MAPPING PROPERTIES OF CERTAIN AVERAGING OPERATORS

Thesis by

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To the memory of Thomas H. Wolff

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Abstract

In this thesis, we investigate the mapping properties of two averaging operators.

In the first part, we consider a model rigid well-curved line complex G_d in $\mathbb{R}^d$. The X-ray transform, X, restricted to G_d is defined as an operator from functions on $\mathbb{R}^d$ to functions on G_d in the following way:

$$Xf(l) = \int_l f, \qquad l \in G_d.$$

We obtain sharp mixed norm estimates for X in $\mathbb{R}^4$ and $\mathbb{R}^5$.

In the second part, we consider the elliptic maximal function M. Let $\mathcal{E}$ be the set of all ellipses in $\mathbb{R}^2$ centered at the origin with axial lengths in $[1/2, 2]$. Let $f : \mathbb{R}^2 \to \mathbb{R}$, then $Mf : \mathbb{R}^2 \to \mathbb{R}$ is defined in the following way:

$$Mf(x) = \sup_{E \in \mathcal{E}} \frac{1}{|E|} \int_E f(x+s)\,d\sigma(s),$$

where $d\sigma$ is the arclength measure on E and $|E|$ is the length of E.

In this part of the thesis, we investigate the L^p mapping properties of M.

Contents

Chapter 1 Introduction

This thesis consists of two distinct results [10], [11] about the mapping properties of two averaging operators.

I) **Restricted X-ray transform**

The full X-ray transform, X_{full}, is an operator from the functions on $\mathbb{R}^d$ to the functions on $\mathcal{G}_d$, the space of all lines in $\mathbb{R}^d$. It is defined as

$$X_{full}f(l) = \int_l f, \quad l \in \mathcal{G}_d.$$

Since $\mathcal{G}_d$ is a $(2d-2)$-dimensional space, X_{full} is over-determined, and it is of interest to investigate the restriction of X_{full} to lower dimensional subsets of $\mathcal{G}_d$.

We are interested in the subsets that are called rigid well-curved line complexes (see [17] for a definition as a member of a general family of line complexes, some properties and applications). We work with the model line complex of this type:

Let γ_d be the curve $\{\gamma_d(t) : \gamma_d(t) = (1, t, t^2, ..., t^{d-1}), t \in (-1, 1)\}$ in $\mathbb{R}^d$. Let $l(t, x)$ denote the line $\{x + s\gamma_d(t) : s \in \mathbb{R}\}$. The model rigid well-curved line complex, G_d, is defined via $G_d = \{l(t, x) : t \in [-1, 1], x \perp \gamma_d(t)\}$. The term well-curved refers to the fact that for all t the first $d-1$ derivatives of $\gamma_d(t)$

are linearly independent, and rigid refers to the fact that for any direction e in the direction set γ_d, all the lines with direction e are in the line complex [16].

We define the restricted X-ray transform, X, as the restriction of X_{full} to G_d.

In this part of the thesis, we obtain almost sharp mixed norm estimates for X in $\mathbb{R}^4$ and $\mathbb{R}^5$.

II) Elliptic Maximal Function

In this part of the thesis, we consider a natural generalization of the circular maximal function by taking maximal averages over ellipses instead of circles.

More explicitly, let $\mathcal{E}$ be the set of all ellipses in $\mathbb{R}^2$ centered at the origin with axial lengths in $[\frac{1}{2}, 2]$. Note that we do not restrict ourselves to the ellipses whose axes are parallel to the co-ordinate axes. The *elliptic maximal function*, M_E^δ, is defined in the following way: Let $f : \mathbb{R}^2 \to \mathbb{R}$, then

$$M_E^\delta f(x) = \sup_{E \in \mathcal{E}} \frac{1}{|E^\delta|} \int_{x+E^\delta} f(u)du, \tag{1.1}$$

where E^δ is the δ-neighborhood of E and $|E^\delta|$ is the measure of E^δ.

In Chapter 3, we obtain some estimates about the asymptotic behavior of the best constant $A_{p,q}(\delta)$ in the inequalities

$$||M_E^\delta f||_q \leq A_{p,q}(\delta) ||f||_p.$$

Notation

$|A|$: Cardinality or the measure of the set A or the length of the vector A.

