Commuting Equivalence Relations and Scales on Differentiable Functions

Thesis by

Janet Pavelich

In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

California Institute of Technology
Pasadena, California

2001
(Submitted the 5th of December 2001)

For mk,

The best cheering squad a person could wish for.

Acknowledgements

I would like to thank my thesis advisor, Alekos Kechris, for his patience, advice, support, and enthusiasm while helping me with this work. Also for showing me, by his example, that virtually all problems are somehow interesting, and that any task worth doing is worth doing well.

I would like to express my gratitude and appreciation for the support, mathematical and otherwise, of the other students who entered the Caltech graduate program in math in 1996; particularly Rowan Killip, Clint White, Brian McKeever, and Greg Łabedzki. Miss you, Greg.

Thanks also to the administrative staff of the Caltech Math Department, past and present, for helping me to sort out all sorts of minor crises, from software headaches to jury duty to late registration, and for always doing so without even hinting that my ignorance of so many things computer-oriented, administrative, and/or time sensitive could be fixed with a good thrashing.

Very, very special thanks to Mark Keel, for keeping the faith and not hesitating to preach it. That's exactly what I needed.

Finally, I want to thank my parents, Mike and Pat Pavelich, who always provided me with blank paper for drawing and writing, rather than coloring books for staying within the lines.

This work was partially funded by NSF grants DMS-9619880 and DMS-9987437.

Abstract

This work consists of two independent chapters:

The first is a study of commuting countable Borel equivalence relations, where two equivalence relations R and S are said to *commute* if, as binary relations, they commute with respect to the composition operator, i.e., $R \circ S = S \circ R$. The primary problem considered is, to what extent does the complexity of $E = R \vee S$ depend on the complexity of R and S, if R and S commute? This is considered both in the case where the underlying space supports no E-invariant probability measure, and the case where it supports at least one such measure. In the first case, the answer is 'not very much': any such aperiodic equivalence relation E can be written as $R \vee S$, where R and S are smooth aperiodic. In the second case, we frame our study within the context of *costs*, a system of invariants for countable Borel equivalence relations with invariant probability measures, developed by G. Levitt [12] and D. Gaboriau [5]. One aspect of costs which is not well understood is the extent to which 'commutativity' within an equivalence relation (in a more general sense than the definition given above) trivializes its cost. We have shown that, under certain conditions, this is in fact the case. One of the consequences of these investigations is a new, elementary proof of the fact the group $SL_2(\mathbb{Z}[\frac{1}{2}])$ is anti-treeable.

The second chapter is motivated by the well known theorem of descriptive set theory that every $\mathbf{\Pi}^1_1$ subset of a Polish (separable, completely metrizable) space admits a $\mathbf{\Pi}^1_1$ scale. We construct a $\mathbf{\Pi}^1_1$ scale on the set of differentiable functions with domain $[0, 1]$, which is a $\mathbf{\Pi}^1_1$ subset of the Polish space $C([0, 1])$. This construction is based on the $\mathbf{\Pi}^1_1$ rank of differentiable functions given by Kechris and Woodin in [4], and, like this rank, is meant to reflect the intrinsic nature of DIFF, and so give a 'natural' criterion for determining whether the uniform limit of differentiable functions is itself differentiable. We then attempt to further analyze this 'scale criterion' for a sequence

of differentiable functions (f_n) by comparing it to the criterion that the sequence (f_n') converges.

Contents

Chapter 1 Summary

Chapter 2 is a study of certain types of countable Borel equivalence relations on standard Borel spaces. Here a space X is *standard Borel* if it has associated to it a σ-algebra of subsets S which is generated by a Polish (seperable, completely metrizable) topology on X. An equivalence relation E on X is *Borel* if, when viewed as a subset of $X \times X$, it is an element of the σ-algebra generated by $S \times S = \{A \times B \in \mathbf{P}(X^2) \,|\, A, B \in S\}$. E is *countable* if every equivalence class is countable.

Definition 2.1.1. *Let R and S be equivalence relations on a set X. R and S are said to* commute, *written $R \square S$, if $R \circ S = S \circ R$, where*

$$R \circ S = \{(x, y) \in X^2 \mid \exists z \in X\, ((x, z) \in R \wedge (z, y) \in S)\}$$

i.e., $R \circ S$ is the standard composition of relations.

Definition 1.0.1. *Let R and S be equivalence relations on a set X. The* join *of R and S, written $R \vee S$, is the equivalence relation on X generated by R and S. That is, $R \vee S$ is the $\subseteq$-least equivalence relation $K \subseteq X^2$ such that $R \subseteq K$ and $S \subseteq K$.*

The general motivating question for this chapter is: for a countable Borel equivalence relations R and S on a standard Borel space X, how does $R \vee S$ relate to R and S, if R and S commute?

In Section 2 we introduce some standard terminology and review some known results in the study of countable Borel equivalence relations, and examine the case of commuting equivalence relations R and S, where at least one of them is *finite*, i.e., each class is finite. It follows directly from the definition that if R is finite and R and S commute, then each $R \vee S$ class contains only finitely many S classes (see Proposition 2.1.2(iii)). $R \vee S$ is then said to have *finite index* over S. From this we get Proposition 2.2.8:

Proposition 2.2.8. *Suppose R and S are commuting Borel equivalence relations on the standard Borel space X.*

i) *If R and S are finite, then $R \vee S$ is finite.*

ii) *If R is finite and S is smooth, then $R \vee S$ is smooth.*

iii) *If R is finite and S is hyperfinite, then $R \vee S$ is hyperfinite.*

It turns out that the converse of this finite index property is also true:

Theorem 2.2.9. *Suppose E and F are aperiodic countable Borel equivalence relations, $F \subseteq E$, and E has finite index over F. Then there is a finite equivalence relation $R \subseteq E$ such that $R \square F$ and $R \vee F = E$.*

(Here *aperiodic* means that each class is infinite. This isn't a restriction, as the theorem is trivial when one of E or F is finite, since then both E and F must be finite, and we can just take $R = E$.)

Theorem 2.2.9 was proved jointly with A. Kechris.

Lastly in Section 2, we consider $R \vee S$, where $R \square S$ and each of R and S are *smooth*; in a sense which is made precise in the discussion preceding Theorem 2.2.7, smooth aperiodic equivalence relations are the simplest among the aperiodic equivalence relations. Proposition 2.2.11 essentially shows that any countable Borel equivalence relation E which contains a smooth aperiodic equivalence relation can be written as the join of two commuting smooth aperiodic equivalence relations. Since the converse is clearly true, this gives a complete characterization of countable Borel equivalence relations which contain a smooth aperiodic equivalence relation, in terms of commutativity.

Starting in Section 3 and continuing through Section 5, we consider countable Borel equivalence relations which do not contain a smooth aperiodic sub-equivalence relation, and continue the investigaton of possible decompositions into the join of commuting sub-equivalence relations. Section 3 reviews various known characterizations of countable Borel equivalence relations which do (resp., do not) contain an aperiodic smooth equivalence relation. One characterization of those that do not is that they admit an invariant probability measure (defined in Section 3) on the underlying space.

In Section 4 (and Section 6) we review a new system of invariants for countable Borel equivalence relations with invariant measures (and countable groups) developed by Levitt [12] and Gaboriau [5]. We frame our question of commutativity within the context of costs. This is relevant because the extent to which 'commutativity', in a more general sense than the definition given above, trivializes cost is not well understood. (For an aperiodic countable Borel equivalence relation E on X with invariant measure μ, its *cost* $C_\mu(E)$ is an extended positive real value, $1 \leq C_\mu(E) \leq \infty$. By 'trivial' we mean $C_\mu(E) = 1$.) As results we obtain:

Theorem 2.5.7. *Suppose that E is an aperiodic countable Borel equivalence relation on a Borel probability space (X, μ), and that μ is an invariant measure for E. If $E = R \vee S$, where $R \Box S$ and R and S are aperiodic, then*

$$C_\mu(E) \leq C_\mu(R) + 2C_\mu(S) - 2.$$

Therefore if one of $C_\mu(R)$ or $C_\mu(S)$ equals 1, then

$$C_\mu(E) \leq \max\left(C_\mu(R), C_\mu(S)\right).$$

As a corollary, we have

Theorem 2.4.12. *Suppose that E is an aperiodic countable Borel equivalence relation on a standard Borel probability space (X, μ) and that μ is an invariant measure for E. If $E = R \vee S$, where R and S are hyperfinite and $R \Box S$, then $C_\mu(E) = 1$. In particular, if E is μ-treeable, it is actually μ-hyperfinite.*

Remark: S. Solecki has since shown that a modification of the proof of Theorem 2.5.7 yields the stronger result

$$C_\mu(E) \leq C_\mu(R) + C_\mu(S) - 1.$$

In the final section, we apply Theorem 2.5.7, along with previously known results of Gaboriau to obtain concrete examples of groups with trivial cost. As consequences of Theorem 2.5.7 we have:

Theorem 2.6.12. *If R is a countably infinite commutative ring and $n \geq 3$, then*

$$C(SL_n(R)) = 1.$$

Hence if $SL_n(R)$ is non-amenable (e.g., in the case where R has characteristic 0), it is anti-treeable.

Proposition 2.6.15. *If K is any countable infinite field, then $GL_n(K)$, the group of all invertible $n \times n$ matrices with entries in K, has cost 1.*

And as a consequence of previously known facts on costs we also obtain

Proposition 2.6.13. *If R is a countable commutative ring with unity which has infinitely many units, then $C(E_2(R)) = 1$.*

Here $E_2(R)$ is the subgroup of $SL_2(R)$ generated by the elementary transvections. In the cases where R not only satisfies the hypotheses of Proposition 2.6.13, but is additionally a Euclidean domain, a commutative semi-local ring, or a ring of integers of a real quadratic field extension $\mathbb{Q}(\sqrt{d})$, $d \in \mathbb{Z}^+$, we have $SL_2(R) = E_2(R)$ (see [6], 4.3.9), so $SL_2(R) = 1$. In particular, from 2.6.13 we obtain a new proof that $SL_2(\mathbb{Z}[\frac{1}{2}])$ is anti-treeable, which, aside from cost machinery, is entirely elementary.

In Chapter 3, the general objects of study are *Polish* (separable, completely metrizable) spaces, and $\mathbf{\Pi}^1_1$ subsets of Polish spaces, where a subset A of a Polish space X is $\mathbf{\Pi}^1_1$ if its complement $X \backslash A$ is $\mathbf{\Sigma}^1_1$, i.e., the continuous image of a Borel set. In particular, we will be interested in the Polish space $C([0,1])$, the space of all real-valued continuous functions on $[0,1]$, topologized by the supnorm metric $\| f - g \|_\infty = \max_{x \in [0,1]} |f(x) - g(x)|$, and its $\mathbf{\Pi}^1_1$ subset DIFF $= \{f \in C([0,1]) \mid f'(x)$ *exists for all* $x \in [0,1]\}$.

Definition 3.1.1. *A* rank *on a set A is any map from A into Ord, the class of ordinals.*

Definition 3.1.3. *Let X be a Polish space, and $A \subseteq X$. A* scale *on A is any countable set of ranks $\{\phi_n : A \to Ord \mid n \in \mathbb{N}\}$ with the following property: for any sequence of points (x_k) in A converging to a point $x \in X$, if $\forall n \in \mathbb{N}, \exists \alpha_n \in Ord$, with* $\lim_{k\to\infty} \phi_n(x_k) = \alpha_n$*, then $x \in A$ and $\forall n \in \mathbb{N}, \phi_n(x) \leq \alpha_n$.*

Definition 1.0.2. *Let X be a Polish space and $A \subseteq X$. A* $\mathbf{\Pi}^1_1$-rank *on the set A is a rank $\phi : A \to Ord$ whose initial segments are uniformly in $\mathbf{\Pi}^1_1 \cap \mathbf{\Sigma}^1_1$, in the following sense: there exist relations $\leq^P_\phi$, $\leq^S_\phi$ with $\leq^P_\phi \in \mathbf{\Pi}^1_1$, $\leq^S_\phi \in \mathbf{\Sigma}^1_1$ (as subsets of $X \times X$) such that $\forall y \in A$,*

$$\phi(x) \leq \phi(y) \Leftrightarrow x \leq^P_\phi y \Leftrightarrow x \leq^S_\phi y.$$

Definition 1.0.3. *A* $\mathbf{\Pi}^1_1$-scale *is one in which all the ranks are $\mathbf{\Pi}^1_1$-ranks.*

Definition 3.1.6. *Let X be a Polish space, $A \subseteq X$, and $\{\phi_n \mid n \in \mathbb{N}\}$ a scale on A. We say that a sequence of points (x_k) of points in A* converges in the scale $\{\phi_n | n \in \mathbb{N}\}$ *if* $\lim_{k\to\infty} \phi_n(x_k)$ *exists for all $n \in \mathbb{N}$.*

It is a well known theorem of descriptive set theory that every $\mathbf{\Pi}^1_1$ set admits a $\mathbf{\Pi}^1_1$ scale (see [3], §36.D). Since a scaled set is closed under sequences which converge both in the topology and in the scale, it is desirable to find a scale which is explicitly related to the Polish topology, in the hopes that convergence in the scale can be used to solve problems of an analytical or topological nature.

With this motivation, in Section 3 of Chapter 3 we define a countable set of 'natural' $\mathbf{\Pi}^1_1$ ranks $\{\psi_{\epsilon,U}\}$ on DIFF (ϵ, U are each elements of a countable parameter set) and show that it is in fact a scale. The ranks $\psi_{\epsilon,U}$ which are used are based on a $\mathbf{\Pi}^1_1$ rank $|\cdot|_{\mathrm{DIFF}}$ developed by Kechris and Woodin in [4].

Although $\{\psi_{\epsilon,U}\}$ is intended to be a natural scale for DIFF, convergence in the scale $\{\psi_{\epsilon,U}\}$ is a complicated property to check for an arbitrary sequence (f_n). Hence in Section 4 we attempt to quantify convergence in the scale by asking whether it implies that the sequence of derivatives (f_n') converges (under suitable hypotheses). If a sequence (f_n) converges in the scale $\{\psi_{\epsilon,U}\}$, then the sequence $(|f_n|_{\mathrm{DIFF}})$ of Kechris-Woodin ranks is eventually constant. Thus it's reasonable to assume that $(|f_n|_{\mathrm{DIFF}})$ is actually constant, and to split the problem into cases according to this constant value. This is how we proceed.

The minimal case, $|f_n|_{\mathrm{DIFF}} = 1$, is also the simplest. In Proposition 3.4.2, we show that if f_n converges pointwise to f, (f_n) converges in the scale $\{\psi_{\epsilon,U}\}$, and $|f_n|_{\mathrm{DIFF}} = 1$ for all n, then f_n' converges uniformly to f'. Proposition 3.4.3 is a partial converse to this: if f_n converges uniformly to $f \in \mathrm{DIFF}$, f_n' converges uniformly to f', and $|f_n|_{\mathrm{DIFF}} = 1$ for all n, then $lim_{n\to\infty}\psi_{\epsilon,U}(f_n)$ exists for 'most' n, in a sense to be made precise. An exact biconditional statement relating the two types of convergence is then given in Corollary 3.4.6.

In the case where $|f_n|_{\mathrm{DIFF}} > 1$, the relation between the two types of convergence is no longer so direct. Using the characterization given in [4], that $|f|_{\mathrm{DIFF}} = 1$ if and only if $f \in C^1([0,1])$, it is easy to find an example which illustrates this. Let $f \in$ DIFF be any function whose derivative isn't continuous, say at the point $x_0 \in [0,1)$. We can then find a sequence (ϵ_n) such that $\epsilon_n \to 0$ and $\lim_{n\to\infty} f'(x_0+\epsilon_n)$ does not exist. If we then define f_n by $f_n(x) = f(x+\epsilon_n)$ (modifying this appropriately for $x \in (1-\epsilon_n, 1])$, then $f_n \to f$, and (f_n) converges in the scale, but clearly $f_n' \not\to f'$, since the real sequence $(f_n'(x_0))$ has no limit. Since a set $S \subseteq [0,1]$ can be the set of discontinuity points of a derivative f' if and only if it is a nowhere dense, countable union of closed sets (see [1], p. 34), this suggests that we weaken the question, and instead ask whether, for a given sequence (f_n), $f_n \to f$, convergence in the scale $\{\psi_n\}$ implies that $f_n'(x) \to f'(x)$ for all x in some G_δ (countable intersection of open subsets of $[0,1]$) which is dense in $[0,1]$.
Without additional hypotheses, the answer is negative:

Proposition 3.4.7. *There exists a sequence (f_n) in DIFF with the following properties:*

i) *(f_n) converges in the scale $\{\psi_{\epsilon,U} \mid \epsilon \in Q, U \in \mathcal{U}\}$,*

ii) $f_n \to 0$ *uniformly,*

iii) $\forall n \in \mathbb{N}$, $|f_n|_{DIFF} = 2$,

iv) $\forall^* x \in [0,1], \lim_{n\to\infty} f_n'(x)$ *does not exist.*

(Here we write '$\forall^* x \in [0,1]$' for 'for all x in a dense G_δ subset of $[0,1]$'.)

The key property of this example seems to be that $|f|_{\text{DIFF}} < |f_n|_{\text{DIFF}}$. When we add the additional hypothesis that, roughly speaking, $|f|_{\text{DIFF}} = |f_n|_{\text{DIFF}}$ on every neighborhood of $[0,1]$, we get an affirmative answer, at least in the case that $|f_n|_{\text{DIFF}} < \infty$:

Theorem 3.4.12. *Let (f_n) be a sequence in DIFF which converges in the scale and converges uniformly to the function $f \in C([0,1])$. Then, as $\{\psi_{\epsilon,U}\}$ is a scale, $f \in$ DIFF. If there exists $k < \omega$ such that f and each f_n are everywhere rank k, then $\forall^* x f_n'(x) \to f'(x)$.*

In the case where $|f_n|_{\text{DIFF}} = 2$, this can be given a more analytical characterization:

Corollary 3.4.17. *Let (f_n) be a sequence in DIFF which converges in the scale and converges uniformly to the function $f \in C([0,1])$, and suppose that $|f_n|_{DIFF} = 2$ for all n. Then, as $\{\psi_{\epsilon,U}\}$ is a scale, $f \in$ DIFF. If, additionally, the set $\{x \in [0,1] \mid f'$ is discontinuous at $x\}$ is dense in $[0,1]$, then $\forall^* x \in [0,1], \lim_{n\to\infty} f_n'(x) = f'(x)$.*

Question: Can Theorem 3.4.12 be generalized to the case in which each f_n and f are everywhere rank α, for any given $\alpha < \omega_1$? That is, if (f_n) is a sequence which converges in the scale and uniformly to $f \in C([0,1])$, and f and each f_n are everywhere rank α, is it true that $\forall^* x f_n'(x) \to f'(x)$?

Question: Is there an analogue to Corollary 3.4.17 for a sequence (f_n) in DIFF whose elements have $|\cdot|_{\text{DIFF}}$-rank greater than 2? That is, if (f_n) is a sequence which converges in the scale and uniformly to $f \in C([0,1])$, and $|f_n|_{\text{DIFF}} = \alpha$, is there an analytical condition C_α which would guarantee that $\forall^* x, \lim_{n\to\infty} f_n'(x)$ exists?

Chapter 2 Costs and Commuting Equivalence Relations

2.1 Commuting Equivalence Relations

Definition 2.1.1. *Let R and S be equivalence relations on a set X. R and S are said to* commute, *written $R\Box S$, if $R \circ S = S \circ R$, where*

$$R \circ S = \{(x, y) \in X^2 \mid \exists z \in X\ ((x, z) \in R \wedge (z, y) \in S)\},$$

i.e., $R \circ S$ is the standard composition of relations.

The following characterizations of commutativity were shown in [1]:

Proposition 2.1.2. *Let R and S be equivalence relations on a set X. Then the following are equivalent:*

i) *$R\Box S$.*

ii) *$R \circ S = R \vee S$, where $R \vee S$, the* join *of R and S, is the $\subseteq$-least equivalence relation $K \subseteq X^2$ such that $R \subseteq K$ and $S \subseteq K$.*

iii) *Within each $R \vee S$-class, every R-class meets every S-class.*

Proof. (i)⇒(ii): We have

$$R \vee S = \{(x, y) \in X^2 \mid \exists k \in \mathbb{N} \exists z_1, \ldots, z_{2k+1} \in X \text{ such that } xRz_1 \wedge z_1S \cdots \wedge z_{2k+1}Sy\},$$

so clearly $R \circ S \subseteq R\Box S$. Conversely, suppose $(x, y) \in R \vee S$, $R\Box S$, and k is minimal for which there exist $z_1, \ldots, z_{2k+1}$ with

$$xRz_1 \wedge z_1S \cdots \wedge z_{2k+1}Sy. \qquad (*)$$

Then $k = 0$. For suppose $k \geq 1$, so that

$$z_1 S z_2 \wedge z_2 R z_3 \wedge z_3 S \cdots$$

is a subformula of $(*)$. Since $R \Box S$, there is $w \in X$ such that $z_1 R w$ and $w S z_3$. But then the subformula

$$x R z_1 \wedge z_1 S z_2 \wedge z_2 R z_3 \wedge z_3 S u$$

(where $u = z_4$ or $u = y$, as appropriate) can be replaced by

$$x R w \wedge w S u$$

which would contradict the minimality of k.

(ii)⇒(iii): Fix $x \in X$, and let $[x]_{R \vee S}$ denote the $R \vee S$-class of x, i.e., $[x]_{R \vee S} = \{y \in X \mid x(R \vee S)y\}$. Let $C \subseteq [x]_{R \vee S}$ be any R-class, and let $D \subseteq [x]_{R \vee S}$ any S-class; fix $y_1 \in C, y_2 \in D$. Since $y_1(R \vee S)y_2$, by (ii) there exists $w \in X$ such that $y_1 R w$ and $w S y_2$. But this just means that $w \in C \cap D \neq \emptyset$.

(iii)⇒(i): By the symmetry of the argument it will suffice to show that $R \circ S \subseteq S \circ R$. Let $(x, y) \in R \circ S$ be given; then $(x, y) \in R \vee S$, so by (iii) there exists $w \in [x]_S \cap [y]_R$. Any such w witnesses that $(x, y) \in S \circ R$. □

Remark: For the subject of this chapter, criterion (iii) appears to be the most useful characterization of commuting equivalence relations.

2.2 Countable Borel Equivalence Relations

We now restrict our attention to the case in which X is a *standard Borel space*; that is, X which has associated to it a σ-algebra of sets S which is generated by a Polish (seperable, completely metrizable) topology. In the following definitions we assume that E is an equivalence relation on a standard Borel space (X, S). By abuse of notation, we will usually write X instead of (X, S).

Definition 2.2.1. *E is* Borel *if, when viewed as a subset of $X \times X$, it is an element of the σ-algebra generated by $S \times S = \{A \times B \in \mathbf{P}(X^2) \mid A, B \in S\}$.*

Definition 2.2.2. *E is* finite *(resp.* countable*) if each equivalence class of E is finite (resp. countable). It is* aperiodic *if each class is infinite.*

For example, if G is a countable group acting on X, then the resulting orbit equivalence relation E_G, defined by

$$xE_Gy \Leftrightarrow \exists g \in G(g \cdot x = y),$$

is countable. It is also Borel provided the action is a Borel function, where the Borel structure of G is given by the discrete topology. A result by Feldman and Moore [4] shows that the converse is also true.

Theorem 2.2.3. *(Feldman-Moore). If E is a countable Borel equivalence relation on the standard Borel space X, then there is a countable group G of Borel automorphisms of X for which $E = E_G$. Moreover, G can be chosen so that the involutions in G generate E_G, i.e., $\forall x, y \in X$ $(xEy \Leftrightarrow \exists g \in G(g^2 = 1$ and $g \cdot x = y))$.*

There are two other classes of Borel equivalence relations which are important to our work here. As above, in the next two definitions E is a Borel equivalence relation on a standard Borel space X.

Definition 2.2.4. *E is* smooth *if there exists a Borel map $f : X \to \mathbb{R}$ such that*

$$\forall x, y \in X, \ xEy \Leftrightarrow f(x) = f(y).$$

Definition 2.2.5. *E is* hyperfinite *if there exist finite Borel equivalence relations*

$$F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots$$

on X such that $E = \bigcup_n F_n$. Equivalently (see, e.g., Dougherty-Jackson-Kechris [2], 5.2), E is hyperfinite if $E = E_{\mathbb{Z}}$ for some Borel action of the group $(\mathbb{Z}, +)$ on X.

Clearly every hyperfinite equivalence relation is countable, but the converse is not true.

If E is smooth and also countable, then in fact it has a Borel *transversal*, i.e., a set $T \subseteq X$ which meets each E-class in exactly one point. (This follows from the fact that any countable-to-one Borel function has a one-sided Borel inverse; see [3], §18.C.) Using this fact it is easy to see the following standard fact:

Proposition 2.2.6. *Every countable smooth equivalence relation is hyperfinite.*

Proof. Let E be a countable smooth equivalence relation on the Borel space X, and let $X' = \{x \in X \mid [x]_E \textit{ is infinite}\}$. If $E\restriction_{X'}$ is hyperfinite, then clearly E is too, so without loss of generality we assume that E is aperiodic. By Theorem 2.2.3, $E = E_G$ for some countable group G. Since E is aperiodic, G must be infinite; let $\{g_n \mid n \in \mathbb{N}\}$ be an enumeration of its elements. Let $T \subseteq X$ be a Borel transversal for E, and define the Borel function $s : X \to X$ by

$$s(x) = \textit{the unique } y \in T \textit{ such that } xEy.$$

We begin by defining a Borel map $r : X \to \mathbb{N}$ which uniformly enumerates the elements of each E-class, i.e., for all $x \in X$, $r\restriction_{[x]_E}$ will be a bijection. Set

$$\begin{aligned} r(x) = 0 &\Leftrightarrow s(x) = x \\ r(x) = n+1 &\Leftrightarrow g_i \cdot s(x) = x, \textit{ where } i \textit{ is minimal} \\ &\qquad \textit{such that } g_i \cdot s(x) \notin r^{-1}(\{0, \cdots, n\}). \end{aligned}$$

Now for each $n \in \mathbb{N}$ let F_n be the finite Borel equivalence relation given by

$$xF_ny \Leftrightarrow xEy \textit{ and } r(x), r(y) \leq n.$$

The F_n's form an increasing sequence of finite Borel equivalence relations and $E = \bigcup_n F_n$. □

Remark: The converse of Proposition 2.2.6 is not true. One example of a non-smooth hyperfinite equivalence relation is E_0, the equivalence relation on the space of all binary sequences $2^{\mathbb{N}}$, given by

$$xE_0y \Leftrightarrow \exists N \forall n \geq N(x(n) = y(n)).$$

Note that this is essentially the Vitali equivalence relation E_V on $\mathbb{R}$, given by $xE_Vy \Leftrightarrow x - y \in \mathbb{Q}$. Just as E_V cannot have a Lebesgue measurable transversal, E_0 cannot have a transversal which is measurable with respect to the $(\frac{1}{2}, \frac{1}{2})$ product measure on $2^{\mathbb{N}}$. Thus we have

$$\textit{countable smooth} \subsetneq \textit{hyperfinite} \subsetneq \textit{countable.}$$

A more general way of comparing Borel equivalence relations is via *Borel reducibility.* Let E and F be two equivalence relations on the Borel spaces X and Y, respectively. E is *Borel reducible* to F, written $E \leq_B F$, if there exists a Borel function $f : X \to Y$ such that

$$\forall x, y \in X, \ xEy \Leftrightarrow f(x)Ff(y).$$

If X and Y are uncountable spaces (so that any countable equivalence relation has uncountably many classes), it follows that if E is smooth and F is hyperfinite, then $E \leq_B F$ and if F is non-smooth then $F \not\leq_B E$. Also, if each of E and F are non-smooth and hyperfinite, then they are *bireducible*, i.e., $E \leq_B F$ and $F \leq_B E$ (see [2], 7.1). A result by Harrington-Kechris-Louveau [7], known as the General Glimm-Effros Dichotomy, states that, in the sense of $\leq_B$, the non-smooth hyperfinite equivalence relations are the immediate successors to the smooth ones.

Theorem 2.2.7. *(Harrington-Kechris-Louveau) If E is a Borel equivalence relation on a standard Borel space X, then exactly one of the following holds:*

i) *E is smooth.*

ii) *$E_0 \leq_B E$.*

Moreover, if case (ii) holds, the function which witnesses the reduction can be taken *to be one-to-one.*

We now return to the subject of commuting equivalence relations, and note a basic consequence of Proposition 2.1.2:

Proposition 2.2.8. *Suppose R and S are commuting Borel equivalence relations on the standard Borel space X.*

i) *If R and S are finite, then $R \vee S$ is finite.*

ii) *If R is finite and S is smooth, then $R \vee S$ is smooth.*

iii) *If R is finite and S is hyperfinite, then $R \vee S$ is hyperfinite.*

Proof. If $R \square S$ and R is finite, then it follows from Proposition 2.1.2(iii) that $R \vee S$ has *finite index* over S, i.e., there are only finitely many S-classes contained in each $R \vee S$-class. Thus (i) is immediate. Also, it is a general fact that a *finite-by-smooth* Borel equivalence relation is smooth, and a *finite-by-hyperfinite* Borel equivalence relation is hyperfinite. (Here, by 'finite-by-($*$)', we mean an equivalence relation E which has finite index over some equivalence relation F with property ($*$).)

