ON PION NUCLEON RESONANCES

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Egon Marx-Oberländer

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ABSTRACT

This thesis shows a method for setting up a numerical computation to find resonances in the pion-nucleon system. It takes into account inelastic reactions through a generalized N/D method, and problems introduced by unstable particles are analyzed. General formulae are calculated for the kinematics of reactions involving the πN, ρN, ωN and πN^* channels, relating sets of Lorentz invariant amplitudes (free of kinematical singularities) to helicity amplitudes in a form that exhibits the reflection symmetry when the total energy W is replaced by $-W$, and a simple form is found for the partial wave expansions. A detailed discussion of the isospin part of the equations is included.

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I. INTRODUCTION

In the last few years several resonances have been discovered in the pion-nucleon system, and attempts have been made to predict their properties.[†] The general purpose of this thesis is to provide the framework for a calculation to determine such resonances. The use of a generalized N/D method, for a problem with several channels, is proposed to take into account the effect of inelastic scattering, related to the main process of pion-nucleon scattering through unitarity.

The channels with several pions are approximated by combining several particles in a resonant state or unstable particle. Specifically, the ρ and ω vector mesons are included in our formulae together with a nucleon. Another resonance with a relatively low mass is the $J = 3/2$, $I = 3/2$ p-wave pion nucleon resonance, denoted by N^*. This should be obtained in a calculation where it is included in a crossed channel, in a self consistent way. The N^* can also be put into the calculation as an external unstable particle in a πN^* channel. There is no theoretical reason to exclude this channel, but uncertainties in a theory with spin 3/2 particles and the corresponding experimental data do not permit an unambiguous approach to its interactions, so it might be convenient to try a first calculation at least without it. The exact problem obviously has an infinite number of channels, and the usual procedure is to keep only those with the lowest masses.

In part II the general kinematics for the reactions under consideration is developed. The main results are the partial wave expansions of

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[†] See, for instance, references 16, 18, 19 and 21; further references are given in these.

certain amplitudes, related to the helicity amplitudes but coming from states with definite parity, and their connection to Lorentz invariant amplitudes. Their form is such that they can be advantageously used in calculations with Regge poles.

In part III a "kinematics" is developed for the isospin part of the matrix elements, that was not included in part II. It is patterned in a form similar to that of ordinary kinematics.

In part IV, the problems related to the dynamics of the reactions are discussed. Both the use of the generalized N/D method, with a possible set of approximations, and the evaluation of simple diagrams that would be used as input data, are examined.

The appendices form also an important part of this thesis, and many general results and formulae used in the main body are derived there. They are more or less self-contained units, and a moderate familiarity with the matters discussed there might facilitate the reading of the thesis itself.

In appendix A we examine the effects of invariance of the strong interactions under several symmetry operations. Special attention is devoted to the reflection symmetry that relates amplitudes at positive and negative energies.

In appendix B, the determination of state vectors with definite helicities is shown.

In appendix C, tables for the rotation matrices $d^J_{\lambda\mu}(\theta)$ for small values of λ and μ are given, and proofs of two general expansions, using either Legendre polynomials or their derivatives, are shown.

In appendix D, two methods to get projection operators for partial wave amplitudes are discussed in detail, and a general expression for amplitudes like those used in part II is obtained.

In appendix E, a pseudo-vector algebra for four-vectors, similar to the one for ordinary three-vectors, is developed. It is used then to find a way of choosing Lorentz invariant amplitudes for the $\rho + N \rightarrow \rho + N$ reaction without introducing kinematical singularities.

In appendix F, a field theory for spin 3/2 particles is partially developed in order to find state vectors, propagator, etc.

In appendix G, the N/D method for several channels is explained, together with a discussion of the difficulties arising when time reversal invariance (expressed by the symmetry of the scattering matrix) is to be built into an approximate solution besides unitarity.

In appendix H, a discussion of the problems presented by unstable external particles and the related complex singularities is given. One way to take them into account is shown in detail.

Finally, in appendix I, some of the notations used in this thesis are stated, specially when books and papers on modern physics show a great variety in their choice of conventions and definitions.

II. KINEMATICS

In this part, all particles will be assumed to have zero isotopic spin. The isotopic spin part of the amplitudes will be left for part III.

a) $\pi + N \to \pi + N$

Four-momentum conservation gives the relation

$$p + q = p' + q' \tag{1}$$

that reduces the number of independent momenta to three. We will use the combinations

$$P = \frac{1}{2}(p + p') \quad Q = \frac{1}{2}(q + q') \quad \Delta = \frac{1}{2}(q - q') = \frac{1}{2}(p' - p) \tag{2}$$

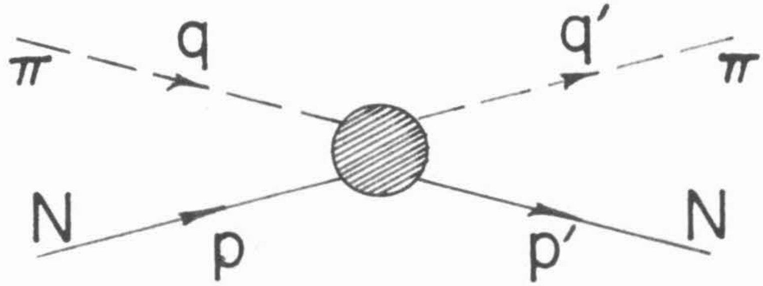

Fig. 1. General diagram for the reaction $\pi + N \to \pi + N$

From the three vectors we can form six scalars. Four of them are fixed, corresponding to the external masses

$$p^2 = p'^2 = M^2 \qquad q^2 = q'^2 = \mu^2 \tag{3}$$

These relations give

$$P \cdot \Delta = Q \cdot \Delta = 0 \tag{4}$$

$$P^2 + \Delta^2 = M^2 \qquad Q^2 + \Delta^2 = \mu^2 \tag{5}$$

One possible choice of independent scalar variables is

$$s = (p + q)^2 = (P + Q)^2 \tag{6a}$$

$$t = (p' - p)^2 = 4\Delta^2 \tag{6b}$$

We can also define

$$u = (p' - q)^2 = (P - Q)^2 \tag{6c}$$

that is related to s and t by

$$s + t + u = 2(M^2 + \mu^2) \tag{6d}$$

Any matrix element for this reaction can be written in the form

$$\mathcal{Q} = \bar{u}(\vec{p}\,')\mathcal{Q}_i(s,t)M_i u(\vec{p}) = \bar{u}(\vec{p}\,')Tu(\vec{p}) \tag{7}$$

where the $\mathcal{Q}_i$ are Lorentz invariant scalar amplitudes, the M_i are invariants characteristic of the reaction and $u(\vec{p})$ is a spinor for a nucleon of momentum $\vec{p}$. The invariants are scalars in space-time and 4×4 matrices in spin space, and it is convenient to build into them the symmetries of the problem.

Lorentz scalars can be formed with the momenta and γ_μ. The spinors $u(\vec{p})$, $u(\vec{p}\,')$ obey

$$(\not{p} - M)u(\vec{p}) = 0 \tag{8a}$$

$$(\not{p}' - M)u(\vec{p}\,') = 0 \tag{8b}$$

and from 8b

$$\bar{u}(\vec{p}\,')(\not{p}' - M) = 0 \tag{8c}$$

Hence the possible factors $\not{P}$, $\not{\Delta}$ in invariants can be eliminated by moving $\not{p}$ to the right and $\not{p}'$ to the left. Factors like $P \cdot Q$ and Δ^2 are included in the G_i.

Parity conservation under strong interactions eliminates invariants like γ_5 and $\gamma_5 \not{Q}$, which change sign under the parity operation, since the interactions are such that the matrix element is a true scalar and not a pseudo-scalar.

In appendix E it is shown how invariants containing the antisymmetric tensor $\epsilon_{\lambda\mu\nu\rho}$ can be expressed in terms of others that do not use it.

It is now easy to convince ourselves that all terms in a matrix amplitude can be written using the two invariants 1 and $\not{Q}$, and T is of the general form

$$T = A + B\not{Q} \qquad (9)$$

It might be desirable to write G in terms of invariants constructed from Pauli spin matrices and 3-vectors. In order to do this we choose the center-of-mass system, where

$$\vec{p} + \vec{q} = \vec{p}' + \vec{q}' = 0 \qquad (10)$$

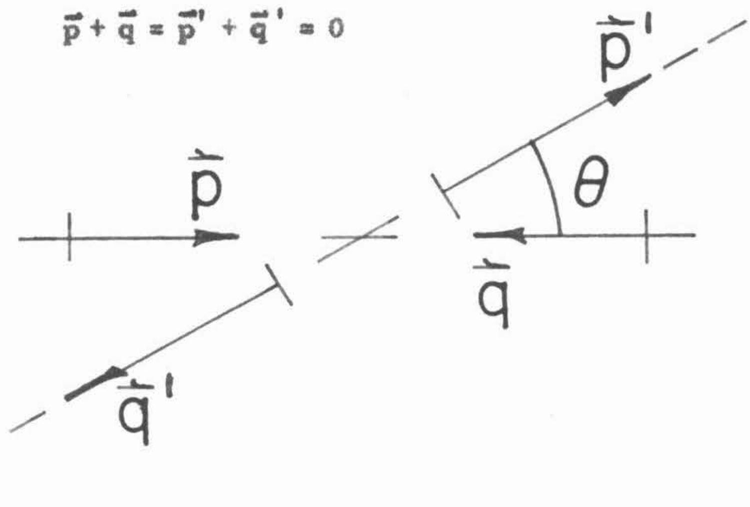

Fig. 2. Incoming and outgoing 3-momenta

We call

$$p_o = p_o' = \text{nucleon energy} \tag{11}$$

$$p_o + q_o = W = \text{total energy} \tag{12}$$

$$x = \cos\theta = \hat{p}\cdot\hat{p}' = \hat{q}\cdot\hat{q}' \tag{13}$$

Then

$$u(\vec{p}) = \left(\frac{E+M}{2M}\right)^{\frac{1}{2}}\left(1 + \frac{\vec{\alpha}\cdot\vec{p}}{E+M}\right)u_o \tag{14a}$$

$$u(\vec{p}\,') = \left(\frac{E+M}{2M}\right)^{\frac{1}{2}}\left(1 + \frac{\vec{\alpha}\cdot\vec{p}\,'}{E+M}\right)u_o' \tag{14b}$$

where

$$u_o = \begin{pmatrix} |i\rangle \\ 0 \end{pmatrix} \qquad u_o' = \begin{pmatrix} |f\rangle \\ 0 \end{pmatrix} \tag{15a, b}$$

$|i\rangle$, $|f\rangle$ being the initial and final 2-component spinors, and

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \tag{16}$$

From equations 15 and 16 it is obvious that relations like

$$u_o'^{\dagger}\,\vec{\alpha}\cdot\vec{a}\;\vec{\alpha}\cdot\vec{b}\;u_o = \langle f|\vec{\sigma}\cdot\vec{a}\;\vec{\sigma}\cdot\vec{b}\,|i\rangle \tag{17}$$

hold, and that any term with an odd number of $\vec{\alpha}$'s is zero.

Hence

$$\bar{u}(\vec{p}\,')Tu(\vec{p}) = \frac{E+M}{2M}u_o'^{\dagger}\left(1 + \frac{\vec{\alpha}\cdot\vec{p}\,'}{E+M}\right)\beta\left(A + B\beta(Q_o - \vec{\alpha}\cdot\vec{Q})\right)\left(1 + \frac{\vec{\alpha}\cdot\vec{p}}{E+M}\right)u_o$$

$$= \langle f|\left(\frac{E+M}{2M}A + \frac{(W-M)(E+M)}{2M}B\right)$$

$$+ \left(-\frac{E-M}{2M}A + \frac{(W+M)(E-M)}{2M}B\right)\vec{\sigma}\cdot\hat{q}'\;\vec{\sigma}\cdot\hat{q}\,|i\rangle \tag{18}$$

where equations 10 to 12 and

$$Q_o = \frac{1}{2}(q_o + q_o') = q_o \qquad \vec{Q} = \frac{1}{2}(\vec{q} + \vec{q}\,') \tag{19}$$

$$\vec{p}^{\,2} = \vec{p}\,'^2 = \vec{q}^{\,2} = \vec{q}\,'^2 = E^2 - M^2 \tag{20}$$

$$\hat{q} = \frac{\vec{q}}{|\vec{q}|} \qquad \hat{q}' = \frac{\vec{q}\,'}{|\vec{q}\,'|} \tag{21}$$

have been used.

The relation between the matrix element and the cross section is as follows:[‡]

$$T_{fi} = (2\pi)^4 \delta^4(p + q - p' - q')\left(\frac{M^2}{4EE'q_o q_o'}\right)^{1/2} \bar{u}(\vec{p}\,')Tu(\vec{p}) \tag{22}$$

then the transition probability is

$$w_{fi} = (2\pi)^4 \delta^4(p + q - p' - q')\,\frac{M^2}{4E^2 q_o^2}\,|G|^2 \tag{23}$$

and the differential cross section

$$d\sigma = \int \frac{d^3\vec{p}\,'}{(2\pi)^3}\,\frac{\vec{q}\,'^2 d|\vec{q}\,'|d\Omega}{(2\pi)^3}\,\frac{1}{|\vec{v}_p - \vec{v}_q|}\,w_{fi} \tag{24}$$

We have to use the formula

$$\delta[f(x)] = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|}\,, \qquad f(x_i) = 0 \tag{25}$$

Since p_o and q_o are fixed, and

$$p_o'^2 = M^2 + \vec{q}\,'^2 \qquad q_o'^2 = \mu^2 + \vec{q}\,'^2 \tag{26a, b}$$

[‡] See Ref. 1 section 14d or Ref. 2 chapter 15.

we get

$$\frac{\partial(p_o + q_o - p_o' - q_o')}{\partial|\vec{q}'|} = -\left(\frac{|\vec{q}'|}{p_o'} + \frac{|\vec{q}'|}{q_o'}\right) = -\frac{|\vec{q}'|W}{Eq_o} \qquad (27)$$

Also

$$|\vec{v}_p - \vec{v}_q| = \left|\frac{\vec{p}}{p_o} - \frac{\vec{q}}{q_o}\right| = \frac{|\vec{q}'|W}{Eq_o} \qquad (28)$$

and

$$\frac{d\sigma}{d\Omega} = \frac{1}{(2\pi)^2}\frac{M^2}{4W^2}|\alpha|^2 \qquad (29)$$

We can define new amplitudes $\mathcal{F}_i$ by

$$\frac{d\sigma}{d\Omega} = |\langle f|\mathcal{F}_1 + \vec{\sigma}\cdot\hat{q}' \ \vec{\sigma}\cdot\hat{q}\ \mathcal{F}_2|i\rangle|^2 \qquad (30)$$

and combining equations 18, 29, and 30 we obtain

$$\mathcal{F}_1 = \frac{E+M}{2W}\ \frac{A + (W-M)B}{4\pi} \qquad (31a)$$

$$\mathcal{F}_2 = \frac{E-M}{2W}\ \frac{-A + (W+M)B}{4\pi} \qquad (31b)$$

as in Ref. 3.

Formulae 31 exhibit the symmetry pointed out in Ref. 4, that is

$$\mathcal{F}_1(-W) = -\mathcal{F}_2(W) \qquad (32)$$

(see appendix A). If a sum over final spin and average over initial spin is performed, we get

$$\frac{d\sigma}{d\Omega} = \frac{1}{2} \sum_{\text{spins}} |\langle f|\mathcal{F}|i\rangle|^2$$

$$= \frac{1}{2} \sum_{\text{spins}} \langle f|\mathcal{F}|i\rangle \langle i|\mathcal{F}^\dagger|f\rangle$$

$$= \frac{1}{2} \text{Tr} (\mathcal{F}\mathcal{F}^\dagger) \tag{33}$$

where we have written

$$\mathcal{F} = \mathcal{F}_1 + \vec{\sigma}\cdot\hat{q}' \ \vec{\sigma}\cdot\hat{q} \ \mathcal{F}_2 \tag{34}$$

and the completeness relation

$$\sum_{\text{spin}} |\lambda\rangle\langle\lambda| = 1 \tag{35}$$

has been used.

Since

$$\text{Tr}\, 1 = 2 \tag{36a}$$

$$\text{Tr}\,(\vec{\sigma}\cdot\vec{a} \ \vec{\sigma}\cdot\vec{b}) = 2\vec{a}\cdot\vec{b} \tag{36b}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{2}\left(2\mathcal{F}_1\mathcal{F}_1^* + 2\hat{q}\cdot\hat{q}'(\mathcal{F}_1\mathcal{F}_2^* + \mathcal{F}_2\mathcal{F}_1^*) + 2\mathcal{F}_2\mathcal{F}_2^*\right)$$

$$= |\mathcal{F}_1|^2 + |\mathcal{F}_2|^2 + 2x \,\text{Re}\, (\mathcal{F}_1\mathcal{F}_2^*) \tag{37}$$

and the total cross section is

$$\sigma = \int \frac{d\sigma}{d\Omega}\, d\Omega$$

$$= 2\pi \int_{-1}^{1} dx \left(|\mathcal{F}_1|^2 + |\mathcal{F}_2|^2 + 2x\, \text{Re}\, (\mathcal{F}_1\mathcal{F}_2^*)\right) \tag{38}$$

If we define a "scalar product" of 2-component vectors

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \tag{39}$$

by

$$\varphi \circ \psi = \int_{-1}^{1} dx\, \varphi^{\dagger} G \psi \tag{40}$$

where[‡]

$$G = \begin{pmatrix} 1 & x \\ x & 1 \end{pmatrix} \tag{41}$$

we have

$$\sigma = 2\pi\, \mathcal{F} \circ \mathcal{F} \tag{42}$$

Here we have written $\mathcal{F}$ for

$$\mathcal{F} = \begin{pmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \end{pmatrix} \tag{43}$$

It is essentially equivalent to the $\mathcal{F}$ defined in equation 34, and it should be evident from the context which one is used.

Comparing the two expressions for the cross section, equations 30 and A-6, we obtain for the helicity amplitudes

$$f_{\frac{1}{2}0;\frac{1}{2}0}(W, \theta, \varphi) = \langle \tfrac{1}{2} | \mathcal{F} | \tfrac{1}{2} \rangle \tag{44a}$$

$$f_{-\frac{1}{2}0;\frac{1}{2}0}(W, \theta, \varphi) = \langle -\tfrac{1}{2} | \mathcal{F} | \tfrac{1}{2} \rangle \tag{44b}$$

where the first spinor refers to the outgoing nucleon of momentum $\vec{p}\,'$ in the direction $\hat{p}\,'(\theta, \varphi)$, and the second to the incoming nucleon of momentum $\vec{p}$ in the direction $\hat{p}(0, 0)$.

[‡] This G differs from the one in equation D7 by a factor of 2.

Following the phase conventions of Ref. 5, we determine in appendix B

$$|\theta,\varphi;\tfrac{1}{2}> = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\, e^{i\varphi} \end{pmatrix} \tag{45a}$$

$$|\theta,\varphi;-\tfrac{1}{2}> = \begin{pmatrix} -\sin\frac{\theta}{2}\, e^{-i\varphi} \\ \cos\frac{\theta}{2} \end{pmatrix} \tag{45b}$$

Either by using

$$\vec{\sigma}\cdot\hat{q}' \;\; \vec{\sigma}\cdot\hat{q}' = \begin{pmatrix} \cos\theta & -\sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & \cos\theta \end{pmatrix} \tag{46}$$

or noticing that, since $\hat{q} = -\hat{p}$ and $\hat{q}' = -\hat{p}$,

$$\tfrac{1}{2}\vec{\sigma}\cdot\hat{q}\,|\,i> = -\lambda_a\,|\,i> \tag{47a}$$

$$<f|\,\tfrac{1}{2}\vec{\sigma}\cdot\hat{q}' = -\lambda_c <f| \tag{47b}$$

we get from equations 44

$$f_{\frac{1}{2}0;\frac{1}{2}0} = \cos\frac{\theta}{2}(\mathcal{F}_1 + \mathcal{F}_2) \tag{48a}$$

$$f_{-\frac{1}{2}0;\frac{1}{2}0} = -\sin\frac{\theta}{2}\, e^{i\varphi}(\mathcal{F}_1 - \mathcal{F}_2) \tag{48b}$$

The elements of the submatrix $T^J(W)$ defined in Ref. 5 and appendix A can be written in the form

$$\frac{1}{|\vec{p}_r|}\, T^J = \begin{array}{c|cc} {}^{\text{final}}_{\text{initial}} & \frac{1}{2}0 & -\frac{1}{2}0 \\ \hline \frac{1}{2}0 & a_1^J & a_2^J \\ -\frac{1}{2}0 & a_2^J & a_1^J \end{array}$$

where the partial wave helicity amplitudes are

$$a^J_{\frac{1}{2}0;\frac{1}{2}0} = a^J_{-\frac{1}{2}0;-\frac{1}{2}0} = a^J_1 \qquad (49b)$$

$$a^J_{-\frac{1}{2}0;\frac{1}{2}0} = a^J_{\frac{1}{2}0;-\frac{1}{2}0} = a^J_2 \qquad (49c)$$

The first equalities in equations 49b and 49c come from parity conservation, expressed for these partial wave helicity amplitudes by R5-43:

$$\langle -\lambda_c, -\lambda_d | S^J | -\lambda_a, -\lambda_b \rangle = \eta_g \langle \lambda_c \lambda_d | S^J | \lambda_a \lambda_b \rangle$$

$$\eta_g = \frac{\eta_c \eta_d}{\eta_a \eta_b} (-1)^{s_c + s_d - s_a - s_b} \qquad (50)$$

where the η_i and s_i are the intrinsic parities and spins of the four particles a, b, c and d. For the reaction in this section

$$\eta_g = 1 \qquad (50a)$$

Using equations A-5, C-9, C-11 and C-6 we get the following partial wave expansions

$$f_{\frac{1}{2}0;\frac{1}{2}0} = \sum_{J=\ell+\frac{1}{2}} a^J_1 \cos\frac{\theta}{2} (P'_{\ell+1} - P'_\ell) \qquad (51a)$$

$$f_{-\frac{1}{2}0;\frac{1}{2}0} = \sum_{j=\ell+\frac{1}{2}} a^J_2 e^{i\varphi} \sin\frac{\theta}{2} \; (-P'_{\ell+1} - P'_\ell) \qquad (51b)$$

Hence

$$\mathfrak{F}_1 + \mathfrak{F}_2 = \sum_{J=\ell+\frac{1}{2}} a^J_1 \; (P'_{\ell+1} - P'_\ell) \qquad (52a)$$

$$\mathfrak{F}_1 - \mathfrak{F}_2 = \sum_{J=\ell+\frac{1}{2}} a_2^J (P'_{\ell+1} + P'_\ell) \tag{52b}$$

$$\mathfrak{F}_1 = \sum_{J=\ell+\frac{1}{2}} \left[\frac{1}{2}(a_1^J + a_2^J) P'_{\ell+1} - \frac{1}{2}(a_1^J - a_2^J) P'_\ell \right] \tag{52c}$$

$$\mathfrak{F}_2 = \sum_{J=\ell+\frac{1}{2}} \left[\frac{1}{2}(a_1^J - a_2^J) P'_{\ell+1} - \frac{1}{2}(a_1^J + a_2^J) P'_\ell \right] \tag{52d}$$

$$\mathfrak{F}_1 = \sum_{J=\ell+\frac{1}{2}} (f_{\ell+} P'_{\ell+1} - f_{\ell+1,-} P'_\ell) \tag{52e}$$

$$\mathfrak{F}_2 = \sum_{J=\ell+\frac{1}{2}} (f_{\ell+1,-} P'_{\ell+1} - f_{\ell+} P'_\ell) \tag{52f}$$

where the $f_{\ell\pm}$ are defined as usual in terms of phase shifts for states of parity $(-1)^\ell$, $J = \ell \pm \frac{1}{2}$

$$f_{\ell\pm} = \frac{1}{|\vec{p}|} e^{i\delta_{\ell\pm}} \sin \delta_{\ell\pm} \tag{53}$$

and we can write the formulae in the form

$$\mathfrak{F}_1 = \sum_\ell (f_{\ell+} P'_{\ell+1} - f_{\ell-} P'_{\ell-1}) \tag{54a}$$

$$\mathfrak{F}_2 = \sum_\ell (f_{\ell-} - f_{\ell+}) P'_\ell \tag{54b}$$

Obviously,

$$f_{\ell+} = \frac{1}{2}(a_1^J + a_2^J) \qquad f_{\ell-} = \frac{1}{2}(a_1^{J-1} - a_2^{J-1}) \tag{55a, b}$$

Proceeding as indicated in appendix D, we obtain from formulae 54

$$\varphi_{\ell+} = \begin{pmatrix} P'_{\ell+1} \\ -P'_{\ell} \end{pmatrix} \qquad \varphi_{\ell-} = \begin{pmatrix} -P'_{\ell-1} \\ P'_{\ell} \end{pmatrix} \tag{56a, b}$$

Then we have from D-15

$$f_{\ell\pm} = \frac{1}{2(\ell+1)} \varphi_{\ell\pm} \circ \mathcal{F} \tag{57}$$

Actually we will use a slightly different notation, and the procedure to get the partial wave expansion and projection operators, as well as the equations connecting A and B to T_1 and T_2 will be shown in detail for this simple case; the more complicated ones to follow being straightforward generalizations of this.

Following the procedure in appendix D, according to equation D-23 we define

$$T_1(W, x) = \frac{f_{\frac{1}{2}0;\frac{1}{2}0}(W, x, \varphi)}{\cos\frac{\theta}{2}} + \frac{f_{-\frac{1}{2}0;\frac{1}{2}0}(W, x, \varphi)}{-\sin\frac{\theta}{2}\, e^{i\varphi}} \tag{58a}$$

$$T_2(W, x) = \frac{f_{\frac{1}{2}0;\frac{1}{2}0}(W, x, \varphi)}{\cos\frac{\theta}{2}} - \frac{f_{-\frac{1}{2}0;\frac{1}{2}0}(W, x, \varphi)}{-\sin\frac{\theta}{2}\, e^{i\varphi}} \tag{58b}$$

$$x = \cos\theta$$

The minus sign in the denominator is there because $d^J_{\frac{1}{2},-\frac{1}{2}}(\theta)$ has one. In general it will be a factor of $(-1)^{\lambda+\mu}$, where λ and μ refer to the first term, i.e., $\frac{1}{2}$ and $\frac{1}{2}$ in this case. The partial wave expansion is then

$$T_1(W,x) = \sum_{J=\ell+\frac{1}{2}} \left[\left(a_1^J(W) + a_2^J(W)\right) P'_{\ell+1}(x) - \left(a_1^J(W) - a_2^J(W)\right) P'_{\ell}(x) \right] \tag{59a}$$

$$T_2(W,x) = \sum_{J=\ell+\frac{1}{2}} \left[\left(a_1^J(W) - a_2^J(W)\right) P'_{\ell+1}(x) - \left(a_1^J(W) + a_2^J(W)\right) P'_{\ell}(x) \right] \tag{59b}$$

Following D-25 we define

$$\beta_1^J(W) = a_1^J(W) + a_2^J(W) \qquad \beta_2^J(W) = a_1^J(W) - a_2^J(W) \tag{60a, b}$$

We notice that β_1^J and β_2^J correspond to transitions between states of definite parity. In fact, from equation R5-41:

$$P\,|JM;\lambda_1\lambda_2> = \eta_1\eta_2(-1)^{J-s_1-s_2}\,|J,M;-\lambda_1,-\lambda_2> \tag{61}$$

where η_i, s_i are the intrinsic parity and spin of the particles, and the definitions:

$$|J,M;\pm> = \frac{1}{\sqrt{2}}\left(|JM;\tfrac{1}{2}0> \mp |JM;-\tfrac{1}{2}0>\right) \tag{62}$$

we get

$$P\,|JM;\pm> = \pm(-1)^{\ell}\,|JM;\pm> \tag{63}$$

since $J = \ell + \frac{1}{2}$, $s_1 = \frac{1}{2}$, $s_2 = 0$, $\eta_1 = 1$, $\eta_2 = -1$.

Recalling 49a, we find that

$$< JM; + | \frac{1}{p} T^J | JM; + > = \beta_2^J \tag{64a}$$

$$< JM; - | \frac{1}{p} T^J | JM; - > = \beta_1^J \tag{64b}$$

$$< JM; + | \frac{1}{p} T^J | JM; - > = 0 \tag{64c}$$

$$< JM; - | \frac{1}{p} T^J | JM; + > = 0 \tag{64d}$$

Equations 64c and 64d are of course expected from parity conservation.

Also from A-14 we have

$$a_1^J(-W) = - a_1^J(W) \qquad a_2^J(-W) = a_2^J(W) \tag{65a, b}$$

and hence

$$\beta_1^J(-W) = - \beta_2^J(W) \qquad T_1(-W, x) = - T_2(W, x) \tag{65c, d}$$

this last form of reflection symmetry being obvious from rewriting 59 as

$$T_1(W, x) = \sum_{J=\ell+\frac{1}{2}} \left(\beta_1^J(W) P'_{\ell+1}(x) - \beta_2^J(W) P'_\ell(x) \right) \tag{66a}$$

$$T_2(W, x) = \sum_{J=\ell+\frac{1}{2}} \left(\beta_2^J(W) P'_{\ell+1}(x) - \beta_1^J(W) P'_\ell(x) \right) \tag{66b}$$

To obtain the projection operators, we continue as in appendix D.

Rewriting equations 51 as

$$f_1(W, x, \varphi) = \sum_{J=\ell+\frac{1}{2}} (J + \tfrac{1}{2}) a_1^J(W)\, d^J_{\frac{1}{2}\frac{1}{2}}(x) \tag{67a}$$

$$f_2(W, x, \varphi) = \sum_{J=\ell+\frac{1}{2}} (J + \tfrac{1}{2})\, a_2^J(W) e^{i\varphi} d^J_{\frac{1}{2}, -\frac{1}{2}}(x) \tag{67b}$$

we get from D-5

$$\alpha_1^J(W) = \int_{-1}^{1} f_1(W,x,\varphi) d^J_{\frac{1}{2}\frac{1}{2}}(x)\, dx \tag{68a}$$

$$\alpha_2^J(W) = e^{-i\varphi}\int_{-1}^{1} f_2(W,x,\varphi) d^J_{\frac{1}{2},-\frac{1}{2}}(x)\, dx \tag{68b}$$

From equations 58,

$$f_1(W,x,\varphi) = \frac{1}{2}\cos\frac{\theta}{2}\,[\,T_1(W,x) + T_2(W,x)\,] \tag{69a}$$

$$f_2(W,x,\varphi) = -\frac{1}{2}\sin\frac{\theta}{2}\, e^{i\varphi}[\,T_1(W,x) - T_2(W,x)] \tag{69b}$$

Using this time equations C10, C12 and C6, from 68 and 69 we get

$$\alpha_1^J(W) = \frac{1}{4}\int_{-1}^{1}[\,T_1(W,x) + T_2(W,x)]\,[\,P_{\ell+1}(x) + P_\ell(x)]\, dx \tag{70a}$$

$$\alpha_2^J(W) = \frac{1}{4}\int_{-1}^{1}[\,-T_1(W,x) + T_2(W,x)]\,[\,P_{\ell+1}(x) - P_\ell(x)]\, dx \tag{70b}$$

and from 60

$$\beta_1^J(W) = \frac{1}{2}\int_{-1}^{1}[\,T_1(W,x)P_\ell(x) + T_2(W,x)P_{\ell+1}(x)]\, dx \tag{71a}$$

$$\beta_2^J(W) = \frac{1}{2}\int_{-1}^{1}[\,T_1(W,x)P_{\ell+1}(x) + T_2(W,x)P_\ell(x)]\, dx \tag{71b}$$

which are the required projection formulae. From equations 48 and 58 we derive

$$T_1(W,x) = 2\mathcal{F}_1(W,x) \qquad\qquad T_2(W,x) = 2\mathcal{F}_2(W,x) \tag{72a, b}$$

and combining these with 31,

$$T_1(W,x) = \frac{E+M}{4\pi W}A(W^2,x) + \frac{(E+M)(W-M)}{4\pi W}B(W^2,x) \tag{73a}$$

$$T_2(W, x) = -\frac{E-M}{4\pi W} A(W^2, x) + \frac{(E-M)(W+M)}{4\pi W} B(W^2, x) \qquad (73b)$$

In general, equations 72 will not be trivial as in this case.

A final remark about time reversal: for πN scattering, parity conservation already demands the symmetry of the scattering matrix, and no new relations are imposed by time reversal.

b) $\rho + N \rightarrow \pi + N$

Momentum conservation gives in this case

$$p + k = p' + q \qquad (74)$$

We take as independent momenta

$$k, \; q \; \text{ and } \; P = \frac{1}{2}(p + p') \qquad (75)$$

Here we have

$$p^2 = p'^2 = M^2 \qquad k^2 = m^2 \qquad q^2 = \mu^2 \qquad (76)$$

We can choose as independent scalar variables

$$s = (P + \frac{1}{2}k + \frac{1}{2}q)^2 \qquad t = \frac{1}{4}(k - q)^2 \qquad (77a, b)$$

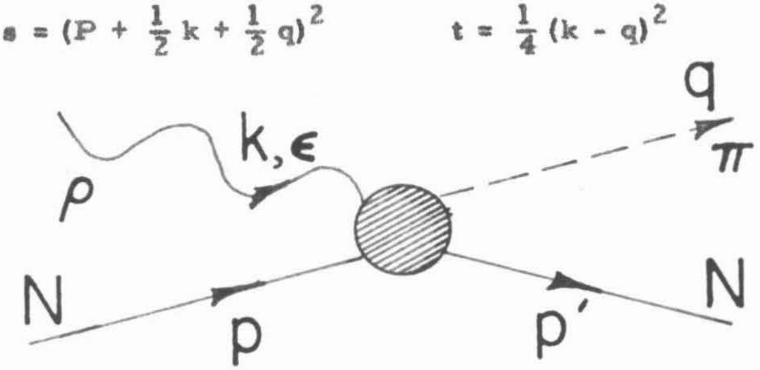

Fig. 3. General diagram for the reaction $\rho + N \rightarrow \pi + N$

Equations 8 are still valid:

$$(\not{p} - M)u(\vec{p}) = 0 \qquad \bar{u}(\vec{p}\,')(\not{p}' - M) = 0 \tag{78a, b}$$

Lorentz scalars that can be formed are $\not{\epsilon}$, $\not{k}$, $\not{q}$, $\not{P}$, $k \cdot \epsilon$, $P \cdot \epsilon$, $q \cdot \epsilon$, s, t. The last two can be included in the invariant amplitudes (see 7). Equation 74 gives

$$\not{q} = \not{p} - \not{p}' + \not{k} \tag{79}$$

and equations 78 allow it to reduce to a combination of $\not{k}$ and 1. They also allow the elimination of $\not{P}$ from a matrix element.

For vector fields, it is necessary to impose a subsidiary condition, corresponding to the continuity equation for the 4-current, R9-4.3,

$$\frac{\partial A_\mu}{\partial x_\mu} = 0 \tag{80}$$

For plane waves

$$A_\mu = \epsilon_\mu e^{-ik \cdot x} \tag{81}$$

and 80 becomes

$$k \cdot \epsilon = 0 \tag{82}$$

which eliminates another scalar. The relation

$$\not{k}^2 = m^2 \tag{83}$$

shows that the matrix element can contain only $\not{k}$ to the powers 0 and 1. A final condition is that it has to be linear and homogeneous in ϵ.

To find the number of independent invariants, we apply the appropriate symmetry conditions to the T matrix (in this case, parity conservation only):

$$\frac{1}{|\bar{P}_r|} T^J = \begin{array}{c|cc} \text{initial} \backslash \text{final} & \frac{1}{2}0 & -\frac{1}{2}0 \\ \hline \frac{1}{2}1 & a_1^J & a_2^J \\ \frac{1}{2}0 & a_3^J & a_4^J \\ \frac{1}{2}-1 & a_5^J & a_6^J \\ -\frac{1}{2}1 & -a_6^J & -a_5^J \\ -\frac{1}{2}0 & -a_4^J & -a_3^J \\ -\frac{1}{2}-1 & -a_2^J & -a_1^J \end{array} \times i(|\vec{q}|/|\vec{k}|)^{1/2} \quad (84)$$

The additional factor $i(|\vec{q}|/|\vec{k}|)^{1/2}$ serves to simplify some equations. Notice that in this case, for equation 50,

$$\eta_a = \eta_c = 1 \qquad \eta_b = \eta_d = -1 \qquad s_a = s_c = \tfrac{1}{2} \quad s_b = 1 \quad s_d = 0$$

and hence

$$\eta_g = -1 \qquad (85)$$

There are, then, six independent amplitudes, and the corresponding invariants can be chosen

$$\begin{aligned} M_1 &= \gamma_5 \not{\epsilon} & M_4 &= \gamma_5 \not{k}\, P\cdot\epsilon \\ M_2 &= \gamma_5 \not{\epsilon}\not{k} & M_5 &= \gamma_5\, q\cdot\epsilon \\ M_3 &= \gamma_5 P\cdot\epsilon & M_6 &= \gamma_5 \not{k}\, q\cdot\epsilon \end{aligned} \qquad (86)$$

The γ_5 are included due to the pseudoscalar nature of the π meson; the matrix element should be a true scalar as in section a).

In analogy to equations 30 and 34, we expect to write the cross section in the form

$$\frac{d\sigma}{d\Omega} = \frac{|\vec{q}|}{|k|} \, |< f|\mathfrak{F}|i >|^2 \tag{87a}$$

$$\mathfrak{F} = \sum_{i=1}^{6} \mathfrak{F}_i N_i \tag{87b}$$

The N_i are invariants involving Pauli matrices; we choose

$$\begin{aligned} N_1 &= i\,\vec{\sigma}\cdot\vec{\epsilon} & N_4 &= i\,\hat{q}\cdot\vec{\epsilon}\;\vec{\sigma}\cdot\hat{q} \\ N_2 &= i\,\vec{\sigma}\cdot\hat{q}\;\vec{\sigma}\cdot\vec{\epsilon}\;\vec{\sigma}\cdot\hat{k} & N_5 &= i\,\hat{k}\cdot\vec{\epsilon}\;\vec{\sigma}\cdot\hat{k} \\ N_3 &= i\,\hat{q}\cdot\vec{\epsilon}\;\vec{\sigma}\cdot\hat{k} & N_6 &= i\,\hat{k}\cdot\vec{\epsilon}\;\vec{\sigma}\cdot\hat{q} \end{aligned} \tag{88}$$

The polarization vectors

$$\epsilon = (\epsilon_o, \vec{\epsilon}) \tag{89a}$$

have to be normalized so that

$$\epsilon^{*}\cdot\epsilon = -1 \tag{89b}$$

From equation 82 we derive

$$\epsilon_o = \frac{\vec{k}\cdot\vec{\epsilon}}{k_o} \tag{89c}$$

and introducing this in 90b we get

$$|\vec{\epsilon}|^2 = 1 + \left|\frac{\vec{k}\cdot\vec{\epsilon}}{k_o}\right|^2$$

and if we define the direction of $\vec{\epsilon}$ relative to $\vec{k}$ by

$$\xi = \frac{|\vec{k}\cdot\vec{\epsilon}|}{|\vec{k}|\,|\vec{\epsilon}|} \tag{89d}$$

we obtain

$$|\vec{\epsilon}|^2 = \frac{k_o^2}{k_o^2 - \xi^2 \vec{k}^2} \tag{89e}$$

and in particular for the different helicity states,

$$|\vec{\epsilon}(1)|^2 = 1 \qquad |\vec{\epsilon}(0)|^2 = \frac{k_o^2}{m^2} \qquad |\vec{\epsilon}(-1)|^2 = 1 \tag{89f}$$

Equation 24 becomes

$$d\sigma = \int \frac{d^3\vec{p}\,'}{(2\pi)^3} \frac{\vec{q}^2 d|\vec{q}|\, d\Omega}{(2\pi)^3} \frac{1}{|\vec{v}_p - \vec{v}_k|} w_{fi} \tag{90a}$$

$$w_{fi} = (2\pi)^4 \delta^4(p + k - p' - q) \frac{M^2}{4EE' q_o k_o} |\mathfrak{a}|^2 \tag{90b}$$

Now p_o and k_o are fixed, and

$$p_o'^2 = M^2 + \vec{q}^2 \tag{91a}$$

$$q_o^2 = \mu^2 + \vec{q}^2 \tag{91b}$$

$$\frac{\partial(p_o + k_o - p_o' - q_o)}{\partial|\vec{q}|} = -\frac{|\vec{q}|W}{E' q_o} \tag{92}$$

$$|\vec{v}_p - \vec{v}_k| = \frac{|\vec{k}|W}{Ek_o} \tag{93}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{(2\pi)^2} \frac{M^2}{4W^2} \frac{|\vec{q}|}{|\vec{k}|} |\mathfrak{a}|^2 \tag{94}$$

We have to calculate next the equations connecting the $\bar{u}(\vec{p}\,')M_i u(\vec{p})$ with the N_i.

With the conventions used here, γ_5 turns out to be

$$\gamma_5 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{95}$$

and hence

$$\gamma_5 u_o' = i \begin{pmatrix} 0 \\ |f> \end{pmatrix} \tag{96}$$

We also use 89c.

$$\bar{u}(\vec{p}\,')M_1 u(\vec{p}) = \sqrt{\frac{E'+M}{2M}}\; u_o'^{\dagger}\left(1+\frac{\vec{\alpha}\cdot\vec{p}\,'}{E'+M}\right)\beta\gamma_5\beta(\epsilon_o - \vec{\alpha}\cdot\vec{\epsilon})\left(1+\frac{\vec{\alpha}\cdot\vec{p}}{E+M}\right)u_o\sqrt{\frac{E+M}{2M}}$$

$$= -\sqrt{\frac{(E+M)(E'+M)}{4M^2}}\; i(0, <f|)\left(1-\frac{\vec{\alpha}\cdot\vec{q}}{E'+M}\right)\left(\frac{\vec{\epsilon}\cdot\vec{k}}{k_o} - \vec{\alpha}\cdot\vec{\epsilon}\right)\left(1-\frac{\vec{\alpha}\cdot\vec{k}}{E+M}\right)\begin{pmatrix} |i> \\ 0 \end{pmatrix}$$

It is obvious that only odd powers of $\vec{\alpha}$ will give non-zero terms. Other invariants are reduced similarly

Using

$$\vec{k}^2 = \vec{p}^2 = E^2 - M^2 \tag{97a}$$

$$\vec{q}^2 = \vec{p}\,'^2 = E'^2 - M^2 \tag{97b}$$

we get

$$\bar{u}(\vec{p}\,')M_1 u(\vec{p}) = <f|\; \frac{(E+M)^{\frac{1}{2}}(E'+M)^{\frac{1}{2}}}{2M} N_1 + \frac{(E-M)^{\frac{1}{2}}(E'-M)^{\frac{1}{2}}}{2M} N_2$$

$$+ \frac{(E+M)^{\frac{1}{2}}(E'+M)^{\frac{1}{2}}(E-M)}{2Mk_o} N_5 + \frac{(E-M)^{\frac{1}{2}}(E'-M)^{\frac{1}{2}}(E+M)}{2Mk_o} N_6\, |i> \tag{98a}$$

$$\overline{u}(\vec{p}\,')M_2u(\vec{p}) = \langle f|\ \frac{(E+M)^{\frac{1}{2}}(E'+M)^{\frac{1}{2}}(E-M+k_o)}{2M}N_1 - \frac{(E-M)^{\frac{1}{2}}(E'-M)^{\frac{1}{2}}(E+M+k_o)}{2M}N_2$$

$$-\frac{(E+M)^{\frac{1}{2}}(E'+M)^{\frac{1}{2}}(E-M)(E+M+k_o)}{2Mk_o}N_5 + \frac{(E-M)^{\frac{1}{2}}(E'-M)^{\frac{1}{2}}(E+M)(E-M+k_o)}{2Mk_o}N_6|i\rangle$$

(98b)

$$\overline{u}(\vec{p}\,')M_3u(\vec{p}) = \langle f|-\frac{(E-M)^{\frac{1}{2}}(E'-M)^{\frac{1}{2}}(E'+M)}{4M}N_3 + \frac{(E+M)^{\frac{1}{2}}(E'+M)^{\frac{1}{2}}(E'-M)}{4M}N_4$$

$$-\frac{(E+M)^{\frac{1}{2}}(E'+M)^{\frac{1}{2}}(E-M)(E+E'+k_o)}{4Mk_o}N_5 + \frac{(E-M)^{\frac{1}{2}}(E'-M)^{\frac{1}{2}}(E+M)(E+E'+k_o)}{4Mk_o}N_6|i\rangle$$

(98c)

$$\overline{u}(\vec{p}\,')M_4u(\vec{p}) = \langle f|\ \frac{(E-M)^{\frac{1}{2}}(E'-M)^{\frac{1}{2}}(E'+M)(E+M+k_o)}{4M}N_3$$

$$+\frac{(E+M)^{\frac{1}{2}}(E+M)^{\frac{1}{2}}(E'-M)(E-M+k_o)}{4M}N_4$$

$$+\frac{(E+M)^{\frac{1}{2}}(E'+M)^{\frac{1}{2}}(E-M)(E+M+k_o)(E+E'+k_o)}{4Mk_o}N_5$$

$$+\frac{(E-M)^{\frac{1}{2}}(E'-M)^{\frac{1}{2}}(E+M)(E-M+k_o)(E+E'+k_o)}{4Mk_o}N_6|i\rangle \qquad (98d)$$

$$\overline{u}(\vec{p}\,')M_5u(\vec{p}) = \langle f|\ \frac{(E-M)^{\frac{1}{2}}(E'-M)^{\frac{1}{2}}(E'+M)}{2M}N_3 - \frac{(E+M)^{\frac{1}{2}}(E'+M)^{\frac{1}{2}}(E'-M)}{2M}N_4$$

$$-\frac{(E+M)^{\frac{1}{2}}(E'+M)^{\frac{1}{2}}(E-M)q_o}{2Mk_o}N_5 + \frac{(E-M)^{\frac{1}{2}}(E'-M)^{\frac{1}{2}}(E+M)q_o}{2Mk_o}N_6|i\rangle \qquad (98e)$$

$$\bar{u}(\vec{p}\,')M_6u(\vec{p}) = \langle f| - \frac{(E-M)^{\frac{1}{2}}(E'-M)^{\frac{1}{2}}(E'+M)(E+M+k_o)}{2M} N_3$$

$$- \frac{(E+M)^{\frac{1}{2}}(E'+M)^{\frac{1}{2}}(E'-M)(E-M+k_o)}{2M} N_4 + \frac{(E+M)^{\frac{1}{2}}(E'+M)^{\frac{1}{2}}(E-M)(E+M+k_o)q_o}{2Mk_o} N_5$$

$$+ \frac{(E-M)^{\frac{1}{2}}(E'-M)^{\frac{1}{2}}(E+M)(E-M+k_o)q_o}{2Mk_o} N_6 |i\rangle \qquad (98f)$$

and hence

$$\mathfrak{F}_1 = \frac{\sqrt{(E+M)(E'+M)}}{8\pi W} [\, \mathfrak{a}_1 + (W-M)\mathfrak{a}_2] \qquad (99a)$$

$$\mathfrak{F}_2 = \frac{\sqrt{(E-M)(E'-M)}}{8\pi W} [\, \mathfrak{a}_1 - (W+M)\mathfrak{a}_2] \qquad (99b)$$

$$\mathfrak{F}_3 = \frac{\sqrt{(E-M)(E'-M)}\,(E'+M)}{16\pi W} [\, -\mathfrak{a}_3 + (W+M)\mathfrak{a}_4 + 2\mathfrak{a}_5 - 2(W+M)\mathfrak{a}_6] \qquad (99c)$$

$$\mathfrak{F}_4 = \frac{\sqrt{(E+M)(E'+M)}\,(E'-M)}{16\pi W} [\, \mathfrak{a}_3 + (W-M)\mathfrak{a}_4 - 2\mathfrak{a}_5 - 2(W-M)\mathfrak{a}_6] \qquad (99d)$$

$$\mathfrak{F}_5 = \frac{\sqrt{(E+M)(E'+M)}\,(E-M)}{16\pi W k_o} [\, 2\mathfrak{a}_1 - 2(W+M)\mathfrak{a}_2 - (W+E')\mathfrak{a}_3 + (W+E')(W+M)\mathfrak{a}_4$$

$$- 2q_o\mathfrak{a}_5 + 2q_o(W+M)\mathfrak{a}_6] \qquad (99e)$$

$$\mathfrak{F}_6 = \frac{\sqrt{(E-M)(E'-M)}\,(E+M)}{16\pi W k_o} [\, 2\mathfrak{a}_1 + 2(W-M)\mathfrak{a}_2 + (W+E')\mathfrak{a}_3$$

$$+ (W+E')(W-M)\mathfrak{a}_4 + 2q_o\mathfrak{a}_5 + 2q_o(W-M)\mathfrak{a}_6] \qquad (99f)$$

Here the reflection symmetry is evident:

$$\mathfrak{F}_1(-W) = +\mathfrak{F}_2(W)\,; \qquad \mathfrak{F}_3(-W) = +\mathfrak{F}_4(W)\,; \quad \mathfrak{F}_5(-W) = +\mathfrak{F}_6(W) \qquad (100a, b, c)$$

The unpolarized cross section is equal to:

$$\left(\frac{d\sigma}{d\Omega}\right)_{unpol} = \frac{|\vec{q}|}{|\vec{k}|}\,\frac{1}{6}\sum_{\substack{spin\\ polarization}} |\langle f\,|\,\mathcal{F}\,|\,i\rangle|^2$$

$$= \frac{|\vec{q}|}{|\vec{k}|}\,\frac{1}{6}\sum_{pol.} Tr\,(\mathcal{F}^\dagger\mathcal{F}) \tag{101}$$

In addition to the relations in 36a and 36b, we have

$$Tr(\vec{\sigma}\cdot\vec{a}\ \ \vec{\sigma}\cdot\vec{b}\ \ \vec{\sigma}\cdot\vec{c}\ \ \vec{\sigma}\cdot\vec{d}) = 2(\vec{a}\cdot\vec{b}\ \ \vec{c}\cdot\vec{d} - \vec{a}\cdot\vec{c}\ \ \vec{b}\cdot\vec{d} + \vec{a}\cdot\vec{d}\ \ \vec{b}\cdot\vec{c}) \tag{102}$$

and these allow us to calculate

$$\begin{aligned}
\frac{1}{2}\,Tr(\mathcal{F}^\dagger\mathcal{F}) = {} & \vec{\epsilon}^{\,2}|\mathcal{F}_1|^2 + \vec{\epsilon}^{\,2}|\mathcal{F}_2|^2 + (\hat{q}\cdot\vec{\epsilon})^2|\mathcal{F}_3|^2 + (\hat{q}\cdot\vec{\epsilon})^2|\mathcal{F}_4|^2 + (\hat{k}\cdot\vec{\epsilon})|\mathcal{F}_5|^2 \\
& + (\hat{k}\cdot\vec{\epsilon})^2|\mathcal{F}_6|^2 + 2(2\hat{k}\cdot\vec{\epsilon}\ \ \hat{q}\cdot\vec{\epsilon} - \hat{k}\cdot\hat{q}\,\vec{\epsilon}^{\,2})\,Re(\mathcal{F}_1^*\mathcal{F}_2) \\
& + 2\,\hat{q}\cdot\vec{\epsilon}\ \ \hat{k}\cdot\vec{\epsilon}\ Re(\mathcal{F}_1^*\mathcal{F}_3) + 2(\hat{q}\cdot\vec{\epsilon})^2 Re(\mathcal{F}_1^*\mathcal{F}_4) + 2(\hat{k}\cdot\vec{\epsilon})^2 Re(\mathcal{F}_1^*\mathcal{F}_5) \\
& + 2\,\hat{k}\cdot\vec{\epsilon}\ \ \hat{q}\cdot\vec{\epsilon}\ Re(\mathcal{F}_1^*\mathcal{F}_6) + 2\,(\hat{q}\cdot\vec{\epsilon})^2 Re(\mathcal{F}_2^*\mathcal{F}_3) + 2\,\hat{k}\cdot\vec{\epsilon}\ \ \hat{q}\cdot\vec{\epsilon}\ Re(\mathcal{F}_2^*\mathcal{F}_4) \\
& + 2\,\hat{k}\cdot\vec{\epsilon}\ \ \hat{q}\cdot\vec{\epsilon}\ Re(\mathcal{F}_2^*\mathcal{F}_5) + 2(\hat{k}\cdot\vec{\epsilon})^2 Re(\mathcal{F}_2^*\mathcal{F}_6) + 2(\hat{q}\cdot\vec{\epsilon})^2\,\hat{k}\cdot\hat{q}\ Re(\mathcal{F}_3^*\mathcal{F}_4) \\
& + 2\,\hat{k}\cdot\vec{\epsilon}\ \hat{q}\cdot\vec{\epsilon}\ Re(\mathcal{F}_3^*\mathcal{F}_5) + 2\,\hat{k}\cdot\vec{\epsilon}\ \ \hat{q}\cdot\vec{\epsilon}\ \ \hat{k}\cdot\hat{q}\ Re(\mathcal{F}_3^*\mathcal{F}_6) \\
& + 2\,\hat{k}\cdot\vec{\epsilon}\ \ \hat{q}\cdot\vec{\epsilon}\ \ \hat{k}\cdot\hat{q}\ Re(\mathcal{F}_4^*\mathcal{F}_5) + 2\,\hat{k}\cdot\vec{\epsilon}\ \ \hat{q}\cdot\vec{\epsilon}\ Re\,(\mathcal{F}_4^*\mathcal{F}_6) \\
& + 2(\hat{k}\cdot\vec{\epsilon})^2\,\hat{k}\cdot\hat{q}\ Re(\mathcal{F}_5^*\mathcal{F}_6)
\end{aligned} \tag{103}$$

where we have assumed $\vec{\epsilon}$ to be real and also

$$\hat{k}^2 = \hat{q}^2 = 1 \tag{104a}$$

We write in what follows

$$\hat{k}\cdot\hat{q} = x = \cos\theta \tag{104b}$$

The sum over polarizations is done by using

$$\sum_{\text{pol.}} \vec{\epsilon}^{\,2} = 3 + \frac{\vec{k}^2}{m^2}$$

$$= 3 + \kappa \tag{105a}$$

$$\sum_{\text{pol.}} \vec{a}\cdot\vec{\epsilon}\;\; \vec{b}\cdot\vec{\epsilon} = \vec{a}\cdot\vec{b} + \kappa\,\hat{k}\cdot\vec{a}\;\;\hat{k}\cdot\vec{b} \tag{105b}$$

and gives

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol.}} = \frac{1}{3}\Big[&(3+\kappa)|\mathfrak{F}_1|^2 + (3+\kappa)|\mathfrak{F}_2|^2 + (1+\kappa x^2)|\mathfrak{F}_3|^2 + (1+\kappa x^2)|\mathfrak{F}_4|^2 \\ &+ (1+\kappa)|\mathfrak{F}_5|^2 + (1+\kappa)|\mathfrak{F}_6|^2 - 2(1-\kappa)x\,\mathrm{Re}\,(\mathfrak{F}_1^*\mathfrak{F}_2) \\ &+ 2(1+\kappa)x\,\mathrm{Re}\,(\mathfrak{F}_1^*\mathfrak{F}_3) + 2(1+\kappa x^2)\mathrm{Re}(\mathfrak{F}_1^*\mathfrak{F}_4) + 2(1+\kappa)\,\mathrm{Re}\,(\mathfrak{F}_1^*\mathfrak{F}_5) \\ &+ 2(1+\kappa)x\,\mathrm{Re}\,(\mathfrak{F}_1^*\mathfrak{F}_6) + 2(1+\kappa x^2)\,\mathrm{Re}\,(\mathfrak{F}_2^*\mathfrak{F}_3) + 2(1+\kappa)x\,\mathrm{Re}\,(\mathfrak{F}_2^*\mathfrak{F}_4) \\ &+ 2(1+\kappa)x\,\mathrm{Re}\,(\mathfrak{F}_2^*\mathfrak{F}_5) + 2(1+\kappa)\,\mathrm{Re}\,(\mathfrak{F}_2^*\mathfrak{F}_6) + 2(1+\kappa x^2)x\,\mathrm{Re}\,(\mathfrak{F}_3^*\mathfrak{F}_4) \\ &+ 2(1+\kappa)x\,\mathrm{Re}\,(\mathfrak{F}_3^*\mathfrak{F}_5) + 2(1+\kappa)x^2\,\mathrm{Re}\,(\mathfrak{F}_3^*\mathfrak{F}_6) + 2(1+\kappa)x^2\,\mathrm{Re}\,(\mathfrak{F}_4^*\mathfrak{F}_5) \\ &+ 2(1+\kappa)x\,\mathrm{Re}\,(\mathfrak{F}_4^*\mathfrak{F}_6) + 2(1+\kappa)x\,\mathrm{Re}\,(\mathfrak{F}_5^*\mathfrak{F}_6)\Big]\end{aligned} \tag{106a}$$

which can be expressed, according to D-7, by

$$G = 2\begin{pmatrix} 3+\kappa & (-1+\kappa)x & (1+\kappa)x & 1+\kappa x^2 & 1+\kappa & (1+\kappa)x \\ (-1+\kappa)x & 3+\kappa & 1+\kappa x^2 & (1+\kappa)x & (1+\kappa)x & 1+\kappa \\ (1+\kappa)x & 1+\kappa x^2 & 1+\kappa x^2 & (1+\kappa x^2)x & (1+\kappa)x & (1+\kappa)x^2 \\ 1+\kappa x^2 & (1+\kappa)x & (1+\kappa x^2)x & 1+\kappa x^2 & (1+\kappa)x^2 & (1+\kappa)x \\ 1+\kappa & (1+\kappa)x & (1+\kappa)x & (1+\kappa)x^2 & 1+\kappa & (1+\kappa)x \\ (1+\kappa)x & 1+\kappa & (1+\kappa)x^2 & (1+\kappa)x & (1+\kappa)x & 1+\kappa \end{pmatrix} \tag{106b}$$

where κ is defined in equation 105a

$$\kappa = \frac{\vec{k}^2}{m^2} \qquad 1+\kappa = \frac{k_o^2}{m^2} \tag{106c}$$

We next have to determine the quantities $\langle \lambda_c \lambda_d | N_i | \lambda_a \lambda_b \rangle$, and we use equations B-7, B-13 (with $\theta = \varphi = 0$) and the ones corresponding to 47:

$$\vec{\sigma}\cdot\hat{k}\,|i\rangle = -\lambda_a\,|i\rangle \tag{107a}$$

$$\vec{\sigma}\cdot\hat{q}\,|f\rangle = -\lambda_c\,|f\rangle \tag{107b}$$

We can write then

$$\begin{aligned} f_{\lambda_c\lambda_d;\lambda_a\lambda_b} &= \sqrt{\frac{|\vec{q}|}{|\vec{k}|}}\,\langle \lambda_c\lambda_d|\mathcal{F}|\lambda_a\lambda_b\rangle \\ &= \sqrt{\frac{|\vec{q}|}{|\vec{k}|}}\sum_{i=1}^{6}\mathcal{F}_i\langle \lambda_c\lambda_d|N_i|\lambda_a\lambda_b\rangle \end{aligned} \tag{108}$$

$$f_{10;11} = i\sqrt{\frac{|\vec{q}|}{|\vec{k}|}}\sin\frac{\theta}{2}\,e^{-i\varphi}\left(\sqrt{2}\,\mathcal{F}_1 + \sqrt{2}\,\mathcal{F}_2 + \sqrt{2}\cos^2\frac{\theta}{2}\,\mathcal{F}_3 + \sqrt{2}\cos^2\frac{\theta}{2}\,\mathcal{F}_4\right) \tag{108a}$$

$$f_{-\frac{1}{2}0;\frac{1}{2}1} = i\sqrt{\frac{|\vec{q}|}{|\vec{k}|}}\cos\frac{\theta}{2}\left(\sqrt{2}\,\mathcal{F}_1 - \sqrt{2}\,\mathcal{F}_2 - \sqrt{2}\sin^2\frac{\theta}{2}\,\mathcal{F}_3 + \sqrt{2}\sin^2\frac{\theta}{2}\,\mathcal{F}_4\right) \tag{108b}$$

$$f_{\frac{1}{2}0;\frac{1}{2}0} = i\sqrt{\frac{|\vec{q}|}{|\vec{k}|}}\cos\frac{\theta}{2}\left(\mathcal{F}_1 + \mathcal{F}_2 + \cos\theta\,\mathcal{F}_3 + \cos\theta\,\mathcal{F}_4 + \mathcal{F}_5 + \mathcal{F}_6\right)\frac{k_o}{m} \tag{108c}$$

$$f_{-\frac{1}{2}0;\frac{1}{2}0} = -i\sqrt{\frac{|\vec{q}|}{|\vec{k}|}}\sin\frac{\theta}{2}\,e^{i\varphi}\left(\mathcal{F}_1 - \mathcal{F}_2 + \cos\theta\,\mathcal{F}_3 - \cos\theta\,\mathcal{F}_4 + \mathcal{F}_5 - \mathcal{F}_6\right)\frac{k_o}{m} \tag{108d}$$

$$f_{\frac{1}{2}0;\frac{1}{2}-1} = i\sqrt{\frac{|\vec{q}|}{|\vec{k}|}}\;2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}\,e^{i\varphi}\left(-\frac{1}{\sqrt{2}}\,\mathcal{F}_3 - \frac{1}{\sqrt{2}}\,\mathcal{F}_4\right) \tag{108e}$$

$$f_{-\frac{1}{2}0;\frac{1}{2}-1} = i\sqrt{\frac{|\vec{q}|}{|\vec{k}|}}\;2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\,e^{2i\varphi}\left(\frac{1}{\sqrt{2}}\mathcal{F}_3 - \frac{1}{\sqrt{2}}\,\mathcal{F}_4\right) \tag{108f}$$

Using A5 and 84,

$$a_1 = \sum_{J=\ell+\frac{1}{2}} a_1^J(P'_{\ell+1} + P'_\ell) = \sqrt{2}\,\mathcal{F}_1 + \sqrt{2}\,\mathcal{F}_2 + \sqrt{2}\,\frac{1+x}{2}\mathcal{F}_3 + \sqrt{2}\,\frac{1+x}{2}\,\mathcal{F}_4 \tag{109a}$$

$$a_2 = \sum_{J=\ell+\frac{1}{2}} a_2^J(P'_{\ell+1} - P'_\ell) = \sqrt{2}\,\mathcal{F}_1 - \sqrt{2}\,\mathcal{F}_2 - \sqrt{2}\frac{1-x}{2}\,\mathcal{F}_3 + \sqrt{2}\,\frac{1-x}{2}\mathcal{F}_4 \tag{109b}$$

$$a_3 = \sum_{J=\ell+\frac{1}{2}} \frac{m}{k_o}\,a_3^J(P'_{\ell+1} - P'_\ell) = \mathcal{F}_1 - \mathcal{F}_2 + x\mathcal{F}_3 + x\mathcal{F}_4 + \mathcal{F}_5 + \mathcal{F}_6 \tag{109c}$$

$$a_4 = \sum_{J=\ell+\frac{1}{2}} \frac{m}{k_o}\;a_4^J\,(P'_{\ell+1} + P'_\ell) = \mathcal{F}_1 - \mathcal{F}_2 + x\mathcal{F}_3 \quad x\mathcal{F}_4 + \mathcal{F}_5 - \mathcal{F}_6 \tag{109d}$$

$$a_5 = \sum_{J=\ell+\frac{1}{2}} a_5^J\;\frac{-P''_{\ell+1} + P''_\ell}{\sqrt{\ell(\ell+2)}} = -\frac{1}{\sqrt{2}}\,\mathcal{F}_3 - \frac{1}{\sqrt{2}}\,\mathcal{F}_4 \tag{109e}$$

$$a_6 = \sum_{J=\ell+\frac{1}{2}} a_6^J\;\frac{P''_{\ell+1} + P''_\ell}{\sqrt{\ell(\ell+2)}} = \frac{1}{\sqrt{2}}\,\mathcal{F}_3 - \frac{1}{\sqrt{2}}\,\mathcal{F}_4 \tag{109f}$$

By inverting these linear equations we obtain the partial wave expansions of the $\mathcal{F}_i$:

$$\mathcal{F}_1 = \frac{1}{2\sqrt{2}}\left[a_1 + a_2 + (1+x)a_5 + (1-x)a_6\right] \tag{110a}$$

$$\mathcal{F}_2 = \frac{1}{2\sqrt{2}}\left[a_1 - a_2 + (1+x)a_5 - (1-x)a_6\right] \tag{110b}$$

$$\mathcal{F}_3 = -\frac{1}{\sqrt{2}}\left[a_5 - a_6\right] \tag{110c}$$

$$\mathcal{F}_4 = -\frac{1}{\sqrt{2}}\left[a_5 + a_6\right] \tag{110d}$$

$$\mathcal{F}_5 = \frac{1}{2\sqrt{2}}\left[-a_1 - a_2 + \sqrt{2}a_3 + \sqrt{2}\,a_4 - (1-x)a_5 - (1+x)a_6\right] \tag{110e}$$

$$\mathcal{F}_6 = \frac{1}{2\sqrt{2}}\left[-a_1 + a_2 + \sqrt{2}\,a_3 - \sqrt{2}\,a_4 - (1-x)a_5 + (1+x)a_6\right] \tag{110f}$$

From these equations we derive the φ_i^J (equation D-12)

$$\varphi_1^J = \frac{P'_{\ell+1} + P'_\ell}{2\sqrt{2}}\begin{pmatrix}1\\1\\0\\0\\-1\\-1\end{pmatrix} \qquad \varphi_2^J = \frac{P'_{\ell+1} - P'_\ell}{2\sqrt{2}}\begin{pmatrix}1\\-1\\0\\0\\-1\\1\end{pmatrix}$$

$$\varphi_3^J = \frac{P'_{\ell+1} - P'_\ell}{2}\,\frac{m}{k_o}\begin{pmatrix}0\\0\\0\\0\\1\\1\end{pmatrix} \qquad \varphi_4^J = \frac{P'_{\ell+1} + P'_\ell}{2}\,\frac{m}{k_o}\begin{pmatrix}0\\0\\0\\0\\1\\-1\end{pmatrix} \tag{111}$$

$$\varphi_5^J = \frac{-P''_{\ell+1} + P''_\ell}{2\sqrt{2\ell(\ell+2)}}\begin{pmatrix}1+x\\1+x\\-2\\-2\\-1+x\\-1+x\end{pmatrix} \qquad \varphi_6^J = \frac{P''_{\ell+1} + P''_\ell}{2\sqrt{2\ell(\ell+2)}}\begin{pmatrix}1-x\\-1+x\\2\\-2\\-1-x\\1+x\end{pmatrix}$$

and using 106b we get the χ_i^J (equation D-16)

$$\chi_1^J = \frac{-P_{\ell+1} + P_\ell}{2\sqrt{2}} \begin{pmatrix} 2 \\ 2 \\ 1+x \\ 1+x \\ 0 \\ 0 \end{pmatrix} \qquad \chi_2^J = \frac{P_{\ell+1} + P_\ell}{2\sqrt{2}} \begin{pmatrix} 2 \\ -2 \\ -1+x \\ -1+x \\ 0 \\ 0 \end{pmatrix}$$

$$\chi_3^J = \frac{P_{\ell+1} + P_\ell}{2} \frac{k_o}{m} \begin{pmatrix} 1 \\ 1 \\ x \\ x \\ 1 \\ 1 \end{pmatrix} \qquad \chi_4^J = \frac{-P_{\ell+1} + P_\ell}{2} \frac{k_o}{m} \begin{pmatrix} 1 \\ -1 \\ x \\ -x \\ 1 \\ -1 \end{pmatrix} \tag{112}$$

$$\chi_5^J = \frac{\sqrt{\ell(\ell+2)}}{2\sqrt{2}} \left(\frac{P_{\ell+2} - P_\ell}{2\ell+3} + \frac{P_{\ell+1} - P_{\ell-1}}{2\ell+1} \right) \begin{pmatrix} 0 \\ 0 \\ -1 \\ -1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

$$\chi_6^J = \frac{\sqrt{\ell(\ell+2)}}{2\sqrt{2}} \left(\frac{P_{\ell+2} - P_\ell}{2\ell+3} - \frac{P_{\ell+1} - P_{\ell-1}}{2\ell+1} \right) \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

From appendix C section b we have used the relations obtained by equating both expressions for the $d^J_{\lambda\mu}$ to eliminate the derivatives of the Legendre polynomials in χ_i^J.

Also in χ_3^J and χ_4^J we have used the definition of κ that gives

$$(1+\kappa)\frac{m}{k_o} = \left(1 + \frac{\bar{k}^2}{m^2}\right)\frac{m}{k_o}$$

$$= \frac{k_o^2}{m^2}\,\frac{m}{k_o}$$

$$= \frac{k_o}{m}$$

We also define

$$
\begin{aligned}
T_1 &= \left[\frac{f_{\frac{1}{2}0;\frac{1}{2}1}}{\sin\frac{\theta}{2}\, e^{-i\varphi}} + \frac{f_{-\frac{1}{2}0;\frac{1}{2}1}}{\cos\frac{\theta}{2}} \right] \frac{1}{i} \sqrt{\frac{|\vec{k}|}{|\vec{q}|}} \\
T_4 &= \left[\frac{f_{\frac{1}{2}0;\frac{1}{2}1}}{\sin\frac{\theta}{2}\, e^{-i\varphi}} - \frac{f_{-\frac{1}{2}0;\frac{1}{2}1}}{\cos\frac{\theta}{2}} \right] \frac{1}{i} \sqrt{\frac{|\vec{k}|}{|\vec{q}|}} \\
T_2 &= \left[\frac{f_{\frac{1}{2}0;\frac{1}{2}0}}{\cos\frac{\theta}{2}} + \frac{f_{-\frac{1}{2}0;\frac{1}{2}0}}{-\sin\frac{\theta}{2}\, e^{i\varphi}} \right] \frac{1}{i} \sqrt{\frac{|\vec{k}|}{|\vec{q}|}} \\
T_5 &= \left[\frac{f_{\frac{1}{2}0;\frac{1}{2}0}}{\cos\frac{\theta}{2}} - \frac{f_{-\frac{1}{2}0;\frac{1}{2}0}}{-\sin\frac{\theta}{2}\, e^{i\varphi}} \right] \frac{1}{i} \sqrt{\frac{|\vec{k}|}{|\vec{q}|}} \\
T_3 &= \left[\frac{f_{\frac{1}{2}0;\frac{1}{2}-1}}{2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}\, e^{i\varphi}} + \frac{f_{-\frac{1}{2}0;\frac{1}{2}-1}}{2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\, e^{2i\varphi}} \right] \frac{1}{i} \sqrt{\frac{|\vec{k}|}{|\vec{q}|}} \\
T_6 &= \left[\frac{f_{\frac{1}{2}0;\frac{1}{2}-1}}{2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}\, e^{i\varphi}} - \frac{f_{-\frac{1}{2}0;\frac{1}{2}-1}}{2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\, e^{2i\varphi}} \right] \frac{1}{i} \sqrt{\frac{|\vec{k}|}{|\vec{q}|}}
\end{aligned}
\tag{113}
$$

$$
\begin{aligned}
\beta_1^J &= \alpha_1^J + \alpha_2^J & \beta_2^J &= \alpha_3^J + \alpha_4^J & \beta_3^J &= \alpha_5^J + \alpha_6^J \\
\beta_4^J &= \alpha_1^J - \alpha_2^J & \beta_5^J &= \alpha_3^J - \alpha_4^J & \beta_6^J &= \alpha_5^J - \alpha_6^J
\end{aligned}
\tag{114}
$$

From these we find, by using A-5 and 84, the partial wave expansion of the T_i:

$$
\begin{aligned}
T_{1,4} &= \sum_{J=\ell+\frac{1}{2}} \left(\beta_{1,4}^J P'_{\ell+1} + \beta_{4,1}^J P'_{\ell} \right) \\
T_{2,5} &= \sum_{J=\ell+\frac{1}{2}} \left(\beta_{2,5}^J P'_{\ell+1} - \beta_{5,2}^J P'_{\ell} \right) \\
T_{3,6} &= \sum_{J=\ell+\frac{1}{2}} \left(\beta_{3,6}^J \frac{P''_{\ell}}{\sqrt{\ell(\ell+2)}} - \beta_{6,3}^J \frac{P''_{\ell+1}}{\sqrt{\ell(\ell+2)}} \right)
\end{aligned}
\tag{115}
$$

Combining with equations 108, we obtain

$$
\begin{aligned}
T_1 &= 2\sqrt{2}\,\mathcal{F}_1 + \sqrt{2}\,x\,\mathcal{F}_3 + \sqrt{2}\,\mathcal{F}_4 \\
T_4 &= 2\sqrt{2}\,\mathcal{F}_2 + \sqrt{2}\,\mathcal{F}_3 + \sqrt{2}\,x\,\mathcal{F}_4 \\
T_2 &= \frac{k_o}{m}\left(2\mathcal{F}_1 + 2x\,\mathcal{F}_3 + 2\mathcal{F}_5\right) \\
T_5 &= \frac{k_o}{m}\left(2\mathcal{F}_2 + 2x\,\mathcal{F}_4 + 2\mathcal{F}_6\right) \\
T_3 &= -\sqrt{2}\,\mathcal{F}_4 \\
T_6 &= -\sqrt{2}\,\mathcal{F}_3
\end{aligned}
\tag{116}
$$

and using equations 99:

$$
\begin{aligned}
T_1 = {} & \frac{\sqrt{(E+M)(E'+M)}}{8\sqrt{2}\,\pi W}\left[4\mathcal{A}_1 + 4(W-M)\mathcal{A}_2 + (E'-M)\{\mathcal{A}_3 + (W-M)\mathcal{A}_4 - 2\mathcal{A}_5 - 2(W-M)\mathcal{A}_6\}\right] \\
& + x\,\frac{\sqrt{(E-M)(E'-M)}\;(E'+M)}{8\sqrt{2}\,\pi W}\left[-\mathcal{A}_3 + (W+M)\mathcal{A}_4 + 2\mathcal{A}_5 - 2(W+M)\mathcal{A}_6\right]
\end{aligned}
$$

$$
\begin{aligned}
T_2 = {} & \frac{\sqrt{(E+M)(E'+M)}}{8\pi W m}\left[2(W-M)\mathcal{A}_1 + 2m^2\mathcal{A}_2 + (E-M)\{-(W+E')\mathcal{A}_3 + (W+E')(W+M)\mathcal{A}_4\right. \\
& \left. -2q_o\mathcal{A}_5 + 2q_o(W+M)\mathcal{A}_6\}\right] + x\,\frac{\sqrt{(E-M)(E'-M)}(E'+M)k_o}{8\pi W m}\left[-\mathcal{A}_3 + (W+M)\mathcal{A}_4\right. \\
& \left. + 2\mathcal{A}_5 - 2(W+M)\mathcal{A}_6\right]
\end{aligned}
\tag{117}
$$

$$
T_3 = \frac{\sqrt{(E+M)(E'+M)}\;(E'-M)}{8\sqrt{2}\,\pi W}\left[-\mathcal{A}_3 - (W-M)\mathcal{A}_4 + 2\mathcal{A}_5 + 2(W-M)\mathcal{A}_6\right]
$$

$$
\begin{aligned}
T_4 = {} & \frac{\sqrt{(E-M)(E'-M)}}{8\sqrt{2}\,\pi W}\left[4\mathcal{A}_1 - 4(W+M)\mathcal{A}_2 + (E'+M)\{-\mathcal{A}_3 + (W+M)\mathcal{A}_4 + 2\mathcal{A}_5 - 2(W+M)\mathcal{A}_6\}\right] \\
& + x\,\frac{\sqrt{(E+M)(E'+M)}\;E'-M)}{8\sqrt{2}\,\pi W}\left[\mathcal{A}_3 + (W-M)\mathcal{A}_4 - 2\mathcal{A}_5 - 2(W-M)\mathcal{A}_6\right]
\end{aligned}
$$

$$T_5 = \frac{\sqrt{(E-M)(E'-M)}}{8\pi W m}\left[2(W+M)a_1 - 2m^2 a_2 + (E+M)\{(W+E')a_3 + (W+E')(W-M)a_4 + 2q_o a_5 + 2q_o(W-M)a_6\}\right] + x\frac{\sqrt{(E+M)(E'+M)}(E'-M)k_o}{8\pi W m}\left[a_3 + (W-M)a_4 - 2a_5 - 2(W-M)a_6\right] \quad (117)$$

$$T_6 = \frac{\sqrt{(E-M)(E'-M)}\;(E'+M)}{8\pi W}\left[a_3 - (W+M)a_4 - 2a_5 + 2(W+M)a_6\right]$$

The reflection symmetry is now expressed by

$$\beta_i^J(-W) = \eta_{(i)}\beta_{i+3}^J(W) \qquad i = 1, 2, 3 \quad (118a)$$

$$T_i(-W) = \eta_{(i)}T_{i+3}(W) \qquad i = 1, 2, 3 \quad (118b)$$

where

$$\eta_i = 1 \qquad i = 1, 3 \qquad \eta_i = -1 \qquad i = 2 \quad (118c)$$

This is in agreement with equation A-14.

