MINIMAL TIME DEADBEAT REGULATION AND CONTROL OF LINEAR, STATIONARY, SAMPLED-DATA SYSTEMS

Thesis by

Jacques J. de Barbeyrac

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ABSTRACT

The problem of minimal time deadbeat regulation and control of linear, stationary, sampled-data systems is studied in this dissertation, assuming that only a limited number of the state variables are directly observable. The problem is first solved for the usual one-input one-output systems. The existing techniques for deadbeat digital compensation are all derived under the assumption that a specific initial state always exists; it will be shown that if this condition is violated and a digital controller is designed using the existing methods, the system has a transient response with time constants corresponding to the stable poles of the open-loop system. A technique to overcome this difficulty is developed using both a state-space and a z-transform approach to the problem. A digital controller which in a sense first identifies the complete state and then proceeds to control it in a deadbeat fashion is synthesized.

The problem is next solved for multi-input, multi-output systems, using a state-space approach different from the one used for the one-input, one-output systems. It is first shown that if all the state variables are directly observable and the system is completely controllable in N sampling periods, there always exists at least one stationary, linear feedback law which will regulate the system in N sampling periods. If only a limited number of the state variables are directly observable, but the system is completely observable in N' sampling periods, then there exist "discrete compensators" which will regulate the system in $(N + N')$ sampling periods.

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CHAPTER I

TIME OPTIMAL CONTROL

1.1 Introduction

Feedback control systems have been the subject of a great deal of investigation in the past decade or two. The early investigators concentrated primarily on the analysis of these systems and this in turn led to a study of how such systems could be synthesized. In the 1950's the synthesis aspect of control systems was pursued and this led to the inevitable question of optimization which is to be studied in detail in this dissertation for a specific case.

For the most part, until quite recently, the study of control systems has been carried out using frequency domain techniques, like the Laplace and Fourier transforms. These so called classical techniques were the basic tools for both analysis and synthesis, and in a restricted sense, for optimization also. For example, the classical problem of choosing the value of an open-loop gain can be looked at as an optimization problem, because generally speaking too large a gain will cause instability and too small a gain will decrease the accuracy of the system. There is a trade-off to be considered here in the problem of finding an "optimum" gain. There are, however, two main reasons why this classical approach to the design of a control system did not lead exactly to an optimization problem: first, the performance criterion was generally more qualitative than quantitative, and second, the configuration of the controller was generally more or less fixed (e.g. the controller is assumed to be a time-invariant linear system).

The modern approach to the synthesis of control systems leads, on the contrary, to real optimization problems. This approach may be summarized as follows: - given a system to be controlled, or regulated, and a certain number of constraints, which must be satisfied,

(a) define a performance index which is to be maximized or minimized.

(b) determine the optimum control law, or control policy, (i.e. a mathematical description of the controlling element) which will maximize (or minimize) the performance criterion and satisfy all the constraints.

(c) find a realizable controller which will implement, or at least approximate, the optimum control law.

It is possible that the optimum controller is very difficult to realize practically; however if one decides to trade optimality for simplicity of realization, for example by restricting the controller to be linear and stationary (i.e. time-invariant), it is then possible to determine how much is sacrificed in the performance of the system by adding this restriction. Thus, one of the main results of the optimization problems is to present an absolute upper (or lower) limit for the chosen performance index.

1.2 Time Optimal Control for Continuous Systems.

One of the first optimization problems which was triggered by the study of relay-controlled linear systems is the so-called time optimal control. This problem can be stated as follows: given a linear system, what is the minimum amount of time necessary for bringing this system starting from arbitrary initial conditions back to its equilibrium position? In the case of continuous systems, it is always (mathematically) possible to reduce the error to zero instantaneously by application of an appropriate impulsive input. The problem is therefore meaningless unless some additional constraint is added. Generally the absolute value of the forcing function is constrained to be smaller than a fixed quantity.

Various methods have been devised for solving this problem at least theoretically. Russian control engineers use L. Pontriagin's (1) (2) maximum principle, other people use Bellman's (3) principle of optimality and dynamic programming, finally others like Kalman (4) start from the classical calculus of variations. Although all these methods lead eventually to the same final result, they give different insights into the problem and suggest different ways for realizing the optimum control.

Even when assuming that the characteristics of the system to be controlled are perfectly known, the difficulties encountered are of two types. First, in order to control in a truly optimum fashion, one needs to know the initial conditions of the system exactly. This problem is generally called the identification problem. Second, even if we assume that the identification problem is solved, there are still great practical difficulties to be overcome.

One way of realizing optimum control is to compute in advance the optimal forcing function which will bring the system back to its equilibrium position. This optimal forcing function will be a function of time (over a finite length of time) dependent on the initial condition of the system. One could conceptually store this information in a digital computer and then use the measured initial position of the system to choose the optimum control law. Since this optimal control law is dependent only upon the initial condition, we really have an open-loop control, and there is little hope of realizing exactly the desired equilibrium position at the end of the minimum time interval for several reasons: first, a digital computer has a finite number of storage elements so that one can store the forcing functions only for a finite number of initial conditions and then approximate for the other ones; second, if any external disturbance acts on the system during the period of control, it is unnoticed until the end of that period. One way of improving the effectiveness would be to measure, from time to time, the actual position of the system and then correct, if necessary, the forcing function to be applied at later times. That would be more of a closed-loop type control.

Another way of getting time optimal control of linear systems is to use the adjoint system. The problem has been shown to be equivalent to a determination of the initial conditions of this adjoint system as a function of the initial conditions of the system to be controlled (5). Once the initial conditions of the adjoint system are known, the variation of the forcing function with time is known. Here too, one can use a digital computer to store the results of computations made in advance and then operate either in a completely open-loop manner or introduce some kind of feedback by recomputing from time to time new initial conditions for the adjoint system as a function of the new initial conditions of the controlled system.

It is clear that the storage problem increases very rapidly with the order of the system to be controlled so that there is little hope of getting practical realizations based on these two methods for systems of average complexity.

It turns out, however, that for a large class of linear systems operated from a limited source of power the time optimal solution (in the continuous case) is obtained by at all times utilizing properly all of the power available. This is called the "bang-bang principle." More specifically, we want to bring a linear system from arbitrary initial conditions back to its equilibrium position, in minimum time using a forcing function which is limited in amplitude. The "bang-bang principle" states that if there exists a forcing function which stays within the prescribed bounds and takes the system back to its equilibrium condition, then there exists a time optimal forcing function which is equal either to its maximum or to its minimum value during all the time necessary to bring back the system. In other words, there exists an optimum relay controller which will implement this optimum "bang-bang" control law. Moreover if the n poles of the system to be controlled are real and distinct, it is known that if a solution exists, it will require at most $(n-1)$ switches, i.e. changes in sign of the forcing function.

It is therefore feasible, at least for this last class of systems to try the following procedure: rather than compute in advance the sequence of forcing functions to be applied as a function of the initial conditions, and storing the results in a computer, use the computer to find as quickly as possible the sequence of forcing functions, or more exactly, the n switching times for each set of initial conditions. Of course, the position of the controlled system changes while the computer searches for the optimal solution and this could be taken into account but if the computing time is short with respect to the time constants of the system, one will get a good approximation of the optimal forcing function. The equations to be solved are generally transcendental equations and therefore difficult to solve. However a priori knowledge of the number of unknown switching times is of great help. In case the open-loop poles are not real, the number of switching times is not known a priori, so that the complexity of the computations is greatly increased.

Instead of trying to find all the switching times for given initial conditions, which amounts to finding the complete optimal forcing function, one can try to find the instantaneous value of the optimal forcing function as a function of the instantaneous position of the system. For bang-bang systems, this leads to the concept of switching surfaces such that the instantaneous value of the optimal control can be found as a function of the instantaneous position of the system.

This method is quite interesting, because it corresponds to a closed-loop type control. However, these switching surfaces are difficult to obtain for high order systems. Considerable effort has been spent by many investigators to obtain approximate representations of the optimum switching surfaces, at least (6) for low-order systems.

The overall picture for the time optimal control of continuous systems with power limitation is that although the problem is theoretically solved, any practical realization will involve an approximation which will quite often result in the system oscillating around its equilibrium position. One suggestion then is to use a dual-mode system in which time optimal control is used far from the equilibrium position but when the system is close to its equilibrium position the control becomes a classical linear control. This linear control if properly chosen will drive the system back to its equilibrium position in a theoretically infinite length of time. Such a procedure would possibly eliminate the limit cycles but would seem rather strange to a mathematician: one looks for time optimal solutions and ends up with a practical system having an infinite response time, while there exists solutions in a finite time. It turns out, however, that a satisfactory solution to the problem of getting to the equilibrium position from a position near to it can be found, if one uses a sampled data mode of operation instead of a continuous one.

1.3 Time Optimal Control for Sampled-Data Systems.

Every time we speak of a sampled-data system in this report, it will mean a system in which the variables can be measured only at the sampling instants and also, one in which control is exerted by means of piecewise constant signals which can change only at the sampling instants. This is a restricted kind of sampled-data systems, but it corresponds to many systems which are controlled by a digital computer.

The optimal time regulating control is the control which will bring a system starting from an arbitrary initial position back to its equilibrium position in the minimum number of sampling instants. It turns out that the optimal time control without amplitude limitation is a meaningful problem in this case, because for a fixed sampling period, it is no longer possible to have the unrealizable solution of an impulsive forcing function. Of course if one considers the sampling period as variable and lets it go towards zero, the sampled-data solution will lead towards some kind of impulsive forcing function, as in the continuous case. The problem with fixed sampling period was first solved by Kalman and Bertram (7) (13), in the case of a system with only one forcing function. However, the identification of the initial conditions of the system was not solved very satisfactorily.

If the amplitude of the forcing function is limited, Desoer and Wing (8) have found an optimal strategy which becomes identical with the switching surface technique as the sampling period goes to zero. It amounts to defining at each sampling instant the value of the next forcing function as a function of the actual position of the system with respect to a critical surface. This critical surface, for an n th order system, is made up of parts of (n-1) hyperplanes, so that it can be a little bit easier to implement than the switching surfaces for continuous systems. However the difficulty here is of the same order as for continuous systems and the tendency is towards finding some approximation to the optimum control so as to get the system near its equilibrium position. Once near the equilibrium position the amplitude constraint will not have to be considered.

It is essentially that part of the optimal time control problem that will be studied in this dissertation, namely time-optimal regulation (and control) of a linear stationary system; the linearity excludes the existence of amplitude limitations, so that the problem is only practically meaningful for sampled-data systems. It will be shown that the identification problem can be solved in a simple manner and that there exists a discrete compensator which both identifies the position of the system and implements the optimum control law.

As already noted, such a theory could be useful for a dual-mode continuous system, in which, if the initial perturbation is large, one would use an approximation to the optimum control law for continuous systems (bang-bang) and then switch to a linear sampled-data mode of operation near the equilibrium position. This would eliminate some difficulties inherent to bang-bang systems, like the existence of limit cycles around the equilibrium position.

Another aim of this dissertation is to determine what happens to the above stated problem when the system is a multi-input, multi-output system. The words "single-variable systems" and "multi-variable systems" will be avoided here, because systems which are basically one-input, one-output are often studied using mathematical models which look like multi-variable systems. It will be shown that the only important factor, at least for the minimal time regulation problem, is to know whether the system has one or several independent inputs (or forcing functions).

The theory of multi-input, multi-output systems is still in its early stages, perhaps because one is tempted to try to extend the results and methods worked out for one-input, one-output systems. This tendency is well illustrated by the number of papers written on the following subject: given two single-input, single-output systems with mutual coupling, is it possible to find compensating circuits which will uncouple them? In some cases this uncoupling may be desirable, but certainly not in all cases. After all, why should the control be better, easier or more efficient when the systems are artificially uncoupled? The underlying reason for this approach is that once the systems are uncoupled, they can be studied separately as one-input, one-output systems and then there are a number of well known mathematical techniques at the disposal of the designer.

Some people suggest that trying to extend the results and methods of the simple case to the complicated case is a basically wrong attitude and that one should try to attack the problems concerning multi-input, multi-output systems with a fresh mind. This is certainly a valid remark but it asks also for much more imagination. To solve the problem considered here, the method used in the one-input, one-output case does not lead to the solution in the multi-input case. However, the method used for solving the multi-input case can of course be applied to the one-input case. With this introduction in mind, a summary of what is to follow is given below.

In Chapter II, we will review what has been done for deadbeat compensation of one-input one-output linear sampled-data systems, using z-transform theory. In Chapter III, we will indicate one method for describing mathematically linear sampled-data systems. This mathematical formulation applies very well to systems having any number of inputs and outputs, and is quite suited to the study of time optimal control. The solution to this problem for the one-input one-output case will be explained in Chapter IV. And, finally the multi-input, multi-output case will be solved in Chapter V.

CHAPTER II

CLASSICAL DEADBEAT COMPENSATION AND ITS DRAWBACKS

2.1 Digital Compensation for Deadbeat Response to Specific Inputs

The problem of determining digital compensators for systems with only one input and one output has been extensively studied by many people (9). The approach has always been made using z-transform theory and for the purpose of completeness the main results and the idea from which they originated will be presented here. In addition, by looking at this problem now it will be possible to place in the proper perspective the results which follow in later chapters.

To become specific consider the block diagram of the given system shown in Figure 1. In this diagram a zero order hold circuit is used as the data reconstruction device; in all of work which is to follow, only zero order hold circuits will be used for this purpose. The classical design problem can be stated in the following manner. Find a discrete compensator $D(z)$ such that the output of the system $c(t)$ becomes identically equal to the input $r(t)$ after a certain finite transient time, when $r(t)$ belongs to some class of deterministic functions of time; the sampling period T is assumed to be known and the duration of the transient time is to be considered in the design of $D(z)$. The types of inputs considered in determining $D(z)$ are generally taken to be polynomials in t, i.e., step inputs, ramp inputs, etc. and the system is assumed to start from rest when computing its response to one of these specific inputs.

Referring to Figure 1, the over-all pulse transfer function is:

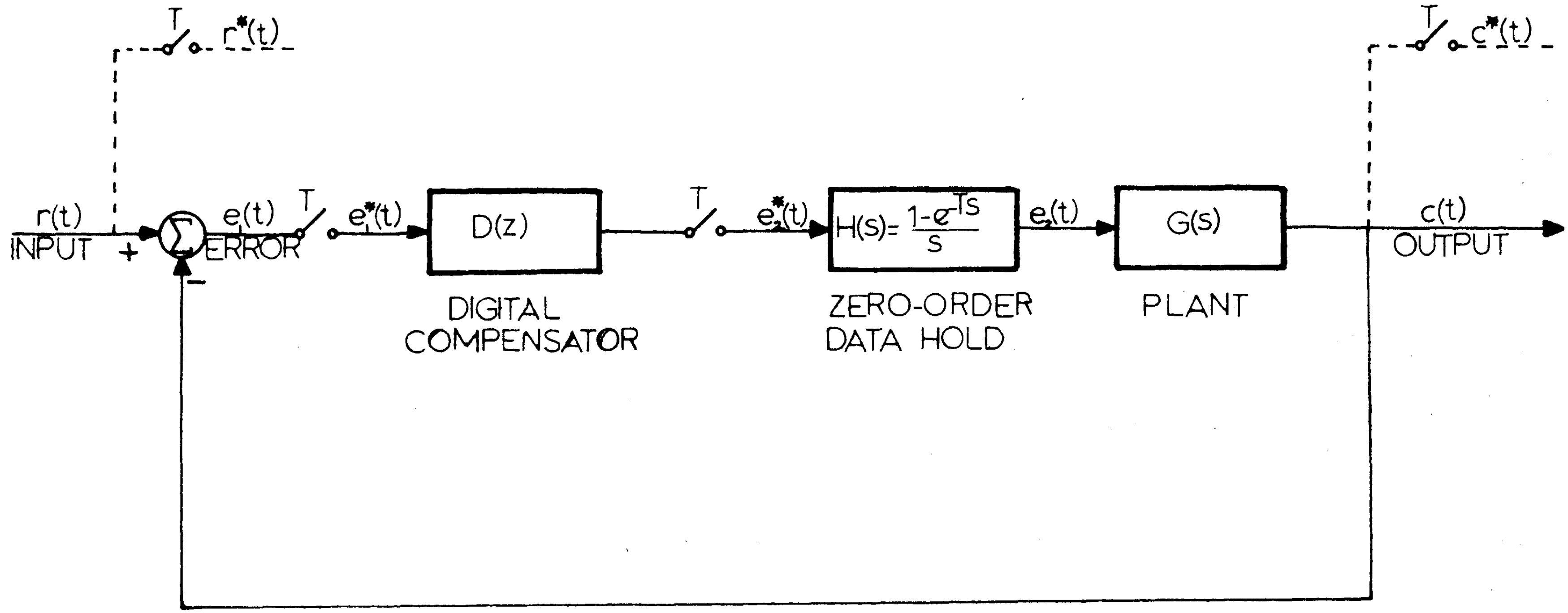

Figure 1: An error-sampled feedback control system with digital compensator and data hold.

$$K(z) = \frac{C(z)}{R(z)} = \frac{D(z)\ GH(z)}{1+D(z)\ GH(z)} \tag{2.1.1}$$

The error sequence is:

$$E_1^*(z) = R(z) - C(z) = R(z)\ [1 - K(z)] \tag{2.1.2}$$

If the reference input $r(t)$ can be represented as an m th order polynomial in t (starting at $t=0$), then the z-transform of this input is given by

$$R(z) = \frac{N(z)}{(1-z^{-1})^{m+1}} \tag{2.1.3}$$

where $N(z)$ is a finite polynomial in z^{-1}.

If the steady-state error is to be zero, we must have (9)

$$1 - K(z) = (1-z^{-1})^{m+1}\ F(z)$$

where $F(z)$ is generally an unspecified ratio of polynomials in z^{-1}. If the error sequence is to be of finite length, i.e., the transient time is of finite duration, $F(z)$ can only be a finite polynomial in z^{-1} that we will denote by $P(z^{-1})$. Therefore

$$1 - K(z) = (1-z^{-1})^{m+1}\ P(z^{-1}) \tag{2.1.4}$$

This equation shows that $K(z)$ must be a finite polynomial in z^{-1}, which means that all of the poles of $K(z)$ are at the origin in the z-plane. To minimize the transient time (often referred to as the settling time of the system), the order of $P(z^{-1})$ in z^{-1} should be as low as possible.

If these two conditions on $[1-K(z)]$ are satisfied, then the input and the output of this system are equal at all sampling instants after a finite length of time; this does not guarantee, however, that the continuous output becomes identical to the continuous input. The conditions under which the continuous output equals the polynomial type input can be readily determined. First, if the continuous output is to equal the input, then the input to the plant $e_2(t)$ must be a polynomial in t for all $t \geq t_s$, where t_s is the duration of the transient. Because of this property, $e_2(t)$ cannot be discontinuous for $t \geq t_s$. Now, since $e_2(t)$ is the output of the zero-order hold circuit, $e_2(t)$ can only be a constant function of time (possibly zero) for all $t \geq t_s$. The significance of this fact becomes clear from what follows.

$E_2(z)$ can be expressed as

$$E_2(z) = \frac{C(z)}{GH(z)} = \frac{K(z)}{GH(z)} R(z) \qquad (2.1.5)$$

Because the steady-state value of $e_2(t)$ is a constant, $E_2(z)$ contains a first-order pole at $z=1$ and has all of its other poles at the origin. Since equation 2.1.5 is an identity and $K(z)$ cannot have a zero at $z=1$ because of the equation 2.1.4, $GH(z)$ must contain a pole of at least m th order at $z = 1$ simply because of the form of $R(z)$. This means that such a feedback system which is to follow polynomial type inputs of order m in t must contain at least m integrators in its plant. Also equation 2.1.5 shows that $K(z)$ must contain as its zeros all the zeros of $GH(z)$.

In summary, then, the classical design of digital deadbeat compensation requires that four conditions be satisfied. These conditions for deadbeat response to inputs expressible as polynomials in t of order m are:

(1) $K(z)$ is a finite polynomial in z^{-1}

(2) $K(z)$ must contain as its zeros all the zeros of $GH(z)$

(3) the plant contains at least m integrations

(4) $1-K(z)$ must contain the factor $(1-z^{-1})^{m+1}$

However for practical reasons the additional following condition must also be satisfied

(5) $1-K(z)$ contains as zeros all of the poles of $GH(z)$ which lie on or outside the unit circle in the z-plane.

These practical reasons become apparent in consideration of the discrete compensator to be implemented. In terms of $K(z)$, the pulse transfer function of the discrete compensator is given by

$$D(z) = \frac{1}{GH(z)} \frac{K(z)}{1-K(z)}$$

$D(z)$ will not contain as poles any zero of $GH(z)$, since these are also zeros of $K(z)$. But $D(z)$ will generally contain as zeros all the poles of $GH(z)$ except those at $z=1$. In other words, the discrete compensator cancels all undesired poles of $GH(z)$. Practically such a cancellation will never be perfect; this is not very important if the poles of $GH(z)$ to be cancelled are inside the unit circle in the z-plane; but if $GH(z)$ contains poles outside this unit circle, an imperfect cancellation would introduce in $K(z)$ an unstable pole which cannot be tolerated.

Before proceeding further, two features of this type of system should be noted. First, if the plant contains exactly m integrations and responds in a deadbeat manner to an input expressible as a polynomial in t of order m, the driving function $e_2(t)$ becomes, after a transient period, a non-zero constant function of time. Since the error signal is zero after this transient period, the discrete compensator must contain an integrator. Second, the four theoretical requirements given previously, though necessary, are not quite sufficient. One should add another condition on the plant to be controlled (10): none of the poles of $G(s)$ are such that $s_j - s_k = 2r\pi i/T$, $r = \pm 1, \pm 2, \ldots$ Physically this suppresses the possibility of having a transient term which vanishes at all sampling instants but not in between. Mathematically, this condition is directly related to the question of controllability of the sampled-data system and will be discussed in detail in a later section.

2.2 Introduction of Initial Conditions in a Sampled-Data System

When the problem of digital compensation is being studied through the use of the z-transform theory, the general problem consists in finding a suitable compensator such that:

1. The closed-loop system is stable, and
2. The response to some specific inputs, generally step or ramp, fulfills some predetermined requirements.

The response to these specific inputs is always computed assuming the system starts from rest, i.e., from zero initial conditions. When one asks about the effect of non-zero initial conditions, the usual answer is: since the closed loop system is stable, the effect of initial conditions will eventually disappear. This is not always true, and it is interesting to determine the time constants of the decay of initial conditions in particular for the case of deadbeat compensation.

Let us first recall the general method for introducing initial conditions. Consider the plant being controlled. Let $x(t)$ represent the continuous output of the plant and $f(t)$ the input to this plant; assume these two quantities are related to each other by a constant coefficient differential equation of the form

$$\frac{d^n x}{dt^n} + b_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \dots + b_0 x(t) = a_m \frac{d^m f}{dt^m} + \dots + a_1 \frac{df}{dt} + a_0 f(t)$$

In other words, let the transfer function of the plant be

$$G(s) = \frac{P(s)}{Q(s)} = \frac{a_m s^m + \dots + a_1 s + a_0}{s^n + b_{n-1}s^{n-1} + \dots + b_0}$$

If initial conditions at $t = 0$ are to be considered, it is convenient to introduce a new function of time $x_1(t)$ which is equal to $x(t)$ for all $t > 0$ and obeys the following differential equation

$$\begin{aligned} \frac{d^n x_1}{dt^n} + b_{n-1}\frac{d^{n-1}x_1}{dt^{n-1}} + \dots + b_0 x_1(t) = {} & a_m \frac{d^m f}{dt^m} + \dots + a_1 \frac{df}{dt} + a_0 f(t) \\ & + [b_1 x(0) + b_2 x^{(1)}(0) + \dots + x^{(n-1)}(0)]\ \delta_0(t) \\ & + [b_2 x(0) + \dots + x^{(n-2)}(0)]\ \delta_1(t) \\ & + \dots\dots\dots\dots\dots\dots \\ & + x(0)\ \delta_{n-1}(t) \end{aligned} \qquad (2.2.1)$$

where all initial conditions on x_1 are identically zero (at $t = 0^-$),

$$x^{(k)}(0) \triangleq \left. \frac{d^k x}{dt^k} \right|_{t = 0^-}$$

and $\delta_0(t)$ is a unit impulse, $\delta_1(t)$ a unit doublet, etc. This is the standard result which says that initial conditions can be viewed as additional forcing functions. There is nothing unusual here but the interpretation of this result allows the effect of non-zero initial conditions to be brought into a z-transform analysis. If $N(s)$ is defined as the Laplace transform of all the terms, except those involving $f(t)$ and its derivatives, on the right side of equation 2.2.1, divided by $P(s)$, the numerator of the plant transfer function, the situation can be interpreted as that shown in Figure 2. For the complete error-sampled system, we would have the situation shown in Figure 3.

To determine how the output is modified by the initial conditions (i.e. by $N(s)$), the sampled output will be computed. The Laplace transform of the output, which will be denoted by $C(s)$, is

$$C(s) = N(s)\,G(s) + F(s)\,G(s)$$

but

$$F(s) = E_1^*(s)\,D^*(s)\,H(s)$$

$$E_1^*(s) = R^*(s) - C^*(s)$$

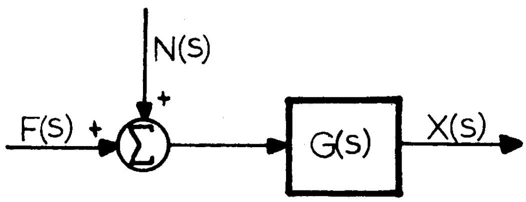

Figure 2: Initial conditions in an open-loop system.

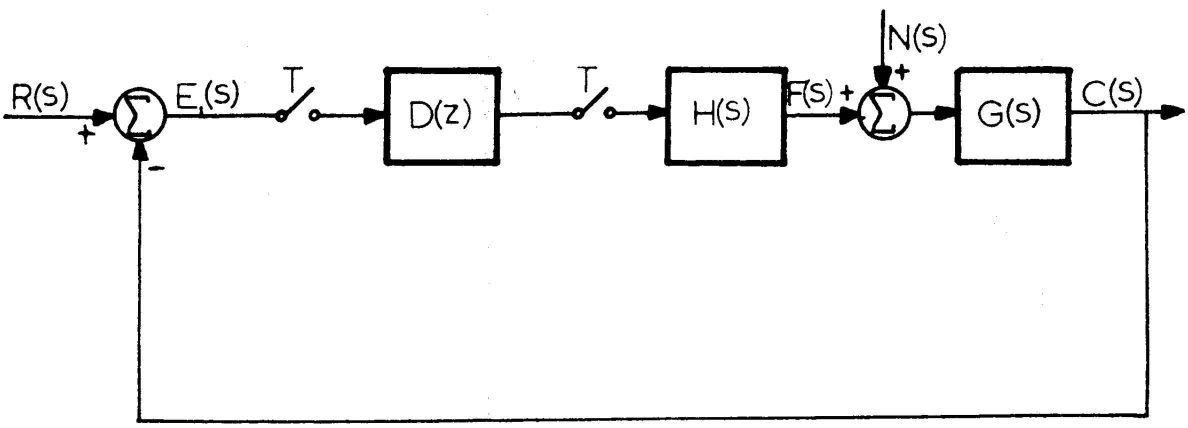

Figure 3: Initial conditions in a feedback system.