For a proof that a finite-by-smooth equivalence relation is smooth, let E and F be countable Borel equivalence relations on X, with F smooth and E of finite index over F. If $f : X \to \mathbb{R}$ witnesses the smoothness of F, then $g : X \to \mathbb{R}$ defined by

$$g(x) = \min(f(y) : xEy)$$

witnesses the smoothness of E.

For a proof that a finite-by-hyperfinite Borel equivalence relation is hyperfinite, see [8], 1.3. $\square$

In the proof of Proposition 2.2.8, we noted that if $R \square S$ and S is finite, then $E = R \vee S$ has finite index over R. It turns out that the converse is also true. This is a joint result with A. Kechris.

Theorem 2.2.9. *Suppose E and F are aperiodic countable Borel equivalence relations, $F \subseteq E$, and E has finite index over F. Then there is a finite equivalence relation $R \subseteq E$ such that $R \square F$ and $R \vee F = E$.*

In the proof of Theorem 2.2.9 we will use the following result, which can be found in [8].

Proposition 2.2.10. *(Marker Lemma) If E is an aperiodic countable Borel equivalence relation on a standard Borel space X, then there exists a sequence $A_0, A_1, \cdots$ of Borel subsets of X such that*

i) $A_0 \supseteq A_1 \supseteq \cdots$ *and each* A_k *is a complete section for* E.

ii) $\bigcap_k A_k = \emptyset$.

Proof of Theorem 2.2.9. Without loss of generality, by treating each subset

$$U_i = \{x \in X \mid \textit{ there are exactly } i \ F\textit{-classes in } [x]_E \ \}$$

separately, we may assume that E has fixed index i over F, for some $i \geq 1$; following the notation of group theory, we will usually write $[E : F] = i$. Let G be a countable group and $G \times X \to X$ a Borel action such that $E = E_G$, and such that E is generated by the involutions in G (see Theorem 2.2.3). Let $\{g_n \mid n \geq 1\}$ be an enumeration of the involutions.

We begin with the case $i = 2$; the analogous assertion for larger values of i will follow by an inductive argument. This base case argument is analogous to the proof of Theorem 3.12 in [8].

$i = 2$.

Claim 1: We can assume that F has the following form: for some complete sections A, B for E with $A \cap B = \emptyset$ and $A \cup B = X$, F is the equivalence relation determined by A and B. That is, xFy if and only if xEy and either $x, y \in A$ or $x, y \in B$.

Proof of Claim 1: Define

$$X_1 = \{x \in X \mid g_1 \cdot x \not F x\}$$
$$X_{n+1} = \{x \in X \mid x \in (X \setminus \bigcup_{k=1}^{n} X_k) \textit{ and } g_{n+1} \cdot x \in (X \setminus \bigcup_{k=1}^{n} X_k) \textit{ and } g_{n+1} \cdot x \not F x\}.$$

Notice that $x \in X_n \Leftrightarrow g_n \cdot x \in X_n$, because g_n is an involution. Also, if x and y are both in $X \setminus \bigcup_{n=1}^{\infty} X_n$, then $xEy \Rightarrow xFy$: because E is generated by $\{g_n \mid n \geq 1\}$, if xEy then there exists n for which $g_n \cdot x = y$. So suppose $x, y \in X \setminus \bigcup_{n=1}^{\infty} X_n$, xEy, and $x \not F y$, and let n_0 be minimal for which $g_{n_0} \cdot x = y$. Then, as $x, y \notin \bigcup_{k<n_0} X_k$, we have $x, y \in X_{n_0}$, a contradiction.

Now let

$$C = \{x \in X \mid [x]_E \setminus \bigcup_{n=1}^{\infty} X_n \neq \emptyset\}$$

$$D = \{x \in X \mid [x]_E \subseteq \bigcup_{n=1}^{\infty} X_n\}.$$

On D Theorem 2.2.9 is true, since we can just define R by

$$xRy \Leftrightarrow \exists n (x, y \in X_n \textit{ and } g_n \cdot x = y).$$

Hence, for the purpose of proving the theorem, we may assume that $X = C$. So let

$$A = \{x \in X \mid [x]_F \setminus \bigcup_{n=1}^{\infty} X_n \neq \emptyset\}$$

$$B = \{x \in X \mid [x]_F \subseteq \bigcup_{n=1}^{\infty} X_n\}.$$

□ (Claim 1)

Thus without loss of generality, we may assume that $X = A \cup B$, where $A \cap B = \emptyset$, A and B are complete sections for E, and where F is determined by A and B. By assumption $E\lceil_A$, $E\lceil_B$ are aperiodic, so by the Marker Lemma fix complete sections $A = A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$ for $E\lceil_A$ and $B = B_0 \supseteq B_1 \supseteq B_2 \supseteq \cdots$ for $E\lceil_B$ such that $\bigcap_k A_k = \bigcap_k B_k = \emptyset$. Also, for each $(x, y) \in E$ let

$$l(x, y) = \textit{ the least } n \in \mathbb{N} \textit{ such that } g_n \cdot x = y.$$

(Notice that $l(x, y) = l(y, x)$.) Call $l(x, y)$ the *label* of the pair (x, y).

Claim 2: There is a decreasing sequence of complete sections $A = A'_0 \supseteq A'_1 \supseteq A'_2 \supseteq \cdots$ for $E\restriction_A$ with $\bigcap_k A'_k = \emptyset$ and the following additional property:

For each n and each $x \in A\backslash A'_n$, there is a sequence $x = x_0, x_1, \cdots, x_{2m} \in A'_n$ such that $m \leq n^2$, and for each j, $1 \leq j \leq 2m$,

i) $l(x_{j-1}, x_j) \leq n^2$,

ii) $x_j \in A$ if j is even,

iii) $x_j \in B$ if j is odd.

Similarly for $E\restriction_B$.

Proof of Claim 2: Let $A'_0 = A$ and for $n \geq 1$ let

$$A'_n = A_n \cup \bigcup_{k<n} \{x \in A_k \backslash A_{k+1} \mid \textit{there is no sequence } x = x_0, x_1, \cdots x_{2m} \in A_{k+1} \textit{ for which } m \leq n, \textit{ and which satisfies (i), (ii), and (iii) above.}\}$$

Clearly $A = A'_0 \supseteq A'_1 \supseteq A'_2 \supseteq \cdots$ is a sequence of markers for $E\restriction_A$. To see that $\bigcap_k A'_k = \emptyset$, fix $x \in A$ and k with $x \in A_k \backslash A_{k+1}$. Then find $n > k$ sufficiently large that there is a sequence $x = x_0, x_1, x_2$ with $x_1 \in B$, $x_2 \in A_{k+1}$, and $l(x_{j-1}, x_j) \leq n^2$ for $j = 1, 2$. Clearly $x \notin A'_n$. Finally, to verify the main property of Claim 2, let $x \in A\backslash A'_n$, and let k be such that $x \in A_k \backslash A_{k+1}$. Then $k < n$ so, as $x \notin A'_n$, there must be some sequence $x = x_0, x_1, x_2, \cdots, x_{2l} = x' \in A_{k+1}$ with $l \leq n$, $l(x_{j-1}, x_j) \leq n^2$ for $1 \leq j \leq 2l$, $x_j \in A$ for j even, and $x_j \in B$ for j odd. If $x' \in A'_n$ we are done. Otherwise we repeat the process with x'. After at most n many steps we obtain a sequence $x = x_0, x_1, x_2, \cdots, x_{2m}$ with $m \leq n^2$, $x_{2m} \in A'_n$, $x_j \in A$ for j even, $x \in B$ for j odd, and $l(x_{j-1}, x_j) \leq n^2$ for $1 \leq j \leq 2m$.

□ (Claim 2)

For each $n \in \mathbb{N}$ and each $x \in A\backslash A'_n$, fix a sequence $x = x_0, x_1, \cdots x_{2m} \in A'_n$ as guaranteed by Claim 2, and define the Borel function $p_n^A : A \to A_n'$ by

$$p_n^A = \begin{cases} x_{2m}, & \text{if } x \in A \setminus A_n' \\ x, & \text{if } x \in A_n'. \end{cases}$$

In a similar manner, define $p_n^B : B \to B_n'$ for each $n \in \mathbb{N}$. Finally, define a bipartite graph G on A and B as follows: for each $x \in A, y \in B$, let

$$\begin{aligned}(x,y) \in G \Leftrightarrow xEy \textit{ and } \exists n \in \mathbb{N}, x' \in A, y' \in B \\ \textit{such that } p_n^A(x') = x, p_n^B(y') = y, \textit{and} \\ n^2 < l(x', y') \leq (n+1)^2.\end{aligned}$$

Claim 3: G is locally finite, and the connected components of G are exactly the equivalence classes of E.

Proof of Claim 3: To check that G is locally finite, fix $x \in A$ (the argument for $y \in B$ being analogous), and let n be such that $x \in A_n' \setminus A_{n+1}'$. If $(x,y) \in G$ (so $y \in B$), there is $m \leq n$ and $x' \in A$, $y' \in B$, such that $p_m^A(x') = x$, $p_m^B(y') = y$, and $m^2 < l(x', y') \leq (m+1)^2$. So there is a sequence $x = x_0, x_1, \cdots, x_{2k+1} = y$ of length at most $4n^2 + 1$ where $l(x_j, x_{j+1}) \leq (n+1)^2$ for each $j \leq 2k$. Since there are only finitely many such sequences, there are only finitely many such y.

For connectedness, it is enough to find a G-path between each $x \in A$ and $y \in B$ with xEy. We proceed by induction on $l(x,y)$. If $l(x,y) = 1$, then $(x,y) \in G$ (in the notation used in the definition of G, this is witnessed by $n = 0$, $x' = x$, and $y' = y$). Now suppose $l(x,y) \geq 2$, and let $n \in \mathbb{N}^+$ be such that $n^2 < l(x,y) \leq (n+1)^2$. If $p_n^A(x) = x'$, $p_n^B(y) = y'$, then there are sequences $x = x_0, x_1, \cdots, x_{2l} = x'$ and $y = y_0, y_1, \cdots, y_{2m} = y'$ with $0 \leq l, m \leq n^2$ as guaranteed by Claim 2. By definition, $(x', y') \in G$, and by construction $l(x_{j-1}, x_j) \leq n^2 < l(x,y)$ for each $j \leq 2l$; likewise $l(y_{k-1}, y_k) \leq n^2 < l(x,y)$ for each $k \leq 2m$. So by induction hypothesis there is a G-path between each x_{j-1}, x_j and each y_{k-1}, y_k, and we are done.

$\square$ (Claim 3)

The existence of the graph G allows us to define R as needed; for each $x \in A$, there is at least one, but only finitely many, $y \in B$ for which $(x, y) \in G$. Hence there is a finite-to-1 Borel function $f : A \to B$ such that for all $x \in A$, $(x, f(x)) \in G$. Let $B' = B \backslash ran(f)$, and similarly fix a Borel function $g : B' \to A$ such that $(g(y), y) \in G$ for all $y \in B'$. Now define R by:

$$xRy \Leftrightarrow x \textit{ and } y \textit{ are in the same connected component of the graph } G' \textit{ which is generated by } f \textit{ and } g.$$

It is easily checked that R is finite. Because G' is bipartite over A and B, and each connected component contains at least 2 elements, $R \Box F$, and $R \vee F = E$.

i>2.

Define the sequence $X_1, X_2, \cdots$ exactly as in the previous case. As before, if $x, y \in X \backslash \bigcup_{n=1}^{\infty} X_n$, then $xEy \Rightarrow xFy$, so if

$$A = \{x \in X \mid [x]_E \backslash \bigcup_{n=1}^{\infty} X_n \neq \emptyset\},$$

then $C = \{x \in A \mid [x]_F \not\subseteq \bigcup_{n=1}^{\infty} X_n\}$ is a set which contains exactly one F-class from each E-class in A.

Claim 1′: On the E-invariant set A, the theorem is true of E.

Proof of Claim 1′: Because C contains exactly one F-class from each E-class in A, $[E\restriction_{A\backslash C} : F\restriction_{A\backslash C}] = i-1$. By induction hypothesis there is a finite equivalence relation S' on $A \backslash C$ such that $S' \Box F\restriction_{A\backslash C}$ and $S' \vee F\restriction_{A\backslash C} = E\restriction_{A\backslash C}$. Let $S = S' \oplus \Delta_C$ be the equivalence relation on A obtained by extending S' in the trivial way to C, and let Q^0 be the standard Borel space whose elements are the equivalence classes of S. Define equivalence relations E^* and F^* on Q^0 by:

$$[x]_S E^* [y]_S \Leftrightarrow \big[\, [x]_S \big]_E = \big[\, [y]_S \big]_E$$

$$[x]_S F^* [y]_S \Leftrightarrow \big[\, [x]_S \big]_F = \big[\, [y]_S \big]_F$$

(so $[x]_S E^* [y]_S$ if and only if xEy, and $[x]_S F^* [y]_S$ if and only if $[x]_S$ and $[y]_S$ meet the same F-classes). $[E^* : F^*] = 2$, so by induction hypothesis there exists a finite equivalence relation R^* on Q^0 such that $R^* \Box F^*$ and $R^* \vee F^* = E^*$. Now define the Borel equivalence relation R on E by

$$xRy \Leftrightarrow [x]_S R^* [y]_S$$

Then R is as needed: it's finite because each of S and R are; also $R \vee F \subseteq E$. To see that both $R \Box F$ and $R \vee F = E$, let $(x, y) \in E$ be given; we'll show that there exists $u \in X$ for which $xRuFy$.

Since E^* is generated by the commuting equivalence relations R^* and F^*, there exists $z \in X$ such that

$$[x]_S R^* [z]_S F^* [y]_S.$$

Hence xRz' for all $z' \in [z]_S$. Since $[z]_S F^* [y]_S$, either both z and y are in C, in which case $xRzFy$, or both z and y are in $A \backslash C$. In this second case, because $S\restriction_{A\backslash C} \Box F\restriction_{A\backslash C}$, there exists $z' \in X$ such that $zSz'Fy$. But then $xRz'Fy$.

$\Box$ (Claim $1'$)

By Claim $1'$, without loss of generality we may assume that $\bigcup_n X_n = X$. Let R^1 be the finite Borel equivalence relation defined on X by

$$xR^1y \Leftrightarrow \exists n\,(x, y \in X_n \textit{ and } g_n \cdot x = y)\,.$$

Let P^1 be the standard Borel space whose elements are the equivalence classes of R^1, and define the Borel equivalence relations E^1 and F^1 by

$$[x]_{R^1} E^1 [y]_{R^1} \Leftrightarrow \left[\, [x]_{R^1} \right]_E = \left[\, [y]_{R^1} \right]_E$$

$$[x]_{R^1} F^1 [y]_{R^1} \Leftrightarrow \left[\, [x]_{R^1} \right]_F = \left[\, [y]_{R^1} \right]_F.$$

Just as we were able to obtain a 'pairing' between E-equivalent, non-F-equivalent elements of X (given by R^1), we now construct such a pairing of the elements of P^1 with respect to E^1 and F^1: let G^1 be a countable group and $G^1 \times P^1 \to P^1$ a Borel action such that the involutions $\{g_n^1 \,|\, n \geq 1\}$ of G^1 generate E^1. Exactly as we defined the sequence $X_1, X_2, \cdots$, define

$$P_1^1 = \{p \in P^1 \mid g_1^1 \cdot p \not{F}^1 p\}$$

$$P_{n+1}^1 = \{p \in P^1 \mid p \in (P^1 \setminus \bigcup_{k=1}^{n} P_k^1) \text{ and } g_{n+1}^1 \cdot p \in (P^1 \setminus \bigcup_{k=1}^{n} P_k^1) \text{ and } g_{n+1}^1 \cdot p \not{F}^1 p\}.$$

Now suppose that $\bigcup_{n=1}^{\infty} P_n^1 \neq P^1$. Then $\forall p, q \in P^1$, $pE^1q \Rightarrow pF^1q$, so on the Borel set

$$A^1 = \{p \in P^1 \mid [p]_{E^1} \setminus \bigcup_{n=1}^{\infty} P_n^1 \neq \emptyset\},$$

the Borel set $C^1 = \{p \in P^1 \mid [p]_{F^1} \setminus \bigcup_{n=1}^{\infty} P_n^1 \neq \emptyset\}$ contains exactly one F^1-class from each E^1-class. Now let

$$A_1 = \{x \in X \mid [x]_{R^1} \in A^1\}$$
$$C_1 = \{x \in X \mid [x]_{R^1} \in C^1\}$$

Then C_1 is an F-invariant set which contains exactly two F-classes from each E-class in A_1. Just as in the proof of Claim $1'$, we can use the induction hypothesis to find a finite equivalence relation R on A, such that $R \Box F\restriction_{A_1}$, and $R \vee F\restriction_{A_1} = E\restriction_{A_1}$:

Since $[E\restriction_{A_1 \setminus C_1} : F\restriction_{A_1 \setminus C_1}] < i$ and $[E\restriction_{C_1} : F\restriction_{C_1}] < i$, there exist finite equivalence relations S_1 on $A_1 \setminus C_1$ and S_2 on C_1 such that S_1 commutes with $F\restriction_{A_1 \setminus C_1}$, and $S_1 \vee F\restriction_{A_1 \setminus C_1} = E\restriction_{A_1 \setminus C_1}$, and similarly for S_2, $F\restriction_{C_1}$, $E\restriction_{C_1}$. Let $S = S_1 \oplus S_2$, and let Q^1 be the standard Borel space whose elements are the equivalence classes of S. As before, define the 'quotient' equivalence relations E^*, F^* by

$$[x]_S E^* [y]_S \Leftrightarrow \left[[x]_S\right]_E = \left[[y]_S\right]_E$$

$$[x]_S F^* [y]_S \Leftrightarrow \left[[x]_S\right]_F = \left[[y]_S\right]_F.$$

$[E^* : F^*] = 2 < i$, so there is a finite equivalence relation R^* on Q^1 such that R^* and F^* commute, and their join is E^*. Now define the equivalence relation R on A_1 by

$$xRy \Leftrightarrow [x]_S R^* [y]_S.$$

Then R commutes with $F\lceil_{A_1}$, and their join is $E\lceil_{A_1}$; the proof of this is entirely analogous to the argument given in Claim 1′. Hence, without loss of generality, we may assume that $\bigcup_{n=1}^{\infty} P_n^1 = P^1$.

Now define R^2 on X by

$$xR^2y \Leftrightarrow xEy \textit{ and } \exists n \left(x, y \in P_n^1 \textit{ and } g_n^1 \cdot [x]_{R^1} = [y]_{R^1}\right).$$

Each R^2 class contains four elements and meets at least three distinct F-classes. If $R^2 \not\Box F$, then iterate the above process, obtaining an equivalence relation R^3 such that each R^3-class contains eight elements and meets at least four distinct F-classes. After at most $i - 1$ many steps, a finite equivalence relation R is produced such that $R \Box F$ and $R \vee F = E$. □

From Proposition 2.2.8 it appears that, at least under certain circumstances, if $R \Box S$ then $R \vee S$ is no more complicated than R or S. This should be contrasted with the general case, where no commutativity is assumed. For example, it is known that any hyperfinite equivalence relation can be described as $R \vee S$, where R and S are finite Borel equivalence relations and each R-class and each S-class have size at most 2 (see [8], 1.21). We next investigate the case in which $R \Box S$, and each of R and S are countable, Borel, and aperiodic. As noted in Theorem 2.2.7, the 'simplest' subcase is when R and S are smooth.

Notation: Given a Borel space Y, let $I(Y)$ denote the trivial Borel equivalence relation which has only one equivalence class, namely $Y \times Y$.

Proposition 2.2.11. *If E is any countable Borel equivalence relation on a standard Borel space X, then there exist smooth aperiodic equivalence relations R and S on*

$X \times \mathbb{N}$ such that $R \Box S$ and $R \vee S = E \times I(\mathbb{N})$.

Proof. Using Theorem 2.2.3, fix a countable group G and an action $G \times X \to X$ which generates E. Without loss of generality we may assume that G is infinite; let $\{g_n : n \in \mathbb{N}\}$ be an enumeration of its elements. Define equivalence relations R and S on $X \times \mathbb{N}$ by

$$(x, m)R(y, n) \Leftrightarrow g_m \cdot x = g_n \cdot y$$
$$(x, m)S(y, n) \Leftrightarrow x = y$$

Clearly each of R and S are smooth, and each is contained in $E \times I(\mathbb{N})$. To see that $E \times I(\mathbb{N}) \subseteq R \vee S$, let (x, m) and (y, n) be such that $(x, m)E \times I(\mathbb{N})(y, n)$, and let k be such that $g_k \cdot x = y$. Then

$$(x, m)R(g_k \cdot x, l)S(g_k \cdot x, n) = (y, n)$$

where l is chosen so that $g_l = g_m \cdot g_k^{-1}$. This also shows that $R \Box S$, by Proposition 2.1.2(ii). □

We will provide a context for Proposition 2.2.11 in the next section.

2.3 Compressible Equivalence Relations. Nadkarni's Theorem

We begin with some terminology.

Notation: If E is a countable Borel equivalence relation on a Borel space X, let $[[E]]$ denote the collection of all Borel bijections $f : A \to B$, where A and B are Borel subsets of X and $\forall x \in A,\ f(x)Ex$.

Definition 2.3.1. *A* complete section *of a Borel equivalence relation E on X is a Borel subset $A \subseteq X$ which meets every E-class.*

Definition 2.3.2. *A countable Borel equivalence relation E on a standard Borel space X is* compressible *if there exists $f \in [[E]]$ such that $f : X \to A$ (i.e., the domain of f is all of X) and $X \backslash A$ is a complete section for E.*

So E is compressible if there is an injective Borel map $f : X \to X$ which maps every E-class into a proper subset of itself. Clearly E can be compressible only if it is aperiodic.

Definition 2.3.3. *Let E be a countable Borel equivalence relation on a standard Borel space X, and let μ be a σ-finite Borel measure on X. μ is an* invariant measure *for E if for all $f \in [[E]], f : A \to B$, we have $\mu(A) = \mu(B)$.*

That is, μ is invariant for E if every Borel bijection which preserves E also preserves μ.

Definition 2.3.4. *Let E and F be countable Borel equivalence relations on the standard Borel spaces X and Y, respectively. E and F are* Borel isomorphic, *written $E \cong_B F$, if there exists a Borel bijection $f : X \to Y$ such that for all $x, y \in X$,*

$$xEy \Leftrightarrow f(x)Ff(y).$$

Proposition 2.3.5. *The following are equivalent, for E a countable Borel equivalence relation on a standard Borel space X.*

i) *E is compressible.*

ii) *$E \cong_B E \times I(\mathbb{N})$.*

iii) *There is a smooth aperiodic equivalence relation $F \subseteq E$.*

For a proof, see [2], 2.5.

There is also the following characterization of compressible equivalence relations, due to M.G. Nadkarni [14], which we state here without proof.

Theorem 2.3.6. *(Nadkarni): A countable Borel equivalence relation E is compressible if and only if it has no invariant Borel probability measure.*

Via the characteristics given in Proposition 2.3.5, the following is an immediate corollary of Propostion 2.2.11:

Corollary 2.3.7. *A countable Borel equivalence relation E on a standard Borel space X is compressible if and only if it is aperiodic and there exist countable smooth equivalence relations R and S on X such that $R \Box S$ and $E = R \vee S$.*

Proof. By Proposition 2.3.5, if E is compressible then $E \cong_B E \times I(\mathbb{N})$, and the assertion holds for $E \times I(\mathbb{N})$ by Proposition 2.2.11. Now suppose E is aperiodic and $E = R \vee S$, where $R \Box S$ and each of R and S are countable and smooth. Let

$$X' = \{x \in X \mid [x]_E \textit{ contains infinitely many } R\textit{-classes}\}.$$

$S\lceil_{X'}$ must be aperiodic, by Proposition 2.1.2(iii), and $E\lceil_{X\setminus X'}$ is smooth, since on $X \setminus X'$, E has finite index over R. Thus $E' = E\lceil_{X\setminus X'} \oplus S\lceil_{X'}$ is a smooth aperiodic equivalence relation contained in E, so by Proposition 2.1.2(iii) E is compressible. $\Box$

Compressible equivalence relations are cofinal in the partial order of countable Borel equivalence relations by $\leq_B$, since for any countable Borel equivalence relation E, $E \leq_B E \times I(\mathbb{N})$ and $E \times I(\mathbb{N})$ is compressible. Hence Corollary 2.3.7 shows that, in some sense, the join of commuting countable smooth equivalence relations can be arbitrarily complex. Another way to view this is that compressible equivalence relations have a complete characterization in terms of commuting smooth equivalence relations.

We now investigate the case in which E is countable and *not* compressible, i.e., has an invariant probability measure. Such E cannot contain an aperiodic smooth subrelation. We instead ask when E can be written as the join of two commuting non-smooth hyperfinite equivalence relations.

2.4 Costs

The notion of the cost of an equivalence relation was introduced by G. Levitt in [12], and developed by D.Gaboriau in [5]. In this section X represents a standard Borel space and μ a Borel probability measure on X which has no point-masses.

Notation: Let $PAut(X,\mu)$ denote the collection of all partial Borel bijections $f : A \to B$, where A and B are Borel subsets of X, which preserve μ in the sense that for all $U \subseteq A$ Borel, $\mu(U) = \mu(f[U])$.

Definition 2.4.1. *A* graphing *on the measure space (X,μ) is any countable subset of $PAut(X,\mu)$. If $\Phi = \{\phi_i : A_i \to B_i \mid i \in I \subseteq \mathbb{N}\}$ is any graphing, its* cost *(with respect to μ) is defined to be*

$$C_\mu(\Phi) = \sum_i \mu(A_i)$$

Remark: Each graphing Φ generates a countable Borel equivalence relation E_Φ for which μ is invariant, namely

$$xE_\Phi y \Leftrightarrow \begin{cases} x = y \text{ or} \\ \\ \exists n \in \mathbb{N}\, \exists k_1, \cdots, k_n \in \mathbb{N}\, \exists e_1, \cdots, e_n \in \{-1,1\} \textit{ such that} \\ x \in dom(\phi_{k_n}^{e_n} \circ \cdots \circ \phi_{k_1}^{e_1}) \textit{ and } \phi_{k_n}^{e_n} \circ \cdots \circ \phi_{k_1}^{e_1}(x) = y. \end{cases}$$

Conversely, by Theorem 2.2.3, any countable Borel equivalence relation E for which μ is invariant is generated by some graphing Φ. Thus the following definition makes sense:

Definition 2.4.2. *If E is a countable Borel equivalence relation on X for which μ is an invariant measure, then the* cost *of E (with respect to μ) is*

$$C_\mu(E) = \inf(C_\mu(\Phi) : \Phi \textit{ is a graphing which generates } E).$$

Definition 2.4.3. *A graphing* Φ *on* X *is a* μ-treeing *if for every reduced word* $w = \phi_k^{e_k} \cdots \phi_1^{e_1}$ *with* $\phi_i \in \Phi$ *and* $e_i \in \{-1, 1\}$, *the set*

$$\{x \in X \mid x \in \mathrm{dom}(\phi_k^{e_k} \circ \cdots \circ \phi_1^{e_1}) \text{ and } \phi_k^{e_k} \circ \cdots \circ \phi_1^{e_1}(x) = x\}$$

is μ*-null.*

Definition 2.4.4. *A countable Borel equivalence relation* E *on* X *is* μ-treeable *if it is generated by a* μ*-treeing. It is* μ-hyperfinite *if there exists a* μ*-conull set* $X' \subseteq X$ *such that* $E\restriction_{X'}$ *is hyperfinite.*

One of the most important theorems in the subject of costs is the following:

Theorem 2.4.5. *(Gaboriau [5], I.11, IV.1) If* Φ *is a graphing of* E *and* $C_\mu(\Phi) = C_\mu(E) < \infty$, *then* Φ *is a* μ*-treeing. Conversely, if* E *is* μ*-treeable and* Φ *is any* μ*-treeing of* E, *then* $C_\mu(\Phi) = C_\mu(E)$.

Thus, to determine the cost of a μ-treeable equivalence relation it suffices to find a single μ-treeing.

Examples:

1) If E is aperiodic and hyperfinite, then $C_\mu(E) = 1$. It is generated by a μ-treeing consisting of a single Borel automorphism $f : X \to X$.

2) If E is finite, with each class of cardinality n, then $C_\mu(E) = 1 - \frac{1}{n}$. Again, E is generated by a μ-treeing consisting of a single mapping $\phi \in PAut(X, \mu)$. The domain of ϕ is $X \backslash T$, where T is a Borel tranversal for E.

3) Let $F_n = \langle a_1, \cdots, a_n \rangle$ denote the free group on n generators. If $F_n : X \to X$ is a free Borel action of F_n on X, then the corresponding equivalence relation E_{F_n} has cost n. By abuse of notation, letting $a_i \in F_n$ also denote the automorphism on (X, μ) corresponding to it via the above action, we see that $\Phi = \{a_i : X \to X \mid i = 1 \cdots n\}$ is a treeing for E. This shows that if $n \neq m$, then the groups F_n and F_m cannot generate the same equivalence relation via free actions.

