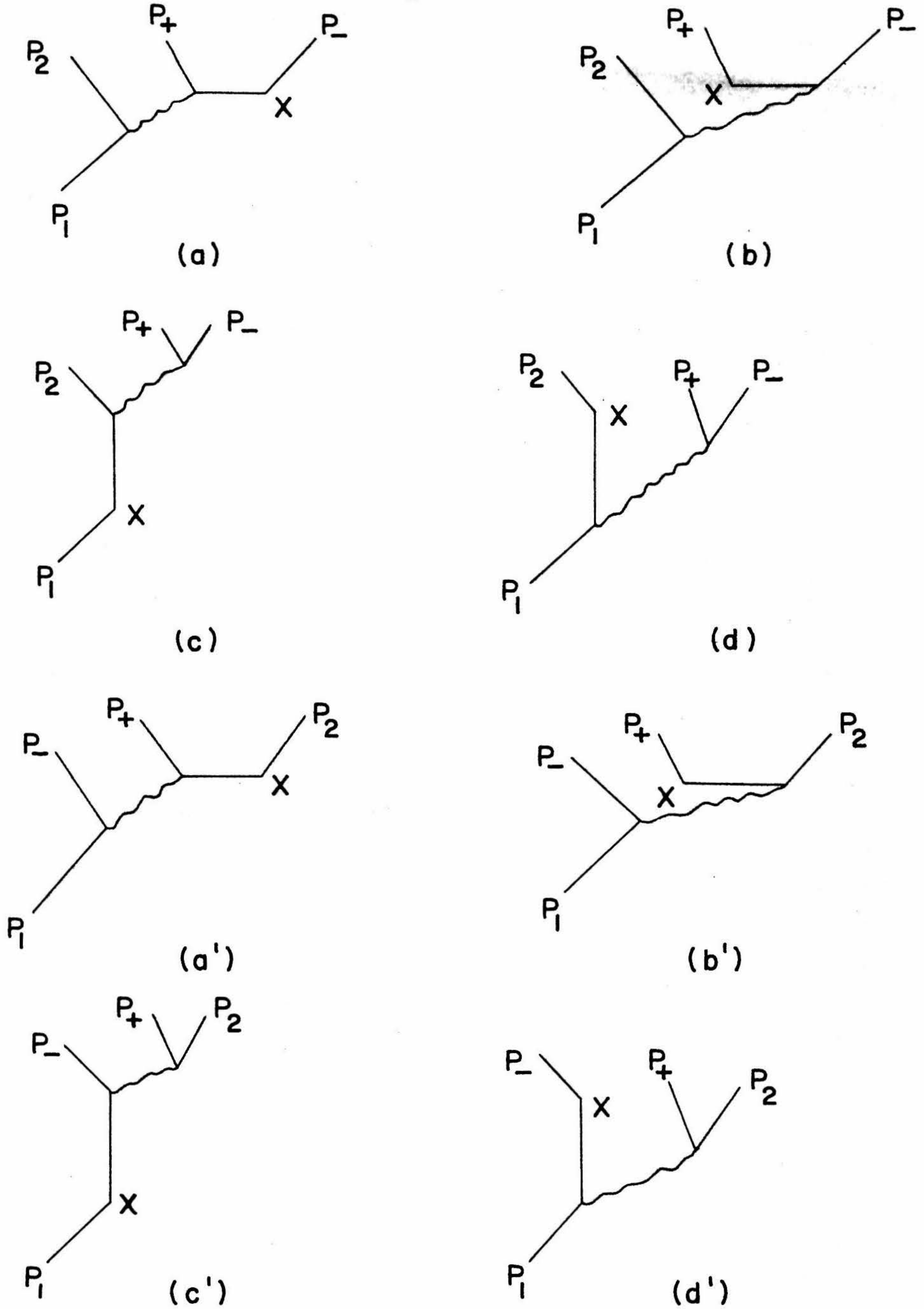

FIG. I FEYNMAN DIAGRAMS FOR DIRECT PAIR PRODUCTION BY AN ELECTRON.(NOTATION EXPLAINED IN SECTION II)

The purpose of the present work is to recalculate the total cross section for direct pair production by electrons, eliminating the defects of the previous calculations. Specifically, we will include the effects of the exchange diagrams and of diagrams c and d, we will improve the treatment of screening by using an analytical representation of the Fermi-Thomas form factor, and we will improve the accuracy with which the differential cross section is integrated by resorting to numerical integration on a high-speed electronic computor.

In this calculation we will, as is customary, neglect the nuclear recoil energy*, and we will treat the nucleus as the point source of a Coulomb field. We will neglect Coulomb corrections, and will treat the Coulomb field in first Born approximation.** We will also neglect radiative corrections. Finally, we will specialize to the case where all particle energies are extremely relativistic, and will neglect terms of order $(m/E)^2$, where E is the energy of any particle.

In Section II we will present an outline of the calculation, and in Section III we will discuss our treatment of screening. Certain terms that cancel in the calculation of the total cross section are discussed in Section IV, and the approximations made in evaluating the traces are discussed in Section V. The magnitude of the various exchange terms is discussed in Section VI, and in Section VII the results obtained for the

*As our results will show, the dominant contribution to the total cross section comes from the region where the nuclear recoil momentum is of the order of 10^{-1} Mev/c or less. This corresponds to recoil energies of the order of 1 ev or less.

**In Section VIII we briefly discuss Coulomb corrections, and estimate that they will be of the order of 10% for the heavy elements.

cross section after integrating over the momenta of the created pair are presented. In Section VIII the final results of the calculation are presented and their significance is discussed. Various details of the calculation are presented in the appendices.

II. OUTLINE OF CALCULATION

Treating the Coulomb field in first Born approximation, the lowest order Feynman diagrams for the direct pair production process with an electron as the incident particle are shown in Figure 1. In these diagrams we denote the incident electron 4-momentum as p_1, the outgoing positron 4-momentum as p_+, and the outgoing electron 4-momenta as p_2 and p_-. Diagrams a', b', c', and d' are the result of the exchange effect.

The matrix element corresponding to these diagrams will be of the form*

$$\mathcal{M} = \frac{(2\pi)^4 e^3}{V^2} \left(\frac{m^4}{E_1 E_2 E_+ E_-}\right)^{1/2} \int d^4q\, \delta^4(p_1+q-p_2-p_+-p_-) \cdot$$

$$\cdot\, a_\eta(q)[\mathcal{M}_a^\eta + \mathcal{M}_b^\eta + \mathcal{M}_c^\eta + \mathcal{M}_d^\eta$$

$$- (\mathcal{M}_a'^\eta + \mathcal{M}_b'^\eta + \mathcal{M}_c'^\eta + \mathcal{M}_d'^\eta)] \qquad (2\text{-}1)$$

where

$$\mathcal{M}_a^\eta = \frac{\overline{u}(p_2)\gamma^\mu u(p_1)\overline{u}(p_-)\gamma^\eta[i\gamma\cdot(p_- - q) - m]\gamma_\mu v(p_+)}{(p_1-p_2)^2[(p_- - q)^2 + m^2]} \qquad (2\text{-}2a)$$

$$\mathcal{M}_b^\eta = \frac{\overline{u}(p_2)\gamma^\mu u(p_1)\overline{u}(p_-)\gamma_\mu[i\gamma\cdot(q - p_+) - m]\gamma^\eta v(p_+)}{(p_1-p_2)^2[(p_+ - q)^2 + m^2]} \qquad (2\text{-}2b)$$

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* We use a metric with $p\cdot q = \vec{p}\cdot\vec{q} - p_0 q_0$, so that $p_1^2 = p_2^2 = p_+^2 = p_-^2 = -m^2$. We are using units such that $\hbar = c = 1$ and $e^2/4\pi = 1/137$. Our Dirac matrices are such that the Dirac equation for electrons is, in momentum space, $(i\gamma\cdot p + m)u(p) = 0$, and our plane-wave spinor normalization is $\overline{u}(p)u(p) = 1$. V is the normalization volume, and q is the 4-momentum contributed by the Coulomb field. (Note that the recoil of the nucleus is -q.)

$$\mathcal{M}_c^\eta = \frac{\bar{u}(p_-)\gamma^\mu v(p_+)\bar{u}(p_2)\gamma_\mu[i\gamma\cdot(p_1+q)-m]\gamma^\eta u(p_1)}{(p_++p_-)^2[(p_1+q)^2+m^2]} \tag{2-2c}$$

$$\mathcal{M}_d^\eta = \frac{\bar{u}(p_-)\gamma^\mu v(p_+)\bar{u}(p_2)\gamma^\eta[i\gamma\cdot(p_2-q)-m]\gamma_\mu u(p_1)}{(p_++p_-)[(p_2-q)^2+m^2]} \tag{2-2d}$$

and the exchange matrix elements $\mathcal{M}'^\eta_a$, etc. are obtained from these by the interchange $p_2 \longleftrightarrow p_-$.

The quantity $a_\eta(q)$ is given in terms of the external field $A_\eta(x)$ by

$$a_\eta(q) = \frac{1}{(2\pi)^4} \int A_\eta(x) e^{-iq\cdot x}\, d^4x \tag{2-3}$$

For a screened Coulomb field, we will have

$$A_\eta(x) = \delta_{\eta o} \frac{Ze}{4\pi|\vec{r}|} \quad G(|\vec{r}|) \tag{2-4}$$

where $G(|\vec{r}|)$ is the atomic screening function and Z is the nuclear charge. This gives

$$a_\eta(q) = \delta(q_o)\delta_{\eta o} \frac{Ze}{(2\pi)^3} \quad \frac{F(\vec{q}^{\,2})}{\vec{q}^{\,2}} \tag{2-5}$$

where the atomic form factor $F(\vec{q}^{\,2})$ is given by

$$\frac{F(\vec{q}^{\,2})}{\vec{q}^{\,2}} = \frac{1}{4\pi} \int \frac{G(|\vec{r}|)}{|\vec{r}|} e^{-i\vec{q}\cdot\vec{r}}\, d^3r \tag{2-6}$$

We will discuss our choice for $G(|\vec{r}|)$ in Section III.

The matrix element will now be of the form

$$\mathcal{M} \equiv \mathcal{N} \int d^4q\,\delta(q_o)\,\delta^4(p_1+q-p_2-p_+-p_-)$$

$$= \mathcal{N}\,\delta(E_1-E_2-E_+-E_-) \qquad (2\text{-}7)$$

where

$$\mathcal{N} = \frac{2\pi Ze^4}{V^2}\left(\frac{m^4}{E_1E_2E_+E_-}\right)^{1/2}\frac{F(\vec{q}^{\,2})}{\vec{q}^{\,2}}\,\delta_{\eta o}(\mathcal{M}^{\eta}-\mathcal{M}'^{\eta}) \qquad (2\text{-}8)$$

where we denote

$$\mathcal{M}^{\eta} \equiv \mathcal{M}_a^{\eta}+\mathcal{M}_b^{\eta}+\mathcal{M}_c^{\eta}+\mathcal{M}_d^{\eta}$$

$$\mathcal{M}'^{\eta} \equiv \mathcal{M}_a'^{\eta}+\mathcal{M}_b'^{\eta}+\mathcal{M}_c'^{\eta}+\mathcal{M}_d'^{\eta} \qquad (2\text{-}9)$$

The differential cross section is now

$$d\sigma = \frac{V}{2\pi\beta_1}\sum \delta(E_1-E_2-E_+-E_-)\,|\mathcal{N}|^2\rho_f \qquad (2\text{-}10)$$

where the summation sign involves an average over initial spins and a sum over final spins, β_1 is the velocity of the initial electron, and the density of final states is

$$\rho_f = \frac{V^3 d^3p_2\,d^3p_+\,d^3p_-}{(2\pi)^9} \qquad (2\text{-}11)$$

To facilitate later stages of the calculation, it will be convenient to re-introduce the d^4q integration, so that we have

$$d\sigma = \frac{V}{2\pi\beta_1}\sum\int dq_o\,\delta(q_o)\int d^3q\,\delta^4(p_1+q-p_2-p_+-p_-)\,|\mathcal{N}|^2\rho_f \qquad (2\text{-}12)$$

We now insert the expressions for $\mathcal{N}$ and ρ_f and integrate over all final states. We must multiply by a factor of 1/2 for the average over initial spins. Since we will integrate $d^3p_2\,d^3p_-$ over the entire available phase space, we must multiply by another factor of 1/2 to correct for the double counting of identical final states. Doing this we obtain for the total cross section*

$$\sigma = \frac{Z^2\alpha^2 r_o^2 m^6}{2\pi^4 E_1\beta_1}\int d^3q \int \frac{d^3p_2}{E_2}\int \frac{d^3p_+}{E_+}\int \frac{d^3p_-}{E_-}\int dq_o \cdot$$

$$\cdot \left[\frac{F(\vec{q}^{\,2})}{\vec{q}^{\,2}}\right]^2 \delta^4(p_1+q-p_2-p_+-p_-)\delta(q_o)\delta_{\eta o}\delta_{\lambda o} \cdot$$

$$\cdot \sum_{\text{spins}} (\mathcal{M}^{\eta *}\mathcal{M}^{\lambda} - \mathcal{M}^{\eta *}\mathcal{M}'^{\lambda}) \qquad (2\text{-}13)$$

where $\alpha = e^2/4\pi = 1/137.04$, and $r_o = e^2/4\pi m = 2.818 \times 10^{-13}$ cm. We note that the term $\mathcal{M}^{\eta *}\mathcal{M}^{\lambda}$ in equation 2-13 is the non-exchange contribution, and the term $\mathcal{M}^{\eta *}\mathcal{M}'^{\lambda}$ is the exchange contribution.

At this point we are left with the task of evaluating the spin summation in equation 2-13, and then carrying out the integration over the differential cross section. The integration is the most difficult part of the calculation, since it involves seven non-trivial integrations. The procedure we follow is to carry out two of the integrations analytically, using a covariant method of integration, and then do the remaining five integrations numerically. In the remainder of this section we outline

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*In this expression we have used the fact that $\mathcal{M}'^{\eta *}\mathcal{M}'^{\lambda}$ will contribute the same amount to the total cross section as $\mathcal{M}^{\eta *}\mathcal{M}^{\lambda}$, and that $\mathcal{M}'^{\eta *}\mathcal{M}^{\lambda}$ will contribute the same amount as $\mathcal{M}^{\eta *}\mathcal{M}'^{\lambda}$.

the steps in this procedure, the details of which are presented in the following sections.

To simplify the discussion, we introduce the notation*

$$I_{ij}^{\eta\lambda} = m^6 \sum_{\text{spins}} \mathcal{M}_i^{\eta *} \mathcal{M}_j^{\lambda}$$

$$I_{ij}^{'\eta\lambda} = m^6 \sum_{\text{spins}} \mathcal{M}_i^{\eta *} \mathcal{M}_j^{'\lambda} \tag{2-14}$$

and

$$K_{ij}^{\eta\lambda} = \int \frac{d^3p_+}{E_+} \int \frac{d^3p_-}{E_-} \delta^4(p_1+q-p_2-p_+-p_-) I_{ij}^{\eta\lambda}$$

$$K_{ij}^{'\eta\lambda} = \int \frac{d^3p_+}{E_+} \int \frac{d^3p_-}{E_-} \delta^4(p_1+q-p_2-p_+-p_-) I_{ij}^{'\eta\lambda} \tag{2-15}$$

The total cross section now becomes

$$\sigma = \frac{Z^2\alpha^2 r_o^2}{2\pi^4 E_1 \beta_1} \int d^3q \int \frac{d^3p_2}{E_2} \left[\frac{F(\vec{q}^{\,2})}{\vec{q}^{\,2}} \right]^2 \cdot$$

$$\cdot \int dq_o \delta(q_o) \delta_{\eta o} \delta_{\lambda o} \left(\sum_{ij} K_{ij}^{\eta\lambda} - \sum_{ij} K_{ij}^{'\eta\lambda} \right) \tag{2-16}$$

The traces, i.e. the $I_{ij}^{\eta\lambda}$ and $I_{ij}^{'\eta\lambda}$, are evaluated in Appendix A. In that appendix we also discuss the various symmetries that exist between the $I_{ij}^{\eta\lambda}$ and $I_{ij}^{'\eta\lambda}$. One of the results of these symmetries is that certain

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*In these expressions the subscripts ij denote the diagram. The factor m^6 makes the $I_{ij}^{\eta\lambda}$, $I_{ij}^{'\eta\lambda}$, $K_{ij}^{\eta\lambda}$, and $K_{ij}^{'\eta\lambda}$ dimensionless.

terms in the total cross section cancel out. This cancellation is discussed in Section IV.

In evaluating the traces we make certain approximations which involve dropping terms of order m^2/E^2 compared to unity, where E is one of the energies E_1, E_2, E_+, E_-. These approximations are discussed in Section V.

We make certain other approximations with respect to the exchange terms, i.e. the $I'^{\eta\lambda}_{ij}$. These approximations are discussed in Section VI.

The next step in the calculation is to evaluate the $K^{\eta\lambda}_{ij}$ and $K'^{\eta\lambda}_{ij}$. These integrals are evaluated analytically by a method which takes advantage of their covariant form.* The integrals are transformed to a special Lorentz system in which they become relatively easy to evaluate, and then the resultant expressions are transformed back into the laboratory coordinate system. This method of evaluating the integrals is illustrated in Appendix B and the results obtained for the various integrals are presented in Appendix C. The final results obtained for the $K^{\eta\lambda}_{ij}$ and $K'^{\eta\lambda}_{ij}$ are presented in Section VII.

Having evaluated the $d^3p_+ d^3p_-$ integrals, i.e. the $K^{\eta\lambda}_{ij}$ and $K'^{\eta\lambda}_{ij}$, we make use of the $\delta_{\eta o}\delta_{\lambda o}\int \delta(q_o)dq_o$ and then carry out the remaining integrals numerically.** To do this, we write

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*It is for this reason that the $I^{\eta\lambda}_{ij}$ and $I'^{\eta\lambda}_{ij}$ were kept in a covariant form.

**We note that the $d^3p_+ d^3p_-$ integrals are the easiest to evaluate analytically. Due to the structure of the denominators in the matrix elements, any other combination, such as $d^3p_2 d^3p_-$, would be much more difficult to evaluate analytically.

$$\int d^3q \int \frac{d^3p_2}{E_2} = 2\pi \int q^2 dq \int \beta_2 E_2 dE_2 \int d(\cos\theta_q) \int d(\cos\theta_2) \int d\phi_2 \tag{2-17}$$

where $\hat{\vec{p}}_1 \cdot \hat{\vec{q}} = \cos\theta_q$ and $\hat{\vec{p}}_1 \cdot \hat{\vec{p}}_2 = \cos\theta_2$ and $|\vec{q}| = q$. The limits on these integrals are determined from the kinematics of the process. This is discussed in Appendix D.

To facilitate the numerical work, we write the cross section as

$$\sigma = Z^2 \alpha^2 r_o^2 \cdot \frac{m^2}{\pi^2 E_1 \beta_1} \int q^2 dq \left[\frac{F(q^2)}{q^2} \right]^2 \sigma(q) \tag{2-18}$$

where

$$\sigma(q) = \frac{1}{\pi m^2} \int \beta_2 E_2 dE_2 \int d(\cos\theta_q) \int d(\cos\theta_2) \int d\phi_2 \cdot$$

$$\cdot \left(\sum_{ij} K_{ij}^{oo} - \sum_{ij} K_{ij}^{\prime oo} \right) \tag{2-19}$$

We leave the integral over q until last, because this is the only integral that involves the screening form factor, and therefore the only one that is a function of Z. By leaving the q integral until last, we can obtain the screened cross section for different values of Z without repeating all of the previous integrations.

In Appendix E we discuss certain details of the numerical methods used, and we give an estimate of the numerical errors involved. The results obtained for $\sigma(q)$ are tabulated in Appendix F for various values of E_1 and q.* In that Appendix we also tabulate various combinations

*The tabulation of $\sigma(q)$ may be useful at a later date for calculating the screened cross section at values of Z not included here, or with a different screening function.

of terms, such as non-exchange and exchange, that enter into $\sigma(q)$. The final results obtained for σ, as a function of E_1 and Z, are tabulated in Appendix G.

In Section VIII we summarize the results of the calculation, and compare these results with previous theories and with experiments. We also discuss the overall accuracy of the present calculation.

III. TREATMENT OF SCREENING

For an accurate calculation of the direct pair production cross section, one must include the screening effect of the atomic electrons on the nuclear Coulomb field. To gain a qualitative idea of the importance of screening, we note that, as shown in Appendix D, the minimum nuclear recoil momentum is $q_{min} = 4m^2/E_1$. This corresponds to a maximum impact parameter $r_{max} \sim 1/q_{min} = E_1/4m^2$. The radius of the atomic electron cloud is, using the Thomas-Fermi model, of the order of $d = 1/me^2Z^{1/3}$. We would now expect that screening will have a significant effect for $r_{max} \gtrsim d$. This corresponds to energies such that $E_1 \gtrsim 137m/Z^{1/3}$. This qualitative argument shows that screening should have an important effect for the energies, 100 Mev to 10 Bev, that we are primarily interested in.

To include the effect of screening in an accurate manner, one must replace the ordinary Coulomb field by a screened Coulomb field, c.f. equation 2-4, where the atomic screening function is

$$G(r) = 1 - \frac{4\pi}{Z}\int_0^r \rho(r')r'^2dr' \qquad (3\text{-}1)$$

where $\rho(r')$ is the electron density distribution. We are now faced with the problem of choosing a model for the electron density distribution of the atom.

The most accurate atomic models available are the Hartree-Fock model, which includes correlation and exchange effects, and the Hartree model, which includes correlation effects, but no exchange effects. For the purposes of this calculation, however, both of these models suffer from two disadvantages: results for the atomic screening function are available only for a few values of Z, and these results are available only in tabular form, which make them awkward to use in a numerical calculation on a computor. Because of these disadvantages, we will use instead in this calculation the Thomas-Fermi atomic model. This model neglects both correlation and exchange effects, but gives a reasonably accurate representation of the atomic electron cloud, omitting, however, the details of the shell structure. For our present purposes, the Thomas-Fermi model has the advantage that the atomic screening function is known for all values of Z, and that several highly accurate analytical representations of Thomas-Fermi results are available.

In this calculation we will use the Moliere representation of the Thomas-Fermi screening function. This is (6)*

$$G(r) = \sum_{i=1}^{3} a_i e^{-\frac{b_i r}{a}} \tag{3-2}$$

where

$$a = \frac{121}{mZ^{1/3}} \tag{3-3}$$

and the coefficients are

$$\begin{aligned} a_1 &= 0.1 \; ; \quad a_2 = 0.55 \; ; \quad a_3 = 0.35 \\ b_1 &= 6 \; ; \quad b_2 = 1.2 \; ; \quad b_3 = 0.3 \end{aligned} \tag{3-4}$$

* We note that Royental (7) has developed a very similar representation of the Thomas-Fermi screening function. The difference between the Moliere and Royental representations would produce a difference in our final result of the order of 0.1%.

Using this expression for $G(r)$, we have for $F(q^2)$, cf. equation 2-6, the result

$$\frac{F(q^2)}{q^2} = \sum_{i=1}^{3} \frac{a_i}{q^2 + \left(\frac{b_i}{a}\right)^2} \tag{3-5}$$

From the form of equation 3-2 we see that the most important contributions to the cross section will come from the region $r/a \sim 1$. By comparing the Moliere representation with the exact value of the Thomas-Fermi screening function as tabulated by Kobayashi, et al. (8), we find that the error in the Moliere representation is less than 0.2% in the region $0 \leq r/a \leq 6$.

The most important error arising from our choice of a screening function will be that due to the inherent inaccuracy of the Thomas-Fermi model. To obtain an estimate of this error, we have used the graphical data of Nelms and Oppenheim (9) to compare the Thomas-Fermi form factor to the Hartree form factor. For the value of recoil momentum $q = mZ^{1/3}/137$, which is the region of maximum contribution to the total cross section, we found the results shown in the following table:

Z	Ratio of Form Factors Thomas-Fermi/Hartree
6 (C)	0.95
26 (Fe)	0.97
80 (Hg)	1.05

These results indicate that, relative to the Hartree form factor, the Thomas-Fermi form factor introduces an error of the order of 5%. Since the cross section involves the square of the form factor, this will introduce a 10% error in the final result for the total cross section.

It should also be noted that the Hartree and Hartree-Fock form factors are expected to differ by a few percent (9).

IV. TERMS THAT CANCEL IN TOTAL CROSS SECTION

In Appendix A certain symmetry relations are obtained connecting various of the $I_{ij}^{\eta\lambda}$ and $I_{ij}^{'\eta\lambda}$. Among these is the relation that under the interchange $p_+ \longleftrightarrow p_-$,

$$I_{ac}^{\eta\lambda} \longleftrightarrow - I_{bc}^{\eta\lambda} \tag{4-1}$$

and

$$I_{ad}^{\eta\lambda} \longleftrightarrow - I_{bd}^{\eta\lambda} \tag{4-2}$$

This symmetry has a particular significance for the calculation of the total cross section. To see this, we note that $K_{ij}^{\eta\lambda}$ is defined by

$$K_{ij}^{\eta\lambda} = \int \frac{d^3p_+}{E_+} \int \frac{d^3p_-}{E_-} \; \delta^4(p_1+q-p_2-p_+-p_-) I_{ij}^{\eta\lambda} \tag{4-3}$$

From this definition and the symmetry relations 4-1 and 4-2 it follows immediately that

$$K_{ac}^{\eta\lambda} = - K_{bc}^{\eta\lambda} \tag{4-4}$$

and

$$K_{ad}^{\eta\lambda} = - K_{bd}^{\eta\lambda} \tag{4-5}$$

This result has the consequence that the non-exchange interference terms between diagrams a or b and c or d cancel each other out and do not contribute to the final result for the total cross section.