Putting these relationships together and taking the z-transform of the first equation gives

$$C(z) = R(z) \frac{D(z)\ GH(z)}{1 + D(z)\ GH(z)} + \frac{NG(z)}{1 + D(z)\ GH(z)}$$

or

$$C(z) = R(z)\ K(z) + NG(z)\ [1 - K(z)] \qquad (2.2.2)$$

In the last equation $NG(z)$ is by definition the z-transform of $N(s)$ $G(s)$. Since $N(s)$ is the Laplace transform of impulses, doublets, etc. divided by the numerator of $G(s)$, it is easily seen that

$$NG(s) = \frac{\text{polynomial in } s \text{ of order } (n-1)}{\text{denominator of } G(s)}$$

and

$$NG(z) = \frac{\text{polynomial in } z^{-1} \text{ of order } (n-1)}{\text{denominator of } G(z)}$$

Since the poles of $G(z)$ and the poles of $GH(z)$ are identical, a general remark can be made about non-zero initial conditions. Namely the effect of initial conditions will gradually disappear if and only if $[1 - K(z)]$ contains among its zeros all of the poles of $GH(z)$ which lie outside or on the unit circle in the z-plane. This statement is identical to the fifth condition (i.e., the one dictated by practical considerations) which was explained in the previous paragraph. But one can see that it is more fundamental than that. Even if it were possible to perfectly cancel the undesirable poles of $GH(z)$ through the use of a discrete compensator, the effect of initial conditions would still not disappear after a certain finite transient period if only the four conditions on $K(z)$ previously presented were used to determine $K(z)$.

2.3 Application to the Case of Deadbeat Compensation

It has been shown in Section 2.1 that for deadbeat compensation $[1 - K(z)]$ must contain as its zeros all those poles of $GH(z)$ which lie on or outside of the unit circle in the z-plane. Thus,

$$NG(z)\,[1 - K(z)] = \frac{\text{polynomial in } z^{-1}}{\prod_{i=1}^{p} (1 - a_i z^{-1})}$$

where the a_i's are all the poles of $GH(z)$ which lie inside the unit circle in the z-plane. This equation shows that the effect of initial conditions will gradually disappear with increasing time, but that the exponential decays associated with the non-zero initial conditions take place with time constants corresponding to the stable poles of the open-loop transfer function $G(s)$, i.e., with time constants that correspond to all the poles of $GH(z)$ inside the unit circle. This is not a very desirable situation.

As far as the effects of initial conditions are concerned, this open-loop behavior can also be understood by looking at the sequence of forcing functions applied to the plant. Refer to Figure 3 again and note that

$$E_2(z) = D(z)\ E_1(z) = D(z)\ [R(z) - C(z)]$$

$$E_2(z) = D(z) \Big[R(z) - R(z)\ K(z) - NG(z)\ [1 - K(z)]\Big]$$

$$E_2(z) = D(z)\ [R(z) - NG(z)]\ [1 - K(z)]$$

$$E_2(z) = [R(z) - NG(z)]\ \frac{K(z)}{GH(z)} \tag{2.2.3}$$

With this last equation it will be possible to show that $E_2(z)$ is a finite polynomial in z^{-1} which means that $e_2(nT)$ goes to zero in a finite time. To show this note that $K(z)$ contains among its zeros all of the zeros of $GH(z)$, this insures the fact that $K(z)/GH(z)$ is a finite polynomial in z^{-1}. The sequence of forcing functions due to the input is $R(z)\ K(z)/GH(z)$, and for the specific inputs for which the system has been designed, this sequence is either finite or reaches some steady-state value after a finite number of sampling periods.

We now want to show that the term $NG(z)$ does not change this, and do so by noting that the poles of $NG(z)$ are also poles of $GH(z)$. Thus the product $NG(z)\ K(z)/GH(z)$ is a finite polynomial in z^{-1}, and the forcing function due to the initial conditions becomes identically zero after a certain number of sampling periods. This explains the open loop behavior of the system after a certain number of sampling periods as far as the effects of initial conditions are concerned.

A simple example will serve to illustrate this situation. In Figure 1 let $G(s) = 1/s(s+1)$. If $T = 1.0$ second, we have

$$GH(z) = \frac{.368(z + .717)}{(z - 1)(z - .368)} = \frac{0.368z^{-1}(1 + .717z^{-1})}{(1 - z^{-1})(1 - 0.368z^{-1})}$$

Since the plant contains an integrator it can certainly respond in a deadbeat fashion to a step input. To find the digital compensator which will give this deadbeat response in the minimum number of sampling periods, it is necessary that

$$\begin{cases} K(z) = a_0 \, z^{-1}(1 + .717z^{-1}) \\ 1 - K(z) = (1 - z^{-1})(1 + b_0 \, z^{-1}) \end{cases}$$

A simple calculation shows

$$a_0 = 0.582$$

$$b_0 = 0.418$$

The corresponding discrete compensator is

$$D(z) = \frac{1}{GH(z)} \frac{K(z)}{1 - K(z)} = 1.582 \, \frac{1 - 0.368z^{-1}}{1 + 0.418z^{-1}}$$

When the input to this system is a step function and the output and its derivative are zero when the step function is applied, the output will become identically equal to the input after two sampling periods.

$$C(z) = K(z)\,R(z) = 0.582z^{-1}(1 + 0.717z^{-1})\,\frac{1}{1 - z^{-1}}$$

$$= 0.582z^{-1} + 1.0z^{-2} + 1.0z^{-3} + \ldots$$

The input to the plant is

$$E_2^*(z) = R(z)\,\frac{K(z)}{GH(z)} = \frac{1}{1 - z^{-1}}\;1.582\,(1 - z^{-1})(1 - 0.368z^{-1})$$

i.e., it becomes identically zero after two sampling periods.

Now assume that when the unit step function was applied there was a non-zero initial velocity $\dot{c}_0$. This initial velocity term can be taken into account by making $N(s)$ equal to an impulse of value $\dot{c}_0$. Then equation 2.2.2 becomes

$$C(z) = \frac{0.582z^{-1}(1 + 0.717z^{-1})}{1 - z^{-1}} + NG(z)\left[(1 - z^{-1})(1 + 0.418z^{-1})\right]$$

where $NG(z) = \sum \left[\frac{\dot{c}_0}{s(s+1)}\right] = \dot{c}_0\,\frac{0.632z^{-1}}{(1 - z^{-1})(1 - 0.368z^{-1})}$

That part of the output due to the initial $\dot{c}_0$ is therefore

$$\dot{c}_0\left[\frac{0.632z^{-1}\,(1 + 0.418z^{-1})}{(1 - 0.368z^{-1})}\right]$$

whose inverse transform can be expressed as

$$c(nT) = 1.347\ \dot{c}_0\ (.368)^{n-1}$$

$$= 3.66\ \dot{c}_0\ (.368)^{n} \qquad n \geq 2$$

Clearly $c(nT)$ decays toward zero with the open-loop time constant. To show that even when $\dot{c}_0 \neq 0$ the input to the plant becomes identically zero after two sampling periods, simply use equation 2.2.3 which shows that

$$E_2(z) = 1.582\ (1 - 0.368z^{-1}) - \dot{c}_0(0.632z^{-1})\ 1.582$$

Therefore

$$e_2(nT) = 0 \qquad \text{for} \qquad n \geq 2$$

Thus the system does operate in an open loop manner after two sampling periods.

This whole chapter has illustrated the fact that for single input, single output systems, the z-transform theory is quite useful because it leads to the solution not only of all problems in which the performances of the system are evaluated from its behavior at the sampling instants, but also of some problems in which the behavior of the system between sampling instants is of interest. However, the direct transposition of this method to multi-input, multi-output systems, although possible, turns out to be impractical. As a consequence very few papers had been published on problems involving multi-input, multi-output systems before the introduction in the control field of a new mathematical tool of which we shall speak in the next chapter.

CHAPTER III

STATE-SPACE TECHNIQUES

3.1 State Description of a Dynamical System

For many years the control systems research problems had been limited to the study or design of linear, time-invariant, single-input, single-output systems. For that class of systems it was possible and quite useful to use transform methods and transfer functions: Laplace transform, giving the usual transfer function, for continuous systems; z-transform, giving the pulse transfer function, for discrete systems. By looking at the properties of the system in this transform plane, it was possible to quickly obtain a general idea of the characteristics of the system; e.g. stability, bandwidth, duration of transients and maximum overshoot in response to specific inputs. However, most of these characteristics were evaluated using "rules of thumb" rather than by exact computations. Moreover, all problems concerning systems which were either multi-input, multi-output or time-varying could not be thoroughly investigated because of a lack of adequate mathematical methods.

Such studies were by and large restricted to linear systems and inevitably people became interested in the study of non-linear systems, especially in a class of non-linearities for which it was impossible to linearize the complete equations, namely relay-controlled linear systems. The transform methods, applied in conjunction with a describing function that characterized the non linear element, were still able to provide partial results such as the existence of limit cycles. But it was no longer easy or even possible to predict the approximate behavior of the system by looking at its characteristics in a transform domain. Because of this it became necessary to study these systems in the time domain and this can be most easily done using mathematical techniques which have been developed by mathematicians interested in non-linear differential equations.

The unusual performances that can be obtained using a relay-controlled linear system triggered a new interest in the optimization of systems and it was then realized that the mathematical techniques which had proved so useful for the study of relay-controlled linear systems were extremely useful for a whole range of control problems. In particular, by formulating the problems involving linear, time-invariant, single-input, single-output systems as a special case of a general formulation valid for a much larger class of problems, it was possible not only to solve a few new problems but also to understand better the difficulties presented by some of the yet unsolved problems.

Although most of the actual research papers on control problems use the above mentioned mathematical techniques (which are referred to as state-space techniques by control engineers), there are very few tutorial accounts of the notion of state, so that a few words on this subject seem warranted. The word state, which has not been mentioned before in this dissertation, can be defined in somewhat abstract terms as follows.

The state of a dynamical system at a certain instant of time, t_0, is the minimum amount of information required about the past history of the system in order to predict its future evolution, i.e. to determine its complete behavior for all $t \geq t_0$.

This definition implies that in the event that the dynamical system is to be subjected to inputs, then inputs are known functions of time for $t > t_0$. This definition of state is quite general and a little bit of thought will convince the reader that it includes all systems that one intuitively considers as being classically dynamical.

For example, consider an RLC circuit. It is well known that if one is given the initial values of the currents through the inductors and of the voltages across the capacitors, then one can find the response of the circuit to any specified input. Consequently the state of this circuit at time t_0 consists of the values of the currents through the inductors and of the voltages across the capacitors at time t_0. Note that when speaking of the response of the circuit, it is not specified as to whether we are referring to the voltage across a resistor, or to the voltage across a terminal pair, or to any other measurable quantity of interest which could also be called an output of the system and which will generally be a function of time. The reason for this is simply that a knowledge of the initial values of the currents through the inductors and of the voltages across the capacitors is sufficient to compute the future values of these same quantities, when the system is subjected to any specified input. On the other hand, at any instant of time $t > t_0$ the value of each of the measurable quantities or outputs can be computed as a function of the values of the basic quantities: currents through inductors, voltages across capacitors. It is now easy to see why these currents and voltages may be called the state variables of the circuit, for the knowledge of their values (i.e. the values of the state variables) at any instant of time, plus the knowledge of the input, is sufficient to compute the value of any physical output of the circuit and moreover to compute the future values of the state variables when the system is subjected to a specified input. The evolution of any physical output of the system is looked at as a consequence of the evolution of the state, which completely characterizes the system. In other words, the dynamic behavior of a system can be represented by the change in state as a function of time.

More generally consider the situation in which a dynamical system can be described by means of a first-order vector differential equation of the form

$$\frac{d\underline{x}}{dt} = \underline{f}(\underline{x}, \underline{u}, t)$$

where $\underline{x}$ is an n-dimensional column vector, $\underline{u}$ an r-dimensional column vector corresponding to r forcing functions, $\underline{f}$ is an n-dimensional vector-valued function and t is the time.

The value of the vector $\underline{x}$ is the state of this system, because a knowledge of $\underline{x}$ at time t_0, plus the knowledge of the forcing functions $(u_1, \dots, u_r)$ for $t > t_0$ is sufficient to compute the values of $\underline{x}$ for all $t > t_0$. The vector $\underline{x}$ is called the state vector and its components $x_1(t)$, $x_2(t)$, ... , $x_n(t)$ the state variables. Another case in which the state may be described by a finite set of numbers corresponds to a discrete system whose input u and output c are related by a difference equation of the form

$$b_0\ c[nT] + b_1\ c[(n-1)T] + \dots + b_n\ c[0] = a_0\ u[nT] + \dots + a_p\ u[(n-p)T]$$

It is clear that the set

$$c[(n-1)T], \dots, c[0]\ ;\ u[(n-1)T], \dots, u[(n-p)T]$$

is the state of the system at the time $t = (n-1)T$ because a knowledge of these variables is sufficient to predict the evolution of the output $c(nT)$ of the system, given the input $u[nT]$.

One can, however, see that there is a difference between these two cases; in the first one, the state variables were instantaneous values of what one could call the "outputs" of the system; in the second, the state variables correspond to past and present values of the input and of the output of the system. In both examples, the future behavior of the system could be predicted once the state and inputs were known.

These two examples show the flexibility of the state concept. It is good at this point to note that it is not always possible to describe the state of a system with a finite set of numbers. The state may also be an infinite set of numbers. For example, consider a pure differentiator whose input is denoted by $u(t)$ and output by $y(t)$. The state at time t_0, denoted by S_{t_0}, is

$$S_{t_0} = u(t),\ t_0 - \tau < t < t_0$$

where τ is an arbitrarily small positive number, but non-zero. The state is now a function of time defined on an arbitrarily small but non-zero interval of time and therefore cannot be described by a finite set of numbers.

Having illustrated the concept of state and hinted at its generality, this concept will now be used to consider a simple problem which has caused some misunderstanding among people interested in control problems. It is the problem of going from a classical transfer function description of a system to a corresponding state description. Consider a linear, lumped-parameter, time-invariant, single input $[u(t)]$, single output $[c(t)]$, system. Such a system is generally characterized by its transfer function

$$G(s) = \frac{C(s)}{U(s)} = \frac{a_m s^m + a_{m-1} s^{m-1} + \dots + a_1 s + a_0}{s^n + b_{n-1} s^{n-1} + \dots + b_1 s + b_0} \qquad (3.1.1)$$

In most systems encountered in practice $m \leq n$.

This transfer function means that the system obeys the following linear differential equation, with constant coefficients

$$c^{(n)}(t) + b_{n-1}\, c^{(n-1)}(t) + \dots + b_0\, c(t) = a_m u^{(m)}(t) + \dots + a_0 u(t)$$

First assume that $a_m = a_{m-1} = \dots = a_1 = 0$, $a_0 \neq 0$. In such a case it is well known that if the values of $c(t)$ and its first $(n-1)$ derivatives are given at $t = t_0$, and $u(t)$ is known for all $t > t_0$, the behavior of the system is perfectly determined for all $t \geq t_0$. In other words, the output and its first $(n-1)$ derivatives can be taken as state variables in this case. Using the conventional notation $\underline{x}$ for a state vector, we would have

$$\underline{x}(t) = [x_1(t), \dots, x_n(t)] = [c(t), \dot{c}(t), \dots, c^{(n-1)}(t)]$$

So that

$$\begin{cases} \dot{x}_1 = x_2 \\ \dots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = -b_{n-1}x_n \dots -b_0 x_1 + a_0 u(t) \end{cases}$$

In matrix form, the n equations written above become

$$\underline{\dot{x}} = F\underline{x} + Gu$$

where

$$F = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & \cdots & & \cdots & . \\ 0 & 0 & 0 & \cdots & 1 \\ -b_0 & -b_1 & -b_2 & \cdots & -b_{n-1} \end{bmatrix} \qquad G = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ a_0 \end{bmatrix}$$

When the plant transfer function contains finite zeros, the situation is a little different. Again assume that the input to the system $u(t)$ is known for $t > t_0$. The knowledge of $u(t)$ for $t > t_0$ implies a knowledge of $\dot{u}(t), \ldots, u^{(m)}(t)$ for $t > t_0$. As before $u(t)$ is not known for $t \leq 0$; however it was implicitely assumed that $u(t)$ was bounded and therefore had at most a finite jump discontinuity at $t = 0$ when the transfer function had no finite zeros. Such a discontinuity could not influence the values of the output and its first $(n-1)$ derivatives at any time $t \geq t_0$, because the output and its derivatives did not depend upon $\dot{u}(t)$, $\ddot{u}(t)$, etc.

With a finite number of zeros in the transfer function the equivalent forcing function $f(t)$ is defined by

$$f(t) = a_0 u(t) + a_1 \dot{u}(t) + \ldots + a_m u^{(m)}(t)$$

and it will contain a delta function and its successive derivatives if $u(t)$ has a finite jump discontinuity. This "infinite" discontinuity of the equivalent forcing function will influence the future values of $c^{(n-1)}(t), \ldots, c^{(n-m)}(t)$.

In this case one sees that a knowledge of $c(t)$ and its first $(n-1)$ derivatives at $t = t_0$, and the knowledge of $u(t)$ for $t > t_0$, are no longer sufficient to completely describe the future behavior of the system; roughly speaking, there is a region of indeterminacy around $t = t_0$. This indeterminacy could be suppressed either by giving $u(t)$ for $t_0 - \tau < t \leq t_0$, or by giving $u(t), \ldots, u^{(m)}(t)$ for $t \geq t_0$.

The second idea for circumventing the difficulty encountered with finite zeros in $G(s)$ leads to the following scheme. Let

$$\begin{cases} x_1 = c(t) \\ x_2 = \dot{c}(t) \\ \cdots\cdots\cdots \\ x_n = c^{(n-1)}(t) \end{cases}$$

Then the system can be described by

$$\begin{cases} \dot{x}_1 = x_2 \\ \cdots\cdots\cdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = -b_{n-1}x_n - \cdots - b_0 x_1 + \sum_{i=0}^{m} a_i u^{(i)}(t) \end{cases}$$

If we define an n dimensional state vector

$$\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

and an (m+1) dimensional input vector

$$\underline{u}(t) = \begin{bmatrix} u(t) \\ \dot{u}(t) \\ \vdots \\ u^{(m)}(t) \end{bmatrix}$$

the system is characterized by the equation

$$\underline{\dot{x}} = F\underline{x} + G\underline{u}$$

where

$$F = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & \cdots & & \cdots & \\ 0 & 0 & 0 & \cdots & 1 \\ -b_0 & -b_1 & -b_2 & \cdots & -b_{n-1} \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \cdots & & \cdots & \\ 0 & 0 & \cdots & 0 \\ a_0 & a_1 & \cdots & a_m \end{bmatrix}$$

In other words, a one-input system has been transformed into an equivalent multi-input system. Although very simple, such a choice of state variables is not really desirable, because the fact that there is only one input has been masked and the result is a "multi-input" system in which the inputs are strongly interrelated. Moreover, in a later section the driving functions will be restricted to be piecewise constant and then $\dot{u}(t)$ and its successive derivatives will be impulse functions and its successive derivatives which are generalized functions and not always very easy to manipulate. Note also that the state variables so defined will be discontinuous when the forcing function will undergo a finite jump discontinuity, so that it will be necessary to know $u(t_0)$, $\dot{u}(t_0)$, ... , $u^{(m)}(t_0)$ in order to compute the state at $t = t_0^+$ from its knowledge at $t = t_0^-$.

An alternate approach at this point seems desirable and it will now be shown that for a transfer function of the form 3.1.1, where $m \leq n$, it is always possible to find a set of n state variables whose values at $t = t_0$, plus $u(t)$ for $t > t_0$, are sufficient to completely describe the future evolution of the system. The output of the system will be a linear combination of the state variables if $m \neq n$, and also of the input if $m = n$.

The easiest way to see this is to consider the implementation (on an analog computer) of a transfer function given by equation 3.1.1. The implementation indicated in Figure 4 clearly shows that one needs only n integrators for an n th order system, in which $m \leq n$. Let the outputs of the n integrators be the state variables.

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \cdots\cdots\cdot \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = - b_{n-1}x_n - b_{n-2}x_{n-1} - \cdots - b_1x_2 - b_0x_1 + u(t) \end{cases}$$

Since $u(t)$ by assumption can contain no infinite discontinuities it is easily seen that a knowledge of $(x_1, x_2, \ldots, x_n)$ at $t = t_0$, and $u(t)$ for $t > t_0$, completely describes the future behavior of the system, i.e., it completely determines $(x_1, x_2, \ldots, x_n)$ for all $t \geq t_0$.

Then at any instant

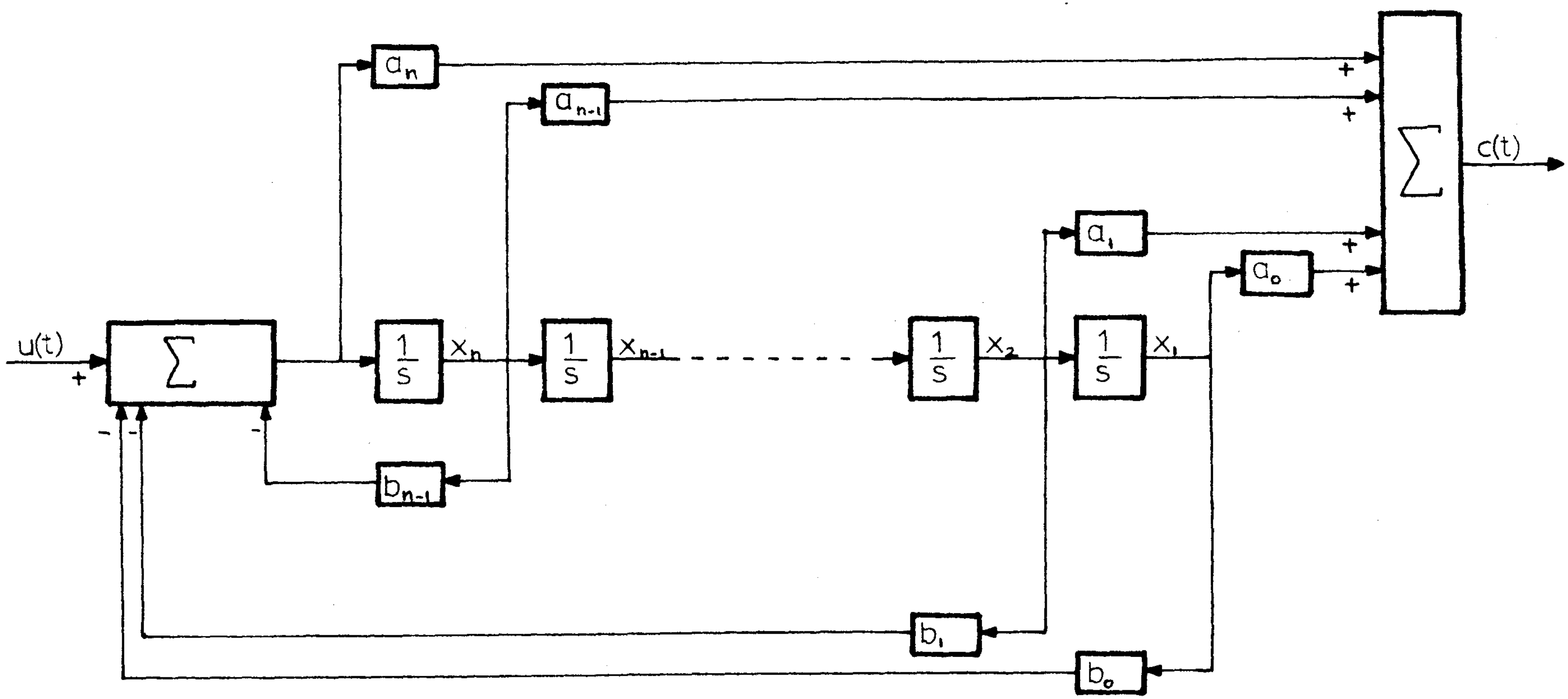

Figure 4: Analog computer implementation for an n th order system described in equation 3.1.1 with $m \leq n$.

$$c(t)=a_0x_1(t)+a_1x_2(t)+\dots+a_{n-1}x_n(t)+a_n\left[u(t)-b_0x_1(t)-b_1x_2(t)-\dots-b_{n-1}x_n(t)\right]$$

To show that this formulation agrees with the given transfer function just write symbolically

$$\begin{cases} sX_1 = X_2 \\ sX_2 = X_3 \\ \dots\dots \\ sX_{n-1} = X_n \\ sX_n = -b_{n-1}X_n-\dots-b_0X_1+U \end{cases} \quad \text{i.e.} \begin{cases} X_2 = sX_1 \\ X_3 = s^2X_1 \\ \dots\dots \\ X_n = s^{n-1}X_1 \\ (s^n+b_{n-1}s^{n-1}+\dots+b_0)\, X_1 = U \end{cases}$$

and also

$$C = (a_0 + a_1s + \dots + a_{n-1}s^{n-1} + a_ns^n)\, X_1$$

so that

$$\frac{C}{U} = \frac{a_0 + a_1s + \dots + a_ns^n}{b_0 + b_1s + \dots + s^n}$$

If the state variables are chosen using this procedure, these state variables are continuous functions of time, even when the input undergoes jump discontinuities. The output and its derivatives are, however, generally discontinuous, as they should be. In case $a_n = 0$, the output is continuous when the input undergoes a first-order discontinuity. Note that in case $m > n$, there is no such way to suppress the indeterminacy around $t = t_0$; one must either give $u(t)$ for $t_0 - \tau < t \leq t_0$, or give $u(t)$ and its successive derivatives for $t \geq t_0$.

The state description given above is by no means unique and for a given transfer function there correspond an infinity of vector differential equations. In particular, one can show that if $a_n = 0$ (i.e. the output is continuous when the input is discontinuous) it is always possible to find a set of state variables such that the output is one of them. More generally if $a_n = a_{n-1} = \dots = a_{n-p} = 0$, the output and its first p derivatives can be taken as state variables. This is also most easily shown by considering an analog computer implementation for the case where $a_n = 0$.

Referring to Figure 5,

$$\begin{cases} \dot{x}_1 = x_2 + \alpha_{n-1}u \\ \dot{x}_2 = x_3 + \alpha_{n-2}u \\ \dots\dots\dots\dots\dots \\ \dot{x}_{n-1} = x_n + \alpha_1 u \\ \dot{x}_n = (-b_{n-1}x_n - b_{n-2}x_{n-1} - \dots - b_0 x_1) + \alpha_0 u \end{cases} \qquad (3.1.2)$$

One can always adjust $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$ in such a way that $x_1(t) = c(t)$.