The projection operators that express the β_i^J in terms of the T_i are

$$\beta_{1,4}^J = \frac{1}{2}\int_{-1}^{1} (T_{1,4}P_\ell - T_{4,1}P_{\ell+1})\,dx$$

$$\beta_{2,5}^J = \frac{1}{2}\int_{-1}^{1} (T_{2,5}P_\ell + T_{5,2}P_{\ell+1})\,dx \quad (119)$$

$$\beta_{3,6}^J = \frac{1}{2}\int_{-1}^{1}\left[T_{3,6}\frac{\sqrt{\ell(\ell+2)}\;P_{\ell+2} - P_\ell)}{2\ell+3} + T_{6,3}\frac{\sqrt{\ell(\ell+2)}\;(P_{\ell+1} - P_{\ell-1})}{2\ell+1}\right]dx$$

Calculations similar to those in this section are shown in detail in section IIa, and a general case is studied in appendix D.

Time reversal invariance relates the matrix elements of the reaction $\rho + N \rightarrow \pi + N$ with those of $\pi + N \rightarrow \rho + N$. It is expressed by R5-55

$$< \lambda_c \lambda_d | S^J | \lambda_a \lambda_b > = < \lambda_a \lambda_b | S^J | \lambda_c \lambda_d > \tag{120}$$

Note: For a photon (mass zero), the invariants M_i appear only in certain combinations due to gauge invariance (see Ref. 10, for instance). We also have $\vec{\epsilon} \cdot \hat{k} = 0$ and the amplitudes $\mathcal{F}_5$ and $\mathcal{F}_6$ are absent; so are T_2 and T_5 (β_2^J and β_5^J, or a_3^J and a_4^J).

c) $\rho + N \rightarrow \rho + N$

Momentum conservation

$$p + k = p' + k' \tag{121a}$$

allows us to choose as independent variables

$$P = \frac{1}{2}(p + p') \qquad K = \frac{1}{2}(k + k') \qquad \Delta = \frac{1}{2}(k - k') = \frac{1}{2}(p' - p) \tag{121b}$$

satisfying relations

$$P \cdot \Delta = K \cdot \Delta = 0 \qquad P^2 + \Delta^2 = 0 \qquad K^2 + \Delta^2 = m^2 \tag{121c, d, e}$$

that leave two independent scalar variables, which can be chosen, for instance

$$s = (P + K)^2 \qquad t = 4\Delta^2 \tag{121f, g}$$

We also have in this case two equations like 82:

$$k \cdot \epsilon = k' \cdot \epsilon' = 0 \tag{122}$$

that are equivalent to

$$\Delta \cdot \epsilon = -K \cdot \epsilon \qquad\qquad \Delta \cdot \epsilon' = K \cdot \epsilon' \tag{123a, b}$$

Also

$$(\not{p} - M)u(\vec{p}) = 0 \qquad \bar{u}(\vec{p}\,')(\not{p}' - M) = 0 \tag{124a, b}$$

permit us to eliminate $\not{P}$ and $\not{Q}$ from invariants. The matrix element also has to be linear and homogeneous in both ϵ and ϵ'.

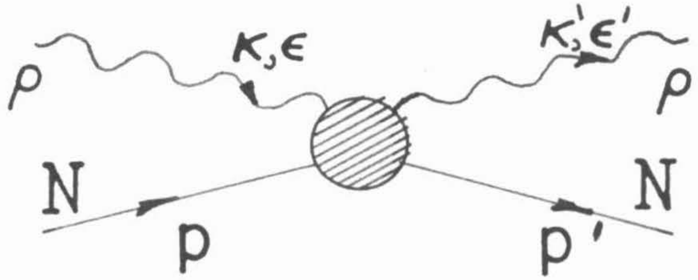

Fig. 4. General diagram for the reaction $\rho + N \rightarrow \rho + N$

In this case, both time reversal invariance and parity conservation reduce the number of invariants, according to the rules derived in appendix A. To find the number of independent invariants, we look at the matrix

$$\frac{1}{|\vec{p}_r|} T^J = \begin{array}{c|cccccc} \text{final} & \tfrac{1}{2}\,1 & \tfrac{1}{2}\,0 & \tfrac{1}{2}\,{-1} & -\tfrac{1}{2}\,1 & -\tfrac{1}{2}\,0 & -\tfrac{1}{2}\,{-1} \\ \text{initial} & & & & & & \end{array}$$

$$\begin{array}{c} \tfrac{1}{2}\,1 \\ \tfrac{1}{2}\,0 \\ \tfrac{1}{2}\,{-1} \\ -\tfrac{1}{2}\,1 \\ -\tfrac{1}{2}\,0 \\ -\tfrac{1}{2}\,{-1} \end{array} \begin{pmatrix} a_1^J & a_2^J & a_3^J & a_4^J & J_5^J & a_6^J \\ a_2^J & a_7^J & a_8^J & a_9^J & a_{10}^J & a_5^J \\ a_3^J & a_8^J & a_{11}^J & a_{12}^J & a_9^J & a_4^J \\ a_4^J & a_9^J & a_{12}^J & a_{11}^J & a_8^J & a_3^J \\ a_5^J & a_{10}^J & a_9^J & a_8^J & a_7^J & a_2^J \\ a_6^J & a_5^J & a_4^J & a_3^J & a_2^J & a_1^J \end{pmatrix} \tag{125}$$

where relations A-35 and A-47 have been used. There are 12 independent elements.

If we then try to write down the 12 invariants, we come up rather with 14 that obey all the imposed conditions.

$$
\begin{aligned}
&M_1 = \not\epsilon\, \not\epsilon' && M_8 = (P\cdot\epsilon\; K\cdot\epsilon' + P\cdot\epsilon'\; K\cdot\epsilon)\not K \\
&M_2 = \not\epsilon\, \not K\, \not\epsilon' && M_9 = K\cdot\epsilon\, \not\epsilon' + K\cdot\epsilon'\, \not\epsilon \\
&M_3 = P\cdot\epsilon\, \not\epsilon' + P\cdot\epsilon'\, \not\epsilon && M_{10} = K\cdot\epsilon\, \not\epsilon'\not K - K\cdot\epsilon'\, \not\epsilon\, \not K \\
&M_4 = \tfrac{1}{2} P\cdot\epsilon(\not\epsilon'\not K - \not K\not\epsilon') && M_{11} = K\cdot\epsilon\; K\cdot\epsilon' \\
&\qquad - \tfrac{1}{2} P\cdot\epsilon'(\not\epsilon\not K - \not K\not\epsilon) && M_{12} = K\cdot\epsilon\; K\cdot\epsilon'\, \not K \qquad (126)\\
&M_5 = P\cdot\epsilon\; P\cdot\epsilon' && M_{13} = \epsilon\cdot\epsilon' \\
&M_6 = P\cdot\epsilon\; P\cdot\epsilon'\, \not K && M_{14} = \epsilon\cdot\epsilon'\, \not K \\
&M_7 = P\cdot\epsilon\; K\cdot\epsilon' + P\cdot\epsilon'\; K\cdot\epsilon &&
\end{aligned}
$$

These invariants are apparently all independent. However it is possible to find two linear relations that they satisfy and this is related to the fact that no more than four vectors can be linearly independent in a four-dimensional vector space.

In appendix E one way of getting these relations is shown. They are the following equalities between spinors:

$$MK\cdot P\, M_1 + P^2 M_2 + MM_4 - M_8 + (K\cdot P - P^2)M_9 - MK\cdot P\, M_{13} + P^2 M_{14} = 0 \qquad (127a)$$

$$[K^2\Delta^2 + (K\cdot P)^2]\, M_1 + MK\cdot P\, M_2 + K\cdot P\, M_4 + K^2 M_7 + M(K^2 - K\cdot P)M_9$$
$$- \Delta^2 M_{10} - 2MM_{12} - [\Delta^2 K^2 + (K\cdot P)^2]\, M_{13} + MK\cdot P\, M_{14} = 0 \qquad (127b)$$

These allow us to eliminate M_8 and M_{12} (or M_4 and M_{12}) without introducing kinematical singularities. This is discussed further in the appendix. (See also Ref. 12.)

We can write, as in equations 30 and 34

$$\frac{d\sigma}{d\Omega} = |\langle f|\mathcal{F}|i\rangle|^2 \tag{128}$$

$$\mathcal{F} = \sum_{i=1}^{14} \mathcal{F}_i' N_i' \tag{129}$$

Time reversal invariance again imposes limitations on the form of the N_i; the rule is analogous to that for the M_i: if initial and final momenta and polarization vectors are interchanged, and the order of the matrices reversed, the N_i should remain invariant. Again there are two linear relations among the 14 invariants, derived also in appendix E.

$$\begin{aligned}
N_1' &= \vec{\epsilon}\cdot\vec{\epsilon}\,' \\
N_2' &= \vec{\sigma}\cdot\vec{\epsilon}\ \vec{\sigma}\cdot\vec{\epsilon}\,' \\
N_3' &= \vec{\epsilon}\cdot\vec{\epsilon}\,'\ \vec{\sigma}\cdot\hat{k}'\ \vec{\sigma}\cdot\hat{k} \\
N_4' &= \hat{k}\cdot\vec{\epsilon}\ \hat{k}'\cdot\vec{\epsilon}\,' \\
N_5' &= \hat{k}\cdot\vec{\epsilon}\ \hat{k}\cdot\vec{\epsilon}\,' + \hat{k}'\cdot\vec{\epsilon}\ \hat{k}'\cdot\vec{\epsilon}\,' \\
N_6' &= \hat{k}'\cdot\vec{\epsilon}\ \hat{k}\cdot\vec{\epsilon}\,' \\
N_7' &= \hat{k}\cdot\vec{\epsilon}\ \vec{\sigma}\cdot\vec{\epsilon}\,'\ \vec{\sigma}\cdot\hat{k}' + \hat{k}'\cdot\vec{\epsilon}\,'\ \vec{\sigma}\cdot\hat{k}\ \vec{\sigma}\cdot\vec{\epsilon} \\
N_8' &= \hat{k}\cdot\vec{\epsilon}\ \vec{\sigma}\cdot\vec{\epsilon}\,'\ \vec{\sigma}\cdot\hat{k} + \hat{k}'\cdot\vec{\epsilon}\,'\ \vec{\sigma}\cdot\hat{k}'\ \vec{\sigma}\cdot\vec{\epsilon} \\
N_9' &= \hat{k}'\cdot\vec{\epsilon}\ \vec{\sigma}\cdot\vec{\epsilon}\,'\ \vec{\sigma}\cdot\hat{k} + \hat{k}\cdot\vec{\epsilon}\,'\ \vec{\sigma}\cdot\hat{k}'\ \vec{\sigma}\cdot\vec{\epsilon} \\
N_{10}' &= \hat{k}'\cdot\vec{\epsilon}\ \vec{\sigma}\cdot\vec{\epsilon}\,'\ \vec{\sigma}\cdot\hat{k}' + \hat{k}\cdot\vec{\epsilon}\,'\ \vec{\sigma}\cdot\hat{k}\ \vec{\sigma}\cdot\vec{\epsilon} \\
N_{11}' &= \vec{\sigma}\cdot\hat{k}'\ \vec{\sigma}\cdot\vec{\epsilon}\ \vec{\sigma}\cdot\vec{\epsilon}\,'\ \vec{\sigma}\cdot\hat{k} \\
N_{12}' &= \hat{k}\cdot\vec{\epsilon}\ \hat{k}'\cdot\vec{\epsilon}\,'\ \vec{\sigma}\cdot\hat{k}'\ \vec{\sigma}\cdot\hat{k} \\
N_{13}' &= \hat{k}'\cdot\vec{\epsilon}\ \hat{k}\cdot\vec{\epsilon}\,'\ \vec{\sigma}\cdot\hat{k}'\ \vec{\sigma}\cdot\hat{k} \\
N_{14}' &= (\hat{k}\cdot\vec{\epsilon}\ \hat{k}\cdot\vec{\epsilon}\,' + \hat{k}'\cdot\vec{\epsilon}\ \hat{k}'\cdot\vec{\epsilon}\,')\ \vec{\sigma}\cdot\hat{k}'\ \vec{\sigma}\cdot\hat{k}
\end{aligned} \tag{130}$$

$$xN_1' - xN_2' - N_3' + 3N_4' + N_6' - N_7' - N_9' + N_{11}' = 0 \quad (131a)$$

$$(1-x^2)N_1' - (1-x^2)N_2' - 3xN_4' + 2N_5' - xN_6' + xN_7' - N_8' + xN_9'$$

$$- N_{10}' + N_{12}' - N_{13}' = 0 \quad (131b)$$

where

$$x = \cos\theta = \hat{k}\cdot\hat{k}'$$

In a manner similar to equations 18 and 98, we derive:

$$\bar{u}(\bar{p}')M_i\, u(\bar{p}) = \langle f|\, a_{ij}N_j'\, |i\rangle \qquad i, j = 1, 2, \ldots, 14 \quad (132)$$

The coefficients a_{ij} are given in Table 1.

From these equations we get

$$\mathcal{F}_j' = \frac{M}{4\pi W}\, \mathcal{A}_i\, a_{ij} \qquad i, j = 1, 2, \ldots, 14 \quad (133)$$

corresponding to equations 31 and 99. We have used the summation convention (not for the i in $|i\rangle$, of course); the table for the coefficients in equation 132 is the transpose of Table 1. There is not much evidence of the reflection symmetry, nor is any expected.

It is now convenient to eliminate the two superfluous invariants and amplitudes. We choose to eliminate N_{11} and N_{13} (for no real good reason; according to the problem it might be better to take any other two that can be determined from equations 131, or a linear combination. Equations D-18 ff. might be useful if a change is made.)

It should be remembered that the sets of $\mathcal{A}_i$ and $\mathcal{F}_i'$ are not unique for a given transition amplitude, due to the superfluous invariants, but if they are only an intermediate step, it might be convenient to use all 14.

TABLE 1

	N_1	N_2	N_3	N_4
M_1	-	$-\frac{E+M}{2M}$	-	$\frac{\vec{k}^2(E+M-4k_o)}{2Mk_o^2}$
M_2	-	$\frac{(E+M)(W-M)}{2M}$	-	$\frac{\vec{k}^2[k_o(3k_o+E-3M)+m^2]}{2Mk_o^2}$
M_3	-	-	-	$\frac{\vec{k}^2(W+E)(E+M+2k_o)}{2Mk_o^2}$
M_4	-	-	-	$\frac{\vec{k}^2(W+E)(W+k_o)}{2Mk_o}$
M_5	-	-	-	$\frac{\vec{k}^2(W+E)^2(E+M)}{8Mk_o^2}$
M_6	-	-	-	$\frac{\vec{k}^2(W+E)^2(W-M)(E+M)}{8Mk_o^2}$
M_7	-	-	-	$\frac{\vec{k}^2(W+E)(E+M)}{4Mk_o}$
M_8	-	-	-	$\frac{\vec{k}^2(W+E)(W-M)(E+M)}{4Mk_o}$
M_9	-	-	-	$\frac{\vec{k}^2(E+M+2k_o)}{2Mk_o}$
M_{10}	-	-	-	$\frac{\vec{k}^2(W+k_o)}{2M}$
M_{11}	-	-	-	$\frac{\vec{k}^2(E+M)}{8M}$
M_{12}	-	-	-	$\frac{\vec{k}^2(W-M)(E+M)}{8M}$
M_{13}	$-\frac{E+M}{2M}$	-	$\frac{E-M}{2M}$	$\frac{\vec{k}^2(E+M)}{2Mk_o^2}$
M_{14}	$-\frac{(W-M)(E+M)}{2M}$	-	$-\frac{(E-M)(W+M)}{2M}$	$\frac{\vec{k}^2(W-M)(E+M)}{2Mk_o^2}$

TABLE 1 (Continued)

	N_5	N_6	N_7	N_8
M_1	-	-	$\frac{\vec{k}^2}{2Mk_o}$	$\frac{\vec{k}^2}{2Mk_o}$
M_2	$-\frac{\vec{k}^2(E+M+2k_o)}{2Mk_o}$	-	$-\frac{\vec{k}^2(k_o-M)}{2Mk_o}$	$\frac{\vec{k}^2(k_o+M)}{2Mk_o}$
M_3	$\frac{\vec{k}^2(E+M+2k_o)}{4Mk_o}$	-	$-\frac{\vec{k}^2(W+E)}{4Mk_o}$	$\frac{\vec{k}^2(W+E)}{4Mk_o}$
M_4	$\frac{\vec{k}^2(W^2+EM+k_o^2)}{4Mk_o}$	$\frac{\vec{k}^2(E+M)}{4M}$	$-\frac{\vec{k}^2W(W+E)}{4Mk_o}$	$-\frac{\vec{k}^2W(W+E)}{4Mk_o}$
M_5	$\frac{\vec{k}^2(W+E)(E+M)}{8Mk_o}$	$\frac{\vec{k}^2(E+M)}{8M}$	-	-
M_6	$\frac{\vec{k}^2(W+E)(W-M)(E+M)}{8Mk_o}$	$\frac{\vec{k}^2(W-M)(E+M)}{8M}$	-	-
M_7	$-\frac{\vec{k}^2E(E+M)}{4Mk_o}$	$-\frac{\vec{k}^2(E+M)}{4M}$	-	-
M_8	$-\frac{\vec{k}^2E(W-M)(E+M)}{4Mk_o}$	$-\frac{\vec{k}^2(W-M)(E+M)}{4M}$	-	-
M_9	$-\frac{\vec{k}^2(E+M+2k_o)}{4Mk_o}$	-	$-\frac{\vec{k}^2}{4M}$	$\frac{\vec{k}^2}{4M}$
M_{10}	$-\frac{\vec{k}^2(E-M+2k_o)}{4M}$	$-\frac{\vec{k}^2(E+M)}{4M}$	$-\frac{\vec{k}^2W}{4M}$	$-\frac{\vec{k}^2W}{4M}$
M_{11}	$-\frac{\vec{k}^2(E+M)}{8M}$	$\frac{\vec{k}^2(E+M)}{8M}$	-	-
M_{12}	$-\frac{\vec{k}^2(W-M)(E+M)}{8M}$	$\frac{\vec{k}^2(W-M)(E+M)}{8M}$	-	-
M_{13}	-	-	-	-
M_{14}	-	-	-	-

TABLE 1 (Continued)

	N_9	N_{10}	N_{11}	N_{12}
M_1	-	-	$\frac{E-M}{2M}$	$-\frac{\vec{k}^2(E-M)}{2Mk_o^2}$
M_2	$-\frac{\vec{k}^2}{2M}$	$\frac{\vec{k}^2}{2M}$	$\frac{(E-M)(W+M)}{2M}$	$-\frac{\vec{k}^2(E-M)(E+M-k_o)}{2Mk_o^2}$
M_3	$\frac{\vec{k}^2}{4M}$	$-\frac{\vec{k}^2}{4M}$	-	$\frac{\vec{k}^2(E-M)(W+E)}{2Mk_o^2}$
M_4	$-\frac{\vec{k}^2 W}{4M}$	$-\frac{\vec{k}^2 W}{4M}$	-	$\frac{\vec{k}^2(E-M)(W+E)}{4Mk_o}$
M_5	-	-	-	$-\frac{\vec{k}^2(W+E)^2(E-M)}{8Mk_o^2}$
M_6	-	-	-	$\frac{\vec{k}^2(W+E)^2(W+M)(E-M)}{8Mk_o^2}$
M_7	-	-	-	$-\frac{\vec{k}^2(W+E)(E-M)}{4Mk_o}$
M_8	-	-	-	$\frac{\vec{k}^2(W+E)(W+M)(E-M)}{4Mk_o}$
M_9	$-\frac{\vec{k}^2}{4M}$	$\frac{\vec{k}^2}{4M}$	-	$\frac{\vec{k}^2(E-M)}{2Mk_o}$
M_{10}	$\frac{\vec{k}^2 W}{4M}$	$\frac{\vec{k}^2 W}{4M}$	-	$\frac{\vec{k}^2(E-M)}{4M}$
M_{11}	-	-	-	$-\frac{\vec{k}^2(E-M)}{8M}$
M_{12}	-	-	-	$\frac{\vec{k}^2(W+M)(E-M)}{8M}$
M_{13}	-	-	-	$-\frac{\vec{k}^2(E-M)}{2Mk_o^2}$
M_{14}	-	-	-	$\frac{\vec{k}^2(W+M)(E-M)}{2Mk_o^2}$

TABLE 1 (Continued)

	N_{13}	N_{14}
M_1	-	-
M_2	-	$-\frac{\vec{k}^2(E-M)}{2Mk_o}$
M_3	-	$\frac{\vec{k}^2(E-M)}{4Mk_o}$
M_4	$-\frac{\vec{k}^2(E-M)}{4M}$	$-\frac{\vec{k}^2E(E-M)}{4Mk_o}$
M_5	$-\frac{\vec{k}^2(E-M)}{8M}$	$-\frac{\vec{k}^2(W+E)(E-M)}{8Mk_o}$
M_6	$\frac{\vec{k}^2(W+M)(E-M)}{8M}$	$\frac{\vec{k}^2(W+E)(W+M)(E-M)}{8Mk_o}$
M_7	$\frac{\vec{k}^2(E-M)}{4M}$	$\frac{\vec{k}^2E(E-M)}{4Mk_o}$
M_8	$-\frac{\vec{k}^2(W+M)(E-M)}{4M}$	$-\frac{\vec{k}^2E(W+M)(E-M)}{4Mk_o}$
M_9	-	$-\frac{\vec{k}(E-M)}{4Mk_o}$
M_{10}	$\frac{\vec{k}^2(E-M)}{4M}$	$-\frac{\vec{k}^2(E-M)}{4M}$
M_{11}	$-\frac{\vec{k}^2(E-M)}{8M}$	$\frac{\vec{k}^2(E-M)}{8M}$
M_{12}	$\frac{\vec{k}^2(W+M)(E-M)}{8M}$	$-\frac{\vec{k}^2(W+M)(E-M)}{8M}$
M_{13}	-	-
M_{14}	-	-

We define

$$N_i = N_i' \quad i = 1, 2, \ldots, 10 \qquad N_{11} = N_{12}' \qquad N_{12} = N_{14}' \tag{134}$$

These equations, together with 131, give us the relations

$$N_i' = b_{ij} N_j \qquad i = 1, 2, \ldots, 14 \qquad j = 1, 2, \ldots, 12 \tag{135}$$

and from

$$\begin{aligned} \mathfrak{F}_j N_j &= \mathfrak{F}_i' N_i' \\ &= \mathfrak{F}_i' b_{ij} N_j \end{aligned}$$

we get

$$\mathfrak{F}_j = \mathfrak{F}_i' b_{ij} \tag{136}$$

A lengthy calculation, similar to the one that gives equation 106, permits us to find the metric G for this case. In the non-relativistic limit, i.e. when $\kappa \ll 1$, we have

$$G=2\begin{pmatrix}
3 & 3 & 3x & x & 2 & x & 2x & 2 & 2x & 2 & x^2 & 2x \\
3 & 9 & 3x & x & 2 & x & -2x & -2 & -2x & -2 & -1+2x^2 & 2x \\
3x & 3x & 3 & x^2 & 2x & x^2 & -2+4x^2 & 2x & 2 & 2x & x & 2 \\
x & x & x^2 & 1 & 2x & x^2 & 2 & 2x & 2x^2 & 2x & x & 2x^2 \\
2 & 2 & 2x & 2x & 2+2x^2 & 2x & 4x & 2+2x^2 & 4x & 2+2x^2 & 2x^2 & 2x+2x^3 \\
x & x & x^2 & x^2 & 2x & 1 & 2x^2 & 2x & 2 & 2x & x^3 & 2x^2 \\
2x & -2x & -2+4x^2 & 2 & 4x & 2x^2 & 8 & 8x & -2+10x^2 & 8x & 2x & -2+6x^2 \\
2 & -2 & 2x & 2x & 2+2x^2 & 2x & 8x & 8 & 8x & 2+6x^2 & 2 & 4x \\
2x & -2x & 2 & 2x^2 & 4x & 2 & -2+10x^2 & 8x & 8 & 8x & 2x & 2+2x^2 \\
2 & -2 & 2x & 2x & 2+2x^2 & 2x & 8x & 2+6x^2 & 8x & 8 & 2x^2 & 4x^3 \\
x^2 & -1+2x^2 & x & x & 2x^2 & x^3 & 2x & 2 & 2x & 2x^2 & 1 & 2x \\
2x & 2x & 2 & 2x^2 & 2x+2x^3 & 2x^2 & -2+6x^2 & 4x & 2+2x^2 & 4x^3 & 2x & 2+2x^2
\end{pmatrix} \qquad (137)$$

We calculate next

$$f_{\lambda_c\lambda_d;\lambda_a\lambda_b} = \sum_{i=1}^{12} < \lambda_c\lambda_d | N_i | \lambda_a\lambda_b > \mathcal{F}_i \qquad (138)$$

using the state vectors from B-7, B-13 (complex conjugated for the final state). We get:

$$f_{\frac{1}{2}1;\frac{1}{2}1} = \cos\frac{\theta}{2}\,[\,\frac{1+x}{2}\mathcal{F}_1 - (1-x)\mathcal{F}_2 + \frac{1+x}{2}\mathcal{F}_3 - \frac{1-x^2}{2}\mathcal{F}_6 - 2(1-x)\mathcal{F}_9 + 2(1-x)\mathcal{F}_{10}]$$

$$f_{\frac{1}{2}0;\frac{1}{2}1} = \sin\frac{\theta}{2}\,e^{-i\varphi}[\,\frac{1+x}{\sqrt{2}}\mathcal{F}_1 + \sqrt{2}\,x\mathcal{F}_2 + \frac{1+x}{\sqrt{2}}\mathcal{F}_3 + \frac{1+x}{\sqrt{2}}\mathcal{F}_5 + \frac{x(1+x)}{\sqrt{2}}\mathcal{F}_6 - \sqrt{2}\,\mathcal{F}_7$$
$$+ \sqrt{2}\,\mathcal{F}_8 + \frac{1+3x}{\sqrt{2}}\mathcal{F}_9 + \frac{1-x}{\sqrt{2}}\mathcal{F}_{10} + \frac{1+x}{\sqrt{2}}\mathcal{F}_{12}]\,\frac{k_o}{m}$$

$$f_{\frac{1}{2}-1;\frac{1}{2}1} = 2\cos\frac{\theta}{2}\sin^2\frac{\theta}{2}\,e^{-2i\varphi}[\,\frac{1}{2}\mathcal{F}_1 + \mathcal{F}_2 + \frac{1}{2}\mathcal{F}_3 + \frac{1+x}{2}\mathcal{F}_6 + \mathcal{F}_9 - \mathcal{F}_{10}]$$

$$(139)$$

$$f_{-\frac{1}{2}1;\frac{1}{2}1} = 2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}\,e^{i\varphi}[\,-\frac{1}{2}\mathcal{F}_1 - \mathcal{F}_2 + \frac{1}{2}\mathcal{F}_3 + \frac{1-x}{2}\mathcal{F}_6 + \mathcal{F}_9 + \mathcal{F}_{10}\,]$$

$$f_{-\frac{1}{2}0;\frac{1}{2}1} = \cos\frac{\theta}{2}\,[\,-\frac{1-x}{\sqrt{2}}\mathcal{F}_1 + \sqrt{2}\,x\mathcal{F}_2 + \frac{1-x}{\sqrt{2}}\mathcal{F}_3 - \frac{1-x}{\sqrt{2}}\mathcal{F}_5 - \frac{x(1-x)}{\sqrt{2}}\mathcal{F}_6 - \sqrt{2}\,\mathcal{F}_7$$
$$- \sqrt{2}\,\mathcal{F}_8 + \frac{1-3x}{\sqrt{2}}\mathcal{F}_9 - \frac{1+x}{\sqrt{2}}\mathcal{F}_{10} + \frac{1-x}{\sqrt{2}}\mathcal{F}_{12}\,]\,\frac{k_o}{m}$$

$$f_{-\frac{1}{2}-1;\frac{1}{2}1} = \sin\frac{\theta}{2}\,e^{-i\varphi}[\,-\frac{1-x}{2}\mathcal{F}_1 + (1+x)\mathcal{F}_2 + \frac{1-x}{2}\mathcal{F}_3 - \frac{1-x^2}{2}\mathcal{F}_6 - 2(1+x)\mathcal{F}_9 - 2(1+x)\mathcal{F}_{10}]$$

$$f_{\frac{1}{2}0;\frac{1}{2}0} = \cos\frac{\theta}{2}\,[\,x\mathcal{F}_1 + (2x-1)\mathcal{F}_2 + x\mathcal{F}_3 + \mathcal{F}_4 + 2x\mathcal{F}_5 + x^2\mathcal{F}_6 + 2\mathcal{F}_7 + 2\mathcal{F}_8 + 2x\mathcal{F}_9$$
$$+ 2x\mathcal{F}_{10} + \mathcal{F}_{11} + 2x\mathcal{F}_{12}]\,\frac{k_o^2}{m^2}$$

$$f_{\frac{1}{2}-1;\frac{1}{2}0} = -2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}\, e^{-i\varphi}\Big[-\frac{1}{\sqrt{2}}\mathcal{F}_1 - \sqrt{2}\,\mathcal{F}_2 - \frac{1}{\sqrt{2}}\mathcal{F}_3 - \frac{1}{\sqrt{2}}\mathcal{F}_5 - \frac{x}{\sqrt{2}}\mathcal{F}_6$$
$$-\frac{1}{\sqrt{2}}\mathcal{F}_9 - \frac{1}{\sqrt{2}}\mathcal{F}_{10} - \frac{1}{\sqrt{2}}\mathcal{F}_{12}\Big]\frac{k_o}{m}$$

$$f_{-\frac{1}{2}1;\frac{1}{2}0} = 2\cos\frac{\theta}{2}\sin^2\frac{\theta}{2}\, e^{2i\varphi}\Big[\frac{1}{\sqrt{2}}\mathcal{F}_1 + \sqrt{2}\,\mathcal{F}_2 - \frac{1}{\sqrt{2}}\mathcal{F}_3 + \frac{1}{\sqrt{2}}\mathcal{F}_5 + \frac{x}{\sqrt{2}}\mathcal{F}_6 - \frac{1}{\sqrt{2}}\mathcal{F}_9$$
$$+\frac{1}{\sqrt{2}}\mathcal{F}_{10} - \frac{1}{\sqrt{2}}\mathcal{F}_{12}\Big]\frac{k_o}{m}$$

(139)

$$f_{-\frac{1}{2}0;\frac{1}{2}0} = -\sin\frac{\theta}{2}\, e^{i\varphi}\Big[x\mathcal{F}_1 + (2x+1)\mathcal{F}_2 - x\mathcal{F}_3 + \mathcal{F}_4 + 2x\mathcal{F}_5 + x^2\mathcal{F}_6 + 2\mathcal{F}_7 - 2\mathcal{F}_8 - 2x\mathcal{F}_9$$
$$+ 2x\mathcal{F}_{10} - \mathcal{F}_{11} - 2x\mathcal{F}_{12}\Big]\frac{k_o^2}{m^2}$$

$$f_{\frac{1}{2}-1;\frac{1}{2}-1} = 2\cos^3\frac{\theta}{2}\Big[\frac{1}{2}\mathcal{F}_1 + \mathcal{F}_2 + \frac{1}{2}\mathcal{F}_3 - \frac{1-x}{2}\mathcal{F}_6\Big]$$

$$f_{-\frac{1}{2}1;\frac{1}{2}-1} = -2\sin^3\frac{\theta}{2}\, e^{3i\varphi}\Big[\frac{1}{2}\mathcal{F}_1 + \mathcal{F}_2 - \frac{1}{2}\mathcal{F}_3 + \frac{1+x}{2}\mathcal{F}_6\Big]$$

We follow the same steps leading to equations 111 and 112 to get the partial wave expansion of the $\mathcal{F}_i$ and the projection operators (these again in a non-relativistic approximation). If we write

$$\mathcal{F}_k(W, x) = \sum_{\substack{J=\ell+\frac{1}{2} \\ i}} a_{ki}(x)\, \Pi_i^J(x)\, a_i^J(W) \qquad (140)$$

we find that the Π_i^J and a_{ki} are given by equations 140a and Table 2.

$$\Pi_1^J = \frac{1}{4}(P'_{\ell+1} - P'_\ell)$$

$$\Pi_2^J = \frac{1}{2\sqrt{2}}(P'_{\ell+1} + P'_\ell)$$

$$\Pi_3^J = \frac{1}{2\sqrt{\ell(\ell+2)}}(P''_{\ell+1} + P''_\ell)$$

$$\Pi_4^J = \frac{1}{2\sqrt{\ell(\ell+2)}}(-P''_{\ell+1} + P''_\ell)$$

$$\Pi_5^J = \frac{1}{2\sqrt{2}}(P'_{\ell+1} - P'_\ell)$$

$$\Pi_6^J = \frac{1}{4}(P'_{\ell+1} + P'_\ell)$$

(140a)

$$\Pi_7^J = \frac{1}{2}(P'_{\ell+1} - P'_\ell)$$

$$\Pi_8^J = \frac{1}{2\sqrt{2\ell(\ell+2)}}(-P''_{\ell+1} + P''_\ell)$$

$$\Pi_9^J = \frac{1}{2\sqrt{2\ell(\ell+2)}}(P''_{\ell+1} + P''_\ell)$$

$$\Pi_{10}^J = \frac{1}{2}(P'_{\ell+1} + P'_\ell)$$

$$\Pi_{11}^J = \frac{\ell(2\ell+1)P'''_{\ell+2} - 3\ell(2\ell+3)P'''_{\ell+1} + 3(\ell+2)(2\ell+1)P'''_\ell - (\ell+2)(2\ell+3)P'''_{\ell-1}}{4\ell(\ell+2)(2\ell+1)(2\ell+3)}$$

$$\Pi_{12}^J = \frac{\ell(2\ell+1)P'''_{\ell+2} + 3\ell(2\ell+3)P'''_{\ell+1} + 3(\ell+2)(2\ell+1)P'''_\ell + (\ell+2)(2\ell+3)P'''_{\ell-1}}{4\ell(\ell+2)(2\ell+1)(2\ell+3)}$$

TABLE 2

	a_1^J	a_2^J	a_3^J	a_4^J	a_5^J	a_6^J	a_7^J	a_8^J	a_9^J	a_{10}^J	a_{11}^J	a_{12}^J
$\bar{a}_1$	$2+x$	-	$(2+x)(1-x)$	$-(2-x)(1+x)$	-	$-(2-x)$	-	-	-	-	$-2(2-x)(1+x)$	$-(2+x)(1-x)$
$\bar{a}_2$	$-(1+x)$	-	$-(1-x^2)$	$1-x^2$	-	$1-x$	-	-	-	-	$(3-x)(1+x)$	$(3+x)(1-x)$
$\bar{a}_3$	1	-	$1-x$	$1+x$	-	1	-	-	-	-	$1+x$	$-(1-x)$
$\bar{a}_4$	-	2	$-2x$	$2x$	2	-	1	$2(1+3x)$	$2(1-3x)$	1	$8x$	$8x$
$\bar{a}_5$	-	-	1	-1	-	-	-	-2	2	-	-6	-6
$\bar{a}_6$	1	-	$1-x$	$1+x$	-	1	-	-	-	-	$-(3-x)$	$3+x$
$\bar{a}_7$	-	-1	x	$-x$	-1	-	-	$-(1+x)$	$-(1-x)$	-	$-2x$	$-2x$
$\bar{a}_8$	-1	1	-1	1	-1	1	-	$1+x$	$-(1-x)$	-	$3+x$	$3-x$
$\bar{a}_9$	-1	-	x	$-x$	-	-1	-	-	-	-	$1-x$	$-(1+x)$
$\bar{a}_{10}$	-	-	-1	1	-	-	-	-	-	-	2	2
$\bar{a}_{11}$	1	-2	$1+x$	$-(1-x)$	2	-1	1	$-2(1-x)$	$2(1+x)$	-1	$-(3-x)$	$-(3+x)$
$\bar{a}_{12}$	-	-	-1	-1	-	-	-	-2	-2	-	-2	2

(140b)

From D-12 we have then

$$\varphi_i^J = \Pi_{(i)}^J \, a_{\cdot i} \tag{141}$$

where the parenthesis indicates that there is no sum over i. The $a_{\cdot i}$ are of course the columns in 140b.

Using D-16 we determine

$$\chi_i^J = K_{(i)}^J \, b_{\cdot i} \tag{142}$$

where the K_i^J and b_{mi} are given by equations 142a and Table 3.

$$K_1^J = \frac{1}{4}(P_{\ell+1} + P_\ell)$$

$$K_2^J = \frac{1}{2\sqrt{2}}(-P_{\ell+1} + P_\ell)$$

$$K_3^J = \frac{\sqrt{\ell(\ell+2)}}{4}\left(\frac{P_{\ell+2}}{2\ell+3} - \frac{P_{\ell+1}}{2\ell+1} - \frac{P_\ell}{2\ell+3} + \frac{P_{\ell-1}}{2\ell+1}\right)$$

$$K_4^J = \frac{\sqrt{\ell(\ell+2)}}{4}\left(\frac{P_{\ell+2}}{2\ell+3} + \frac{P_{\ell+1}}{2\ell+1} - \frac{P_\ell}{2\ell+3} - \frac{P_{\ell-1}}{2\ell+1}\right)$$

$$K_5^J = \frac{1}{2\sqrt{2}}(P_{\ell+1} + P_\ell)$$

$$K_6^J = \frac{1}{4}(-P_{\ell+1} + P_\ell)$$

$$K_7^J = \frac{1}{2}(P_{\ell+1} + P_\ell)$$

$$K_8^J = \frac{\sqrt{\ell(\ell+2)}}{2\sqrt{2}}\left(\frac{P_{\ell+2}}{2\ell+3} + \frac{P_{\ell+1}}{2\ell+1} - \frac{P_\ell}{2\ell+3} - \frac{P_{\ell-1}}{2\ell+1}\right)$$

$$K_9^J = \frac{\sqrt{\ell(\ell+2)}}{2\sqrt{2}}\left(\frac{P_{\ell+2}}{2\ell+3} - \frac{P_{\ell+1}}{2\ell+1} - \frac{P_\ell}{2\ell+3} + \frac{P_{\ell-1}}{2\ell+1}\right)$$

$$K_{10}^J = \frac{1}{2}(-P_{\ell+1} + P_\ell)$$

$$K_{11}^J = \frac{1}{4}\left[\frac{\ell P_{\ell+2}}{2\ell+3} + \frac{3\ell P_{\ell+1}}{2\ell+1} + \frac{3(\ell+2)P_\ell}{2\ell+3} + \frac{(\ell+2)P_{\ell-1}}{2\ell+1}\right]$$

$$K_{12}^J = \frac{1}{4}\left[-\frac{\ell P_{\ell+2}}{2\ell+3} + \frac{3\ell P_{\ell+1}}{2\ell+1} - \frac{3(\ell+2)P_\ell}{2\ell+3} + \frac{(\ell+2)P_{\ell-1}}{2\ell+1}\right]$$

TABLE 3

	X_1^J	X_2^J	X_3^J	X_4^J	X_5^J	X_6^J	X_7^J	X_8^J	X_9^J	X_{10}^J	X_{11}^J	X_{12}^J
1	1+x	1+x	1	-1	-(1-x)	-(1-x)	x	-1	1	x	1	1
2	-2(1-x)	2x	2	-2	2x	2(1+x)	-(1-2x)	-2	2	1+2x	2	2
3	1+x	1+x	1	1	1-x	1-x	x	-1	-1	-x	1	-1
4	-	-	-	-	-	-	1	-	-	1	-	-
5	-	1+x	-	-	-(1-x)	-	2x	-1	1	2x	-	-
6	$-(1-x^2)$	x(1+x)	1+x	1-x	-x(1-x)	$-(1-x^2)$	x^2	-x	x	x^2	-(1-x)	1+x
7	-	-2	-	-	-2	-	2	-	-	2	-	-
8	-	2	-	-	-2	-	2	-	-	-2	-	-
9	-4(1-x)	1+3x	2	2	1-3x	-4(1+x)	2x	-1	-1	-2x	-	-
10	4(1-x)	1-x	-2	2	-(1+x)	-4(1+x)	2x	-1	1	2x	-	-
11	-	-	-	-	-	-	1	-	-	-1	-	-
12	-	1+x	-	-	1-x	-	2x	-1	-1	-2x	-	-

(142b)

Now, according to D-23 and D-25 (see also 113 and 114), we define:

$$T_{1,7} = \frac{f_{\frac{1}{2}1;\frac{1}{2}1}}{\cos\frac{\theta}{2}} \pm \frac{f_{-\frac{1}{2}-1;\frac{1}{2}1}}{\sin\frac{\theta}{2}\, e^{-i\varphi}}$$

$$T_{2,8} = \frac{f_{\frac{1}{2}-1;\frac{1}{2}-1}}{2\cos^3\frac{\theta}{2}} \pm \frac{f_{-\frac{1}{2}1;\frac{1}{2}-1}}{-2\sin^3\frac{\theta}{2}\, e^{3i\varphi}}$$

$$T_{3,9} = \frac{f_{\frac{1}{2}0;\frac{1}{2}0}}{\cos\frac{\theta}{2}} \pm \frac{f_{-\frac{1}{2}0;\frac{1}{2}0}}{-\sin\frac{\theta}{2}\, e^{i\varphi}} \qquad (143)$$

$$T_{4,10} = \frac{f_{\frac{1}{2}-1;\frac{1}{2}1}}{2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\, e^{-2i\varphi}} \pm \frac{f_{-\frac{1}{2}1;\frac{1}{2}1}}{2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}\, e^{i\varphi}}$$

$$T_{5,11} = \frac{f_{\frac{1}{2}0;\frac{1}{2}1}}{\sin\frac{\theta}{2}\, e^{-i\varphi}} \pm \frac{f_{-\frac{1}{2}0;\frac{1}{2}1}}{\cos\frac{\theta}{2}}$$

$$T_{6,12} = \frac{f_{\frac{1}{2}-1;\frac{1}{2}0}}{-2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}\, e^{-i\varphi}} \pm \frac{f_{-\frac{1}{2}1;\frac{1}{2}0}}{2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\, e^{2i\varphi}}$$

$$\beta^J_{1,7} = a^J_1 \pm a^J_6 \qquad \beta^J_{4,10} = a^J_3 \pm a^J_4$$

$$\beta^J_{2,8} = a^J_{11} \pm a^J_{12} \qquad \beta^J_{5,11} = a^J_2 \pm a^J_5 \qquad (144)$$

$$\beta^J_{3,9} = a^J_7 \pm a^J_{10} \qquad \beta^J_{6,12} = a^J_8 \pm a^J_9$$

It should be remembered that the β^J_i correspond to transitions between states of a definite parity (see equations 61 through 64). In the present case we have

$$P\,|JM;\lambda_1\lambda_2> = (-1)^{\ell}\,|JM;-\lambda_1-\lambda_2> \qquad (145a)$$

$$|JM;\lambda_1\lambda_2 \pm> \equiv \frac{1}{\sqrt{2}}\left(|JM;\lambda_1\lambda_2> \pm |JM;-\lambda_1-\lambda_2>\right) \qquad (145b)$$

Using A-5 and 125 we find the partial wave expansion for the T_i

$$
\begin{aligned}
T_{1,7} &= \sum_{J=\ell+\frac{1}{2}} \left[\beta^J_{1,7} P'_{\ell+1} - \beta^J_{7,1} P'_\ell \right] \\
T_{2,8} &= \sum_{J=\ell+\frac{1}{2}} \left[\beta^J_{2,8} \left(\frac{P'''_{\ell+2}}{(\ell+2)(2\ell+3)} + \frac{3P'''_\ell}{\ell(2\ell+3)} \right) \right. \\
&\qquad \left. - \beta^J_{8,2} \left(\frac{3P'''_{\ell+1}}{(\ell+2)(2\ell+1)} + \frac{P'''_{\ell-1}}{\ell(2\ell+1)} \right) \right] \\
T_{3,9} &= \sum_{J=\ell+\frac{1}{2}} \left[\beta^J_{3,9} P'_{\ell+1} - \beta^J_{9,3} P'_\ell \right] \\
T_{4,10} &= \sum_{J=\ell+\frac{1}{2}} \left[\beta^J_{4,10} \frac{P''_\ell}{\sqrt{\ell(\ell+2)}} + \beta^J_{10,4} \frac{P''_{\ell+1}}{\sqrt{\ell(\ell+2)}} \right] \\
T_{5,11} &= \sum_{J=\ell+\frac{1}{2}} \left[\beta^J_{5,11} P'_{\ell+1} + \beta^J_{11,5} P'_\ell \right] \\
T_{6,12} &= \sum_{J=\ell+\frac{1}{2}} \left[\beta^J_{6,12} \frac{P''_\ell}{\sqrt{\ell(\ell+2)}} - \beta^J_{12,6} \frac{P''_{\ell+1}}{\sqrt{\ell(\ell+2)}} \right]
\end{aligned}
\tag{146}
$$

Using equations 139 we get:

$$
\begin{aligned}
T_1 &= x\mathfrak{F}_1 + 2x\mathfrak{F}_2 + \mathfrak{F}_3 - (1-x^2)\mathfrak{F}_6 - 4\mathfrak{F}_9 - 4x\mathfrak{F}_{10} \\
T_7 &= \mathfrak{F}_1 - 2\mathfrak{F}_2 + x\mathfrak{F}_3 + 4x\mathfrak{F}_9 + 4\mathfrak{F}_{10} \\
T_2 &= \mathfrak{F}_1 + 2\mathfrak{F}_2 + x\mathfrak{F}_6 \\
T_8 &= \mathfrak{F}_3 - \mathfrak{F}_6 \\
T_3 &= (2x\mathfrak{F}_1 + 4x\mathfrak{F}_2 + 2\mathfrak{F}_4 + 4x\mathfrak{F}_5 + 2x^2\mathfrak{F}_6 + 4\mathfrak{F}_7 + 4x\mathfrak{F}_{10}) \frac{k_o^2}{m^2} \\
T_9 &= (-2\mathfrak{F}_2 + 2x\mathfrak{F}_3 + 4\mathfrak{F}_8 + 4x\mathfrak{F}_9 + 2\mathfrak{F}_{11} + 4x\mathfrak{F}_{12}) \frac{k_o^2}{m^2}
\end{aligned}
\tag{147}
$$

$$T_4 = \mathfrak{F}_3 + \mathfrak{F}_6 + 2\mathfrak{F}_9$$

$$T_{10} = \mathfrak{F}_1 + 2\mathfrak{F}_2 + x\mathfrak{F}_6 - 2\mathfrak{F}_{10}$$

$$T_5 = (\sqrt{2}\, x\mathfrak{F}_1 + 2\sqrt{2}\, x\mathfrak{F}_2 + \sqrt{2}\, \mathfrak{F}_3 + \sqrt{2}\, x\mathfrak{F}_5 + \sqrt{2}\, x^2\mathfrak{F}_6 - 2\sqrt{2}\, \mathfrak{F}_7 + \sqrt{2}\, \mathfrak{F}_9$$
$$- \sqrt{2}\, x\mathfrak{F}_{10} + \sqrt{2}\, \mathfrak{F}_{12}) \frac{k_o}{m} \qquad (147)$$

$$T_{11} = (\sqrt{2}\, \mathfrak{F}_1 + \sqrt{2}\, x\mathfrak{F}_3 + \sqrt{2}\, \mathfrak{F}_5 + \sqrt{2}\, x\mathfrak{F}_6 + 2\sqrt{2}\, \mathfrak{F}_8 + 3\sqrt{2}\, x\mathfrak{F}_9 + \sqrt{2}\, \mathfrak{F}_{10} + \sqrt{2}\, x\mathfrak{F}_{12}) \frac{k_o}{m}$$

$$T_6 = (-\sqrt{2}\, \mathfrak{F}_3 - \sqrt{2}\, \mathfrak{F}_9 - \sqrt{2}\, \mathfrak{F}_{12}) \frac{k_o}{m}$$

$$T_{12} = (-\sqrt{2}\, \mathfrak{F}_1 - 2\sqrt{2}\, \mathfrak{F}_2 - \sqrt{2}\, \mathfrak{F}_5 - \sqrt{2}\, x\mathfrak{F}_6 - \sqrt{2}\, \mathfrak{F}_{10}) \frac{k_o}{m}$$

From equations 133 and 136 we have

$$\mathfrak{F}_k = \frac{M}{4\pi W}\, \mathfrak{a}_i a_{ij} b_{jk} \qquad (148a)$$

and using equations 147, we calculate

$$T_\ell = \mathfrak{a}_i c_{i\ell} \qquad (148b)$$

It is found that we can write the $c_{i\ell}$ in the form

$$c_{i\ell} = c_{i\ell}^{(0)} + c_{i\ell}^{(1)}\, x + c_{i\ell}^{(2)}\, x^2 \qquad (148c)$$

The coefficients $c_{i\ell}^{(0)}$, $c_{i\ell}^{(1)}$, $c_{i\ell}^{(2)}$ are shown in Tables 4, 5 and 6 respectively. The reflection symmetry is now clearly exhibited, and it is expressed by

$$\beta_i^J(-W) = \eta_{(i)} \beta_{i+6}^J(W) \qquad i = 1, 2, \ldots, 6 \qquad (149a)$$

$$T_i(-W) = \eta_{(i)} T_{i+6}(W) \qquad i = 1, 2, \ldots, 6 \qquad (149b)$$

$$\eta_i = 1 \quad i = 5, 6 \qquad \eta_i = -1 \quad i = 1, 2, 3, 4 \qquad (149c)$$

The projection operators for the β_i^J are (see D-28):

$$\beta^J_{1,7} = \frac{1}{2}\int_{-1}^{1}\left[T_{1,7}P_\ell + T_{7,1}P_{\ell+1}\right]dx$$

$$\beta^J_{2,8} = \frac{1}{2}\int_{-1}^{1}\left[T_{2,8}\,\frac{3\ell P_{\ell+1} + (\ell+2)P_{\ell-1}}{2\ell+1} + T_{8,2}\frac{\ell P_{\ell+2} + 3(\ell+2)P_\ell}{2\ell+3}\right]dx$$

$$\beta^J_{3,9} = \frac{1}{2}\int_{-1}^{1}\left[T_{3,9}P_\ell + T_{9,3}P_{\ell+1}\right]dx$$

$$\beta^J_{4,10} = \frac{1}{2}\int_{-1}^{1}\left[T_{4,10}\frac{\sqrt{\ell(\ell+2)}(P_{\ell+2} - P_\ell)}{2\ell+3} - T_{10,4}\frac{\sqrt{\ell(\ell+2)}(P_{\ell+1} - P_{\ell-1})}{2\ell+1}\right]dx \tag{150}$$

$$\beta^J_{5,11} = \frac{1}{2}\int_{-1}^{1}\left[T_{5,11}P_\ell - T_{11,5}P_{\ell+1}\right]dx$$

$$\beta^J_{6,12} = \frac{1}{2}\int_{-1}^{1}\left[T_{6,12}\,\frac{\sqrt{\ell(\ell+2)}\,(P_{\ell+2} - P_\ell)}{2\ell+3} + T_{12,6}\frac{\sqrt{\ell(\ell+2)}\,(P_{\ell+1} - P_{\ell-1})}{2\ell+1}\right]dx$$

For a mass zero vector particle we are led to the case of Compton scattering. Gauge invariance demands then that only certain combinations of the invariants M_i appear; and the matrix elements of T^J for which λ_b or λ_d are zero are absent.

TABLE 4

	G_1	G_2
T_1	$-\frac{E-M}{4\pi W}$	$\frac{(E-M)(E+M-k_o)}{4\pi W}$
T_2	$-\frac{E+M}{4\pi W}$	$\frac{(E+M)(W-M)}{4\pi W}$
T_3	$\frac{(E-M)[(E+M)^2-2k_o(E+M)-k_o^2]}{4\pi W m^2}$	$\frac{m^2(E-M)(E+M-k_o)}{4\pi W m^2}$
T_4	$\frac{E-M}{4\pi W}$	$\frac{(E-M)k_o}{4\pi W}$
T_5	$-\frac{\vec{k}^2}{2\sqrt{2}\,\pi W m}$	$-\frac{\vec{k}^2(E+M-k_o)}{4\sqrt{2}\,\pi W m}$
T_6	$-\frac{k_o(E-M)}{2\sqrt{2}\,\pi W m}$	$\frac{(E-M)[m^2+k_o(W+M)]}{4\sqrt{2}\,\pi W m}$
T_7	$\frac{E+M}{4\pi W}$	$\frac{(E+M)(E-M-k_o)}{4\pi W}$
T_8	$\frac{E-M}{4\pi W}$	$\frac{(E-M)(W+M)}{4\pi W}$
T_9	$-\frac{(E+M)[(E-M)^2-2k_o(E-M)-k_o^2]}{2\pi W m^2}$	$\frac{m^2(E+M)(E-M-k_o)}{4\pi W m^2}$
T_{10}	$-\frac{E+M}{4\pi W}$	$\frac{(E+M)k_o}{4\pi W}$
T_{11}	$\frac{\vec{k}^2}{2\sqrt{2}\,\pi W m}$	$-\frac{\vec{k}^2(E-M-k_o)}{4\sqrt{2}\,\pi W m}$
T_{12}	$\frac{k_o(E+M)}{2\sqrt{2}\,\pi W m}$	$-\frac{(E+M)[m^2+k_o(W-M)]}{4\sqrt{2}\,\pi W m}$

TABLE 4 (Continued)

	G_3	G_4	G_5
T_1	$-\frac{\vec{k}^2}{4\pi W}$	$\frac{\vec{k}^2(3E-M+4k_o)}{16\pi W}$	$-\frac{\vec{k}^2(E+M)}{32\pi W}$
T_2	-	$\frac{\vec{k}^2(E-M)}{16\pi W}$	$\frac{\vec{k}^2(E-M)}{32\pi W}$
T_3	$\frac{\vec{k}^2(W+E)(W+M)}{4\pi W m^2}$	$\frac{k_o\vec{k}^2(W+E)(E-M+2k_o)}{8\pi W m^2}$	$\frac{\vec{k}^2(W+E)^2(E+M)}{16\pi W m^2}$
T_4	$\frac{\vec{k}^2}{8\pi W}$	$-\frac{\vec{k}^2(E-M+2k_o)}{16\pi W}$	$\frac{\vec{k}^2(E+M)}{32\pi W}$
T_5	$\frac{\vec{k}^2(5E-M+3k_o)}{8\sqrt{2}\,\pi W m}$	$\frac{\vec{k}^2[W^2+E(2E+M+3k_o)]}{8\sqrt{2}\,\pi W m}$	$-\frac{\vec{k}^2(W+E)(E-M)}{16\sqrt{2}\,\pi W m}$
T_6	$-\frac{\vec{k}^2(W-M)}{8\sqrt{2}\,\pi W m}$	$\frac{\vec{k}^2[W^2-E(k_o+M)]}{8\sqrt{2}\pi W m}$	$\frac{\vec{k}^2(W+E)(E-M)}{16\sqrt{2}\,\pi W m}$
T_7	$-\frac{\vec{k}^2}{4\pi W}$	$-\frac{\vec{k}^2(3E+M+4k_o)}{16\pi W}$	$\frac{\vec{k}^2(E-M)}{32\pi W}$
T_8	-	$-\frac{\vec{k}^2(E+M)}{16\pi W}$	$-\frac{\vec{k}^2(E+M)}{32\pi W}$
T_9	$\frac{\vec{k}^2(W+E)(W-M)}{4\pi W m^2}$	$-\frac{k_o\vec{k}^2(W+E)(E+M+2k_o)}{8\pi W m^2}$	$-\frac{\vec{k}^2(W+E)^2(E-M)}{16\pi W m^2}$
T_{10}	$\frac{\vec{k}^2}{8\pi W}$	$\frac{\vec{k}^2(E+M+2k_o)}{16\pi W}$	$-\frac{\vec{k}^2(E-M)}{32\pi W}$
T_{11}	$\frac{\vec{k}^2(5E+M+3k_o)}{8\sqrt{2}\,\pi W m}$	$-\frac{\vec{k}^2[W^2+E(2E-M+3k_o)]}{8\sqrt{2}\,\pi W m}$	$\frac{\vec{k}^2(W+E)(E+M)}{16\sqrt{2}\,\pi W m}$
T_{12}	$-\frac{\vec{k}^2(W+M)}{8\sqrt{2}\,\pi W m}$	$-\frac{\vec{k}^2[W^2-E(k_o-M)]}{8\sqrt{2}\,\pi W m}$	$-\frac{\vec{k}^2(W+E)(E+M)}{16\sqrt{2}\,\pi W m}$

TABLE 4 (Continued)

	a_6	a_7	a_8
T_1	$-\frac{\vec{k}^2(E+M)(W-M)}{32\pi W}$	$\frac{\vec{k}^2(E+M)}{16\pi W}$	$\frac{\vec{k}^2(E+M)(W-M)}{16\pi W}$
T_2	$-\frac{\vec{k}^2(E-M)(W+M)}{32\pi W}$	$-\frac{\vec{k}^2(E-M)}{16\pi W}$	$\frac{\vec{k}^2(E-M)(W+M)}{16\pi W}$
T_3	$\frac{\vec{k}^2(W+E)^2(E+M)(W-M)}{16\pi W m^2}$	$\frac{k_o\vec{k}^2(W+E)(E+M)}{8\pi W m^2}$	$\frac{k_o\vec{k}^2(W+E)(E+M)(W-M)}{8\pi W m^2}$
T_4	$\frac{\vec{k}^2(E+M)(W-M)}{32\pi W}$	$-\frac{\vec{k}^2(E+M)}{16\pi W}$	$-\frac{\vec{k}^2(E+M)(W-M)}{16\pi W}$
T_5	$\frac{\vec{k}^2(W+E)(E-M)(W+M)}{16\sqrt{2}\,\pi W m}$	$\frac{\vec{k}^2 E(E-M)}{8\sqrt{2}\,\pi W m}$	$-\frac{\vec{k}^2 E(E-M)(W+M)}{8\sqrt{2}\,\pi W m}$
T_6	$-\frac{\vec{k}^2(W+E)(E-M)(W+M)}{16\sqrt{2}\,\pi W m}$	$-\frac{\vec{k}^2 E(E-M)}{8\sqrt{2}\,\pi W m}$	$\frac{\vec{k}^2 E(E-M)(W+M)}{8\sqrt{2}\,\pi W m}$
T_7	$-\frac{\vec{k}^2(E-M)(W+M)}{32\pi W}$	$-\frac{\vec{k}^2(E-M)}{16\pi W}$	$\frac{\vec{k}^2(E-M)(W+M)}{16\pi W}$
T_8	$-\frac{\vec{k}^2(E+M)(W-M)}{32\pi W}$	$\frac{\vec{k}^2(E+M)}{16\pi W}$	$\frac{\vec{k}^2(E+M)(W-M)}{16\pi W}$
T_9	$\frac{\vec{k}^2(W+E)^2(E-M)(W+M)}{16\pi W m^2}$	$-\frac{k_o\vec{k}^2(W+E)(E-M)}{8\pi W m^2}$	$\frac{k_o\vec{k}^2(W+E)(E-M)(W+M)}{8\pi W m^2}$
T_{10}	$\frac{\vec{k}^2(E-M)(W+M)}{32\pi W}$	$\frac{\vec{k}^2(E-M)}{16\pi W}$	$-\frac{\vec{k}^2(E-M)(W+M)}{16\pi W}$
T_{11}	$\frac{\vec{k}^2(W+E)(E+M)(W-M)}{16\sqrt{2}\,\pi W m}$	$-\frac{\vec{k}^2 E(E+M)}{8\sqrt{2}\,\pi W m}$	$-\frac{\vec{k}^2 E(E+M)(W-M)}{8\sqrt{2}\,\pi W m}$
T_{12}	$-\frac{\vec{k}^2(W+E)(E+M)(W-M)}{16\sqrt{2}\,\pi W m}$	$\frac{\vec{k}^2 E(E+M)}{8\sqrt{2}\,\pi W m}$	$\frac{\vec{k}^2 E(E+M)(W-M)}{8\sqrt{2}\,\pi W m}$

TABLE 4 (Continued)

	a_9	a_{10}	a_{11}
T_1	$\frac{\vec{k}^2}{4\pi W}$	$-\frac{\vec{k}^2(3E-M+4k_o)}{16\pi W}$	$-\frac{\vec{k}^2(E+M)}{32\pi W}$
T_2	-	$-\frac{\vec{k}^2(E-M)}{16\pi W}$	$\frac{\vec{k}^2(E-M)}{32\pi W}$
T_3	$\frac{k_o\vec{k}^2(W+M)}{4\pi W m^2}$	$\frac{k_o^2\vec{k}^2(E-M+2k_o)}{8\pi W m^2}$	$\frac{k_o^2\vec{k}^2(E+M)}{16\pi W m^2}$
T_4	$-\frac{\vec{k}^2}{8\pi W}$	$\frac{\vec{k}^2(E-M+2k_o)}{16\pi W}$	$\frac{\vec{k}^2(E+M)}{32\pi W}$
T_5	$-\frac{\vec{k}^2(E-M-k_o)}{8\sqrt{2}\,\pi W m}$	$\frac{k_o\vec{k}^2(2E+M+3k_o)}{8\sqrt{2}\,\pi W m}$	$\frac{k_o\vec{k}^2(E-M)}{16\sqrt{2}\,\pi W m}$
T_6	$\frac{\vec{k}^2(W-M)}{8\sqrt{2}\pi W m}$	$-\frac{k_o\vec{k}^2(k_o+M)}{8\sqrt{2}\,\pi W m}$	$-\frac{k_o\vec{k}^2(E-M)}{16\sqrt{2}\,\pi W m}$
T_7	$\frac{\vec{k}^2}{4\pi W}$	$\frac{\vec{k}^2(3E+M+4k_o)}{16\pi W}$	$\frac{\vec{k}^2(E-M)}{32\pi W}$
T_8	-	$\frac{\vec{k}^2(E+M)}{16\pi W}$	$-\frac{\vec{k}^2(E+M)}{32\pi W}$
T_9	$\frac{k_o\vec{k}^2(W-M)}{4\pi W m^2}$	$-\frac{k_o^2\vec{k}^2(E+M+2k_o)}{8\pi W m^2}$	$-\frac{k_o^2\vec{k}^2(E-M)}{16\pi W m^2}$
T_{10}	$-\frac{\vec{k}^2}{8\pi W}$	$-\frac{\vec{k}^2(E+M+2k_o)}{16\pi W}$	$-\frac{\vec{k}^2(E-M)}{32\pi W}$
T_{11}	$-\frac{\vec{k}^2(E+M-k_o)}{8\sqrt{2}\,\pi W m}$	$-\frac{k_o\vec{k}^2(2E-M+3k_o)}{8\sqrt{2}\,\pi W m}$	$-\frac{k_o\vec{k}^2(E+M)}{16\sqrt{2}\,\pi W m}$
T_{12}	$\frac{\vec{k}^2(W+M)}{8\sqrt{2}\,\pi W m}$	$\frac{k_o\vec{k}^2(k_o-M)}{8\sqrt{2}\,\pi W m}$	$\frac{k_o\vec{k}^2(E+M)}{16\sqrt{2}\,\pi W m}$

TABLE 4 (Continued)

	a_{12}	a_{13}	a_{14}
T_1	$-\frac{\vec{k}^2(E+M)(W-M)}{32\pi W}$	$\frac{E-M}{8\pi W}$	$-\frac{(E-M)(W+M)}{8\pi W}$
T_2	$-\frac{\vec{k}^2(E-M)(W+M)}{32\pi W}$	$-\frac{E+M}{8\pi W}$	$-\frac{(E+M)(W-M)}{8\pi W}$
T_3	$\frac{k_o^2\vec{k}^2(E+M)(W-M)}{16\pi W m^2}$	$\frac{\vec{k}^2(E+M)}{4\pi W m^2}$	$\frac{\vec{k}^2(E+M)(W-M)}{4\pi W m^2}$
T_4	$\frac{\vec{k}^2(E+M)(W-M)}{32\pi W}$	$\frac{E-M}{8\pi W}$	$-\frac{(E-M)(W+M)}{8\pi W}$
T_5	$-\frac{k_o\vec{k}^2(E-M)(W+M)}{16\sqrt{2}\,\pi W m}$	$\frac{k_o(E-M)}{4\sqrt{2}\,\pi W m}$	$-\frac{k_o(E-M)(W+M)}{4\sqrt{2}\,\pi W m}$
T_6	$\frac{k_o\vec{k}^2(E-M)(W+M)}{16\sqrt{2}\,\pi W m}$	$-\frac{k_o(E-M)}{4\sqrt{2}\,\pi W m}$	$\frac{k_o(E-M)(W+M)}{4\sqrt{2}\,\pi W m}$
T_7	$-\frac{\vec{k}^2(E-M)(W+M)}{32\pi W}$	$-\frac{E+M}{8\pi W}$	$-\frac{(E+M)(W-M)}{8\pi W}$
T_8	$-\frac{\vec{k}^2(E+M)(W-M)}{32\pi W}$	$\frac{E-M}{8\pi W}$	$-\frac{(E-M)(W+M)}{8\pi W}$
T_9	$\frac{k_o^2\vec{k}^2(E-M)(W+M)}{16\pi W m^2}$	$-\frac{\vec{k}^2(E-M)}{4\pi W m^2}$	$\frac{\vec{k}^2(E-M)(W+M)}{4\pi W m^2}$
T_{10}	$\frac{\vec{k}^2(E-M)(W+M)}{32\pi W}$	$-\frac{E+M}{8\pi W}$	$-\frac{(E+M)(W-M)}{8\pi W}$
T_{11}	$-\frac{k_o\vec{k}^2(E+M)(W-M)}{16\sqrt{2}\,\pi W m}$	$-\frac{k_o(E+M)}{4\sqrt{2}\,\pi W m}$	$-\frac{k_o(E+M)(W-M)}{4\sqrt{2}\,\pi W m}$
T_{12}	$\frac{k_o\vec{k}^2(E+M)(W-M)}{16\sqrt{2}\,\pi W m}$	$\frac{k_o(E+M)}{4\sqrt{2}\,\pi W m}$	$\frac{k_o(E+M)(W-M)}{4\sqrt{2}\,\pi W m}$

TABLE 5

	a_1	a_2	a_3
T_1	$-\frac{E+M}{4\pi W}$	$-\frac{(E+M)(E-M-k_o)}{4\pi W}$	$\frac{\vec{k}^2}{4\pi W}$
T_2	-	-	-
T_3	$-\frac{k_o^2(E+M)}{2\pi W m^2}$	$\frac{k_o(E+M)}{2\pi W}$	$\frac{k_o\vec{k}^2(W+M)}{4\pi W m^2}$
T_4	-	-	-
T_5	$-\frac{k_o(E+M)}{2\sqrt{2}\,\pi W m}$	$-\frac{(E+M)[k_o(E-M-k_o)-m^2]}{4\sqrt{2}\,\pi W m}$	$\frac{\vec{k}^2(E+M+3k_o)}{8\sqrt{2}\,\pi W m}$
T_6	-	-	-
T_7	$\frac{E-M}{4\pi W}$	$-\frac{(E-M)(E+M-k_o)}{4\pi W}$	$\frac{\vec{k}^2}{4\pi W}$
T_8	-	-	-
T_9	$\frac{k_o^2(E-M)}{2\pi W m^2}$	$\frac{k_o(E-M)}{2\pi W}$	$\frac{k_o\vec{k}^2(W-M)}{4\pi W m^2}$
T_{10}	-	-	-
T_{11}	$\frac{k_o(E-M)}{2\sqrt{2}\,\pi W m}$	$-\frac{(E-M)[k_o(E+M-k_o)-m^2]}{4\sqrt{2}\,\pi W m}$	$\frac{\vec{k}^2(E-M+3k_o)}{8\sqrt{2}\,\pi W m}$
T_{12}	-	-	-

TABLE 5 (Continued)

	G_4	G_5	G_6
T_1	$\frac{\vec{k}^2}{4\pi}$	-	-
T_2	$\frac{\vec{k}^2(E+M)}{16\pi W}$	$\frac{\vec{k}^2(E+M)}{16\pi W}$	$\frac{\vec{k}^2(E+M)(W-M)}{32\pi W}$
T_3	$\frac{k_o\vec{k}^2[W^2-E(k_o-M)]}{4\pi Wm^2}$	$\frac{k_o\vec{k}^2(W+E)(E+M)}{8\pi Wm^2}$	$\frac{k_o\vec{k}^2(W+E)(E+M)(W-M)}{8\pi Wm^2}$
T_4	$-\frac{\vec{k}^2(E-M)}{16\pi W}$	$-\frac{\vec{k}^2(E-M)}{32\pi W}$	$\frac{\vec{k}^2(E-M)(W+M)}{32\pi W}$
T_5	$\frac{\vec{k}^2[W(W+M)+2k_o^2]}{8\sqrt{2}\,\pi Wm}$	$\frac{\vec{k}^2(E^2+MW)}{8\sqrt{2}\,\pi Wm}$	$\frac{\vec{k}^2[k_o(2EW+EM-M^2)-m^2E]}{8\sqrt{2}\,\pi Wm}$
T_6	$\frac{k_o(E-M)}{8\sqrt{2}\,\pi Wm}$	$\frac{k_o(E-M)}{16\sqrt{2}\,\pi Wm}$	$-\frac{k_o(E-M)(W+M)}{16\sqrt{2}\,\pi Wm}$
T_7	$-\frac{\vec{k}^2}{4\pi}$	-	-
T_8	$-\frac{\vec{k}^2(E-M)}{16\pi W}$	$-\frac{\vec{k}^2(E-M)}{32\pi W}$	$\frac{\vec{k}^2(E-M)(W+M)}{32\pi W}$
T_9	$-\frac{k_o\vec{k}^2[W^2-E(k_o+M)]}{4\pi Wm^2}$	$-\frac{k_o\vec{k}^2(W+E)(E-M)}{8\pi Wm^2}$	$\frac{k_o\vec{k}^2(W+E)(E-M)(W+M)}{8\pi Wm^2}$
T_{10}	$\frac{\vec{k}^2(E+M)}{16\pi W}$	$\frac{\vec{k}^2(E+M)}{32\pi W}$	$\frac{\vec{k}^2(E+M)(W-M)}{32\pi W}$
T_{11}	$-\frac{\vec{k}^2[W(W-M)+2k_o^2]}{8\sqrt{2}\,\pi Wm}$	$-\frac{\vec{k}^2(E^2-MW)}{8\sqrt{2}\,\pi Wm}$	$\frac{\vec{k}^2[k_o(2EW-EM-M^2)-m^2E]}{8\sqrt{2}\,\pi Wm}$
T_{12}	$-\frac{k_o(E+M)}{8\sqrt{2}\,\pi Wm}$	$-\frac{k_o(E+M)}{16\sqrt{2}\,\pi Wm}$	$-\frac{k_o(E+M)(W-M)}{16\sqrt{2}\,\pi Wm}$

TABLE 5 (Continued)

	a_7	a_8	a_9
T_1	-	-	$-\frac{\vec{k}^2}{4\pi W}$
T_2	$-\frac{\vec{k}^2(E+M)}{16\pi W}$	$-\frac{\vec{k}^2(E+M)(W-M)}{16\pi W}$	-
T_3	$-\frac{k_o\vec{k}^2E(E+M)}{4\pi Wm^2}$	$-\frac{k_o\vec{k}^2E(E+M)(W-M)}{4\pi Wm^2}$	$-\frac{k_o\vec{k}^2(W+M)}{4\pi Wm^2}$
T_4	$\frac{\vec{k}^2(E-M)}{16\pi W}$	$-\frac{\vec{k}^2(E-M)(W+M)}{16\pi W}$	-
T_5	$\frac{\vec{k}^2[E(k_o-E)-MW]}{8\sqrt{2}\,\pi Wm}$	$-\frac{\vec{k}^2[(E-M)W(W+M)+2MEk_o]}{8\sqrt{2}\,\pi Wm}$	$-\frac{\vec{k}^2(E+M+3k_o)}{8\sqrt{2}\,\pi Wm}$
T_6	$-\frac{k_o(E-M)}{8\sqrt{2}\,\pi Wm}$	$\frac{k_o(E-M)(W+M)}{8\sqrt{2}\,\pi Wm}$	-
T_7	-	-	$-\frac{\vec{k}^2}{4\pi W}$
T_8	$\frac{\vec{k}^2(E-M)}{16\pi W}$	$-\frac{\vec{k}^2(E-M)(W+M)}{16\pi W}$	-
T_9	$\frac{k_o\vec{k}^2E(E-M)}{4\pi Wm^2}$	$-\frac{k_o\vec{k}^2E(E-M)(W+M)}{4\pi Wm^2}$	$-\frac{k_o\vec{k}^2(W-M)}{4\pi Wm^2}$
T_{10}	$-\frac{\vec{k}^2(E+M)}{16\pi W}$	$-\frac{\vec{k}^2(E+M)(W-M)}{16\pi W}$	-
T_{11}	$-\frac{\vec{k}^2[E(k_o-E)+MW]}{8\sqrt{2}\,\pi Wm}$	$-\frac{\vec{k}^2[(E+M)W(W-M)-2MEk_o]}{8\sqrt{2}\,\pi Wm}$	$-\frac{\vec{k}^2(E-M+3k_o)}{8\sqrt{2}\,\pi Wm}$
T_{12}	$\frac{k_o(E+M)}{8\sqrt{2}\,\pi Wm}$	$\frac{k_o(E+M)(W-M)}{8\sqrt{2}\,\pi Wm}$	-

TABLE 5 (Continued)

	a_{10}	a_{11}	a_{12}
T_1	$-\frac{\vec{k}^2}{4\pi}$	-	-
T_2	$-\frac{\vec{k}^2(E+M)}{16\pi W}$	$\frac{\vec{k}^2(E+M)}{32\pi W}$	$\frac{\vec{k}^2(E+M)(W-M)}{32\pi W}$
T_3	$-\frac{k_o^2\vec{k}^2(k_o-M)}{4\pi W m^2}$	$-\frac{k_o^2\vec{k}^2(E+M)}{8\pi W m^2}$	$-\frac{k_o^2\vec{k}^2(E+M)(W-M)}{8\pi W m^2}$
T_4	$\frac{\vec{k}^2(E-M)}{16\pi W}$	$-\frac{\vec{k}^2(E-M)}{32\pi W}$	$\frac{\vec{k}^2(E-M)(W+M)}{32\pi W}$
T_5	$-\frac{k_o\vec{k}^2(E+3k_o)}{8\sqrt{2}\,\pi W m}$	$-\frac{k_o\vec{k}^2 E}{8\sqrt{2}\,\pi W m}$	$-\frac{\vec{k}^2 k_o^2 M}{8\sqrt{2}\,\pi W m}$
T_6	$-\frac{k_o(E-M)}{8\sqrt{2}\,\pi W m}$	$\frac{k_o(E-M)}{16\sqrt{2}\,\pi W m}$	$-\frac{k_o(E-M)(W+M)}{16\sqrt{2}\,\pi W m}$
T_7	$\frac{\vec{k}^2}{4\pi}$	-	-
T_8	$\frac{k^2(E-M)}{16\pi W}$	$-\frac{\vec{k}^2(E-M)}{32\pi W}$	$\frac{\vec{k}^2(E-M)(W+M)}{32\pi W}$
T_9	$\frac{k_o^2\vec{k}^2(k_o+M)}{4\pi W m^2}$	$\frac{k_o^2\vec{k}^2(E-M)}{8\pi W m^2}$	$-\frac{k_o^2\vec{k}^2(E-M)(W+M)}{8\pi W m^2}$
T_{10}	$-\frac{\vec{k}^2(E+M)}{16\pi W}$	$\frac{\vec{k}^2(E+M)}{32\pi W}$	$\frac{\vec{k}^2(E+M)(W-M)}{32\pi W}$
T_{11}	$\frac{k_o\vec{k}^2(E+3k_o)}{8\sqrt{2}\,\pi W m}$	$\frac{k_o\vec{k}^2 E}{8\sqrt{2}\,\pi W m}$	$\frac{\vec{k}^2 k_o^2 M}{8\sqrt{2}\,\pi W m}$
T_{12}	$\frac{k_o(E+M)}{8\sqrt{2}\,\pi W m}$	$-\frac{k_o(E+M)}{16\sqrt{2}\,\pi W m}$	$-\frac{k_o(E+M)(W-M)}{16\sqrt{2}\,\pi W m}$

TABLE 5 (Continued)

	a_{13}	a_{14}
T_1	$-\frac{E+M}{8\pi W}$	$-\frac{(E+M)(W-M)}{8\pi W}$
T_2	-	-
T_3	$-\frac{k_o^2(E+M)}{4\pi W m^2}$	$-\frac{k_o^2(E+M)(W-M)}{4\pi W m^2}$
T_4	-	-
T_5	$-\frac{k_o(E+M)}{4\sqrt{2}\,\pi W m}$	$-\frac{k_o(E+M)(W-M)}{4\sqrt{2}\,\pi W m}$
T_6	-	-
T_7	$\frac{E-M}{8\pi W}$	$-\frac{(E-M)(W+M)}{8\pi W}$
T_8	-	-
T_9	$\frac{k_o^2(E-M)}{4\pi W m^2}$	$-\frac{k_o^2(E-M)(W+M)}{4\pi W m^2}$
T_{10}	-	-
T_{11}	$\frac{k_o(E-M)}{4\sqrt{2}\,\pi W m}$	$-\frac{k_o(E-M)(W+M)}{4\sqrt{2}\,\pi W m}$
T_{12}	-	-

TABLE 6

	G_1	G_2	G_3	G_4	G_5
T_1	-	-	-	$\frac{\vec{k}^2(E+M)}{16\pi W}$	$\frac{\vec{k}^2(E+M)}{32\pi W}$
T_2	-	-	-	-	-
T_3	-	-	-	$\frac{k_o^2\vec{k}^2(E+M)}{8\pi W m^2}$	$\frac{k_o^2\vec{k}^2(E+M)}{16\pi W m^2}$
T_4	-	-	-	-	-
T_5	-	-	-	$-\frac{k_o\vec{k}^2(E+M)}{8\sqrt{2}\,\pi W m}$	$\frac{k_o\vec{k}^2(E+M)}{16\sqrt{2}\,\pi W m}$
T_6	-	-	-	-	-
T_7	-	-	-	$-\frac{\vec{k}^2(E-M)}{16\pi W}$	$-\frac{\vec{k}^2(E-M)}{32\pi W}$
T_8	-	-	-	-	-
T_9	-	-	-	$-\frac{k_o^2\vec{k}^2(E-M)}{8\pi W m^2}$	$-\frac{k_o^2\vec{k}^2(E-M)}{16\pi W m^2}$
T_{10}	-	-	-	-	-
T_{11}	-	-	-	$-\frac{k_o\vec{k}^2(E-M)}{8\sqrt{2}\,\pi W m}$	$-\frac{k_o\vec{k}^2(E-M)}{16\sqrt{2}\,\pi W m}$
T_{12}	-	-	-	-	-

TABLE 6 (Continued)

	a_6	a_7	a_8	a_9
T_1	$\frac{\vec{k}^2(E+M)(W-M)}{32\pi W}$	$-\frac{\vec{k}^2(E+M)}{16\pi W}$	$-\frac{\vec{k}^2(E+M)(W-M)}{16\pi W}$	-
T_2	-	-	-	-
T_3	$\frac{k_o^2\vec{k}^2(E+M)(W-M)}{16\pi W m^2}$	$-\frac{k_o^2\vec{k}^2(E+M)}{8\pi W m^2}$	$-\frac{k_o^2\vec{k}^2(E+M)(W-M)}{8\pi W m^2}$	-
T_4	-	-	-	-
T_5	$\frac{k_o\vec{k}^2(E+M)(W-M)}{16\sqrt{2}\,\pi W m}$	$-\frac{k_o\vec{k}^2(E+M)}{8\sqrt{2}\,\pi W m}$	$-\frac{k_o\vec{k}^2(E+M)(W-M)}{8\sqrt{2}\,\pi W m}$	-
T_6	-	-	-	-
T_7	$\frac{\vec{k}^2(E-M)(W+M)}{32\pi W}$	$\frac{\vec{k}^2(E-M)}{16\pi W}$	$-\frac{\vec{k}^2(E-M)(W+M)}{16\pi W}$	-
T_8	-	-	-	-
T_9	$\frac{k_o^2\vec{k}^2(E-M)(W+M)}{16\pi W m^2}$	$\frac{k_o^2\vec{k}^2(E-M)}{8\pi W m^2}$	$-\frac{k_o^2\vec{k}^2(E-M)(W+M)}{8\pi W m^2}$	-
T_{10}	-	-	-	-
T_{11}	$\frac{k_o\vec{k}^2(E-M)(W+M)}{16\sqrt{2}\,\pi W m}$	$\frac{k_o\vec{k}^2(E-M)}{8\sqrt{2}\,\pi W m}$	$-\frac{k_o\vec{k}^2(E-M)(W+M)}{8\sqrt{2}\,\pi W m}$	-
T_{12}	-	-		

TABLE 6 (Continued)

	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}
T_1	$-\frac{\vec{k}^2(E+M)}{16\pi W}$	$\frac{\vec{k}^2(E+M)}{32\pi W}$	$\frac{\vec{k}^2(E+M)(W-M)}{32\pi W}$	-	-
T_2	-	-	-		
T_3	$-\frac{k_o^2\vec{k}^2(E+M)}{8\pi W m^2}$	$\frac{k_o^2\vec{k}^2(E+M)}{16\pi W m^2}$	$\frac{k_o^2\vec{k}^2(E+M)(W-M)}{16\pi W m^2}$	-	-
T_4	-	-	-		
T_5	$-\frac{k_o\vec{k}^2(E+M)}{8\sqrt{2}\,\pi W m}$	$\frac{k_o\vec{k}^2(E+M)}{16\sqrt{2}\,\pi W m}$	$\frac{k_o\vec{k}^2(E+M)(W-M)}{16\sqrt{2}\,\pi W m}$	-	-
T_6	-	-	-	-	-
T_7	$\frac{\vec{k}^2(E-M)}{16\pi W}$	$-\frac{\vec{k}^2(E-M)}{32\pi W}$	$\frac{\vec{k}^2(E-M)(W+M)}{32\pi W}$	-	-
T_8	-	-	-	-	-
T_9	$\frac{k_o^2\vec{k}^2(E-M)}{8\pi W m^2}$	$-\frac{k_o^2\vec{k}^2(E-M)}{16\pi W m^2}$	$\frac{k_o^2\vec{k}^2(E-M)(W+M)}{16\pi W m^2}$	-	-
T_{10}	-	-	-	-	-
T_{11}	$\frac{k_o\vec{k}^2(E-M)}{8\sqrt{2}\,\pi W m}$	$-\frac{k_o\vec{k}^2(E-M)}{16\sqrt{2}\,\pi W m}$	$\frac{k_o\vec{k}^2(E-M)(W+M)}{16\sqrt{2}\,\pi W m}$	-	-
T_{12}	-	-	-	-	-

d) $\rho + N \rightarrow \omega + N$

The kinematics of this reaction is very similar to that of ρN scattering, the difference arising from the fact that time reversal no longer reduces the number of independent amplitudes, since the ρ and ω mesons are different particles. The masses are also slightly different, so that some of the equations 121 change.

$$p + k = p' + k' \tag{149a}$$

$$p^2 = p'^2 = M^2 \qquad k^2 = m^2 \qquad k'^2 = m'^2 \tag{149b}$$

$$P = \frac{1}{2}(p + p') \qquad K = \frac{1}{2}(k + k') \qquad \Delta = \frac{1}{2}(k - k') = \frac{1}{2}(p' - p) \tag{149c}$$

$$P \cdot \Delta = 0 \tag{149d}$$

$$K \cdot \Delta = \frac{1}{4}(m^2 - m'^2) \tag{149e}$$

$$P^2 + \Delta^2 = M^2 \tag{149f}$$

$$K^2 + \Delta^2 = \frac{1}{2}(m^2 + m'^2) \tag{149g}$$

$$s = (P + K)^2 \tag{149h}$$

$$t = 4\Delta^2 \tag{149i}$$

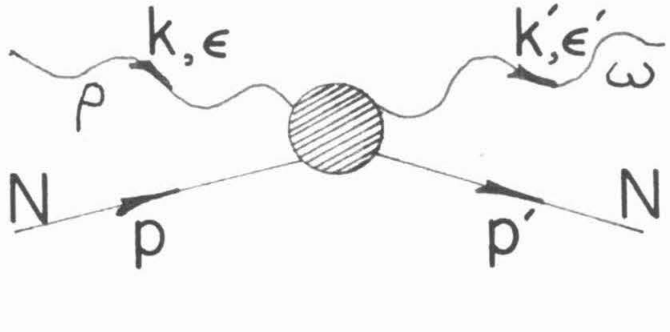

Fig. 5. General diagram for the reaction $\rho + N \rightarrow \omega + N$

Equations 122, 123 and 124 are still valid:

$$k\cdot\epsilon = k'\cdot\epsilon' = 0 \tag{150a}$$

$$\Delta\cdot\epsilon = -K\cdot\epsilon \tag{150b}$$

$$\Delta\cdot\epsilon' = K\cdot\epsilon' \tag{150c}$$

$$(\not{p}-M)u(\vec{p}) = 0 \tag{151a}$$

$$\overline{u}(\vec{p}\,')(\not{p}'-M) = 0 \tag{151b}$$

We now get 20 invariants:

$$\begin{aligned}
M_1 &= \epsilon\cdot\epsilon' & M_{11} &= P\cdot\epsilon\not{\epsilon}' \\
M_2 &= \epsilon\cdot\epsilon'\not{K} & M_{12} &= P\cdot\epsilon\not{\epsilon}'\not{K} \\
M_3 &= \not{\epsilon}\not{\epsilon}' & M_{13} &= K\cdot\epsilon\; K\cdot\epsilon' \\
M_4 &= \not{\epsilon}\not{K}\not{\epsilon}' & M_{14} &= K\cdot\epsilon\; K\cdot\epsilon'\not{K} \\
M_5 &= K\cdot\epsilon'\not{\epsilon} & M_{15} &= P\cdot\epsilon\; P\cdot\epsilon' \\
M_6 &= K\cdot\epsilon'\not{\epsilon}\not{K} & M_{16} &= P\cdot\epsilon\; P\cdot\epsilon'\not{K} \\
M_7 &= K\cdot\epsilon\not{\epsilon}' & M_{17} &= P\cdot\epsilon\; K\cdot\epsilon' \\
M_8 &= K\cdot\epsilon\not{\epsilon}'\not{K} & M_{18} &= P\cdot\epsilon\; K\cdot\epsilon'\not{K} \\
M_9 &= P\cdot\epsilon'\not{\epsilon} & M_{19} &= K\cdot\epsilon\; P\cdot\epsilon' \\
M_{10} &= P\cdot\epsilon'\not{\epsilon}\not{K} & M_{20} &= K\cdot\epsilon\; P\cdot\epsilon'\not{K}
\end{aligned} \tag{152}$$

Two equations similar to 127 can be found; the procedure is the same, as indicated in appendix E. The important difference is that $K\cdot\Delta$ is no longer zero, and this will probably introduce additional terms.