Besides simplifying the calculation, this cancellation is of interest in conjunction with estimates made by Murota, et al. (5), of the error involved in completely neglecting diagrams c and d. They showed that the leading term in the contribution to the total cross section from the square of diagrams a and b was, apart from constant factors, of order $\ln^3(E_1/m)$. They also showed that the leading term coming from the square of diagrams c and d was of order $\ln(E_1/m)$. They then inferred from these results that the leading contribution coming from the non-exchange interference terms between diagrams a or b and c or d should be of the order $\ln^2(E_1/m)$.

The results we obtained in equations 4-4 and 4-5 show that, while these interference terms may make a contribution to the differential cross section of this relative order of magnitude, they in fact do not contribute to the total cross section. This means that, as long as one does not consider the exchange effect, the error involved in completely neglecting diagrams c and d is of order $\ln(E_1/m)$, and not of order $\ln^2(E_1/m)$ as suggested by Murota, et al.*

*The present calculation will still contain correction terms to the previous calculations of order $\ln^2(E_1/m)$. These will come from the increased accuracy of integration and elimination of arbitrary parameters in the final result.

V. APPROXIMATION MADE IN EVALUATING TRACES

The calculation of the traces can be appreciably simplified because of the limitation to extremely relativistic energies. This simplification is possible because of the fact that at very high energies the dominant contribution to the total cross section comes from small angles about the forward direction. This "small-angle effect" is a familiar feature of all electromagnetic processes in the ultra-relativistic energy region, and for direct pair production it was pointed out in the original theoretical work (3,10). Before discussing the approximation that we make based on the small-angle effect, we will digress briefly to give a qualitative explanation of its occurrence.

The concentration of the total cross section at very small forward angles is due to the form of the various denominators in the matrix elements, i.e. the Feynman propagators. These propagators have poles at angles slightly beyond the forward direction ($\cos\theta > 1$), and in the limit of infinite energy these poles approach the physical region.

To see explicitly what we are talking about, consider one of the denominator factors, for example $(p_+ + p_-)^2$. We have

$$(p_+ + p_-)^2 = 2(p_+ \cdot p_- - m^2)$$

$$= -2[m^2 + E_+E_-(1 - \beta_+\beta_- \cos\theta_{+-})] \qquad (5\text{-}1)$$

For $m/E_+ \ll 1$, $m/E_- \ll 1$, this becomes

$$(p_+ + p_-)^2 = -\frac{m^2(E_+ + E_-)^2}{E_+E_-} - 2E_+E_-(1 - \cos\theta_{+-}) \qquad (5\text{-}2)$$

We see from this expression that the pole occurs for

$$\cos \theta_{+-} = 1 + \frac{m^2(E_+ + E_-)^2}{2E_+^2 E_-^2} \tag{5-3}$$

We also see from equation 5-2 that a term containing this denominator factor will be large only in the region $\theta_{+-} \lesssim (m^2/E_+E_-)^{1/2}$, or what is the same thing, in the region $p_+ \cdot p_- \lesssim m^2$.

Similar considerations hold for the various other denominator factors. We see from this qualitative discussion that the cross section will be large only for small angles in the forward direction, as has been pointed out previously (3, 5, 10). We also see that this has the consequence that the dominant contribution occurs in the region where all of the four-vector dot products are of the order of m^2, i.e. $p_1 \cdot p_2 \sim m^2$, $p_1 \cdot p_+ \sim m^2$, etc.

Having discussed this small-angle effect, let us now consider the form of the various terms arising in the evaluation of the spin summation, i.e. the traces. As can be seen from the expressions in Appendix A, a typical term is of the form

$$\mathrm{Tr}[\gamma^\mu \gamma \cdot p_a \gamma^\nu \gamma \cdot p_b] \cdot \mathrm{Tr}[\gamma_\mu \gamma \cdot p_c \gamma^\eta \gamma \cdot p_d \gamma^\lambda \gamma \cdot p_e \gamma_\nu \gamma \cdot p_f]$$

where p_a, p_b, etc. are various four-momenta, the μ, ν indices are to be summed over, and the γ^η, γ^λ arise from the interaction with the external field. When this expression is evaluated, and the η, λ indices are set equal to zero, corresponding to a static external field, cf. equation 2-16, the result will contain terms of the two types:

Type 1 $\quad p_a \cdot p_b \, p_c \cdot p_d \, E_e \, E_f$

Type 2 $\quad p_a \cdot p_b \, p_c \cdot p_d \, p_e \cdot p_f$

The terms of type 1 arise from γ^η and γ^λ contracting on two of the four-momentum vectors in the trace, and the terms of type 2 arise from γ^η and γ^λ contracting on each other.

We now see that, because of the concentration of the cross section at small angles, terms of type 2 will be of order m^2/E^2 compared to terms of type 1. In the evaluation of the traces in Appendix A, we will neglect the terms of type 2. This will appreciably simplify the calculation of the traces.

VI. MAGNITUDE OF EXCHANGE TERMS

Murota, et al. (5), discussed the magnitude of the exchange terms, and concluded that they were of order $(m/E_1)^2 \ln^2(E_1/m)$ relative to the leading non-exchange terms and therefore negligible for $m/E_1 << 1$. However, they only considered the exchange terms arising from diagrams a and b in Figure 1. In actual fact, the exchange terms arising from diagrams c and d, or from the interference terms between diagrams a or b and c or d, are much larger than the above estimate, and contribute an appreciable amount to the final result.

In order to understand qualitatively the difference in magnitude of the various exchange terms, it is useful to view the exchange effect as a repulsion in the phase space of the two final state electrons. One can say that, if in the absence of exchange the two electrons tend to come off together in phase space, then the exchange effect will be large. However, if in the absence of exchange the two electrons tend to come off far apart in phase space, then the exchange effect will be small.

In our present problem the bulk of the total cross section comes from small forward angles, cf. section V. Therefore we can suppose that the electrons come off at the same angle, and the magnitude of the exchange effect will then depend only on their energies. We can then say:

(1) If in the absence of exchange the two electron energies tend to be equal, then the exchange effect will be large;

(2) If in the absence of exchange the two electron energies tend to be very different, then the exchange effect will be small.

We can now use this approach to understand the difference in magnitude of the various exchange terms in the present problem.

Consider first diagram b. The two denominator factors of this diagram are

$$(p_1 - p_2)^2 \qquad \text{and} \qquad (p_1 - p_2 - p_-)^2 + m^2$$

In the forward direction we find, using $E_1 >> m$, etc.,

$$(p_1 - p_2)^2 = \frac{m^2(E_1 - E_2)^2}{E_1 E_2} \qquad (6\text{-}1)$$

and

$$(p_1 - p_2 - p_-)^2 + m^2 = \frac{m^2(E_1 - E_2)}{E_1 E_2 E_-} \left[E_-(E_1 - E_2) + E_1 E_2 - E_-^2 \right] \qquad (6\text{-}2)$$

We see that the product of these two factors will be a minimum, and the contribution to the cross section a maximum, when $E_2 \sim E_1$ and $E_- << E_2$. Thus, with respect to diagram b, the energies of the two final state electrons tend to be very different, and the exchange effect will be small. A similar result can be obtained for diagram a.

Now let us consider diagram c. The two denominator factors are

$$(p_+ + p_-)^2 \qquad \text{and} \qquad (p_2 + p_+ + p_-)^2 + m^2$$

In the forward direction we find, using $E_1 >> m$, etc.

$$(p_+ + p_-)^2 = -\frac{m^2(E_+ + E_-)^2}{E_+ E_-} \qquad (6\text{-}3)$$

$$(p_2 + p_+ + p_-)^2 = - \frac{m^2}{E_2 E_+ E_-} [E_2(E_+ + E_-)^2 + E_-(E_2 + E_+)^2 + E_+(E_2 + E_-)^2 - 4E_2E_+E_-] \qquad (6\text{-}4)$$

We see that the product of these two factors will be minimum, and the contribution to the cross section maximum, when $E_+ \sim E_- \sim E_2 \sim E_1/3$. Thus, with respect to diagram c, the two final state electrons tend to have the same energy, and the exchange effect will therefore be large. Consideration of diagram d leads to a similar result.

The above arguments enable us to understand qualitatively the difference in magnitude of the various exchange terms. We can understand why, as pointed out by Murota, et al., the exchange terms arising from diagrams a and b are very small. On the other hand, the above results indicate that the exchange terms arising from diagrams c or d should be much larger.

The arguments used above do not give a quantitative estimate of the magnitude of the various exchange terms. However they suggest that for those cases where the final state electrons tend to have the same energy, the exchange terms should have roughly the same magnitude as the non-exchange terms. With respect to the terms arising from diagrams c and d, this implies that the exchange terms should be of order $\ln(E_1/m)$, as are the non-exchange terms (5). With respect to the interference terms between diagrams a or b and c or d, the situation is much less clear, both because these terms contain denominator factors which are a mixture of the two types discussed above, and because the non-exchange terms, due to certain symmetries, cancel each other out. However a plausible estimate of the magnitude of the exchange interference terms is that they will be roughly of the order of magnitude that the non-exchange interference terms would have been if they had not cancelled, namely $\ln^2(E_1/m)$.

The above discussion indicates that the total exchange contribution to the cross section will be much larger than the estimate of Murota, et al., which was $(m/E_1)^2 \ln^2(E_1/m)$, and which was obtained on the basis of considering only diagrams a and b. The discussion indicates that the exchange contribution will in fact be dominated by the exchange terms involving diagrams c and d and by the exchange interference terms between diagrams a or b and c or d. These dominant exchange terms are estimated to be of order $\ln(E_1/m)$ or $\ln^2(E_1/m)$.

The qualitative arguments of this section are confirmed by the detailed numerical results of the present calculation. From the results presented in Appendices F and G and summarized in Section VIII, we can see that the leading exchange terms are much larger than the Murota estimate, and are in fact of order $\ln(E_1/m)$.

We close this section by noting that in the remainder of the calculation we will neglect the exchange terms arising solely from diagrams a and b, i.e. the terms $K'^{\eta\lambda}_{aa}$, $K'^{\eta\lambda}_{bb}$, $K'^{\eta\lambda}_{ab}$, $K'^{\eta\lambda}_{ba}$, since, as shown by Murota, et al. and by the arguments in this section, they are certainly negligible, being only of order $(m/E_1)^2 \ln^2(E_1/m)$.

VII. RESULTS OF INTEGRATION OVER $d^3p_+d^3p_-$

In this section we present the results obtained for the $K_{ij}^{\eta\lambda}$ and $K_{ij}^{'\eta\lambda}$ after integrating over $d^3p_+d^3p_-$. This calculation was carried out starting with the expressions obtained for the $I_{ij}^{\eta\lambda}$ and $I_{ij}^{'\eta\lambda}$ in Appendix A, and using the results for the various integrals given in Appendix C.

In the results presented in this section we have, after the integration, set $\eta = \lambda = 0$ and $q_o = 0$, since these are the only terms contributing to the final result. (See equation 2-16.) We use the notation $k = p_1 + q - p_2$ and $E_k = E_1 - E_2$. The coefficients A_i, B_i, D_i, D_i', F_i, G_i, V_i, W_i which appear in the results arise as a result of the covariant integration process, and are presented in Appendix C.

We present the results for the non-exchange terms first, and then the results for the exchange terms.

A. Non-Exchange Terms

$$K_{aa}^{oo} = \frac{m^2\pi}{2(p_1\cdot p_2+m^2)^2}\Big(4E_k^2\{B_{11}-2B_{17}+\frac{m^2}{2}(B_3-B_1+2B_{15})$$

$$-B_7[p_1\cdot q\,p_2\cdot k+p_1\cdot k\,p_2\cdot q+m^2(p_1\cdot p_2+m^2+q\cdot k)]\}$$

$$+E_k\{E_1[2p_2\cdot k(B_3-B_1+2B_{15})+p_2\cdot q(B_4+4B_{15})-8B_{22}]$$

$$+E_2[2p_1\cdot k(B_3-B_1+2B_{15})+p_1\cdot q(B_4+4B_{15})-8B_{23}]\}$$

$$+4E_1E_2(B_6-2B_{21})\Big)$$

$$K^{oo}_{ab} = \frac{m^2\pi}{2(p_1\cdot p_2 + m^2)^2} \Big(E_1E_2\{4B_6 - A_o - 8D_{17} + q^2[2B_o + (k^2-q^2)D_o]\}$$

$$+2E_k^2\{(D_3 - D_1)[2p_1\cdot q\, p_2\cdot q - p_1\cdot q\, p_2\cdot k - p_2\cdot q\, p_1\cdot k - (p_1\cdot p_2 + m^2)(q^2 + 2m^2)$$

$$+ m^2(k^2 - q^2)] + p_1\cdot k(6D_7 - 4D_3p_2\cdot k - D_4p_2\cdot q) + p_1\cdot q(2D_8 - D_4p_2\cdot k)$$

$$+ 2D_7'p_2\cdot k + 2D_{11}p_1\cdot p_2 - 4D_{13} + m^2(2B_3 - B_1 - \frac{q^2}{2} D_o)\}$$

$$+E_k\{E_1[p_2\cdot q(B_4 - 2B_1) + 4p_2\cdot k(D_{11}' - D_6 + \frac{1}{2} B_3)$$

$$+ 8(D_{10} - D_{18}) + q^2(p_2\cdot q - p_2\cdot k)D_o]$$

$$+ E_2[p_1\cdot q(B_4 - 2B_1) + 4p_1\cdot k(2D_{11} - D_6 + \frac{1}{2} B_3) - 8D_{19}$$

$$+ q^2(p_1\cdot q - p_1\cdot k)D_o]\}\Big)$$

$$K^{oo}_{cc} = \frac{4\pi m^2E_1}{k^4(q^2 + 2p_1\cdot q)^2} \Big(A_o\{E_1[k^2(p_2\cdot k + m^2) - 2m^2p_2\cdot k$$

$$+ E_2m^2(q^2 + 2p_1\cdot q)\}$$

$$+ 2(p_2\cdot k)(A_1 - A_2)[2E_1(p_2\cdot k) - E_k(q^2 + 2p_1\cdot q)]$$

$$+ 2A_3[E_2(q^2 + 2p_1\cdot q) + 2E_1m^2]\Big)$$

and

$$K_{cd}^{oo} = \frac{2\pi m^2}{k^4} \Big(E_k^2(A_1 - A_2)$$

$$+ \frac{1}{(q^2 + 2p_1\cdot q)} \{ E_1[m^2 A_o(E_1+E_2)+2A_3E_1]$$

$$+ E_k(A_2-A_1)[2E_1(p_1\cdot k+q\cdot k) + q^2E_k]\}$$

$$+ \frac{1}{(q^2 - 2p_2\cdot q)} \{ E_2[m^2 A_o(E_1+E_2)+2A_3E_2]$$

$$+ E_k(A_2-A_1)[2E_2(p_2\cdot k-q\cdot k) + q^2E_k]\}$$

$$+ \frac{1}{(q^2+2p_1\cdot q)(q^2-2p_2\cdot q)} \{ m^2A_o[2E_1E_2(k^2-2p_1\cdot p_2-2m^2)-q^2(E_1^2+E_2^2)]$$

$$+ q^2k^2[A_oE_1E_2+2E_k^2(A_1-2A_2)]$$

$$+ (A_2-A_1)[4E_1E_2(k^2q\cdot k-2p_1\cdot kp_2\cdot k)-q^2E_k^2(2p_1\cdot p_2+2m^2+q^2)]$$

$$+ 4A_3[E_1E_2(q\cdot k-2p_1\cdot p_2)-q^2E_k^2)\}\Big)$$

K_{dd}^{oo} is obtained from K_{cc}^{oo} by the interchange $p_1 \longleftrightarrow p_2$, $q \longleftrightarrow -q$. Because of the symmetry relations discussed in Appendix A, $K_{bb}^{oo} = K_{aa}^{oo}$. The remaining non-exchange terms (K_{ac}^{oo}, etc.) cancel out, as is discussed in Section IV.

B. Exchange Terms

$$K'^{oo}_{cc} = \frac{2\pi m^2 E_1}{k^2(q^2+2p_1\cdot q)^2} \Big(E_1 \{ -m^2 V_o (q^2+2p_1\cdot q+2p_2\cdot k)$$

$$+ 2V_1[(2p_2\cdot k+3m^2)(p_1\cdot k+q\cdot k)+m^2 p_2\cdot k]$$

$$+ 2V_2[(2p_2\cdot k+3m^2)(p_1\cdot p_2+p_2\cdot q)-m^4]\}$$

$$- 2(V_1E_k+V_2E_2)(q^2+2p_1\cdot q)(p_2\cdot k+m^2)$$

$$+ (2V_3p_2\cdot k-V_4m^2)[E_k(q^2+2p_1\cdot q)-2E_1(p_1\cdot k+q\cdot k)]$$

$$+ (V_4p_2\cdot k - 2V_5m^2+ 2V_6)[E_2(q^2\cdot 2p_1\cdot q)- 2E_1'(p_1\cdot p_2+ p_2\cdot q)]\Big)$$

$$K_{dc}^{'00} = \frac{2\pi m^2}{k^2(q^2+2p_1\cdot q)(q^2-2p_2\cdot q)} \cdot$$

$$\cdot\Big(V_0\{m^2[E_1^2(q\cdot k-p_2\cdot q)+E_2^2(p_1\cdot q+q\cdot k)-E_1E_k(p_1\cdot q+p_2\cdot q)+E_2E_kq\cdot k]$$

$$+E_1E_2[m^2(q\cdot k-k^2-2p_1\cdot p_2-2m^2-2p_1\cdot k-2p_2\cdot k)+\frac{1}{2}k^2q^2]\}$$

$$+V_1\{E_1E_k[2p_2\cdot q(p_1\cdot k+q\cdot k+m^2)+2m^2p_1\cdot q-q^2p_2\cdot k]$$

$$+E_k^2[q^2(p_1\cdot p_2+m^2)-2p_1\cdot q\,p_2\cdot q]$$

$$+2E_1E_2[(p_1\cdot k+q\cdot k)(p_2\cdot k-q\cdot k+2m^2)+p_1\cdot k\,p_2\cdot k]-2m^2q\cdot k(E_1^2+E_2^2)$$

$$+E_2E_k[2p_1\cdot q(q\cdot k-p_2\cdot k)-q^2p_1\cdot k-2m^2q\cdot k]\}$$

$$+V_2\{2E_1E_2[(p_1\cdot p_2+p_2\cdot q)(p_2\cdot k-q\cdot k)+p_1\cdot p_2\,p_2\cdot k$$

$$+m^2(2p_1\cdot p_2+p_1\cdot q+p_2\cdot q-2m^2+\frac{1}{2}q\cdot k)]$$

$$+E_2E_k[q^2(p_1\cdot p_2+m^2)+p_2\cdot q(m^2-2p_1\cdot q)]$$

$$+E_1E_k[p_2\cdot q(2p_1\cdot p_2+2p_2\cdot q-m^2)+q^2m^2]$$

$$+E_2^2[2p_1\cdot q(q\cdot k-p_2\cdot k)-q^2p_1\cdot k-m^2(2p_1\cdot q+2q^2+q\cdot k)]-2E_1^2m^2p_2\cdot q\}$$

$$+V_3\{2E_1E_2(k^2q\cdot k-2p_1\cdot k\,p_2\cdot k)+E_1E_k[q^2p_2\cdot k-2p_2\cdot q(p_1\cdot k+q\cdot k)]$$

$$+E_2E_k[q^2p_1\cdot k+2p_1\cdot q(p_2\cdot k-q\cdot k)]$$

$$+E_k^2[2p_1\cdot q\,p_2\cdot q-q^2(p_1\cdot p_2+m^2)]\}$$

$$+V_4\{E_1E_2[q\cdot k(p_1\cdot p_2+m^2+p_2\cdot q)+p_2\cdot k(\frac{q^2}{2}-p_2\cdot q-2p_1\cdot p_2)+2p_1\cdot k m^2]$$

$$+E_2E_k[p_1\cdot q(p_2\cdot q-m^2)-\frac{q^2}{2}(p_1\cdot p_2+2m^2)]$$

$$+E_2^2[\frac{q^2}{2}p_1\cdot k+p_1\cdot q(p_2\cdot k-q\cdot k)]-E_1E_k[\frac{q^2m^2}{2}+p_2\cdot q(p_1\cdot p_2+p_2\cdot q)]\}$$

$$+V_5E_2m^2[E_1(4p_1\cdot p_2+2p_2\cdot q-q^2)-E_2(2p_1\cdot q+q^2)]$$

$$+2V_6[E_1E_2(2q^2+p_1\cdot q-p_2\cdot q-2p_1\cdot p_2)+E_2^2p_1\cdot q-E_1^2p_2\cdot q]\Big)$$

$$K'^{oo}_{dd} = \frac{\pi m^2}{k^2(q^2-2p_2q)} \Big(E_2[E_2(A_o-4B_6-q^2B_o)-3A_1E_1]$$

$$-2E_k^2B_3p_2\cdot q+2E_2E_k[B_1(p_2\cdot q+k^2+\frac{q^2}{2}-2q\cdot k-2m^2)-2B_3p_2\cdot k-B_4p_2\cdot q]$$

$$+V_o\{E_k[E_2(2p_2\cdot q+k^2-4p_2\cdot k-q^2-m^2)-2m^2E_1-E_k(2p_2\cdot q+m^2)]$$

$$-E_2^2(k^2+q^2+2m^2)\}$$

$$+V_1\{E_k[4p_2\cdot qE_k+2m^2E_1-E_2(2p_2\cdot q+k^2-4p_2\cdot k+2q\cdot k-q^2)]+2q\cdot kE_2^2\}$$

$$+V_2E_2[2m^2E_1+2p_2\cdot qE_k-E_2(k^2-4p_2\cdot k-q^2)]+(2V_3E_k+V_4E_2)(q\cdot kE_2-p_2qE_k)$$

$$+F_o\{E_2^2[2m^2(p_1\cdot q+q\cdot k)-q^2(k^2+q^2+2m^2-2q\cdot k)]$$

$$+2m^2[E_1E_2(q^2+q\cdot k-p_2\cdot q)+E_1E_kq^2-E_2E_k(\frac{q^2}{2}+p_1\cdot q+p_2\cdot q)]\}$$

$$+2(F_1E_k+F_3E_2)\{E_2[(k^2+q^2+2m^2-2q\cdot k)(p_2\cdot q-\frac{q^2}{2}+2m^2-2p_2\cdot k)$$

$$+m^2(2p_1\cdot p_2-p_1\cdot q-k^2-6m^2+4p_2\cdot k+4q\cdot k-2q^2)]$$

$$+m^2[E_1(p_2\cdot q+q\cdot k-q^2)-E_k(p_1\cdot q+p_2\cdot q+\frac{q^2}{2})]\}$$

$$+2(F_4E_k^2+F_8E_2^2+F_9E_2E_k)[m^2(2p_1\cdot q+q\cdot k)+p_2\cdot q(2q\cdot k-k^2-q^2)]\Big)$$

and

$$K'^{oo}_{db} = \frac{\pi m^2}{k^2(q^2-2p_2\cdot q)} \Big(E_2[E_1A_o+E_k(A_o-A_1)] - q^2E_1^2B_o$$

$$+ 2B_1E_k[E_2(p_1\cdot q-q\cdot k-2p_1\cdot k+2p_2\cdot k+k^2-2m^2)+\frac{q^2}{2}E_1+p_2\cdot qE_k]$$

$$- 2E_2E_k(2B_3p_1\cdot k+B_4p_1\cdot q)-2B_3E_k^2p_2\cdot q-4B_6E_1E_2$$

$$+W_o[2E_2E_k(2p_1\cdot q-p_1\cdot k-\frac{3m^2}{2})-2E_1E_2(\frac{k^2}{2}+m^2)-m^2E_k(E_k+2E_1)]$$

$$+2(W_1E_k-W_2E_1)[E_2(p_1\cdot k-2p_1\cdot q+2m^2)+\frac{m^2}{2}E_k+E_1(p_2\cdot q+m^2-\frac{q^2}{2})]$$

$$+2G_o\{q^2E_1E_k[p_2\cdot q-p_1\cdot k+\frac{1}{2}(k^2-q^2)]$$

$$+m^2E_2[E_2(p_1\cdot q+q\cdot k)+E_k(\frac{q^2}{2}-p_1\cdot q-p_2\cdot q)+E_1(q\cdot k-p_2\cdot q)]\}$$

$$+2(G_1E_k-G_3E_1)\{E_k[p_1\cdot p_2q^2+p_2\cdot q(2p_1\cdot k-2p_1\cdot q+m^2)+m^2(\frac{q^2}{2}-p_1\cdot q)]$$

$$+E_1[m^2(q\cdot k-p_2\cdot q)+q^2(p_1\cdot k+\frac{q^2}{2}-\frac{k^2}{2}-p_2\cdot q]$$

$$+2E_2[m^2(3p_2\cdot k-5p_1\cdot k+p_1\cdot p_2+\frac{3}{2}p_1\cdot q+\frac{1}{2}(k^2-q^2)-q\cdot k-5m^2)$$

$$+p_1\cdot k(p_2\cdot k-p_1\cdot k-\frac{q^2}{2}) + p_1\cdot q(q^2+\frac{k^2}{2} - 2q\cdot k)]\}$$

$$+2(G_4E_k^2+G_8E_1^2-G_9E_1E_k)[m^2(2p_1\cdot q+q\cdot k-q^2)$$

$$- q^2p_1\cdot p_2 + 2p_2\cdot q(p_1\cdot q - p_1\cdot k)]\Big)$$

The terms K'^{oo}_{da}, K'^{oo}_{ca}, and K'^{oo}_{cb} are obtained from the above expressions by the substitutions

$$P_1 \longleftrightarrow - P_2$$

$$V_i \longleftrightarrow W_i$$

$$F_i \longleftrightarrow G_i$$

Under this substitution we have

$$K'^{oo}_{cc} \rightarrow K'^{oo}_{da}$$

$$K'^{oo}_{dc} \rightarrow K'^{oo}_{ca}$$

$$K'^{oo}_{dd} \rightarrow K'^{oo}_{cb}$$

The remaining exchange terms (K'^{oo}_{aa}, etc.) are neglected, since, as indicated in Section VI, they are expected to be negligible.

