# METHODS FOR DERIVING CONSERVATION LAWS

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# ABSTRACT

Systematic methods are used to find all possible conservation laws of a given type for certain systems of partial differential equations, including some from fluid mechanics. The necessary and sufficient conditions for a vector to be divergence-free are found in the form of a system of first order, linear, homogeneous partial differential equations, usually overdetermined. Incompressible, inviscid fluid flow is treated in the unsteady two-dimensional and steady three-dimensional cases. A theorem about the degrees of freedom of partial differential equations, needed for finding conservation laws, is proven. Derivatives of the dependent variables are then included in the divergence-free vectors. Conservation laws for Laplace's equation are found with the aid of complex variables, used also to treat the two-dimensional steady flow case when first derivatives are included in the vectors. Conservation laws, depending on an arbitrary number of derivatives, are found for a general first order quasi-linear equation in two independent variables, using two differential operators, which are associated with the derivatives with respect to the two independent variables. Linear totally hyperbolic systems are then treated using an obvious generalization of the above operators.

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# Chapter I

## Introduction

By a conservation law, we mean the expression stating that the divergence of a vector (or tensor) is zero. In the problems considered, we will be given a set of independent variables, say $x_i$, $i = 1, \ldots, n$, and a set of dependent variables, say $u_j$, $j = 1, \ldots, m$, such that the dependence of $u_j$ on $x_i$ is given through a set of partial differential equations. We then look for all vectors $\underline{V}$ depending explicitly on $x_i$, $u_j$, and derivatives of $u_j$ with respect to $x_i$, such that as an implicit function of $x_i$, the divergence of $\underline{V}$ is zero for all $u_j(x_1, \ldots, x_n)$ that satisfy the set of partial differential equations.

Given any two surfaces $s_1$ and $s_2$ such that $s_1$ encloses $s_2$, we know, by Gauss' theorem, that if $\underline{V}$ is divergence-free in the region between $s_1$ and $s_2$, then

$$\int_{s_1} \underline{V} \cdot \underline{n} \, ds = \int_{s_2} \underline{V} \cdot \underline{n} \, ds$$

where $\underline{n}$ is normal to the surface. This property of a divergence-free vector can be used in singular perturbation theory to relate information from one region of space to another, e.g. I-Dee Chang [1], in a study of Navier-Stokes flow at a large distance from a finite body, related unknown constants in the perturbation expansion (far away from the body) to quantities defined at the body such as lift, drag, and torque. This same property could also be used as some sort of check on a numerical solution of a set of partial differential equations. Whitham [2] has also found it useful to obtain conservation laws for the formalism of his averaging theory for nonlinear dispersive waves.

Osborn [3] has discussed the existence of conservation laws by use of pfaffian forms. However, only constant coefficient partial differential equations are considered. The more general case was to be treated in a later paper. The problem of transforming a set of first order equations, where the independent variables did not appear explicitly, to conservation form was considered by Loewner [4]. He considered mainly elliptical systems and by use of certain mappings obtained inequalities for the behavior of stationary, two-dimensional, compressible flow on the boundary of the flow region. A nonlinear wave equation, $y_{tt} - (1+\epsilon y_x)^\alpha y_{xx} = 0$, was treated by Kruskal and Zabusky [5], and an infinite number of polynomial invariants and conservation laws were found. In a series of papers on the Korteweg-deVries equation [6], [7], [8], all the polynomial conservation laws were found. The conservation laws were obtained by use of a certain nonlinear transformation depending on an arbitrary parameter. By use of operators comparable to those in Chapter VII a uniqueness theorem is then proven and other recursion formulas derived for the divergence-free vectors. The techniques used in this paper are similar to those of Howard [9] who found all possible divergence formulas involving vorticity, i.e. all formulas of the type

$$f(\underline{x}, \underline{q}, \underline{\Omega}) = \operatorname{div} \underline{V}(\underline{x}, \underline{q}, \underline{\Omega})$$

where $\underline{\Omega} = \operatorname{curl} \underline{q}$. Lagerstrom [10] made more precise the formulation of the problem of finding divergence formulas (conservation laws included) and the type of theorem one needs to get the equations for which the solutions give <u>all</u> possible divergence formulas. The results were applied to two-dimensional incompressible, inviscid, steady flow. Methods were also developed for obtaining conservation laws for less restricted flows, but not all conservation laws were found.

By forming the divergence expression and setting it equal to zero, we can deduce necessary conditions for a vector to be divergence-free. The idea is to get enough of these necessary conditions so that they will also be sufficient conditions. These conditions will always be in the form of linear, homogeneous, first order partial differential equations for the components of the vector. This system of equations will almost always be overdetermined, restricting the number of solutions. The general solution of these equations will yield <u>all</u> possible divergence-free vectors.

In Chapters II and III we extend the results of Lagerstrom to the time-dependent, two-dimensional case and the steady three-dimensional case, respectively. The vectors arrived at in Chapter II are just slight generalizations of the physically meaningful ones. In Chapter III we find the same vectors as Lagerstrom did, i.e. the physically meaningful ones. The only difference is in the dimension of the vectors. In Chapter IV a general theorem is proven concerning what we may specify about a solution to a system of partial differential equations and still have it exist. This type of theorem is needed in getting the necessary conditions for a vector to be divergence-free. This theorem is used in the following chapters to incorporate derivatives of the dependent variables in the divergence-free vectors. Laplace's equation is considered in Chapter V with the use of complex variables and the introduction of an operator associated with the derivative of an analytic function. The results are used to conclude all possible one-parameter continuous transformations which leave the action integral invariant. In Chapter VI we incorporate first derivatives in the divergence-free vectors for the steady two-dimensional flow previously discussed by Lagerstrom, and because analytic functions appear, we make use of the above operator to solve the resultant equations. A general first order quasi-linear equation in two independent variables is considered in Chapter VII which motivates the introduction of two operators associated with the derivatives with respect to each independent variable. It is found that functions of the dependent and independent variables and derivatives of the dependent variables which are constant along characteristics play an essential role. We generalize the above results in Chapter VIII in which a system of first order totally hyperbolic linear equations is studied. This motivates a simple generalization of the operators in the previous chapter. Again, functions which are constant along characteristics enter, but because we are dealing with a system and not just one equation, it is possible that no such functions exist that depend on a certain order derivative of the dependent variables. In that case, quadratic functions of these variables appear. Always appearing in the general divergence-free vector $\underline{V}$ is the trivial one

$$\underline{V} = (V_1, V_2) = \left( - \frac{df}{dx_2} , \frac{df}{dx_1} \right)$$

where $x_1$ and $x_2$ are the independent variables, $\frac{d}{dx_1}$ and $\frac{d}{dx_2}$ represent total partial derivatives, and $f$ is an arbitrary function of the independent variables, dependent variables, and derivatives of the dependent variables.

It appears that the method in Chapter VIII can be used to handle other systems of equations. For the hyperbolic case for non-distinct characteristics the results might generalize nicely, but for the parabolic case where the matrix cannot be diagonalized, as exemplified by the Korteweg-deVries equation

$$u_t + uu_x + u_{xxx} = 0,$$

a general result might be difficult to attain. It seems if the elliptic case could be handled in a manner similar to that as in Chapter VIII, but some generalization to complex variables would be necessary.

## Chapter II

## Incompressible, Inviscid, Time-dependent, Two-dimensional Flow.

The equations describing two-dimensional inviscid, incompressible, time-dependent flow are given as

$$q_{1_{x_1}} + q_{2_{x_2}} = 0 \tag{1}$$

$$q_{1_t} + q_1 q_{1_{x_1}} + q_2 q_{1_{x_2}} + p_{x_1} = 0 \tag{2'}$$

$$q_{2_t} + q_1 q_{2_{x_1}} + q_2 q_{2_{x_2}} + p_{x_2} = 0 \tag{3'}$$

where $\underline{q} = (q_1, q_2)$ is the velocity and $p$ the pressure. We first prove a lemma concerning the existence of solutions of the above equations.

Lemma: There exists a solution of (1), (2'), (3') such that at any fixed, but arbitrary point $(t_0, x_{10}, x_{20})$ we may prescribe arbitrarily the values of $\underline{q}$, p, $\underline{q}_{\underline{x}}$ (where $(q_{1_{x_1}})_0 + (q_{2_{x_2}})_0 = 0$), $p_{\underline{x}}$, and $p_t$.

This can be done by considering $\underline{q}$ as a linear function in $\underline{x}$ and t, i.e.

$$\underline{q} = \underline{q}_0 + A(\underline{x}-\underline{x}_0) - (A\underline{q}_0 + \underline{a})(t-t_0)$$

$$p = p_0 + b(t-t_0) + \underline{a} \cdot (\underline{x}-\underline{x}_0) + \frac{|A|}{2} (\underline{x}-\underline{x}_0)^2$$

$$+ (A\underline{a} - |A|\underline{q}_0) \cdot (\underline{x}-\underline{x}_0)(t-t_0)$$

where

A: matrix $A_{ij}$, arbitrary except that $A_{11} + A_{22} = 0$

$|A|$ = determinant of A

$\underline{q}_0$, $p_0$, $\underline{a}$, b arbitrary constants

Note: For a $2 \times 2$ matrix whose trace is zero $A^2 = -|A|I$, I being the identity matrix. The above functions $\underline{q}$ and p satisfy (1), (2'), and (3'). Since at $\underline{x} = \underline{x}_0$ and $t = t_0$,

$$\underline{q} = \underline{q}_0$$
$$p = p_0$$
$$q_{i\,x_j} = A_{ij}$$
$$p_{x_i} = a_i$$
$$p_t = b$$

the lemma is proven.

For our purposes it is more convenient to use the total head, $h = p + \frac{q_1^2 + q_2^2}{2}$, instead of the pressure. Equations (2') and (3') become

$$q_{1\,t} - (q_{2\,x_1} - q_{1\,x_2})\, q_2 + h_{x_1} = 0 \tag{2}$$

$$q_{2\,t} + (q_{2\,x_1} - q_{1\,x_2})\, q_1 + h_{x_2} = 0 \tag{3}$$

From the previous lemma it can be seen that h, $h_{x_i}$, and $h_t$ can be chosen arbitrarily at $(t_0, \underline{x}_0)$.

We now look for all vectors $\underline{V}(\underline{x}, t, \underline{q}, h) = (V_0, V_1, V_2)$ such that div $\underline{V} = 0$ for <u>all</u> $\underline{q}$, h satisfying (1), (2), and (3).

Writing out the divergence formula we get

$$\frac{\partial V_0}{\partial t} + \frac{\partial V_0}{\partial q_1} q_{1_t} + \frac{\partial V_0}{\partial q_2} q_{2_t} + \frac{\partial V_0}{\partial h} h_t + \frac{\partial V_1}{\partial x_1} + \frac{\partial V_1}{\partial q_1} q_{1_{x_1}} + \frac{\partial V_1}{\partial q_2} q_{2_{x_1}}$$

$$+ \frac{\partial V_1}{\partial h} h_{x_1} + \frac{\partial V_2}{\partial x_2} + \frac{\partial V_2}{\partial q_1} q_{1_{x_2}} + \frac{\partial V_2}{\partial q_2} q_{2_{x_2}} + \frac{\partial V_2}{\partial h} h_{x_2} = 0$$

$\Longrightarrow$

$$\frac{\partial V_0}{\partial t} + \frac{\partial V_1}{\partial x_1} + \frac{\partial V_2}{\partial x_2} + \frac{\partial V_0}{\partial h} h_t + \left[\frac{\partial V_1}{\partial q_1} - \frac{\partial V_2}{\partial q_2}\right] q_{1_{x_1}}$$

$$+ \left[- \frac{\partial V_0}{\partial q_1} q_2 + \frac{\partial V_0}{\partial q_2} q_1 + \frac{\partial V_2}{\partial q_1}\right] q_{1_{x_2}} + \left[\frac{\partial V_0}{\partial q_1} q_2 - \frac{\partial V_0}{\partial q_2} q_1 + \frac{\partial V_1}{\partial q_2}\right] q_{2_{x_1}}$$

$$+ \left[- \frac{\partial V_0}{\partial q_1} + \frac{\partial V_1}{\partial h}\right] h_{x_1} + \left[- \frac{\partial V_0}{\partial q_2} + \frac{\partial V_2}{\partial h}\right] h_{x_2} = 0$$

using (1), (2), and (3)

Let us now choose a solution of (1), (2), (3) where the values of $t$, $\underline{x}$, $\underline{q}$, and $h$ are fixed but arbitrary. For these fixed values let $h_t$, $h_{\underline{x}}$, and $\underline{q}_{\underline{x}}$ be zero, which the above lemma allows. Then we must have

$$\frac{\partial V_0}{\partial t} + \frac{\partial V_1}{\partial x_1} + \frac{\partial V_2}{\partial x_2} = 0$$

Since $V_0$, $V_1$, $V_2$ do not depend on any gradients of $\underline{q}$ and $h$ we must have

$$\frac{\partial V_0}{\partial h} h_t + \left[ \frac{\partial V_1}{\partial q_1} - \frac{\partial V_2}{\partial q_2} \right] q_{1_{x_1}} + \left[ - \frac{\partial V_0}{\partial q_1} q_2 + \frac{\partial V_0}{\partial q_2} q_1 + \frac{\partial V_2}{\partial q_1} \right] q_{1_{x_2}}$$

$$+ \left[ \frac{\partial V_0}{\partial q_1} q_2 - \frac{\partial V_0}{\partial q_2} q_1 + \frac{\partial V_1}{\partial q_2} \right] q_{2_{x_1}} + \left[ - \frac{\partial V_0}{\partial q_1} + \frac{\partial V_1}{\partial h} \right] h_{x_1}$$

$$+ \left[ - \frac{\partial V_0}{\partial q_2} + \frac{\partial V_2}{\partial h} \right] h_{x_2} = 0 \tag{4}$$

For the same values of $t$, $\underline{x}$, $\underline{q}$, and $h$ choose the following six solutions of (1), (2), (3) (whose existence are guaranteed by the above lemma):

I) A solution with $h_t = 1$; all other gradients zero

II) A solution with $q_{1_{x_1}} = 1$; all other gradients zero

III) A solution with $q_{1_{x_2}} = 1$; all other gradients zero

IV) A solution with $q_{2_{x_1}} = 1$; all other gradients zero

V) A solution with $h_{x_1} = 1$; all other gradients zero

VI) A solution with $h_{x_2} = 1$; all other gradients zero

From equation (4), we then have

$$\frac{\partial V_0}{\partial h} = 0 \tag{a}$$

$$\frac{\partial V_1}{\partial q_1} - \frac{\partial V_2}{\partial q_2} = 0 \tag{b}$$

$$- \frac{\partial V_0}{\partial q_1} q_2 + \frac{\partial V_0}{\partial q_2} q_1 + \frac{\partial V_2}{\partial q_1} = 0 \tag{c}$$

$$\frac{\partial V_0}{\partial q_1} q_2 - \frac{\partial V_0}{\partial q_2} q_1 + \frac{\partial V_1}{\partial q_2} = 0 \tag{d}$$

$$- \frac{\partial V_0}{\partial q_1} + \frac{\partial V_1}{\partial h} = 0 \tag{e}$$

$$- \frac{\partial V_0}{\partial q_2} + \frac{\partial V_2}{\partial h} = 0 \tag{f}$$

Along with

$$\frac{\partial V_0}{\partial t} + \frac{\partial V_1}{\partial x_1} + \frac{\partial V_2}{\partial x_2} = 0 \,, \qquad \text{(g)}$$

derived previously, (a)-(g) are equations which must be satisfied at the fixed values $t, \underline{x}, \underline{q}$, and $h$. However, since these values were arbitrary, (a)-(g) are partial differential equations in the 6 independent variables $t, \underline{x}, \underline{q}$, and $h$ which must be satisfied at all points $t, \underline{x}, \underline{q}, h$. (a)-(g) are necessary conditions for $\operatorname{div} \underline{V}(t, \underline{x}, \underline{q}, h) = 0$ for all $\underline{q}$ and $h$ which are solutions to (1), (2), and (3); they are also seen to be sufficient conditions. Hence, the general solution of (a)-(g) will yield all possible divergence-free vectors $\underline{V}(t, \underline{x}, \underline{q}, h)$ of the system (1), (2), and (3).

Using (e) and (f) in (c) and (d), we obtain

$$\frac{\partial V_1}{\partial h} q_2 = \frac{\partial V_2}{\partial h} q_1 + \frac{\partial V_2}{\partial q_1} \qquad \text{(c')}$$

$$\frac{\partial V_2}{\partial h} q_1 = \frac{\partial V_1}{\partial h} q_2 + \frac{\partial V_1}{\partial q_2} \qquad \text{(d')}$$

(c'), (d') and (b) are equations one gets from the steady state flow case and are solved in [10], pp. 12-14. In the non-steady case, we treat $t$ along with $\underline{x}$ as a parameter in the solution of (c'), (d'), and (b) with the result that

$$V_1 = F(t, \underline{x}, h) q_1 + c_1(t, \underline{x}) \left( \frac{q_1^2 - q_2^2}{2} \right) + c_2(t, \underline{x}) q_1 q_2 + c_1(t, \underline{x}) h + d_1(t, \underline{x})$$

$$V_2 = F(t, \underline{x}, h) q_2 + c_1(t, \underline{x}) q_1 q_2 + c_2(t, \underline{x}) \left( \frac{q_2^2 - q_1^2}{2} \right) + c_2(t, \underline{x}) h + d_2(t, \underline{x}) \,.$$

Equations (e) and (f) imply

$$\frac{\partial V_0}{\partial q_1} = \frac{\partial V_1}{\partial h} = \frac{\partial F}{\partial h} q_1 + c_1 \Longrightarrow V_0 = \frac{\partial F}{\partial h} \frac{q_1^2}{2} + c_1 q_1 + f(q_2, h, t, \underline{x})$$

$$\frac{\partial V_0}{\partial q_2} = \frac{\partial V_2}{\partial h} = \frac{\partial F}{\partial h} q_2 + c_2$$

$$\therefore \frac{\partial f}{\partial q_2} = \frac{\partial F}{\partial h} q_2 + c_2 \Longrightarrow f = \frac{\partial F}{\partial h} \frac{q_2^2}{2} + c_2 q_2 + d_0(h, t, \underline{x})$$

Thus

$$V_0 = \frac{\partial F}{\partial h}\left(\frac{q_1^2 + q_2^2}{2}\right) + c_1 q_1 + c_2 q_2 + d_0 .$$

But from equation (a)

$$\frac{\partial V_0}{\partial h} = 0 \Longrightarrow \begin{cases} \dfrac{\partial^2 F}{\partial h^2} = 0 \Longrightarrow F = a(t, \underline{x}) h + b(t, \underline{x}) \\ \dfrac{\partial d_0}{\partial h} = 0 \Longrightarrow d_0 = d_0(t, \underline{x}) \end{cases}$$

Thus after satisfying (a)-(f), we have

$$V_0 = a\left(\frac{q_1^2 + q_2^2}{2}\right) + c_1 q_1 + c_2 q_2 + d_0$$

$$V_1 = (ah+b) q_1 + c_1\left(\frac{q_1^2 - q_2^2}{2}\right) + c_2 q_1 q_2 + c_1 h + d_1$$

$$V_2 = (ah+b) q_2 + c_1 q_1 q_2 + c_2\left(\frac{q_2^2 - q_1^2}{2}\right) + c_2 h + d_2$$

where all unknown coefficients are functions of $t$ and $\underline{x}$ only.