Again we can write

$$\frac{d\sigma}{d\Omega} = \frac{|\vec{k}\,'|}{|\vec{k}|}\, |\langle f|\mathfrak{F}|i\rangle|^2 \tag{153}$$

$$\mathfrak{F} = \sum_{i=1}^{20} \mathfrak{F}_i' N_i' \tag{154}$$

$$
\begin{aligned}
N'_1 &= \vec{\epsilon}\cdot\vec{\epsilon}\,' & N'_{11} &= \hat{k}\cdot\vec{\epsilon}\;\; \vec{\sigma}\cdot\vec{\epsilon}\,'\;\vec{\sigma}\cdot\hat{k} \\
N'_2 &= \hat{k}\cdot\vec{\epsilon}\;\hat{k}'\cdot\vec{\epsilon}\,' & N'_{12} &= \hat{k}\cdot\vec{\epsilon}\;\; \vec{\sigma}\cdot\hat{k}'\;\vec{\sigma}\cdot\vec{\epsilon}\,' \\
N'_3 &= \hat{k}\cdot\vec{\epsilon}\;\; \hat{k}\cdot\vec{\epsilon}\,' & N'_{13} &= \hat{k}'\cdot\vec{\epsilon}\;\; \vec{\sigma}\cdot\vec{\epsilon}\,'\;\vec{\sigma}\cdot\hat{k} \\
N'_4 &= \hat{k}'\cdot\vec{\epsilon}\;\hat{k}'\cdot\vec{\epsilon}\,' & N'_{14} &= \hat{k}'\cdot\vec{\epsilon}\;\; \vec{\sigma}\cdot\hat{k}'\;\vec{\sigma}\cdot\vec{\epsilon}\,' \\
N'_5 &= \hat{k}'\cdot\vec{\epsilon}\;\; \hat{k}\cdot\vec{\epsilon}\,' & N'_{15} &= \hat{k}'\cdot\vec{\epsilon}\,'\;\vec{\sigma}\cdot\hat{k}'\;\vec{\sigma}\cdot\vec{\epsilon} \\
N'_6 &= \vec{\epsilon}\cdot\vec{\epsilon}\,'\;\vec{\sigma}\cdot\hat{k}'\;\vec{\sigma}\cdot\hat{k} & N'_{16} &= \hat{k}'\cdot\vec{\epsilon}\,'\;\vec{\sigma}\cdot\vec{\epsilon}\;\vec{\sigma}\cdot\hat{k} \\
N'_7 &= \hat{k}\cdot\vec{\epsilon}\;\; \hat{k}'\cdot\vec{\epsilon}\,'\;\vec{\sigma}\cdot\hat{k}'\;\vec{\sigma}\cdot\hat{k} & N'_{17} &= \hat{k}\cdot\vec{\epsilon}\,'\;\vec{\sigma}\cdot\hat{k}'\;\vec{\sigma}\cdot\vec{\epsilon} \\
N'_8 &= \hat{k}\cdot\vec{\epsilon}\;\; \hat{k}\cdot\vec{\epsilon}\,'\;\vec{\sigma}\cdot\hat{k}'\;\vec{\sigma}\cdot\hat{k} & N'_{18} &= \hat{k}\cdot\vec{\epsilon}\,'\;\vec{\sigma}\cdot\vec{\epsilon}\;\vec{\sigma}\cdot\hat{k} \\
N'_9 &= \hat{k}'\cdot\vec{\epsilon}\;\hat{k}'\cdot\vec{\epsilon}\,'\;\vec{\sigma}\cdot\hat{k}'\;\vec{\sigma}\cdot\hat{k} & N'_{19} &= \vec{\sigma}\cdot\vec{\epsilon}\;\; \vec{\sigma}\cdot\vec{\epsilon}\,' \\
N'_{10} &= \hat{k}'\cdot\vec{\epsilon}\;\; \hat{k}\cdot\vec{\epsilon}\,'\;\vec{\sigma}\cdot\hat{k}'\;\vec{\sigma}\cdot\hat{k} & N'_{20} &= \vec{\sigma}\cdot\hat{k}'\;\vec{\sigma}\cdot\vec{\epsilon}\;\; \vec{\sigma}\cdot\vec{\epsilon}\,'\;\vec{\sigma}\cdot\hat{k}
\end{aligned}
\tag{155}
$$

The two linear relations that can be used to eliminate two invariants are:

$$
xN'_1 - N'_2 + N'_5 - N'_6 + N'_{12} - N'_{13} + N'_{16} - N'_{17} - xN'_{19} + N'_{20} = 0 \tag{156a}
$$

$$
\begin{aligned}
(1-x^2)N'_1 + xN'_2 - xN'_5 + N'_7 - N'_{10} - N'_{11} - xN'_{12} + xN'_{13} + N'_{14} \\
- N'_{15} - xN'_{16} + xN'_{17} + N'_{18} - (1-x^2)N'_{19} = 0
\end{aligned}
\tag{156b}
$$

and they are obtained the same way equations 131 were, as shown in appendix E.

The relation between the two sets of invariants is

$$
\bar{u}(\vec{p}\,')M_i u(\vec{p}) = \langle f|a_{ij}N'_j|i\rangle \qquad i, j = 1, 2, \ldots, 20 \tag{157}
$$

The coefficients a_{ij} are given in Table 7 where we have set

$$
R^{\pm} = \sqrt{(E\pm M)(E'\pm M)} \tag{157a}
$$

TABLE 7

	N_1	N_2	N_3	N_4	N_5
M_1	$-\frac{R^+}{2M}$	$\frac{(E+M)(E'+M)R^-}{2Mk_o k'_o}$	-	-	-
M_2	$-\frac{(W-M)R^+}{2M}$	$\frac{(W-M)(E+M)(E'+M)R^-}{2Mk_o k'_o}$	-	-	-
M_3	-	$\frac{(E+M)(E'+M)R^-}{2Mk_o k'_o}$	-	-	-
M_4	-	$-\frac{(E+M)(E'+M)(E-M-k'_o)R^-}{2Mk_o k'_o}$	$-\frac{\vec{k}^2 R^+}{2Mk_o}$	$-\frac{\vec{k}'^2 R^+}{4Mk'_o}$	-
M_5	-	$\frac{(E+M)(E'+M)R^-}{4Mk'_o}$	$-\frac{\vec{k}^2 R^+}{4Mk_o}$	-	-
M_6	-	$\frac{(E+M)(E'+M)R^-}{4M}$	$-\frac{\vec{k}^2 k'_o R^+}{4Mk_o}$	$-\frac{\vec{k}'^2 k_o R^+}{4Mk'_o}$	$\frac{(E+M)(E'+M)R^-}{4M}$
M_7	-	$\frac{(E+M)(E'+M)R^-}{4Mk_o}$	-	$-\frac{\vec{k}'^2 R^+}{4Mk'_o}$	-
M_8	-	-	-	-	-
M_9	-	$\frac{(W+E)(E+M)(E'+M)R^-}{4Mk_o k'_o}$	$\frac{\vec{k}^2 R^+}{4Mk_o}$	-	-
M_{10}	-	$\frac{(W+E)(E+M)(E'+M)R^-}{4Mk_o}$	$\frac{\vec{k}^2 k'_o R^+}{4Mk_o}$	$-\frac{\vec{k}'^2 (W+E)R^+}{4Mk'_o}$	$-\frac{(E+M)(E'+M)R^-}{4M}$

TABLE 7 (Continued)

	N_6	N_7	N_8	N_9	N_{10}
M_1	$\frac{R^-}{2M}$	$-\frac{(E-M)(E'-M)R^+}{2Mk_o k'_o}$	-	-	-
M_2	$-\frac{(W+M)R^-}{2M}$	$\frac{(W+M)(E-M)(E'-M)R^+}{2Mk_o k'_o}$	-	-	-
M_3	-	$-\frac{(E-M)(E'-M)R^+}{2Mk_o k'_o}$	-	-	-
M_4	-	$-\frac{(E-M)(E'-M)(E+M-k'_o)R^+}{2Mk_o k'_o}$	$-\frac{\vec{k}^2 R^-}{2Mk_o}$	$-\frac{\vec{k}'^2 R^-}{2Mk'_o}$	-
M_5	-	$\frac{(E-M)(E'-M)R^+}{4Mk'_o}$	$-\frac{\vec{k}^2 R^-}{4Mk_o}$	-	-
M_6	-	$-\frac{(E-M)(E'-M)R^+}{4M}$	$\frac{\vec{k}^2 k'_o R^-}{4Mk_o}$	$\frac{\vec{k}'^2 k_o R^-}{4Mk'_o}$	$-\frac{(E-M)(E'-M)R^+}{4M}$
M_7	-	$\frac{(E-M)(E'-M)R^+}{4Mk_o}$	-	$-\frac{\vec{k}'^2 R^-}{4Mk'_o}$	-
M_8	-	-	-	-	-
M_9	-	$\frac{(W+E)(E-M)(E'-M)R^+}{4Mk_o k'_o}$	$\frac{\vec{k}^2 R^-}{4Mk_o}$	-	-
M_{10}	-	$-\frac{(W+E)(E-M)(E'-M)R^+}{4Mk_o}$	$-\frac{\vec{k}^2 k'_o R^-}{4Mk_o}$	$\frac{\vec{k}'^2 (W+E)R^-}{4Mk'_o}$	$\frac{(E-M)(E'-M)R^+}{4M}$

TABLE 7 (Continued)

	N_{11}	N_{12}	N_{13}	N_{14}	N_{15}
M_1	-	-	-	-	-
M_2	-	-	-	-	-
M_3	$\frac{(E-M)R^+}{2Mk_o}$	$-\frac{(E+M)R^-}{2Mk_o}$	-	-	$\frac{(E'-M)R^+}{2Mk'_o}$
M_4	$\frac{(k'_o+M)(E-M)R^+}{2Mk_o}$	$\frac{(k'_o-M)(E+M)R^-}{2Mk_o}$	$-\frac{(E'+M)R^-}{2M}$	$-\frac{(E'-M)R^+}{2M}$	$\frac{(k_o+M)(E'-M)R^+}{2Mk'_o}$
M_5	-	-	-	-	$-\frac{k_o(E'-M)R^+}{4Mk'_o}$
M_6	-	-	-	-	$\frac{k_oW(E'-M)R^+}{4Mk'_o}$
M_7	$\frac{k'_o(E-M)R^+}{4Mk_o}$	$\frac{k'_o(E+M)R^-}{4Mk_o}$	$-\frac{(E'+M)R^-}{4M}$	$-\frac{(E'-M)R^+}{4M}$	-
M_8	$-\frac{k'_oW(E-M)R^+}{4Mk_o}$	$\frac{k'_oW(E+M)R^-}{4Mk_o}$	$\frac{W(E'+M)R^-}{4M}$	$-\frac{W(E'-M)R^+}{4M}$	-
M_9	-	-	-	-	$-\frac{(W+E)(E'-M)R^+}{4Mk'_o}$
M_{10}	-	-	-	-	$\frac{W(W+E)(E'-M)R^+}{4Mk'_o}$

TABLE 7 (Continued)

	N_{16}	N_{17}	N_{18}	N_{19}	N_{20}
M_1	-	-	-	-	-
M_2	-	-	-	-	-
M_3	$-\frac{(E'+M)R^-}{2Mk'_o}$	-	-	$-\frac{R^+}{2M}$	$\frac{R^-}{2M}$
M_4	$\frac{(k_o-M)(E'+M)R^-}{2Mk'_o}$	$-\frac{(E+M)R^-}{2M}$	$-\frac{(E-M)R^+}{2M}$	$\frac{(W-M)R^+}{2M}$	$\frac{(W+M)R^-}{2M}$
M_5	$-\frac{k_o(E'+M)R^-}{4Mk'_o}$	$\frac{(E+M)R^-}{4M}$	$\frac{(E-M)R^+}{4M}$	-	-
M_6	$-\frac{k_oW(E'+M)R^-}{4Mk'_o}$	$-\frac{W(E+M)R^-}{4M}$	$\frac{W(E-M)R^+}{4M}$	-	-
M_7	-	-	-	-	-
M_8	-	-	-	-	-
M_9	$-\frac{(W+E)(E'+M)R^-}{4Mk'_o}$	$-\frac{(E+M)R^-}{4M}$	$-\frac{(E-M)R^+}{4M}$	-	-
M_{10}	$-\frac{W(W+E)(E'+M)R^-}{4Mk'_o}$	$\frac{W(E+M)R^-}{4M}$	$-\frac{W(E-M)R^+}{4M}$	-	-

TABLE 7 (Continued)

	N_1	N_2	N_3	N_4
M_{11}	-	$\frac{(W+E')(E+M)(E'+M)R^-}{4Mk_o k'_o}$	-	$\frac{\vec{k}'^2 R^+}{4Mk'_o}$
M_{12}	-	-	-	-
M_{13}	-	$\frac{(E+M)(E'+M)R^-}{8M}$	$-\frac{\vec{k}^2 k'_o R^+}{8Mk_o}$	$-\frac{\vec{k}'^2 k_o R^+}{8Mk'_o}$
M_{14}	-	$\frac{(W-M)(E+M)(E'+M)R^-}{8M}$	$-\frac{\vec{k}^2 k'_o (W-M)R^+}{8Mk_o}$	$-\frac{\vec{k}'^2 k_o (W-M)R^+}{8Mk'_o}$
M_{15}	-	$\frac{(W+E)(W+E')(E+M)(E'+M)R^-}{8Mk_o k'_o}$	$\frac{\vec{k}^2 (W+E')R^+}{8Mk_o}$	$\frac{\vec{k}'^2 (W+E)R^+}{8Mk'_o}$
M_{16}	-	$\frac{(W+E)(W+E')(W-M)(E+M)(E'+M)R^-}{8Mk_o k'_o}$	$\frac{\vec{k}^2 (W+E')(W-M)R^+}{8Mk_o}$	$\frac{\vec{k}'^2 (W+E)(W-M)R^+}{8Mk'_o}$
M_{17}	-	$\frac{(W+E')(E+M)(E'+M)R^-}{8Mk'_o}$	$-\frac{\vec{k}^2 (W+E')R^+}{8Mk_o}$	$\frac{\vec{k}'^2 k_o R^+}{8Mk'_o}$
M_{18}	-	$\frac{(W+E')(W-M)(E+M)(E'+M)R^-}{8Mk'_o}$	$-\frac{\vec{k}^2 (W+E')(W-M)R^+}{8Mk_o}$	$\frac{\vec{k}'^2 k_o (W-M)R^+}{8Mk'_o}$
M_{19}	-	$\frac{(W+E)(E+M)(E'+M)R^-}{8Mk_o}$	$\frac{\vec{k}^2 k'_o R^+}{8Mk_o}$	$-\frac{\vec{k}'^2 (W+E)R^+}{8Mk'_o}$
M_{20}	-	$\frac{(W+E)(W-M)(E+M)(E'+M)R^-}{8Mk_o}$	$\frac{\vec{k}^2 k'_o (W-M)R^+}{8Mk_o}$	$-\frac{\vec{k}'^2 (W+E)(W-M)R^+}{8Mk'_o}$

TABLE 7 (Continued)

	N_5	N_6	N_7	N_8
M_{11}	-	-	$\frac{(W+E')(E-M)(E'-M)R^+}{4Mk_ok'_o}$	-
M_{12}	-	-	-	-
M_{13}	$\frac{(E+M)(E'+M)R^-}{8M}$	-	$-\frac{(E-M)(E'-M)R^+}{8M}$	$\frac{\vec{k}^2 k'_o R^-}{8Mk_o}$
M_{14}	$\frac{(W-M)(E+M)(E'+M)R^-}{8M}$	-	$\frac{(W+M)(E-M)(E'-M)R^+}{8M}$	$-\frac{\vec{k}^2 k'_o (W+M)R^-}{8Mk_o}$
M_{15}	$\frac{(E+M)(E'+M)R^-}{8M}$	-	$-\frac{(W+E)(W+E')(E-M)(E'-M)R^+}{8Mk_ok'_o}$	$-\frac{\vec{k}^2 (W+E')R^-}{8Mk_o}$
M_{16}	$\frac{(W-M)(E+M)(E'+M)R^-}{8M}$	-	$\frac{(W+E)(W+E')(W+M)(E-M)(E'-M)R^+}{8Mk_ok'_o}$	$\frac{\vec{k}^2 (W+E')(W+M)R^-}{8Mk_o}$
M_{17}	$-\frac{(E+M)(E'+M)R^-}{8M}$	-	$-\frac{(W+E')(E-M)(E'-M)R^+}{8Mk'_o}$	$\frac{\vec{k}^2 (W+E')R^-}{8Mk_o}$
M_{18}	$-\frac{(W-M)(E+M)(E'+M)R^-}{8M}$	-	$\frac{(W+E')(W+M)(E-M)(E'-M)R^+}{8Mk'_o}$	$-\frac{\vec{k}^2 (W+E')(W+M)R^-}{8Mk_o}$
M_{19}	$-\frac{(E+M)(E'+M)R^-}{8M}$	-	$-\frac{(W+E)(E-M)(E'-M)R^+}{8Mk_o}$	$-\frac{\vec{k}^2 k'_o R^-}{8Mk_o}$
M_{20}	$-\frac{(W-M)(E+M)(E'+M)R^-}{8M}$	-	$\frac{(W+E)(W+M)(E-M)(E'-M)R^+}{8Mk_o}$	$\frac{\vec{k}^2 k'_o (W+M)R^-}{8Mk_o}$

TABLE 7 (Continued)

	N_9	N_{10}	N_{11}	N_{12}
M_{11}	$\frac{\vec{k}'^2 R^-}{4Mk'_o}$	-	$\frac{(W+E')(E-M)R^+}{4Mk_o}$	$\frac{(W+E')(E+M)R^-}{4Mk_o}$
M_{12}	-	-	$-\frac{W(W+E')(E-M)R^+}{4Mk_o}$	$\frac{W(W+E')(E+M)R^-}{4Mk_o}$
M_{13}	$\frac{\vec{k}'^2 k_o R^-}{8Mk'_o}$	$-\frac{(E-M)(E'-M)R^+}{8M}$	-	-
M_{14}	$-\frac{\vec{k}'^2 k_o (W+M)R^-}{8Mk'_o}$	$\frac{(W+M)(E-M)(E'-M)R^+}{8M}$	-	-
M_{15}	$-\frac{\vec{k}'^2 (W+E)R^-}{8Mk'_o}$	$-\frac{(E-M)(E'-M)R^+}{8M}$	-	-
M_{16}	$\frac{\vec{k}'^2 (W+E)(W+M)R^-}{8Mk'_o}$	$\frac{(W+M)(E-M)(E'-M)R^+}{8M}$	-	-
M_{17}	$-\frac{\vec{k}'^2 k_o R^-}{8Mk'_o}$	$\frac{(E-M)(E'-M)R^+}{8M}$	-	-
M_{18}	$\frac{\vec{k}'^2 k_o (W+M)R^-}{8Mk'_o}$	$-\frac{(W+M)(E-M)(E'-M)R^+}{8M}$	-	-
M_{19}	$\frac{\vec{k}'^2 (W+E)R^-}{8Mk'_o}$	$\frac{(E-M)(E'-M)R^+}{8M}$	-	-
M_{20}	$-\frac{\vec{k}'^2 (W+E)(W+M)R^-}{8Mk'_o}$	$-\frac{(W+M)(E-M)(E'-M)R^+}{8M}$	-	-

TABLE 7 (Continued)

	N_{13}	N_{14}	N_{15}	N_{16}	N_{17}	N_{18}	N_{19}	N_{20}
M_{11}	$\frac{(E'+M)R^-}{4M}$	$\frac{(E'-M)R^+}{4M}$	-	-	-	-	-	-
M_{12}	$-\frac{W(E'+M)R^-}{4M}$	$\frac{W(E'-M)R^+}{4M}$	-	-	-	-	-	-
M_{13}	-	-	-	-	-	-	-	-
M_{14}	-	-	-	-	-	-	-	-
M_{15}	-	-	-	-	-	-	-	-
M_{16}	-	-	-	-	-	-	-	-
M_{17}	-	-	-	-	-	-	-	-
M_{18}	-	-	-	-	-	-	-	-
M_{19}	-	-	-	-	-	-	-	-
M_{20}	-	-	-	-	-	-	-	-

Since we have, as in equation 93,

$$\frac{d\sigma}{d\Omega} = \left(\frac{M}{4\pi W}\right)^2 \frac{|\vec{k}'|}{|\vec{k}|} |a|^2 \tag{158}$$

we derive, corresponding to equation 133:

$$\mathfrak{F}_j' = \frac{M}{4\pi W} \, a_i a_{ij} \qquad i, j = 1, 2, \ldots, 20 \tag{159}$$

These $\mathfrak{F}_j'$ show the reflection symmetry, as is easy to see from Table 7. This comes from the fact that the factor $\sigma \cdot k$ is always on the right, and $\sigma \cdot k'$ on the left.

We will eliminate the invariants N_{19} and N_{20}, and we set

$$N_i = N_i' \qquad i = 1, 2, \ldots, 18 \tag{160a}$$

which together with equations 156 give

$$N_i' = b_{ij} N_j \qquad i = 1, 2, \ldots, 20 \qquad j = 1, 2, \ldots, 18 \tag{160b}$$

and hence

$$\mathfrak{F}_j = \mathfrak{F}_i' b_{ij} \tag{160c}$$

We next calculate the matrix G that gives the unpolarized cross section, and we get Table 8 of $\frac{1}{2} G_{ij}$ (with $K \ll 1$).

TABLE 8

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	3	x	1	1	x	$3x$	x^2	x	x	x^2	1	x	x	1	1	x	x	1
2	x	1	x	x	x^2	x^2	x	x^2	x^2	x^3	x	1	x^2	x	x	1	x^2	x
3	1	x	1	x^2	x	x	x^2	x	x^3	x^2	1	x	x	x^2	x^2	x	x	1
4	1	x	x^2	1	x	x	x^2	x^3	x	x^2	x^2	x	x	1	1	x	x	x^2
5	x	x^2	x	x	1	x^2	x^3	x^2	x^2	x	x	x^2	1	x	x	x^2	1	x
6	$3x$	x^2	x	x	x^2	3	x	1	1	x	x	1	1	x	x	1	1	x
7	x^2	x	x^2	x^2	x^3	x	1	x	x	x^2	1	x	x	x^2	1	x	x	x^2
8	x	x^2	x	x^3	x^2	1	x	1	x^2	x	x	1	x^2	x	x	x^2	1	x
9	x	x^2	x^3	x	x^2	1	x	x^2	1	x	x	x^2	1	x	x	1	x^2	x
10	x^2	x^3	x^2	x^2	x	x	x^2	x	x	1	x^2	x	x	1	x^2	x	x	1
11	1	x	1	x^2	x	x	1	x	x	x^2	3	$-x$	$3x$	$-x^2$	1	x	x	1
12	x	1	x	x	x^2	1	x	1	x^2	x	$-x$	3	$-x^2$	$3x$	x	1	1	x
13	x	x^2	x	x	1	1	x	x^2	1	x	$3x$	$-x^2$	3	$-x$	x	1	1	x
14	1	x	x^2	1	x	x	x^2	x	x	1	$-x^2$	$3x$	$-x$	3	1	x	x	1
15	1	x	x^2	1	x	x	1	x	x	x^2	1	x	x	1	3	$-x$	$3x$	$-x^2$
16	x	1	x	x	x^2	1	x	x^2	1	x	x	1	1	x	$-x$	3	$-x^2$	$3x$
17	x	x^2	x	x	1	1	x	1	x^2	x	x	1	1	x	$3x$	$-x^2$	3	$-x$
18	1	x	1	x^2	x	x	x^2	x	x	1	1	x	x	1	$-x^2$	$3x$	$-x$	3

The matrix $\frac{1}{p} T^J$ now has the form

$$\frac{1}{|\vec{p}_r|} T^J = \begin{array}{c|cccccc} \text{initial} \backslash \text{final} & \frac12 1 & \frac12 0 & \frac12 -1 & -\frac12 1 & -\frac12 0 & -\frac12 -1 \\ \hline \frac12 1 & a_1^J & a_2^J & a_3^J & a_4^J & a_5^J & a_6^J \\ \frac12 0 & a_7^J & a_8^J & a_9^J & a_{10}^J & a_{11}^J & a_{12}^J \\ \frac12 -1 & a_{13}^J & a_{14}^J & a_{15}^J & a_{16}^J & a_{17}^J & a_{18}^J \\ -\frac12 1 & a_{18}^J & a_{17}^J & a_{16}^J & a_{15}^J & a_{14}^J & a_{13}^J \\ -\frac12 0 & a_{12}^J & a_{11}^J & a_{10}^J & a_9^J & a_8^J & a_7^J \\ -\frac12 -1 & a_6^J & a_5^J & a_4^J & a_3^J & a_2^J & a_1^J \end{array} \sqrt{\frac{|\vec{k}'|}{|\vec{k}|}} \qquad (162)$$

In the same way equations 139 were obtained, we get

$$f_{\frac12 1;\frac12 1} = \cos\frac{\theta}{2}\Big[\frac{1+x}{2}\bar{a}_1 - \frac{1-x^2}{2}\bar{a}_5 + \frac{1+x}{2}\bar{a}_6 - \frac{1-x^2}{2}\bar{a}_{10} - (1-x)\bar{a}_{13} - (1-x)\bar{a}_{14} - (1-x)\bar{a}_{17} - (1-x)\bar{a}_{18}\Big]$$

$$f_{\frac12 0;\frac12 1} = \sin\frac{\theta}{2} e^{-i\varphi}\Big[\frac{1+x}{\sqrt2}\bar{a}_1 + \frac{1+x}{\sqrt2}\bar{a}_4 + \frac{x(1+x)}{\sqrt2}\bar{a}_5 + \frac{1+x}{\sqrt2}\bar{a}_6 + \frac{1+x}{\sqrt2}\bar{a}_9 + \frac{x(1+x)}{\sqrt2}\bar{a}_{10} + \frac{1+x}{\sqrt2}\bar{a}_{13} + \frac{1+x}{\sqrt2}\bar{a}_{14} + \sqrt2\,\bar{a}_{15} + \sqrt2\,\bar{a}_{16} + \sqrt2\,x\bar{a}_{17} + \sqrt2\,x\bar{a}_{18}\Big]\frac{k_o'}{m},$$

$$f_{\frac12 -1;\frac12 1} = 2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2} e^{-2i\varphi}\Big[\frac12\bar{a}_1 + \frac{1+x}{2}\bar{a}_5 + \frac12\bar{a}_6 + \frac{1+x}{2}\bar{a}_{10} + \bar{a}_{17} + \bar{a}_{18}\Big] \qquad (163)$$

$$f_{-\frac12 1;\frac12 1} = 2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2} e^{i\varphi}\Big[-\frac12\bar{a}_1 + \frac{1-x}{2}\bar{a}_5 + \frac12\bar{a}_6 - \frac{1-x}{2}\bar{a}_{10} + \bar{a}_{17} - \bar{a}_{18}\Big]$$

$$f_{-\frac12 0;\frac12 1} = \cos\frac{\theta}{2}\Big[-\frac{1-x}{\sqrt2}\bar{a}_1 - \frac{1-x}{\sqrt2}\bar{a}_4 - \frac{x(1-x)}{\sqrt2}\bar{a}_5 + \frac{1-x}{\sqrt2}\bar{a}_6 + \frac{1-x}{\sqrt2}\bar{a}_9 + \frac{x(1-x)}{\sqrt2}\bar{a}_{10} + \frac{1-x}{\sqrt2}\bar{a}_{13} - \frac{1-x}{\sqrt2}\bar{a}_{14} - \sqrt2\,\bar{a}_{15} + \sqrt2\,\bar{a}_{16} - \sqrt2\,x\bar{a}_{17} + \sqrt2\,x\bar{a}_{18}\Big]\frac{k_o'}{m},$$

$$f_{-\frac{1}{2}-1;\frac{1}{2}1} = \sin\frac{\theta}{2}\, e^{-i\varphi}\Big[-\frac{1-x}{2}\mathfrak{Z}_1 - \frac{1-x^2}{2}\mathfrak{Z}_5 + \frac{1-x}{2}\mathfrak{Z}_6 + \frac{1-x^2}{2}\mathfrak{Z}_{10} - (1+x)\mathfrak{Z}_{13} + (1+x)\mathfrak{Z}_{14} - (1+x)\mathfrak{Z}_{17} + (1+x)\mathfrak{Z}_{18}\Big]$$

$$f_{\frac{1}{2}1;\frac{1}{2}0} = -\sin\frac{\theta}{2}\, e^{i\varphi}\Big[\frac{1+x}{\sqrt{2}}\mathfrak{Z}_1 + \frac{1+x}{\sqrt{2}}\mathfrak{Z}_3 + \frac{x(1+x)}{\sqrt{2}}\mathfrak{Z}_5 + \frac{1+x}{\sqrt{2}}\mathfrak{Z}_6 + \frac{1+x}{\sqrt{2}}\mathfrak{Z}_8 + \frac{x(1+x)}{2}\mathfrak{Z}_{10} + \sqrt{2}\,\mathfrak{Z}_{11} + \sqrt{2}\,\mathfrak{Z}_{12} + \sqrt{2}\,x\mathfrak{Z}_{13} + \sqrt{2}\,x\mathfrak{Z}_{14} + \frac{1+x}{\sqrt{2}}\mathfrak{Z}_{17} + \frac{1+x}{\sqrt{2}}\mathfrak{Z}_{18}\Big]\frac{k_o}{m}$$

$$f_{\frac{1}{2}0;\frac{1}{2}0} = \cos\frac{\theta}{2}\Big[x\mathfrak{Z}_1 + \mathfrak{Z}_2 + x\mathfrak{Z}_3 + x\mathfrak{Z}_4 + x^2\mathfrak{Z}_5 + x\mathfrak{Z}_6 + \mathfrak{Z}_7 + x\mathfrak{Z}_8 + x\mathfrak{Z}_9 + x^2\mathfrak{Z}_{10} + \mathfrak{Z}_{11} + \mathfrak{Z}_{12} + x\mathfrak{Z}_{13} + x\mathfrak{Z}_{14} + \mathfrak{Z}_{15} + \mathfrak{Z}_{16} + x\mathfrak{Z}_{17} + x\mathfrak{Z}_{18}\Big]\frac{k_o k'_o}{mm'}$$

$$f_{\frac{1}{2}-1;\frac{1}{2}0} = -2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}\, e^{-i\varphi}\Big[\frac{1}{\sqrt{2}}\mathfrak{Z}_1 + \frac{1}{\sqrt{2}}\mathfrak{Z}_3 + \frac{x}{\sqrt{2}}\mathfrak{Z}_5 + \frac{1}{\sqrt{2}}\mathfrak{Z}_6 + \frac{1}{\sqrt{2}}\mathfrak{Z}_8 + \frac{x}{\sqrt{2}}\mathfrak{Z}_{10} + \frac{1}{\sqrt{2}}\mathfrak{Z}_{17} + \frac{1}{\sqrt{2}}\mathfrak{Z}_{18}\Big]\frac{k_o}{m}$$

$$(163)$$

$$f_{-\frac{1}{2}1;\frac{1}{2}0} = 2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\, e^{2i\varphi}\Big[\frac{1}{\sqrt{2}}\mathfrak{Z}_1 + \frac{1}{\sqrt{2}}\mathfrak{Z}_3 + \frac{x}{\sqrt{2}}\mathfrak{Z}_5 - \frac{1}{\sqrt{2}}\mathfrak{Z}_6 - \frac{1}{\sqrt{2}}\mathfrak{Z}_8 - \frac{1}{\sqrt{2}}\mathfrak{Z}_{10} - \frac{1}{\sqrt{2}}\mathfrak{Z}_{17} + \frac{1}{\sqrt{2}}\mathfrak{Z}_{18}\Big]\frac{k_o}{m}$$

$$f_{-\frac{1}{2}0;\frac{1}{2}0} = -\sin\frac{\theta}{2}\, e^{i\varphi}\Big[x\mathfrak{Z}_1 + \mathfrak{Z}_2 + x\mathfrak{Z}_3 + x\mathfrak{Z}_4 + x^2\mathfrak{Z}_5 - x\mathfrak{Z}_6 - \mathfrak{Z}_7 - x\mathfrak{Z}_8 - x\mathfrak{Z}_9 - x^2\mathfrak{Z}_{10} - \mathfrak{Z}_{11} + \mathfrak{Z}_{12} - x\mathfrak{Z}_{13} + x\mathfrak{Z}_{14} - \mathfrak{Z}_{15} + \mathfrak{Z}_{16} - x\mathfrak{Z}_{17} + x\mathfrak{Z}_{18}\Big]\frac{k_o k'_o}{mm'}$$

$$f_{-\frac{1}{2}-1;\frac{1}{2}0} = \cos\frac{\theta}{2}\Big[-\frac{1-x}{\sqrt{2}}\mathfrak{Z}_1 - \frac{1-x}{\sqrt{2}}\mathfrak{Z}_3 - \frac{x(1-x)}{\sqrt{2}}\mathfrak{Z}_5 + \frac{1-x}{\sqrt{2}}\mathfrak{Z}_6 + \frac{1-x}{\sqrt{2}}\mathfrak{Z}_8 + \frac{x(1-x)}{\sqrt{2}}\mathfrak{Z}_{10} - \sqrt{2}\,\mathfrak{Z}_{11} + \sqrt{2}\,\mathfrak{Z}_{12} - \sqrt{2}\,x\mathfrak{Z}_{13} + \sqrt{2}\,x\mathfrak{Z}_{14} + \frac{1-x}{\sqrt{2}}\mathfrak{Z}_{17} - \frac{1-x}{\sqrt{2}}\mathfrak{Z}_{18}\Big]\frac{k_o}{m}$$

$$f_{\frac{1}{2}1;\frac{1}{2}-1} = 2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\, e^{2i\varphi}\Big[\frac{1}{2}\mathfrak{Z}_1 + \frac{1+x}{2}\mathfrak{Z}_5 + \frac{1}{2}\mathfrak{Z}_6 + \frac{1+x}{2}\mathfrak{Z}_{10} + \mathfrak{Z}_{13} + \mathfrak{Z}_{14}\Big]$$

$$f_{\frac{1}{2}0;\frac{1}{2}-1} = 2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}\, e^{i\varphi}\Big[-\frac{1}{\sqrt{2}}\mathcal{F}_1 - \frac{1}{\sqrt{2}}\mathcal{F}_4 - \frac{x}{\sqrt{2}}\mathcal{F}_5 - \frac{1}{\sqrt{2}}\mathcal{F}_6 - \frac{1}{\sqrt{2}}\mathcal{F}_9 - \frac{x}{\sqrt{2}}\mathcal{F}_{10} - \frac{1}{\sqrt{2}}\mathcal{F}_{13} - \frac{1}{\sqrt{2}}\mathcal{F}_{14}\Big]\frac{k'_o}{m'}$$

$$f_{\frac{1}{2}-1;\frac{1}{2}-1} = 2\cos^3\frac{\theta}{2}\Big[\frac{1}{2}\mathcal{F}_1 - \frac{1-x}{2}\mathcal{F}_5 + \frac{1}{2}\mathcal{F}_6 - \frac{1-x}{2}\mathcal{F}_{10}\Big] \tag{163}$$

$$f_{-\frac{1}{2}1;\frac{1}{2}-1} = -2\sin^3\frac{\theta}{2}\, e^{3i\varphi}\Big[\frac{1}{2}\mathcal{F}_1 + \frac{1+x}{2}\mathcal{F}_5 - \frac{1}{2}\mathcal{F}_6 - \frac{1+x}{2}\mathcal{F}_{10}\Big]$$

$$f_{-\frac{1}{2}0;\frac{1}{2}-1} = 2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\, e^{2i\varphi}\Big[\frac{1}{\sqrt{2}}\mathcal{F}_1 + \frac{1}{\sqrt{2}}\mathcal{F}_4 + \frac{x}{\sqrt{2}}\mathcal{F}_5 - \frac{1}{\sqrt{2}}\mathcal{F}_6 - \frac{1}{\sqrt{2}}\mathcal{F}_9 - \frac{x}{\sqrt{2}}\mathcal{F}_{10} - \frac{1}{\sqrt{2}}\mathcal{F}_{13} + \frac{1}{\sqrt{2}}\mathcal{F}_{14}\Big]\frac{k'_o}{m'}$$

$$f_{-\frac{1}{2}-1;\frac{1}{2}-1} = 2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}\, e^{i\varphi}\Big[-\frac{1}{2}\mathcal{F}_1 + \frac{1-x}{2}\mathcal{F}_5 + \frac{1}{2}\mathcal{F}_6 - \frac{1-x}{2}\mathcal{F}_{10} + \mathcal{F}_{13} - \mathcal{F}_{14}\Big]$$

As in equation 140, we write

$$\mathcal{F}_k(W,x) = \sum_{\substack{J=\ell+\frac{1}{2}\\ i}} a_{ki}(x)\,\Pi_i^J(x)\,\alpha_i^J(W) \tag{164}$$

where

$$\Pi_1^J = \frac{P'_{\ell+1} - P'_\ell}{4(1-x)} \qquad \Pi_5^J = \frac{P'_{\ell+1} - P'_\ell}{2\sqrt{2}}$$

$$\Pi_2^J = \frac{P'_{\ell+1} + P'_\ell}{2\sqrt{2}} \qquad \Pi_6^J = \frac{P'_{\ell+1} + P'_\ell}{4(1+x)}$$

$$\Pi_3^J = \frac{P''_{\ell+1} + P''_\ell}{4\sqrt{\ell(\ell+2)}} \qquad \Pi_7^J = \frac{P'_{\ell+1} + P'_\ell}{2\sqrt{2}} \tag{164a}$$

$$\Pi_4^J = \frac{-P''_{\ell+1} + P''_\ell}{4\sqrt{\ell(\ell+2)}} \qquad \Pi_8^J = \frac{P'_{\ell+1} - P'_\ell}{2}$$

$$\Pi_9^J = \frac{-P''_{\ell+1} + P''_\ell}{2\sqrt{2\ell(\ell+2)}}$$

$$\Pi_{10}^J = \frac{P''_{\ell+1} + P''_\ell}{2\sqrt{2\ell(\ell+2)}}$$

$$\Pi_{11}^J = \frac{P'_{\ell+1} + P'_\ell}{2}$$

$$\Pi_{12}^J = \frac{P'_{\ell+1} - P'_\ell}{2\sqrt{2}}$$

$$\Pi_{13}^J = \frac{P''_{\ell+1} + P''_\ell}{4\sqrt{\ell(\ell+2)}} \tag{164a}$$

$$\Pi_{14}^J = \frac{-P''_{\ell+1} + P''_\ell}{2\sqrt{2\ell(\ell+2)}}$$

$$\Pi_{15}^J = \frac{\ell(2\ell+1)P'''_{\ell+2} - 3\ell(2\ell+3)P'''_{\ell+1} + 3(\ell+2)(2\ell+1)P'''_\ell - (\ell+2)(2\ell+3)P'''_{\ell-1}}{4\ell(\ell+2)(2\ell+1)(2\ell+3)(1-x)}$$

$$\Pi_{16}^J = \frac{\ell(2\ell+1)P'''_{\ell+2} + 3\ell(2\ell+3)P'''_{\ell+1} + 3(\ell+2)(2\ell+1)P'''_\ell + (\ell+2)(2\ell+3)P'''_{\ell-1}}{4\ell(\ell+2)(2\ell+1)(2\ell+3)(1+x)}$$

$$\Pi_{17}^J = \frac{P''_{\ell+1} + P''_\ell}{2\sqrt{2\ell(\ell+2)}}$$

$$\Pi_{18}^J = \frac{-P''_{\ell+1} + P''_\ell}{4\sqrt{\ell(\ell+2)}}$$

and the a_{ki} are given in Table 9 for $\kappa \ll 1$.

TABLE 9

	a_1^J	a_2^J	a_3^J	a_4^J	a_5^J	a_6^J	a_7^J	a_8^J	a_9^J	a_{10}^J	a_{11}^J	a_{12}^J	a_{13}^J	a_{14}^J	a_{15}^J	a_{16}^J	a_{17}^J	a_{18}^J
$\mathfrak{z}_1$	1-x	-	1-x	-1-x	-	-1-x	-	-	-	-	-	-	1-x	-	$1-x^2$	$1-x^2$	-	-1-x
$\mathfrak{z}_2$	-x	-1	x	-x	-1	x	-1	1	-1+x	-1-x	1	-1	x	-1+x	$3x-x^2$	$3x+x^2$	-1-x	-x
$\mathfrak{z}_3$	-	-	-2	2	-	-	-	-	-2	2	-	-	-	-	-2+2x	-2-2x	-	-
$\mathfrak{z}_4$	-	-	-	-	-	-	-	-	-	-	-	-	-2	-2	-2+2x	-2-2x	2	2
$\mathfrak{z}_5$	1	-	1	1	-	1	-	-	-	-	-	-	1	-	-3+x	3+x	-	1
$\mathfrak{z}_6$	1-x	-	1-x	1+x	-	1+x	-	-	-	-	-	-	1-x	-	$1-x^2$	$-1+x^2$	-	1+x
$\mathfrak{z}_7$	-x	-1	x	x	1	-x	-1	1	-1+x	1+x	-1	1	x	-1+x	$3x-x^2$	$-3x-x^2$	1+x	x
$\mathfrak{z}_8$	-	-	-2	-2	-	-	-	-	-2	-2	-	-	-	-	-2+2x	2+2x	-	-
$\mathfrak{z}_9$	-	-	-	-	-	-	-	-	-	-	-	-	-2	-2	-2+2x	2+2x	-2	-2
$\mathfrak{z}_{10}$	1	-	1	-1	-	-1	-	-	-	-	-	-	1	-	-3+x	-3-x	-	-1
$\mathfrak{z}_{11}$	x	-	x	x	-	x	1	-	1+x	-1+x	-	-1	-x	-	$-x-x^2$	$-x+x^2$	-	-x
$\mathfrak{z}_{12}$	x	-	x	-x	-	-x	1	-	1+x	1-x	-	1	-x	-	$-x-x^2$	$x-x^2$	-	x
$\mathfrak{z}_{13}$	-1	-	-1	-1	-	-1	-	-	-	-	-	-	1	-	1+x	-1+x	-	1
$\mathfrak{z}_{14}$	-1	-	-1	1	-	1	-	-	-	-	-	-	1	-	1+x	1-x	-	-1
$\mathfrak{z}_{15}$	x	1	-x	-x	-1	x	-	-	-	-	-	-	x	1+x	$-x-x^2$	$x-x^2$	-1+x	x
$\mathfrak{z}_{16}$	x	1	-x	x	1	-x	-	-	-	-	-	-	x	1+x	$-x-x^2$	$-x+x^2$	1-x	-x
$\mathfrak{z}_{17}$	-1	-	1	1	-	-1	-	-	-	-	-	-	-1	-	1+x	-1+x	-	-1
$\mathfrak{z}_{18}$	-1	-	1	-1	-	1	-	-	-	-	-	-	-1	-	1+x	1-x	-	1

From D12, we have then

$$\varphi_i^J = \Pi^J_{(i)} a_{\cdot i} \tag{165}$$

and by D-16

$$\chi_i^J = K^J_{(i)} b_{\cdot i} \tag{166}$$

where the K_i^J are

$$K_1^J = \frac{1}{4}(P_{\ell+1} + P_\ell)$$

$$K_2^J = \frac{1}{2\sqrt{2}}(-P_{\ell+1} + P_\ell)$$

$$K_3^J = \frac{\sqrt{\ell(\ell+2)}}{4}\left(\frac{P_{\ell+2}}{2\ell+3} - \frac{P_{\ell+1}}{2\ell+1} - \frac{P_\ell}{2\ell+3} + \frac{P_{\ell-1}}{2\ell+1}\right)$$

$$K_4^J = \frac{\sqrt{\ell(\ell+2)}}{4}\left(\frac{P_{\ell+2}}{2\ell+3} + \frac{P_{\ell+1}}{2\ell+1} - \frac{P_\ell}{2\ell+3} - \frac{P_{\ell-1}}{2\ell+1}\right)$$

$$K_5^J = \frac{1}{2\sqrt{2}}(P_{\ell+1} + P_\ell)$$

$$K_6^J = \frac{1}{4}(-P_{\ell+1} + P_\ell)$$

$$K_7^J = \frac{1}{2\sqrt{2}}(-P_{\ell+1} + P_\ell)$$

$$K_8^J = \frac{1}{2}(P_{\ell+1} + P_\ell)$$

$$K_9^J = \frac{\sqrt{\ell(\ell+2)}}{2\sqrt{2}}\left(\frac{P_{\ell+2}}{2\ell+3} + \frac{P_{\ell+1}}{2\ell+1} - \frac{P_\ell}{2\ell+3} - \frac{P_{\ell-1}}{2\ell+1}\right)$$

$$K_{10}^J = \frac{\sqrt{\ell(\ell+2)}}{2\sqrt{2}}\left(\frac{P_{\ell+2}}{2\ell+3} - \frac{P_{\ell+1}}{2\ell+1} - \frac{P_\ell}{2\ell+3} + \frac{P_{\ell-1}}{2\ell+1}\right)$$

$$K_{11}^J = \frac{1}{2}(-P_{\ell+1} + P_\ell)$$

$$K_{12}^J = \frac{1}{2\sqrt{2}}(P_{\ell+1} + P_\ell)$$

$$K_{13}^{J} = \frac{\sqrt{\ell(\ell+2)}}{4}\left(\frac{P_{\ell+2}}{2\ell+3} - \frac{P_{\ell+1}}{2\ell+1} - \frac{P_{\ell}}{2\ell+3} + \frac{P_{\ell-1}}{2\ell+1}\right)$$

$$K_{14}^{J} = \frac{\sqrt{\ell(\ell+2)}}{2\sqrt{2}}\left(\frac{P_{\ell+2}}{2\ell+3} + \frac{P_{\ell+1}}{2\ell+1} - \frac{P_{\ell}}{2\ell+3} - \frac{P_{\ell-1}}{2\ell+1}\right)$$

$$K_{15}^{J} = \frac{1}{4}\left[\frac{\ell P_{\ell+2}}{2\ell+3} + \frac{3\ell P_{\ell+1}}{2\ell+1} + \frac{3(\ell+2)P_{\ell}}{2\ell+3} + \frac{(\ell+2)P_{\ell-1}}{2\ell+1}\right] \tag{166a}$$

$$K_{16}^{J} = \frac{1}{4}\left[-\frac{\ell P_{\ell+2}}{2\ell+3} + \frac{3\ell P_{\ell+1}}{2\ell+1} - \frac{3(\ell+2)P_{\ell}}{2\ell+3} + \frac{(\ell+2)P_{\ell-1}}{2\ell+1}\right]$$

$$K_{17}^{J} = \frac{\sqrt{\ell(\ell+2)}}{2\sqrt{2}}\left(\frac{P_{\ell+2}}{2\ell+3} - \frac{P_{\ell+1}}{2\ell+1} - \frac{P_{\ell}}{2\ell+3} + \frac{P_{\ell-1}}{2\ell+1}\right)$$

$$K_{18}^{J} = \frac{\sqrt{\ell(\ell+2)}}{4}\left(\frac{P_{\ell+2}}{2\ell+3} + \frac{P_{\ell+1}}{2\ell+1} - \frac{P_{\ell}}{2\ell+3} - \frac{P_{\ell-1}}{2\ell+1}\right)$$

and the b_{ki} are given in Table 10 for $\kappa \ll 1$.

We define now the following amplitudes:

$$T_{1,10} = \frac{f_{\frac{1}{2}1;\frac{1}{2}1}}{\cos\frac{\theta}{2}} \pm \frac{f_{-\frac{1}{2}-1;\frac{1}{2}1}}{\sin\frac{\theta}{2}\, e^{-i\varphi}}$$

$$T_{2,11} = \frac{f_{\frac{1}{2}-1;\frac{1}{2}-1}}{2\cos^{3}\frac{\theta}{2}} \pm \frac{f_{-\frac{1}{2}1;\frac{1}{2}-1}}{-2\sin^{3}\frac{\theta}{2}\, e^{3i\varphi}}$$

$$T_{3,12} = \frac{f_{\frac{1}{2}0;\frac{1}{2}0}}{\cos\frac{\theta}{2}} \pm \frac{f_{-\frac{1}{2}0;\frac{1}{2}0}}{-\sin\frac{\theta}{2}\, e^{i\varphi}} \tag{167}$$

$$T_{4,13} = \frac{f_{\frac{1}{2}1;\frac{1}{2}-1}}{2\sin^{2}\frac{\theta}{2}\cos\frac{\theta}{2}\, e^{2i\varphi}} \pm \frac{f_{-\frac{1}{2}-1;\frac{1}{2}-1}}{2\sin\frac{\theta}{2}\cos^{2}\frac{\theta}{2}\, e^{i\varphi}}$$

$$T_{5,14} = \frac{f_{\frac{1}{2}-1;\frac{1}{2}1}}{2\sin^{2}\frac{\theta}{2}\cos\frac{\theta}{2}\, e^{-2i\varphi}} \pm \frac{f_{-\frac{1}{2}1;\frac{1}{2}1}}{2\sin\frac{\theta}{2}\cos^{2}\frac{\theta}{2}\, e^{i\varphi}}$$

$$T_{6,15} = \frac{f_{\frac{1}{2}1;\frac{1}{2}0}}{-\sin\frac{\theta}{2}\, e^{i\varphi}} \pm \frac{f_{-\frac{1}{2}-1;\frac{1}{2}0}}{\cos\frac{\theta}{2}}$$

TABLE 10

	X^J_1	X^J_2	X^J_3	X^J_4	X^J_5	X^J_6	X^J_7	X^J_8	X^J_9	X^J_{10}	X^J_{11}	X^J_{12}	X^J_{13}	X^J_{14}	X^J_{15}	X^J_{16}	X^J_{17}	X^J_{18}
1	1+x	1+x	1	-1	-1+x	-1+x	1+x	x	-1	1	x	-1+x	1	-1	1-x	1+x	1	-1
2	-	-	-	-	-	-	-	1	-	-	1	-	-	-	-	-	-	-
3	-	-	-	-	-	-	1+x	x	-1	1	x	-1+x	-	-	-	-	-	-
4	-	1+x	-	-	-1+x	-	-	x	-	-	x	-	-	-1	-	-	1	-
5	$-1+x^2$	$x+x^2$	1+x	1-x	$-x+x^2$	$-1+x^2$	$x+x^2$	x^2	-x	x	x^2	$-x+x^2$	1+x	-x	$-(1-x)^2$	$(1+x)^2$	x	1-x
6	1+x	1+x	1	1	1	1-x	1+x	x	-1	-1	-x	1-x	1	-1	1-x	-1-x	-1	1
7	-	-	-	-	-	-	-	1	-	-	-1	-	-	-	-	-	-	-
8	-	-	-	-	-	-	1+x	x	-1	-1	-x	1-x	-	-	-	-	-	-
9	-	1+x	-	-	1-x	-	-	x	-	-	-x	-	-	-1	-	-	-1	-
10	$-1+x^2$	$x+x^2$	1+x	-1+x	$x-x^2$	$1-x^2$	$x+x^2$	x^2	-x	-x	$-x^2$	$x-x^2$	1+x	-x	$-(1-x)^2$	$-(1+x)^2$	-x	-1+x
11	-	-	-	-	-	-	2	1	-	-	-1	-2	-	-	-	-	-	-
12	-	-	-	-	-	-	2	1	-	-	1	2	-	-	-	-	-	-
13	-2+2x	1+x	-	-	1-x	-2-2x	2x	x	-	-	-x	-2x	2	-1	-	-	-1	2
14	-2+2x	1+x	-	-	-1+x	2+2x	2x	x	-	-	x	2x	2	-1	-	-	1	-2
15	-	2	-	-	-2	-	-	1	-	-	-1	-	-	-	-	-	-	-
16	-	2	-	-	2	-	-	1	-	-	1	-	-	-	-	-	-	-
17	-2+2x	2x	2	2	-2x	-2-2x	1+x	x	-1	-1	-x	1-x	-	-	-	-	-	-
18	-2+2x	2x	2	-2	2x	2+2x	1+x	x	-1	1	x	-1+x	-	-	-	-	-	-

(166b)

$$T_{7,16} = \frac{f_{\frac{1}{2}-1;\frac{1}{2}0}}{-2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}e^{-i\varphi}} \pm \frac{f_{-\frac{1}{2}1;\frac{1}{2}0}}{2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}e^{2i\varphi}}$$

$$T_{8,17} = \frac{f_{\frac{1}{2}0;\frac{1}{2}1}}{\sin\frac{\theta}{2}e^{-i\varphi}} \pm \frac{f_{-\frac{1}{2}0;\frac{1}{2}1}}{\cos\frac{\theta}{2}} \tag{167}$$

$$T_{9,18} = \frac{f_{\frac{1}{2}0;\frac{1}{2}-1}}{2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}e^{i\varphi}} \pm \frac{f_{-\frac{1}{2}0;\frac{1}{2}-1}}{2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}e^{2i\varphi}}$$

$$\beta^J_{1,10} = a^J_1 \pm a^J_6 \qquad \beta^J_{5,14} = a^J_3 \pm a^J_4$$

$$\beta^J_{2,11} = a^J_{15} \pm a^J_{16} \qquad \beta^J_{6,15} = a^J_7 \pm a^J_{12}$$

$$\beta^J_{3,12} = a^J_8 \pm a^J_{11} \qquad \beta^J_{7,16} = a^J_9 \pm a^J_{10} \tag{168}$$

$$\beta^J_{4,13} = a^J_{13} \pm a^J_{18} \qquad \beta^J_{8,17} = a^J_2 \pm a^J_5$$

$$\beta^J_{9,18} = a^J_{14} \pm a^J_{17}$$

The partial wave expansion of the T_i's is then

$$T_{1,10} = \sum_{J=\ell+\frac{1}{2}} \left[\beta^J_{1,10} P'_{\ell+1} - \beta^J_{10,1} P'_\ell \right]$$

$$T_{2,11} = \sum_{J=\ell+\frac{1}{2}} \left[\beta^J_{2,11} \left(\frac{P'''_{\ell+2}}{(\ell+2)(2\ell+3)} + \frac{3P'''_\ell}{\ell(2\ell+3)} \right) - \beta^J_{11,2} \left(\frac{3P'''_{\ell+1}}{(\ell+2)(2\ell+1)} + \frac{P'''_{\ell-1}}{\ell(2\ell+1)} \right) \right] \tag{169}$$

$$T_{3,12} = \sum_{J=\ell+\frac{1}{2}} \left[\beta^J_{3,12} P'_{\ell+1} - \beta^J_{12,3} P'_\ell \right]$$

$$T_{4,13} = \sum_{J=\ell+\frac{1}{2}} \left[\beta^J_{4,13} \frac{P''_\ell}{\sqrt{\ell(\ell+2)}} + \beta^J_{13,4} \frac{P''_{\ell+1}}{\sqrt{\ell(\ell+2)}}\right]$$

$$T_{5,14} = \sum_{J=\ell+\frac{1}{2}} \left[\beta^J_{5,14} \frac{P''_\ell}{\sqrt{\ell(\ell+2)}} + \beta^J_{14,5} \frac{P''_{\ell+1}}{\sqrt{\ell(\ell+2)}}\right]$$

$$T_{6,15} = \sum_{J=\ell+\frac{1}{2}} \left[\beta^J_{6,15} P'_{\ell+1} + \beta^J_{15,6} P'_\ell\right] \tag{169}$$

$$T_{7,16} = \sum_{J=\ell+\frac{1}{2}} \left[\beta^J_{7,16} \frac{P''_\ell}{\sqrt{\ell(\ell+2)}} - \beta^J_{16,7} \frac{P''_{\ell+1}}{\sqrt{\ell(\ell+2)}}\right]$$

$$T_{8,17} = \sum_{J=\ell+\frac{1}{2}} \left[\beta^J_{8,17} P'_{\ell+1} + \beta^J_{17,8} P'_\ell\right]$$

$$T_{9,18} = \sum_{J=\ell+\frac{1}{2}} \left[\beta^J_{9,18} \frac{P''_\ell}{\sqrt{\ell(\ell+2)}} - \beta^J_{18,9} \frac{P''_{\ell+1}}{\sqrt{\ell(\ell+2)}}\right]$$

Using equations 163 we get:

$$T_1 = x\mathfrak{J}_1 - (1-x^2)\mathfrak{J}_5 + \mathfrak{J}_6 - 2\mathfrak{J}_{13} + 2x\mathfrak{J}_{14} - 2\mathfrak{J}_{17} + 2x\mathfrak{J}_{18}$$

$$T_{10} = \mathfrak{J}_1 + x\mathfrak{J}_6 - (1-x^2)\mathfrak{J}_{10} + 2x\mathfrak{J}_{13} - 2\mathfrak{J}_{14} + 2x\mathfrak{J}_{17} - 2\mathfrak{J}_{18}$$

$$T_2 = \mathfrak{J}_1 + x\mathfrak{J}_5 - \mathfrak{J}_{10} \tag{170}$$

$$T_{11} = -\mathfrak{J}_5 + \mathfrak{J}_6 + x\mathfrak{J}_{10}$$

$$T_3 = (2x\mathfrak{J}_1 + 2\mathfrak{J}_2 + 2x\mathfrak{J}_3 + 2x\mathfrak{J}_4 + 2x^2\mathfrak{J}_5 + 2\mathfrak{J}_{12} + 2x\mathfrak{J}_{14} + 2\mathfrak{J}_{16} + 2x\mathfrak{J}_{18})\frac{k_o k'_o}{mm'}$$

$$T_{12} = (2x\mathfrak{J}_6 + 2\mathfrak{J}_7 + 2x\mathfrak{J}_8 + 2x\mathfrak{J}_9 + 2x^2\mathfrak{J}_{10} + 2\mathfrak{J}_{11} + 2x\mathfrak{J}_{13} + 2\mathfrak{J}_{15} + 2x\mathfrak{J}_{17})\frac{k_o k'_o}{mm'}$$

$$T_4 = \mathfrak{J}_5 + \mathfrak{J}_6 + x\mathfrak{J}_{10} + 2\mathfrak{J}_{13}$$

$$T_{13} = \mathfrak{a}_1 + x\mathfrak{a}_5 + \mathfrak{a}_{10} + 2\mathfrak{a}_{14}$$

$$T_5 = \mathfrak{a}_5 + \mathfrak{a}_6 + x\mathfrak{a}_{10} + 2\mathfrak{a}_{17}$$

$$T_{14} = \mathfrak{a}_1 + x\mathfrak{a}_5 + \mathfrak{a}_{10} + 2\mathfrak{a}_{18}$$

$$T_6 = (\sqrt{2}\,x\mathfrak{a}_1 + \sqrt{2}\,x\mathfrak{a}_3 + \sqrt{2}\,x^2\mathfrak{a}_5 + \sqrt{2}\,\mathfrak{a}_6 + \sqrt{2}\,\mathfrak{a}_8 + \sqrt{2}\,x\mathfrak{a}_{10} + 2\sqrt{2}\,\mathfrak{a}_{12} + 2\sqrt{2}\,x\mathfrak{a}_{14} + \sqrt{2}\,\mathfrak{a}_{17} + \sqrt{2}\,x\mathfrak{a}_{18})\frac{k_o}{m}$$

$$T_{15} = (\sqrt{2}\,\mathfrak{a}_1 + \sqrt{2}\,\mathfrak{a}_3 + \sqrt{2}\,x\mathfrak{a}_5 + \sqrt{2}\,x\mathfrak{a}_6 + \sqrt{2}\,x\mathfrak{a}_8 + \sqrt{2}\,x^2\mathfrak{a}_{10} + 2\sqrt{2}\,\mathfrak{a}_{11} + 2\sqrt{2}\,x\mathfrak{a}_{13} + \sqrt{2}\,x\mathfrak{a}_{17} + \sqrt{2}\,\mathfrak{a}_{18})\frac{k_o}{m}$$

$$T_7 = (\sqrt{2}\,\mathfrak{a}_1 + \sqrt{2}\,\mathfrak{a}_3 + \sqrt{2}\,x\mathfrak{a}_5 + \sqrt{2}\,\mathfrak{a}_{18})\frac{k_o}{m} \tag{170}$$

$$T_{16} = (\sqrt{2}\,\mathfrak{a}_6 + \sqrt{2}\,\mathfrak{a}_8 + \sqrt{2}\,x\mathfrak{a}_{10} + \sqrt{2}\,\mathfrak{a}_{17})\frac{k_o}{m}$$

$$T_8 = (\sqrt{2}\,x\mathfrak{a}_1 + \sqrt{2}\,x\mathfrak{a}_4 + \sqrt{2}\,x^2\mathfrak{a}_5 + \sqrt{2}\,\mathfrak{a}_6 + \sqrt{2}\,\mathfrak{a}_9 + \sqrt{2}\,x\mathfrak{a}_{10} + \sqrt{2}\,\mathfrak{a}_{13} + \sqrt{2}\,x\mathfrak{a}_{14} + 2\sqrt{2}\,\mathfrak{a}_{16} + 2\sqrt{2}\,x\mathfrak{a}_{18})\frac{k_o'}{m'}$$

$$T_{17} = (\sqrt{2}\,\mathfrak{a}_1 + \sqrt{2}\,\mathfrak{a}_4 + \sqrt{2}\,x\mathfrak{a}_5 + \sqrt{2}\,x\mathfrak{a}_6 + \sqrt{2}\,x\mathfrak{a}_9 + \sqrt{2}\,x^2\mathfrak{a}_{10} + \sqrt{2}\,x\mathfrak{a}_{13} + \sqrt{2}\,\mathfrak{a}_{14} + 2\sqrt{2}\,\mathfrak{a}_{15} + 2\sqrt{2}\,x\mathfrak{a}_{17})\frac{k_o'}{m'}$$

$$T_9 = (-\sqrt{2}\,\mathfrak{a}_6 - \sqrt{2}\,\mathfrak{a}_9 - \sqrt{2}\,x\mathfrak{a}_{10} - \sqrt{2}\,\mathfrak{a}_{13})\frac{k_o'}{m'}$$

$$T_{18} = (-\sqrt{2}\,\mathfrak{a}_1 - \sqrt{2}\,\mathfrak{a}_4 - \sqrt{2}\,x\mathfrak{a}_5 - \sqrt{2}\,\mathfrak{a}_{14})\frac{k_o'}{m'}$$

From equations 159, 160c and 170 we can find

$$T_\ell = \mathfrak{a}_i c_{i\ell} \tag{171}$$

where we can write

$$c_{i\ell} = c_{i\ell}^{(0)} + c_{i\ell}^{(1)}x + c_{i\ell}^{(2)}x^2 \tag{171a}$$

The coefficients $c_{i\ell}^{(0)}$, $c_{i\ell}^{(1)}$ and $c_{i\ell}^{(2)}$ are given in tables 11, 12 and 13 respectively where we have again set

$$R^{\pm} = \sqrt{(E\pm M)(E'\pm M)} \tag{171b}$$

The reflection symmetry in this case is expressed by

$$\beta_i^J(-W) = \eta_{(i)}\beta_{i+9}^J(W) \qquad i = 1, 2, \ldots, 9 \tag{172a}$$

$$T_i(-W) = \eta_{(i)} T_{i+9}^J(W) \qquad i = 1, 2, \ldots, 9 \tag{172b}$$

$$\eta_i = 1 \quad i = 6, 7, 8, 9 \qquad n_i = -1 \quad i = 1, 2, 3, 4, 5 \tag{172c}$$

The projection operators for β_i^J are:

$$\beta_{1,10}^J = \frac{1}{2}\int_{-1}^{1}\left(T_{1,10}P_\ell + T_{10,1}P_{\ell+1}\right)dx$$

$$\beta_{2,11}^J = \frac{1}{2}\int_{-1}^{1}\Big(T_{2,11}\frac{3\ell P_{\ell+1}+(\ell+2)P_{\ell-1}}{2\ell+1} + T_{11,2}\frac{\ell P_{\ell+2} + 3(\ell+2)P_\ell}{2\ell+3}\Big)dx$$

$$\beta_{3,12}^J = \frac{1}{2}\int_{-1}^{1}\left(T_{3,12}P_\ell + T_{12,3}P_{\ell+1}\right)dx \tag{173}$$

$$\beta_{4,13}^J = \frac{1}{2}\int_{-1}^{1}\Big(T_{4,13}\frac{\sqrt{\ell(\ell+2)}\,(P_{\ell+2} - P_\ell)}{2\ell+3} - T_{13,4}\frac{\sqrt{\ell(\ell+2)}\,(P_{\ell+1} - P_{\ell-1})}{2\ell+1}\Big)dx$$

$$\beta^J_{5,14} = \frac{1}{2}\int_{-1}^{1}\left(T_{5,14}\,\frac{\sqrt{\ell(\ell+2)}\,(P_{\ell+2}-P_\ell)}{2\ell+3} - T_{14,5}\,\frac{\sqrt{\ell(\ell+2)}\,(P_{\ell+1}-P_{\ell-1})}{2\ell+1}\right)dx$$

$$\beta^J_{6,15} = \frac{1}{2}\int_{-1}^{1}\left(T_{6,15}P_\ell - T_{15,6}P_{\ell+1}\right)dx$$

$$\beta^J_{7,16} = \frac{1}{2}\int_{-1}^{1}\left(T_{7,16}\,\frac{\sqrt{\ell(\ell+2)}\,(P_{\ell+2}-P_\ell)}{2\ell+3} + T_{16,7}\,\frac{\sqrt{\ell(\ell+2)}\,(P_{\ell+1}-P_{\ell-1})}{2\ell+1}\right)dx \qquad (173)$$

$$\beta^J_{8,17} = \frac{1}{2}\int_{-1}^{1}\left(T_{8,17}P_\ell - T_{17,8}P_{\ell+1}\right)dx$$

$$\beta^J_{9,18} = \frac{1}{2}\int_{-1}^{1}\left(T_{9,18}\,\frac{\sqrt{\ell(\ell+2)}\,(P_{\ell+2}-P_\ell)}{2\ell+3} + T_{18,9}\,\frac{\sqrt{\ell(\ell+2)}\,(P_{\ell+1}-P_{\ell-1})}{2\ell+1}\right)dx$$

Remarks similar to those at the end of section b apply in case one of the particles has zero mass. It would correspond, for instance, to photoproduction of 2 or 3 pions in a resonant state.