VIII. DISCUSSION OF RESULTS

The analytical expressions obtained in the previous sections were integrated numerically over the remaining part of the final state phase space to obtain the final result for the direct pair production cross section. The numerical calculation was carried out for various values of Z and for nine values of primary energy between 10 Mev and 100 Bev. The results of this calculation, both for the total cross section and for various terms that contribute to it, are tabulated in Appendix G. In this section we will summarize these results and discuss their significance.

In Table 1 we present numerical results for both the unscreened and screened total cross sections. The table shows the behavior of the cross section as a function of primary energy, and also shows the effect of screening. As has been shown by previous calculations (1, 3, 5), we see that the screening effect begins to be noticeable at about 100 Mev, and becomes of increasing importance as the primary energy is increased above that value.

In Table 2 we present results for the exchange contribution to the total cross section, $\sigma_{Exchange}$, and for the non-exchange contribution from diagrams c and d, σ_{CD}. We see that σ_{CD} becomes negligible, i.e., less than one per cent of the total contribution, for energies greater than or of the order of 100 Mev, and that $\sigma_{Exchange}$ becomes negligible for energies greater than or of the order of 1 Bev.

The results in Table 2 are of interest because of the disagreement with the estimates of Murota, et al. (5). These authors

Table 1

Total Cross Section for Direct Pair Production

Primary Energy (Bev)	Total Cross Section (in units of $Z^2\alpha^2 r_0^2$)		
	Unscreened	Screened	
		Z = 26	Z = 82
0.01	1.070	1.068	1.066
0.03	5.798	5.753	5.713
0.1	19.89	19.51	19.24
0.3	43.73	41.95	41.02
1.0	79.87	72.14	69.37
3.0	121.5	100.9	94.90
10.0	194.6	118.5	106.7
30.0	273.2	133.6	116.2
100.0	384.2	156.6	131.2

$(\alpha^2 r_0^2 = 4.228 \times 10^{-30}\ cm^2)$

Table 2

Effect of Exchange and of Diagrams c and d

Primary Energy (Bev)	Unscreened		Screened $Z = 82$	
	$\sigma_{CD}/\sigma_{Total}$	$\sigma_{Exchange}/\sigma_{Total}$	$\sigma_{CD}/\sigma_{Total}$	$\sigma_{Exchange}/\sigma_{Total}$
0.01	0.0506	0.1620	0.0505	0.1600
0.1	0.0074	0.0173	0.0072	0.0148
1.0	0.0023	0.0069	0.0020	0.0047
10.0	0.0009	0.0053	0.0006	0.0031
100.0	0.0003	--	0.0001	--

σ_{CD} = Non-exchange contribution from diagrams c and d

$\sigma_{Exchange}$ = Total exchange contribution (absolute magnitude) estimated that σ_{CD} would be much larger than we have found it to be, and that $\sigma_{Exchange}$ would be much smaller. As we have discussed in Sections IV and VI, this disagreement arises because of the cancellation of the non-exchange interference terms between diagrams a, b and c, d, which Murota, et al. were not aware of, and because these previous authors considered only those exchange diagrams arising solely from diagrams a and b.

Table 2 also shows that the dominant contribution arises from the non-exchange terms involving diagrams a and b, and that for energies of the order of or greater than 100 Mev the contribution from all other terms is less than ten per cent of the total.

In Figures 2 and 3 we compare our results with those of previous calculations*. This comparison is of particular interest because of the disagreement among these earlier calculations. As we indicated in Section I, this disagreement is primarily due to the various approximations made by each author during the integration of the differential cross section, and also due to the various arbitrary parameters that these approximations introduced. The present work may be thought of as an attempt, by means of an accurate numerical calculation, to discriminate between the results of the previous calculations.

The unscreened cross section is presented in Figure 2. For energies up to 1 Bev the results of the present calculation agree

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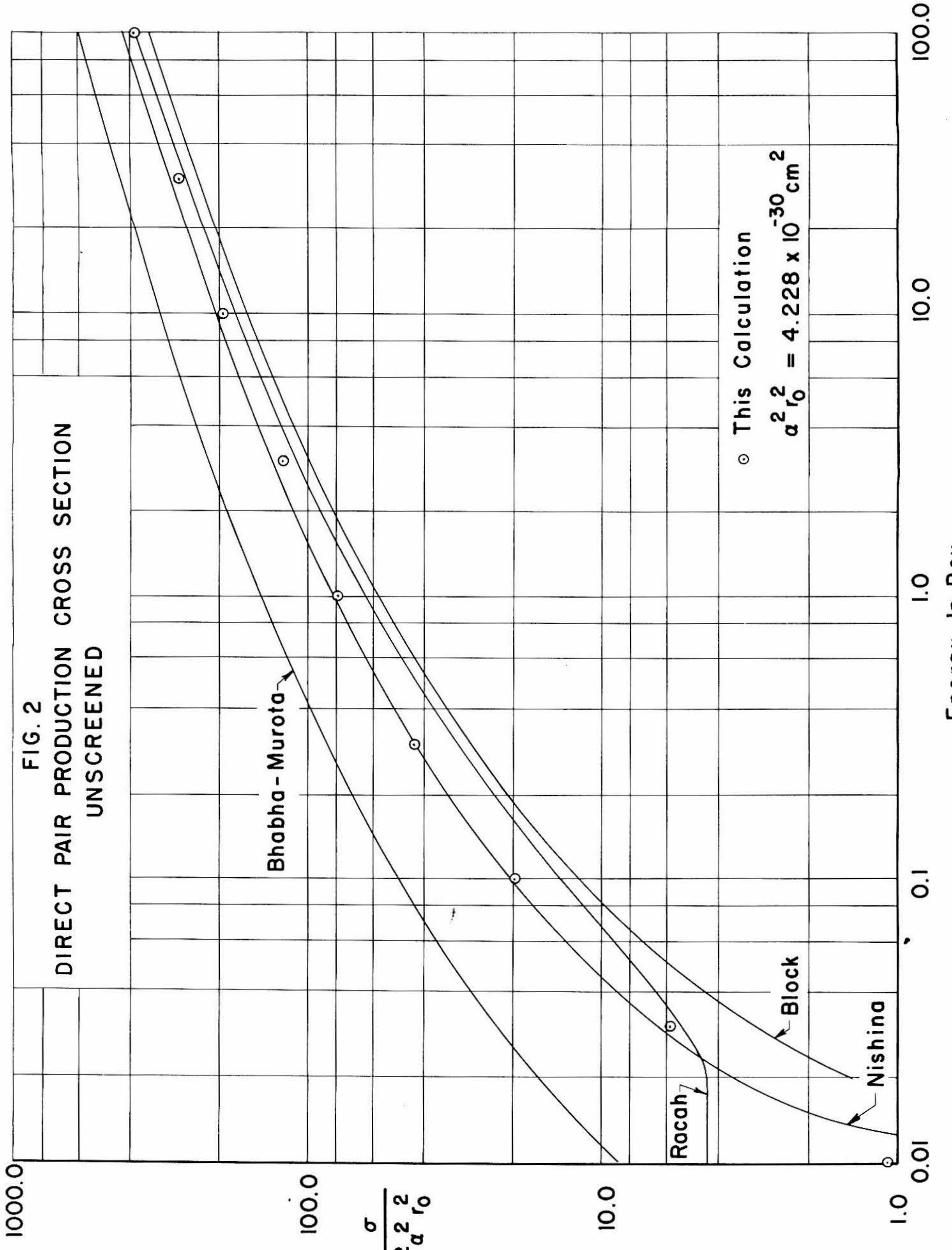
* The small circles in Figures 2 and 3 are approximately of the size of the numerical errors in the present calculation.

FIG. 2
DIRECT PAIR PRODUCTION CROSS SECTION
UNSCREENED



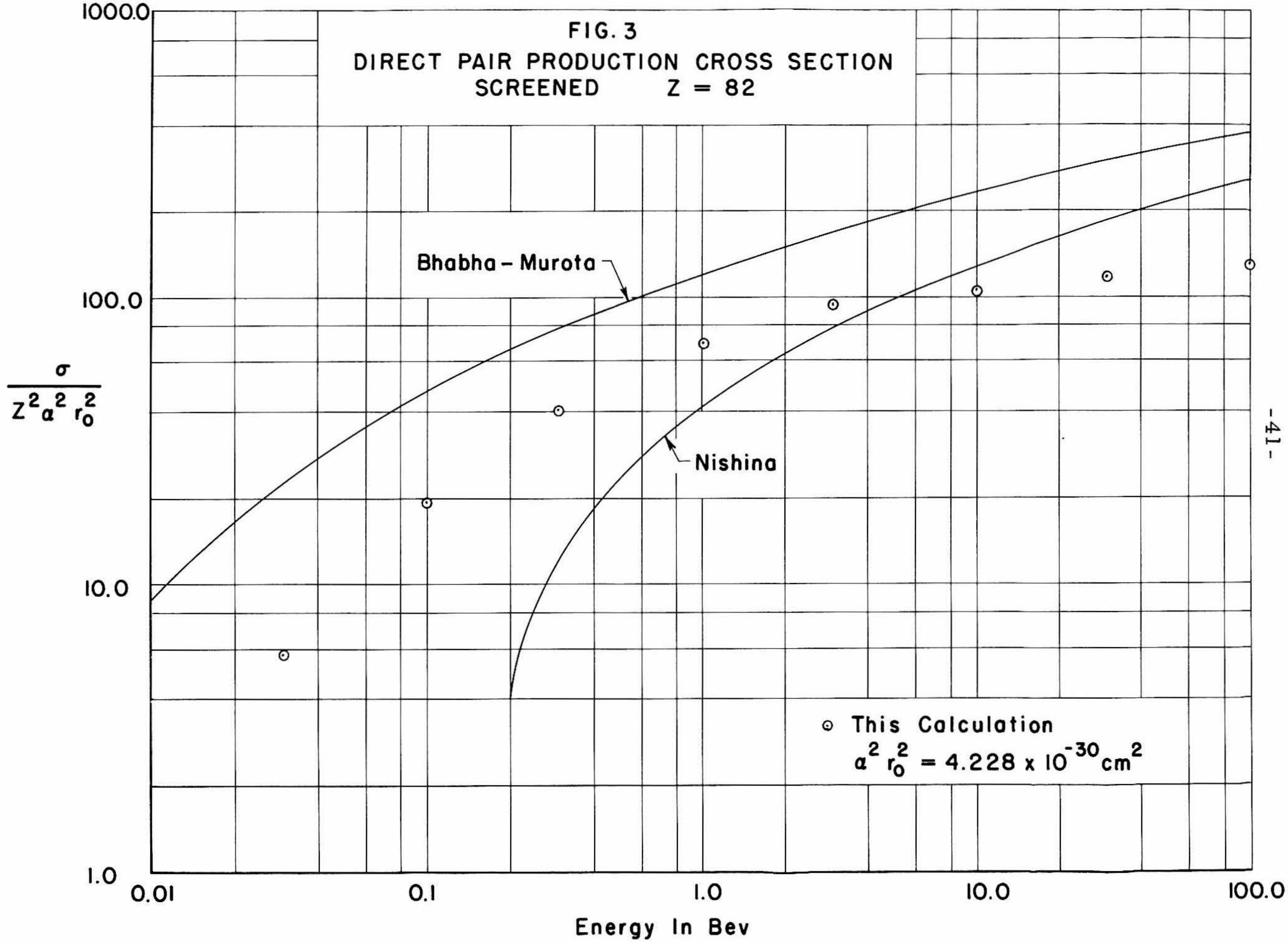


FIG. 3
DIRECT PAIR PRODUCTION CROSS SECTION
SCREENED Z = 82

fairly well with those of Nishina, et al. (1). For energies above 1 Bev the present calculation gives results which lie between the Nishina cross section and the Racah cross section, and which approach the Racah cross section at the highest energy considered (100 Bev).

The comparison of results for the screened cross section presented in Figure 3 are much less informative, primarily due to the fact that not all of the previous theories included screening. The only definite conclusion that can be reached is that the Bhabha-Murota, et al., screened cross section is definitely too large, a conclusion that had been reached previously (4).

The accuracy of the present calculation is limited by a number of factors, the most important of which are: (1) numerical errors in the integration, which are of the order of 3-4%, c.f. Appendix E, (2) the error inherent in the Thomas-Fermi screening function, (3) the neglect of Coulomb corrections, and (4) the neglect of direct pair production by the atomic electrons. Of these errors, the last three are common to all existing calculations of the direct pair production cross section.

The error in the Thomas-Fermi screening function can be estimated by comparison with the Hartree screening function. This is done in Section V, where it is concluded that the Thomas-Fermi screening function introduces an error in the total cross section of the order of 10%.

The error due to neglect of Coulomb corrections is much harder to estimate. Coulomb corrections have been extensively investigated for bremsstrahlung and for ordinary pair production

(11, 12) where it has been found that the correction is quadratic in Z and for lead decreases the cross section by about 10%, for energies greater than 50 Mev. However, the direct pair production process differs in an important way from the bremsstrahlung or ordinary pair production processes. Namely, the direct pair production matrix element involves four charged-particle wave functions which must be Coulomb-corrected, whereas the other processes involve only two. Because of this difference it is not clear what the relation will be between the Coulomb correction to direct pair production and to the other processes. All we do here is conjecture that the Coulomb correction to direct pair production will also be quadratic in Z, and will be roughly the same order of magnitude, i.e., 10% for lead.

The situation with respect to direct pair production by the atomic electrons is similarly uncertain. However, it is plausible to conjecture that, as in the case of bremsstrahlung and ordinary pair production (13), the contribution due to the atomic electrons can be included approximately by replacing the multiplicative factor Z^2 in the total cross section by $Z(Z+1)$.

From the above discussion we conclude that the total error in the present calculation is, for heavy elements such as lead, of the order of 25%, and for lighter elements for which the Coulomb correction is not as important, of the order of 10-15%.

We conclude this section by presenting, in Figure 4, a comparison of the results of the present calculation with various experimental results for the direct pair production cross section

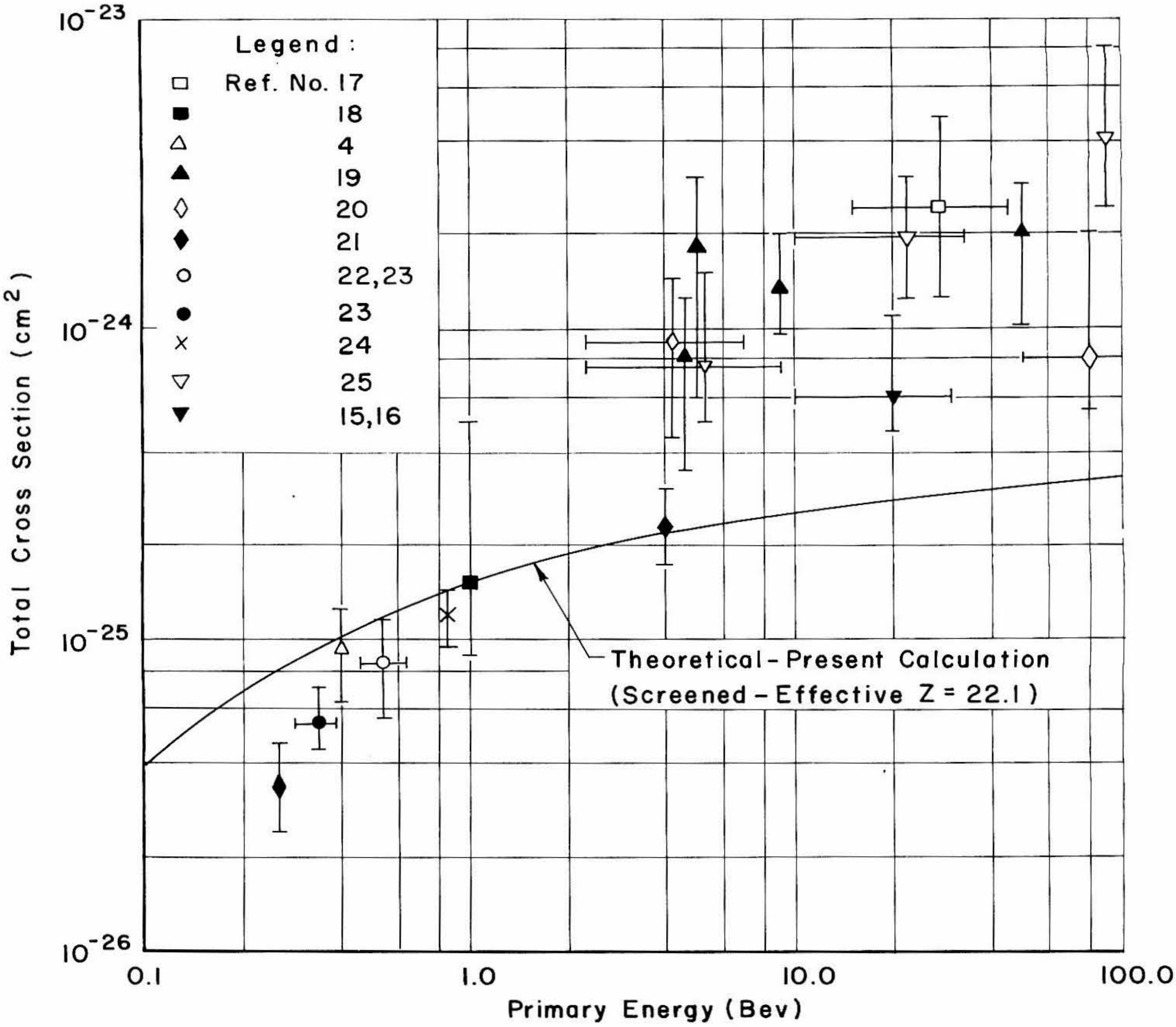


FIG. 4 COMPARISON OF THEORY AND EXPERIMENT

(4, 15-25)*. The theoretical curve shown is calculated for an effective Z of 22.1, which is appropriate to the nuclear emulsions used in these experiments (4).

From the results shown in Figure 4, we see that for energies up to about 1 Bev, there is reasonably good agreement between theory and experiment. For higher energies, however, the situation is unclear, due partially to the fact that the experiments disagree amongst themselves. The experiments in this higher energy region were all carried out using nuclear emulsions exposed to cosmic rays, and were all beset by two serious difficulties: the determination of the primary electron energy, and the correction for so-called pseudo-tridents.

Various methods have been used for the energy determination, including multiple scattering measurements, and, for those cases where the primary electron arises from a previous pair in a shower, measurements of the opening angle of the previous pair. For energies in the multi-Bev region these methods are all subject to large errors, and it has been suggested several times in the literature that the primary electron energies may have been seriously underestimated (4, 15).

In addition to direct pair production, a high energy electron may initiate a two-step process in which it emits a bremsstrahlung photon which then produces an electron-positron pair. If the bremsstrahlung and the pair production occur close enough together in the

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*The experimental points shown were obtained from Fig. 1 of the paper of Roe and Ozaki (14), with the addition of the more recent work by Tumanyan, et al. (15, 16).

emulsion, the resulting two-step process is indistinguishable from direct pair production. These spurious direct pair events, which must be subtracted out to obtain the true direct pair production cross section, are referred to as pseudo-tridents. Numerous calculations of the pseudo-trident correction have been carried out, with varying results (25, 26, 27)*. The results of the most recent calculation (15), which uses a Monte Carlo technique, indicate that the previous calculations contain large errors. These errors would introduce large systematic errors in the experimental values of the cross section, particularly in the very high energy range, where the pseudo-trident correction is very large.

In view of the experimental uncertainties discussed above, and in view of the inconsistency of the experimental points above 1 Bev, the disagreement between theory and experiment for these energies is probably not significant. It would appear that there is a need for experiments of increased accuracy in the energy region above 1 Bev.

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*The number of pseudo-tridents relative to the total number of observed tridents (pseudo plus real) is of the order of 10% for primary electron energies of about 1 Bev, and is of the order of 90% for primary energies of about 100 Bev (27).

APPENDIX A

Traces and Symmetries

In this appendix we evaluate the traces, i.e. the $I_{ij}^{\eta\lambda}$ and $I_{ij}^{'\eta\lambda}$, and we discuss the symmetries that exist between the various terms.

The calculation is simplified if we note that the final result will be real, and that $\mathcal{M}_i^{'\eta *}\mathcal{M}_j^{\lambda}$ will contribute the same amount to the total cross section as $\mathcal{M}_i^{\eta *}\mathcal{M}_j^{'\lambda}$, since they differ only in the interchange $p_2 \longleftrightarrow p_-$, and p_2 and p_- are both dummy variables of integration insofar as the total cross section is concerned. From these facts it follows that

$$I_{ij}^{\eta\lambda} = I_{ji}^{\lambda\eta}$$

and

$$I_{ij}^{'\eta\lambda} = I_{ji}^{'\lambda\eta}$$

where the last equation is true in the sense that the two terms contribute equally to the final result. This means that we only have to calculate half of the cross terms.

We now proceed to evaluate the traces, first the non-exchange terms, and then the exchange terms.

1. Non-Exchange Traces

Using standard methods, we sum over the spins and obtain

$$I_{aa}^{\eta\lambda} = \frac{m^2 J_{aa}^{\eta\lambda}}{(p_1-p_2)^4 (q^2-2p_-\cdot q)^2}$$

$$I_{bb}^{\eta\lambda} = \frac{m^2 J_{bb}^{\eta\lambda}}{(p_1-p_2)^4 (q^2-2p_+\cdot q)^2}$$

$$I_{ab}^{\eta\lambda} = \frac{m^2 J_{ab}^{\eta\lambda}}{(p_1-p_2)^4 (q^2-2p_-\cdot q)(q^2-2p_+\cdot q)}$$

$$I_{cc}^{\eta\lambda} = \frac{m^2 J_{cc}^{\eta\lambda}}{(p_++p_-)^4 (q^2+2p_1\cdot q)^2}$$

$$I_{dd}^{\eta\lambda} = \frac{m^2 J_{dd}^{\eta\lambda}}{(p_++p_-)^4 (q^2-2p_2\cdot q)^2}$$

$$I_{cd}^{\eta\lambda} = \frac{m^2 J_{cd}^{\eta\lambda}}{(p_++p_-)^4 (q^2+2p_1\cdot q)(q^2-2p_-\cdot q)}$$

$$I_{ac}^{\eta\lambda} = \frac{m^2 J_{ac}^{\eta\lambda}}{(p_1-p_2)^2 (p_++p_-)^2 (q^2-2p_-\cdot q)(q^2+2p_1\cdot q)}$$

$$I_{ad}^{\eta\lambda} = \frac{m^2 J_{ad}^{\eta\lambda}}{(p_1-p_2)^2 (p_++p_-)^2 (q^2-2p_-\cdot q)(q^2-2p_2\cdot q)}$$

$$I_{bc}^{\eta\lambda} = \frac{m^2 J_{bc}^{\eta\lambda}}{(p_1-p_2)^2 (p_++p_-)^2 (q^2-2p_+\cdot q)(q^2+2p_1\cdot q)}$$

$$I_{bd}^{\eta\lambda} = \frac{m^2 J_{bd}^{\eta\lambda}}{(p_1-p_2)^2 (p_++p_-)^2 (q^2-2p_+\cdot q)(q^2-2p_2\cdot q)}$$

where the $J_{ij}^{\eta\lambda}$ are given by

$$J_{aa}^{\eta\lambda} = \frac{1}{16} Tr\{\gamma^\mu(-i\gamma\cdot p_2+m)\gamma^\nu(-i\gamma\cdot p_1+m)\} \cdot$$

$$\cdot Tr\{\gamma_\mu[i\gamma\cdot(p_--q)-m]\gamma^\eta(-i\gamma\cdot p_-+m)\gamma^\lambda[i\gamma\cdot(p_--q)-m]\gamma_\nu(i\gamma\cdot p_++m)\}$$

$$J_{bb}^{\eta\lambda} = \frac{1}{16}\{\gamma^\mu(-i\gamma\cdot p_2+m)\gamma^\nu(-i\gamma\cdot p_1+m)\} \cdot$$

$$\cdot Tr\{\gamma^\eta[i\gamma\cdot(q-p_+)-m]\gamma_\mu(-i\gamma\cdot p_-+m)\gamma_\nu[i\gamma\cdot(q-p_+)-m]\gamma^\lambda(i\gamma\cdot p_++m)\}$$

$$J_{ab}^{\eta\lambda} = \frac{1}{16} Tr\{\gamma^\mu(-i\gamma\cdot p_2+m)\gamma^\nu(-i\gamma\cdot p_1+m)\} \cdot$$

$$\cdot Tr\{\gamma_\mu[i\gamma\cdot(p_--q)-m]\gamma^\eta(-i\gamma\cdot p_-+m)\gamma_\nu[i\gamma\cdot(q-p_+)-m]\gamma^\lambda(i\gamma\cdot p_++m)\}$$

$$J_{cc}^{\eta\lambda} = \frac{1}{16} Tr\{\gamma^\mu(-i\gamma\cdot p_-+m)\gamma^\nu(i\gamma\cdot p_++m)\} \cdot$$

$$\cdot Tr\{\gamma^\eta[i\gamma\cdot(p_1+q)-m]\gamma_\mu(-i\gamma\cdot p_2+m)\gamma_\nu[i\gamma\cdot(p_1+q)-m]\gamma^\lambda(-i\gamma\cdot p_1+m)\}$$

$$J_{dd}^{\eta\lambda} = \frac{1}{16} Tr\{\gamma^\mu(-i\gamma\cdot p_-+m)\gamma^\nu(i\gamma\cdot p_++m)\} \cdot$$

$$\cdot Tr\{\gamma_\mu[i\gamma\cdot(p_2-q)-m]\gamma^\eta(-i\gamma\cdot p_2+m)\gamma^\lambda[i\gamma\cdot(p_2-q)-m]\gamma_\nu(-i\gamma\cdot p_1+m)\}$$

$$J_{cd}^{\eta\lambda} = \frac{1}{16} Tr\{\gamma^\mu(-i\gamma\cdot p_-+m)\gamma^\nu(i\gamma\cdot p_++m)\} \cdot$$

$$\cdot Tr\{\gamma^\eta[i\gamma\cdot(p_1+q)-m]\gamma_\mu(-i\gamma\cdot p_2+m)\gamma^\lambda[i\gamma\cdot(p_2-q)-m]\gamma_\nu(-i\gamma\cdot p_1+m)\}$$

$$J_{ac}^{\eta\lambda} = \frac{1}{16} Tr\{\gamma^\mu(-i\gamma\cdot p_2+m)\gamma_\nu[i\gamma\cdot(p_1+q)-m]\gamma^\lambda(-i\gamma\cdot p_1+m)\} \cdot$$

$$\cdot Tr\{\gamma^\nu(i\gamma\cdot p_++m)\gamma_\mu[i\gamma\cdot(p_--q)-m]\gamma^\eta(-i\gamma\cdot p_-+m)\}$$

$$J_{ad}^{\eta\lambda} = \frac{1}{16} Tr\{\gamma^\mu(-i\gamma\cdot p_2+m)\gamma^\lambda[i\gamma\cdot(p_2-q)-m]\gamma_\nu(-i\gamma\cdot p_1+m)\} \cdot$$

$$\cdot Tr\{\gamma^\nu(i\gamma\cdot p_++m)\gamma_\mu[i\gamma\cdot(p_--q)-m]\gamma^\eta(-i\gamma\cdot p_-+m)\}$$

$$J_{bc}^{\eta\lambda} = \frac{1}{16} \mathrm{Tr}\{ \gamma^{\mu}(-i\gamma\cdot p_2+m)\gamma_{\nu}[i\gamma\cdot(p_1+q)-m]\gamma^{\lambda}(-i\gamma\cdot p_1+m)\} \cdot$$

$$\cdot \mathrm{Tr}\{\gamma^{\nu}(i\gamma\cdot p_+ +m)\gamma^{\eta}[i\gamma\cdot(q-p_+)-m]\gamma_{\mu}(-i\gamma\cdot p_- +m)\}$$

$$J_{bd}^{\eta\lambda} = \frac{1}{16} \mathrm{Tr}\{\gamma^{\mu}(-i\gamma\cdot p_2+m)\gamma^{\lambda}[i\gamma\cdot(p_2-q)-m]\gamma_{\nu}(-i\gamma\cdot p_1+m)\} \cdot$$

$$\cdot \mathrm{Tr}\{\gamma^{\nu}(i\gamma\cdot p_+ +m)\gamma^{\eta}[i\gamma\cdot(q-p_+)-m]\gamma_{\mu}(-i\gamma\cdot p_- +m)\}$$