Using (g) and equating coefficients of independent functions equal to zero, we get

$$a_t + c_{1_{x_1}} - c_{2_{x_2}} = 0 \quad \text{(i)} \qquad c_{1_{x_1}} + c_{2_{x_2}} = 0 \quad \text{(vi)}$$

$$a_t - c_{1_{x_1}} + c_{2_{x_2}} = 0 \quad \text{(ii)} \qquad c_{1_t} + b_{x_1} = 0 \quad \text{(vii)}$$

$$c_{2_{x_1}} + c_{1_{x_2}} = 0 \quad \text{(iii)} \qquad c_{2_t} + b_{x_2} = 0 \quad \text{(viii)}$$

$$a_{x_1} = 0 \quad \text{(iv)} \qquad d_{0_t} + d_{1_{x_1}} + d_{2_{x_2}} \quad \text{(ix)}$$

$$a_{x_2} = 0 \quad \text{(v)}$$

(i), (ii), (iv), and (v) yield $a$ = constant.

(i), (ii), (vi) imply $c_1 = c_1(t, x_2)$, $c_2 = c_2(t, x_1)$. Therefore by (iii)
$c_{2_{x_1}} = -c_{1_{x_2}} = c(t)$

$$c_1 = -c(t)x_2 + r(t)$$

$$c_2 = c(t)x_1 + s(t)$$

(vii) then gives

$$b_{x_1} = c'(t)x_2 - r'(t)$$

$$b = c'(t)x_1 x_2 - r'(t)x_1 + w(x_2, t)$$

From (viii)

$$c'(t)x_1 + s'(t) + c'(t)x_1 + w_{x_2} = 0$$

$$\Longrightarrow c'(t) = 0 \quad \Longrightarrow c = \text{constant}$$

$$w = -s'(t)x_2 + v(t)$$

$$b = -r'(t)x_1 - s'(t)x_2 + v(t)$$

Since (a)-(g) were the necessary and sufficient conditions for $\underline{V}$ to be divergence free, the most general divergence free vector for equations (1), (2), and (3) depending only on $t$, $\underline{x}$, $\underline{q}$, and $h$ is

$$V_0 = \frac{a}{2}\left(q_1^2 + q_2^2\right) + \left[-cx_2 + r(t)\right]q_1 + \left[cx_1 + s(t)\right]q_2 + d_0(t,\underline{x})$$

$$V_1 = \left[ah - r'(t)x_1 - s'(t)x_2 + v(t)\right]q_1 + \left[-cx_2 + r(t)\right]\left[\frac{q_1^2 - q_2^2}{2}\right]$$

$$+ \left[cx_1 + s(t)\right]q_1 q_2 + \left[-cx_2 + r(t)\right]h + d_1(t,\underline{x})$$

$$V_2 = \left[ah - r'(t)x_1 - s'(t)x_2 + v(t)\right]q_2 + \left[-cx_2 + r(t)\right]q_1 q_2$$

$$+ \left[cx_1 + s(t)\right]\left[\frac{q_2^2 - q_1^2}{2}\right] + \left[cx_1 + s(t)\right]h + d_2(t,\underline{x})$$

where $r(t)$, $s(t)$, $v(t)$, $a$, $c$ are arbitrary and $\underline{d} = (d_0, d_1, d_2)$ is an arbitrary vector function of $t$, $x_1$, $x_2$ such that $\operatorname{div} \underline{d} = 0$.

$\underline{V}$ can then be seen to be a linear combination of the following six vectors (returning to the variable p):

$$V_0^{(1)} = \tfrac{1}{2}(q_1^2 + q_2^2)$$
$$V_1^{(1)} = \left[p + \tfrac{1}{2}(q_1^2 + q_2^2)\right] q_1$$
$$V_2^{(1)} = \left[p + \tfrac{1}{2}(q_1^2 + q_2^2)\right] q_2$$

$$V_0^{(2)} = -x_2 q_1 + x_1 q_1$$
$$V_1^{(2)} = -x_2 (q_1^2 + p) + x_1 q_1 q_2$$
$$V_2^{(2)} = -x_2 q_1 q_2 + x_1 (q_2^2 + p)$$

$$V_0^{(3)} = r(t) q_1$$
$$V_1^{(3)} = -r'(t) x_1 q_1 + r(t)(q_1^2 + p)$$
$$V_2^{(3)} = -r'(t) x_1 q_2 + r(t) q_1 q_2$$

$$V_0^{(4)} = s(t) q_2$$
$$V_1^{(4)} = -s'(t) x_2 q_1 + s(t) q_1 q_2$$
$$V_2^{(4)} = -s'(t) x_2 q_2 + s(t)(q_2^2 + p)$$

$$V_0^{(5)} = 0$$
$$V_1^{(5)} = v(t) q_1$$
$$V_2^{(5)} = v(t) q_2$$

$$V_0^{(6)} = d_0(t, \underline{x})$$
$$V_1^{(6)} = d_1(t, \underline{x}) \qquad \text{where} \quad d_{0_{t_1}} + d_{1_{x_1}} + d_{2_{x_2}} = 0$$
$$V_2^{(6)} = d_2(t, \underline{x})$$

$\underline{V}^{(1)}$ and $\underline{V}^{(2)}$ express conservation of energy and angular momentum, respectively. $\underline{V}^{(3)}$ and $\underline{V}^{(4)}$ are generalizations of the conservation of momentum vector in the $x_1$ and $x_2$ directions, respectively. $\underline{V}^{(5)}$ is essentially the conservation of mass vector, and $\underline{V}^{(6)}$ is a trivial vector not dependent on system (1), (2) and (3).

## Chapter III

### Incompressible, Inviscid, Steady-state, Three-dimensional Flow

The equations describing three-dimensional steady, incompressible, inviscid flow are

$$q_{j,j} = 0 \qquad \left( ,j = \frac{\partial}{\partial x_j} \right) \tag{1}$$

$$q_j q_{i,j} + p_{,i} = 0 \qquad i = 1, 2, 3 \tag{2}$$

where, in this chapter, the summation convention is used ( j is summed over 1, 2, 3), $\underline{q} = (q_1, q_2, q_3)$ is the velocity, and p is the pressure.

Lemma: There exists a solution of the above equations such that at any fixed but arbitrary point $(x_{1_0}, x_{2_0}, x_{3_0})$ we may prescribe arbitrarily the values of $\underline{q}$, p, and $q_{i,k}$ as long as $\left(q_{i,i}\right)_0 = 0$ and $\left(q_{i,j}\, q_{j,k}\right)_0 = \left(q_{k,j}\, q_{j,i}\right)_0$.

Proof: The second equation is equivalent to (by using $p_{,ik} = p_{,ki}$)

$$q_{i,j}\, q_{j,k} + q_j\, q_{i,jk} = q_{k,j}\, q_{j,i} + q_j\, q_{k,ji} \tag{*}$$

Assume

$$\underline{q} = \underline{q}_0 + A(\underline{x} - \underline{x}_0) \qquad A = \text{matrix } (A_{ij})$$

$$q_{i,j} = A_{ij} \qquad \underline{q}_0 \text{ arbitrary vector}$$

(All second derivatives of $\underline{q}$ are equal to zero.)

The first equation becomes $A_{ii} = 0$

$$(*) \text{ becomes } A_{ij} A_{jk} = A_{kj} A_{ji}$$

As long as the two above conditions are satisfied by A, then $\underline{q} = \underline{q}_0 + A(\underline{x} - \underline{x}_0)$ is a solution. We can then solve for p which will be a quadratic function of $\underline{x} - \underline{x}_0$, and since there will be an arbitrary constant in p we take that to be the prescribed value of p at $\underline{x} = \underline{x}_0$. We also note that at $\underline{x} = \underline{x}_0$, $\underline{q} = \underline{q}_0$.

Instead of p we will use, for convenience, h, the total head

$$h = p + \tfrac{1}{2}(q_1^2 + q_2^2 + q_3^2) \tag{3}$$

From the above lemma we note that there exists a solution with h chosen arbitrarily at $\underline{x}_0$.

We now look for vectors $\underline{V}(\underline{x}, \underline{q}, h)$ such that div$\underline{V}$ = 0 for all $\underline{q}$, h satisfying (1), (2), and (3). Writing out div$\underline{V}$ we get

$$\frac{\partial V_i}{\partial x_i} + \frac{\partial V_i}{\partial q_j} q_{j,i} + \frac{\partial V_i}{\partial h} h_{,i} = 0$$

Since $h_{,i} = p_{,i} + q_j q_{j,i} = q_j(q_{j,i} - q_{i,j})$ we have

$$\frac{\partial V_i}{\partial x_i} + \left(\frac{\partial V_i}{\partial q_j} + \frac{\partial V_i}{\partial h} q_j - \frac{\partial V_j}{\partial h} q_i\right) q_{j,i} = 0$$

For any arb. $\underline{x}$, $\underline{q}$, h we first pick $q_{j,i} = 0$ for all j,i. Thus

$$\frac{\partial V_i}{\partial x_i} = 0 \tag{4}$$

at $\underline{x}, \underline{q}, h$ since $\underline{V}$ does not depend on $q_{j,i}$. Therefore

$$\left(\frac{\partial V_i}{\partial q_j} + \frac{\partial V_i}{\partial h} q_j - \frac{\partial V_j}{\partial h} q_i\right) q_{j,i} = 0$$

For the same values of $\underline{x}, \underline{q}, h$ and for each $\ell, m, \ell \neq m$, choose $q_{j,i} = 1$ for $j = \ell$, $i = m$ and $q_{j,i} = 0$ otherwise. This can be done by the above lemma since

$$q_{j,k}\, q_{k,i} = q_{i,k}\, q_{k,j} = 0 \qquad \text{and} \quad q_{i,i} = 0$$

This yields the equation

$$\frac{\partial V_m}{\partial q_\ell} + \frac{\partial V_m}{\partial h} q_\ell - \frac{\partial V_\ell}{\partial h} q_m = 0 \qquad \ell \neq m \tag{5}$$

Next we choose $q_{1,1} = 1$, $q_{2,2} = -1$, $q_{3,3} = 0$ which, again, is allowed by the lemma. This gives the equations

$$\frac{\partial V_1}{\partial q_1} - \frac{\partial V_2}{\partial q_2} = 0 \tag{6}$$

Similarly

$$\frac{\partial V_2}{\partial q_2} - \frac{\partial V_3}{\partial q_3} = 0, \quad \frac{\partial V_1}{\partial q_1} - \frac{\partial V_3}{\partial q_3} = 0 \qquad (7), \quad (7')$$

Equations (7) and (7') are equivalent when (6) is used. Since the above equations are to be satisfied at any arbitrary point $\underline{x}, \underline{q}, h$, they must be satisfied for all $\underline{x}, \underline{q}, h$. Thus (4), (5), (6), (7) or (7') are the necessary, and also seen to be sufficient, conditions for $\operatorname{div} \underline{V} = 0$.

For $(\ell, m) = (2,1)$ and $(1,2)$, respectively, we get

$$\frac{\partial V_1}{\partial q_2} + \frac{\partial V_1}{\partial h} q_2 - \frac{\partial V_2}{\partial h} q_1 = 0 \tag{8}$$

$$\frac{\partial V_2}{\partial q_1} + \frac{\partial V_2}{\partial h} q_1 - \frac{\partial V_1}{\partial h} q_2 = 0 \tag{9}$$

(6), (8), (9) are equations obtained for the two-dimensional case and are solved in [10], pp. 12-14, as was previously mentioned in Chapter II. The solution of (6), (8), (9) is then

$$V_1 = F(\underline{x}, h, q_3) q_1 + c_1(\underline{x}, q_3)\left(\frac{q_1^2 - q_2^2}{2}\right) + c_2(\underline{x}, q_3) q_1 q_2 + c_1(\underline{x}, q_3) h + \tilde{d}_1(\underline{x}, q_3) \tag{10}$$

$$V_2 = F(\underline{x}, h, q_3) q_2 + c_1(\underline{x}, q_3) q_1 q_2 + c_2(\underline{x}, q_3)\left(\frac{q_2^2 - q_1^2}{2}\right) + c_2(\underline{x}, q_3) h + \tilde{d}_2(\underline{x}, q_3) \tag{11}$$

Substituting into

$$\frac{\partial V_2}{\partial q_3} + \frac{\partial V_2}{\partial h} q_3 - \frac{\partial V_3}{\partial h} q_2 = 0 \qquad (\ell = 3,\ m = 2)$$

yields

$$F_{q_3} q_2 + c_{1_{q_3}} q_1 q_2 + c_{2_{q_3}}\left(\frac{q_2^2 - q_1^2}{2}\right) + c_{2_{q_3}} h + \tilde{d}_{2_{q_3}} + F_h q_2 q_3 + c_2 q_3 = V_{3h} q_2$$

Let $q_2 = 0$ which implies

$$c_{2_{q_3}}\left(-\tfrac{1}{2}\, q_1^2\right) + c_{2_{q_3}} h + \tilde{d}_{2_{q_3}} + c_2 q_3 = 0$$

Therefore, we must have

$$c_{2_{q_3}} = 0 \quad \Longrightarrow \quad c_2 = c_2(\underline{x}) \tag{12}$$

$$\tilde{d}_{2_{q_3}} + c_2 q_3 = 0 \quad \Longrightarrow \quad \tilde{d}_2 = -c_2(\underline{x})\frac{q_3^2}{2} + d_2(\underline{x}) \tag{13}$$

Using

$$\frac{\partial V_1}{\partial q_3} + \frac{\partial V_1}{\partial h} q_3 - \frac{\partial V_3}{\partial h} q_1 = 0 \qquad (\ell = 3, \quad m = 1)$$

we have

$$F_{q_3} q_1 + c_{1_{q_3}}\left(\frac{q_1^2 - q_2^2}{2}\right) + c_{1_{q_3}} h + \tilde{d}_{1_{q_3}} + F_h q_1 q_3 + c_1 q_3 = V_{3h} q_1$$

Thus, we must have (similar to the preceding argument)

$$c_1 = c_1(\underline{x}) \tag{14}$$

$$\tilde{d}_1 = -c_1(\underline{x})\frac{q_3^2}{2} + d_1(\underline{x}) \tag{15}$$

Hence $V_{3h} = F_{q_3} + F_h q_3$ (16)

We now use

$$\frac{\partial V_3}{\partial q_2} + \frac{\partial V_3}{\partial h} q_3 - \frac{\partial V_2}{\partial h} q_2 = 0 \qquad (\ell = 2,\ m = 3)$$

giving from (11) and (16)

$$V_{3_{q_2}} + q_2 F_{q_3} = c_2 q_3$$

so that

$$V_3 = -\frac{q_2^2}{2} F_{q_3} + c_2 q_3 q_2 + M(\underline{x}, q_1, q_3, h) \tag{17}$$

However, (16) implies

$$F_{q_3} + F_h q_3 = -\frac{q_2^2}{2} F_{q_3 h} + M_h$$

Therefore, we must have

$$F_{q_3 h} = 0 \implies F = g(\underline{x}, q_3) + f(\underline{x}, h) \tag{18}$$

and

$$M_h = g_{q_3} + f_h q_3 \implies M = h g_{q_3} + f q_3 + N(\underline{x}, q_1, q_3) \tag{19}$$

And now using

$$\frac{\partial V_3}{\partial q_1} + \frac{\partial V_3}{\partial h} q_3 - \frac{\partial V_1}{\partial h} q_2 = 0 \qquad (\ell = 1,\ m = 3)$$

with (10), (16), (17), (18), (19) we get

$$N_{q_1} + g_{q_3} q_1 = c_1 q_3 \Longrightarrow N = -\frac{q_1^2}{2} g_{q_3} + c_1 q_1 q_3 + P(\underline{x} \;\; q_3)$$

Equation ( 7) gives, using the above results

$$g = -\frac{q_1^2 + q_2^2}{2} g_{q_3 q_3} + h g_{q_3 q_3} + P_{q_3}$$

Therefore, we must have

$$g_{q_3 q_3} = 0 \Rightarrow g = c_3(\underline{x}) q_3 , \tag{20}$$

the arbitrary function of $\underline{x}$ being absorbed in $F_2(\underline{x}, q_3)$, and

$$P_{q_3} = g \Longrightarrow P = c_3(\underline{x}) \frac{q_3^2}{2} + d_3(\underline{x}) \tag{21}$$

Thus, we have

$$V_1 = f(\underline{x}, h) q_1 + c_1(\underline{x}) h + c_1(\underline{x}) \left( \frac{q_1^2 - q_2^2 - q_3^2}{2} \right) + c_2(\underline{x}) q_1 q_2 + c_3(\underline{x}) q_1 q_3 + d_1(\underline{x}) \tag{22}$$

$$V_2 = f(\underline{x}, h) q_2 + c_2(\underline{x}) h + c_1(\underline{x}) q_1 q_2 + c_2(\underline{x}) \left( \frac{q_2^2 - q_1^2 - q_3^2}{2} \right) + c_3(\underline{x}) q_2 q_3 + d_2(\underline{x}) \tag{23}$$

$$V_3 = f(\underline{x}, h) q_3 + c_3(\underline{x}) h + c_1(\underline{x}) q_1 q_3 + c_2(\underline{x}) q_2 q_3 + c_3(\underline{x}) \left( \frac{q_3^2 - q_1^2 - q_2^2}{2} \right) + d_3(\underline{x}) \tag{24}$$