χ_A: Characteristic function of the set A.

$\mathcal{N}(A, \eta)$: η neighborhood of the set A.

C, K: Constants that may vary from line to line.

$A \lesssim B$: $A \leq CB$.

$A \approx B$: $A \lesssim B$ and $B \lesssim A$.

$A << B$: $A \leq C^{-1}B$ where C is a large enough constant.

Chapter 2 Restricted X-ray Transform

2.1 Overview and General Discussion

Let $\mathcal{G}_{k,d}$ be the space of all k-planes in $\mathbb{R}^d$. The Radon transform or the k-plane transform $\mathcal{R}_{k,d}$ is defined as an operator from the functions defined on $\mathbb{R}^d$ to the functions defined on $\mathcal{G}_{k,d}$ via

$$\mathcal{R}_{k,d}f(p) = \int_p f, \quad p \in \mathcal{G}_{k,d}.$$

The Radon transform found important applications in integral geometry and in the study of PDE's.

$\mathcal{R}_{1,d}$ is often called the X-ray transform due to its applications in radiology; we denote it by X_{full}. It is well-known [33], [19] that the sharp mixed norm estimates for the full X-ray transform implies the Kakeya conjecture and it is related to some of the main problems in the summability of Fourier transform, Fourier restriction and more generally to oscillatory integrals, non-linear P.D.E.'s and number theory [13], [2], [3], [34], [4], [28]. For some mapping properties of X_{full}, see, e.g., [9], [5], [33] and [19].

Note that $\mathcal{G}_{1,d}$ is a $(2d-2)$-dimensional manifold, thus X_{full} is overdetermined for $d \geq 3$, and it is of interest to consider its restrictions to lower dimensional subspaces of $\mathcal{G}_{1,d}$. For the definition of the restricted X-ray transforms as part of a more general class of transformations and some of its properties, see [16].

One particular example is the restriction of X_{full} to the space of light rays (lines in $\mathbb{R}^d$ making a 45 degree angle with the plane $x_d = 0$). Recently, Wolff [35] obtained mixed norm estimates for this operator (almost sharp in $\mathbb{R}^3$) and used this information to prove almost sharp bilinear cone restriction estimates in all dimensions.

We are interested in the restriction of X_{full} to d dimensional line complexes in $\mathbb{R}^d$. Let $d \geq 3$; the subspace G_d of $\mathcal{G}_{1,d}$ we are interested in is defined as follows: Let γ_d be the curve $\{\gamma_d(t) : \gamma_d(t) = (1, t, t^2, ..., t^{d-1}), t \in [-1, 1]\}$ in $\mathbb{R}^d$. Let $l(t, x)$ denote the line $\{x + s\gamma_d(t) : s \in \mathbb{R}\}$, where $x \in H_t := \{x : x \perp \gamma_d(t)\}$. We identify G_d with $[-1, 1] \times \mathbb{R}^{d-1}$ via $G_d = \{l(t, x) : t \in [-1, 1], x \in H_t\}$. This line complex is a model case for a general class called *rigid well-curved line complexes* (see, e.g., [15], [17] and [16]). It is called well-curved since $\gamma_d'(t), ..., \gamma_d^{(d-1)}(t)$ are linearly independent for any $t \in [-1, 1]$, and the term rigid is used to describe the fact that for any point $\gamma_d(t)$ in the "direction set" γ_d, G_d contains all the lines in $\mathbb{R}^d$ having the direction $\gamma_d(t)$. We call the lines in G^d the γ_d-rays.