By using the above expressions for the $J_{ij}^{\eta\lambda}$, and certain rules for manipulating traces of γ-matrices, in particular

$$\mathrm{Tr}(\gamma_{\alpha}\gamma_{\beta}\gamma_{\delta}\cdots\gamma_{\eta}\gamma_{\nu}\gamma_{\mu}) = \mathrm{Tr}(\gamma_{\mu}\gamma_{\nu}\gamma_{\eta}\cdots\gamma_{\delta}\gamma_{\beta}\gamma_{\alpha})$$

one can obtain the following symmetry rules:

(1) Under the interchange $p_+ \longleftrightarrow -p_-$, $q \rightarrow -q$

$$J_{aa}^{\eta\lambda} \longleftrightarrow J_{bb}^{\eta\lambda}$$

$$J_{ab}^{\eta\lambda} \rightarrow J_{ab}^{\lambda\eta}$$

(2) Under the interchange $p_1 \longleftrightarrow p_2$, $q \rightarrow -q$

$$J_{cc}^{\eta\lambda} \longleftrightarrow J_{dd}^{\eta\lambda}$$

$$J_{cd}^{\eta\lambda} \rightarrow J_{cd}^{\lambda\eta}$$

(3) Under the interchange $p_1 \longleftrightarrow p_-$, $p_2 \longleftrightarrow -p_+$, $q \rightarrow -q$

$$J_{aa}^{\eta\lambda} \longleftrightarrow J_{cc}^{\eta\lambda}$$

$$J_{bb}^{\eta\lambda} \longleftrightarrow J_{dd}^{\eta\lambda}$$

$$J^{\eta\lambda}_{ab} \longleftrightarrow J^{\eta\lambda}_{cd}$$

$$J^{\eta\lambda}_{ad} \longleftrightarrow J^{\lambda\eta}_{bc}$$

$$J^{\eta\lambda}_{ac} \longleftrightarrow J^{\lambda\eta}_{ac}$$

$$J^{\eta\lambda}_{bd} \longleftrightarrow J^{\lambda\eta}_{bd}$$

(4) Under the interchange $p_1 \longleftrightarrow p_2$, $p_+ \longleftrightarrow -p_-$, $q \rightarrow -q$

$$J^{\eta\lambda}_{aa} \longleftrightarrow J^{\eta\lambda}_{bb}$$

$$J^{\eta\lambda}_{cc} \longleftrightarrow J^{\eta\lambda}_{dd}$$

$$J^{\eta\lambda}_{ac} \longleftrightarrow J^{\eta\lambda}_{bd}$$

$$J^{\eta\lambda}_{ad} \longleftrightarrow J^{\eta\lambda}_{bc}$$

$$J^{\eta\lambda}_{ab} \rightarrow J^{\eta\lambda}_{ab}$$

$$J^{\eta\lambda}_{cd} \rightarrow J^{\eta\lambda}_{cd}$$

The use of these symmetry rules greatly simplifies the evaluation of the traces and is an aid in checking the results.

We now proceed to evaluate the traces using standard methods. We drop terms involving $g^{\eta\lambda}$, since they correspond to the terms of type 2 discussed in Section V. We also drop terms involving q^{η} or q^{λ}, since they will not contribute to the final result, and we drop terms which are antisymmetric in η, λ for the same reason.

We obtain

$$J_{aa}^{\eta\lambda} = 8p_-^{\eta}p_-^{\lambda}\{p_1\cdot p_+p_2\cdot(p_--q)+p_2\cdot p_+p_1\cdot(p_--q)+m^2[p_+\cdot(p_--q)-p_1\cdot p_2-2m^2]\}$$

$$-2(q^2-2p_-\cdot q)[m^2(p_+^{\eta}p_-^{\lambda}+p_+^{\lambda}p_-^{\eta})+p_1\cdot p_+(p_2^{\eta}p_-^{\lambda}+p_2^{\lambda}p_-^{\eta})$$

$$+p_2\cdot p_+(p_1^{\eta}p_-^{\lambda}+p_1^{\lambda}p_-^{\eta})]$$

$$J_{ab}^{\eta\lambda} = -(p_1^{\eta}p_2^{\lambda}+p_1^{\lambda}p_2^{\eta})(q^2-2p_+\cdot q)(q^2-2p_-\cdot q)-4(p_+^{\eta}p_-^{\lambda}+p_+^{\lambda}p_-^{\eta})p_1\cdot q\,p_2\cdot q$$

$$+(q^2-2p_+\cdot q)[2m^2p_-^{\eta}(p_-^{\lambda}-p_+^{\lambda})+2p_-^{\eta}(p_1^{\lambda}p_2\cdot p_-+p_2^{\lambda}p_1\cdot p_-)+q^2(p_1^{\eta}p_2^{\lambda}+p_1^{\lambda}p_2^{\eta})$$

$$-p_1\cdot q(p_2^{\eta}p_-^{\lambda}+p_2^{\lambda}p_-^{\eta})-p_2\cdot q(p_1^{\eta}p_-^{\lambda}+p_1^{\lambda}p_-^{\eta})]$$

$$+(q^2-2p_-\cdot q)[2m^2p_+^{\lambda}(p_+^{\eta}-p_-^{\eta})+2p_+^{\lambda}(p_1^{\eta}p_2\cdot p_++p_2^{\eta}p_1\cdot p_+)+q^2(p_1^{\eta}p_2^{\lambda}+p_1^{\lambda}p_2^{\eta})$$

$$-p_1\cdot q(p_2^{\eta}p_+^{\lambda}+p_2^{\lambda}p_+^{\eta})-p_2\cdot q(p_1^{\eta}p_+^{\lambda}+p_1^{\lambda}p_+^{\eta})]$$

$$+q^2\{4p_+^{\eta}p_-^{\lambda}(p_1\cdot p_2+m^2)-2m^2(p_+^{\eta}p_+^{\lambda}+p_-^{\eta}p_-^{\lambda})+(p_1^{\eta}p_2^{\lambda}+p_1^{\lambda}p_2^{\eta})(2p_+\cdot p_--2m^2-q^2)$$

$$-2p_1\cdot p_+p_2^{\eta}(p_+^{\lambda}+p_-^{\lambda})-2p_1\cdot p_-p_2^{\lambda}(p_+^{\eta}+p_-^{\eta})-2p_2\cdot p_+p_1^{\eta}(p_+^{\lambda}+p_-^{\lambda})-2p_2\cdot p_-p_1^{\lambda}(p_+^{\eta}+p_-^{\eta})$$

$$+p_2\cdot q[p_1^{\eta}(p_+^{\lambda}+p_-^{\lambda})+p_1^{\lambda}(p_+^{\eta}+p_-^{\eta})]+p_1\cdot q[p_2^{\eta}(p_+^{\lambda}+p_-^{\lambda})+p_2^{\lambda}(p_+^{\eta}+p_-^{\eta})]\}$$

$$+4p_+^{\lambda}p_-^{\eta}[2m^2(p_1\cdot p_2-p_+\cdot p_-+2m^2)-2(p_1\cdot p_+p_2\cdot p_-+p_1\cdot p_-p_2\cdot p_+)$$

$$+p_1\cdot q\,p_2\cdot(p_++p_-)+p_2\cdot q\,p_1\cdot(p_++p_-)]$$

$$J_{ac}^{\eta\lambda} = \frac{1}{2}(q^2+2p_1\cdot q)(q^2-2p_-\cdot q)(p_2{}^{\eta}p_+{}^{\lambda}+p_2{}^{\lambda}p_+{}^{\eta})$$

$$+(q^2+2p_1\cdot q)[2m^2p_-^{\eta}p_2^{\lambda}+p_2\cdot q(p_+^{\eta}p_-^{\lambda}+p_+^{\lambda}p_-^{\eta})-2p_2\cdot p_+p_-^{\eta}(p_1^{\lambda}+p_-^{\lambda})-2p_-^{\eta}p_+^{\lambda}p_2\cdot p_-$$

$$-\frac{q^2}{2}(p_2{}^{\eta}p_+{}^{\lambda}+p_2{}^{\lambda}p_+{}^{\eta})]$$

$$+(q^2-2p_-\cdot q)[-2m^2p_+^{\eta}p_1^{\lambda}-p_+\cdot q(p_1^{\eta}p_2^{\lambda}+p_1^{\lambda}p_2^{\eta})-2p_2\cdot p_+p_1^{\lambda}(p_1^{\eta}+p_-^{\eta})-2p_2^{\eta}p_1^{\lambda}p_1\cdot p_+$$

$$-\frac{q^2}{2}(p_2{}^{\eta}p_+{}^{\lambda}+p_2{}^{\lambda}p_+{}^{\eta})]$$

$$+q^2[2p_+^{\eta}p_2^{\lambda}p_1\cdot p_--2p_-^{\eta}p_1^{\lambda}p_2\cdot p_++2p_-^{\eta}p_+^{\lambda}p_2\cdot p_-+2p_2^{\eta}p_1^{\lambda}p_1\cdot p_++2m^2(p_+^{\eta}p_1^{\lambda}-p_-^{\eta}p_2^{\lambda})$$

$$-2p_1^{\eta}p_2^{\lambda}(p_+\cdot p_--m^2)-2p_+^{\eta}p_-^{\lambda}(p_1\cdot p_2+m^2)+p_+\cdot q(p_1^{\eta}p_2^{\lambda}+p_1^{\lambda}p_2^{\eta})-p_2\cdot q(p_+^{\eta}p_-^{\lambda}+p_+^{\lambda}p_-^{\eta})$$

$$+2p_2\cdot p_+(p_1^{\eta}p_1^{\lambda}+p_-^{\eta}p_-^{\lambda}+2p_-^{\eta}p_1^{\lambda})+\frac{q^2}{2}(p_2^{\eta}p_+^{\lambda}+p_2^{\lambda}p_+^{\eta})]$$

$$+4p_1^{\lambda}p_-^{\eta}[2p_1\cdot p_+p_2\cdot p_-+2p_1\cdot p_-p_2\cdot p_++p_+\cdot q(p_2\cdot p_--m^2)-p_2\cdot q(p_1\cdot p_++m^2)$$

$$+2m^2(p_+\cdot p_--p_1\cdot p_2-2m^2)]-2p_2\cdot q\,p_+\cdot q(p_1{}^{\eta}p_-{}^{\lambda}+p_1{}^{\lambda}p_-{}^{\eta})$$

$$J_{ad}^{\eta\lambda} = \frac{1}{2}(q^2-2p_2\cdot q)(q^2-2p_-\cdot q)(p_1{}^{\eta}p_+{}^{\lambda}+p_1{}^{\lambda}p_+{}^{\eta})$$

$$+(q^2-2p_2\cdot q)[2m^2p_-{}^{\eta}p_1{}^{\lambda}+p_1\cdot q(p_+{}^{\eta}p_-{}^{\lambda}+p_+{}^{\lambda}p_-{}^{\eta})+2p_1\cdot p_+p_-^{\eta}(p_2^{\lambda}-p_-^{\lambda})-2p_-^{\eta}p_+^{\lambda}p_1\cdot p_-$$

$$-\frac{q^2}{2}(p_1{}^{\eta}p_+{}^{\lambda}+p_1{}^{\lambda}p_+{}^{\eta})]$$

$$+(q^2-2p_-\cdot q)[-2m^2p_+^{\eta}p_2^{\lambda}+p_+\cdot q(p_1^{\eta}p_2^{\lambda}+p_1^{\lambda}p_2^{\eta})+2p_1\cdot p_+p_2^{\lambda}(p_-^{\eta}-p_2^{\eta})-2p_1^{\eta}p_2^{\lambda}p_2\cdot p_+$$

$$-\frac{q^2}{2}(p_1{}^{\eta}p_+{}^{\lambda}+p_1{}^{\lambda}p_+{}^{\eta})]$$

$$+q^2[2p_-^{\eta}p_2^{\lambda}p_1\cdot p_+-2p_+^{\eta}p_1^{\lambda}p_2\cdot p_-+2p_-^{\eta}p_+^{\lambda}p_1\cdot p_-+2p_1^{\eta}p_2^{\lambda}p_2\cdot p_++2m^2(p_+^{\eta}p_2^{\lambda}-p_-^{\eta}p_1^{\lambda})$$

$$+2p_1^{\lambda}p_2^{\eta}(p_+\cdot p_--m^2)+2p_+^{\eta}p_-^{\lambda}(p_1\cdot p_2+m^2)-p_1\cdot q(p_+^{\eta}p_-^{\lambda}+p_+^{\lambda}p_-^{\eta})-p_+\cdot q(p_1^{\eta}p_2^{\lambda}+p_1^{\lambda}p_2^{\eta})$$

$$+2p_1\cdot p_+(p_2{}^{\eta}p_2{}^{\lambda}+p_-{}^{\eta}p_-{}^{\lambda}-2p_-{}^{\eta}p_2{}^{\lambda})+\frac{q^2}{2}(p_1^{\eta}p_+^{\lambda}+p_1^{\lambda}p_+^{\eta})]$$

$$+4p_-{}^{\eta}p_2{}^{\lambda}[2p_1\cdot p_+p_2\cdot p_-+2p_1\cdot p_-p_2\cdot p_+-p_+\cdot q(p_1\cdot p_-+m^2)-p_1\cdot q(p_2\cdot p_+-m^2)$$

$$+2m^2(p_+\cdot p_--p_1\cdot p_2-2m^2)]+2p_1\cdot qp_+\cdot q(p_2{}^{\eta}p_-{}^{\lambda}+p_2{}^{\lambda}p_-{}^{\eta})$$

The remaining $J_{ij}^{\eta\lambda}$ can be obtained from these using the symmetry relations discussed above.

From the expressions obtained above for the $I_{ij}^{\eta\lambda}$ and $J_{ij}^{\eta\lambda}$ additional symmetry relations can be obtained. In particular, one can show that under the interchange $p_+ \longleftrightarrow p_-$,

$$I_{aa}^{\eta\lambda} \longleftrightarrow I_{bb}^{\eta\lambda}$$

$$I_{ac}^{\eta\lambda} \longleftrightarrow -I_{bc}^{\eta\lambda}$$

$$I_{ad}^{\eta\lambda} \longleftrightarrow -I_{bd}^{\eta\lambda}$$

The significance of the last two of these symmetries is discussed in Section IV.

2. Exchange Traces

Proceeding as before we have

$$I_{aa}^{'\eta\lambda} = \frac{m^2 J_{aa}^{'\eta\lambda}}{(p_1-p_2)^2(p_1-p_-)^2(q^2-2p_-\cdot q)(q^2-2p_2\cdot q)}$$

$$I_{bb}^{'\eta\lambda} = \frac{m^2 J_{bb}^{'\eta\lambda}}{(p_1-p_2)^2(p_1-p_-)^2(q^2-2p_+\cdot q)^2}$$

$$I_{ba}^{'\eta\lambda} = \frac{m^2 J_{ba}^{'\eta\lambda}}{(p_1-p_2)^2(p_1-p_-)^2(q^2-2p_+\cdot q)(q^2-2p_2\cdot q)}$$

$$I_{cc}^{'\eta\lambda} = \frac{m^2 J_{cc}^{'\eta\lambda}}{(p_++p_-)^2(p_2+p_+)^2(q^2+2p_1\cdot q)^2}$$

$$I'^{\eta\lambda}_{dd} = \frac{m^2 J'^{\eta\lambda}_{dd}}{(p_+ + p_-)^2 (p_2 + p_+)^2 (q^2 - 2p_2\cdot q)(q^2 - 2p_-\cdot q)}$$

$$I'^{\eta\lambda}_{dc} = \frac{m^2 J'^{\eta\lambda}_{dc}}{(p_+ + p_-)^2 (p_2 + p_+)^2 (q^2 + 2p_1\cdot q)(q^2 - 2p_2\cdot q)}$$

$$I'^{\eta\lambda}_{ca} = \frac{m^2 J'^{\eta\lambda}_{ca}}{(p_+ + p_-)^2 (p_1 - p_-)^2 (q^2 + 2p_1\cdot q)(q^2 - 2p_2\cdot q)}$$

$$I'^{\eta\lambda}_{cb} = \frac{m^2 J'^{\eta\lambda}_{cb}}{(p_+ + p_-)^2 (p_1 - p_-)^2 (q^2 + 2p_1\cdot q)(q^2 - 2p_+\cdot q)}$$

$$I'^{\eta\lambda}_{da} = \frac{m^2 J'^{\eta\lambda}_{da}}{(p_+ + p_-)^2 (p_1 - p_-)^2 (q^2 - 2p_2\cdot q)^2}$$

$$I'^{\eta\lambda}_{db} = \frac{m^2 J'^{\eta\lambda}_{db}}{(p_+ + p_-)^2 (p_1 - p_-)^2 (q^2 - 2p_2\cdot q)(q^2 - 2p_+\cdot q)}$$

where the $J'^{\eta\lambda}_{ij}$ are given by

$$J'^{\eta\lambda}_{aa} = \frac{1}{16}\,\mathrm{Tr}\{\,\gamma^{\mu}(-i\gamma\cdot p_2+m)\gamma^{\lambda}[\,i\gamma\cdot(p_2-q)-m]\,\gamma_{\nu}\,(i\gamma\cdot p_+ + m)\,\cdot$$

$$\cdot\;\gamma_{\mu}[\,i\gamma\cdot(p_- - q)-m]\,\gamma^{\eta}(-i\gamma\cdot p_- + m)\gamma^{\nu}\,(-i\gamma\cdot p_1+m)\}$$

$$J'^{\eta\lambda}_{bb} = \frac{1}{16}\,\mathrm{Tr}\{\,\gamma^{\mu}(-i\gamma\cdot p_2+m)\gamma_{\nu}\,[i\gamma\cdot(q-p_+)-m]\,\gamma^{\lambda}(i\gamma\cdot p_+ + m)\,\cdot$$

$$\cdot\;\gamma^{\eta}[i\gamma\cdot(q-p_+)-m]\,\gamma_{\mu}(-i\gamma\cdot p_- + m)\gamma^{\nu}(-i\gamma\cdot p_1+m)\}$$

$$J'^{\eta\lambda}_{ba} = \frac{1}{16}\,\mathrm{Tr}\{\,\gamma^{\mu}(-i\gamma\cdot p_2+m)\gamma^{\lambda}[i\gamma\cdot(p_2-q)-m]\,\gamma_{\nu}(i\gamma\cdot p_+ + m)\,\cdot$$

$$\cdot\;\gamma^{\eta}\,[i\gamma\cdot(q-p_+)-m]\,\gamma_{\mu}(-i\gamma\cdot p_- + m)\gamma^{\nu}(-i\gamma\cdot p_1+m)\}$$

$$J'^{\eta\lambda}_{cc} = \frac{1}{16}\,\mathrm{Tr}\{\, \gamma^{\mu}(-i\gamma\cdot p_{-}+m)\gamma_{\nu}[\,i\gamma\cdot(p_1+q)-m]\,\gamma^{\lambda}(-i\gamma\cdot p_1+m)\cdot$$

$$\cdot\ \gamma^{\eta}[\,i\gamma\cdot(p_1+q)-m]\,\gamma_{\mu}(-i\gamma\cdot p_2+m)\gamma^{\nu}(i\gamma\cdot p_{+}+m)\}$$

$$J'^{\eta\lambda}_{dd} = \frac{1}{16}\,\mathrm{Tr}\{\gamma^{\mu}(-i\gamma\cdot p_{-}+m)\gamma^{\lambda}[\,i\gamma\cdot(p_{-}-q)-m]\,\gamma_{\nu}(-i\gamma\cdot p_1+m)\cdot$$

$$\cdot\ \gamma_{\mu}[\,i\gamma\cdot(p_2-q)-m]\,\gamma^{\eta}(-i\gamma\cdot p_2+m)\gamma^{\nu}(i\gamma\cdot p_{+}+m)\}$$

$$J'^{\eta\lambda}_{dc} = \frac{1}{16}\,\mathrm{Tr}\{\gamma^{\mu}(-i\gamma\cdot p_{-}+m)\gamma_{\nu}[\,i\gamma\cdot(p_1+q)-m]\,\gamma^{\lambda}(-i\gamma\cdot p_1+m)\cdot$$

$$\cdot\ \gamma_{\mu}[\,i\gamma\cdot(p_2-q)-m]\,\gamma^{\eta}(-i\gamma\cdot p_2+m)\gamma^{\nu}(i\gamma\cdot p_{+}+m)\}$$

$$J'^{\eta\lambda}_{ca} = \frac{1}{16}\,\mathrm{Tr}\{\gamma^{\mu}(-i\gamma\cdot p_{-}+m)\gamma^{\nu}(-i\gamma\cdot p_1+m)\gamma^{\eta}[\,i\gamma\cdot(p_1+q)-m]\,\cdot$$

$$\cdot\,\gamma_{\mu}(-i\gamma\cdot p_2+m)\gamma^{\lambda}[\,i\gamma\cdot(p_2-q)-m]\,\gamma^{\nu}(i\gamma\cdot p_{+}+m)\}$$

$$J'^{\eta\lambda}_{cb} = \frac{1}{16}\,\mathrm{Tr}\{\gamma^{\mu}(-i\gamma\cdot p_{-}+m)\gamma^{\nu}(-i\gamma\cdot p_1+m)\gamma^{\eta}[\,i\gamma\cdot(p_1+q)-m]\,\cdot$$

$$\cdot\,\gamma_{\mu}(-i\gamma\cdot p_2+m)\gamma^{\nu}[\,i\gamma\cdot(q-p_{+})-m]\,\gamma^{\lambda}(i\gamma\cdot p_{+}+m)\}$$

$$J'^{\eta\lambda}_{da} = \frac{1}{16}\,\mathrm{Tr}\{\gamma^{\mu}(-i\gamma\cdot p_{-}+m)\gamma^{\nu}(-i\gamma\cdot p_1+m)\gamma_{\mu}[\,i\gamma\cdot(p_2-q)-m]\,\cdot$$

$$\cdot\ \gamma^{\eta}(-i\gamma\cdot p_2+m)\gamma^{\lambda}[\,i\gamma\cdot(p_2-q)-m]\,\gamma^{\nu}(i\gamma\cdot p_{+}+m)\}$$

$$J'^{\eta\lambda}_{db} = \frac{1}{16}\,\mathrm{Tr}\{\gamma^{\mu}(-i\gamma\cdot p_{-}+m)\gamma^{\nu}(-i\gamma\cdot p_1+m)\gamma_{\mu}[\,i\gamma\cdot(p_2-q)-m]\,\cdot$$

$$\cdot\ \gamma^{\eta}(-i\gamma\cdot p_2+m)\gamma^{\nu}[i\gamma\cdot(q-p_{+})-m]\,\gamma^{\lambda}(i\gamma\cdot p_{+}+m)\}$$