TABLE II

	G_1	G_2	G_3
T_1	$\frac{R^-}{8\pi W}$	$-\frac{(W+M)R^-}{8\pi W}$	$-\frac{R^-}{4\pi W}$
T_2	$-\frac{R^+}{8\pi W}$	$-\frac{(W-M)R^+}{8\pi W}$	$-\frac{R^+}{4\pi W}$
T_3	$\frac{(E+M)(E'+M)R^-}{4\pi W m m'}$	$\frac{(W-M)(E+M)(E'+M)R^-}{4\pi W m m'}$	$\frac{[(E+M)(E'+M-k_o')-k_o(W+M)]R^-}{4\pi W m m'}$
T_4	$\frac{R^-}{8\pi W}$	$-\frac{(W+M)R^-}{8\pi W}$	$\frac{R^-}{4\pi W}$
T_5	$\frac{R^-}{8\pi W}$	$-\frac{(W+M)R^-}{8\pi W}$	$\frac{R^-}{4\pi W}$
T_6	$\frac{k_o R^-}{4\sqrt{2}\,\pi W m}$	$-\frac{k_o(W+M)R^-}{4\sqrt{2}\,\pi W m}$	$-\frac{(E+M)R^-}{2\sqrt{2}\,\pi W m}$
T_7	$-\frac{k_o R^+}{4\sqrt{2}\,\pi W m}$	$-\frac{k_o(W-M)R^+}{4\sqrt{2}\,\pi W m}$	$-\frac{k_o R^+}{2\sqrt{2}\,\pi W m}$
T_8	$\frac{k_o' R^-}{4\sqrt{2}\,a W m'}$	$-\frac{k_o'(W+M)R^-}{4\sqrt{2}\,\pi W m'}$	$-\frac{(E'+M)R^-}{2\sqrt{2}\,\pi W m'}$
T_9	$-\frac{k_o' R^-}{4\sqrt{2}\,\pi W m'}$	$\frac{k_o'(W+M)R^-}{4\sqrt{2}\,\pi W m'}$	$-\frac{k_o' R^-}{2\sqrt{2}\,\pi W m'}$

TABLE 11 (Continued)

	G_4	G_5	G_6
T_1	$\frac{(E+M-k_o')R^-}{4\pi W}$	$-\frac{(E+M)R^-}{8\pi W}$	$\frac{(E+M)(W-M+k_o')R^-}{16\pi W}$
T_2	$\frac{(W-M)R^+}{4\pi W}$	-	$\frac{(E-M)(E'-M)R^+}{16\pi W}$
T_3	$\frac{[m^2(E'+M)+(W+M)(k_o'W-Mk_o)+2Mk_ok_o']R^-}{4\pi W mm'}$	$\frac{k_o(E'+M)(E+M-k_o)R^-}{8\pi W mm'}$	$\frac{k_o(E'+M)[k_o'(E+M)-Wk_o]R^-}{8\pi W mm'}$
T_4	$\frac{k_o'R^-}{4\pi W}$	-	$\frac{(E+M)(E'+M)R^-}{16\pi W}$
T_5	$\frac{k_oR^-}{4\pi W}$	$\frac{(E+M)R^-}{8\pi W}$	$-\frac{(E+M)(W-M+k_o')R^-}{16\pi W}$
T_6	$-\frac{(E+M)(E'+M-k_o')R^-}{4\sqrt{2}\,\pi W m}$	$-\frac{(E+M)(E-M-k_o)R^-}{8\sqrt{2}\,\pi W m}$	$\frac{(E+M)[k_o'(E-M)-Wk_o]R^-}{8\sqrt{2}\,\pi W m}$
T_7	$\frac{[m^2+k_o(W-M)]R^+}{4\sqrt{2}\,\pi W m}$	$-\frac{(E+M)(E+M-k_o)R^+}{8\sqrt{2}\,\pi W m}$	$-\frac{(E-M)[k_o'(E+M)-Wk_o]R^+}{8\sqrt{2}\,\pi W m}$
T_8	$-\frac{(E'+M)(E+M-k_o)R^-}{4\sqrt{2}\,\pi W m'}$	$-\frac{k_o(E'+M)R^-}{4\sqrt{2}\,\pi W m'}$	$-\frac{k_o(E'+M)(W+M+k_o')R^-}{8\sqrt{2}\,\pi W m'}$
T_9	$-\frac{[m'^2+k_o'(W+M)]R^-}{4\sqrt{2}\,\pi W m'}$	-	$-\frac{\bar{k}'^2 k_o R^-}{8\sqrt{2}\,\pi W m'}$

TABLE 11 (Continued)

	$\mathcal{a}_7$	$\mathcal{a}_8$	$\mathcal{a}_9$	$\mathcal{a}_{10}$
T_1	$\frac{(E'+M)R^-}{8\pi W}$	$-\frac{(E'+M)R^-}{8\pi}$	$\frac{(E+M)R^-}{8\pi W}$	$-\frac{(E+M)(W-M+k'_o)R^-}{16\pi W}$
T_2	-	-	-	$-\frac{(E-M)(E'-M)R^+}{16\pi W}$
T_3	$\frac{k'_o(W+M)(E+M)R^-}{8\pi W m m'}$	$\frac{k'^2_o(E+M)R^-}{8\pi m m'}$	$\frac{(W+E)(E'+M)(E+M-k_o)R^-}{8\pi W m m'}$	$\frac{(W+E)(E'+M)[k'_o(E+M)-Wk_o]R^-}{8\pi W m m'}$
T_4	$-\frac{(E'+M)R^-}{8\pi W}$	$\frac{(E'+M)R^-}{8\pi}$	-	$-\frac{(E+M)(E'+M)R^-}{16\pi W}$
T_5	-	-	$-\frac{(E+M)R^-}{8\pi W}$	$\frac{(E+M)(W-M+k'_o)R^-}{16\pi W}$
T_6	$\frac{k'_o(E+M)R^-}{4\sqrt{2}\,\pi W m}$	$\frac{k'_o(E+M)R^-}{4\sqrt{2}\,\pi m}$	$\frac{(E+M)(E-M-k_o)R^-}{8\sqrt{2}\,\pi W m}$	$-\frac{(E+M)[k'_o(E-M)-Wk_o]R^-}{8\sqrt{2}\,\pi W m}$
T_7	-	-	$\frac{(E-M)(E+M-k_o)R^+}{8\sqrt{2}\,\pi W m}$	$\frac{(E-M)[k'_o(E+M)-Wk_o]R^+}{8\sqrt{2}\,\pi W m}$
T_8	$-\frac{(W-M)(E'+M)R^-}{8\sqrt{2}\,\pi W m'}$	$\frac{k'_o(E'+M)R^-}{8\sqrt{2}\,\pi m'}$	$-\frac{(W+E)(E'+M)R^-}{4\sqrt{2}\,\pi W m'}$	$-\frac{(W+E)(E'+M)(W+M+k'_o)R^-}{8\sqrt{2}\,\pi W m'}$
T_9	$\frac{(W-M)(E'+M)R^-}{8\sqrt{2}\,\pi W m'}$	$-\frac{k'_o(E'+M)R^-}{8\sqrt{2}\,\pi m'}$	-	$-\frac{E'^2(W+E)R^-}{8\sqrt{2}\,\pi W m'}$

TABLE 11 (Continued)

	a_{11}	a_{12}	a_{13}	a_{14}
T_1	$-\frac{(E'+M)R^-}{8\pi W}$	$\frac{(E'+M)R^-}{8\pi}$	$-\frac{(E+M)(E'+M)R^-}{32\pi W}$	$-\frac{(W-M)(E+M)(E'+M)R^-}{32\pi W}$
T_2	-	-	$\frac{(E-M)(E'-M)R^+}{32\pi W}$	$-\frac{(W+M)(E-M)(E'-M)R^+}{32\pi W}$
T_3	$\frac{(W+E')(W+M)(E+M)R^-}{8\pi W m m'}$	$\frac{k_o'(W+E')(E+M)R^-}{8\pi m m'}$	$\frac{k_o k_o'(E+M)(E'+M)R^-}{16\pi W m m'}$	$\frac{k_o k_o'(W-M)(E+M)(E'+M)R^-}{16\pi W m m'}$
T_4	$\frac{(E'+M)R^-}{8\pi W}$	$-\frac{(E'+M)R^-}{8\pi}$	$\frac{(E+M)(E'+M)R^-}{32\pi W}$	$\frac{(W-M)(E+M)(E'+M)R^-}{32\pi W}$
T_5	-	-	$\frac{(E+M)(E'+M)R^-}{32\pi W}$	$\frac{(W-M)(E+M)(E'+M)R^-}{32\pi W}$
T_6	$\frac{(W+E')(E+M)R^-}{4\sqrt{2}\,\pi W m}$	$\frac{(W+E')(E+M)R^-}{4\sqrt{2}\,\pi m}$	$\frac{\vec{k}^2 k_o' R^-}{16\sqrt{2}\,\pi W m}$	$-\frac{\vec{k}^2 k_o'(W+M)R^-}{16\sqrt{2}\,\pi W m}$
T_7	-	-	$-\frac{\vec{k}^2 k_o' R^+}{16\sqrt{2}\,\pi W m}$	$-\frac{\vec{k}^2 k_o'(W-M)R^+}{16\sqrt{2}\,\pi W m}$
T_8	$\frac{(W-M)(E'+M)R^-}{8\sqrt{2}\,\pi W m'}$	$-\frac{k_o'(E'+M)R^-}{8\sqrt{2}\,\pi m'}$	$\frac{\vec{k}'^2 k_o R^-}{16\sqrt{2}\,\pi W m'}$	$-\frac{\vec{k}'^2 k_o(W+M)R^-}{16\sqrt{2}\,\pi W m'}$
T_9	$-\frac{(W-M)(E'+M)R^-}{8\sqrt{2}\,\pi W m'}$	$\frac{k_o'(E'+M)R^-}{8\sqrt{2}\,\pi m'}$	$-\frac{\vec{k}'^2 k_o R^-}{16\sqrt{2}\,\pi W m'}$	$\frac{\vec{k}'^2 k_o(W+M)R^-}{16\sqrt{2}\,\pi W m'}$

TABLE 11 (Continued)

	q_{15}	q_{16}	q_{17}
T_1	$-\frac{(E+M)(E'+M)R^-}{32\pi W}$	$-\frac{(W-M)(E+M)(E'+M)R^-}{32\pi W}$	$\frac{(E+M)(E'+M)R^-}{32\pi W}$
T_2	$\frac{(E-M)(E'-M)R^-}{32\pi W}$	$-\frac{(W+M)(E-M)(E'-M)R^+}{32\pi W}$	$-\frac{(E-M)(E'-M)R^+}{32\pi W}$
T_3	$\frac{(W+E)(W+E')(E+M)(E'+M)R^-}{16\pi W m m'}$	$\frac{(W+E)(W+E')(W-M)(E+M)(E'+M)R^-}{16\pi W m m'}$	$\frac{k_o(W+E')(E+M)(E'+M)R^-}{16\pi W m m'}$
T_4	$\frac{(E+M)(E'+M)R^-}{32\pi W}$	$\frac{(W-M)(E+M)(E'+M)R^-}{32\pi W}$	$-\frac{(E+M)(E'+M)R^-}{32\pi W}$
T_5	$\frac{(E+M)(E'+M)R^-}{32\pi W}$	$\frac{(W-M)(E+M)(E'+M)R^-}{32\pi W}$	$-\frac{(E+M)(E'+M)R^-}{32\pi W}$
T_6	$-\frac{\vec{k}^2(W+E')R^-}{16\sqrt{2}\,\pi W m}$	$\frac{\vec{k}^2(W+E')(W+M)R^-}{16\sqrt{2}\,\pi W m}$	$\frac{\vec{k}^2(W+E')R^-}{16\sqrt{2}\,\pi W m}$
T_7	$\frac{\vec{k}^2(W+E')R^+}{16\sqrt{2}\,\pi W m}$	$\frac{\vec{k}^2(W+E')(W-M)R^+}{16\sqrt{2}\,\pi W m}$	$-\frac{\vec{k}^2(W+E')R^+}{16\sqrt{2}\,\pi W m}$
T_8	$-\frac{\vec{k}'^2(W+E)R^-}{16\sqrt{2}\,\pi W m'}$	$\frac{\vec{k}'^2(W+E)(W+M)R^-}{16\sqrt{2}\,\pi W m'}$	$-\frac{\vec{k}'^2 k_o R^-}{16\sqrt{2}\,\pi W m'}$
T_9	$\frac{\vec{k}'^2(W+E)R^-}{16\sqrt{2}\,\pi W m'}$	$-\frac{\vec{k}'^2(W+E)(W+M)R^-}{16\sqrt{2}\,\pi W m'}$	$\frac{\vec{k}'^2 k_o R^-}{16\sqrt{2}\,\pi W m'}$

TABLE 11 (Continued)

	a_{18}	a_{19}	a_{20}
T_1	$\frac{(W-M)(E+M)(E'+M)R^-}{32\pi W}$	$\frac{(E+M)(E'+M)R^-}{32\pi W}$	$\frac{(W-M)(E+M)(E'+M)R^-}{32\pi W}$
T_2	$\frac{(W+M)(E-M)(E'-M)R^+}{32\pi W}$	$-\frac{(E-M)(E'-M)R^+}{32\pi W}$	$\frac{(W+M)(E-M)(E'-M)R^+}{32\pi W}$
T_3	$\frac{k_o(W+E')(W-M)(E+M)(E'+M)R^-}{16\pi W m m'}$	$\frac{k_o'(W+E)(E+M)(E'+M)R^-}{16\pi W m m'}$	$\frac{k_o'(W+E)(W-M)(E+M)(E'+M)R^-}{16\pi W m m'}$
T_4	$-\frac{(W-M)(E+M)(E'+M)R^-}{32\pi W}$	$-\frac{(E+M)(E'+M)R^-}{32\pi W}$	$-\frac{(W-M)(E+M)(E'+M)R^-}{32\pi W}$
T_5	$-\frac{(W-M)(E+M)(E'+M)R^-}{32\pi W}$	$-\frac{(E+M)(E'+M)R^-}{32\pi W}$	$-\frac{(W-M)(E+M)(E'+M)R^-}{32\pi W}$
T_6	$-\frac{\vec{k}^2(W+E')(W+M)R^-}{16\sqrt{2}\,\pi W m}$	$-\frac{\vec{k}^2 k_o' R^-}{16\sqrt{2}\,\pi W m}$	$\frac{\vec{k}^2 k_o'(W+M)R^-}{16\sqrt{2}\,\pi W m}$
T_7	$-\frac{\vec{k}^2(W+E')(W-M)R^+}{16\sqrt{2}\,\pi W m}$	$\frac{\vec{k}^2 k_o' R^+}{16\sqrt{2}\,\pi W m}$	$\frac{\vec{k}^2 k_o'(W-M)R^+}{16\sqrt{2}\,\pi W m}$
T_8	$\frac{\vec{k}'^2 k_o(W+M)R^-}{16\sqrt{2}\,\pi W m'}$	$\frac{\vec{k}'^2(W+E)R^-}{16\sqrt{2}\,\pi W m'}$	$-\frac{\vec{k}'^2(W+E)(W+M)R^-}{16\sqrt{2}\,\pi W m'}$
T_9	$-\frac{\vec{k}'^2 k_o(W+M)R^-}{16\sqrt{2}\,\pi W m'}$	$-\frac{\vec{k}'^2(W+E)R^-}{16\sqrt{2}\,\pi W m'}$	$\frac{\vec{k}'^2(W+E)(W+M)R^-}{16\sqrt{2}\,\pi W m'}$

TABLE 11 (Continued)

	G_1	G_2	G_3
T_{10}	$-\frac{R^+}{8\pi W}$	$-\frac{(W-M)R^+}{8\pi W}$	$\frac{R^+}{4\pi W}$
T_{11}	$\frac{R^-}{8\pi W}$	$-\frac{(W+M)R^-}{8\pi W}$	$\frac{R^-}{4\pi W}$
T_{12}	$-\frac{(E-M)(E'-M)R^+}{4\pi W m m'}$	$\frac{(W+M)(E-M)(E'-M)R^+}{4\pi W m m'}$	$\frac{[(E-M)(E'-M-k_o')-k_o(W-M)]R^+}{4\pi W m m'}$
T_{13}	$-\frac{R^+}{8\pi W}$	$-\frac{(W-M)R^+}{8\pi W}$	$-\frac{R^+}{4\pi W}$
T_{14}	$-\frac{R^+}{8\pi W}$	$-\frac{(W-M)R^+}{8\pi W}$	$-\frac{R^+}{4\pi W}$
T_{15}	$-\frac{k_o R^+}{4\sqrt{2}\pi W m}$	$-\frac{k_o(W-M)R^+}{4\sqrt{2}\pi W m}$	$\frac{(E-M)R^+}{2\sqrt{2}\pi W m}$
T_{16}	$\frac{k_o R^-}{4\sqrt{2}\pi W m}$	$-\frac{k_o(W+M)R^-}{4\sqrt{2}\pi W m}$	$\frac{k_o R^-}{2\sqrt{2}\pi W m}$
T_{17}	$-\frac{k_o' R^+}{4\sqrt{2}\pi W m'}$	$-\frac{k_o'(W+M)R^+}{4\sqrt{2}\pi W m'}$	$\frac{(E'-M)R^+}{2\sqrt{2}\pi W m'}$
T_{18}	$\frac{k_o' R^+}{4\sqrt{2}\pi W m'}$	$\frac{k_o'(W+M)R^+}{4\sqrt{2}\pi W m'}$	$\frac{k_o' R^+}{2\sqrt{2}\pi W m'}$

TABLE 11 (Continued)

	G_4	G_5	G_6
T_{10}	$\frac{(E-M-k_o')R^+}{4\pi W}$	$-\frac{(E-M)R^+}{8\pi W}$	$-\frac{(E-M)(W+M+k_o')R^+}{16\pi W}$
T_{11}	$\frac{(W+M)R^-}{4\pi W}$	-	$-\frac{(E+M)(E'+M)R^-}{16\pi W}$
T_{12}	$\frac{[m^2(E'-M)+(W-M)(Wk_o'+Mk_o)-2Mk_ok_o']R^+}{4\pi W mm'}$	$\frac{k_o(E'-M)(E-M-k_o)R^+}{8\pi W mm'}$	$-\frac{k_o(E'-M)[k_o'(E-M)-Wk_o]R^+}{8\pi W mm'}$
T_{13}	$\frac{k_o' R^+}{4\pi W}$	-	$-\frac{(E-M)(E'-M)R^+}{16\pi W}$
T_{14}	$\frac{k_o R^+}{4\pi W}$	$\frac{(E-M)R^+}{8\pi W}$	$\frac{(E-M)(W+M+k_o')R^+}{16\pi W}$
T_{15}	$-\frac{(E-M)(E'-M-k_o')R^+}{4\sqrt{2}\,\pi W m}$	$-\frac{(E-M)(E+M-k_o)R^+}{8\sqrt{2}\,\pi W m}$	$-\frac{(E-M)[k_o'(E+M)-Wk_o]R^+}{8\sqrt{2}\,\pi W m}$
T_{16}	$\frac{[m^2+k_o(W+M)]R^-}{4\sqrt{2}\,\pi W m}$	$-\frac{(E+M)(E-M-k_o)R^-}{8\sqrt{2}\,\pi W m}$	$\frac{(E+M)[k_o'(E-M)-Wk_o]R^-}{8\sqrt{2}\,\pi W m}$
T_{17}	$-\frac{(E'-M)(E-M-k_o)R^+}{4\sqrt{2}\,\pi W m'}$	$-\frac{k_o(E'-M)R^+}{4\sqrt{2}\,\pi W m'}$	$\frac{k_o(E'-M)(W-M+k_o)R^+}{8\sqrt{2}\,\pi W m'}$
T_{18}	$-\frac{[m'^2+k_o'(W-M)]R^+}{4\sqrt{2}\,\pi W m'}$	-	$\frac{\vec{k}'^2 k_o R^+}{8\sqrt{2}\,\pi W m'}$

TABLE 11 (Continued)

	$\mathcal{G}_7$	$\mathcal{G}_8$	$\mathcal{G}_9$	$\mathcal{G}_{10}$
T_{10}	$\frac{(E'-M)R^-}{8\pi W}$	$\frac{(E'-M)R^+}{8\pi}$	$\frac{(E-M)R^+}{8\pi W}$	$\frac{(E+M)(W+M+k_o')R^+}{16\pi W}$
T_{11}	-	-	-	$\frac{(E+M)(E'+M)R^-}{16\pi W}$
T_{12}	$\frac{k_o'(W-M)(E-M)R^+}{8\pi W m m'}$	$-\frac{k_o'^2(E-M)R^+}{8\pi m m'}$	$\frac{(W+E)(E'-M)(E-M-k_o)R^+}{8\pi W m m'}$	$-\frac{(W+E)(E'-M)[k_o'(E-M)-Wk_o]R^+}{8\pi W m m'}$
T_{13}	$-\frac{(E'-M)R^+}{8\pi W}$	$-\frac{(E'-M)R^+}{8\pi}$	-	$\frac{(E-M)(E'-M)R^+}{16\pi W}$
T_{14}	-	-	$-\frac{(E-M)R^+}{8\pi W}$	$-\frac{(E-M)(W+M+k_o')R^+}{16\pi W}$
T_{15}	$\frac{k_o'(E-M)R^+}{4\sqrt{2}\,\pi W m}$	$-\frac{k_o'(E-M)R^+}{4\sqrt{2}\,\pi W m}$	$\frac{(E-M)(E+M-k_o)R^+}{8\sqrt{2}\,\pi W m}$	$\frac{(E-M)[k_o'(E+M)-Wk_o]R^+}{8\sqrt{2}\,\pi W m}$
T_{16}	-	-	$\frac{(E+M)(E-M-k_o)R^-}{8\sqrt{2}\,\pi W m}$	$-\frac{(E+M)[k_o'(E-M)-Wk_o]R^-}{8\sqrt{2}\,\pi W m}$
T_{17}	$-\frac{(W+M)(E'-M)R^+}{8\sqrt{2}\,\pi W m'}$	$-\frac{k_o'(E'-M)R^+}{8\sqrt{2}\,\pi W m'}$	$-\frac{(W+E)(E'-M)R^+}{4\sqrt{2}\,\pi W m'}$	$\frac{(W+E)(E'-M)(W-M+k_o')R^+}{8\sqrt{2}\,\pi W m'}$
T_{18}	$\frac{(W+M)(E'-M)R^+}{8\sqrt{2}\,\pi W m'}$	$\frac{k_o'(E'-M)R^+}{8\sqrt{2}\,\pi W m'}$	-	$\frac{\bar{k}'^2(W+E)R^+}{8\sqrt{2}\,\pi W m'}$

TABLE 11 (Continued)

	a_{11}	a_{12}	a_{13}	a_{14}
T_{10}	$-\frac{(E'-M)R^+}{8\pi W}$	$-\frac{(E'-M)R^+}{8\pi}$	$\frac{(E-M)(E'-M)R^+}{32\pi W}$	$-\frac{(W+M)(E-M)(E'-M)R^+}{32\pi W}$
T_{11}	-	-	$-\frac{(E+M)(E'+M)R^-}{32\pi W}$	$-\frac{(W-M)(E+M)(E'+M)R^-}{32\pi W}$
T_{12}	$\frac{(W+E')(W-M)(E-M)R^+}{8\pi W m m'}$	$-\frac{k'_o(W+E')(E-M)R^+}{8\pi m m'}$	$-\frac{k_o k'_o(E-M)(E'-M)R^+}{16\pi W m m'}$	$\frac{k_o k'_o(W+M)(E-M)(E'-M)R^+}{16\pi W m m'}$
T_{13}	$\frac{(E'-M)R^+}{8\pi W}$	$\frac{(E'-M)R^+}{8\pi}$	$-\frac{(E-M)(E'-M)R^-}{32\pi W}$	$\frac{(W+M)(E-M)(E'-M)R^+}{32\pi W}$
T_{14}	-	-	$-\frac{(E-M)(E'-M)R^-}{32\pi W}$	$\frac{(W+M)(E-M)(E'-M)R^+}{32\pi W}$
T_{15}	$\frac{(W+E')(E-M)R^+}{4\sqrt{2}\,\pi W m}$	$-\frac{(W+E')(E-M)R^+}{4\sqrt{2}\,\pi m}$	$-\frac{\vec{k}^2 k'_o R^+}{16\sqrt{2}\,\pi W m}$	$-\frac{\vec{k}^2 k'_o (W-M)R^+}{16\sqrt{2}\,\pi W m}$
T_{16}	-	-	$\frac{\vec{k}^2 k'_o R^-}{16\sqrt{2}\,\pi W m}$	$-\frac{\vec{k}^2 k'_o (W-M)R^+}{16\sqrt{2}\,\pi W m}$
T_{17}	$\frac{(W+M)(E'-M)R^+}{8\sqrt{2}\,\pi W m'}$	$\frac{k'_o(E'-M)R^+}{8\sqrt{2}\,\pi m'}$	$-\frac{\vec{k}'^2 k_o R^+}{16\sqrt{2}\,\pi W m'}$	$-\frac{\vec{k}'^2 k_o (W-M)R^+}{16\sqrt{2}\,\pi W m'}$
T_{18}	$-\frac{(W+M)(E'-M)R^+}{8\sqrt{2}\,\pi W m'}$	$-\frac{k'_o(E'-M)R^+}{8\sqrt{2}\,\pi m'}$	$\frac{\vec{k}'^2 k_o R^+}{16\sqrt{2}\,\pi W m'}$	$\frac{\vec{k}'^2 k_o (W-M)R^+}{16\sqrt{2}\,\pi W m'}$

TABLE 11 (Continued)

	a_{15}	a_{16}	a_{17}
T_{10}	$\frac{(E-M)(E'-M)R^+}{32\pi W}$	$-\frac{(W+M)(E-M)(E'-M)R^+}{32\pi W}$	$-\frac{(E-M)(E'-M)R^+}{32\pi W}$
T_{11}	$-\frac{(E+M)(E'+M)R^-}{32\pi W}$	$-\frac{(W-M)(E+M)(E'+M)R^-}{32\pi W}$	$\frac{(E+M)(E'+M)R^-}{32\pi W}$
T_{12}	$-\frac{(W+E)(W+E')(E-M)(E'-M)R^+}{16\pi W m m'}$	$\frac{(W+E)(W+E')(W+M)(E-M)(E'-M)R^+}{16\pi W m m'}$	$-\frac{k_o(W+E')(E-M)(E'-M)R^+}{16\pi W m m'}$
T_{13}	$-\frac{(E-M)(E'-M)R^+}{32\pi W}$	$\frac{(W+M)(E-M)(E'-M)R^+}{32\pi W}$	$\frac{(E-M)(E'-M)R^+}{32\pi W}$
T_{14}	$-\frac{(E-M)(E'-M)R^+}{32\pi W}$	$\frac{(W+M)(E-M)(E'-M)R^+}{32\pi W}$	$\frac{(E-M)(E'-M)R^+}{32\pi W}$
T_{15}	$\frac{\vec{k}^2(W+E')R^+}{16\sqrt{2}\,\pi W m}$	$\frac{\vec{k}^2(W+E')(W-M)R^+}{16\sqrt{2}\,\pi W m}$	$-\frac{\vec{k}^2(W+E')R^+}{16\sqrt{2}\,\pi W m}$
T_{16}	$-\frac{\vec{k}^2(W+E')R^-}{16\sqrt{2}\,\pi W m}$	$\frac{\vec{k}^2(W+E')(W+M)R^-}{16\sqrt{2}\,\pi W m}$	$\frac{\vec{k}^2(W+E')R^-}{16\sqrt{2}\,\pi W m}$
T_{17}	$\frac{\vec{k}'^2(W+E)R^+}{16\sqrt{2}\,\pi W m'}$	$\frac{\vec{k}'^2(W+E)(W-M)R^+}{16\sqrt{2}\,\pi W m'}$	$\frac{\vec{k}'^2 k_o R^+}{16\sqrt{2}\,\pi W m'}$
T_{18}	$-\frac{\vec{k}'^2(W+E)R^+}{16\sqrt{2}\,\pi W m'}$	$-\frac{\vec{k}'^2(W+E)(W-M)R^+}{16\sqrt{2}\,\pi W m'}$	$-\frac{\vec{k}'^2 k_o R^+}{16\sqrt{2}\,\pi W m'}$

TABLE 11 (Continued)

	G_{18}	G_{19}	G_{20}
T_{10}	$\frac{(W+M)(E-M)(E'-M)R^{+}}{32\pi W}$	$-\frac{(E-M)(E'-M)R^{+}}{32\pi W}$	$\frac{(W+M)(E-M)(E'-M)R^{+}}{32\pi W}$
T_{11}	$\frac{(W-M)(E+M)(E'+M)R^{-}}{32\pi W}$	$\frac{(E+M)(E'+M)R^{-}}{32\pi W}$	$\frac{(W-M)(E+M)(E'+M)R^{-}}{32\pi W}$
T_{12}	$\frac{k_o(W+E')(W+M)(E-M)(E'-M)R^{+}}{16\pi W m m'}$	$-\frac{k_o'(W+E)(E-M)(E'-M)R^{+}}{16\pi W m m'}$	$\frac{k_o'(W+E)(W+M)(E-M)(E'-M)R^{+}}{16\pi W m m'}$
T_{13}	$-\frac{(W+M)(E-M)(E'-M)R^{+}}{32\pi W}$	$\frac{(E-M)(E'-M)R^{+}}{32\pi W}$	$-\frac{(W+M)(E-M)(E'-M)R^{+}}{32\pi W}$
T_{14}	$-\frac{(W+M)(E-M)(E'-M)R^{+}}{32\pi W}$	$\frac{(E-M)(E'-M)R^{+}}{32\pi W}$	$-\frac{(W+M)(E-M)(E'-M)R^{+}}{32\pi W}$
T_{15}	$-\frac{\vec{k}^{2}(W+E')(W-M)R^{+}}{16\sqrt{2}\,\pi W m}$	$\frac{\vec{k}^{2}k_o'R^{+}}{16\sqrt{2}\,\pi W m}$	$\frac{\vec{k}^{2}k_o'(W-M)R^{+}}{16\sqrt{2}\,\pi W m}$
T_{16}	$-\frac{\vec{k}^{2}(W+E')(W+M)R^{-}}{16\sqrt{2}\,\pi W m}$	$-\frac{\vec{k}^{2}k_o'R^{-}}{16\sqrt{2}\,\pi W m}$	$\frac{\vec{k}^{2}k_o'(W+M)R^{+}}{16\sqrt{2}\,\pi W m}$
T_{17}	$\frac{\vec{k}'^{2}k_o(W-M)R^{+}}{16\sqrt{2}\,\pi W m'}$	$-\frac{\vec{k}'^{2}(W+E)R^{+}}{16\sqrt{2}\,\pi W m'}$	$-\frac{\vec{k}'^{2}(W+E)(W-M)R^{+}}{16\sqrt{2}\,\pi W m'}$
T_{18}	$-\frac{\vec{k}'k_o(W-M)R^{+}}{16\sqrt{2}\,\pi W m'}$	$\frac{\vec{k}'^{2}(W+E)R^{+}}{16\sqrt{2}\,\pi W m'}$	$\frac{\vec{k}'^{2}(W+E)(W-M)R^{+}}{16\sqrt{2}\,\pi W m'}$

TABLE 12

	G_1	G_2	G_3	G_4	G_5
T_1	$-\frac{R^+}{8\pi W}$	$-\frac{(W-M)R^+}{8\pi W}$	$-\frac{R^+}{4\pi W}$	$-\frac{(E-M-k'_o)R^2}{4\pi W}$	$\frac{(E-M)R^+}{8\pi W}$
T_2	-	-	-	-	-
T_3	$-\frac{k_o k'_o R^+}{4\pi W m m'}$	$-\frac{k_o k'_o (W-M) R^+}{4\pi W m m'}$	$-\frac{k_o k'_o R^+}{2\pi W m m'}$	$\frac{(m^2 k'_o + m'^2 k_o) R^+}{4\pi W m m'}$	$-\frac{k'_o (E-M)(E+M-k_o) R^+}{8\pi W m m'}$
T_4	-	-	-	-	-
T_5	-	-	-	-	-
T_6	$-\frac{k'_o R^+}{4\sqrt{2}\,\pi W m}$	$-\frac{k_o (W-M) R^+}{4\sqrt{2}\,\pi W m}$	$-\frac{k_o R^+}{2\sqrt{2}\,\pi W m}$	$-\frac{[\vec{k}^2 + k_o(E-M-2k'_o)]\,R^+}{4\sqrt{2}\,\pi W m}$	$-\frac{(E-M)(E+M-k_o) R^+}{8\sqrt{2}\,\pi W m}$
T_7	-	-	-	-	-
T_8	$-\frac{k'_o R^+}{4\sqrt{2}\,\pi W m'}$	$-\frac{k'_o (W-M) R^+}{4\sqrt{2}\,\pi W m'}$	$-\frac{k'_o R^+}{2\sqrt{2}\,\pi W m'}$	$-\frac{[\vec{k}'^2 + k'_o(E'-M-2k_o)]\,R^+}{4\sqrt{2}\,\pi W m'}$	$\frac{k'_o (E-M) R^+}{4\sqrt{2}\,\pi W m'}$
T_9	-	-	-	-	-

TABLE 12 (Continued)

	a_6	a_7	a_8	a_9
T_1	$\frac{(E-M)R^+}{8\pi}$	$-\frac{(E'-M)R^+}{8\pi W}$	$-\frac{(E'-M)R^+}{8\pi}$	$-\frac{(E-M)R^+}{8\pi W}$
T_2	$\frac{(E+M)(E'+M)R^-}{16\pi W}$	-	-	-
T_3	$\frac{[W(E-M)k_o k'_o - \vec{k}^2 k'^2_o - \vec{k}'^2 k_o^2] R^+}{8\pi W m m'}$	$-\frac{k_o(W+M)(E'-M)R^+}{8\pi W m m'}$	$-\frac{k_o k'_o (E'-M)R^+}{8\pi m m'}$	$\frac{k'_o(E-M)(E+M-k_o)R^+}{8\pi W m m'}$
T_4	$-\frac{(E-M)(E'-M)R^+}{16\pi W}$	-	-	-
T_5	$-\frac{(E-M)(E'-M)R^+}{16\pi W}$	-	-	-
T_6	$\frac{[k_o(k'_o+M)-k'_o(E+M)](E-M)R^+}{8\sqrt{2}\,\pi W m}$	$-\frac{k_o(E'-M)R^+}{4\sqrt{2}\,\pi W m}$	$-\frac{k_o(E'-M)R^+}{4\sqrt{2}\,\pi m}$	$\frac{(E-M)(E+M)-k_o)R^+}{8\sqrt{2}\,\pi W m}$
T_7	$\frac{k_o(E+M)(E'+M)R^-}{8\sqrt{2}\,\pi W m}$	-	-	-
T_8	$\frac{[k'_o(W+M+k'_o)(E-M)-\vec{k}'^2 k_o]R^-}{8\sqrt{2}\,\pi W m'}$	$-\frac{(W+M)(E'-M)R^-}{8\sqrt{2}\,\pi W m'}$	$-\frac{k'_o(E'-M)R^-}{8\sqrt{2}\,\pi m'}$	$-\frac{k'_o(E-M)R^-}{4\sqrt{2}\,\pi W m'}$
T_9	$\frac{k'_o(E-M)(E'-M)R^+}{8\sqrt{2}\,\pi W m'}$	-	-	-

TABLE 12 (Continued)

	a_{10}	a_{11}	a_{12}
T_1	$-\frac{(E-M)R^+}{8\pi}$	$\frac{(E'-M)R^+}{8\pi W}$	$\frac{(E'-M)R^+}{8\pi}$
T_2	$-\frac{(E+M)(E'+M)R^-}{16\pi W}$	-	-
T_3	$\frac{\{k_o'(E-M)[k_o'(E+M)-Wk_o]-\bar{k}'^2k_o(W+E)\}R^+}{8\pi Wmm'}$	$\frac{k_o(W+M)(E'-M)R^+}{8\pi Wmm'}$	$\frac{k_ok_o'(E'-M)R^+}{8\pi mm'}$
T_4	$\frac{(E-M)(E'-M)R^+}{16\pi W}$	-	-
T_5	$\frac{(E-M)(E'-M)R^+}{16\pi W}$	-	-
T_6	$-\frac{[k_o(k_o'+M)-k_o'(E+M)](E-M)R^+}{8\sqrt{2}\,\pi Wm}$	$\frac{k_o(E'-M)R^+}{4\sqrt{2}\,\pi Wm}$	$\frac{k_o(E'-M)R^+}{4\sqrt{2}\,\pi m}$
T_7	$-\frac{k_o(E+M)(E'+M)R^-}{8\sqrt{2}\,\pi Wm}$	-	-
T_8	$-\frac{[\bar{k}^2(W+E)+k_o'(E-M)(W+M+k_o')]R^+}{8\sqrt{2}\,\pi Wm'}$	$\frac{(W+M)(E'-M)R^+}{8\sqrt{2}\,\pi Wm'}$	$\frac{k_o'(E'-M)R^+}{8\sqrt{2}\,\pi m'}$
T_9	$-\frac{k_o'(E-M)(E'-M)R^+}{8\sqrt{2}\,\pi Wm'}$	-	-

TABLE 12 (Continued)

	q_{13}	q_{14}
T_1	-	-
T_2	$\frac{(E+M)(E'+M)R^-}{32\pi W}$	$\frac{(W-M)(E+M)(E'+M)R^-}{32\pi W}$
T_3	$-\frac{(\vec{k}'^2 k_o'^2+\vec{k}'^2 k_o^2)R^+}{16\pi W m m'}$	$-\frac{(\vec{k}^2 k_o'^2+\vec{k}'^2 k_o^2)(W-M)R^+}{16\pi W m m'}$
T_4	$-\frac{(E-M)(E'-M)R^+}{32\pi W}$	$\frac{(W+M)(E-M)(E'-M)R^+}{32\pi W}$
T_5	$-\frac{(E-M)(E'-M)R^+}{32\pi W}$	$\frac{(W+M)(E-M)(E'-M)R^+}{32\pi W}$
T_6	$-\frac{[k_o(E'-M)+k_o'(E+M)](E-M)R^+}{16\sqrt{2}\,\pi W m}$	$\frac{[k_o(W+M)(E'-M)-k_o'(W-M)(E+M)](E-M)R^+}{16\sqrt{2}\,\pi W m}$
T_7	$\frac{k_o(E+M)(E'+M)R^-}{16\sqrt{2}\,\pi W m}$	$\frac{k_o(W-M)(E+M)(E'+M)R^-}{16\sqrt{2}\,\pi W m}$
T_8	$-\frac{[k_o(E'+M)+k_o'(E-M)](E'-M)R^+}{16\sqrt{2}\,\pi W m'}$	$\frac{[k_o(W-M)(E'+M)-k_o'(W+M)(E-M)](E'-M)R^+}{16\sqrt{2}\,\pi W m'}$
T_9	$\frac{k_o'(E-M)(E'-M)R^+}{16\sqrt{2}\,\pi W m'}$	$-\frac{k_o'(W+M)(E-M)(E'-M)R^+}{16\sqrt{2}\,\pi W m'}$

TABLE 12 (Continued)

	a_{15}	a_{16}
T_1	-	-
T_2	$\frac{(E+M)(E'+M)R^-}{32\pi W}$	$\frac{(W-M)(E+M)(E'+M)R^-}{32\pi W}$
T_3	$\frac{[\vec{k}^2 k_o'(W+E')+\vec{k}'^2(W+E)]R^+}{16\pi W m m'}$	$\frac{[\vec{k}^2 k_o'(W+E')+\vec{k}'^2 k_o(W+E)](W-M)R^+}{16\pi W m m'}$
T_4	$-\frac{(E-M)(E'-M)R^+}{32\pi W}$	$\frac{(W+M)(E-M)(E'-M)R^+}{32\pi W}$
T_5	$-\frac{(E-M)(E'-M)R^+}{32\pi W}$	$\frac{(W+M)(E-M)(E'-M)R^+}{32\pi W}$
T_6	$-\frac{[k_o(E'-M)-(W+E')(E+M)](E-M)R^+}{16\sqrt{2}\,\pi W m}$	$\frac{[k_o(W+M)(E'-M)+(W+E')(W-M)](E-M)R^+}{16\sqrt{2}\,\pi W m}$
T_7	$\frac{k_o(E+M)(E'+M)R^-}{16\sqrt{2}\,\pi W m}$	$\frac{k_o(W-M)(E+M)(E'+M)R^-}{16\sqrt{2}\,\pi W m}$
T_8	$\frac{[(W+E)(E'+M)-k_o'(E-M)](E'-M)R^+}{16\sqrt{2}\,\pi W m'}$	$\frac{[(W+E)(W-M)(E'+M)+k_o'(W+M)(E-M)](E'-M)R^+}{16\sqrt{2}\,\pi W m'}$
T_9	$\frac{k_o'(E-M)(E'-M)R^+}{16\sqrt{2}\,\pi W m'}$	$-\frac{k_o'(W+M)(E-M)(E'-M)R^+}{16\sqrt{2}\,\pi W m'}$

TABLE 12 (Continued)

	$\mathcal{Q}_{17}$	$\mathcal{Q}_{18}$
T_1	-	-
T_2	$-\frac{(E+M)(E'+M)R^-}{32\pi W}$	$-\frac{(W-M)(E+M)(E'+M)R^-}{32\pi W}$
T_3	$-\frac{[\vec{k}^2 k_o'(W+E')-\vec{k}'^2 k_o^2]R^+}{16\pi W m m'}$	$-\frac{[\vec{k}^2 k_o'(W+E')-\vec{k}'^2 k_o^2](W-M)R^+}{16\pi W m m'}$
T_4	$\frac{(E-M)(E'-M)R^+}{32\pi W}$	$-\frac{(W+M)(E-M)(E'-M)R^+}{32\pi W}$
T_5	$\frac{(E-M)(E'-M)R^+}{32\pi W}$	$-\frac{(W+M)(E-M)(E'-M)R^+}{32\pi W}$
T_6	$\frac{[k_o(E'-M)-(W+E')(E+M)](E-M)R^+}{16\sqrt{2}\,\pi W m}$	$-\frac{[k_o(W+M)(E'-M)+(W+E')(W-M)](E-M)R^+}{16\sqrt{2}\,\pi W m}$
T_7	$-\frac{k_o(E+M)(E'+M)R^-}{16\sqrt{2}\,\pi W m}$	$-\frac{k_o(W-M)(E+M)(E'+M)R^-}{16\sqrt{2}\,\pi W m}$
T_8	$\frac{[k_o(E'+M)+k_o'(E-M)](E'-M)R^+}{16\sqrt{2}\,\pi W m'}$	$\frac{[k_o(W-M)(E'+M)-k_o'(W+M)(E-M)](E'-M)R^+}{16\sqrt{2}\,\pi W m'}$
T_9	$-\frac{k_o'(E-M)(E'-M)R^+}{16\sqrt{2}\,\pi W m'}$	$\frac{k_o'(W+M)(E-M)(E'-M)R^+}{16\sqrt{2}\,\pi W m'}$

TABLE 12 (Continued)

	α_{19}	α_{20}
T_1	-	-
T_2	$-\frac{(E+M)(E'+M)R^-}{32\pi W}$	$-\frac{(W-M)(E+M)(E'+M)R^-}{32\pi W}$
T_3	$\frac{[\vec{k}^2 k_o'^2 - \vec{k}'^2 k_o(W+E)] R^+}{16\pi W m m'}$	$\frac{[\vec{k}^2 k_o'^2 - \vec{k}'^2 k_o(W+E)](W-M)R^+}{16\pi W m m'}$
T_4	$\frac{(E-M)(E'-M)R^+}{32\pi W}$	$-\frac{(W+M)(E-M)(E'-M)R^+}{32\pi W}$
T_5	$\frac{(E-M)(E'-M)R^+}{32\pi W}$	$-\frac{(W+M)(E-M)(E'-M)R^+}{32\pi W}$
T_6	$\frac{[k_o(E'-M)+k_o'(E+M)](E-M)R^+}{16\sqrt{2}\,\pi W m}$	$-\frac{[k_o(W+M)(E'-M)-k_o'(W-M)(E+M)](E-M)R^+}{16\sqrt{2}\,\pi W m}$
T_7	$-\frac{k_o(E+M)(E'+M)R^-}{16\sqrt{2}\,\pi W m}$	$-\frac{k_o(W-M)(E+M)(E'+M)R^-}{16\sqrt{2}\,\pi W m}$
T_8	$-\frac{[(W+E)(E'+M)-k_o'(E-M)](E'-M)R^+}{16\sqrt{2}\,\pi W m'}$	$-\frac{[(W+E)(W-M)(E'+M)+k_o'(W+M)(E-M)](E'-M)R^+}{16\sqrt{2}\,\pi W m'}$
T_9	$-\frac{k_o'(E-M)(E'-M)R^+}{16\sqrt{2}\,\pi W m'}$	$\frac{k_o'(W+M)(E-M)(E'-M)R^+}{16\sqrt{2}\,\pi W m'}$

TABLE 12 (Continued)

	G_1	G_2	G_3	G_4	G_5
T_{10}	$\frac{R^-}{8\pi W}$	$-\frac{(W+M)R^-}{8\pi W}$	$\frac{R^-}{4\pi W}$	$-\frac{(E+M-k_o')R^-}{4\pi W}$	$\frac{(E+M)R^-}{8\pi W}$
T_{11}	-	-	-	-	-
T_{12}	$\frac{k_o k_o' R^-}{4\pi W m m'}$	$-\frac{k_o k_o'(W+M)R^-}{4\pi W m m'}$	$\frac{k_o k_o' R^-}{2\pi W m m'}$	$\frac{(m^2 k_o' + m'^2 k_o)R^-}{4\pi W m m'}$	$-\frac{k_o'(E+M)(E-M-k_o)R^-}{8\pi W m m'}$
T_{13}	-	-	-	-	-
T_{14}	-	-	-	-	-
T_{15}	$\frac{k_o R^-}{4\sqrt{2}\,\pi W m}$	$-\frac{k_o(W+M)R^-}{4\sqrt{2}\,\pi W m}$	$\frac{k_o R^-}{2\sqrt{2}\,\pi W m}$	$-\frac{[\vec{k}^2 + k_o(E+M-2k_o')]R^-}{4\sqrt{2}\,\pi W m}$	$-\frac{(E+M)(E-M-k_o)R^-}{8\sqrt{2}\,\pi W m}$
T_{16}	-	-	-	-	-
T_{17}	$\frac{k_o' R^-}{4\sqrt{2}\,\pi W m'}$	$-\frac{k_o'(W+M)R^-}{4\sqrt{2}\,\pi W m'}$	$\frac{k_o' R^-}{2\sqrt{2}\,\pi W m'}$	$-\frac{[\vec{k}'^2 + k_o'(E'+M-2k_o)]R^-}{4\sqrt{2}\,\pi W m'}$	$\frac{k_o'(E+M)R^-}{4\sqrt{2}\,\pi W m'}$
T_{18}	-	-	-	-	-

TABLE 12 (Continued)

	G_6	G_7	G_8	G_9
T_{10}	$-\frac{(E+M)R^-}{8\pi}$	$-\frac{(E'+M)R^-}{8\pi W}$	$\frac{(E'+M)R^-}{8\pi}$	$-\frac{(E+M)R^-}{8\pi W}$
T_{11}	$-\frac{(E-M)(E'-M)R^+}{16\pi W}$	-	-	-
T_{12}	$-\frac{[W(E+M)k_o k_o' - \vec{k}^2 k_o'^2 - \vec{k}'^2 k_o^2] R^-}{8\pi W m m'}$	$-\frac{k_o(W-M)(E'+M)R^-}{8\pi W m m'}$	$\frac{k_o k_o'(E'+M)(R^-}{8\pi m m'}$	$\frac{k_o'(E+M)(E-M-k_o)R^-}{8\pi W m m'}$
T_{13}	$\frac{(E+M)(E'+M)R^-}{16\pi W}$	-	-	-
T_{14}	$\frac{(E+M)(E'+M)R^-}{16\pi W}$	-	-	-
T_{15}	$-\frac{[k_o(k_o'-M)-k_o'(E-M)](E+M)R^-}{8\sqrt{2}\,\pi W m}$	$-\frac{k_o(E'+M)R^-}{4\sqrt{2}\,\pi W m}$	$\frac{k_o(E'+M)R^-}{4\sqrt{2}\,\pi m}$	$\frac{(E+M)(E-M-k_o)R^-}{8\sqrt{2}\,\pi W m}$
T_{16}	$-\frac{k_o(E-M)(E'-M)R^+}{8\sqrt{2}\,\pi W m}$	-	-	-
T_{17}	$-\frac{[k_o'(W-M+k_o')-\vec{k}'^2 k_o] R^-}{8\sqrt{2}\,\pi W m'}$	$-\frac{(W-M)(E'+M)R^-}{8\sqrt{2}\,\pi W m'}$	$\frac{k_o'(E'+M)R^-}{8\sqrt{2}\,\pi m'}$	$-\frac{k_o'(E+M)R^-}{4\sqrt{2}\,\pi W m'}$
T_{18}	$-\frac{k_o'(E+M)(E'+M)R^-}{8\sqrt{2}\,\pi W m'}$	-	-	-

TABLE 12 (Continued)

	a_{10}	a_{11}	a_{12}
T_{10}	$\frac{(E+M)R^-}{8\pi}$	$\frac{(E'+M)R^-}{8\pi W}$	$-\frac{(E'+M)R^-}{8\pi}$
T_{11}	$\frac{(E-M)(E'-M)R^+}{16\pi W}$	-	-
T_{12}	$-\frac{\{k_o'(E+M)[k_o'(E-M)-Wk_o]-\vec{k}'^2 k_o(W+E)\} R^-}{8\pi W m m'}$	$\frac{k_o(W-M)(E'+M)R^-}{8\pi W m m'}$	$-\frac{k_o k_o'(E'+M)R^-}{8\pi m m'}$
T_{13}	$-\frac{(E+M)(E'+M)R^-}{16\pi W}$	-	-
T_{14}	$-\frac{(E+M)(E'+M)R^-}{16\pi W}$	-	-
T_{15}	$\frac{[k_o(k_o'-M)-k_o'(E-M)](E+M)R^-}{8\sqrt{2}\,\pi W m}$	$\frac{k_o(E'+M)R^-}{4\sqrt{2}\,\pi W m}$	$-\frac{k_o(E'+M)R^-}{4\sqrt{2}\,\pi m}$
T_{16}	$\frac{k_o(E-M)(E'-M)R^+}{8\sqrt{2}\,\pi W m}$	-	-
T_{17}	$\frac{[\vec{k}'^2(W+E)+k_o'(E+M)(W-M+k_o')] R^-}{8\sqrt{2}\,\pi W m'}$	$\frac{(W-M)(E'+M)R^-}{8\sqrt{2}\,\pi W m'}$	$-\frac{k_o'(E'+M)R^-}{8\sqrt{2}\,\pi W m'}$
T_{18}	$\frac{k_o'(E+M)(E'+M)R^-}{8\sqrt{2}\,\pi W m'}$	-	-

TABLE 12 (Continued)

	a_{13}	a_{14}
T_{10}	-	-
T_{11}	$-\frac{(E-M)(E'-M)R^{+}}{32\pi W}$	$\frac{(W+M)(E-M)(E'-M)R^{+}}{32\pi W}$
T_{12}	$\frac{(\vec{k}^2 k_o'^2+\vec{k}'^2 k_o^2)R^{-}}{16\pi W m m'}$	$-\frac{(\vec{k}^2 k_o'^2+\vec{k}'^2 k_o^2)(W+M)R^{-}}{16\pi W m m'}$
T_{13}	$\frac{(E+M)(E'+M)R^{-}}{32\pi W}$	$\frac{(W-M)(E+M)(E'+M)R^{-}}{32\pi W}$
T_{14}	$\frac{(E+M)(E'+M)R^{-}}{32\pi W}$	$\frac{(W-M)(E+M)(E'+M)R^{-}}{32\pi W}$
T_{15}	$\frac{[k_o(E'+M)+k_o'(E-M)](E+M)R^{-}}{16\sqrt{2}\,\pi W m}$	$\frac{[k_o(W-M)(E'+M)-k_o'(W+M)(E-M)](E+M)R^{-}}{16\sqrt{2}\,\pi W m}$
T_{16}	$-\frac{k_o(E-M)(E'-M)R^{+}}{16\sqrt{2}\,\pi W m}$	$\frac{k_o(W+M)(E-M)(E'-M)R^{+}}{16\sqrt{2}\,\pi W m}$
T_{17}	$\frac{[k_o(E'-M)+k_o'(E+M)](E'+M)R^{-}}{16\sqrt{2}\,\pi W m'}$	$-\frac{[k_o(W+M)(E'-M)-k_o'(W-M)(E+M)](E+M)R^{-}}{16\sqrt{2}\,\pi W m'}$
T_{18}	$-\frac{k_o'(E+M)(E'+M)R^{-}}{16\sqrt{2}\,\pi W m'}$	$-\frac{k_o'(W-M)(E+M)(E'+M)R^{-}}{16\sqrt{2}\,\pi W m'}$

TABLE 12 (Continued)

	a_{15}	a_{16}
T_{10}	-	-
T_{11}	$-\frac{(E-M)(E'-M)R^{+}}{32\pi W}$	$\frac{(W+M)(E-M)(E'-M)R^{+}}{32\pi W}$
T_{12}	$-\frac{[\vec{k}^{2}k_o'(W+E')+\vec{k}'^{2}k_o(W+E)]R^{-}}{16\pi W m m'}$	$\frac{[\vec{k}^{2}k_o'(W+E')+\vec{k}'^{2}k_o(W+E)](W+M)R^{-}}{16\pi W m m'}$
T_{13}	$\frac{(E+M)(E'+M)R^{-}}{32\pi W}$	$\frac{(W-M)(E+M)(E'+M)R^{-}}{32\pi W}$
T_{14}	$\frac{(E+M)(E'+M)R^{-}}{32\pi W}$	$\frac{(W-M)(E+M)(E'+M)R^{-}}{32\pi W}$
T_{15}	$\frac{[k_o(E'+M)-(W+E')(E-M)](E+M)R^{-}}{16\sqrt{2}\,\pi W m}$	$\frac{[k_o(W-M)(E'+M)+(W+E')(W+M)(E-M)](E+M)R^{-}}{16\sqrt{2}\,\pi W m}$
T_{16}	$-\frac{k_o(E-M)(E'-M)R^{+}}{16\sqrt{2}\,\pi W m}$	$\frac{k_o(W+M)(E-M)(E'-M)R^{+}}{16\sqrt{2}\,\pi W m}$
T_{17}	$-\frac{[(W+E)(E'-M)-k_o'(E+M)](E'+M)R^{-}}{16\sqrt{2}\,\pi W m'}$	$\frac{[(W+E)(W+M)(E'-M)+k_o'(W-M)(E+M)](E'+M)R^{-}}{16\sqrt{2}\,\pi W m'}$
T_{18}	$-\frac{k_o'(E+M)(E'+M)R^{-}}{16\sqrt{2}\,\pi W m'}$	$-\frac{k_o'(W-M)(E+M)(E'+M)R^{-}}{16\sqrt{2}\,\pi W m'}$

TABLE 2 (Continued)

	q_{17}	q_{18}
T_{10}	-	-
T_{11}	$\frac{(E-M)(E'-M)R^{+}}{32\pi W}$	$-\frac{(W+M)(E-M)(E'-M)R^{+}}{32\pi W}$
T_{12}	$\frac{[\vec{k}^{2}k_o'(W+E')-\vec{k}'^{2}k_o^{2}]R^{-}}{16\pi W mm'}$	$-\frac{[\vec{k}^{2}k_o'(W+E')-\vec{k}'^{2}k_o^{2}](W+M)R^{-}}{16\pi W mm'}$
T_{13}	$-\frac{(E+M)(E'+M)R^{-}}{32\pi W}$	$-\frac{(W-M)(E+M)(E'+M)R^{-}}{32\pi W}$
T_{14}	$-\frac{(E+M)(E'+M)R^{-}}{32\pi W}$	$-\frac{(W-M)(E+M)(E'+M)R^{-}}{32\pi W}$
T_{15}	$-\frac{[k_o(E'+M)-(W+E')(E-M)](E+M)R^{-}}{16\sqrt{2}\,\pi W m}$	$-\frac{[k_o(W-M)(E'+M)+(W+E')(W+M)(E-M)](E+M)R^{-}}{16\sqrt{2}\,\pi W m}$
T_{16}	$\frac{k_o(E-M)(E'-M)R^{+}}{16\sqrt{2}\,\pi W m}$	$-\frac{k_o(W+M)(E-M)(E'-M)R^{+}}{16\sqrt{2}\,\pi W m}$
T_{17}	$-\frac{[k_o(E'-M)+k_o'(E+M)](E'+M)R^{-}}{16\sqrt{2}\,\pi W m'}$	$\frac{[k_o(W+M)(E'-M)-k_o'(W-M)(E+M)](E'+M)R^{-}}{16\sqrt{2}\,\pi W m'}$
T_{18}	$\frac{k_o'(E+M)(E'+M)R^{-}}{16\sqrt{2}\,\pi W m'}$	$\frac{k_o'(W-M)(E+M)(E'+M)R^{-}}{16\sqrt{2}\,\pi W m'}$

TABLE 12 (Continued)

	$\mathcal{G}_{19}$	$\mathcal{G}_{20}$
T_{10}	-	-
T_{11}	$\frac{(E-M)(E'-M)R^{+}}{32\pi W}$	$-\frac{(W+M)(E-M)(E'-M)R^{+}}{32\pi W}$
T_{12}	$-\frac{[\vec{k}^{2}k_{o}'^{2}-\vec{k}'^{2}k_{o}(W+E)]R^{-}}{16\pi W m m'}$	$\frac{[\vec{k}^{2}k_{o}'^{2}-\vec{k}'^{2}k_{o}(W+E)](W+M)R^{-}}{16\pi W m m'}$
T_{13}	$-\frac{(E+M)(E'+M)R^{-}}{32\pi W}$	$-\frac{(W-M)(E+M)(E'+M)R^{-}}{32\pi W}$
T_{14}	$-\frac{(E+M)(E'+M)R^{-}}{32\pi W}$	$-\frac{(W-M)(E+M)(E'+M)R^{-}}{32\pi W}$
T_{15}	$-\frac{[k_{o}(E'+M)+k_{o}'(E-M)](E+M)R^{-}}{16\sqrt{2}\,\pi W m}$	$-\frac{[k_{o}(W-M)(E'+M)-k_{o}'(W+M)(E-M)](E+M)R^{-}}{16\sqrt{2}\,\pi W m}$
T_{16}	$\frac{k_{o}(E-M)(E'-M)R^{+}}{16\sqrt{2}\,\pi W m}$	$-\frac{k_{o}(W+M)(E-M)(E'-M)R^{+}}{16\sqrt{2}\,\pi W m}$
T_{17}	$\frac{[(W+E)(E'-M)-k_{o}'(E+M)](E'+M)R^{-}}{16\sqrt{2}\,\pi W m'}$	$-\frac{[(W+E)(W+M)(E'-M)+k_{o}'(W-M)(E+M)](E'+M)R^{-}}{16\sqrt{2}\,\pi W m'}$
T_{18}	$\frac{k_{o}'(E+M)(E'+M)R^{-}}{16\sqrt{2}\,\pi W m'}$	$\frac{k_{o}'(W-M)(E+M)(E'+M)R^{-}}{16\sqrt{2}\,\pi W m'}$

TABLE 13

	a_1	a_2	a_3	a_4	a_5	a_6	a_7
T_1	-	-	-	-	-	$\frac{(E+M)(E'+M)R^-}{16\pi W}$	-
T_2	-	-	-	-	-	-	-
T_3	-	-	-	-	-	$\frac{k_o k'_o(E+M)(E'+M)R^-}{8\pi W m m'}$	-
T_4	-	-	-	-	-	-	-
T_5	-	-	-	-	-	-	-
T_6	-	-	-	-	-	$\frac{k_o(E+M)(E'+M)R^-}{8\sqrt{2}\,\pi W m}$	-
T_7	-	-	-	-	-	-	-
T_8	-	-	-	-	-	$\frac{k'_o(E+M)(E'+M)R^-}{8\sqrt{2}\,\pi W m'}$	-
T_9	-	-	-	-	-	-	-
T_{10}	-	-	-	-	-	$-\frac{(E-M)(E'-M)R^+}{16\pi W}$	-
T_{11}	-	-	-	-	-	-	-
T_{12}	-	-	-	-	-	$-\frac{k_o k'_o(E-M)(E'-M)R^+}{8\pi W m m'}$	-
T_{13}	-	-	-	-	-	-	-
T_{14}	-	-	-	-	-	-	-
T_{15}	-	-	-	-	-	$-\frac{k_o(E-M)(E'-M)R^+}{8\sqrt{2}\,\pi W m}$	-
T_{16}	-	-	-	-	-	-	-
T_{17}	-	-	-	-	-	$-\frac{k'_o(E-M)(E'-M)R^+}{8\sqrt{2}\,\pi W m'}$	-
T_{18}	-	-	-	-	-	-	-

TABLE 13 (Continued)

	a_8	a_9	a_{10}	a_{11}	a_{12}
T_1	-	-	$-\frac{(E+M)(E'+M)R^-}{16\pi W}$	-	-
T_2	-	-	-	-	-
T_3	-	-	$-\frac{k_o k'_o(E+M)(E'+M)R^-}{8\pi W m m'}$	-	-
T_4	-	-	-	-	-
T_5	-	-	-	-	-
T_6	-	-	$-\frac{k_o(E+M)(E'+M)R^-}{8\sqrt{2}\,\pi W m}$	-	-
T_7	-	-	-	-	-
T_8	-	-	$-\frac{k'_o(E+M)(E'+M)R^-}{8\sqrt{2}\,\pi W m'}$	-	-
T_9	-	-	-	-	-
T_{10}	-	-	$\frac{(E-M)(E'-M)R^+}{16\pi W}$	-	-
T_{11}	-	-	-	-	-
T_{12}	-	-	$\frac{k_o k'_o(E-M(E'-M)R^+}{8\pi W m m'}$	-	-
T_{13}	-	-	-	-	-
T_{14}	-	-	-	-	-
T_{15}	-	-	$\frac{k_o(E-M)(E'-M)R^+}{8\sqrt{2}\,\pi W m}$	-	-
T_{16}	-	-	-	-	-
T_{17}	-	-	$\frac{k'_o(E-M)(E'-M)R^+}{8\sqrt{2}\,\pi W m'}$	-	-
T_{18}	-	-	-	-	-

TABLE 13 (Continued)

	q_{13}	q_{14}
T_1	$\frac{(E+M)(E'+M)R^-}{32\pi W}$	$\frac{(W-M)(E+M)(E'+M)(R^-}{32\pi W}$
T_2	-	-
T_3	$\frac{k_o k'_o(E+M)(E'+M)\ R^-}{16\pi W m m'}$	$\frac{k_o k'_o(W-M)(E+M)(E'+M)\ R^-}{16\pi W m m'}$
T_4	-	-
T_5	-	-
T_6	$\frac{k_o(E+M)(E'+M)\ R^-}{16\sqrt{2}\ \pi W m}$	$\frac{k_o(W-M)(E+M)(E'+M)\ R^-}{16\sqrt{2}\ \pi W m}$
T_7	-	-
T_8	$\frac{k'_o(E+M)(E'+M)\ R^-}{16\sqrt{2}\ \pi W m'}$	$\frac{k'_o(W-M)(E+M)(E'+M)R^-}{16\sqrt{2}\ \pi W m'}$
T_9	-	-
T_{10}	$-\frac{(E-M)(E'-M)R^+}{32\pi W}$	$\frac{(W+M)(E-M)(E-M')R^+}{32\pi W}$
T_{11}	-	-
T_{12}	$-\frac{k_o k'_o(E-M)(E'-M)R^+}{16\pi W m m'}$	$\frac{k_o k'_o(W+M)(E-M)(E'-M)R^+}{16\pi W m m'}$
T_{13}	-	-
T_{14}	-	-
T_{15}	$-\frac{k_o(E-M)(E'-M)R^+}{16\sqrt{2}\ \pi W m}$	$\frac{k_o(W+M)(E-M)(E'-M)R^+}{16\sqrt{2}\ \pi W m}$
T_{16}	-	-
T_{17}	$-\frac{k'_o(E-M)(E'-M)R^+}{16\sqrt{2}\ \pi W m'}$	$\frac{k'_o(W+M)(E-M)(E'-M)R^+}{16\sqrt{2}\ \pi W m'}$
T_{18}	-	-

TABLE 13 (Continued)

	a_{15}	a_{16}
T_1	$\frac{(E+M)(E'+M)R^-}{32\pi W}$	$\frac{(W-M)(E+M)(E'+M)R^-}{32\pi W}$
T_2	-	-
T_3	$\frac{k_o k_o'(E+M)(E'+M)R^-}{16\pi W m m'}$	$\frac{k_o k_o'(W-M)(E+M)(E'+M)R^-}{16\pi W m m'}$
T_4	-	-
T_5	-	-
T_6	$\frac{k_o(E+M)(E'+M)R^-}{16\sqrt{2}\,\pi W m}$	$\frac{k_o(W-M)(E+M)(E'+M)R^-}{16\sqrt{2}\,\pi W m}$
T_7	-	-
T_8	$\frac{k_o'(E+M)(E'+M)R^-}{16\sqrt{2}\,\pi W m'}$	$\frac{k_o'(W-M)(E+M)(E'+M)R^-}{16\sqrt{2}\,\pi W m'}$
T_9	-	-
T_{10}	$-\frac{(E-M)(E'-M)R^+}{32\pi W}$	$\frac{(W+M)(E-M)(E'-M)R^+}{32\pi W}$
T_{11}	-	-
T_{12}	$-\frac{k_o k_o'(E-M)(E'-M)R^+}{16\pi W m m'}$	$\frac{k_o k_o'(W+M)(E-M)(E'-M)R^+}{16\pi W m m'}$
T_{13}	-	-
T_{14}	-	-
T_{15}	$-\frac{k_o(E-M)(E'-M)R^+}{16\sqrt{2}\,\pi W m}$	$\frac{k_o(W+M)(E-M)(E'-M)R^+}{16\sqrt{2}\,\pi W m}$
T_{16}	-	-
T_{17}	$-\frac{k_o'(E-M)(E'-M)R^+}{16\sqrt{2}\,\pi W m'}$	$\frac{k_o'(W+M)(E-M)(E'-M)R^+}{16\sqrt{2}\,\pi W m'}$
T_{18}	-	-

TABLE 13 (Continued)

	a_{17}	a_{18}
T_1	$-\frac{(E+M)(E'+M)R^-}{32\pi W}$	$-\frac{(W-M)(E+M)(E'+M)R^-}{32\pi W}$
T_2	-	-
T_3	$-\frac{k_o k'_o(E+M)(E'+M)R^-}{16\pi W m m'}$	$-\frac{k_o k'_o(W-M)(E+M)(E'+M)R^-}{16\pi W m m'}$
T_4	-	-
T_5	-	-
T_6	$-\frac{k_o(E+M)(E'+M)R^-}{16\sqrt{2}\,\pi W m}$	$-\frac{k_o(W-M)(E+M)(E'+M)R^-}{16\sqrt{2}\,\pi W m}$
T_7	-	-
T_8	$-\frac{k'_o(E+M)(E'+M)R^-}{16\sqrt{2}\,\pi W m'}$	$-\frac{k'_o(W-M)(E+M)(E'+M)R^-}{16\sqrt{2}\,\pi W m'}$
T_9	-	-
T_{10}	$\frac{(E-M)(E'-M)R^+}{32\pi W}$	$-\frac{(W+M)(E-M)(E'-M)R^+}{32\pi W}$
T_{11}	-	-
T_{12}	$\frac{k_o k'_o(E-M)(E'-M)R^+}{16\pi W m m'}$	$-\frac{k_o k'_o(W+M)(E-M)(E'-M)R^+}{16\pi W m m'}$
T_{13}	-	-
T_{14}	-	-
T_{15}	$\frac{k_o(E-M)(E'-M)R^+}{16\sqrt{2}\,\pi W m}$	$-\frac{k_o(W+M)(E-M)(E'-M)R^+}{16\sqrt{2}\,\pi W m}$
T_{16}	-	-
T_{17}	$\frac{k'_o(E-M)(E'-M)R^+}{16\sqrt{2}\,\pi W m'}$	$-\frac{k'_o(W+M)(E-M)(E'-M)R^+}{16\sqrt{2}\,\pi W m'}$
T_{18}	-	-

TABLE 13 (Continued)

	a_{19}	a_{20}
T_1	$-\frac{(E+M)(E'+M)R^-}{32\pi W}$	$-\frac{(W-M)(E+M)(E'+M)R^-}{32\pi W}$
T_2	-	-
T_3	$-\frac{k_o k_o'(E+M)(E'+M)R^-}{16\pi W m m'}$	$-\frac{k_o k_o'(W-M)(E+M)(E'+M)R^-}{16\pi W m m'}$
T_4	-	-
T_5	-	-
T_6	$-\frac{k_o(E+M)(E'+M)R^-}{16\sqrt{2}\,\pi W m}$	$-\frac{k_o(W-M)(E+M)(E'+M)R^-}{16\sqrt{2}\,\pi W m}$
T_7	-	-
T_8	$-\frac{k_o'(E+M)(E'+M)R^-}{16\sqrt{2}\,\pi W m'}$	$-\frac{k_o'(W-M)(E+M)(E'+M)R^-}{16\sqrt{2}\,\pi W m'}$
T_9	-	-
T_{10}	$\frac{(E-M)(E'-M)R^+}{32\pi W}$	$-\frac{(W+M)(E-M)(E'-M)R^+}{32\pi W}$
T_{11}	-	-
T_{12}	$\frac{k_o k_o'(E-M)(E'-M)R^+}{16\pi W m m'}$	$-\frac{k_o k_o'(W+M)(E-M)(E'-M)R^+}{16\pi W m m'}$
T_{13}	-	-
T_{14}	-	-
T_{15}	$\frac{k_o(E-M)(E'-M)R^+}{16\sqrt{2}\,\pi W m}$	$-\frac{k_o(W+M)(E-M)(E'-M)R^+}{16\sqrt{2}\,\pi W m}$
T_{16}	-	-
T_{17}	$\frac{k_o'(E-M)(E'-M)R^+}{16\sqrt{2}\,\pi W m'}$	$-\frac{k_o'(W+M)(E-M)(E'-M)R^+}{16\sqrt{2}\,\pi W m'}$
T_{18}	-	-

e) The πN^* channel

As has already been indicated in the introduction, the inclusion of the πN^* channel in a first calculation is unlikely, so we will not develop the general kinematics for processes involving it. Nevertheless, in appendix F we have obtained all the necessary formulae for the calculation of matrix elements for this channel.

With the propagator given by F-19 and the vertices discussed in part IV the amplitudes corresponding to graphs involving internal N^* can be determined, and if one or two external $N^{*'}$s are involved, the state vectors given in F-15 together with formulae B-7 and B-13 permit us to calculate the helicity amplitudes $f_{\lambda_c \lambda_d ; \lambda_a \lambda_b}$. Then equation D-23 defines the $T^{\pm}_{\lambda_c \lambda_d ; \lambda_a \lambda_b}$ that have a definite parity, and the corresponding partial wave expansions and projection operators are given by D-26 and D-28 respectively, and thus the matrix elements that are needed can be calculated.

As an example, we will calculate the necessary formulae for the kinematics of the reaction $\pi + N^* \rightarrow \pi + N^*$.

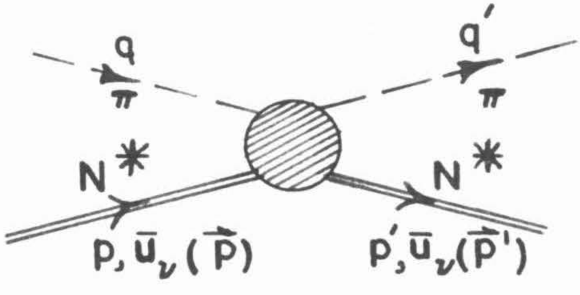

Fig. 6. General diagram for the reaction $\pi + N^* \rightarrow \pi + N^*$

As usual, we have

$$p + q = p' + q' \tag{174a}$$

$$P = \frac{1}{2}(p + p') \quad Q = \frac{1}{2}(q + q') \quad \Delta = \frac{1}{2}(q-q') = \frac{1}{2}(p' - p) \tag{174b}$$

$$P\cdot\Delta = Q\cdot\Delta = 0 \tag{174c}$$

$$P^2 + \Delta^2 = M'^2 \tag{174d}$$

$$Q^2 + \Delta^2 = \mu^2 \tag{174e}$$

where M' is the mass of the N^*.

The matrix element $\mathcal{Q}$ is now of the form

$$\mathcal{Q} = \bar{u}_\nu(\vec{p}\,')T_{\nu\mu}u_\mu(\vec{p}) \tag{175a}$$

$$T_{\nu\mu} = \mathcal{Q}_i M_{i,\nu\mu} \tag{175b}$$

The number of independent invariants is 6, as can be seen from

$$\frac{1}{|\vec{p}_r|}T^J = \begin{array}{c|cccc} {}^{\text{final}}_{\text{initial}} & \frac{3}{2}\,0 & \frac{1}{2}\,0 & -\frac{1}{2}\,0 & -\frac{3}{2}\,0 \\ \frac{3}{2}\,0 & a_1^J & a_2^J & a_3^J & a_4^J \\ \frac{1}{2}\,0 & a_2^J & a_5^J & a_6^J & a_3^J \\ -\frac{1}{2}\,0 & a_3^J & a_6^J & a_5^J & a_2^J \\ -\frac{3}{2}\,0 & a_4^J & a_3^J & a_2^J & a_1^J \end{array} \tag{176}$$

where parity conservation and time reversal invariance have been taken into account.

We will contract the two vector indices μ, ν with two "polarization vectors" ϵ_μ, ϵ'_ν to have the equations in a more familiar form.

Equations F-14a, c indicate that $\not{P}$ and $\not{Q}$ can be eliminated from the invariants, equations F-14b, d, that $\not{\epsilon}$ and $\not{\epsilon}'$, operating to the right and left respectively, give zero, and from F-2b we can derive two further relations. All these can be written

$$\overline{\not{P}} = M' \qquad \overline{\not{P}}' = M' \tag{177a}$$

$$\overline{\not{\epsilon}} = 0 \qquad \overline{\not{\epsilon}}' = 0 \tag{177b}$$

$$\Delta\cdot\epsilon = P\cdot\epsilon \qquad \Delta\cdot\epsilon' = -P\cdot\epsilon' \tag{177c}$$

The invariants that we obtain, after taking into account all obvious constraints, are

$$\begin{aligned} M_1 &= \epsilon\cdot\epsilon' & M_5 &= Q\cdot\epsilon\, Q\cdot\epsilon' \\ M_2 &= \epsilon\cdot\epsilon'\not{Q} & M_6 &= Q\cdot\epsilon\, Q\cdot\epsilon'\not{Q} \\ M_3 &= P\cdot\epsilon\, P\cdot\epsilon' & M_7 &= P\cdot\epsilon\, Q\cdot\epsilon' + Q\cdot\epsilon\, P\cdot\epsilon' \\ M_4 &= P\cdot\epsilon\, P\cdot\epsilon'\not{Q} & M_8 &= (P\cdot\epsilon\, Q\cdot\epsilon' + Q\cdot\epsilon\, P\cdot\epsilon')\not{Q} \end{aligned} \tag{178}$$

There are two superfluous invariants, and in the way indicated in appendix E, with the help of equations 177 we find

$$M'P\cdot QM_1 - P^2M_2 + 2M_4 - M'M_7 = 0 \tag{179a}$$

$$[\Delta^2Q^2 + (P\cdot Q)^2]M_1 - M'P\cdot QM_2 + 2Q^2M_3 - 2\Delta^2M_5 - 2P\cdot QM_7 + M'M_8 = 0 \tag{179b}$$

These show that M_4 and M_8, or M_7 and M_8, can be eliminated without introducing kinematical singularities.

If we define

$$T_{1,4} = \frac{f_{\frac{3}{2}0;\frac{3}{2}0}}{2\cos^3\frac{\theta}{2}} \pm \frac{f_{-\frac{3}{2}0;\frac{3}{2}0}}{-2\sin^3\frac{\theta}{2}e^{3i\varphi}}$$

$$T_{2,5} = \frac{f_{\frac{1}{2}0;\frac{1}{2}0}}{\cos\frac{\theta}{2}} \pm \frac{f_{-\frac{1}{2}0;\frac{1}{2}0}}{-\sin\frac{\theta}{2}e^{i\varphi}} \tag{180}$$

$$T_{3,6} = \frac{f_{\frac{3}{2}0;\frac{1}{2}0}}{-2\cos^2\frac{\theta}{2}\sin\frac{\theta}{2}e^{-i\varphi}} \pm \frac{f_{-\frac{3}{2}0;\frac{1}{2}0}}{2\cos\frac{\theta}{2}\sin^2\frac{\theta}{2}e^{2i\varphi}}$$

$$\beta^J_{1,4} = a^J_1 \pm a^J_4$$

$$\beta^J_{2,5} = a^J_5 \pm a^J_6 \tag{181}$$

$$\beta^J_{3,6} = a^J_3 \pm a^J_4$$

we find that the partial wave expansions are

$$T_{1,4} = \sum_{J=\ell+\frac{1}{2}} \left[\beta^J_{1,4}\left(\frac{P'''_{\ell+2}}{(\ell+2)(2\ell+3)} + \frac{3P'''_{\ell}}{\ell(2\ell+3)}\right) - \beta^J_{4,1}\left(\frac{3P'''_{\ell+1}}{(\ell+2)(2\ell+1)} + \frac{P'''_{\ell-1}}{\ell(2\ell+1)}\right)\right]$$

$$T_{2,5} = \sum_{J=\ell+\frac{1}{2}} \left[\beta^J_{2,5}P'_{\ell+1} - \beta^J_{5,2}P'_{\ell} \right] \tag{182}$$

$$T_{3,6} = \sum_{J=\ell+\frac{1}{2}} \left[\beta^J_{3,6}\frac{P''_{\ell}}{\sqrt{\ell(\ell+2)}} - \beta^J_{6,3}\frac{P''_{\ell+1}}{\sqrt{\ell(\ell+2)}} \right]$$

and the projection operators

$$\beta^J_{1,4} = \frac{1}{2}\int_{-1}^{1}\left[T_{1,4}\,\frac{3\ell P_{\ell+1}+(\ell+2)P_{\ell-1}}{2\ell+1} + T_{4,1}\frac{\ell P_{\ell+2}+3(\ell+2)P_{\ell}}{2\ell+3}\right]dx$$

$$\beta^J_{2,5} = \frac{1}{2}\int_{-1}^{1}\left[T_{2,5}P_\ell + T_{5,2}P_{\ell+1}\right]dx \tag{183}$$

$$\beta^J_{3,6} = \frac{1}{2}\int_{-1}^{1}\left[T_{3,6}\frac{\sqrt{\ell(\ell+2)}\,(P_{\ell+2}-P_\ell)}{2\ell+3} + T_{6,3}\frac{\sqrt{\ell(\ell+2)}\,(P_{\ell+1}-P_{\ell-1})}{2\ell+1}\right]dx$$

If we use the state vector from F-15 in equations 175, we can find the $f_{\lambda_c\lambda_d;\lambda_a\lambda_b}$ that appear in equations 180, and from them we get the relations

$$T_j = \mathcal{G}_i c_{ij} \tag{184}$$

$$c_{ij} = c^{(0)}_{ij} + c^{(1)}_{ij}x + c^{(2)}_{ij}x^2 \tag{185}$$

The coefficients $c^{(0)}_{ij}$, $c^{(1)}_{ij}$ and $c^{(2)}_{ij}$ are shown in tables 14, 15 and 16 respectively.