This is easily seen by eliminating $x_2, x_3, \dots, x_n$ in the above equations. Using symbolic calculus and multiplying by appropriate polynomials in s, equations 3.1.2 can be put in the following form:

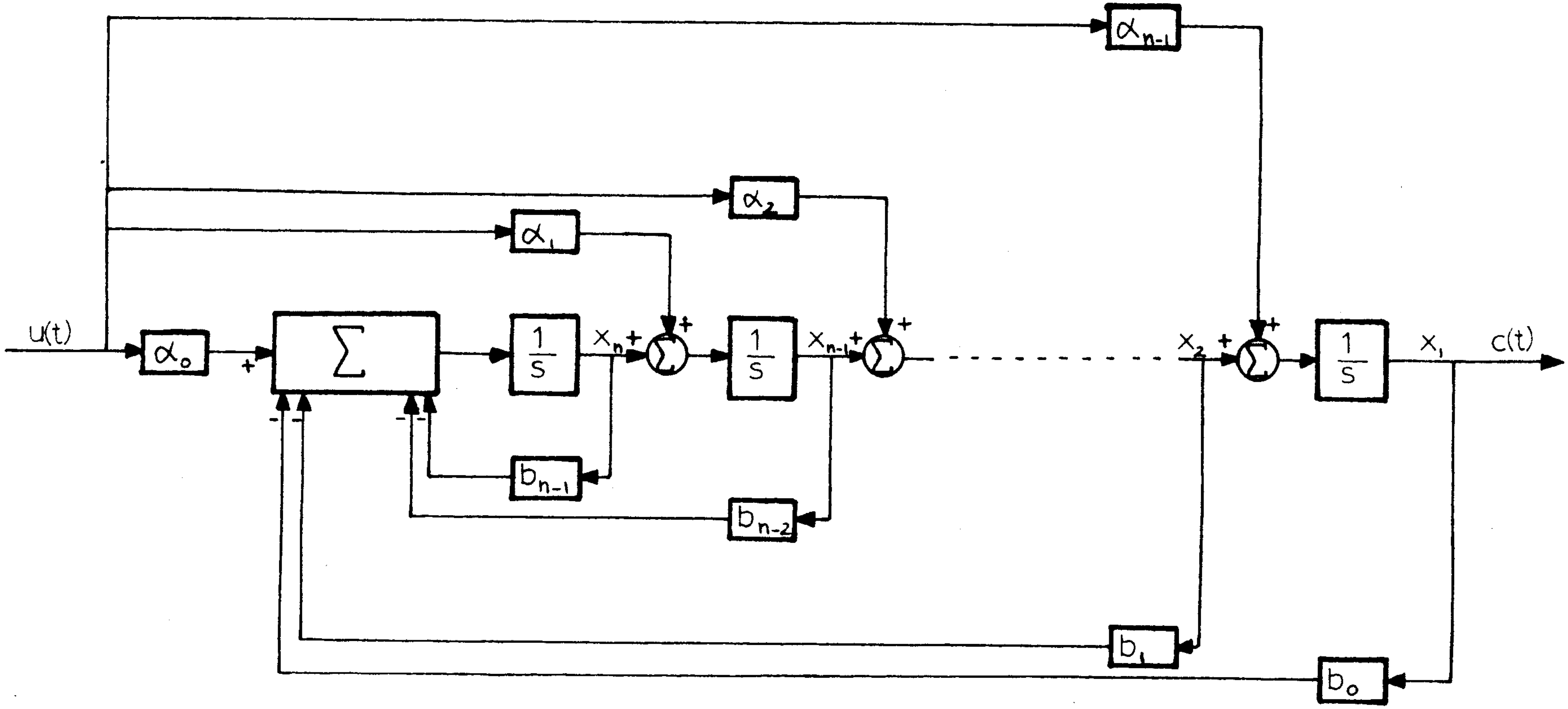

Figure 5: Other analog computer implementation for an n th order system having at most (n-1) finite zeros.

$$\begin{cases} (b_1+b_2s+\ldots+b_{n-1}s^{n-2}+s^{n-1})sX_1 = (b_1+b_2s+\ldots+b_{n-1}s^{n-2}+s^{n-1})(X_2+\alpha_{n-1}U) \\ (b_2+\ldots+b_{n-1}s^{n-3}+s^{n-2})sX_2 = (b_2+\ldots+b_{n-1}s^{n-3}+s^{n-2})(X_3+\alpha_{n-2}U) \\ \ldots \qquad \ldots \\ (b_{n-1}+s)sX_{n-1} = (b_{n-1}+s)(X_n+\alpha_1 U) \\ sX_n = (-b_{n-1}X_n-b_{n-2}X_{n-1}-\ldots-b_0X_1)+\alpha_0 U \end{cases}$$

Add together these n equations to obtain after simplification

$$(b_1+b_2s+\ldots+b_{n-1}s^{n-2}+s^{n-1})sX_1 = -b_0X_1+U\left[\alpha_0+\alpha_1(b_{n-1}+s)+\ldots+\alpha_{n-1}(b_1+\ldots+s^{n-1})\right]$$

Rearrangement of the various terms leads to the final form

$$(s^n+b_{n-1}s^{n-1}+\ldots+b_1s+b_0)X_1 = U\left[\alpha_{n-1}s^{n-1}+(\alpha_{n-1}b_{n-1}+\alpha_{n-2})s^{n-2}+\ldots \right.$$

$$\left. \ldots+(\alpha_{n-1}b_1+\ldots+\alpha_1b_{n-1}+\alpha_0)\right]$$

so that $c(t) = x_1(t)$ if the α's are chosen such that

$$\begin{cases} \alpha_{n-1} = a_{n-1} \\ \alpha_{n-1}b_{n-1}+\alpha_{n-2} = a_{n-2} \\ \ldots\ldots\ldots\ldots \\ \alpha_{n-1}b_1+\alpha_{n-2}b_2+\ldots+\alpha_1b_{n-1}+\alpha_0 = a_0 \end{cases} \qquad (3.1.3)$$

This system of equations clearly always has a solution and it is immediately seen that if $a_{n-1} = \cdots = a_{n-p} = 0$, then $\alpha_{n-1} = \cdots = \alpha_{n-p} = 0$, so that $x_1 = c$, $x_2 = \dot{c}$, ... , $x_{p+1} = c^{(p)}$.

When the transfer function contains no finite zeros, $a_1 = a_2 = \cdots = a_n = 0$, this formulation shows that the state variables are the output and its successive derivatives.

This last analog computer setup and the corresponding vector differential equations may be considered as being the most useful ones because they preserve as state variables the output and all the successive derivatives which can be taken as state variables. Therefore, in what is to follow it will be assumed that given a transfer function the state variable formulation will be derived from this last analog computer setup. The number of finite zeros will be assumed to be smaller than the number of poles $(m \leq n-1)$, and therefore the output will be one of the state variables for every single-input, single-output system.

3.2 General Time Solution

Whether or not the system is characterized by one or several transfer functions we will always assume that the system may be described by a vector differential equation which can be written

$$\frac{d\underline{x}}{dt} = \underline{f}(\underline{x}, \underline{u}, t)$$

Moreover we will assume that we are dealing with a linear system with constant coefficients (i.e. stationary system) so that in matrix form the system can be described by

$$\dot{\underline{x}} = F\underline{x} + G\underline{u} \tag{3.2.1}$$

where $\underline{x}$ is an n-dimensional vector representing the state of the system, $\underline{u}$ is an r dimensional vector representing the r inputs, F is a constant $(n \times n)$ matrix and G is a constant $(n \times r)$ matrix. The outputs of the system will be linear combinations of the state variables, so that if we have m outputs $y_i (i = 1, 2, \dots, m)$, we define an m-dimensional output vector $\underline{y}$ such that

$$\underline{y}(t) = M\underline{x}(t) \tag{3.2.2}$$

where M is an $(m \times n)$ constant matrix. To solvė equation 3.2.1 we first solve the homogeneous equation

$$\dot{\underline{x}} = F\underline{x} \quad ; \qquad x(0) = x_0$$

The solution of this equation can be expressed as (11)

$$\underline{x} = e^{Ft}\underline{x}_0$$

where e^{Ft} is a matrix exponential function defined by means of the infinite series

$$e^{Ft} = I + Ft + \dots + \frac{F^n t^n}{n!} + \dots$$

This matrix series converges for all F for any fixed value of t and for all t for any fixed F.

Another property of the exponential matrix can be expressed as

$$e^{F(s+t)} = e^{Fs}e^{Ft}$$

so that letting $s = -t$, we get the result

$$e^{F(-t+t)} = I = e^{-Ft}e^{Ft}$$

Hence e^{Ft} is never singular and its inverse is e^{-Ft}.

In the general case when the initial condition is $\underline{x}(t_0) = \underline{x}_0$ we will write

$$\underline{x}(t) = e^{F(t-t_0)}\underline{x}_0 = \phi(t-t_0)\underline{x}_0$$

where $\phi(t-t_0)$ is a matrix depending on only one variable, namely $(t-t_0)$. From what has been said about $e^{F\alpha}$, it can be shown that ϕ has the following properties:

1. $\phi(0) = I$
2. ϕ is never singular, and $\phi^{-1}(t-t_0) = \phi(t_0-t)$

We turn now to the solution of the inhomogeneous equation

$$\dot{\underline{x}} = F\underline{x} + G\underline{u} \qquad ; \qquad \underline{x}(t_0) = \underline{x}_0$$

write

$$e^{-F(t-t_0)}[\dot{\underline{x}} - F\underline{x}] = \frac{d}{dt}\left[e^{-F(t-t_0)}\underline{x}(t)\right] = e^{-F(t-t_0)}G\underline{u}(t)$$

Hence

$$e^{-F(t-t_0)}\underline{x}(t) = x_0 + \int_{t_0}^{t} e^{-F(s-t_0)} G\underline{u}(s)ds$$

$$\underline{x}(t) = e^{F(t-t_0)}\underline{x}_0 + \int_{t_0}^{t} e^{F(t-s)} G\underline{u}(s)ds$$

$$\underline{x}(t) = \phi(t-t_0)x_0 + \int_{t_0}^{t} \phi(t-s)G\underline{u}(s)ds$$

In later work we will be using driving functions $\underline{u}$ which will be held constant during each sampling period. To obtain a mathematical description for this condition one notes that if $\underline{u}(t) = \underline{u}(kT)$ is a constant for $kT \leq t < (k+1)T$, we can write

$$\underline{x}[(k+1)T] = \phi(T)\ \underline{x}(kT) + \int_{kT}^{(k+1)T} \phi[(k+1)T-s]Gds\ \underline{u}(kT)$$

If we let

$$\Delta(T) = \int_{kT}^{(k+1)T} \phi[(k+1)T-s]Gds = \int_{0}^{T} \phi(T-s)Gds$$

and

$$\underline{x}(iT) = \underline{x}_i \qquad\qquad \underline{u}(iT) = \underline{u}_i$$

we then get the state transition equation

$$\underline{x}_{k+1} = \phi(T)\underline{x}_k + \Delta(T)\underline{u}_k \qquad (3.2.3)$$

where ϕ is a non-singular constant $(n \times n)$ matrix and Δ is a constant $(n \times r)$ matrix. This last equation expresses the fact that because of the linearity of the system, its state at $t = (k+1)T$ can be expressed as the sum of two terms. The first one, independent of the driving function, is the state the system would have reached if there had been no driving function (i.e. corresponds to the free motion of the dynamical system); the second term, independent of the initial state, represents the effect of a constant driving function $\underline{u}_k$ over a length of time T. The fact that the system is time-invariant implies that ϕ and Δ are only dependent upon the time interval T and not upon the exact location of this time interval. Equation 3.2.3 together with $\underline{y}_k = M\underline{x}_k$ completely describe the dynamical system at the sampling instants, i.e. at instants of time which are T seconds apart.

In what follows, we will mainly be working with this discrete-time state transition equation because we are interested in digital control. However our goal will be to design a system having deadbeat performance, i.e. one in which the state or the output becomes identical to some given state or input for all time after a certain transient period. If the given reference state or input is zero, it is conventional to speak of deadbeat regulation. If it is not zero, one speaks of deadbeat follow-up.

One can immediately see that there is a difficulty in using a discrete-time description of a system for solving such a problem; nothing can be said about the behavior of the system in between the sampling instants. However since we always assume that this discrete-time dynamic system arises from a continuous-time dynamic system by simply considering the latter only at discrete instants of time, this difficulty can be removed in a straightforward manner. For example, if we search for deadbeat regulation of the state it is evident from the discrete-time transition equation that $\underline{x}_k \equiv 0$, plus $\underline{u}_{k+p} \equiv 0$ for all $p > 0$ implies $\underline{x}_{k+p} \equiv 0$; i.e., the system is at its equilibrium position at all later sampling instants. But knowing that we are dealing with a continuous-time dynamic system described by $\dot{\underline{x}} = F\underline{x} + G\underline{u}$, is enough to ascertain that if $\underline{x}_k \equiv 0$ and $\underline{u}(t) = 0$ for $t > kT$, then $\underline{x}(t) \equiv 0$ for $t > kT$. In other words, regulation of the system at all sampling instants is sufficient to insure deadbeat regulation of the state.

Another difficulty appears when looking at the output vector $\underline{y}$. Assume that we want the output vector to become some predetermined function of time $\underline{y}_d(t)$ after a transient period. Considering the continuous time description of the system, we have

$$\underline{y}(t) = M\underline{x}(t) \tag{3.2.2}$$

where M is generally a non-square matrix, because the output vector generally has a smaller dimensionality than the state vector. If this is the case, then for a given $\underline{y}_d(t)$ there exists an infinity of $\underline{x}_d(t)$ satisfying equation 3.2.2. Among this infinity of solutions of equation 3.2.2 there may exist several solutions of the vector differential equation $\dot{\underline{x}} = F\underline{x} + G\underline{u}$ where u is a stepwise constant function of time. Therefore to a given $\underline{y}_d(t)$, there may correspond several states of the dynamic system each of which would result in an output equal to

$\underline{y}_d(t)$. This means the solution to a problem may not be unique. However, in the case of deadbeat regulation of the output, i.e. the desired output is $\underline{y}_d(t) \equiv 0$, there is always at least the solution $\underline{x}_d(t) \equiv 0$, which amounts to regulating the state. There may exist other solutions but we will limit ourselves to this particular one and avoid the difficulties of a multiple solution problem.

As an extreme example, consider the system shown in Fig. 6. It is clear that $y_1(t)$ can vanish for all times, although $\underline{x}(t) = [x_1(t), x_2(t)]$ is non-zero. For example, choose $u_1(t)$ and $u_2(t)$ so that $\underline{x}_1(t) = -x_2(t) = C$ after a certain transient period, where C is an arbitrary constant.

We will even show that for a one-input, one-output system described by an ordinary transfer function and driven by a stepwise constant function of time, deadbeat regulation of the output can be obtained without having deadbeat regulation of the state. Take the usual description given in 3.1.2 for an n th order system

$$\begin{cases} \dot{x}_1 = x_2 + \alpha_{n-1}u \\ \dot{x}_2 = x_3 + \alpha_{n-2}u \\ \cdots\cdots\cdots\cdots\cdots \\ \dot{x}_{n-1} = x_n + \alpha_1 u \\ \dot{x}_n = (-b_0x_1 - b_1x_2 - \dots - b_{n-1}x_n) + \alpha_0 u \end{cases}$$

and

$$x_1(t) \equiv c(t), \text{ the output of the system}$$

We want deadbeat regulation of the output after a finite transient period, i.e.

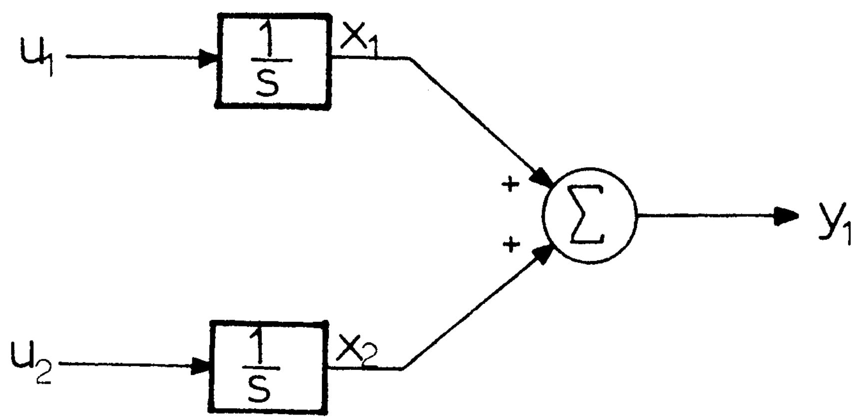

Figure 6: A very special multiple input, single output system.

$$x_1(t) \equiv 0 \qquad \text{for} \qquad t \geq t_s$$

In particular

$$\dot{x}_1 \equiv 0 \qquad \text{for} \qquad t > t_s$$

i.e. $$x_2 + \alpha_{n-1} u \equiv 0 \qquad \text{for} \qquad t > t_s$$

But by previous restriction $u(t)$ must be a stepwise constant function of time. Therefore $\dot{x}_2$ must be zero, except perhaps at the sampling instants. Continuing with the same type of reasoning, one gets the following conditions for deadbeat regulation

$$\begin{cases} x_1 = 0 \\ x_2 + \alpha_{n-1} u = 0 \\ x_3 + \alpha_{n-2} u = 0 \\ \cdots\cdots\cdots\cdots \\ x_n + \alpha_1 u = 0 \\ -b_1 x_2 - b_2 x_3 - \cdots - b_{n-1} x_n + \alpha_0 u = 0 \end{cases}$$

The last n linear equations contain the n unknown $x_2, x_3, \ldots, x_n$ and u. There will exist at least one non-trivial solution if and only if the determinant is zero. Equivalently write

$$\begin{cases} x_2 = -\alpha_{n-1} u \\ \cdots\cdots\cdots\cdots \\ x_n = -\alpha_1 u \end{cases}$$

and substitute these values into the last equation

$$(b_1\alpha_{n-1} + b_2\alpha_{n-2} + \cdots + b_{n-1}\alpha_1 + \alpha_0)u = 0$$

or using 3.1.3

$$a_0 u = 0$$

If $a_0 \neq 0$, then the only solution is $u = 0$ which leads to the trivial solution $\underline{x} = 0$.

But if $a_0 = 0$, i.e. for a transfer function of the form

$$G(s) = \frac{a_1 s + \ldots + a_m s^m}{b_0 + b_1 s + \ldots + s^n}$$

there are an infinity of solutions. Namely, the states to be reached for getting an output vanishing identically are all on a straight line in the state space. Notice that this can only happen if the plant contains "numerator dynamics."

If it is desired that the output vector becomes identical to some non-zero function of time $\underline{y}_d(t)$, the problem is more complicated, because then there is no obvious solution for $\dot{\underline{x}}_d(t)$. This problem will be discussed in a later section of this dissertation.

CHAPTER IV

DEADBEAT REGULATION AND CONTROL OF SINGLE INPUT, SINGLE OUTPUT SYSTEMS

4.1 Continuous Compensation in an Optimal Regulating System.

Rather than consider the problem of digital compensation at the outset, it is more instructive to first look at continuous compensation. This will give us a much better understanding of the role of the discrete compensator which, as we shall see, can also be directly determined using the z-transform theory. In the material which immediately follows the problem solved by Kalman and Bertram (7) will be discussed. The arguments and explanations used by the authors will not be repeated here but rather we shall use a different and somewhat heuristic approach to obtain a solution.

We want to consider a single input, linear, time invariant sampled-data system. Let $\underline{x}_k$, an n-dimensional vector, represent the state of the system at time $t = kT$, when T is the sampling period. The input to the plant, referred to as the control, is constant during each sampling interval and may be expressed as

$$u_k = u(kT) \qquad\qquad kT \leq t < (k+1)T$$

The state transition equation can be written as

$$\underline{x}_{k+1} = \phi\underline{x}_k + \Delta u_k \quad .$$

when ϕ is a non-singular constant $(n \times n)$ matrix and Δ is a constant $(n \times 1)$ matrix, i.e. a vector in the n-dimensional state space.

With this description we wish to solve the following problem. Given an arbitrary initial state $\underline{x}_0$, find the control sequence (specified by $u_0, u_1, u_2, \ldots$) which will force the system to its equilibrium position (i.e. to the state $\underline{x} \equiv 0$) in the minimum number of sampling periods. We shall also assume that all of the state variables are instantaneously available.

We assume that the system is controllable, i.e. that the vectors $\Delta, \phi\Delta, \phi^2\Delta, \ldots, \phi^{(n-1)}\Delta$ span the n-dimensional space, and will show shortly why this assumption is necessary. Then, if the system is controllable, the problem we wish to solve does have a solution and furthermore the solution will show that it is possible to reach the equilibrium state in at most "n" sampling periods. In other words, we can always determine a control sequence $u_0, u_1, \ldots, u_{n-1}$, such that $\underline{x}_k \equiv 0$ for $k \geq n$, starting from an arbitrary $\underline{x}_0$. In addition, the required control can be obtained using a linear, time invariant feedback scheme in which

$$u_k = \alpha_1 x_{k,1} + \alpha_2 x_{k,2} + \cdots + \alpha_n x_{k,n}$$

A heuristic proof of the above statements proceeds as follows. The state of the system at successive sampling instants is given by

$$\underline{x}_1 = \phi\underline{x}_0 + u_0\Delta$$

$$\underline{x}_2 = \phi^2\underline{x}_0 + u_0\phi\Delta + u_1\Delta$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\underline{x}_n = \phi^n\underline{x}_0 + u_0\phi^{n-1}\Delta + u_1\phi^{n-2}\Delta + \cdots + u_{n-1}\Delta$$

Since we desire that $\underline{x}_n \equiv 0$ and since ϕ is non-singular, we must have

$$\underline{x}_0 = - u_0\phi^{-1}\Delta - u_1\phi^{-2}\Delta - \dots - u_{n-1}\phi^{-n}\Delta \qquad (4.1.1)$$

If we let $\underline{e}_k = -\phi^{-k}\Delta$, then the above equation can be written

$$\underline{x}_0 = u_0\underline{e}_1 + u_1\underline{e}_2 + \dots + u_{n-1}\underline{e}_n \qquad (4.1.2)$$

Thus equation 4.1.2 represents n scalar equations with n unknowns, $u_0, u_1, \dots, u_{n-1}$, for any $\underline{x}_0 = (x_{0,1}, x_{0,2}, \dots, x_{0,n})$. This equation has a solution for any $\underline{x}_0$ if and only if the determinant of the coefficients of the u's is different from zero. This is equivalent to saying that the n vectors $\underline{e}_k$ $(k = 1, 2, \dots, n)$ are linearly independent. Using the definition of $\underline{e}_k$ and the fact that ϕ is non-singular, this means that the vectors Δ, $\phi\Delta$, ... , $\phi^{(n-1)}\Delta$ must be linearly independent. In other words, a solution exists if the system is controllable. Note also that a solution in less than n sampling periods is generally impossible. Assuming this to be the case, equation 4.1.2 has one and only one solution for each $\underline{x}_0$, say

$$\begin{cases} u_0 = \alpha_{1,1}x_{0,1} + \alpha_{1,2}x_{0,2} + \dots + \alpha_{1,n}x_{0,n} \\ u_1 = \alpha_{2,1}x_{0,1} + \alpha_{2,2}x_{0,2} + \dots + \alpha_{2,n}x_{0,n} \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ u_{n-1} = \alpha_{n,1}x_{0,1} + \alpha_{n,2}x_{0,2} + \dots + \alpha_{n,n}x_{0,n} \end{cases} \qquad (4.1.3)$$

Thus we have found a control sequence which depends only upon the initial value of the state variables. But let us look at this control sequence more closely and in particular consider it with respect to the state $\underline{x}_1$ which is the state at the end of the first sampling period. If we consider $\underline{x}_1$ as a new initial condition, we know that we can reach the equilibrium point in at most n sampling periods with a uniquely determined control sequence, u_0^1, u_1^1, ... , u_{n-1}^1, with

$$u_0^1 = \alpha_{1,1}x_{1,1} + \alpha_{1,2}x_{1,2} + \cdots + \alpha_{1,n}x_{1,n}$$

Since $\underline{x}_1$ is an intermediate position and since the control sequence is unique, it follows that

$$u_0^1 = u_1$$
$$u_1^1 = u_2$$
$$\cdots\cdots$$
$$u_{n-2}^1 = u_{n-1}$$
$$u_{n-1}^1 = u_n = 0$$

Hence u_1 can be written

$$u_1 = \alpha_{1,1}x_{1,1} + \alpha_{1,2}x_{1,2} + \cdots + \alpha_{1,n}x_{1,n}$$

.

A comparison of this equation with u_0 in equation 4.1.3 shows that the control can be obtained through linear time invariant feedback since the same type of reasoning can be applied to u_2. This means that if u_0 is some linear combination of the state variables measured at time $t = 0$, $u_k (k = 1, 2, \ldots)$ is given by the same linear combination of the state variables measured at time $t = kT$. Thus, the feedback required is both linear and time invariant.

At this time the regulator problem has been conceptually solved; all that is needed is a knowledge of the state variables at each sampling instant. If only one state variable is directly measurable, as it is the case for systems with only one output, it is theoretically possible to find a continuous compensator whose output would be the required linear combination of the instantaneous values of the state variables. For example, in the case of a simple system in which the n-state variables are assumed to be the output and its first (n-1) derivatives, the minimal time system can be represented as shown in Figure 7. However, the difficulties associated with the physical realization of a continuous compensator of this type are such that it seems desirable to look for another type of compensation and an alternative procedure will be studied in the next section.