The symmetry rules for the exchange terms are:

(1) Under the interchange $p_1 \longleftrightarrow -p_+$

$$J'^{\eta\lambda}_{aa} \longleftrightarrow J'^{\lambda\eta}_{dd}$$

$$J'^{\eta\lambda}_{bb} \longleftrightarrow J'^{\lambda\eta}_{cc}$$

$$J'^{\eta\lambda}_{ba} \longleftrightarrow J'^{\lambda\eta}_{dc}$$

$$J'^{\eta\lambda}_{db} \longleftrightarrow J'^{\lambda\eta}_{ca}$$

$$J'^{\eta\lambda}_{cb} \rightarrow J'^{\lambda\eta}_{cb}$$

$$J'^{\eta\lambda}_{da} \rightarrow J'^{\lambda\eta}_{da}$$

(2) Under the interchange $p_2 \longleftrightarrow p_-$

$$J'^{\eta\lambda}_{aa} \rightarrow J'^{\lambda\eta}_{aa}$$

$$J'^{\eta\lambda}_{bb} \rightarrow J'^{\lambda\eta}_{bb}$$

$$J'^{\eta\lambda}_{cc} \rightarrow J'^{\lambda\eta}_{cc}$$

$$J'^{\eta\lambda}_{dd} \rightarrow J'^{\lambda\eta}_{dd}$$

(3) Under the interchange $p_1 \longleftrightarrow p_2$, $p_+ \longleftrightarrow -p_-$, $q \rightarrow -q$

$$J'^{\eta\lambda}_{ca} \longleftrightarrow J'^{\eta\lambda}_{dc}$$

$$J'^{\eta\lambda}_{da} \longleftrightarrow J'^{\eta\lambda}_{cc}$$

$$J'^{\eta\lambda}_{cb} \longleftrightarrow J'^{\eta\lambda}_{dd}$$

We now evaluate the traces and obtain

$$J'^{\eta\lambda}_{aa} = 2p_2^{\ \eta}p_2^{\ \lambda}[m^2(p_-\cdot q-p_1\cdot q-p_+\cdot q)-2p_1\cdot p_+p_-\cdot q]$$

$$+2p_-^{\ \eta}p_-^{\ \lambda}[m^2(p_2\cdot q-p_1\cdot q-p_+\cdot q)-2p_1\cdot p_+p_2\cdot q]$$

$$+(p_2^{\ \eta}p_-^{\ \lambda}+p_2^{\ \lambda}p_-^{\ \eta})[2p_1\cdot p_+(p_2\cdot q+p_-\cdot q-2p_2\cdot p_--q^2)+m^2(2p_2\cdot p_-+2p_1\cdot p_-+2p_1\cdot p_2$$

$$-2p_+\cdot p_--2p_2\cdot p_++2p_+\cdot q-2p_1\cdot q-2p_1\cdot p_+-p_-\cdot q-p_2\cdot q+4m^2)]$$

$$+m^2\{(p_2\cdot q+p_-\cdot q-p_+\cdot q)[p_1^\eta(p_2^\lambda+p_-^\lambda)+p_1^\lambda(p_2^\eta+p_-^\eta)]+q^2(p_1^\eta p_+^\lambda+p_1^\lambda p_+^\eta)$$

$$+(p_1\cdot q+p_2\cdot q-p_-\cdot q-q^2)(p_2^\eta p_+^\lambda+p_2^\lambda p_+^\eta)+(p_1\cdot q+p_-\cdot q-p_2\cdot q-q^2)(p_+^\eta p_-^\lambda+p_+^\lambda p_-^\eta)\}$$

$$J'^{\eta\lambda}_{bb} = (q^2-2p_+\cdot q)\{(2p_2\cdot p_-+m^2)(p_1^\eta p_+^\lambda+p_1^\lambda p_+^\eta)$$

$$+m^2[p_+^{\ \eta}(p_2^{\ \lambda}+p_-^{\ \lambda})+p_+^{\ \lambda}(p_2^{\ \eta}+p_-^{\ \eta})]\}$$

$$+4p_+^{\ \eta}p_+^{\ \lambda}\{m^2[(p_2+p_-)\cdot(p_1+q-p_+)+p_2\cdot p_-+2m^2]$$

$$-(2p_2\cdot p_-+m^2)p_1\cdot(p_+-q)\}$$

$$J'^{\eta\lambda}_{ba} = 2m^2[p_+^\eta\, p_+^{\ \lambda}(p_2\cdot q-p_1\cdot q-p_-\cdot q)+p_2^{\ \eta}p_2^{\ \lambda}(p_+\cdot q-p_1\cdot q-p_-\cdot q)]$$

$$+2(p_+^\eta p_2^\lambda+p_+^\lambda p_2^\eta)\{p_1\cdot p_+p_-\cdot(p_2-q)+p_2\cdot p_-p_1\cdot(p_+-q)+p_1\cdot qp_-\cdot q-\tfrac{1}{2}q^2p_1\cdot p_-$$

$$+m^2[p_2\cdot p_+-p_1\cdot p_--(p_1+p_-)\cdot(p_2-p_+)-\tfrac{1}{2}(p_2\cdot q+p_+\cdot q)-2m^2]\}$$

$$+(p_+^{\ \eta}p_-^{\ \lambda}+p_+^{\ \lambda}p_-^{\ \eta})[p_1\cdot p_2q^2+2p_2\cdot qp_1\cdot(p_+-q)+m^2(p_1\cdot q+p_+\cdot q-p_2\cdot q)]$$

$$+(p_1^\eta p_2^\lambda+p_1^\lambda p_2^\eta)[p_+\cdot p_-q^2+2p_+\cdot qp_-\cdot(p_2-q)+m^2(p_+\cdot q+p_2\cdot q-p_-\cdot q-q^2)]$$

$$+m^2[(p_2^\eta p_-^\lambda+p_2^\lambda p_-^\eta)(p_1\cdot q+p_2\cdot q+p_+\cdot q-q^2)+(p_1^\eta p_+^\lambda+p_1^\lambda p_+^\eta)(p_+\cdot q-p_2\cdot q-p_-\cdot q)]$$

$$+(p_1^{\ \eta}p_-^{\ \lambda}+p_1^{\ \lambda}p_-^{\ \eta})[2p_2\cdot qp_+\cdot q+q^2(m^2-p_2\cdot p_+)]$$

The remaining $J_{ij}^{'\eta\lambda}$ can be obtained from these using the symmetry relations discussed above.

From the final expressions obtained for the $I_{ij}^{'\eta\lambda}$ additional symmetry relations can be obtained. In particular, one can show that under the interchange $p_1 \longleftrightarrow -p_2$, $p_+ \longleftrightarrow p_-$,

$$I_{ca}^{'\eta\lambda} \longleftrightarrow I_{dc}^{'\eta\lambda}$$

$$I_{da}^{'\eta\lambda} \longleftrightarrow I_{cc}^{'\eta\lambda}$$

$$I_{cb}^{'\eta\lambda} \longleftrightarrow I_{dd}^{'\eta\lambda}$$

These symmetry relations are useful in evaluating the $K_{ij}^{'\eta\lambda}$.

APPENDIX B

Covariant Method of Integration

We will illustrate the method of integration by evaluating the integral

$$I_1^{\mu} = \int \frac{d^3p_+}{E_+} \int \frac{d^3p_-}{E_-} \frac{\delta^4(k-p_+-p_-)\, p_-^{\mu}}{q^2-2p_-\cdot q}$$

where $k = p_1 + q - p_2$. We write this as

$$I_1^{\mu} = 4\int d^4p_+ \int d^4p_- \,\theta(p_{+0})\theta(p_{-0})\delta(p_+^2+m^2)\delta(p_-^2+m^2) \frac{\delta^4(k-p_+-p_-)\, p_-^{\mu}}{q^2-2p_-\cdot q}$$

or

$$I_1^{\mu} = 4\int d^4p_- \,\theta(k_0-p_{-0})\theta(p_{-0})\delta(k^2-2k\cdot p_-)\delta(p_-^2+m^2) \frac{p_-^{\mu}}{q^2-2p_-\cdot q}$$

I_1^{μ} is covariant and may be evaluated in any coordinate system. We choose the system with $k = (0, k_0)$. In this system we have

$$I_1^{\mu} = \frac{1}{k_0}\int dp_{-0} \int |\vec{p}_-|\, dp_- \int d\Omega_-\, \delta(p_{-0}-\frac{k_0}{2})\delta[|\vec{p}_-| - (\frac{k_0^2}{4}-m^2)^{\frac{1}{2}}] \frac{p_-^{\mu}}{q^2+2p_{-0}q_0-2\vec{p}_-\cdot\vec{q}}$$

Taking $\hat{q}$ as the polar axis, the only two non-zero components are I_1^0 and I_1^1. Evaluating the integrals, we have

$$I_1^0 = -\frac{\pi k_0}{2|\vec{q}|} \ln \left[\frac{q^2+k_0q_0-2|\vec{q}|(\frac{k_0^2}{4}-m^2)^{\frac{1}{2}}}{q^2+k_0q_0+2|\vec{q}|(\frac{k_0^2}{4}-m^2)^{\frac{1}{2}}} \right]$$

and

$$I_1^1 = \frac{\pi\hat{q}}{2k_0\vec{q}^2}\left\{-4|\vec{q}|(\frac{k_0^2}{4}-m^2)^{\frac{1}{2}} - (q^2+k_0q_0)\ln\left[\frac{q^2+k_0q_0-2|\vec{q}|(\frac{k_0^2}{4}-m^2)^{\frac{1}{2}}}{q^2+k_0q_0+2|\vec{q}|(\frac{k_0^2}{4}-m^2)^{\frac{1}{2}}}\right]\right.$$

To rewrite these in covariant form, we note that

$$|\vec{q}| = \left[q^2 - \frac{(q\cdot k)^2}{k^2} \right]^{\frac{1}{2}}$$

and

$$\hat{\vec{q}} = \frac{1}{|\vec{q}|} \left(q^{\mu} - \frac{q\cdot k}{k^2} k^{\mu} \right)$$

We obtain as the final result

$$I_1^{\mu} = \pi \left(B_1 k^{\mu} + B_2 q^{\mu} \right)$$

where

$$B_1 = \frac{1}{2[(q\cdot k)^2 - k^2 q^2]^{\frac{1}{2}}} \left\{ - \frac{2q\cdot k\left(1 + \frac{4m^2}{k^2}\right)^{\frac{1}{2}}}{[(q\cdot k)^2 - k^2 q^2]^{\frac{1}{2}}} \right.$$

$$\left. + \left[\frac{(q\cdot k)(q^2 - q\cdot k)}{k^2 q^2 - (q\cdot k)^2} - 1 \right] \cdot \ln \left\{ \frac{q^2 - q\cdot k - \left(1 + \frac{4m^2}{k^2}\right)^{\frac{1}{2}} [(q\cdot k)^2 - k^2 q^2]^{\frac{1}{2}}}{q^2 - q\cdot k + \left(1 + \frac{4m^2}{k^2}\right)^{\frac{1}{2}} [(q\cdot k)^2 - k^2 q^2]^{\frac{1}{2}}} \right\} \right\}$$

and

$$B_2 = \frac{k^2}{2[(q\cdot k)^2 - k^2 q^2]^{3/2}} \left\{ 2\left(1 + \frac{4m^2}{k^2}\right)^{\frac{1}{2}} [(q\cdot k)^2 - k^2 q^2]^{\frac{1}{2}} \right.$$

$$\left. + (q^2 - q\cdot k) \ln \left\{ \frac{q^2 - q\cdot k - \left(1 + \frac{4m^2}{k^2}\right)^{\frac{1}{2}} [(q\cdot k)^2 - k^2 q^2]^{\frac{1}{2}}}{q^2 - q\cdot k + \left(1 + \frac{4m^2}{k^2}\right)^{\frac{1}{2}} [(q\cdot k)^2 - k^2 q^2]^{\frac{1}{2}}} \right\} \right\}$$

APPENDIX C

Integrals

In this appendix we present the results obtained for all of the analytical integrals encountered in the present calculation.

The required integrals are, where we use $k = p_1+q-p_2$,

$$H_0 = \int \frac{d^3p_+}{E_+} \int \frac{d^3p_-}{E_-} \delta^4(k-p_+-p_-)$$

$$H_1{}^{\mu} = \int \frac{d^3p_+}{E_+} \int \frac{d^3p_-}{E_-} \delta^4(k-p_+-p_-)\, p_-{}^{\mu}$$

$$H_2^{\mu\nu} = \int \frac{d^3p_+}{E_+} \int \frac{d^3p_-}{E_-} \delta^4(k-p_+-p_-)\, p_-{}^{\mu} p_-{}^{\nu}$$

$$I_0 = \int \frac{d^3p_+}{E_+} \int \frac{d^3p_-}{E_-} \delta^4(k-p_+-p_-) \frac{1}{q^2-2p_-\cdot q}$$

$$I_1{}^{\mu} = \int \frac{d^3p_+}{E_+} \int \frac{d^3p_-}{E_-} \delta^4(k-p_+-p_-) \frac{p_-{}^{\mu}}{q^2-2p_-\cdot q}$$

$$I_2{}^{\mu\nu} = \int \frac{d^3p_+}{E_+} \int \frac{d^3p_-}{E_-} \delta^4(k-p_+-p_-) \frac{p_-{}^{\mu} p_-{}^{\nu}}{q^2-2p_-\cdot q}$$

$$I_3{}^{\mu\nu} = \int \frac{d^3p_+}{E_+} \int \frac{d^3p_-}{E_-} \delta^4(k-p_+-p_-) \frac{p_-{}^{\mu} p_-{}^{\nu}}{(q^2-2p_-\cdot q)^2}$$

$$I_4{}^{\mu\nu} = \int \frac{d^3p_+}{E_+} \int \frac{d^3p_-}{E_-} \delta^4(k-p_+-p_-) \frac{p_-{}^{\mu} p_-{}^{\nu} p_-\cdot L}{(q^2-2p_-\cdot q)^2}$$

where $L^{\mu} = p_1{}^{\mu}(p_2\cdot k+p_2\cdot q) + p_2{}^{\mu}(p_1\cdot k+p_1\cdot q)+m^2(k^{\mu}+q^{\mu})$

$$I_5^{\mu\nu} = \int \frac{d^3p_+}{E_+} \int \frac{d^3p_-}{E_-} \delta^4(k-p_+-p_-) \frac{p_-^{\mu}p_-^{\nu}p_1\cdot p_-p_2\cdot p_-}{(q^2-2p_-\cdot q)^2}$$

$$J_0 = \int \frac{d^3p_+}{E_+} \int \frac{d^3p_-}{E_-} \delta^4(k-p_+-p_-) \frac{1}{2m^2-2p_2\cdot p_+}$$

$$J_1^{\mu} = \int \frac{d^3p_+}{E_+} \int \frac{d^3p_-}{E_-} \delta^4(k-p_+-p_-) \frac{p_+^{\mu}}{2m^2-2p_2\cdot p_+}$$

$$J_2^{\mu\nu} = \int \frac{d^3p_+}{E_+} \int \frac{d^3p_-}{E_-} \delta^4(k-p_+-p_-) \frac{p_+^{\mu}p_+^{\nu}}{2m^2-2p_2\cdot p_+}$$

$$K_0 = \int \frac{d^3p_+}{E_+} \int \frac{d^3p_-}{E_-} \delta^4(k-p_+-p_-) \frac{1}{2p_1\cdot p_-+m^2}$$

$$K_1^{\mu} = \int \frac{d^3p_+}{E_+} \int \frac{d^3p_-}{E_-} \delta^4(k-p_+-p_-) \frac{p_-^{\mu}}{2p_1\cdot p_-+m^2}$$

$$K_2^{\mu\nu} = \int \frac{d^3p_+}{E_+} \int \frac{d^3p_-}{E_-} \delta^4(k-p_+-p_-) \frac{p_-^{\mu}p_-^{\nu}}{2p_1\cdot p_-+m^2}$$

$$N_0 = \int \frac{d^3p_+}{E_+} \int \frac{d^3p_-}{E_-} \delta^4(k-p_+-p_-) \frac{1}{(q^2-2p_+\cdot q)(q^2-2p_-\cdot q)}$$

$$N_1^{\mu} = \int \frac{d^3p_+}{E_+} \int \frac{d^3p_-}{E_-} \delta^4(k-p_+-p_-) \frac{p_-^{\mu}}{(q^2-2p_+\cdot q)(q^2-2p_-\cdot q)}$$

$$N_2^{\mu\nu} = \int \frac{d^3p_+}{E_+} \int \frac{d^3p_-}{E_-} \delta^4(k-p_+-p_-) \frac{p_-^{\mu}p_-^{\nu}}{(q^2-2p_+\cdot q)(q^2-2p_-\cdot q)}$$

$$N_3^{\mu\nu} = \int \frac{d^3p_+}{E_+} \int \frac{d^3p_-}{E_-} \delta^4(k-p_+-p_-) \frac{p_-^{\mu} p_-^{\nu} p_2 \cdot p_-}{(q^2-2p_+\cdot q)(q^2-2p_-\cdot q)}$$

$$N_3'^{\mu\nu} = \int \frac{d^3p_+}{E_+} \int \frac{d^3p_-}{E_-} \delta^4(k-p_+-p_-) \frac{p_-^{\mu} p_-^{\nu} p_1 \cdot p_-}{(q^2-2p_+\cdot q)(q^2-2p_-\cdot q)}$$

$$N_4^{\mu\nu} = \int \frac{d^3p_+}{E_+} \int \frac{d^3p_-}{E_-} \delta^4(k-p_+-p_-) \frac{p_-^{\mu} p_-^{\nu} p_1 \cdot p_- p_2 \cdot p_-}{(q^2-2p_+\cdot q)(q^2-2p_-\cdot q)}$$

$$Q_0 = \frac{1}{2} \int \frac{d^3p_+}{E_+} \int \frac{d^3p_-}{E_-} \delta^4(k-p_+-p_-) \frac{1}{(q^2-2p_-\cdot q)(p_2\cdot p_+-m^2)}$$

$$Q_1^{\mu} = \frac{1}{2} \int \frac{d^3p_+}{E_+} \int \frac{d^3p_-}{E_-} \delta^4(k-p_+-p_-) \frac{p_-^{\mu}}{(q^2-2p_-\cdot q)(p_2\cdot p_+-m^2)}$$

$$Q_2^{\mu\nu} = \frac{1}{2} \int \frac{d^3p_+}{E_+} \int \frac{d^3p_-}{E_-} \delta^4(k-p_+-p_-) \frac{p_-^{\mu} p_-^{\nu}}{(q^2-2p_-\cdot q)(p_2\cdot p_+-m^2)}$$

$$R_0 = -\frac{1}{2} \int \frac{d^3p_+}{E_+} \int \frac{d^3p_-}{E_-} \delta^4(k-p_+-p_-) \frac{1}{(q^2-2p_+\cdot q)(p_1\cdot p_-+m^2)}$$

$$R_1^{\mu} = -\frac{1}{2} \int \frac{d^3p_+}{E_+} \int \frac{d^3p_-}{E_-} \delta^4(k-p_+-p_-) \frac{p_+^{\mu}}{(q^2-2p_+\cdot q)(p_1\cdot p_-+m^2)}$$

$$R_2^{\mu\nu} = -\frac{1}{2} \int \frac{d^3p_+}{E_+} \int \frac{d^3p_-}{E_-} \delta^4(k-p_+-p_-) \frac{p_+^{\mu} p_+^{\nu}}{(q^2-2p_+\cdot q)(p_1\cdot p_-+m^2)}$$

These integrals are evaluated using the method illustrated in Appendix B. In order to conveniently express the results, we introduce the following notation:

$$g = \left(1 + \frac{4m^2}{k^2}\right)^{\frac{1}{2}}$$

$$a_1 = k^2 q^2 - (q \cdot k)^2$$

$$a_2 = -k^2 m^2 - (p_2 \cdot k)^2$$

$$a_3 = -k^2 m^2 - (p_1 \cdot k)^2$$

$$c_1 = (-a_1)^{\frac{1}{2}}$$

$$c_2 = (-a_2)^{\frac{1}{2}}$$

$$c_3 = (-a_3)^{\frac{1}{2}}$$

$$b_1 = c_1 g$$

$$b_2 = c_2 g$$

$$b_3 = c_3 g$$

$$\delta_0 = k^2 (p_1 \cdot p_2) - (p_1 \cdot k)(p_2 \cdot k)$$

$$\delta_1 = k^2 (p_1 \cdot q) - (p_1 \cdot k)(q \cdot k)$$

$$\delta_2 = k^2 (p_2 \cdot q) - (p_2 \cdot k)(q \cdot k)$$

$$N_1 = g^2(\delta_2{}^2 - a_1 a_2) - a_2(q^2 - q \cdot k)^2 - a_1(p_2 \cdot k - 2m^2)^2$$
$$+ 2\delta_2(q^2 - q \cdot k)(p_2 \cdot k - 2m^2)$$

$$N_2 = g^2(\delta_1{}^2 - a_1 a_3) - a_3(q^2 - q \cdot k)^2 - a_1(p_1 \cdot k + 2m^2)^2$$
$$+ 2\delta_1(q^2 - q \cdot k)(p_1 \cdot k + 2m^2)$$

$$L_1 = \frac{q^2-q\cdot k-b_1}{q^2-q\cdot k+b_1}$$

$$L_2 = \frac{2m^2-p_2\cdot k-b_2}{2m^2-p_2\cdot k+b_2}$$

$$L_3 = \frac{2m^2+p_1\cdot k-b_3}{2m^2+p_1\cdot k+b_3}$$

$$L_4 = \frac{(q^2-q\cdot k)(p_2\cdot k-2m^2) + g^2\delta_2+g(N_1)^{\frac{1}{2}}}{(q^2-q\cdot k)(p_2\cdot k-2m^2) + g^2\delta_2-g(N_1)^{\frac{1}{2}}}$$

$$L_5 = \frac{-(q^2-q\cdot k)(p_1\cdot k+2m^2)-g^2\delta_1+g(N_2)^{\frac{1}{2}}}{-(q^2-q\cdot k)(p_1\cdot k+2m^2)-g^2\delta_1-g(N_2)^{\frac{1}{2}}}$$

$$\zeta_1 = 2b_1+(q^2-q\cdot k)\ \ln L_1$$

$$\zeta_2 = 2b_2+(2m^2-p_2\cdot k)\ \ln L_2$$

$$\zeta_3 = 2b_3+(2m^2+p_1\cdot k)\ \ln L_3$$

$$\eta = \frac{b(q^2-q\cdot k)}{(q^2-q\cdot k)^2-b^2}$$

We now write the expressions obtained for the integrals in the form

$$H_0 = \pi A_0$$

$$H_1^{\mu} = \pi k^{\mu} A_1$$

$$H_2^{\mu\nu} = \pi[A_2 k^\mu k^\nu + A_3 g^{\mu\nu}]$$

$$I_0 = \pi B_0$$

$$I_1^\mu = \pi(B_1 k^\mu + B_2 q^\mu)$$

$$I_2^{\mu\nu} = \pi[B_3 k^\mu k^\nu + \tfrac{1}{2}B_4(k^\mu q^\nu + k^\nu q^\mu) + B_5 q^\mu q^\nu + B_6 g^{\mu\nu}]$$

$$I_3^{\mu\nu} = \pi[B_7 k^\mu k^\nu + \tfrac{1}{2}B_8(k^\mu q^\nu + k^\nu q^\mu) + B_9 q^\mu q^\nu + B_{10} g^{\mu\nu}]$$

$$I_4^{\mu\nu} = \pi[B_{11} k^\mu k^\nu + \tfrac{1}{2}B_{12}(k^\mu q^\nu + 1^\nu q^\mu) + B_{13} q^\mu q^\nu$$

$$+ B_{14} g^{\mu\nu} + \tfrac{1}{2}B_{15}(k^\mu L^\nu + k^\nu L^\mu) + \tfrac{1}{2}B_{16}(q^\mu L^\nu + q^\nu L^\mu)]$$

$$I_5^{\mu\nu} = \pi[B_{17} k^\mu k^\nu + \tfrac{1}{2}B_{18}(k^\mu q^\nu + k^\nu q^\mu) + B_{19} q^\mu q^\nu$$

$$+ B_{20}\, g^{\mu\nu} + \tfrac{1}{2}B_{21}(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu) + \tfrac{1}{2}B_{22}(k^\mu p_1^\nu + k^\nu p_1^\mu)$$

$$+ \tfrac{1}{2}B_{23}(k^\mu p_2^\nu + k^\nu p_2^\mu) + \tfrac{1}{2}B_{24}(q^\mu p_1^\nu + q^\nu p_1^\mu) + \tfrac{1}{2}B_{25}(q^\mu p_2^\nu + q^\nu p_2^\mu)]$$

$$J_0 = \pi V_0$$

$$J_1^\mu = \pi(V_1 k^\mu + V_2 p_2^\mu)$$

$$J_2^{\mu\nu} = \pi[V_3 k^\mu k^\nu + \tfrac{1}{2}V_4(p_2^\mu k^\nu + p_2^\nu k^\mu) + V_5 p_2^\mu p_2^\nu + V_6\, g^{\mu\nu}]$$

$$K_0 = \pi W_0$$

$$K_1^{\mu} = \pi(W_1 k^{\mu} - W_2 p_1^{\mu})$$

$$K_2^{\mu\nu} = \pi[W_3 k^{\mu}k^{\nu} - \tfrac{1}{2}W_4(p_1^{\mu}k^{\nu} + p_1^{\nu}k^{\mu}) + W_5 p_1^{\mu}p_1^{\nu} + W_6 g^{\mu\nu}]$$