The last equation to be solved is (4), which becomes, using (22)-(24),

$$f_{x_1} q_1 + c_{1\,x_1} h + c_{1\,x_1}\left(\frac{q_1^2 - q_2^2 - q_3^2}{2}\right) + c_{2\,x_1} q_1 q_2 + c_{3\,x_1} q_1 q_3 + d_{x_1}$$

$$+ f_{x_2} q_2 + c_{2\,x_2} h + c_{1\,x_2} q_1 q_2 + c_{2\,x_2}\left(\frac{q_2^2 - q_1^2 - q_3^2}{2}\right) + c_{3\,x_2} q_2 q_3 + d_{2\,x_2}$$

$$+ f_{x_3} q_3 + c_{3\,x_3} h + c_{1\,x_3} q_1 q_3 + c_{2\,x_3} q_2 q_3 + c_{3\,x_3}\left(\frac{q_3^2 - q_1^2 - q_2^2}{2}\right) + d_{3\,x_3} = 0$$

Again, using the independence of the coefficients of the functions of $\underline{x}$, we have

$$d_{1\,x_1} + d_{2\,x_2} + d_{3\,x_3} = 0 \qquad (25)$$

$$f_{x_i} = 0 \Longrightarrow f = f(h) \qquad (26)$$

$$c_{1\,x_1} = c_{2\,x_2} = c_{3\,x_3} = 0 \qquad (27)$$

$$c_{2\,x_1} + c_{1\,x_2} = 0, \quad c_{3\,x_1} + c_{1\,x_3} = 0, \quad c_{3\,x_2} + c_{2\,x_3} = 0 \qquad (28),(29),(30)$$

Equations (27)-(30) give as solutions

$$\begin{aligned} c_1 &= -a_3 x_2 + a_2 x_3 + e_1 \\ c_2 &= a_3 x_1 - a_1 x_3 + e_2 \\ c_3 &= -a_2 x_1 + a_1 x_2 + e_3 \end{aligned} \qquad (31)$$

where $a_1, a_2, a_3, e_1, e_2, e_3$ are arbitrary constants. We can write (31) as

$$\underline{c} = \underline{a} \times \underline{x} + \underline{e}$$

and $\underline{V}$ can now be written

$$\underline{V} = f(h)\underline{q} + (\underline{q} \circ \underline{q} + pI)(\underline{a} \times \underline{x} + \underline{e}) + \underline{d}(\underline{x})$$

or, letting

$$\underline{\underline{L}} = \underline{q} \circ \underline{q} + pI, \quad (L_{ij} = q_i q_j + p\,\delta_{ij})$$

$$\underline{V} = f(h)\underline{q} + (\underline{x} \times \underline{\underline{L}})^T \underline{a} + \underline{\underline{L}}\,\underline{e} + \underline{d}(\underline{x})$$

where

$f(h)$ is an arbitrary function of $h$

$\underline{a}$, $\underline{e}$ are arbitrary constant vectors

$\underline{d}(\underline{x})$ is an arbitrary divergence-free vector function of $\underline{x}$

Note: $\underline{x} \times \underline{\underline{L}}$ is defined as $\epsilon_{imn} x_m L_{nj}$ so that $\underline{x} \times \underline{\underline{L}}$ is a second order tensor. We then see that $\underline{\underline{L}}(\underline{a} \times \underline{x}) = (\underline{x} \times \underline{\underline{L}})^T \underline{a}$ where "T" denotes the transpose.

$\underline{V}$ thus consists of a sum of vectors which represent, respectively, constancy of total head along streamlines and conservation of mass, conservation of angular momentum, conservation of momentum, and a trivial divergence-free vector. We see that the only conservation laws obtained are the physically familiar ones.

# Chapter IV

## Degrees of Freedom in Partial Differential Equations

In obtaining conservation laws, we make very strong use of what is allowed to be prescribed in a solution to a P. D. E. To this end the Cauchy-Kovalevskaya theorem is considered. The theorem for a system of m first order equations for the functions $u_1, \ldots, u_m$ of the independent variables $x_1, \ldots, x_n$ states that the system

$$\frac{\partial u_k}{\partial x_1} = f_k\left(x_i, u_i, \frac{\partial u_i}{\partial x_j}\right) \qquad (k = 1, \ldots, m)$$

$\left(\frac{\partial u_i}{\partial x_1}\right.$ does not appear in $\left.f_k\right)$ with Cauchy initial data

$$u_k\Big|_{x_1=0} = \phi_k(x_2, \ldots, x_m) \quad (k = 1, \ldots, m),$$

where $\phi_k$ is assumed regular when its arguments become zero and $f_k$ is assumed regular when its arguments become the initial values, possesses a unique regular solution. This theorem can obviously be generalized to the case where the initial data are evaluated on $x_1 = c_1$ (constant) and $\phi_k$ is assumed regular when its arguments take on constant (not necessarily zero) values. In particular, we could choose $\phi_k$ to be a terminating Taylor series where the choice for

$$\frac{\partial^p u_k}{\partial x_2^{j_1} \partial x_3^{j_2} \cdots \partial x_n^{j_{n-1}}} \qquad \begin{array}{l} j_1 + j_2 + \cdots j_{n-1} = p \\ p = 0, 1, \ldots, N \qquad N \text{ arbitrary} \end{array}$$

evaluated at $\underline{x} = \underline{c}$ is completely arbitrary. We now state the above in a theorem.

<u>Theorem</u>: For the system of $m$ equations for the functions $u_1, \dots, u_m$ of the independent variables $x_1, \dots, x_n$

$$\frac{\partial u_k}{\partial x_1} = f_k\left( x_i,\ u_i,\ \frac{\partial u_i}{\partial x_j} \right) \qquad (k = 1,\ \dots,\ m)$$

$\left(\frac{\partial u_i}{\partial x_1}\right.$ does not appear in $\left. f_k \right)$, a solution exists, which at an arbitrary point $\underline{x} = \underline{c}$, the values of $u_k$ and all its derivatives (excluding $x_1$ derivatives) up to a finite order may be arbitrarily prescribed as long as $f_k$ is regular for these initial conditions.

## Chapter V

## Laplace's Equation in Two Dimensions

In finding conservation laws, it is sometimes convenient to make use of complex variables to aid in the computations. Before proceeding with Laplace's equation we define an operator and consider some of its properties. Let $F(x,y) = f(x,y) + ig(x,y)$, $f$ and $g$ real, be any sufficiently smooth complex function of the real variables $x$ and $y$. Let $z = x + iy$ and define an operator $L$:

$$L_z F = \frac{\partial F}{\partial x} - i\frac{\partial F}{\partial y} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + i\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)$$

$$L_{\bar{z}} F = \frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y} = \frac{\partial f}{\partial x} - \frac{\partial g}{\partial y} + i\left(\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\right)$$

Since $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$, we could think of $F$ as a function of $z$ and $\bar{z}$, and $L_z F$ and $L_{\bar{z}} F$ as associated with $\frac{\partial F}{\partial z}$ and $\frac{\partial F}{\partial \bar{z}}$ respectively. In fact, we note that if $L_{\bar{z}} F = 0$ then $\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y}$ and $\frac{\partial g}{\partial x} = -\frac{\partial f}{\partial g}$ and thus $F$ is an analytic function of $z$ and hence does not depend on $\bar{z}$. Similarly $L_z F = 0 \implies F$ is an analytic function of $\bar{z}$.

The following is a list of other properties of $L$ which can be easily verified:

1. $\overline{L_z F} = L_{\bar{z}} \bar{F}$

2. $L_z \bar{z} = L_{\bar{z}} z = 0$

3. $L_z z = L_{\bar{z}} \bar{z} = 2$

4. $L_{\bar{z}} L_z F = L_z L_{\bar{z}} F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = \nabla^2 F$

5. If $F$ is an analytic function of $z$, then $L_z F = 2F'(z)$

6. $L_z(FG) = (L_z F)G + F(L_z G)$

7. $\mathrm{Re}(L_{\bar{z}} F) = 0 \implies \frac{\partial f}{\partial x} = \frac{\partial g}{\partial y} \implies f = \frac{\partial h}{\partial y}, \quad g = \frac{\partial h}{\partial x}$ so that

$F = \frac{\partial h}{\partial y} + i\,\frac{\partial h}{\partial x} = iL_z h$, $h$ real. Conversely, if

$F = iL_z h$, where $h$ is real, then $L_{\bar{z}} F = iL_{\bar{z}} L_z h =$

$i\nabla^2 h$ so that $\mathrm{Re}(L_{\bar{z}} F) = 0$. Also $\mathrm{Re}(L_z F) = 0$

$\iff F = iL_{\bar{z}} k$ where $k$ is real.

8. $L_{\bar{z}} F = 0 \iff F$ is an analytic function of $z$

$L_z F = 0 \iff F$ is an analytic function of $\bar{z}$

9. There exists an $f(x,y)$ (may be complex) such that

$$F = L_z L_{\bar{z}} f = L_{\bar{z}} L_z f = \nabla^2 f$$

for any complex $F(x,y)$. (This is just a statement that Poisson's equation always has a solution).

10. There exists a $g(x,y)$ such that

$$F = L_z g$$

or an $h(x,y)$ such that

$$F = L_{\bar{z}} h$$

(Follows directly from 9.)

11. If $F$ is a sufficiently smooth function of four variables $x_1$, $y_1$, $x_2$, $y_2$, letting $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, we have

$$L_{z_1} L_{z_2} F = L_{z_2} L_{z_1} F$$

We now look for conservation laws for Laplace's equation: $\phi_{xx} + \phi_{yy} = 0$. We will look for vectors which depend on $x, y, \phi$, and up to second order derivatives of $\phi$, i.e. $\phi_x$, $\phi_y$, $\phi_{xx}$, $\phi_{xy}$. Since $\phi_{yy}$ can be solved for in terms of $\phi_{xx}$ ($\phi_{yy} = -\phi_{xx}$), the vectors need not depend on $\phi_{yy}$. We also note that we may prescribe the values of $\phi$, $\phi_x$, $\phi_y$, $\phi_{xx}$, $\phi_{xy}$, $\phi_{xxx}$, $\phi_{xxy}$ arbitrarily at some arbitrary point $(x, y)$ and a solution to Laplace's equation will exist. This can be shown from the Cauchy-Kovalevskaya theorem.

Let $\underline{V}(x, y, \phi, \phi_x, \phi_y, \phi_{xx}, \phi_{xy}) = (V_1, V_2)$ be a vector which is divergence-free, i.e.

$$\frac{\partial V_1}{\partial x} + \frac{\partial V_1}{\partial \phi}\phi_x + \frac{\partial V_1}{\partial \phi_x}\phi_{xx} + \frac{\partial V_1}{\partial \phi_y}\phi_{yx} + \frac{\partial V_1}{\partial \phi_{xx}}\phi_{xxx} + \frac{\partial V_1}{\partial \phi_{xy}}\phi_{xyx}$$

$$+ \frac{\partial V_2}{\partial y} + \frac{\partial V_2}{\partial \phi}\phi_y + \frac{\partial V_2}{\partial \phi_x}\phi_{xy} + \frac{\partial V_2}{\partial \phi_y}\phi_{yy} + \frac{\partial V_2}{\partial \phi_{xx}}\phi_{xxy} + \frac{\partial V_2}{\partial \phi_{xy}}\phi_{xyy} = 0$$

Since $V_1$ and $V_2$ do not depend on $\phi_{xxx}$ and $\phi_{xxy}$ and $\phi_{xyy} = -\phi_{xxx}$ we obtain, by the independence of $x, y, \phi, \phi_x, \phi_y, \phi_{xx}, \phi_{xy}$, $\phi_{xxx}$, and $\phi_{xxy}$

$$\frac{\partial V_1}{\partial \phi_{xx}} = \frac{\partial V_2}{\partial \phi_{xy}}, \qquad \frac{\partial V_1}{\partial \phi_{xy}} = -\frac{\partial V_2}{\partial \phi_{xx}} \qquad (1), \quad (2)$$

and

$$\frac{\partial V_1}{\partial x} + \frac{\partial V_1}{\partial \phi}\phi_x + \frac{\partial V_1}{\partial \phi_x}\phi_{xx} + \frac{\partial V_1}{\partial \phi_y}\phi_{xy}$$

$$+ \frac{\partial V_2}{\partial y} + \frac{\partial V_2}{\partial \phi}\phi_y + \frac{\partial V_2}{\partial \phi_x}\phi_{xy} - \frac{\partial V_2}{\partial \phi_y}\phi_{xx} = 0 \qquad (3)$$

(1), (2) and (3) are also seen to be sufficient conditions for $\operatorname{div}\underline{V} = 0$.

Let $V = V_1 + iV_2$

$z_1 = x + iy$

$z_2 = \phi_x + i\phi_y$

$z_3 = \phi_{xx} + i\phi_{xy}$

Then (1) and (2) imply $V$ is an analytic function of $z_3$. Now,

$$\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} = \operatorname{Re} L_{z_1} V$$

$$\frac{\partial V_1}{\partial \phi}\phi_x + \frac{\partial V_2}{\partial \phi}\phi_y = \operatorname{Re}\left[\frac{\partial V}{\partial \phi}\,\overline{z}_2\right]$$

$$\frac{\partial V_1}{\partial \phi_x}\phi_{xx} + \frac{\partial V_2}{\partial \phi_x}\phi_{xy} + \frac{\partial V_1}{\partial \phi_y}\phi_{xy} - \frac{\partial V_2}{\partial \phi_y}\phi_{xx} = \operatorname{Re}\left[(L_{\overline{z}_2} V)\overline{z}_3\right]$$

so that (3) becomes

$$\operatorname{Re}\left[L_{z_1} V + \frac{\partial V}{\partial \phi}\,\overline{z}_2 + (L_{\overline{z}_2} V)\,\overline{z}_3\right] = 0 \qquad (4)$$

If we take the Laplacian of (4) with respect to $\phi_{xx}$, $\phi_{xy}$ $(\nabla^2_{\phi_{xx},\phi_{xy}})$ we get

$$\operatorname{Re}\left[\frac{\partial}{\partial z_3}(L_{\overline{z}_2} V)\right] = 0$$

by using the fact that the first two terms in the bracketed expression of (4) are analytic functions of $z_3$ (since V is an analytic function of $z_3$) and hence they disappear leaving

$$0 = \nabla^2_{\phi_{xx},\phi_{xy}} \mathrm{Re}\left[(L_{\bar{z}_2} V)\, \bar{z}_3\right] = \mathrm{Re}\left\{\nabla^2_{\phi_{xx},\phi_{xy}}\left[(L_{\bar{z}_2} V)\, \bar{z}_3\right]\right\}$$

$$= \mathrm{Re}\left\{ L_{z_3} L_{\bar{z}_3}\left[(L_{\bar{z}_2} V)\bar{z}_3\right]\right\} = \mathrm{Re}\left[2L_{z_3}(L_{\bar{z}_2} V)\right]$$

$$= 4\,\mathrm{Re}\frac{\partial}{\partial z_3}(L_{\bar{z}_2} V)$$

using properties 4, 6, 8, 5, and $L_{\bar{z}_2} V$ is an analytic function of $z_3$.

Since $\frac{\partial}{\partial z_3}(L_{\bar{z}_2} V)$ is an analytic function of $z_3$ whose real part is zero, we must have

$$\frac{\partial}{\partial z_3} L_{\bar{z}_2} V = iA(x, y, \phi, \phi_x, \phi_y) \quad \text{where } A \text{ is real and independent of } \phi_{xx} \text{ and } \phi_{xy}$$

$$\Longrightarrow \; L_{\bar{z}_2} V = iAz_3 + B(x, y, \phi, \phi_x, \phi_y) \qquad B \text{ complex}$$

There exists a real $a(x, y, \phi, \phi_x, \phi_y)$ such that $\nabla^2_{\phi_x,\phi_y} a = A$ or $L_{\bar{z}_2} L_{z_2} a = A$. Similarly there exists a $b(x, y, \phi, \phi_x, \phi_y)$ such that $L_{\bar{z}_2} b = B$. These two statements follow from properties 9 and 10 respectively.

We now have, using property 8 and that everything in the bracket is analytic in $z_3$,

$$L_{\overline{z}_2}\left[V - i(L_{z_2}a)z_3 - b\right] = 0$$

$$\Longrightarrow V = i(L_{z_2}a)z_3 + b + c(x, y, \phi, z_2, z_3) \qquad (5)$$

where c is analytic in $z_2$ for fixed $z_3$ and analytic in $z_3$ for fixed $z_2$.