Now, we define the restricted X-ray transform as an operator from the functions defined on $\mathbb{R}^d$ to the functions defined on G_d in the following way:

$$Xf(l(t,x)) = \int_{l(t,x)} f, \quad t \in [-1,1], x \in H_t.$$

We work with the following mixed norms for the functions defined on G_d:

$$\|f\|_{L^q(L^r)} = \|f\|_{q,r} = (\int_{-1}^{1} (\int_{H_t} |f(l(t,x))|^r dx)^{q/r} dt)^{1/q}.$$

We are interested in the estimates of the following type: If $f : \mathbb{R}^d \to \mathbb{R}$ is supported in the unit cube Q_1, then

$$\|Xf\|_{q,r} \leq C_{pqr} \|f\|_p. \tag{2.1}$$

Proposition 2.1.1. The following conditions for p, q and r are necessary for (2.1) to hold

$$\tfrac{d}{p} \leq \tfrac{d-1}{r} + 1, \tag{2.2}$$

$$\tfrac{(d-1)d}{p} \leq \tfrac{2}{q} + \tfrac{(d-1)d}{r}, \tag{2.3}$$

$$\tfrac{(d-2)(d+1)}{p} \leq \tfrac{(d-1)d}{r}. \tag{2.4}$$

Proof. The following counter-examples prove Proposition 2.1.1; they are quite standard (see, e.g., [5], [15], [17] and [16]). The restriction (2.2) can be obtained by applying X to the characteristic function of a δ-ball. To obtain (2.3), let f be the characteristic function of the set $|x_1| \leq 1, |x_2| \leq \delta, ..., |x_d| \leq \delta^{d-1}$.

Note that $\|f\|_p \approx \delta^{d(d-1)/(2p)}$ and for all $|t| < \delta$, we have $Xf \approx 1$ on a subset of H_t of measure $\gtrsim \delta^{d(d-1)/2}$. Hence $\|Xf\|_{q,r} \gtrsim \delta^{1/q}\delta^{d(d-1)/(2r)}$, which proves the necessity of (2.3). Finally, divide γ_d into M ($\approx 1/\delta$) segments $s_1, ..., s_M$ of length δ centered at $t_1, ..., t_M$, respectively. For any segment s_i, consider the parallelogram $P_i \subset \mathbb{R}^{d-1}$ with dimensions $\delta \times \delta^2 \times ... \times \delta^{d-1}$, whose longest axis is tangent to γ_d at $\gamma_d(t_i)$ and whose other axes are in the directions $\gamma_d''(t_i), ..., \gamma_d^{(d-1)}(t_i)$, respectively. Let f be the characteristic function of the set $\{(x_1, x_2, ..., x_d) \in \mathbb{R}^d : x_1 \in (1,2), (x_2/x_1, ..., x_d/x_1) \in \cup_i^M P_i\}$. Note that $\|f\|_p \approx \delta^{(d^2-d-2)/(2p)}$ and for all t, $Xf \approx 1$ on a subset of H_t of measure $\gtrsim \delta^{d(d-1)/2}$. Hence $\|Xf\|_{q,r} \gtrsim \delta^{d(d-1)/(2r)}$, which proves the necessity of (2.4). □

In light of Proposition 2.1.1, one may conjecture that

Conjecture. [1] If p, q and r satisfy the inequalities (2.2), (2.3) and (2.4), then (2.1) holds.

We have the following theorem that contains one of the main results of this thesis.

Theorem 2.1.2. *The conjecture is true in $\mathbb{R}^d$ for $d = 3$, 4 or 5 except the end-point issues. More explicitly, if p, q and r satisfy (2.2), (2.3) and (2.4) with inequalities replaced with strict inequalities, then (2.1) holds in $\mathbb{R}^d$ for $d = 3$, 4 or 5.*

The case $d = 3$ follows from Wolff's above-mentioned mixed norm estimates

[1] Recently, the conjecture is settled in all dimensions by Michael Christ and the author [7].

for the X-ray transform restricted to light rays [35], since in $\mathbb{R}^3$ the space of light rays is a rigid well-curved line complex.

If one considers the case $q = r$ only, the conjecture had been settled for $d = 3$ in [14] and [31], and for the case $q = r$ and $d = 4$, it had been verified except the endpoint issues in [17]. In higher dimensions, the conjecture was verified for $p = d/(d-1)$ and $q = r = (d-1)/(d-2)$ in [23] and for $q = r = 2$ and $p = (2d^2 - 2d)/(d^2 - d + 2)$ in [15]. Note that the results mentioned here are valid for all rigid well-curved line complexes whereas Theorem 2.1.2 is valid only in the model case.