$$N_0 = \pi D_0$$

$$N_1^{\mu} = \pi(D_1 k^{\mu} + D_2 q^{\mu})$$

$$N_2^{\mu\nu} = \pi[D_3 k^{\mu}k^{\nu} + \tfrac{1}{2}D_4(k^{\mu}q^{\nu} + k^{\nu}q^{\mu}) + D_5 q^{\mu}q^{\nu} + D_6 g^{\mu\nu}]$$

$$N_3^{\mu\nu} = \pi[D_7 k^{\mu}k^{\nu} + \tfrac{1}{2}D_8(k^{\mu}q^{\nu} + k^{\nu}q^{\mu}) + D_9 q^{\mu}q^{\nu}$$

$$+ D_{10} g^{\mu\nu} + \tfrac{1}{2}D_{11}(k^{\mu}p_2^{\nu} + k^{\nu}p_2^{\mu}) + \tfrac{1}{2}D_{12}(q^{\mu}p_2^{\nu} + q^{\nu}p_2^{\mu})]$$

$$N_3'^{\mu\nu} = \pi[D_7' k^{\mu}k^{\nu} + \tfrac{1}{2}D_8'(k^{\mu}q^{\nu} + k^{\nu}q^{\mu}) + D_9' q^{\mu}q^{\nu}$$

$$+ D_{10}' g^{\mu\nu} + \tfrac{1}{2}D_{11}'(k^{\mu}p_1^{\nu} + k^{\nu}p_1^{\mu}) + \tfrac{1}{2}D_{12}'(q^{\mu}p_1^{\nu} + q^{\nu}p_1^{\mu})]$$

$$N_4^{\mu\nu} = \pi[D_{13} k^{\mu}k^{\nu} + \tfrac{1}{2}D_{14}(k^{\mu}q^{\nu} + k^{\nu}q^{\mu}) + D_{15} q^{\mu}q^{\nu}$$

$$+ D_{16} g^{\mu\nu} + \tfrac{1}{2}D_{17}(p_1^{\mu}p_2^{\nu} + p_1^{\nu}p_2^{\mu}) + \tfrac{1}{2}D_{18}(k^{\mu}p_1^{\nu} + k^{\nu}p_1^{\mu})$$

$$+ \tfrac{1}{2}D_{19}(k^{\mu}p_2^{\nu} + k^{\nu}p_2^{\mu}) + \tfrac{1}{2}D_{20}(q^{\mu}p_1^{\nu} + q^{\nu}p_1^{\mu}) + \tfrac{1}{2}D_{21}(q^{\mu}p_2^{\nu} + q^{\nu}p_2^{\mu})]$$

$$Q_0 = \pi F_0$$

$$Q_1^{\mu} = \pi(F_1 k^{\mu} + F_2 q^{\mu} + F_3 p_2^{\mu})$$

$$Q_2^{\mu\nu} = \pi[F_4 k^{\mu} k^{\nu} + \tfrac{1}{2}F_5(k^{\mu}q^{\nu} + k^{\nu}q^{\mu}) + F_6 q^{\mu}q^{\nu}$$

$$+ F_7 g^{\mu\nu} + F_8 p_2^{\mu}p_2^{\nu} + \tfrac{1}{2}F_9(k^{\mu}p_2^{\nu} + k^{\nu}p_2^{\mu})$$

$$+ \tfrac{1}{2}F_{10}(q^{\mu}p_2^{\nu} + q^{\nu}p_2^{\mu})]$$

$$R_0 = \pi G_0$$

$$R_1^{\mu} = \pi(G_1 k^{\mu} + G_2 q^{\mu} - G_3 p_1^{\mu})$$

$$R_2^{\mu\nu} = \pi[G_4 k^{\mu}k^{\nu} + \tfrac{1}{2}G_5(k^{\mu}q^{\nu} + k^{\nu}q^{\mu}) + G_6 q^{\mu}q^{\nu}$$

$$+ G_7 g^{\mu\nu} + G_8 p_1^{\mu}p_1^{\nu} - \tfrac{1}{2}G_9(k^{\mu}p_1^{\nu} + k^{\nu}p_1^{\mu})$$

$$- \tfrac{1}{2}G_{10}(q^{\mu}p_1^{\nu} + q^{\nu}p_1^{\mu})]$$

The coefficients in the above expressions are given by

$$A_0 = 2g$$

$$A_1 = g$$

$$A_2 = \frac{g}{2}\left(1 + \frac{g^2}{3}\right)$$

$$A_3 = -\frac{k^2 g^3}{6}$$

$$B_0 = -\frac{\ln L_1}{c_1}$$

$$B_1 = \frac{1}{2c_1}\left(\frac{q\cdot k}{a_1}\zeta_1 - \ln L_1\right)$$

$$B_2 = -\frac{k^2 \zeta_1}{2a_1 c_1}$$

$$B_3 = \frac{1}{8a_1^2 c_1}\{\zeta_1[4a_1(q\cdot k) + (2a_1 - 3k^2q^2)(q^2 - q\cdot k)]$$

$$+ (k^2q^2b_1^2 - 2a_1^2)\ln L_1\}$$

$$B_4 = \frac{k^2}{4a_1^2 c_1}\{\zeta_1[3(q\cdot k)(q^2 - q\cdot k) - 2a_1] - (q\cdot k)b_1^2 \ln L_1\}$$

$$B_5 = \frac{k^4}{8a_1^2 c_1}[b_1^2 \ln L_1 - 3\zeta_1(q^2 - q\cdot k)]$$

$$B_6 = \frac{k^2}{8a_1 c_1}[\zeta_1(q^2 - q\cdot k) - b_1^2 \ln L_1]$$

$$B_7 = \frac{1}{2a_1^2 c_1}\{\frac{a_1^2 \eta}{q^2 - q\cdot k} + \tfrac{1}{2}k^2q^2\zeta_1 + (q\cdot k)^2[\zeta_1 - b_1 + \eta(q^2 - q\cdot k)]$$

$$-a_1(q\cdot k)(2\eta + \ln L_1)\}$$

$$B_8 = \frac{k^2}{2a_1^2c_1}\{a_1(2\eta+\ln L_1) + q\cdot k\,[\,2b_1-3\zeta_1-2\eta(q^2-q\cdot k)]\}$$

$$B_9 = \frac{k^4}{4a_1^2c_1}[\,3\zeta_1-2b_1+2\eta(q^2-q\cdot k)]$$

$$B_{10} = -\frac{k^2\zeta_1}{4a_1c_1}$$

$$B_{11} = \frac{1}{16a_1^3c_1}[k^2(L\cdot q)-(q\cdot k)(L\cdot k)]\Big\{2a_1^2(2\eta+\ln L_1)+k^2q^2[3\zeta_1(q^2-q\cdot k)-b_1^2\ln L_1]$$

$$+2(q\cdot k)\{\zeta_1[\,3(q\cdot k)(q^2-q\cdot k)-4a_1]+2[\,b_1-\eta(q^2-q\cdot k)][\,2a_1-(q\cdot k)(q^2-q\cdot k)]\}\Big\}$$

$$+\frac{L\cdot k}{8a_1^2c_1}\Big\{\frac{2a_1^2\eta}{q^2-q\cdot k}+k^2q^2\zeta_1+2(q\cdot k)\{(q\cdot k)[\zeta_1-b_1+\eta(q^2-q\cdot k)]-a_1(2\eta+\ln L_1)\}\Big\}$$

$$+\frac{1}{8a_1^2c_1}\{\frac{q\cdot k}{a_1}[\,k^2(L\cdot q)-(q\cdot k)(L\cdot k)]-L\cdot k\}\cdot\{\zeta_1[\,3(q\cdot k)(q^2-q\cdot k)-2a_1]$$

$$-b_1^2(q\cdot k)\ln L_1\}$$

$$B_{12} = \frac{k^2}{8a_1^3c_1}\Big\{[\,k^2(L\cdot q)-(q\cdot k)(L\cdot k)\{2q^2(k^2-q\cdot k)[\,3\zeta_1-2b_1+2\eta(q^2-q\cdot k)]$$

$$+3(q\cdot k)[b_1^2\ln L_1-3\zeta_1(q^2-q\cdot k)]\}$$

$$+a_1(L\cdot k)\{2a_1(2\eta+\ln L_1)+3\zeta_1(q^2-q\cdot k)-b_1^2\ln L_1$$

$$-2(q\cdot k)[\,2\zeta_1-2b_1+2\eta\,(q^2-q\cdot k)]\}\Big\}$$

$$B_{13} = \frac{k^4}{8a_1^3 c_1}\left\{\frac{3}{2}[k^2(L\cdot q)-(q\cdot k)(L\cdot k)]\cdot[3\zeta_1(q^2-q\cdot k)-b_1{}^2\ln L_1]\right.$$

$$\left.+[3\zeta_1-2b_1+2\eta(q^2-q\cdot k)]\{(q^2-q\cdot k)[k^2(L\cdot q)-(q\cdot k)(L\cdot k)]+a_1(L\cdot k)\}\right\}$$

$$B_{14} = \frac{k^2}{16a_1{}^2 c_1}\ \{[k^2(L\cdot q)-(q\cdot k)(L\cdot k)][b_1{}^2\ln L_1-3\zeta_1(q^2-q\cdot k)]-2a_1(L\cdot k)\zeta_1\}$$

$$B_{15} = \frac{k^2}{8a_1{}^2 c_1}\ \{\zeta_1[3(q\cdot k)(q^2-q\cdot k)-2a_1]-b_1{}^2(q\cdot k)\ln L_1\}$$

$$B_{16} = \frac{k^4}{8a_1{}^2 c_1}\ [b_1{}^2\ln L_1-3\zeta_1(q^2-q\cdot k)]$$

$$B_{17} = \frac{1}{16a_1{}^3 c_1}\left\{(\frac{3\delta_1\delta_2}{a_1}-\delta_0)\{(q\cdot k)^2[\frac{b_1{}^3}{3}+\zeta_1 q^2(k^2-q\cdot k)]\right.$$

$$+q^4(k^2-q\cdot k)[(k^2-q\cdot k)(\zeta_1-b_1+\eta(q^2-q\cdot k))-\zeta_1(q\cdot k)\}$$

$$+[2a_1(p_1\cdot k)(p_2\cdot k)+b_1{}^2(\delta_0-\frac{\delta_1\delta_2}{a_1})]\ \{a_1[\frac{a_1\eta}{q^2-q\cdot k}-(q\cdot k)(2\eta+\ln L_1)]$$

$$+(q\cdot k)^2[\zeta_1-b_1+\eta_1(q^2-q\cdot k)]+\frac{k^2q^2}{2}\zeta_1\}$$

$$+\frac{k^2q^2}{4}\{(\delta_0-\frac{5\delta_1\delta_2}{a_1})[b_1{}^2(\zeta_1-\frac{4b_1}{3})-2\zeta_1(q^2-q\cdot k)^2]-\zeta_1 b_1{}^2(\delta_0-\frac{\delta_1\delta_2}{a_1})\}$$

$$+(p_1\cdot k\delta_2+p_2\cdot k\delta_1)\{\frac{3k^2q^2}{2}[3\zeta_1(q^2-q\cdot k)-b_1{}^2\ln L_1]+a_1{}^2(2\eta+\ln L_1)$$

$$+(q\cdot k)[b_1^2(3\zeta_1-\frac{10b_1}{3})-5\zeta_1(q^2-q\cdot k)^2]+(q\cdot k)[(q\cdot k)(q^2-q\cdot k)-2a_1][3\zeta_1-2b_1+2\eta(q^2-q\cdot k)]$$

$$+\frac{\delta_1\delta_2(q\cdot k)}{a_1}\{(q\cdot k)[9\zeta_1(q^2-q\cdot k)^2+b_1^2(6b_1-5\zeta_1)]+2a_1[b_1^2\ln L_1-3\zeta_1(q^2-q\cdot k)]\}$$

$$\left.+a_1(p_1\cdot k)(p_2\cdot k)\{\zeta_1[4a_1-b_1^2+(q^2-q\cdot k)^2-6(q\cdot k)(q^2-q\cdot k)]+2b_1^2[(q\cdot k)\ln L_1+\frac{b_1}{3}]\}\right\}$$

$$B_{18} = \frac{k^2}{16a_1^3 c_1}\Big\{ \Big(\frac{3\delta_1\delta_2}{a_1} - \delta_0\Big)\{a_1(q^2-q\cdot k)[3\zeta_1 - 2b_1 + 2\eta(q^2-q\cdot k)]$$

$$-2(q\cdot k)[\tfrac{1}{3}b_1^3 + (q^2-q\cdot k)^2(2\zeta_1 - b_1) + \eta(q^2-q\cdot k)^3]\}$$

$$+\frac{q\cdot k}{2}\{b_1^2\zeta_1(\delta_0 - \frac{\delta_1\delta_2}{a_1}) + (\delta_0 - \frac{5\delta_1\delta_2}{a_1})[2\zeta_1(q^2-q\cdot k)^2 - b_1^2(\zeta_1 - \frac{4b_1}{3})]\}$$

$$+[2a_1(p_1\cdot k)(p_2\cdot k) + b_1^2(\delta_0 - \frac{\delta_1\delta_2}{a_1})]\{a_1(2\eta + \ln L_1) - (q\cdot k)[3\zeta_1 - 2b_1 + 2\eta(q^2-q\cdot k)]\}$$

$$+(p_1\cdot k\,\delta_2 + p_2\cdot k\,\delta_1)\{3(q\cdot k)[b_1^2\ln L_1 - 3\zeta_1(q^2-q\cdot k)] + 2q^2(k^2-q\cdot k)[3\zeta_1 - 2b_1 + 2\eta(q^2-q\cdot k)]$$

$$+[5\zeta_1(q^2-q\cdot k)^2 + b_1^2(\frac{10b_1}{3} - 3\zeta_1)]\}$$

$$+\frac{2\delta_1\delta_2}{a_1}\{(q\cdot k)[b_1^2(5\zeta_1 - 6b_1) - 9\zeta_1(q^2-q\cdot k)^2] + a_1[3\zeta_1(q^2-q\cdot k) - b_1^2\ln L_1]\}$$

$$+2a_1(p_1\cdot k)(p_2\cdot k)[3\zeta_1(q^2-q\cdot k) - b_1^2\ln L_1]\Big\}$$

$$B_{19} = \frac{k^4}{64a_1^3 c_1}\Big\{ 2[2a_1(p_1\cdot k)(p_2\cdot k) + b_1^2(\delta_0 - \frac{\delta_1\delta_2}{a_1})][3\zeta_1 - 2b_1 + 2\eta(q^2-q\cdot k)]$$

$$+4\Big(\frac{3\delta_1\delta_2}{a_1} - \delta_0\Big)[\tfrac{1}{3}b_1^3 + (q^2-q\cdot k)^2(2\zeta_1 - b_1) + \eta(q^2-q\cdot k)^3]$$

$$+\Big(\frac{5\delta_1\delta_2}{a_1} - \delta_0\Big)[2\zeta_1(q^2-q\cdot k)^2 - b_1^2(\zeta_1 - \frac{4b_1}{3}] - \zeta_1 b_1^2(\delta_0 - \frac{\delta_1\delta_2}{a_1})$$

$$+2(p_1\cdot k\delta_2 + p_2\cdot k\delta_1)\{2(q^2-q\cdot k)[3\zeta_1 - 2b_1 + 2\eta(q^2-q\cdot k)] + 3[3\zeta_1(q^2-q\cdot k) - b_1^2\ln L_1]\}$$

$$+\frac{4\delta_1\delta_2}{a_1}[9\zeta_1(q^2-q\cdot k)^2 + b_1^2(6b_1 - 5\zeta_1)]\Big\}$$

$$B_{20} = \frac{k^2}{64a_1{}^2c_1}\{(\delta_0 - \frac{5\delta_1\delta_2}{a_1})[2\zeta_1(q^2-q\cdot k)^2 - b_1{}^2(\zeta_1 - \frac{4b_1}{3})]$$

$$+2(p_1\cdot k\delta_2 + p_2\cdot k\delta_1)[b_1{}^2 \ln L_1 - 3\zeta_1(q^2-q\cdot k)]$$

$$-\zeta_1[4a_1(p_1\cdot k)(p_2\cdot k) + b_1{}^2(\delta_0 - \frac{\delta_1\delta_2}{a_1})]\}$$

$$B_{21} = \frac{k^4}{16a_1{}^2c_1}[\zeta_1(q^2-q\cdot k)^2 - b_1{}^2(\zeta_1 - \frac{2b_1}{3})]$$

$$B_{22} = \frac{k^2}{16a_1{}^2c_1}\left\{\delta_2\{\frac{q\cdot k}{a_1}[5\zeta_1(q^2-q\cdot k)^2 - b_1{}^2(3\zeta_1 - \frac{10b_1}{3})]\right.$$

$$+b_1{}^2\ln L_1 - 3\zeta_1(q^2-q\cdot k)\}$$

$$\left.+p_2\cdot k\{\zeta_1[(q^2-q\cdot k)(4q\cdot k-q^2)+b_1{}^2-2a_1] - b_1{}^2[\frac{2b_1}{3} + (q\cdot k)\ln L_1]\}\right\}$$

$$B_{23} = \frac{k^2}{16a_1{}^2c_1}\left\{\delta_1\{\frac{q\cdot k}{a_1}[5\zeta_1(q^2-q\cdot k)^2 - b_1{}^2(3\zeta_1 - \frac{10b_1}{3})]\right.$$

$$+b_1{}^2\ln L_1 - 3\zeta_1(q^2-q\cdot k)\}$$

$$\left.+p_1\cdot k\{\zeta_1[(q^2-q\cdot k)(4q\cdot k-q^2)+b_1^2-2a_1] - b_1{}^2[\frac{2b_1}{3} + (q\cdot k)\ln L_1]\}\right\}$$

$$B_{24} = \frac{k^4}{16a_1{}^3c_1}\{\delta_2[b_1{}^2(3\zeta_1 - \frac{10b_1}{3}) - 5\zeta_1(q^2-q\cdot k)^2]$$

$$+a_1(p_2\cdot k)[b_1{}^2\ln L_1 - 3\zeta_1(q^2-q\cdot k)]\}$$

$$B_{25} = \frac{k^4}{16a_1^3 c_1} \{\delta_1[b_1^2(3\zeta_1 - \frac{10b_1}{3}) - 5\zeta_1(q^2 - q\cdot k)^2]$$

$$+ a_1(p_1\cdot k)[b_1^2 \ln L_1 - 3\zeta_1(q^2 - q\cdot k)]\}$$

$$V_0 = -\frac{\ln L_2}{c_2}$$

$$V_1 = \frac{1}{2c_2}(\frac{p_2\cdot k}{a_2}\zeta_2 - \ln L_2)$$

$$V_2 = -\frac{k^2\zeta_2}{2a_2c_2}$$

$$V_3 = \frac{1}{8a_2^2c_2}\{\zeta_2[4a_2(p_2\cdot k) + (2m^2 - p_2\cdot k)(2a_2 + 3k^2m^2)]$$

$$-(k^2m^2b_2^2 + 2a_2^2)\ln L_2\}$$

$$V_4 = \frac{k^2}{4a_2^2c_2}\{\zeta_2[3(p_2\cdot k)(2m^2 - p_2\cdot k) - 2a_2] - b_2^2(p_2\cdot k)\ln L_2\}$$

$$V_5 = \frac{k^4}{8a_2^2c_2}[b_2^2 \ln L_2 - 3\zeta_2(2m^2 - p_2\cdot k)]$$

$$V_6 = \frac{k^2}{8a_2c_2}[\zeta_2(2m^2 - p_2\cdot k) - b_2^2 \ln L_2]$$

$$D_0 = -\frac{\ln L_1}{c_1(q^2 - q\cdot k)}$$

$$D_1 = -\frac{\ln L_1}{2c_1(q^2 - q\cdot k)}$$

$$D_2 = 0$$

$$D_3 = \frac{1}{8a_1^2 c_1}\left[\zeta_1(2a_1 - 3k^2q^2) + (b_1^2k^2q^2 - 2a_1^2)\frac{\ln L_1}{(q^2 - q\cdot k)}\right]$$

$$D_4 = \frac{k^2(q\cdot k)}{4a_1^2 c_1}\left(3\zeta_1 - \frac{b_1^2 \ln L_1}{q^2 - q\cdot k}\right)$$

$$D_5 = \frac{k^4}{8a_1^2 c_1}\left(\frac{b_1^2 \ln L_1}{q^2 - q\cdot k} - 3\zeta_1\right)$$

$$D_6 = \frac{k^2}{8a_1 c_1}\left(\zeta_1 - \frac{b_1^2 \ln L_1}{q^2 - q\cdot k}\right)$$

$$D_7 = \frac{1}{8a_1^2 c_1}\left\{\delta_2(q\cdot k)\left(3\zeta_1 - \frac{b_1^2 \ln L_1}{q^2 - q\cdot k}\right)\right.$$

$$\left. - p_2\cdot k\left[\tfrac{1}{2}(k^2q^2 + 2a_1)\left(\zeta_1 - \frac{b_1^2 \ln L_1}{q^2 - q\cdot k}\right) + \zeta_1(q\cdot k)^2 + \frac{a_1^2 \ln L_1}{q^2 - q\cdot k}\right]\right\}$$

$$D_8 = \frac{k^2}{8a_1^2 c_1}\left(3\zeta_1 - \frac{b_1^2 \ln L_1}{q^2 - q\cdot k}\right)\left[(p_2\cdot k)(q\cdot k) - \delta_2\right]$$

$$D_9 = -\frac{k^4(p_2\cdot k)}{16a_1^2 c_1}\left(3\zeta_1 - \frac{b_1^2 \ln L_1}{q^2 - q\cdot k}\right)$$

$$D_{10} = \frac{k^2(p_2\cdot k)}{16a_1 c_1}\left(\zeta_1 - \frac{b_1^2 \ln L_1}{q^2 - q\cdot k}\right)$$

$$D_{11} = \frac{k^2}{8a_1 c_1}\left(\zeta_1 - \frac{b_1^2 \ln L_1}{q^2 - q\cdot k}\right)$$

$$D_{12} = 0$$

$$D_{13} = \frac{1}{64a_1^2c_1}\Big\{2a_1(\delta_0 - \frac{\delta_1\delta_0}{a_1})\{\zeta_1 + \frac{(q\cdot k)^2}{a_1^2}[\frac{2b_1^3}{3} + \zeta_1(q^2-q\cdot k)^2]\}$$

$$+ \frac{k^2q^2}{2a_1}(\delta_0 - \frac{5\delta_1\delta_2}{a_1})\{\frac{2b_1^3}{3} + \zeta_1[(q^2-q\cdot k)^2-b_1^2]\}$$

$$+ \tfrac{1}{2}[\frac{b_1^2}{a_1}(\delta_0 - \frac{\delta_1\delta_2}{a_1}) + 2(p_1\cdot k)(p_2\cdot k)]\cdot[\zeta_1(4a_1-5k^2q^2) + (k^2q^2b_1^2-4a_1^2)\frac{\ln L_1}{q^2-q\cdot k}]$$

$$+ \frac{(q\cdot k)}{a_1}(\delta_1 p_2\cdot k + \delta_2 p_1\cdot k)\{(4a_1-b_1^2)(\zeta_1 - \frac{b_1^2\ln L_1}{q^2-q\cdot k}) + 5\zeta_1[(q^2-q\cdot k)^2-b_1^2] + 8a_1\zeta_1 + \frac{10b_1^3}{3}\}$$

$$+ (p_1\cdot k)(p_2\cdot k)\{(b_1^2-k^2q^2-8a_1)(\zeta_1 - \frac{b_1^2\ln L_1}{q^2-q\cdot k}) + \zeta_1[b_1^2-(q^2-q\cdot k)^2] - \frac{2b_1^3}{3}\}$$

$$+ \frac{\delta_1\delta_2(q\cdot k)^2}{a_1^2}\{9\zeta_1[b_1^2-(q^2-q\cdot k)^2] + b_1^2(\zeta_1 - \frac{b_1^2\ln L_1}{q^2-q\cdot k}) - 6b_1^3\}\Big\}$$

$$D_{14} = \frac{k^2}{64a_1^3c_1}\Big\{(q\cdot k)\{(5\zeta_1 - \frac{b_1^2\ln L_1}{q^2-q\cdot k})[b_1^2(\delta_0 - \frac{\delta_1\delta_2}{a_1}) + 2a_1(p_1\cdot k)(p_2\cdot k)]$$

$$+ (\frac{17\delta_1\delta_2}{a_1} - 5\delta_0)[\frac{2b_1^3}{3} + \zeta_1(q^2-q\cdot k)^2] + 2a_1(p_1\cdot k)(p_2\cdot k)(\zeta_1 - \frac{b_1^2\ln L_1}{q^2-q\cdot k})$$

$$-b_1^2\zeta_1(\frac{5\delta_1\delta_2}{a_1} - \delta_0) + \frac{2\delta_1\delta_2}{a_1}[\zeta_1(9(q^2-q\cdot k)^2-10b_1^2) + b_1^3(6 + \frac{b_1\ln L_1}{q^2-q\cdot k})]\}$$

$$-(\delta_1 p_2\cdot k + \delta_2 p_1\cdot k)\{\zeta_1[12a_1 + 5(q^2-q\cdot k)^2 - 6b_1^2]$$

$$+ b_1^2[\frac{10b_1}{3} + \frac{(b_1^2-4a_1)\ln L_1}{q^2-q\cdot k}]\}\Big\}$$

$$D_{15} = \frac{k^4}{128a_1^3 c_1} \Big\{ (5\delta_0 - \frac{17\delta_1\delta_2}{a_1}) [\frac{2b_1^3}{3} + \zeta_1(q^2 - q\cdot k)^2]$$