Definition 2.4.6. *Let E and F be countable Borel equivalence relations on the standard Borel probability spaces (X, μ) and (Y, ν), respectively. E and F are* orbit equivalent *if there are Borel conull sets $X' \subseteq X$ and $Y' \subseteq Y$ and a Borel bijection $f : X' \to Y'$ such that $f\mu$ and ν are equivalent and for all $x, y \in X'$, $xEy \Leftrightarrow f(x)Ff(y)$.*

We now list, without proof, a few more facts about cost due to Levitt and Gaboriau. For these we assume E is a countable Borel equivalence relation on a standard Borel probability space (X, μ), and that μ is invariant for E.

Theorem 2.4.7. *(Levitt [12], 2) If E is aperiodic, then $C_\mu(E) \geq 1$ and E is μ-hyperfinite if and only if it is μ-treeable and $C_\mu(E) = 1$.*

Theorem 2.4.8. *(Gaboriau [5], II.6) If $U \subseteq X$ is a μ-a.e. complete section for E (i.e., $\mu([U]_E) = 1$), then the cost of the equivalence relation $E\restriction_U$ on U with respect to the measure $\mu' = \frac{\mu\restriction_U}{\mu(U)}$ is*

$$C_{\mu'}(E\restriction_U) = \frac{C_\mu(E) - 1 + \mu(U)}{\mu(U)}.$$

Theorem 2.4.9. *(Gaboriau [5], III.5) Suppose F is a hyperfinite equivalence relation on X and $F \subseteq E$ (and hence μ is F-invariant). Let Φ be a μ-treeing for F. Then for all $\epsilon > 0$ there exists a graphing Ψ_ϵ for E such that $\Phi \subseteq \Psi_\epsilon$ and $C_\mu(\Psi_\epsilon) \leq C_\mu(E) + \epsilon$.*

Corollary 2.4.10. *(Gaboriau [5], IV.15, IV.36) Suppose that E_1 and E_2 are countable Borel equivalence relations on X and μ is E_i-invariant, $i = 1, 2$. If $F \subseteq E_1 \cap E_2$ is hyperfinite, then*

$$C_\mu(E_1 \vee E_2) \leq C_\mu(E_1) + C_\mu(E_2) - C_\mu(F).$$

In particular, if $E_1 \cap E_2$ is aperiodic, then

$$C_\mu(E_1 \vee E_2) \leq C_\mu(E_1) + C_\mu(E_2) - 1.$$

Proof. By Theorem 2.4.9, given $\epsilon > 0$ and a μ-treeing Φ of F, there exists a graphing Ψ_i of E_i, $i = 1, 2$, which contains Φ and such that $C_\mu(\Psi_i) \leq C_\mu(E_i) + \frac{\epsilon}{2}$. Then $\Psi_1 \vee (\Psi_2 \backslash \Phi)$ is a graphing for $E_1 \vee E_2$, so

$$\begin{aligned} C_\mu(E_1 \vee E_2) &\leq C_\mu(\Psi_1 \vee (\Psi_2 \backslash \Phi)) \\ &\leq C_\mu(\Psi_1) + C_\mu(\Psi_2) - C_\mu(\Phi) \\ &\leq C_\mu(E_1) + C_\mu(E_2) - C_\mu(F) + \epsilon. \end{aligned}$$

The second assertion follows from the first, and the fact that every aperiodic countable Borel equivalence relation contains an aperiodic hyperfinite one over the same underlying space (see [8], 3.25). □

Corollary 2.4.11. *If $E_1, E_2, \cdots, E_i, \cdots$ are countable Borel equivalence relations on X, μ is E_i invariant for each i, and $\bigcap_i E_i$ is aperiodic, then*

$$C_\mu(\bigvee_i E_i) \leq 1 + \sum_i (C_\mu(E_i) - 1)$$

Proof. Let $F \subseteq \bigcap_i E_i$ be an aperiodic hyperfinite equivalence relation on X and Φ a treeing for F. Given $\epsilon > 0$, for each i let Ψ_i be a graphing of E_i with $\Phi \subseteq \Psi_i$ and $C_\mu(\Psi_i) \leq C_\mu(E) + \frac{\epsilon}{2^i}$. $\Phi \vee (\bigvee_i (\Psi_i \backslash \Phi))$ is a graphing of E, so

$$\begin{aligned} C_\mu(E) &\leq C_\mu(\Phi) + \sum_i C_\mu(\Psi_i \backslash \Phi) \\ &\leq 1 + \sum_i (C_\mu(E_i) - 1) + \epsilon. \end{aligned}$$

□

We now return to the question of which aperiodic countable Borel equivalence relations with an invariant probability measure can be written as the join of two commuting hyperfinite ones. By Theorem 2.4.7 the cost of such an equivalence relation is between 1 and 2. In fact, commutativity implies that the cost is exactly 1. (Of course the join of two non-commuting hyperfinite equivalence relations can have cost exactly 2.)

Theorem 2.4.12. *Suppose that E is an aperiodic countable Borel equivalence relation on a standard Borel probability space (X, μ) and that μ is an invariant measure for E. If $E = R \vee S$, where R and S are hyperfinite and $R \square S$, then $C_\mu(E) = 1$. In particular, if E is μ-treeable, it is actually μ-hyperfinite.*

We would like to thank D. Gaboriau for a useful suggestion which helped to complete the proof of Theorem 2.4.12.

2.5 Proof of Theorem 2.4.12

Definition 2.5.1. *Let E be a Borel equivalence relation on a standard Borel space X. A set $A \subseteq X$ is E-*invariant *if $\forall x, y \in X (x \in A$ and $xEy) \Rightarrow (y \in E)$. Given any $B \subseteq X$, let $[B]_E = \{y \in X \mid \exists x \in B(xEy)\}$. Thus $[B]_E$ is the smallest E-invariant set containing B.*

Definition 2.5.2. *Let E be a Borel equivalence relation on a standard Borel probability space (X, μ). μ is E-*ergodic *if each E-invariant subset of X is either μ-null or μ-conull.*

We will first show that it suffices to prove the theorem in the case where μ is E-ergodic. For this we will use the Uniform Ergodic Decomposition Theorem, due independently to Farrell [3] and Varadarajan [16], which relates E-invariant Borel probability measures on X to those which are additionally E-ergodic.

Notation: Given a countable Borel equivalence relation E on a standard Borel space X, let $\mathbf{I}_E$ be the set of all E-invariant Borel probability measures on X, and let $\mathbf{EI}_E$ be the set of all measures in $\mathbf{I}_E$ which are additionally E-ergodic.

Theorem 2.5.3. *(Farrell, Varadarajan) Let E be a countable Borel equivalence relation on a standard Borel space X, and suppose that $\mathbf{I}_E \neq \emptyset$. Then $\mathbf{EI}_E \neq \emptyset$, and there is a Borel surjection $e : X \to \mathbf{EI}_E$ such that*

i) $\forall x, y \in X,\ xEy \Rightarrow e(x) = e(y)$

ii) $\forall \nu \in \mathbf{EI}_E,\ \nu\left(e^{-1}(\{\nu\})\right) = 1$

iii) $\forall \mu \in \mathbf{I}_E,\ \mu = \int_X e(x)d\mu(x)$. *That is, for each bounded Borel function* $f : X \to \mathbb{R}$, $\int_X f d\mu = \int_X \left(\int_X f de(x)\right) d\mu(x)$.

Notation: For e and ν as in the above theorem, let $X_\nu = e^{-1}(\{\nu\})$.

Theorem 2.5.4. *Let E be a countable Borel equivalence relation on a standard Borel space X, and $\mu \in \mathbf{I}_E$. Let $\mu = \int_X e(x)d\mu(x)$ be the ergodic decomposition of μ with respect to E. Then*

$$C_\mu(E) = \int_X C_{e(x)}(E)d\mu(x).$$

Proof. First note that the analogous assertion is true for graphings, i.e., $C_\mu(\Phi) = \int_X C_{e(x)}(\Phi)d\mu(x)$ for any graphing Φ of E. This follows from the fact that, for any $\nu \in \mathbf{I}_E$, $C_\nu(\Phi) = \frac{1}{2}\int_X v_\Phi(x)d\nu$, where $v_\Phi(x)$ is the valence of x in the combinatorial graph given by Φ. From this it follows easily that $C_\mu(E) \geq \int_X C_{e(x)}(E)d\mu(x)$: given $\epsilon > 0$, if Φ is a graphing of E such that $C_\mu(E) \geq C_\mu(\Phi) - \epsilon$, then

$$C_\mu(E) \geq C_\mu(\Phi) - \epsilon = \int_X C_{e(x)}(\Phi)d\mu(x) - \epsilon \geq \int_X C_{e(x)}(E)d\mu(x) - \epsilon.$$

For the reverse inequality: let $\epsilon \geq 0$ be given. For each $\nu \in \mathbf{EI}_E$, there is a graphing $\Phi^\nu = \{\phi_i^\nu : A_i^\nu \to B_i^\nu \mid i \in \mathbb{N}\}$ defined on X_ν which generates $E\restriction_{X_\nu}$ and such that $C_\nu(E) \geq C_\nu(\Phi^\nu) - \epsilon$. We seek to choose the Φ^ν's in such a way that each ϕ_i, defined by

$$\phi_i = \bigoplus_{\nu \in \mathbf{EI}_E} \phi_i^\nu : \bigcup_{\nu \in \mathbf{EI}_E} A_i^\nu \to \bigcup_{\nu \in \mathbf{EI}_E} B_i^\nu,$$

is μ-measurable. We begin with a few observations on graphings and cost calculations:

First note that if $\Phi = \{\phi : A_i \to B_i \,|\, i \in \mathbb{N}\}$ is any graphing for E and Φ' is a graphing of the form $\Phi' = \{\phi' : A_i' \to B_i' \mid i \in \mathbb{N}\}$, where, for all $i \in \mathbb{N}$, $\nu(A_i \triangle A_i') = 0$ and $\phi_i\restriction_{A_i \cap A_i'} = \phi_i'\restriction_{A_i \cap A_i'}$, then there exists an E-invariant ν-conull set X' such that for all $(x, y) \in X' \times X'$ and all $i \in \mathbb{N}$, $y = \phi_i(x) \Leftrightarrow y = \phi_i'(x)$. (Hence $C_\nu(\Phi) = C_\nu(\Phi')$.) Call such a pair Φ, Φ' *ν-equivalent.* Then when making cost calculations with respect to a measure ν, we can always replace a graphing Φ by one to which it is ν-equivalent.

Secondly, fix any countable group $G = \{g_j \,|\, j \in \mathbb{N}\}$ and a Borel group action $G \times X \to X$ which generates E, and view the elements of G as Borel bijections on X via this action. If $\Phi = \{\phi_i : A_i \to B_i \mid i \in \mathbb{N}\}$ is any graphing for E, then for each i and each $x \in A_i$ there is j such that $\phi_i(x) = g_j \cdot x$; thus A_i may be partitioned into countably many Borel sets $A_i^0, \cdots, A_i^j, \cdots$ so that $\phi\restriction_{A_i^j} = g_j\restriction_{A_i^j}$ for each $j \in \mathbb{N}$. Doing this for each i, and then letting $A^j = \bigcup_i A_i^j$, it's clear that $\Phi^* = \{g_j\restriction_{A^j} \,|\, j \in \mathbb{N}\}$ and Φ have the same edgeset (i.e., for each pair (x, y), $\exists i\ \phi_i(x) = y$ if and only if $\exists j\ x \in A_j$ and $g_j \cdot x = y$) and for any $\nu \in \mathbf{I}_E$ $C_\nu(\Phi^*) \leq C_\nu(\Phi)$. When calculating the cost of E, we may take the infimum simply over the costs of all such graphings $\Phi = \{g_j\restriction_{A_j} \mid j \in \mathbb{N}\}$ which generate E. Note that any such graph is parametrized by the sequence of Borel sets $(A_j)_{j \in \mathbb{N}}$.

Finally, fix a Polish topology τ on X whose Borel σ-algebra is S; let $\Phi = \{g_j\restriction_{A_j} | j \in \mathbb{N}\}$ be a graphing which generates E. For each $\nu \in \mathbf{I}_E$ we can replace Φ by the ν-equivalent graphing $\Phi' = \{g_j\restriction_{C_j} \mid j \in \mathbb{N}\}$, where $C^j = \bigcup_k C_k^j$ is the increasing union of compact sets, $C^j \subseteq A^j$, and $\nu(A^j \backslash C^j) = 0$. Thus when determining $C_\nu(E)$, we only need to consider the graphings of E which are parametrized by elements of the Borel space

$$K(X)^{\mathbb{N} \times \mathbb{N}} = \{(C_k^j)_{j,k \in \mathbb{N}} \mid \forall j, k\ C_k^j \text{ is a compact subset of } X\}.$$

For ease of notation, we will denote a general element of $K(X)^{\mathbb{N} \times \mathbb{N}}$ by $\bar{C}$, rather than $(C_k^j)_{j,k \in \mathbb{N}}$. We will continue to use the notation $C^j = \bigcup_k C_k^j$.

Now consider the function $f : \mathbf{EI}_E \to \mathbb{R}$ given by

$$f(\nu) = C_\nu(E).$$

Lemma 2.5.5. *For each $r \in \mathbb{R}$, the set $\{\nu \in \mathbf{EI}_E \mid f(\nu) \geq r\}$ is $\mathbf{\Pi}_1^1$.*

Proof. Using the above remarks, we have, for each $r \in \mathbb{R}$,

$$
\begin{aligned}
f(\nu) \geq r \Leftrightarrow & \textit{ For all graphings } \Phi \textit{ of } E, C_\nu(\Phi) \geq r \\
\Leftrightarrow & \textit{ For all graphings } \Phi^* \textit{ of } E \textit{ which have the form} \\
& \Phi^* = \{g_j{\restriction}_{A^j} \mid j \in \mathbb{N}, A_j \in S\}, C_\nu(\Phi^*) \geq r \\
\Leftrightarrow & \forall \bar{C} \in K(X)^{\mathbb{N}\times\mathbb{N}} \, (\textit{If } \{g_j{\restriction}_{C^j} \mid j \in \mathbb{N}\} \textit{is } \nu\textit{-equivalent to a graphing} \\
& \textit{of } E, \textit{ then } \sum_j \nu(C^j) \geq r \Big).
\end{aligned}
$$

The parenthetical part of the last condition can be written as

$$
\begin{aligned}
&\forall^*_\nu x \in X \forall j \in \mathbb{N} \exists j_1, \cdots, j_n \in \mathbb{N} \exists \epsilon_1, \cdots, \epsilon_n \in \{-1, 1\} \\
&\quad (x \in C^{j_1}(\epsilon_1) \wedge g_{j_1} \cdot x \in C^{j_2}(\epsilon_2) \wedge \cdots \wedge g_{j_{n-1}} \cdots g_{j_1} \cdot x \in C^{j_n}(\epsilon_n) \\
&\qquad\qquad \wedge g_{j_n} \cdots g_{j_1} \cdot x = g_j \cdot x),
\end{aligned}
$$

where '$\forall^*_\nu x$' means 'for ν a.e. x', and $C^{j_i}(\epsilon_i)$ is C^{j_i} if $\epsilon_i = 1$ and is $g_{j_i} \cdot C^{j_i}$ if $\epsilon_i = -1$. This is a Borel condition on $(\bar{C}, \nu)$, so the full condition is $\mathbf{\Pi}^1_1$ on ν.

□

By Lemma 2.5.5, f is $e\mu$-measurable, where $e\mu$ is the image measure of μ under e, i.e., for each Borel set $V \subseteq \mathbf{EI}_E$, $\tilde{\mu}(V) = \mu\big(e^{-1}(V)\big)$. Hence we can find a Borel function $\tilde{f} : \mathbf{EI}_E \to \mathbb{R}$ such that

$$
e\mu\Big(\{\nu \in \mathbf{EI}_E \mid \tilde{f}(\nu) \neq f(\nu)\}\Big) = 0.
$$

Now define $Q \subseteq \mathbf{EI}_E \times K(X)^{\mathbb{N}\times\mathbb{N}}$ by

$$
\begin{aligned}
(\nu, \bar{C}) \in Q \Leftrightarrow \Phi_{\bar{C}} = \{g_j{\restriction}_{C^j} \mid j \in \mathbb{N}\} & \text{ is } \nu\text{-equivalent to a} \\
& \text{graphing of } E{\restriction}_{X_\nu} \text{ and } \sum_j \nu(C^j) \leq \tilde{f}(\nu) + \epsilon.
\end{aligned}
$$

Q is Borel (which follows from the proof of Lemma 2.5.5 above), and the projection of Q onto $\mathbf{EI}_E$,

$$\{\nu \in \mathbf{EI}_E \mid \exists \bar{C}(\nu, \bar{C}) \in Q\}$$

is all of $\mathbf{EI}_E$. Hence, by the Jankov - Von Neumann Theorem (see [3], §18.A) there is a $\sigma(\mathbf{\Sigma}^1_1)$-measurable function $s : \mathbf{EI}_E \to K(X)^{\mathbb{N}\times\mathbb{N}}$ which uniformizes Q, i.e., for all $\nu \in \mathbf{EI}_E, (\nu, s(\nu)) \in Q$.

Thus, denoting $s(\nu)$ by the sequence $(C^j_{s(\nu)})_{j\in\mathbb{N}}$, if for each j we define the set $A_j \subseteq X$ by

$$\begin{aligned} x \in A_j &\Leftrightarrow \exists \nu \in \mathbf{EI}_E \left(x \in e^{-1}(\{\nu\}) \cap C^j_{s(\nu)} \right) \\ &\Leftrightarrow x \in \text{ the } j\text{th element of the sequence } s\big(e(x)\big), \end{aligned}$$

then A_j is in $\sigma(\mathbf{\Sigma}^1_1)$, and hence is μ-measurable. Let $A'_j \subseteq X$ be a Borel set such that $\mu(A_j \triangle A'_j) = 0$. If we then let $\Phi = \{g_j\restriction_{A'_j} \mid j \in \mathbb{N}\}$, we have

$$\begin{aligned} C_\mu(E) \leq C_\mu(\Phi) &= \Sigma_j \mu(A'_j) = \Sigma_j \mu(A_j) \\ &= \int_X \Sigma_j e(x)(A_j) d\mu(x) = \int_X \Sigma_j e(x)(K_j(e(x))) d\mu(x) \\ &\leq \int_X C_{e(x)}(E) + \epsilon d\mu(x) = \int_X C_{e(x)}(E) d\mu(x) + \epsilon, \end{aligned}$$

which establishes the reverse inequality. □

Definition 2.5.6. *Let μ be a Borel probability measure on a standard Borel space X, and E a Borel equivalence relation on X. μ is E-*quasi-ergodic*, or* quasi-ergodic *for E, if there exists a finite partition $X_1 \cup X_2 \cup \cdots X_n$ of X such that for each i, X_i is Borel and E-invariant, $\mu(X_i) > 0$, and $\frac{1}{\mu(X_i)} \cdot \mu\restriction_{X_i}$ is $E\restriction_{X_i}$-ergodic.*

Notation: For Y a standard Borel space, let Δ_Y denote the Borel equivalence relation which is simply equality on Y.

Notation: When there is no room for confusion, given $A \subseteq X$ and F any equivalence relation on X, we will denote the equivalence relation $F\restriction_A \oplus \Delta\restriction_{X\setminus A}$ on X, which is defined by $x(F\restriction_A \oplus \Delta\restriction_{X\setminus A})y \Leftrightarrow x = y$ *or* $x, y \in A$ *and* xFy, simply by $F\restriction_A$.

Proof of Theorem 2.4.12. This theorem is a consequence of the following, more general formula for the cost of the join of commuting countable aperiodic Borel equivalence relations.

Theorem 2.5.7. *Suppose that E is an aperiodic countable Borel equivalence relation on a Borel probability space (X, μ), and that μ is an invariant measure for E. If $E = R \vee S$, where $R \square S$ and R and S are aperiodic, then*

$$C_\mu(E) \leq C_\mu(R) + 2C_\mu(S) - 2.$$

Therefore if one of $C_\mu(R)$ or $C_\mu(S)$ equals 1, then

$$C_\mu(E) \leq \max\left(C_\mu(R), C_\mu(S)\right).$$

Proof. If one of $C_\mu(R)$ or $C_\mu(S)$ is infinite, then the assertion is trivial, so suppose otherwise. By Lemma 2.5.5, we may assume that μ is E-ergodic.

We split the proof into two cases; this is in fact redundant, because the first case is covered by the second, but we include it because it permits a much stronger result.

Case 1: μ is not quasi-ergodic for either of R and S (and, as stated above, $C_\mu(R)$, $C_\mu(S)$ are each finite). Then $C_\mu(R \vee S) = 1$.

Proof of Case 1: When we restrict our attention to invariant sets, cost is 'additive', in the sense that if A and B are any two disjoint, R-invariant sets, then

$$C_\mu(R\restriction_{A\cup B}) = C_\mu(R\restriction_A) + C_\mu(R\restriction_B).$$

Since $C_\mu(R) < \infty$ and μ isn't quasi-ergodic for R, one can therefore find a sequence $\{U_n \mid n \in \mathbb{N}\}$ of R-invariant sets of positive measure with $C_\mu(R\restriction_{U_n}) \to 0$. Let $\epsilon > 0$ be given, and fix an R-invariant set U for which $C_\mu(R\restriction_U) < \frac{\epsilon}{2}$. By adding an E-invariant null set to U (which will have no effect on cost calculations), we may assume that $[U]_E = X$. Similarly, fix an S-invariant set V such that $C_\mu(S) < \frac{\epsilon}{2}$, and such that $[V]_E = X$.

We claim that $E\restriction_{U\cup V} = R\restriction_U \vee S\restriction_V$. Clearly $E\restriction_{U\cup V} \supseteq R\restriction_U \vee S\restriction_V$. For the other direction, let $x, y \in U \cup V$ be given, with xEy. If $x \in U$ and $y \in V$, then, because $R\square S$, $[x]_R \cap [y]_S \neq \emptyset$, and so there exists z such that $x(R\restriction_U)z(S\restriction_V)y$. Thus $x\,(R\restriction_U \vee S\restriction_V)\,y$. If $x, y \in U$, then, because V meets each E-class, there exists z such that xEz, yEz, and $[z]_S \subseteq V$. By commutativity, $[x]_R \cap [z]_S \neq \emptyset$, $[y]_R \cap [z]_S \neq \emptyset$, so there exist z_1, z_2 such that

$$x(R\restriction_U)z_1(S\restriction_V)z_2(R\restriction_U)y.$$

Thus, again, $x\,(R\restriction_U \vee S\restriction_V)\,y$. The case in which $x, y \in V$ is entirely analogous, so the claim is proven, and by Theorem 2.4.8,

$$\begin{aligned} C_\mu(E) &= C_\mu(E\restriction_{U\cup V}) - \mu(U \cup V) + \mu(X) \\ &\leq C_\mu(R\restriction_U) + C_\mu(S\restriction_V) - \mu(U \cup V) + \mu(X) \\ &< 1 + \epsilon. \end{aligned}$$

General Case: If $R \cap S$ is μ-a.e. aperiodic, then we are done by Corollary 2.4.10, so suppose that

$$A = \{x \mid [x]_{R\cap S} \textit{ is finite }\}$$

has positive μ-measure. Because $\mu(A) > 0$ and μ is R-invariant, $R\restriction_A$ must be μ-a.e. aperiodic. Hence by Proposition 2.2.10, we can fix a countable sequence $B_1, B_2, \cdots$ of Borel sets such that $A \supseteq B_1 \supseteq B_2 \supseteq \cdots$, each B_n is complete section for R, and $\bigcap_n B_n = \emptyset$ (in particular, $\mu(B_n) \to 0$).

For each $n = 1, 2\cdots$, let $R_n = R \vee S\restriction_{B_n}$. We claim that for each n, $R_n \cap S$ is μ-a.e. aperiodic. Let

$$C_n = \{x \mid [x]_{R_n\cap S} \textit{ is finite }\}.$$

Clearly $C_n \subseteq A$. Also, if $x \in C_n$, then $[x]_{R_n\cap S}$ contains only finitely many $R \cap S$

classes. We'll show that for μ-a.e. $x \in A$, $[x]_{R_n \cap S}$ contains infinitely many $R \cap S$ classes, implying that $\mu(C_n) = 0$.

Because μ is S-invariant, $S\restriction_{B_n}$ must be μ-a.e. aperiodic (on B_n); hence for μ-a.e. $x \in A$, we have

$$\forall y \in B_n,\ xRy \Rightarrow [y]_{S\restriction_{B_n}} \textit{ is infinite }.$$

Fix any such $x \in A$ for which the above statement is true, and fix $y \in B_n \cap [x]_R$. Because $[y]_{S\restriction_{B_n}} \subseteq B \subseteq A$, $[y]_{S\restriction_{B_n}}$ must contain infinitely many $R \cap S$ classes. Let $z_1, z_2 \cdots$ be a sequence of elements in $[y]_{S\restriction_{B_n}}$ such that $j \neq k \Rightarrow z_j \not R z_k$. By commutativity, $[z_j]_R \cap [x]_S \neq \emptyset$, so fix $u_j \in [z_j]_R \cap [x]_S$. Clearly $j \neq k \Rightarrow u_j \neq u_k$. But for each j, $u_j \in [x]_{R_n} \cap [x]_S = [x]_{R_n \cap S}$, showing that $x \notin C_n$. Thus $\mu(C_n) = 0$, and so by Corollary 2.4.10, for each n we have

$$\begin{aligned} C_\mu(E) = C_\mu(R_i \vee S) &\leq C_\mu(R_i) + C_\mu(S) - 1 \\ &\leq C_\mu(R) + C_\mu(S\restriction_{B_n}) + C_\mu(S) - 1 \\ &= C_\mu(R) + 2C_\mu(S) - 2 + \mu(B_n), \end{aligned}$$

from which we achieve the desired result.

□

Remark: S. Solecki has shown that a modification of the above argument yields the stronger result

$$C_\mu(E) \leq C_\mu(R) + C_\mu(S) - 1$$

given the same hypotheses as in Theorem 2.5.7.

An application of Theorem 2.5.7 is the following.

Theorem 2.5.8. *Let E be a countable Borel equivalence relation on the Borel probability space (X, μ), and suppose μ is E-invariant. Suppose also that $E = \bigvee_i E_i$, where each E_i is an aperiodic countable Borel equivalence relation, $C_\mu(E_i) = 1$, and for each i, $E_i \square E_{i+1}$. Then $C_\mu(E) = 1$.*

Proof. By Theorem 2.5.7, $C_\mu(E_i \vee E_{i+1}) = 1$ for each i. By induction, using Corollary

2.4.10, we can therefore show that $F_n = \bigvee_{i=1}^n E_i$ has cost 1 for all n. The assertion is clearly true for the base cases $n = 1$ and $n = 2$; for the inductive step, note that $F_{n+1} = F_n \vee (E_n \vee E_{n+1})$, where $C_\mu(F_n) = C_\mu(E_n \vee E_{n+1}) = 1$, and $F_n \cap (E_n \vee E_{n+1}) = E_n$ is aperiodic.