From these tables we see that the reflection symmetry in this case is expressed by

$$T_i(-W) = \eta_{(i)}T_{i+3}(W) \qquad i = 1, 2, 3 \tag{186a}$$

$$\beta^J_i(-W) = \eta_{(i)}\beta^J_{i+3}(W) \qquad i = 1, 2, 3 \tag{186b}$$

$$\eta_i = 1 \quad i = 3 \qquad \eta_i = -1 \quad i = 1, 2 \tag{186c}$$

TABLE 14

	G_1	G_2
T_1	$\frac{E-M'}{8\pi W}$	$-\frac{(W+M')(E-M')}{8\pi W}$
T_2	$\frac{(E+M')(4E^2+4EM'-5M'^2)}{24\pi WM'^2}$	$\frac{(W-M')(E+M')(4E^2+4EM'-5M'^2)}{24\pi WM'^2}$
T_3	$\frac{2E^2-EM'+M'^2}{4\sqrt{3}\,\pi WM'}$	$-\frac{2E(W+M')(E-M')-M'(W-M')(E+M')}{4\sqrt{3}\,\pi WM'}$
T_4	$-\frac{E+M'}{8\pi W}$	$-\frac{(W-M')(E+M')}{8\pi W}$
T_5	$-\frac{(E-M')(4E^2-4EM'-5M'^2)}{24\pi WM'^2}$	$\frac{(W+M')(E-M')(4E^2-4EM'-5M'^2)}{24\pi WM'^2}$
T_6	$-\frac{2E^2+EM'+M'^2}{4\sqrt{3}\,\pi WM'}$	$-\frac{2E(W-M')(E+M')+M'(W+M')(E-M')}{4\sqrt{3}\,\pi WM'}$

TABLE 14 (Continued)

	G_3	G_4
T_1	$\frac{\vec{p}^2(E-M')}{16\pi W}$	$-\frac{\vec{p}^2(W+M')(E-M)}{16\pi W}$
T_2	$\frac{\vec{p}^2(E+M')(4E^2+4EM'-M'^2)}{96\pi WM'^2}$	$\frac{\vec{p}^2(W-M')(E+M'(4E^2+4EM'-M'^2)}{96\pi WM'^2}$
T_3	$\frac{\vec{p}^2(2E^2-EM'+M'^2)}{16\sqrt{3}\,\pi WM'}$	$-\frac{\vec{p}^2[2E(W+M')(E-M')-M'(W-M')(E+M')]}{16\sqrt{3}\,\pi WM'}$
T_4	$-\frac{\vec{p}^2(E+M')}{16\pi W}$	$-\frac{\vec{p}^2(W-M')(E+M')}{16\pi W}$
T_5	$-\frac{\vec{p}^2(E-M')(4E^2-4EM'-M'^2)}{96\pi WM'^2}$	$\frac{\vec{p}^2(W+M')(E-M')(4E^2-4EM'-M'^2)}{96\pi WM'^2}$
T_6	$-\frac{\vec{p}^2(2E^2+EM'+M'^2)}{16\sqrt{3}\,\pi WM'}$	$-\frac{\vec{p}^2[2E(W-M')(E+M')+M'(W+M')(E-M')]}{16\sqrt{3}\,\pi WM'}$

TABLE 14 (Continued)

	G_5	G_6
T_1	$\frac{\vec{p}^2(E-M')}{16\pi W}$	$-\frac{\vec{p}^2(W+M')(E-M')}{16\pi W}$
T_2	$\frac{\vec{p}^2(E+M')[4(W+q_o)^2-4M'(W+q_o)-M'^2]}{96\pi W M'^2}$	$\frac{\vec{p}^2(W-M')(E+M')[4(W+q_o)^2-4M'(W+q_o)-M'^2]}{96\pi W M'^2}$
T_3	$-\frac{\vec{p}^2[2(W+q_o)(E-M')-M'(E+M')]}{16\sqrt{3}\,\pi W M'}$	$\frac{\vec{p}^2[2(W+q_o)(W+M')(E-M')+M'(W-M')(E+M')]}{16\sqrt{3}\,\pi W M'}$
T_4	$-\frac{\vec{p}^2(E+M')}{16\pi W}$	$-\frac{\vec{p}^2(W-M')(E+M')}{16\pi W}$
T_5	$-\frac{\vec{p}^2(E-M')[4(W+q_o)^2-4M'(W+q_o)-M'^2]}{96\pi W M'^2}$	$\frac{\vec{p}^2(W+M')(E-M')[4(W+q_o)^2+4M'(W+q_o)-M'^2]}{96\pi W M'^2}$
T_6	$\frac{\vec{p}^2[2(W+q_o)(E+M')+M'(E-M')]}{16\sqrt{3}\,\pi W M'}$	$\frac{\vec{p}^2[2(W+q_o)(W-M')(E+M')-M'(W+M')(E-M')]}{16\sqrt{3}\,\pi W M'}$

TABLE 14 (Continued)

	a_7	a_8
T_1	$-\frac{\vec{p}^2(E-M')}{8\pi W}$	$\frac{\vec{p}^2(W+M')(E-M')}{8\pi W}$
T_2	$\frac{\vec{p}^2[(E+M')\{4E(W+q_o)+M'^2\}-4EM'(E-M')]}{48\pi W M'^2}$	$\frac{\vec{p}^2[(W-M')(E+M')\{4E(W+q_o)+M'^2\}+4EM'(W+M')(E-M')]}{48\pi W M'^2}$
T_3	$\frac{\vec{p}^2[2q_o(E-M')-M'(E+M')]}{8\sqrt{3}\,\pi W M'}$	$-\frac{\vec{p}^2[2q_o(W+M')(E-M')+M'(W-M')(E+M')]}{8\sqrt{3}\,\pi W M'}$
T_4	$\frac{\vec{p}^2(E+M')}{8\pi W}$	$\frac{\vec{p}^2(W-M')(E+M')}{8\pi W}$
T_5	$-\frac{\vec{p}^2[(E-M')\{4E(W+q_o)+M'^2\}+4EM'(E+M')]}{48\pi W M'^2}$	$\frac{\vec{p}^2[(W+M')(E-M')\{4E(W+q_o)+M'^2\}-4EM'(W-M')(E+M')]}{48\pi W M'^2}$
T_6	$-\frac{\vec{p}^2[2q_o(E+M')+M'(E-M')]}{8\sqrt{3}\,\pi W M'}$	$-\frac{\vec{p}^2[2q_o(W-M')(E+M')-M'(W+M')(E-M')]}{8\sqrt{3}\,\pi W M'}$

TABLE 15

	G_1	G_2
T_1	-	-
T_2	$-\frac{4E^2(E+M')+M'(4E+M')(E-M')}{24\pi W M'^2}$	$-\frac{4E^2(W-M')(E+M')-M'(4E+M')(W+M')(E-M')}{24\pi W M'^2}$
T_3	-	-
T_4	-	-
T_5	$\frac{4E^2(E-M')-M'(4E-M')(E+M')}{24\pi W M'^2}$	$-\frac{4E^2(W+M')(E-M')+M'(4E-M')(W-M')(E+M')}{24\pi W M'^2}$
T_6	-	-

TABLE 15 (Continued)

	G_3	G_4
T_1	$\frac{\vec{p}^2(E+M')}{16\pi W}$	$\frac{\vec{p}^2(W-M')(E+M')}{16\pi W}$
T_2	$\frac{\vec{p}^2E^3}{12\pi WM'^2}$	$\frac{\vec{p}^2[E^2(W-M')(E+M')-q_oM'^2E]}{12\pi WM'^2}$
T_3	$-\frac{\vec{p}^2(2E^2-EM'+M'^2)}{16\sqrt{3}\,\pi WM'}$	$\frac{\vec{p}^2[2E(W+M')(E-M')-M'(W-M')(E+M')]}{16\sqrt{3}\,\pi WM'}$
T_4	$-\frac{\vec{p}^2(E-M')}{16\pi W}$	$\frac{\vec{p}^2(W+M')(E-M')}{16\pi W}$
T_5	$-\frac{\vec{p}^2E^3}{12\pi WM'^2}$	$\frac{\vec{p}^2[E^2(W+M')(E-M')-q_oM'^2E]}{12\pi WM'^2}$
T_6	$\frac{\vec{p}^2(2E^2+EM'+M'^2)}{16\sqrt{3}\,\pi WM'^2}$	$\frac{\vec{p}^2[2E(W-M')(E+M')+M'(W+M')(E-M')]}{16\sqrt{3}\,\pi WM'}$

TABLE 15 (Continued)

	G_5	G_6
T_1	$\frac{\vec{p}^2(E+M')}{16\pi W}$	$\frac{\vec{p}^2(W-M')(E+M')}{16\pi W}$
T_2	$\frac{\vec{p}^2[E(W+q_o)(E+M')+M'(q_oE-M'W)]}{12\pi W M'^2}$	$\frac{\vec{p}^2[E(W+q_o)(W-M')(E+M')+M'(M'q_o^2+W^2E-WM'^2)]}{12\pi W M'^2}$
T_3	$-\frac{\vec{p}^2(2E^2-EM'+M'^2)}{16\sqrt{3}\,\pi W M'}$	$\frac{\vec{p}^2[2E(W+M')(E-M')-M'(W-M')(E+M')]}{16\sqrt{3}\,\pi W M'}$
T_4	$-\frac{\vec{p}^2(E-M')}{16\pi W}$	$\frac{\vec{p}^2(W+M')(E-M')}{16\pi W}$
T_5	$-\frac{\vec{p}^2[E(W+q_o)(E-M')-M'(q_oE+M'W)]}{12\pi W M'^2}$	$\frac{\vec{p}^2[E(W+q_o)(W+M')(E-M')+M'(M'q_o^2-W^2E+WM'^2)]}{12\pi W M'^2}$
T_6	$\frac{\vec{p}^2(2E^2+EM'+M'^2)}{16\sqrt{3}\,\pi W M'^2}$	$\frac{\vec{p}^2[2E(W-M')(E+M')+M'(W+M')(E-M')]}{16\sqrt{3}\,\pi W M'}$

TABLE 15 (Continued)

	G_7	G_8
T_1	$-\frac{\vec{p}^2(E+M')}{8\pi W}$	$-\frac{\vec{p}^2(W-M')(E+M')}{8\pi W}$
T_2	$-\frac{\vec{p}^2[2WE(E+M')+M'E(E-q_o)-M'^2W]}{12\pi WM'^2}$	$-\frac{\vec{p}^2[2WE(W-M')(E+M')+M'^2q_o(E-q_o)-M'W(WE-M'^2)]}{12\pi WM'^2}$
T_3	$\frac{\vec{p}^2(2E^2-EM'+M'^2)}{8\sqrt{3}\,\pi WM'}$	$-\frac{\vec{p}^2[2E(W+M')(E-M')-M'(W-M')(E+M')]}{8\sqrt{3}\,\pi WM'}$
T_4	$\frac{\vec{p}^2(E-M')}{8\pi W}$	$-\frac{\vec{p}^2(W+M')(E-M')}{8\pi W}$
T_5	$\frac{\vec{p}^2[2WE(E-M')-M'E(E-q_o)-M'^2W]}{12\pi WM'^2}$	$-\frac{\vec{p}^2[2WE(W+M')(E-M')+M'^2q_o(E-q_o)+M'W(WE-M'^2)]}{12\pi WM'^2}$
T_6	$-\frac{\vec{p}^2(2E^2+EM'+M'^2)}{8\sqrt{3}\,\pi WM'^2}$	$-\frac{\vec{p}^2[2E(W-M')(E+M')+M'(W+M')(E-M')]}{8\sqrt{3}\,\pi WM'}$

TABLE 16

	G_1	G_2	G_3	G_4
T_1			-	-
T_2	-	-	$\frac{\vec{p}^2[E+M')(4E^2+M'^2)+4M'E(E-M')]}{96\pi WM'^2}$	$\frac{\vec{p}^2[(W-M')(E+M')(4E^2+M'^2)-4M'E(W+M')(E-M')]}{96\pi WM'^2}$
T_3	-	-	-	-
T_4	-	-	-	-
T_5	-	-	$-\frac{\vec{p}^2[(E-M')(4E^2+M'^2)-4M'E(E+M')]}{96\pi WM'^2}$	$\frac{\vec{p}^2[(W+M')(E-M')(4E^2+M'^2)+4M'E(W-M')(E+M')]}{96\pi WM'^2}$
T_6	-	-	-	-

	G_5	G_6
T_1	-	-
T_2	$\frac{\vec{p}^2[(E+M')(4E^2+M'^2)+4M'E(E-M')]}{96\pi WM'^2}$	$\frac{\vec{p}^2[(W-M')(E+M')(4E^2+M'^2)-4M'E(W+M')(E-M')]}{96\pi WM'^2}$
T_3	-	-
T_4	-	-
T_5	$-\frac{\vec{p}^2[(E-M')(4E^2+M'^2)-4M'E(E+M')]}{96\pi WM'^2}$	$\frac{\vec{p}^2[(W+M')(E-M')(4E^2+M'^2)+4M'E(W-M')(E+M')]}{96\pi WM'^2}$
T_6	-	-

TABLE 16 (Continued)

	G_7	G_8
T_1	-	-
T_2	$-\frac{\vec{p}^2(4E^2-4EM'+M'^2)(E+M')}{48\pi WM'^2}$	$-\frac{\vec{p}^2(4E^2-4EM'+M'^2)(W-M')(E+M')}{48\pi WM'^2}$
T_3	-	-
T_4	-	-
T_5	$\frac{\vec{p}^2(4E^2+4EM'+M'^2)}{48\pi WM'^2}$	$-\frac{\vec{p}^2(4E^2+4EM'+M'^2)(W+M')(E-M')}{48\pi WM'^2}$
T_6	-	-

III. ISOSPIN "KINEMATICS"

In part II we have developed the kinematics for the different processes assuming that all particles have isospin zero. The effect of isospin will be to increase the multiplicity of each state, but it does not interfere with the kinematics and so it can be superimposed on it.

a) $\pi + N \rightarrow \pi + N$

Pions have isospin 1 and nucleons $\frac{1}{2}$, so that the πN system can be either in a state with $I = \frac{3}{2}$ or $I = \frac{1}{2}$. Since only strong interactions are included, charge independence is assumed and it is sufficient to specify the magnitude of the isospin.

The isospin space for the πN system will be a direct product of the spaces for the π and N, and the state vectors for states with a definite total isospin are given by the corresponding Clebsch-Gordan coefficients, i.e.

$$|I, I_z> = \sum_{I_{2z}} C(I_1, I_2, I; I_z - I_{2z}, I_{2z}) \, |I_1, I_z - I_{2z}> |I_2, I_{2z}> \tag{1}$$

(See ref. 6, section 10, for instance.) As an example we will give

$$|\tfrac{3}{2}, \tfrac{1}{2}> = \sqrt{\tfrac{2}{3}} \, |1, 0> | \tfrac{1}{2}, \tfrac{1}{2}> + \sqrt{\tfrac{1}{3}} \, |1, 1> | \tfrac{1}{2}, -\tfrac{1}{2}> \tag{1a}$$

or

$$| \tfrac{3}{2}, \tfrac{1}{2}> = \sqrt{\tfrac{2}{3}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sqrt{\tfrac{1}{3}} \begin{pmatrix} -1 \\ -i \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1b}$$

We have taken for spin 1 the usual representation described in ref. 1

section 7h, where

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad T_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \qquad T_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2a}$$

or

$$(T_k)_{ij} = -i\epsilon_{ijk} \tag{2b}$$

$$\pi^+ \doteq |1,1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ -i \\ 0 \end{pmatrix} \qquad \pi^0 \doteq |1,0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \pi^- \doteq |1,-1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \tag{3}$$

For spin $\frac{1}{2}$ we get the usual Pauli spin matrices, and we will call them τ_i. We have then

$$\vec{I} = \vec{T} + \frac{1}{2}\vec{\tau} \tag{4}$$

where each term contains implicitly a unit operator that acts on the other state vector of the direct product.

In general

$$I_z\,|I, I_z\rangle = I_z\,|I, I_z\rangle \tag{5a}$$

where I_z on the left hand side is an operator, and on the right a number, and

$$\vec{I}^2\,|I, I_z\rangle = I(I+1)\,|I, I_z\rangle \tag{5b}$$

In πN scattering, we can always write the amplitudes in the form

$$\mathfrak{a} = \langle N_c\,\pi_d\,|A|\,N_a\,\pi_b\rangle \tag{6}$$

where the state vectors correspond to the isospin of the particles, and

$$A = a_i M_i \qquad i = 1, 2 \tag{7a}$$

$$M_1 = 1 = 1_3\, 1_2 \qquad M_2 = \vec{T}\cdot\vec{\tau} = T_1\tau_1 + T_2\tau_2 + T_3\tau_3 \tag{7b}$$

These two are the only "scalars" (scalars under rotations in a three-dimensional isospin space) that can be formed.

If we want the matrix elements for states with a definite total isospin, we calculate

$$a(I) = a_i < I \,|M_i|\, I > \tag{8}$$

To calculate $< I|M_2|I >$ we can write

$$\vec{I}^2 = (\vec{T} + \tfrac{1}{2}\vec{\tau})^2$$

$$I(I+1) = T(T+1) + \vec{T}\cdot\vec{\tau} + \tfrac{1}{4}\tau(\tau+1)$$

$$\vec{T}\cdot\vec{\tau} = I(I+1) - \frac{11}{4} \tag{9}$$

and equation 8 gives

$$a(\tfrac{5}{2}) = a_1 + a_2 \tag{10a}$$

$$a(\tfrac{1}{2}) = a_1 - 2a_2 \tag{10b}$$

If the cross section is needed for a particular reaction, we can find it from equation 6, or by using the $a(I)$ of equation 10 and the inverse of equation 1 (see R6-3.8):

$$|I_1 I_{1z} > |I_2, I_z - I_{1z} > = \sum_I C(I_1 I_2 I; I_{1z}, I_z - I_{1z})\, |I\, I_z > \tag{11}$$

and remembering that total isospin is conserved.

b) Reactions involving vector mesons

The ρ-meson has isospin 1 and the ρN channel has the same structure as the πN channel.

The ω-meson has isospin 0, and hence the ωN channel has always $I = \frac{1}{2}$.

We have for the $\omega + N \rightarrow \pi + N$ reaction

$$\mathcal{A} = \langle \pi N_c | A | N_a \rangle \tag{12a}$$

since the dependence on the ω is trivial.

It can also be written

$$\mathcal{A} = \langle N_c | A(\pi) | N_a \rangle \tag{12b}$$

where

$$A(\pi) = \mathcal{A}_1 M(\pi) \tag{13a}$$

$$M(\pi) = \vec{\pi} \cdot \vec{\tau} \tag{13b}$$

We have written here $\vec{\pi}$ for the isospin state vector of the pion. Obviously we have

$$\mathcal{A}(\tfrac{1}{2}) = \mathcal{A}_1 \tag{14}$$

For the reaction $\omega + N \rightarrow \omega + N$ we find

$$\mathcal{A} = \langle N_c | A | N_a \rangle \tag{15a}$$

$$A = \mathcal{A}_1 M_1 \tag{15b}$$

$$M_1 = 1 \tag{15c}$$

$$\mathcal{A}(\tfrac{1}{2}) = \mathcal{A}_1 \tag{15d}$$

Equation 15c expresses the fact that the ω meson is neutral and the charge of the nucleon has to be conserved.

c) Reactions involving the πN^* channel

We can construct the states for the N^*, that is, for a particle of isospin $\frac{3}{2}$, out of those for isospin 1 and $\frac{1}{2}$, by using the corresponding Clebsch-Gordan coefficients. We write this state $N^*(I_z)$, or $N_k^*(I_z)$ if we want to make explicit the three-vector nature of $N^*(I_z)$; then each $N_k^*(I_z)$, $k = 1, 2, 3$, is a two-component spinor.

We have, then,

$$N^*(\tfrac{3}{2}) = \begin{pmatrix} -\frac{1}{\sqrt{2}}\binom{1}{0} \\ -\frac{i}{\sqrt{2}}\binom{1}{0} \\ 0 \end{pmatrix} \qquad N^*(\tfrac{1}{2}) = \begin{pmatrix} -\frac{1}{\sqrt{6}}\binom{0}{1} \\ -\frac{i}{\sqrt{6}}\binom{0}{1} \\ \sqrt{\frac{2}{3}}\binom{1}{0} \end{pmatrix}$$

$$N^*(-\tfrac{1}{2}) = \begin{pmatrix} \frac{1}{\sqrt{6}}\binom{1}{0} \\ -\frac{i}{\sqrt{6}}\binom{1}{0} \\ \sqrt{\frac{2}{3}}\binom{0}{1} \end{pmatrix} \qquad N^*(-\tfrac{3}{2}) = \begin{pmatrix} \frac{1}{\sqrt{2}}\binom{0}{1} \\ -\frac{i}{\sqrt{2}}\binom{0}{1} \\ 0 \end{pmatrix} \tag{16}$$

It is easy to check that all four states obey the equation

$$\tau_k N_k^* = 0 \tag{17}$$

in analogy to equation F-1b for the wave function of a spin $\frac{3}{2}$ particle.

For $\pi + N^* \rightarrow \pi + N^*$ we can write then

$$\mathcal{G} = \langle N_{cj}^* \pi_d | A_{ik} | N_{ak}^* \pi_b \rangle \tag{18}$$

$$A_{jk} = \mathcal{G}_i M_{i,jk} \qquad i = 1, 2, 3 \tag{19}$$

There are only three independent invariants because the total isospin can take the values $\frac{5}{2}$, $\frac{3}{2}$ and $\frac{1}{2}$ and the invariants are second rank tensors in the form we have written them.

We can choose

$$M_{1,jk} = \delta_{jk} 1_3 1_2 \tag{20a}$$

$$M_{2,jk} = \delta_{jk} \vec{T}\cdot\vec{\tau} \tag{20b}$$

$$M_{3,jk} = T_j T_k \tag{20c}$$

Other invariants are reduced to combinations of these with the help of the relations

$$i\epsilon_{ijk}\tau_i = \tau_j\tau_k - \delta_{jk} \tag{21a}$$

$$i\epsilon_{ijk} = \tau_i\tau_j\tau_k - \delta_{ij}\tau_k + \delta_{ik}\tau_j - \delta_{jk}\tau_i \tag{21b}$$

$$T_k T_j = T_j T_k - i\epsilon_{ijk} T_i \tag{21c}$$

or are zero between state vectors due to equation 17.

Next we can calculate the $\mathcal{A}(I)$ by using any of the states with total isospin I. We get

$$\mathcal{A}(\tfrac{5}{2}) = \mathcal{A}_1 + \mathcal{A}_2 \tag{22a}$$

$$\mathcal{A}(\tfrac{3}{2}) = \mathcal{A}_1 - \tfrac{2}{3}\mathcal{A}_2 + \tfrac{5}{3}\mathcal{A}_3 \tag{22b}$$

$$\mathcal{A}(\tfrac{1}{2}) = \mathcal{A}_1 - \tfrac{5}{3}\mathcal{A}_2 + \tfrac{2}{3}\mathcal{A}_3 \tag{22c}$$

For $\pi + N^* \rightarrow \pi + N$ we have

$$\mathcal{A} = < N\pi_d \,|A_k\, |N_k^*\pi_b > \tag{23a}$$

$$A_k = a_i M_{i,k} \qquad i = 1, 2 \tag{23b}$$

In this case the two invariants are vectors in isospin space.

$$M_{1,k} = 1_2 T_k \tag{24a}$$

$$M_{2,k} = \vec{T} \cdot \vec{\tau}\ T_k \tag{24b}$$

$$a(\tfrac{3}{2}) = -\sqrt{\tfrac{5}{3}}\ a_1 - \sqrt{\tfrac{5}{3}}\ a_2 \tag{25a}$$

$$a(\tfrac{1}{2}) = \sqrt{\tfrac{2}{3}}\ a_1 - 2\sqrt{\tfrac{2}{3}}\ a_2 \tag{25b}$$

For $\pi + N^* \rightarrow \omega + N$ there is only one invariant:

$$a = \langle N | A_k(\pi) | N_k^* \rangle \tag{26}$$

$$A_k(\pi) = a_1 M_{1,k}(\pi) \tag{27a}$$

$$M_{1,k}(\pi) = 1_2 \pi_k \tag{27b}$$

$$a(\tfrac{1}{2}) = -\sqrt{2}\ a_1 \tag{27c}$$

This is all the isospin "kinematics" that is needed for the four different channels. Matrix elements where total isospin and its z-component are not both conserved are zero for strong interactions.

Finally we would like to point out that this section shows only one way of writing down the different amplitudes; and other equivalent ways are possible. In particular, if the analogy to part II is carried one step further, the isovectors corresponding to particles of isospin 1 can be included in the invariants.

For instance, the invariants for $\pi + N \rightarrow \pi + N$ can be chosen

$$M_1 = \vec{\pi}_d \cdot \vec{\pi}_b \qquad M_2 = \vec{\tau} \cdot \vec{\pi}_d\ \vec{\tau} \cdot \vec{\pi}_b \tag{28a, b}$$

IV. DYNAMICS

a) General approach

The solutions to problems in relativistic quantum mechanics proposed so far all demand that some approximation be used in order to get a practical calculation. It is desirable to build into the approximate solution as many properties that the exact solution is known to have as is possible without getting into undue complications.

One such solution that incorporates unitarity into the scattering matrix is the N/D method presented in appendix G. Essentially, the submatrix of the scattering matrix corresponding to a (conserved) total angular momentum J when a partial wave expansion is made is seen to obey the equation

$$S^J S^{J\dagger} = 1 \tag{1}$$

as in reference 5. There a matrix T^J is defined by

$$iT^J = S^J - 1 \tag{2}$$

and hence T^J obeys

$$2 \operatorname{Im} T^J = T^{J*} T^J \tag{3}$$

if T^J is symmetric (as is demanded by time reversal invariance).

Scattering amplitudes, in a theory where the S-matrix is unitary, obey dispersion relations that make explicit certain cuts in the complex energy (or energy squared) plane. These are called unitarity cuts.

Each T^J is then written in the form

$$T^J = N^J (D^J)^{-1} \tag{4}$$

so that D^J has all the unitarity cuts and no others, and N^J the remaining singularities of T^J.

From equations 3 and 4 we can find relations for Im N^J and Im D^J like those in G-14a, b, which can be written as integral equations. Two of the possible approaches to a numerical computation will be briefly discussed.

It is possible to assume a certain set of singularities given for T, not including the unitarity cuts, such as cuts arising from the exchange of a certain particle in a crossed channel. The integral equations for N and D can then be solved as is done in references 16 and 23 for a very simple choice of the singularities, or the solution can be calculated on a computer.

It is also possible to choose an approximate solution for N, from the Born approximation for instance, and then calculate D by performing the integrals in equation G-14c.

The inelastic channels are related to pion nucleon scattering by the unitarity equations, and their presence can produce resonances as is discussed in references 16, 18 and 23. It is then desirable to include these channels, at least in an approximate way. This is done by considering several particles represented by one unstable particle or isobar when there is a resonance in the corresponding system.

If the information we want to use as input for the solution is fairly extensive, it looks advisable to use the second suggested approximation for the N/D method, since involves only the calculation of integrals and not the numerical solution of integral equations. This method has the disadvantage of giving a T-matrix that is not symmetric, as required by time reversal invariance, and hence unitarity is also violated even in the approximation of considering only a few channels (see the discussion in appendix G).

The Feynman diagrams of the Born approximation, including the nucleon pole in the main channel and other poles in the crossed channels are shown in section IVb, and a few examples are worked out in section IVd.

Isospin is conserved by strong interactions, and hence the states with different total isospin can be treated separately. The way to project out the corresponding amplitudes is shown in part III.

Total angular momentum is also conserved; this leads to the use of a partial wave expansion. Moreover, parity is also conserved by strong interactions; this permits one to separate the sets of amplitudes corresponding to a given J in two groups, one for transitions between states of parity + and the other for parity -, matrix elements between states of different parity are always zero. This separation is built into the amplitudes called β_i^J in part II, and the simple relation between states of definite parity and those of definite helicity is shown for instance in equation II-62.

The sets of amplitudes or matrix elements are related by an equation of the form

$$T'^J = CT^J C^{-1} \qquad (5a)$$

For instance, for πN scattering alone,

$$\begin{pmatrix} a_1^J + a_2^J & 0 \\ 0 & a_1^J - a_2^J \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} a_1^J & a_2^J \\ a_2^J & a_1^J \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad (5b)$$

This does not change equation 3. (It should be noticed that $\frac{1}{2}\,T^J$ and $\frac{1}{2}\,T'^J$ are the matrices that obey equation G-12. This agrees with equations II-49, II-52 and II-53 for πN scattering.)

Singularities like branch cuts coming from partial waves expansions and density of states factors can be eliminated from the equations by redefining the T-matrix as is shown for instance in reference 19. One obvious change is the choice of the total energy W instead of $s = W^2$ as the independent variable. Other changes like those indicated in R19-28 to 32 have the purpose of giving the amplitudes the right threshold behavior, when the substraction point for D is taken at threshold, as in equation G-14d.

The inclusion in the problem of unstable external particles and the resulting complex singularities give rise to the need of modifying the unitarity relations. This is shown in detail in appendix H for the case discussed in reference 16 and a discussion of the application to the problem in consideration is also included. References 18 and 19 also deal with these difficulties, in a different way.

The matrix elements can then be calculated for πN scattering for small values of J and the two values of parity and isospin; and from the phase shifts (equations II-53) the possible resonances can be found.

b) Feynman diagrams

If we exclude external N^* isobars, we have to consider the pole terms indicated in figures 1 to 6. To write down the amplitudes corresponding to the different graphs we follow Feynman's rules as described in reference 1 section 14c, for instance. It should be noticed that factors like $\frac{1}{\sqrt{2\omega_{\vec{k}}}}$ and $\sqrt{\frac{M}{E(\vec{p})}}$ are not included in $\mathcal{A}$ but added later when the cross section is computed. Neither are the powers of 2π, and momentum conservation at each vertex can be taken into account by choosing adequately the momenta of internal particles.

The propagators for particles of spin 0, $\frac{1}{2}$ and 1 are well known, and for spin $\frac{3}{2}$ it is given by equation F-19. The form of the interactions is more uncertain, and some principle of minimal complication is usually invoked to rule out certain terms and leave others. The final justification comes of course from the check with experiment, but detailed data and a good theory to make calculations are required to reach sound conclusions.

The values of all the effective coupling constants must also be determined from experiment, unless a theory like one of higher symmetries of strong interactions is used. The vertices involved in figures 1 to 6 will be taken as follows:

$$V(\pi NN) = g_{\pi NN}\gamma_5 \qquad \text{(see fig. 7a)} \qquad (6)$$

$$V(\pi NN^*) = g_{\pi NN^*}(p_\mu + q_\mu) \qquad \text{(see fig. 7b)} \qquad (7)$$

$$V(\rho NN) = g_{\rho NN}\gamma_\mu \qquad \text{(see fig. 7c)} \qquad (8)$$

$$V(\rho\pi\pi) = g_{\rho\pi\pi}(q_\mu + q'_\mu) \qquad \text{(see fig. 7d)} \qquad (9)$$

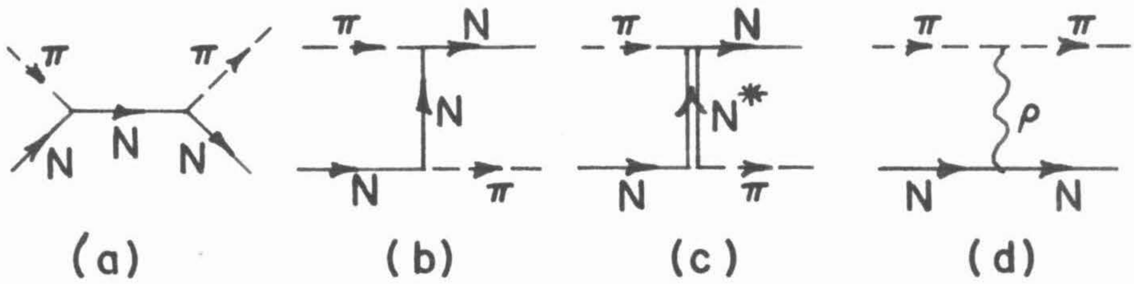

Fig. 1. Graphs for $\pi + N \rightarrow \pi + N$

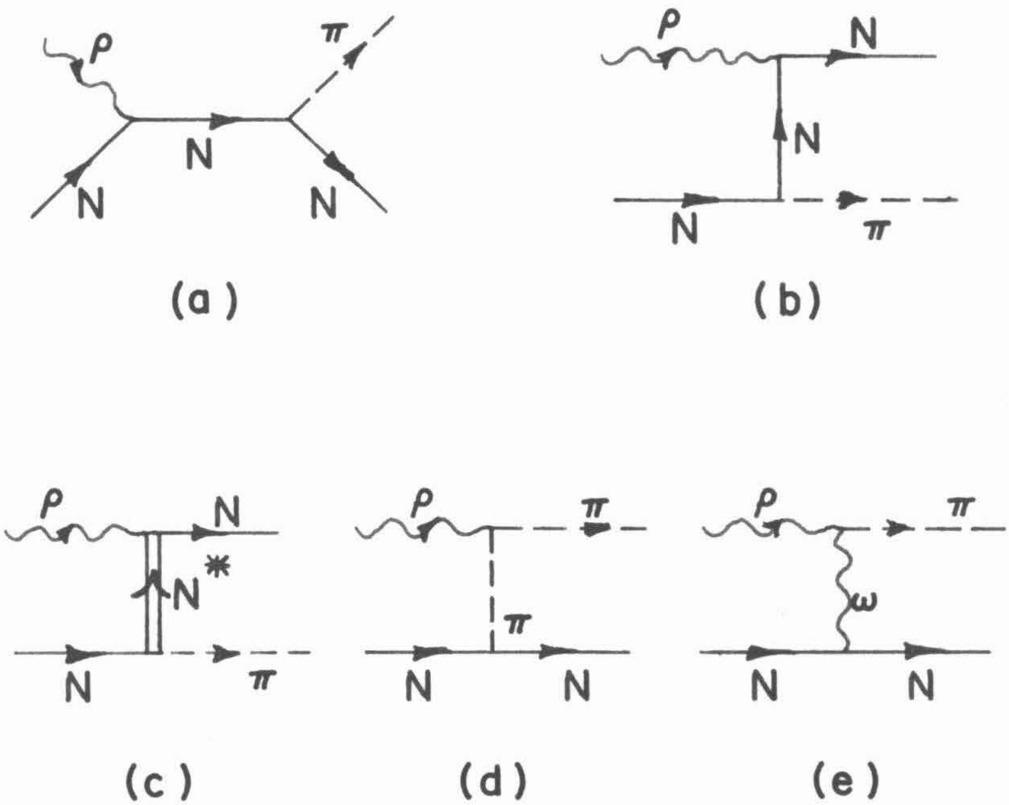

Fig. 2. Graphs for $\rho + N \rightarrow \pi + N$

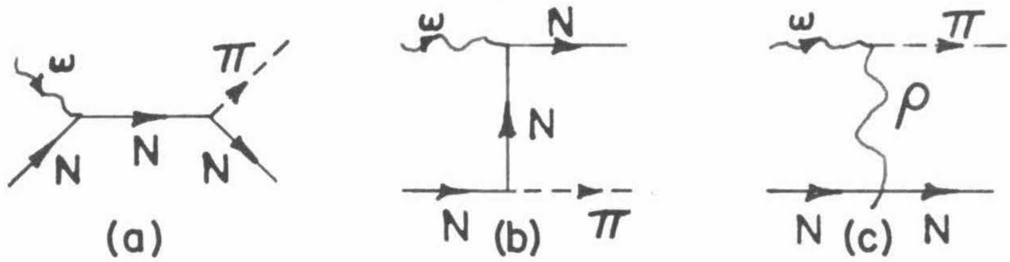

Fig. 3. Graphs for $\omega + N \rightarrow \pi + N$

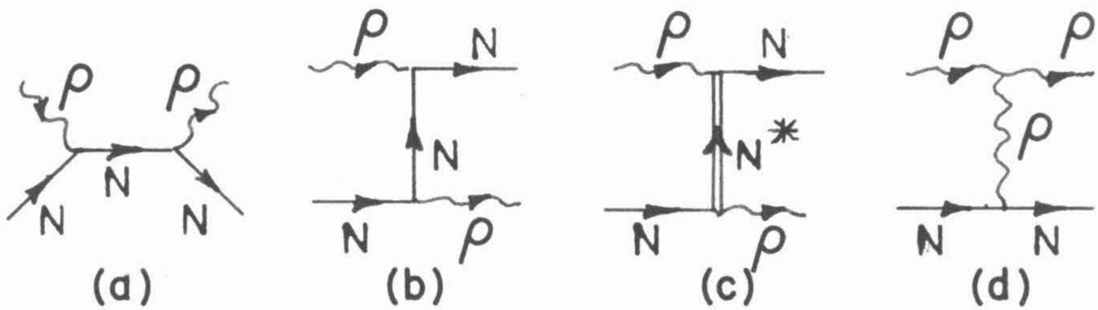

Fig. 4. Graphs for $\rho + N \rightarrow \rho + N$

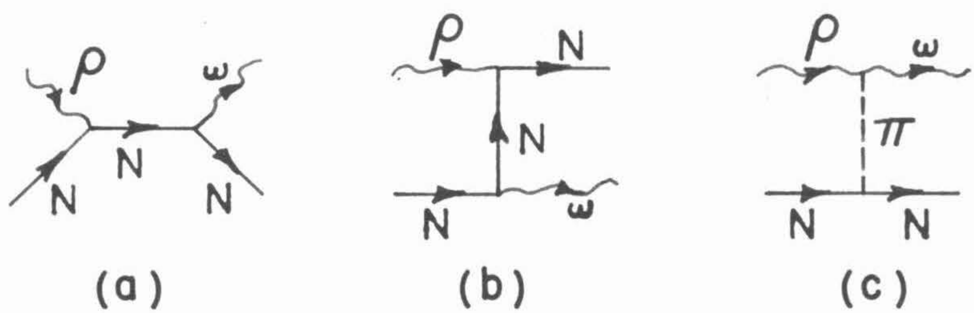

Fig. 5. Graphs for $\omega + N \rightarrow \rho + N$

Fig. 6. Graphs for $\omega + N \rightarrow \omega + N$

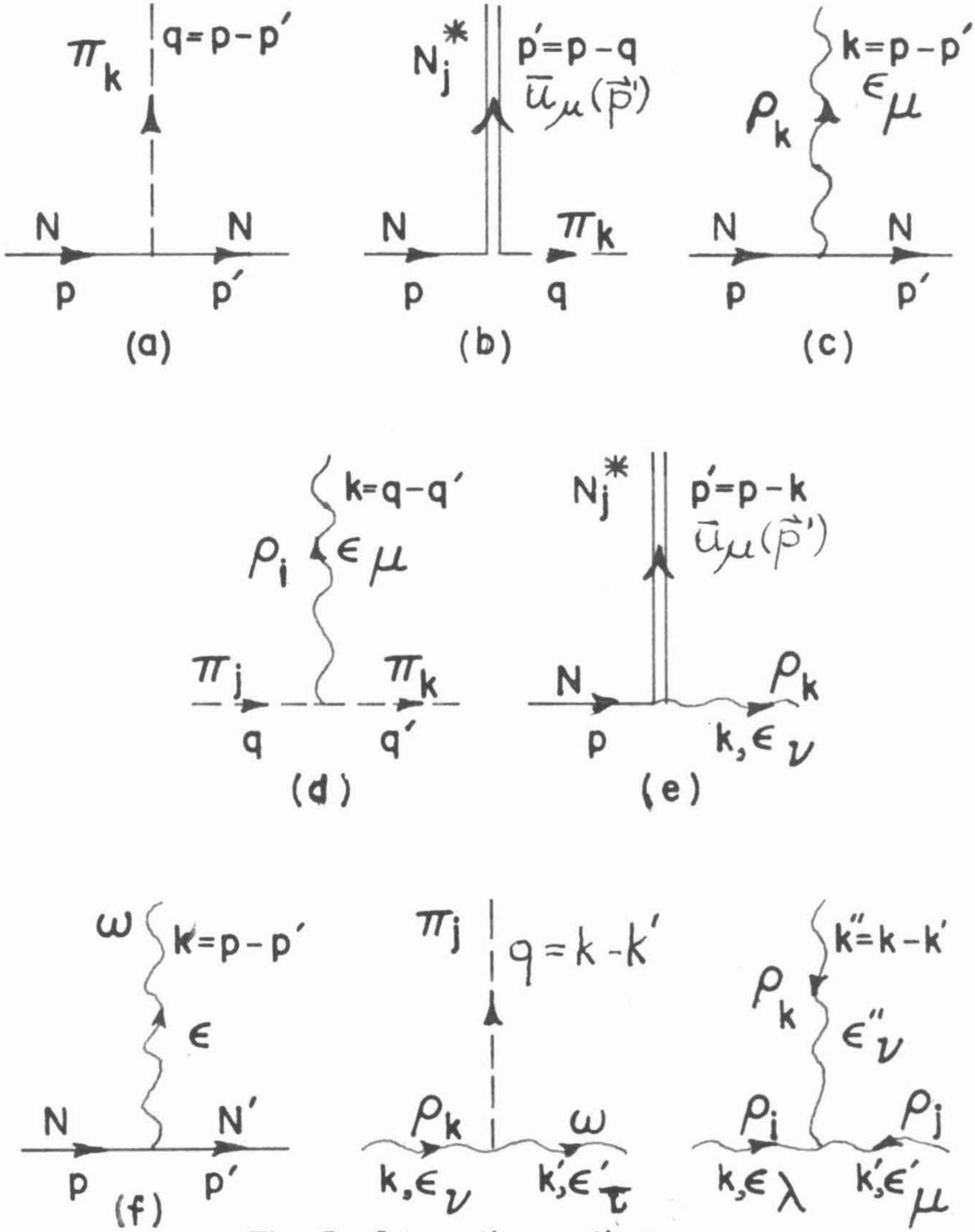

Fig. 7. Interaction vertices

$$V(\rho NN^*) = g_{\rho NN}\, \gamma_\nu (p_\mu + k_\mu)\gamma_5 \qquad \text{(see fig. 7e)} \qquad (10)$$

$$V(\omega NN) = g_{\omega NN}\gamma_\mu \qquad \text{(see fig. 7f)} \qquad (11)$$

$$V(\pi\rho\omega) = g_{\pi\rho\sigma}\epsilon_{\lambda\mu\nu\tau}k_\lambda k'_\mu \qquad \text{(see fig. 7g)} \qquad (12)$$

$$V(\rho\rho\rho) = g_{3\rho}(k''_\lambda \delta_{\mu\nu} + k_\mu \delta_{\nu\lambda} + k'_\nu \delta_{\lambda\mu}) \qquad \text{(see fig. 7h)} \qquad (13)$$

The πNN vertex is the usual γ_5 coupling. In the πNN^* vertex, the choice is rather arbitrary. It corresponds to the interaction Lagrangian given in reference 21, and is the only term that is non-zero when the N^* is on the mass shell.

A general vertex could be written

$$V(\pi NN^*) = (a_1 + b_1\gamma_5)(p_\mu + q_\mu) + (a_2 + b_2\gamma_5)p'_\mu + (a_3 + b_3\gamma_5)\gamma_\mu$$
$$+ (a_4 + b_4\gamma_5)\sigma_{\mu\nu}(p_\nu + q_\nu) + (a_5 + b_5\gamma_5)\sigma_{\mu\nu}p'_\nu \qquad (14)$$

The terms with γ_5 are ruled out because the N^* is known to be a p-wave resonance of a (pseudoscalar) pion and a nucleon, so that u_μ transforms like a pseudovector with respect to the index μ; and so does the combination of the pseudoscalar pion and the vectors in equation 14. It is easy to see from equations F-1b and F-2b

$$\gamma_\mu u_\mu(\bar{p}) = 0 \qquad (15a)$$

$$p_\mu u_\mu(\bar{p}) = 0 \qquad (15b)$$

that all terms but the first give zero when acting on an external N^*. But it is quite possible that they give important contributions when the

N^* is an intermediate particle.

The ρNN vertex is the one that is found in reference 20 and corresponds to the coupling of the ρ meson to the nucleon current. The same can be said about the $\rho\pi\pi$ vertex.

For the ρNN^* vertex there is also a more or less arbitrary choice to be made. In equation 10 we have taken a combination of 7 and 8, the γ_5 being added by parity considerations. As in equation 14, there are many other tensors that can be formed with the momenta and γ matrices, most of which give zero contribution with the particles on the mass shell. Equation 11 is similar to 8.

The $\pi\rho\omega$ vertex does not appear in the Lagrangian discussed in reference 20, but it does not violate the known conservation laws of strong interactions. Parity conservation demands that a pseudoscalar formed with the momenta and polarization vectors of the mesons be the form of the interaction; the only one that can be formed is $\epsilon_{\lambda\mu\nu\rho} k_\lambda k'_\mu \epsilon_\nu \epsilon'_\rho$ where $\epsilon_{\lambda\mu\nu\rho}$ is the antisymmetric (Levi-Civita) tensor.

In the 3ρ vertex we have taken what results from the unitary symmetry of reference 20; it should be remembered that the ρ meson couples to the isospin current, of which it is also a source.

Other vertices are forbidden by conservation laws of strong interactions, like G-parity, isospin, parity.

If the πN^* channel is also included in the problem, many other graphs with either one or two external N^*'s have to be considered. Also new vertices, shown in fig. 8, introduce new coupling constants into the problem.

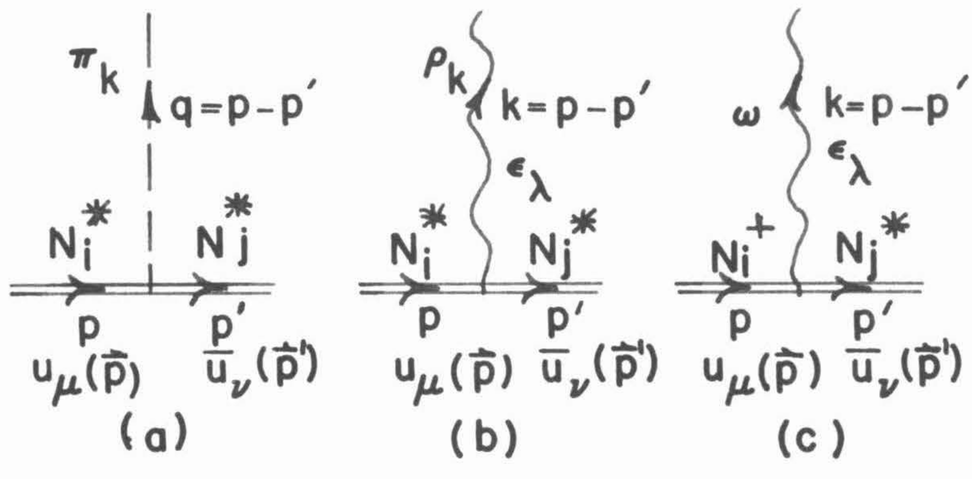

Fig. 8. Vertices with two N^*'s

There are many possible forms for the interactions; a simple choice could be

$$V(\pi N^* N^*) = g_{\pi N^* N^*} \gamma_5 \delta_{\mu\nu} \qquad \text{(see fig. 8a)} \qquad (16)$$

$$V(\rho N^* N^*) = g_{\rho N^* N^*} \gamma_\lambda \delta_{\mu\nu} \qquad \text{(see fig. 8b)} \qquad (17)$$

$$V(\omega N^* N^*) = g_{\omega N^* N^*} \gamma_\lambda \delta_{\mu\nu} \qquad \text{(see fig. 8c)} \qquad (18)$$

Equations 17 and 18 agree with the proposition that the vector mesons are coupled to the current of the N^*, equation F-20c.

c) Isospin dynamics

In section IVb we have not written down the isospin part in vertices and propagators, and we propose to do so here. It should be remembered that the π and ρ have $I = 1$ (vectors in isospin space), the ω has $I = 0$ (scalar), the N has $I = \frac{1}{2}$ (two component spinor) and the N^* has $I = \frac{3}{2}$ (direct product of a vector and a spinor). By considering the possible expressions of adequate rank in isospin space, we find

$$V(\pi NN) = \tau_k \tag{19}$$

$$V(\pi NN^*) = \delta_{jk} 1_2 \tag{20}$$

$$V(\rho NN) = \tau_k \tag{21}$$

$$V(\rho\pi\pi) = -i\epsilon_{ijk} \tag{22}$$

$$V(\rho NN^*) = \delta_{jk} 1_2 \tag{23}$$

$$V(\omega NN) = 1_2 \tag{24}$$

$$V(\pi\rho\omega) = \delta_{jk} \tag{25}$$

$$V(\rho\rho\rho) = -\, i\epsilon_{ijk} \tag{26}$$

$$V(\pi N^* N^*) = \delta_{ij}\tau_k \tag{27}$$

$$V(\rho N^* N^*) = \delta_{ij}\tau_k \tag{28}$$

$$V(\omega N^* N^*) = \delta_{ij} 1_2 \tag{29}$$

Equations III-21a, b, and III-17 when applied to the N^*, are used to reduce the number of possible terms.

The propagators for the isospin part of the wave functions are determined by summing over the isospin states of the particle

$$P = \sum_{\lambda} |\lambda > < \lambda| \tag{30}$$

where $|\lambda>$ is the state vector for $I_z = \lambda$. We get

$$P(\pi) = \delta_{jk} \hat{=} 1_3 \tag{31}$$

$$P(\rho) = \delta_{jk} \hat{=} 1_3 \tag{32}$$

$$P(\omega) = 1 \tag{33}$$

$$P(N) = 1_2 \tag{34}$$

$$P(N^*) = \frac{2}{3}\,\delta_{jk} 1_2 - \frac{1}{3}\, i\epsilon_{ijk}\tau_i \triangleq \frac{2}{3}\, 1_3 1_2 + \frac{1}{3}\, \vec{T}\cdot\vec{\tau} \tag{35}$$

d) Examples

We will show how the determination of the amplitudes corresponding to the graphs in figures 1 to 6 is carried out. For instance, we will take the graph from 1b (see fig. 9). The isospin part goes as follows: with the notation of equation II-6, and using equations 19 and 34,

$$\mathcal{C} = \langle N_c\, \pi_{dj} \,|\tau_k 1_2 \tau_j|\, N_a\, \pi_{bk} \rangle \tag{36}$$

Notice the importance of choosing the indices in a correct way.

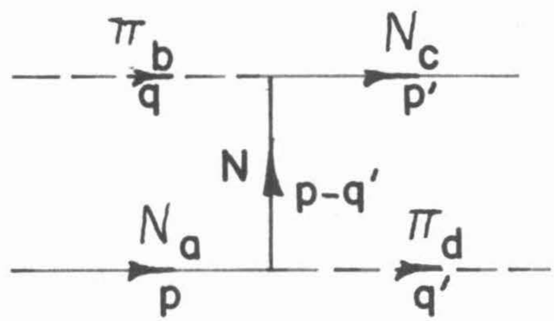

Fig. 9. Nucleon exchange diagram

Equation 17 gives

$$\begin{aligned} \tau_k\tau_j &= \delta_{jk} - i\epsilon_{ijk}\tau_i \\ &= (1 + \vec{T}\cdot\vec{\tau})_{jk} \end{aligned} \tag{37}$$

Accordingly,

$$\mathcal{C}_1 = 1,\ \mathcal{C}_2 = 1 \qquad \mathcal{C}(\tfrac{3}{2}) = 2,\ \ \mathcal{C}(\tfrac{1}{2}) = -1 \tag{38a, b}$$

For the rest of the amplitude, we have

$$\mathcal{A} = \frac{-ig^2_{\pi NN}}{(2\pi)^6}\bar{u}(\vec{p}\,')\gamma_5 \frac{\not{p} - \not{q}' + M}{(p-q')^2 - M^2 + i\epsilon}\gamma_5 u(\vec{p}) \tag{39a}$$

$$\mathcal{A} = -\frac{ig^2_{\pi NN}}{(2\pi)^6}\bar{u}(\vec{p})\frac{\not{p} - \not{q}' - M}{(p-q')^2 - M^2}u(\vec{p})$$

$$\begin{aligned}(p-q')^2 &= M^2 - 2p\cdot q' + \mu^2 \\ &= M^2 - 2p_o q_o' + 2\,\vec{p}\cdot\vec{q}\,' + \mu^2 \\ &= M^2 - 2\,\frac{s+M^2-\mu^2}{2\sqrt{s}}\;\frac{s-M^2+\mu^2}{2\sqrt{s}} - 2\,\frac{s^2-2(M^2+\mu^2)s + (M^2-\mu^2)^2}{4s}\,x \\ &\qquad + \mu^2\end{aligned}$$

Between spinors,

$$\begin{aligned}\not{p} - \not{q}' - M &= M - \not{Q} - M \\ &= -\not{Q}\end{aligned}$$

So, in equation II-9

$$A = 0 \tag{39b}$$

$$B = -\frac{ig_{\pi NN}}{(2\pi)^2}\;\frac{2W^2}{W^4 - 2\mu^2 W^2 - (M^2-\mu^2)^2 + [W^4 - 2(M^2+\mu^2)W^2 + (M^2-\mu^2)^2]\,x} \tag{39c}$$

We write B in the form

$$B = \frac{1}{a + bx} \tag{39d}$$

If we want to find resonances in the $p_{3/2}$ channel, i.e. $J = \frac{3}{2}$ positive parity (see equations II-61 and 63), we have to determine $\beta_1^{3/2}(W)$. To this effect, we determine T_1 and T_2 from equation II-73

$$T_1 = \frac{(E+M)(W-M)}{4\pi W} \frac{1}{a+bx} \tag{40a}$$

$$T_2 = \frac{(E-M)(W+M)}{4\pi W} \frac{1}{a+bx} \tag{40b}$$

and from equation II-71a

$$\beta_1^{\frac{3}{2}} = \frac{1}{2}\int_{-1}^{1}\left[\frac{(E+M)(W-M)}{4\pi W} \frac{P_1(x)}{a+bx} + \frac{(E+M)(W-M)}{4\pi W} \frac{P_2(x)}{a+bx}\right] dx$$

$$= -\frac{(E+M)(W-M)}{4\pi Wb} Q_1(-\frac{a}{b}) - \frac{(E+M)(W-M)}{4\pi Wb} Q_2(-\frac{a}{b})$$

$$= -\frac{(E+M)(W-M)}{4\pi Wb}\left[\frac{a}{2b}\log\left(\frac{a+b}{a-b}\right) - 1\right]$$

$$+ \frac{(E+M)(W-M)}{4\pi WB}\left[\frac{1}{4}\left(\frac{3a^2}{b^2} - 1\right)\log\left(\frac{a+b}{a-b}\right) - \frac{3a}{b}\right] \tag{41}$$

We will also do the calculation for the graph in figure 2e (see fig. 10).

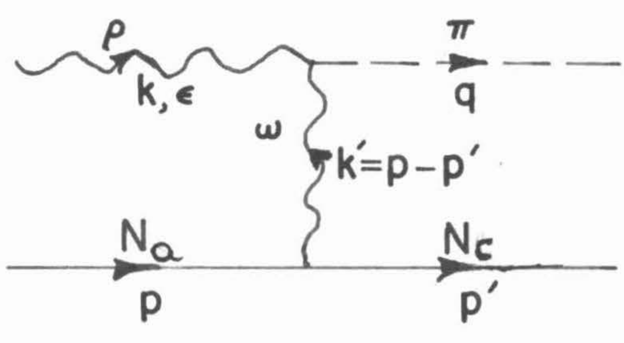

Fig. 10 ω meson exchange diagram

Isospin part:

$$G = \langle N_c \pi_k | 1_2 \delta_{jk} | N_a \rho_j \rangle \tag{42}$$

Hence

$$\mathcal{G}_1 = 1 \qquad \mathcal{G}_2 = 0 \tag{43a}$$

$$\mathcal{G}(\tfrac{3}{2}) = 1 \quad \mathcal{G}(\tfrac{1}{2}) = 1 \tag{43b}$$

Space part:

$$G = \frac{f g_{\pi\rho\omega} g_{\omega NN}}{(2\pi)^6} \bar{u}(\vec{p}\,')\gamma_\mu u(\vec{p}) \frac{\delta_{\mu\nu} - \frac{k'_\mu k'_\nu}{m'^2}}{k'^2 - m'^2 + i\epsilon} \epsilon_{\lambda\sigma\rho\nu} k_\lambda k'_\sigma \epsilon_\rho \tag{44}$$

Between spinors,

$$\not{k}' = \not{p} - \not{p}' = 0$$

$$\gamma_\mu k_\lambda (q_\sigma - k_\sigma)\epsilon_\rho \epsilon_{\lambda\sigma\rho\mu} = \gamma \cdot k \wedge q \wedge \epsilon$$

$$= \gamma_5 (\not{k}\not{q}\not{\epsilon} - q\cdot\epsilon\not{k} - k\cdot q\not{\epsilon})$$

We have used equations E-18 and $k\cdot\epsilon = 0$.

$$\gamma_5 \not{k}\not{q}\not{\epsilon} = \gamma_5 \not{k}(\not{p} + \not{k} - \not{p}')\not{\epsilon}$$

$$= m^2\gamma_5\not{\epsilon} - 2p'\cdot k\,\gamma_5\not{\epsilon} + M\gamma_5\not{\epsilon}\not{k} + 2p\cdot\epsilon\,\gamma_5\not{k} + M\gamma_5\not{\epsilon}\not{k}$$

$$= (m^2 - 2p'\cdot k)\gamma_5\not{\epsilon} + 2M\gamma_5\not{\epsilon}\not{k} + 2P\cdot\epsilon\,\gamma_5\not{k} + q\cdot\epsilon\,\gamma_5\not{k}$$

Hence

$$\gamma\cdot k \wedge q \wedge \epsilon = (m^2 - 2p'\cdot k - k\cdot q)\gamma_5\not{\epsilon} + 2M\gamma_5\not{\epsilon}\not{k} + 2\gamma_5\not{k}P\cdot\epsilon \tag{45}$$

$$\begin{aligned} \mathcal{G}_1 &= (m^2 - 2p'\cdot k - k\cdot q)G & \mathcal{G}_4 &= 2G \\ \mathcal{G}_2 &= 2MG & \mathcal{G}_5 &= 0 \\ \mathcal{G}_3 &= 0 & \mathcal{G}_6 &= 0 \end{aligned} \tag{46}$$

where

$$G(W,x) = \frac{ig_{\pi\rho\omega}g_{\omega NN}}{(2\pi)^6[(p-p')^2-m'^2]} \qquad (46a)$$

If we are still interested in $J = \frac{3}{2}$, positive parity, we have to determine $\beta_1^{3/2}$, $\beta_2^{3/2}$ and $\beta_3^{3/2}$. This can be done if we use equations II-117 and II-119 after expressing the G_i's in terms of W and x.

It should also be remembered that the $a_i^{J'}$s in this case are the elements of $-i\sqrt{\frac{|\vec{q}|}{|k|}}\,\frac{1}{|\vec{p}_r|}T^J$ and not of T^J itself.

e) Discussion of the input data

To define the left hand cuts in our approximation of the N/D method, the different masses and coupling constants have to be determined from experiment or by some theoretical calculation.

The masses of the stable particles (under strong interactions), the pions and nucleons, have been known for a long time, as has the πNN coupling constant. The unstable particles, ρ, ω and N^*, have been found as resonances in 2π, 3π and πN systems in scattering experiments. The energy at which the resonance is observed gives the mass of the corresponding particle. The width of the N^* is related to the πNN^* coupling constant, and that of the ρ to the $\rho\pi\pi$ coupling constant (see reference 21).

If we consider that the ρ meson is coupled to the isospin current with a universal coupling constant, we have a relation between the $\rho\pi\pi$, ρNN, 3ρ and ρN^*N^* coupling constants. The same is true for the ω meson in relation to the hypercharge current, relating the coupling constants for ωNN and ωN^*N^*.

If the mesons and nucleons obey approximately a higher symmetry, like unitary symmetry (see reference 20), the coupling constants involving the ρ or the ω are essentially the same. Further information about the ρNN coupling constant can be extracted from πN scattering data, and about the ωNN one, from the isoscalar nucleon form factor or nucleon nucleon scattering data.

A discussion about the $\omega\pi\rho$ interaction can be found in reference 22.

This would leave without even an approximate determination the ρNN^* and $\pi N^* N^*$ coupling constants, and they should be considered as free parameters to be adjusted so as to give the best agreement of the results of the calculation with experimental data about the πN resonances. Other data that are rather uncertain can also be varied around the assumed value.

APPENDIX A

SYMMETRY OPERATIONS

a) Reflection symmetry

We will make a general analysis of the symmetry pointed out by MacDowell in reference 4, where he finds the transformation properties of the transition amplitudes in the $\pi + N \rightarrow \pi + N$ reaction when $W \rightarrow -W$, that is, under a change of sign in the total energy.

Our definitions of positive and negative energy spinors for particles will be the following:

$$u_+(\vec{p},\lambda) = \sqrt{\frac{E+M}{2M}}\left(1 + \frac{\vec{\alpha}\cdot\vec{p}}{E+M}\right)\begin{pmatrix} |\hat{p},\lambda> \\ 0 \end{pmatrix} \tag{1a}$$

$$u_-(\vec{p},\lambda) = \sqrt{\frac{-E+M}{2M}}\left(1 + \frac{\vec{\alpha}\cdot\vec{p}}{-E+M}\right)\begin{pmatrix} |\hat{p},\lambda> \\ 0 \end{pmatrix}$$

where the two-component spinors $|\hat{p},\lambda>$ satisfy

$$\frac{1}{2}\vec{\sigma}\,\hat{p}\,|\hat{p},\lambda> = \lambda|\hat{p},\lambda> \tag{2}$$

so that λ is the helicity of the state.

Also

$$E = +\sqrt{\vec{p}^2 + M^2} \tag{3}$$

For $\vec{p}$ non-zero, we have $E > M$ and it should be realized that a change of E into $-E$ carries u_+ into u_- and $u_+^\dagger$ into $-u_-^\dagger$.

We next compute

$$\gamma_o\gamma_5 u_+(\vec{p},\lambda) = \gamma_o\gamma_5\sqrt{\frac{E+M}{2M}}\left(1+\sqrt{\frac{E-M}{E+M}}\,\vec{\alpha}\cdot\hat{p}\right)\begin{pmatrix} |\hat{p},\lambda> \\ 0 \end{pmatrix}$$

$$= \sqrt{\frac{E+M}{2M}}\left(1-\sqrt{\frac{E-M}{E+M}}\,\vec{\alpha}\cdot\hat{p}\right)\gamma_o\gamma_5\begin{pmatrix} |\hat{p},\lambda> \\ 0 \end{pmatrix}$$

$$= i\sqrt{\frac{E-M}{2M}}\left(\sqrt{\frac{E+M}{E-M}}-\vec{\alpha}\cdot\hat{p}\right)\begin{pmatrix} 0 \\ -|\hat{p},\lambda> \end{pmatrix}$$

$$= \sqrt{\frac{-E+M}{2M}}\left(-1+\sqrt{\frac{E+M}{E-M}}\,\vec{\alpha}\cdot\hat{p}\right)\vec{\alpha}\cdot\hat{p}\begin{pmatrix} 0 \\ -|\hat{p},\lambda> \end{pmatrix}$$

$$= \sqrt{\frac{-E+M}{2M}}\left(1+\frac{\vec{\alpha}\cdot\vec{p}}{-E+M}\right)\begin{pmatrix} 2\lambda|\hat{p},\lambda> \\ 0 \end{pmatrix}$$

Hence

$$\gamma_o\gamma_5 u_+(\vec{p},\lambda) = 2\lambda u_-(\vec{p},\lambda) \tag{4a}$$

$$\overline{u}_+(\vec{p},\lambda)\gamma_5\gamma_o = 2\lambda\overline{u}_-(\vec{p},\lambda) \tag{4b}$$

For scattering of mesons and nucleons, the following helicity amplitudes are defined as in reference 5:

$$f_{\lambda_c\lambda_d;\lambda_a\lambda_b}(W,\theta,\varphi) = \sum_J (J+\tfrac{1}{2})a^J_{\lambda_c\lambda_d;\lambda_a\lambda_b}(W)e^{i(\lambda-\mu)\varphi}d^J_{\lambda\mu}(\theta) \tag{5}$$

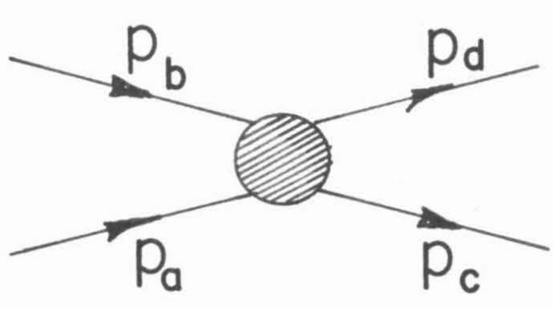

Fig. 1. General diagram

where

$$\lambda = \lambda_a - \lambda_b \qquad\qquad \mu = \lambda_c - \lambda_d \tag{5a}$$

$$a^J_{\lambda_c\lambda_d;\lambda_a\lambda_b}(W) = \langle\lambda_c\lambda_d|\ \frac{1}{|\vec{p}|}\ T^J(W)|\lambda_a\lambda_b\rangle \tag{5b}$$

$$\vec{p} = \frac{m_b\vec{p}_a - m_a\vec{p}_b}{m_a + m_b} \tag{5c}$$

$\vec{p}$ is the relative momentum of the incoming particles. The differential cross section is

$$d\sigma_{(\lambda)}(W,\theta) = |f_{\lambda_c\lambda_d;\lambda_a\lambda_b}(W,\theta,\varphi)|^2 d\Omega \tag{6}$$

Comparing with the expressions of the cross section in terms of Lorentz-invariant amplitudes (II-29, II-94, etc.)

$$f_{\lambda_c\lambda_d;\lambda_a\lambda_b} = \frac{M}{4\pi W}\sqrt{\frac{|\vec{p}_a|}{|\vec{p}_b|}}\ \mathcal{a}_{\lambda_c\lambda_d;\lambda_a\lambda_b} \tag{7}$$

where

$$\mathcal{a}_{\lambda_c\lambda_d;\lambda_a\lambda_b} = \bar{u}_+(\vec{p}_c,\lambda_c)T(W,\lambda_b,\lambda_d)u_+(\vec{p}_a,\lambda_a) \tag{8}$$

From

$$p_{oa} = \frac{W^2 + m_a^2 - m_b^2}{2W} \tag{9}$$

etc., it can be seen that the transformation of $W \to -W$ changes the sign of the energies of all the particles.

From equations B-13 we see that $\vec{\epsilon}(0)$ changes sign when $W \to -W$, but not $\vec{\epsilon}(\pm 1)$. In addition we have

$$\gamma_5\gamma_o(-a_o\gamma_o - \vec{a}\cdot\vec{\gamma}) = (a_o\gamma_o - \vec{a}\cdot\vec{\gamma})\gamma_5\gamma_o \tag{10}$$

$$\epsilon_o(-W) = \frac{\vec{\epsilon}(-W)\cdot\vec{k}}{-k_o} = -(-1)^{1+\lambda}\epsilon_o(W) \tag{11}$$

and, remembering that the amplitudes a_i are always functions of $s = W^2$, it is easy to prove that for the reactions with nucleons and mesons

$$\gamma_5\gamma_o T(-W,\lambda_b,\lambda_d)\gamma_o\gamma_5 = -(-1)^{\lambda_b+\lambda_d} T(W,\lambda_b,\lambda_d) \tag{12}$$

where the T's are defined in part II.

Combining equations 4, 7, 8, 12 and remembering the remark after equation 3 we get

$$f_{\lambda_c\lambda_d;\lambda_a\lambda_b}(-W) = \frac{M}{4\pi W}\sqrt{\frac{|\vec{p}_d|}{|\vec{p}_b|}}\ \bar{u}_-(\vec{p}_c,\lambda_c)T(-W,\lambda_b\lambda_d)u_-(\vec{p}_a,\lambda_a)$$

$$= 4\lambda_a\lambda_c\frac{M}{4\pi W}\sqrt{\frac{|\vec{p}_d|}{|\vec{p}_b|}}\ \bar{u}_+(\vec{p}_c,\lambda_c)\gamma_5\gamma_o T(-W,\lambda_b\lambda_d)\gamma_o\gamma_5 u_+(\vec{p}_a,\lambda_a)$$

$$= -4\lambda_a\lambda_c(-1)^{\lambda_b+\lambda_d}\frac{M}{4\pi W}\sqrt{\frac{|\vec{p}_d|}{|\vec{p}_b|}}\ \bar{u}_+(\vec{p}_c,\lambda_c)T(W,\lambda_b,\lambda_d)u_+(\vec{p}_a,\lambda_a)$$

$$= -4\lambda_a\lambda_c(-1)^{\lambda_b+\lambda_d} f_{\lambda_c\lambda_d;\lambda_a\lambda_b}(W) \tag{13}$$

and hence

$$a^J_{\lambda_c\lambda_d;\lambda_a\lambda_b}(-W) = -4\lambda_a\lambda_c(-1)^{\lambda_b+\lambda_d} a^J_{\lambda_c\lambda_d;\lambda_a\lambda_b}(W)$$

$$= -(-1)^{\lambda-\mu} a^J_{\lambda_c\lambda_d;\lambda_a\lambda_b}(W) \tag{14}$$

If parity is not conserved, then T has factors of the form $a + b\gamma_5$, and

$$\gamma_o(a + b\gamma_5)\gamma_o = a - b\gamma_5 \tag{15}$$

and equation 12 is no longer true.

But it is still possible to relate amplitudes at negative energy with those at positive energy. This is accomplished by using only γ_5 instead of $\gamma_0\gamma_5$. We have, for instance, that

$$\gamma_5 u_+(\vec{p},\lambda) = \gamma_5\sqrt{\frac{E+M}{2M}}\left(1+\sqrt{\frac{E-M}{E+M}}\,\vec{\alpha}\cdot\hat{p}\right)\begin{pmatrix}|\hat{p},\lambda>\\ 0\end{pmatrix}$$

$$= i\sqrt{\frac{E+M}{2M}}\left(1+\sqrt{\frac{E-M}{E+M}}\,\vec{\alpha}\cdot\hat{p}\right)\begin{pmatrix}0\\ |\hat{p},\lambda>\end{pmatrix}$$

$$= \sqrt{\frac{-E+M}{2M}}\left(1+\sqrt{\frac{E+M}{E-M}}\,\vec{\alpha}\cdot\hat{p}\right)\vec{\alpha}\cdot\hat{p}\begin{pmatrix}0\\ |\hat{p},\lambda>\end{pmatrix}$$

$$= 2\lambda\sqrt{\frac{-E+M}{2M}}\left(1+\frac{\vec{\alpha}\cdot(-\vec{p})}{-E+M}\right)\begin{pmatrix}2\lambda e^{2i\lambda\varphi}\,|-\hat{p},-\lambda>\\ 0\end{pmatrix}$$

Hence

$$\gamma_5 u_+(\vec{p},\lambda) = e^{2i\lambda\varphi}u_-(-\vec{p},-\lambda) \tag{16a}$$

$$\bar{u}_+(\vec{p},\lambda)\gamma_5 = e^{-2i\lambda\varphi}\,\bar{u}_-(-\vec{p},-\lambda) \tag{16b}$$

where the relation

$$|\hat{p},\lambda> = 2\lambda e^{2i\lambda\varphi}|-\hat{p},-\hat{\lambda}> \tag{17a}$$

can be derived from equations B-7 by substituting $\theta \to \pi - \theta$, $\varphi \to \varphi + \pi$.

From equation B-12 we get

$$\hat{\epsilon}(\hat{k},\lambda) = (-1)^{1+\lambda}e^{-2i\lambda\varphi}\hat{\epsilon}(-\hat{k},-\lambda) \tag{17b}$$

and also

$$\vec{\epsilon}(\hat{k},\lambda) = e^{-2i\lambda\varphi}\vec{\epsilon}(-\hat{k},-\lambda) \tag{17c}$$

Other needed relations are

$$\lambda_5\not{\epsilon} = -\not{\epsilon}\lambda_5 \tag{18}$$

$$\epsilon(-k,-\lambda) = e^{2i\lambda\varphi}\epsilon(k,\lambda) \tag{19}$$

If we write $\{p_i\}$ to denote the set of the four-momenta of the particles, we can verify for the different reactions that

$$\gamma_5 T(\{-p_i\}, -\lambda_b, -\lambda_d)\gamma_5 = -\eta_g e^{2i(\lambda_b\varphi_o - \lambda_d\varphi)} T(\{p_i\};\lambda_b\lambda_d) \qquad (20)$$

where η_g is the intrinsic parity factor (see equation II-50) and φ_o is arbitrary (the final result will be independent of φ_o).

Equations 7, 8, 16 and 20 give

$$f_{-\lambda_c-\lambda_d;-\lambda_a-\lambda_b}(-W, \pi-\theta, \varphi+\pi) = \frac{M}{4\pi W}\bar{u}_-(-\vec{p}_c, -\lambda_c)T(\{-p_i\}, -\lambda_b, -\lambda_d)u_-(-\vec{p}_a, -\lambda_a)$$

$$= \frac{M}{4\pi W} e^{2i(\lambda_c\varphi - \lambda_a\varphi_o)}\bar{u}_+(\vec{p}_c, \lambda_c)\gamma_5 T(\{-p_i\}, -\lambda_b - \lambda_d)\gamma_5 u_+(\vec{p}_a, \lambda_a)$$

$$= -\eta_g e^{2i(\mu\varphi - \lambda\ \varphi_o)} f_{\lambda_c\lambda_d;\lambda_a\lambda_b}(W, \theta, \varphi) \qquad (21)$$

Actually the function on the left-hand side has a different initial state also, corresponding to a relative momentum in the direction $(\pi, \varphi_o+\pi)$, and equation A-5 can no longer be used. We have to calculate the equivalent of R5-30, which becomes

$$< \pi-\theta, \varphi+\pi; -\lambda_c, -\lambda_d | S(-W) | \pi, \varphi_o+\pi; -\lambda_a, -\lambda_b >$$

$$= \sum_{JJ', MM'} < \pi-\theta, \varphi+\pi; -\lambda_c, -\lambda_d | JM; -\lambda_c-\lambda_d >< JM; -\lambda_c-\lambda_d | S(-W) | J'M'; -\lambda_a-\lambda_b >$$

$$< J'M'; -\lambda_a-\lambda_b | \pi, \varphi_o+\pi; -\lambda_a, -\lambda_b > \qquad (22)$$

From R5-24

$$< \pi-\theta, \varphi+\pi; -\lambda_c, -\lambda_d | JM; -\lambda_c-\lambda_d >$$

$$= \sqrt{\frac{2J+1}{4\pi}}\, e^{i(M+\mu)(\varphi+\pi)} d^J_{M, -\mu}(\pi-\theta) \qquad (23a)$$

$$<\pi, \varphi_o+\pi; -\lambda_a, -\lambda_b \, |J'M'; -\lambda_a -\lambda_b>$$

$$= \sqrt{\frac{2J'+1}{4\pi}} \; e^{i(M'+\lambda)(\varphi_o+\pi)} (-1)^{J'+M'} \delta_{M'\lambda} \qquad (23b)$$

$$\lambda = \lambda_a - \lambda_b \qquad \mu = \lambda_c - \lambda_d$$

$$< JM; -\lambda_c, -\lambda_d \, | S(-W) \, | J'M'; -\lambda_a -\lambda_b >$$

$$= \; < JM; -\lambda_c -\lambda_d \, | S^J(-W) \, | JM; -\lambda_a -\lambda_b > \delta_{JJ'} \delta_{MM'} \qquad (23c)$$

and substituting in 22:

$$<\pi-\theta, \varphi+\pi; -\lambda_c, -\lambda_d \, | S(-W) \, | \pi, \varphi_o+\pi; -\lambda_a, -\lambda_b >$$

$$= \sum_J \frac{2J+1}{4\pi} e^{i(\lambda+\mu)(\varphi+\pi)} d^J_{\lambda, -\mu}(\pi-\theta) < JM; -\lambda_c, -\lambda_d \, | S^J(-W) \, | JM; -\lambda_a -\lambda_b >$$

$$\times \; e^{-2i\lambda(\varphi_o+\pi)} (-1)^{J+\lambda} \qquad (24)$$

and since

$$\frac{d\sigma}{d\Omega} = \left(\frac{2\pi}{|\vec{p}|} \right)^2 | < \theta\varphi\lambda_c\lambda_d \, | T(E) \, | 00\lambda_a\lambda_b > |^2 \qquad (25)$$

we obtain for the modified $f_{(-\lambda)}$:

$$f_{-\lambda_c, -\lambda_d; -\lambda_a -\lambda_b}(-W, \pi-\theta, \varphi+\pi)$$

$$= \sum_J (J+\tfrac{1}{2})(-1)^{\mu-\lambda} e^{i(\lambda+\mu)\varphi} e^{-2i\lambda\varphi_o} d^J_{\lambda\mu}(\theta) a^J_{-\lambda_c -\lambda_d, -\lambda_a -\lambda_b}(-W) \qquad (26)$$

where C-4 has been used.