$$-(5\zeta_1 - \frac{b_1^2 \ln L_1}{q^2 - q\cdot k}) [b_1^2(\delta_0 - \frac{\delta_1\delta_2}{a_1}) + 2a_1(p_1\cdot k)(p_2\cdot k)]$$

$$+ b_1^2\zeta_1 (\frac{5\delta_1\delta_2}{a_1} - \delta_0) - 2a_1(p_1\cdot k)(p_2\cdot k)(\zeta_1 - \frac{b_1^2 \ln L_1}{q^2 - q\cdot k})$$

$$+ \frac{2\delta_1\delta_2}{a_1} \{\zeta_1[10b_1^2 - 9(q^2 - q\cdot k)^2] - b_1^3(6 + \frac{b_1 \ln L_1}{q^2 - q\cdot k})\} \Big\}$$

$$D_{16} = \frac{k^2}{128a_1^2 c_1} \Big\{ (\frac{5\delta_1\delta_2}{a_1} - \delta_0)\{\frac{2b_1^3}{3} + \zeta_1[(q^2 - q\cdot k)^2 - b_1^2]\}$$

$$+ (\zeta_1 - \frac{b_1^2 \ln L_1}{q^2 - q\cdot k}) [b_1^2(\delta_0 - \frac{\delta_1\delta_2}{a_1}) + 4a_1(p_1\cdot k)(p_2\cdot k)] \Big\}$$

$$D_{17} = \frac{k^4}{64a_1^2 c_1} \{\zeta_1[2b_1^2 - (q^2 - q\cdot k)^2] - b_1^3(\frac{2}{3} + \frac{b_1 \ln L_1}{q^2 - q\cdot k})\}$$

$$D_{18} = \frac{k^2}{64a_1^2 c_1} \Big\{ \frac{\delta_2 q\cdot k}{a_1} \{\zeta_1[6b_1^2 - 5(q^2 - q\cdot k)^2] - b_1^3(\frac{10}{3} + \frac{b_1 \ln L_1}{q^2 - q\cdot k})\}$$

$$+ p_2\cdot k\{\zeta_1[4a_1 - 2b_1^2 + (q^2 - q\cdot k)^2] + b_1^2[\frac{2b_1}{3} + \frac{(b_1^2 - 4a_1)\ln L_1}{q^2 - q\cdot k}]\} \Big\}$$

$$D_{19} = \frac{k^2}{64a_1^2 c_1} \Big\{ \frac{\delta_1 q\cdot k}{a_1} \{\zeta_1[6b_1^2 - 5(q^2 - q\cdot k)^2] - b_1^3(\frac{10}{3} + \frac{b_1 \ln L_1}{q^2 - q\cdot k})\}$$

$$+ p_1\cdot k \{\zeta_1[4a_1 - 2b_1^2 + (q^2 - q\cdot k)^2] + b_1^2[\frac{2b_1}{3} + \frac{(b_1^2 - 4a_1)\ln L_1}{q^2 - q\cdot k}]\} \Big\}$$

$$D_{20} = \frac{k^4\delta_2}{64a_1^3c_1}\{\zeta_1[5(q^2-q\cdot k)^2-6b_1^2]+b_1^3(\frac{10}{3}+\frac{b_1\ln L_1}{q^2-q\cdot k})\}$$

$$D_{21} = \frac{k^4\delta_1}{64a_1^3c_1}\{\zeta_1[5(q^2-q\cdot k)^2-6b_1^2] + b_1^3(\frac{10}{3}+\frac{b_1\ln L_1}{q^2-q\cdot k})\}$$

$$F_0 = \frac{\ln L_4}{(N_1)^{\frac{1}{2}}}$$

$$F_1 = \frac{1}{2(a_1a_2-\delta_2^2} \{\frac{\ln L_1}{c_1}(\delta_2 q\cdot k - a_1 p_2\cdot k) + \frac{\ln L_2}{c_2}(a_2 q\cdot k - \delta_2 p_2\cdot k)$$

$$+ \frac{\ln L_4}{(N_1)^{\frac{1}{2}}}[(2m^2-p_2\cdot k)(a_1p_2\cdot k-\delta_2q\cdot k)+(q^2-q\cdot k)(\delta_2p_2\cdot k-a_2q\cdot k)$$

$$+a_1a_2-\delta_2^2]\}$$

$$F_2 = \frac{k^2}{2(a_1a_2-\delta_2^2)}\{c_2\ln L_2 - \frac{\delta_2}{c_1}\ln L_1$$

$$+ \frac{\ln L_4}{(N_1)^{\frac{1}{2}}}[a_2(q^2-q\cdot k) + \delta_2(2m^2-p_2\cdot k)]\}$$

$$F_3 = \frac{k^2}{a(a_1a_2-\delta_2^2)}\{\frac{\delta_2}{c_2}\ln L_2 - c_1\ln L_1$$

$$- \frac{\ln L_4}{(N_1)^{\frac{1}{2}}}[a_1(2m^2-p_2\cdot k) + \delta_2(q^2-q\cdot k)]\}$$

$$F_4 = \frac{1}{4a_1^{\,2}c_1(a_1a_2-\delta_2^{\,2})}\Bigg\{ \frac{(q\cdot k)^2 c_1\delta_2\zeta_2(\delta_2^{\,2}-a_1a_2)}{a_2c_2}$$

$$+c_1(q\cdot k)(a_1a_2-\delta_2^{\,2})\left[\frac{(q^2-q\cdot k)\ln L_4}{(N_1)^{\frac{1}{2}}} + \frac{\ln L_2}{c_2}\right]\cdot\{ q\cdot k)(q^2-q\cdot k)-2a$$

$$+\frac{2(\delta_2 q\cdot k-a_1p_2\cdot k)}{(a_1a_2-\delta_2^{\,2})}\left[\delta_2(q^2-q\cdot k)+a_1(2m^2-p_2\cdot k)\right]\}$$

$$+\frac{c_1\ln L_4(\delta_2 q\cdot k-a_1p_2\cdot k)}{(N_1)^{\frac{1}{2}}}\left[\delta_2(q^2-q\cdot k)+a_1(2m^2-p_2\cdot k)\right]\cdot\{-2a_1$$

$$+\frac{(\delta_2 q\cdot k-a_1p_2\cdot k)}{(a_1a_2-\delta_2^{\,2})}\left[\delta_2(q^2-q\cdot k)+a_1(2m^2-p_2\cdot k)\right]\}$$

$$+\frac{a_1c_1\ln L_4}{2(N_1)^{\frac{1}{2}}}\left\{2a_1(a_1a_2-\delta_2^{\,2})-N_1\left[k^2q^2+\frac{(\delta_2 q\cdot k-a_1p_2\cdot k)^2}{(a_1a_2-\delta_2^{\,2})}\right]\right\}$$

$$+2(\delta_2 q\cdot k-a_1p_2\cdot k)\left[a_1\left(a_1\ln L_1-\frac{\delta_2c_1\ln L_2}{c_2}\right)-q\cdot k\left(a_1\zeta_1+\frac{\delta_2^{\,2}c_1\zeta_2}{a_2c_2}\right)\right]$$

$$+\frac{(\delta_2 q\cdot k-a_1p_2\cdot k)^2}{(a_1a_2-\delta_2^{\,2})}\left\{\frac{c_1\ln L_2}{c_2}\left[(\delta_2^{\,2}+a_1a_2)(q^2-q\cdot k)+3a_1\delta_2(2m^2-p_2\cdot k)\right]\right.$$

$$\left.-2a_1\left[a_1\ln L_1(2m^2-p_2\cdot k)+\delta_2(\zeta_1+b_1)\right]-\frac{\delta_2^{\,3}\zeta_2}{a_2}\right\}$$

$$+a_1k^2q^2\left\{\frac{c_1\ln L_2}{c_2}\left[a_2(q^2-q\cdot k)+\delta_2(2m^2-p_2\cdot k)\right]-\delta_2(\zeta_1+2b_1)\right.$$

$$\left.-a_1\ln L_1(2m^2-p_2\cdot k)\right\}\Bigg\}$$

$$F_5 = \frac{k^2}{2a_1^{\,2}(a_1a_2-\delta_2^{\,2})}\left\{\frac{a_1\delta_2}{c_1}\left(\frac{\delta_2 c_1 \ln L_2}{c_2} - a_1 \ln L_1\right)\right.$$

$$+\left[\frac{(q^2-q\cdot k)\ln L_4}{(N_1)^{\frac{1}{2}}}+\frac{\ln L_2}{c_2}\right]\{(a_1a_2-\delta_2^{\,2})[a_1-(q\cdot k)(q^2-q\cdot k)]$$

$$-(2\delta_2 q\cdot k-a_1p_2\cdot k)[\delta_2(q^2-q\cdot k)+a_1(2m^2-p_2\cdot k)]\}$$

$$+\frac{\delta_2 \ln L_4}{(N_1)^{\frac{1}{2}}}[\delta_2(q^2-q\cdot k)+a_1(2m^2-p_2\cdot k)]\ \{a_1$$

$$-\frac{(\delta_2 q\cdot k-a_1p_2\cdot k)}{(a_1a_2-\delta_2^{\,2})}[\delta_2(q^2-q\cdot k)+a_1(2m^2-p_2\cdot k)]\}$$

$$+\frac{a_1(N_1)^{\frac{1}{2}}\ln L_4}{2}\left[q\cdot k+\frac{\delta_2(\delta_2 q\cdot k-a_1p_2\cdot k)}{(a_1a_2-\delta_2^{\,2})}\right]$$

$$+\frac{a_1(q\cdot k)}{c_1}\{\delta_2(\zeta_1+2b_1)+a_1\ln L_1(2m^2-p_2\cdot k)-\frac{c_1\ln L_2}{c_2}[a_2(q^2-q\cdot k)+\delta_2(2m^2-p_2\cdot k)]\}$$

$$+\frac{(\delta_2 q\cdot k-a_1p_2\cdot k)}{c_1(a_1a_2-\delta_2^{\,2})}\left\{(a_1a_2-\delta_2^{\,2})\left(a_1\zeta_1+\frac{b_1\delta_2^{\,2}\zeta_2}{a_2b_2}\right)+\frac{\delta_2^{\,2}\zeta_2}{a_2}\right.$$

$$+2a_1\delta_2[a_1\ln L_1(2m^2-p_2\cdot k)+\delta_2(\zeta_1+b_1)]$$

$$\left.-\frac{\delta_2c_1\ln L_2}{c_2}[(\delta_2^{\,2}+a_1a_2)(q^2-q\cdot k)+3a_1\delta_2(2m^2-p_2\cdot k)]\right\}$$

$$\left.+\frac{a_1\delta_2(q\cdot k)}{c_1}\left(\zeta_1+\frac{c_1\zeta_2}{c_2}\right)\right\}$$

$$F_6 = \frac{k^4}{4a_1^2(a_1a_2-\delta_2^2)^2}\Bigg\{\frac{\ln L_4}{(N_1)^{\frac{1}{2}}}\{\delta_2^2[a_1(2m^2-p_2\cdot k)+\delta_2(q^2-q\cdot k)]^2-\frac{a_1^2a_2N_1}{2}\}$$

$$+(a_1a_2-\delta_2^2)\left[\frac{\ln L_4(q^2-q\cdot k)}{(N_1)^{\frac{1}{2}}}+\frac{\ln L_2}{c_2}\right]\{(a_1a_2-\delta_2^2)(q^2-q\cdot k)$$

$$+2\delta_2[a_1(2m^2-p_2\cdot k)+\delta_2(q^2-q\cdot k)]\}$$

$$+\frac{\ln L_2}{c_2}\{\delta_2^2[(\delta_2^2+a_1a_2)(q^2-q\cdot k)+3a_1\delta_2(2m^2-p_2\cdot k)]$$

$$+a_1(a_1a_2-\delta_2^2)[a_2(q^2-q\cdot k)+\delta_2(2m^2-p_2\cdot k)]\}$$

$$-\frac{a_1^2\ln L_1}{c_1}(2m^2-p_2\cdot k)(\delta_2^2+a_1a_2)-\frac{a_1\delta_2}{c_1}[\zeta_1(\delta_2^2+a_1a_2)+2b_1a_1a_2]$$

$$-\frac{\delta_2\zeta_2}{a_2c_2}[a_1^2a_2^2+\delta_2^4(\frac{c_2}{c_1}-1)]+\frac{2a_1\delta_2\zeta_1}{c_1}(\delta_2^2-a_1a_2)\Bigg\}$$

$$F_7 = \frac{k^2}{4c_1(a_1a_2-\delta_2^2)}\{\delta_2(\zeta_1+2b_1)+\frac{c_1}{2}(N_1)^{\frac{1}{2}}\ln L_4+a_1\ln L_1(2m^2-p_2\cdot k)$$

$$-\frac{c_1\ln L_2}{c_2}[a_2(q^2-q\cdot k)+\delta_2(2m^2-p_2\cdot k)]\}$$

$$F_8 = \frac{k^4}{4c_1(a_1a_2-\delta_2{}^2)^2}\int_0^\rho \Big\{ \frac{c_1 \ln L_4}{(N_1)^{\frac{1}{2}}} \{[a_1(2m^2-p_2\cdot k)+\delta_2(q^2-q\cdot k)]^2 - \frac{a_1N_1}{2}\}$$

$$+ \frac{c_1 \ln L_2}{c_2}[(\delta_2{}^2+a_1a_2)(q^2-q\cdot k)+3a_1\delta_2(2m^2-p_2\cdot k)]$$

$$-2a_1[a_1 \ln L_1(2m^2-p_2\cdot k)+\delta_2(\zeta_1+b_1)] - \frac{\delta_2{}^3\zeta_2}{a_2} \Big\}$$

$$F_9 = \frac{k^2}{2a_1(a_1a_2-\delta_2{}^2)^2}\int_0^\rho \Big\{ \frac{\ln L_4}{(N_1)^{\frac{1}{2}}}[a_1(2m^2-p_2\cdot k)+\delta_2(q^2-q\cdot k)] \cdot$$

$$\cdot \{(\delta_2 q\cdot k - a_1p_2\cdot k)[a_1(2m^2-p_2\cdot k)+\delta_2(q^2-q\cdot k)] + (a_1a_2-\delta_2{}^2)[(q\cdot k)(q^2-q\cdot k)-a_1]\}$$

$$+ \frac{\ln L_2}{c_2}\{(\delta_2 q\cdot k-a_1p_2\cdot k)[(\delta_2{}^2+a_1a_2)(q^2-q\cdot k)+3a_1\delta_2(2m^2-p_2\cdot k)]$$

$$+(q\cdot k)(a_1a_2-\delta_2{}^2)[a_1(2m^2-p_2\cdot k)+\delta_2(q^2-q\cdot k)] - a_1\delta_2(a_1a_2-\delta_2{}^2)\}$$

$$- \frac{a_1}{2}(N_1)^{\frac{1}{2}}\ln L_4(\delta_2 q\cdot k-a_1p_2\cdot k) - \frac{2a_1}{c_1}(\delta_2 q\cdot k-a_1p_2\cdot k)[a_1 \ln L_1(2m^2-p_2\cdot k)$$

$$+\delta_2(\zeta_1+b_1)]$$

$$+ \frac{a_1(a_1a_2-\delta_2^2)}{c_1}[a_1 \ln L_1 - \zeta_1(q\cdot k)] - \frac{\delta_2^2\zeta_2}{a_2c_1}[\delta_2(\delta_2 q\cdot k-a_1p_2\cdot k)+\frac{c_1(q\cdot k)}{c_2}(a_1a_2-\delta_2^2)] \Big\}$$

$$F_{10} = \frac{k^4}{2a_1(a_1a_2-\delta_2^2)^2}\Bigg\{\frac{2a_1\delta_2}{c_1}[a_1\ln L_1(2m^2-p_2\cdot k)+\delta_2(\zeta_1+b_1)]$$

$$-\frac{\delta_2\ln L_4}{(N_1)^{\frac{1}{2}}}\{[a_1(2m^2-p_2\cdot k)+\delta_2(q^2-q\cdot k)]^2-\frac{a_1N_1}{2}\}$$

$$-\frac{\delta_2\ln L_2}{c_2}[(\delta_2^2+a_1a_2)(q^2-q\cdot k)+3a_1\delta_2(2m^2-p_2\cdot k)]$$

$$+(\delta_2^2-a_1a_2)[a_1(2m^2-p_2\cdot k)+\delta_2(q^2-q\cdot k)][\frac{\ln L_2}{c_2}+\frac{\ln L_4(q^2-q\cdot k)}{(N_1)^{\frac{1}{2}}}]$$

$$+\frac{\delta_2^2\zeta_2}{a_2}[\frac{\delta_2^2}{c_1}+\frac{(a_1a_2-\delta_2^2)}{c_2}]+\frac{a_1\zeta_2}{c_1}(a_1a_2-\delta_2^2)\Bigg\}$$

The W_i and G_i can be obtained from the corresponding V_i and F_i, respectively, by the substitutions

$p_2 \rightarrow -p_1$

$a_2 \rightarrow a_3$

$c_2 \rightarrow c_3$

$b_2 \rightarrow b_3$

$L_2 \rightarrow L_3$

$\zeta_2 \rightarrow \zeta_3$

$\delta_2 \rightarrow -\delta_1$

$N_1 \rightarrow N_2$

$L_4 \rightarrow L_5$

The D'_7, D'_8, . . . , D'_{12} can be obtained from D_7, D_8, . . . , D_{12}, respectively, by the substitutions

$P_2 \rightarrow P_1$

$\delta_2 \rightarrow \delta_1$

APPENDIX D

Kinematics and Limits of Integration

The region of integration for the $d^3q\, d^3p_2$ integration is determined by the kinematics of the process. To discuss this, we write

$$\int d^3q \int \frac{d^3p_2}{E_2} = 2\pi \int q^2 dq \int \beta_2 E_2 dE_2 \int d(\cos\theta q) \int d(\cos\theta_2) \int d\phi_2 \qquad \text{(D-1)}$$

where the angles are defined by

$$\hat{\vec{p}}_1 \cdot \hat{\vec{q}} = \cos\theta q$$

$$\hat{\vec{p}}_1 \cdot \hat{\vec{p}}_2 = \cos\theta_2 \qquad \text{(D-2)}$$

$$\hat{\vec{p}}_2 \cdot \hat{\vec{q}} = \cos\theta q \cos\theta_2 + \sin\theta_2 \sin\theta q \cos\phi_2$$

We must now determine the limits on the integrals in equation D-1. The integrals are to be carried out in the order indicated, namely the ϕ_2 integral first and the q integral last.

We will first discuss the limits on the q and E_2 integrals. If we for convenience consider E_2 first, we see that since we are assuming the target nucleus to be infinitely heavy, the laboratory system and the center-of-momentum system are identical. Also, the recoil energy $q^2/2M$ is completely negligible. For these reasons we have immediately as the limits on E_2,

$$m \leq E_2 \leq E_1 - 2m \qquad \text{(D-3)}$$

Now let us consider the limits on $q \equiv |\vec{q}|$. We have

$$q = |\vec{p}_1 - \vec{p}_2 - \vec{p}_+ - \vec{p}_-| \tag{D-4}$$

The maximum value of q will occur when all of the final-state particles come off in the backward direction. This gives

$$\begin{aligned} q_{max} &= |\vec{p}_1| + |\vec{p}_2| + |\vec{p}_+| + |\vec{p}_-| \\ &= (E_1{}^2-m^2)^{\frac{1}{2}} + (E_2{}^2-m^2)^{\frac{1}{2}} + (E_+{}^2-m^2)^{\frac{1}{2}} + (E_-{}^2-m^2)^{\frac{1}{2}} \end{aligned} \tag{D-5}$$

Dropping terms of order $(m/E)^2$, we have

$$q_{max} = E_1 + E_2 + E_+ + E_- = 2E_1 \tag{D-6}$$

The minimum value of q will occur when all the final-state particles come off in the forward direction. Then we have

$$\begin{aligned} q_{min} &= |\vec{p}_1| - |\vec{p}_2| - |\vec{p}_+| - |\vec{p}_-| \\ &= (E_1{}^2-m^2)^{\frac{1}{2}} - (E_2{}^2-m^2)^{\frac{1}{2}} - (E_+{}^2-m^2)^{\frac{1}{2}} - (E_-{}^2-m^2)^{\frac{1}{2}} \end{aligned} \tag{D-7}$$

Since we have already integrated over p_+ and p_-, we must minimize the above expression with respect to E_+, E_- subject to the constraint $E_+ + E_- = E_1 - E_2$. From the symmetry involved, we see that the minimum will occur for $E_+ = E_- = (E_1 - E_2)/2$. This gives for q_{min} as a function of E_2,

$$q_{min}(E_2) = (E_1{}^2-m^2)^{\frac{1}{2}} - (E_2{}^2-m^2)^{\frac{1}{2}} - [(E_1-E_2)^2-4m^2]^{\frac{1}{2}} \tag{D-8}$$

From this expression we find for the end points $E_2 = m$, $E_2 = E_1 - 2m$, the values

$$q_{min}\,(E_2 = m) = m, \qquad q_{min}\,(E_2 = E_1 - 2m) = 2m\;, \tag{D-9}$$

where we have dropped terms of order $(m/E_1)^2$ compared to one.

For the minimum of $q_{min}(E_2)$ with respect to E_2, we find from the symmetry involved that the minimum occurs for $E_2 = E_+ = E_- = E_1/3$, and is

$$q_{min} = \frac{4m^2}{E_1} \tag{D-10}$$

where we have again dropped terms of order $(m/E_1)^2$.

From the above results, we find that the region of integration in the q, E_2 plane has the form shown in Figure 5a.

If we now wanted to integrate over q first, and then E_2, the region of integration would be given by equations D-3, D-6, and D-8. However, for convenience in treating screening, we wish to do the integration over E_2 before the integration over q. To find the limits for this order of integration, we must solve the equation $q = q_{min}(E_2)$ for E_2 as a function of q. The final result we obtain for the limits is

$$\frac{4m^2}{E_1} < q < 2E_1 \tag{D-11}$$

and

$$\epsilon_1(q) < E_2 < \epsilon_2(q) \tag{D-12}$$

where

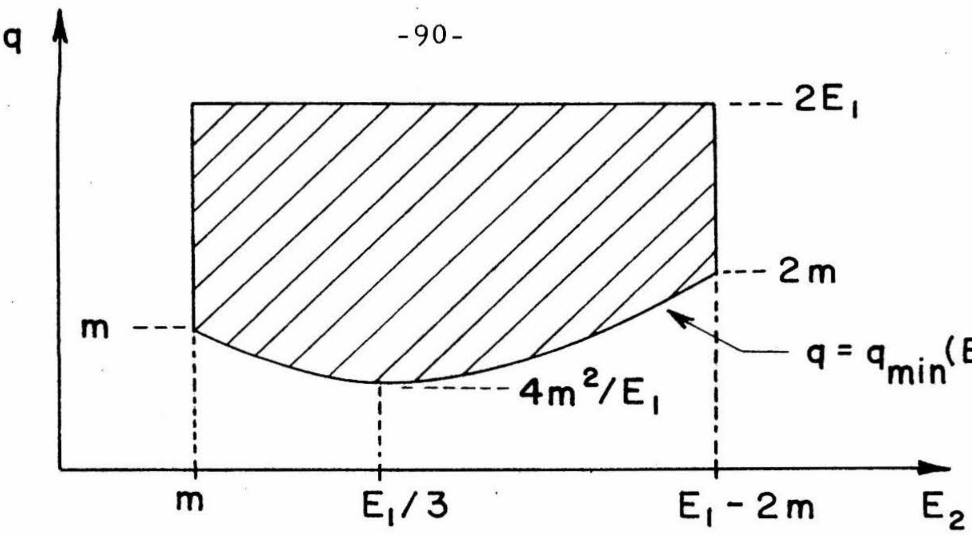

FIG. 5a REGION OF INTEGRATION IN q, E_2 PLANE

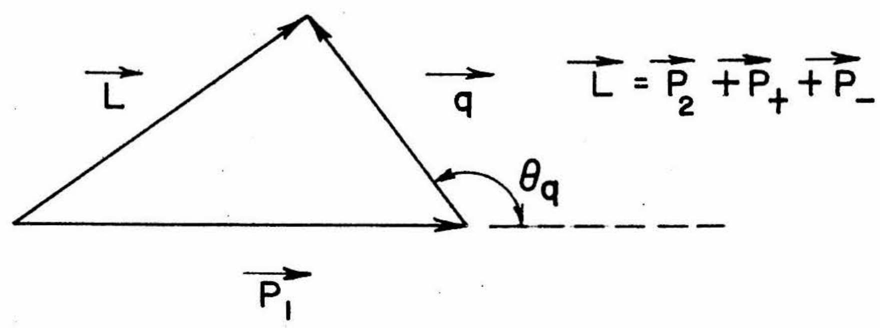

FIG. 5b VECTOR RELATIONSHIP FOR DETERMINING $(\cos\theta_q)_{max}$

$$\epsilon_1(q) = \begin{cases} m & \text{for } q > m \\ \frac{E_1}{2}\left\{1 + \frac{1}{(2qE_1+m^2-q^2)}\ \{-3m^2 - [4qE_1^2(q - \frac{4m^2}{E_1} - \frac{3q^2}{E_1})]^{\frac{1}{2}}\}\right\} & \text{for } q < m \end{cases} \qquad \text{(D-13)}$$

and

$$\epsilon_2(q) = \begin{cases} E_1 - 2m & \text{for } q > 2m \\ \frac{E_1}{2}\left\{1 + \frac{1}{(2qE_1+m^2-q^2)}\ \{-3m^2 + [4qE_1^2(q - \frac{4m^2}{E_1} - \frac{3q^2}{E_1})]^{\frac{1}{2}}\}\right\} & \text{for } q < 2m \end{cases} \qquad \text{(D-14)}$$

In the expressions for $\epsilon_1(q)$ and $\epsilon_2(q)$ we have again dropped terms of order $(m/E_1)^2$.