Thus $E = \bigcup_n F_n$ is the union of an increasing sequence of cost 1 equivalence relations. It follows that E must also have cost 1 (see [5], IV.25). Indeed, since $\bigcap_n F_n = E_1$ is aperiodic, it must contain an aperiodic hyperfinite equivalence relation E'; let Φ be a μ-treeing for E'. By Theorem 2.4.9, for each $\epsilon > 0$ and each n there exists a graphing $\Psi_{n,\epsilon}$ of F_n such that $\Phi \subseteq \Psi_{n,\epsilon}$ and $C_\mu(\Psi_{n,\epsilon}) \leq 1 + \frac{\epsilon}{2^n}$. But then $\Phi \vee (\bigvee_n (\Psi_{n,\epsilon} \backslash \Phi))$ is a graphing for E and

$$C_\mu\left(\Phi \vee \left(\bigvee_n (\Psi_{n,\epsilon} \backslash \Phi)\right)\right) \leq C_\mu(\Phi) + \sum_n C_\mu(\Psi_{n,\epsilon} \backslash \Phi)$$
$$< 1 + \epsilon.$$

□

Corollary 2.5.9. *Suppose that E and F are countable Borel equivalence relations on (X, μ), with μ invariant for each. Further suppose that there are aperiodic hyperfinite equivalence relations $E' \subseteq E$ and $F' \subseteq F$, and decompositions $E = \bigvee_i E_i$ and $F = \bigvee_i F_i$, where each E_i and F_i are finite and such that $E' \square F_i$, $F' \square E_i$ for all i. Then $C_\mu(E \vee F) = 1$.*

Proof. By Proposition 2.2.8, $E' \vee F_i$ and $F' \vee E_i$ are hyperfinite for each i. Then, as in the proof of Theorem 2.5.8, we can use an inductive argument and Corollary 2.4.10 to show that $C_\mu(E' \vee (\bigvee_{i=1}^n F_i)) = C_\mu(F' \vee (\bigvee_{i=1}^n E_i)) = 1$ for all n. Consequently, as was also shown in Theorem 2.5.8, $C_\mu(E' \vee (\bigvee_{i=1}^\infty F_i)) = C_\mu(F' \vee (\bigvee_{i=1}^\infty E_i)) = 1$. Finally, since $E = (E' \vee (\bigvee_{i=1}^\infty F_i)) \vee (F' \vee (\bigvee_{i=1}^\infty E_i))$ and $(E' \vee (\bigvee_{i=1}^\infty F_i)) \cap (F' \vee (\bigvee_{i=1}^\infty E_i))$ is aperiodic,

$$1 \leq C_\mu(E) \leq C_\mu(E' \vee (\bigvee_{i=1}^\infty F_i)) + C_\mu(F' \vee (\bigvee_{i=1}^\infty E_i)) - 1 = 1.$$

□

Corollary 2.5.10. *(Gaboriau [5], V.1) Let E and F be aperiodic countable Borel equivalence relations on the Borel probability spaces (X,μ) and (Y,ν), respectively, such that μ is E-invariant and ν is F-invariant. Then the product equivalence relation $E \times F$ on $(X \times Y, \mu \times \nu)$ has cost 1.*

Proof. Let G and H be countable groups of Borel automorphisms for X and Y respectively, such that $E = E_G$ and $F = E_H$. By Theorem 2.2.3, we may assume that E and F are, respectively, generated by the involutions of G and H. Thus, if we let $\{g_i \mid i \in \mathbb{N}\}$ be an enumeration of the involutions of G, and for all $i \in \mathbb{N}$, let $E_i = E_{<g_i>}$, then E_i is a finite equivalence relation where each equivalence class contains at most 2 elements, and $E = \bigvee_i E_i$. Analogously define $F_i = E_{<h_i>}$ for each $i \in \mathbb{N}$, and let $E' \subseteq E$ and $F' \subseteq F$ be any aperiodic hyperfinite equivalence relations. $E \times F = (E \times \Delta_Y) \vee (\Delta_X \times F)$, and the decompositions $E \times \Delta_Y = \bigvee_i (E_i \times \Delta_Y)$ and $\Delta_X \times F = \bigvee_i (\Delta_X \times F_i)$ are such that $(E_i \times \Delta_Y) \square (\Delta_X \times F')$, $(E' \times \Delta_Y) \square (\Delta_X \times F_i)$ for all i. Thus by Corollary 2.5.9, $C_\mu(E \times F) = 1$. □

As a final remark in this section, we point out that the known examples of a countable Borel equivalence relation E with invariant measure μ, such that $C_\mu(E) > 1$ and which can be non-trivially decomposed into the join of two commuting aperiodic Borel equivalence relations, are rather limited. More precisely, suppose that $E = R \vee S$, where R and S commute, and each of R and S have infinite index in E. It is possible that $C_\mu(E) > 1$ if at least one of R and S have infinite cost. (One such example is the following: let E be any equivalence relation generated by a free μ-invariant Borel action α of the free group F_2; as noted in example (3) of §4, $C_\mu(E) = 2$. Now let $f : F_2 \to \mathbb{Z}$ be any surjective homomorphism, let $N = \ker f$, and $H = < x >$, where x is any element of F_2 with $f(x) = 1$. Then $N \cong F_\infty$, $H \cong \mathbb{Z} = F_1$, and $NH = HN = F_2$. Thus if R is the equivalence relation generated by the restriction of the action α to N, and S is the equivalence relation generated by the restriction of α to H, then $C_\mu(R) = \infty$, $C_\mu(S) = 1$, $R \square S$, and $E = R \vee S$.) However, there is no known example in which each of $C_\mu(R)$ and $C_\mu(S)$ are finite and $C_\mu(E) > 1$. Does such an example exist, or can Theorem 2.5.7 be further strengthened?

2.6 Costs of Groups

Gaboriau has also extended the notion of cost to countable groups. We present here an introduction to the subject, followed by an application of Theorem 2.4.12.

Definition 2.6.1. *An action of a countable group G on a σ-finite measure space (X, μ) is* invariant *if for every $g \in G$ and every Borel set $A \subseteq X$, $\mu(A) = \mu(g \cdot A)$. Equivalently, the action is invariant if μ is an invariant measure for the orbit equivalence relation E_G.*

Definition 2.6.2. *If G is a countable group, then its* cost *is defined to be*

$$C(G) = \inf\left(C_\mu(E_G)\right)$$

where the infimum is taken over all free invariant actions of G on Borel probability spaces (X, μ).

Definition 2.6.3. *A countable group G has* fixed price *if all orbit equivalence relations E_G which result from a free invariant action of G on a Borel probability space have the same cost.*

Definition 2.6.4. *A countable group G is* treeable *if every orbit equivalence relation E_G resulting from a free invariant action of G on a Borel probability space (X, μ) is μ-treeable. G is* anti-treeable *if no such E_G is μ-treeable.*

Currently all known examples are fixed priced groups which are either treeable or anti-treeable.

The following theorems simplify certain cost calculations:

Proposition 2.6.5. *(Gaboriau [5], VI.21) If Φ is a treeing for any orbit equivalence relation E_G induced by a free invariant action, then $C(G) = C_\mu(\Phi)$. Hence any* *treeable group is fixed price. Moreover for any group G there is a free invariant action such that the corresponding orbit equivalence relation E_G realizes the cost of G, i.e., $C(G) = C_\mu(E_G)$.*

Proposition 2.6.6. *(Gaboriau [5], VI.24) If G is a countable group and H is any fixed price infinite normal subgroup of G, then $C(G) \leq C(H)$.*

Definition 2.6.7. *A group G is* amenable *if it supports a finitely additive probability measure defined on all of its subsets which is invariant under the action of G on itself by left multiplication.*

The class of amenable groups includes all solvable groups, and excludes all groups which have a subgroup isomorphic to F_2 (see [17]).

Theorem 2.6.8. *Let G be a group, and suppose G acts freely and invariantly on a Borel probabability space (X, μ); if the action generates a μ-hyperfinite equivalence relation, then G is amenable. Any amenable group is treeable and has fixed price* 1.

The first assertion above is essentially folklore; for a reference, see [11]. In [15] Orstein and Weiss showed that amenable groups are treeable, and Levitt established the cost of such groups (see [12]).

By Proposition 2.6.5 and Theorem 2.6.8, costs yield a method of determining whether a group is anti-treeable; specifically, any cost 1 non-amenable group must be anti-treeable. In turn this provides a method of determining whether an equivalence relation can be treeable.

From Corollary 2.4.10 and Theorem 2.5.7, we have the following consequences regarding the costs of groups.

Corollary 2.6.9. *If H and K are any countable fixed price subgroups of a given group G and $H \cap K$ is infinite, then*

$$C(<H, K>) \leq C(H) + C(K) - 1.$$

Proof. Denote the group $<H, K>$ by G', and let $G' \times X \to X$ be any free action of G' on a Borel probability space (X, μ). Let $E_{G'}$ be the corresponding orbit equivalence relation, and let E_H (resp. E_K) be the equivalence relation obtained by restricting the action to $H \leq G'$ (resp. $K \leq G'$). Since $G' = <H, K>$, $E_{G'} = E_H \vee E_K$; since $H \cap K$ is infinite, $E_{H \cap K} = E_H \cap E_K$ is aperiodic. Thus by Theorem 2.5.7,

$$\begin{aligned} C(<H, K>) \leq C_\mu(E_{G'}) &\leq C_\mu(E_H) + C_\mu(E_K) - 1 \\ &= C(H) + C(K) - 1 \end{aligned}$$

since H and K are fixed price. □

Corollary 2.6.10. *If H and K are countably infinite fixed price subgroups of a given group G such that $HK = KH$, then*

$$C(<H, K>) \leq C(H) + 2C(K) - 2.$$

Therefore, if one of H and K has cost 1, then

$$C(<H, K>) \leq \max\left(C(H), C(K)\right).$$

In particular, if $G = HK$ with H, K countably infinite subgroups of fixed price 1*, then G also has fixed price* 1.

Proof. Using the same notation as in the proof of Corollary 2.6.9, we have $E_{G'} = E_H \vee E_K$ and, because $HK = KH$, $E_H \Box E_K$. Thus by Corollary **??**,

$$\begin{aligned} C(<H, K>) \leq C_\mu(E_{G'}) &\leq C_\mu(E_H) + 2C_\mu(E_K) - 2 \\ &= C(H) + 2C(K) - 2 \end{aligned}$$

since H and K are fixed price. The second assertion follows similarly. □

Definition 2.6.11. *If R is any commutative ring with unity, then* $SL_n(R)$, the special linear group over R of rank n, *is the group of all $n \times n$ matrices with entries* *in R which have determinant 1. (Here both the determinant function and matrix multiplication are defined in terms of the operations on R.)*

The next result generalizes example VI.26(b) of [5]; a proof can also be given using criterion VI.24(3) of the same work.

Theorem 2.6.12. *If R is a countably infinite commutative ring and $n \geq 3$, then*

$$C(SL_n(R)) = 1.$$

Hence if $SL_n(R)$ is non-amenable (e.g., in the case where R has characteristic 0), it is anti-treeable.

Proof. Let $E_n(R)$ be the subgroup of $SL_n(R)$ generated by the elementary transvections, i.e.,

$$E_n(R) = <e_{ij}(r) \mid i, j \leq n, i \neq j, r \in R>,$$

where $e_{ij}(r)$ denotes the element of $SL_n(R)$ with 1's along the diagonal, r in the ijth entry, and 0's elsewhere. For $n \geq 3$, $E_n(R) \trianglelefteq SL_n(R)$ (see [6], 1.2.13), so by Proposition 2.6.6 it will suffice to show that $C(E_n(R)) = 1$. For each pair $i, j \leq n, i \neq j$, let

$$T_{ij} = <e_{ij}(r) \,|\, r \in R> .$$

T_{ij} is infinite abelian, so it has cost 1. Now put the T_{ij}'s into a sequence so that for each (i,j) and (i',j'), if T_{ij} and $T_{i'j'}$ are adjacent in the sequence, then either $i = i'$ or $j = j'$. If T_{ij} and $T_{i'j'}$ are an adjacent pair, then the commutator group $[T_{ij}, T_{i'j'}] = \{1_R\}$, so in particular $T_{ij}T_{i'j'} = T_{i'j'}T_{ij}$. For notational simplicity, denote these groups by their position in the sequence, rather than the matrix entry where their elements differ from I_n. Then we have a sequence of subgroups

$$T_1, T_2, \cdots, T_{n^2-n}$$

such that $E_n(R) = <T_i \mid i \leq n^2 - n>$, $C(T_i) = 1$, and $T_iT_{i+1} = T_{i+1}T_i$ for all $i \leq n^2 - n - 1$. By Theorem 2.5.8, $C(<T_i, T_{i+1}>) = 1$ for all $i \leq n^2 - n - 1$. It then follows by induction that $C(<T_i\,|\,i \leq k>) = 1$ for all $k \leq n^2 - n$: $k = 1$ is trivial and $k = 2$ is above. For the inductive step, note that $<T_i|i \leq k+1> = <T_i|i \leq k> \vee <T_k, T_{k+1}>$. By induction hypothesis and Theorem 2.5.8, $C(<T_i\,|\,i \leq k>) = C(<T_k, T_{k+1}>) = 1$, and $<T_i\,|\,i \leq k> \cap <T_k, T_{k+1}> = T_k$ is infinite, so by Corollary 2.6.9, $<T_i\,|\,i \leq k+1>$ has cost 1 also. Hence $C(E_n(R)) = C(SL_n(R)) = 1$. □

For $n = 2$ the above argument fails; in fact the analogous assertion for $n = 2$ is false: Gaboriau has shown that $C(SL_2(\mathbb{Z})) > 1$. Nevertheless, Corollary 2.6.9 can be used to give a partial result in the same direction.

Proposition 2.6.13. *If R is a countable commutative ring with unity which has infinitely many units, then $C(E_2(R)) = 1$.*

Proof. $E_2(R)$ is generated by the subgroups U and L, where

$$U = \left\langle \begin{pmatrix} u & r \\ 0 & u^{-1} \end{pmatrix} \mid u \in R^*, r \in R \right\rangle$$

$$L = \left\langle \begin{pmatrix} u & 0 \\ r & u^{-1} \end{pmatrix} \mid u \in R^*, r \in R \right\rangle.$$

Clearly $E_2(R) \leq < U, L >$; that each of U and L are subgroups of $E_2(R)$ can be seen by the calculations

$$\begin{pmatrix} u & r \\ 0 & u^{-1} \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 \\ u^{-1} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & u^{-1}(r-1) \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} u & 0 \\ r & u^{-1} \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 \\ u^{-1}(r-1) & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -u^{-1} \\ 0 & 1 \end{pmatrix}.$$

Each of U and L have cost 1 (they are solvable); since $U \cap L$ is infinite, $C(E_2(R)) = 1$, by Corollary 2.6.9. □

In the case $n = 2$, it's possible that $E_n(R)$ may not be normal in $SL_n(R)$ (see [6], p. 27). Also in [6] (4.3.9), several criteria are given which guarantee that $SL_2(R) = E_2(R)$. These include the cases in which R is a Euclidean domain, R is a commutative semi-local ring, and all rings of integers of real quadratic field extensions $\mathbb{Q}(\sqrt{d})$, $d \in \mathbb{Z}^+$. So the following generalizes example VI.29 from [5].

Corollary 2.6.14. *Let R be a commutative ring with unity which has infinitely many units. If, additionally, R satisfies one of the conditions listed above, then $C(SL_2(R)) = 1$. Hence if $SL_2(R)$ is non-amenable (e.g., if the characteristic of R is 0), it is anti-treeable.*

This includes the case in which $R = \mathbb{Z}[\frac{1}{2}] = \{\frac{k}{2^l} |\ k, l \in \mathbb{Z}\}$, the ring of dyadic integers (it is a Euclidean domain), and so gives a new proof that $SL_2(\mathbb{Z}[\frac{1}{2}])$ is anti-treeable, which, aside from the cost machinery, uses entirely elementary methods. For another proof, see [10], §3.

We conclude this section with a proof that the general linear group $GL_n(K)$ has cost 1, for any $n \geq 1$ and any countably infinite field K. Although this result follows from Theorem 2.6.12 and Corollary 2.6.14 (and also by a result of Gaboriau), the proof to follow illustrates a different technique. It also, in the case $n = 2$, yields an example of an equivalence relation which can be written as the join of two commuting hyperfinite equivalence relations, but is not itself hyperfinite. We would like to thank M. Aschbacher for the conversations which led to Proposition 2.6.15.

Question: Suppose E is a countable Borel equivalence relation on the Borel probability space (X,μ), μ is E-invariant, and $C_\mu(E)=1$. Can E be written as the join of two commuting μ-hyperfinite equivalence relations?

Proposition 2.6.15. *If K is any countable infinite field, then $GL_n(K)$, the group of all invertible $n\times n$ matrices with entries in K, has cost 1.*

Proof. Let L be any nth degree field extension over K; since the multiplicative group of units L^* acts on the n-dimensional K-vector space $(L,+)$ by left multiplication, we can view L^* as a subgroup of $GL_n(K)$. Fix a basis $(e_1,\cdots,e_n)$ for $(L,+)$. Because L^* acts transitively on $(L\backslash\{0\},+)$, we have $GL_n(K)=L^*H$, where

$$\begin{aligned} H &= \textit{the stabilizer of } e_1 \\ &= \left\{ \begin{bmatrix} 1 & a_1 & \dots & a_{n-1} \\ 0 & & & \\ \vdots & & A_{n-1} & \\ 0 & & & \end{bmatrix} \mid a_1,\cdots,a_{n-1}\in K; A_{n-1}\in GL_{n-1}(K) \right\} \\ &= (K^{n-1},+)\rtimes GL_{n-1}(K). \end{aligned}$$

Since each of L^* and $(K^{n-1},+)$ are infinite abelian groups, they are fixed price with cost 1; thus by using Corollary 2.6.10 twice, we get $C(GL_n(K))\le C(GL_{n-1}(K))$. By induction, $C(GL_n(K))=C(GL_1(K))=C(K^*)=1$ for all n. □

As mentioned above, in the case $n=2$ we have $GL_2(K)=L^*\left((K,+)\rtimes K^*\right)$. Each of L^* and $(K,+)\rtimes K^*$ are amenable, but if K has characteristic 0, then $GL_2(K)$ is not amenable. Thus if $GL_2(K)$ acts freely and invariantly on the Borel probability space (X,μ), then the resulting orbit equivalence relation E is not μ-hyperfinite (by Theorem 2.6.8), but $E=E_{L^*}\vee E_{(K,+)\rtimes K^*}$, where $E_{L^*}\square E_{(K,+)\rtimes K^*}$ and each of these sub-equivalence relations is μ-hyperfinite (again by Theorem 2.6.8).

Bibliography

[1] P. Dubreil, M.L. Dubreil-Jacotin, *Théorie algebrique des relations d'équivalence*, J. Math. **18** (1939), 63-95.

[2] R. Dougherty, S. Jackson, A.S. Kechris, *The structure of hyperfinite Borel equivalence relations*, Trans. Amer. Math. Soc. **341** (1994), 193-225.

[3] R.H. Farrell, *Representations of invariant measures*, Ill. J. Math. **6** (1962), 447-467.

[4] J. Feldman, C. Moore, *Ergodic equivalence relations, cohomology, and Von Neumann algebras, I*, Trans. Amer. Math. Soc. **234** (1977), 289-324.

[5] D. Gaboriau, *Coût des relations d'équivalence et des groupes*, Invent. Math. **139** (2000), 41-98.

[6] A.J. Hahn, O.T. O'Meara, *The classical groups of K-theory*, Springer-Verlag (1989).

[7] L.A. Harrington, A.S. Kechris, A. Louveau, *A Glimm-Effros dichotomy for Borel equivalence relations*, J. Amer. Math. Soc. **3** (1990), 903-927.

[8] S. Jackson, A.S. Kechris, A. Louveau, *Countable Borel equivalence relations*, preprint (2000).

[9] A.S. Kechris, *Classical descriptive set theory*, Springer-Verlag (1995).

[10] A.S. Kechris, *On the classification problem for rank 2 torsion-free abelian groups*, J. London Math. Soc., to appear.

[11] A.S. Kechris, *Amenable equivalence relations and Turing degrees*, J. Symb. Logic **56** (1991), 182-194.

[12] G. Levitt, *On the cost of generating an equivalence relation*, Erg. Theory and Dynam. Syst. **15** (1995), 1173-1181.

[13] Y.N. Moschovakis, *Descriptive set theory*, North Holland (1980).

[14] M.G. Nadkarni, *On the existence of a finite invariant measure*, Proc. Indian Acad. Soc. (Math. Sci.) **100** (1990), 203-220.

[15] D. Ornstein, B. Weiss, *Ergodic theory of amenable group actions I. The Rohlin lemma.*, Bull. Amer. Math. Soc. **2** (1980), 161-164.

[16] V.S. Varadarajan, *Groups of automorphisms of Borel spaces*, Trans. Amer. Math. Soc. **109** (1963), 191-220.

[17] S. Wagon, *The Banach-Tarski paradox*, Cambridge Univ. Press (1985).

Chapter 3 A Natural $\mathbf{\Pi}^1_1$ Scale on DIFF

3.1 Preliminaries

Definition 3.1.1. *A* rank *on a set S is any map from S into Ord, the class of ordinals.*

Definition 3.1.2. *Let Γ be a class of subsets of Polish spaces (e.g., the closed sets, the Borel sets, etc.), and let $\check{\Gamma}$ denote the complement class, i.e., the class of sets whose complements are in Γ. If X is a Polish space and $A \subseteq X$, a* Γ-rank *on the set A is a rank $\phi : A \to Ord$ whose initial segments are uniformly in $\Gamma \cap \check{\Gamma}$, in the following sense: there exist relations $\leq^{\Gamma}_{\phi}$, $\leq^{\check{\Gamma}}_{\phi}$ with $\leq^{\Gamma}_{\phi} \in \Gamma$, $\leq^{\check{\Gamma}}_{\phi} \in \check{\Gamma}$ (as subsets of $X \times X$) such that $\forall y \in A$,*

$$\phi(x) \leq \phi(y) \Leftrightarrow x \leq^{\Gamma}_{\phi} y \Leftrightarrow x \leq^{\check{\Gamma}}_{\phi} y.$$

Γ is called a ranked class *if each $A \in \Gamma$ admits a Γ-rank.*

Definition 3.1.3. *Let X be a Polish space, and $A \subseteq X$. A* scale *on A is any countable set of ranks $\{\phi_n : A \to Ord \mid n \in \mathbb{N}\}$ with the following property: for any sequence of points (x_k) in A converging to a point $x \in X$, if $\forall n \in \mathbb{N}, \exists \alpha_n \in Ord$, with $\lim_{k\to\infty} \phi_n(x_k) = \alpha_n$, then $x \in A$ and $\forall n \in \mathbb{N}, \phi_n(x) \leq \lim_{k\to\infty} \phi_n(x_k)$.*

Remark: When defining convergence in Ord, we use the discrete topology, so $\lim_{k\to\infty} \phi_n(x_k) = \alpha_n$ means that the sequence $(\phi_n(x_k))_k$ is eventually constant, with value α_n.

Definition 3.1.4. *A* Γ-scale *is one in which all the ranks are Γ-ranks. Γ is a* scaled class *if each $A \in \Gamma$ has a Γ-scale.*

Theorem 3.1.5. *The class $\mathbf{\Pi}^1_1$ is scaled; moreover, for each $A \in \mathbf{\Pi}^1_1$ there is a $\mathbf{\Pi}^1_1$-scale $\{\phi_n \mid n \in \mathbb{N}\}$ on A with the property that $ran(\phi_n) \leq \omega_1$ for all n, with equality if and only if A is not Borel.*

For a proof, see [3] § 36.D.

Definition 3.1.6. *Let X be a Polish space, $A \subseteq X$, and $\{\phi_n \mid n \in \mathbb{N}\}$ a scale on A. We say that a sequence of points (x_k) of points in A* converges in the scale $\{\phi_n | n \in \mathbb{N}\}$ *if $\lim_{k\to\infty} \phi_n(x_k)$ exists for all $n \in \mathbb{N}$.*

If there is no risk of ambiguity, we will simply use the term 'converges in the scale', without explicitly mentioning the particular scale $\{\phi_n \mid n \in \mathbb{N}\}$.

The proof of Theorem 3.1.5 gives a general method for constructing a scale for each $\mathbf{\Pi}^1_1$ subset of $\mathbb{N}^{\mathbb{N}}$, the space of all infinite sequences of natural numbers, with the usual product topology. The general case can be reduced to this, because any two uncountable Polish spaces are indistinguishable up to Borel isomorphism (the theorem is trivial for countable spaces). In most cases, however, even if a concrete scale for a given $\mathbf{\Pi}^1_1$ set A can be obtained in this way, it will reflect very little of the intrinsic nature of A. Since a scaled set is closed under sequences which converge both in the topology and in the scale, it is desirable to find a scale which is explicitly related to the Polish topology, in the hopes that convergence in the scale can be used to solve problems of an analytical or topological nature.

One general approach to finding a 'natural' $\mathbf{\Pi}^1_1$ scale for a given $\mathbf{\Pi}^1_1$ set is to find a single 'natural' $\mathbf{\Pi}^1_1$ rank, and then find a way to localize it with respect to a fixed countable basis of the underlying Polish space. That is the approach taken here: we show that a natural $\mathbf{\Pi}^1_1$ rank on the set of differentiable functions with domain $[0, 1]$, developed by Kechris and Woodin in [4], can be successfully 'localized' to produce a scale. We then attempt to quantify the analytical strength of convergence in this scale, by comparing, for a sequence of differentiable functions (f_n) which converges in the supnorm to a differentiable function f, convergence in the scale to pointwise convergence of the derivatives (f'_n).

3.2 A Natural $\mathbf{\Pi}^1_1$ Rank on DIFF

This section is a summary of results from [4], by Kechris and Woodin.

Let

$$C([0,1]) = \{f \mid f \textit{ a real-valued continuous function with domain } [0,1]\},$$

$$\mathrm{DIFF} = \{f \in C([0,1]) \mid f'(x) \textit{ exists for all } x \in [0,1]\},$$

where $f'(0)$ and $f'(1)$ are understood to be one-sided derivatives. $C([0,1])$ is a Polish space with the usual supnorm metric $\| f - g \|_\infty = \max_{x \in [0,1]} |f(x) - g(x)|$. We'll see that DIFF is a $\mathbf{\Pi}^1_1$ subset of $C([0,1])$.

Definition 3.2.1. *Let X be a set and $\mathcal{D}$ a collection of subsets of X which is closed under non-empty intersections. A* derivative *on $\mathcal{D}$ is a map $D : \mathcal{D} \to \mathcal{D}$ with the following properties:*

i) $\forall A \in \mathcal{D}, D(A) \subseteq A$.

ii) $\forall A, B \in \mathcal{D}, A \subseteq B \Rightarrow D(A) \subseteq D(B)$.

If $D : \mathcal{D} \to \mathcal{D}$ is a derivative and $A \in \mathcal{D}$, then we can define, by transfinite recursion, the *iterated derivatives* $D^\alpha(A)$ of A by

$$\begin{aligned} D^0(A) &= A \\ D^{\beta+1} &= D\left(D^\beta(A)\right) \\ D^\lambda(A) &= \bigcap_{\beta<\lambda} D^\beta(A), \quad \text{if } \lambda \text{ is a limit ordinal.} \end{aligned}$$

There will be a least ordinal $\alpha < card(A)^+$ with the property that $D^{\alpha+1}(A) = D^\alpha(A)$; this is called the *D-rank* of A, and is denoted by $|A|_D$. The set $D^{|A|_D}(A)$ will be denoted by $D^\infty(A)$.

Theorem 3.2.2. *Let X be a Polish space and let $\mathcal{D} = K(X)$, the hyperspace of compact subsets of X (which is also Polish). Let Y be any standard Borel space, and let $\mathbb{D} : Y \times \mathcal{D} \to \mathcal{D}$ be a Borel map with the property that for each $y \in Y$, the section map $\mathbb{D}_y : \mathcal{D} \to \mathcal{D}$ is a derivative. Then*

$$\Omega_{\mathbb{D}} = \{(y, K) \mid \mathbb{D}^\infty_y(K) = \emptyset\}$$

is a $\mathbf{\Pi}^1_1$ *set and the map* $(y, K) \mapsto |K|_{\mathbb{D}_y}$ *is a* $\mathbf{\Pi}^1_1$ *rank on* $\Omega_{\mathbb{D}}$.

(See [3], § 34.E.)

Toward defining a $\mathbf{\Pi}^1_1$ rank on DIFF, let

$$Q = (0,1) \cap \mathbb{Q},$$

$$\begin{aligned}\mathcal{U} &= \{U \mid U \text{ a } [0,1]\text{-relatively open interval with rational endpoints }\} \\ &= \{[0,r) \mid r \in Q\} \cup \{(r,s) \mid r,s \in Q\} \cup \{(r,1] \mid r \in Q\} \cup \{[0,1]\},\end{aligned}$$

and define a map $\mathbb{D} : Q \times C([0,1]) \times K([0,1]) \to K([0,1])$ by

$$\begin{aligned}\mathbb{D}(\epsilon, f, K) = \{x \in K \mid \forall U \in \mathcal{U} \text{ with } x \in U,\ \exists p,q,r,s \in U \cap Q,\ p < q,\\ r < s, \text{ such that } [p,q] \cap [r,s] \cap K \neq \emptyset \text{ and}\\ |\Delta_f(p,q) - \Delta_f(r,s)| > \epsilon\},\end{aligned}$$

where $\Delta_f(p,q) = \frac{f(q)-f(p)}{q-p}$.

It's not difficult to check that $\mathbb{D}$ is a Borel map, and that for each pair $(\epsilon, f) \in Q \times C([0,1])$, $\mathbb{D}_{\epsilon,f} : \mathcal{D} \to \mathcal{D}$ is a derivative. Hence by Theorem 3.2.2, the set $\Omega_{\mathbb{D}} = \{(\epsilon, f, K) \mid \mathbb{D}^{\infty}_{\epsilon,f}(K) = \emptyset\}$ is $\mathbf{\Pi}^1_1$, and the map $(\epsilon, f, K) \mapsto |K|_{\mathbb{D}_{\epsilon,f}}$ is a $\mathbf{\Pi}^1_1$ rank on it.

Theorem 3.2.3. *(Kechris-Woodin)* $\forall f \in C([0,1])$,

$$f \in \mathit{DIFF} \Leftrightarrow \forall \epsilon \in Q, \mathbb{D}^{\infty}_{\epsilon,f}([0,1]) = \emptyset.$$

Because $\Omega_{\mathbb{D}}$ is $\mathbf{\Pi}^1_1$, for each pair (ϵ, K), the section

$$\Omega_{\epsilon,K} = \{f \in C([0,1]) \mid (\epsilon, f, K) \in \Omega_{\mathbb{D}}\} = \{f \in C([0,1]) \mid \mathbb{D}^{\infty}_{\epsilon,f}(K) = \emptyset\}$$

is also $\mathbf{\Pi}^1_1$. The corresponding rank $f \mapsto |K|_{\mathbb{D}_{\epsilon,f}}$ is $\mathbf{\Pi}^1_1$ as well: if $\leq^S$, $\leq^P$ are relations in $(Q \times C([0,1]) \times K([0,1]))^2$ which are, respectively, $\mathbf{\Sigma}_1^1$ and $\mathbf{\Pi}_1^1$ and witness the definability of the rank of $\Omega_{\mathbb{D}}$, then the section relations $\leq^S_{\epsilon,K}$, $\leq^P_{\epsilon,K}$ will be, respectively, $\mathbf{\Sigma}_1^1$ and $\mathbf{\Pi}_1^1$ in $(C([0,1]))^2$ and will witness the definability of the section rank on $\Omega_{(\epsilon,K)}$. (Here $f \leq^S_{(\epsilon,K)} g$ if and only if $(\epsilon, f, K) \leq^S (\epsilon, f, K)$. $\leq^P_{(\epsilon,K)}$ is defined analogously.)