Substituting into (4), we obtain

$$\mathrm{Re}\Big[i(L_{z_1}L_{z_2}a)z_3 + L_{z_1}b + L_{z_1}c + i\frac{\partial}{\partial\phi}(L_{z_2}a)\overline{z}_2 z_3 + \frac{\partial b}{\partial\phi}\overline{z}_2$$

$$+ \frac{\partial c}{\partial\phi}\overline{z}_2 + (L_{z_2}\overline{b})z_3\Big] = 0 \qquad (6)$$

using the analyticity of c in $z_2$, the realness of a, and the fact that

$$\mathrm{Re}\left[(L_{\overline{z}_2}b)\overline{z}_3\right] = \mathrm{Re}\left[\overline{(L_{\overline{z}_2}b)\overline{z}_3}\right] = \mathrm{Re}\left[(L_{z_2}\overline{b})z_3\right]$$

The term inside the bracket of (6) is an analytic function of $z_3$ and by (6) its real part is zero. Hence it is equal to an imaginary constant relative to $z_3$ (does not depend on $\phi_{xx}$ or $\phi_{xy}$) i.e.

$$L_{z_1}c + \frac{\partial c}{\partial\phi}\overline{z}_2 + \left[iL_{z_1}L_{z_2}a + i\frac{\partial}{\partial\phi}(L_{z_2}a)\overline{z}_2 + L_{z_2}\overline{b}\right]z_3$$

$$+ L_{z_1}b + \frac{\partial b}{\partial\phi}\overline{z}_2 = iI(x, y, \phi, \phi_x, \phi_y), \quad I \text{ real} \qquad (7)$$

Differentiating twice with respect to $z_3$ yields

$$L_{z_1}\frac{\partial^2 c}{\partial z_3^2} + \frac{\partial^3 c}{\partial\phi\,\partial z_3^2}\overline{z}_2 = 0 \qquad (8)$$

Since $c$ is analytic in $z_2$, we must have

$$\frac{\partial^3 c}{\partial \phi \partial z_3^2} = 0 \implies \frac{\partial^2 c}{\partial z_3^2} = \text{fcn}\,(x, y, z_2, z_3)$$

Therefore, there must exist a $D(x, y, z_2, z_3)$ such that

$$\frac{\partial^2 c}{\partial z_3^2} = \frac{\partial^2 D}{\partial z_3^2}$$

so that

$$c = D(x, y, z_2, z_3) + E(x, y, \phi, z_2)\, z_3 \quad ,$$

the constant term in $z_3$ being absorbed into $b(x, y, \phi, \phi_x, \phi_y)$. From (8) we also have

$$L_{z_1} \frac{\partial^2 c}{\partial z_3^2} = 0 \implies L_{z_1} \frac{\partial^2 D}{\partial z_3^2} = 0$$

$$\implies \frac{\partial^2 D}{\partial z_3^2} = \text{analytic function of } \bar{z}_1,\, z_2, \text{ and } z_3$$

From a theorem in complex variables we can find a $d(\bar{z}_1, z_2, z_3)$ such that

$$\frac{\partial^2 D}{\partial z_3^2} = \frac{\partial^2 d}{\partial z_3^2}$$

$$\implies \quad D = d(\bar{z}_1, z_2, z_3)\ ,$$

the linear and constant term in $z_3$ being absorbed into $E(x, y, \phi, z_2) z_3$

and $b(x, y, \phi, \phi_x, \phi_y)$ respectively. We now have for $c$:

$$c = d(\overline{z}_1, z_2, z_3) + E(x, y, \phi, z_2) z_3$$

We can always find an $e(x, y, \phi, z_2)$ such that

$$E(x, y, \phi, z_2) = i \frac{\partial e}{\partial z_2} = \frac{i}{2} L_{z_2} e = \frac{i}{2} L_{z_2} (e + \overline{e})$$

by properties 5 and 8 and the fact that $\overline{e}$ is an analytic function of $\overline{z}_2$. Thus $E(x, y, \phi, z_2) = iL_{z_2} \mathrm{Re}\, e$ so that the term $E(x, y, \phi, z_2) z_3$ in (9) can be absorbed in the first term of $V$ in (5) so that finally we have for $c$:

$$c = d(\overline{z}_1, z_2, z_3) \tag{10}$$

Substituting (10) in (7) we have

$$\left[iL_{z_1} L_{z_2} a + i\frac{\partial}{\partial \phi}(L_{z_2} a)\overline{z}_2 + L_{z_2}\overline{b}\right] z_3 + L_{z_1} b + \frac{\partial b}{\partial \phi}\ \overline{z}_2 = iI(x, y, \phi, \phi_x, \phi_y) \tag{11}$$

Thus the bracketed term in front of $z_3$ must vanish, yielding

$$L_{z_2}\left[iL_{z_1} a + i\frac{\partial a}{\partial \phi}\ \overline{z}_2 + \overline{b}\right] = 0$$

This, then, implies

$$\overline{b} = -iL_{z_1} a - i\frac{\partial a}{\partial \phi}\ \overline{z}_2 + f(x, y, \phi, \overline{z}_2)$$

or

$$b = iL_{\overline{z}_1} a + i\frac{\partial a}{\partial \phi}\ z_2 + g(x, y, \phi, z_2) \tag{12}$$

where $g = \overline{f}$ is an analytic function of $z_2$. (11) also gives

$$\mathrm{Re}\left[ L_{z_1} b + \frac{\partial b}{\partial \phi} \overline{z}_2 \right] = 0$$

which when (12) is substituted in becomes

$$\mathrm{Re}\left[ L_{z_1} g + \frac{\partial g}{\partial \phi} \overline{z}_2 \right] = 0 \tag{13}$$

since $iL_{z_1} L_{\overline{z}_1} a = i\nabla^2_{x,y} a$,

$$iL_{z_1} \frac{\partial a}{\partial \phi} z_2 + iL_{\overline{z}_1} \frac{\partial a}{\partial \phi} \overline{z}_2 = 2i\,\mathrm{Re}\left(L_{z_1} \frac{\partial a}{\partial \phi} z_2\right),$$

and $\quad i \dfrac{\partial^2 a}{\partial \phi^2} z_2 \overline{z}_2$

are all imaginary.

With an argument similar to the one used in (4) for $V$ an analytic function of $z_3$, we obtain for $g$, an analytic function of $z_2$,

$$\mathrm{Re}\left[ \frac{\partial}{\partial z_2} \frac{\partial g}{\partial \phi} \right] = 0 \implies$$ there exists a real $m(x, y, \phi)$ and a complex $n(x, y, \phi)$ such that

$$\frac{\partial g}{\partial \phi} = i \frac{\partial^2 m(x, y, \phi)}{\partial \phi^2} z_2 + \frac{\partial n(x, y, \phi)}{\partial \phi}$$

$$\implies g = \frac{i\partial m(x, y, \phi)}{\partial \phi} z_2 + n(x, y, \phi) + p(x, y, z_2),$$

$p$ analytic in $z_2$ (14)

Since we can extract from $n(x, y, \phi)$ a term $iL_{\overline{z}_1} m$ and this term together with the first term in (14) can be absorbed into the first two terms of (12), we now have for $g$ :

$$g = n(x, y, \phi) + p(x, y, z_2) \tag{15}$$

Substituting (15) into (13) yields

$$\mathrm{Re}\left[ L_{z_1} n + L_{z_1} p + \frac{\partial \overline{n}}{\partial \phi} z_2 \right] = 0 \tag{16}$$

so that the bracketed term must be equal to an imaginary constant relative to $z_2$, which, then implies

$$\frac{\partial^2}{\partial z_2^2} L_{z_1} p = 0 \implies L_{z_1} \frac{\partial^2 p}{\partial z_2^2} = 0$$

$$\implies \frac{\partial^2 p}{\partial z_2^2} = \text{analytic function of } \overline{z}_1$$

Thus there exists an analytic function of $\overline{z}_1$ and $z_2$ such that its second derivative equals $\frac{\partial^2 p}{\partial z_2^2}$, so that we have for $p$:

$$p = \mathrm{fcn}(\overline{z}_1, z_2) + Q(x, y) z_2 + \mathrm{fcn}(x, y) \tag{17}$$

The first term in (17) can be absorbed in $d(\overline{z}_1, z_2, z_3)$ and the last term can be absorbed in $n(x, y, \phi)$ thus giving for $p$:

$$p = Q(x, y) z_2 \tag{18}$$

Substituting back into (16) we find

$$\mathrm{Re}\left[ L_{z_1} n + (L_{z_1} Q) z_2 + \frac{\partial \overline{n}}{\partial \phi} z_2 \right] = 0$$

so that we must have

$$L_{z_1} Q + \frac{\partial \overline{n}}{\partial \phi} = 0 \tag{19}$$

$$\mathrm{Re}\left[ L_{z_1} n \right] = 0 \tag{20}$$

Integrating (19), we have

$$n = -(L_{\overline{z}_1} \overline{Q})\phi + R(x, y) \tag{21}$$

and now substituting into (20)

$$\mathrm{Re}\left[ -(L_{z_1} L_{\overline{z}_1} \overline{Q})\phi + L_{z_1} R \right] = 0$$

Since Q and R do not depend on $\phi$

$$\mathrm{Re}\left[ L_{z_1} L_{\overline{z}_1} \overline{Q} \right] = 0 \tag{22}$$

$$\mathrm{Re}\left[ L_{z_1} R \right] = 0 \tag{23}$$

From property 7, (22) states

$$L_{\overline{z}_1} \overline{Q} = iL_{\overline{z}_1} q(x, y) \qquad \text{where } q \text{ is real}$$

$$\Longrightarrow \quad \overline{Q} = iq + \text{ analytic function of } z_1$$

$$Q = -iq + \text{ analytic function of } \overline{z}_1$$

The last term in $Q$ will drop out in the expression for $n$, (21), and in the expression for $p$, (18), as it can be absorbed into $d(\overline{z}_1, z_2, z_3)$, so that we have for $Q$:

$$Q = -iq \tag{24}$$

Again using property 7, (23) yields

$$R = iL_{\overline{z}_1} r(x, y) \ , \qquad r \text{ real} \tag{25}$$

Hence the expression for $g$ becomes (from (15) ),

$$g = -i(L_{\overline{z}_1} q)\phi + iL_{\overline{z}_1} r - iqz_2$$

We can absorb $g$ completely in the first two terms of $b$, (12), by absorbing $-q\phi + r$ into $a$, so that $b$ finally becomes

$$b = iL_{\overline{z}_1} a + i \frac{\partial a}{\partial \phi} z_2 \tag{26}$$

And now we can finally write down for $V$:

$$V = i\left[L_{z_2} a(x, y, \phi, \phi_x, \phi_y)\right] z_3 + iL_{\overline{z}_1} a + i\frac{\partial a}{\partial \phi} z_2 + d(\overline{z}_1, z_2, z_3) \tag{27}$$

where $a(x, y, \phi, \phi_x, \phi_y)$ is any real function of $x, y, \phi, \phi_x, \phi_y$ and $d(\overline{z}_1, z_2, z_3)$ is any jointly analytic function of $\overline{z}_1, z_2, z_3$.

Transforming back to real variables we have for $V_1$ and $V_2$ :

$$V_1 = -\frac{\partial a}{\partial \phi_x}\phi_{xy} + \frac{\partial a}{\partial \phi_y}\phi_{xx} - \frac{\partial a}{\partial y} - \frac{\partial a}{\partial \phi}\phi_y + \mathrm{Re}\, d(\overline{z}_1, z_2, z_3)$$

$$V_2 = \frac{\partial a}{\partial \phi_x}\phi_{xx} + \frac{\partial a}{\partial \phi_y}\phi_{xy} + \frac{\partial a}{\partial x} + \frac{\partial a}{\partial \phi}\phi_x + \ell m\, d(\overline{z}_1, z_2, z_3)$$

or

$$V_1 = -\frac{d}{dy} a(x, y, \phi, \phi_x, \phi_y) + \mathrm{Re}\, d(\overline{z}_1, z_2, z_3) \tag{28}$$

$$V_2 = \frac{d}{dx} a(x, y, \phi, \phi_x, \phi_y) + \ell m\, d(\overline{z}_1, z_2, z_3) \tag{29}$$

where $\frac{d}{dx}$ , $\frac{d}{dy}$ denote partial derivatives when $\phi$, $\phi_x$, $\phi_y$ are functions of $x$ and $y$. The first terms in (28) and (29) make up a trivial divergence-free vector because it's divergence-free for any function $\phi(x, y)$ whether or not it satisfies Laplace's equation. The interesting terms relative to Laplace's equation are the last terms in (28) and (29). We can see how they came about as follows: The statement that $\underline{V}$ be divergence-free is equivalent to

$$\mathrm{Re}\left[L_{z_1} V\right] = 0 \tag{30}$$

where total derivatives with respect to $x$ and $y$ are now taken

We note that if $V$ is any analytic function of $\overline{z}_1$, (30) will be satisfied since

$$L_{z_1} f(\overline{z}_1) = 0$$

where $f$ is any analytic function of $\overline{z}_1$. Now Laplace's equation states

$$\nabla^2 \phi = 0 \implies L_{z_1} L_{\overline{z}_1} \phi = 0 \implies L_{\overline{z}_1} \phi = f(\overline{z}_1)$$

But $L_{\overline{z}_1} \phi = \phi_x + i\phi_y = z_2$ so that $z_2$ is an analytic function of $\overline{z}_1$. Also any derivative of an analytic function of $\overline{z}_1$ will be an analytic function of $\overline{z}_1$, so that $\frac{dz_2}{d\overline{z}_1} = \frac{d}{dx}(\phi_x + i\phi_y) = \phi_{xx} + i\phi_{xy} = z_3$ is also an analytic function of $\overline{z}_1$. Therefore any analytic function of $\overline{z}_1, z_2$ and $z_3$ will be an analytic function of $\overline{z}_1$ and hence will give a divergence-free vector.

We could generalize our results to include higher derivatives of $\phi$, but from the above discussion it is not hard to see what the results would be.

It is interesting to compare these results with those of Noether's theorem concerning equations which come from a variational principle. Let $u_a$, $a = 1, \ldots, n$, be functions of the $m$ independent variables $x_1, \ldots, x_m$. Let $J[\underline{u}]$ be a functional of $\underline{u}$:

$$J[\underline{u}] = \int_D L(x_i, u_a, u_{a,j}) \, d\underline{x}$$

where $u_{a,j} = \frac{\partial u_a}{\partial x_j}$ and the integral is a volume integral over some arbitrary domain $\mathcal{D}$ in $\underline{x}$-space. If we consider the first variation in $J[\underline{u}]$ and require it to vanish, $u_a$ must satisfy

$$\sum_{k=1}^{m} \frac{d}{dx_k}\left(\frac{\partial L}{\partial u_{a,k}}\right) - \frac{\partial L}{\partial u_a} = 0 \qquad a = 1, \ldots, n \qquad (*)$$

whichare the Euler-Lagrange equations. Assuming that there exists an L such that the equations for $u_a$ take the above form, these equations are said to come from a variational principle and any solution $u_a(\underline{x})$ is said to be an extremal.

Consider now a one-parameter group of transformations:

$$x_i' = x_i + X_i\left(\underline{x}, \underline{u}, \frac{\partial \underline{u}}{\partial \underline{x}} ; \epsilon\right)$$

$$u_a' = u_a + \Pi_a\left(\underline{x}, \underline{u}, \frac{\partial \underline{u}}{\partial \underline{x}} ; \epsilon\right)$$

such that at $\epsilon = 0$, $X_i = \Pi_a = 0$. If $\underline{u}$ is a function of $\underline{x}$, then the above transformations induce a function $\underline{u}'$ of $\underline{x}'$ and $\epsilon$; the domain $\mathcal{D}$ in $\underline{x}$-space goes into a domain $\mathcal{D}'(\epsilon)$ in $\underline{x}'$-space. Noether's theorem tells us that if $J[\underline{u}]$ is invariant under this group of transformations, i. e.

$$\int_{\mathcal{D}} L\left(\underline{x}, \underline{u}, \frac{\partial \underline{u}}{\partial \underline{x}}\right) d\underline{x} = \int_{\mathcal{D}'} L\left(\underline{x}', \underline{u}', \frac{\partial \underline{u}'}{\partial \underline{x}'}\right) d\underline{x}'$$

for any $\underline{u}(\underline{x})$ which is an extremal, we obtain a conservation law for the system of equations (*). What's more, every conservation law of

(*), involving no higher than first derivatives, can be obtained from some one-parameter group of transformations which leaves $J[\underline{u}]$ invariant. To exhibit the conservation law associated with the above transformation, let

$$X_i = \left.\frac{\partial \mathfrak{X}_i}{\partial \epsilon}\right|_{\epsilon=0}, \quad U_a = \left.\frac{\partial \mathfrak{U}_a}{\partial \epsilon}\right|_{\epsilon=0}$$

($x_i + X_i\epsilon$ and $u_a + U_a\epsilon$ are called the infinitesimal transformations) and

$$T_k = \sum_{a=1}^{n} \sum_{j=1}^{m} \left[ \frac{\partial L}{\partial u_{a,k}} \left( U_a - u_{a,j} X_j \right) \right] + L X_k$$

Then

$$\sum_{k=1}^{m} \frac{dT_k}{dx_k} = 0$$

For Laplace's equation, $\phi_{xx} + \phi_{yy} = 0$, we have for $J[\phi]$ :

$$J[\phi] = \int \int \tfrac{1}{2} \left( \phi_x^2 + \phi_y^2 \right) dx\, dy$$

with the corresponding infinitesimal transformations

$$x' = x + X(x, y, \phi, \phi_x, \phi_y)\, \epsilon$$

$$y' = y + Y(x, y, \phi, \phi_x, \phi_y)\, \epsilon$$

$$\phi' = \phi + \Phi(x, y, \phi, \phi_x, \phi_y)\, \epsilon$$

$J[\phi]$ is invariant under the above transformations for all $\phi$ satisfying Laplace's equation if and only if the following vector $\underline{T} = (T_1, T_2)$ is divergence-free:

$$T_1 = \phi_x(\Phi - \phi_x X - \phi_y Y) + \tfrac{1}{2}(\phi_x^2 + \phi_y^2) X$$

$$= \phi_x \Phi - \tfrac{1}{2}(\phi_x^2 - \phi_y^2) X - \phi_x \phi_y Y$$

$$T_2 = \phi_y(\Phi - \phi_x X - \phi_y Y) + \tfrac{1}{2}(\phi_x^2 + \phi_y^2) Y$$

$$= \phi_y \Phi - \phi_x \phi_y X - \tfrac{1}{2}(\phi_y^2 - \phi_x^2) Y$$

Letting $T = T_1 + iT_2$, we have

$$T = (\phi_x + i\phi_y)\Phi - \tfrac{1}{2}(\phi_x + i\phi_y)^2 (X - iY)$$

From our previous work $\underline{T}$ will be a divergence-free vector depending $x, y, \phi, \phi_x$, and $\phi_y$ if and only if $T$ has the form

$$T = F(x-iy, \phi_x + i\phi_y) + i\left(\frac{da}{dx}(x, y, \phi) + i\,\frac{da}{dy}(x, y, \phi)\right)$$

where $F$ is analytic in $x-iy$ and $\phi_x + i\phi_y$ and $a(x, y, \phi)$ is a real function. We can then solve for $X, Y, \Phi$ to obtain groups of transformations leaving $J[\phi]$ invariant. Notice that since there are two equations for the three unknowns $X, Y$, and $\Phi$, we will have infinitely many transformations corresponding to the same conservation law.

Consider now the surface integral of $\underline{T}$:

$$\int_c T_1\,dy - T_2\,dx = \int_c (\mathrm{Re}F)\,dy - (\ell m F)\,dx \qquad \text{c: closed curve in } (x,y) \text{ plane}$$

$$= \ell m \int_c \overline{F}(x-iy, \phi_x + i\phi_y)\,dz \quad \text{where } z = x+iy$$

$$= \ell m \int_c G(x+iy, \phi_x - i\phi_y)\,dz$$

where $\overline{F}(x-iy, \phi_x+i\phi_y) = G(x+iy, \phi_x - i\phi_y)$. Note that if there are no singularities in the solution on and within $c$, then $\phi_x - i\phi_y$ is analytic in $x+iy$ and hence the surface integral is zero. If we let

$$s_1 = \ell m \int_c -iG\,dz = \mathrm{Re}\int_c G\,dz \qquad \text{since } -iG \text{ is analytic in } x+iy \text{ and } \phi_x - i\phi_y$$

$$s_2 = \ell m \int_c G\,dz\ ,$$

we then associate two constants $s = s_1 + is_2 = \int_c G\,dz$ with a given function $G(x+iy, \phi_x - i\phi_y)$ and a given curve $c$.