In the following remark, we discuss simple estimates for X:

Remark. i) Note that (2.1) holds for all q and r if $p = \infty$, since we are interested in local estimates.

ii) Fubini's theorem implies that (2.1) holds for $p = q = r = 1$.

iii) X is bounded from $W^{2,-\frac{1}{2d-2}}$ to L^2 (see, e.g., [17] and [15]). Here $W^{p,\alpha}(Q_1)$ is the Sobolev space consisting of all functions f supported in Q_1 such that $\|(1-\triangle)^{\alpha/2} f\|_p < \infty$.

Proof of (iii). We prove this using the method of stationary phase. We modify the definition of X in the following way:

$$Xf(l(t,x)) = h(t) \int_l f, \qquad t \in \mathbb{R}, \quad x \in H_t,$$

here h is a smooth cut off function. A straightforward calculation shows that

X^*, the adjoint of X, is the transform

$$X^*g(x) = \int_{-\infty}^{\infty} g(l(t,x))h(t)dt, \quad x \in \mathbb{R}^d,$$

where g is a function on G^d.

Using the T^*T method, it suffices to prove that $X^*X : L^2 \to W^{2,\frac{1}{d-1}}$.

We have

$$\begin{aligned} X^*Xf(x) &= \int f(x+s\gamma(t))h^2(t)dt\,ds, \\ &= \int \hat{f}(\xi)m(\xi)e^{2\pi i x.\xi}d\xi, \end{aligned}$$

where

$$m(\xi) = \int e^{2\pi i s\xi.\gamma(t)}h^2(t)dt\,ds.$$

Thus, it suffices to prove that $|m(\xi)| \lesssim |\xi|^{-1/(d-1)}$. This follows from the stationary phase estimate (see, e.g., [27] p. 342)

$$\left|\int e^{2\pi i s\xi.\gamma(t)}h^2(t)dt\right| \lesssim |s\xi|^{-1/(d-1)}. \qquad \square$$

In light of these remarks, Theorem 2.1.2. (the cases $d=4$ and $d=5$) can be obtained from the following theorem by interpolation.

Theorem 2.1.3. *Let $d = 4$ or 5. Let $p = q = (d+2)/d$ and $r = (d^2+d-2)/(d^2-d-2)$. Then, the restricted X-ray transform X is bounded from the Sobolev space $W^{p,\varepsilon}(Q_1)$ to $L^q(L^r)$ for any $\varepsilon > 0$, where Q_1 is the unit cube in*

$\mathbb{R}^d$.

In [35], Wolff used the "bush" construction. It was introduced by Bourgain in [2] and used by several other authors (see, e.g., [29]). A bush is a family of tubes passing through a common point. The basic observation there was the following; in the case of light rays the intersection of a bush with a tube passing through a point far from the bush is at most a small ball.

As in [35], in the proof of Theorem 2.1.3, we use the bush construction. The basic property of the bushes in our case is the following transversality property: Let $d \geq 4$. If the basepoint of one bush is far from another bush, then their intersection is at most a finite union of small balls. This is consequence of well-curvedness. This property yields the proof in $\mathbb{R}^4$.

However, for $d = 5$, this property by itself is not enough. The reason for this is that in $\mathbb{R}^5$ two generic bushes do not intersect at all. We overcome this difficulty by collecting the bushes into groups that we denote by bushfields. A bushfield is a set of tubes intersecting a given tube that we call the basetube. In some aspects, this object is similar to that used in [32], that came to be recognized as the "hairbrush" (see, e.g., [19]). The main difference is that a bushfield behaves like a disjoint union of bushes. This is because of the following basic properties:

i) The tubes in a bushfield are disjoint away from the base tube.

ii) If the basepoint of a given bush β in $\mathbb{R}^5$ is far from a given bushfield bf, then $\beta \cap bf$ consists of at most finitely many small balls, as in the case of two bushes in $\mathbb{R}^4$.

To make use of these properties, we use a standard technique that is usually called the bilinear reduction (see, e.g., [30], [29], [19] and [35]) together with the rescaling argument in [35].