By Theorem 3.2.3, DIFF$= \bigcap_{\epsilon \in Q} \Omega_{\epsilon,[0,1]}$. But for each $f \in C([0,1])$, $K \in K([0,1])$, $\epsilon_1 < \epsilon_2 \Rightarrow \mathbb{D}_{\epsilon_1,f}(K) \supseteq \mathbb{D}_{\epsilon_2,f}(K)$, so in fact we can write

$$\text{DIFF} = \bigcap_n \Omega_{\frac{1}{n},[0,1]},$$

i.e., we can take the intersection over any sequence (ϵ_i) in Q which has 0 as a limit point. Since the class $\mathbf{\Pi}_1^1$ is closed under countable intersections and unions, this shows that DIFF is $\mathbf{\Pi}_1^1$ in $C([0,1])$. Moreover, the map

$$f \mapsto \sup_n |[0,1]|_{\mathbb{D}_{\frac{1}{n},f}}$$

is a $\mathbf{\Pi}_1^1$ rank on DIFF (see [4]).

Notation: In the sequel, we will write $|f|_{\text{DIFF}}$ for $\sup_n |[0,1]|_{\mathbb{D}_{\frac{1}{n},f}}$.

To summarize, we have

Theorem 3.2.4. *(Kechris-Woodin)*

i) *$\forall \epsilon \in Q$, the map $f \mapsto |[0,1]|_{\mathbb{D}_{\epsilon,f}}$ is a $\mathbf{\Pi}_1^1$ rank on the $\mathbf{\Pi}_1^1$ set $\Omega_{\epsilon,[0,1]}$.*

ii) *The map $f \mapsto |f|_{DIFF}$ is a $\mathbf{\Pi}_1^1$ rank on the $\mathbf{\Pi}_1^1$ set DIFF.*

Definition 3.2.5. *Let X be a topological space, Y a metric space with metric d, $A \subseteq X$, and $f : A \to Y$ any function from A into Y. The* oscillation *of f at a point $x \in A$ is*

$$\begin{aligned} osc_f(x) &= \inf \left\{ diam(f[A \cap U]) \mid U \text{ an open neighborhood of } x \right\} \\ &= \inf \left\{ \sup \left\{ d(f(u), f(v)) \mid u, v \in A \cap U \right\} \mid U \text{ an open neighborhood of } x \right\}. \end{aligned}$$

So f is continuous at x if and only if $osc_f(x) = 0$.

Proposition 3.2.6. *If $f \in DIFF$, then for any $\epsilon \in Q$,*

$$x \in \mathbb{D}(\epsilon, f, [0,1]) \Rightarrow osc_{f'}(x) > \epsilon,$$
$$x \notin \mathbb{D}(\epsilon, f, [0,1]) \Rightarrow osc_{f'}(x) \leq 2\epsilon.$$

Proof. The first assertion follows directly from the definition of $\mathbb{D}(\epsilon, f, [0,1])$ and the Mean Value Theorem. For the second, suppose that $osc_{f'}(x) > 2\epsilon$, and for a given neighborhood $U \in \mathcal{U}$ of x, let $u, v \in U$ be such that $|f'(u) - f'(v)| > 2\epsilon$. Fix $p, q, r, s \in U \cap Q$, $p < q$, $r < s$, such that $p \leq u \leq q$, $r \leq v \leq s$ and $|\Delta_f(p,q) - \Delta_f(r,s)| > 2\epsilon$. If $[p,q] \cap [r,s] \neq \emptyset$, then we're done, so suppose otherwise, say $p < q < r < s$. Since

$$|\Delta_f(p,q) - \Delta_f(q,r)| + |\Delta_f(q,r) - \Delta_f(r,s)| > 2\epsilon,$$

there is a triple $(a,b,c) \in \{(p,q,r),(q,r,s)\}$ such that $|\Delta_f(a,b) - \Delta_f(b,c)| > \epsilon$, and trivially $[a,b] \cap [b,c] \neq \emptyset$. Since U was arbitrary, this shows that $x \in \mathbb{D}(\epsilon, f, [0,1])$. □

With Proposition 3.2.6 and Theorem 3.2.4, we recover another property of the sets $\mathbb{D}(\epsilon, f, [0,1])$, given in [4]:

Theorem 3.2.7. *(Kechris-Woodin) $\forall f \in DIFF$,*

$$f \in C^1([0,1]) \Leftrightarrow \forall \epsilon \in Q \ \mathbb{D}(\epsilon, f, [0,1]) = \emptyset,$$

where $C^1([0,1]) = \{f \in DIFF \mid f' \in C([0,1])\}$.

Example: Let

$$f(x) = \begin{cases} 0 & \text{if } x \in [0, \frac{1}{2}], \\ (x - \frac{1}{2})^2 \sin\left(\frac{1}{x - \frac{1}{2}}\right) & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

$f \in$ DIFF, and f' has a single discontinuity, at $x = \frac{1}{2}$. $osc_{f'}(\frac{1}{2}) = 2$, so for each $\epsilon \in Q$, $\mathbb{D}(\epsilon, f, [0,1]) = \{\frac{1}{2}\}$. For any ϵ, for any differentiable function g, and any set $K \in K([0,1])$ with Cantor-Bendixson rank 1 (i.e., any set with no limit points),

$\mathbb{D}(\epsilon, g, K) = \emptyset$. Hence $\mathbb{D}^2(\epsilon, f, [0,1]) = \emptyset$, i.e., $|[0,1]|_{\mathbb{D}_{\epsilon,f}} = 2$ for each $\epsilon \in Q$, so $|f|_{\mathrm{DIFF}} = 2$ as well.

A method of inductively constructing functions of any rank $\alpha < \omega_1$ is outlined in [4] (pp. 262-264).

3.3 A Natural $\mathbf{\Pi}_1^1$ Scale on DIFF

In this section we will construct a countable collection of $\mathbf{\Pi}_1^1$ ranks on DIFF, based largely on the ranks defined in the previous section, namely the ranks

$$f \mapsto |[0,1]|_{\mathbb{D}_{\epsilon,f}}$$

on $\Omega_{\epsilon,[0,1]}$, for each $\epsilon \in Q$, and the rank

$$f \mapsto |f|_{\mathrm{DIFF}} = \sup_n |[0,1]|_{\mathbb{D}_{\frac{1}{n},f}}$$

on DIFF. However we will also require that these new ranks contain additional information, which will guarantee them to form a scale.

Definition 3.3.1. *Let $K \in K([0,1])$, $f \in C([0,1])$, and $\epsilon \in Q$. A closed interval $I \subseteq [0,1]$ is $\epsilon - K$* good for f *if $\forall p, q, r, s \in int(I) \cap Q$, $p < q$, $r < s$,*

$$[p,q] \cap [r,s] \cap K \neq \emptyset \Rightarrow |\Delta_f(p,q) - \Delta_f(r,s)| \leq \epsilon,$$

where $int(I)$ is the interior of I in the $[0,1]$-topology.

Remark: If I is $\epsilon - K$ good for f, then $\mathbb{D}(\epsilon, f, K) \cap int(I) = \emptyset$. It's possible, however, that $\mathbb{D}(\epsilon, f, K) \cap I \neq \emptyset$. For an example, let f be the function from the example in the previous section, let $K = [0,1]$, $I = [0, \frac{1}{2}]$, and $\epsilon \in (0, \frac{2}{\pi}) \cap Q$.

Definition 3.3.2. *Let $f \in C([0,1])$, $K \in K([0,1])$, $\epsilon \in Q$, and $m \in \mathbb{N}^+$. An $\epsilon - m$* covering of K for f *is a finite sequence $I_0, \cdots, I_{n-1}$ of closed intervals such that:*

i) *$\forall j \leq n-1$, I_j is $\epsilon - K$ good for f.*

ii) $\forall j \leq n-1$, I_j *has length at least* $\frac{3}{m}$.

iii) $\forall x \in K, \exists j \leq n-1$ *such that* $B(x, \frac{1}{m}) = \{y \in [0,1] \mid |x-y| < \frac{1}{m}\} \subseteq I_j$.

Lemma 3.3.3. *For each* $f \in C([0,1])$, $K \in K([0,1])$, $\epsilon \in Q$, $\mathbb{D}(\epsilon, f, K) = \emptyset$ *if and only if for some* $m \in \mathbb{N}^+$ *there is an* $\epsilon - m$ *cover of* K *for* f.

Proof:

$\Leftarrow$: If $I_0, \cdots, I_{n-1}$ is an $\epsilon - m$ cover of K for f, then each I_j is $\epsilon - K$ good for f and $K \subseteq \bigcup_{j=0}^{n-1} int(I_j)$. As was noted in the above remark, $\mathbb{D}(\epsilon, f, K) \subseteq K \setminus \bigcup_{j=0}^{n-1} int(I_j)$, so $\mathbb{D}(\epsilon, f, K) = \emptyset$.

$\Rightarrow$: By definition of the map $\mathbb{D}$, if $x \in K \setminus \mathbb{D}(\epsilon, f, K)$ then there exists an open interval U_x containing x whose closure is $\epsilon - K$ good for f. By compactness, K is covered by finitely many of these, $U_{x0}, \cdots, U_{x_{n-1}}$; without loss of generality, omitting certain of these U_{x_j} if necessary, we may assume that no U_{x_j} is contained in any other. We claim that their closures form an $\epsilon - m$ cover of K for f, for some $m \in \mathbb{N}^+$.

Toward determining a value m for which $\bar{U}_{x0}, \cdots, \bar{U}_{x_{n-1}}$ is an $\epsilon - m$ cover of K for f, let

$$\eta_1 = \min\{b_j - a_k \mid j, k \leq n-1 \text{ } and \text{ } U_{x_j} \cap U_{x_k} \neq \emptyset\}$$
$$\eta_2 = \min\{d(p, K) \mid j \leq n-1, \ p \in \{a_j, b_j\} \text{ } and \text{ } p \notin \overline{U_{x_j} \cap K}\},$$

where $d(p, K)$ is defined to be $\min\{|p-q| \mid q \in K\}$. Let $\eta = \min\{\frac{\eta_1}{2}, \eta_2\}$. Then we claim that for each $x \in K$ there exists j such that $\mathcal{B}(x, \eta) \subseteq U_{x_j}$. Toward a contradiction, suppose otherwise, and fix a value x for which the assertion is false. We first note that it cannot be the case that $x \in U_{x_j} \cap U_{x_k}$, for any pair j, k: otherwise, switching the roles of j and k if necessary, $a_j < a_k < x < b_j < b_k$. But then if $x \in (a_k, \frac{a_k+b_j}{2}]$, $B(x, \eta_1) \subseteq U_{x_k}$ and similarly, if $x \in [\frac{a_k+b_j}{2}, b_j)$, then $B(x, \eta_1) \subseteq U_{x_j}$, a contradiction.

Hence there must be a unique j for which $x \in U_{x_j}$. If $a_j \notin \overline{U_{x_j} \cap K}$, then $x - a_j \geq d(a_j, \overline{U_{x_j} \cap K}) \geq \eta_1$. On the other hand, if $a_j \in \overline{U_{x_j} \cap K}$, then there must be a $k \leq n-1$ such that $a_j \in U_{x_k}$. Since $x \notin U_{x_k}$, we must have $x - a_j \geq b_k - b_j \geq \eta_2$.

Thus in either case, $x - a_j \geq \eta$, and we can similarly show that $b_j - x \geq \eta$, so $B(x, \eta) \subseteq U_{x_j}$.

Thus if m is any integer greater than

$$\max\left\{\frac{1}{\eta}, \max_{j \leq n-1}\left\{\frac{3}{b_j - a_j}\right\}\right\},$$

then $\bar{U}_{x_0}, \cdots, \bar{U}_{x_{n-1}}$ is an $\epsilon - m$ cover of K for f.

□

Before giving the next definition we note that, by compactness of the interval $[0,1]$, for any $f \in \Omega_{\epsilon,[0,1]}$, $K \in K([0,1])$ and $\epsilon \in Q$, the ordinal $|K|_{\mathbb{D}_{\epsilon,f}}$ must be a successor, since it is the least β for which $\mathbb{D}^{\beta}(\epsilon, f, K) = \emptyset$.

Definition 3.3.4. *Fix a bijection $\mathcal{B} : \omega^2 \to \omega$ with the property that for all $l_1, l_2, m \in \omega$, $l_1 \leq l_2 \Rightarrow \mathcal{B}(l_1, m) \leq \mathcal{B}(l_2, m)$. For each $\epsilon \in Q$, let $\rho_{\epsilon,[0,1]} : \Omega_{\epsilon,[0,1]} \to Ord$ be the rank given by*

$$\begin{aligned}\rho_{\epsilon,[0,1]}(f) &= |[0,1]|_{\mathbb{D}_{\epsilon,f}} - 1 \\ &= \textit{the least } \alpha \textit{ such that } \mathbb{D}^{\alpha+1}(\epsilon, f, [0,1]) = \emptyset,\end{aligned}$$

and let $r_{\epsilon,[0,1]} : \Omega_{\epsilon,[0,1]} \to \omega$ be the rank given by

$$r_{\epsilon,[0,1]}(f) = \mathcal{B}(l, m)$$

where $\mathcal{B}(l, m)$ is least such that $\mathbb{D}^{\alpha}(\epsilon, f, [0,1])$ has an $\epsilon - m$ cover for f consisting of l intervals, where $\alpha = \rho_{\epsilon,[0,1]}(f)$.

Definition 3.3.5. *For each $\epsilon \in Q$, let $\phi_{\epsilon,[0,1]} : \Omega_{\epsilon,[0,1]} \to Ord$ be the rank defined by*

$$\phi_{\epsilon,[0,1]}(f) = \rho_{\epsilon,[0,1]}(f) \cdot \omega + r_{\epsilon,[0,1]}(f)$$

Let $\psi_{\epsilon,[0,1]} : DIFF \to Ord$ *be the rank defined by*

$$\psi_{\epsilon,[0,1]}(f) = |f|_{DIFF} \cdot \omega_1 + \phi_{\epsilon,[0,1]}(f)$$

So $\phi_{\epsilon,[0,1]}(f) \leq \phi_{\epsilon,[0,1]}(g)$ *if and only if* $\rho_{\epsilon,[0,1]}(f) < \rho_{\epsilon,[0,1]}(g)$ *or* $\rho_{\epsilon,[0,1]}(f) = \rho_{\epsilon,[0,1]}(g)$ *and* $r_{\epsilon,[0,1]}(f) \leq r_{\epsilon,[0,1]}(g)$. *Likewise* $\psi_{\epsilon,[0,1]}$ *has the lexicographical ordering of the pair* $(|\cdot|_{DIFF}, \phi_{\epsilon,[0,1]})$.

Notation: We will denote $\phi_{\epsilon,[0,1]}$ by $< \rho_{\epsilon,[0,1]}(f), r_{\epsilon,[0,1]}(f) >$, rather than $\rho_{\epsilon,[0,1]}(f) \cdot \omega + r_{\epsilon,[0,1]}(f)$, to emphasize the lexicographical ordering. Similarly, we will denote $\psi_{\epsilon,[0,1]}$ by $< |f|_{\mathrm{DIFF}}, \phi_{\epsilon,[0,1]}(f) >$.

Lemma 3.3.6. *For each* $\epsilon \in Q$, $\phi_{\epsilon,[0,1]}$ *is a* $\mathbf{\Pi}^1_1$ *rank on* $\Omega_{\epsilon,[0,1]}$, *and* $\psi_{\epsilon,[0,1]}$ *is a* $\mathbf{\Pi}^1_1$ *rank on DIFF.*

Proof: Let $\epsilon \in Q$ be given. Following the proof of Theorem (34.10) in [3], we can fix an effective 'coding' of the countable ordinals: as given in [3], p. 273, let $WO^* \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ be the set of characteristic functions of countable well-orderings, and let $x \mapsto |x|^*$ be the map on WO^* which takes a point to the order type that it represents. WO^* is $\mathbf{\Pi}^1_1$, and $|\cdot|^* : WO^* \to \omega_1$ is a $\mathbf{\Pi}^1_1$ rank.

Again following the principles of the proof of (34.10), in [3], we can find a $\mathbf{\Sigma}^1_1$ relation $R(x, f) \subseteq X \times C([0,1])$ such that for each $f \in \Omega_{\epsilon,[0,1]}$,

$$R(x, f) \Leftrightarrow x \in WO^* \text{ and } |x|^* \leq \rho_{\epsilon,[0,1]}(f).$$

Then the relations $\leq_\epsilon^S$ and $\leq_\epsilon^P$ on $C([0,1])$ given by

$$
\begin{aligned}
&f \leq_\epsilon^S g \Leftrightarrow \\
&\exists x\Big(R(x,g) \textit{ and } \mathbb{D}^{|x|^*+1}(\epsilon, g, [0,1]) = \emptyset \textit{ and } \big(\textit{either } \mathbb{D}^{|x|^*}(\epsilon, f, [0,1]) = \emptyset \textit{ or} \\
&\qquad\qquad \big(\mathbb{D}^{|x|^*+1}(\epsilon, f, [0,1]) = \emptyset \textit{ and } \forall l, m \in \mathbb{N}, \\
&\qquad\qquad\quad \textit{if there exists an } \epsilon - m \textit{ cover of } \mathbb{D}^{|x|^*}(\epsilon, g, [0,1]) \textit{ for} \\
&\qquad\qquad\quad g \textit{ consisting of } l \textit{ intervals, then there exist } l', m' \in \mathbb{N} \\
&\qquad\qquad\quad \textit{such that } \mathcal{B}(l'm') \leq \mathcal{B}(l,m) \textit{ and such that there exists} \\
&\qquad\qquad\quad \textit{an } \epsilon - m' \textit{ cover of } \mathbb{D}^{|x|^*}(\epsilon, f, [0,1]) \textit{ for } f \textit{ consisting} \\
&\qquad\qquad\quad \textit{of } l' \textit{ intervals}\big)\big)\Big).
\end{aligned}
$$

$$
\begin{aligned}
&f \leq_\epsilon^P g \Leftrightarrow \\
&\forall x\Big(\big(R(x,g) \textit{ and } \mathbb{D}^{|x|^*+1}(\epsilon, g, [0,1]) = \emptyset\big) \Rightarrow \big(\textit{either } \mathbb{D}^{|x|^*}(\epsilon, f, [0,1]) = \emptyset \textit{ or} \\
&\qquad\qquad \big(\mathbb{D}^{|x|^*+1}(\epsilon, f, [0,1]) = \emptyset \textit{ and } \forall l, m \in \mathbb{N}, \\
&\qquad\qquad\quad \textit{if there exists an } \epsilon - m \textit{ cover of } \mathbb{D}^{|x|^*}(\epsilon, g, [0,1]) \textit{ for} \\
&\qquad\qquad\quad g \textit{ consisting of } l \textit{ intervals, then there exists } l', m' \in \mathbb{N} \\
&\qquad\qquad\quad \textit{such that } \mathcal{B}(l'm') \leq \mathcal{B}(l,m) \textit{ and such that there exists} \\
&\qquad\qquad\quad \textit{an } \epsilon - m' \textit{ cover of } \mathbb{D}^{|x|^*}(\epsilon, f, [0,1]) \textit{ for } f \textit{ consisting} \\
&\qquad\qquad\quad \textit{of } l' \textit{ intervals}\big)\big)\Big).
\end{aligned}
$$

are $\mathbf{\Sigma_1^1}$ and $\mathbf{\Pi_1^1}$, respectively, and for $g \in \Omega_{\epsilon,[0,1]}$,

$$\phi_{\epsilon,[0,1]}(f) \leq \phi_{\epsilon,[0,1]}(g) \iff f \leq_\epsilon^S g \iff f \leq_\epsilon^P g.$$

Hence $\phi_{\epsilon,[0,1]}$ is a $\mathbf{\Pi_1^1}$ rank, for each $\epsilon \in Q$. To see that each $\psi_{\epsilon,[0,1]}$ is as well, let $\leq_{\text{DIFF}}^S$ and $\leq_{\text{DIFF}}^P$ be $\mathbf{\Sigma_1^1}$ and $\mathbf{\Pi_1^1}$ be relations on $C([0,1])$ (respectively) which witness the fact that $|\cdot|_{\mathrm{DIFF}}$ is a $\mathbf{\Pi}^1_1$ rank, and let $\leq^S_\epsilon$, $\leq^P_\epsilon$ be as above. Then the relations

$$f \preceq^S_\epsilon g \iff \left(f \leq^S_{\mathrm{DIFF}} g \wedge (g \leq^P_{\mathrm{DIFF}} f \Rightarrow f \leq^S_\epsilon g)\right),$$

$$f \preceq^P_\epsilon g \iff \left(f \leq^P_{\mathrm{DIFF}} g \wedge (g \leq^S_{\mathrm{DIFF}} f \Rightarrow f \leq^P_\epsilon g)\right),$$

are $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$, respectively, and demonstrate the fact that $\psi_{\epsilon,[0,1]}$ is a $\mathbf{\Pi}^1_1$ rank. □

We now proceed to construct 'local' versions of each $\phi_{\epsilon,[0,1]}$:

Definition 3.3.7. *For each $V \in \mathcal{U}$, let*

$$\begin{aligned}\mathcal{U}_V =&\{W \mid W \text{ a } \bar{V}\text{-relatively open interval with rational endpoints }\}\\ =&\{W \mid W = U \cap \bar{V}, \text{ for some } U \in \mathcal{U}\},\end{aligned}$$

and define $\mathbb{D}_V : Q \times C([0,1]) \times K(\bar{V}) \to K(\bar{V})$ by

$$\begin{aligned}\mathbb{D}_V(\epsilon, f, K) = \{x \in K \mid \forall W \in \mathcal{U}_V \text{ with } x \in W, \exists p,q,r,s \in W \cap Q,\ p<q,\\ r<s, \text{ such that } [p,q]\cap[r,s]\cap K \neq \emptyset \text{ and}\\ |\Delta_f(p,q) - \Delta_f(r,s)| > \epsilon\}.\end{aligned}$$

Also, for each $\epsilon \in Q$, $V \in \mathcal{U}$, let

$$\Omega_{\epsilon,V} = \{f \in C([0,1]) \mid \exists \alpha \in Ord, \mathbb{D}^\alpha_V(\epsilon, f, \bar{V}) = \emptyset\}.$$

As in the case $\bar{V} = [0,1]$, $\Omega_{\epsilon,V}$ is a $\mathbf{\Pi}^1_1$ subset of $C([0,1])$ and $f \mapsto |\bar{V}|_{\mathbb{D}_{\epsilon,f}}$ is a $\mathbf{\Pi}^1_1$ rank on $\Omega_{\epsilon,V}$. Analogously, we have

Definition 3.3.8. *For each $\epsilon \in Q$, $V \in \mathcal{U}$, let $\rho_{\epsilon,V} : \Omega_{\epsilon,V} \to Ord$ be the rank given by*

$$\rho_{\epsilon,V}(f) = |\bar{V}|_{\mathbb{D}_V(\epsilon,f)} - 1.$$

As before, this makes sense because $|\bar{V}|_{\mathbb{D}_V(\epsilon,f)}$ will always be a successor.

Definition 3.3.9. *Let* $V \in \mathcal{U}$, $K \in K(\bar{V})$, $f \in C([0,1])$, *and* $\epsilon \in Q$. *A closed interval* $I \subseteq \bar{V}$ *is* $\epsilon - K$ good for f in $\bar{V}$ *if* $\forall p, q, r, s \in int_{\bar{V}}(I) \cap Q$, $p < q$, $r < s$,

$$[p,q] \cap [r,s] \cap K \neq \emptyset \Rightarrow |\Delta_f(p,q) - \Delta_f(r,s)| < \epsilon,$$

where $int_{\bar{V}}(I)$ *is the interior of* I *in the relative* $\bar{V}$*-topology.*

Definition 3.3.10. *Let* $V \in \mathcal{U}$, $K \in K(\bar{V})$, $f \in C([0,1])$, $\epsilon \in Q$, *and* $m \in \mathbb{N}^+$. *An* $\epsilon - m$ covering of K for f in $\bar{V}$ *is a finite sequence* $I_0, \cdots, I_{n-1}$ *of closed intervals such that*

i) $\forall j \leq n-1$, I_j *is* $\epsilon - K$ *good for* f *in* $\bar{V}$.

ii) $\forall j \leq n-1$, I_j *has length at least* $\frac{3}{m}$.

iii) $\forall x \in K, \exists j \leq n-1$ *such that* $B(x, \frac{1}{m}) \cap \bar{V} \subseteq I_j$.

Lemma 3.3.11. *For each* $V \in \mathcal{U}$, $K \in K(\bar{V})$, $f \in C([0,1])$, *and* $\epsilon \in Q$, $\mathbb{D}_V(\epsilon, f, K) = \emptyset$ *if and only if for some* $m \in \mathbb{N}^+$ *there is an* $\epsilon - m$ *cover of* K *for* f *in* $\bar{V}$.

Definition 3.3.12. *For each* $\epsilon \in Q$, $V \in \mathcal{U}$, *define the rank* $r_{\epsilon,V} : \Omega_{\epsilon,V} \to \omega$ *by*

$$r_{\epsilon,v}(f) = \mathcal{B}(l,m)$$

where $\mathcal{B} : \omega^2 \to \omega$ *is the bijection fixed in Definition 3.3.4, and* $\mathcal{B}(l,m)$ *is least such that* $\mathbb{D}^{\alpha}_{V}(\epsilon, f, K)$ *has an* $\epsilon - m$ *cover for* f *in* $\bar{V}$ *consisting of* l *intervals, for* $\alpha = \rho_{\epsilon,V}(f)$.

Define $\phi_{\epsilon,V} : \Omega_{\epsilon,V} \to \omega_1$ *and* $\psi_{\epsilon,V}$: *DIFF* $\to \omega_2$ *by*

$$\phi_{\epsilon,V} = < \rho_{\epsilon,V}, r_{\epsilon,V} >,$$

$$\psi_{\epsilon,V} = < |\cdot|_{DIFF}, \phi_{\epsilon,V} > .$$

As in the case $\bar{V} = [0,1]$, $\phi_{\epsilon,V}$ is a $\mathbf{\Pi}^1_1$ rank on $\Omega_{\epsilon,V}$, and $\psi_{\epsilon,V}$ is a $\mathbf{\Pi}^1_1$ rank on DIFF.

Theorem 3.3.13. *The set* $\{\phi_{\epsilon,V} \mid \epsilon \in Q, V \in \mathcal{U}\}$ *is a scale on DIFF.* $\{\psi_{\epsilon,V} \mid \epsilon \in Q, V \in \mathcal{U}\}$ *is a* $\mathbf{\Pi}^1_1$ *scale on DIFF.*

Before beginning the proof of Theorem 3.3.13, we introduce two lemmas which will be useful both in the proof and the sequel, along with a fact about the derived sets for which we currently have no application, but which appears potentially useful:

Lemma 3.3.14. *For each $U, V \in \mathcal{U}$ with $U \cap V \neq \emptyset$, for each $\epsilon \in Q$, $K \in K(\bar{V})$, $f \in C([0,1])$, $\beta \in Ord$,*

$$U \cap \mathbb{D}^{\beta}_{V}(\epsilon, f, K) \subseteq \mathbb{D}^{\beta}_{V \cap U}(\epsilon, f, K \cap \bar{U}) \subseteq \bar{U} \cap \mathbb{D}^{\beta}_{V}(\epsilon, f, K).$$

Proof. The proof is by induction on β. Fix U, V, ϵ, K, f as above.

For $\beta = 0$, the assertion becomes

$$U \cap K \subseteq \bar{U} \cap K \subseteq \bar{U} \cap K,$$

and so is clearly true.