As an example consider irrotational flow past a body with a uniform velocity at infinity. Then the velocity components $u, v$ are $\phi_x$ and $\phi_y$ respectively. The flow near infinity is assumed asymptotically expanded as

$$W = u - iv = A_0 + \frac{A_1}{z} + \frac{A_2}{z^2} + \cdots \qquad \text{as } z \to \infty$$

If we take a curve $c_0$ around the body and another curve $c_1$ enclosing the body far away from it, then since there are no singularities between $c_0$ and $c_1$

$$\int_{c_0} G\,dz = \int_{c_1} G\,dz$$

Letting $G = (u-iv)\,z^n$, we get

$$\int_{c_0} (u-iv)\,z^n\,dz = (A_{n+1})\;2\pi i$$

so that if we know the flow at the body or simply the values of $\int_{c_0} (u-iv)\,z^n dz$, we can find the flow far away from the body.

# Chapter VI

## Incompressible, Inviscid, Steady-state Two-dimensional Flow, Incorporating First Derivatives

The equations given for the two-dimensional flow are

$$q_{1,1} + q_{2,2} = 0 \tag{1}$$

$$q_1 q_{1,1} + q_2 q_{1,2} + p_{,1} = 0 \tag{2}$$

$$q_1 q_{2,1} + q_2 q_{2,2} + p_{,2} = 0 \tag{3}$$

For convenience we change from $p$ to $h = p + \frac{1}{2}(q_1^2 + q_2^2)$ so that (2) and (3) become

$$h_{,1} = q_2(q_{2,1} - q_{1,2}) \tag{2'}$$

$$h_{,2} = q_1(q_{1,2} - q_{2,1}) \tag{3'}$$

If we solve for the $x_2$ derivatives of $q_1$, $q_2$, and $h$ we get

$$q_{1,2} = q_{2,1} - \frac{h_{,1}}{q_2} \tag{4}$$

$$q_{2,2} = -q_{1,1} \tag{5}$$

$$h_{,2} = -\frac{q_1}{q_2} h_{,1} \tag{6}$$

We also note that by differentiating (2) with respect to $x_2$ and (3) with respect to $x_1$ we obtain

$$q_1(q_{2,11} - q_{1,21}) + q_2(q_{2,12} - q_{1,22}) = 0 \qquad (7)$$

(7) states that the gradient of the vorticity, $(q_{2,1} - q_{1,2})$, is perpendicular to the velocity field.

We may, according to our previous theorem, choose $q_1, q_2 \neq 0$, $h$, $q_{1,1}$, $q_{2,1}$, $h_{,1}$, $q_{1,11}$, $q_{2,11}$, and $h_{,11}$ arbitrarily at some arbitrary point $x_1, x_2$ and a solution to (4), (5), and (6) will exist. The choice of $h_{,1}$ is equivalent to a choice of $q_{1,2}$. Also, a choice of $h_{,11}$ is equivalent to a choice of $q_{2,11} - q_{1,21}$ so that a choice of $q_{2,11}$ is equivalent to a choice of $q_{1,21}$. We could have also solved (1), (2'), and (3') for the $x_1$ derivatives of the dependent variables. With a similar argument as above we could choose arbitrarily at some arbitrary point $x_1, x_2$, the values of $q_1 \neq 0$, $q_2$, $h$, $q_{1,1}$, $q_{2,1}$, $q_{1,2}$, $q_{1,11}$, $q_{1,21}$, and $q_{2,12} - q_{1,22}$. Summing up we state:

Theorem: There exists a solution of (1), (2), (3) where at some arbitrary point $x_1, x_2$ the values of $q_1$, $q_2$ $(q_1^2 + q_2^2 \neq 0)$ $h$, $q_{1,1}$, $q_{2,1}$, $q_{1,2}$, $q_{1,11}$, $q_{1,21}$, and the magnitude of the gradient of $q_{2,1} - q_{1,2}$ may be chosen arbitrarily. The last conclusion makes use of equation (7).

We now look for a vector $\underline{V} = \text{fcn}(x_1, x_2, q_1, q_2, h, q_{1,1}, q_{2,1}, q_{1,2})$ (all other gradients can be solved for in terms of the above) such that $\text{div}\,\underline{V} = 0$ for all $q_1(x_1, x_2)$, $q_2(x_1, x_2)$, and $h(x_1, x_2)$ which are solutions of (1), (2), and (3). Since $\underline{V}$ will be considered not only a continuous function of its variables but as smooth a function as necessary, we may exclude the case $q_1 = q_2 = 0$, because this can be obtained by taking the limit as $q_1 \to 0$ and $q_2 \to 0$. Forming the divergence and setting it equal to zero we have

$$\frac{\partial V_i}{\partial x_i} + \frac{\partial V_i}{\partial q_j} q_{j,i} + \frac{\partial V_i}{\partial h} h_{,i} + \frac{\partial V_i}{\partial q_{j,k}} q_{j,ki} = 0$$

j and k not equal to 2 simultaneously

$$= \frac{\partial V_i}{\partial x_i} + \left(\frac{\partial V_i}{\partial q_j} + \frac{\partial V_i}{\partial h} q_j - \frac{\partial V_j}{\partial h} q_i\right) q_{j,i} + \frac{\partial V_i}{\partial q_{j,k}} q_{j,ki} = 0$$

We keep all the variables that $\underline{V}$ depends on fixed but arbitrary. By choosing $q_i$ to be a linear function so that $q_{j,ki} = 0$ we obtain

$$\frac{\partial V_i}{\partial x_i} + \left(\frac{\partial V_i}{\partial q_j} + \frac{\partial V_i}{\partial h} q_j - \frac{\partial V_j}{\partial h} q_i\right) q_{j,i} = 0 \tag{8}$$

and

$$\frac{\partial V_i}{\partial q_{j,k}} q_{j,ki} = 0 \qquad \text{j and k not equal to 2 simultaneously} \tag{9}$$

By the above theorem, we may choose $q_{1,11} = -1$, $q_{1,21} = 0$ and the gradient of $q_{2,1} - q_{1,2}$ equal to zero so that

$$1 = -q_{1,11} = q_{2,21} = q_{2,12} = q_{1,22}$$

$$0 = q_{1,21} = q_{2,11}$$

$$= q_{1,12} = -q_{2,22}$$

Substituting into (9) gives

$$- \frac{\partial V_1}{\partial q_{1,1}} + \frac{\partial V_2}{\partial q_{2,1}} + \frac{\partial V_2}{\partial q_{1,2}} = 0 . \tag{10}$$

Similarly, we now choose $q_{1,11} = 0$, $q_{1,21} = 1$ and the gradient of $q_{2,1} - q_{1,2}$ equal to zero and obtain

$$\frac{\partial V_2}{\partial q_{1,1}} + \frac{\partial V_1}{\partial q_{2,1}} + \frac{\partial V_1}{\partial q_{1,2}} = 0 \tag{11}$$

Substituting (10) and (11) back into (9) we are left with

$$\frac{\partial V_1}{\partial q_{2,1}} (q_{2,11} - q_{1,21}) - \frac{\partial V_2}{\partial q_{1,2}} (q_{2,12} - q_{1,22}) = 0 \tag{12}$$

The fact that we may choose the magnitude of the gradient of $q_{2,1} - q_{1,2}$ to be non-zero combined with equations (7) and (12) allows us to conclude

$$\frac{\partial V_1}{\partial q_{2,1}} q_2 + \frac{\partial V_2}{\partial q_{1,2}} q_1 = 0 \tag{13}$$

Equations (8), (10), (11), and (13) are the necessary conditions for div $\underline{V} = 0$. They are also seen to be sufficient conditions.

Equations (10) and (11) can best be understood by a change of variables. Let

$$\xi = q_{1,1} \qquad \frac{\partial}{\partial q_{1,1}} = \frac{\partial}{\partial \xi}$$

$$\eta = \frac{q_{2,1} + q_{1,2}}{2} \qquad \frac{\partial}{\partial q_{1,2}} = \frac{1}{2} \frac{\partial}{\partial \eta} - \frac{1}{2} \frac{\partial}{\partial \zeta}$$

$$\zeta = \frac{q_{2,1} - q_{1,2}}{2} \qquad \frac{\partial}{\partial q_{2,1}} = \frac{1}{2} \frac{\partial}{\partial \eta} + \frac{1}{2} \frac{\partial}{\partial \zeta}$$

Then (10) and (11) become, respectively,

$$\frac{\partial V_1}{\partial \xi} = \frac{\partial V_2}{\partial \eta} \tag{14}$$

$$\frac{\partial V_2}{\partial \xi} = - \frac{\partial V_1}{\partial \eta} \tag{15}$$

(14) and (15) state that $V = V_1 + iV_2$ is an analytic function of $z = \xi + i\eta$, $V = V(\underline{x}, \underline{q}, h, \zeta, z)$. Let $x = x_1 + ix_2$, $q = q_1 + iq_2$; $\underline{x}$ is the vector $(x_1, x_2)$, $x$ is the complex variable $x_1 + ix_2$, and $\overline{x}$ is its conjugate; the same applies to $\underline{q}$, $q$, and $\overline{q}$. Equation (13) becomes

$$\mathrm{Re}\left[ \frac{\partial V}{\partial z} q + i \frac{\partial V}{\partial \zeta} \overline{q} \right] = 0 \tag{16}$$

Since $V$ is analytic in $z$, (16) implies there exists a real $H(\underline{x}, \underline{q}, h, \zeta)$ such that

$$\frac{\partial V}{\partial z} q + i \frac{\partial V}{\partial \zeta} \overline{q} = i \frac{\partial H}{\partial \zeta} (\underline{x}, \underline{q}, h, \zeta) \, q \overline{q} \tag{17}$$

(17) can be solved like a first order P. D. E. in real variables. Let $F(\underline{x}, \underline{q}, h, \zeta, z) = V - Hq$. Note: $Hq$ is a particular solution of (17). The equation for $F$ is

$$\frac{\partial F}{\partial z} q + i \frac{\partial F}{\partial \zeta} \overline{q} = 0 \tag{18}$$

Change variables from $\zeta, z \rightarrow \zeta, W$ where

$\zeta = \zeta$

$W = z + i\frac{q}{\bar{q}}\zeta$ Note: F is analytic in W for fixed $\zeta$

$$\left.\frac{\partial F}{\partial z}\right|_{\zeta \text{ fixed}} = \left.\frac{\partial F}{\partial W}\right|_{\zeta \text{ fixed}}$$

$$\left.\frac{\partial F}{\partial \zeta}\right|_{z \text{ fixed}} = i\frac{q}{\bar{q}}\left.\frac{\partial F}{\partial W}\right|_{\zeta \text{ fixed}} + \left.\frac{\partial F}{\partial \zeta}\right|_{W \text{ fixed}}$$

Equation (18) becomes

$$\frac{\partial F}{\partial \zeta} = 0 \quad \text{where we are in } \zeta, W \text{ coordinates}$$

$$\Longrightarrow F = F(\underline{x}, \underline{q}, h, W), \quad F \text{ analytic in } W$$

We now have for V:

$$V = H(\underline{x}, \underline{q}, h, \zeta)q + F(\underline{x}, \underline{q}, h, W), \quad H \text{ real} \tag{19}$$

Equation (8) becomes after transforming to $\xi, \eta$, and $\zeta$ variables

$$\frac{\partial V_1}{\partial x_1} + \frac{\partial V_2}{\partial x_2} + \left(\frac{\partial V_1}{\partial q_1} - \frac{\partial V_2}{\partial q_2}\right)\xi + \left(\frac{\partial V_1}{\partial q_2} + \frac{\partial V_2}{\partial q_1}\right)\eta$$

$$+ \left(\frac{\partial V_1}{\partial q_2} - \frac{\partial V_2}{\partial q_1} + 2\frac{\partial V_1}{\partial h}q_2 - 2\frac{\partial V_2}{\partial h}q_1\right)\zeta = 0 \tag{20}$$

We now use the L operator notation and (20) transforms to

$$\operatorname{Re}\left\{ L_x V + (L_{\overline{q}} V)\overline{z} + i(L_q V + 2\frac{\partial V}{\partial h}\overline{q})\zeta \right\} = 0 \tag{21}$$

which after substitution of (19) becomes

$$\operatorname{Re}\left\{ \overline{q} L_{\overline{x}} H + L_x F + \left[ q L_{\overline{q}} H + L_{\overline{q}} F - 2i \frac{q}{\overline{q}} \zeta \frac{\partial F}{\partial W} \right] \overline{W} \right.$$

$$\left. + i \frac{\overline{q}}{q} \zeta L_{\overline{q}} F + i\zeta L_q F + 2i\, \overline{q} q \zeta \frac{\partial F}{\partial h} \right\} = 0 \tag{22}$$

We eliminate all terms involving $F$ by taking the $\zeta$ derivative of (22) twice yielding

$$\operatorname{Re}\left\{ \overline{q}\, L_{\overline{x}} \frac{\partial^2 H}{\partial \zeta^2} + \overline{q} L_q \frac{\partial^2 H}{\partial \zeta^2}\, W \right\} = 0 \tag{23}$$

Since $H$ does not involve $W$, we must have

$$L_q \frac{\partial^2 H}{\partial \zeta^2} = 0 \tag{24}$$

$$\operatorname{Re}\left\{ \overline{q}\, L_{\overline{x}} \frac{\partial^2 H}{\partial \zeta^2} \right\} = 0 \tag{25}$$

Since $H$ is real (24) implies

$$\frac{\partial^2 H}{\partial \zeta^2} = \text{fcn}(\underline{x}, h, \zeta)$$

This statement coupled with (25) gives

$$L_{\overline{x}} \frac{\partial^2 H}{\partial \zeta^2} = 0 \implies \frac{\partial^2 H}{\partial \zeta^2} = \text{fcn}(h, \zeta) \implies H = a(h, \zeta) + b(\underline{x}, \underline{q}, h)\zeta \tag{26}$$

where a and b are both real; the constant term in $\zeta$ can be incorporated in F.

When (26) is substituted back into (22), we get a linear expression in $\zeta$ equated to zero, so that the coefficient of the $\zeta$ term along with terms not involving $\zeta$ must each be zero, i.e.

$$\mathrm{Re}\left\{ \bar{q}L_{\underline{x}}b + \bar{q}WL_{\underline{q}}b - 2i\frac{q}{\bar{q}^2}\bar{W}\frac{\partial F}{\partial W} + i\frac{\bar{q}}{q}L_{\underline{q}}F + iL_qF + 2i\bar{q}\frac{\partial F}{\partial h}\right\} = 0 \tag{27}$$

$$\mathrm{Re}\left\{ L_xF + \bar{W}L_{\underline{q}}F \right\} = 0 \tag{28}$$

Except for the term involving $\bar{W}$, each term in (27) is an analytic function of W. As in preceding arguments, we must have

$$\mathrm{Re}\left[ -2i\frac{q}{\bar{q}^2}\frac{\partial^2 F}{\partial W^2}\right] = 0$$

Thus there exists a real $A(\underline{x}, \underline{q}, h)$ such that

$$-2i\frac{q}{\bar{q}^2}\frac{\partial^2 F}{\partial W^2} = -2iAq\bar{q}$$

or $$\frac{\partial^2 F}{\partial W^2} = A\bar{q}^3$$

$$\Longrightarrow \quad F = \tfrac{1}{2}A(\underline{x}, \underline{q}, h)W^2 + B(\underline{x}, \underline{q}, h)W + C(\underline{x}, \underline{q}, h) \tag{29}$$

where B and C are complex. Similarly (28) implies

$$\mathrm{Re}\left[\frac{\partial}{\partial W}L_{\underline{q}}F\right] = 0$$

which becomes using (29)

$$\mathrm{Re}\left[ W L_{\bar{q}}(A\bar{q}^3) + L_{\bar{q}} B \right] = 0$$

so that

$$L_{\bar{q}}(A\bar{q}^3) = 0 \qquad (30)$$

$$\mathrm{Re}\, L_{\bar{q}} B = 0 \qquad (31)$$

(31) implies that

$$B = iL_q \alpha(\underline{x}, \underline{q}, h), \quad \alpha \text{ real} \qquad (32)$$

When (29) is substituted back into (27) we get

$$\mathrm{Re}\Big\{ \bar{q} L_{\bar{x}} b + \bar{q} W L_q b + 2i \frac{\bar{q}}{q^2} W\bar{B} + i \frac{\bar{q}}{q}(L_{\bar{q}} B) W + i \frac{\bar{q}}{q} L_{\bar{q}} C$$

$$+ i\left[ \tfrac{1}{2} L_q(A\bar{q}^3) W^2 + (L_q B) W + L_q C \right]$$

$$+ 2i\bar{q}\left( \tfrac{1}{2} \frac{\partial A}{\partial h} \bar{q}^3 W^2 + \frac{\partial B}{\partial h} W + \frac{\partial C}{\partial h} \right) \Big\} = 0 \qquad (33)$$

This is a quadratic expression in W so that the coefficient of the $W^2$ term must be zero, i. e.

$$\frac{i}{2} L_q(A\bar{q}^3) + i\bar{q}^4 \frac{\partial A}{\partial h} = 0 \quad \Longrightarrow \quad L_q A + 2\bar{q} \frac{\partial A}{\partial h} = 0 \qquad (34)$$

Now, (30) states

$$qL_q A + 6A = 0 \tag{35}$$

so that by (34)

$$\frac{\partial A}{\partial h} = \frac{3}{q\overline{q}} A \implies A = \tilde{A}(\underline{x}, \underline{q}) e^{\frac{3}{q\overline{q}} h}$$

But by (35)

$$qL_q \tilde{A} - \frac{6h}{q\overline{q}} \tilde{A} + 6\tilde{A} = 0$$

which implies that $\tilde{A} = 0$, so that $A = 0$. Thus $F$ is a linear function of $W$,

$$F = B(\underline{x}, \underline{q}, h) W + C(\underline{x}, \underline{q}, h) \tag{36}$$

The coefficient of the $W$ term of (33) must also vanish giving

$$L_q b + 2i \frac{\overline{B}}{q^2} + \frac{i}{q} L_{\overline{q}} B + \frac{i}{\overline{q}} L_q B + 2i \frac{\partial B}{\partial h} = 0 \tag{37}$$

Since $\mathrm{Re}\, L_{\overline{q}} B = 0$, $\overline{L_{\overline{q}} B} = -L_{\overline{q}} B$ and $L_{\overline{q}}(\frac{B}{\overline{q}}) = \frac{1}{\overline{q}} L_{\overline{q}} B - \frac{2}{\overline{q}^2} B$

so that by taking the complex conjugate of (37) we have

$$L_{\overline{q}} \left[ b + \frac{iB}{\overline{q}} - \frac{i\overline{B}}{q} \right] - 2i \frac{\partial \overline{B}}{\partial h} = 0 \tag{38}$$

Now, using (32), (38) becomes

$$L_{\bar{q}}\left[ b - \frac{L_q \alpha}{\bar{q}} - \frac{L_{\bar{q}} \alpha}{q} - 2\frac{\partial \alpha}{\partial h} \right] = 0$$

so that

$$b = \frac{L_q \alpha}{\bar{q}} + \frac{L_{\bar{q}} \alpha}{q} + 2\frac{\partial \alpha}{\partial h} \qquad \text{Note: } \frac{L_q \alpha}{\bar{q}} + \frac{L_{\bar{q}} \alpha}{q} \text{ is real} \tag{39}$$

where the function of $\underline{x}$ and $h$ coming from the integration can be absorbed into $\alpha(\underline{x}, \underline{q}, h)$. After the elimination of the $W^2$ and $W$ terms in (33), we have

$$\mathrm{Re}\left\{ \bar{q} L_{\bar{x}} b + i\frac{\bar{q}}{q} L_{\bar{q}} C + i L_q C + 2i\bar{q}\frac{\partial C}{\partial h} \right\} = 0 \quad , \tag{40}$$

which we shall come back to later.