Equation 21 then becomes:

$$\sum_J (J+\tfrac{1}{2})(-1)^{\mu-\lambda} e^{i(\lambda+\mu)\varphi} e^{-2i\lambda\varphi_o} d^J_{\lambda\mu}(\theta) a^J_{-\lambda_c-\lambda_d;-\lambda_a-\lambda_b}(-W)$$

$$= -\sum_J (J+\tfrac{1}{2}) e^{i(\lambda-\mu)\varphi} d^J_{\lambda\mu}(\theta) a^J_{\lambda_c\lambda_d;\lambda_a\lambda_b}(\theta) e^{2i(\mu\varphi-\lambda\varphi_o)}$$

and hence

$$a^J_{-\lambda_c-\lambda_d;-\lambda_a-\lambda_b}(-W) = -\eta_g(-1)^{\lambda-\mu} a^J_{\lambda_c\lambda_d;\lambda_a\lambda_b}(W) \tag{27}$$

If parity is conserved, we have

$$a^J_{-\lambda_c,-\lambda_d;-\lambda_a-\lambda_b}(W) = \eta_g a^J_{\lambda_c\lambda_d;\lambda_a\lambda_b}(W) \tag{28}$$

and we get back equation 14.

In this second form, the symmetry corresponds to a space-time reflection, i.e., a change of direction of all four axes in space-time. Energy and momentum change sign, since they are components of a four-vector. Spin can be considered either as a pseudovector or components of a second rank tensor, and does not change sign. Hence the helicity, that is, the projection of the spin on the momentum of a particle, does change sign.

We notice, finally, that the exponential in the state vector for plane waves is not changed by this operation, since the coordinate four-vector also changes sign:

$$e^{-ip\cdot x} = e^{-i(-p)\cdot(-x)} \tag{29}$$

In the first part of the discussion of the reflection symmetry we have combined this operation with the parity operation (to be discussed briefly in section b of this appendix), the combination of both being obviously equivalent to time reflection (not to be confused with time reversal).

Sometimes in reactions like $\rho + \rho \rightarrow N + \overline{N}$ we have matrix amplitudes of the form

$$\mathcal{C} = \overline{u}(\vec{p})\, T\, v(\vec{p}\,') \tag{30}$$

where $v(\vec{p})$ is a spinor for the antiparticle; and the effect of the symmetry operators can be seen from

$$v_+(\vec{p}, \lambda) = \sqrt{\frac{E+M}{2M}}\left(1 + \frac{\vec{\alpha}\cdot\vec{p}}{E+M}\right)\begin{pmatrix} 0 \\ |\hat{p}, -\lambda> \end{pmatrix}$$

$$\begin{aligned}
\gamma_5 v_+(\vec{p}, \lambda) &= i\sqrt{\frac{E+M}{2M}}\left(1 + \sqrt{\frac{E-M}{E+M}}\,\vec{\alpha}\cdot\hat{p}\right)\begin{pmatrix} |\hat{p}, -\lambda> \\ 0 \end{pmatrix} \\
&= \sqrt{\frac{-E+M}{2M}}\left(1 - \frac{\vec{\alpha}\cdot\vec{p}}{-E+M}\right)\vec{\alpha}\cdot\hat{p}\begin{pmatrix} |\hat{p}, -\lambda> \\ 0 \end{pmatrix} \\
&= \sqrt{\frac{-E+M}{2M}}\left(1 - \frac{\vec{\alpha}\cdot\vec{p}}{-E+M}\right)\begin{pmatrix} 0 \\ -2\lambda|\hat{p}, -\lambda> \end{pmatrix} \\
&= -2\lambda\sqrt{\frac{-E+M}{2M}}\left(1 + \frac{\vec{\alpha}\cdot(-\vec{p})}{-E+M}\right)\begin{pmatrix} 0 \\ -2\lambda e^{-2i\lambda\varphi}|-\hat{p}, \lambda> \end{pmatrix}
\end{aligned} \tag{31}$$

Hence

$$\gamma_5 v_+(\vec{p}, \lambda) = e^{-2i\lambda\varphi}\, v_-(-\vec{p}, -\lambda) \tag{32a}$$

$$\overline{v}_+(\vec{p}, \lambda)\gamma_5 = e^{2i\lambda\varphi}\, \overline{v}_-(-\vec{p}, -\lambda) \tag{32b}$$

Notice the difference in sign of the phase factors when compared to those in equations 16.

b) Parity

This operation corresponds to a change of direction of all three space axes; position vectors and momenta change sign, spin does not, and, hence, helicity also changes sign.

The operator for spinors is γ_o:

$$\gamma_o u_+(\vec{p},\lambda) = \gamma_o \sqrt{\frac{E+M}{2M}} \left(1 + \frac{\vec{a}\cdot\vec{p}}{E+M}\right)\begin{pmatrix} |\hat{p},\lambda > \\ 0 \end{pmatrix}$$

$$= \sqrt{\frac{E+M}{2M}} \left(1 - \frac{\vec{a}\cdot\vec{p}}{E+M}\right)\begin{pmatrix} |\hat{p},\lambda > \\ 0 \end{pmatrix}$$

$$= 2\lambda e^{2i\lambda\varphi} \sqrt{\frac{E+M}{2M}} \left(1 + \frac{\vec{a}\cdot(-\vec{p})}{E+M}\right)\begin{pmatrix} |-\hat{p},-\lambda > \\ 0 \end{pmatrix}$$

$$\gamma_o u_+(\vec{p},\lambda) = 2\lambda e^{2i\lambda\varphi} u_+(-\vec{p},-\lambda) \tag{33}$$

As to the effect on the matrix in spin space T, it can be seen from

$$\gamma_o(a_o\gamma_o - \vec{a}\cdot\vec{\gamma}) = (a_o\gamma_o - (-\vec{a})\cdot\vec{\gamma})\gamma_o \tag{34a}$$

$$a_o b_o - \vec{a}\cdot\vec{b} = a_o b_o - (-\vec{a})\cdot(-\vec{b}) \tag{34b}$$

$$\gamma_o\gamma_5 = -\gamma_5\gamma_o \tag{34c}$$

It becomes obvious that if parity is to be conserved, the matrix T should not contain terms like that in equation 15.

The effect on helicity amplitudes is studied in reference 5 and is summarized by equation R5-43.

$$< -\lambda_c -\lambda_d | S^J | -\lambda_a -\lambda_b > = \eta_g < \lambda_c \lambda_d | S^J | \lambda_a \lambda_b > \tag{35}$$

c) Time reversal

This symmetry operation, more aptly called "reversal of the direction of motion" by Wigner on p. 325 of reference 11, corresponds to the operation of time inversion in classical mechanics. Here, together with the direction of time, the signs of momenta, spins and quantities whose classical expression contains the time to an odd power, are changed. Consequently, helicity is unaffected.

In this case, care has to be taken with the exponential in the plane wave, so we will write

$$\psi(\vec{x}, t; \vec{p}, \lambda) = u_+(\vec{p}, \lambda) e^{-ip\cdot x} \tag{36}$$

The operator for time reversal is antiunitary (see reference 11, page 326 ff.), and we have for spinors

$$\eta\psi(\vec{x}, -t; -\vec{p}, \lambda) = \gamma_0\gamma_2\gamma_5 K\psi(\vec{x}, t; \vec{p}, \lambda) \tag{36a}$$

where K stands for the operation of complex conjugation.

Indeed we have

$$\gamma_0\gamma_2\gamma_5 K\psi(\vec{x}, t; \vec{p}, \lambda) = \gamma_0\gamma_2\gamma_5 K\sqrt{\frac{E+M}{2M}}\left(1 + \frac{\vec{\alpha}\cdot\vec{p}}{E+M}\right)\begin{pmatrix} |\hat{p}, \lambda> \\ 0 \end{pmatrix} e^{-ip\cdot x}$$

$$= \gamma_0\gamma_2\gamma_5\sqrt{\frac{E+M}{2M}}\left(1 + \frac{\vec{\alpha}^*\cdot\vec{p}}{E+M}\right)\begin{pmatrix} |\hat{p}, \lambda>^* \\ 0 \end{pmatrix} e^{ip\cdot x}$$

$$= i\gamma_0\gamma_2\sqrt{\frac{E+M}{2M}}\left(1 + \frac{\vec{\alpha}^*\cdot\vec{p}}{E+M}\right)\begin{pmatrix} 0 \\ |\hat{p}, \lambda>^* \end{pmatrix} e^{i(Et-\vec{p}\cdot\vec{x})}$$

$$= i\alpha_2\sqrt{\frac{E+M}{2M}}\left(1 + \frac{\vec{\alpha}\cdot\vec{p}}{E+M}\right)\begin{pmatrix} 0 \\ |\hat{p}, \lambda>^* \end{pmatrix} e^{-i[E(-t)-(-\vec{p})\cdot\vec{x}]}$$

$$= \sqrt{\frac{E+M}{2M}}\left(1 - \frac{\vec{\alpha}\cdot\vec{p}}{E+M}\right)\begin{pmatrix} i\sigma_2|\hat{p}, \lambda>^* \\ 0 \end{pmatrix} e^{-i[E(-t)-(-\vec{p})\cdot\vec{x}]}$$

$$= e^{-2i\lambda\varphi}\sqrt{\frac{E+M}{2M}}\left(1+\frac{\vec{\alpha}\cdot(-\vec{p})}{E+M}\right)\begin{pmatrix} |-\hat{p},\lambda> \\ 0 \end{pmatrix} e^{-i[E(-t)-(-\vec{p})\cdot\vec{x}]}$$

$$= e^{-2i\lambda\varphi}\psi(\vec{x}, -t; -\vec{p}, \lambda)$$

Hence the phase factor η is determined:

$$\eta = e^{-2i\lambda\varphi} \qquad (36b)$$

We have used relations such as

$$\gamma_5\vec{\alpha}^* = \vec{\alpha}^*\gamma_5 \qquad (37a)$$

$$\alpha_2\vec{\alpha}^* = -\vec{\alpha}\,\alpha_2 \qquad (37b)$$

$$i\sigma_2|\hat{p},\lambda>^* = e^{-2i\lambda\varphi}|-\hat{p},\lambda> \qquad (37c)$$

which are easy to check.

If we call

$$U = \gamma_o\gamma_2\gamma_5 \qquad (38a)$$

we have

$$U^\dagger = U^{-1} = \gamma_5\gamma_2\gamma_o \qquad (38b)$$

$$U^* = U \qquad (38c)$$

$$Uu_+(\vec{p},\lambda) = e^{2i\lambda\varphi}u_+(-\vec{p},\lambda) \qquad (39a)$$

$$\bar{u}_+(\vec{p},\lambda)U^{-1} = e^{-2i\lambda\varphi}\bar{u}_+^*(-\vec{p},\lambda) \qquad (39b)$$

Then, remembering that $\mathcal{C}$ is a number, we have:

$$\mathcal{C} = \bar{u}(\vec{p}\,',\lambda')Tu(\vec{p},\lambda)$$

$$= \bar{u}(\vec{p}\,',\lambda')U^{-1}UTU^{-1}Uu(\vec{p},\lambda)$$

$$= e^{2i(\lambda-\lambda')\varphi}\,\bar{u}^*(-\vec{p}\,',\lambda')UTU^{-1}u^*(-\vec{p},\lambda)$$

$$= u^\dagger(-\vec{p},\lambda)(UTU^{-1})^T\gamma_o u(-\vec{p}\,',\lambda')$$

$$= \bar{u}(-\vec{p},\lambda)\gamma_2\gamma_5 T^T\gamma_5\gamma_2 u(-\vec{p}\,',\lambda') \qquad (40)$$

With our particular choice for γ_μ, we obviously have

$$\gamma_m^T = \gamma_m \qquad m = 0,\ 2,\ 5 \qquad (41a)$$

$$\gamma_m^T = -\gamma_m \qquad m = 1,\ 3 \qquad (41b)$$

and hence

$$\gamma_5\gamma_2\gamma_o^T\gamma_2\gamma_5 = \gamma_o \qquad (42a)$$

$$\gamma_5\gamma_2\gamma_m^T\gamma_2\gamma_5 = -\gamma_m \qquad m = 1, 2, 3, 5 \qquad (42b)$$

$$\gamma_5\gamma_2\not{a}^T\gamma_2\gamma_5 = a_o\gamma_o - (-\vec{a})\cdot\vec{\gamma} \qquad (42c)$$

$$\gamma_5\gamma_2(\not{a}\not{b}\ldots\not{c})^T\gamma_2\gamma_5 = (c_o\gamma_o + \vec{c}\cdot\vec{\gamma})\ldots(b_o\gamma_o + \vec{b}\cdot\vec{\gamma})(a_o\gamma_o + \vec{a}\cdot\vec{\gamma}) \qquad (42d)$$

Also from B-12 we get

$$\vec{\epsilon}(\hat{k},\lambda) = -e^{-2i\lambda\varphi}\vec{\epsilon}^{\,*}(-\hat{k},\lambda) \qquad (43a)$$

$$\epsilon_o(\vec{k},\lambda) = e^{-2i\lambda\varphi}\epsilon_o^*(-\vec{k},\lambda) \qquad (43b)$$

and hence

$$p\cdot\epsilon(\vec{k},\lambda) = p_o\epsilon_o(\vec{k},\lambda) - \vec{p}\cdot\vec{\epsilon}(\hat{k},\lambda)$$

$$= e^{-2i\lambda\varphi}[p_o\epsilon_o^*(-\vec{k},\lambda) - (-\vec{p})\cdot\vec{\epsilon}^{\,*}(-\hat{k},\lambda)] \qquad (43c)$$

Equations 42d and 43c show what the operator does to T: it essentially reverses the order of the factors and changes the signs of the three-momenta. The effect of the change of sign in t is to interchange the initial and final states; this shows already for the spinors in equation 40 and the complex conjugate of the polarization vector in 43c.

If a theory is invariant under time reversal, the matrix element G should at most be multiplied by a phase factor when initial and final states are interchanged and the sign of all momenta changed, leaving the helicities unaffected. If we express the results of our calculations by

$$\gamma_2\gamma_5[T(\vec{p}_i, \vec{p}_f; \lambda_b\lambda_d)]^T\gamma_5\gamma_2 = \eta' T_R(-\vec{p}_i, -\vec{p}_f; \lambda_b\lambda_d) \qquad (44)$$

where the subscript R is a reminder of the reverse order of the factors.

Invariance then demands that

$$T(-\vec{p}_f, -\vec{p}_i; \lambda_d, \lambda_b) = T_R(-\vec{p}_i, -\vec{p}_f; \lambda_b, \lambda_d) \qquad (45)$$

Then the invariants M_i of equation II-7, where

$$T = G_i M_i \qquad (46)$$

should be chosen so that an exchange of the initial and final momenta and polarization vector, together with a reversal in the order of the matrices involved, leaves them unchanged.

Notice that the phase factors in equations 39 and 43 will produce a factor $e^{2i(\lambda-\mu)\varphi}$ necessary to get the right φ dependence in equation 5 when λ and μ have been interchanged.

The effect on the partial wave helicity amplitudes is given in R5-55

$$< \lambda_c\lambda_d|S^J|\lambda_a\lambda_b > = < \lambda_a\lambda_b|S^J|\lambda_c\lambda_d > \qquad (47)$$

APPENDIX B

DETERMINATION OF HELICITY STATES FOR PARTICLES OF SPIN ½ AND 1

a) Spin ½

We follow the conventions in reference 5, and compute first the rotation matrix $R_{\varphi,\theta,0}$ [‡]

$$R_{\alpha\beta\gamma} = e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z} \tag{1}$$

$$J_k = \tfrac{1}{2}\sigma_k \tag{2}$$

Remembering that

$$\sigma_k^2 = 1 \tag{3}$$

we get by series expansion

$$e^{-i\frac{1}{2}\alpha\sigma_k} = \cos\frac{\alpha}{2} - i\sigma_k \sin\frac{\alpha}{2} \tag{4}$$

Hence

$$R_{\varphi,\theta,0} = \begin{pmatrix} e^{-i\frac{\varphi}{2}} & 0 \\ 0 & e^{i\frac{\varphi}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\frac{\varphi}{2}} & -\sin\frac{\theta}{2}\, d^{-i\frac{\varphi}{2}} \\ \sin\frac{\theta}{2}\, e^{i\frac{\varphi}{2}} & \cos\frac{\theta}{2}\, e^{i\frac{\varphi}{2}} \end{pmatrix} \tag{5}$$

The helicity states are

$$|\theta,\varphi;\lambda\rangle = R_{\varphi,\theta,-\varphi}\psi_\lambda = e^{i\lambda\varphi} R_{\varphi,\theta,0}\psi_\lambda \tag{6}$$

$$|\theta,\varphi,\tfrac{1}{2}\rangle = e^{i\frac{1}{2}\varphi} R_{\varphi,\theta,0}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\, e^{i\varphi} \end{pmatrix} \tag{7a}$$

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‡ See reference 6 section 13.

$$|\theta, \varphi, -\tfrac{1}{2}> = e^{-\frac{1}{2}i\varphi} R_{\varphi, \theta, 0} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\frac{\theta}{2} e^{-i\varphi} \\ \cos\frac{\theta}{2} \end{pmatrix} \qquad (7b)$$

b) **Spin 1**

The vector mesons will be labeled as particles b and d, and the

$$\chi_\lambda = (-1)^{s-\lambda} e^{-i\pi J_y} \psi_\lambda \qquad (8)$$

have to be used (see R5-13).

Since we are dealing here with ordinary three-vectors, the rotation matrices are well known; but they are also easy to determine from equation 1:

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad (9a)$$

$$J_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \qquad (9b)$$

$$J_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (9c)$$

from R6-5.42; the corresponding ψ_λ are the usual polarization vectors given in R6-5.44.

$$e^{i\alpha J_k} = (1 - J_k^2) + J_k^2 \cos\alpha - i\, J_k \sin\alpha \qquad (10)$$

$$R_{\varphi,\theta,0} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta\cos\varphi & -\sin\varphi & \sin\theta\cos\varphi \\ \cos\theta\sin\varphi & \cos\varphi & \sin\theta\sin\varphi \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \qquad (11)$$

We then calculate

$$\hat{\epsilon}(+1) = e^{-i\varphi} R_{\theta,\varphi,0} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

$$= \frac{e^{-i\varphi}}{\sqrt{2}} \begin{pmatrix} \cos\theta\cos\varphi + i\sin\varphi \\ \cos\theta\sin\varphi - i\cos\varphi \\ -\sin\theta \end{pmatrix} \qquad (12a)$$

$$\hat{\epsilon}(0) = R_{\theta,\varphi,0} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} \qquad (12b)$$

$$\hat{\epsilon}(-1) = e^{i\varphi} R_{\theta,\varphi,0} \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ -i \\ 0 \end{pmatrix}$$

$$= \frac{e^{i\varphi}}{\sqrt{2}} \begin{pmatrix} -\cos\theta\cos\varphi + i\sin\varphi \\ -\cos\theta\sin\varphi - i\cos\varphi \\ \sin\theta \end{pmatrix} \qquad (12c)$$

and hence, using II-89f to obtain the correct normalization for the four-vector ϵ,

$$\bar{\epsilon}(+1) = \frac{e^{-i\varphi}}{\sqrt{2}} \begin{pmatrix} \cos\theta\cos\varphi + i\sin\varphi \\ \cos\theta\sin\varphi - i\cos\varphi \\ -\sin\theta \end{pmatrix} \qquad (13a)$$

$$\vec{\epsilon}(0) = \frac{k_o}{m} \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} \tag{13b}$$

$$\vec{\epsilon}(-1) = \frac{e^{i\varphi}}{\sqrt{2}} \begin{pmatrix} -\cos\theta\cos\varphi + i\sin\varphi \\ -\cos\theta\sin\varphi - i\cos\varphi \\ \sin\theta \end{pmatrix} \tag{13c}$$

It should be remembered that the helicities correspond to particles moving in the direction of $-\vec{p}$, and that θ and φ define the direction of $+\vec{p}$.

APPENDIX C

$d^J_{\lambda\mu}$ FUNCTIONS

a) Useful symmetries:

$$d^J_{\lambda\mu}(\theta) = (-1)^{\lambda-\mu} d^J_{\mu\lambda}(\theta) \tag{1}$$

$$= (-1)^{J-\mu} d^J_{-\lambda,\,\mu}(\pi-\theta) \tag{2}$$

$$= (-1)^{J+\lambda} d^J_{\mu,\,-\lambda}(\pi-\theta) \tag{3}$$

$$= (-1)^{J+\lambda} d^J_{\lambda,\,-\mu}(\pi-\theta) \tag{4}$$

$$= (-1)^{J-\mu} d^J_{-\mu,\,\lambda}(\pi-\theta) \tag{5}$$

$$= (-1)^{\lambda-\mu} d^J_{-\lambda,\,-\mu}(\theta) \tag{6}$$

$$= d^J_{-\mu,\,-\lambda}(\theta) \tag{7}$$

b) Half integer indices (table)

$$x = \cos\theta \qquad J = \ell + \frac{1}{2} \tag{8a, b}$$

$$d^J_{\frac{1}{2},\frac{1}{2}}(\theta) = \frac{\cos\frac{\theta}{2}}{\ell+1}\left[P'_{\ell+1}(x) - P'_{\ell}(x)\right] \tag{9}$$

$$= \frac{1}{2\cos\frac{\theta}{2}}\left[P_{\ell+1}(x) + P_{\ell}(x)\right] \tag{10}$$

$$d^J_{-\frac{1}{2},\frac{1}{2}}(\theta) = \frac{\sin\frac{\theta}{2}}{\ell+1}\left[P'_{\ell+1}(x) + P'_{\ell}(x)\right] \tag{11}$$

$$= \frac{1}{2\sin\frac{\theta}{2}}\left[-P_{\ell+1}(x) + P_{\ell}(x)\right] \tag{12}$$

$$d^J_{\frac{3}{2},\frac{1}{2}}(\theta) = \frac{2\cos^2\frac{\theta}{2}\sin\frac{\theta}{2}}{(\ell+1)\sqrt{\ell(\ell+2)}}\left[-P''_{\ell+1}(x) + P''_{\ell}(x)\right] \tag{13}$$

$$= \frac{\sqrt{\ell(\ell+2)}}{4\cos^2\frac{\theta}{2}\sin\frac{\theta}{2}}\left[\frac{P_{\ell+2}(x)}{2\ell+3} + \frac{P_{\ell+1}(x)}{2\ell+1} - \frac{P_{\ell}(x)}{2\ell+3} - \frac{P_{\ell-1}(x)}{2\ell+1}\right] \tag{14}$$

$$d^J_{-\frac{3}{2},\frac{1}{2}}(\theta) = \frac{2\cos\frac{\theta}{2}\sin^2\frac{\theta}{2}}{(\ell+1)\sqrt{\ell(\ell+2)}}\left[P''_{\ell+1}(x) + P''_{\ell}(x)\right] \tag{15}$$

$$= \frac{\sqrt{\ell(\ell+2)}}{4\cos\frac{\theta}{2}\sin^2\frac{\theta}{2}}\left[\frac{P_{\ell+2}(x)}{2\ell+3} - \frac{P_{\ell+1}(x)}{2\ell+1} - \frac{P_{\ell}(x)}{2\ell+3} + \frac{P_{\ell-1}(x)}{2\ell+1}\right] \tag{16}$$

$$d^J_{\frac{3}{2},\frac{3}{2}}(\theta) = \frac{2\cos^3\frac{\theta}{2}}{\ell(\ell+1)(\ell+2)}\left[\frac{\ell P'''_{\ell+2}(x)}{2\ell+3} - \frac{3\ell P'''_{\ell+1}(x)}{2\ell+1} + \frac{3(\ell+2)P'''_{\ell}(x)}{2\ell+3} - \frac{(\ell+2)P'''_{\ell-1}(x)}{2\ell+1}\right] \tag{17}$$

$$= \frac{1}{4\cos^3\frac{\theta}{2}}\left[\frac{\ell P_{\ell+2}(x)}{2\ell+3} + \frac{3\ell P_{\ell+1}(x)}{2\ell+1} + \frac{3(\ell+2)P_{\ell}(x)}{2\ell+3} + \frac{(\ell+2)P_{\ell-1}(x)}{2\ell+1}\right] \tag{18}$$

$$d^J_{-\frac{3}{2},\frac{3}{2}}(\theta) = \frac{2\sin^3\frac{\theta}{2}}{\ell(\ell+1)(\ell+2)}\left[\frac{\ell P'''_{\ell+2}(x)}{2\ell+3} + \frac{3\ell P'''_{\ell+1}(x)}{2\ell+1} + \frac{3(\ell+2)P'''_{\ell}(x)}{2\ell+3} + \frac{(\ell+2)P'''_{\ell-1}(x)}{2\ell+1}\right] \tag{19}$$

$$= \frac{1}{4\sin^3\frac{\theta}{2}}\left[-\frac{\ell P_{\ell+2}(x)}{2\ell+3} + \frac{3\ell P_{\ell+1}(x)}{2\ell+1} - \frac{3(\ell+2)P_{\ell}(x)}{2\ell+3} + \frac{(\ell+2)P_{\ell-1}(x)}{2\ell+1}\right] \tag{20}$$

$$d^J_{\frac{5}{2},\frac{1}{2}}(\theta) = \frac{4\cos^3\frac{\theta}{2}\sin^2\frac{\theta}{2}}{(\ell+1)\sqrt{(\ell-1)\ell(\ell+2)(\ell+3)}}\left[P'''_{\ell+1}(x) - P'''_{\ell}(x)\right] \tag{21}$$

$$= \frac{\sqrt{(\ell-1)\ell(\ell+2)(\ell+3)}}{8\cos^3\frac{\theta}{2}\sin^2\frac{\theta}{2}}\left[\frac{P_{\ell+3}(x)}{(2\ell+5)(2\ell+3)} + \frac{P_{\ell+2}(x)}{(2\ell+3)(2\ell+1)} - \frac{2P_{\ell+1}(x)}{(2\ell+5)(2\ell+1)} - \frac{2P_{\ell}(x)}{(2\ell+3)(2\ell-1)} + \frac{P_{\ell-1}(x)}{(2\ell+3)(2\ell+1)} + \frac{P_{\ell-2}(x)}{(2\ell+1)(2\ell-1)}\right] \tag{22}$$

$$d^J_{-\frac{5}{2},\frac{1}{2}}(\theta) = \frac{4\cos^2\frac{\theta}{2}\sin^3\frac{\theta}{2}}{(\ell+1)\sqrt{(\ell-1)\ell(\ell+2)(\ell+3)}}\left[P'''_{\ell+1}(x) + P'''_{\ell}(x)\right] \tag{23}$$

$$= \frac{\sqrt{(\ell-1)\ell(\ell+2)(\ell+3)}}{8\cos^2\frac{\theta}{2}\sin^3\frac{\theta}{2}}\left[-\frac{P_{\ell+3}(x)}{(2\ell+5)(2\ell+3)} + \frac{P_{\ell+2}(x)}{(2\ell+3)(2\ell+1)} + \frac{2P_{\ell+1}(x)}{(2\ell+5)(2\ell+1)} - \frac{2P_{\ell}(x)}{(2\ell+3)(2\ell-1)} - \frac{P_{\ell-1}(x)}{(2\ell+3)(2\ell+1)} + \frac{P_{\ell-2}(x)}{(2\ell+1)(2\ell-1)}\right] \tag{24}$$

$$d^J_{\frac{5}{2},\frac{3}{2}}(\theta) = \frac{4\cos^4\frac{\theta}{2}\sin\frac{\theta}{2}}{\ell(\ell+1)(\ell+2)\sqrt{(\ell-1)(\ell+3)}}\left[-\frac{\ell P^{(4)}_{\ell+2}(x)}{2\ell+3} + \frac{3\ell P^{(4)}_{\ell+1}(x)}{2\ell+1} - \frac{3(\ell+2)P^{(4)}_{\ell}(x)}{2\ell+3} + \frac{(\ell+2)P^{(4)}_{\ell-1}(x)}{2\ell+1}\right] \tag{25}$$

$$= \frac{\sqrt{(\ell-1)(\ell+3)}}{8\cos^4\frac{\theta}{2}\sin\frac{\theta}{2}}\left[\frac{\ell P_{\ell+3}}{(2\ell+5)(2\ell+3)} + \frac{3\ell P_{\ell+2}}{(2\ell+3)(2\ell+1)} + \frac{2(\ell+5)P_{\ell+1}}{(2\ell+5)(2\ell+1)} - \frac{2(\ell-3)P_{\ell}}{(2\ell+3)(2\ell-1)} - \frac{3(\ell+2)P_{\ell-1}}{(2\ell+3)(2\ell+1)} - \frac{(\ell+2)P_{\ell-2}}{(2\ell+1)(2\ell-1)}\right] \tag{26}$$

$$d^J_{-\frac{5}{3},\frac{3}{2}}(\theta) = \frac{4\cos\frac{\theta}{2}\sin^4\frac{\theta}{2}}{\ell(\ell+1)(\ell+2)\sqrt{(\ell-1)(\ell+3)}}\left[\frac{\ell P^{(4)}_{\ell+2}(x)}{2\ell+3} + \frac{3\ell P^{(4)}_{\ell+1}(x)}{2\ell+1} + \frac{3(\ell+2)P^{(4)}_{\ell-1}}{2\ell+3} + \frac{(\ell+2)P^{(4)}_{\ell-1}}{2\ell+1}\right] \tag{27}$$

$$= \frac{\sqrt{(\ell-1)(\ell+3)}}{8\cos\frac{\theta}{2}\sin^4\frac{\theta}{2}}\left[\frac{\ell P_{\ell+3}(x)}{(2\ell+5)(2\ell+3)} - \frac{3\ell P_{\ell+2}(x)}{(2\ell+3)(2\ell+1)} + \frac{2(\ell+5)P_{\ell+1}(x)}{(2\ell+5)(2\ell+1)} + \frac{2(\ell-3)P_{\ell}(x)}{(2\ell+3)(2\ell-1)} - \frac{3(\ell+2)P_{\ell-1}(x)}{(2\ell+3)(2\ell+1)} + \frac{(\ell+2)P_{\ell-2}}{(2\ell+1)(2\ell-1)}\right] \tag{28}$$

$$d^J_{\frac{5}{2},\frac{5}{2}}(\theta) = \frac{4\cos^5\frac{\theta}{2}}{(\ell-1)\ell(\ell+1)(\ell+2)(\ell+3)}\left[\frac{(\ell-1)\ell P^{(5)}_{\ell+3}(x)}{(2\ell+5)(2\ell+3)} - \frac{5(\ell-1)\ell P^{(5)}_{\ell+2}(x)}{(2\ell+3)(2\ell+1)}\right.$$
$$+ \frac{10(\ell-1)(\ell+3)P^{(5)}_{\ell+1}(x)}{(2\ell+5)(2\ell+1)} - \frac{10(\ell-1)(\ell+3)P^{(5)}_{\ell}(x)}{(2\ell+3)(2\ell-1)}$$
$$\left. + \frac{5(\ell+2)(\ell+3)P^{(5)}_{\ell-1}(x)}{(2\ell+3)(2\ell+1)} - \frac{(\ell+2)(\ell+3)P^{(5)}_{\ell-2}(x)}{(2\ell+1)(2\ell-1)}\right] \tag{29}$$

$$= \frac{1}{8\cos^5\frac{\theta}{2}}\left[\frac{(\ell-1)\ell P_{\ell+3}(x)}{(2\ell+5)(2\ell+3)} + \frac{5(\ell-1)\ell P_{\ell+2}(x)}{(2\ell+3)(2\ell+1)}\right.$$
$$+ \frac{10(\ell-1)(\ell+3)P_{\ell+1}(x)}{(2\ell+5)(2\ell+1)} + \frac{10(\ell-1)(\ell+3)P_{\ell}(x)}{(2\ell+3)(2\ell-1)}$$
$$\left. + \frac{5(\ell+2)(\ell+3)P_{\ell-1}(x)}{(2\ell+3)(2\ell+1)} + \frac{(\ell+2)(\ell+3)P_{\ell}(x)}{(2\ell+1)(2\ell-1)}\right] \tag{30}$$

$$d^J_{-\frac{5}{2},\frac{5}{2}}(\theta) = \frac{4\sin^5\frac{\theta}{2}}{(\ell-1)\ell(\ell+1)(\ell+2)(\ell+3)}\left[\frac{(\ell-1)\ell P^{(5)}_{\ell+3}(x)}{(2\ell+5)(2\ell+3)} + \frac{5(\ell-1)\ell P^{(5)}_{\ell+2}}{(2\ell+3)(2\ell+1)}\right.$$
$$+ \frac{10(\ell-1)(\ell+3)P^{(5)}_{\ell+1}(x)}{(2\ell+5)(2\ell+1)} + \frac{10(\ell-1)(\ell+3)P^{(5)}_{\ell}(x)}{(2\ell+3)(2\ell-1)}$$
$$\left. + \frac{5(\ell+2)(\ell+3)P^{(5)}_{\ell-1}(x)}{(2\ell+3)(2\ell+1)} + \frac{(\ell+2)(\ell+3)P^{(5)}_{\ell-2}(x)}{(2\ell+1)(2\ell-1)}\right] \tag{31}$$

$$= \frac{1}{8\sin^5\frac{\theta}{2}}\left[-\frac{(\ell-1)\ell P_{\ell+3}(x)}{(2\ell+5)(2\ell+3)} + \frac{5(\ell-1)\ell P_{\ell+2}(x)}{(2\ell+3)(2\ell+1)}\right.$$
$$- \frac{10(\ell-1)(\ell+3)P_{\ell+1}(x)}{(2\ell+5)(2\ell+1)} + \frac{10(\ell-1)(\ell+3)P_{\ell}(x)}{(2\ell+3)(2\ell-1)}$$
$$\left. - \frac{5(\ell+2)(\ell+3)P_{\ell-1}(x)}{(2\ell+3)(2\ell+1)} + \frac{(\ell+2)(\ell+3)P_{\ell-2}(x)}{(2\ell+1)(2\ell-1)}\right] \tag{32}$$

c) Integer indices (table)

$$J = \ell \tag{33}$$

$$d^J_{00}(\theta) = P_\ell(x) \tag{34}$$

$$d^J_{10}(\theta) = \frac{-2\cos\frac{\theta}{2}\sin\frac{\theta}{2}}{\sqrt{\ell(\ell+1)}} P'_\ell(x) \tag{35}$$

$$= \frac{\sqrt{\ell(\ell+1)}}{2\cos\frac{\theta}{2}\sin\frac{\theta}{2}}\left[\frac{P_{\ell+1}(x)}{2\ell+1} - \frac{P_{\ell-1}(x)}{2\ell+1}\right] \tag{36}$$

$$d^J_{-10}(\theta) = -d^J_{10}(\theta) \tag{37}$$

$$d^J_{11}(\theta) = \frac{2\cos^2\frac{\theta}{2}}{\ell(\ell+1)}\left[\frac{\ell P''_{\ell+1}(x)}{2\ell+1} - P''_\ell(x) + \frac{(\ell+1)P''_{\ell-1}(x)}{2\ell+1}\right] \tag{38}$$

$$= \frac{1}{2\cos^2\frac{\theta}{2}}\left[\frac{\ell P_{\ell+1}(x)}{2\ell+1} + P_\ell(x) + \frac{(\ell+1)P_{\ell-1}(x)}{2\ell+1}\right] \tag{39}$$

$$d^J_{-11}(\theta) = \frac{2\sin^2\frac{\theta}{2}}{\ell(\ell+1)}\left[\frac{\ell P''_{\ell+1}(x)}{2\ell+1} + P''_\ell(x) + \frac{(\ell+1)P''_{\ell-1}(x)}{2\ell+1}\right] \tag{40}$$

$$= \frac{1}{2\sin^2\frac{\theta}{2}}\left[\frac{\ell P_{\ell+1}(x)}{2\ell+1} - P_\ell(x) + \frac{(\ell+1)P_{\ell-1}(x)}{2\ell+1}\right] \tag{41}$$

$$d^J_{20}(\theta) = \frac{4\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2}}{\sqrt{(\ell-1)\ell(\ell+1)(\ell+2)}} P''_\ell(x) \tag{42}$$

$$= \frac{\sqrt{(\ell-1)\ell(\ell+1)(\ell+2)}}{4\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2}}\left[\frac{P_{\ell+2}(x)}{(2\ell+3)(2\ell+1)} - \frac{2P_\ell(x)}{(2\ell+3)(2\ell-1)} + \frac{P_{\ell-2}(x)}{(2\ell+1)(2\ell-1)}\right] \tag{43}$$

$$d^J_{-20}(\theta) = d^J_{20}(\theta) \tag{44}$$

$$d^J_{21}(\theta) = \frac{4\cos^3\frac{\theta}{2}\sin\frac{\theta}{2}}{\ell(\ell+1)\sqrt{(\ell-1)(\ell+2)}}\left[-\frac{\ell P'''_{\ell+1}(x)}{2\ell+1} + P'''_{\ell}(x) - \frac{(\ell+1)P'''_{\ell-1}(x)}{2\ell+1}\right] \tag{45}$$

$$= \frac{\sqrt{(\ell-1)(\ell+2)}}{4\cos^3\frac{\theta}{2}\sin\frac{\theta}{2}}\left[\frac{\ell P_{\ell+2}(x)}{(2\ell+3)(2\ell+1)} + \frac{P_{\ell+1}(x)}{2\ell+1} + \frac{3P_{\ell}(x)}{(2\ell+3)(2\ell-1)} - \frac{P_{\ell-1}(x)}{2\ell+1} - \frac{(\ell+1)P_{\ell-2}(x)}{(2\ell+1)(2\ell-1)}\right] \tag{46}$$

$$d^J_{-21}(\theta) = \frac{4\cos\frac{\theta}{2}\sin^3\frac{\theta}{2}}{\ell(\ell+1)\sqrt{(\ell-1)(\ell+2)}}\left[\frac{\ell P'''_{\ell+1}(x)}{2\ell+1} + P'''_{\ell}(x) + \frac{(\ell+1)P'''_{\ell-1}(x)}{2\ell+1}\right] \tag{47}$$

$$= \frac{\sqrt{(\ell-1)(\ell+2)}}{4\cos\frac{\theta}{2}\sin^3\frac{\theta}{2}}\left[-\frac{\ell P_{\ell+2}(x)}{(2\ell+3)(2\ell+1)} + \frac{P_{\ell+1}(x)}{2\ell+1} - \frac{3P_{\ell}(x)}{(2\ell+3)(2\ell-1)} - \frac{P_{\ell-1}(x)}{2\ell+1} + \frac{(\ell+1)P_{\ell-2}(x)}{(2\ell+1)(2\ell-1)}\right] \tag{48}$$

$$d^J_{22}(\theta) = \frac{4\cos^4\frac{\theta}{2}}{(\ell-1)\ell(\ell+1)(\ell+2)}\left[\frac{(\ell-1)\ell P^{(4)}_{\ell+2}(x)}{(2\ell+3)(2\ell+1)} - \frac{2(\ell-1)P^{(4)}_{\ell+1}(x)}{2\ell+1} + \frac{6(\ell-1)(\ell+2)P^{(4)}_{\ell}(x)}{(2\ell+3)(2\ell-1)} - \frac{2(\ell+2)P^{(4)}_{\ell-1}(x)}{2\ell+1} + \frac{(\ell+1)(\ell+2)P^{(4)}_{\ell-2}(x)}{(2\ell+1)(2\ell-1)}\right] \tag{49}$$

$$= \frac{1}{4\cos^4\frac{\theta}{2}}\left[\frac{(\ell-1)\ell P_{\ell+2}(x)}{(2\ell+3)(2\ell+1)} + \frac{2(\ell-1)P_{\ell+1}(x)}{2\ell+1} + \frac{6(\ell-1)(\ell+2)P_{\ell}(x)}{(2\ell+3)(2\ell-1)} + \frac{2(\ell+2)P_{\ell-1}(x)}{2\ell+1} + \frac{(\ell+1)(\ell+2)P_{\ell-2}(x)}{(2\ell+1)(2\ell-1)}\right] \tag{50}$$

$$d^J_{-22}(\theta) = \frac{4\sin^4\frac{\theta}{2}}{(\ell-1)\ell(\ell+1)(\ell+2)}\left[\frac{(\ell-1)\ell P^{(4)}_{\ell+2}(x)}{(2\ell+3)(2\ell+1)} + \frac{2(\ell-1)P^{(4)}_{\ell+1}(x)}{2\ell+1} + \frac{6(\ell-1)(\ell+1)P^{(4)}_{\ell}(x)}{(2\ell+3)(2\ell-1)} + \frac{2(\ell+2)P^{(4)}_{\ell-1}(x)}{2\ell+1} + \frac{(\ell+1)(\ell+2)P^{(4)}_{\ell-2}(x)}{(2\ell+1)(2\ell-1)}\right] \tag{51}$$

$$= \frac{1}{4\sin^4\frac{\theta}{2}}\left[\frac{(\ell-1)\ell P_{\ell+2}(x)}{(2\ell+3)(2\ell+1)} - \frac{2(\ell-1)P_{\ell+1}(x)}{2\ell+1} + \frac{6(\ell-1)(\ell+2)P_\ell(x)}{(2\ell+3)(2\ell-1)}\right.$$
$$\left. - \frac{2(\ell+2)P_{\ell-1}(x)}{2\ell+1} + \frac{(\ell+1)(\ell+2)P_{\ell-2}(x)}{(2\ell+1)(2\ell-1)}\right] \qquad (52)$$

d) **Recursion relations and general formulae**

One convenient way of calculating one of these functions is to start with

$$d^\ell_{\lambda 0}(\theta) = (-1)^\lambda \sqrt{\frac{(\ell-\lambda)!}{(\ell+\lambda)!}}\ \sin^\lambda\theta \frac{d^\lambda}{dx^\lambda} P_\ell(x) \qquad (53)$$

for an integer ℓ. $d^\ell_{\lambda\mu}(\theta)$ can be determined by repeated use of

$$d^\ell_{\lambda,\mu+1}(\theta) = \frac{1}{\sqrt{(\ell+\mu+1)(\ell-\mu)}}\left(-\frac{\lambda}{\sin\theta} + \mu\cot\theta - \frac{d}{d\theta}\right) d^\ell_{\lambda\mu}(\theta) \qquad (54)$$

For a half integer $J = \ell + \frac{1}{2}$, one can start from equation 53, then use

$$d^J_{\lambda\frac{1}{2}}(\theta) = \frac{1}{\sqrt{\ell+1}}\left[\sqrt{J+\lambda}\, d^\ell_{\lambda-\frac{1}{2},0}(\theta)\cos\frac{\theta}{2} + \sqrt{J-\lambda}\, d^\ell_{\lambda+\frac{1}{2},0}(\theta)\sin\frac{\theta}{2}\right] \qquad (55)$$

and equation 54.

Other recursion relations that might be useful, especially if only small values of J are needed, are

$$\sqrt{J+\mu}\, d^J_{\lambda\mu}(\theta) = \sqrt{J+\lambda}\, d^{J-\frac{1}{2}}_{\lambda-\frac{1}{2},\mu-\frac{1}{2}}(\theta)\cos\frac{\theta}{2} + \sqrt{J-\lambda}\, d^{J-\frac{1}{2}}_{\lambda+\frac{1}{2},\mu-\frac{1}{2}}(\theta)\sin\frac{\theta}{2} \qquad (56)$$

$$\sqrt{(J+\mu)(J+\mu-1)}\, d^J_{\lambda\mu}(\theta) = \sqrt{(J+\lambda)(J+\lambda-1)}\,(1+\cos\theta) d^{J-1}_{\lambda-1,\mu-1}(\theta)$$
$$+ 2\sqrt{J^2-\lambda^2}\,\sin\theta\, d^{J-1}_{\lambda,\mu-1}(\theta) + \sqrt{(J-\lambda)(J-\lambda-1)}\,(1-\cos\theta) d^{J-1}_{\lambda+1,\mu-1}(\theta)$$
$$(57)$$

For proofs and further relations, see references 5 and 6.

We will prove next that the forms of the expansions in sections b and c are general. R6-4.14 expresses the $d^j_{m'm}$ in terms of hypergeometric functions, related by R7-10.8.16 to Jacobi polynomials:

$$d^j_{m'm}(\theta) = \left[\frac{(j-m)!(j+m')!}{(j+m)!(j-m')!} \right]^{\frac{1}{2}} \frac{(\cos\frac{\theta}{2})^{2j+m-m'}(-\sin\frac{\theta}{2})^{m'-m}}{(m'-m)!}$$

$$\times F(m'-j, -m-j, m'-m+1; -\tan^2\frac{\theta}{2}) \tag{59}$$

$$m' \geq m \geq 0 \qquad \cos\theta = x$$

($m' \geq m \geq 0$ is a trivial limitation; relations 1 through 7 can be used.)

$$d^j_{m'm}(x) = \left[\frac{(j-m)!(j+m')!}{(j+m)!(j-m')!} \right]^{\frac{1}{2}} \frac{(1+x)^{j+\frac{m-m'}{2}}(-1+x)^{\frac{m'-m}{2}}}{2^j(m'-m)!}$$

$$\times F(m'-j, -m-j, m'-m+1; \frac{x-1}{x+1})$$

$$= \left[\frac{(j-m)!(j+m')!}{(j+m)!(j-m')!} \right]^{\frac{1}{2}} \frac{(1+x)^{\frac{m'+m}{2}}(-1+x)^{\frac{m'-m}{2}}}{2^{m'}(m'-m)!}$$

$$\times \binom{j-m}{j-m'}^{-1} P^{(m'-m,\, m'+m)}_{j-m'}(x)$$

$$= (-1)^{\frac{m'-m}{2}} \left[\frac{(j+m')!(j-m')!}{(j+m)!(j-m)!} \right]^{\frac{1}{2}} \frac{(1+x)^{\frac{m'+m}{2}}(1-x)^{\frac{m'-m}{2}}}{2^{m'}}$$

$$\times P^{(m'-m,\, m'+m)}_{j-m'}(x) \tag{60a}$$

$$= c(1+x)^{\frac{m'+m}{2}}(1-x)^{\frac{m'-m}{2}} P^{(m'-m,m'+m)}_{j-m'}(x) \tag{60b}$$

where

$$c = (-1)^{\frac{m'-m}{2}} \left[\frac{(j+m')!(j-m')!}{(j+m)!(j-m)!}\right]^{\frac{1}{2}} 2^{-m'} \tag{60c}$$

We will determine the coefficients in the following expansion in Legendre polynomials:

$$d^j_{m'm}(x)\ (1+x)^{\frac{m'+m}{2}} (1-x)^{\frac{m'-m}{2}} = \sum_{k=0}^{\infty} a_k P_k(x) \tag{61a}$$

From R8-6.3.42

$$a_k = \frac{2k+1}{2} \int_{-1}^{1} d^j_{m'm}(x)\ (1+x)^{\frac{m'+m}{2}} (1-x)^{\frac{m'-m}{2}} P_k(x)\ dx \tag{61b}$$

and using equation 60b

$$a_k = \frac{2k+1}{2} c \int_{-1}^{1} (1+x)^{m'+m} (1-x)^{m'-m} P^{(m'-m,\, m'+m)}_{j-m'}(x) P_k(x)\ dx \tag{61c}$$

Rodrigues' formulae for Jacobi and Legendre polynomials, R7-10.8.10 and R7-10.10.7, are

$$2^n n!\ P_n^{(\alpha,\beta)}(x) = (-1)^n (1-x)^{-\alpha} (1+x)^{-\beta} \frac{d^n}{dx^n}\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right] \tag{62a}$$

$$2^n n!\ P_n(x) = \frac{d^n}{dx^n}\left[(x^2-1)^n\right] \tag{62b}$$

and hence

$$a_k = c_{1k} \int_{-1}^{1} \frac{d^{j-m'}}{dx^{j-m'}}\left[(1-x)^{j-m}(1+x)^{j+m}\right] \frac{d^k}{dx^k}\left[(1-x^2)^k\right] \tag{63a}$$

$$c_{1k} = (-1)^{j+k-\frac{m'+m}{2}}\ 2^{-j-k}\left[\frac{(j+m')!}{(j+m)!(j-m)!(j-m')!}\right] \frac{1}{k!}\ (k+\tfrac{1}{2}) \tag{63b}$$

Integrating by parts, we get the following two expressions:

$$a_k = (-1)^k c_{1k} \int_{-1}^{1} \frac{d^{j-m'+k}}{dx^{j-m'+k}} \left[(1-x)^{j-m} (1+x)^{j+m} \right] (1-x^2)^k \tag{64a}$$

$$a_k = (-1)^{j-m'} c_{1k} \int_{-1}^{1} (1-x)^{j-m} (1+x)^{j+m} \frac{d^{k+j-m'}}{dx^{k+j-m'}} \left[(1-x^2)^k \right] \tag{64b}$$

But $(1-x)^{j-m}(1+x)^{j+m}$ is a polynomial of degree $2j$ in x, and $(1-x^2)^k$ is of degree $2k$; hence

$$a_k = 0 \qquad \text{for } k > j + m' \tag{65a}$$

$$a_k = 0 \qquad \text{for } k < j - m' \tag{65b}$$

$$d^j_{m'm}(x) = \frac{1}{(1+x)^{\frac{m'+m}{2}} (1-x)^{\frac{m'-m}{2}}} \sum_{k=j-m'}^{j+m'} a_k P_k(x) \tag{66}$$

This result agrees with those in sections b and c.

In reference 7, page 164, (i) it is stated that derivatives of orthogonal polynomials also form a system of orthogonal polynomials. We will use the Gegenbauer polynomials T^{α}_{ℓ}, defined in terms of derivatives of Legendre polynomials:

$$d^j_{m'm}(x)\,(1+x)^{-\frac{m'+m}{2}}(1-x)^{-\frac{m'-m}{2}} = \sum_{k=0}^{\infty} b_k T^{m+m'}_{k-m-m'}(x) \tag{67a}$$

where

$$T^{\alpha}_{\ell}(x) = \frac{d^{\alpha}}{dx^{\alpha}} P_{\ell+\alpha} \tag{67b}$$

and, using a formula in reference 8 page 783,

$$b_k = \frac{2k+1}{2} \frac{(k+m+m')!}{(k-m-m')!} \int_{-1}^{1} d^j_{m'm}(x)(1+x)^{-\frac{m'+m}{2}}(1-x)^{-\frac{m'-m}{2}}$$

$$\times (1-x^2)^{m'+m}\, T^{m'+m}_{k-m'-m}(x)\, dx \qquad (67c)$$

Using equations 60, 62a and 62b, we get

$$b_k = c_{2k} \int_{-1}^{1} (1-x)^{2m} \frac{d^{j-m'}}{dx^{j-m'}} \left[(1-x)^{j-m}(1+x)^{j+m} \right]$$

$$\times \frac{d^{k+m+m'}}{dx^{k+m+m'}} \left[(1-x^2)^k \right] dx \qquad (68a)$$

$$c_{2k} = (-1)^{j+k-(m'+m)/2}\, 2^{-j-k} \left[\frac{(j+m')}{(j+m)!(j-m)!(j-m')!} \right]^{\frac{1}{2}}$$

$$\times \frac{(k+m+m')!}{(k-m-m')!} (k+\tfrac{1}{2}) \qquad (68b)$$

and, integrating by parts,

$$b_k = (-1)^{k+m+m'}\, c_{2k}$$

$$\times \int_{-1}^{1} \frac{d^{k+m+m'}}{dx^{k+m+m'}} \left[(1-x)^{2m} \frac{d^{j-m'}}{dx^{j-m'}} \left((1-x)^{j-m}(1+x)^{j+m} \right) \right] (1-x^2)^k dx \qquad (69a)$$

$$b_k = (-1)^{j-m'} c_{2k}$$

$$\times \int_{-1}^{1} (1-x)^{j-m}(1+x)^{j+m} \frac{d^{j-m'}}{dx^{j-m'}} \left[(1-x)^{2m} \frac{d^{k+m+m'}}{dx^{k+m+m'}} \left((1-x^2)^k \right) \right] dx \qquad (69b)$$

Hence

$$b_k = 0 \qquad \text{for } k > j+m \qquad (70a)$$

$$b_k = 0 \qquad \text{for } k < j-m \qquad (70b)$$

$$d^{j}_{m'm}(x) = (1+x)^{\frac{m'+m}{2}}(1-x)^{\frac{m'-m}{2}}\sum_{k=j-m}^{j+m} b_k T^{m+m'}_{k-m-m'}(x) \qquad (71)$$

also in agreement with results in sections b and c.

The integrals in equations 64 and 69 can be reduced to beta functions $B(m, m)$ (see R8-4.5.54) to determine a_k and b_k; in practice, though, it might be preferable to use recursion relations.

APPENDIX D

DETERMINATION OF PROJECTION OPERATORS

One way to determine the projection operators for a partial wave expansion is shown in the first part of this appendix. By summing over final helicities and averaging over initial ones, equation A-6 gives[‡]

$$\left(\frac{d\sigma}{d\Omega}\right)_{unpol} = \frac{1}{(2s_a+1)(2s_b+1)} \sum_{(\lambda)} \left|f_{\lambda_c\lambda_d;\lambda_a\lambda_b}\right|^2 \tag{1}$$

$$\sum_{(\lambda)} = \sum_{\lambda_a\lambda_b\lambda_c\lambda_d}$$

Using A-5

$$\left(\frac{d\sigma}{d\Omega}\right)_{unpol} = \frac{1}{(2s_a+1)(2s_b+1)} \sum_{(\lambda)} \sum_{JJ'} (J+\tfrac{1}{2})(J'+\tfrac{1}{2})$$

$$\times\; a^{J}_{\lambda_c\lambda_d;\lambda_a\lambda_b} \left(a^{J'}_{\lambda_c\lambda_d;\lambda_a\lambda_b}\right)^{*} d^{J}_{\lambda\mu}(\theta) d^{J'}_{\lambda\mu}(\theta) \tag{2}$$

Hence the total cross section is

$$\sigma = \int \frac{d\sigma}{d\Omega}\, d\Omega$$

$$\sigma = 2\pi \int_{-1}^{1} dx\, \frac{d\sigma(x)}{d\Omega}$$

$$x = \cos\theta$$

$$\sigma = \frac{1}{(2s_a+1)(2s_b+1)} \sum_{(\lambda)} \sum_{JJ'} (J+\tfrac{1}{2})(J'+\tfrac{1}{2})\, a^{J}_{(\lambda)} a^{J'\,*}_{(\lambda)} \int_0^{\pi} \sin\theta\, d\theta\, d^{J}_{\lambda\mu}(\theta)\, d^{J'}_{\lambda\mu}(\theta) \tag{4}$$

[‡] 2s+1 is to be replaced by 2 if the mass of the corresponding particle is zero.

R6-4.59 gives

$$\int_0^\pi d\theta \sin\theta \, d^J_{\lambda\mu}(\theta)\, d^{J'}_{\lambda\mu}(\theta) = \frac{2}{2J+1}\,\delta_{JJ'} \tag{5}$$

Carrying out the integration and summation over J' in 4,

$$\sigma = \frac{2\pi}{(2s_a+1)(2s_b+1)} \sum_{(\lambda)} \sum_J (J+\tfrac{1}{2})\, |a^J_{(\lambda)}|^2$$

$$= \frac{2\pi}{(2s_a+1)(2s_b+1)} \sum_{J,i} (J+\tfrac{1}{2})\, n_i\, |a^J_i|^2 \tag{6}$$

where a^J_i are the independent elements of $\frac{1}{|\vec{p}|} T^J$ and n_i the number of times a_i appears in it.

On the other hand, if $\mathcal{F}_i$ is a set of amplitudes like those in II-30 for instance, we can write

$$\left(\frac{d\sigma}{d\Omega}\right)_{unpol} = \frac{1}{(2s_a+1)(2s_b+1)} \sum_{ij} \mathcal{F}^*_i G_{ij} \mathcal{F}_j$$

$$= \frac{1}{(2s_a+1)(2s_b+1)}\, \mathcal{F}^\dagger G \mathcal{F} \tag{7}$$

where $G_{ij} = G^*_{ji}$ is a "metric" determined by summing over spins and polarizations. This is how equation II-37 was derived. (Notice that $\mathcal{F}$ is used both for a column vector of $\mathcal{F}_i$'s as in equation 7 and for a 2×2 matrix as in II-34; they are essentially the same entity.)

The $\mathcal{F}_k$ are next expanded in partial waves, giving

$$\mathcal{F}_k = \sum_{J,i} A^J_{ki}\, a^J_i \tag{8a}$$

or, using matrix notation

$$\mathcal{F} = \sum_J A^J a^J \tag{8b}$$

Substituting 8b into 7,

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol}} = \frac{1}{(2s_a+1)(2s_b+1)} \sum_{J,\,J'} a^{J\dagger} A^{J\dagger} G A^{J'} a^{J'} \tag{9}$$

$$\sigma = \frac{2\pi}{(2s_a+1)(2s_b+1)} \sum_{JJ'} a^{J\dagger} \left(\int_{-1}^{1} dx\, A^{J\dagger} G A^{J'} \right) a^{J'} \tag{10}$$

By comparing this equation with 6, we obtain

$$\int_{-1}^{1} A^{J\dagger} G A^{J'} dx = (J+\tfrac{1}{2}) n_i \delta_{JJ'} 1 \tag{11}$$

If we define

$$\varphi_i^J = A^J_{\cdot i} \tag{12}$$

and a generalization of the "scalar product" of II-40

$$\varphi \circ \psi \equiv \int_{-1}^{1} dx\, \varphi^{\dagger} G \psi \tag{13}$$

we get from 11:

$$\varphi_i^J \circ \varphi_j^{J'} = n_i (J+\tfrac{1}{2}) \delta_{JJ'} \delta_{ij} \tag{14}$$

Then, if a certain "vector" $\mathfrak{F}$ is given,

$$\begin{aligned} \varphi_i^J \circ \mathfrak{F} &= \sum_{J'} \varphi_i^J \circ A^{J'} a^{J'} \\ &= \sum_{J',\,j} \varphi_i^J \circ (\varphi_j a_j^{J'}) \\ &= \sum_{J',\,j} n_i (J+\tfrac{1}{2}) \delta_{JJ'} \delta_{ij} a_j^{J'} \\ &= n_i (J+\tfrac{1}{2})\, a_i^J \end{aligned}$$

or

$$a_i^J = \frac{1}{n_i(J+\frac{1}{2})} \varphi_i^J \circ \mathfrak{F} \tag{15}$$

We can calculate the vectors

$$\chi_i^J = \frac{1}{n_i(J+\frac{1}{2})} G\varphi_i^J \tag{16}$$

and 15 can be rewritten

$$a_i^J = \int_{-1}^{1} \chi_i^{J\dagger} \mathfrak{F}\, dx \tag{17}$$

and we can call the χ_i^J projection vectors.

If a different set of amplitudes

$$\mathfrak{F}' = B\mathfrak{F} \tag{18}$$

is then chosen, we have from 7

$$\left(\frac{d\sigma}{d\Omega}\right)_{unpol} = \frac{1}{(2s_a+1)(2s_b+1)} \mathfrak{F}'^{\dagger} G' \mathfrak{F}'$$

$$= \frac{1}{(2s_a+1)(2s_b+1)} \mathfrak{F}^{\dagger} B^{\dagger} G' B \mathfrak{F}$$

and

$$G' = (B^{-1})^{\dagger} G B^{-1} \tag{19}$$

From 8b,

$$\mathfrak{F}' = \sum_J BA^J a^J$$

and

$$A'^J = BA^J \tag{20}$$

$$\varphi_i'^J = BA^J_{\cdot i} = B\varphi_i^J \tag{21}$$

$$\chi_i^{\prime J} = \frac{1}{n_i(J+\frac{1}{2})} (B^{-1})^\dagger G\varphi_i^J$$

$$= (B^{-1})^\dagger \chi_i^J \tag{22}$$

Check:

$$a_i^J = \int_{-1}^{1} \chi_i^{J\dagger} \mathcal{F}' \, dx$$

$$= \int_{-1}^{1} \chi_i^{J\dagger} B^{-1} B \mathcal{F} \, dx$$

$$= \int_{-1}^{1} \chi_i^{J\dagger} \mathcal{F} \, dx \qquad \text{which is equation 17}$$

Sometimes it is easy to compute the projection operators directly. For instance, if we define the amplitudes $T^{\pm}_{(\lambda)}$

$$T^{\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b} = \frac{f_{\lambda_c\lambda_d;\lambda_a\lambda_b}}{2^{\lambda-\frac{1}{2}}(\cos\frac{\theta}{2})^{\lambda+\mu}(\sin\frac{\theta}{2})^{\lambda-\mu}e^{i(\lambda-\mu)\varphi}}$$

$$\pm \frac{f_{-\lambda_c-\lambda_d;\lambda_a\lambda_b}}{(-1)^{\lambda+\mu}2^{\lambda-\frac{1}{2}}(\cos\frac{\theta}{2})^{\lambda-\mu}(\sin\frac{\theta}{2})^{\lambda+\mu}e^{i(\lambda+\mu)\varphi}} \tag{23}$$

$$\lambda = \lambda_a - \lambda_b \qquad \mu = \lambda_c - \lambda_d \qquad \lambda \geq \mu \geq 0$$

Then, using equations A-5, C-71 and C-4, we can express them in terms of Gegenbauer polynomials (derivatives of Legendre polynomials)

$$T^{\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b} = \sqrt{2} \sum_J (J+\tfrac{1}{2}) \sum_{n=0}^{2\mu} b_n(J,\lambda,\mu) \Big[a^J_{\lambda_c\lambda_d;\lambda_a\lambda_b}$$

$$\pm (-1)^{n+\lambda+\mu} a^J_{-\lambda_c-\lambda_d;\lambda_a\lambda_b} \Big] T^{\lambda+\mu}_{J-\lambda-2\mu+n}(x) \tag{24}$$

or, if we define

$$\beta^{J\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b} = a^J_{\lambda_c\lambda_d;\lambda_a\lambda_b} \pm (-1)^{\lambda+\mu} a^J_{-\lambda_c-\lambda_d;\lambda_a\lambda_b} \tag{25}$$

$$T^{\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b} = \sqrt{2}\sum_J (J+\tfrac{1}{2})\Big[\beta^{J\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}\sum_{m=0}^{\mu_o}\Big(b_{2m}(J,\lambda,\mu)\,T^{\lambda+\mu}_{J-\lambda-2\mu+2m}(x)\Big)$$

$$+\beta^{J\mp}_{\lambda_c\lambda_d;\lambda_a\lambda_b}\sum_{m=0}^{\mu_o'}\Big(b_{2m+1}(J,\lambda,\mu)\,T^{\lambda+\mu}_{J-\lambda-2\mu+2m+1}(x)\Big)\Big] \tag{26}$$

where μ_o and μ_o' are either μ and $\mu-1$, or both $\mu-\frac{1}{2}$, whichever are integers.

From equation A-5, using D-5, we get

$$a^J_{\lambda_c\lambda_d;\lambda_a\lambda_b} = e^{-i(\lambda-\mu)\varphi}\int_{-1}^{1} f_{\lambda_c\lambda_d;\lambda_a\lambda_b}(x)\, d^J_{\lambda\mu}(x)\, dx \tag{27}$$

We will use next the other expression for the $d^J_{\lambda\mu}$ in terms of Legendre polynomials, equation C-66.

$$\beta^{J\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b} = \int_{-1}^{1}\Big[e^{-i(\lambda-\mu)\varphi} f_{\lambda_c\lambda_d;\lambda_a\lambda_b}(x)\, d^J_{\lambda\mu}(x)$$

$$\pm (-1)^{\lambda+\mu} e^{-i(\lambda+\mu)\varphi} f_{-\lambda_c-\lambda_d;\lambda_a\lambda_b}(x)\, d^J_{\lambda,-\mu}(x)\Big]\, dx$$

$$= 2^{-\frac{1}{2}}\int_{-1}^{1}\Big[\Big(T^{+}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(x)+T^{-}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(x)\Big)(1+x)^{\frac{\lambda+\mu}{2}}(1-x)^{\frac{\lambda-\mu}{2}} d^J_{\lambda\mu}(x)$$

$$\pm\Big(T^{+}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(x)-T^{-}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(x)\Big)(1+x)^{\frac{\lambda-\mu}{2}}(1-x)^{\frac{\lambda+\mu}{2}} d^J_{\lambda,-\mu}(x)\Big]\, dx$$

$$= 2^{-\frac{1}{2}} \int_{-1}^{1} \Big[\Big(T^{+}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(x) + T^{-}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(x) \Big) \sum_{n=0}^{2\lambda} a_n(J,\lambda,\mu) P_{J-\lambda+n}(x)$$

$$\pm \Big(T^{+}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(x) - T^{-}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(x) \Big) \sum_{n=0}^{2\lambda} a_n(J,\lambda,\mu)(-1)^{2\lambda+n}$$

$$\times P_{J-\lambda+n}(x) \Big] dx$$

$$= \sqrt{2} \int_{-1}^{1} \Big[T^{\pm(-)^{2\lambda}}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(x) \sum_{m=0}^{\lambda_o} a_{2m}(J,\lambda,\mu) P_{J-\lambda+2m}(x)$$

$$+ T^{\mp(-)^{2\lambda}}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(x) \sum_{m=0}^{\lambda_o'} a_{2m+1}(J,\lambda,\mu) P_{J-\lambda+2m+1}(x) \Big] dx \qquad (28)$$

where λ_o and λ_o' are either λ and $\lambda-1$, or both $\lambda-\frac{1}{2}$. Equation 28 exhibits the projection operators, showing that for these amplitudes only $2\lambda+1$ Legendre polynomials (and no derivatives or powers of x) are involved.

From the properties of the helicity amplitudes and helicity states under the parity transformation it is easy to see that the amplitudes $\beta^{J\pm}_{(\lambda)}$ correspond to transitions between states of definite parity. Consequently, their coefficients in the partial wave expansion (equation 26) are either even or odd functions of x.

The properties of the $\beta^{J\pm}_{(\lambda)}$ under the reflection discussed in section a) of appendix A can be derived using equation A-14:

$$\beta^{J\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(-W) = a^J_{\lambda_c\lambda_d;\lambda_a\lambda_b}(-W) \pm (-1)^{\lambda+\mu} a^J_{-\lambda_c-\lambda_d;\lambda_a\lambda_b}(-W)$$

$$= -\left[(-1)^{\lambda-\mu} a^J_{\lambda_c\lambda_d;\lambda_a\lambda_b}(W) \pm a^J_{-\lambda_c-\lambda_d;\lambda_a\lambda_b}(W) \right]$$

$$= (-1)^{\lambda+\mu)}\left[a^J_{\lambda_c\lambda_d;\lambda_a\lambda_b}(W) \mp (-1)^{\lambda+\mu} a_{-\lambda_c-\lambda_d;\lambda_a\lambda_b}(W) \right]$$

$$= (-1)^{\lambda+\mu} \beta^{J\mp}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(W) \tag{29a}$$

and from equation 26

$$T^{\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(-W, x) = (-1)^{\lambda+\mu} T^{\mp}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(W, x) \tag{29b}$$

APPENDIX E

LINEAR RELATIONS BETWEEN INVARIANTS

As was pointed out for equations II-126 and II-130, in the more complicated cases there are relations among invariants that are not obvious and that are related to the fact that no more than n vectors can be linearly independent in an n-dimensional vector space. The straightforward algebraic statement of this fact involves the antisymmetric tensor of the corresponding rank, and is not immediately useful to find these relations. In this appendix a way of finding them will be shown, and the problem of the introduction of kinematical singularities for the invariant amplitudes will be discussed

First we will show how equations II-131 are obtained for the three-dimensional case, since the other case is a direct generalization of this one. Incidentally, we will also derive some useful relations for reducing terms containing the antisymmetric tensors to a combination of the invariants used here. The vector product will be indicated by $\wedge$.

By expanding $(\vec{a} \wedge \vec{b}) \wedge (\vec{c} \wedge \vec{d})$ in two different ways, or otherwise, we get

$$[\vec{a}\ \vec{b}\ \vec{c}]\,\vec{d} = [\vec{d}\ \vec{a}\ \vec{b}]\,\vec{c} + [\vec{d}\ \vec{c}\ \vec{a}]\,\vec{b} + [\vec{d}\ \vec{b}\ \vec{c}]\,\vec{a} \tag{1}$$

where $[\vec{a}\ \vec{b}\ \vec{c}]$ is the "box product" of the three vectors, that is

$$[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a} \wedge \vec{b} \cdot \vec{c} \tag{2a}$$

or

$$[\vec{a}\ \vec{b}\ \vec{c}] = \epsilon_{ijk} a_i b_j c_k \qquad i, j, k = 1, 2, 3 \tag{2b}$$

By making the substitutions

$$a \sim a \wedge b \qquad b \sim b \wedge c \qquad c \sim c \wedge a$$

in equation 1, we find after some algebraic reductions

$$[\vec{a}\,\vec{b}\,\vec{c}]\vec{d} = \vec{d}\cdot\vec{a}\;\vec{b}\wedge\vec{c} + \vec{d}\cdot\vec{b}\;\vec{c}\wedge\vec{a} + \vec{d}\cdot\vec{c}\;\vec{a}\wedge\vec{b} \tag{3}$$

From the property of the Pauli spin matrices

$$\sigma_\ell \sigma_j = \delta_{\ell j} + i\epsilon_{\ell jk}\sigma_k \tag{4a}$$

(see R1-1.101), we get

$$\vec{\sigma}\cdot\vec{a}\;\vec{\sigma}\cdot\vec{b} = \vec{a}\cdot\vec{b} + i\vec{\sigma}\cdot\vec{a}\wedge\vec{b} \tag{4b}$$

and

$$\vec{\sigma}\cdot\vec{a}\;\vec{\sigma}\cdot\vec{b} + \vec{\sigma}\cdot\vec{b}\;\vec{\sigma}\cdot\vec{a} = 2\vec{a}\cdot\vec{b} \tag{4c}$$

By repeated use of these equations, we have

$$i\sigma\cdot[(\hat{k}'\wedge\vec{\epsilon}')\wedge(\hat{k}\wedge\vec{\epsilon})] = \vec{\sigma}\cdot(\hat{k}'\wedge\vec{\epsilon}')\vec{\sigma}\cdot(\hat{k}\wedge\vec{\epsilon}) - \hat{k}'\wedge\vec{\epsilon}'\cdot\hat{k}\wedge\vec{\epsilon}$$

$$= -(\vec{\sigma}\cdot\hat{k}'\;\vec{\sigma}\cdot\vec{\epsilon}' - \hat{k}'\cdot\vec{\epsilon}')(\vec{\sigma}\cdot\hat{k}\;\vec{\sigma}\cdot\vec{\epsilon} - \hat{k}\cdot\vec{\epsilon}) - \hat{k}\cdot\hat{k}'\;\vec{\epsilon}\cdot\vec{\epsilon}' + \hat{k}'\cdot\vec{\epsilon}\;\hat{k}\cdot\vec{\epsilon}'$$

$$= -xN_1' + 2N_3' - 3N_4' + N_6' + N_7' \tag{5a}$$

where

$$x = \hat{k}\cdot\hat{k}' = \cos\theta$$

On the other hand, and using equation 3, we have

$$(\hat{k}'\wedge\vec{\epsilon}')\wedge(\hat{k}\wedge\vec{\epsilon}) = [\hat{k}'\hat{k}\,\vec{\epsilon}]\vec{\epsilon}' - [\vec{\epsilon}'\hat{k}\,\vec{\epsilon}]\hat{k}'$$

$$= \hat{k}\cdot\vec{\epsilon}'\vec{\epsilon}\wedge\hat{k}' + \vec{\epsilon}\cdot\vec{\epsilon}'\;\hat{k}'\wedge\hat{k} - \hat{k}\cdot\hat{k}'\;\vec{\epsilon}\wedge\vec{\epsilon}' - \hat{k}'\cdot\vec{\epsilon}\;\vec{\epsilon}'\wedge\hat{k} \tag{5b}$$

and hence, using equation 4b again

$$i\vec{\sigma}\cdot[\hat{k}'\wedge\vec{\epsilon}'] \wedge (\hat{k}\wedge\vec{\epsilon})] = \hat{k}\cdot\vec{\epsilon}'\, i\vec{\sigma}\cdot\vec{\epsilon}\wedge\hat{k}' + \vec{\epsilon}\cdot\vec{\epsilon}'\, i\vec{\sigma}\cdot\hat{k}'\wedge\hat{k}$$

$$-\hat{k}\wedge\hat{k}'\,\vec{\sigma}\cdot\vec{\epsilon}\wedge\vec{\epsilon}' - \hat{k}'\cdot\vec{\epsilon}\,\vec{\sigma}\cdot\vec{\epsilon}'\wedge\hat{k}'$$

$$= -xN'_2 + N'_3 + 2N'_6 - N'_9 \qquad (5c)$$

Combining 5a and 5c, we get

$$xN'_1 - xN'_2 - N'_3 + 3N'_4 + N'_6 - N'_7 - N'_9 + N'_{11} = 0 \qquad (6)$$

which is equation II-131a. (The N'_i are defined in II-130.)

If the same procedure is applied to a slightly different expression like $(\hat{k}\wedge\vec{\epsilon})\wedge(\hat{k}\wedge\vec{\epsilon}') + (\hat{k}'\wedge\vec{\epsilon})\wedge(\hat{k}'\wedge\vec{\epsilon}')$, we get the same equation 6. (The combination of terms has to be chosen so that time invariance is satisfied.)

We try a more complicated expression next. By expanding $(\vec{a}\wedge\vec{b})\wedge[(\vec{c}\wedge\vec{d})\wedge(\vec{d}\wedge\vec{f})]$ in two different ways, we get:

$$(\vec{a}\cdot\vec{c}\,\vec{b}\cdot\vec{d} - \vec{a}\cdot\vec{d}\,\vec{b}\cdot\vec{c})\vec{e}\wedge\vec{f} + (\vec{a}\cdot\vec{e}\,\vec{b}\cdot\vec{c} - \vec{b}\cdot\vec{e}\,\vec{a}\cdot\vec{c})\vec{d}\wedge\vec{f}$$

$$+ (\vec{a}\cdot\vec{e}\,\vec{b}\cdot\vec{d} - \vec{b}\cdot\vec{c}\,\vec{a}\cdot\vec{d})\vec{f}\wedge\vec{c} + (\vec{b}\cdot\vec{f}\,\vec{a}\cdot\vec{c} - \vec{a}\cdot\vec{f}\,\vec{b}\cdot\vec{c})\vec{d}\wedge\vec{e}$$

$$+ (\vec{b}\cdot\vec{f}\,\vec{a}\cdot\vec{d} - \vec{a}\cdot\vec{f}\,\vec{b}\cdot\vec{d})\vec{e}\wedge\vec{c} + (\vec{b}\cdot\vec{f}\,\vec{a}\cdot\vec{e} - \vec{a}\cdot\vec{f}\,\vec{b}\cdot\vec{e})\vec{c}\wedge\vec{d} = 0 \qquad (7a)$$

If we set now $\vec{a} = \hat{k}$, $\vec{b} = \hat{k}'$, $\vec{c} = \vec{\epsilon}$, $\vec{d} = \vec{\epsilon}'$, $\vec{e} = \hat{k}$, $\vec{f} = \hat{k}'$, multiply by $i\sigma$ and use 4b, after some algebraic reductions we obtain:

$$(1-x^2)N'_1 - (1-x^2)N'_2 - 3xN'_4 + 2N'_5 - xN'_6 + xN'_7 - N'_8 + xN'_9$$

$$- N'_{10} + N'_{12} - N'_{13} = 0 \qquad (7b)$$

which is equation II-131b.

A useful relation obtained from equations 4a, 4c and 4b is

$$i[\vec{a}\,\vec{b}\,\vec{c}] = \vec{\sigma}\cdot\vec{a}\,\vec{\sigma}\cdot\vec{b}\,\vec{\sigma}\cdot\vec{c} - \vec{a}\cdot\vec{b}\,\vec{\sigma}\cdot\vec{c} + \vec{c}\cdot\vec{a}\,\vec{\sigma}\cdot\vec{b} - \vec{b}\cdot\vec{c}\,\vec{\sigma}\cdot\vec{a} \tag{8}$$

We will adapt now these calculations to four-vectors. It should be remembered that we have to use the Minkowski metric when a summation over an index whose range is 0, 1, 2, 3 is performed. For instance

$$a\cdot b = a_\mu b_\mu = g^{\mu\nu} a_\mu b_\nu \qquad \mu, \nu = 0, 1, 2, 3 \tag{9a}$$

where the metric tensor is given by

$$\begin{aligned} g^{00} &= -g^{11} = -g^{22} = -g^{33} = 1 \\ g^{\mu\nu} &= 0 \qquad \mu \neq \nu \end{aligned} \tag{9b}$$

We also use

$$\delta_{\mu\nu} = g^{\mu\nu} \tag{9c}$$

We define a "triple vector product"

$$v = a \wedge b \wedge c \tag{10a}$$

by the equation

$$v_\rho = \epsilon_{\lambda\mu\nu\rho} a_\lambda b_\mu c_\nu \tag{10b}$$

where $\epsilon_{\lambda\mu\nu\rho}$ is the antisymmetric tensor and

$$\epsilon_{0123} = 1 \tag{10c}$$

Obviously, the triple vector product is zero when any two factors are equal, and changes sign when two of the vectors are interchanged, while a cyclic permutation does not change it.