Suppose now that β is a successor, say $\beta = \gamma + 1$. For each $W \in \mathcal{U}$, say that an interval $I \subseteq [0,1]$ (open, half-open, or closed) *satisfies condition* $C(\gamma, W)$ if there exist $p, q, r, s \in I \cap Q$, $p < q$, $r < s$, such that

$$[p,q] \cap [r,s] \cap \mathbb{D}^{\gamma}_{W}(\epsilon, f, K \cap \bar{W}) \neq \emptyset \text{ and } |\Delta_f(p,q) - \Delta_f(r,s)| > \epsilon.$$

For the leftmost relation, if $x \in U \cap \mathbb{D}^{\beta}_{V}(\epsilon, f, K)$ then, since each $I \in \mathcal{U}_V$ with $x \in I$ satisfies condition $C(\gamma, V)$, each $I' \in \mathcal{U}_{U \cap V}$ with $x \in I'$ does as well. So given $I' \in \mathcal{U}_{U \cap V}$ with $x \in I'$, fix $p, q, r, s \in I' \cap Q$ such that $[p,q] \cap [r,s] \cap \mathbb{D}^{\gamma}_{V}(\epsilon, f, K) \neq \emptyset$ and $|\Delta_f(p,q) - \Delta_f(r,s)| > \epsilon$. By induction hypothesis, $U \cap \mathbb{D}^{\gamma}_{V}(\epsilon, f, K) \subseteq \mathbb{D}^{\gamma}_{U \cap V}(\epsilon, f, K \cap \bar{U})$, so $[p,q] \cap [r,s] \cap \mathbb{D}^{\gamma}_{U \cap V}(\epsilon, f, K \cap \bar{U}) \neq \emptyset$ as well, showing that I' in fact satisfies condition $C(\gamma, U \cap V)$. By definition, since I' was an arbitrary $\mathcal{U}_{U \cap V}$-neighborhood of x, $x \in \mathbb{D}^{\beta}_{U \cap V}(\epsilon, f, K \cap \bar{U})$.

For the rightmost relation, if $x \in \mathbb{D}^{\beta}_{U \cap V}(\epsilon, f, K \cap \bar{U})$ then certainly $x \in \bar{U}$, and for each $I \in \mathcal{U}_{U \cap V}$, if $x \in I$ then I satisfies condition $C(\gamma, U \cap V)$. Thus if $I' \in \mathcal{U}_V$ contains x, it will satisfy condition $C(\gamma, U \cap V)$ because $I'' = \bar{U} \cap I' \in \mathcal{U}_{U \cap V}$ does. By induction hypothesis $\mathbb{D}^{\gamma}_{U\cap V}(\epsilon, f, K\cap\bar{U}) \subseteq \mathbb{D}^{\gamma}_{V}(\epsilon, f, K)$, so in fact I' satisfies condition $C(\gamma, V)$. Since the same argument works for any neighborhood of x in $\mathcal{U}_V$, $x \in \bar{U}\cap\mathbb{D}^{\beta}_{V}(\epsilon, f, K)$.

Finally, if β is a limit ordinal, then we have, from the definition of $\mathbb{D}^{\beta}_{V}$ and the induction hypothesis,

$$\begin{aligned} U\cap\mathbb{D}^{\beta}_{V}(\epsilon, f, K) &= \bigcap_{\gamma<\beta} U\cap\mathbb{D}^{\gamma}_{V}(\epsilon, f, K) \\ &\subseteq \bigcap_{\gamma<\beta} \mathbb{D}^{\gamma}_{U\cap V}(\epsilon, f, K\cap\bar{U}) \\ &= \mathbb{D}^{\beta}_{U\cap V}(\epsilon, f, K\cap\bar{U}), \end{aligned}$$

and

$$\begin{aligned} \mathbb{D}^{\beta}_{U\cap V}(\epsilon, f, K\cap\bar{U}) &= \bigcap_{\gamma<\beta} \mathbb{D}^{\gamma}_{U\cap V}(\epsilon, f, K\cap\bar{U}) \\ &\subseteq \bigcap_{\gamma<\beta} \bar{U}\cap\mathbb{D}^{\gamma}_{V}(\epsilon, f, K) \\ &= \bar{U}\cap\mathbb{D}^{\beta}_{V}(\epsilon, f, K). \end{aligned}$$

□

Lemma 3.3.15. *Let $f_n \in DIFF$ and $\epsilon \in Q$. If $\lim_{n\to\infty}\phi_{\epsilon,U}(f_n)$ exists for all $U \in \mathcal{U}$, then $\lim_{n\to\infty}\mathbb{D}^{\alpha}_{U}(\epsilon, f_n, \bar{U})$ exists in $K([0,1])$ for each $U \in \mathcal{U}$ and each $\alpha \in Ord$.*

Proof. Let $U \in \mathcal{U}$ and $\alpha \in Ord$ be given. It suffices to show that the topological limit of the sequence $\left(\mathbb{D}^{\alpha}_{U}(\epsilon, f_n, \bar{U})\right)_n$ exists, i.e., that

$$\begin{aligned} \mathrm{Tlim}\sup_{n\to\infty}\mathbb{D}^{\alpha}_{U}(\epsilon, f_n, \bar{U}) = \{x \in [0,1] \mid & \textit{ for each neighborhood } W \textit{ of } x, \\ & W\cap\mathbb{D}^{\alpha}_{U}(\epsilon, f_n, \bar{U}) \neq \emptyset \textit{ for infinitely many } n\} \end{aligned}$$

is equal to

$$\operatorname{Tlim\,inf}_{n\to\infty} \mathbb{D}^{\alpha}_{\bar{U}}(\epsilon, f_n, \bar{U}) = \{x \in [0,1] \mid \textit{for each neighborhood } W \textit{ of } x, \; W \cap \mathbb{D}^{\alpha}_{\bar{U}}(\epsilon, f_n, \bar{U}) \neq \emptyset \textit{ for all but finitely many } n\}.$$

(See [3], § 4.F, pp.24-28.) Toward a contradiction, suppose

$$x \in \operatorname{Tlim\,sup}_{n\to\infty} \mathbb{D}^{\alpha}_{\bar{U}}(\epsilon, f_n, \bar{U}) \setminus \operatorname{Tlim\,inf}_{n\to\infty} \mathbb{D}^{\alpha}_{\bar{U}}(\epsilon, f_n, \bar{U}).$$

Then there exists $W \in \mathcal{U}$ and subsequences $(f_{n_k}), (f_{n_l})$ such that

$$\forall k \; W \cap \mathbb{D}^{\alpha}_{\bar{U}}(\epsilon, f_{n_k}, \bar{U}) \neq \emptyset,$$

while

$$\forall l \; \bar{W} \cap \mathbb{D}^{\alpha}_{\bar{U}}(\epsilon, f_{n_l}, \bar{U}) = \emptyset.$$

By Lemma 3.3.14, this implies

$$\mathbb{D}^{\alpha}_{W\cap U}(\epsilon, f_{n_k}, \overline{W \cap U}) \neq \emptyset$$

for each k, while

$$\mathbb{D}^{\alpha}_{W\cap U}(\epsilon, f_{n_l}, \overline{W \cap U}) = \emptyset$$

for each l. But from this it follows that for each k, $\rho_{\epsilon,W\cap U}(f_{n_k}) \geq \alpha$, while for each l, $\rho_{\epsilon,W\cap U}(f_{n_l}) < \alpha$, i.e., $\lim_{n\to\infty} \rho_{\epsilon,W\cap U}(f_n)$ does not exist. Hence $\lim_{n\to\infty} \phi_{\epsilon,W\cap U}(f_n)$ does not exist either, a contradiction. □

Proposition 3.3.16. *Let (f_n) be a sequence in DIFF which converges in the scale $\{\phi_{\epsilon,U} \mid \epsilon \in Q, U \in \mathcal{U}\}$; let $\epsilon \in Q$ and $U \in \mathcal{U}$ be given. If $\lambda \leq \lim_n \rho_{\epsilon,U}(f_n)$ is a limit ordinal, then*

$$\lim_{n\to\infty} \mathbb{D}^{\lambda}_{\bar{U}}(\epsilon, f_n, \bar{U}) = \bigcap_{\alpha<\lambda} \lim_{n\to\infty} \mathbb{D}^{\alpha}_{\bar{U}}(\epsilon, f_n, \bar{U}),$$

i.e., we may switch the order of the limit and the intersection.

Proof. By Lemma 3.3.15, $\lim_{n\to\infty} \mathbb{D}^\alpha_{\bar{U}}(\eta, f_n, \bar{V})$ exists for each $\eta \in Q$, $V \in \mathcal{U}$, $\alpha \in Ord$.
$\subseteq$: For all $\alpha < \lambda$ and each n, $\mathbb{D}^\lambda_{\bar{U}}(\epsilon, f_n, \bar{U}) \subseteq \mathbb{D}^\alpha_{\bar{U}}(\epsilon, f_n, \bar{U})$. Hence, for all $\alpha < \lambda$, $\lim_{n\to\infty} \mathbb{D}^\lambda_{\bar{U}}(\epsilon, f_n, \bar{U}) \subseteq \lim_{n\to\infty} \mathbb{D}^\alpha_{\bar{U}}(\epsilon, f_n, \bar{U})$, i.e.,

$$\lim_{n\to\infty} \mathbb{D}^\lambda_{\bar{U}}(\epsilon, f_n, \bar{U}) \subseteq \bigcap_{\alpha<\lambda} \lim_{n\to\infty} \mathbb{D}^\alpha_{\bar{U}}(\epsilon, f_n, \bar{U}).$$

$\supseteq$: Suppose $x \notin \lim_{n\to\infty} \mathbb{D}^\lambda_{\bar{U}}(\epsilon, f_n, \bar{U})$. By Lemma 3.3.14, there exists $V \in \mathcal{U}$, such that $x \in V$ and $\lim_{n\to\infty} \mathbb{D}^\lambda_{\bar{V}}(\epsilon, f_n, \bar{V}) = \emptyset$; hence $\lim_{n\to\infty} \rho_{\epsilon,V}(f_n) < \lambda$, denote it by α_V. Since $\lim_{n\to\infty} \mathbb{D}^{\alpha_V+1}_{\bar{V}}(\epsilon, f_n, \bar{V}) = \emptyset$, $x \notin \lim_{n\to\infty} \mathbb{D}^{\alpha_V+1}_{\bar{U}}(\epsilon, f_n, \bar{U})$, again by Lemma 3.3.14. So $x \notin \bigcap_{\alpha<\lambda} \lim_{n\to\infty} \mathbb{D}^\alpha_{\bar{U}}(\epsilon, f_n, \bar{U})$. □

Proof of Theorem 3.3.13. To see that $\{\phi_{\epsilon,V} \mid \epsilon \in Q, V \in \mathcal{U}\}$ is a scale on DIFF, let (f_n) be a sequence in DIFF; suppose that $\lim_{n\to\infty} f_n = f$ and that $\lim_{n\to\infty} \phi_{\epsilon,V}(f_n)$ exists for each $\epsilon \in Q$ and $V \in \mathcal{U}$. Since $\lim_{n\to\infty} \phi_{\epsilon,V}(f_n)$ exists if and only if each of $\lim_{n\to\infty} \rho_{\epsilon,V}(f_n)$ and $\lim_{n\to\infty} r_{\epsilon,V}(f_n)$ exist, we can write $\lim_n \phi_{\epsilon,V}(f_n) =< \alpha_{\epsilon,V}, \mathcal{B}(l_{\epsilon,V}, m_{\epsilon,V}) >$, where

$$\alpha_{\epsilon,V} = \lim_{n\to\infty} \rho_{\epsilon,V}(f_n)$$
$$\mathcal{B}(l_{\epsilon,V}, m_{\epsilon,V}) = \lim_{n\to\infty} r_{\epsilon,V}(f_n).$$

By induction on $\alpha_{\epsilon,V}$, we'll show that $\mathbb{D}^\infty_{\bar{V}}(\epsilon, f, \bar{V}) = \emptyset$ for all $\epsilon \in Q$ and for all $V \in \mathcal{U}$; hence, by Theorem 3.2.3, $f \in$ DIFF. Also in the induction, we'll show that for all $\epsilon \in Q$ and $V \in \mathcal{U}$, $\phi_{\epsilon,V}(f) \leq < \alpha_{\epsilon,V}, \mathcal{B}(l_{\epsilon,V}, m_{\epsilon,V}) >$, completing the proof that $\{\phi_{\epsilon,V} \mid \epsilon \in Q, V \in \mathcal{U}\}$ is a scale.

$\alpha_{\epsilon,V} = 0$.
Let $N_{\epsilon,V} \in \mathbb{N}$ be such that $n \geq N_{\epsilon,V}$ implies $\phi_{\epsilon,V}(f_n) =< 0, \mathcal{B}(l_{\epsilon,V}, m_{\epsilon,V}) >$; that is, for each $n \geq N_{\epsilon,V}$, $\mathbb{D}_V(\epsilon, f_n, \bar{V}) = \emptyset$ and $\bar{V}$ has an $\epsilon - m_{\epsilon,V}$ covering for f_n in $\bar{V}$ which consists of $l_{\epsilon,V}$ many intervals. Fix such a cover for each f_n, and denote it by $[a^n_0, b^n_0], \cdots, [a^n_{l-1}, b^n_{l-1}]$ (for ease of notation, in subscripts we will write l for $l_{\epsilon,V}$).

Then fix a subsequence (n_k) with the property that $\lim_{k\to\infty} a_i^{n_k}$ and $\lim_{k\to\infty} b_i^{n_k}$ exist for each $i = 0, 1, \cdots l_{\epsilon,V} - 1$. Denote these limits by a_i and b_i, respectively, for each i.

Claim: $[a_0, b_0], \cdots, [a_{l-1}, b_{l-1}]$ is an $\epsilon - m_{\epsilon,V}$ cover of $\bar{V}$ for f in $\bar{V}$. Hence $\mathbb{D}_V(\epsilon, f, \bar{V}) = \emptyset$ and $\phi_{\epsilon,V}(f) \leq << 0, < l_{\epsilon,V}, m_{\epsilon,V} >>$.

Proof of Claim: Firstly, it's clear that for each j, $b_j - a_j \geq \frac{3}{m}$ and that for each $x \in \bar{V}$, there is $j \leq l_{\epsilon,V}$ such that $B(x, \frac{1}{m_{\epsilon,V}}) \subseteq [a_j, b_j]$, because the analogous facts are true for each of the coverings $[a_0^{n_k}, b_0^{n_k}], \cdots, [a_{l-1}^{n_k}, b_{l-1}^{n_k}]$. So we need only show that each $[a_j, b_j]$ is $\epsilon - \bar{V}$ good for f in $\bar{V}$.

Toward this end, fix $j \leq l_{\epsilon,V} - 1$ and suppose that $p, q, r, s \in int_{\bar{V}}([a_j, b_j]) \cap Q$ are such that $p < q$, $r < s$, and $[p, q] \cap [r, s] \neq \emptyset$.

As a preliminary case, suppose additionally that $a_j < p, q, r, s < b_j$ (which may fail if either of a_j or b_j is an endpoint of $\bar{V}$). Then, because $\lim_{k\to\infty} a_j^{n_k} = a_j$ and $\lim_k b_j^{n_k} = b_j$, for k sufficiently large we have $p, q, r, s \in [a_j^{n_k}, b_j^{n_k}]$; since $[a_j^{n_k}, b_j^{n_k}]$ is $\epsilon - \bar{V}$ good for f_{n_k} in $\bar{V}$, we then have, for infinitely many k,

$$|\Delta_{f_{n_k}}(p, q) - \Delta_{f_{n_k}}(r, s)| \leq \epsilon.$$

Since $f_{n_k} \to f$, it follows that $|\Delta_f(p, q) - \Delta_f(r, s)| \leq \epsilon$ as well. By the continuity of f, this must also be true when $a_j \leq p, q, r, s \leq b_j$ (i.e., we can omit the preliminary constraint). So $[a_j, b_j]$ is $\epsilon - \bar{V}$ good for f in $\bar{V}$. □(Claim)

$\alpha_{\epsilon,V} > 0$: As in the previous case, let $N_{\epsilon,V} \in \mathbb{N}$ be such that $n \geq N_{\epsilon,V}$ implies $\phi_{\epsilon,V}(f_n) = < \alpha_{\epsilon,V}, \mathcal{B}(l_{\epsilon,V}, m_{\epsilon,V}) >$; for each $n \geq N_{\epsilon,V}$ fix an $\epsilon - m_{\epsilon,V}$ cover $[a_0^n, b_0^n], \cdots, [a_{l-1}^n, b_{l-1}^n]$ of $\mathbb{D}_V^{\alpha_{\epsilon,V}}(\epsilon, f_n, \bar{V})$ for f_n in $\bar{V}$. (Again, for ease of notation, in subscripts we will write l for $l_{\epsilon,V}$.) Also fix a subsequence (n_k) with the property that for each $i \leq l_{\epsilon,V} - 1$ there exist a_i and b_i such that $\lim_{k\to\infty} a_i^{n_k} = a_i$ and $\lim_{k\to\infty} b_i^{n_k} = b_i$. By Lemma 3.3.15, $\lim_{n\to\infty} \mathbb{D}_V^{\alpha_{\epsilon,V}}(\epsilon, f_n, \bar{V})$ exists.

Notation: Let $\mathbf{D} = \lim_{n\to\infty} \mathbb{D}_V^{\alpha_{\epsilon,V}}(\epsilon, f_n, \bar{V})$.

By definition of convergence in $K([0,1])$, for each $\delta > 0$ and all n sufficiently large,

$$\mathbf{D} \subseteq B\big(\mathbb{D}_V^{\alpha_{\epsilon,V}}(\epsilon, f_n, \bar{V}), \delta\big) \quad \text{and} \quad \mathbb{D}_V^{\alpha_{\epsilon,V}}(\epsilon, f_n, \bar{V}) \subseteq B(\mathbf{D}, \delta),$$

where $B(S,\eta) = \{x \in [0,1] \mid \min_{y \in S} |x-y| < \eta\}$. So for each $x \in \mathbf{D}$, there exists $j \leq l_{\epsilon,V} - 1$ such that $B(x, \frac{1}{m_{\epsilon,V}}) \subseteq [a_j, b_j]$, because the analogous fact is true for each $x \in \mathbb{D}_V^{\alpha_{\epsilon,V}}(\epsilon, f_{n_k}, \bar{V})$, with respect to the covering $[a_0^{n_k}, b_0^{n_k}], \cdots, [a_{l-1}^{n_k}, b_{l-1}^{n_k}]$. Also, for each j, the interval $[a_j, b_j]$ has length at least $\frac{3}{m}$, because the corresponding fact is true for each $[a_j^{n_k}, b_j^{n_k}]$.

We complete the inductive step by verifying the following two facts:

i) For each j, $[a_j, b_j]$ is $\epsilon - \mathbf{D}$ good for f. Hence $[a_0, b_0], \cdots, [a_{l-1}, b_{l-1}]$ is an $\epsilon - m$ cover of $\mathbf{D}$ for f.

ii) $\mathbb{D}_V^{\alpha_{\epsilon,V}}(\epsilon, f, \bar{V}) \subseteq \mathbf{D}$.

Proof of (ii): Toward a contradiction, suppose $x \in \mathbb{D}_V^{\alpha_{\epsilon,V}}(\epsilon, f, \bar{V}) \backslash \mathbf{D}$, and let $W \in \mathcal{U}$ be such that $x \in W$, $\bar{W} \cap \mathbf{D} = \emptyset$. Then for n sufficiently large, we have $\bar{W} \cap \mathbb{D}_V^{\alpha_{\epsilon,V}}(\epsilon, f_n, \bar{V}) = \emptyset$ as well, so by Lemma 3.3.14,

$$\lim_{n \to \infty} \rho_{\epsilon, V \cap W}(f_n) < \alpha_{\epsilon,V}.$$

By induction hypothesis we should then have $\rho_{\epsilon,V\cap W}(f) < \alpha_{\epsilon,V}$, but again by Lemma 3.3.14, $\bar{W} \cap \mathbb{D}_V^{\alpha_{\epsilon,V}}(\epsilon, f, \bar{V}) \neq \emptyset$ implies $\rho_{\epsilon,V\cap W}(f) \geq \alpha_{\epsilon,V}$, a contradiction.

Proof of (i): We split the argument into two cases. Denote the closed interval $\bar{V}$ by $[u, v]$.

Case 1: $\mathbf{D} \subseteq (u, v)$.

Fix $j \leq l_{\epsilon,V} - 1$, and suppose $p, q, r, s \in int_{\bar{V}}([a_j, b_j]) \cap Q$ are such that $p < q$, $r < s$, and $[p,q] \cap [r,s] \cap \mathbf{D} \neq \emptyset$.

As a preliminary case, assume additionally that

$$a_j < p, q, r, s < b_j \quad \text{and} \quad (p,q) \cap (r,s) \cap \mathbf{D} \neq \emptyset.$$

Then for k sufficiently large we'll also have

$$a_j^{n_k} < p, q, r, s < b_j^{n_k} \text{ and } (p, q) \cap (r, s) \cap \mathbb{D}_V^{\alpha_\epsilon, V}(\epsilon, f_{n_k}, \bar{V}) \neq \emptyset.$$

Because $[a_j^{n_k}, b_j^{n_k}]$ is $\epsilon - \mathbb{D}_V^{\alpha_\epsilon, V}(\epsilon, f_{n_k}, \bar{V})$ good for f_{n_k} in $\bar{V}$, we have

$$|\Delta_{f_{n_k}}(p, q) - \Delta_{f_{n_k}}(r, s)| \leq \epsilon.$$

Since $f_{n_k} \to f$, it follows that

$$|\Delta_f(p, q) - \Delta_f(r, s)| \leq \epsilon.$$

For the general Case 1, because $[p, q] \cap [r, s] \cap \mathbf{D} \subseteq (a_j, b_j)$, we can find sequences $(p_n), (q_n), (r_n), (s_n)$ in $int_{\bar{V}}([a_j, b_j]) \cap Q$ such that $p_n \to p$, $q_n \to q$, $r_n \to r$, $s_n \to s$, and for each n,

$$a_j < p_n, q_n, r_n, s_n < b_j \text{ and } (p_n, q_n) \cap (r_n, s_n) \cap \mathbf{D} \neq \emptyset.$$

By the preliminary case, $|\Delta_f(p_n, q_n) - \Delta_f(r_n, s_n)| \leq \epsilon$ for each n, so by the continuity of f, $|\Delta_f(p, q) - \Delta_f(r, s)| \leq \epsilon$ as well.

Case 2: $\{u, v\} \cap \mathbf{D} \neq \emptyset$.

We begin by making a slight modification to the covering $[a_0, b_0], \cdots [a_{l-1}, b_{l-1}]$. If $u \in \mathbf{D}$, consider each $[a_i, b_i]$ for which $a_i = u$ (there must be at least one). If $a_i^{n_k} = u$ for infinitely many k, then we leave a_i unchanged (it remains the left endpoint of the ith interval). However, if $a_i^{n_k} > u$ for all but finitely many k, then we replace a_i, as the left endpoint of the ith interval, by

$$a_i' = \min\{a_i + \frac{1}{2m_{\epsilon,V}}, b_i - \frac{3}{m_{\epsilon,V}}\}$$

unless $a_i = b_i - \frac{3}{m_{\epsilon,V}}$ (i.e., $[a_i, b_i] = [u, u + \frac{3}{m_{\epsilon,V}}]$), in which case we omit $[a_i, b_i]$ from the covering. Because for each k there exists $i \leq n-1$ such that $a_i^{n_k} = u$, there must be at least one i such that $a_i = u$ and $a_i^{n_k} = u$ for infinitely many k. Hence this modified cover still has the property that for each $x \in \mathbf{D}$, there is a j such that $B(x, \frac{1}{m_{\epsilon,V}}) \subseteq [a_j, b_j]$.

Similarly, if $v \in \mathbf{D}$, after modifying the a_i's as needed, we consider each $[a_j, b_j]$ for which $b_j = v$. If $b_j = v$ but $b_j^{n_k} < v$ for all but finitely many k, then we replace b_j, as the right endpoint of the jth interval, by

$$b_j' = \max\{b_j - \frac{1}{2m_{\epsilon,V}}, a_j + \frac{3}{m_{\epsilon,V}}\}$$

unless $[a_j, b_j] = [v - \frac{3}{m_{\epsilon,V}}, v]$, in which case we omit it from the covering.

Now fix $j \leq l_{\epsilon,V} - 1$ and let $p, q, r, s \in int_{\bar{V}}([a_j, b_j]) \cap Q$ with $p < q$, $r < s$ and $[p, q] \cap [r, s] \cap \mathbf{D} \neq \emptyset$. If $[p, q] \cap [r, s] \cap \mathbf{D} \subseteq (a_j, b_j)$, then the argument from Case 1 applies, and we may conclude that $|\Delta_f(p, q) - \Delta_f(r, s)| \leq \epsilon$. Otherwise $\{u, v\} \cap [p, q] \cap [r, s] \cap \mathbf{D} \neq \emptyset$, since the only way that $[p, q] \cap [r, s] \cap \mathbf{D} \not\subseteq (a_j, b_j)$ is if $a_j = p = r = u$ or $b_j = q = s = v$. For concreteness, suppose that $u \in [p, q] \cap [r, s] \cap \mathbf{D}$, so that $a_j = p = r = u$. Then $a_j^{n_k} = u$ for infinitely many k. If $b_j \neq v$, then $p, q, r, s < b_j$, so for all sufficiently large k with $a_j^{n_k} = u$, we have $p, q, r, s \in int_{\bar{V}}([a_j^{n_k}, b_j^{n_k}])$. Also, because $u \in \lim_{k\to\infty} \mathbb{D}_V^{\alpha_{\epsilon,V}}(\epsilon, f_{n_k}, \bar{V}) = \mathbf{D}$, we must have $[p, q] \cap [r, s] \cap \mathbb{D}_V^{\alpha_{\epsilon,V}}(\epsilon, f_{n_k}, \bar{V}) \neq \emptyset$, if k is sufficiently large.

Thus for infinitely many k we have $|\Delta_{f_{n_k}}(p, q) - \Delta_{f_{n_k}}(r, s)| \leq \epsilon$, so

$$|\Delta_f(p, q) - \Delta_f(r, s)| \leq \epsilon.$$

If $b_j = v$, then possibly $q = v$ or $s = v$, in which case there may not be infinitely many k for which $p, q, r, s \in int_{\bar{V}}([a_j^{n_k}, b_j^{n_k}])$ (there would be infinitely many k for which $b_j^{n_k} = v$, but they may not be the same k for which $a_j^{n_k} = u$). But we can find sequences $(q_m), (s_m)$, $q_m \to q$, $s_m \to s$, such that for each m there exist infinitely many k for which $p, q_m, r, s_m \in int_{\bar{V}}([a_j^{n_k}, b_j^{n_k}])$. Using the above argument we get

$$|\Delta_f(p, q_m) - \Delta_f(r, s_m)| \leq \epsilon \text{ for all } m,$$

so $|\Delta_f(p,q) - \Delta_f(r,s)| \leq \epsilon$ as well. The case in which $v \in [p,q] \cap [r,s] \cap \mathbf{D}$ is handled analogously.

Thus each interval $[a_j, b_j]$ must be $\epsilon - \mathbf{D}$ good for f in $\bar{V}$. This completes both the inductive step and the proof that $\{\phi_{\epsilon,V} \mid \epsilon \in Q, V \in \mathcal{U}\}$ is a scale.

As in Lemma 3.3.6, each $\psi_{\epsilon,V}$ is a $\mathbf{\Pi}^1_1$ rank. Suppose (f_n) is a sequence in DIFF, $\lim_{n\to\infty} f_n = f$, and $\lim_{n\to\infty} \psi_{\epsilon,V}(f_n)$ exists for each $\epsilon \in Q$ and each $V \in \mathcal{U}$. Then $\lim_{n\to\infty} \phi_{\epsilon,V}(f_n)$ exists for all $\epsilon \in Q$, $V \in \mathcal{U}$; since $\{\phi_{\epsilon,V} \mid \epsilon \in Q, V \in \mathcal{U}\}$ is a scale, $f \in DIFF$ and $\phi_{\epsilon,V}(f) \leq \lim_{n\to\infty} \phi_{\epsilon,V}(f_n)$ for all $\epsilon \in Q$, $V \in \mathcal{U}$. This implies that $\rho_{\epsilon,V}(f) \leq \lim_{n\to\infty} \rho_{\epsilon,V}(f_n)$ for all $\epsilon \in Q$, $V \in \mathcal{U}$; in particular,

$$
\begin{aligned}
|f|_{\text{DIFF}} &= \sup_m(\rho_{\frac{1}{m},[0,1]}(f) + 1) \\
&\leq \sup_m(\lim_{n\to\infty} \rho_{\frac{1}{m},[0,1]}(f_n) + 1) \\
&\leq \lim_{n\to\infty}(\sup_m \rho_{\frac{1}{m},[0,1]}(f_n) + 1) \\
&= \lim_{n\to\infty} |f_n|_{\text{DIFF}}
\end{aligned}
$$

(which exists, by hypothesis). Thus for all $\epsilon \in Q$, $V \in \mathcal{U}$, $\psi_{\epsilon,V}(f) \leq \lim_n \psi_{\epsilon,V}(f_n)$, so $\{\psi_{\epsilon,V} \mid \epsilon \in Q, V \in \mathcal{U}\}$ is also a scale. □

3.4 Analytical Strength of Scale Convergence

Here we attempt to quantify the analytical strength of convergence in the scale $\{\psi_{\epsilon,U} \mid \epsilon \in Q, U \in \mathcal{U}\}$, by investigating the conditions under which convergence in the scale implies pointwise convergence of the derivatives. As was shown at the end of the proof of Theorem 3.3.13, a sequence (f_n) in DIFF converges in the scale $\{\phi_{\epsilon,U} \mid \epsilon \in Q, U \in \mathcal{U}\}$ if it converges in the scale $\{\psi_{\epsilon,U} \mid \epsilon \in Q, U \in \mathcal{U}\}$, so, out of convenience, we will generally work with $\{\phi_{\epsilon,U} \mid \epsilon \in Q, U \in \mathcal{U}\}$ instead.