Before leaving this question of continuous compensation, let us indicate a formal way of computing the coefficients of the optimal linear combination of state variables. In matrix notation, we will write

$$u_0 = A\underline{x}_0$$

where A is an n-dimensional row vector to be determined. To do this first write equation 4.1.1 in the form

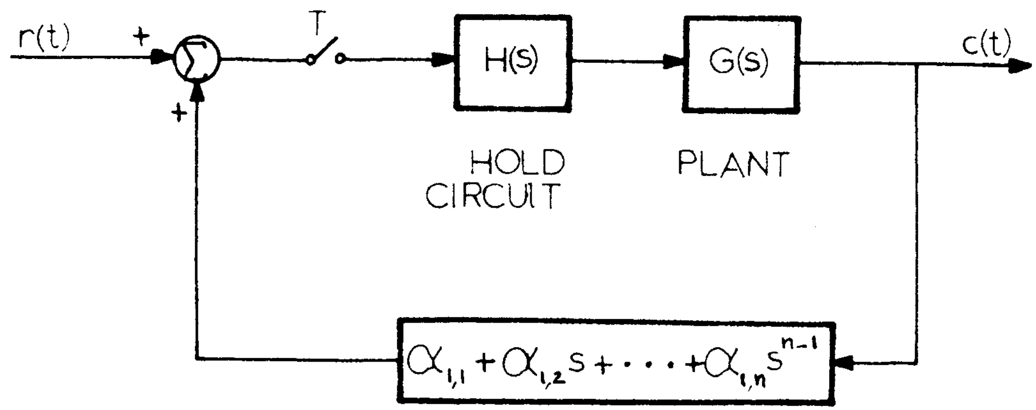

Figure 7: A time-optimal regulating system using continuous compensation.

$$-\underline{x}_0 = [\phi^{-1}\Delta|\phi^{-2}\Delta| \;\dots\; |\phi^{-n}\Delta] \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-1} \end{bmatrix}$$

so that, with the assumptions already made,

$$\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-1} \end{bmatrix} = - [\phi^{-1}\Delta|\phi^{-2}\Delta| \;\dots\; |\phi^{-n}\Delta]^{-1} \underline{x}_0$$

The row vector A is therefore the first row of the matrix $-[\phi^{-1}\Delta|\phi^{-2}\Delta| \;\dots\; |\phi^{-n}\Delta]^{-1}$ and it can be uniquely characterized by n scalar equations

$$\begin{cases} A\phi^{-1}\Delta = -1 \\ A\phi^{-2}\Delta = 0 \\ \dots\dots\dots\dots \\ A\phi^{-n}\Delta = 0 \end{cases}$$

or by the matrix equation

$$A[\phi^{-1}\Delta|\phi^{-2}\Delta| \;\dots\; |\phi^{-n}\Delta] = [-1 \; 0 \; \dots \; 0]$$

With this equation the row matrix A can be readily computed.

4.2 Discrete Compensation.

The ideas just presented reveal one major difficulty in the practical realization of a minimal time regulating system, namely, that all state variables must be measured. If they are not directly measureable then it is necessary to simulate or approximate them (7). Realizing this it then seems natural to ask if it is possible to design a deadbeat regulating system when all components of the state are not directly measurable without using synthetic state variables. If so just what is required, or what is the penalty that is imposed in such a case?

First the answer as to whether or not such a system can be designed is "yes" and the solution involves the use of a digital compensator which, in addition to implementing the control sequence, identifies or determines the state variables which are not directly measurable. For example, in the case in which only one state variable is available, the digital compensator drives (more or less arbitrarily) the n th order system for $(n-1)$ sampling periods in order to determine all state variables and once having a complete description of the state of the system, the digital compensator proceeds to force the system to the equilibrium position in an additional n sampling periods. Thus it requires $(2n-1)$ sampling periods to reach the equilibrium position of the system. In general, if "p" $(1 \leq p \leq n)$ components of the state, or "p" linearly independent combinations of the state variables are directly observable then at least $(n-p)/p$ sampling periods (if $(n-p)/p$ is not an integer replace it by the first larger integer) will be necessary to completely identify the state and at least $n+(n-p)/p$ sampling periods will be required to reach the equilibrium position.

Let us now show that a digital compensator can do just this and also show how such a compensator can be realized. The system starts from some initial state $\underline{x}_0$.

$$\underline{x}_0 = \begin{bmatrix} x_{0,1} \\ x_{0,2} \\ \vdots \\ x_{0,n} \end{bmatrix}$$

Assume that only $x_{0,1}$ is measurable at $t = 0$ and at future times kT, $(k = 1, 2, \ldots)$ only $x_{k,1}$ is measurable. The state of the system at successive sampling instants can be written

$$\begin{aligned} \underline{x}_1 &= \phi \underline{x}_0 + u_0 \Delta \\ \underline{x}_2 &= \phi^2 \underline{x}_0 + u_0 \phi \Delta + u_1 \Delta \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \underline{x}_{n-1} &= \phi^{n-1} \underline{x}_0 + u_0 \phi^{n-2} \Delta + \cdots + u_{n-2} \Delta \end{aligned} \qquad (4.2.1)$$

Consider the scalar equations corresponding to the first component of each of these vector equations. There are $(n-1)$ such equations and they are linear in the $(n-1)$ unknown quantities $x_{0,2}, x_{0,3}, \ldots, x_{0,n}$. In these equations $x_{k,1}$ $(k = 0, 1, \ldots, n-1)$ and $u_k (k=0,1,\ldots,n-2)$ are known since u_k is the forcing function at $t = kT$ and $x_{k,1}$ is the measurable component of the state of the system. Generally speaking (more exactly: if the system is completely observable), this system of equations has a solution which gives $x_{0,2}, x_{0,3}, \ldots, x_{0,n}$ as a linear combination of $x_{k,1}$ and u_k. Once the initial state variables are known, the last vector equation of equation 4.2.1 can be used to compute

$\underline{x}_{n-1}$. Thus, $\underline{x}_{n-1}$ can be written as

$$\underline{x}_{n-1} = B \begin{bmatrix} x_{0,1} \\ x_{1,1} \\ \vdots \\ x_{n-1,1} \end{bmatrix} + C \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-2} \end{bmatrix} \qquad (4.2.2)$$

where B is a constant $(n \times n)$ matrix, and C is an $[nx(n-1)]$ constant matrix. Notice that the determination of $\underline{x}_{n-1}$ does not depend upon the choice of u_i but rather on a knowledge of the value of u_i.

Also since B and C depend only upon the system being regulated and the sampling period, equation 4.2.2 may be expressed as

$$\underline{x}_k = B \begin{bmatrix} x_{k-n+1,1} \\ \cdots\cdots\cdots\cdots \\ x_{k-1,1} \\ x_{k,1} \end{bmatrix} + C \begin{bmatrix} u_{k-n+1} \\ \cdots\cdots \\ u_{k-2} \\ u_{k-1} \end{bmatrix} ; \; k \geq (n-1) \qquad (4.2.3)$$

Thus after $(n-1)$ sampling periods, all components of the state vector are known. However, for a minimal time deadbeat response we must have

$$u_k = \alpha_{1,1}x_{k,1} + \alpha_{1,2}x_{k,2} + \cdots + \alpha_{1,n}x_{k,n}$$

Using equation 4.2.3, u_k can be expressed as

$$u_k = \text{L.C.}\, [x_{k,1}, x_{k-1,1}, \cdots , x_{k-n+1,1}; u_{k-1}, \cdots , u_{k-n+1}] \qquad (4.2.4)$$

where L.C. [ ] indicates a "linear combination of." Since equation 4.2.4 indicates linear combinations, it can also be written as

$$\text{L.C. } [u_k, u_{k-1}, \dots , u_{k-n+1}] = \text{L.C. } [x_{k,1}, x_{k-1,1}, \dots , x_{k-n+1,1}] \tag{4.2.5}$$

and equation 4.2.5 can be interpreted as meaning that the desired control sequence can be obtained as a linear combination of the values of the measurable state variable at successive sampling instants. In terms of the z-transform, equation 4.2.5 characterizes a discrete compensator $D(z)$ having a pulse transfer function of the form

$$D(z) = \frac{\text{polynomial in } z^{-1} \text{ of order } (n-1)}{\text{polynomial in } z^{-1} \text{ of order } (n-1)}$$

If the measurable state variable is the output of the system, the compensated system has the classical aspect shown in Figure 1.

Such a regulator will take the system from an arbitrary initial position to its equilibrium position in at most $(n-1) + n$ sampling periods, the first $(n-1)$ sampling periods being used in a certain sense to identify the state of the system, the last n sampling periods being used to reach the equilibrium position. To show that this represents the minimal time solution it is only necessary to note that it is impossible to achieve the equilibrium position without a knowledge of the state variables. Equation 4.2.2 shows that at least $(n-1)$ sampling periods are required to compute all components of the state; the fact that it requires n sampling periods to reach the equilibrium state, once the state is known, has already been shown to be the optimal solution. Hence the solution just presented is the minimal-time solution.

Example:

To clarify and illustrate the ideas just presented, consider the following simple problem. Referring again to Figure 1 in which $G(s) = 1/s(s+1)$, find the discrete compensator which will force the system from an arbitrary initial state to the equilibrium state $(\underline{x} \equiv 0)$ in the minimum time, assuming that only one state variable (referred to as the output) is observable.

In this case, it is easily seen that the output of the plant and its first derivative are the state variables, so that the conventional vector differential equation corresponding to this plant is

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_2+u \end{cases} \quad \text{where} \quad \begin{cases} x_1 = c \\ x_2 = \dot{c} \end{cases}$$

The continuous time solution is

$$\underline{x}(t) = \begin{bmatrix} 1 & 1-e^{-(t-t_0)} \\ 0 & e^{-(t-t_0)} \end{bmatrix} \underline{x}(t_0) + \int_{t_0}^{t} \begin{bmatrix} 1 & 1-e^{-(t-s)} \\ 0 & e^{-(t-s)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(s)ds$$

The discrete-time solution, including the effect of the first-order hold, is

$$\underline{x}_{k+1} = \begin{bmatrix} 1 & 1-e^{-T} \\ 0 & e^{-T} \end{bmatrix} \underline{x}_k + u_k \begin{bmatrix} T - 1+e^{-T} \\ 1 - e^{-T} \end{bmatrix}$$

It is easily verified that this system is controllable for any T.

Although we are really interested in the case where only one state variable, namely $x_1 = c$, is measurable, we assume first that both state variables are known at each sampling instant. To determine the coefficients of the optimal feedback scheme start from an arbitrary $\underline{x}_k$, and recall that $\underline{x}_{k+2}$ has to be zero. Thus,

$$\underline{x}_{k+2} \equiv 0 = \phi^2 \underline{x}_k + u_k \phi \Delta + u_{k+1} \Delta$$

Let us carry out the computations for $T = 1.0$ second.

$$\phi = \begin{bmatrix} 1 & .632 \\ 0 & .368 \end{bmatrix} \qquad \Delta = \begin{bmatrix} .368 \\ .632 \end{bmatrix}$$

$$0 = \begin{bmatrix} 1 & .863 \\ 0. & .135 \end{bmatrix} \begin{bmatrix} x_{k,1} \\ x_{k,2} \end{bmatrix} + u_k \begin{bmatrix} .768 \\ .231 \end{bmatrix} + u_{k+1} \begin{bmatrix} .368 \\ .632 \end{bmatrix}$$

Solving this vector equation for u_k shows that

$$-u_k = 1.580_{k,1} + 1.242x_{k,2} \tag{4.2.6}$$

If only $x_{k,1}$ is observable, $x_{k,2}$ must be eliminated from this equation if the discrete compensator is to depend only upon the observable state variable. Since we are dealing with a second-order system and can only observe directly one state variable, it will be necessary to wait one sampling period to have a complete description of the state.

The equation

$$x_{k+1,1} = x_{k,1} + 0.632x_{k,2} + 0.368u_k$$

shows that the unknown initial state variable is given by

$$x_{k,2} = (0.632)^{-1}[x_{k+1,1} - x_{k,1} - 0.368u_k]$$

Therefore at $t = (k+1)T$, the complete state of the system is known. The state variable $x_{k+1,1}$ is measured and the other state variable may be expressed as

$$x_{k+1,2} = .368x_{k,2} + 0.632u_k$$

These last two equations may be combined and the inobservable state variable can be expressed as a function of measurable quantities

$$x_{k+1,2} = .583(x_{k+1,1} - x_{k,1} - 0.368u_k) + 0.632u_k$$

$$x_{k+1,2} = 0.583(x_{k+1,1} - x_{k,1}) + 0.418u_k$$

Combining this last equation and equation 4.2.6 we get

$$-u_{k+1} = 1.580x_{k+1,1} + 1.242 \times 0.583(x_{k+1,1} - x_{k,1}) + 1.242 \times 0.418u_k$$

$$-u_{k+1} - 0.520u_k = 2.304x_{k+1,1} - 0.725x_{k,1}$$

Noting that u_k is the force applied at $t = kT$, that the input $r(t)$ to be followed by the system is assumed to be identically zero, and that in terms of the z-transform the transfer function of the discrete compensator, $D(z)$, is given by $U(z)/-X_1(z)$, the discrete compensator is found to be

$$D(z) = \frac{2.304 - 0.725z^{-1}}{1 + 0.520z^{-1}}$$

If we use the pulse transfer function $GH(z)$, computed in Section 2.3, to find the closed-loop pulse transfer function of the system so compensated we get

$$K(z) = \frac{D(z)GH(z)}{1 + D(z)GH(z)} = .848z^{-1}(1 + .717z^{-1})(1 - .314z^{-1})$$

The point to observe about $K(z)$ at this time is that it is of third order in z^{-1}, one order higher than it would be if the standard techniques of compensation for deadbeat response to step input had been used (see Section 2.3). More will be said about this a little later.

4.3 Non-Zero Inputs.

Rather than discuss separately step inputs, then ramp inputs, etc.

we will consider the class C_m of inputs expressible as polynomials in t of maximum order m

$$\begin{cases} r(t) = \sum_{i=0}^{m} r_i t^i & \text{for} \quad t \geq 0 \\ r(t) = 0 & \text{for} \quad t < 0 \end{cases}$$

The question now is whether it is possible to find a compensator which will allow the system, starting from arbitrary initial conditions, to follow in a deadbeat manner any input belonging to the class C_m.

Since we want the output to be identically equal to the input after a certain transient period, the plant must be capable of generating a smooth output identical to the input, when driven by a stepwise constant function of time (the output of the zero-order hold). For inputs of the class C_m, the plant must contain at least m integrators.

Consider, therefore, the state description of a system having n poles, among which at least m would be at the origin, in the s-plane, and having at most $(n-1)$ finite zeros so that the output can be taken as one of the state variables. Using the state variable description corresponding to the analog computer setup of Figure 5 we can write

$$c = x_1$$

and

$$\begin{cases} \dot{x}_1 = x_2 + \alpha_{n-1} u \\ \dot{x}_2 = x_3 + \alpha_{n-2} u \\ \cdots\cdots\cdots\cdots\cdots \\ \dot{x}_{n-1} = x_n + \alpha_1 u \\ \dot{x}_n = (-b_{n-1} x_n - b_{n-2} x_{n-1} - \cdots - b_m x_{m+1}) + \alpha_0 u \end{cases} \tag{4.3.1}$$

If $b_m \neq 0$, the plant has exactly m integrators. When solving the follow-up problem, we will try to use as much as possible of the results obtained for the regulator problem. With this idea in mind, we form the modified state variables

$$\begin{cases} \hat{x}_1 = x_1 - r \\ \hat{x}_2 = x_2 - \frac{dr}{dt} \\ \dots\dots\dots\dots \\ \hat{x}_m = x_m - \frac{d^{m-1}r}{dt^{m-1}} \\ \hat{x}_{m+1} = x_{m+1} - \frac{d^m r}{dt^m} \\ \hat{x}_k = x_k \qquad \text{for} \qquad (m+1) < k \leq n \end{cases} \tag{4.3.2}$$

Solving these equations for x_j $(j = 1, \dots , n)$ and substituting in equation 4.3.1 yields

$$\begin{cases} \frac{d\hat{x}_1}{dt} = \hat{x}_2 + \alpha_{n-1}u \\ \frac{d\hat{x}_2}{dt} = \hat{x}_3 + \alpha_{n-2}u \\ \dots\dots\dots\dots\dots\dots \\ \frac{d\hat{x}_{n-1}}{dt} = \hat{x}_n + \alpha_1 u \\ \frac{d\hat{x}_n}{dt} = (-b_{n-1}\hat{x}_n - b_{n-2}\hat{x}_{n-1} - \dots - b_{m+1}\hat{x}_{m+2} - b_m\hat{x}_{m+1}) + \alpha_0 u - b_m \frac{d^m r}{dt^m} \end{cases} \tag{4.3.3}$$

If we compare equations 4.3.1 and 4.3.3, we notice that the presence of b_m introduces a difficulty.

Let us first assume that $b_m = 0$, i.e. the plant contains at least $(m+1)$ integrators. In this case, the modified state variables $\hat{x}_i$ obey the same differential equations as the unmodified ones (equation 4.3.1) in the case of regulation. Therefore, the previously determined analog feedback, operating on the modified state variables, namely $u_k = \alpha_{1,1}\hat{x}_{k,1} + \alpha_{1,2}\hat{x}_{k,2} + \cdots + \alpha_{1,n}\hat{x}_{k,n}$, will drive the system to the equilibrium position

$$\underline{\hat{x}} \equiv 0, \qquad \text{or} \qquad c(t) = x_1(t) \equiv r(t)$$

after n sampling periods.

If we assume that only $\hat{x}_{k,1} = x_{k,1} - r_k = c_k - r_k = -e_k$ is measurable at each sampling instant $t = kT$, the identification procedure is also unchanged; therefore the discrete compensator which was previously determined for the regulation, will correctly identify the state after $(n-1)$ sampling instants and then drive the error to zero in the next n sampling instants, if it operates on the sampled-error signal.

Assume that now $b_m \neq 0$ in which case the plant contains exactly m integrators and is to respond in a deadbeat manner to inputs

$$r(t) = \sum_{i=0}^{m} r_i t^i$$

so that $d^m r/dt^m$ is generally a non-zero constant function of time. In this case, the modified state variables $\hat{x}_i$'s do not obey the same vector differential equation as the unmodified ones x_i's, i.e. regulation of the x_i's and regulation of the $\hat{x}_i$'s are different problems.

That was to be expected because if such a plant is to follow an input belonging to the class C_m the steady-state value of the forcing function $u(t)$ must be different of zero. In order to motivate the transformations that will be made on the vector differential equation, let us find quickly the value of the steady-state forcing function using the transfer function description of the plant considered here, namely

$$G(s) = \frac{a_{n-1}s^{n-1} + \dots + a_1 s + a_0}{s^n + b_{n-1}s^{n-1} + \dots + b_m s^m}$$

where $b_m \neq 0$ (and $a_0 \neq 0$) so that the plant contains exactly m integrators. If the steady-state value of the forcing function is U, the transform of the output of the plant will have the following form in steady-state

$$C(s) = \frac{a_{n-1}s^{n-1} + \dots + a_1 s + a_0}{(s^{n-m} + \dots + b_m)\, s^m}\,\frac{U}{s} + \frac{\text{some polynomial in } s}{(s^{n-m} + \dots + b_m)\, s^m}$$

If $c(t) \equiv r(t) = \sum_{i=0}^{m} r_i t^i$ in steady-state, application of the final value theorem to the m th derivative of $c(t)$ shows that

$$\frac{d^m c}{dt^m} = \frac{d^m r}{dt^m} = \frac{a_0}{b_m}\, U, \qquad \text{i.e.} \qquad U = \frac{b_m}{a_0}\,\frac{d^m r}{dt^m} \quad (a_0 \neq 0)$$

In other words, if we want to transform as previously the deadbeat follow-up problem into an already solved deadbeat regulation problem for some modified variables, it is clear that we will have to introduce a modified forcing function which will characterize the variations of u around its steady-state value.

To that effect, we first define

$$u(t) = u_{s.s.} + \check{u}(t)$$

where $u_{s.s.}$ is a constant function of time, to be determined. Define also another set of modified state variables

$$\begin{cases} \check{x}_1 = \hat{x}_1 = x_1 - r \\ \check{x}_2 = \hat{x}_2 + \alpha_{n-1} u_{s.s.} \\ \check{x}_3 = \hat{x}_3 + \alpha_{n-2} u_{s.s.} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \check{x}_n = \hat{x}_n + \alpha_1 u_{s.s.} \end{cases} \qquad (4.3.4)$$

Substituting in equation 4.3.3 yields

$$\begin{cases} \dfrac{d\check{x}_1}{dt} = \check{x}_2 + \alpha_{n-1}\check{u} \\ \dfrac{d\check{x}_2}{dt} = \check{x}_3 + \alpha_{n-2}\check{u} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{d\check{x}_{n-1}}{dt} = \check{x}_n + \alpha_1\check{u} \\ \dfrac{d\check{x}_n}{dt} = (-b_{n-1}\check{x}_n - b_{n-2}\check{x}_{n-1} - \dots - b_m\check{x}_{m+1}) + \alpha_0\check{u} \\ \qquad + (\alpha_0 + \alpha_1 b_{n-1} + \dots + \alpha_{n-m} b_m)\, u_{s.s.} - b_m \dfrac{d^m r}{dt^m} \end{cases} \qquad (4.3.5)$$

If we choose $u_{s.s.}$ such that

$$(\alpha_0 + \alpha_1 b_{n-1} + \cdots + \alpha_{n-m} b_m)\, u_{s.s.} - b_m \frac{d^m r}{dt^m} = 0$$

which using equation 3.1.3, can be written as

$$u_{s.s.} = \frac{b_m}{a_0} \frac{d^m r}{dt^m} \qquad (4.3.6)$$

we obtain exactly the vector differential equation 4.3.1 for these modified state variables with the modified forcing function $\check{u}$. Regulation of the $\check{x}$'s corresponds to having

$$\check{x}_1(t) = \hat{x}_1(t) = c(t) - r(t) \equiv 0 \qquad \text{with} \qquad \check{u}(t) \equiv 0$$

and this is the result we wanted to obtain. Thus, the discrete-time "solution" of equation 4.3.5 is the same as the one used for the solution of the regulator problem, namely

$$\check{\underline{x}}_{k+1} = \phi\check{\underline{x}}_k + \Delta\check{u}_k = \phi\check{\underline{x}}_k + \Delta(u_k - u_{s.s.}) \qquad (4.3.7)$$

and deadbeat follow-up will be obtained in minimum time if the control law has the following form

$$u_k - u_{s.s.} = \check{u}_k = \alpha_{1,1}\check{x}_{k,1} + \alpha_{1,2}\check{x}_{k,2} + \cdots + \alpha_{1,n}\check{x}_{k,n} \qquad (4.3.8)$$

On the other hand, in terms of the physical system described by equation

4.3.2, it is possible to re-write equations 4.3.7 and 4.3.8 in the following forms:

$$\begin{bmatrix} \hat{x}_{k+1,1} \\ \hat{x}_{k+1,2} + \alpha_{n-1}u_{s.s.} \\ \cdots\cdots\cdots \\ \hat{x}_{k+1,n} + \alpha_1 u_{s.s.} \end{bmatrix} = \phi \begin{bmatrix} \hat{x}_{k,1} \\ \hat{x}_{k,2} + \alpha_{n-1}u_{s.s.} \\ \cdots\cdots\cdots \\ \hat{x}_{k,n} + \alpha_1 u_{s.s.} \end{bmatrix} + \Delta(u_k - u_{s.s.})$$

(4.3.9)

$$(u_k - u_{s.s.}) = \alpha_{1,1}\hat{x}_{k,1} + \alpha_{1,2}[\hat{x}_{k,2} + \alpha_{n-1}u_{s.s.}] + \cdots + \alpha_{1,n}[\hat{x}_{k,n} + \alpha_1 u_{s.s.}]$$

This last equation can be rearranged to take the form

$$u_k = \alpha_{1,1}\hat{x}_{k,1} + \alpha_{1,2}\hat{x}_{k,2} + \cdots + \alpha_{1,n}\hat{x}_{k,n} + \beta u_{s.s.} \qquad (4.3.10)$$

Thus, at each sampling instant, if we want to compute the optimal forcing function we need to know not only the complete modified state $\check{x}_k$ (or $\hat{x}_k$) but also $u_{s.s.}$ which characterizes the input signal.

An implementation of this control law could be theoretically obtained using a continuous compensator as indicated in Figure 8 for a system in which the output and its first (n-1) derivatives are assumed to be the n state-variables (the parameter β of equation 4.3.10 is equal to 1.0 because in such a case $\alpha_i = 0$ for $i = 1, 2, \ldots, n-1$). Generally $\hat{x}_{k,1} = \check{x}_{k,1}$ will be the only measurable state variable, so that we will have to identify not only the remaining (n-1) state

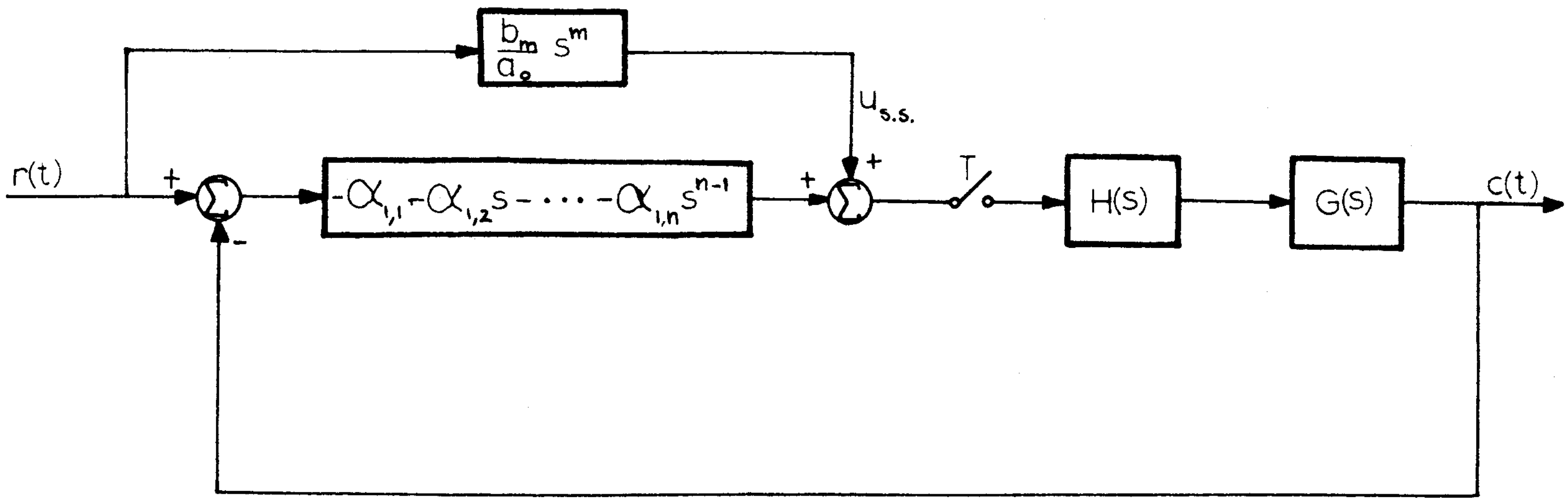

Figure 8: Time-optimal continuous compensation allowing a plant $G(s)$ containing m integrations to follow any input of the class C_m.

variables, but also $u_{s.s.}$. Note that $u_{s.s.}$ can be identified because equation 4.3.7 shows that it acts roughly like an additional forcing function for the modified state $\check{x}_k$. We previously needed (n-1) sampling instants to identify the complete state of the system by observing the first component of the state and the forcing function. It is easy to see that now at the end of (n-1) sampling instants, the complete state will be known as a linear function of $u_{s.s.}$. An extra sampling period will allow one to determine $u_{s.s.}$ and therefore to identify completely all the unknowns. Using this information in equation 4.3.8, the corresponding discrete compensator will be

$$\text{L.C.}\ [u_k,\ u_{k-1},\ \dots\ ,\ u_{k-n}] = \text{L.C.}\ [\check{x}_{k,1}\check{x}_{k-1,1},\ \dots\ ,\ \check{x}_{k-n,1}]$$

The transfer function of the discrete compensator D(z) is now the ratio of two polynomials in z^{-1}, each of order n, instead of order (n-1) as it is for the regulator problem. Note that this D(z) must always have a pole at z = 1, that is, it must contain the digital equivalent of one integrator, simply because when the error is reduced to zero, we must have $u_k = u_{s.s.}$ and this is generally a non-zero term.