We define the "box product" for four four-vectors by

$$[abcd] = \epsilon_{\lambda\mu\nu\rho} a_\lambda b_\mu c_\nu d_\rho \tag{11a}$$

Hence

$$[abcd] = (a \wedge b \wedge c) \cdot d = d \cdot (a \wedge b \wedge c) \qquad (11b)$$

(The parentheses around the vector product are actually superfluous.)

$$[abcd] = - \begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & c_3 \\ d_0 & d_1 & d_2 & d_3 \end{vmatrix} \qquad (11c)$$

We would like to emphasize the different behavior of box products of three-vectors and four-vectors under a cyclic permutation; the latter change sign, and hence

$$[abcd] = - a \cdot b \wedge c \wedge d \qquad (11d)$$

so that the position of the dot is not irrelevant after a certain definition has been chosen.

The box product of four-vectors is zero when any two vectors are equal, or, more generally, when the four vectors are linearly dependent.

An equation analogous to 1 can be obtained. Since any five four-vectors are linearly dependent, we can write

$$\alpha_1 a^1 + \alpha_2 a^2 + \alpha_3 a^3 + \alpha_4 a^4 + \alpha_5 a^5 = 0$$

Multiplications by $a^1 \wedge a^2 \wedge a^3$, $a^1 \wedge a^2 \wedge a^4$, $a^1 \wedge a^3 \wedge a^4$ and $a^2 \wedge a^3 \wedge a^4$ give

$$\alpha_4[a^1a^2a^3a^4] + \alpha_5[a^1a^2a^3a^5] = 0$$

$$\alpha_3[a^1a^2a^4a^3] + \alpha_5[a^1a^2a^4a^5] = 0$$

$$\alpha_2[a^1a^3a^4a^2] + \alpha_5[a^1a^3a^4a^5] = 0$$

$$\alpha_1[a^2a^3a^4a^1] + \alpha_5[a^2a^3a^4a^5] = 0$$

and hence

$$[a^1a^2a^3a^4]a^5 + [a^5a^1a^2a^3]a^4 + [a^4a^5a^1a^2]a^3 + [a^3a^4a^5a^1]a^2 + [a^2a^3a^4a^5]a^1 = 0 \qquad (12)$$

We get the equation equivalent to 3 by the analogous substitutions; these give

$$[a^1a^2a^3a^4]a^5 = a^4\cdot a^5\, a^1\wedge a^2\wedge a^3 - a^3\cdot a^5\, a^4\wedge a^1\wedge a^2 + a^2\cdot a^5\, a^3\wedge a^4\wedge a^1 - a^1\cdot a^5\, a^2\wedge a^3\wedge a^4 \qquad (13)$$

We have substituted $a^1\wedge a^2\wedge a^3 \sim (a^2\wedge a^3\wedge a^4)\wedge(a^3\wedge a^4\wedge a^1)\wedge(a^4\wedge a^1\wedge a^3)$, etc. By direct computation we can show that

$$\epsilon_{\lambda\mu\nu\rho}\epsilon_{\alpha\beta\gamma\rho} = -\delta_{\lambda\alpha}\delta_{\mu\beta}\delta_{\nu\gamma} - \delta_{\lambda\beta}\delta_{\mu\gamma}\delta_{\nu\alpha} - \delta_{\lambda\gamma}\delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\lambda\alpha}\delta_{\mu\gamma}\delta_{\nu\beta} + \delta_{\lambda\beta}\delta_{\mu\alpha}\delta_{\nu\gamma} + \delta_{\lambda\gamma}\delta_{\mu\beta}\delta_{\nu\alpha} \qquad (14)$$

and by its use

$$[(a^1\wedge a^2\wedge a^3)\wedge a^4\wedge a^5]_\gamma = \epsilon_{\lambda\mu\nu\rho}a^1_\lambda a^2_\mu a^3_\rho \epsilon_{\rho\alpha\beta\gamma}a^4_\alpha a^5_\beta$$
$$= -\epsilon_{\lambda\mu\nu\rho}\epsilon_{\alpha\beta\gamma\rho}a^1_\lambda a^2_\mu a^3_\rho a^4_\alpha a^5_\beta$$
$$= a^1\cdot a^4\; a^2\cdot a^5\; a^3_\gamma + a^1\cdot a^5\; a^2_\gamma a^3\cdot a^4 + a^1_\gamma a^2\cdot a^4\; a^3\cdot a^5$$
$$- a^1\cdot a^4\; a^2_\gamma a^3\cdot a^5 - a^1\cdot a^5\; a^2\cdot a^4\; a^3_\gamma - a^1_\gamma a^2\cdot a^5\; a^3\cdot a^4$$

or

$$(a^1 \wedge a^2 \wedge a^3) \wedge a^4 \wedge a^5 = (a^1 \cdot a^4 \; a^2 \cdot a^5 - a^1 \cdot a^5 \; a^2 \cdot a^4) a^3$$
$$- (a^1 \cdot a^4 \; a^3 \cdot a^5 - a^1 \cdot a^5 \; a^3 \cdot a^4) a^2$$
$$+ (a^2 \cdot a^4 \; a^3 \cdot a^5 - a^2 \cdot a^5 \; a^3 \cdot a^4) a^1 \qquad (15)$$

Equation 15 can also be proven by noticing that the vector product is perpendicular to all three vectors, and hence $(a^1 \wedge a^2 \wedge a^3) \wedge a^4 \wedge a^5$ is perpendicular to $a^1 \wedge a^2 \wedge a^3$ and lies in the "plane" of a^1, a^2, a^3. We can then write

$$(a^1 \wedge a^2 \wedge a^3) \wedge a^4 \wedge a^5 = \alpha_1 a^1 + \alpha_2 a^2 + \alpha_3 a^3$$

and multiplying by a^4 and a^5

$$0 = \alpha_1 a^1 \cdot a^4 + \alpha_2 a^2 \cdot a^4 + \alpha_3 a^3 \cdot a^4$$
$$0 = \alpha_1 a^1 \cdot a^5 + \alpha_2 a^2 \cdot a^5 + \alpha_3 a^3 \cdot a^5$$

and hence

$$\alpha_1 \propto a^2 \cdot a^4 \; a^3 \cdot a^5 - a^2 \cdot a^5 \; a^3 \cdot a^4$$
$$\alpha_2 \propto a^1 \cdot a^5 \; a^3 \cdot a^4 - a^1 \cdot a^4 \; a^3 \cdot a^5$$
$$\alpha_3 \propto a^1 \cdot a^4 \; a^2 \cdot a^5 - a^1 \cdot a^5 \; a^2 \cdot a^4$$

The constant of proportionality can be found from a particular case (with unit vectors, for instance).

Another useful relation is obtained in the following way:

$$a^1 \wedge a^2 \wedge a^3 \cdot a^4 \wedge a^5 \wedge a^6 = - [\, a^4 \wedge a^5 \wedge a^6 \; a^1 a^2 a^3]$$
$$= - (a^4 \wedge a^5 \wedge a^6) \wedge a^1 \wedge a^2 \cdot a^3$$

and using 15 we get

$$a^1\wedge a^2\wedge a^3\cdot a^4\wedge a^5\wedge a^6 = -a^1\cdot a^4\ a^2\cdot a^5\ a^3\cdot a^6 - a^1\cdot a^5\ a^2\cdot a^6\ a^3\cdot a^4$$
$$- a^1\cdot a^6\ a^2\cdot a^4\ a^3\cdot a^5 + a^1\cdot a^4\ a^2\cdot a^6\ a^3\cdot a^5$$
$$+ a^1\cdot a^5\ a^2\cdot a^4\ a^3\cdot a^6 + a^1\cdot a^6\ a^2\cdot a^5\ a^3\cdot a^4 \quad (16)$$

Or it can be derived directly from equation 14.

By direct computation we can show that

$$\epsilon_{\lambda\mu\nu\rho}\gamma_\lambda\gamma_\mu\gamma_\nu\gamma_5 = 6\gamma_\rho \quad (17)$$

We use it and 14 to derive the exact relation:

$$\gamma\cdot a\wedge b\wedge c = \epsilon_{\lambda\mu\nu\rho}a_\lambda b_\mu c_\nu\gamma_\rho$$
$$= \frac{1}{6}\epsilon_{\lambda\mu\nu\rho}a_\lambda b_\mu c_\nu \epsilon_{\alpha\beta\tau\rho}\gamma_\alpha\gamma_\beta\gamma_\tau\gamma_5$$
$$= \frac{1}{6}\gamma_5(\delta_{\lambda\alpha}\delta_{\mu\beta}\delta_{\nu\tau} + \delta_{\lambda\beta}\delta_{\mu\tau}\delta_{\nu\alpha} + \delta_{\lambda\tau}\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\lambda\alpha}\delta_{\mu\tau}\delta_{\nu\beta}$$
$$- \delta_{\lambda\beta}\delta_{\mu\alpha}\delta_{\nu\tau} - \delta_{\lambda\tau}\delta_{\mu\beta}\delta_{\nu\alpha})a_\lambda b_\mu c_\nu\gamma_\alpha\gamma_\beta\gamma_\tau$$

$$\gamma\cdot a\wedge b\wedge c = \gamma_5(\not a\not b\not c - b\cdot c\not a + c\cdot a\not b - a\cdot b\not c) \quad (18)$$

Using 18 we prove the following:

$$[abcd] = a\wedge b\wedge c\cdot d$$
$$= \frac{1}{2}\gamma\cdot a\wedge b\wedge c\not d + \frac{1}{2}\not d\gamma\cdot a\wedge b\wedge c$$
$$= \frac{1}{2}\gamma_5\{(\not a\not b\not c - b\cdot c\not a + c\cdot a\not b - a\cdot b\not c)\not d$$
$$- \not d(\not a\not b\not c - b\cdot c\not a + c\cdot a\not b - a\cdot b\not c)\}$$

and after some reductions and permutations

$$[abcd] = \gamma_5(\not a\not b\not c\not d - c\cdot d\not a\not b + b\cdot d\not a\not c - b\cdot c\not a\not d - a\cdot d\not b\not c + a\cdot c\not b\not d$$
$$- a\cdot b\not c\not d + a\cdot b\ c\cdot d - a\cdot c\ b\cdot d + a\cdot d\ b\cdot c) \quad (19)$$

It is easy to prove equations R2-A38:

$$\gamma_\mu \gamma_\mu = 4 \qquad (20a)$$

$$\gamma_\mu \not{a} \gamma_\mu = -2\not{a} \qquad (20b)$$

$$\gamma_\mu \not{a} \not{b} \gamma_\mu = 4a \cdot b \qquad (20c)$$

$$\gamma_\mu \not{a} \not{b} \not{c} \gamma_\mu = -2\not{c} \not{b} \not{a} \qquad (20d)$$

Using the first three and 14 and 16, we find

$$\epsilon_{\lambda\mu\nu\rho} a_\lambda b_\mu \gamma_\nu \gamma_\rho = \gamma_5 (\not{b} \not{a} - \not{a} \not{b}) \qquad (21)$$

Using 17 we have:

$$\epsilon_{\lambda\mu\nu\rho} a_\lambda \gamma_\mu \gamma_\nu \gamma_\rho = - \gamma_5 \epsilon_{\mu\nu\rho\lambda} \gamma_\mu \gamma_\nu \gamma_\rho \gamma_5 a_\lambda$$

$$= - 6 \gamma_5 \gamma_\lambda a_\lambda$$

$$\epsilon_{\lambda\mu\nu\rho} a_\lambda \gamma_\mu \gamma_\nu \gamma_\rho = -6 \gamma_5 \not{a} \qquad (22)$$

Equations 18, 19, 21 and 22 show that any terms containing the antisymmetric tensor can be eliminated from matrix elements.

Our next task is to find the linear relations among the M_i of II-126, following the methods that produced equations 6 and 7b. Use of rather simple expressions like $(\epsilon \wedge K \wedge P) \wedge \epsilon' \wedge K + (\epsilon' \wedge K \wedge P) \wedge \epsilon \wedge K$ produce only identities.

Non-trivial results are obtained as follows. By repeated use of 18, 16, II-121, II-123, II-124 and rearranging of factors, we get

$$\gamma\cdot(\epsilon\wedge\Delta\wedge P)\wedge(\epsilon'\wedge\Delta\wedge K)\wedge(K\wedge\Delta\wedge P)$$

$$= \gamma_5(\gamma\cdot\epsilon\wedge\Delta\wedge P\,\gamma\cdot\epsilon'\wedge\Delta\wedge K\,\gamma\cdot K\wedge\Delta\wedge P$$

$$- \epsilon\wedge\Delta\wedge P\cdot\epsilon'\wedge\Delta\wedge K\,\gamma\cdot K\wedge\Delta\wedge P$$

$$+ \epsilon\wedge\Delta\wedge P\cdot K\wedge\Delta\wedge P\,\gamma\cdot\epsilon'\wedge\Delta\wedge K$$

$$- \epsilon'\wedge\Delta\wedge K\cdot K\wedge\Delta\wedge P\,\gamma\cdot\epsilon\wedge\Delta\wedge P)$$

$$= -(\not\epsilon\,\not\Delta\,\not P - \Delta\cdot\epsilon\,\not P + P\cdot\epsilon\,\not\Delta)(\not\epsilon'\not\Delta\,\not K - \Delta\cdot\epsilon'\not K + K\cdot\epsilon'\not\Delta)(\not K\not\Delta\not P + K\cdot P\,\not\Delta)$$

$$+ (-\epsilon\cdot\epsilon'\,\Delta^2K\cdot P + K\cdot\epsilon\,\Delta^2P\cdot\epsilon' + \Delta\cdot\epsilon\,\Delta\cdot\epsilon'\,K\cdot P)(\not K\not\Delta\not P + K\cdot P\,\not\Delta)$$

$$- (-K\cdot\epsilon\,\Delta^2P^2 + P\cdot\epsilon\,\Delta^2K\cdot P)(\not\epsilon'\not\Delta\,\not K - \Delta\cdot\epsilon'\not K + K\cdot\epsilon'\not\Delta)$$

$$+ (-K\cdot\epsilon'\Delta^2K\cdot P + P\cdot\epsilon'\Delta^2K^2)(\not\epsilon\,\not\Delta\,\not P - \Delta\cdot\epsilon\,\not P + P\cdot\epsilon\,\not\Delta)$$

$$= K\cdot P\,\Delta^2\{MK\cdot PM_1 + P^2M_2 + MM_4 - M_8 + (K\cdot P - P^2)M_9$$

$$- MK\cdot PM_{13} + P^2M_{14}\} \qquad (23a)$$

On the other hand, using 15 and II-124 we get

$$\gamma\cdot(\epsilon\wedge\Delta\wedge P)\wedge(\epsilon'\wedge\Delta\wedge K)\wedge(K\wedge\Delta\wedge P)$$

$$= [\epsilon'\Delta KP][K\Delta P\epsilon]\not\Delta$$

$$= 0 \qquad (23b)$$

and hence

$$MK\cdot PM_1 + P^2M_2 + MM_4 - M_8 + (K\cdot P - P^2)M_9 - MK\cdot PM_{13}$$

$$+ P^2M_{14} = 0 \qquad (23c)$$

In a similar way

$$\gamma\cdot(\epsilon \wedge K \wedge \Delta) \wedge (\epsilon' \wedge K \wedge P) \wedge (\Delta \wedge K \wedge P)$$

$$= \gamma_5(\gamma\cdot\epsilon \wedge K \wedge \Delta\ \gamma\cdot\epsilon' \wedge K \wedge P\ \gamma\cdot\Delta \wedge K \wedge P$$

$$- \epsilon \wedge K \wedge \Delta\cdot\epsilon' \wedge K \wedge P\ \gamma\cdot\Delta \wedge K \wedge P$$

$$+ \epsilon \wedge K \wedge \Delta\cdot\Delta \wedge K \wedge P\ \gamma\cdot\epsilon' \wedge K \wedge P$$

$$- \epsilon' \wedge K \wedge P\cdot\Delta \wedge K \wedge P\ \gamma\cdot\epsilon \wedge K \wedge \Delta)$$

$$= \Delta^2 \{(K\cdot P)^2 - K^2P^2\} M_2 - MK^2\Delta^2 M_4 - \Delta^2K^2M_6 - MK^2K\cdot PM_7$$

$$+ \{\Delta^2K\cdot P + \Delta^2K^2 + (K\cdot P)^2\} M_8 + \{K^2P^2K\cdot P + P^2K^2\Delta^2 - \Delta^2(K\cdot P)^2$$

$$- (K\cdot P)^3\} M_9 - M\Delta^2K\cdot PM_{10} + 2M(K\cdot P)^2 M_{11}$$

$$+ \{P^2K^2 - 2\Delta^2K\cdot P - P^2\Delta^2 - 2P^2K\cdot P - (K\cdot P)^2\} M_{12} \qquad (24a)$$

$$\gamma\cdot(\epsilon \wedge K \wedge \Delta) \wedge (\epsilon' \wedge K \wedge P) \wedge (\Delta \wedge K \wedge P)$$

$$= [\epsilon\Delta KP][\Delta\epsilon' KP] \not{K}$$

$$= [\epsilon' KP\Delta]\gamma\cdot(K^2P \wedge \epsilon \wedge \Delta - K\cdot P\epsilon \wedge \Delta \wedge K + K\cdot\epsilon\ \Delta \wedge K \wedge P)$$

$$= \gamma\cdot\{K^2P \wedge \epsilon \wedge (-\Delta\cdot\epsilon' K \wedge P \wedge \Delta + \Delta^2\epsilon' \wedge K \wedge P)$$

$$- P\cdot K\epsilon \wedge (-\Delta\cdot\epsilon' K \wedge P \wedge \Delta + \Delta^2\epsilon' \wedge K \wedge P) \wedge K$$

$$+ K\cdot\epsilon(-\Delta\cdot\epsilon' K \wedge \Delta \wedge P + \Delta^2\epsilon' \wedge K \wedge P) \wedge K \wedge P\}$$

$$= -K^2\Delta^2M_6 + \Delta^2K\cdot PM_8 + \{K^2P^2 - (K\cdot P)^2 - \Delta^2P^2\} M_{12}$$

$$+ \Delta^2\{K^2P^2 - (K\cdot P)^2\} M_{14} \qquad (24b)$$

where 13 and 15 have been used to get 24b. From these two:

$$\Delta^2\{K^2P^2 - (K\cdot P)^2\} M_2 + MK^2\Delta^2M_4 + MK^2K\cdot PM_7 - \{\Delta^2K^2 + (K\cdot P)^2\} M_8$$

$$+ (\Delta^2 + K\cdot P)\{(K\cdot P)^2 - K^2P^2\} M_9 + M\Delta^2K\cdot PM_{10} + 2M^2K\cdot PM_{12}$$

$$+ \Delta^2\{K^2P^2 - (K\cdot P)^2\} M_{14} = 0 \qquad (24c)$$

Also

$$\gamma\cdot(\epsilon \wedge K \wedge P) \wedge (\epsilon' \wedge \Delta \wedge P) \wedge (K \wedge \Delta \wedge P)$$

$$= - M\Delta^2\{K^2P^2 - (K\cdot P)^2\}M_1 + M\Delta^2 K\cdot PM_4 + M\Delta^2K^2M_5$$

$$+ M(K^2P^2 - \Delta^2K\cdot P)M_7 - M^2K\cdot PM_8 - M^2\{K^2P^2 - (K\cdot P)^2\}M_9$$

$$- MP^2\Delta^2M_{10} + M\{\Delta^2P^2 + (K\cdot P)^2 - K^2P^2\}M_{11} + 2M^2P^2M_{12} \quad (25a)$$

$$\gamma\cdot(\epsilon \wedge K \wedge P) \wedge (\epsilon' \wedge \Delta \wedge P) \wedge (K \wedge \Delta \wedge P)$$

$$= - [\epsilon K\Delta P][K\epsilon'\Delta P]\not{P}$$

$$= MK^2\Delta^2M_5 - M\Delta^2K\cdot PM_7 + M\{P^2\Delta^2 - K^2P^2 + (K\cdot P)^2\}M_{11}$$

$$- M\Delta^2\{K^2P^2 - (K\cdot P)^2\}M_{13} \quad (25b)$$

$$\Delta^2\{K^2P^2 - (K\cdot P)^2\}M_1 - \Delta^2K\cdot PM_4 - K^2P^2M_7 + MK\cdot PM_8$$

$$+ M\{K^2P^2 - (K\cdot P)^2\}M_9 - P^2\Delta^2M_{10} - 2MP^2M_{12}$$

$$- \Delta^2\{K^2P^2 - (K\cdot P)^2\}M_{13} = 0 \quad (25c)$$

One more relation that can be derived from 11c and the rules for multiplication of determinants is

$$[a^1a^2a^3a^4][a^5a^6a^7a^8] = - \begin{vmatrix} a_0^1 & -a_1^1 & -a_2^1 & -a_3^1 \\ a_0^2 & -a_1^2 & -a_2^2 & -a_3^2 \\ a_0^3 & -a_1^3 & -a_2^3 & -a_3^3 \\ a_0^4 & -a_1^4 & -a_2^4 & -a_3^4 \end{vmatrix} \begin{vmatrix} a_0^5 & a_1^5 & a_2^5 & a_3^5 \\ a_0^6 & a_1^6 & a_2^6 & a_3^6 \\ a_0^7 & a_1^7 & a_2^7 & a_3^7 \\ a_0^8 & a_1^8 & a_2^8 & a_3^8 \end{vmatrix}$$

$$= - \begin{vmatrix} a^1\cdot a^5 & a^1\cdot a^6 & a^1\cdot a^7 & a^1\cdot a^8 \\ a^2\cdot a^5 & a^2\cdot a^6 & a^2\cdot a^7 & a^2\cdot a^8 \\ a^3\cdot a^5 & a^3\cdot a^6 & a^3\cdot a^7 & a^3\cdot a^8 \\ a^4\cdot a^5 & a^4\cdot a^6 & a^4\cdot a^7 & a^4\cdot a^8 \end{vmatrix} \quad (26)$$

This gives an alternative, and maybe easier, way of finding equations 24b and 25b. The three equations 23c, 24 and 25c are not independent, as should be expected. This then serves as a check, since they were obtained independently. A little further manipulation gives equations II-127.

It must be remembered that equations II-124 were used repeatedly and hence these equalities are only true when multiplied on both sides by the corresponding spinors $\bar{u}(\vec{p}\,')$ and $u(\vec{p})$. This is a basic difference between them and equations II-131.

These calculations, unfortunately, do not show a systematic way of obtaining relations between invariants that can be shown to work in all cases. The general idea is to obtain non-trivial results by two different expansions of a sufficiently complicated expression. The method outlined by Hearn in reference 12 does not seem to lead to the right number of invariants in an obvious way, and further relations like equations II-127 are necessary.

We will now briefly discuss the problem of kinematical singularities. In short, they can be defined as those singularities not present in the reactions with spinless particles.

In the same reference 12 it is pointed out that, in perturbation theory, the only difference between matrix elements for spinless reactions and the corresponding ones for reactions involving particles with spin lies in the additional factors that appear in the numerator. These have to be reduced to the chosen set of invariants without introducing singularities.

It is obvious that no denominators are involved in the reduction of the matrix element to a linear combination of invariants like those 14 in II-126. Any terms involving the antisymmetric tensor can be reduced by using equations 18, 19, 21 and 22.

In the particular case of ρN elastic scattering, equations II-127 show that M_8 and M_{12}, for instance, can be eliminated without introducing new singularities. This proves, to all orders in perturbation theory, that the remaining 12 invariants are associated with amplitudes free of kinematical singularities.

This gives no way of telling if a set of amplitudes free of kinematical singularities can be found until the actual relations are calculated; but it is presumed that this is always possible.

APPENDIX F

SPIN $\frac{3}{2}$ FORMALISM

This problem has been treated in several different ways; we will follow more or less reference 13. See also reference 14 and other references quoted there.

The wave functions for spin $\frac{3}{2}$ particles will be written as a set of four four-component spinors ψ_μ, $\mu = 0, 1, 2, 3$; the index μ is a tensor index in space-time. From the general theory of particles with spin we can derive R13-4.56a:

$$(\not{p} - M)\psi_\mu = 0 \tag{1a}$$

$$\gamma_\mu \psi_\mu = 0 \tag{1b}$$

where p_μ is the usual differential operator when acting on a function of x_μ,

$$p_\mu = -i\,\partial_\mu = -i\,\frac{\partial}{\partial x_\mu} \tag{2}$$

The subsidiary condition 1b serves to eliminate the spin $\frac{1}{2}$ part in the ψ_μ.

Multiplying 1a on the left by $\not{p} + M$, we get

$$(p^2 - M^2)\psi_\mu = 0 \tag{2a}$$

Multiplying it on the left by γ_μ and using 1b and the commutation rules for γ matrices,

$$p_\mu \psi_\mu = 0 \tag{2b}$$

Both equations 1a and 1b can be included in one of the form

$$\Lambda_{\mu\nu}(p)\psi_\nu = 0 \tag{3a}$$

where $\Lambda_{\mu\nu}$ is given by R13-4.56b:

$$\Lambda_{\mu\nu}(p) = (\not{p}-M)\delta_{\mu\nu} - \frac{1}{3}(\gamma_\mu p_\nu + \gamma_\nu p_\mu) + \frac{1}{3}\gamma_\mu(\not{p}+M)\gamma_\nu \tag{3b}$$

This can be checked by multiplying 3a on the left by γ_μ or p_μ; we get respectively

$$\left(\frac{2}{3}p_\nu + \frac{1}{3}M\gamma_\nu\right)\psi_\nu = 0 \tag{3c}$$

$$\left(-Mp_\nu + \frac{2}{3}\not{p}p_\nu + \frac{1}{3}M\not{p}\gamma_\nu\right)\psi_\nu = 0 \tag{3d}$$

and by combining these with 3a we get back equations 1.

Equation 3a can be obtained from a Lagrangian density by the usual equations (R9-2.4)

$$\frac{\partial \mathcal{L}}{\partial u_i} - p_k \frac{\partial \mathcal{L}}{\partial(p_k u_i)} = 0 \tag{4}$$

From R13-7.124

$$\mathcal{L} = \overline{\psi}_\mu \overrightarrow{\Lambda}_{\mu\nu}\psi_\nu \tag{5a}$$

where the arrow indicates the direction in which $\Lambda_{\mu\nu}$ operates. Or, to bring it into a form closer to R9-7.27:

$$\begin{aligned} \mathcal{L} = \frac{1}{2}\{ &\overline{\psi}_\mu \overrightarrow{\not{p}}\,\psi_\mu - \frac{1}{3}\overline{\psi}_\mu(\gamma_\mu \overrightarrow{p}_\nu + \gamma_\nu \overrightarrow{p}_\mu)\psi_\nu + \frac{1}{3}\overline{\psi}_\mu\gamma_\mu\overrightarrow{\not{p}}\gamma_\nu\psi_\nu\} \\ &- \frac{1}{2}\{\overline{\psi}_\mu\overleftarrow{\not{p}}\psi_\mu - \frac{1}{3}\overline{\psi}_\mu(\gamma_\mu\overleftarrow{p}_\nu + \gamma_\nu\overleftarrow{p}_\mu)\psi_\nu + \frac{1}{3}\overline{\psi}_\mu\gamma_\mu\overleftarrow{\not{p}}\gamma_\nu\psi_\nu\} \\ &- M(\overline{\psi}_\mu\psi_\mu - \frac{1}{3}\overline{\psi}_\mu\gamma_\mu\gamma_\nu\psi_\nu) \end{aligned} \tag{5b}$$

From the Lagrangian density we can obtain the energy momentum tensor and angular momentum tensor.

If in R-2. 20

$$T_{\alpha\beta} = \frac{\partial \mathcal{L}}{\partial(p_\alpha u_i)} p_\beta u_i - \mathcal{L}\delta_{\alpha\beta} \tag{6a}$$

we sum over the spinor index implicitly, we get

$$T_{\alpha\beta} = \frac{\partial \mathcal{L}}{\partial(p_\alpha \psi_\mu)} \overrightarrow{p}_\beta \psi_\mu + \overline{\psi}_\mu \overleftarrow{p}_\beta \frac{\partial \mathcal{L}}{\partial(p_\alpha \overline{\psi}_\mu)} - \mathcal{L}\delta_{\alpha\beta} \tag{6b}$$

We carry out these operations on 5b and use equations 1; $T_{\alpha\beta}$ is reduced to

$$T_{\alpha\beta} = \frac{1}{2}(\overline{\psi}_\mu \gamma_k \overrightarrow{p}_\ell \psi_\mu - \overline{\psi}_\mu \overleftarrow{p}_\ell \gamma_k \psi_\mu) \tag{6c}$$

It should be noticed that the adjoint state vector $\overline{\psi}_\mu$ satisfies the equations

$$\overline{\psi}_\mu(\overleftarrow{\not p} + M) = 0 \tag{7a}$$

$$\overline{\psi}_\mu \gamma_\mu = 0 \tag{7b}$$

which are directly derived from 1a and 1b; and that for functions that obey these field equations the Lagrangian density reduces to zero.

Equation R9-2. 27 for the spin tensor

$$S_{\kappa,\lambda\rho} = i \frac{\partial \mathcal{L}}{\partial(p_\kappa u_i)} u_j A_{ij,\lambda\rho} \tag{8a}$$

can be written

$$S_{\kappa,\lambda\rho} = i \frac{\partial \mathcal{L}}{\partial(p_\kappa \psi_\mu)} A^{\psi}_{\mu\nu,\lambda\rho} \psi_\nu + i\overline{\psi}_\nu A^{\overline{\psi}}_{\mu\nu,\lambda\rho} \frac{\partial \mathcal{L}}{\partial(p_\kappa \overline{\psi}_\mu)} \tag{8b}$$

The $A_{ij,\lambda\rho}$ are defined in terms of infinitesimal rotations $\delta\omega_{\lambda\rho}$ by

$$\delta u_i = \sum_{j;\lambda<\rho} A_{ij,\lambda\rho} u_j \delta\omega_{\sigma\rho} \tag{9a}$$

ψ_μ has actually two indices: μ, which is a vector index, and an implicit spinor index. We determine then, by using R9-2.24 for a vector field and the equations above R9-7.30 for a spinor field, for instance, that

$$A^{\psi}_{\mu\nu,\lambda\rho} = \delta_{\mu\nu}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\lambda} + \frac{i}{2}\sigma_{\lambda\rho}\delta_{\mu\nu} \tag{9b}$$

$$A^{\overline{\psi}}_{\mu\nu,\lambda\rho} = \delta_{\mu\lambda}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\lambda} - \frac{i}{2}\sigma_{\lambda\rho}\delta_{\mu\nu} \tag{9c}$$

where

$$\sigma_{\lambda\rho} = \frac{\gamma_\lambda\gamma_\rho - \gamma_\rho\gamma_\lambda}{2i} \tag{9d}$$

Carrying out the calculations in 8b and using equations 1, we obtain

$$\begin{aligned} S_{\kappa,\lambda\rho} = {} & -i(\overline{\psi}_\lambda\gamma_\kappa\psi_\rho - \overline{\psi}_\rho\gamma_\kappa\psi_\lambda) - \frac{1}{4}\overline{\psi}_\mu(\gamma_\kappa\sigma_{\lambda\rho} + \sigma_{\lambda\rho}\gamma_\kappa)\psi_\mu \\ & + \frac{i}{6}(\overline{\psi}_\kappa\gamma_\lambda\psi_\rho - \overline{\psi}_\kappa\gamma_\rho\psi_\lambda + \overline{\psi}_\rho\gamma_\lambda\psi_\kappa - \overline{\psi}_\lambda\gamma_\rho\psi_\kappa) \end{aligned} \tag{10}$$

Of special interest will be the z-component of spin

$$\begin{aligned} S_3 &= \int S_{0,12} d^3x \\ &= \int \{-i(\psi_1^\dagger\psi_2 - \psi_2^\dagger\psi_1) - \frac{1}{2}\psi_\mu^\dagger\sigma_{12}\psi_\mu \\ &\qquad + \frac{i}{6}(\overline{\psi}_0\gamma_1\psi_2 - \overline{\psi}_0\gamma_2\psi_1 + \overline{\psi}_2\gamma_1\psi_0 - \overline{\psi}_1\gamma_2\psi_0)\} d^3x \end{aligned} \tag{11}$$

We will study next the plane wave solutions of the spin $\frac{3}{2}$ equation. We write

$$\psi_\mu(x) = \eta u_\mu(\vec{p},\lambda)e^{-ip\cdot x} \tag{12}$$

where η is a normalization factor such that if the u_μ are normalized to

$$\bar{u}_\mu u_\mu = -1 \tag{13a}$$

$$\int \psi_\mu^\dagger \psi_\mu d^3x = -1 \tag{13b}$$

the integration being carried out over a volume of normalization V. Hence

$$|\eta|^2 = \frac{1}{V}\cdot\frac{M}{E} \tag{13c}$$

where $E = p_o$ is the energy of the particle. We notice that u_μ and $\bar{u}_\mu$ obey the equations

$$(\not{p} - M)u_\mu(\vec{p}) = 0 \tag{14a}$$

$$\gamma_\mu u_\mu(\vec{p}) = 0 \tag{14b}$$

$$\bar{u}_\mu(\not{p} - M) = 0 \tag{14c}$$

$$\bar{u}_\mu \gamma_\mu = 0 \tag{14d}$$

where p_μ is the momentum four-vector and no longer an operator.

We will prove next that the following state vectors correspond to different helicity states for the spin $\frac{3}{2}$ particle:

$$u_\mu(\tfrac{3}{2}) = \epsilon_\mu(1)u(\tfrac{1}{2}) \tag{15a}$$

$$u_\mu(\tfrac{1}{2}) = \sqrt{\tfrac{2}{3}}\,\epsilon_\mu(0)u(\tfrac{1}{2}) + \sqrt{\tfrac{1}{3}}\,\epsilon_\mu(1)u(-\tfrac{1}{2}) \tag{15b}$$

$$u_\mu(-\tfrac{1}{2}) = \sqrt{\tfrac{1}{3}}\,\epsilon_\mu(-1)u(\tfrac{1}{2}) + \sqrt{\tfrac{2}{3}}\,\epsilon_\mu(0)u(-\tfrac{1}{2}) \tag{15c}$$

$$u_\mu(-\tfrac{3}{2}) = \epsilon_\mu(-1)u(-\tfrac{1}{2}) \tag{15d}$$

The momentum $\vec{p}$ has been omitted, and the labels correspond to helicities of the corresponding spin $\frac{3}{2}$, spin 1 and spin $\frac{1}{2}$ states. The spin $\frac{3}{2}$ states have been considered a direct product of spin 1 and spin $\frac{1}{2}$ states, and the corresponding Clebsch-Gordan coefficients have been used. The $u_\mu(\lambda)$ satisfy trivially equation 1a, since the Dirac spinors $u(\lambda')$ do.

In what follows we will choose the z-axis in the direction of $\vec{p}$, to avoid unnecessary algebra. Then

$$\epsilon(1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ -i \\ 0 \end{pmatrix} \qquad \epsilon(0) = \begin{pmatrix} \frac{|\vec{p}|}{M} \\ 0 \\ 0 \\ \frac{E}{M} \end{pmatrix} \qquad \epsilon(-1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix} \tag{16a}$$

$$u(\pm\tfrac{1}{2}) = \sqrt{\frac{E+M}{2M}}\left(1 + \frac{|\vec{p}|}{E+M}\alpha_3\right) \begin{pmatrix} |\pm\frac{1}{2}> \\ 0 \end{pmatrix} \tag{16b}$$

where

$$|\tfrac{1}{2}> = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |-\tfrac{1}{2}> = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Equation 1b for $u_\mu(\frac{3}{2})$ gives

$$\begin{aligned} \gamma_\mu u_\mu(\tfrac{3}{2}) &= \frac{1}{\sqrt{2}} \sqrt{\frac{E+M}{2M}}(\gamma_1 + i\gamma_2)\left(1 + \frac{|\vec{p}|}{E+M}\alpha_3\right) \begin{pmatrix} |\frac{1}{2}> \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{|\vec{p}|}{E+M}(\sigma_1 + i\sigma_2) & \sigma_1 + i\sigma_2 \\ -(\sigma_1 + i\sigma_2) & \frac{|\vec{p}|}{E+M}(\sigma_1 + i\sigma_2) \end{pmatrix} \begin{pmatrix} |\frac{1}{2}> \\ 0 \end{pmatrix} \sqrt{\frac{E+M}{4M}} \\ &= 0 \end{aligned}$$

since

$$\begin{aligned} (\sigma_1 + i\sigma_2)|\tfrac{1}{2}> &= \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= 0 \end{aligned}$$

For $u_\mu(\frac{1}{2})$ we have

$$\gamma_\mu u_\mu(\tfrac{1}{2}) = \sqrt{\tfrac{2}{3}}\sqrt{\frac{E+M}{2M}}\left(\frac{|\vec{p}|}{M}\gamma_0 - \frac{E}{M}\gamma_3\right)\left(1 + \frac{|\vec{p}|}{E+M}\alpha_3\right)\begin{pmatrix} |\frac{1}{2}> \\ 0 \end{pmatrix}$$

$$+ \sqrt{\tfrac{1}{3}}\,\frac{1}{\sqrt{2}}\sqrt{\frac{E+M}{2M}}(\gamma_1 + i\gamma_2)\left(1 + \frac{|\vec{p}|}{E+M}\alpha_3\right)\begin{pmatrix} |-\frac{1}{2}> \\ 0 \end{pmatrix}$$

$$= \sqrt{\tfrac{1}{6}}\sqrt{\frac{E+M}{2M}}\left\{ 2\begin{pmatrix} \frac{|\vec{p}|}{E+M} & -\sigma_3 \\ \sigma_3 & -\frac{|\vec{p}|}{E+M} \end{pmatrix}\begin{pmatrix} |\frac{1}{2}> \\ 0 \end{pmatrix} \right.$$

$$\left. + \begin{pmatrix} -\frac{|\vec{p}|}{E+M}(\sigma_1 + i\sigma_2) & \sigma_1 + i\sigma_2 \\ -(\sigma_1 + i\sigma_2) & \frac{|\vec{p}|}{E+M}(\sigma_1 + i\sigma_2) \end{pmatrix}\begin{pmatrix} |-\frac{1}{2}> \\ 0 \end{pmatrix} \right\}$$

$$= \sqrt{\tfrac{1}{6}}\sqrt{\frac{E+M}{2M}}\left\{ 2\begin{pmatrix} \frac{|\vec{p}|}{E+M}\begin{vmatrix} 1 \\ 0 \end{vmatrix} \\ \begin{vmatrix} 1 \\ 0 \end{vmatrix} \end{pmatrix} + \begin{pmatrix} -\frac{|\vec{p}|}{E+M}\begin{vmatrix} 2 \\ 0 \end{vmatrix} \\ -\begin{vmatrix} 2 \\ 0 \end{vmatrix} \end{pmatrix} \right\}$$

$$= 0$$

and in a similar way we can prove that the other two states also satisfy equation 1b.

We can check the helicities of the states by substituting 12 into 11, and after performing a trivial integration we get

$$S_3(\tfrac{3}{2}) = \frac{M}{E}\left\{ -i\left(\tfrac{1}{2}\, i\, u^\dagger(\tfrac{1}{2})u(\tfrac{1}{2}) + \tfrac{1}{2}\, i u^\dagger(\tfrac{1}{2})u(\tfrac{1}{2})\right) - \tfrac{1}{2}\, u^\dagger(\tfrac{1}{2})\epsilon^*_\mu(1)\sigma_{12}\epsilon_\mu(1)u(\tfrac{1}{2}) \right\}$$

$$= \frac{M}{E}\,(1 + \tfrac{1}{2})\,\frac{E}{M}$$

$$= \frac{3}{2}$$

since

$$\sigma_{12} = \Sigma_3 = \begin{vmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{vmatrix}$$

and

$$\Sigma_3 u(\pm \tfrac{1}{2}) = \pm \tfrac{1}{2} u(\pm \tfrac{1}{2})$$

Also the $u(\lambda)$ are normalized so that $\overline{u}u = 1$, and hence $u^\dagger u = \frac{E}{M}$

$$S_3(\tfrac{1}{2}) = \frac{M}{E}\Big\{-i\Big(\tfrac{1}{6} i\, u^\dagger(-\tfrac{1}{2})u(-\tfrac{1}{2}) + \tfrac{1}{6} i u^\dagger(-\tfrac{1}{2})u(-\tfrac{1}{2})\Big) - \tfrac{1}{2}\Big(\tfrac{2}{3}\frac{\vec{p}^2}{M^2}u^\dagger(\tfrac{1}{2})\sigma_{12}u(\tfrac{1}{2})$$

$$- \tfrac{1}{6}u^\dagger(-\tfrac{1}{2})\sigma_{12}u(-\tfrac{1}{2}) - \tfrac{1}{6}u^\dagger(-\tfrac{1}{2})\sigma_{12}u(-\tfrac{1}{2}) - \tfrac{2}{3}\frac{E^2}{M^2}u^\dagger(\tfrac{1}{2})\sigma_{12}u(\tfrac{1}{2})\Big)$$

$$+ \tfrac{i}{6}\Big(-\tfrac{i}{3}\overline{u}(\tfrac{1}{2})\gamma_1 u(-\tfrac{1}{2}) + \tfrac{1}{3}\overline{u}(\tfrac{1}{2})\gamma_2 u(-\tfrac{1}{2}) + \tfrac{i}{3} u(-\tfrac{1}{2})\gamma_1 u(\tfrac{1}{2})$$

$$+ \tfrac{1}{3}\overline{u}(-\tfrac{1}{2})\gamma_2 u(\tfrac{1}{2})\Big)\Big\}$$

$$= \frac{M}{E}\Big\{\frac{1}{3}\frac{E}{M} - \frac{1}{2}\Big(\frac{2}{3}\frac{\vec{p}^2}{M^2} + \frac{1}{6} + \frac{1}{6} - \frac{2}{3}\frac{E^2}{M^2}\Big)\frac{E}{M}\Big\}$$

$$= \frac{1}{2}$$

The third term in S_3 again gives no contribution, as can be seen from the computation:

$$\overline{u}(\tfrac{1}{2})\gamma_1 u(-\tfrac{1}{2}) = (\langle\tfrac{1}{2}|, 0)(1 + \frac{|\vec{p}|}{E+M}\alpha_3)\alpha_1\Big(1 + \frac{|\vec{p}|}{E+M}\Big)\begin{pmatrix} |-\tfrac{1}{2}\rangle \\ 0 \end{pmatrix}$$

$$= (\langle\tfrac{1}{2}|, 0)\Big\{\alpha_1 + \frac{|\vec{p}|}{E+M}(\alpha_1\alpha_3 + \alpha_3\alpha_1) + \frac{\vec{p}^2}{(E+M)^2}\alpha_3\alpha_1\alpha_3\Big\}\begin{pmatrix} |-\tfrac{1}{2}\rangle \\ 0 \end{pmatrix}$$

$$= \frac{2M}{E+M}(\langle\tfrac{1}{2}|, 0)\alpha_1\begin{pmatrix} |-\tfrac{1}{2}\rangle \\ 0 \end{pmatrix}$$

$$= 0$$

and similarly for the other products.

The helicities of the other two state vectors can be checked in the same way. All the results obtained so far are of course unchanged when the fields are quantized.

Our next task is to determine the propagator for a spin $\frac{3}{2}$ particle. To this effect, we have to find an operator $d_{\lambda\mu}(p)$ such that

$$d_{\lambda\mu}(p)\Lambda_{\mu\nu}(p) = (p^2-M^2)\delta_{\lambda\nu} \tag{17}$$

It is straightforward to check that

$$\begin{aligned} d_{\lambda\mu}(p) = (\not p+M)[\,\delta_{\lambda\mu} - \tfrac{1}{3}\gamma_\lambda\gamma_\mu - \tfrac{1}{3M}(\gamma_\lambda p_\mu - \gamma_\mu p_\lambda) - \tfrac{2}{3M^2} p_\lambda p_\mu\,] \\ + \frac{2}{3M^2}(p^2-M^2)[\,(\gamma_\lambda p_\mu - \gamma_\mu p_\lambda) + (\not p+M)\gamma_\lambda\gamma_\mu\,] \end{aligned} \tag{18a}$$

(This operator is incorrectly given by R13-4.56a.) It can also be written in a slightly more symmetric form

$$\begin{aligned} d_{\lambda\mu}(p) = (\not p+M)(\delta_{\lambda\mu} - \frac{p_\lambda p_\mu}{M^2}) + \tfrac{1}{3}(\gamma_\lambda + \frac{p_\lambda}{M})(\not p-M)(\gamma_\mu + \frac{p_\mu}{M}) \\ + \frac{2}{3M^2}(p^2-M^2)[\,\gamma_\lambda p_\mu + \gamma_\mu p_\lambda - \gamma_\lambda(\not p-M)\gamma_\mu\,] \end{aligned} \tag{18b}$$

The first term is, of course, the direct product of the corresponding operator for spin $\frac{1}{2}$ and spin 1 particles.

The propagator is then, with the conventions used in reference 1 page 478 and $h = c = 1$

$$P_{\mu\nu} = -\frac{i}{(2\pi)^4}\frac{d_{\mu\nu}(p)}{p^2-M^2+i\epsilon} \tag{19}$$

where p is the four-momentum. The minus sign is related to the normalization of the u_μ, equation 13a.

Another quantity of interest is the current vector, given by R9-2.31

$$J_\mu = \frac{\partial \mathcal{L}}{\partial(p_\mu u_i^*)} u_i^* - \frac{\partial \mathcal{L}}{\partial(p_\mu u_i)} u_i \tag{20a}$$

Written in terms of spinors, it becomes

$$J_\mu = \bar{\psi}_\nu \frac{\partial \mathcal{L}}{\partial(p_\mu \bar{\psi}_\nu)} - \frac{\partial \mathcal{L}}{\partial(p_\mu \psi_\nu)} \psi_\nu \tag{20b}$$

If we take the Lagrangian density given by 5b, and use equations 1 and 7 in the reduction of terms, we get

$$J_\mu = - \bar{\psi}_\nu \gamma_\mu \psi_\nu \tag{20c}$$

for the current of spin $\frac{3}{2}$ particles.

APPENDIX G

N/D METHOD

In section IVa we point out that the T matrix can be written

$$T = \frac{N}{D} \tag{1}$$

in order to separate it in two parts such that D has only the right-hand cuts, arising through unitarity, and N the rest of the cuts; the equations they obey will insure that unitarity be satisfied. In this appendix we will not be concerned with the complications arising from anomalous thresholds and complex singularities; these are left to appendix H.

The T matrix is defined from the scattering matrix by

$$S = 1 + (2\pi)^4 i\delta^4(P_\mu - P'_\mu)T \tag{2}$$

where P_μ and P'_μ are the initial and final total four-momenta.

It can be shown that the S matrix is unitary for physical values of the variables (see reference 1, section IId, for potential scattering, reference 9, section 17.4, for a general case), i.e.

$$S^\dagger S = 1 \tag{3a}$$

$$SS^\dagger = 1 \tag{3b}$$

(For infinite and continuous matrices both equations have to be proven separately; one is not a consequence of the other.)

Relation 3a can be written

$$\sum_c \langle b|S^\dagger|c\rangle\langle c|S|a\rangle = \langle b|a\rangle \tag{3c}$$

where the sum over intermediate states becomes an integration for continuous spectra.

Substituting 2 into 3c we get

$$<b|T-T^{\dagger}|a> = (2\pi)^4 i \sum_{c} \delta^4(P^a_\mu - P^c_\mu)<b|T^{\dagger}|c><c|T|a> \qquad (4)$$

valid when

$$P^a_\mu = P^b_\mu \qquad (4a)$$

For the case of scattering of two spinless particles in the center of mass system, equation 4 becomes

$$<b|T-T^{\dagger}|a> = <b|(2\pi)^4 i\int \frac{d^3\vec{q}}{(2\pi)^3}\,\frac{d^3\vec{q}\,'}{(2\pi)^3}\,\delta^3(\vec{q}+\vec{q}\,')\delta(q_o+q_o'-W)T^{\dagger}T|a> \quad (4b)$$

where $\vec{q}$, $\vec{q}\,'$ are the three-momenta of the intermediate particles; q_o, q_o', their energies, and W, the total energy.

$$T-T^{\dagger} = (2\pi)^4 i\int \frac{\vec{q}^{\,2} d|\vec{q}|d\Omega_{\hat{q}}}{(2\pi)^6}\,\delta[q_o(|\vec{q}|)+q_o'(|\vec{q}|)-W]\,T^{\dagger}T$$

and using II-25

$$T - T^{\dagger} = 2\pi i\int d\Omega_{\hat{q}}\,\rho(W)T^{\dagger}T \qquad (5)$$

where

$$\rho(W) = \frac{|\vec{q}|q_o q_o'}{(2\pi)^3 W} \qquad (6)$$

$$q_o^2 = \vec{q}^{\,2} + m^2 \qquad q_o' = \vec{q}^{\,2} + m'^2 \qquad q_o + q_o' = W \qquad (6a, b, c)$$

We write now a partial wave expansion for T

$$T(W,x) = \frac{1}{2\pi\rho(W)}\sum_{\ell}(2\ell+1)T_\ell(W)P_\ell(x) \qquad (7)$$

and substituting into equation 5

$$\frac{1}{2\pi\rho}\sum_{\ell}(2\ell+1)(T_\ell - T_\ell^\dagger)P_\ell(x)$$

$$= 2\pi i\int d\Omega_{\hat{q}}\,\rho\left(\frac{1}{2\pi\rho}\right)^2\sum_{\ell}(2\ell+1)T_\ell^\dagger P_\ell(x')\sum_{\ell'}(2\ell'+1)T_{\ell'}P_{\ell'}(x'')$$

where the angles θ, θ' and θ'' are shown in figure 1, and obey the equation

$$\cos\theta'' = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\varphi'$$

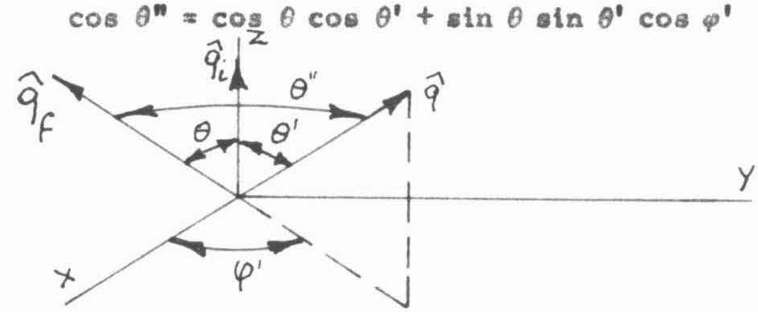

Fig. 1. Direction of momenta

Reference 8 on page 1327 gives the addition theorem for Legendre polynomials

$$P_\ell(x'') = P_\ell(x)P_\ell(x') + 2\sum_{m=1}^{\ell}\frac{(\ell-m)!}{(\ell+m)!}P_\ell^m(x)P_\ell^m(x')\cos m\varphi'$$

and using this and the orthogonality of Legendre polynomials we get finally

$$T_\ell - T_\ell^\dagger = 2iT_\ell^\dagger T \tag{8a}$$

or

$$\mathrm{Im}\,T_\ell = |T_\ell|^2 \tag{8b}$$

$$T_\ell = e^{i\delta_\ell}\sin\delta_\ell \tag{8c}$$

We will drop the subscript in what follows. We can use equation 1 and get

$$D^* \,\mathrm{Im}\, N - N^* \,\mathrm{Im}\, D = |N|^2$$

and along the right-hand cut where $\mathrm{Im}\, N = 0$ we have

$$\mathrm{Im}\, D = - N \tag{9a}$$

and along the left-hand cut where $\mathrm{Im}\, D = 0$, we have

$$\mathrm{Im}\, N = D \,\mathrm{Im}\, T \tag{9b}$$

Now we can use equations 9 to write dispersion relations. For instance, from 9a we get

$$D = 1 - \frac{s-s_o}{\pi} \int_{s_1}^{\infty} \frac{N(s')\, ds'}{(s'-s_o)(s'-s-i\epsilon)} \tag{9c}$$

where $s = W^2$; s_1 corresponds to the threshold for the reaction, and s_o is usually taken equal to s_1 since this would give the right threshold behavior.

Equation 8a is easily generalized to the case of several channels of two particles with spin:

$$T^J - T^{J\dagger} = 2i T^{J\dagger} T^J \tag{10}$$

where the T^J are now matrices with elements corresponding to the helicity amplitudes in the different channels, for instance. The super index J will be dropped in what follows.

Time reversal invariance requires that

$$T^T = T \tag{11}$$

so that equation 10 becomes

$$\text{Im } T = T^* T \tag{12}$$

We can write equation 1 now in the form

$$T = ND^{-1} \tag{13}$$

and we get the equations corresponding to 9a and 9b:

$$\text{Im } D = -N \tag{14a}$$

$$\text{Im } N = (\text{Im } T)D \tag{14b}$$

where the elements involved are now matrices.

From 14a we get again

$$D = 1 - \frac{s-s_o}{\pi} \int \frac{\Theta(s'-s_i)N(s')\,ds'}{(s'-s_o)(s'-s-i\epsilon)} \tag{14c}$$

where $\Theta(s'-s_i)$ is a diagonal matrix whose ith element is $\theta(s'-s_i)$, the step function with s_i equal to the threshold of the corresponding reaction. This gives the actual lower limits of integration; and we observe that Θ does not commute with N.

We can also define the substractions at different points for different channels:

$$D_{ij} = \delta_{ij} - \frac{s-s_i}{\pi} \int_{s_i}^{\infty} \frac{N_{ij}(s')\,ds'}{(s'-s_i)(s'-s-i\epsilon)} \tag{14d}$$

(no sum over i)

With the definition of D in 14c with $s_o = 0$ we can easily find a connection with the form factor for the simplified case we are assuming (see reference 15).

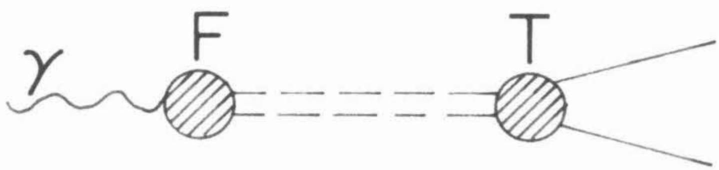

Fig. 2. Diagram for form factors

Unitarity applied to graphs like that of figure 2 gives the following equation for the column vector F of form factors for the different channels:

$$\text{Im } F(s) = T^{\dagger}(s)\, F(s) \tag{15}$$

We also know that

$$F(0) = Q \tag{16}$$

where Q is the "charge vector."

We will prove that

$$F = (D^T)^{-1} Q \tag{17}$$

satisfies both equations. From 14c we get $D(0) = 1$, so that 16 is obtained trivially. Also from 17

$$\begin{aligned} \text{Im } F &= \text{Im } (D^T)^{-1} Q \\ &= -(D^{\dagger})^{-1} (\text{Im } D^T)(D^T)^{-1} Q \\ &= (D^{\dagger})^{-1} N^T F \qquad \text{(using 14a)} \end{aligned}$$

and since N is real in the region under consideration, we get back equation 15.

Also in reference 15 it is proved that equations 14a and b are solved as integral equations with a symmetric Im T at the left-hand cut, then T is symmetric. The only cut $T-T^T$ could have is the unitarity cut. But in this region

$$\text{Im}\,(T-T^T) = T^\dagger T - T^T T^*$$

If we take the complex conjugate of this equation,

$$\text{Im}\,(T-T^T) = T^T T^* - T^\dagger T = -\,\text{Im}\,(T-T^T)$$

and hence $T-T^T$ has no cuts (nor other singularities), and since it goes to zero at infinity, it is zero everywhere and T is symmetric.

Unfortunately, if the determinantal approximation is used, taking N equal to the Born approximation for instance, T turns out not to be symmetric and, although equation 12 is satisfied, unitarity as expressed by equation 10 is not.

Several ways to insure the symmetry of the T matrix have been tried. The more obvious possibility of writing

$$T = \frac{1}{2}\,[\,ND^{-1} + (D^T)^{-1}N^T] \tag{18}$$

does not lead to a tractable expression for Im T.

Next it has been suggested to use

$$T = 2[\,DN^{-1} + (N^T)^{-1}D^T]^{-1} \tag{19}$$

$$\text{Im}\;T = \frac{1}{2}\,T^*\,\text{Im}\,[\,DN^{-1} + (N^T)^{-1}D^T]$$

Along the right-hand cut we have

$$\text{Im}\;N = 0 \tag{20a}$$

$$\text{Im}\;D = -N \tag{20b}$$

Hence

$$\text{Im } T = -\frac{1}{2} T^{*}[(\text{Im } D)N^{-1} + (N^{T})^{-1}\text{Im } D^{\gamma}]\,T$$

$$= -\frac{1}{2} T^{*}[-NN^{-1} - (N^{T})^{-1}N^{T}]\,T$$

$$= T^{*}T$$

and equation 12 is satisfied by this symmetric T. Nevertheless, there is a serious difficulty that shows up when a limiting process is carried out to reduce a two channel problem to one channel by letting a coupling constant go to zero.

Assume for instance that we start with a symmetric N of the form

$$N(s) = \begin{vmatrix} a_1(s) & \epsilon b(s) \\ \epsilon b(s) & \epsilon^2 a_2(s) \end{vmatrix} \tag{21}$$

as in a case like that in figure 3.

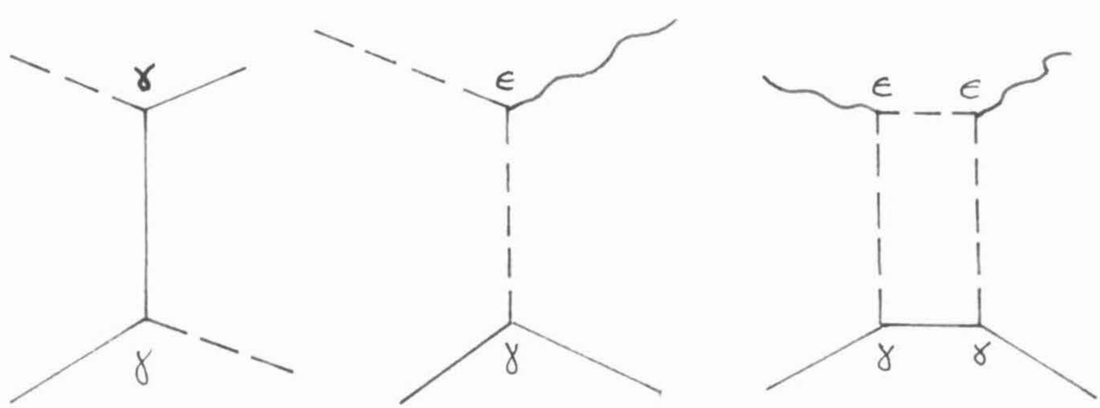

Fig. 3. Two channel problem

We will simplify the notation by writing equation 14c as

$$D = 1 - \int \Theta N \tag{22}$$

Equation 19 can be written

$$T = N[\tfrac{1}{2} D + \tfrac{1}{2} N^{-1} D^T N]^{-1}$$

$$= N[1 - \tfrac{1}{2}\int \Theta N - \tfrac{1}{2} N^{-1}(\int N\Theta) N]^{-1}$$

$$\int \Theta N = \begin{pmatrix} \int \theta_1 a_1 & \epsilon \int \theta_1 b \\ \epsilon \int \theta_2 b & \epsilon^2 \int \theta_2 a_2 \end{pmatrix}$$

$$N^{-1} = \frac{1}{\epsilon^2(a_1 a_2 - b^2)} \begin{pmatrix} \epsilon^2 a_2 & -\epsilon b \\ -\epsilon b & a_1 \end{pmatrix}$$

If T is calculated we get

$$T = \frac{1}{\Delta} \begin{pmatrix} a_1 + O^2(\epsilon) & \epsilon(b - \frac{b}{2}\int \theta_1 a_1 + \frac{a_1}{2}\int \theta_1 b) + O^3(\epsilon) \\ \epsilon(b - \frac{b}{2}\int \theta_1 a_1 + \frac{a_1}{2}\int \theta_1 b) + O^3(\epsilon) & \epsilon^2(a_2 - a_2\int \theta_1 a_1 + b\int \theta_1 b) + O^4(\epsilon) \end{pmatrix}$$

where

$$\Delta = 1 - \int \theta_1 a_1 + \frac{b}{4(a_1 a_2 - b^2)}(\int \theta_1 a_1)(b\int \theta_1 a_1 - a_1 \int \theta_1 b) + O^2(\epsilon)$$

and when $\epsilon \to 0$ we get additional terms in the denominator for the scattering in the first channel.

A slight variation where $N = N^T$,

$$T = 2(ND^{-1} + D^{-1}N)^{-1}$$

$$D = 1 - \tfrac{1}{2}\int \Theta N - \tfrac{1}{2}\int N\Theta$$

presents the same problem.

This problem is not present when we write

$$T = N^{\frac{1}{2}} D^{-1} N^{\frac{1}{2}} \tag{23}$$

$$D = 1 - \int N^{\frac{1}{2}} \circledast N^{\frac{1}{2}} \tag{24}$$

so that if N is symmetric, so are D and T, but there are problems of a different kind. In the first place, along the unitarity cut where N is real, it is not clear if all its eigenvalues, and hence $N^{\frac{1}{2}}$, are real. Besides the signs of the elements of the diagonal matrix similar to $N^{\frac{1}{2}}$ are not determined, and in general a different choice will produce a different T matrix.

One way of obtaining a symmetric T matrix without solving integral equations is the following procedure by iteration. We set, for instance, with N_B equal to the Born approximation,

$$\frac{1}{2}(N + N^T) = N_B \tag{25a}$$

$$T = ND^{-1} \tag{25b}$$

$$D = 1 - \int \circledast N \tag{25c}$$

If we define X by

$$N = N_B + X \tag{26a}$$

from 25a we get, remembering that N_B is symmetric

$$X^T = -X \tag{26a}$$

We want to impose the condition that T is symmetric, i.e.,

$$ND^{-1} = (D^T)^{-1} N^T \tag{27a}$$

and using equations 26a and b, we get

$$XD^{-1} + (D^T)^{-1}X = (D^T)^{-1}N_B - N_B D^{-1} \tag{27b}$$

The iteration procedure would be to use

$$N_1 = N_B \tag{28a}$$

$$D = 1 - \int \otimes N_i \tag{28b}$$

Then X_i is determined from 27b and we take

$$N_{i+1} = N_B + \text{Re}\, X_i \tag{28c}$$

Equation 27b insures that X is antisymmetric but in general it will not be real. If the final X has an appreciable imaginary part, then T will not be symmetric and the procedure is not of much use.

Since the whole method is generally only an approximation, it might be best go give up the exact fulfillment of time reversal invariance and unitarity and use equation 14d; or to solve equations 14a and 14b with some appropriate input for Im T at the left-hand cut as integral equations.

APPENDIX H

COMPLEX SINGULARITIES

It is well known that certain graphs involving unstable particles or anomalous thresholds have complex singularities, and that these demand great care in the use of concepts like unitarity and a straightforward application of the N/D method of appendix G is impossible.

We will follow closely the procedure developed by Ball, Frazer and Nauenberg in reference 16, although we will change some definitions of branch cuts, though presumably their choice is arbitrary. Their approach is to consider the inelastic reaction $\pi + N \to \pi + \pi + N$ and approximate the two pion system by a resonance or unstable particle. We will work with the amplitudes T_{11} for $\pi + N \to \pi + N$, T_{21} for $\pi + N \to \pi + \pi + N$ and T_{22} for $\pi + \pi + N \to \pi + \pi + N$.

T_{11} is a function of the usual variables s and t, T_{21} will be a function of s, t and ω, the square of the energy of the two pions in their own center of mass system; the other two variables necessary to define the relative motion of the two pions are eliminated by the assumption that the pions are in a resonant state or unstable particle. In T_{22} we will have two additional variables, ω_1 and ω_2. Time reversal invariance demands that T_{12} be equal to T_{21}, and that T_{22} be symmetric in ω_1 and ω_2. The particles will be considered to be spinless, and the two pion system, in an S-wave resonance.

We will also use the conventions of reference 16, where the T matrix obeys the unitarity relations as formulated by Blankenbecler in reference 15.

In the s channel, for all variables having physical values ($t < 0$, ω, ω_1, $\omega_2 > 4\mu^2$, $s > (M + \mu)^2$) we have

$$\text{Im } T_{11}(s_+, t) = \sum \{ T_{11}(s_+, t')T_{11}(s_-, t'') + T_{21}(s_+, t', \omega'_+)T_{21}(s_-, t'', \omega'_-)\} \tag{1a}$$

$$\text{Im } T_{21}(s_+, t, \omega_+) = \sum \{T_{21}(s_+, t', \omega_+)T_{11}(s_-, t'') + T_{22}(s_+, t', \omega_+, \omega'_+)T_{21}(s_-, t'', \omega'_-)\} \tag{1b}$$

$$\text{Im } T_{22}(s_+, t, \omega_{1+}, \omega_{2+}) = \sum \{T_{21}(s_+, t', \omega_{1+})T_{21}(s_-, t'', \omega_{2-}) + T_{22}(s_+, t', \omega_{1+}, \omega'_+)T_{22}(s_-, t'', \omega_{2-}, \omega'_-)\} \tag{1c}$$

where the primes indicate intermediate variables, and the $\sum$ corresponds to an integral over the four momenta q_i of the intermediate particles,

$$\sum = \frac{1}{2} \int \prod_i [\frac{d^4q_i}{(4\pi)^4} 2\pi\delta_p(q_i^2 - m_i^2)] (2\pi)^4 \delta^4(\sum_i q_i - P) \tag{2}$$

where P is the total four-momentum of the system.

The integration indicated in equation 2 differs from that in C-4b by factors of $2q_{oi}$ resulting from the $\delta_p(q_i^2 - m_i^2)$; this is due to a different definition of the T matrix that is Lorentz invariant. Also $s_\pm$ stands for $s \pm i\epsilon$, etc., and the transition amplitudes satisfy

$$T^*_{ij}(x, y, \ldots) = T_{ij}(x^*, y^*, \ldots) \tag{3}$$

Equations 1, in dispersion graph language, correspond to the relations in figure 1.

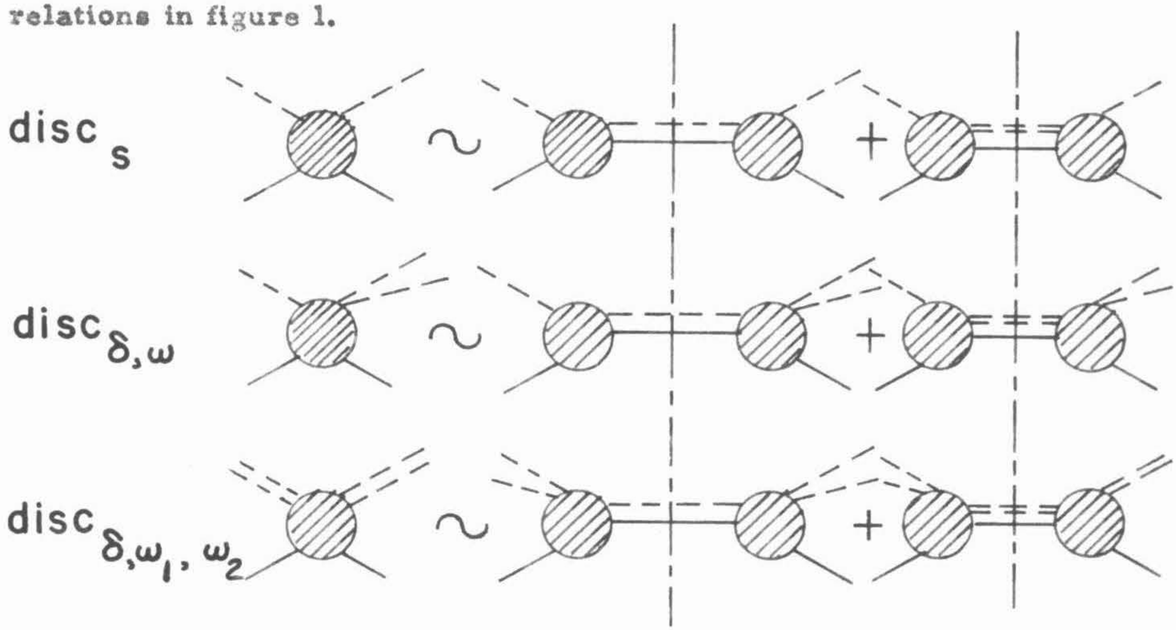

Fig. 1. Dispersion graphs

From equation 3 we derive obviously

$$2i \operatorname{Im} T_{11}(s_+, t) = T_{11}(s_+, t) - T_{11}(s_-, t) \tag{4a}$$

$$2i \operatorname{Im} T_{21}(s_+, t, \omega_+) = T_{21}(s_+, t, \omega_+) - T_{21}(s_-, t, \omega_-) \tag{4b}$$

$$2i \operatorname{Im} T_{22}(s_+, t, \omega_{1+}, \omega_{2+}) = T_{22}(s_+, t, \omega_{1+}, \omega_{2+}) - T_{22}(s_-, t, \omega_{1-}, \omega_{2-}) \tag{4c}$$

For our calculations we would like to know the discontinuities in s for constant ω, ω_1, ω_2, for instance,

$$T_{21}(s_+, t, \omega_+) - T_{21}(s_-, t, \omega_+) = 2i \operatorname{Im} T_{21}(s_+, t, \omega_+) - [T_{21}(s_-, t, \omega_+) - T_{21}(s_-, t, \omega_-)] \tag{5}$$

The discontinuity in ω of T_{21} for constant s, t can be derived from the equation corresponding to figure 2

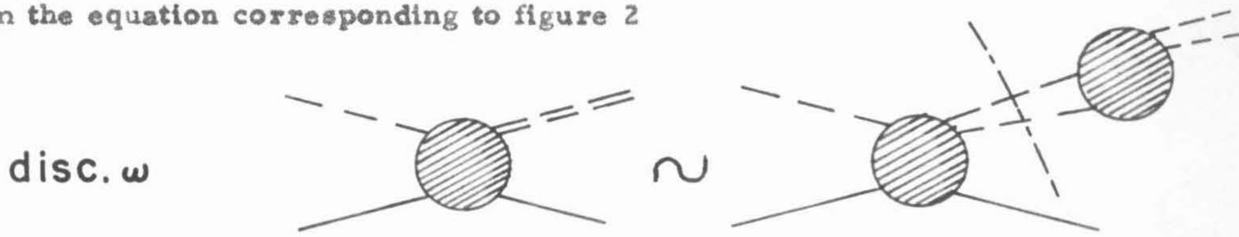

Fig. 2. Dispersion graphs for the ω-reaction

$$T_{21}(s,t,\omega_+) - T_{21}(s,t,\omega_-) = 2i \sum T(\omega_+, t')T_{21}(s,t'',\omega_-) \tag{6}$$

where T is the scattering amplitude for π-π scattering. Unitarity applied to pion-pion scattering gives for the S-wave scattering amplitude

$$f(\omega) = \frac{1}{\rho(\omega)} e^{i\delta(\omega)} \sin \delta(\omega) \tag{7}$$

$$\rho(\omega) = \frac{1}{16\pi} \sqrt{\frac{\omega - 4\mu^2}{\omega}} \tag{8}$$

The derivation of equation 7 goes as follows

$$\mathrm{Im}\ T(\omega, x) = \sum T^*(\omega, x')T(\omega, x'') \tag{9}$$

Using equation 2 we can write, for the C. M. system:

$$\mathrm{Im}\ T(\omega, x) = \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \frac{d^4q'}{(2\pi)^4} 2\pi\delta_p(q^2-\mu^2) 2\pi\delta_p(q'^2-\mu^2(2\pi)^4\delta^3(\vec{q}+\vec{q}\,')$$

$$\times\ \delta(q_o + q_o' - \sqrt{\omega})\, T^*(\omega, x')T(\omega, x'')$$

Proceeding in a similar form to the derivation of equation G-5, we get

$$T(\omega, x) = \frac{\rho(\omega)}{4\pi} \int d\Omega_{\hat{q}}\, T^*(\omega, x')\, T(\omega, x'')$$

Then we write the partial wave expansion

$$T(\omega, x) = \sum_{\ell} (2\ell + 1) f_\ell(\omega) P_\ell(x)$$

and like equation G-8 we get

$$\text{Im}\, f_\ell(\omega) = \rho(\omega) f_\ell^*(\omega) f_\ell(\omega) \tag{10}$$

and equation 7 follows immediately, for $\ell = 0$.

If we take the two pions in equation 6 to be in an S-wave state, we project out, like above,

$$T_{21}(s, t, \omega_+) - T_{21}(s, t, \omega_-) = 2ie^{i\delta(\omega_+)} \sin \delta(\omega_+) T_{21}(s, t, \omega_-) \tag{11}$$

We define T_{22}^D as the contribution to T_{22} due to the disconnected process of figure 3, we have then

$$T_{22}^D = (2\pi)^3 \delta^3(\vec{p} - \vec{p}\,') f(\omega) \tag{12}$$

where $\vec{p}$ and $\vec{p}\,'$ are the initial and final nucleon three-momenta.

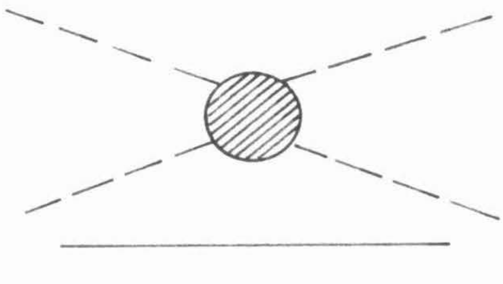

Fig. 3. Disconnected graph

Equation 6 can then also be written

$$T_{21}(s, t, \omega_+) - T_{21}(s, t, \omega_-)$$

$$= 2i \sum T^D_{22}(s, t', \omega_+, \omega''_+) T_{21}(s, t'', \omega''_-) \tag{13}$$

where now the nucleon is also included in the intermediate state.

Substituting this expression and equation 1b in equation 5 we obtain

$$T_{21}(s_+, t, \omega) - T_{21}(s_-, t, \omega)$$

$$= 2i \sum \{T_{21}(s_+, t', \omega) T_{11}(s_-, t'')$$

$$+ T^C_{22}(s_+, t', \omega, \omega''_+) T_{21}(s_-, t'', \omega_-)\} \tag{14}$$

where

$$T^C_{22} = T_{22} - T^D_{22} \tag{14a}$$

A similar reasoning applied to T_{22} gives

$$T_{22}(s, t, \omega_{1+}, \omega_2) - T_{22}(s, t, \omega_{1-}, \omega_2)$$

$$= 2ie^{i\delta(\omega_{1+})} \sin \delta(\omega_{1+}) T_{22}(s, t, \omega_{1-}, \omega_2) \tag{15a}$$

$$T_{22}(s, t, \omega_1, \omega_{2+}) - T_{22}(s, t, \omega_1, \omega_{2-})$$

$$= 2ie^{i\delta(\omega_{2-})} \sin \delta(\omega_{2-}) T_{22}(s, t, \omega_1, \omega_{2-}) \tag{15b}$$

and the equivalent to equation 14

$$T^C_{22}(s_+, t, \omega_1, \omega_2) - T^C_{22}(s_-, t, \omega_1, \omega_2) = 2i \sum \{T_{21}(s_+, t', \omega_1) T_{21}(s_-, t'', \omega_2)$$

$$+ T^C_{22}(s_+, t', \omega_1, \omega''_+) T^C_{22}(s_-, t'', \omega_2, \omega''_-) \tag{16}$$

If we introduce the functions

$$M_{11}(s,t) = T_{11}(s,t) \tag{17a}$$

$$M_{21}(s,t,\omega) = \frac{T_{21}(s,t,\omega)}{f(\omega)} \tag{17b}$$

$$M_{22}(s,t,\omega_1,\omega_2) = \frac{T^{C}_{22}(s,t,\omega_1,\omega_2)}{f(\omega_1)f(\omega_2)} \tag{17c}$$

we can prove that M_{21} and M_{22} have no discontinuities in $\omega, \omega_1, \omega_2$ for these variables greater than 0. For instance,

$$M_{21}(s,t,\omega_+) - M_{21}(s,t,\omega_-) = \frac{T_{21}(s,t,\omega_+) - T_{21}(s,t,\omega_-)}{f(\omega_+)} - \frac{f(\omega_+) - f(\omega_-)}{f(\omega_+)f(\omega_-)} T_{21}(s,t,\omega_-)$$

and using equation 11, 10 and 7

$$M_{21}(s,t,\omega_+) - M_{21}(s,t,\omega_-) = \frac{2if(\omega_+)\rho(\omega_+)T_{21}(s,t,\omega_-)}{f(\omega_+)} - \frac{2i\rho(\omega_+)f(\omega_-)f(\omega_+)}{f(\omega_+)f(\omega_-)} T_{21}(s,t,\omega_-)$$

$$= 0$$

In our approximation we will not have to consider the branch cuts in t. It will be shown that, to satisfy the equations

$$\text{disc}_{\omega}\, M_{21}(s,t,\omega) = 0 \tag{18a}$$

$$\text{disc}_{\omega_1}\, M_{22}(s,t,\omega_1,\omega_2) = 0 \tag{18b}$$

Diagrams like those in figure 4 have to be included, giving contributions

$$T_{21}(s,t,\omega) = \frac{gT_{11}(s,t)}{\sigma - M^2} \tag{19a}$$

$$T_{22}(s,t,\omega_1,\omega_2) = \frac{gT_{21}(s,t,\omega_2)}{\sigma_1 - M^2} + \frac{gT_{21}(s,t,\omega_1)}{\sigma_2 - M^2} \tag{19b}$$

where the σ's correspond to the four-momentum squared of the nucleon propagator.