To begin, we note that the proof of Theorem 3.3.13 shows that the full strength of the hypotheses was not used. This gives some indication of the strength of convergence in the scale.

Corollary 3.4.1. *Suppose $f \in C([0,1])$ and there exists a sequence (f_n) in DIFF which converges in the scale $\{\phi_{\epsilon,U} \mid \epsilon \in Q, U \in \mathcal{U}\}$ and which converges pointwise to f. Then $f \in$ DIFF and $\phi_{\epsilon,U}(f) \leq \lim_{n\to\infty} \phi_{\epsilon,U}(f_n)$ for each $\epsilon \in Q$, $U \in \mathcal{U}$. Similarly for the scale $\{\psi_{\epsilon,U} \mid \epsilon \in Q, U \in \mathcal{U}\}$.*

The subsequent results of this investigation of (f_n) in DIFF depend on $(|f_n|_{\text{DIFF}})$. We begin with the simplest case, where $|f_n|_{\text{DIFF}} = 1$, i.e., each f_n is in $C^1([0,1])$.

Proposition 3.4.2. *Suppose (f_n) is a sequence of $C^1([0,1])$ functions which converges in the scale*

$$\{\phi_{\epsilon,U} \mid \epsilon \in Q, U \in \mathcal{U}\}$$

and which converges pointwise to a function $f \in C([0,1])$. Then $f \in DIFF$ and $f_n' \to f'$ uniformly.

Proof. By Corollary 3.4.1 and Theorem 3.2.7, $f \in DIFF$ and

$$\rho_{\epsilon,[0,1]}(f_n) \leq \lim_{n\to\infty} \rho_{\epsilon,[0,1]}(f_n) = 0.$$

Again by Theorem 3.2.7, it follows that $f \in C^1([0,1])$.

Towards a contradiction, suppose that (f_n') fails to converge to f' uniformly, so that there exist $\delta > 0$, a subsequence (f_{n_j}), and a sequence of points (x_j) in $[0,1]$ such that $|f_{n_j}(x_j) - f(x_j)| > \delta$ for each j. Fix such δ, (f_{n_j}), (x_j); without loss of generality, we may assume that $x_j \longrightarrow x$ for some $x \in [0,1]$. Also fix $\epsilon > \frac{\delta}{3}$. Because $\lim_{n\to\infty} r_{\epsilon,[0,1]}(f_n)$ exists, there is an interval $[a,b]$ containing x in its $[0,1]$-interior which is $\epsilon - [0,1]$ good for all but finitely many f_n. Indeed, say $\lim_{n\to\infty} r_{\epsilon,[0,1]}(f_n) = \mathcal{B}(l,m)$. For each f_n with $r_{\epsilon,[0,1]}(f_n) = \mathcal{B}(l,m)$, f_n has an $\epsilon - m$ cover of $[0,1]$. Hence there is an interval $[a_n, b_n]$ which is $\epsilon - [0,1]$ good for f_n and with $B(x, \frac{1}{m}) \subseteq [a_n, b_n]$. Take $[a,b] = \bigcap_n [a_n, b_n]$. Let $M \in \mathbb{N}$ be such that for all $j \geq M$,

i) $x_j \in int([a,b])$,

ii) $[a,b]$ is $\epsilon - [0,1]$ good for f_{n_j}.

Now fix $p, q \in int([a,b])$ such that

$$p \leq q \text{ and } |f'(x) - \Delta_f(p,q)| < \frac{\epsilon}{2},$$

and for each $j \geq M$ fix $p_j, q_j \in int_{[0,1]}[a,b]$ such that

$$p_j \leq q_j \text{ and } |f'_{n_j}(x_j) - \Delta_{f_{n_j}}(p_j, q_j)| < \frac{\epsilon}{2}.$$

Then, for j sufficiently large, we have

$$\begin{aligned} |f'_{n_j}(x_j) - f'(x_j)| &\leq |f_{n_j}(x_j) - \Delta_{f_{n_j}}(p_j,q_j)| + |\Delta_{f_{n_j}}(p_j,q_j) - \Delta_{f_{n_j}}(p,q)| + \\ &\qquad |\Delta_{f_{n_j}}(p,q) - \Delta_f(p,q)| + |\Delta_f(p,q) - f'(x)| + |f'(x) - f'(x_j)| \\ &< |\Delta_{f_{n_j}}(p,q) - \Delta_f(p,q)| + |f'(x) - f'(x_j)| + 2\epsilon, \end{aligned}$$

by our choice of p, q, p_j, q_j, and because $[a,b]$ is $\epsilon - [0,1]$ good for f_j (noting that if $j \geq M$ is sufficiently large, then $[p,q] \cap [p_j, q_j] \neq \emptyset$). Because $f_{n_j} \to f$ and because f' is continuous, the remaining two terms on the left-hand side tend to zero, hence $\limsup_{j\to\infty} |f'_{n_j}(x_j) - f'(x_j)| \leq 2\epsilon < \delta$, a contradiction.

□

Proposition 3.4.3. *Suppose (f_n) is a sequence in $C^1([0,1])$, $f \in DIFF$, $f_n \to f$ uniformly and $f'_n \to f'$ uniformly. Then, for each $U \in \mathcal{U}$, there is a nowhere dense set $\mathcal{E}_U \subseteq Q$ such that $\lim_{n\to\infty} \phi_{\epsilon,U}(f_n)$ exists for each $\epsilon \notin \mathcal{E}_U$.*

Lemma 3.4.4. *Suppose (f_n) is a sequence in $C^1([0,1])$, $f \in DIFF$, $f_n \to f$ uniformly and $f'_n \to f'$ uniformly. Let $\epsilon \in Q$, $U \in \mathcal{U}$ be given. If $[a,b]$ is $\epsilon - \bar{U}$ good for f in $\bar{U}$, then for each $\eta > 0$ there exists N_η such that for all $n \geq N_\eta$, $[a,b]$ is $(1+\eta)\epsilon - \bar{U}$ good for f_n in $\bar{U}$.*

Proof. The proof is given for $\bar{U} = [0,1]$; the same argument works for any choice of $U \in \mathcal{U}$. Let ϵ and $[a,b]$ be as in the hypothesis of the lemma, and let $\eta > 0$ be given.

Because f' is continuous, the function $F : [0,1] \times [0,1] \to \mathbb{R}$ given by

$$F(p,q) = \begin{cases} f'(p) & \text{if } p = q, \\ \Delta_f(p,q) & \text{if } p \neq q \end{cases}$$

is continuous (and hence uniformly continuous). Fix δ such that for any $(p,q), (r,s) \in [0,1] \times [0,1]$,

$$\sqrt{(r-p)^2 + (s-q)^2} < 2\sqrt{2} \;\Rightarrow\; |F(p,q) - F(r,s)| < \tfrac{\eta\epsilon}{3}.$$

Let $N_\eta \in \mathbb{N}$ be such that $n \geq N_\eta$ implies both

i) $\| f_n' - f' \|_\infty < \frac{\eta\epsilon}{3}$,

ii) $\forall u, v \in [0,1]$, if $|u - v| \geq \delta$, then $|\Delta_{f_n}(u,v) - \Delta_f(u,v)| < \frac{\eta\epsilon}{3}$.

Now let $p, q, r, s \in int([a,b]) \cap Q$, $[p,q] \cap [r,s] \neq 0$. We'll show by cases that for all $n \geq N_\eta$, $|\Delta_{f_n}(p,q) - \Delta_{f_n}(r,s)| < (1+\eta)\epsilon$.

Case 1: $q - p \geq \delta$ and $s - r \geq \delta$. Then for all $n \geq N_\eta$,

$$\begin{aligned} |\Delta_{f_n}(p,q) - \Delta_{f_n}(r,s)| \leq & |\Delta_{f_n}(p,q) - \Delta_f(p,q)| + |\Delta_f(p,q) - \Delta_f(r,s)| \\ & + |\Delta_f(r,s) - \Delta_{f_n}(r,s)| \\ < & \tfrac{\eta\epsilon}{3} + \epsilon + \tfrac{\eta\epsilon}{3} \\ < & (1+\eta)\epsilon. \end{aligned}$$

Case 2: $q - p < \delta$ and $s - r < \delta$.

Using the Mean Value Theorem, for each $n \geq N_\eta$, fix $p_n \in (p,q)$, $r_n \in (r,s)$ such that

$f'_n(p_n) = \Delta_{f_n}(p,q)$, $f'_n(r_n) = \Delta_{f_n}(r,s)$. Then for all $n \geq N_\eta$,

$$\begin{aligned}
|\Delta_{f_n}(p,q) - \Delta_{f_n}(r,s)| &= |f'_n(p_n) - f'_n(r_n)| \\
&\leq |f'_n(p_n) - f'(p_n)| + |f'(p_n) - f'(r_n)| + |f'(r_n) - f'_n(r_n)| \\
&< \tfrac{\eta\epsilon}{3} + \tfrac{\eta\epsilon}{3} + \tfrac{\eta\epsilon}{3} \\
&< (1+\eta)\epsilon.
\end{aligned}$$

The bound for the term $|f'(p_n) - f'(r_n)|$ comes from the fact that $|r_n - p_n| < 2\delta$: then $\sqrt{(r_n - p_n)^2 + (r_n - p_n)^2} < 2\sqrt{2}\delta$, so $|F(r_n, r_n) - F(p_n, p_n)| < \frac{\eta\epsilon}{3}$.

Case 3: $q - p \geq \delta$ and $s - r < \delta$.

For each $n \geq N_\eta$, fix $r_n \in (r,s)$ such that $f'_n(r_n) = \Delta_{f_n}(r,s)$. Then for all $n \geq N_\eta$,

$$\begin{aligned}
|\Delta_{f_n}(p,q) - \Delta_{f_n}(r,s)| \leq &|\Delta_{f_n}(p,q) - \Delta_f(p,q)| + |\Delta_f(p,q) - \Delta_f(r,s)| \\
&+ |\Delta_f(r,s) - f'(r_n)| + |f'(r_n) - f'_n(r_n)| \\
<&\tfrac{\eta\epsilon}{3} + \epsilon + \tfrac{\eta\epsilon}{3} + \tfrac{\eta\epsilon}{3} \\
=&(1+\eta)\epsilon.
\end{aligned}$$

The bound for the second term comes from the fact that $[a,b]$ is $\epsilon - [0,1]$ good for f. The bound for the third term follows from the fact that $\sqrt{(r-r_n)^2 + (s-r_n)^2} < \sqrt{2}\delta$, since then $|F(r,s) - F(r_n, r_n)| < \frac{\eta\epsilon}{3}$.

□

Definition 3.4.5. *Let $\epsilon \in Q$, $U \in \mathcal{U}$, and let $g \in C^1([0,1])$. We say that ϵ is U-*sharp *for g if, for each $\epsilon' < \epsilon$, $r_{\epsilon',U}(g) > r_{\epsilon,U}(g)$.*

Proof of Proposition 3.4.3. Let $U \in \mathcal{U}$ be given. Because each f_n is in $C^1([0,1])$, $\phi_{\epsilon,U}(f_n) =< 0, r_{\epsilon,U}(f_n) >= r_{\epsilon,U}(f_n)$; that is, we need to determine the values $\epsilon \in Q$ for which $\lim_{n\to\infty} r_{\epsilon,U}(f_n)$ exists. We first note that $\lim_{n\to\infty} r_{\epsilon,U}(f_n)$ exists if ϵ is not U-sharp for f. For suppose that $\epsilon \in Q$ isn't U-sharp for f, and fix $\epsilon' \in Q$ such that $\epsilon' < \epsilon$ and $r_{\epsilon',U}(f) = r_{\epsilon,U}(f) = \mathcal{B}(l,m)$. If $I_0, \cdots, I_{l-1}$ is an $\epsilon' - m$ cover of $\bar{U}$ for f in $\bar{U}$, then by Lemma 3.4.4, it will be an $\epsilon - m$ cover of $\bar{U}$ for all but finitely many f_n in

$\bar{U}$. Hence $\limsup_{n\to\infty} r_{\epsilon,U}(f)$ and $\liminf_{n\to\infty} r_{\epsilon,U}(f)$ both exist and are bounded by $r_{\epsilon,U}(f)$. Moreover, it's impossible to have $\liminf_{n\to\infty} r_{\epsilon,U}(f) < r_{\epsilon,U}(f)$. If (f_{n_j}) were a subsequence with $r_{\epsilon,U}(f_{n_j}) = \mathcal{B}(l', m') < \mathcal{B}(l, m)$ for all j, then, as in the proof of Theorem 3.3.13 (i.e., the proof of the base case $\alpha_{\epsilon,U} = 0$), we could produce an $\epsilon - m'$ cover of $\bar{U}$ for f in $\bar{U}$ consisting of l' many intervals, a contradiction. Hence $\liminf_{n\to\infty} r_{\epsilon,U}(f) = \limsup_{n\to\infty} r_{\epsilon,U}(f) = r_{\epsilon,U}(f)$. Secondly, we note that for each fixed $\epsilon_0 \in Q$, there are only finitely many points $\epsilon \in Q$ such that $\epsilon > \epsilon_0$ and ϵ is U-sharp for f. This is because the function $\epsilon \mapsto r_{\epsilon,U}(f)$ is a decreasing map into $\mathbb{N}$. Thus if $\mathcal{E}_U$ is the set of U-sharp points for f, it is as needed.

□

Corollary 3.4.6. *Suppose (f_n) is a sequence in $C^1([0,1])$, $f \in DIFF$, and $f_n \to f$ uniformly. Then $f_n' \to f'$ uniformly if and only if for each U there is a dense set $Q_U \subseteq Q$ such that $\lim_{n\to\infty} \phi_{\epsilon,U}(f_n)$ exists for each $\epsilon \in Q_U$.*

Proof.
$\Leftarrow$: This follows from the proof of Proposition 3.4.2, which shows that the proposition's hypothesis that (f_n) converges in the scale $\{\phi_{\epsilon,U} \,|\, \epsilon \in Q, U \in \mathcal{U}\}$ can be replaced by the weaker hypothesis that $\lim_{n\to\infty} r_{\epsilon,[0,1]}(f_n)$ exists for all ϵ in some dense subset $Q_{[0,1]}$ of Q.
$\Rightarrow$: Proposition 3.4.3

□

Thus, where sequences of $C^1([0,1])$ functions are concerned (in other words, functions with $|\cdot|_{\text{DIFF}}$-rank 1), uniform convergence of the derivatives is roughly equivalent to convergence in the scale. However, once we allow $\lim_{n\to\infty} |f_n|_{\text{DIFF}} \geq 2$, the relationship between convergence in the scale and convergence of the sequence (f_n') is not as strong. As the following proposition shows, even if the functions f_n are required to have $|\cdot|_{\text{DIFF}}$-rank 2, convergence in the scale no longer guarantees convergence of (f_n'), even pointwise.

Notation: We write "$\forall^* x \in U$" for "for all x in a comeager subset of U" (see [3], §8.G).

Proposition 3.4.7. *There exists a sequence (f_n) in DIFF with the following properties:*

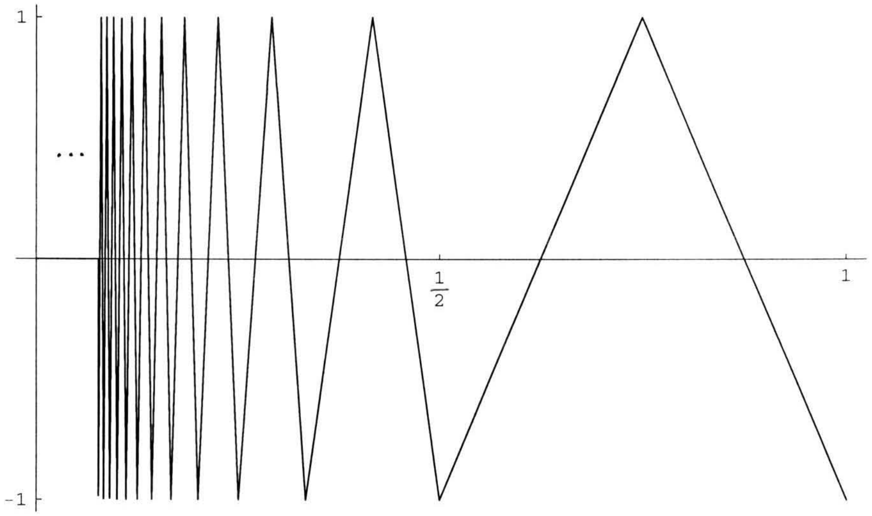
i) *(f_n) converges in the scale $\{\phi_{\epsilon,U} \mid \epsilon \in Q, U \in \mathcal{U}\}$,*

ii) *$f_n \to 0$ uniformly,*

iii) *$\forall n \in \mathbb{N}$, $|f_n|_{DIFF} = 2$,*

iv) *$\forall^* x \in [0,1], \lim_{n\to\infty} f_n'(x)$ does not exist.*

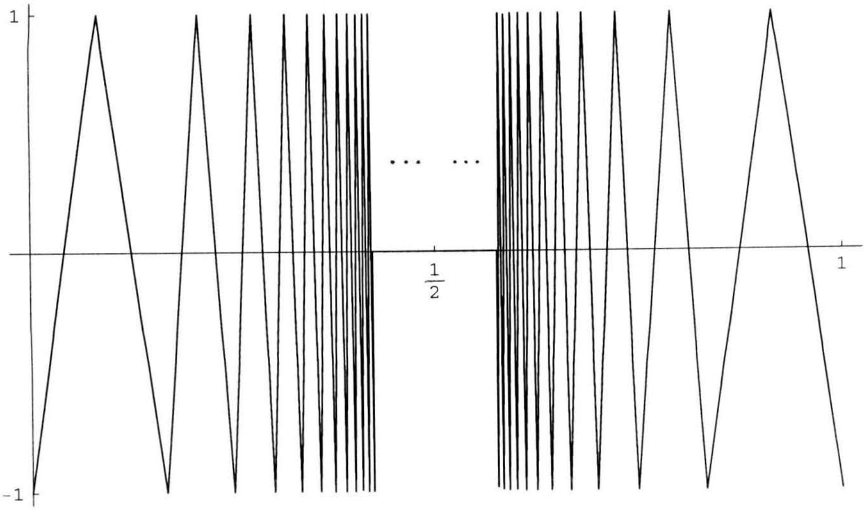
Proof. Define the auxiliary function $g : [0,1] \to \mathbb{R}$ in the following manner: let $g(0) = 0$, for each $k \in \mathbb{N}^+$ let $g(\frac{1}{k}) = -1$, $g(\frac{1}{2}(\frac{1}{k} + \frac{1}{k+1})) = 1$, and on each of the intervals $[\frac{1}{k+1}, \frac{1}{2}(\frac{1}{k} + \frac{1}{k+1})]$, $[\frac{1}{2}(\frac{1}{k} + \frac{1}{k+1}), \frac{1}{k}]$, define g to be linear.



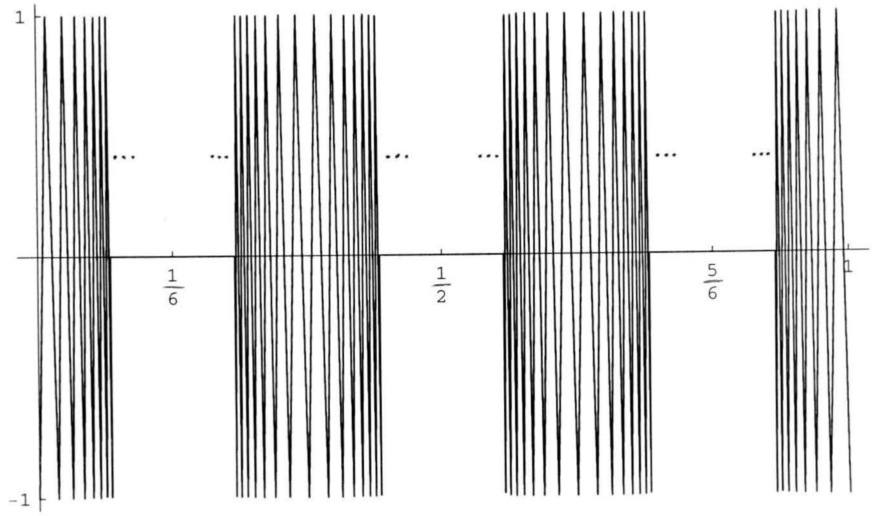
graph of g

Thus g is a piecewise linear function which is discontinuous exactly at 0. Also, g is a derivative, i.e., the function $f(x) = \int_0^x g(t)dt$ is differentiable, and $f'(x) = g(x)$ for each $x \in [0,1]$ (see, e.g., [1], p. 27). Now, for each $n \in \mathbb{N}$, define the function $g_n : [0,1] \to \mathbb{R}$ as follows: for each $k \in \{0, \cdots, 3^n - 1\}$, let $m_{n,k}$ be the midpoint of the interval $[\frac{k}{3^n}, \frac{k+1}{3^n}]$ and set

$$g_n(x) = \begin{cases} -1 & \text{if } x \in \{0, \frac{1}{3^n}, \cdots, \frac{3^n-1}{3^n}, 1\}, \\ g(|x - m_{n,k}|) & \text{if } x \in (\frac{k}{3^n}, \frac{k+1}{3^n}). \end{cases}$$



graph of g_0



graph of g_1

g_n is discontinuous exactly on the set $\{m_{n,k} \mid k = 0, \cdots, 3^n - 1\}$, and the function $f_n : [0, 1] \to \mathbb{R}$ given by

$$f_n(x) = \int_0^x g_n(t)dt$$

is differentiable, and $f_n'(x) = g_n(x)$ for all $x \in [0, 1]$. This sequence (f_n) is the example which proves Proposition 3.4.7:

It is straightforward to check that $f_n \to 0$ uniformly. To check that properties (i) and (iii) hold, we first note that for any $n \in \mathbb{N}$ and any $k \in \{0, \cdots, 3^n - 1\}$, $osc_{f_n'}(m_{n,k}) = 2$, so by Proposition 3.2.6,

$$\mathbb{D}_U(\epsilon, f_n, \bar{U}) = \{m_{n,k} \mid k = 0, \cdots, 3^n - 1\} \cap \bar{U}$$

for each $\epsilon \in Q$ and each $U \in \mathcal{U}$. In particular, for n sufficiently large, $\mathbb{D}_U(\epsilon, f_n, \bar{U})$ will be finite and nonempty. Because the $\mathbb{D}$-derivative of a finite set is empty, $\mathbb{D}^2_U(\epsilon, f_n, \bar{U}) = \emptyset$, so $\lim_{n\to\infty} \rho_{\epsilon,U}(f_n) = 1$ for each $\epsilon \in Q$ and each $U \in \mathcal{U}$; also $|f_n|_{\text{DIFF}} = 2$ for each $n \in \mathbb{N}$. Thus (iii) holds, and to verify (i), we need only check that $\lim_{n\to\infty} r_{\epsilon,U}(f_n)$ exists for each $\epsilon \in Q$ and each $U \in \mathcal{U}$.

Lemma 3.4.8. *Let (f_n) be the sequence of functions defined above, and let $n_0 \in \mathbb{N}$, $\epsilon \in Q$, $U \in \mathcal{U}$ be given. Suppose that $[a, b]$ is $\epsilon - \mathbb{D}_U(\epsilon, f_{n_0}, \bar{U})$ good for f_{n_0} in $\bar{U}$, and that $int_{\bar{U}}([a, b])$ contains a discontinuity x of f'_{n_0}. If $n \geq n_0$ is sufficiently large that $B(x, \frac{1}{2\cdot 3^n}) \cap \bar{U} \subseteq [a, b]$, then $[a, b]$ is also $\epsilon - \mathbb{D}_U(\epsilon, f_n, \bar{U})$ good for f_n in $\bar{U}$.*

Proof. Toward a contradiction, suppose that $n_1 \geq n_0$ is such that $B(x, \frac{1}{2\cdot 3^{n_1}}) \cap \bar{U} \subseteq [a, b]$, but $[a, b]$ fails to be $\epsilon - \mathbb{D}_U(\epsilon, f_{n_1}, \bar{U})$ good for f_{n_1} in $\bar{U}$. Witnessing this, let $p, q, r, s \in Q \cap int_{\bar{U}}([a, b])$, $p < q$, $r < s$, be such that

$$[p, q] \cap [r, s] \cap \mathbb{D}_U(\epsilon, f_{n_1}, \bar{U}) \neq \emptyset \text{ and } |\Delta_{f_{n_1}}(p, q) - \Delta_{f_{n_1}}(r, s)| > \epsilon.$$

Essentially, we'll show that if p, q, r, s exist as above, then there also exist $p', q', r', s' \in B(x, \frac{1}{2\cdot 3^{n_1}}) \cap \bar{U}$ which witness the failure of $[a, b]$ to be $\epsilon - \mathbb{D}_U(\epsilon, f_{n_1}, \bar{U})$ good for f_{n_1} in $\bar{U}$. Since $\mathbb{D}_U(\epsilon, f_{n_1}, \bar{U}) \cap B(x, \frac{1}{2\cdot 3^{n_1}}) = \mathbb{D}_U(\epsilon, f_{n_0}, \bar{U}) \cap B(x, \frac{1}{2\cdot 3^{n_1}})$ and $f_{n_1}\restriction_{B(x,\frac{1}{2\cdot 3^{n_1}})\cap\bar{U}} = f_{n_0}\restriction_{B(x,\frac{1}{2\cdot 3^{n_1}})\cap\bar{U}}$, this leads to a contradiction.

Because $[p, q] \cap [r, s] \cap \mathbb{D}_U(\epsilon, f_{n_1}, \bar{U}) \neq \emptyset$, there exists $k \leq 3^{n_1}$ for which $m_{n_1,k} \in [p, q] \cap [r, s]$. In particular, $r, p \leq m_{n_1}$ and $q, s \geq m_{n_1}$. Because

$$ran\Big(f_{n_1}\restriction_{[\frac{k}{3^{n_1}}, m_{n_1,k}]}\Big) = ran\Big(f_{n_1}\restriction_{[m_{n_1}, \frac{k+1}{3^{n_1}}]}\Big) = ran\,(f_{n_1}),$$

we can find points $p', r' \in [\frac{k}{3^{n_1}}, m_{n_1,k}]$ and $q', s' \in [m_{n_1}, \frac{k+1}{3^{n_1}}]$ such that

i) $f_{n_1}(p') = f_{n_1}(p)$,

ii) if $p \in [\frac{k}{3^{n_1}}, m_{n_1,k}]$, then $p' = p$

and similarly for q', r', s'. Thus $|q' - p'| \le |q - p|$ and $|s' - r'| \le |s - r|$, so

$$|\Delta_{f_{n_1}}(p', q')| \ge |\Delta_{f_{n_1}}(p, q)| \text{ and } |\Delta_{f_{n_1}}(r', s')| \ge |\Delta_{f_{n_1}}(r, s)|.$$

If $sgn\Delta_{f_{n_1}}(p, q) = -sgn\Delta_{f_{n_1}}(r, s)$, then we have

$$\begin{aligned}|\Delta_{f_{n_1}}(p, q) - \Delta_{f_{n_1}}(r, s)| =& |\Delta_{f_{n_1}}(p, q)| + |\Delta_{f_{n_1}}(r, s)| \\ \le& |\Delta_{f_{n_1}}(p', q')| + |\Delta_{f_{n_1}}(r', s')| \\ =& |\Delta_{f_{n_1}}(p', q') - \Delta_{f_{n_1}}(r', s')|,\end{aligned}$$

since then $sgn\Delta_{f_{n_1}}(p', q') = -sgn\Delta_{f_{n_1}}(r', s')$ as well. If $sgn\Delta_{f_{n_1}}(p, q) = sgn\Delta_{f_{n_1}}(r, s)$, then $\max(|\Delta_{f_{n_1}}(p, q)|, |\Delta_{f_{n_1}}(r, s)|) > \epsilon$; without loss of generality, suppose that $|\Delta_{f_{n_1}}(p, q)| > \epsilon$. Because $f'_{n_1}(m_{n_1,k}) = 0$, we can find $r'', s'' \in Q$ such that $r'' < m_{n_1,k} < s''$ and $|\Delta_{f_{n_1}}(p', q') - \Delta_{f_{n_1}}(r'', s'')| > \epsilon$. In either case, if $[a, b]$ isn't $\bar{U}$ good for f_{n_1} in $\bar{U}$, we are able to find points $p*, q*, r*, s*$ in $Q \cap B(m_{n_1,k}, \frac{1}{2\cdot 3^{n_1}}) \cap \bar{U}$ such that

$$[p*, q*] \cap [r*, s*] \cap \mathbb{D}_U(\epsilon, f_{n_1}, \bar{U}) \ne \emptyset \text{ and } |\Delta_{f_{n_1}}(p*, q*) - \Delta_{f_{n_1}}(r*, s*)| > \epsilon.$$

By the periodicity of f_{n_1}, we would therefore also be able to find points in $B(x, \frac{1}{2\cdot 3^{n_1}}) \cap \bar{U}$ which witness the failure of $[a, b]$ to be $\epsilon - \mathbb{D}_U(\epsilon, f_{n_1}, \bar{U})$ good for f_{n_1} in $\bar{U}$. But, as noted above, $f_{n_1}\restriction_{B(x, \frac{1}{2\cdot 3^{n_1}}) \cap \bar{U}} = f_{n_0}\restriction_{B(x, \frac{1}{2\cdot 3^{n_1}}) \cap \bar{U}}$ and $\mathbb{D}_U(\epsilon, f_{n_1}, \bar{U}) \cap B(x, \frac{1}{2\cdot 3^{n_1}}) = \mathbb{D}_U(\epsilon, f_{n_0}, \bar{U}) \cap B(x, \frac{1}{2\cdot 3^{n_1}}) = \{x\}$, so this would imply that $[a, b]$ fails to be $\epsilon - \mathbb{D}_U(\epsilon, f_{n_0}, \bar{U})$ good for f_{n_0} in $\bar{U}$, a contradiction.