Example. Let $G(s) = 1/s(s+1)$ in Figure 1 and let us look for a digital compensator which will make the output c(t) follow any ramp input. For the regulator problem, we had chosen the following state variables

$$\begin{cases} x_1 = c \\ x_2 = \dot{c} \end{cases}$$

which obeyed the differential equations

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_2 + u \end{cases}$$

The discrete time transition equation (with $T = 1$) was

$$\begin{cases} x_{k,1} = x_{k-1,1} + 0.632x_{k-1,2} + 0.368u_{k-1} \\ x_{k,2} = 0.368x_{k-1,2} + 0.632u_{k-1} \end{cases}$$

By observing u and $x_{i,1}$ we were able to identify the state after one sampling period and find

$$x_{k,2} = 0.583x_{k,1} - 0.583x_{k-1,1} + 0.418u_{k-1}$$

The optimal feedback was defined by

$$u_k = -1.580x_{k,1} - 1.242x_{k,2}$$

and we computed the following relationship between the forcing function and the observable output

$$u_k + 0.520u_{k-1} = -2.304c_k + 0.725c_{k-1}$$

$$D(z) = \frac{U(z)}{-C(z)} = \frac{2.304 - 0.725z^{-1}}{1 + 0.520z^{-1}}$$

Let us know consider the problem of following ramp inputs; i.e. any input $r(t)$ for which

$$\frac{d^2r}{dt^2} = 0 \qquad ; \qquad t \geq 0$$

Using equation 4.3.4 we define

$$\begin{cases} \check{x}_1 = x_1 - r \\ \check{x}_2 = x_2 - \frac{dr}{dt} \end{cases}$$

The differential equations obeyed by these modified state variables are

$$\begin{cases} \dot{\check{x}}_1 = \check{x}_2 \\ \dot{\check{x}}_2 = - \check{x}_2 + u - \frac{dr}{dt} \end{cases}$$

We can immediately write equation 4.3.7 in the form

$$\begin{cases} \check{x}_{k,1} = \check{x}_{k-1,1} + 0.632\check{x}_{k-1,2} + 0.368 \, (u_{k-1} - \frac{dr}{dt}) \\ \check{x}_{k,2} = 0.368\check{x}_{k-1,2} + 0.632 \, (u_{k-1} - \frac{dr}{dt}) \end{cases}$$

The optimal control law (equation 4.3.8) takes the form

$$u_k - \frac{dr}{dt} = - 1.580\check{x}_{k,1} - 1.242\check{x}_{k,2}$$

It is apparent that two sampling periods are needed to identify the complete state and $u_{s.s.} = dr/dt$.

After one sampling period, $\check{x}_{k,2}$ is known as a linear function of $u_{s.s.}$

$$\check{x}_{k,2} = 0.583\check{x}_{k,1} - 0.583\check{x}_{k-1,1} + 0.418\,(u_{k-1} - u_{s.s.})$$

After an extra-sampling period, $u_{s.s.}$ can be identified using the following equation

$$\check{x}_{k+1,1} = \check{x}_{k,1} + 0.632\check{x}_{k,2} + 0.368\,(u_k - u_{s.s.})$$

$$0.632u_{s.s.} = -\check{x}_{k+1,1} + 1.368\check{x}_{k,1} - 0.368\check{x}_{k-1,1} + 0.264u_k + 0.368u_{k-1}$$

Using this result to compute $\check{x}_{k+1,2}$ and putting the complete results of the identification procedure in the equation of the optimal function, it is easy to obtain the following finite difference equation

$$u_k - 0.365u_{k-1} - 0.635u_{k-2} = -(4.710\check{x}_{k,1} - 4.015\check{x}_{k-1,1} + 0.885\check{x}_{k-2,1})$$

Noting that $\check{x}_{i,1} = -r_i + c_i = -e_i$, the corresponding discrete compensator is given by

$$D(z) = \frac{U(z)}{E(z)} = \frac{4.710 - 4.015z^{-1} + 0.885z^{-2}}{1 - 0.365z^{-1} - 0.635z^{-2}}$$

To indicate that $D(z)$ does contain an integrator, we note that

$$1 - 0.365z^{-1} - 0.635z^{-2}\Big|_{z=1} = 0$$

4.4 Design Through the Z-Transform Theory.

4.4.1 Design of the Discrete Compensator for Regulation

We showed in Section 2.2 that, in the absence of any input signal, the output sequence due to initial conditions was given by

$$C(z) = NG(z)\ [1 - K(z)]$$

Therefore the effect of initial conditions will disappear after a finite time, only if $[1 - K(z)]$ is a finite polynomial in z^{-1} and contains as zeros all the poles of $NG(z)$, i.e. all the poles of $GH(z)$.

This condition insures that the output due to initial conditions becomes zero at all sampling instants after a finite transient period. But this is not sufficient for deadbeat regulation. As in Section 2.1, we look at the sequence of forcing functions

$$E_2(z) = - D(z)\ C(z) = - \frac{1}{GH(z)} \frac{K(z)}{1 - K(z)} NG(z)\ [1 - K(z)]$$

$$E_2(z) = - \frac{NG(z)}{GH(z)} K(z)$$

This sequence of forcing functions must be finite: since the poles of

$NG(z)$ are always poles of $GH(z)$, the only requirement is that the polynomial $K(z^{-1})$ contains among its zeros all the zeros of $GH(z)$.

Assuming the plant to be controllable, the necessary and sufficient conditions for deadbeat regulation are:

1. $K(z)$ is a finite polynomial in z^{-1}.
2. $K(z)$ contains among its zeros all zeros of $GH(z)$.
3. $1-K(z)$ contains among its zeros all the poles of $GH(z)$.

The minimal time requirement will be satisfied if we take $K(z)$ of minimal order in z^{-1}.

Let us show that if only the output of the n th order plant (or the error-signal) is instantaneously observable, the minimal time is $(2n-1)$ sampling periods. For an n th order plant, $GH(z)$ is generally expressible as the ratio of two polynomials in z^{-1}, each of order n

$$GH(z) = \frac{P_1(z^{-1})}{P_2(z^{-1})}$$

$K(z)$ is determined by the two conditions

$$\begin{cases} K(z^{-1}) = P_1(z^{-1}) \times P_3(z^{-1}) \\ 1-K(z^{-1}) = P_2(z^{-1}) \times P_4(z^{-1}) \end{cases}$$

where $P_3(z^{-1})$ and $P_4(z^{-1})$ are two polynomials in z^{-1} to be determined of order as low as possible.

Combining these two equations, one gets

$$P_1(z^{-1})\, P_3(z^{-1}) + P_2(z^{-1})\, P_4(z^{-1}) = 1.0$$

This equation cannot generally be satisfied if $P_3(z^{-1})$ and $P_4(z^{-1})$ are polynomials of order less than $(n-1)$. If these polynomials are of order $(n-1)$, then $P_1P_3 + P_2P_4 = 1.0$ gives $2n$ equations in $z^0, z^{-1}, \ldots, z^{-(2n-1)}$ which are linear in the $2n$ unknown coefficients of $P_3(z^{-1})$ and $P_4(z^{-1})$. The solution, if it exists, is therefore unique. The over-all pulse transfer function $K(z^{-1})$ is then of order $(2n-1)$ in z^{-1}. The corresponding discrete compensator is the ratio of the two polynomials of order $(n-1)$ in z^{-1}, as shown below

$$D(z) = \frac{1}{GH(z)} \frac{K(z)}{1 - K(z)} = \frac{P_3(z^{-1})}{P_4(z^{-1})}$$

The polynomials $P_3(z^{-1})$ and $P_4(z^{-1})$ could have been allowed to be of order higher than $(n-1)$ allowing some arbitrariness in the choice of coefficients; but this would increase the response time and since we are interested in minimal time systems, $P_3(z^{-1})$ and $P_4(z^{-1})$ are chosen as polynomials of order $(n-1)$.

The most general $NG(z)$ is of the form

$$NG(z) = \frac{\text{polynomial in } z^{-1} \text{ of order } (n-1)}{P_2(z^{-1})}$$

and the corresponding output sequence due to $NG(z)$ (i.e., to initial conditions)

$$C(z) = NG(z)\,[1 - K(z)] = P_4(z^{-1}) \times \text{polynomial in } z^{-1} \text{ of order } (n-1)$$

becomes zero at the $(2n-1)$ sampling instant. One can also determine that the forcing function vanishes after this time, insuring deadbeat regulation.

4.4.2 Non-Zero Inputs.

If the plant contains m integrations, the discrete compensator designed for regulation is such that $[1 - K(z)]$ contains the factor $(1 - z^{-1})^m$, according to condition 3. If one looks back at the conditions for deadbeat response to specific inputs (Section 2.1), it is immediately seen that they are satisfied for all inputs belonging to the class C_{m-1}. Therefore the discrete compensator previously determined will make the system, starting from arbitrary initial conditions, follow exactly any input of the class C_{m-1}.

We know that the effect of initial conditions disappears after the $(2n-1)$ sampling instant. As for the response to an input of the class C_{m-1}, the corresponding error sequence is

$$E_1(z) = R(z)\ [1 - K(z)]$$

The z-transform of the input, $R(z)$, has the general form

$$\frac{\text{polynomial in } z^{-1} \text{ of order } (m-1)}{(1 - z^{-1})^m}$$

so that

$$E_1(z) = \text{polynomial in } z^{-1} \text{ of order } [(m-1) + (2n-1) - m]$$

$$= \text{polynomial in } z^{-1} \text{ of order } (2n-2)$$

and the error signal becomes zero at the $(2n-1)$ sampling instant. Similarly for the sequence of forcing functions, due to the input signal,

$$E_2(z) = \frac{K(z)}{GH(z)} R(z) = P_3(z^{-1})\ P_2(z^{-1})\ R(z)$$

$P_2(z^{-1})$ contains the factor $(1-z^{-1})^m$, so that

$$E_2(z) = P_3(z^{-1}) \times [\text{polynomial in } z^{-1} \text{ or order } (n-m) + (m-1)]$$

$$E_2(z) = \text{polynomial in } z^{-1} \text{ of order } (2n-2)$$

and deadbeat response is insured at the $(2n-1)$ sampling instant.

Retaining the assumption that the plant contains m integrators, we now consider all inputs belonging to the class C_m but not to C_{m-1}. Let us look at what happens if we use the previously determined discrete compensator. Because the system is linear, we can forget about the effect of initial conditions since they will disappear after $(2n-1)$ sampling periods. The z-transform of the input, $R(z)$, has now the general form

$$R(z) = \frac{\text{polynomial in } z^{-1} \text{ of order } m}{(1 - z^{-1})^{m+1}}$$

The error sequence is

$$E_1(z) = \frac{\text{polynomial in } z^{-1} \text{ or order } (2n-1)}{1 - z^{-1}}$$

so that $e_1(kT)$ reaches and stays at a steady-state value for $k \geq 2n-1$. The sequence of forcing functions is

$$E_2(z) = \frac{\text{polynomial in } z^{-1} \text{ of order } (2n-1)}{1 - z^{-1}}$$

so that $e_2(kT)$ also reaches and stays at a steady-state value for $k \geq 2n-1$. We see that the output becomes identical to the input after $(2n-1)$ sampling periods, except for a constant steady-state error.

The only solution for obtaining zero steady-state error is to have $[1 - K(z)]$ contain the additional factor $(1 - z^{-1})$. In such a case

$$\begin{cases} K(z^{-1}) = P_1(z^{-1})\, P_3'(z^{-1}) \\ 1-K(z^{-1}) = (1 - z^{-1})\, P_2(z^{-1})\, P_4'(z^{-1}) \end{cases}$$

or

$$P_1(z^{-1})\, P_3'(z^{-1}) + (1 - z^{-1})\, P_2(z^{-1})\, P_4'(z^{-1}) = 1.0$$

If we take P_3' of order n in z^{-1} and P_4' of order $(n-1)$, we get $(2n + 1)$ equations in $z^0, z^{-1}, \ldots, z^{-2n}$, which are linear in the $(2n + 1)$ unknown coefficients of $P_3'(z^{-1})$ and $P_4'(z^{-1})$. The pulsed transfer function $K(z^{-1})$ will then be of order $2n$ in z^{-1}, one order higher than before. The corresponding discrete compensator will contain the digital equivalent of an integrator and will be of the form

$$D'(z) = \frac{P_3'(z^{-1})}{(1 - z^{-1})\ P_4'(z^{-1})}$$

It is clear that we still have deadbeat regulation but generally after $2n$ sampling instants, instead of $(2n-1)$, so that we have lost the minimal time property. However, we have now deadbeat response to any input belonging to the class C_m after at most $2n$ sampling instants.

The digital compensator designed for minimal time deadbeat regulation or deadbeat response to inputs of the class C_m, using the above techniques, is identical to the one which would be found using the state variable approach to this problem, because the solution to this problem is unique. This equivalence will be demonstrated with the running example.

Example:

Let $$G(s) = \frac{1}{s(s+1)} \quad \text{or} \quad GH(z) = \frac{.368z^{-1}\ (1 + .717z^{-1})}{(1 - z^{-1})\ (1 - 0.368z^{-1})}$$

For minimal time deadbeat regulation

$$\begin{cases} K(z^{-1}) = z^{-1}(1 + .717z^{-1})\ (a_0 + a_1 z^{-1}) \\ 1\text{-}K(z^{-1}) = (1 - z^{-1})\ (1 - 0.368z^{-1})\ (b_0 + b_1 z^{-1}) \end{cases}$$

Solving for the unknowns shows

$$a_0 = .848 \quad ; \quad a_1 = -\,0.267 \quad ; \quad b_0 = 1.0 \quad ; \quad b_1 = 0.520$$

and

$$D(z) = \frac{1}{GH(z)} \frac{K(z)}{1 - K(z)} = \frac{0.848 - 0.267z^{-1}}{1 + 0.520z^{-1}} \frac{1}{0.368}$$

$$D(z) = \frac{2.304 - 0.725z^{-1}}{1 + 0.520z^{-1}}$$

This is the same compensator as was found using the state variable approach. Knowing

$$K(z) = .848z^{-1}\,(1 + .717z^{-1})\,(1 - .314z^{-1})$$

one can plot the response to a step input with zero or arbitrary initial conditions. In any case, the settling time is 3T.

The previously determined discrete compensator gives deadbeat response to step inputs, but not to ramp inputs. Let us look for a discrete compensator giving non-minimal time deadbeat regulation but deadbeat response to ramp inputs. The conditions are:

$$\begin{cases} K(z^{-1}) = z^{-1}\,(1 + .717z^{-1}).\,\overbrace{(a_0 + a_1z^{-1} + a_2z^{-2})}^{P_3'(z^{-1})} \\ 1-K(z^{-1}) = (1 - z^{-1})\,(1 - 0.368z^{-1})\,(1 - z^{-1})\,\overbrace{(b_0 + b_1z^{-1})}^{P_4'(z^{-1})} \end{cases}$$

Solving for the 5 unknown yields

$b_0 = 1$ $\qquad$ $a_0 = 1.733$

$b_1 = 0.635$ $\qquad$ $a_1 = -1.476$

$a_2 = 0.326$

and therefore the corresponding discrete compensator is given by

$$D(z) = \frac{1.733 - 1.476z^{-1} + 0.326z^{-2}}{0.368(1 - z^{-1})(1 + 0.635z^{-1})}$$

$$D(z) = \frac{4.710 - 4.015z^{-1} + 0.885z^{-2}}{1 - 0.365z^{-1} - 0.635z^{-2}}$$

This is also the same compensator as was found using the state variable approach.

CHAPTER V

DEADBEAT REGULATION AND CONTROL OF MULTIPLE INPUT, MULTIPLE OUTPUT SYSTEMS

5.1 General Considerations of Controllability

As explained in Chapter III, this dissertation is concerned with an investigation of plants whose dynamic behavior may be described or approximated by the linear, time-invariant vector difference equation

$$\underline{x}_{k+1} = \phi \underline{x}_k + \Delta \underline{u}_k$$

In the multiple input case, Δ becomes a $(n \times r)$ matrix, and the r components of the input vector $\underline{u}$ can now be chosen arbitrarily because they correspond to r distinct physical inputs. The output of the plant is related to the state by an equation of the form

$$\underline{y}_k = M\underline{x}_k$$

A study of single input systems was presented in Chapter IV and before we consider the multi-input case and investigate some specific problems such as regulation of the plant, we will make some general remarks.

Definition. A plant is defined to be state-(output-) controllable if it is possible to find a sequence of forcing functions which will force it from any initial state (output) to any desired final state (output) in a finite number of sampling periods.

Bertram and Sarachik (12) have shown that:

"A plant of order n is state controllable if and only if it is possible to find n linearly independent vectors among the

nr vectors which are the columns of the matrix

$$\psi_n = [\phi^{n-1}\Delta | \phi^{n-2}\Delta | \ \cdots \ |\Delta] \qquad "$$

A sketch of the proof given by the authors may be helpful to understand the significance of this condition. Let the initial state of the plant be $\underline{x}(t_0) = \underline{x}_0$. After N sampling periods, the state is

$$\underline{x}(t_0 + NT) \triangleq \underline{x}_N = \phi^N \underline{x}_0 + \phi^{N-1}\Delta\underline{u}_0 + \phi^{N-2}\Delta\underline{u}_1 + \cdots + \Delta\underline{u}_{N-1}$$

The plant is state-controllable in N sampling periods if given any $\underline{x}_0$, it is possible to find a sequence of forcing functions $\underline{u}_0, u_1, \cdots, \underline{u}_{N-1}$, such that $\underline{x}_N$ is equal to any desired state. The above equation written under the form

$$\underline{x}_N - \phi^N \underline{x}_0 \triangleq \underline{\epsilon}_N = [\phi^{N-1}\Delta | \phi^{N-2}\Delta | \ \cdots \ |\Delta] \begin{bmatrix} \underline{u}_0 \\ \underline{u}_1 \\ \vdots \\ \underline{u}_{N-1} \end{bmatrix}$$

will have at least one solution for any $\underline{\epsilon}_N$ if and only if the rank of ψ_N is n. Using the Cayley-Hamilton theorem, it is easy to show that the rank of ψ_N for $N \geq n$ is equal to the rank of ψ_n. Therefore a plant is state-controllable iff it is state controllable in n sampling periods, i.e. if the rank of ψ_n is n.

The smallest positive integer N for which the rank of ψ_N is n is denoted by N_S and corresponds to the minimum number of sampling periods necessary to bring the system from any initial state to any desired final state. Since Δ contains r columns and ϕ^i is an $(n \times n)$ matrix, it is clear that $n/r \leq N_S$, and the results of Bertram and Sarachik show that if a plant is state-controllable, it must be state-controllable in at most n sampling periods so that $N_S \leq n$. Therefore, it is true that

$$\frac{n}{r} \leq N_S \leq n$$

for a state-controllable plant.

In the paper previously cited, it is also shown that for any plant to be output-controllable, the matrix M, which is $(m \times n)$, must be of rank m. If the plant is state-controllable and if the rank of M is m, then the plant is also output-controllable. It is not always necessary, however, for a plant to be state-controllable in order to be output-controllable.

Before proceeding to the control problem, let us recall that the matrix ϕ is assumed to be non-singular, and then derive some mathematical results which will be useful later on. Let $\underline{\delta}_1, \ldots , \underline{\delta}_r$ denote the column vectors of the matrix Δ and let us assume that they are linearly independent. For otherwise this implies that the r inputs are dependent and that a linear change of variables can reduce the number of inputs. Since this change of variables can always be made before starting, we can always formulate the problem using independent column vectors of Δ, without loss of generality.

Let us now indicate a normal procedure for checking state-controllability and finding N_S at the same time. Because the matrix ϕ

is non-singular, we can replace the matrix ψ_n by $\psi_n' = \phi^{-n}\psi_n$, and use ψ_n' when considering the question of controllability. In terms of ϕ and Δ, ψ_n' can be written

$$\psi_n' = [\phi^{-1}\Delta | \phi^{-2}\Delta | \ \cdots \ | \phi^{-n}\Delta]$$

The procedure is the following:

(a) Compute the r vectors $\phi^{-1}\underline{\delta}_1, \ \ldots \ , \phi^{-1}\underline{\delta}_r$, i.e. the columns of $\phi^{-1}\Delta$. With the assumptions previously made, they are linearly independent.

(b) Next consider $\phi^{-2}\underline{\delta}_1, \ \ldots \ , \phi^{-2}\underline{\delta}_r$; $\phi^{-3}\underline{\delta}_1, \ \ldots$ one by one in turn and test it for linear dependence on previously selected vectors. If it is dependent discard it from consideration; if not, include it with the group of vectors selected in (a).

(c) Repeat this until a total of (n-r) vectors has been selected from the sequence in (b). On reaching $\phi^{-n}\underline{\delta}_r$, if less than (n-r) vectors have been selected, the plant is not state-controllable. If the plant is state controllable, the last vector kept is denoted by $\phi^{-N_s}\underline{\delta}_i (1 \le i \le r)$ and in turn defines N_s.

Note that if $\phi^{-i}\underline{\delta}_j$ is linearly dependent on previously considered vectors, i.e.

$$\phi^{-i}\underline{\delta}_j = \sum_{\alpha=1}^{i-1} \sum_{\beta=1}^{r} a_{\alpha\beta}\phi^{-\alpha}\underline{\delta}_\beta + \sum_{\beta=1}^{j-1} a_{i\beta}\phi^{-i}\underline{\delta}_\beta$$

then each $\phi^{-(i+k)}\underline{\delta}_j$ for $k = 1, 2, \ \ldots \ , (n-i)$, will also be linearly dependent on vectors considered before it. This linear dependence is immediately exhibited by premultiplying each side of the above equation by ϕ^{-k}. Thus if we discard a vector $\phi^{-i}\underline{\delta}_j$, we can immediately discard also all vectors of the form $\phi^{-(i+k)}\underline{\delta}_j$ where k is a positive integer. Therefore we have shown that if the plant is state-controllable in $N_S \leq n$ sampling periods, it is always possible to select at least one group of n linearly independent vectors of the form $\phi^{-i}\underline{\delta}_j$ $(1 \leq j \leq r)$ such that

(1) $1 \leq i < N_S$

(2) The sequence of vectors associated with any particular $\underline{\delta}_j$ is "connected," which means that if $\phi^{-i}\underline{\delta}_j$ is among the n vectors selected, $\phi^{-i'}\underline{\delta}_j$ belongs also to the set of n selected vectors for $i' = 1, 2, \ldots, (i-1)$.

These properties will be essential for the derivation of the main result of this dissertation but they can also be used immediately to make an interesting remark. Because of the assumptions made on ϕ and Δ, namely ϕ has rank n and Δ has rank r, there are always r possible sequences, each one of which contains at least one element, namely $\phi^{-1}\underline{\delta}_i$ $(i = 1, 2, \ldots, r)$. Let n_i $(i = 1, 2, \ldots, r)$ denote the number of connected vectors belonging to the sequence starting with $\phi^{-1}\underline{\delta}_i$. These positive integers n_i's satisfy the following relationships

$$\sum_{i=1}^{r} n_i = n$$

and $\qquad n_i \geq 1 \qquad$ for $\qquad i = 1, 2, \ldots, r$

It is therefore easy to see that

$$\text{maximum } [n_i] \leq n - r + 1 \qquad i = 1, 2, \ldots , r$$

Remembering that

$$N_s = \text{maximum } [n_i] \qquad i = 1, 2, \ldots , r$$

the following inequality is obtained

$$N_s \leq n - r + 1$$

For a single input system, it is already known that complete controllability cannot be obtained in less than n sampling periods. The availability of $(r-1)$ supplementary independent inputs makes the system controllable in $n-(r-1)$ sampling periods, if it is controllable at all. In other words, these $(r-1)$ supplementary inputs save at least $(r-1)$ sampling periods.

We can now formulate the problem to be solved in the next section. We know that in the case $r = 1$, i.e. a system with only one input, the controllability of the plant insures the existence of a unique, stationary linear feedback which, operating on the sampled values of the state variables, takes the plant from any initial state to its equilibrium state in $n = n/r = N_s$ sampling periods. We now wish to determine if an equivalent statement can be made for the general case of r inputs. In particular, does there exist a linear stationary feedback which, operating on the instantaneous values of the sampled state variables, will take the system from any initial state to its equilibrium position and if it does exist, how many sampling periods are needed to do this? If a solution exists, is it unique and how can one compute the optimal feedback law or laws?

The solution to this problem will be obtained through an approach quite different from the one used in the one-input case. Instead of determining a sequence of forcing functions which would regulate the system and showing that it corresponds to a linear stationary processing of the state variables, we will assume here that a linear stationary feedback does exist and attempt to adjust it in such a way as to obtain deadbeat regulation of the state.

5.2 Deadbeat Regulation of the State.

Consider any linear stationary feedback operating on the instantaneous values of the state variables which are assumed for the moment to be available. For this case we may consider the input as being identically zero, and write

$$\underline{u}_k = A\underline{x}_k$$

where A is an $(r \times n)$ constant matrix to be determined.

The discrete-time state transition equation becomes

$$\underline{x}_{k+1} = (\phi + \Delta A)\underline{x}_k$$

and after N sampling periods

$$\underline{x}_N = (\phi + \Delta A)^N \underline{x}_0$$

The desired final state is the equilibrium state $\underline{x} \equiv 0$, and we will reach it from any initial state $\underline{x}_0$ in N sampling periods if and only if

$$(\phi + \Delta A)^N \underline{x}_0 \equiv 0$$

for any $\underline{x}_0$.

Since $(\phi + \Delta A)$ is an $(n \times n)$ matrix and $\underline{x}_0$ is any n-dimensional vector, we must have

$$(\phi + \Delta A)^N \equiv 0$$

Therefore, this control problem can be abstracted and the problem can be reformulated mathematically as a matrix problem in the following manner

<u>Given</u>: a matrix ϕ which is $[n \times n]$ and non-singular and a matrix Δ which is $[n \times r]$ and of rank r <u>does there exist</u> a matrix A, which is $[r \times n]$, such that $(\phi + \Delta A)^N \equiv 0$ for some positive integer $N \geq 1$.

The solution of this abstracted mathematical problem will now be considered. Let us denote by N_S' the smallest N for which $(\phi + \Delta A)^N \equiv 0$. Assuming this equality to be true for some N, we show that $N_S' \leq n$.

Proof: If $(\phi + \Delta A)^N \equiv 0$ for some positive integer N, it means in particular that all the eigenvalues, λ_i, of $(\phi + \Delta A)^N$ are at the origin, i.e. $\lambda_i = 0$ $(i = 1, 2, \ldots , n)$. But the eigenvalues of $(\phi + \Delta A)^N$ are those of $(\phi + \Delta A)$ raised to the power N, so that all the eigenvalues of $(\phi + \Delta A)$ must also be at the origin. Using the Cayley-Hamilton theorem which states that every square matrix satisfies its characteristic equation,

$$(\phi + \Delta A)^n \equiv 0$$

and therefore $N_S' \leq n$.