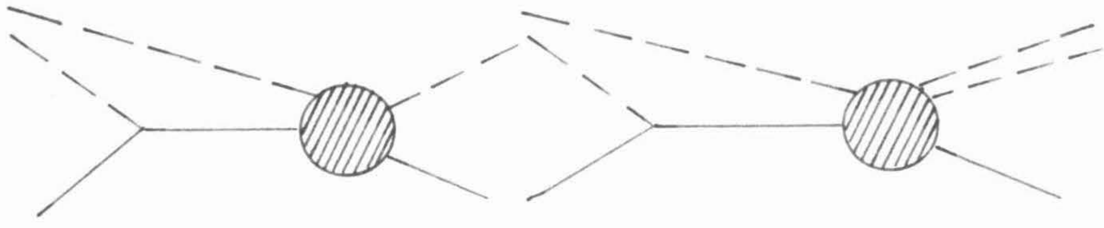

Fig. 4. Additional diagrams

If we now make a partial wave expansion

$$M_{ij} = \sum_{\ell=0}^{\infty} (2\ell+1) M_{ij}^{\ell} P_{\ell}(x) \tag{20}$$

From equations 1a, 14 and 16 we get, after integrating over some of the variables in $\sum$

$$\frac{1}{2i}\{M_{11}^{\ell}(s_+) - M_{11}^{\ell}(s_-)\} = M_{11}^{\ell}(s_+)\rho_1(s_+)M_{11}^{\ell}(s_-)$$
$$+ \sum{}' M_{21}^{\ell}(s_+, \omega')|f(\omega')|^2 M_{21}^{\ell}(s_- \omega') \tag{21a}$$

$$\frac{1}{2i}\{M_{21}^{\ell}(s_+, \omega) - M_{21}^{\ell}(s_-, \omega)\} = M_{21}^{\ell}(s_+, \omega)\rho_1(s_+)M_{11}^{\ell}(s_-)$$
$$+ \sum{}' M_{22}^{\ell}(s_+, \omega, \omega')|f(\omega')|^2 M_{21}^{\ell}(s_-, \omega') \tag{21b}$$

$$\frac{1}{2i}\{M_{22}^{\ell}(s_+, \omega_1, \omega_2) - M_{22}^{\ell}(s_-, \omega_1, \omega_2)\} = M_{21}^{\ell}(s_+, \omega_1)\rho_1(s_+)M_{21}^{\ell}(s_-, \omega_2)$$
$$+ \sum{}' M_{22}^{\ell}(s_+, \omega_1, \omega')|f(\omega')|^2 M_{22}^{\ell}(s_-, \omega_2, \omega') \tag{21c}$$

all valid for $s \geq (M+\mu)^2$; where $\sum{}'$ now stands for

$$\sum{}' \theta[s-(M+2\mu)^2]\int_{4\mu^2}^{(\sqrt{s}-M)^2} d\omega' \rho_2(s,\omega') \tag{22}$$

and

$$\rho_1(s) = \frac{1}{8\pi}\,\frac{q_1(s)}{\sqrt{s}}\,\theta[s-(M+\mu)^2] \tag{23a}$$

$$\rho_2(s,\omega) = \frac{1}{8\pi^2}\,\frac{q_2(s,\omega)}{\sqrt{s}}\,\rho(\omega) \tag{23b}$$

$$q_1(s) = \frac{\sqrt{[s-(M+\mu)^2][s-(M-\sqrt{\omega})^2]}}{2\sqrt{s}} \tag{23c}$$

$$q_2(s,\omega) = \frac{\sqrt{[s-(M+\sqrt{\omega})^2][s-(M-\sqrt{\omega})^2]}}{2\sqrt{s}} \tag{23d}$$

$\rho(\omega)$ being defined in equation 8.

We will indicate some steps in the integration over a three-particle intermediate state.

$$\sum T_{21}(s,t',\omega')T^*_{21}(s,t'',\omega')$$

$$= \frac{1}{2}\int \frac{d^4q}{(2\pi)^4}\,\frac{d^4q'}{(2\pi)^4}\,\frac{d^4q''}{(2\pi)^4}\,2\pi\delta_p(q^2-M^2)2\pi\delta_p(q'^2-\mu^2)2\pi\delta_p(q''^2-\mu^2)$$

$$\times (2\pi)^4\delta^4(q+q'+q''-P)T_{21}(s,t',\omega')T^*_{21}(s,t'',\omega')$$

$$= \frac{1}{2(2\pi)^5}\int d^4q'd^4q''\delta_p[(p-q'-q'')^2-M^2]\,\delta_p(q'^2-\mu^2)\delta_p(q''^2-\mu^2)$$

$$\times\, T_{21}(s,t',\omega')T^*_{21}(s,t'',\omega')$$

We define

$$k = q' + q'' \qquad \kappa = q' - q''$$

$$\sum T_{21}T^*_{21} = \frac{1}{2(2\pi)^5}\int d^4k\, d^4\kappa\, \delta[(P-k)^2-M^2]\,\delta[(k+\kappa)^2-4\mu^2]\,\delta[(k-\kappa)^2-4\mu^2]$$

$$\times\, \theta(P_o-k_o)\theta(k_o+\kappa_o)\theta(k_o-\kappa_o)T_{21}(s,t',\omega')T^*_{21}(s,t'',\omega')$$

$$= \frac{1}{4(2\pi)^5}\int d^4k\, d^4\kappa\, \delta[(P-k)^2-M^2]\,\delta(k^2+\kappa^2-4\mu^2)\delta(2k\cdot\kappa)\theta(P_o-k_o)$$

$$\times\, \theta(k_o+\kappa_o)\theta(k_o-\kappa_o)T_{21}(s,t',\omega')T^*_{21}(s,t'',\omega')$$

We will take momentarily a system where $\vec{k} = 0$. Then $k\cdot\kappa = 0$ implies $\kappa_o = 0$ and from $k^2+\kappa^2-4\mu^2 = 0$ we get $\vec{\kappa}^2 = k^2-4\mu^2$

$$\sum T_{21}T^*_{21} = \frac{1}{4(2\pi)^5}\int d^4k\, d\kappa_o\, \vec{\kappa}^2\, d|\vec{\kappa}|\, d\Omega_{\hat{k}}\, \delta[(P-k)^2-M^2]\,\delta[\vec{\kappa}^2-(k^2-4\mu^2)]$$

$$\times\, \delta(2k_o\kappa_o)\theta(P_o-k_o)\theta(k_o+\kappa_o)\theta(k_o-\kappa_o)T_{21}(s,t',\omega')T^*_{21}(s,t'',\omega')$$

$$= \frac{1}{4(2\pi)^5}\int d^4k\, \frac{1}{2k_o}\sqrt{\frac{k^2-\mu^2}{2}}\; 4\pi\, \delta[(P-k)^2-M^2]\,\theta(P_o-k_o)\theta(k_o)$$

$$\times\, T_{21}(s,t',\omega')T^*_{21}(s,t'',\omega')$$

and writing it again in invariant form

$$\sum T_{21}T^*_{21} = \frac{1}{8(2\pi)^4}\int d^4k\, \sqrt{\frac{k^2-\mu^2}{k^2}}\; \delta[(P-k)^2-M^2]\,\theta(P_o-k_o)\theta(k_o)$$

$$\times\, T_{21}(s,t',\omega')T^*_{21}(s,t'',\omega')$$

We notice that t' is a function of s and $x' = \hat{q}_i\cdot\hat{q}$, t'' of s and $x'' = \hat{q}_f\cdot\hat{q}$, $\vec{q} = \vec{P} - \vec{k}$, and $\omega' = k^2$, so T_{21} does not depend on κ.

We choose now the C.M. system, where $\vec{P} = 0$

$$\sum T_{21}T_{21}^{*} = \frac{1}{8(2\pi)^4}\int dk_o \vec{k}^2 d|\vec{k}|\, d\Omega_{\hat{k}}\, \delta[(\sqrt{s}-k_o)^2-\vec{k}^2-M^2]\,\theta(P_o-k_o)$$

$$\times\, \theta(k_o)\sqrt{\frac{k^2-\mu^2}{k^2}}\sum_{\ell}(2\ell+1)P_\ell(x')M_{21}^{\ell}(s,\omega')$$

$$\times \sum_{\ell'}(2\ell'+1)P_{\ell'}(x'')M_{21}^{\ell'*}(s,\omega')\,|f(\omega')|^2$$

$$= \frac{1}{8(2\pi)^4}\int \frac{\vec{k}^2 d|\vec{k}|}{2\sqrt{\vec{k}^2+M^2}}\sqrt{\frac{k^2-\mu^2}{k^2}}\; 4\pi \sum_{\ell}(2\ell+1)P_\ell(x)M_{21}^{\ell}(s,\omega')$$

$$\times\, M_{21}^{\ell *}(s,\omega')\,|f(\omega')|^2$$

From $\vec{q}^{\,2} \geq 0$, $\vec{q}^{\,\prime 2} \geq 0$, $\vec{q}^{\,\prime\prime 2} \geq 0$ we get: $\sqrt{s} = q_o + q_o' + q_o'' \geq M + 2\mu$, hence the $\theta[s-(M+2\mu)]^2$ factor. We have seen that $\kappa^2 \leq 0$, and from $k^2+\kappa^2 = 4\mu^2$ we conclude $k^2 \geq 4\mu^2$. Also $\vec{k}^2 \geq 0$ implies, from

$$\vec{k}^2 = \frac{[s-(M+\sqrt{\omega'})^2][s-(M-\sqrt{\omega'})^2]}{4s} = \frac{s^2-2s(M^2+\omega')+(M^2-\omega')^2}{4s}$$

$$\sqrt{s} > M + \sqrt{\omega'} \qquad \text{or} \qquad \omega' < (\sqrt{s} - M)^2$$

Also

$$2\,|\vec{k}|\,d|\vec{k}| = \frac{2\omega'-2(M^2+s)}{4s}\,d\omega' = -\frac{2q_o}{2\sqrt{s}}\,d\omega'$$

$$\sum T_{21}T_{21}^{*} = \sum_{\ell} (2\ell+1)P_{\ell}(x)\int_{4\mu^2}^{(\sqrt{s}-M)^2} d\omega' \frac{1}{16(2\pi)^3} \frac{|\vec{k}|}{\sqrt{s}} \sqrt{\frac{\omega'-4\mu^2}{\omega'}} \, |f(\omega')|^2$$

$$M_{21}^{\ell}(s,\omega')M_{21}^{\ell *}(s,\omega')$$

and equation 23b is obtained. Since $|f(\omega)|$ is large only in the neighborhood of the resonance at $\omega = m_{\rho}^2$, we can use the values of M_{21} and M_{22} at this point in the integrals. We get then

$$\frac{1}{2i}[M_{11}^{\ell}(s_+)-M_{11}^{\ell}(s_-)] = M_{11}^{\ell}(s_+)\rho_1(s_+)M_{11}^{\ell}(s_-)$$
$$+ M_{21}^{\ell}(s_+,m_{\rho}^2)\rho_2(s_+)M_{21}^{\ell}(s_-,m_{\rho}^2) \qquad (24a)$$

$$\frac{1}{2i}[M_{21}^{\ell}(s_+,\omega)-M_{21}^{\ell}(s_-,\omega)] = M_{21}^{\ell}(s_+,\omega)\rho_1(s_+)M_{11}^{\ell}(s_-)$$
$$+ M_{22}^{\ell}(s_+,\omega,m_{\rho}^2)\rho_2(s_+)M_{21}^{\ell}(s_-,m_{\rho}^2) \qquad (24b)$$

$$\frac{1}{2i}[M_{22}^{\ell}(s_+,\omega_1,\omega_2)-M_{22}^{\ell}(s_-,\omega_1,\omega_2)] = M_{21}^{\ell}(s_+,\omega_1)\rho_1(s_+)M_{21}^{\ell}(s_-,\omega_2)$$
$$+ M_{22}^{\ell}(s_+,\omega_1,m_{\rho}^2)\rho_2(s_+)M_{22}^{\ell}(s_-,\omega_2,m_{\rho}^2) \qquad (24c)$$

where

$$\rho_2(s) = \theta[s-(M+2\mu)^2]\int_{4\mu^2}^{(\sqrt{s}-M)^2} d\omega'\rho_2(s,\omega')\,|f(\omega')|^2 \qquad (25)$$

Equations 25 have the same form as the partial wave unitarity relations for channels with two stable particles; the difference is all contained in the generalized density of states of equation 26.

We follow reference 16 in the analysis of the behavior of the singularities when ω is varied. The one pion exchange of figure 5 will give a pole in the t variable, with residue $gf(\omega)$ where g is the pion nucleon coupling constant and $f(\omega)$ the π-π scattering amplitude.

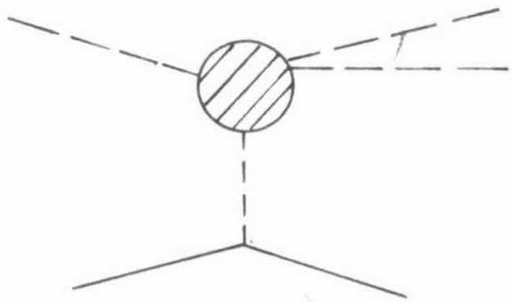

Fig. 5. One pion exchange graph

We will study the S-wave projection of this pole:

$$T^{0}_{21}(s,\omega) = f(\omega)B(s,\omega) \tag{26}$$

$$B(s,\omega) = \frac{\alpha(s,\omega)}{\pi} \log \frac{\beta(s,\omega)+\alpha(s,\omega)}{-\beta(s,\omega)+\alpha(s,\omega)} \tag{27}$$

$$\alpha(s,\omega) = \frac{\pi s g}{\sqrt{[s-(M+\mu)^2][s-(M-\mu)^2][s-(M+\sqrt{\omega})^2][s-(M-\sqrt{\omega})^2]}} \tag{28a}$$

$$\beta(s,\omega) = \frac{\pi s g}{s^2 - s(2M^2+\omega-\mu^2) + (M^2-\mu^2)(M^2-\omega)} \tag{28b}$$

From equation 27 we see that $B(s,\omega)$ has singularities where the logarithm is either zero or infinite, that is, where

$$[\beta(x,\omega)]^2 = [\alpha(s,\omega)]^2 \tag{29}$$

The solutions are

$$s = 0, \quad s = \infty \tag{30a}$$

$$s_{\pm}(\omega) = M^2 + \frac{\omega}{2} \pm \frac{\sqrt{4M^2-\mu^2}}{2\mu} \sqrt{\omega(4\mu^2-\omega)} \tag{30b}$$

$$\log \frac{\beta+\alpha}{-\beta+\alpha} = \log \frac{1+\frac{\beta}{\alpha}}{1-\frac{\beta}{\alpha}}$$

$$\frac{\beta}{\alpha} = \frac{\sqrt{[s-(M+\mu)^2][s-(M-\mu)^2][s-(M+\sqrt{\omega})^2][s-(M-\sqrt{\omega})^2]}}{s^2 - s(2M^2+\omega-\mu^2) + (M^2-\mu^2)(M^2-\omega)} \tag{31}$$

The branch points in α are $(M+\mu)^2$, $(M-\mu)^2$, $(M+\sqrt{\omega})^2$ and $(M-\sqrt{\omega})^2$. The branch cuts will be chosen as in figure 6; this implies

$$\alpha^*(s^*, \omega) = -\alpha(s, \omega) \tag{32}$$

We also choose $\alpha > 0$ above the right-hand cut. We next give ω a small positive imaginary part, and take the branch points $s = 0$ and $s = \infty$ below and above the left-hand cut. Hence for $s = 0$ the numerator of the logarithm is zero, and for $s = \infty$, the denominator, and we can join both branch points by a branch cut.

It should be remembered that the Riemann surface for the logarithmic function has an infinite number of sheets, and not only two as the square root, and care has to be exercised to compute the discontinuities across two branch cuts running together.

For small $\delta > 0$, and $s_{\pm}(\omega)$ real

$$\operatorname{Im} s_{\pm}(\omega + i\delta) = \delta \frac{ds_{\pm}(\omega)}{d\omega}$$

$$= \left[\frac{1}{2} \pm \frac{\sqrt{4M^2-\mu^2}}{2\mu} \frac{2\mu^2-\omega}{\sqrt{\omega(4\mu^2-\omega)}} \right] \delta \tag{33}$$

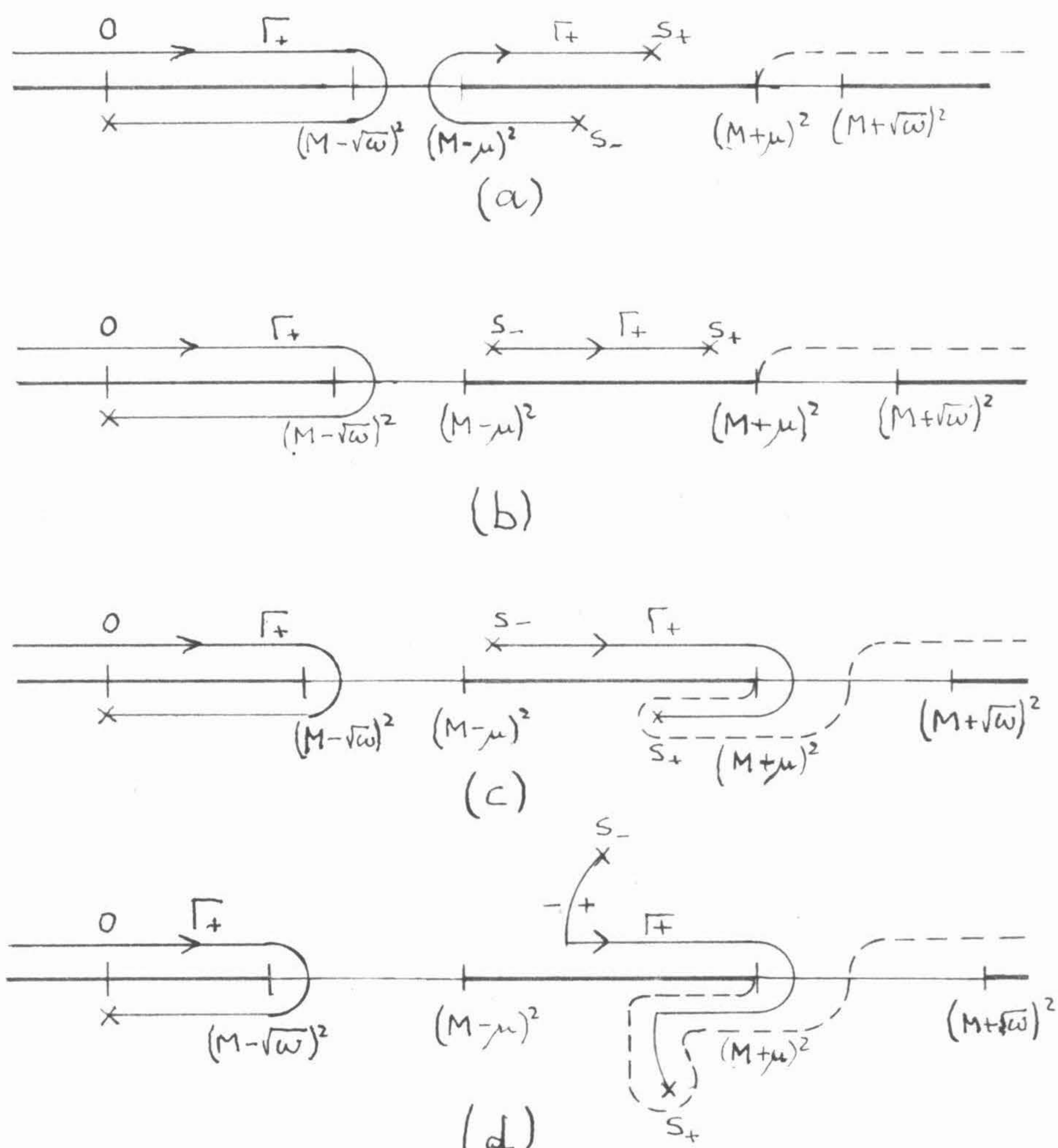

Fig. 6. Singularities of $B(s, \omega)$

We notice that $\operatorname{Im} s(\omega)$ changes sign at the same time $\frac{ds(\omega)}{d\omega}$ does.

It is easy to check that for increasing ω we find the situation in figures 6a, b, c. The turning points are reached

for $\omega = 2\mu^2(1 - \frac{\mu}{2M})$ $\qquad \operatorname{Im} s_- = 0$ (34a)

$$s_- = (M-\mu)^2 \tag{34b}$$

$$s_+ = (M+\mu)^2 - \frac{\mu^3}{M} \tag{34c}$$

and for $\omega = 2\mu^2(1 + \frac{\mu}{2M})$ $\qquad \operatorname{Im} s_+ = 0$ (35a)

$$s_+ = (M+\mu)^2 \tag{35b}$$

$$s_- = (M-\mu)^2 + \frac{\mu^3}{M} \tag{35c}$$

It can be seen that both α and β change sign when $\operatorname{Im} s_\pm$ changes sign, so that we always have

$$\frac{\alpha(s_+, \omega)}{\beta(s_+, \omega)} < 0 \qquad \frac{\alpha(s_-, \omega)}{\beta(s_-, \omega)} > 0 \tag{36}$$

and these inequalities show that this branch cut in $B(s, \omega)$ can be limited to the segment between s_+ and s_-.

In the physical region, $\omega > 4\mu^2$, $s_\pm$ becomes complex. If the branch cuts of $\sqrt{\omega(4\mu^2-\omega)}$ are chosen to run from $4\mu^2$ to ∞ and from 0 to $-\infty$, and its value in the region from 0 to $4\mu^2$ is chosen positive, then

$$s_\pm(\omega) = M^2 + \frac{\omega}{2} \mp i\,\frac{\sqrt{4M^2-\mu^2}}{2\mu}\,\sqrt{\omega(\omega-4\mu^2)} \tag{37}$$

and the branch cuts of $B(s, \omega)$ appear as shown in figure 6d.

Also from equation 27 we see that the discontinuity of $B(s, \omega)$ across the cuts of $\alpha(s, \omega)$ is zero.

A convenient representation for $B(s,\omega)$ is

$$B(s,\omega+i\delta) = \frac{1}{\pi}\int_{\Gamma_+} \frac{a(s',\omega+i\delta)}{s'-s}\, ds' \tag{38}$$

since

$$\text{disc } B(s,\omega) = 2ia(s,\omega) \tag{38a}$$

across the branch cuts of the logarithm that form the path Γ_+ and are indicated in figure 6.

For real ω we have

$$B^*(s,\omega) = \frac{1}{\pi}\int_{\Gamma_+} \frac{a^*(s',\omega+i\delta)}{s'^*-s^*}\, ds'^* \qquad \delta\to 0$$

$$= -\frac{1}{\pi}\int_{\Gamma_+^*} \frac{a(s',\omega+i\delta)}{s'-s^*}\, ds' \qquad \text{(equation 32 is used)}$$

$$= \frac{1}{\pi}\int_{\Gamma_+} \frac{a(s',\omega+i\delta)}{s'-s^*}\, ds'$$

$$= B(s^*,\omega)$$

and for real s, $B(s,\omega)$ is real even for $\omega > 4\mu^2$.

If the only interaction we consider is the one pion exchange (see figure 5) and if there are no anomalous thresholds, or complex singularities, equations 24 and 26 give for the S-wave amplitudes M_{ij}^{o} (written without the superscript in what follows):

$$M_{11}(s) = \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} ds'\, \frac{M_{11}(s'_+)\rho_1(s'_+)M_{11}(s'_-)}{s'-s}$$

$$+ \frac{1}{\pi}\int_{(M+2\mu)^2}^{\infty} ds'\, \frac{M_{21}(s'_+,m_\rho^2)\rho_2(s'_+)M_{21}(s'_-,m_\rho^2)}{s'-s} \tag{39a}$$

$$M_{21}(s,\omega) = B(s,\omega) + \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} ds' \frac{M_{21}(s'_+,\omega)\rho_1(s'_+)M_{11}(s'_-)}{s'-s}$$

$$+\frac{1}{\pi}\int_{(M+2\mu)^2}^{\infty} ds' \frac{M_{22}(s'_+,\omega,m_\rho^2)\rho_2(s'_+)M_{21}(s'_-,m_\rho^2)}{s'-s} \qquad (39b)$$

$$M_{22}(s,\omega_1,\omega_2) = \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} ds' \frac{M_{21}(s'_+,\omega_1)\rho_1(s'_+)M_{21}(s'_-,\omega_2)}{s'-s}$$

$$+\frac{1}{\pi}\int_{(M+2\mu)^2}^{\infty} ds' \frac{M_{22}(s'_+,\omega_1,m_\rho^2)\rho_2(s'_+)M_{22}(s'_-,\omega_2,m_\rho^2)}{s'-s} \qquad (39c)$$

Since for ω, ω_1, $\omega_2 > 2\mu^2(1+\frac{\mu}{2M})$ anomalous thresholds appear, we should write down equations 39 for ω, ω_1, ω_2 less than this value and then continue the equations analytically to the value of interest, m_ρ^2.

But, as is pointed out in reference 16, this procedure leads to the wrong discontinuity equations in ω, and the contributions of the graphs in figure 4 indicated in equations 19 have to be included. We have to project out the contribution to S-wave scattering.

According to R16-5.2 this gives

$$\frac{M_{11}(s)}{\pi f(\omega)\rho(\omega)}\int_{s_+(\omega)}^{s_-(\omega)} ds' \frac{\rho_1(s')a(s',\omega)}{s'-s}$$

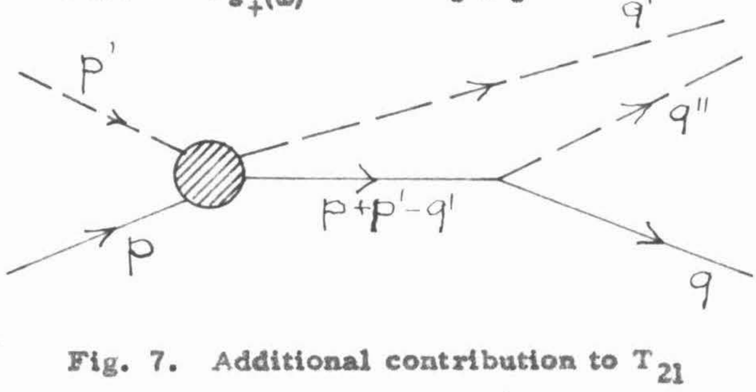

Fig. 7. Additional contribution to T_{21}

From equation 19

$$T'_{21} = \frac{gT_{11}(s,t)}{(p+p'-q')^2-M^2}$$

$$= \frac{gT_{11}}{s-2q'\cdot(p+p')+\mu^2-M^2}$$

We choose the pion C.M. system, and we define $k = q' + q''$. We have then

$$\vec{k} = 0 \qquad q'_o = q''_o = \frac{1}{2}\sqrt{\omega} \qquad k^2 = k_o^2 = \omega$$

$$|\vec{q}\,'| = |\vec{q}\,''| = \frac{1}{2}\sqrt{\omega-4\mu^2} \qquad (p+p')^2 = (k+q)^2 = s$$

$$q_o = \frac{s-\omega-M^2}{2\sqrt{\omega}} \qquad \vec{q}^{\,2} = (\vec{p}+\vec{p}\,')^2 = \frac{s^2-2s(M^2+\omega)+(M^2-\omega)^2}{4\omega}$$

$$p_o + p'_o = \frac{s-M^2+\omega}{2\sqrt{\omega}}$$

We choose $\vec{p}+\vec{p}\,'$ as the polar axis, then

$$T'_{21} = \frac{gT_{11}}{s-2q'_o(p_o+p'_o)+\mu^2-M^2+2|\vec{q}\,'|\,|\vec{p}+\vec{p}\,'|x}$$

$$= \frac{2gT_{11}}{s-M^2-\omega+2\mu^2+\sqrt{s^2-2(M^2+\omega)s+(M^2-\omega)^2}\sqrt{\frac{\omega-4\mu^2}{\omega}}\;x}$$

Taking the S-wave projection of the two pion system,

$$T''_{21} = \frac{gT_{11}}{\sqrt{s^2-2(M^2+\omega)+(M^2-\omega)^2}\sqrt{\frac{\omega-4\mu^2}{\omega}}}$$

$$\times \log \frac{s-M^2-\omega+2\mu^2+\sqrt{s^2-2(M^2+\omega)s+(M^2-\omega)^2}\sqrt{\frac{\omega-4\mu^2}{\omega}}}{s-M^2-\omega+2\mu^2-\sqrt{s^2-2(M^2+\omega)s+(M^2-\omega)^2}\sqrt{\frac{\omega-4\mu^2}{\omega}}}$$

We next notice that the branch points from the logaritm are precisely $s_{\pm}(\omega)$ as given in equation 30b, and hence

$$T''_{21} = \frac{gT_{11}}{16\pi\rho(\omega)} \int_{s_+(\omega)}^{s_-(\omega)} ds' \frac{1}{\sqrt{s^2 - 2(M^2+\omega) + (M^2-\omega)^2}\,(s'-s)}$$

$$= \frac{T_{11}(s,t)}{\pi\rho(\omega)} \int_{s_+(\omega)}^{s_-(\omega)} ds' \; \frac{\rho_1(s')a(s',\omega)}{s'-s}$$

and next we can take the S-wave projection in the C.M. system and use 17b to arrive at the desired result. The branch cut in $\rho_1(s)$ was chosen between $(M-\mu)^2$ and $(M+\mu)^2$.

Since the physical cuts are already included in the integrals in equation 39b, we want only the contribution of the left-hand singularities in s, which is done by including M_{11} under the integral. Similar terms are added to equation 39c.

Equations 39 become

$$M_{11}(s) = \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} ds' \; \frac{M_{11}(s'_+)\rho_1(s'_+)M_{11}(s'_-)}{s'-s}$$

$$+ \frac{1}{\pi}\int_{(M+2\mu)^2}^{\infty} ds' \frac{M_{21}(s'_+, m_\rho^2)\rho_2(s'_+)M_{21}(s'_-, m_\rho^2)}{s'-s} \qquad (40a)$$

$$M_{21}(s,\omega) = \frac{1}{\pi}\int_{\Gamma(\omega)} ds' \frac{a(s',\omega)}{s'-s} + \frac{1}{\pi I(\omega)\rho(\omega)} \int_{s_+(\omega)}^{s_-(\omega)} ds' \frac{\rho_1(s')a(s',\omega)M_{11}(s')}{s'-s}$$

$$+ \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} ds' \; \frac{M_{21}(s'_+,\omega)\rho_1(s'_+)M_{11}(s'_-)}{s'-s}$$

$$+ \frac{1}{\pi}\int_{(M+2\mu)^2}^{\infty} ds' \; \frac{M_{22}(s'_+,\omega, m_\rho^2)\rho_2(s'_+)M_{21}(s'_-, m_\rho^2)}{s'-s} \qquad (40b)$$

$$M_{22}(s,\omega_1,\omega_2) = \frac{1}{\pi f(\omega_1)\rho(\omega_1)} \int_{s_+(\omega_1)}^{s_-(\omega_1)} \frac{\rho_1(s')a(s',\omega_1)M_{21}(s',\omega_2)}{s'-s}$$

$$+ \frac{1}{\pi f(\omega_2)\rho(\omega_2)} \int_{s_+(\omega_2)}^{s_-(\omega_2)} ds' \frac{\rho_1(s')a(s',\omega_2)M_{21}(s',\omega_1)}{s'-s}$$

$$+ \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} ds' \frac{M_{21}(s'_+,\omega_1)\rho_1(s'_+)M_{21}(s'_-,\omega_2)}{s'-s}$$

$$+ \frac{1}{\pi}\int_{(M+2\mu)^2}^{\infty} ds' \frac{M_{22}(s'_+,\omega_1,m_\rho^2)\rho_2(s'_+)M_{22}(s'_-,m_\rho^2,\omega_2)}{s'-s} \qquad (40c)$$

Equations 40 have to be modified when the singularity $s_+(\omega)$ of $B(s,\omega)$ in M_{21} reaches the point $s = (M+\mu)^2$ and turns around. The integrals beginning at $s = (M+2\mu)^2$ are unaffected by this singularity, and hence equation 40a is unchanged, and there are no anomalous thresholds or complex singularities in the $\pi + N \to \pi + N$ reaction, at least in this approximation.

We will continue equation 40b both for ω with a small positive and negative imaginary part, and we will show that M_{21} has no discontinuity in ω.

First for $\omega \to \omega + i\delta$, we define the continuation of M_{21} to the non-physical sheet reached going downwards through the unitarity cut for $(M+\mu)^2 < s < (M+2\mu)^2$, by

$$M_{21}^{II}(s-i\epsilon) = M_{21}(s+i\epsilon) \qquad \epsilon \to 0 \qquad (41)$$

Unitarity gives for M_{j1}, $j = 1, 2$

$$M_{j1}(s+i\epsilon)-M_{j1}(s-i\epsilon) = 2i\rho_1(s+i\epsilon)M_{11}(s-i\epsilon)M_{j1}(s+i\epsilon) \qquad (42a)$$

and, remembering that the cut in ρ_1 stops at $(M+\mu)^2$,

$$M_{j1}(s+i\epsilon) = \frac{M_{j1}(s-i\epsilon)}{1-2i\rho_1(s-i\epsilon)M_{11}(s-i\epsilon)} \tag{42b}$$

and hence

$$M^{II}_{j1}(s) = \frac{M_{j1}(s)}{1-2i\rho_1(s)M_{11}(s)} \tag{43}$$

we can write

$$\int_{(M+\mu)^2}^{\infty} ds' \frac{M_{21}(s'_+, \omega)\rho_1(s'_+)M_{11}(s'_-)}{s'-s}$$

$$= \int_{(M+\mu)^2}^{\infty} ds' \frac{M^{II}_{21}(s'_-, \omega)\rho_1(s'_-)M_{11}(s'_-)}{s'-s} \tag{44}$$

We continue next to $\text{Re}\,\omega > 4\mu^2$, noting that we get into an unphysical sheet for M_{11}, corresponding to M^{II}_{11}:

$$M_{21}(s,\omega_+) = B(s,\omega_+) + \frac{1}{\pi}\int_{s_+(\omega)}^{(M+\mu)^2} \frac{[\,\text{disc}\, M^{II}_{21}(s',\omega_+)]\,\rho_1(s')M_{11}(s')}{s'-s}$$

$$+ \frac{1}{\pi f(\omega_+)\rho(\omega_+)}\int_{s_+(\omega_+)}^{(M+\mu)^2} ds' \frac{\rho_1(s')a(s',\omega_+)M^{II}_{11}(s')}{s'-s}$$

$$+ \frac{1}{\pi f(\omega_+)\rho(\omega_+)}\int_{(M+\mu)^2}^{s_-(\omega_+)} ds' \frac{\rho_1(s')a(s',\omega_+)M_{11}(s')}{s'-s} + \int_{(M+\mu)^2}^{\infty} + \int_{(M+2\mu)^2}^{\infty} \tag{45}$$

where $\text{disc}\, M^{II}_{21}(s',\omega_+)$ is the discontinuity of this function across the cut of $B(s,\omega_+)$ with the direction shown in figure 6. From equation 45 we get, remembering that $\text{disc}\, M_{11} = 0$,

$$\text{disc } M_{21}(s,\omega_+) = 2i\alpha(s,\omega_+) - 2i[\text{disc } M^{II}_{21}(s,\omega_+)]\,\rho_1(s)M_{11}(s) - 2i\,\frac{\rho_1(s)\alpha(s,\omega_+)M^{II}_{11}(s)}{f(\omega_+)\rho(\omega_+)} \tag{46a}$$

and from equation 43

$$\text{disc } M^{II}_{21}(s,\omega_+) = \frac{\text{disc } M_{21}(s,\omega_+)}{1 - 2i\rho_1(s)M_{11}(s)} \tag{46b}$$

and combining these equations

$$\text{disc } M^{II}_{21}(s,\omega_+) = 2i\alpha(s,\omega_+)\left[1 - \frac{\rho_1(s)M^{II}_{11}(s)}{f(\omega_+)\rho(\omega_+)}\right] \tag{46c}$$

and substituting this into equation 45, and simplifying with the aid of equation 43,

$$M_{21}(s,\omega_+) = B(s,\omega_+) + \frac{1}{\pi f(\omega_+)\rho(\omega_+)}\int_{s_+(\omega_+)}^{(M+\mu)^2} ds'\,\frac{\alpha(s',\omega_+)\rho_1(s')M_{11}(s')[1+2if(\omega_+)\rho(\omega_+)]}{s'-s} + \int_{(M+\mu)^2}^{s_-(\omega)} + \int_{(M+\mu)^2}^{\infty} + \int_{(M+2\mu)^2}^{\infty} \tag{47}$$

Unitarity applied to π-π scattering gives

$$f(\omega+i\epsilon)-f(\omega-i\epsilon) = 2i\rho(\omega+i\epsilon)f(\omega-i\epsilon)f(\omega+i\epsilon) \tag{48a}$$

and, since $f^*(\omega^*) = f(\omega)$

$$\frac{1}{f(\omega+i\epsilon)} - \frac{1}{f^*(\omega+i\epsilon)} = -2i\rho(\omega+i\epsilon) \tag{48b}$$

$$\frac{1}{f^*(\omega)} = \frac{1}{f(\omega)} + 2i\rho(\omega) \tag{48c}$$

and we get

$$M_{21}(s,\omega_+) = B(s,\omega_+) + \frac{1}{\pi f^*(\omega_+)\rho(\omega_+)} \int_{s_+(\omega_+)}^{(M+\mu)^2} ds' \frac{a(s',\omega_+)\rho_1(s')M_{11}(s')}{s'-s}$$

$$+ \frac{1}{\pi f(\omega_+)\rho(\omega_+)} \int_{(M+\mu)^2}^{s_-(\omega_+)} ds' \frac{\rho_1(s')M_{11}(s')a(s',\omega_+)}{s'-s}$$

$$+ \frac{1}{\pi} \int_{(M+\mu)^2}^{\infty} ds' \frac{\rho_1(s'_+)M_{21}(s'_+,\omega_+)M_{11}(s_-)}{s'-s}$$

$$+ \frac{1}{\pi} \int_{(M+2\mu)^2}^{\infty} ds' \frac{M_{22}(s'_+,\omega_+,m_\rho^2)\rho_2(s'_+)M_{21}(s'_-,m_\rho^2)}{s'-s} \qquad (49)$$

Equation 7 can be written

$$f(\omega)\rho(\omega) = e^{i\delta(\omega)} \sin \delta(\omega)$$

For $\omega = m_\rho^2$ there is a resonance, i.e. $\delta(m_\rho^2) = \frac{\pi}{2}$, and hence

$$f(m_\rho^2)\rho(m_\rho^2) = i \qquad (50a)$$

and from equation 48c,

$$f^*(m_\rho^2)\rho(m_\rho^2) = -i \qquad (50b)$$

and introducing equations 50 in 49,

$$M_{21}(s, m_\rho^2) = B(s, m_\rho^2) + \frac{1}{\pi}\left(\int_{s_+(\omega_+)}^{(M+\mu)^2} ds' - \int_{(M+\mu)^2}^{s_-(\omega_+)} ds' \right) \frac{a(s', m_\rho^2)\rho_1(s')M_{11}(s')}{s'-s}$$

$$+ \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} ds' \, \frac{\rho_1(s'_+)M_{21}(s'_+, m_\rho^2)M_{11}(s_-)}{s'-s}$$

$$+ \frac{1}{\pi}\int_{(M+2\mu)^2}^{\infty} ds' \, \frac{M_{22}(s'_+, m_\rho^2, m_\rho^2)\rho_2(s'_+)M_{21}(s'_-, m_\rho^2)}{s'-s} \qquad (51)$$

We will repeat this continuation, but now with $\omega \to \omega - i\delta$.

From equation 33 we see that

$$s_\pm(\omega_-) = s_\mp(\omega_+) \qquad (52)$$

which is true also for $\mathrm{Re}\,\omega > 4\mu^2$ according to the choice of branch cuts mentioned above equation 37. This implies that the contour is deformed now into the upper half-plane, and we have to define the analytic continuations of M_{j1} going upward through the cut

$$M_{j1}^{III}(s) = \frac{M_{j1}(s)}{1+2i\rho_1(s)M_{11}(s)} \qquad (53)$$

By taking the complex conjugate of equation 48c and replacing ω by ω^*, we get, using equation 48b

$$\frac{1}{f^*(\omega)} = \frac{1}{f(\omega)} - 2i\rho^*(\omega^*) \qquad (54a)$$

and hence

$$\rho^*(\omega^*) = -\rho(\omega) \qquad (54b)$$

so that the cuts for $\rho(\omega)$ have to be taken from $-\infty$ to 0 and from $4\mu^2$ to ∞, which agrees with the choice of cuts for $s_\pm(\omega)$.

Continuing equation 40b to $\operatorname{Re}\omega > 4\mu^2$, we get

$$M_{21}(s,\omega_-) = B(s,\omega_-) + \frac{1}{\pi f(\omega_-)\rho(\omega_-)}\int_{s_+(\omega_-)}^{(M+\mu)^2} ds' \frac{\rho_1(s')a(s',\omega_-)M_{11}^{III}(s')}{s'-s}$$

$$+ \frac{1}{\pi f(\omega_-)\rho(\omega_-)}\int_{(M+\mu)^2}^{s_-(\omega_-)} ds' \frac{\rho_1(s')a(s',\omega_-)M_{11}(s')}{s'-s}$$

$$- \frac{1}{\pi}\int_{s_+(\omega_-)}^{(M+\mu)^2} ds' \frac{[\operatorname{disc} M_{21}(s',\omega_-)]\rho_1(s')M_{11}^{III}(s')}{s'-s}$$

$$+ \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} ds' \frac{M_{21}(s'_+,\omega_-)\rho_1(s'_+)M_{11}(s'_-)}{s'-s}$$

$$+ \frac{1}{\pi}\int_{(M+2\mu)^2}^{\infty} ds' \frac{M_{22}(s'_+,\omega_-,m_\rho^2)\rho_2(s'_+)M_{21}(s'_-,m_\rho^2)}{s'-s} \tag{55}$$

Taking the discontinuities of equation 55 across the path from $s_+(\omega_-)$ to $(M+\mu)^2$:

$$\operatorname{disc} M_{21}(s,\omega_-) = -2ia(s,\omega_-) - 2i\frac{\rho_1(s)a(s,\omega_-)M_{11}^{III}(s)}{f(\omega_-)\rho(\omega_-)}$$

$$+ 2i[\operatorname{disc} M_{21}(s,\omega_-)]\,\rho_1(s)M_{11}^{III}(s) \tag{56a}$$

and hence

$$\operatorname{disc} M_{21}(s,\omega_-) = -2ia(s,\omega_-)\frac{1+\dfrac{\rho_1(s)M_{11}^{III}(s)}{f(\omega_-)\rho(\omega_-)}}{1-2i\rho_1(s)\,M_{11}^{III}(s)} \tag{56b}$$

For $\operatorname{Re}\omega > 4\mu^2$, $\rho(\omega)$ is real and equations 48b and 54b give

$$\frac{1}{f(\omega_-)\rho(\omega_-)} = -\frac{1}{f^*(\omega_+)\rho(\omega_+)} \tag{57a}$$

$$-\frac{1}{f(\omega_+)\rho(\omega_+)} = 2i + \frac{1}{f(\omega_-)\rho(\omega_-)} \tag{57b}$$

We also have

$$\alpha(s,\omega_+) = \alpha(s,\omega_-) \qquad\qquad B(s,\omega_+) = B(s,\omega_-) \tag{57c}$$

and using equations 52, 56, 57, we get from equation 55

$$M_{21}(s,\omega_-) = B(s,\omega_+) + \frac{1}{\pi f^*(\omega_+)\rho(\omega_+)}\int_{s_+(\omega_+)}^{(M+\mu)^2} ds' \frac{\alpha(s',\omega_+)\rho_1(s')M_{11}(s')}{s'-s}$$

$$+ \frac{1}{\pi f(\omega_+)\rho(\omega_+)}\int_{(M+\mu)^2}^{s_-(\omega_+)} ds' \frac{\alpha(s',\omega_+)\rho_1(s')M_{11}(s')}{s'-s}$$

$$+ \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} ds' \frac{\rho_1(s'_+)M_{21}(s'_+,\omega_-)M_{11}(s'_-)}{s'-s}$$

$$+ \frac{1}{\pi}\int_{(M+2\mu)^2}^{\infty} ds' \frac{M_{22}(s'_+,\omega_-,m_\rho^2)\rho_2(s'_+)M_{21}(s'_-,m_\rho^2)}{s'-s} \tag{58}$$

This equation shows that the discontinuity in ω of M_{21} is zero, as required by equation 18a.

The equation for M_{22} is handled in much the same way. We will continue first in ω_1, giving it a small positive imaginary part, and keeping ω_2 fixed at a value less than $2\mu^2(1 + \frac{\mu}{2M})$

$$\begin{aligned}
M_{22}(s,\omega_{1+},\omega_2) = {} & \frac{1}{\pi f(\omega_{1+})\rho(\omega_{1+})}\left\{\int_{s_+(\omega_{1+})}^{(M+\mu)^2} ds' \, \frac{\rho_1(s')\alpha(s',\omega_{1+})M_{21}^{II}(s',\omega_2)}{s'-s}\right. \\
& \left. + \int_{(M+\mu)^2}^{s_-(\omega_{1+})} ds' \, \frac{\rho_1(s')\alpha(s',\omega_{1+})M_{21}(s',\omega_2)}{s'-s}\right\} \\
& + \frac{1}{\pi f(\omega_2)\rho(\omega_2)} \int_{s_+(\omega_2)}^{s_-(\omega_2)} ds' \, \frac{\rho_1(s')\alpha(s',\omega_2)M_{21}(s',\omega_{1+})}{s'-s} \\
& + \frac{1}{\pi}\int_{s_+(\omega_{1+})}^{(M+\mu)^2} ds' \, \frac{[\text{disc } M_{21}^{II}(s',\omega_{1+})]\rho_1(s')M_{21}(s',\omega_2)}{s'-s} \\
& + \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} ds' \, \frac{M_{21}(s'_+,\omega_{1+})\rho_1(s'_+)M_{21}(s'_-,\omega_2)}{s'-s} \\
& + \frac{1}{\pi}\int_{(M+2\mu)^2}^{\infty} ds' \, \frac{M_{22}(s'_+,\omega_{1+},m_\rho^2)\rho_2(s'_+)M_{22}(s'_-,m_\rho^2,\omega_2)}{s'-s}
\end{aligned} \tag{59}$$

and using equations 43, 46c and 48c, we get

$$\begin{aligned}
M_{22}(s,\omega_{1+},\omega_2) = {} & \frac{1}{\pi f(\omega_{1+})\rho(\omega_{1+})}\int_{s_+(\omega_{1+})}^{(M+\mu)^2} ds' \, \frac{\alpha(s',\omega_{1+})\rho_1(s')M_{21}(s',\omega_2)}{s'-s} \\
& + \frac{1}{\pi f(\omega_{1+})\rho(\omega_{1+})}\int_{(M+\mu)^2}^{s_-(\omega_{1+})} ds' \, \frac{\alpha(s',\omega_{1+})\rho_1(s')M_{21}(s',\omega_2)}{s'-s} \\
& + \frac{1}{\pi f(\omega_2)\rho(\omega_2)}\int_{s_+(\omega_2)}^{s_-(\omega_2)} ds' \, \frac{\rho_1(s')\alpha(s',\omega_2)M_{21}(s',\omega_{1+})}{s'-s} \\
& + \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} ds' \, \frac{M_{21}(s'_+,\omega_{1+})\rho_1(s'_+)M_{21}(s'_-,\omega_2)}{s'-s} \\
& + \frac{1}{\pi}\int_{(M+2\mu)^2}^{\infty} ds' \, \frac{M_{22}(s'_+,\omega_{1+},m_\rho^2)\rho_2(s'_+)M_{22}(s'_-,m_\rho^2,\omega_2)}{s'-s}
\end{aligned} \tag{60}$$

We continue now in ω_2, keeping $\omega_2 \leq \omega_1$. The third integral in equation 60 is affected, and the path from $s_+(\omega_2)$ to $(M+\mu)^2$ lies in the region II defined by the cut in M_{11}; and another dent is made in the third integral in equation 40c (see figure 8). Equation 60 then becomes:

$$M_{22}(s,\omega_{1+},\omega_{2+}) = \frac{1}{\pi f^*(\omega_{1+})\rho(\omega_{1+})}\int_{s_+(\omega_{1+})}^{(M+\mu)^2} ds' \frac{a(s',\omega_{1+})\rho_1(s')M_{21}(s'_+,\omega_{2+})}{s'-s}$$

$$+ \frac{1}{\pi f(\omega_{1+})\rho(\omega_{1+})}\int_{(M+\mu)^2}^{s_-(\omega_{1+})} ds' \frac{a(s',\omega_{1+})\rho_1(s')M_{21}(s'_+,\omega_{2+})}{s'-s}$$

$$+ \frac{1}{\pi}\int_{s_+(\omega_{2+})}^{(M+\mu)^2} ds' \frac{M_{21}^{II}(s'_-,\omega_{1+})\rho_1(s')\,\mathrm{disc}\; M_{21}(s'_-,\omega_{2+})}{s'-s}$$

$$+ \frac{1}{\pi f(\omega_{2+})\rho(\omega_{2+})}\int_{s_+(\omega_{2+})}^{(M+\mu)^2} ds' \frac{\rho_1(s')a(s',\omega_{2+})M_{21}^{II}(s'_-,\omega_{1+})}{s'-s}$$

$$+ \frac{1}{\pi f(\omega_{2+})\rho(\omega_{2+})}\int_{(M+\mu)^2}^{s_-(\omega_{2+})} ds' \frac{\rho_1(s')a(s',\omega_{2+})M_{21}(s'_-,\omega_{1+})}{s'-s}$$

$$+ \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} + \int_{(M+2\mu)^2}^{\infty} \tag{61}$$

Using 46b and c we can find disc M_{21}, and then 48c allows some simplifications; we get

$$\mathrm{disc}\; M_{21}(s'_-,\omega_{2+}) + \frac{a(s',\omega_{2+})}{f(\omega_{2+})\rho(\omega_{2+})} = \frac{a(s',\omega_{2+})[1-2i\rho_1(s')M_{11}(s')]}{f^*(\omega_{2+})\rho(\omega_{2+})} \tag{62}$$

and using 43 and 62 we can write 61 in the form

$$M_{22}(s, \omega_{1+}, \omega_{2+}) = \frac{1}{\pi f^*(\omega_{1+})\rho(\omega_{1+})} \int_{s_+(\omega_{1+})}^{(M+\mu)^2} + \frac{1}{\pi f(\omega_{1+})\rho(\omega_{1+})} \int_{(M+\mu)^2}^{s_-(\omega_{1+})}$$

$$+ \frac{1}{\pi f^*(\omega_{2+})\rho(\omega_{2+})} \int_{s_+(\omega_{2+})}^{(M+\mu)^2} ds' \, \frac{M_{21}(s'_-, \omega_{1+})\rho_1(s')a(s', \omega_{2+})}{s'-s}$$

$$+ \frac{1}{\pi f(\omega_{2+})\rho(\omega_{2+})} \int_{(M+\mu)^2}^{s_-(\omega_{2+})} + \int_{(M+\mu)^2}^{\infty} + \int_{(M+2\mu)^2}^{\infty} \qquad (63)$$

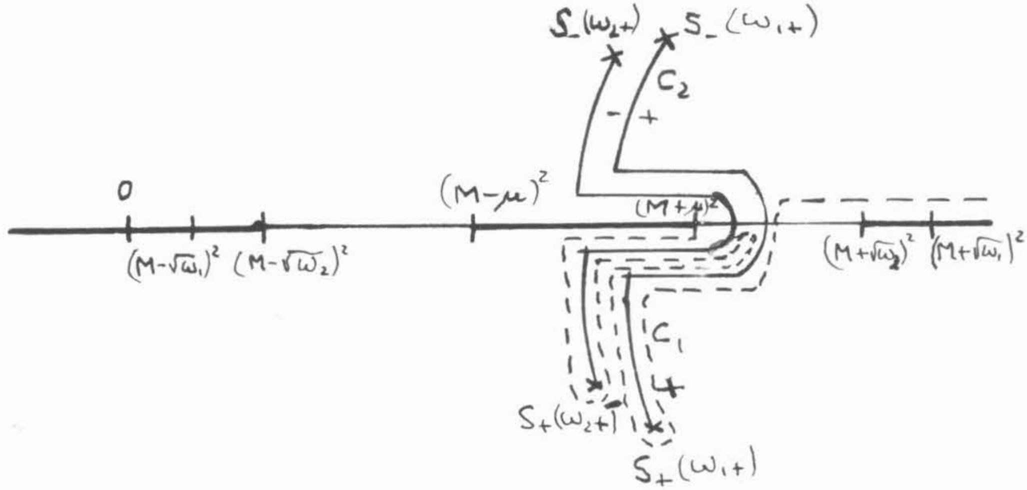

Fig. 8. Continuation of M_{22}

If we want to express all integrals in terms of $M_{21}(s_+, \omega_{1+})$, we have to find the disc of this function both across C_1 and C_2 in figure 8. The first can be found from 46b and c, and the second directly from equation 49. We get

$$\mathrm{disc}_1\, M_{21}(s_+, \omega_+) = 2ia(s, \omega_+)\left[1 - \frac{\rho_1(s)M_{11}(s)}{f^*(\omega)\rho(\omega)} \right] \qquad (64a)$$

$$\text{disc}_2\, M_{21}(s_+, \omega_+) = 2i\alpha(s, \omega_+)\left[1 - \frac{\rho_1(s)M_{11}(s)}{f(\omega)\rho(\omega)}\right] \tag{64b}$$

and equation 63 becomes

$$\begin{aligned} M_{22}(s, \omega_{1+}, \omega_{2+}) &= \frac{1}{\pi f^*(\omega_{1+})\rho(\omega_{1+})}\int_{s_+(\omega_{1+})}^{(M+\mu)^2} ds'\, \frac{\alpha(s', \omega_{1+})\rho_1(s')M_{21}(s'_+, \omega_{2+})}{s'-s} \\ &+ \frac{1}{\pi f(\omega_{1+})\rho(\omega_{1+})}\int_{(M+\mu)^2}^{s_-(\omega_{1+})} ds'\, \frac{\alpha(s', \omega_{1+})\rho_1(s')M_{21}(s'_+, \omega_{2+})}{s'-s} \\ &+ \frac{1}{\pi f^*(\omega_{2+})\rho(\omega_{2+})}\int_{s_+(\omega_{2+})}^{(M+\mu)^2} ds'\, \frac{A}{s'-s} \\ &+ \frac{1}{\pi f(\omega_{2+})\rho(\omega_{2+})}\int_{(M+\mu)^2}^{s_-(\omega_{2+})} ds'\, \frac{A'}{s'-s} \\ &+ \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} ds'\, \frac{M_{21}(s'_+, \omega_{1+})\rho_1(s'_+)M_{21}(s'_-, \omega_{2+})}{s'-s} \\ &+ \frac{1}{\pi}\int_{(M+2\mu)^2}^{\infty} ds'\, \frac{M_{22}(s'_+, \omega_{1+}, m_\rho^2)\rho_2(s'_+)M_{22}(s'_-, m_\rho^2, \omega_{2+})}{s'-s} \end{aligned} \tag{65}$$

where

$$A = \alpha(s', \omega_{2+})\rho_1(s')\left[M_{21}(s'_+, \omega_{1+}) - 2i\alpha(s', \omega_{1+})\left(1 - \frac{\rho_1(s')M_{11}(s')}{f^*(\omega_{1+})\rho(\omega_{1+})}\right)\right] \tag{65a}$$

and A' is obtained from A by replacing f^* by f.

If we take now $\omega_1 = \omega_2 = m_\rho^2 + i\delta$ and use equations 50,

$$M_{22}(s, m_\rho^2, m_\rho^2)$$

$$= \frac{2i}{\pi} \int_{s_+(m_\rho^2)}^{(M+\mu)^2} ds' \frac{a(s', m_\rho^2)\rho_1(s')[M_{21}(s'_+, m_\rho^2) - ia(s', m_\rho^2)\{1 - i\rho_1(s')M_{11}(s')\}]}{s'-s}$$

$$- \frac{2i}{\pi} \int_{(M+\mu)^2}^{s_-(m_\rho^2)} ds' \frac{a(s', m_\rho^2)\rho_1(s')[M_{21}(s'_+, m_\rho^2) - ia(s', m_\rho^2)\{1 + i\rho_1(s')M_{11}(s')\}]}{s'-s}$$

$$+ \frac{1}{\pi} \int_{(M+\mu)^2}^{\infty} ds' \frac{M_{21}(s'_+, m_\rho^2)\rho_1(s'_+)M_{21}(s'_-, m_\rho^2)}{s'-s}$$

$$+ \frac{1}{\pi} \int_{(M+2\mu)^2}^{\infty} ds' \frac{M_{22}(s'_+, m_\rho^2, m_\rho^2)\rho_2(s'_+)M_{22}(s'_-, m_\rho^2, m_\rho^2)}{s'-s} \qquad (66)$$

In equations 61 and 66 we notice the additional terms due to the complex singularities, that is, the integrals from $s_+(m_\rho^2)$ to $(M+\mu)^2$ and from $(M+\mu)^2$ to $s_-(m_\rho^2)$.

Still following reference 16, we write

$$M = ND^{-1} \qquad (67a)$$

$$N = (n_{ij}) \qquad (67b)$$

$$D = (\delta_{ij} + d_{ij}) \qquad (67c)$$

Then, from 67a, we can write

$$N = MD \qquad (67d)$$

and if we restrict ourselves to $\omega = \omega_1 = \omega_2 = m_\rho^2$ and we drop them as variables, this is written out

$$n_{11}(s) = M_{11}(s)[1 + d_{11}(s)] + M_{12}(s)d_{21}(s) \qquad (68a)$$

$$n_{12}(s) = M_{11}(s)d_{12}(s) + M_{12}[1 + d_{22}(s)] \tag{68b}$$

$$n_{21}(s) = M_{21}(s)[1 + d_{11}(s)] + M_{22}(s)d_{21}(s) \tag{68c}$$

$$n_{22}(s) = M_{21}(s)d_{12}(s) + M_{22}(s)[1 + d_{22}(s)] \tag{68d}$$

In reference 16 they next eliminate the M_{ij} and find the relations between the n_{ij} and d_{ij} by using equations 68 and the integral equations 40a, 51 and 66 for the M_{ij}. We will merely quote the results:

$$d_{11}(s) = -\frac{1}{\pi}\left(\int_{s_+(m_\rho^2)}^{(M+\mu)^2} - \int_{(M+\mu)^2}^{s_-(m_\rho^2)}\right) ds' \frac{\rho_1(s')a(s')d_{21}(s')}{s'-s}$$

$$-\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} ds' \frac{\rho_1(s'_+)n_{11}(s')}{s'-s} \tag{69a}$$

$$d_{12}(s) = -\frac{1}{\pi}\left(\int_{s_+(m_\rho^2)}^{(M+\mu)^2} - \int_{(M+\mu)^2}^{s_-(m_\rho^2)}\right) ds' \frac{\rho_1(s')a(s')[1 + d_{22}(s')]}{s'-s}$$

$$-\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} ds' \frac{\rho_1(s'_+)n_{12}(s')}{s'-s} \tag{69b}$$

$$d_{21}(s) = -\frac{1}{\pi}\int_{(M+2\mu)^2}^{\infty} ds' \frac{\rho_2(s'_+)n_{21}(s')}{s'-s} \tag{69c}$$

$$d_{22}(s) = -\frac{1}{\pi}\int_{(M+2\mu)^2}^{\infty} ds' \frac{\rho_2(s'_+)n_{22}(s')}{s'-s} \tag{69d}$$

$$n_{11}(s) = \frac{1}{\pi}\int_{\Gamma} ds' \frac{a(s')d_{21}(s')}{s'-s} \tag{70a}$$

$$n_{12}(s) = B(s) + \frac{1}{\pi}\int_{\Gamma} ds' \frac{a(s')d_{22}(s')}{s'-s} \tag{70b}$$

$$n_{21}(s) = B(s) + \frac{1}{\pi}\int_{\Gamma} ds' \, \frac{\alpha(s')d_{11}(s')}{s'-s}$$

$$+ \frac{1}{\pi}\left(\int_{s_+(m_\rho^2)}^{(M+\mu)^2} - \int_{(M+\mu)^2}^{s_-(m_\rho^2)}\right) ds' \, \frac{\rho_1(s')\alpha(s')n_{11}(s')}{s'-s} \qquad (70c)$$

$$n_{22}(s) = \frac{1}{\pi}\int_{\Gamma} ds' \, \frac{\alpha(s')d_{12}(s')}{s'-s}$$

$$+ \frac{1}{\pi}\left(\int_{s_+(m_\rho^2)}^{(M+\mu)^2} - \int_{(M+\mu)^2}^{s_-(m_\rho^2)}\right) ds' \, \frac{\rho_1(s')\alpha(s')n_{12}(s')}{s'-s} \qquad (70d)$$

We observe that d_{11} and d_{12} have cuts along $(C_1) + (C_2)$ and from $(M+\mu)^2$ to ∞; d_{21} and d_{22} only from $(M+2\mu)^2$ to ∞; and the n_{ij} have only cuts along Γ. It is then straightforward to prove that the right equations for the discontinuities are obtained.

For instance, from 68c and d

$$M_{21}(s) = \frac{n_{21}(s)[1 + d_{22}(s)] - n_{22}(s)d_{21}(s)}{[1 + d_{11}(s)][1 + d_{22}(s)] - d_{12}(s)d_{21}(s)} \qquad (71a)$$

we will take the discontinuity across C_2

$$\text{disc } M_{21} = \frac{\text{disc }[n_{21}(1 + d_{22}) - n_{22}d_{21}]}{\Delta(s_+)}$$

$$- \frac{[n_{21}(1 + d_{22}) - n_{22}d_{21} - \text{disc }\{n_{21}(1+d_{22}) - n_{22}d_{22}\}]\,\text{disc }\Delta}{\Delta(s_+)\Delta(s_-)}$$

where $\Delta = (1 + d_{11})(1 + d_{22}) - d_{12}d_{21}$.

From equations 69 and 70 we get

$$\text{disc }\{n_{21}(1 + d_{22}) - n_{22}d_{21}\} = (2i\alpha + 2i\alpha d_{11} - 2\rho_1\alpha n_{11})(1 + d_{22})$$

$$- (2i\alpha d_{12} - 2\rho_1 d n_{12})d_{21}$$

$$\text{disc}\,\Delta = 2\rho_1 a d_{21}(1+d_{22}) - 2\rho_1 a(1+d_{22})d_{21} = 0$$

and hence

$$\text{disc}\,M_{21} = \frac{2ia[(1+d_{11})(1+d_{22})-d_{12}d_{21}] - 2\rho_1 a n_{11}(1+d_{22})+2\rho_1 n_{12}d_{21}}{\Delta}$$

$$\text{disc}\,M_{21} = 2ia - 2a\rho_1 M_{11} \tag{71b}$$

This agrees with what is obtained from equation 51. We observe that the singularities of the n_{ij} and d_{ij} are no longer separate, that is, there no longer is a unitarity right-hand cut, and left-hand cuts.

There are now two possibilities: to solve the integral equations, as it is done in reference 16, or to make an approximation in the perturbation theory spirit.

$B(s)$ is of first order in g, and so is $a(s)$, hence, if we keep only g to the first order, we are left with

$$d_{11}(s) = -\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} ds'\,\frac{\rho_1(s'_+)n_{11}(s')}{s'-s} \tag{72a}$$

$$d_{12}(s) = -\frac{i}{\pi}\left(\int_{s_+(m_\rho^2)}^{(M+\mu^2)} - \int_{(M+\mu)^2}^{s_-(m_\rho^2)}\right) ds'\,\frac{\rho_1(s')a(s')}{s'-s}$$

$$-\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} ds'\,\frac{\rho_1(s'_+)n_{12}(s')}{s'-s} \tag{72b}$$

$$d_{21}(s) = -\frac{1}{\pi}\int_{(M+2\mu)^2}^{\infty} ds'\,\frac{\rho_2(s'_+)n_{21}(s')}{s'-s} \tag{72c}$$

$$d_{22}(s) = -\frac{1}{\pi}\int_{(M+2\mu)^2}^{\infty} ds'\,\frac{\rho_2(s'_+)n_{22}(s')}{s'-s} \tag{72d}$$

$$n_{11}(s) = 0 \tag{73a}$$

$$n_{12}(s) = B(s) \tag{73b}$$

$$n_{21}(s) = B(s) \tag{73c}$$

$$n_{22}(s) = 0 \tag{73d}$$

The only difference with the equations for the case without anomalous thresholds is the additional term in equation 72b.

The analysis of the problem of complex singularities has been restricted so far to a simple example, although it probably has all, or at least most of the features, of the real problem. Of course we could try to repeat the analysis in the general case of particles with spin, but the amount of algebra involved is rather forbidding. We will take then the alternative of assuming that no fundamental changes are introduced by the different complications.

One generalization will have to be the inclusion of other than the S-wave components of the M_{ij}. The analytic properties of the interaction term can be deduced from the properties of the Legendre functions of the second kind, since R8-6.3.44 gives

$$\int_{-1}^{1} \frac{P_\ell(x)\,dx}{a-x} = 2Q_\ell(a) \tag{74}$$

and the same reference, on p. 1328 gives

$$Q_\ell(x) = \frac{1}{2} P_\ell(x) \log \frac{x+1}{x-1} - \frac{2\ell-1}{1\cdot\ell} P_{\ell-1}(x) - \frac{2\ell-5}{3(\ell-1)} P_{\ell-3}(x) \ldots \tag{75}$$

so that the branch points coming from the log are the same for all partial waves.

On the other hand,

$$Q_\ell(-x) = (-1)^{\ell+1} Q_\ell(x) \tag{76}$$

and the discontinuities of the $B^\ell(s,\omega)$ across the branch cuts of $\alpha(s,\omega)$ is not zero for odd ℓ. This can be taken care of by defining

$$\left(M^\ell_{ij}(s,\omega)\right)' = \alpha(s,\omega) M^\ell_{ij}(s,\omega) \tag{77}$$

for odd ℓ (see equation 57c).

The inclusion of particles with spin has the effect of increasing the number of partial wave transition amplitudes for a given process, but all projection operators have the form of equation 74 as can be seen in part II and appendix D. Moreover, for the amplitudes that correspond to states of definite parity only every other P_ℓ appears, and equation 77 can still be used where necessary.

Thus, the analytic properties of the partial wave amplitudes for the same reaction are the same for each submatrix characterized by its total J and parity, and presumably obey the same equations.

We also observe that equations 69 and 70 are linear in the d_{ij} and n_{ij}, so that contribution to N coming from other graphs can be included without disturbing the equations. It should be remembered that α is the discontinuity of B across its branch cuts, so, unless the branch cuts coincide, the new graph will not contribute to the integral involving $\alpha(s,\omega)$. Other extraneous cuts coming from the kinematics can be eliminated by changing the definitions of the M_{ij}, as is pointed out in part IV.

A more delicate matter is the inclusion of several unstable particles. For instance, the N^* is a p-wave resonance in the πN system, so that the $\pi + \pi + N$ state studied so far can also be approximated by $\pi + N^*$, as is done in reference 18. In this article also a different treatment of unstable external particles is used, which might be useful. One reasonable assumption is that the phase spaces for the $\pi + N^*$ and the $\rho + N$ do not overlap to any significant extent, so that we are not counting states twice when we consider both.

The analysis made in this appendix can then be repeated by considering the πN^* channel instead of or in addition to the ρN channel. This is however not exempt of new problems. If η is the energy of the πN system in its center of mass squared, the discontinuity equations for η are associated with diagrams like that in figure 9, which involves the πN scattering amplitude, itself a part of the problem, and the justification for introducing experimental data or results of an independent calculation, as could be done with the $\pi\pi$ scattering amplitude $f(\omega)$, is dubious. One possibility is making a calculation to determine the πN scattering in some approximation, and feeding this back into the appropriate integrals.

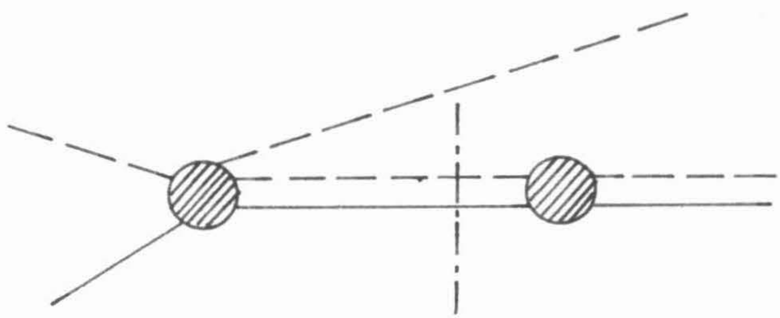

Fig. 9. Diagram for the N^*

The graph corresponding to the interaction singularities is shown in figure 10, it will give a pole in the t variable with residue $g\tau(\eta)$ where $\tau(\eta)$ is in the πN scattering amplitude.

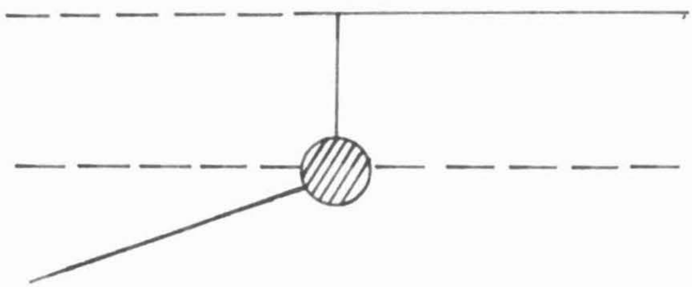

Fig. 10. Nucleon exchange graph

Taking the S-wave projection of this pole, we have

$$T^{o}_{21}(s,\eta) = \tau(\eta)B(s,\eta) \tag{78}$$

$$B(s,\eta) = -\frac{\alpha(s,\eta)}{\pi}\log\frac{\beta(s,\eta)+\alpha(s,\eta)}{-\beta(s,\eta)+\alpha(s,\eta)} \tag{79}$$

$$\alpha(s,\eta) = \frac{\pi s g}{\sqrt{[s-(M+\mu)^2][s-(M-\mu)^2][s-(\sqrt{\eta}+\mu)^2][s-(\sqrt{\eta}-\mu)^2]}} \tag{80a}$$

$$\beta(s,\eta) = \frac{\pi s g}{s^2 - s(\eta - M^2 + 2\mu^2) - (M^2-\mu^2)(\eta-\mu^2)} \tag{80b}$$

and we can determine

$$s_{\pm}(\eta) = \frac{\mu^2}{2}(1+\frac{\mu^2}{M^2}) + (1-\frac{\mu^2}{2M^2})\eta \pm \frac{\mu\sqrt{4M^2-\mu^2}}{2M^2}\sqrt{[\eta-(M-\mu)^2][(M+\mu)^2-\eta]} \tag{81}$$

As expected, these branch points of $B(s, \eta)$ become complex when the N^* becomes unstable, i.e., when $\eta > (M + \mu)^2$. If both unstable particles are considered simultaneously, one has to worry about two variable masses, and diagrams with two unstable vertices occur, like that in Fig. 11.

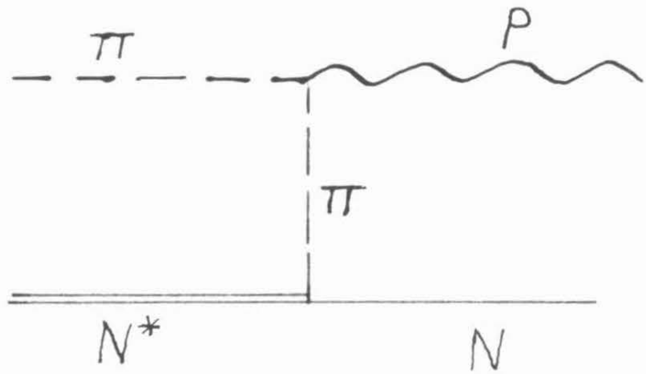

Fig. 11. Diagram with two unstable vertices

The other unstable meson considered, the ω, has only one mode of decay, into three pions. But these diagrams have been neglected, and the $\omega\rho\pi$ vertex is certainly not unstable since the masses of the two vector mesons are almost equal. Moreover, the width corresponding to the ω meson is small, and hence it can be treated as a stable particle without making any obvious big errors.

APPENDIX I

NOTATIONS AND CONVENTIONS

a) Summation convention

When an index is repeated in a monomial expression, a summation over the full range of the index is implied. For instance

$$A = a_i b_i \qquad i = 1, 2, \ldots, n \tag{1a}$$

means

$$A = \sum_{i=1}^{n} a_i b_i \tag{1b}$$

This is not so when the index is a number, and also when parentheses are put around one of the indices. That is, we would write

$$A_i = a_{(i)} b_i \tag{2}$$

and no summation over i is implied.

There is also an important variation of the convention; when we use a greek index whose range is 0, 1, 2, 3, indicating the components of a vector or tensor in Lorentz space, we write

$$p \cdot k = p_\mu k_\mu = p_o k_o - \vec{p} \cdot \vec{k} = p_o k_o - p_1 k_1 - p_2 k_2 - p_3 k_3 \tag{3a}$$

$$p^2 = p_\mu p_\mu = p_o^2 - \vec{p}^2 \tag{3b}$$

Correspondingly, $\delta_{\mu\nu}$ is not the ordinary Kronecker delta, but the metric tensor in a space with a Minkowski metric, that is

$$\delta_{oo} = -\delta_{11} = -\delta_{22} = -\delta_{33} = 1$$

$$\delta_{\mu\nu} = 0 \qquad \mu \neq \nu \tag{4}$$

It is straightforward to check that the ordinary rules for sums carry through without change. For instance

$$p_\mu \delta_{\mu\nu} k_\nu = p_\mu k_\mu = p \cdot k$$

This is the scalar product of four-vectors as defined in equation 3a, and in other references use is made of the metric tensor $g^{\mu\nu}$ that is the same as our $\delta_{\mu\nu}$, and upper and lower indices are used as in tensor calculus.

The antisymmetric or Levi-Civita tensor $\epsilon_{\lambda\mu\nu\rho}$ is defined in equation E10, and this modified summation convention applies to it too.

A vector product is indicated by the symbol $\wedge$.

b) γ matrices

In our convention, the γ matrices obey the anticommutation relations

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu} \tag{5a}$$

Notice that

$$\gamma_0^2 = -\gamma_1^2 = -\gamma_2^2 = -\gamma_3^2 = 1 \tag{5b}$$

They are related to the matrices $\vec{\alpha}, \beta$ by

$$\gamma_0 = \beta \tag{6a}$$

$$\gamma_i = \beta \alpha_i \qquad i = 1, 2, 3 \tag{6b}$$

The matrices $\beta, \vec{\alpha}$ are Hermitean, and hence γ_0 is Hermitean and $\vec{\gamma}$ antihermitean. This can be expressed by

$$\gamma_\mu^\dagger = \gamma_0 \gamma_\mu \gamma_0 \tag{7}$$

If A is a matrix, A^* is its complex conjugate, A^T its transpose and $A^\dagger$ its Hermitean conjugate, that is

$$A^\dagger = (A^*)^T \tag{8}$$

When an explicit representation is needed for the γ matrices, we use

$$\gamma_o = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \tag{9a}$$

where the σ_i are the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{9b}$$

We also define

$$\gamma_5 = \gamma_o\gamma_1\gamma_2\gamma_3 \tag{10a}$$

and in the representation of 9a

$$\gamma_5 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{10b}$$

Also

$$\gamma_5^\dagger = -\gamma_5 \tag{10c}$$

so we can give to μ the value 5 in equation 7, since γ_5 anticommutes with γ_o. In fact, it is easy to see that

$$\gamma_5\gamma_\mu = -\gamma_\mu\gamma_5 \qquad \mu = 0, 1, 2, 3 \tag{10d}$$

$$\gamma_5\alpha_i = \alpha_i\gamma_5 \qquad i = 1, 2, 3 \tag{10e}$$

The unit matrix is more often than not omitted, but its presence and rank is easily deduced from the context. For instance, when we write $p_\mu \gamma_\mu + M$, the M is multiplying a unit matrix of fourth rank (1_4), and in the expression for γ_0 in equation 9a the 1 stands for a second rank unit matrix (1_2).

A slash will be used in connection with a four vector to mean

$$\not{p} = p_\mu \gamma_\mu \tag{11}$$

c) General conventions

Equations and figures are numbered consecutively in each part or appendix. When a reference is made, the part or appendix is indicated unless it is an equation in the same part. When an equation from an outside reference is used, its number is indicated also. So R9-22.13 means equation 22.13 in reference 9.

Conventions that have not been mentioned in this appendix are either the usual ones or they are explained in the text.

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