We must now consider the limits on the cos θq integration. We wish to determine these limits for fixed values of q and E_2, and for arbitrary values of cos θ_2, ϕ_2, and p_+, p_-. We first note that, because of the definition $\vec{p}_1 + \vec{q} = \vec{p}_2 + \vec{p}_+ + \vec{p}_-$, the nuclear recoil is $-\vec{q}$. Since the nuclear recoil can certainly be in the forward direction, this means that the lower limit on cos θq is

$$(\cos\theta q)_{min} = -1 \tag{D-15}$$

To determine the upper limit on cos θq, we note that from momentum conservation we have the vector relationship shown in Figure 5b, where $\vec{L} = \vec{p}_2 + \vec{p}_+ + \vec{p}_-$. Since $|\vec{L}| < |\vec{p}_1|$, we see that $(\cos\theta q)_{max} < 0$. From the cosine law we now have

$$\begin{aligned}\vec{L}^2 &= \vec{p}_1^{\,2} + \vec{q}^{\,2} - 2\,|\vec{p}_1|\,|\vec{q}|\cos(\pi - \theta q) \\ &= \vec{p}_1^{\,2} + \vec{q}^{\,2} + 2\,|\vec{p}_1|\,|\vec{q}|\cos\theta q\end{aligned} \tag{D-16}$$

From this we obtain

$$(\cos\theta q)_{max} = -\frac{\vec{p}_1^{\,2} + \vec{q}^{\,2} - (\vec{L}^2)_{max}}{2\,|\vec{p}_1|\,|\vec{q}|} \tag{D-17}$$

where $\vec{L}^2$ is to be maximized for fixed $|\vec{q}|$ and E_2. It can easily be demonstrated that the maximum value of $\vec{L}^2$ will occur when $\vec{p}_2$, $\vec{p}_+$ and $\vec{p}_-$ all come off in the same direction, and when $E_+ = E_- = (E_1 - E_2)/2$. Using this fact, we obtain finally, again denoting $q \equiv |\vec{q}|$,

$$(\cos\theta q)_{max} = -\frac{q^2 + 4m^2 + 2E_2(E_1 - E_2) - 2(E_2^{\,2} - m^2)^{\frac{1}{2}}[(E_1 - E_2)^2 - 4m^2]^{\frac{1}{2}}}{2q(E_1^{\,2} - m^2)^{\frac{1}{2}}} \tag{D-18}$$

Now we must determine the limits on cos θ_2. These limits must be determined for fixed values of q, E_2 and cos θq. We first note that $\vec{p}_2$ can certainly come off in the forward direction, which means that

$$(\cos\theta_2)_{max} = 1 \tag{D-19}$$

The lower limit on $\cos\theta_2$ is most easily determined if one notes that the following kinematical inequality must always be satisfied:

$$k^2 \leq -4m^2 \tag{D-20}$$

where $k = p_1 + q - p_2$. To prove this inequality, we note that, because of energy-momentum conservation, $k = p_+ + p_-$. In the coordinate system in which $\vec{k} = 0$, we then have $k^2 = -(E_+ + E_-)^2 \leq -4m^2$. Since k^2 is a scalar invariant, this inequality must hold in all coordinate systems, which completes the proof of equation D-20.

The accessible region of final-state phase space is then characterized by $k^2 < -4m^2$, and the boundary of this region is determined by $k^2 = -4m^2$. This means that $(\cos\theta_2)_{min}$ is determined by solving the equation $k^2 = -4m^2$ for $\cos\theta_2$ as a function of E_1, E_2, q, $\cos\theta q$, and ϕ_2, and then minimizing this with respect to ϕ_2. Proceeding to do this, we have

$$\begin{aligned} k^2 &= 2p_1 \cdot q - 2p_1 \cdot p_2 - 2p_2 \cdot q + q^2 - 2m^2 \\ &= 2p_1 \cdot q + q^2 - 2m^2 + 2E_1E_2(1-\beta_1\beta_2\cos\theta_2) \\ &\quad -2E_2\beta_2 q[\cos\theta q\cos\theta_2 + \sin\theta q\sin\theta_2\cos\phi_2] \end{aligned} \tag{D-21}$$

Setting $k^2 = -4m^2$ then gives

$$\begin{aligned} \cos\theta_2 &= \frac{1}{(E_1\beta_1 + q\cos\theta q)^2 + q^2\sin^2\theta q\cos^2\phi_2} \cdot \\ &\quad \cdot \Big\{ (\cos\theta_2)_0 (E_1\beta_1 + q\cos\theta q)^2 \\ &\quad - q\sin\theta q\cos\phi_2 \{(E_1\beta_1 + q\cos\theta q)^2[1-(\cos\theta_2)_0^2] + q^2\sin^2\theta q\cos^2\phi_2\}^{\frac{1}{2}} \Big\} \end{aligned} \tag{D-22}$$

where we have denoted

$$(\cos\theta_2)_0 = \frac{2E_1E_2 + 2p_1\cdot q + q^2 + 2m^2}{2E_2\beta_2(E_1\beta_1 + q\cos\theta q)} \qquad \text{(D-23)}$$

Minimizing the expression in equation D-22 with respect to ϕ_2, we obtain finally

$$(\cos\theta_2)_{min} = \frac{1}{(E_1\beta_1 + q\cos\theta q)^2 + q^2\sin^2\theta q} \cdot$$

$$\cdot \Big\{ (\cos\theta_2)_0 (E_1\beta_1 + q\cos\theta q)^2 \qquad \text{(D-24)}$$

$$- q\sin\theta q \{ (E_1\beta_1 + q\cos\theta q)^2 [1 - (\cos\theta_2)_0^2] + q^2\sin^2\theta q \}^{\frac{1}{2}} \Big\}$$

All that remains now is to determine the limits on ϕ_2. In principle these could also be determined from the condition $k^2 \leq -4m^2$. However, in practice we have found it more convenient to extend the ϕ_2 integral over the entire range $0 \leq \phi_2 \leq 2\pi$, and then apply the requirement $k^2 \leq -4m^2$ as a subsidiary condition, i.e. we set the integrand equal to zero for $k^2 > -4m^2$.

This completes our discussion of the limits of integration. The final results are given in equations D-11, D-12, D-15, D-18, D-19, and D-24.

APPENDIX E

Details of Numerical Calculation

As indicated in Section II, the five dimensional numerical integration over ϕ_2, cos θ_2, cos θq, E_2, and q was done in two parts: first a four dimensional multiple integration over ϕ_2, cos θ_2, cos θq, and E_2, the result of which, $\sigma(q)$, was tabulated as a function of E_1 and q, and finally the single integration over q.

The four-fold multiple integration was treated numerically as a succession of single variable integrations, each of which was done using Gaussian quadrature. For the E_2, cos θq, and cos θ_2 integrals, the integration interval was divided into four unit cells with four-point Gaussian quadrature used in each cell. For the ϕ_2 interval one unit cell was used, with two-point Gaussian quadrature. This arrangement resulted in a distribution of 8192 points throughout the four dimensional region of integration.

The numerical accuracy of the four-fold multiple integration was tested for elected values of E_1 and q by doubling the number of points. The error in $\sigma(q)$ was found to be of the order of 1%.

For the final integration over q, it was necessary to interpolate in the $\sigma(q)$ tabulation. For this purpose it was found adequate to tabulate $\sigma(q)$ at four values per power of ten, i.e., q = 1.0, 2.5, 5.0, 7.5, 10.0, etc. Using four-point Aitken-Lagrangian interpolation then resulted in an interpolation accuracy of the order of 2%.

The final integration over q was carried out using a special Gaussian quadrature program which continuously subdivided the integration interval until the desired accuracy was obtained. The truncation error in this integration was of the order of 0.1%. In addition to this error, the final integration was cut off at $q_{max} = 10$ mev/c instead of being carried all the way to $q_{max} = 2E_1$. This introduced a further error of the order of 0.1%.

The total numerical error in the calculation was of the order of 3-4%.

APPENDIX F

Numerical Results for $\sigma(q)$

We introduce the notation

$$\sigma_{AB}(q) = \frac{1}{\pi m^2} \int \beta_2 E_2 dE_2 \int d(\cos\theta q) \int d(\cos\theta_2) \int d\phi_2 \cdot$$

$$\cdot\ (K^{00}_{aa} + K^{00}_{bb} + 2K^{00}_{ab})$$

$$\sigma_{CD}(q) = \frac{1}{\pi m^2} \int \beta_2 E_2 dE_2 \int d(\cos\theta q) \int d(\cos\theta_2) \int d\phi_2 \cdot$$

$$\cdot\ (K^{00}_{cc} + K^{00}_{dd} + 2K^{00}_{cd})$$

$$\sigma_{Exchange}(q) = \frac{1}{\pi m^2} \int \beta_2 E_2 dE_2 \int d(\cos\theta q) \int d(\cos\theta_2) \int d\phi_2 \cdot$$

$$\cdot\ [K'^{00}_{cc} + K'^{00}_{dd} + 2(K'^{00}_{dc} + K'^{00}_{ca} + K'^{00}_{cb} + K'^{00}_{da} + K'^{00}_{db})]$$

These three terms represent, respectively, the non-exchange contribution from diagrams a and b, the non-exchange contribution from diagrams c and d, and the total exchange contribution (neglecting the terms discussed in Section VI).

We tabulate the numerical results for these terms on the following pages*. We use Mev for energy units, and Mev/c for momentum units. The $\sigma(q)$ are dimensionless and their values do not depend

* Since $\sigma_{CD}(q)$ and $\sigma_{Exchange}(q)$ become negligible in comparison to $\sigma_{AB}(q)$ at high energy, we have not calculated $\sigma_{CD}(q)$ for primary energies of 3×10^3 Mev or 3×10^4 Mev, and we have not calculated $\sigma_{Exchange}(q)$ for primary energies of 3×10^3, 3×10^4, or 1×10^5 Mev.

on the units.

$E_1 = 1.0 \times 10^1$ Mev

q(Mev/c)	$\sigma_{AB}(q)$	$\sigma_{CD}(q)$	$\sigma_{Exchange}(q)$
2.5×10^{-1}	9.342×10^{0}	5.275×10^{-1}	4.566×10^{0}
5.0	8.710×10^{1}	2.619×10^{0}	1.743×10^{1}
7.5	1.848×10^{2}	5.216	2.387
1.0×10^{0}	2.534	7.856	2.307
2.5	2.372	1.735×10^{1}	2.079
5.0	8.537×10^{1}	1.850	1.163
7.5	2.977	1.592	7.034×10^{0}
1.0×10^{2}	8.399×10^{0}	1.317	3.169

$E_1 = 3.0 \times 10^1$ Mev

q(Mev/c)	$\sigma_{AB}(q)$	$\sigma_{CD}(q)$	$\sigma_{Exchange}(q)$
5.0×10^{-2}	3.862×10^{-1}	4.727×10^{-2}	3.141×10^{-1}
7.5	6.547×10^{0}	3.900×10^{-1}	2.858×10^{0}
1.0×10^{-1}	2.352×10^{1}	9.089	6.892
2.5	3.559×10^{2}	5.119×10^{0}	3.303×10^{1}
5.0	1.360×10^{3}	1.430×10^{1}	4.858
7.5	2.293	2.452	4.891
1.0×10^{0}	2.905	3.449	5.037
2.5	2.737	6.767	2.890
5.0	1.579	7.385	1.381
7.5	8.718×10^{2}	6.868	1.075
1.0×10^{1}	5.666	6.057	9.990×10^{0}

$E_1 = 1.0 \times 10^2$ Mev

q(Mev/c)	$\sigma_{AB}(q)$	$\sigma_{CD}(q)$	$\sigma_{Exchange}(q)$
2.5×10^{-2}	1.110×10^{1}	5.532×10^{-1}	3.972×10^{0}
5.0	1.323×10^{2}	2.576×10^{0}	1.861×10^{1}
7.5	3.877	5.005	3.206
1.0×10^{-1}	7.598	7.657	4.203
2.5	4.644×10^{3}	2.616×10^{1}	5.507
5.0	1.391×10^{4}	6.413	9.237
7.5	2.207	1.065×10^{2}	1.461×10^{2}
1.0×10^{0}	2.716	1.492	1.882
2.5	2.717	2.966	2.429
5.0	1.416	3.165	2.023
7.5	1.002	2.832	7.154×10^{1}
1.0×10^{1}	7.610×10^{3}	2.318	5.472

$E_1 = 3.0 \times 10^2$ Mev

q(Mev/c)	$\sigma_{AB}(q)$	$\sigma_{CD}(q)$	$\sigma_{Exchange}(q)$
5.0×10^{-3}	3.944×10^{-1}	4.769×10^{-2}	2.999×10^{-1}
7.5	6.690×10^{0}	3.924×10^{-1}	2.770×10^{0}
1.0×10^{-2}	2.418×10^{1}	9.127	6.698
2.5	3.919×10^{2}	4.995×10^{0}	3.207×10^{1}
5.0	1.840×10^{3}	1.328×10^{1}	5.125
7.5	4.083	2.244	5.457
1.0×10^{-1}	6.984	3.216	5.953
2.5	3.361×10^{4}	9.611	1.393×10^{2}

$E_1 = 3.0 \times 10^2$ Mev (Cont'd)

q(Mev/c)	$\sigma_{AB}(q)$	$\sigma_{CD}(q)$	$\sigma_{Exchange}(q)$
5.0	8.504	2.098×10^2	3.561
7.5	1.226×10^5	3.198	5.740
1.0×10^0	1.426	4.296	7.331
2.5	1.440	8.225	1.138×10^3
5.0	9.062×10^4	1.013×10^3	1.049
7.5	6.387	1.018	7.186×10^2
1.0×10^1	4.305	9.211×10^2	6.699

$E_1 = 1.0 \times 10^3$ Mev

q(Mev/c)	$\sigma_{AB}(q)$	$\sigma_{CD}(q)$	$\sigma_{Exchange}(q)$
2.5×10^{-3}	1.176×10^1	5.524×10^{-1}	4.200×10^0
5.0	1.391×10^2	2.586×10^0	1.942×10^1
7.5	4.071	5.012	4.071
1.0×10^{-2}	8.011	7.660	4.355
2.5	5.181×10^3	2.569×10^1	5.779
5.0	1.828×10^4	5.883	8.631
7.5	3.610	9.286	1.304×10^2
1.0×10^{-1}	5.683	1.270×10^2	1.841
2.5	2.005×10^5	3.121	5.916
5.0	4.135	5.468	1.406×10^3
7.5	5.498	7.137	2.165
1.0×10^0	6.390	8.592	2.957
2.5	6.803	1.448×10^3	4.666
5.0	4.529	2.069	4.361
7.5	3.318	2.798	4.192
1.0×10^1	2.641	2.346	3.321

$E_1 = 3.0 \times 10^3$ Mev

q(Mev/c)	$\sigma_{AB}(q)$
5.0×10^{-4}	5.931×10^{-1}
7.5	9.753×10^{0}
1.0×10^{-3}	3.330×10^{1}
2.5	5.495×10^{2}
5.0	2.432×10^{3}
7.5	5.534
1.0×10^{-2}	9.271
2.5	4.435×10^{4}
5.0	1.175×10^{5}
7.5	1.976
1.0×10^{-1}	2.807
2.5	7.646
5.0	1.454×10^{6}
7.5	1.945
1.0×10^{0}	2.240
2.5	1.824
5.0	1.694
7.5	1.401
1.0×10^{1}	8.732×10^{5}

$E_1 = 1.0 \times 10^4$ Mev

q(Mev/c)	$\sigma_{AB}(q)$	$\sigma_{CD}(q)$	$\sigma_{Exchange}(q)$
2.5×10^{-4}	3.289×10^{1}	4.302×10^{-1}	1.770×10^{1}
5.0	4.657×10^{2}	2.904×10^{0}	9.293
7.5	1.436×10^{3}	5.193	1.364×10^{2}
1.0×10^{-3}	2.464	8.165	1.731
2.5	1.654×10^{4}	2.638×10^{1}	2.643
5.0	4.455	5.719	3.372
7.5	9.664	8.587	5.610
1.0×10^{-2}	1.284×10^{5}	1.190×10^{2}	7.494
2.5	3.244	2.946	9.978
5.0	5.022	4.881	1.347×10^{3}
7.5	8.740	6.584	2.441
1.0×10^{-1}	1.179×10^{6}	7.902	3.578
2.5	2.536	1.197×10^{3}	5.971
5.0	4.263	1.660	1.204×10^{4}
7.5	5.153	1.983	1.448
1.0×10^{0}	5.947	1.931	2.104
2.5	6.798	2.706	2.880
5.0	5.278	3.451	2.768
7.5	2.913	2.821	1.970
1.0×10^{1}	2.554	2.670	1.778

$E_1 = 3.0 \times 10^4$ Mev

q(Mev/c)	$\sigma_{AB}(q)$
5.0×10^{-5}	1.324×10^{0}
7.5	1.988×10^{1}
1.0×10^{-4}	4.648
2.5	1.075×10^{2}
5.0	2.252×10^{3}
7.5	1.189×10^{4}
1.0×10^{-3}	3.216
2.5	8.873
5.0	2.698×10^{5}
7.5	3.563
1.0×10^{-2}	5.554
2.5	1.403×10^{6}
5.0	2.451
7.5	3.105
1.0×10^{-1}	4.735
2.5	5.652
5.0	1.051×10^{7}
7.5	1.175
1.0×10^{0}	1.300
2.5	1.545
5.0	1.402
7.5	7.739×10^{6}
1.0×10^{1}	6.778×10^{6}

$E_1 = 1.0 \times 10^5$ Mev

q(Mev/c)	$\sigma_{AB}(q)$	$\sigma_{CD}(q)$
2.5×10^{-5}	2.658×10^{1}	1.595×10^{-1}
5.0	6.028	8.969
7.5	9.033	1.660×10^{0}
1.0×10^{-4}	1.204×10^{2}	2.569
2.5	3.015	8.460
5.0	7.200×10^{3}	1.566×10^{1}
7.5	3.799×10^{4}	6.714
1.0×10^{-3}	2.259×10^{5}	1.190×10^{2}
2.5	4.643	3.134
5.0	1.186×10^{6}	3.172
7.5	1.900	4.137
1.0×10^{-2}	3.360	5.203
2.5	6.714	7.877
5.0	1.198×10^{7}	1.032×10^{3}
7.5	1.364	1.393
1.0×10^{-1}	1.786	1.419
2.5	2.153	1.648
5.0	2.305	2.181
7.5	2.622	2.348
1.0×10^{0}	2.068	2.035
2.5	1.749	1.953
5.0	1.472	2.909
7.5	1.196	3.124
1.0×10^{1}	9.201×10^{6}	3.242

APPENDIX G

Numerical Results for σ

We introduce the notation

$$\sigma_{AB} = z^2\alpha^2 r_0^{\,2} \cdot \frac{m^2}{\pi^2 E_1\beta_1} \int q^2 dq \left[\frac{F(q^2)}{q^2}\right]^2 \sigma_{AB}(q)$$

$$\sigma_{\text{Non-exchange}} = z^2\alpha^2 r_0^{\,2} \cdot \frac{m^2}{\pi^2 E_1\beta_1} \int q^2 dq \left[\frac{F(q^2)}{q^2}\right]^2 \cdot \left[\sigma_{AB}(q) + \sigma_{CD}(q)\right]$$

$$\sigma_{\text{Total}} = z^3\alpha^2 r_0^{\,2} \cdot \frac{m^2}{\pi^2 E_1\beta_1} \int q^2 dq \left[\frac{F(q^2)}{q^2}\right]^2 \cdot$$

$$\cdot \left[\sigma_{AB}(q) + \sigma_{CD}(q) - \sigma_{\text{Exchange}}(q)\right]$$

These three expressions represent, respectively, the non-exchange contribution from diagrams a and b, the total non-exchange contribution, and the total contribution including exchange.

On the following pages we tabulate the numerical results for these terms for various values of E_1 and z*. We tabulate the combination $\sigma/z^2\alpha^2 r_0^{\,2}$, which is dimensionless and does not depend on the choice of units. The z = 0 case on the following pages is the unscreened case.

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* As noted in Appendix F, $\sigma_{AB}(q)$ is the dominating term at high energies. For that reason we have not calculated $\sigma_{\text{Non-exchange}}$ for primary energies of 3×10^3 or 3×10^4 Mev, and we have not calculated σ_{Total} for energies of 3×10^3, 3×10^4, or 1×10^5 Mev.

$E_1 = 1.0 \times 10^1$ Mev

z	$\sigma_{AB}/z^2\alpha^2r_0^2$	$\sigma_{Non-Ex.}/z^2\alpha^2r_0^2$	$\sigma_{Total}/z^2\alpha^2r_0^2$
0	1.190×10^0	1.244×10^0	1.070×10^0
6	1.188	1.242	1.070
13	1.187	1.242	1.069
26	1.186	1.240	1.068
33	1.186	1.240	1.068
47	1.185	1.239	1.067
56	1.184	1.238	1.067
67	1.183	1.237	1.066
78	1.183	1.237	1.066
80	1.183	1.236	1.066
82	1.183	1.236	1.066

$E_1 = 3.0 \times 10^1$ Mev

z	$\sigma_{AB}/z^2\alpha^2r_0^2$	$\sigma_{Non-Ex.}/z^2\alpha^2r_0^2$	$\sigma_{Total}/z^2\alpha^2r_0^2$
0	5.954×10^0	6.049×10^0	5.798×10^0
6	5.933	6.027	5.779
13	5.921	6.014	5.768
26	5.903	5.997	5.753
33	5.896	5.989	5.747
47	5.883	5.976	5.735
56	5.876	5.969	5.729
67	5.868	5.960	5.722
78	5.860	5.953	5.715
80	5.859	5.952	5.714
82	5.858	5.950	5.713

$E_1 = 1.0 \times 10^2$ Mev

z	$\sigma_{AB}/z^2\alpha^2r_0^2$	$\sigma_{Non-Ex.}/z^2\alpha^2r_0^2$	$\sigma_{Total}/z^2\alpha^2r_0^2$
0	2.009×10^1	2.023×10^1	1.989×10^1
6	1.989	2.004	1.971
13	1.980	1.994	1.962
26	1.968	1.982	1.951
33	1.962	1.977	1.947
47	1.954	1.968	1.939
56	1.950	1.964	1.935
67	1.945	1.959	1.930
78	1.940	1.954	1.926
80	1.940	1.954	1.925
82	1.939	1.953	1.924

$E_1 = 3.0 \times 10^2$ Mev

z	$\sigma_{AB}/z^2\alpha^2r_0^2$	$\sigma_{Non-Ex.}/z^2\alpha^2r_0^2$	$\sigma_{Total}/z^2\alpha^2r_0^2$
0	4.398×10^1	4.416×10^1	4.373×10^1
6	4.297	4.314	4.278
13	4.256	4.276σ	4.237
26	4.211	4.228	4.195
33	4.194	4.210	4.178
47	4.165	4.182	4.153
56	4.152	4.168	4.138
67	4.136	4.152	4.121
78	4.121	4.137	4.107
80	4.118	4.134	4.104
82	4.116	4.132	4.102

$E_1 = 1.0 \times 10^3$ Mev

z	$\sigma_{AB}/z^2\alpha^2r_0^2$	$\sigma_{Non-Ex.}/z^2\alpha^2r_0^2$	$\sigma_{Total}/z^2\alpha^2r_0^2$
0	8.024×10^1	8.042×10^1	7.987×10^1
6	7.487	7.502	7.464
13	7.364	7.379	7.346
26	7.234	7.248	7.214
33	7.181	7.195	7.164
47	7.094	7.108	7.075
56	7.052	7.066	7.033
67	7.008	7.021	6.989
78	6.968	6.982	6.950
80	6.962	6.975	6.943
82	6.955	6.969	6.937

$E_1 = 3.0 \times 10^3$ Mev

z	$\sigma_{AB}/z^2\alpha^2r_0^2$
0	1.215×10^2
6	1.071
13	1.039
26	1.009
33	0.997
47	0.979
56	0.970
67	0.960
78	0.952
80	0.950
82	0.949

$E_1 = 1.0 \times 10^4$ Mev

z	$\sigma_{AB}/z^2\alpha^2 r_0^2$	$\sigma_{Non\text{-}Ex.}/z^2\alpha^2 r_0^2$	$\sigma_{Total}/z^2\alpha^2 r_0^2$
0	1.954×10^2	1.956×10^2	1.946×10^2
6	1.336	1.337	1.333
13	1.258	1.259	1.255
26	1.188	1.189	1.185
33	1.163	1.164	1.161
47	1.127	1.128	1.124
56	1.109	1.110	1.106
67	1.091	1.091	1.088
78	1.075	1.076	1.072
80	1.072	1.073	1.070
82	1.070	1.070	1.067

$E_1 = 3.0 \times 10^4$ Mev

z	$\sigma_{AB}/z^2\alpha^2 r_0^2$
0	2.732×10^2
6	1.558
13	1.441
26	1.336
33	1.300
47	1.247
56	1.221
67	1.194
78	1.169
80	1.166
82	1.162

$E_1 = 1.0 \times 10^5$ Mev

z	$\sigma_{AB}/z^2\alpha^2 r_0^2$	$\sigma_{Non-Ex.}/z^2\alpha^2 r_0^2$
0	3.840×10^2	3.842×10^2
6	1.915	1.916
13	1.728	1.728
26	1.565	1.566
33	1.514	1.515
47	1.434	1.435
56	1.395	1.395
67	1.356	1.356
78	1.322	1.322
80	1.317	1.317
82	1.311	1.312

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