Note that reciprocally if all the eigenvalues of $(\phi + \Delta A)$ are at the origin, application of the Cayley-Hamilton theorem insures that $(\phi + \Delta A)^n \equiv 0$. Therefore, a necessary and sufficient condition to have $(\phi + \Delta A)^n \equiv 0$ is that all the eigenvalues of $(\phi + \Delta A)$ be at the origin. The best way of expressing this condition for the problem to be solved here is to use the Jordan canonical form theorem which can be stated as follows. Given any square matrix B with eigenvalues $\lambda_1, \lambda_2, \ldots , \lambda_p$, each one of respective multiplicity $n_1, n_2, \ldots , n_p$ $(n_1 + n_2 + \ldots + n_p = n)$, there exists a non-singular matrix J such that

$$J^{-1}BJ = \begin{bmatrix} \underbrace{\begin{matrix} \lambda_1 & * & \cdot \\ & \ddots & * \\ & & \lambda_1 \end{matrix}}_{n_1} & & 0 \\ & \ddots & \\ 0 & & \underbrace{\begin{matrix} \lambda_p & * & \cdot \\ & \ddots & * \\ & & \lambda_p \end{matrix}}_{n_p} \end{bmatrix}$$

where each asterisk stands for either a zero or a one.

In our case, if $(\phi + \Delta A)^n \equiv 0$, there exists a non-singular matrix J such that

$$J^{-1}(\phi + \Delta A)J = \begin{bmatrix} 0 & * & & & & 0 \\ & 0 & * & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & 0 & & & \cdot & * \\ & & & & & 0 \end{bmatrix} = D$$

We now wish to look at the problem of finding an A such that an equation of this type will be satisfied.

To do this let $\underline{j}_1, \underline{j}_2, \ldots \underline{j}_n$ denote the n (linearly independent) column vectors of J and write the equation

$$(\phi + \Delta A)J = JD$$

by columns.

$$\begin{aligned} (\phi + \Delta A)\underline{j}_1 &= 0 \\ (\phi + \Delta A)\underline{j}_2 &= *\underline{j}_1 \\ &\cdots\cdots\cdots\cdots \\ (\phi + \Delta A)\underline{j}_n &= *\underline{j}_{n-1} \end{aligned} \qquad (5.2.1)$$

where $* = 0$ or 1

Any non-zero vector $\underline{j}_i$ such that $(\phi + \Delta A)\underline{j}_i = 0$ is an eigenvector of $(\phi + \Delta A)$. The vector $\underline{j}_1$ is certainly one of them and there are as many other linearly independent eigenvectors $\underline{j}_i$ as there are asterisks equal to zero.

Lemma 1. For whatever A is chosen, there cannot be more than r linearly independent eigenvectors.

Proof: The rank of $\emptyset$ is n, and the rank of $\Delta = r \leq n$, by assumption. Now rank $\Delta A \leq$ minimum [rank Δ, rank A] so that rank $\Delta A \leq r$.

The dimension of the null space of $(\emptyset + \Delta A)$ is the nullity of $(\emptyset + \Delta A)$, and the nullity is connected to the rank by the following equality:

$$\text{nullity of } (\emptyset + \Delta A) = n - \text{rank } (\emptyset + \Delta A).$$

Since rank $\emptyset$ is n and rank ΔA is at most r, we have

$$\text{rank } (\emptyset + \Delta A) \geq n-r$$

and

$$\text{nullity of } (\emptyset + \Delta A) \leq n-(n-r) = r$$

and thus there cannot exist more than r linearly independent eigenvectors.

This lemma will be used to show that one cannot hope to find an A such that $N_s' < n/r$, where N_s' is the smallest N for which $(\emptyset + \Delta A)^N \equiv 0$.

Lemma 2. For whatever A is chosen, any eigenvector $\underline{x}$ of

$(\phi + \Delta A)$ must lie in the r-dimensional subspace spanned by the r linearly independent vectors

$$\phi^{-1}\underline{\delta}_1, \ \phi^{-1}\underline{\delta}_2, \ \ldots , \ \phi^{-1}\underline{\delta}_r$$

<u>Proof</u>: Let $\underline{x}$ be an eigenvector. Then

$$(\phi + \Delta A)\underline{x} = 0 \qquad \text{or} \qquad \underline{x} = -\phi^{-1}\Delta A\underline{x} \qquad (\underline{x} \neq 0)$$

Let

$$A\underline{x} = (r \times n)(n \times 1) \triangleq \begin{bmatrix} -\alpha_1 \\ -\alpha_2 \\ \vdots \\ -\alpha_r \end{bmatrix}$$

Note that $A\underline{x}$ cannot be identically zero, because if it were so, we would have $\phi\underline{x} = 0$ which is impossible for any $\underline{x} \neq 0$.

We can write

$$\underline{x} = -\phi^{-1}\Delta \begin{bmatrix} -\alpha_1 \\ -\alpha_2 \\ \vdots \\ -\alpha_r \end{bmatrix}$$

or $\underline{x} = \alpha_1\phi^{-1}\underline{\delta}_1 + \alpha_2\phi^{-1}\underline{\delta}_2 + \ldots + \alpha_r\phi^{-1}\underline{\delta}_r$, where at least one $\alpha_i \neq 0$.

Note that this lemma checks the previous one, and makes it more precise at the same time. In other words, we have shown that any eigenvector can be expressed as a linear combination of $\phi^{-1}\underline{\delta}_1, \ldots \phi^{-1}\underline{\delta}_r$.

Assume now that we want to take as an eigenvector a certain linear combination of these r independent vectors; i.e. we wish to let $\underline{x} = \alpha_1\phi^{-1}\underline{\delta}_1 + \ldots + \alpha_r\phi^{-1}\underline{\delta}_r$, where at least one α is non-zero, be an eigenvector of $(\phi + \Delta A)$. How must we choose A for this to be true?

Lemma 3. A necessary and sufficient condition on the matrix A for the vector

$$\underline{x} = \alpha_1\phi^{-1}\underline{\delta}_1 + \ldots + \alpha_r\phi^{-1}\underline{\delta}_r = \phi^{-1}\Delta \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix} \tag{5.2.2}$$

to be an eigenvector of $(\phi + \Delta A)$ is that

$$A\underline{x} = - \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix} \quad \text{i.e.} \quad A\phi^{-1}\Delta \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix} = - \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix} \tag{5.2.3}$$

where at least one α is different from zero.

Proof: Necessary condition

$\underline{x}$ is an eigenvector and hence $(\phi + \Delta A)\underline{x} = 0$

$$(\phi + \Delta A)\phi^{-1}\Delta \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix} = 0$$

$$\Delta \left\{ \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix} + A\phi^{-1}\Delta \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix} \right\} = 0$$

But the rank of Δ is exactly r (by assumption) so that the above equality can only hold if

$$A\phi^{-1}\Delta \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix} = - \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix}$$

Sufficient condition

We wish to show that if

$$\underline{x} = \phi^{-1}\Delta \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix} \quad \text{and } A \text{ is chosen such that } A\underline{x} = \begin{bmatrix} -\alpha_1 \\ \vdots \\ -\alpha_r \end{bmatrix}$$

then $\underline{x}$ is an eigenvector

$$(\phi + \Delta A)\underline{x} = (\phi + \Delta A)\phi^{-1}\Delta \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix} = \Delta \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix} + \Delta A\phi^{-1}\Delta \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix}$$

$$= \Delta \left\{ \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix} + \begin{bmatrix} -\alpha_1 \\ \vdots \\ -\alpha_r \end{bmatrix} \right\} = 0$$

The condition that at least one α must be different from zero insures that $\underline{x} \neq 0$, and the proof of the lemma is now complete.

As an application of these results concerning eigenvectors, consider the special case in which one looks for matrices A such that the corresponding matrices $(\phi + \Delta A)$ have exactly r linearly independent eigenvectors. Since all eigenvectors lie in the subspace spanned by $\phi^{-1}\underline{\delta}_1, \ldots, \phi^{-1}\underline{\delta}_r$, this is equivalent to saying that these r linearly independent vectors are eigenvectors; then according to Lemma 3, a necessary and sufficient condition on A is that it satisfies the r vectorial relations

$$A\phi^{-1}\underline{\delta}_1 = -\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \qquad A\phi^{-1}\underline{\delta}_r = -\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

For notational simplicity these r relations may be grouped into the matrix equation $A\phi^{-1}\Delta = -I$, where I is the $(r \times r)$ identity matrix.

Having characterized the possible eigenvectors of $(\phi + \Delta A)$, we return to its Jordan canonical representation. We have seen that a necessary and sufficient condition for having $(\phi + \Delta A)^n \equiv 0$ is the existence of n linearly independent vectors $\underline{j}_1, \ldots, \underline{j}_n$, satisfying a set of equations as indicated in 5.2.1. But we have just shown that one can have at most r linearly independent eigenvectors, i.e. vectors $\underline{j}_i \neq 0$, such that $(\phi + \Delta A)\underline{j}_i = 0$. Therefore, at least (n-r) unknown * in equation 5.2.1 have to be 1.

Generally speaking, with an eigenvector $\underline{j}_i$ there will be associated vectors $\underline{j}_{i+1}, \ldots, \underline{j}_{i+p_i}$ such that

$$\begin{cases} (\phi + \Delta A)\underline{j}_i = 0 \\ (\phi + \Delta A)\underline{j}_{i+1} = \underline{j}_i \\ \dots\dots\dots\dots\dots\dots \\ (\phi + \Delta A)\underline{j}_{i+p_i} = \underline{j}_{i+p_i-1} \\ (\phi + \Delta A)\underline{j}_{i+p_i+1} = 0 \end{cases}$$

The vector $\underline{j}_{i+1}$ which verifies $(\phi + \Delta A)^2\underline{j}_{i+1} = 0$ with $(\phi + \Delta A)\underline{j}_{i+1} \neq 0$ is called a principal vector of grade 2, and so on until $\underline{j}_{i+p_i}$ which is a principal vector of grade (p_i+1). If the last equation of the above relations is satisfied, i.e. $(\phi + \Delta A)\underline{j}_{i+p_i+1} = 0$, (p_i+1) is the maximum grade associated with $\underline{j}_i$. Note that $\underline{j}_i$, which we generally call an eigenvector, can also be called a principal vector of grade 1.

If we consider the set of all eigenvectors $\underline{j}_i$ and their associated principal vectors of respective maximum grade (p_i+1), it is easy to see considering the Jordan canonical form, that if

$$P = \max. \quad [p_i+1] \qquad\qquad i = 1, 2, \dots, r$$

then

$$(\phi + \Delta A)^P = 0$$

but

$$(\phi + \Delta A)^{P-1} \neq 0$$

or $P = N_s'$ as previously defined.

At this point we see the significance of the asterisk in equation 5.2.1 being a 0 or a 1. If we want N_s' as small as possible, we must minimize P, the largest of the maximum grades (p_i+1) associated with each eigenvector $\underline{j}_i$. The best we can hope for is to be able to pick r eigenvectors (see Lemma 1), r principal vectors of grade 2, ... , r principal vectors of grade n/r (if n/r is an integer), all non-zero and linearly independent. In general, one can see that $P = N_s' \geq n/r$ (if n/r is not an integer, replace it by the first integer larger than n/r). Combining this with a previous result shows that

$$\frac{n}{r} \leq N_s' \leq n$$

In the first section of this chapter, it was shown that N_s, the minimum number of sampling periods necessary for controlling the state, has the same upper and lower bounds. Here N_s' is related to the minimum number of sampling periods necessary to regulate the state using a linear stationary feedback and it will be shown later on that $N_s' = N_s$.

We now turn our attention to the characterization of the principal vectors of various grades associated with a particular eigenvector

$$\underline{j}_i = \phi^{-1}\Delta \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix}$$

where at least one α is different from zero.

We know that such a $\underline{j}_i$ is an eigenvector if and only if

$$A\underline{j}_i = - \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix}$$

Consider first a principal vector of grade 2, $\underline{j}_{i+1}$, associated with $\underline{j}_i$

$$(\phi + \Delta A)\underline{j}_{i+1} = \underline{j}_i$$

$$\underline{j}_{i+1} = \phi^{-1}\underline{j}_i - \phi^{-1}\Delta A\underline{j}_{i+1}$$

Let

$$A\underline{j}_{i+1} = - \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_r \end{bmatrix}$$

then

$$\underline{j}_{i+1} = \phi^{-1}\underline{j}_i + \phi^{-1}\Delta \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_r \end{bmatrix}$$

i.e.

$$\underline{j}_{i+1} = \phi^{-2}\Delta \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix} + \phi^{-1}\Delta \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_r \end{bmatrix} \tag{5.2.4}$$

This is the most general expression for a principal vector of grade 2 associated with $\underline{j}_i$. If it effectively is a principal vector of grade 2, then the following equation must be satisfied.

$$A\underline{j}_{i+1} = - \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_r \end{bmatrix} = A\phi^{-2}\Delta \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix} + A\phi^{-1}\Delta \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_r \end{bmatrix} \qquad (5.2.5)$$

Reciprocally, if A is chosen such that the previous equation is satisfied, as well as equation 5.2.3, then

$$\left\{ \phi^{-2}\Delta \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix} + \phi^{-1}\Delta \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_r \end{bmatrix} \right\} = \underline{j}_{i+1}$$

is certainly a principal vector of grade 2, associated with the eigenvector $\underline{j}_i$, because

$$(\phi + \Delta A) \left[\phi^{-2}\Delta \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix} + \phi^{-1}\Delta \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_r \end{bmatrix} \right] = \phi^{-1}\Delta \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix} +$$

$$\Delta \left\{ A\phi^{-2}\Delta \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix} + A\phi^{-1}\Delta \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_r \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_2 \end{bmatrix} \right\} = \phi^{-1}\Delta \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix} = \underline{j}_i$$

Therefore equation 5.2.5 is a necessary and sufficient condition for the vector $\underline{j}_{i+1}$, as expressed in 5.2.4, to be a principal vector of grade 2 associated with the eigenvector $\underline{j}_i$ (i.e. in addition to equation 5.2.3 where at least one α is different from zero).

There are several particular cases of interest which simplify the mathematical details and which will be useful in our control problem. First, if A is chosen such that $(\phi + \Delta A)$ has exactly r linearly independent eigenvectors, we have seen that A must satisfy the relation

$$A\phi^{-1}\Delta = -I$$

Thus, equation 5.2.5 simply becomes

$$A\phi^{-2}\Delta \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix} = 0$$

independently of the β's. Second, by considering equation 5.2.4, one can see that $\underline{j}_{i+1}$ is the sum of two components. The first one $\sum_{i=1}^{r} \alpha_i \phi^{-2}\underline{\delta}_i$, belongs to the subspace filled by the vectors $\phi^{-2}\underline{\delta}_1, \ldots, \phi^{-2}\underline{\delta}_r$ which is generally distinct from the one filled by the eigenvectors, while the second one, $\sum_{i=1}^{r} \beta_i \phi^{-1}\underline{\delta}_i$, lies in the subspace possibly filled by the eigenvectors. Therefore, even if we chose A in such a way that $(\phi + \Delta A)$ has less than r linearly independent eigenvectors, it will often be convenient and proper to choose all the β's equal to zero, so that the simplified form of 5.2.5 will still be valid.

All the arguments for going from principal vectors of grade 1, i.e. eigenvectors, to principal vectors of grade 2 can be directly carried over for going from one grade to the next higher. However, it will be presently demonstrated that a solution to the control problem being studied in this chapter can be obtained within the mathematical framework of what has been developed above.

We want to regulate the plant in as few sampling periods as possible, i.e. we want to minimize $P = N_s'$. To accomplish this choose A so that $(\emptyset + \Delta A)$ has as many linearly independent eigenvectors as possible; Δ being of rank r, we can take r linearly independent eigenvectors, namely $\emptyset^{-1}\underline{\delta}_1, \ldots, \emptyset^{-1}\underline{\delta}_r$, or any other set of r independent linear combinations of these vectors. In any case, if we have r independent eigenvectors the necessary and sufficient condition on A is that

$$A\emptyset^{-1}\Delta = -I$$

Assuming this to be the case, we next consider the principal vectors of grade 2. We note that in checking the independence of $\underline{j}_{i+1}$, as expressed in equation 5.2.4, with respect to the r eigenvectors, we may as well take all β's equal to zero, since $\sum_{i=1}^{r} \beta_i \emptyset^{-1}\underline{\delta}_i$ is certainly a linear combination of eigenvectors for any set of β's. If we can pick r principal vectors of grade 2 associated with the r eigenvectors, so that they form all together a set of $2r$ linearly independent vectors, A must satisfy

$$A\emptyset^{-2}\Delta \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix} = 0 \qquad \text{for } r \text{ independent vectors } \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix}$$

So this is equivalent to requiring that

$$A\phi^{-2}\Delta = 0$$

This condition is analogous to $A\phi^{-1}\Delta = -I$ which is the condition for obtaining r eigenvectors.

The generalization to principal vectors of higher grades being obvious, we are now in a position to assert that the following theorem is true.

Theorem. If a multiple-input system is controllable in N_s sampling periods, there always exists at least one stationary linear feedback which acting on the instantaneous values of the sampled state variables will take the system from an arbitrary initial state to its equilibrium state in N_s sampling periods. Furthermore, there can exist several different feedbacks giving the same results in the same number of sampling periods N_s, with $n/r \leq N_s \leq n$.

There can also exist several other feedbacks giving the same result in N sampling periods, where $N_s < N \leq n$.

Note that there cannot exist any stationary linear feedback which acting on these instantaneous values of the state variables would regulate the system in more than n sampling periods, because there does not exist any matrix A such that $(\phi + \Delta A)^{n+p} \equiv 0$ (p any positive integer), but $(\phi + \Delta A)^{n} \not\equiv 0$.

Proof. The proof of this theorem is based on what has been previously said about controllability. If the plant is controllable in N_s sampling periods, there exists at least one set of n linearly independent vectors of the form $\phi^{-i}\underline{\delta}_j$ which can be grouped in connected sequences as indicated below

$$n_1 \left\{ \begin{matrix} \phi^{-1}\underline{\delta}_1 \\ \cdot \\ \cdot \\ \cdot \\ \phi^{-n_1}\underline{\delta}_1 \end{matrix} \right. \quad \cdots \quad n_i \left\{ \begin{matrix} \phi^{-1}\underline{\delta}_i \\ \phi^{-2}\underline{\delta}_i \\ \cdot \\ \cdot \\ \cdot \\ \phi^{-N_s}\underline{\delta}_i \end{matrix} \right. \quad \cdots \quad n_r \left\{ \begin{matrix} \phi^{-1}\underline{\delta}_r \\ \cdot \\ \cdot \\ \cdot \\ \phi^{-n_r}\underline{\delta}_r \end{matrix} \right.$$

where

$$n_1 + \cdots + n_i + \cdots + n_r = n$$

and

$$N_s = \text{maximum} \quad [n_1, \ldots , n_i, \ldots , n_r]$$

We want to find an A such that $(\phi + \Delta A)^{N_s} \equiv 0$ and we have seen that this was most easily done by finding at the same time a matrix J such that $J^{-1}(\phi + \Delta A)J$ has the proper Jordan canonical form. From the discussion on possible eigenvectors and principal vectors of various grades associated with $(\phi + \Delta A)$, choose A such that

(1) $\phi^{-1}\underline{\delta}_1$, $\phi^{-1}\underline{\delta}_2$, ... , $\phi^{-1}\underline{\delta}_r$ are all eigenvectors of $(\phi + \Delta A)$, i.e. $A\phi^{-1}\Delta = - I$. Since I is the $[r \times r]$ identity matrix, this represents r^2 scalar equations, linear in the components of A.

(2) $\phi^{-2}\underline{\delta}_1, \ldots, \phi^{-n_1}\underline{\delta}_1$; ... ; $\phi^{-2}\underline{\delta}_i, \ldots, \phi^{-N_s}\underline{\delta}_i$; ... ; $\phi^{-2}\underline{\delta}_r, \ldots, \phi^{-n_r}\underline{\delta}_r$; are principal vectors of grade 2 or higher

(up to N_s).

This will be the case if and only if A satisfies the following equations

$$\begin{cases} A\phi^{-2}\underline{\delta}_1 = 0 \ , \ \dots \ , \ A\phi^{-n_1}\underline{\delta}_1 = 0 \\ \dots\dots\dots\dots\dots\dots \\ A\phi^{-2}\underline{\delta}_i = 0 \ , \ \dots \ , \ A\phi^{-N_s}\underline{\delta}_i = 0 \\ \dots\dots\dots\dots\dots\dots \\ A\phi^{-2}\underline{\delta}_r = 0 \ , \ \dots \ , \ A\phi^{-n_r}\underline{\delta}_r = 0 \end{cases}$$

If we denote by J_k the $(n \times n_k)$ matrix

$$J_k = \left[\phi^{-1}\underline{\delta}_k \,|\, \dots \,|\, \phi^{-n_k}\underline{\delta}_k \right] \qquad (k = 1, 2, \dots , r)$$

all these requirements on A can be expressed as

$$A[J_1 | J_2 | \dots | J_r] = \left[\underbrace{\begin{matrix} -1 & 0 & \dots & 0 \\ 0 & & & . \\ . & & & . \\ . & & 0 & . \\ . & & & . \\ 0 & & & 0 \end{matrix}}_{n_1} \middle| \underbrace{\begin{matrix} 0 & 0 & \dots & 0 \\ -1 & . & & . \\ 0 & . & & . \\ . & . & 0 & . \\ . & . & & . \\ 0 & 0 & & 0 \end{matrix}}_{n_2} \middle| \dots \middle| \underbrace{\begin{matrix} 0 & 0 & \dots & 0 \\ . & & & . \\ . & & & . \\ . & & 0 & . \\ 0 & & & . \\ -1 & 0 & & 0 \end{matrix}}_{n_r} \right]$$

The matrix which multiplies A on the right is an $(n \times n)$ non-singular matrix, which is just the transformation matrix J which will put $(\phi + \Delta A)$ in its Jordan canonical form.

This last matrix equation has one and only one solution in "A". This solution corresponds to one stationary linear feedback which will regulate the plant in $N_s' = N_s$ sampling periods, since the longest sequence of connected vectors contains exactly N_s vectors and therefore $(\phi + \Delta A)^{N_s} \equiv 0$.

There can clearly exist other solutions in N_s sampling periods. For example, suppose that $\phi^{-(N_s-1)} \underline{\delta}_i$ and $\phi^{-(N_s-1)} \underline{\delta}_{i+1}$ are both taken as principal vectors of grade $(N_s - 1)$. Suppose that we do not take $\phi^{-N_s}\underline{\delta}_i$ as principal vector of highest grade N_s, but rather that we are permitted to choose $\phi^{-N_s}\underline{\delta}_{i+1}$ as the last principal vector. It is clear that the matrix A corresponding to $\phi^{-N_s}\underline{\delta}_{i+1}$ as the last principal vector will generally be different from the matrix A corresponding to $\phi^{-N_s}\underline{\delta}_i$, though regulation of the state will still be obtained after N_s sampling periods.

Similarly, if we can find other sequences of connected but linearly independent vectors going up to grades larger than N_s, we will get different feedback matrices A. It should be noted, however, that it is impossible to go up to grades larger than n, as was shown using the Cayley-Hamilton theorem.

<u>Application of the general result to the case $r = 1$.</u>

In the one-input case, the matrix Δ can be considered as an n-dimensional column vector. Since $r = 1$, we have $N_S = n$, if the system is controllable. According to what we have said previously, we are obliged to pick A such that $\phi^{-1}\Delta$ is an eigenvector, and $\phi^{-i}\Delta$ is a principal vector of grade i for $i = 2, \ldots , n$. The row vector

A is therefore obtained from the equation

$$A\,[\phi^{-1}\Delta|\phi^{-2}\Delta|\;\ldots\;|\phi^{-n}\Delta] = [-1\;0\;\ldots\;0]$$

A solution of this equation exists, and is unique, if and only if the matrix $[\phi^{-1}\Delta|\;\ldots\;|\phi^{-n}\Delta]$ is non-singular, i.e. if the system is controllable. The reader should compare this result with the last paragraph of Section 4.1.

5.3 Identification of the Values of the State Variables at the Sampling Instants.

All the previous considerations of deadbeat regulation of the state were made with the assumption that one could measure instantaneously the values of the state variables at each sampling instant. Practically we can observe at each sampling instant only the value of the output vector $\underline{y}_k$, and since the matrix M is quite generally a singular matrix, $\underline{y}_k$ does not uniquely specify $\underline{x}_k$. Therefore we will have to identify the state by observing and recording the values of the output vector and of the forcing functions for a certain number of sampling instants.

We will assume that the rank of M is m, so that the plant which is assumed to be state-controllable is also output-controllable. At each sampling instant, we can measure m linearly independent linear combinations of the values of the state variables at that instant. At $t = kT$, we measure $\underline{y}_k = M\underline{x}_k$. But the state transition equation

$$\underline{x}_k = \phi\underline{x}_{k-1} + \Delta\underline{u}_{k-1}$$

can be written in the form

$$\underline{x}_{k-1} = \phi^{-1}\underline{x}_k - \phi^{-1}\Delta\underline{u}_{k-1}$$

and we measured

$$\underline{y}_{k-1} = M\phi^{-1}\underline{x}_k - M\phi^{-1}\Delta\underline{u}_{k-1}$$

Similarly

$$\underline{y}_{k-2} = M\phi^{-2}\underline{x}_k - M\phi^{-2}\Delta\underline{u}_{k-1} - M\phi^{-1}\Delta\underline{u}_{k-2}$$

At any time $t = kT$, in terms of the preceding $(n-1)$ measurements these various equations may be grouped as follows:

$$\begin{bmatrix} \underline{y}_k \\ \underline{y}_{k-1} \\ \cdot \\ \cdot \\ \cdot \\ \underline{y}_{k-(n-1)} \end{bmatrix} = \begin{bmatrix} M \\ M\phi^{-1} \\ \cdot \\ \cdot \\ \cdot \\ M\phi^{-(n-1)} \end{bmatrix} \begin{bmatrix} x_{k,1} \\ x_{k,2} \\ \cdot \\ \cdot \\ \cdot \\ x_{k,n} \end{bmatrix} + \begin{array}{l} \text{terms depending on the} \\ \text{forcing functions} \\ \underline{u}_{k-1}, \cdots, \underline{u}_{k-(n-1)} \end{array}$$

$$(nm \times 1) = (nm \times n) \times (n \times 1)$$

We say that the system is completely observable if this equation has a unique solution $\underline{x}_k$. This is only possible if one can find n linearly independent lines in the $(nm \times n)$ matrix

$$\begin{bmatrix} M \\ M\phi^{-1} \\ \cdot \\ \cdot \\ \cdot \\ M\phi^{-(n-1)} \end{bmatrix}$$

It is not necessary to go further than $M\phi^{-(n-1)}$ because it is easily seen that if the system is not observable in n sampling instants, it is not observable at all ($M\phi^{-n}$ is a linear combination of $M, M\phi^{-1}, \ldots, M\phi^{-(n-1)}$ by the Cayley-Hamilton theorem).