2.2 Bush Decomposition

Fix $\delta > 0$. We work with tubes $\tau \subset \mathbb{R}^d$ such that the axis of τ is a γ_d-ray and it has dimensions $\delta \times ... \times \delta \times 1$. Two δ-tubes are called δ-separated if the distance between their axis with respect to a (fixed) smooth metric on G_d is greater than δ.

We say two segments of γ_d are disjoint if the distance between them is positive. Fix two disjoint segments W and B of γ_d. We call a tube whose axis direction belongs to W (resp. B) a white (resp. black) tube. Also fix two arbitrary δ-separated families of white and black tubes, $\mathcal{W}$ and $\mathcal{B}$ respectively. Until the end of Lemma 2.5.1, we work with these δ, $\mathcal{W}$ and $\mathcal{B}$.

Let Φ_S denotes the sum of the characteristic functions of the objects in the set S, e.g., $\Phi_{\mathcal{W}}$, $\Phi_{\mathcal{B}}$.

In the Sections 2.2-2.5, we estimate the $L^{p'}$ norm of the function $\min(\Phi_{\mathcal{W}}, \Phi_{\mathcal{B}})$. This can be considered as a bilinear estimate for the adjoint of X. We begin with the following bush decomposition lemma of Wolff [35]. We give a proof for the reader's convenience. A bush [2] is a set of tubes passing through a common point p, that is called a base point for the bush. A white (resp. black) bush means a bush consisting of white (resp. black) tubes. Given a set $\mathcal{W}$ of δ-tubes, we define a μ-fold point for $\mathcal{W}$ to be a point contained in at least μ tubes from $\mathcal{W}$ or equivalently a point x such that $\Phi_{\mathcal{W}}(x) \geq \mu$.

Lemma 2.2.1. Given a set $\mathcal{W}$ of δ-tubes, we have a decomposition

$$\mathcal{W} = \cup_{j=1}^{J} \mathcal{W}_j, \qquad 2^J \approx |\mathcal{W}|,$$

such that

i) $\mathcal{W}_j$ is a union of $\lesssim 2^j$ bushes β_i^j, and any tube in $\mathcal{W}$ belongs to at most one of the bushes β_i^j.

ii) $\mathcal{W}_g^k := \cup_{j>k} \mathcal{W}_j$ does not have any $|\mathcal{W}|/2^k$-fold points, i.e., $\Phi_{\mathcal{W}_g^k} \leq |\mathcal{W}|/2^k$, for all $k \leq J$.

iii) $\mathcal{W}_b^k := \cup_{j \leq k} \mathcal{W}_j$ is a union of $\lesssim 2^k$ bushes.

First, we prove the following lemma:

Lemma 2.2.2. Given a set $\mathcal{W}$ of δ-tubes and a positive number $\mu \leq |\mathcal{W}|$, we can decompose $\mathcal{W}$ as

$$\mathcal{W} = \mathcal{W}_g \cup \mathcal{W}_b,$$

where $\mathcal{W}_b$ is a union of $\lesssim |\mathcal{W}|/\mu$ bushes and $\mathcal{W}_g$ does not have any μ-fold points.

Proof. We construct $\mathcal{W}_b$ inductively. Take any μ-fold point $x_1 \in \mathbb{R}^d$ for $\mathcal{W}$. The tubes in $\mathcal{W}$ containing x_1 forms a bush β_1. Let $\mathcal{W}_b = \beta_1$ and $\mathcal{W}^1 = \mathcal{W} \backslash \beta_1$. Repeat this procedure with $\mathcal{W}_1$ instead of $\mathcal{W}$. This gives another bush β_2. Let $\mathcal{W}_b = \beta_1 \cup \beta_2$ and $\mathcal{W}_2 = \mathcal{W}_1 \backslash \beta_2$. Continue to repeat this procedure until there is no μ-fold points. Since we subtract at least μ tubes from $\mathcal{W}$ in each step, we stop at most in $|\mathcal{W}|/\mu$ steps. Note that this gives $\mathcal{W}_b = \cup_{i=1}^{k} \beta_i$, $k \leq |\mathcal{W}|/\mu$, and $\mathcal{W}_g := \mathcal{W}^k$ has no μ-fold points. □