□

From Lemma 3.4.8 it follows that for each $\epsilon \in Q$, and each $U \in \mathcal{U}$, $\lim_{n\to\infty} r_{\epsilon,U}(f_n)$ exists. Clearly the lemma shows that the sequence $(r_{\epsilon,U}(f_n))_n$ must be bounded. Moreover, we cannot have subsequences $(r_{\epsilon,U}(f_{n_k}))_k$ and $(r_{\epsilon,U}(f_{n_l}))_l$ with $\lim_{k\to\infty} r_{\epsilon,U}(f_{n_k}) <$

$\lim_{l\to\infty} r_{\epsilon,U}(f_{n_l})$, because this would also contradict Lemma 3.4.8: in the notation of the lemma, set n_0 to be any n_{k_0} such that $r_{\epsilon,U}(f_{n_{k_0}}) = \lim_{k\to\infty} r_{\epsilon,U}(f_{n_k})$. Thus (f_n) converges in the scale, i.e., property (i) holds.

Finally, we verify property (iv). For each $x \in [0,1]$, let $\sum_{i=1}^{\infty} \frac{t_i(x)}{3^i}$ denote the ternary expansion of x (so for each i, $t_i(x) \in \{0,1,2\}$), and for each $n \in \mathbb{N}^+$, let $x_n = \sum_{i=n}^{\infty} \frac{t_i(x)}{3^i}$. Then

$$f'_n(x) = g_n(x) = g(|x - m_{n,k}|) = g(|x_n - \tfrac{1}{2\cdot 3^n}|),$$

where $k \in \{0, \cdots, 3^n - 1\}$ is such that $x \in [\frac{k}{3^n}, \frac{k+1}{3^n}]$. Denote $|x_n - \frac{1}{2\cdot 3^n}|$ by x_n^*, and define, for each $N \in \mathbb{N}$,

$$A_N = \left\{x \in (0,1] \mid \exists n_1, n_2 \geq N \textit{ such that } g(x_{n_1}^*) \in [-1, -\tfrac{1}{3}) \textit{ and } g(x_{n_2}^*) \in (\tfrac{1}{3}, 1]\right\}.$$

Let $A = \bigcap_N A_N$. $A \subseteq \{x \in [0,1] \mid \lim_{n\to\infty} f'_n(x) \textit{ does not exist }\}$, and each A_N is open; if we can additionally show that each A_N is dense, then A, and hence $\{x \in [0,1] \mid \lim_{n\to\infty} f'_n(x) \textit{ does not exist }\}$, must be comeager. Toward this end, fix N and let $y \in [0,1]$, $\delta > 0$ be given; we'll show that $A_N \cap B(y,\delta) \neq \emptyset$. Let $M_1 \geq N$ be such that $\frac{1}{3^{M_1}} < \delta$, so that if $t_i(x) = t_i(y)$ for all $i \leq M_1$, then $|x - y| < \delta$. Let $k_1 \in \mathbb{N}^+$ be sufficiently large that $t_i(\frac{1}{k_1}) = 0$ for all $i \leq M_1$, and define

$$x_{M_1} = \frac{1}{k_1} + \frac{1}{2\cdot 3^{M_1+1}}.$$

Then $g(|x_{M_1} - \frac{1}{2\cdot 3^{M_1+1}}|) = g(\frac{1}{k_1}) = -1$. Now let $M_2 \geq M_1$ be such that for any $\eta \in \mathbb{R}$,

$$|\eta| < \frac{1}{3^{M_2}} \Rightarrow g(|x_{M_1} - \frac{1}{2\cdot 3^{M_1+1}}| + \eta) < -\frac{2}{3}$$

(which is possible, because g is continuous at $u = |x_{M_1} - \frac{1}{2\cdot 3^{M_1+1}}|$), and let $k_2 \in N^+$ be sufficiently large that $t_i(\frac{1}{2}(\frac{1}{k_2} + \frac{1}{k_2+1})) = 0$ for all $i \leq M_2$. Let

$$x_{M_2} = \frac{1}{2}\left(\frac{1}{k_2} + \frac{1}{k_2+1}\right) + \frac{1}{2\cdot 3^{M_2+1}}.$$

Then $g(|x_{M_2} - \frac{1}{2\cdot 3^{M_2+1}}|) = g(\frac{1}{2}(\frac{1}{k_2} + \frac{1}{k_2+1})) = 1$. If we let $x \in [0,1]$ be the point

$$\begin{aligned} x &= \sum_{i=1}^{M_1} \frac{t_i(y)}{3^i} + \sum_{i=M_1+1}^{M_2} \frac{t_i(x_{M_1})}{3^i} + x_{M_2} \\ &= \sum_{i=1}^{M_1} \frac{t_i(y)}{3^i} + \sum_{i=M_1+1}^{M_2} \frac{t_i(x_{M_1})}{3^i} + \sum_{i=M_2+1}^{\infty} \frac{t_i(x_{M_2})}{3^i}, \end{aligned}$$

then clearly $x \in B(y, \delta)$, and if $n_1 = M_1 + 1$, $n_2 = M_2 + 1$, then $g(x^*_{n_1}) \in [-1, -\frac{1}{3})$, and $g(x^*_{n_2}) = 1 \in (\frac{1}{3}, 1]$. So $x \in A_N$, as needed.

□

Remark: Motivated by an interest to strengthen property (iv) of Proposition 3.4.7 to

iv') *For co-countably many $x \in [0,1]$, $\lim_{n\to\infty} f'_n(x)$ does not exist,*

we posed the following question to Z. Buczolich:

Let $S^* \subseteq [0,1]$ be the set of all points x which satisfy the following property: for infinitely many integers n, there is some positive integer k such that

$$\left| \frac{\{3^n x\}}{3^n} - \frac{1}{2\cdot 3^n} - 1k \right| < \frac{1}{6 \cdot k^2},$$

where $\{u\}$ denotes the fractional part of u. Is S^* co-countable?

Buczolich has answered this question in the negative, showing that the Cantor set contains uncountably many points not in S^* (see [2]). Although this neither proves nor refutes the truth of (iv'), it seems to suggest that (iv') may be false.

Nevertheless, for any sequence (f_n) in DIFF, there is a general connection between convergence in the scale and pointwise convergence of the derivatives. The main fact which demonstrates this is the following:

Proposition 3.4.9. *Suppose that (f_n) is a sequence in DIFF, $f \in C([0,1])$, $f_n \to f$ pointwise, and (f_n) converges in the scale $\{\phi_{\epsilon,U} \mid \epsilon \in Q, U \in \mathcal{U}\}$. Then $f \in$ DIFF and $\lim_{n\to\infty} f'_n(x) = f'(x)$ for all $x \in \bigcap_{\epsilon\in Q} ([0,1] \setminus \lim_{n\to\infty} \mathbb{D}(\epsilon, f_n, [0,1]))$.*

Proof. By Corollary 3.4.1, $f \in$ DIFF. We show now that, for each $\epsilon \in Q$, if $x \notin \lim_{n\to\infty} \mathbb{D}(\epsilon, f_n, [0,1])$, then $\limsup_{n\to\infty} |f_n'(x) - f'(x)| \leq 2\epsilon$.

Fix $\epsilon \in Q$ and suppose $x \notin \lim_{n\to\infty} \mathbb{D}(\epsilon, f_n, [0,1])$; let $V \in \mathcal{U}$ be such that $x \in V$ and $\bar{V} \cap \mathbb{D}(\epsilon, f_n, [0,1]) = \emptyset$ for n sufficiently large. Then by Lemma 3.3.14, $\mathbb{D}_V(\epsilon, f_n, \bar{V}) = \emptyset$ for all such n, so $\lim_{n\to\infty} \phi_{\epsilon,V}(f_n) =< 0, \mathcal{B}(l,m) >$ for some $l, m \in N$. In particular, for each n sufficiently large, there exist a closed interval $[a_n, b_n] \subseteq \bar{V}$ such that $[a_n, b_n]$ is $\epsilon - \bar{V}$ good for f_n in $\bar{V}$ and $B(x, \frac{1}{m}) \cap \bar{V} \subseteq [a_n, b_n]$. If we let $[a,b] = \bigcap_n [a_n, b_n]$, then $[a,b]$ contains x in its $\bar{V}$-interior and it is uniformly $\epsilon - \bar{V}$ good for each f_n in $\bar{V}$, n sufficiently large (say $n \geq N_1$).

Fix $p, q \in int_{\bar{V}}([a,b])$ such that $p \leq x \leq q$ and $|\Delta_f(p,q) - f'(x)| < \frac{\epsilon}{2}$, and for each $n \geq N_1$, fix $p_n, q_n \in int_{\bar{V}}([a,b])$, such that $p_n \leq x \leq q_n$ and $|\Delta_{f_n}(p,q) - f_n'(x)| < \frac{\epsilon}{2}$. Then for $n \geq N_1$, we have

$$\begin{aligned} |f_n'(x) - f'(x)| <& |\Delta_{f_n}(p_n, q_n) - \Delta_f(p,q)| + \epsilon \\ \leq& |\Delta_{f_n}(p_n, q_n) - \Delta_{f_n}(p,q)| + |\Delta_{f_n}(p,q) - \Delta_f(p,q)| + \epsilon. \end{aligned}$$

Because $[a,b]$ is $\epsilon - \bar{V}$ good for f_n in $\bar{V}$ and $[p_n, q_n] \cap [p,q] \cap \bar{V} \neq \emptyset$, $|\Delta_{f_n}(p_n, q_n) - \Delta_{f_n}(p,q)| \leq \epsilon$; because $f_n(p) \to f(p)$ and $f_n(q) \to f(q)$, $|\Delta_{f_n}(p,q) - \Delta_f(p,q)| \to 0$. Hence it follows that

$$\limsup_{n\to\infty} |f_n'(x) - f'(x)| \leq 2\epsilon$$

for each $x \in [0,1] \backslash \lim_{n\to\infty} \mathbb{D}(\epsilon, f_n, [0,1])$, and so $\lim_{n\to\infty} f_n'(x) = f'(x)$ for each $x \in \bigcap_{\epsilon\in Q} \left([0,1] \backslash \lim_{n\to\infty} \mathbb{D}(\epsilon, f_n, [0,1])\right)$.

□

Remark. If (f_n) is the sequence discussed in Proposition 3.4.7, then we have, for each $\epsilon \in Q$,

$$\begin{aligned} \lim_{n\to\infty} \mathbb{D}(\epsilon, f_n, [0,1]) =& \lim_{n\to\infty} \left\{ \tfrac{2k+1}{2\cdot 3^n} \mid k = 0, \cdots 3^n - 1 \right\} \\ =& [0,1]. \end{aligned}$$

Thus this example shows that, at least in the sense of category, Proposition 3.4.9 is best possible.

We next show that, under certain conditions on functions f_n and f as in the hypothesis of Proposition 3.4.9, for each $\epsilon \in Q$ the set $\lim_{n\to\infty} \mathbb{D}(\epsilon, f_n, [0,1])$ is nowhere dense, and hence $f_n'(x) \to f'(x)\ \forall^* x \in [0,1]$. We begin by stating a useful fact whose proof was given in the proof of Theorem 3.3.13:

Proposition 3.4.10. *Suppose that (f_n) is a sequence in DIFF which converges in the scale and converges pointwise to a function $f \in C([0,1])$. Let $\epsilon \in Q$, $U \in \mathcal{U}$ be given, let $\alpha_{\epsilon,U} = \lim_{n\to\infty} \rho_{\epsilon,U}(f_n)$, and let $\mathcal{B}(l,m) = \lim_{n\to\infty} r_{\epsilon,U}(f_n)$. Then there exists a subsequence (f_{n_k}) and a corresponding sequence (C_k) such that*

i) $\forall k$, $C_k = \{[a_0^k, b_0^k], \cdots, [a_{l-1}^k, b_{l-1}^k]\}$ *is an* $\epsilon - m$ *cover of* $\mathbb{D}_U^{\alpha_{\epsilon,U}}(\epsilon, f_{n_k}, \bar{U})$ *for* f_{n_k} *in* $\bar{U}$ *consisting of* l *intervals.*

ii) $\exists a_0, b_0, \cdots, a_{l-1}, b_{l-1} \in [0,1]$ *such that for each* $i \le l-1$,

$$\lim_{k\to\infty} a_i^k = a_i \text{ and } \lim_{k\to\infty} b_i^k = b_i.$$

iii) *By possibly shrinking or omitting certain of the intervals* $[a_i, b_i]$, *we obtain an* $\epsilon - m$ *cover* $[a_0', b_0'], \cdots, [a_{l'-1}', b_{l'-1}']$ *of* $\lim_{n\to\infty} \mathbb{D}_U^{\alpha_{\epsilon,U}}(\epsilon, f_n, \bar{U})$ *for* f *in* $\bar{U}$.

In particular,

$$\mathbb{D}_U\left(\epsilon, f, \lim_{n\to\infty} \mathbb{D}_U^{\alpha_{\epsilon,U}}(\epsilon, f_n, \bar{U})\right) = \emptyset.$$

Definition 3.4.11. *$f \in$ DIFF is* everywhere ρ-rank α *if:*

i) $\forall \epsilon \in Q$, $U \in \mathcal{U}$ $\rho_{\epsilon,U}(f) \le \alpha$

ii) $\forall U \in \mathcal{U}\ \exists \epsilon \in Q$ *such that* $\rho_{\epsilon,U}(f) = \alpha$ *(and hence, by monotonicity,* $\rho_{\epsilon',U}(f) = \alpha$ $\forall \epsilon' \le \epsilon$*).*

Theorem 3.4.12. *Let (f_n) be a sequence in DIFF which converges in the scale and converges pointwise to the function $f \in C([0,1])$. If there exists $k < \omega$ such that f and each f_n are everywhere rank k, then $\forall^* x f_n'(x) \to f'(x)$.*

Lemma 3.4.13. *Let (f_n) be a sequence in DIFF which converges in the scale and converges pointwise to the function $f \in C([0,1])$, and suppose that f and each f_n are* *everywhere rank k, for some $k < \omega$. Then for each $U \in \mathcal{U}$, and each $\epsilon \in Q$ for which $\rho_{\epsilon,U}(f) = k$,*

$$\mathbb{D}_U^k(\epsilon, f, \lim_{n\to\infty} \mathbb{D}_U(\epsilon, f_n, \bar{U})) = \emptyset.$$

Hence $\lim_{n\to\infty} \mathbb{D}_U(\epsilon, f_n, \bar{U}) \neq \bar{U}$.

Proof. Let $U \in \mathcal{U}$ be given, and fix $\epsilon \in Q$ for which $\rho_{\epsilon,U}(f) = k$, as in the hypothesis of the lemma. If $k = 1$, then $\lim_{n\to\infty} \rho_{\epsilon,U}(f_n) = 1$ also, and so

$$\mathbb{D}_U(\epsilon, f, \lim_{n\to\infty} \mathbb{D}_U(\epsilon, f_n, \bar{U})) = \emptyset$$

by Proposition 3.4.10.

For the case $k > 1$, we use the following fact:

Lemma 3.4.14. *Let (f_n) be a sequence in DIFF which converges in the scale and converges pointwise to the function $f \in C([0,1])$. For each $\epsilon \in Q$, $U \in \mathcal{U}$, let $\alpha_{\epsilon,U} = \lim_{n\to\infty} \rho_{\epsilon,U}(f_n)$; then for any $\beta \leq \alpha_{\epsilon,U}$, we have*

$$\mathbb{D}_U(\epsilon, f, \lim_{n\to\infty} \mathbb{D}^\beta(\epsilon, f, \bar{U})) \subseteq \lim_{n\to\infty} \mathbb{D}_U^{\beta+1}(\epsilon, f, \bar{U}).$$

Proof: If $\beta = \alpha_{\epsilon,U}$, then by Proposition 3.4.10, $\mathbb{D}_U(\epsilon, f, \lim_{n\to\infty} \mathbb{D}_U^\beta(\epsilon, f_n, \bar{U})) = \emptyset$, so the assertion holds. Now suppose that $\beta < \alpha_{\epsilon,U}$; we first note that each of the sets $\mathbb{D}_U(\epsilon, f, \lim_{n\to\infty} \mathbb{D}^\beta(\epsilon, f, \bar{U}))$ and $\lim_{n\to\infty} \mathbb{D}_U^{\beta+1}(\epsilon, f, \bar{U})$ are contained in $\lim_{n\to\infty} \mathbb{D}_U^\beta(\epsilon, f_n, \bar{U})$. So let $x \in \lim_{n\to\infty} \mathbb{D}_U^\beta(\epsilon, f_n, \bar{U})$ be given, and suppose that $x \notin \lim_{n\to\infty} \mathbb{D}_U^{\beta+1}(\epsilon, f, \bar{U})$; we'll show that $x \notin \mathbb{D}_U(\epsilon, f, \lim_{n\to\infty} \mathbb{D}^\beta(\epsilon, f, \bar{U}))$. Since $x \notin \lim_{n\to\infty} \mathbb{D}_U^{\beta+1}(\epsilon, f_n, \bar{U})$, there exists $V \in \mathcal{U}$, $V \subseteq U$, such that for all n sufficiently large, $\bar{V} \cap \mathbb{D}_U^{\beta+1}(\epsilon, f_n, \bar{U}) = \emptyset$. By Lemma 3.3.14, this implies that $\lim_{n\to\infty} \rho_{\epsilon,V}(f_n) = \beta$; but then by Proposition 3.4.10,

$$\mathbb{D}_V(\epsilon, f, \lim_{n\to\infty} \mathbb{D}_V^\beta(\epsilon, f_n, \bar{V})) = \emptyset,$$

and this in turn implies that

$$V \cap \mathbb{D}_U(\epsilon, f, \lim_{n\to\infty} \mathbb{D}_U^{\beta}(\epsilon, f_n, \bar{U})) = \emptyset,$$

because, for all $W \in \mathcal{U}$, $\bar{W} \subseteq V$, we have, by Lemma 3.3.14

$$\begin{aligned} W \cap \mathbb{D}_U(\epsilon, f, \lim_{n\to\infty} \mathbb{D}_U^{\beta}(\epsilon, f_n, \bar{U})) &\subseteq \mathbb{D}_W(\epsilon, f, \bar{W} \cap \lim_{n\to\infty} \mathbb{D}_U^{\beta}(\epsilon, f_n, \bar{U})) \\ &\subseteq \mathbb{D}_V(\epsilon, f, \bar{W} \cap \lim_{n\to\infty} \mathbb{D}_U^{\beta}(\epsilon, f_n, \bar{U})) \\ &\subseteq \mathbb{D}_V(\epsilon, f, \lim_{n\to\infty} \mathbb{D}_V^{\beta}(\epsilon, f_n, \bar{V})) = \emptyset. \end{aligned}$$

(This last inequality follows because, for any $W' \in \mathcal{U}$, $\bar{W} \subseteq W'$, $\bar{W'} \subseteq V$, we have

$$\begin{aligned} \bar{W} \cap \lim_{n\to\infty} \mathbb{D}_U^{\beta}(\epsilon, f_n, \bar{U})) &\subseteq W' \cap \lim_{n\to\infty} \mathbb{D}_U^{\beta}(\epsilon, f_n, \bar{U})) \\ &\subseteq \lim_{n\to\infty} (\bar{W'} \cap \mathbb{D}_U^{\beta}(\epsilon, f_n, \bar{U})) \\ &\subseteq \lim_{n\to\infty} \mathbb{D}_{W'}(\epsilon, f_n \bar{W'}) \\ &\subseteq \lim_{n\to\infty} \mathbb{D}_V(\epsilon, f_n \bar{V}).) \end{aligned}$$

□

Returning to the proof of Lemma 3.4.13, let $\epsilon \in Q$ be such that $\rho_{\epsilon,U}(f) = k$. By Proposition 3.4.10 we have

$$\mathbb{D}_U(\epsilon, f, \lim_{n\to\infty} \mathbb{D}_U^{k}(\epsilon, f_n, \bar{U})) = \emptyset,$$

and by Lemma 3.4.14 we have

$$\mathbb{D}_U^{k-l+1}(\epsilon, f, \lim_{n\to\infty} \mathbb{D}_U^{l}(\epsilon, f_n, \bar{U})) \subseteq \mathbb{D}_U^{k-l}(\epsilon, f, \lim_{n\to\infty} \mathbb{D}_U^{l+1}(\epsilon, f_n, \bar{U}))$$

for each of $l = 1, \cdots, k$. Hence repeated substitution yields

$$\mathbb{D}_U^{k}(\epsilon, f, \lim_{n\to\infty} \mathbb{D}_U(\epsilon, f_n, \bar{U}) = \emptyset,$$

while, because ϵ was chosen such that $\rho_{\epsilon,U}(f) = k$, $\mathbb{D}^k_{\bar{U}}(\epsilon, f, \bar{U}) \neq \emptyset$. Thus

$$\lim_{n\to\infty} \mathbb{D}_U(\epsilon, f_n, \bar{U}) \neq \bar{U}.$$

□

Lemma 3.4.15. *Let (f_n) be a sequence in DIFF which converges in the scale $\{\phi_{\epsilon,U} | \epsilon \in Q, U \in \mathcal{U}\}$, and let $\epsilon \in Q$, $U \in \mathcal{U}$, $K \in K(\bar{U})$, $\alpha \in Ord$ be given. If $V \in \mathcal{U}$ is such that $V \subseteq \lim_{n\to\infty} \mathbb{D}^\alpha_{\bar{U}}(\epsilon, f_n, K)$, then $\lim_{n\to\infty} \mathbb{D}^\alpha_{\bar{V}}(\epsilon, f_n, \bar{V}) = \bar{V}$.*

Proof. Using the fact that $\lim_{n\to\infty} \mathbb{D}^\alpha_{\bar{U}}(\epsilon, f_n, K) = \text{T}\limsup_{n\to\infty} \mathbb{D}^\alpha_{\bar{U}}(\epsilon, f_n, K)$ (see [3], § 4.F) we have

$$\begin{aligned}
&V \subseteq \lim_{n\to\infty} \mathbb{D}^\alpha_{\bar{U}}(\epsilon, f_n, K) \Leftrightarrow \\
&\forall x \in V \exists (x_n) \text{ such that } x_n \to x \text{ and } \forall n\, x_n \in \mathbb{D}^\alpha_{\bar{U}}(\epsilon, f_n, K) \Rightarrow \\
&\forall x \in V \exists (x_n) \text{ such that } x_n \to x \text{ and } \forall^\infty n\, x_n \in \mathbb{D}^\alpha_{\bar{U}}(\epsilon, f_n, K) \cap V \Rightarrow \\
&\forall x \in V \exists (x_n) \text{ such that } x_n \to x \text{ and } \forall^\infty n\, x_n \in \mathbb{D}^\alpha_{\bar{V}}(\epsilon, f_n, K \cap \bar{V}) \Rightarrow \\
&V \subseteq \lim_{n\to\infty} \mathbb{D}^\alpha_{\bar{V}}(\epsilon, f_n, K \cap \bar{V}) \subseteq \lim_{n\to\infty} \mathbb{D}^\alpha_{\bar{V}}(\epsilon, f_n, \bar{V}).
\end{aligned}$$

□

Proof of Theorem 3.4.12. We claim that for each $\epsilon \in Q$, the set $\lim_{n\to\infty} \mathbb{D}(\epsilon, f_n, [0,1])$ is nowhere dense. Toward a contradiction, suppose otherwise: let $\epsilon_0 \in Q$ and $U_0 \in \mathcal{U}$ be such that $U_0 \subseteq \lim_{n\to\infty} \mathbb{D}(\epsilon_0, f_n, [0,1])$; by monotonicity, $U_0 \subseteq \lim_{n\to\infty} \mathbb{D}(\epsilon, f_n, [0,1])$ for each $\epsilon \leq \epsilon_0$ as well, so in particular, $U_0 \subseteq \lim_{n\to\infty} \mathbb{D}(\epsilon, f_n, [0,1])$ for each ϵ such that $\rho_{\epsilon,U}(f) = k$. But then by Lemma 3.4.15, $\lim_{n\to\infty} \mathbb{D}_{U_0}(\epsilon, f_n, \bar{U}_0) = \bar{U}_0$, which contradicts Lemma 3.4.13.

□

We conclude this section with a couple of related results which are specfic to the case $|f_n| \leq 2$ for each n:

Theorem 3.4.16. *Suppose that (f_n) is a sequence in DIFF, $f \in C([0,1])$, $f_n \to f$ pointwise, and (f_n) converges in the scale $\{\phi_{\epsilon,U} \mid \epsilon \in Q, U \in \mathcal{U}\}$. Suppose also that*

$|f_n|_{DIFF} \leq 2$ for all n. Then there exists an open set O such that

i) *$\forall^* x \in int([0,1]\backslash O)$, $\lim_{n\to\infty} f_n'(x) = f'(x)$,*

ii) *$f'\restriction_O$ is continuous.*

Proof. Define

$$O = \bigcup_{\epsilon \in Q} int\left(\lim_{n\to\infty} \mathbb{D}(\epsilon, f_n, [0,1])\right)$$
$$= \bigcup_{m=1}^{\infty} int\left(\lim_{n\to\infty} \mathbb{D}(\tfrac{1}{m}, f_n, [0,1])\right).$$

By hypothesis, for each n, $|f_n|_{DIFF} \leq 2$, so for every $\epsilon \in Q$, $U \in \mathcal{U}$, $\rho_{\epsilon,U}(f_n) \leq 1$. If $\lim_{n\to\infty} \rho_{\epsilon,[0,1]}(f_n) = 0$ for each ϵ, then $O = \emptyset$ and the assertion holds by Proposition 3.4.2 (in fact, f' is continuous on $[0,1]$ and $f_n' \to f'$ uniformly on $[0,1]$). Hence we consider the case in which $\lim_{n\to\infty} \rho_{\epsilon,[0,1]}(f_n) = 1$ for all ϵ sufficiently small. Because $\bigcap_{m=1}^{\infty} \lim_{n\to\infty} \mathbb{D}(\frac{1}{m}, f_n, [0,1])$ is dense in $int([0,1]\backslash O)$, (i) holds by Lemma 3.4.9. It remains to verify (ii). By Proposition 3.4.10 and Lemma 3.3.14,

$$\mathbb{D}(\epsilon, f, [0,1]) \cap int\left(\lim_{n\to\infty} \mathbb{D}(\epsilon, f_n, [0,1])\right) = \emptyset.$$

Hence by Proposition 3.2.6 and the fact that $\bigcup_{m=1}^{\infty} \lim_{n\to\infty} \mathbb{D}(\frac{1}{m}, f_n, [0,1])$ is an increasing union,

$$x \in \bigcup_{m=1}^{\infty} int\left(\lim_{n\to\infty} \mathbb{D}(\tfrac{1}{m}, f_n, [0,1])\right) \Rightarrow \forall m \geq 1, x \notin \mathbb{D}(\tfrac{1}{m}, f, [0,1])$$
$$\Rightarrow osc_{f'}(x) = 0,$$

i.e., $f'\restriction_O$ is continuous.

□

Corollary 3.4.17. *Let (f_n) and f be as described in the hypotheses of Theorem 3.4.16. If, additionally, the set $\{x \in [0,1] \mid f' \text{ is discontinuous at } x\}$ is dense in $[0,1]$, then $\forall^* x \in [0,1], \lim_{n\to\infty} f_n'(x) = f'(x)$.*

Question: Is there an analogue to Corollary 3.4.17 for a sequence (f_n) in DIFF whose elements have $|\cdot|_{\mathrm{DIFF}}$-rank greater than 2? That is, if (f_n) is a sequence which converges in the scale and $\lim_{n\to\infty} |f_n|_{\mathrm{DIFF}} = \alpha > 2$, is there an analytical condition C_α which would guarantee that $\bigcup_{\epsilon\in Q} \lim_{n\to\infty} \mathbb{D}(\epsilon, f_n, [0,1])$ were nowhere dense, and hence that $\forall^* x, \lim_{n\to\infty} f_n'(x)$ exists?

Question: Can Theorem 3.4.12 be generalized to the case in which each f_n and f are everywhere rank α, for any given $\alpha < \omega_1$?

Bibliography

[1] A.M. Bruckner, *Differentiation of real functions*, second edition, American Mathematical Society (1994).

[2] Z. Buczolich, *A question of J. Pavelich related to Diophantine approximation*, preprint (2001).

[3] A.S. Kechris, *Classical descriptive set theory*, Springer-Verlag (1995).

[4] A.S. Kechris, H.W. Woodin, *Ranks of differentiable functions*, Mathematika **33** (1986), 252-278.