We are still left with solving (28), which becomes a linear expression in $W$ when (36) is substituted in. Again, the coefficient of $W$ must be set equal to zero along with the real part of the term not involving $W$, i. e.

$$L_x B + \overline{L_{\bar{q}} C} = 0 \quad \Longrightarrow \quad L_{\bar{x}} \bar{B} + L_{\bar{q}} C = 0 \tag{41}$$

$$\mathrm{Re}\, L_x C = 0 \tag{42}$$

Using (32) in (41), we get

$$C = i L_{\bar{x}} \alpha + M(\underline{x}, h, q), \quad \text{where } M \text{ is analytic in } q \tag{43}$$

We now substitute the expression for b, (39), and the expression for C, (43), into (40) giving

$$\mathrm{Re}\left\{ i \frac{\partial M}{\partial q} + i \bar{q} \frac{\partial M}{\partial h} \right\} = 0 \tag{44}$$

Since M is analytic in q, there must exist a real $N(\underline{x},h)$ such that

$$\frac{\partial^2 M}{\partial q \partial h} = \frac{\partial N}{\partial h} (\underline{x}, h)$$

Thus

$$M = N(\underline{x}, h) q + P(\underline{x}, h) + R(\underline{x}, q) \tag{45}$$

where $P(\underline{x}, h)$ is complex and $R(\underline{x}, q)$ is analytic in q. Substituting (45) back into (44) gives

$$\mathrm{Re}\left[ i \frac{\partial R}{\partial q} - i \frac{\partial \overline{P}}{\partial h} q \right] = 0 \tag{46}$$

which implies

$$\frac{\partial^2 R}{\partial q^2} = 0 \implies R = \tfrac{1}{2} \delta(\underline{x}) q^2 + \beta(\underline{x}) q \tag{47}$$

where $\delta(\underline{x})$, $\beta(\underline{x})$ are complex. The term involving $\underline{x}$ alone was absorbed into $P(\underline{x}, h)$. Using (47) in (46) we are left with a linear function of q so that

$$\delta(\underline{x}) - \frac{\partial \overline{P}}{\partial h} = 0$$

$$\text{Re}\left[ i\beta(\underline{x}) \right] = 0 \implies \beta(\underline{x}) \text{ is real.} \tag{48}$$

$\beta(\underline{x})$ may be taken to be zero as the $\beta(\underline{x})q$ term in (47) can be absorbed in $N(\underline{x}, h)q$ in (45) so that

$$R = \tfrac{1}{2}\delta(\underline{x})q^2 \tag{49}$$

From (48)

$$P = \overline{\delta}(\underline{x})h + \epsilon(\underline{x}), \quad \text{where } \epsilon(\underline{x}) \text{ is complex} \tag{50}$$

The final equation to be solved is (42), which becomes after substitution of (43), (45), (49), and (50)

$$\text{Re}\left\{ qL_x N + hL_x\overline{\delta} + L_x\epsilon + \tfrac{1}{2}q^2 L_x\delta \right\} = 0 \tag{51}$$

so that

$$L_x\delta = 0 \tag{52}$$

$$L_x N = 0 \tag{53}$$

$$\text{Re } L_x\overline{\delta} = 0 \implies \text{Re } L_{\overline{x}}\delta = 0 \tag{54}$$

$$\text{Re } L_x\epsilon = 0 \tag{55}$$

(52) implies that $\delta = \delta(\overline{x})$ is analytic in $\overline{x}$ and hence (54) states

$$\operatorname{Re} \frac{\partial \delta}{\partial \overline{x}} = 0 \implies \frac{\partial \delta}{\partial \overline{x}} = im, \quad \text{where } m \text{ is a real constant}$$

so that

$$\delta = im\overline{x} + i\overline{n}\,, \quad \text{where } n \text{ is a complex constant} \tag{56}$$

(53) implies that N is a function of h alone (since N is real) and hence can be absorbed in $a(h, \zeta)$, (26), so that

$$N = 0 \tag{57}$$

(55) gives $\epsilon = iL_{\overline{x}} e(\underline{x})$ where e is real. e may be taken to be zero as it can be absorbed in $\alpha(\underline{x}, \underline{q}, h)$, so that

$$\epsilon(\underline{x}) = 0\,. \tag{58}$$

Thus the expression for F. is, from (36), (32), (43), (45), (49), (50), (56), (57), and (58),

$$F = iL_q \alpha(\underline{x}, \underline{q}, h)W + iL_{\overline{x}}\alpha(\underline{x}, \underline{q}, h) + (-imx + n)h + \tfrac{1}{2}(im\overline{x} + \overline{n})q^2 \tag{59}$$

and thus for V from (19), (26), (39), (59) and $W = z + i\frac{q}{\overline{q}}\zeta$

$$V = a(h, \zeta)q + (-imx - in)h + \tfrac{1}{2}(im\overline{x} + i\overline{n})q^2 + izL_q\alpha + \zeta L_{\overline{q}}\alpha + 2\frac{\partial \alpha}{\partial h}\zeta q + iL_{\overline{x}}\alpha$$

Letting $n = n_1 + in_2$, we then have for $V_1$ and $V_2$:

$$V_1 = a(h,\zeta)q_1 + (mx_2 + n_2)h + \tfrac{1}{2}(mx_2 + n_2)(q_1^2 - q_2^2) - (mx_1 + n_1)q_1 q_2 - \frac{d}{dx_2}\alpha(\underline{x}, \underline{q}, h)$$

$$V_2 = a(h,\zeta)q_2 - (mx_1 + n_1)h + \tfrac{1}{2}(mx_1 + n_1)(q_1^2 - q_2^2) + (mx_2 + n_2)q_1 q_2 + \frac{d}{dx_1}\alpha(\underline{x}, \underline{q}, h)$$

Except for the addition of a trivial divergence-free vector, $\left(-\frac{d}{dx_2}\alpha, \frac{d}{dx_1}\alpha\right)$, allowing $\underline{V}$ to contain gradients of $\underline{q}$ changes only the term $a(h)\underline{q}$ to $a(h,\zeta)\,\underline{q}$. The fact that the vorticity, $\zeta$, is constant along streamlines accounts for this term.

# Chapter VII

## First Order Equations

In this chapter, we will be considering functions of the two independent variables $x_1$ and $x_2$, the dependent variable $u(x_1, x_2)$, and $x_1$-derivatives of u. Let $F^n(x_1, x_2, u^{(0)}, u^{(1)}, \ldots, u^{(n)})$ be such a function, where $u^{(j)} = \frac{\partial^j u}{\partial x_1{}^j}$ and the superscript n of $F^n$ refers to the fact that $F^n$ depends on $x_1$-derivatives of u up to order n. $F^n$ is assumed to be a sufficiently smooth function of its arguments, which are to be taken as independent variables of $F^n$. For a given function $f(x_1, x_2, u, u^{(1)})$, we define two operators $D^p_{x_1}$ and $D^p_{x_2}$:

$$D^p_{x_1} F^n = \frac{\partial F^n}{\partial x_1} + \sum_{m=0}^{p} u^{(m+1)} \frac{\partial F^n}{\partial u^{(m)}} \tag{1}$$

$$D^p_{x_2} F^n = \frac{\partial F^n}{\partial x_2} + \sum_{m=0}^{p} \frac{d^m f}{dx_1{}^m} \frac{\partial F^n}{\partial u^{(m)}} \tag{2}$$

where

$$\frac{df}{dx_1} = D^1_{x_1} f \qquad\qquad \text{Note: If } p > n$$

$$\frac{d^2 f}{dx_1^2} = D^2_{x_1} \frac{df}{dx_1} \qquad\qquad D^p_{x_1} F^n = D^n_{x_1} F^n$$

$$\frac{d^m f}{dx_1{}^m} = D^m_{x_1} \frac{d^{m-1} f}{dx_1{}^m} \qquad\qquad D^p_{x_2} F^n = D^n_{x_2} F^n$$

If we think of u as being a function of $x_1$, $x_2$, then $D^n_{x_1} F^n$ is the total partial derivative of $F^n$ with respect to $x_1$, $\frac{dF^n}{dx_1}$, and the notation $\frac{df}{dx_1}, \frac{d^2 f}{dx_1^2}, \ldots, \frac{d^n f}{dx_1^n}$ has the obvious interpretation of, respectively, the first, second, ..., and nth total partial derivative of $f$ with respect to $x_1$. If we now know that $u(x_1, x_2)$ is a solution to the first order partial differential equation

$$\frac{\partial u}{\partial x_2} = f\left(x_1, x_2, u, \frac{\partial u}{\partial x_1}\right) , \tag{3}$$

then $D^n_{x_2} F^n = \frac{dF^n}{dx_2}$.

The operators $D_{x_1}$ and $D_{x_2}$ obey the same addition and multiplication rules as the normal differential operator. One other important property is the following:

$$D^{n+1}_{x_1}\left(D^n_{x_2} F^n\right) = D^{n+1}_{x_2}\left(D^n_{x_1} F^n\right) \tag{4}$$

To prove (4) we let

$$D^{n+1}_{x_1} D^n_{x_2} F^n = P^{n+2}\left(x_1, x_2, u, \ldots, u^{(n+2)}\right)$$

$$D^{n+1}_{x_2} D^n_{x_1} F^n = Q^{n+2}\left(x_1, x_2, u, \ldots, u^{(n+2)}\right)$$

If $u(x_1, x_2)$ is any solution of (3) then

$$P^{n+2}\left(x_1,x_2,u(x_1,x_2), \ldots, u^{(n+2)}(x_1,x_2)\right) = Q^{n+2}\left(x_1,x_2,u(x_1,x_2), \ldots, u^{(n+2)}(x_1,x_2)\right) \tag{5}$$

since (4) then states

$$\frac{d}{dx_1}\frac{d}{dx_2}F^n = \frac{d}{dx_2}\frac{d}{dx_1}F^n, \quad F^n \text{ function of } x_1, x_2$$

which is just the commutativity property of the differential operators $\frac{d}{dx_1}$ and $\frac{d}{dx_2}$. By a previous theorem, there exists a solution of (3) such that at any arbitrary but fixed point $x_{1_0}$, $x_{2_0}$, the values of $u, \ldots, u^{(n+2)}$ are equal to $u_0, \ldots, u_0^{(n+2)}$, respectively, $u_0, \ldots, u_0^{(n+2)}$ being completely arbitrary numbers. Thus at $x_{1_0}$, $x_{2_0}$ (5) states

$$\overset{n+2}{P}\left(x_{1_0}, x_{2_0}, u_0, \ldots, u_0^{(n+2)}\right) = \overset{n+2}{Q}\left(x_{1_0}, x_{2_0}, u_0, \ldots, u_0^{(n+2)}\right) \quad (6)$$

The arbitrariness of the arguments of $P^{n+2}$ and $Q^{n+2}$ in (6) then proves (4).

We now look for all vectors $\overset{n}{\underline{V}}\left(x_1, x_2, u, \ldots, u^{(n)}\right), \overset{n}{\underline{V}} = \left(\overset{n}{V}_1, \overset{n}{V}_2,\right)$, such that $\operatorname{div}\overset{n}{\underline{V}} = 0$ for all $u$ which are solutions of (3). We need not include $x_2$-derivatives in $\overset{n}{\underline{V}}$ since they can be solved for in terms of $x_1$-derivatives. Since $u$ is now thought of as a function of $x_1$ and $x_2$, we have for the divergence

$$\overset{n}{D}_{x_1}\overset{n}{V}_1 + \overset{n}{D}_{x_2}\overset{n}{V}_2 = 0 \quad (7)$$

Since $x_1, x_2, u, \ldots, u^{(n+1)}$ can be chosen independently, (7) is an equation for the $n+4$ above independent variables. At this point we will now assume that (3) is a quasi-linear equation, i.e. $\frac{\partial f}{\partial u^{(1)}} = \text{fcn}(x_1, x_2, u)$. $u^{(n+1)}$ will appear linearly in (7). ($\overset{n}{\underline{V}}$ does not depend on $u^{(n+1)}$). Hence its coefficient must vanish identically. The coefficient of $u^{(n+1)}$ in $\overset{n}{D}_{x_1}\overset{n}{V}_1$ is $\frac{\partial V_1^n}{\partial u^{(n)}}$; the coefficient of $u^{(n+1)}$ in $\overset{n}{D}_{x_2}\overset{n}{V}_2$ is

$\frac{\partial f}{\partial u^{(1)}} \frac{\partial \overset{n}{V}_2}{\partial u^{(n)}}$. Thus we must have

$$\frac{\partial \overset{n}{V}_1}{\partial u^{(n)}} + \frac{\partial f}{\partial u^{(1)}} \frac{\partial \overset{n}{V}_2}{\partial u^{(n)}} = 0 \tag{8}$$

In solving (8) we only consider $n \geqslant 1$, so that (8) implies

$$\overset{n}{V}_1 = -\frac{\partial f}{\partial u^{(1)}} \overset{n}{V}_2 + \overset{n-1}{F}\left(x_1, x_2, u, \ldots, u^{(n-1)}\right), \quad n \geqslant 1 \tag{9}$$

Substituting (9) back into (7), we have a first order P. D. E. for $\overset{n}{V}_2$ in terms of $F^{n-1}$:

$$-\frac{\partial f}{\partial u^{(1)}} \overset{n}{D}_{x_1} \overset{n}{V}_2 + \overset{n}{D}_{x_2} \overset{n}{V}_2 - \left(\frac{d}{dx_1} \frac{\partial f}{\partial u^{(1)}}\right) \overset{n}{V}_2 + \overset{n-1}{D}_{x_1} \overset{n-1}{F} = 0. \tag{10}$$

where

$$\frac{d}{dx_1} \frac{\partial f}{\partial u^{(1)}} = \overset{n}{D}_{x_1} \frac{\partial f}{\partial u^{(1)}} = \overset{0}{D}_{x_1} \frac{\partial f}{\partial u^{(1)}}$$

and

$$\overset{n}{D}_{x_1} \overset{n-1}{F} = \overset{n-1}{D}_{x_1} \overset{n-1}{F}$$

(The individual terms in (10) involving $u^{(n+1)}$ will cancel when combined). For each $F^{n-1}$ let $Q^n$ be a solution of (10), i.e.

$$-\frac{\partial f}{\partial u^{(1)}} \overset{n}{D}_{x_1} \overset{n}{Q} + \overset{n}{D}_{x_2} \overset{n}{Q} - \left(\frac{d}{dx_1} \frac{\partial f}{\partial u^{(1)}}\right) \overset{n}{Q} + \overset{n-1}{D}_{x_1} \overset{n-1}{F} = 0 \tag{11}$$

$\overset{n}{Q}$ will be exhibited later. Let $V_2 = \overset{n}{W} + \overset{n}{Q}$. Therefore $\overset{n}{W}$ must satisfy

$$-\frac{\partial f}{\partial u^{(1)}}\,\overset{n}{D}_{x_1}\overset{n}{W} + \overset{n}{D}_{x_2}\overset{n}{W} - \left(\frac{d}{dx_1}\,\frac{\partial f}{\partial u^{(1)}}\right)\overset{n}{W} = 0 \tag{12}$$

Before solving (12), we observe that the solution of

$$-\frac{\partial f}{\partial u^{(1)}}\,\overset{0}{D}_{x_1}\overset{0}{y} + \overset{0}{D}_{x_2}\overset{0}{y} = 0$$

i.e.
$$-\frac{\partial f}{\partial u^{(1)}}\,\frac{\partial \overset{0}{y}}{\partial x_1} + \frac{\partial \overset{0}{y}}{\partial x_2} + \underbrace{\left(f - \frac{\partial f}{\partial u^{(1)}}\,u^{(1)}\right)}_{\substack{\text{independent}\\ \text{of } u^{(1)}}}\frac{\partial \overset{0}{y}}{\partial u} = 0 \tag{13}$$

will have two independent solutions $\overset{0}{y}_1$, $\overset{0}{y}_2$ depending only on $x_1, x_2, u$. We also note that if $y^j$, $j \geqslant 0$, is a solution of

$$-\frac{\partial f}{\partial u^{(1)}}\,\overset{j}{D}_{x_1}\overset{j}{y} + \overset{j}{D}_{x_2}\overset{j}{y} = 0 \tag{14}$$

then by applying the operator $\overset{j+1}{D}_{x_1}$ to (14), we get

$$-\frac{\partial f}{\partial u^{(1)}}\,\overset{j+1}{D}_{x_1}\left(\overset{j}{D}_{x_1}\overset{j}{y}\right) - \left(\frac{d}{dx_1}\,\frac{\partial f}{\partial u^{(1)}}\right)\overset{j}{D}_{x_1}\overset{j}{y} + \overset{j+1}{D}_{x_2}\,\overset{j}{D}_{x_1}\overset{j}{y} = 0$$

using (4). Thus $\overset{j+1}{y} = \overset{j}{D}_{x_1}\overset{j}{y}$ is a solution of

$$-\frac{\partial f}{\partial u^{(1)}}\,\overset{j+1}{D}_{x_1}\overset{j+1}{y} + \overset{j+1}{D}_{x_2}\overset{j+1}{y} - \left(\frac{d}{dx_1}\,\frac{\partial f}{\partial u^{(1)}}\right)\overset{j+1}{y} = 0 \tag{15}$$