If we want to know the state of the system as quickly as possible, we check one after the other the following matrices until we find one with rank n

$$\begin{bmatrix} M \\ M\phi^{-1} \\ \cdot \\ \cdot \\ \cdot \\ M\phi^{-(n/m - 1)} \end{bmatrix}, \quad \begin{bmatrix} M \\ M\phi^{-1} \\ \cdot \\ \cdot \\ \cdot \\ M\phi^{-(n/m - 1)} \\ M\phi^{-n/m} \end{bmatrix}, \ldots$$

For simplicity of notation, we have assumed that n/m was an integer. If it is not we would start with the first integer larger than n/m.

The first matrix of this series having rank n indicates the minimum number of sampling instants N' one has to wait before being able to correctly identify the state. The solution in a given number of sampling instants will generally not be unique and there will also exist several solutions corresponding to different number of sampling periods (unless $m = 1$, one "output" system, in which case if a solution exists after $(n-1)$ sampling periods it is unique).

One can see the great analogy between the problems of controllability and those concerning the observability, as has already been mentioned by Kalman (14). Once the minimal time identification procedure has been determined one can combine it with the optimal feedback A (already computed) and find the set of discrete compensators which will regulate the state in a minimal number of sampling periods.

5.4 Considerations on the Family of Solutions of the Deadbeat Regulation Problem.

We have proved that generally speaking there exists an indefinite number of matrices A_N such that

$$(\emptyset + \Delta A_N)^N \equiv 0$$

but $$(\emptyset + \Delta A_N)^{N-1} \not\equiv 0 \qquad \text{if} \qquad N_S \leq N \leq n$$

With this in mind it is logical to ask if these matrices can be grouped into classes of some kind? In answering this question, we will also determine how arbitrary these matrices are.

Given an N, we know the length of the longest sequence of connected vectors which constitute J; call it $n_1 \geq n_2 \geq \cdots \geq n_r \geq 0$

Since

$$N = n_1$$

and

$$\sum_{i=1}^{r} n_i = n$$

we have

$$\sum_{i=2}^{r} n_i = n - n_1$$

If $r = 2$, n_2 is fixed. However, if $r > 2$ and $n - n_1 > 1$, there are several possibilities. For example let $n - n_1 = 2$; there are then two possibilities,

either

$$\begin{cases} n_2 = 2 \\ n_{i>2} = 0 \end{cases}$$

or

$$\begin{cases} n_2 = 1 \\ n_3 = 1 \\ n_{i>3} = 0 \end{cases}$$

From the discussion of 5.2, it may be concluded that all the matrices A corresponding to a given set of numbers $n_1, \dots, n_r$ lead to matrices $(\emptyset + \Delta A)$ which are similar to each other, since they all have the same Jordan canonical form. It seems reasonable then to group in the same class all the matrices A corresponding to the same ordered sequence $(n_1, n_2, \dots, n_r)$ where

and
$$\begin{cases} n \geq n_1 \geq n_2 \geq \cdots \geq n_r \geq 0 \\ \\ \sum_{i=1}^{r} n_i = n \end{cases}$$

Let us show next that all the matrices A of a same class (as previously defined) are functions of a certain number of independent parameters, this number being only a function of the sequence $(n_1, n_2, \dots, n_r)$. Consider first the case where $N = n$, i.e. $n_1 = n$, $n_2 = \dots = n_r = 0$ so that there is only one sequence of connected vectors. Again, let the eigenvector be

$$\underline{j}_1 = \emptyset^{-1} \Delta \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix}$$

where at least one of the α's is different from zero. Any principal vector of grade 2 is of the form

$$\underline{j}_2 = \phi^{-2}\Delta \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix} + \phi^{-1}\Delta \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_r \end{bmatrix}$$

where the β's are arbitrary. The last vector of the sequence will be

$$\underline{j}_n = \phi^{-n}\Delta \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix} + \phi^{-(n-1)}\Delta \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_r \end{bmatrix} + \dots + \phi^{-1}\Delta \begin{bmatrix} \nu_1 \\ \vdots \\ \nu_r \end{bmatrix}$$

Apparently, there are nr parameters and only the one constraint which expresses the condition of linear independence of these n vectors. However, it is easy to show that the matrix A is really a function of only $n(r-1)$ parameters. First, when choosing the eigenvector in an r-dimensional space, one has r degrees of freedom. However, one can always normalize the eigenvector and the corresponding principal vectors. In that sense, its length really does not matter (as long as it is different from zero). If A satisfies the equation

$$A\left[\underline{j}_1 \middle| \underline{j}_2 \middle| \cdots \middle| \underline{j}_n\right] = \begin{bmatrix} -\alpha_1 & -\beta_1 & & -\nu_1 \\ \vdots & \vdots & \cdots & \vdots \\ -\alpha_r & -\beta_r & & -\nu_r \end{bmatrix}$$

it also satisfies

$$A\left[k\underline{j}_1 \,\middle|\, k\underline{j}_2 \,\middle|\, \cdots \,\middle|\, k\underline{j}_n\right] = \begin{bmatrix} -k\alpha_1 & & -k\nu_1 \\ \vdots & \cdots & \vdots \\ -k\alpha_r & & -k\nu_r \end{bmatrix}$$

Therefore one can consider that these are only $(r-1)$ independent parameters among the α's, as far as A is concerned. Next, when choosing $\underline{j}_2$, there are apparently r new parameters: $\beta_1, \beta_2, \cdots, \beta_r$. However, it is easy to see that A is unchanged if $\underline{j}_2$ is replaced by $(\underline{j}_2 + k\underline{j}_1)$, where k is arbitrary, and at the same time $\underline{j}_i$ is replaced by $(\underline{j}_i + k\underline{j}_{i-1})$ for $i = 3, \cdots, n$.

To simplify the equations to be written, assume all the $\gamma, \cdots, \nu$'s are zero. Then if

$$A\left[\phi^{-1}\Delta\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_r\end{bmatrix} \,\middle|\, \phi^{-2}\Delta\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_r\end{bmatrix} + \phi^{-1}\Delta\begin{bmatrix}\beta_1\\ \vdots\\ \beta_r\end{bmatrix} \,\middle|\, \cdots \,\middle|\, \phi^{-n}\Delta\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_r\end{bmatrix}\right.$$

$$\left. + \phi^{-(n-1)}\Delta\begin{bmatrix}\beta_1\\ \vdots\\ \beta_r\end{bmatrix}\right] = -\begin{bmatrix}\alpha_1 & \beta_1 & 0 & \cdot & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \alpha_r & \beta_r & 0 & \cdot & 0\end{bmatrix}$$

multiply both sides on the right by the matrix

$$\begin{bmatrix} 1 & k & & \\ & \cdot & \cdot & 0 \\ & & \cdot & \cdot \\ 0 & & \cdot & k \\ & & & \cdot \\ & & & 1 \end{bmatrix}$$

to get

$$A \left[\phi^{-1}\Delta \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix} \middle| \; \phi^{-2}\Delta \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix} + \phi^{-1}\Delta \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_r \end{bmatrix} + k\phi^{-1}\Delta \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix} \middle| \right.$$

$$\phi^{-3}\Delta \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix} + \phi^{-2}\Delta \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_r \end{bmatrix} + k\phi^{-2}\Delta \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix} + k\phi^{-1}\Delta \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_r \end{bmatrix} \middle| \cdots$$

$$\left. \phi^{-n}\Delta \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix} + \phi^{-(n-1)}\Delta \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_r \end{bmatrix} + k\phi^{-(n-1)}\Delta \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix} + k\phi^{-(n-2)}\Delta \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_r \end{bmatrix} \right]$$

$$= - \begin{bmatrix} \alpha_1 & \beta_1 + k\alpha_1 & k\beta_1 & 0 & & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \alpha_r & \beta_r + k\alpha_r & k\beta_r & 0 & & 0 \end{bmatrix}$$

This result indicates that in choosing the β's to form $\underline{j}_2$, one can choose them in such a way that the vector $\sum_{i=1}^{r} \beta_i \phi^{-1} \underline{\delta}_i$ is perpendicular to the non-zero vector $\sum_{i=1}^{r} \alpha_i \phi^{-1} \underline{\delta}_i$. In other words A depends on only $(r-1)$ parameters among the β's. This same remark is valid for choosing the γ's, ... , ν's, so that A depends only on $(r-1)\gamma$'s, ... , $(r-1)\nu$'s.

The final result is that when forming the sequence of n connected vectors to compute an A_n, there are only $n(r-1)$ parameters, with the constraint of linear indepencence of the n vectors. As an example, if $r = 1$, there is no possible choice but to take $\underline{j}_1 = \emptyset^{-1}\Delta, \ldots, \underline{j}_n = \emptyset^{-n}\Delta$. If these vectors are linearly independent, there is only one matrix A_n.

Next consider the case in which

$$n > n_1 \geq n_2 > 0, \qquad n_3 = \ldots = n_r = 0$$

i.e. we have two sequences of connected vectors. The previous remarks about a sequence of n connected vectors apply equally well to any sequence of $n' < n$ connected vectors, considered by itself, so that it might seem that we would have an A_{n_1,n_2} depending upon $n_1(r-1) + n_2(r-1) = n(r-1)$ parameters. However, these two sequences are not independently formed, so that A_{n_1,n_2} really depends on a smaller number of parameters.

Assume that we first form the most general sequence corresponding to n_1: $\underline{j}_1, \underline{j}_2, \ldots, \underline{j}_{n_1-n_2}, \ldots, \underline{j}_{n_1} \rightarrow n_1 r$ parameters. As previously explained A would only depend on $n_1(r-1)$ parameters instead of $n_1 r$. Next form $\underline{j}_1', \underline{j}_2', \ldots, \underline{j}_{n_2}'$. Because of the presence of $\underline{j}_1, \underline{j}_2, \ldots, \underline{j}_{n_2}$, A will only depend on $n_2(r-2)$ parameters among these apparent $n_2 r$ parameters. Now with $\underline{j}_1', \underline{j}_2', \ldots, \underline{j}_{n_2}'$ chosen, one can combine them with $\underline{j}_{n_1-n_2+1}, \ldots, \underline{j}_{n_1}$, to show that one arbitrary parameter in the choice of each of the vectors $\underline{j}_{n_1-n_2+1}, \ldots, \underline{j}_{n_1}$ has really no influence on A. Therefore, A_{n_1,n_2} really depends upon the following number of parameters

$$n_1(r-1) + [n_2(r-2) - n_2] = n(r-1) - 2n_2$$

The same argument can be extended to the more complicated cases; for example, if $n_1 \geq n_2 \geq n_3$, $n_4 = \cdots = n_r = 0$, A depends upon the following number of parameters

$$n_1(r-1) + [n_2(r-2) - n_2] + [n_3(r-3) - 2n_3] = n(r-1) - 2n_2 - 4n_3$$

and the general law is that $A_{n_1, n_2, \ldots, n_r}$ depends upon $[n(r-1) - 2n_2 - 4n_3 - \cdots - 2(r-1)n_r]$ parameters, where

$$\sum_{i=1}^{r} n_i = n$$

and

$$n_1 \geq n_2 \geq \cdots \geq n_r$$

To check this formula, let n/r be an integer and consider the case

$$n_1 = n_2 = \cdots = n_r = \frac{n}{r}$$

The number of parameters upon which $A_{n/r,\ldots,n/r}$ will depend is

$$n(r-1) - 2\frac{n}{r}[1 + 2 + \ldots + (r-1)] = n(r-1) - 2\frac{n}{r}\,\frac{r(r-1)}{2} = 0$$

i.e. there is a unique solution, at least if the constraint of linear independence is verified. This solution, if it exists, is given for example by the equation

$$A\left[\phi^{-1}\underline{\delta}_1 | \ldots |\phi^{-1}\underline{\delta}_r |\phi^{-2}\underline{\delta}_1 | \ldots |\phi^{-2}\underline{\delta}_r | \ldots |\phi^{-n/r}\underline{\delta}_1 | \ldots |\phi^{-n/r}\underline{\delta}_r\right]$$

$$= \left[\begin{array}{cccc|c} -1 & & & 0 & \\ & -1 & & & \\ & & \cdot & & 0 \\ 0 & & & \cdot & \\ & & & -1 & \end{array}\right]$$

We have therefore shown that all the matrices A of a same class are functions of a characteristic number of parameters.

Let us take a simple numerical example to illustrate the ideas just presented. In particular let

$$n = 3 \qquad \text{and} \qquad r = 2 \qquad \text{so that} \qquad 2 \leq N_s \leq 3$$

with

$$\phi = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 0 \\ 3 & 1 & 2 \end{bmatrix} \qquad (\text{rank } \phi = 3)$$

and

$$\Delta = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = [\underline{\delta}_1 | \underline{\delta}_2] \qquad (\text{rank } \Delta = 2)$$

We first compute the matrices $\phi^{-1}\Delta$, $\phi^{-2}\Delta$, $\phi^{-3}\Delta$

$$\phi^{-1}\Delta = \begin{bmatrix} -2/3 & -3/3 \\ -1/3 & 0 \\ 5/3 & 6/3 \end{bmatrix} \qquad \phi^{-2}\Delta = \begin{bmatrix} 19/9 & 24/9 \\ 8/9 & 12/9 \\ -25/9 & -33/9 \end{bmatrix}$$

$$\phi^{-3}\Delta = \begin{bmatrix} -134/27 & -174/27 \\ -55/27 & -69/27 \\ 191/27 & 246/27 \end{bmatrix}$$

If we look for a solution in two sampling periods, we must have $n_1 = 2$ and $n_2 = 1$, so that $n_1+n_2 = n$.

The general result previously derived indicates that A_2 will be a function of the following number of parameters

$$n(r-1) - 2n_2 = 1$$

Let us show it directly. The most general A_2 is defined by an equation of the following type

$$A_2 \left[\alpha_1 \phi^{-1} \underline{\delta}_1 + \alpha_2 \phi^{-1} \underline{\delta}_2 \,|\, \beta_1 \phi^{-1} \underline{\delta}_1 + \beta_2 \phi^{-1} \underline{\delta}_2 \,|\, \alpha_1 \phi^{-2} \underline{\delta}_1 + \alpha_2 \phi^{-2} \underline{\delta}_2 + \gamma_1 \phi^{-1} \underline{\delta}_1 + \gamma_2 \phi^{-1} \underline{\delta}_2 \right]$$

$$= - \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{bmatrix}$$

where the 3 column vectors of the matrix multiplying A_2 on the right must be linearly independent (in particular we must have $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$). Apparently A_2 depends on the 6 parameters $\alpha_{1,2}$, $\beta_{1,2}$, $\gamma_{1,2}$. However A_2 depends really on only 1 parameter. To demonstrate this, first multiply both sides on the right by

$$\begin{bmatrix} \frac{\beta_2}{\alpha_1\beta_2 - \alpha_2\beta_1} & \frac{-\beta_1}{\alpha_1\beta_2 - \alpha_2\beta_1} & k_1 \\ \frac{-\alpha_2}{\alpha_1\beta_2 - \alpha_2\beta_1} & \frac{\alpha_1}{\alpha_1\beta_2 - \alpha_2\beta_1} & k_2 \\ 0 & 0 & 1 \end{bmatrix}$$

The equation becomes

$$A_2\left[\phi^{-1}\underline{\delta}_1 \,|\, \phi^{-1}\underline{\delta}_2 \,|\, \alpha_1\phi^{-2}\underline{\delta}_1+\alpha_2\phi^{-2}\underline{\delta}_2+(\gamma_1+k_1\alpha_1+k_2\beta_1)\phi^{-1}\underline{\delta}_1+(\gamma_2+k_1\alpha_2+k_2\beta_2)\phi^{-2}\underline{\delta}_1\right]$$

$$= \begin{bmatrix} -1 & 0 & \gamma_1+k_1\alpha_1+k_2\beta_1 \\ 0 & -1 & \gamma_2+k_1\alpha_2+k_2\beta_2 \end{bmatrix}$$

Then one can always find k_1 and k_2 such that

$$\begin{cases} k_1\alpha_1+k_2\beta_1 = -\gamma_1 \\ k_1\alpha_2+k_2\beta_2 = -\gamma_2 \end{cases}$$

because

$$\alpha_1\beta_2-\alpha_2\beta_1 \neq 0$$

Therefore the most general A_2 is the solution of

$$A_2\left[\phi^{-1}\underline{\delta}_1 \,|\, \phi^{-1}\underline{\delta}_2 \,|\, \alpha_1\phi^{-2}\underline{\delta}_1 + \alpha_2\phi^{-2}\underline{\delta}_2\right] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

There are apparently 2 parameters left, but the amplitude of the vector $(\alpha_1\phi^{-2}\underline{\delta}_1 + \alpha_2\phi^{-2}\underline{\delta}_2)$ has no influence on A_2, since an A_2 which satisfies the above equation satisfies also

$$A_2\left[\phi^{-1}\underline{\delta}_1 \,|\phi^{-2}\underline{\delta}_1\,|k\alpha_1\phi^{-2}\underline{\delta}_1 + k\alpha_2\phi^{-2}\underline{\delta}_2\right] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

Therefore A_2 is really a function of only one parameter, which can be arbitrarily varied, as long as the condition of linear independence is satisfied. To express mathematically the dependence of A_2 on only one parameter, there are several methods. For example, one can keep the two parameters α_1 and α_2, with the constraint that the length of the vector $(\alpha_1\phi^{-2}\underline{\delta}_1 + \alpha_2\phi^{-2}\underline{\delta}_2)$ is to be a constant, for example, unity. Another method consists in saying that given α_1 and α_2, one can always find a "k" such that

$$\begin{cases} k\alpha_1 = \alpha \\ k\alpha_2 = 1-\alpha \end{cases} \qquad \text{i.e.} \qquad k = \frac{1}{\alpha_1+\alpha_2}$$

unless

$$\alpha_1+\alpha_2 = 0$$

However, it is clear that the particular case in which $\alpha_1 = -\alpha_2$ can be obtained by letting $\alpha \to \infty$, ($k \to \infty$ for finite α_1 and α_2) so that the most general A_2 is the solution of the matrix equation

$$A_2 \left[\phi^{-1}\underline{\delta}_1 | \phi^{-1}\underline{\delta}_2 | \alpha\, \phi^{-2}\underline{\delta}_1 + (1-\alpha)\phi^{-2}\underline{\delta}_2 \right] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

The parameter α can take any value between $+\infty$ and $-\infty$, for which the matrix equation has a solution.

In our specific example, this equation takes the form

$$A_2 \begin{bmatrix} -2/3 & -3/3 & 19/9\, \alpha + 24/9\,(1-\alpha) \\ -1/3 & 0 & 8/9\, \alpha + 12/9\,(1-\alpha) \\ 5/3 & 6/3 & -25/9\, \alpha - 33/9\,(1-\alpha) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ \\ 0 & -1 & 0 \end{bmatrix}$$

The determinant of the (3×3) matrix is

$$\Delta = \frac{2\alpha - 9}{9}$$

which indicates that α can take any value except $\alpha = 9/2$ for which value the matrix is singular. Assuming $\alpha \neq 9/2$, we have a whole family of solutions, namely

$$A_2(\alpha) = \begin{bmatrix} \frac{-8(\alpha-3)}{2\alpha-9} & \frac{2\alpha-15}{2\alpha-9} & \frac{-4(\alpha-3)}{2\alpha-9} \\ \frac{4\alpha-9}{2\alpha-9} & \frac{-3(\alpha-6)}{2\alpha-9} & \frac{\alpha}{2\alpha-9} \end{bmatrix}$$

Such an explicit expression for the feedback matrix can be useful, because it shows immediately and directly the dependence of the feedback coefficients on the choice of α.

However, in the general case, the explicit functional dependence of a matrix $A_{n_1,\ldots,n_r}$ on the various parameters is quite complicated. For example, in our specific case, if we want regulation in 3 sampling periods, the most general A_3 will be obtained by taking

$$\begin{cases} \underline{j}_1 = \alpha_1\phi^{-1}\underline{\delta}_1 + \alpha_2\phi^{-1}\underline{\delta}_2 \\ \underline{j}_2 = \alpha_1\phi^{-2}\underline{\delta}_1 + \alpha_2\phi^{-2}\underline{\delta}_2 + \beta_1\phi^{-1}\underline{\delta}_1 + \beta_2\phi^{-1}\underline{\delta}_2 \\ \underline{j}_3 = \alpha_1\phi^{-3}\underline{\delta}_1 + \alpha_2\phi^{-3}\underline{\delta}_2 + \beta_1\phi^{-2}\underline{\delta}_1 + \beta_2\phi^{-2}\underline{\delta}_2 + \gamma_1\phi^{-1}\underline{\delta}_1 + \gamma_2\phi^{-1}\underline{\delta}_2 \end{cases}$$

where $\alpha_{1,2}$, $\beta_{1,2}$, $\gamma_{1,2}$ are such that $\underline{j}_1$, $\underline{j}_2$, $\underline{j}_3$ are linearly independent. Among the 6 parameters $\alpha_{1,2}$, $\beta_{1,2}$, $\gamma_{1,2}$, there are only 3 independent ones, because one can add the following constraints without any loss of generality on A_3

1) $(\alpha_1\phi^{-1}\underline{\delta}_1 + \alpha_2\phi^{-1}\underline{\delta}_2)$ has unit length

2) $(\beta_1 \phi^{-1} \underline{\delta}_1 + \beta_2 \phi^{-1} \underline{\delta}_2)$ is orthogonal to $(\alpha_1 \phi^{-1} \underline{\delta}_1 + \alpha_2 \phi^{-1} \underline{\delta}_2)$

3) $(\gamma_1 \phi^{-1} \underline{\delta}_1 + \gamma_2 \phi^{-1} \underline{\delta}_2)$ is orthogonal to $(\alpha_1 \phi^{-1} \underline{\delta}_1 + \alpha_2 \phi^{-1} \underline{\delta}_2)$

The matrix A_3 is then defined by the equation

$$A_3 \, [\underline{j}_1 | \underline{j}_2 | \underline{j}_3] = - \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{bmatrix}$$

In order to determine the explicit functional dependence of A_3 on the various parameters, one has to invert the matrix $[\underline{j}_1 | \underline{j}_2 | \underline{j}_3]$, and obviously this will lead to quite complicated expressions for the elements of A_3.

On the other hand, one may not be especially interested in getting the explicit form of the feedback matrix, but rather be interested in getting the explicit form of $(\phi + \Delta A)$. The difficulty is the same however, as the following will show. From the above example

$$J^{-1}[\phi + \Delta A_3]J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so that

$$[\phi + \Delta A_3] = [0 | \underline{j}_1 | \underline{j}_2] \; [\underline{j}_1 | \underline{j}_2 | \underline{j}_3]^{-1}$$

and thus it is still necessary to compute J^{-1}.

5.5 Non-Zero Inputs.

Having considered at length the problem of deadbeat regulation, we now consider the following problem. Given a reference state, is it possible to obtain deadbeat follow-up or if this is not possible, what kind of steady-state error results? Is the steady-state error dependent on the choice of the feedback matrix A?

Let us define an input or reference state vector $\underline{r}$, whose value at the sampling instant $t = kT$ is denoted by $\underline{r}_k$. We assume that $\underline{r}$ is the state of a free, linear, stationary dynamical system of order n, so that we can write

$$\underline{r}(t) = \psi(t-t_0)\underline{r}(t_0) \qquad \text{for} \qquad t \geq t_0$$

and

$$\underline{r}_{k+1} = \psi(T)\underline{r}_k$$

This represents a certain class of deterministic inputs, just as in the one-input, one-output case (see Section 4.3).

It is obvious that the deadbeat follow-up problem is only meaningful if the continuous behavior of the plant is known. In other words, knowing its behavior at the sampling instants is not sufficient. Therefore let the given plant correspond to the following equation

$$\underline{x}(t) = \phi(t-t_0)\underline{x}(t_0) + \Delta(t-t_0)\underline{u}_{t_0} \qquad \text{for} \qquad 0 \leq t-t_0 < T$$

Deadbeat follow-up means that we want to have $\underline{x}(t) \equiv \underline{r}(t)$ for all time after a certain finite transient period. For a given reference state, it will be possible to have deadbeat response after the time t_s only if there exists a constant $\underline{u}_{s.s.}$ satisfying the following equation

$$\psi(t-t_s)\underline{r}(t_s) \equiv \phi(t-t_s)\underline{r}(t_s) + \Delta(t-t_s)\underline{u}_{s.s.} \qquad \text{for} \qquad t > t_s \qquad (5.5.1)$$

The fact that $\underline{u}$ must be constant for all $t > t_s$ can be proved in the following way: $\underline{u}$ must be constant during each sampling period (by assumption), in particular for $t_s < t < t_s + T$. But if the above equation holds for $t_s < t < t_s + T$ with some $\underline{u}$, by analytic continuation it will also hold for all later times if $\underline{u}$ is kept constant. An important particular case occurs when $\underline{u}_{s.s.} \equiv 0$ is a solution of the above equation. It means that the reference state corresponds to a free motion of the given dynamical system. An example in the one input, one output case would be a step input for a system containing at least one integration.

Now assume that the reference state is such that deadbeat follow-up is possible. Let us indicate a formal way for computing $\underline{u}_{s.s.}$. To that effect, rewrite equation 5.5.1 with $t-t_s = T$

$$\psi(T)\underline{r}(t_s) = \phi(T)\underline{r}(t_s) + \Delta(T)\underline{u}_{s.s.}$$

or

$$\Delta(T)\underline{u}_{s.s.} = [\psi(T)-\phi(T)]\ \underline{r}(t_s)$$

It has already been explained why there is no loss of generality in assuming that the $(n \times r)$ constant matrix $\Delta(T)$ has rank r. Using that assumption it is a simple matter to give an explicit expression for $\underline{u}_{s.s.}$. Premultiply both sides of the equation written above by $\Delta^*(T)$, where Δ^* denotes the transpose of the matrix Δ.

$$\Delta^*\Delta\underline{u}_{s.s.} = \Delta^*[\psi-\phi]\ \underline{r}(t_s)$$

The matrix $(\Delta^*\Delta)$ is a $(r \times r)$ matrix of rank r, if Δ has rank r, and therefore $(\Delta^*\Delta)$ has an inverse, so that

$$\underline{u}_{s.s.} = (\Delta^*\Delta)^{-1}\ \Delta^*\ [\psi-\phi]\ \underline{r}(t_s) \triangleq A'\underline{r}(t_s)$$

That result indicates in particular that if there exists a $\underline{u}_{s.s.}$ satisfying equation 5.5.1, it is unique (a consequence of the linear independence of the r forcing functions).