Hence a particular solution of (12) will be $w' = D^{0}_{x_1} y^{0}_{1}$. If we now let $W^{n} = w' Y^{n}$, then $Y^{n}$ will satisfy the homogeneous equation

$$- \frac{\partial f}{\partial u^{(1)}} D^{n}_{x_1} Y^{n} + D^{n}_{x_2} Y^{n} = 0 \tag{16}$$

Since (16) is a homogeneous linear first order P. D. E. in the $n+3$ variables $x_1, x_2, u, \ldots, u^{(n)}$, the general solution of (16) will be an arbitrary function of $n+2$ independent solutions of (16). We already know two solutions, $y^{0}_{1}$, $y^{0}_{2}$. From our previous discussion $D^{0}_{x_1} y^{0}_{2}$ will be a solution of (15) and hence $y^{1} = \frac{1}{w'} D^{0}_{x_1} y^{0}_{2}$ will be a solution of (16) since $w'$ is also a solution of (15). In general, if $y^{j}$ is a solution of (16), then

$$y^{j+1} = \frac{1}{w'} D^{j}_{x_1} y^{j} \qquad j = 0, \ldots, n-1 \tag{17}$$

is also a solution. Thus we have $n+2$ solutions, $y^{0}_{1}$, $y^{0}_{2}$, $y^{1}$, ..., $y^{n}$ which can be seen to be independent so that we have for $W^{n}$:

$$W^{n} = w' G\left( y^{0}_{1}, y^{0}_{2}, y^{1}, \ldots, y^{n} \right), \quad \text{G: arbitrary function of its } n+2 \text{ variables} \tag{18}$$

To solve for $Q^{n}$, the form of equation (11) and the preceding discussion suggests that we consider the following equation for $P^{n-1}$

$$- \frac{\partial f}{\partial u^{(1)}} D^{n-1}_{x_1} P^{n-1} + D^{n-1}_{x_2} P^{n-1} + F^{n-1} = 0 \tag{19}$$

We note that for every $F^{n-1}$ there exists a solution $P^{n-1}$ of (19) by the linearity of the equation. Also, since $F^{n-1}$ was an arbitrary function, $P^{n-1}$ is an arbitrary function. Proceeding as above, we apply the operator $D^n_{x_1}$ to (19) giving

$$-\frac{\partial f}{\partial u^{(1)}} D^n_{x_1}\left(\overset{n-1}{D}_{x_1}\overset{n-1}{P}\right) + \overset{n}{D}_{x_2}\left(\overset{n-1}{D}_{x_1}\overset{n-1}{P}\right) - \left(\frac{d}{dx_1}\frac{\partial f}{\partial u^{(1)}}\right)\overset{n-1}{D}_{x_1}\overset{n-1}{P} + \overset{n-1}{D}_{x_1}F^{n-1}=0$$

so that $Q^n = \overset{n-1}{D}_{x_1}\overset{n-1}{P}$ is a solution of (11). Thus the vector $\overset{n}{\underline{V}}$ is, from (9) and the above

$$\overset{n}{V}_1 = -\frac{\partial f}{\partial u^{(1)}} w'G(\overset{0}{y}_1, \overset{0}{y}_2, \overset{1}{y}, \ldots, \overset{n}{y}) - \overset{n-1}{D}_{x_2}\overset{n-1}{P} \tag{20}$$

$$\overset{n}{V}_2 = w'G(\overset{0}{y}_1, \overset{0}{y}_2, \overset{1}{y}, \ldots, \overset{n}{y}) + \overset{n-1}{D}_{x_1}\overset{n-1}{P} \tag{21}$$

Note: The vector $\left(-\overset{n-1}{D}_{x_2}\overset{n-1}{P}, \overset{n-1}{D}_{x_1}\overset{n-1}{P}\right)$ is the trivial divergence-free vector $\left(-\frac{dP^{n-1}}{dx_2}, \frac{dP^{n-1}}{dx_1}\right)$

It is interesting to note that $w'$ is a linear function of $u_{x_1}$.

To give an interpretation to the non-trivial terms of $\underline{V}^n$, we first note that since (3) is quasi-linear it can be written

$$-\frac{\partial f}{\partial u_{x_1}} u_{x_1} + u_{x_2} = f - \frac{\partial f}{\partial u_{x_1}} u_{x_1}, \quad f - \frac{\partial f}{\partial u_{x_1}} u_{x_1} \text{ is independent of } u_{x_1}$$

The characteristics for this equation are the solution of

$$\frac{dx_1}{ds} = -\frac{\partial f}{\partial u_{x_1}} \;, \quad \frac{dx_2}{ds} = 1, \quad \frac{du}{ds} = f - \frac{\partial f}{\partial u_{x_1}} u_{x_1} \tag{22}$$

If the variable $\xi$ is used to denote the characteristic we are on, then (22) has the solution

$$x_1 = X_1(s, \xi), \quad x_2 = X_2(s, \xi), \quad u = U(s, \xi)$$

determined by particular initial data. We could solve $s$ and $\xi$ in terms of $x_1$ and $x_2$ and then substitute into $U(s, \xi|$ to get $u$ as a function of $x_1$ and $x_2$.

Let $u = u(x_1, x_2)$ be a solution of (3). Then if $F(x_1, x_2, u)$ is any function which is constant along characteristics, then

$$\frac{dF}{ds} = 0$$

i.e.

$$-\frac{\partial f}{\partial u_{x_1}} \frac{\partial F}{\partial x_1} + \frac{\partial F}{\partial x_2} + \left( f - \frac{\partial f}{\partial u_{x_1}} u_{x_1} \right) \frac{\partial F}{\partial u} = 0$$

which is the same equation as (13). Thus

$$\overset{0}{y}_1 = F_1(\xi)$$

$$\overset{0}{y}_2 = F_2(\xi)$$

Now

$$\frac{d\overset{0}{y}_1}{dx_1} = F_1'(\xi)\,\frac{d\xi}{dx_1}\ , \quad \frac{d\overset{0}{y}_2}{dx_1} = F_2'(\xi)\,\frac{d\xi}{dx_1}$$

so that

$$y' = \frac{1}{w'}\, \overset{0}{D}_{x_1}\, \overset{0}{y}_2 = \frac{d\overset{0}{y}_2}{dx_1} \Big/ \frac{d\overset{0}{y}_1}{dx_1} = \text{fcn}(\xi)$$

Similarly all $y^j$ , $j = 1, \ldots, n$ are functions of $\xi$. Since

$$\frac{d\xi}{ds} = 0 \implies -\frac{\partial f}{\partial u_{x_1}}\, \frac{d\xi}{dx_1} + \frac{d\xi}{dx_2} = 0\ ,$$

then

$$\frac{d\xi}{dx_2} = \frac{\partial f}{\partial u_{x_1}}\, \frac{d\xi}{dx_1}$$

It can then be seen that the first terms of (20) and (21) can be written respectively as

$$-\frac{\partial f}{\partial u_{x_1}}\, w_1\, f(\overset{0}{y}_1, \overset{0}{y}_2, y', \ldots, y^n) = -\frac{d\xi}{dx_2}\, H(\xi) = -\frac{d}{dx_2}\, h(\xi)$$

$$w_1\, f(\overset{0}{y}_1, \overset{0}{y}_2, y', \ldots, y^n) = \frac{d\xi}{dx_1}\, H(\xi) = \frac{d}{dx_1}\, h(\xi)$$

$$\text{where} \quad h'(\xi) = H(\xi)$$

which are obviously divergence-free. These results can be generalized to the full non-linear case, i.e. $f$ is non-linear in $u_{x_1}$, by solving (8) for $n \geq 2$ and proceeding in a similar fashion.

## Chapter VIII

## Totally Hyperbolic Equations

Consider the system of m first order linear equations:

$$\underline{v}_{x_2} = A(x_1, x_2)\,\underline{v}_{x_1} + B(x_1, x_2)\,\underline{v} + \underline{c}(x_1, x_2) \tag{1}$$

where $\underline{v}$: vector $v_i (i = 1, \ldots, m)$, A: matrix $A_{ij}(i, j = 1, \ldots, m)$, $B$ = matrix $B_{ij}(i, j = 1, \ldots, m)$, $\underline{c}$: vector $c_i (i = 1, \ldots, m)$. If in a given domain of the $x_1, x_2$ plane, the matrix A has distinct real eigenvalues $\lambda_1(x_1, x_2), \ldots, \lambda_m(x_1, x_2)$, then the above system is called totally hyperbolic in that domain. Assuming that this is the case in the entire $x_1, x_2$ plane, then there exists a matrix $D(x_1, x_2)$ such that $D^{-1}AD = \Lambda$, where $\Lambda$ is a diagonal matrix whose diagonal elements are the eigenvalues $\lambda_1, \ldots, \lambda_m$. By making a change of dependent variables, $\underline{v} = D\underline{u}$, we get as an equation for $\underline{u}$:

$$\underline{u}_{x_2} = \Lambda(x_1, x_2)\,\underline{u}_{x_1} + \Gamma(x_1, x_2)\,\underline{u} + \underline{\delta}(x_1, x_2)$$

where $\Gamma = D^{-1}BD + D^{-1}AD_{x_1} - D^{-1}D_{x_2}$

$$\underline{\delta} = D^{-1}\underline{c}$$

The equations we will then consider are

$$u_{i_{x_2}} = f_i(u_{i_{x_1}}, \underline{u}, \underline{x}) = \lambda_i u_{i_{x_1}} + \gamma_i \qquad i = 1, \ldots, m \tag{2}$$

where $\lambda_i$ and $\gamma_i$ are functions of $x_1$ and $x_2$, $\gamma_i$ being linear in $\underline{u}$.

As an obvious extension of the operators introduced in the preceding chapter we define

$$D^p_{x_1} F^n\left(\underline{x}, \underline{u}, \underline{u}^{(1)}, \ldots, \underline{u}^{(n)}\right) = \frac{\partial F^n}{\partial x_1} + \sum_{k=0}^{p} \sum_{j=1}^{m} u_j^{(k+1)} \frac{\partial F^n}{\partial u_j^{(k)}}$$

where $u_j^{(k)} = \dfrac{\partial^k u_j}{\partial {x_1}^k}$, and

$$D^p_{x_2} F^n = \frac{\partial F^n}{\partial x_2} + \sum_{k=0}^{p} \sum_{j=1}^{m} \frac{d^k f_j}{dx_1^k} \frac{\partial F^n}{\partial u_j^{(k)}}$$

with the obvious interpretation of $\dfrac{d^k f_j}{dx_1^k}$. Again, for any function $\underline{u}(x_1,x_2)$ which satisfy (2), the above operators, for $p=n$, are simply the total derivatives of $F^n$ with respect to $x_1$ and $x_2$ respectively. The commutation relation between the two operators still holds, i.e.

$$D^{n+1}_{x_1} D^n_{x_2} F^n = D^{n+1}_{x_2} D^n_{x_1} F^n$$

(The proof in the preceding chapter generalizes in an obvious way.)

We now consider vectors $\underline{V}^n(\underline{x}, \underline{u}, \underline{u}^{(1)}, \ldots, \underline{u}^{(n)})$ whose divergence is zero for all $\underline{u}$ satisfying (2), i.e.

$$D^n_{x_1} V^n_1 + D^n_{x_2} V^n_2 = 0 . \qquad (3)$$

Equating the coefficients of $u_j^{(n+1)}$ to zero, we have

$$\frac{\partial V^n_1}{\partial u_j^{(n)}} + \lambda_j \frac{\partial V^n_2}{\partial u_j^{(n)}} = 0 \qquad (j = 1, \ldots, m) \qquad (4)$$

We will assume that $m \geq 2$. Solving the $j = 1$ equation, we obtain

$$V_1^n = -\lambda_1 V_2^n + G^n(u_2^{(n)}, \ldots, u_m^{(n)})$$

where the dependence of G on $\underline{x}$ and lower derivatives of $\underline{u}$ is not exhibited. Substituting this expression into the $j = 2$ equation and solving, we get

$$V_2^n = \frac{1}{\lambda_1 - \lambda_2}\left[F^n(u_1^{(n)}, u_3^{(n)}, \ldots, u_m^{(n)}) + G^n(u_2^{(n)}, \ldots, u_m^{(n)})\right]$$

Absorbing $\frac{1}{\lambda_1 - \lambda_2}$ into F and G yields for $V_1^n$ and $V_2^n$

$$V_1^n = -\lambda_1 F^n(u_1^{(n)}) - \lambda_2 G^n(u_2^{(n)})$$

$$V_2^n = F^n(u_1^{(n)}) + G^n(u_2^{(n)})$$

where the dependence on $u_3^{(n)}, \ldots, u_m^{(n)}$ is omitted since they acted only as parameters in solving the $j = 1, 2$ equations.

By induction, we show that the general solution of (4) is

$$\begin{aligned} V_1^n &= -\lambda_1 F_1^n(u_1^{(n)}) - \lambda_2 F_2^n(u_2^{(n)}) \cdots - \lambda_m F_m^n(u_m^{(n)}) \\ V_2^n &= F_1^n(u_1^{(n)}) + F_2^n(u_2^{(n)}) + \cdots + F_m^n(u_m^{(n)}) \end{aligned} \tag{5}$$

where $F_j^n$ depends on $u_j^{(n)}$ and not on $u_\ell^{(n)}$, $\ell \neq j$. We assume that the first j equations of (4) have as a solution

$$\overset{n}{V_1} = -\lambda_1 \overset{\sim n}{F_1} - \cdots - \lambda_j \overset{\sim n}{F_j}$$

$$\overset{n}{V_2} = \overset{\sim n}{F_1} + \cdots + \overset{\sim n}{F_j}$$

where $\overset{\sim n}{F_k}$ depends on $u_k^{(n)}$ and not on $u_\ell^{(n)}$, $\ell \leq j$, $\ell \neq k$ and that $u_{j+1}^{(n)}, \ldots, u_m^{(n)}$ appear simply as parameters in $\overset{\sim n}{F_k}$. Substituting into the equation for $j+1$ yields

$$\sum_{k=1}^{j} \left(\lambda_{j+1} - \lambda_k\right) \frac{\partial \overset{\sim n}{F_k}}{\partial u_{j+1}^{(n)}} = 0$$

$$\Longrightarrow \qquad \frac{\partial^2 \overset{\sim n}{F_k}}{\partial u_k^{(n)} \partial u_{j+1}^{(n)}} = 0 \; .$$

$$\Longrightarrow \qquad \overset{\sim n}{F_k} = \overset{n}{F_k}(u_k^{(n)}) + \overset{n}{G_k}(u_{j+1}^{(n)})$$

and

$$\sum_{k=1}^{j} \left(\lambda_{j+1} - \lambda_k\right) \overset{n}{G_k} = 0$$

the integration constant being absorbed into the $\overset{n}{G_k}$. Letting $\overset{n}{F_{j+1}} = \sum_{k=1}^{j} \overset{n}{G_k}$, we get

$$\overset{n}{V_1} = -\lambda_1 \overset{n}{F_1} - \cdots - \lambda_j \overset{n}{F_j} - \lambda_{j+1} \overset{n}{F_{j+1}}$$

$$\overset{n}{V_2} = \overset{n}{F_1} + \cdots + \overset{n}{F_j} + \overset{n}{F_{j+1}}$$

thus establishing (5). Substituting (5) into (3) gives

$$\sum_{j=1}^{m} \left( -D^{n}_{x_1} \lambda_j F^{n}_j + D^{n}_{x_2} F^{n}_j \right) = 0 \tag{6}$$

We note that $-D^{n}_{x_1} \lambda_k F^{n}_k + D^{n}_{x_2} F^{n}_k$ is linear in $u^{(n)}_j$ for $j \neq k$, so that by differentiating (6) twice with respect to $u^{(n)}_k$ yields, letting $G^{n}_k = \frac{\partial^2 F^{n}_k}{\partial u^{(n)2}_k}$,

$$-D^{n}_{x_1} \lambda_k G^{n}_k + D^{n}_{x_2} G^{n}_k + 2\left( n \frac{\partial \lambda_k}{\partial x_1} + \frac{\partial \gamma_k}{\partial u_k} \right) G^{n}_k = 0$$

or

$$-\lambda_k D^{n}_{x_1} G^{n}_k + D^{n}_{x_2} G^{n}_k + \left[ (2n-1) \frac{\partial \lambda_k}{\partial x_1} + 2 \frac{\partial \gamma_k}{\partial u_k} \right] G^{n}_k = 0 \tag{7}$$

Since the coefficient of $G^{n}_k$ is a function of $x_1$ and $x_2$ alone, we can find a particular solution of (7) depending only on $x_1$ and $x_2$, i.e. there exists a $g_k(x_1, x_2; n)$ such that

$$-\lambda_k D^{n}_{x_1} g_k + D^{n}_{x_2} g_k = -\lambda_k \frac{\partial g_k}{\partial x_1} + \frac{\partial g_k}{\partial x_2} = -\left[ (2n-1) \frac{\partial \lambda_k}{\partial x_1} + 2 \frac{\partial \gamma_k}{\partial u_k} \right] g_k \tag{7a}$$

A $g^{n}_k$ satisfying the above equation can always be found. Letting $G^n_k = g_k H^n_k$ we find that $H^n_k$ satisfies the following equation:

$$-\lambda_k D^{n}_{x_1} H^{n}_k + D^{n}_{x_2} H^{n}_k = 0 \tag{8a}$$

(8a) is a first order linear homogeneous partial differential equation for $H^n_k$. However, since $H^n_k$ must not depend on $u^{(n)}_j$, $j \neq k$, the coefficients of $u_j^{(n)}$, $j \neq k$, must be set equal to zero. Hence

$$(\lambda_j - \lambda_k) \frac{\partial H_k^n}{\partial u_j^{(n-1)}} + \frac{\partial \gamma_k}{\partial u_j} \frac{\partial H_k^n}{\partial u_k^{(n)}} = 0 \qquad \begin{array}{l} j = 1, \ldots, m \\ j \neq k \end{array} \tag{8b}$$

The $m-1$ equations (8b) along with equation (8a) constitute a highly overdetermined system. There always exists a solution of (8) which depends only on $x_1$ and $x_2$, because there exists an $h_k(x_1, x_2)$ which satisfies

$$- \lambda_k \frac{\partial h_k}{\partial x_1} + \frac{\partial h_k}{\partial x_2} = 0 \tag{9}$$

Since (8) is a homogeneous system, then an arbitrary function of any set of solutions is also a solution. Thus given a maximal set of independent solutions of (8), the general solution is an arbitrary function of these independent solutions. (Independence, here, means functional independence.) We also note that any solution of (8) for the case $\ell < n$ is also a solution for the case $n$.

The first property of system (8) to be shown is that to a set of maximal solutions of (8) for the case $n-1$ (solutions depending on $u_k^{(n-1)}$ and lower derivatives), only one solution depending on $u_k^{(n)}$ need be added to give a set of maximal solutions of (8) for the case $n$. Of course, it is possible that no solutions need be added as in the case where no solution depending on $u_k^{(n)}$ exists, for if $H^{n-1}$ is a solution depending on no higher derivatives than $\underline{u}^{(n-1)}$, then

$$- \lambda_k D_{x_1}^{n-1} H^{n-1} + D_{x_2}^{n-1} H^{n-1} = 0$$

has a solution only if the coefficients of $\underline{u}^{(n)}$ vanish, so that

$$(\lambda_j - \lambda_k) \frac{\partial H^{(n-1)}}{\partial u_j^{(n-1)}} = 0 \qquad j = 1, \ldots, m$$

and thus $H^{n-1} = H_k^{n-1}$, i.e. does not depend on $u_j^{(n-1)}$, $j \neq k$. If there exists a solution to (8) depending on $u_k^{(n)}$, say $P_k^n$, then make a change of variables from $u_k^{(n)}$ to $P_k^n$ leaving the other variables unchanged. (8a) can be written as

$$-\lambda_k D_{x_1}^{n-1} H_k^n + D_{x_2}^{n-1} H_k^n + \frac{\partial H_k^n}{\partial u_k^{(n)}} \left( D_{x_1}^{n-1} \frac{d^{n-1} f_k}{dx_1^{n-1}} \right) = 0 \tag{10}$$

We note that it is sufficient to consider equation (8a) alone as long as it is remembered that $H_k^n$ does not depend on $u_j^{(n)}$, $j \neq k$. Under the above change of variables

$$H_k^n\left(\underline{x}, \underline{u}, \ldots, \underline{u}^{(n-1)}, u_k^{(n)}\right) \rightarrow H_k^n\left(\underline{x}, \underline{u}, \ldots, \underline{u}^{(n-1)}, P_k^n\right)$$

$$\frac{\partial}{\partial x_i} \rightarrow \frac{\partial}{\partial x_i} + \frac{\partial P_k^n}{\partial x_i} \frac{\partial}{\partial P_k^n} \qquad i = 1, 2$$

$$\frac{\partial}{\partial u_j^{(\ell)}} \rightarrow \frac{\partial}{\partial u_j^{(\ell)}} + \frac{\partial P_k^n}{\partial u_j^{(\ell)}} \frac{\partial}{\partial P_k^n} \qquad \begin{array}{l} \ell = 1, \ldots, n-1 \\ j = 1, \ldots, m \end{array}$$

$$\frac{\partial}{\partial u_k^{(n)}} \rightarrow \frac{\partial P_k^n}{\partial u_k^{(n)}} \frac{\partial}{\partial P_k^n}$$

so that (10) becomes, remembering that $P_k^n$ is a solution of (10),

$$-\lambda_k D_{x_1}^{n-1} H_k^n + D_{x_2}^{n-1} H_k^n = 0 \tag{11}$$

In (11) there are no derivatives with respect to $P_k^n$ nor does $P_k^n$ appear in any of the coefficients, so that $P_k^n$ simply acts as if arbitrarily introduced into $H_k^n$. Using the same reasoning as above, $H_k^n$ cannot depend on $u_j^{(n-1)}$, $j \neq k$ (except through $P_k^n$), because such a dependence would introduce $u_j^{(n)}$ into (11) and the equation could not be satisfied. Thus to a set of maximal independent solutions for the case $n-1$ we need only add $P_k^n$ to get a set of maximal independent solutions for the case $n$.

If we have a solution of (8a) for the case $n-1$, $n \geq 1$, i.e. a function $P_k^{n-1}$ which satisfies

$$-\lambda_k \overset{n}{D}_{x_1} \overset{n-1}{P_k} + \overset{n}{D}_{x_2} \overset{n-1}{P_k} = -\lambda_k \overset{n-1}{D}_{x_1} \overset{n-1}{P_k} + \overset{n-1}{D}_{x_2} \overset{n-1}{P_k} = 0, \qquad (12)$$

and which depends on $u_k^{(n-1)}$, then a solution for the case $n$ can be generated by the following: Apply $\overset{n}{D}_{x_1}$ to (12) and using the commutation relation, we get

$$-\overset{n}{D}_{x_1}\left(\lambda_k \overset{n-1}{D}_{x_1} \overset{n-1}{P_k}\right) + \overset{n}{D}_{x_2}\left(\overset{n-1}{D}_{x_1} \overset{n-1}{P_k}\right) = 0$$

or

$$-\lambda_k \overset{n}{D}_{x_1}\left(\overset{n-1}{D}_{x_1} \overset{n-1}{P_k}\right) + \overset{n}{D}_{x_2}\left(\overset{n-1}{D}_{x_1} \overset{n-1}{P_k}\right) - \frac{\partial \lambda_k}{\partial x_1}\left(\overset{n-1}{D}_{x_1} \overset{n-1}{P_k}\right) = 0 \qquad (13)$$

There exists a function $s_k(x_1, x_2)$ which satisfies

$$-\lambda_k \overset{n}{D}_{x_1} s_k + \overset{n}{D}_{x_2} s_k = -\lambda_k \frac{\partial s_k}{\partial x_1} + \frac{\partial s_k}{\partial x_2} = \frac{\partial \lambda_k}{\partial x_1} s_k$$

In fact, $s_k = \frac{\partial h_k}{\partial x_1}$, where $h_k$ satisfies (9), is such a function. Thus, the quotient $P_k^n = \left(D_{x_1}^{n-1} P_k^{n-1}\right) \Big/ \frac{\partial h_k}{\partial x_1}$ satisfies the homogeneous equation

$$-\lambda_k \overset{n}{D}_{x_1} \overset{n}{P}_k + \overset{n}{D}_{x_2} \overset{n}{P}_k = 0 .$$

Now, it is not necessarily true that if a solution, depending on $u_k^{(n)}$, exists for the case n, that a solution depending on $u_k^{(n-1)}$ exists for the case n-1. An example of this is the system

$$u_{1\,x_2} = u_2$$

$$u_{2\,x_2} = u_{2\,x_1} + \frac{2}{x_2^2} u_1$$

for which no solution of (8) exists for the case $n = 1$, $k = 1$ and depending on $u_1^{(1)}$, while for the case $n = 2$, the function

$$\overset{2}{P}_1 = u_1^{(2)} - u_2^{(1)} - \frac{2}{x_2} u_1^{(1)} + \frac{2}{x_2^2} u_1 + \frac{2}{x_2} u_2$$

satisfies (8). We can sum up the above as follows: Let $\ell_k$ be the first case for which a solution to (8) exists depending on $u_k^{(\ell_k)}$. Then for all $n < \ell_k$ the maximal set of independent solutions consists only of $h_k(x_1, x_2)$. For $n \geqslant \ell_k$ the maximal set of independent solutions consists of $h_k, \overset{\ell_k}{P}_k, \ldots, \overset{n}{P}_k$, where $P_k^{j+1} = \left(\overset{j}{D}_{x_1} \overset{j}{P}_k\right) \Big/ \frac{\partial h_k}{\partial x_1}$, $\ell_k \leqslant j \leqslant n-1$. For the case $n < \ell_k$

$$\frac{\partial^2 \overset{n}{F}_k}{\partial u_k^{(n)2}} = \overset{n}{G}_k = g_k(x_1, x_2; n)\, H_k\left[h_k(x_1, x_2)\right]$$

$$\Longrightarrow \quad \overset{n}{F}_k = g_k H_k(h_k) \frac{u_k^{(n)2}}{2} + a_k^{n-1} u_k^{(n)} + b_k^{n-1} \qquad (14)$$

where $H_k$ is an arbitrary function of $h_k$, and $g_k$ satisfies (7a). For the case $n \geq \ell_k$, it can be shown that $\overset{n}{P}_k$ can be taken to be a linear function of $u_k^{(n)}$, and in fact, the coefficient of $u_k^{(n)}$ can be taken to be a function of $x_1$ and $x_2$ alone. We may also note that $g_k$ can be taken as

$$g_k = \frac{\partial h_k}{\partial x_1}\left(\frac{\partial P_k^n}{\partial u_k^{(n)}}\right)^2$$

which can be seen to satisfy (7a) so that we get for $F_k^n$

$$\overset{n}{F}_k = \frac{\partial h_k}{\partial x_1}\overset{n}{H}_k\left(h_k, \overset{\ell_k}{P}_k, \ldots, \overset{n}{P}_k\right) + \overset{n-1}{a}_k \overset{(n)}{u}_k + \overset{n-1}{b}_k \qquad (15)$$

where

$\overset{n}{H}_k$ is an arbitrary function of $h_k$, $\overset{\ell_k}{P}_k, \ldots, \overset{n}{P}_k$

$a_k^{n-1}$ and $b_k^{n-1}$ are functions of $\underline{x}, \underline{u}, \ldots, \underline{u}^{(n-1)}$, which at this stage in the analysis are arbitrary functions.

We note that $h_k(x_1, x_2)$ = constant is the equation for the characteristic corresponding to $\lambda_k$. Also, as in the preceding chapter, $\overset{\ell_k}{P}_k, \ldots, \overset{n}{P}_k$ are all constants along the characteristic corresponding to $\lambda_k$, so that that first term in (15) can be thought of as $\frac{d}{dx_1}H(h_k)$, where H is an arbitrary function of $h_k$. The first term in $-\lambda_k F_k^n$, from $V_1^n$, would then be $-\frac{d}{dx_2}H(h_k)$. We can divide the solutions into three cases:

Case 1: $n \geq \ell_j$, $j = 1, \ldots, m$.

Then each $F_j^n$ has the form (15). When we substitute (15) into (6), we arrive at

$$\sum_{j=1}^{m} \left[ -D^{n}_{x_1} \lambda_j \left( a_j^{n-1} u_j^{(n)} + b_j^{n-1} \right) + D^{n}_{x_2} \left( a_j^{n-1} u_j^{(n)} + b_j^{n-1} \right) \right] = 0 \qquad (16)$$

since the first term in $F_j^n$ satisfies $-D^n_{x_1} \lambda_j F^n_j + D^n_{x_2} F^n_j = 0$. In (16) we must set all coefficients of $u_j^{(n)}$ and products of $u_j^{(n)}$ equal to zero. In particular, setting the coefficient of $u_k^{(n)} u_j^{(n)}$ equal to zero gives

$$\left(\lambda_k - \lambda_j\right) \frac{\partial a_j^{n-1}}{\partial u_k^{(n-1)}} + \left(\lambda_j - \lambda_k\right) \frac{\partial a_k^{n-1}}{\partial u_j^{(n-1)}} = 0$$

so that

$$\frac{\partial a_j^{n-1}}{\partial u_k^{(n-1)}} = \frac{\partial a_k^{n-1}}{\partial u_j^{(n-1)}} \qquad (17)$$

By induction, we can show that the solution to (17) is given by

$$a_j^{n-1} = \frac{\partial a^{n-1}}{\partial u_j^{(n-1)}} \qquad (18)$$

where $a^{n-1}$ is an arbitrary function of $\underline{x}, \underline{u}, \ldots, \underline{u}^{(n-1)}$. We see that (18) is a solution (17) and that (18) is a generalization of the fact that if the curl of a vector is zero, then the vector can be written as the gradient of a scalar. We can now write, using (18),

$$\sum_{j=1}^{m} \left( a_j^{n-1} u_j^{(n)} + b_j^{n-1} \right) = D^{n-1}_{x_1} a^{n-1} + V_2^{n-1} \qquad (19a)$$

and

$$\sum_{j=1}^{m} \lambda_j \left( \overset{n-1}{a_j} \overset{(n)}{u_j} + \overset{n-1}{b_j} \right) = \overset{n-1}{D_{x_2}} \overset{n-1}{a} - \overset{n-1}{V_1} \tag{19b}$$

where

$$\overset{n-1}{V_1} = \sum_{j=1}^{m} \left( \overset{n-1}{D_{x_1}} \frac{d \overset{n-1}{f_j}}{d \overset{n-1}{x_1}} \right) \frac{\partial \overset{n-1}{a}}{\partial u_j^{(n-1)}} + \overset{n-2}{D_{x_2}} \overset{n-1}{a} - \sum_{j=1}^{m} \lambda_j \overset{n-1}{b_j}$$

$$\overset{n-1}{V_2} = -\overset{n-2}{D_{x_1}} \overset{n-1}{a} + \sum_{j=1}^{m} \overset{n-1}{b}_j$$

Since $\overset{n-1}{b_j}$ is arbitrary at this stage of the analysis, $\overset{n-1}{V_1}$ and $\overset{n-1}{V_2}$ are also arbitrary at this stage. Substituting (19) into (16) and remembering that $D_{x_1}^{n} D_{x_2}^{n-1} a^{n-1} + D_{x_2}^{n} D_{x_1}^{n-1} a^{n-1} = 0$, we find that $V_1^{n-1}$ and $V_2^{n-1}$ satisfy

$$\overset{n-1}{D_{x_1}} \overset{n-1}{V_1} + \overset{n-1}{D_{x_2}} \overset{n-1}{V_2} = 0$$

Thus for the case $n \geq \ell_j$, $j = 1, \ldots, m$, we have

$$\overset{n}{V_1} = -\sum_{j=1}^{m} \lambda_j \frac{\partial h_j}{\partial x_1} \overset{n}{H_j} \left( h_j, \overset{\ell_j}{P_j}, \ldots, \overset{n}{P_j} \right) - \overset{n-1}{D_{x_2}} \overset{n-1}{a} + \overset{n-1}{V_1}$$

$$\overset{n}{V_2} = \sum_{j=1}^{m} \frac{\partial h_j}{\partial x_1} \overset{n}{H_j} \left( h_j, \overset{\ell_j}{P_j}, \ldots, \overset{n}{P_j} \right) + \overset{n-1}{D_{x_1}} \overset{n-1}{a} + \overset{n-1}{V_2} \tag{20}$$

where $\underline{V}^{n-1} = \left( V_1^{n-1}, V_2^{n-1} \right)$ is an arbitrary divergence-free vector depending on $\underline{x}, \underline{u}, \ldots, \underline{u}^{(n-1)}$. The problem is thus reduced to the next lower case. If $n-1 \geq \ell_j$, $j=1, \ldots, m$, then the first two expresions in $\underline{V}^{n-1}$ can be incorporated into the corresponding two expressions in $\underline{V}^n$ so that the solution for $\underline{V}^n$ would now be the same except for the last term which would be replaced by $\underline{V}^{n-2}$. We continue this until we get to the point where $\underline{V}^k$ is such that $k < \ell_j$ for some $j=1,\ldots,m$, which brings us to Case 2.

Case 2: $n < \ell_j$ for some $j=1, \ldots, m$ and $n \geq \ell_k$ for some $k=1, \ldots, m$. Assume for definiteness that $j=1$ and $k=2, \ldots, m$. From this particular case it is easy to generalize the results. When we compute the coefficient of $u_1^{(n)} u_k^{(n)}$, $k>1$, in (6), and set it equal to zero, the result is, using (14),

$$g_1 H_1(h_1) \frac{\partial \gamma_1}{\partial u_k} + \left(\lambda_k - \lambda_1\right) \frac{\partial a_1^{n-1}}{\partial u_k^{(n-1)}} + \left(\lambda_1 - \lambda_k\right) \frac{\partial a_k^{n-1}}{\partial u_1^{(n-1)}} = 0 \tag{21}$$

From the cross product terms $u_i^{(n)} u_k^{(n)}$, $i>1$, $k>1$, we obtain as we did above

$$a_k^{n-1} = \frac{\partial a^{n-1}}{\partial u_k^{(n-1)}} \qquad k > 1$$

After substituting into (21), it can then be shown that

$$a_1^{n-1} = \frac{\partial a^{n-1}}{\partial u_1^{(n-1)}} + \sum_{k=2}^{m} \left[ \frac{1}{\lambda_1 - \lambda_k} \, g_1 H_1(h_1) \frac{\partial \gamma_1}{\partial u_k} \, u_k^{(n-1)} \right]$$

We can then proceed as above and conclude that

$$\overset{n}{V_1} = -\sum_{j=2}^{m} \lambda_j \frac{\partial h_j}{\partial x_1} \overset{n}{H_j}\left(h_j, \overset{\ell_j}{P_j}, \ldots, \overset{n}{P_j}\right) - D_{x_2} \overset{n-1\ n-1}{a} + \text{quadratic function of } u_1^{(n)} + V_1^{n-1}$$

$$\overset{n}{V_2} = \sum_{j=2}^{m} \frac{\partial h_j}{\partial x_1} \overset{n}{H_j}\left(h_j, \overset{\ell_j}{P_j}, \ldots, \overset{n}{P_j}\right) + D_{x_1} \overset{n-1\ n-1}{a} + \text{quadratic function of } u_1^{(n)} + V_2^{n-1}$$

The quadratic functions of $u_1^{(n)}$ have no noteworthy general structure but can be obtained for each particular system (2).

Case 3: $n < \ell_j$, $j = 1, \ldots, m$. In this case, all we can say is that $V_1^n$ and $V_2^n$ are sums of quadratic functions of $u_j^{(n)}$.

The particular case $\ell_j = 0$ occurs if and only if $f_i$ is independent of $u_j$, $j \neq i$, i.e. when the equations are completely uncoupled. The divergence-free vector $\underline{V}^n$ then takes the form

$$\overset{n}{V_1} = -\sum_{j=1}^{m} \lambda_j \frac{\partial h_j}{\partial x_1} \overset{n}{H_j}\left(h_j, \overset{0}{P_j}, \ldots, \overset{n}{P_j}\right) - D_{x_2} \overset{n-1\ n-1}{a}$$

$$\overset{n}{V_2} = \sum_{j=1}^{m} \frac{\partial h_j}{\partial x_1} \overset{n}{H_j}\left(h_j, \overset{0}{P_j}, \ldots, \overset{n}{P_j}\right) + D_{x_1} \overset{n-1\ n-1}{a}$$

which is just the sum of divergence-free vectors, obtained in the previous chapter for single first order equations, with the addition of a trivial divergence-free vector.

In general, the vector $\overset{n}{\underline{V}}$ will consist of: 1) terms in which arbitrary functions of quantities constant along characteristics appear; 2) quadratic terms; and 3) trivial divergence-free vectors.

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