If equation 5.5.1 holds, one can rewrite it replacing t_s by kT and t by $(k+1)T$, to obtain

$$\psi(T)\underline{r}(kT) = \phi(T)\underline{r}(kT) + \Delta(T)\underline{u}_{s.s.}$$

where $\underline{u}_{s.s.} = A'\underline{r}(kT)$ is a constant, independent of k. Apparently these equations are only valid if $kT \geq t_s$, but by the principle of analytic continuation they must also hold for $t_0 \leq kT < t_s$, so that deadbeat follow-up could be instantaneously obtained if the system had the proper initial conditions. In other words, the only reference states belonging to the class of interest for which it is possible to get deadbeat follow-up are such that

$$\underline{r}(t) = \phi(t-t_0)\underline{r}(t_0) + \Delta(t-t_0)\underline{u}_{s.s.} \qquad \text{for} \qquad t > t_0$$

The discrete description of these input states is

$$\underline{r}_{k+1} = \phi\underline{r}_k + \Delta A'\underline{r}_k \tag{5.5.2}$$

where $A'\underline{r}_k$ is a constant forcing function, independent of k.

From now on, we will limit ourselves to this class of inputs. Then, for this class of inputs, we will show that it is possible to find a linear, stationary control law such that the state of the dynamical system, initially arbitrary, will become identical to the reference state after N_s sampling periods. This will be first demonstrated assuming that the complete state of the system and reference state can be instantaneously measured at each sampling instant.

Consider any linear stationary processing of the instantaneous values of the reference state and of the state of the system which can be written as

$$\underline{u}_k = A\underline{x}_k + B\underline{r}_k$$

We want to show that if $\underline{x}_k$ and $\underline{r}_k$ are instantaneously measurable, there exist matrices A and B such that $\underline{x}(t) \equiv \underline{r}(t)$ for $t \geq N_s T$. Let $\underline{x}_0$ be the arbitrary initial state of the system. At the successive sampling instants, we have.

$$\underline{x}_1 = \phi \underline{x}_0 + \Delta(A\underline{x}_0 + B\underline{r}_0) = (\phi + \Delta A)\underline{x}_0 + \Delta B\underline{r}_0$$

$$\underline{x}_2 = (\phi + \Delta A)^2\underline{x}_0 + (\phi + \Delta A)\Delta B\underline{r}_0 + \Delta B\underline{r}_1$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad (5.5.3)$$

$$\underline{x}_N = (\phi + \Delta A)^N\underline{x}_0 + (\phi + \Delta A)^{N-1}\Delta B\underline{r}_0 + \cdots$$

$$\cdots + (\phi + \Delta A)\Delta B\underline{r}_{N-2} + \Delta B\underline{r}_{N-1}$$

For the effect of initial conditions to disappear completely after a finite number of sampling instants, A has to be one of the previously determined matrices, so that $(\phi + \Delta A)^N \equiv 0$. In particular if we want this effect to disappear as quickly as possible, one must take $A = A_{N_s}$. The matrix A being fixed, the problem now becomes one of finding B such that we will obtain a deadbeat follow-up system.

Assume that at $t = NT$, we have $\underline{x}_N = \underline{r}_N$. The state of the controlled system will remain identical to the reference state at all later times if and only if

$$\underline{u}_{s.s.} = A'\underline{r}_k = A\underline{x}_k + B\underline{r}_k = (A + B)\underline{r}_k \qquad \text{for all} \qquad k \geq N$$

Since A' and A are known, this equation always has at least one solution in B, namely $B = A'-A$, although there may exist several other solutions depending on the nature of $\underline{r}_k$. To avoid difficulties, we will use the solution $B = A'-A$, so that

$$\underline{r}_{k+1} = (\phi + \Delta A')\underline{r}_k$$

can be written

$$\underline{r}_{k+1} = (\phi + \Delta A + \Delta B)\underline{r}_k \qquad \text{for all } k\text{'s.} \tag{5.5.4}$$

The only thing left to verify is that if A and B are chosen as indicated, then the state of the controlled system will effectively become identical to the reference state at some sampling instant, i.e. that $\underline{x}_N = \underline{r}_N$ for some N. To show that this is true, rewrite the state of the controlled system in the form

$$\underline{x}_N = (\phi + \Delta A)^N \underline{x}_0 + (\phi + \Delta A)^{N-1}\Delta B\underline{r}_0 + (\phi + \Delta A)^{N-2}\Delta B(\phi + \Delta A + \Delta B)\underline{r}_0 +$$

$$\ldots + \Delta B(\phi + \Delta A + \Delta B)^{N-1}\underline{r}_0$$

Now if P and Q are any two square matrices of the same dimensions

$$(P+Q)^N = P^N + P^{N-1}Q + P^{N-2}Q(P+Q) + \dots + PQ(P+Q)^{N-2} + Q(P+Q)^{N-1}$$

This can be readily demonstrated by recursion. Using this formula, we get

$$\underline{x}_N = (\phi+\Delta A)^N \underline{x}_0 + (\phi+\Delta A+\Delta B)^N \underline{r}_0 - (\phi+\Delta A)^N \underline{r}_0$$

Since A is chosen so that $(\phi+\Delta A)^N \equiv 0$, it is clear that $\underline{x}_N = \underline{r}_N$. In other words, for the considered class of inputs (i.e., inputs defined by $\underline{r}_{k+1} = \phi \underline{r}_k + \Delta \underline{u}_{s.s.}$, with $\underline{u}_{s.s.}$ independent of k) it is always possible to get deadbeat response after N_s sampling instants using a stationary, linear processing of the instantaneous values of the state variables of the controlled system and of the reference input. A schematic block-diagram would be as indicated in Figure 9.

Practically the only variables that one can measure are often the error signal(s). It is then of interest to know under what conditions one can get deadbeat follow-up by processing only the error signals. This amounts to adding the constraint $B = -A$; this means that deadbeat follow-up will only be obtained for those input states which satisfy the equation $\underline{r}_{k+1} = \phi \underline{r}_k$. In other words, if the input state corresponds to a free motion of the dynamical system to be controlled, then any of the preceding feedback laws (determined for regulation in N sampling instants) will allow the system to follow such an input after N sampling instants, if one uses the error signal in place of $\underline{x}_N$.

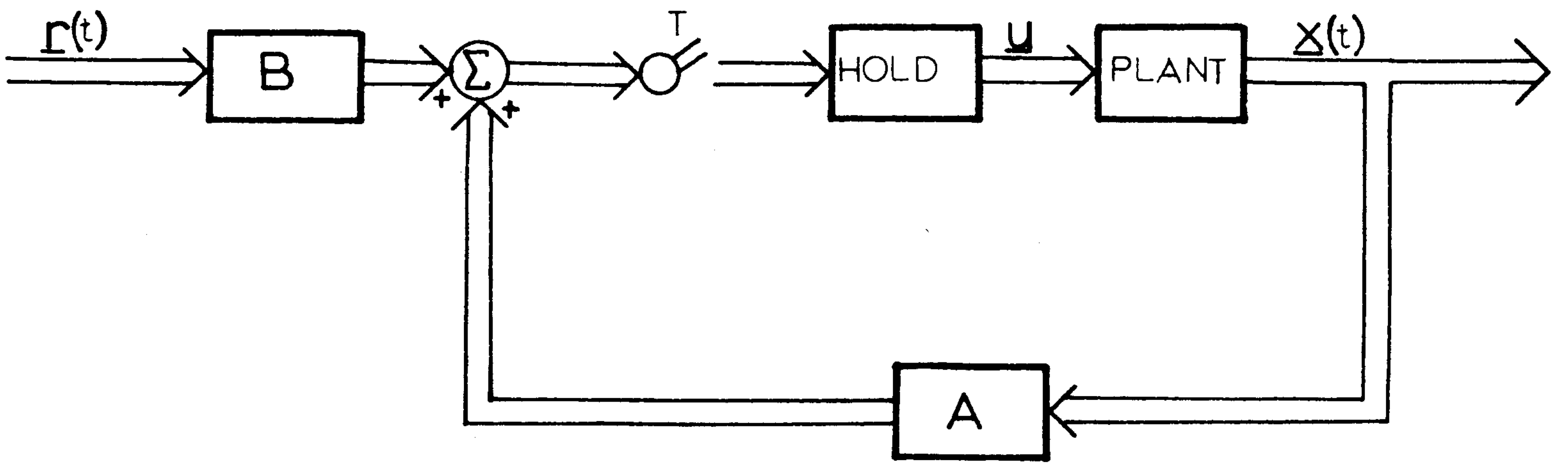

Figure 9: Time-optimal control for deadbeat regulation and follow-up of inputs described by 5.5.4, when all state variables are available.

This result is easily understood, because if the input is such that $\underline{r}_{k+1} = \phi \underline{r}_k$, one can write

$$\underline{r}_{k+1} - \underline{x}_{k+1} = \phi \underline{r}_k - \phi \underline{x}_k - \Delta \underline{u}_k$$

or

$$\underline{e}_{k+1} = \phi \underline{e}_k - \Delta \underline{u}_k$$

The error obeys the same equation (within a sign change) as the state in the regulator case and since we want the error to vanish identically, it is seen that the two problems are identical. A practical configuration would be as indicated in Figure 10.

Now let us consider the identification problem. The observable output of the plant is $\underline{y} = M\underline{x}$. We have assumed until now that we were given a reference state. However, in practice we might be given a desired output, called $\underline{y}_d$, in contrast to a reference input, and what happens inside the system may not be of as much interest as is the specified output state. Therefore we can assume that at each sampling instant we can measure

$$M\underline{e}_k = M(\underline{r}_k - \underline{x}_k) = \underline{y}_{k,d} - \underline{y}_k$$

If $\underline{r}_{k+1} = \phi \underline{r}_k$, we see that the identification problem in this follow-up case is the same as in the regulator case, so that the same discrete

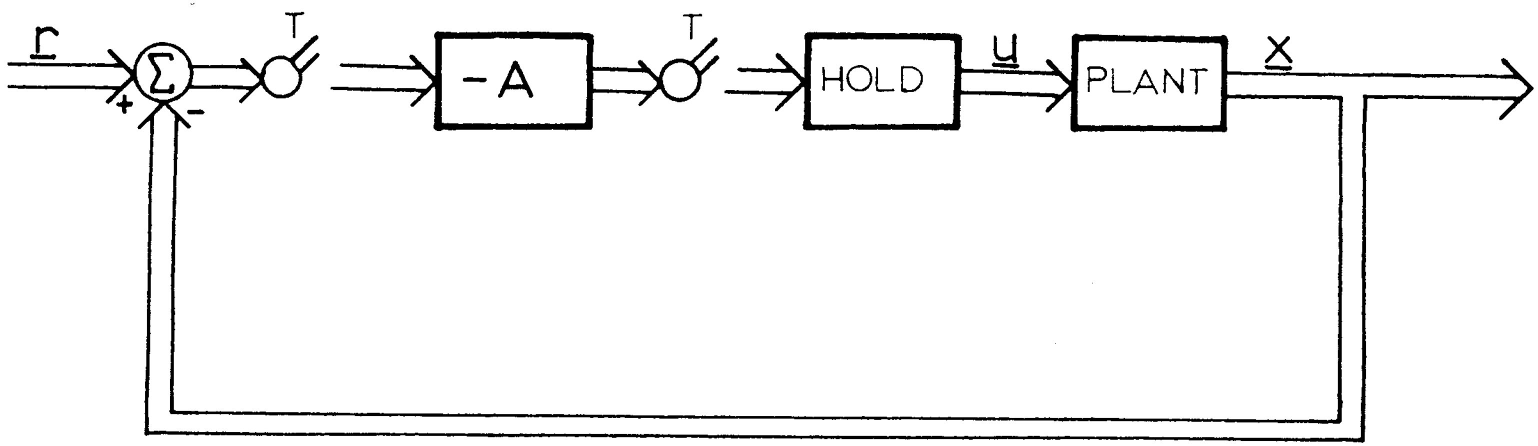

Figure 10: Time-optimal control for deadbeat regulation and follow-up of inputs described by $\underline{r}_{k+1} = \phi \underline{r}_k$, when all error variables are available.

compensators, operating on the observable error signals, will both identify and control the system. This is indicated in Figure 11. Note that the above considerations are simply a generalization of what has been said in Section 4.3 about plants containing m integrators and inputs belonging to class C_{m-1}.

Now if the input obeys $\underline{r}_{k+1} = \phi \underline{r}_k + \Delta A' \underline{r}_k$ with $A' \underline{r}_k = \underline{u}_{s.s.} \neq 0$, the optimal policy is to let

$$\underline{u}_k = A\underline{x}_k + B\underline{r}_k = A\underline{x}_k + (A'-A)\underline{r}_k$$

or

$$\underline{u}_k = - A\underline{e}_k + A'\underline{r}_k = - A\underline{e}_k + \underline{u}_{s.s.}$$

Thus, it is apparently necessary to process both the error signal and the reference state, the processing of the reference state being one way of computing $\underline{u}_{s.s.}$.

However, let us look at the identification problem; now we have

$$\begin{cases} \underline{x}_{k+1} = \phi \underline{x}_k + \Delta \underline{u}_k \\ \underline{r}_{k+1} = \phi \underline{r}_k + \Delta \underline{u}_{s.s.} \end{cases}$$

so that

$$\underline{e}_{k+1} = \phi \underline{e}_k - \Delta \underline{u}_k + \Delta \underline{u}_{s.s.}$$

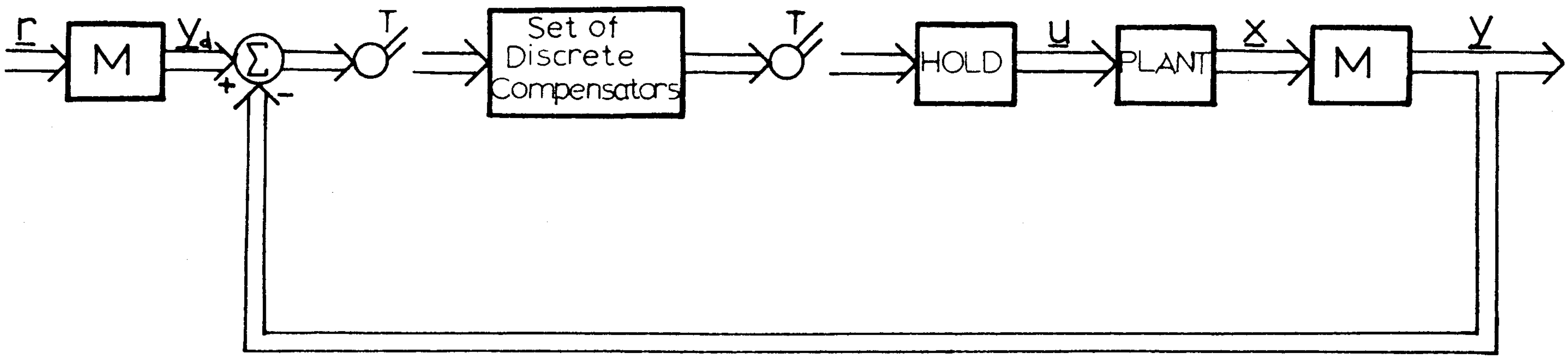

Figure 11: Configuration for deadbeat regulation and follow-up of specific inputs when all state variables are not directly measurable.

and as previously explained, we can only observe at each sampling instant

$$\underline{y}_{k,d} - \underline{y}_k = M(\underline{r}_k - \underline{x}_k) = M\underline{e}_k$$

Therefore in this follow-up problem, the identification procedure has to be modified with respect to the one worked out in the regulator case. By analogy with the case of one-input systems containing exactly m integrators and having to follow inputs of the class C_m, one would consider $\underline{u}_{s.s.}$ as an additional quantity to identify and that would presumably take extra sampling periods. However, it is quite possible to find a set of discrete compensators which by operating only on the error signals would make the system follow such an input. These discrete compensators would, of course, contain the digital equivalents of simple integrators.

There may be two reasons why such a design procedure would not be used. First, the transient duration will generally be increased and second the shape of the transients, say during regulation, may become unacceptable. It is therefore of interest to see what happens when one uses the optimal regulating feedback matrix A operating on the error signal (i.e., $\underline{u}_k = -A\underline{e}_k$) and the reference state is defined by $\underline{r}_{k+1} = \phi\underline{r}_k + \Delta A'\underline{r}_k$ with $A'\underline{r}_k \neq 0$. In the one-input case, we know that the error will reach a steady-state value after a finite transient period. Is it the same in the multi-input case?

Assume that the error reaches some steady-state value after a finite number of sampling periods. It means in particular that

$$\underline{r}_N - \underline{x}_N = \underline{r}_{N+1} - \underline{x}_{N+1}$$

$$= (\phi+\Delta A')\underline{r}_N - (\phi+\Delta A)\underline{x}_N + \Delta A\underline{r}_N$$

A rearrangement of the above equation yields

$$(\phi+\Delta A-I)\underline{x}_N = (\phi+\Delta A-I)\underline{r}_N + \Delta A'\underline{r}_N$$

Since $(\phi+\Delta A)$ has all its eigenvalues equal to zero, $(\phi+\Delta A-I)$ has all its eigenvalues equal to -1 and is therefore non-singular; so that we can write

$$\underline{x}_N = \underline{r}_N + (\phi+\Delta A-I)^{-1}\Delta A'\underline{r}_N$$

and

$$\underline{e}_N = -(\phi+\Delta A-I)^{-1}\Delta A'\underline{r}_N$$

This will be the value of the steady-state error, if there is a steady-state error. From this expression for $\underline{e}_N$, one can see that the steady-state error can only be zero if $A'\underline{r}_N = 0$, i.e. if $\underline{r}_{k+1} = \phi\underline{r}_k$.

Now for the class of inputs considered, $A'\underline{r}_N = \underline{u}_{s.s.}$, so that if the error ever reaches this value, it will stay constant for all later times. The only thing left to check is that the error will reach this value after a finite number of sampling periods. Let us show that if $A = A_N$, i.e., $(\phi+\Delta A)^N \equiv 0$ but $(\phi+\Delta A)^{N-1} \neq 0$, the error at the N th sampling instant will exactly have the steady-state value.

From the transition equations of the reference input and of the state of the system, namely

$$\underline{r}_{k+1} = \phi\underline{r}_k + \Delta A'\underline{r}_k$$

$$\underline{x}_{k+1} = \phi\underline{x}_k - \Delta A\underline{e}_k$$

we obtain the transition equation of the error vector

$$\underline{e}_{k+1} = (\phi+\Delta A)\underline{e}_k + \Delta A'\underline{r}_k$$

By analogy with equation 5.5.3, the error at the N th sampling instant will have the following expression

$$\underline{e}_N = (\phi+\Delta A)^N\underline{e}_0 + (\phi+\Delta A)^{N-1}\Delta A'\underline{r}_0 + (\phi+\Delta A)^{N-2}\Delta A'\underline{r}_1 + \dots + \Delta A'\underline{r}_{N-1}$$

Using the two equations

$$(\phi+\Delta A)^N \equiv 0$$

and

$$A'\underline{r}_k = \underline{u}_{s.s.}$$

the error at the N th sampling instant can be written

$$\underline{e}_N = \left[(\phi+\Delta A)^{N-1} + (\phi+\Delta A)^{N-2} + \ldots + (\phi+\Delta A) + I\right] \Delta\underline{u}_{s.s.}$$

and therefore

$$(\phi+\Delta A-I)\underline{e}_N = -\Delta\underline{u}_{s.s.}$$

$$\underline{e}_N = - (\phi+\Delta A-I)^{-1} \Delta A'\underline{r}_N$$

so that we will effectively get a steady-state error after N sampling instants.

For a given input belonging to the class of interest, i.e. for a given $\underline{u}_{s.s.}$, the steady-state error will generally be a function of $(\phi+\Delta A-I)^{-1}$, and the choice of the feedback matrix A will have an effect on it. One could therefore propose to choose A among the infinity of solutions (see Section 5.4), so that the steady-state error would be minimized for a particular input. One could possibly make use of the relation

$$(\phi+\Delta A-I)^{-1} = - \left[(\phi+\Delta A)^{N-1} + (\phi+\Delta A)^{N-2} + \ldots + (\phi+\Delta A) + I\right]$$

which is a consequence of $(\phi+\Delta A)^N \equiv 0$.

However, we have assumed until now that we were using a "continuous compensation." If we take into account the identification problem, i.e., if we use the set of discrete compensators which identify and regulate the system, the value of the steady-state error in following an input of the class of interest will depend not only on the choice of the regulating feedback matrix A but also on the identification procedure. The real problem would then be to find a set of discrete compensators which would regulate the system and lead to a steady-state error as small as possible when the system tries to follow a particular input. This is suggested as an area for further studies.

CHAPTER VI

FINIS

6.1 Summary and Conclusions.

Previously, it was known from the z-transform theory that very special performances could be obtained from one-input, one-output linear sampled-data systems. In particular there existed a technique to design discrete compensators which would make the output of the system initially at rest become identical to some specific input after a finite number of sampling periods. However, the influence of arbitrary initial conditions had not been thoroughly investigated. It is first shown here that if such a discrete compensator is used, the effect of initial conditions will eventually disappear, but the time constants of this decay will be the time constants of the stable part of the system to be controlled. In other words, the closed-loop system is theoretically quite effective as a "servomechanism," i.e. for following some specific inputs, but rather ineffective as a regulator. Since these two functions cannot be separated in practical applications, the over-all performance cannot be very satisfactory.

It is then shown what modifications have to be introduced in the classical design of the discrete compensator in order to get a closed-loop system having truly deadbeat performances. Quite generally, discrete compensators exist such that the system, which starts from arbitrary initial conditions, either will reach its equilibrium position or will follow some specific inputs after a finite number of sampling periods. However, the design of compensators through the z-transform theory amounts to manipulating polynomials in z, and although this method leads quickly to the solution in the one-input, one-output case, the role of the discrete compensator is not very clear. It would probably become even less clear in the case of multiple input, multiple output systems.

Here the problem is also considered from the state point of view. In this approach, it is first assumed that the state variables are all instantaneously measurable at each sampling instant. The first objective is then to show that if the system is controllable and N_S represents the minimum number of sampling periods necessary to regulate the state of the system, there always exists at least one stationary, linear feedback law which will bring the system starting from an arbitrary initial state back to its equilibrium state in N_S sampling periods. This property is relatively easy to prove for systems with only one input because the sequence of forcing functions which bring the system back to its equilibrium position in the minimum number of sampling periods is unique. Once this sequence of forcing functions has been determined, it is just a matter of showing that it does correspond to a stationary linear processing of the instantaneous values of the state variables.

However, for systems with several independent inputs the same approach does not seem possible because the time optimal solution is generally not unique. Among all these solutions, some correspond to stationary linear feedback laws, but others do not have that property. The process of sorting out the solutions of the first kind is not obvious so that another procedure is followed. It is assumed that the forcing functions are some stationary linear functions of the instantaneous values of the state variables. The problem is then to adjust if possible this stationary feedback law so as to get deadbeat regulation in N_S sampling periods. This is shown to be possible if the system is controllable in N_S sampling periods; at the same time, other stationary linear feedback laws are found which would regulate the system in N sampling periods, where $N_S < N \leq n$.

Next, the identification problem is considered because in practice only some state variables or linear combinations of those are instantaneously measurable at each sampling instant. For example, in systems having only one output one can measure at each sampling instant the value of one state variable or, generally speaking, one linear combination of state variables which depends on the choice of the state vector. Under the general condition of complete observability of the system, it is possible to compute the actual state of the system as a stationary linear function of the present and past values of the outputs and the past values of the forcing functions. The solution to the identification problem in a minimum number of sampling periods $N' \leq n$ may or may not be unique, and there exist other identification schemes corresponding to longer identification times.

The last step consists of putting the results of the time-optimal identification procedure in the optimal control law, so that the forcing functions at each sampling instant become stationary, linear functions of the observable quantities -- the present and the past values of the outputs and the past values of the forcing functions. In case the system has only one input and one output, the equation determining the value of the forcing function at each sampling instant can be looked at as a finite order, linear, stationary difference equation between the two variables input and output. This is just the mathematical description of the operations performed by the very simple type of digital computer called the "discrete compensator." For systems having several independent inputs, the idea is just the same, but instead of one difference equation there are several -- as many equations as there are independent inputs. This set of difference equations must be stored in some simple special purpose digital computer, called here a "set of discrete compensators."

Finally, it is shown that if the previously determined discrete compensators operate not on the outputs of the system, but on the differences between the outputs and some external signals called the desired outputs, then the system will have deadbeat response after $(N_S + N')$ sampling periods if these desired outputs correspond to a free motion of the dynamical system being controlled. If the desired outputs correspond to a motion of the dynamical system when it is driven by constant functions of time, it is still possible to get deadbeat follow-up. However, it is then necessary either to process separately the desired outputs without modifying the discrete compensators which operate on the error signals, or to modify these discrete compensators with possible loss of the minimal time property for regulation. If the discrete compensators determined for minimal time deadbeat regulation are used, and if it is impractical or impossible to process separately the desired output signals, then the error signals will reach some steady-state values after $(N_S + N')$ sampling periods. This steady-state error will be dependent not only on the optimal feedback law chosen, but also on the identification procedure.

6.2 Suggestions for Further Study.

Since the solution to the time-optimal control problem as originally stated is generally not unique, it is relevant to ask if one or several of these feedback laws should be preferred to the others. Since all these feedback laws are equivalent from the point of view of time of response, an additional criterion of performance has to be chosen in order to determine the "best" solution from these time-optimal solutions. It has already been indicated that such a criterion could be the steady-state error in response to some specific inputs. Alternately, it could be of interest to minimize the total energy spent in the regulation process. "Energy" is used here in a generalized sense, and minimization of the energy would be mathematically translated as minimization of a quantity of the form $\sum_k \underline{u}_k^* Q \underline{u}_k$, where Q would be some non-negative definite matrix. Such a criterion is quite often used in control problems because it insures that the forcing functions will not become excessively large, and because at the same time it often leads to relatively simple analytical solutions. On the other hand, if one tries to minimize the maximum value of the forcing functions during a regulation process, the problem is much more involved.

It should be noted that no matter what additional criterion of performance is chosen, the optimization will only be carried out over a very limited set of control laws and hence will generally be difficult. For example, among all these time-optimal control laws, none of them can be expected to be optimum with respect to energy for all initial conditions. For a particular set of initial conditions one solution may be better than another, but there will generally exist other initial conditions for which the situation will be reversed. One would therefore be obliged either to restrict the possible initial conditions or to adopt a probabilistic point of view. A probabilistic approach involves assigning a certain probability to each set of initial conditions and then minimizing the average value of the energy spent to bring the system back to its equilibrium position. Also, an analytical solution might be impossible to find, so that it would be necessary to look for the optimum by using some iterative procedure which could be performed on a digital computer.

Finally, it might be of interest to study the effects of small changes in the characteristics of the system to be controlled or in the characteristics of the controlling element. The best time-optimal solution could be defined as the time-optimal solution for which the sensitivity of the system to small parameter changes in a minimum.

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