Proof of Lemma 2.2.1. Apply Lemma 2.2.2 to $\mathcal{W}$ with $\mu = |\mathcal{W}|/2$. This gives a set $\mathcal{W}_g^1$ with no $|\mathcal{W}|/2$-fold points and a collection $\mathcal{W}_1$ of bushes β_i^1. Then apply Lemma 2.2.2 to $\mathcal{W}_g^1$ with $\mu = |\mathcal{W}|/4$ to obtain $\mathcal{W}_g^2$ with no $|\mathcal{W}|/2^2$-fold points and a collection $\mathcal{W}_2$ of bushes β_i^2. Continue to repeat this procedure taking $\mu = |\mathcal{W}|/2^j$ at the jth step. We stop the procedure at Jth step, where J is the smallest integer such that $|\mathcal{W}|/2^J < 1$. Note that $\mathcal{W} = \cup_{j=1}^J \mathcal{W}_j$ and by Lemma 2.2.2, $\mathcal{W}_j$ is a union of at most $|\mathcal{W}|/(|\mathcal{W}|/2^j) = 2^j$ bushes β_i^j. This yields the part i) of the lemma. Part ii) follows from the construction and part iii) immediately follows from part i). □

Lemma 2.2.1 gives a decomposition of $\mathcal{W}$ into a set of bushes β_i^j. At this point, we fix $\varepsilon > 0$ and a tiling of Q_1 by δ^ε-cubes. The letter Q is reserved for these δ^ε-cubes. The following definitions are from [35].

Definition. A tube w is related to a δ^ε-cube Q, $w \sim Q$, if w belongs to a bush β_i^j whose basepoint is in Q or one of its neighbors. Similarly, a tube w is related to a point x, $w \sim x$, if x is in a cube that is related to w.

Definition.

$$\Phi_{\mathcal{W}}^*(x) := \sum_{w \sim x} \Phi_{\mathcal{W}}(x), \qquad \tilde{\Phi}_{\mathcal{W}}(x) := \sum_{w \not\sim x} \Phi_{\mathcal{W}}(x) = \Phi_{\mathcal{W}}(x) - \Phi_{\mathcal{W}}^*(x).$$

We use Lemma 2.2.1 for $\mathcal{B}$ too and define $\tilde{\Phi}_{\mathcal{B}}$ and $\Phi_{\mathcal{B}}^*$ similarly.

2.3 Main Lemma in $\mathbb{R}^4$; Bushes

The following lemma is the main lemma of the proof in $\mathbb{R}^4$. Let $m = |\mathcal{W}|$, $n = |\mathcal{B}|$.

Lemma 2.3.1. Let $d = 4$. With the notation in Section 2.2, for any μ and ν we have

i) $|\{x \in Q_1 : \tilde{\Phi}_{\mathcal{W}}(x) \geq \mu, \Phi_{\mathcal{B}} \geq \nu\}| \lesssim \delta^{5/2 - C\varepsilon} \frac{nm^{1/2}}{\nu^2 \mu^{3/2}}$,

ii) $|\{x \in Q_1 : \Phi_{\mathcal{W}}(x) \geq \mu, \tilde{\Phi}_{\mathcal{B}} \geq \nu\}| \lesssim \delta^{5/2 - C\varepsilon} \frac{n^{1/2} m}{\nu^{3/2} \mu^2}$.

We begin the proof with the following geometric lemma about the transversality of white and black δ-bushes.

Lemma 2.3.2. Fix $\varepsilon > 0$, and let W and B be two disjoint segments of γ_4. Let x and y be two arbitrary points in $2Q_1$ and S_W (resp. S_B) be the surface consisting of all white (resp. black) rays passing from the point x (resp. y). Let Q_x be the δ^{ε}-cube centered at the point x. Then

i) the measure of the intersection of the δ neighborhood of S_W and a black δ-tube is $\lesssim \delta^4$,

ii) the measure of the set $Q_1 \cap (\mathcal{N}(S_W, \delta) \backslash Q_x) \cap \mathcal{N}(S_B, \delta)$ is $\lesssim \delta^{-C\varepsilon} \delta^4$.

Proof. We use the following parametrizations:

$$S_B = \{y + (a, at, at^2, at^3) : a \in (-2, 2), \gamma_4(t) \in B\},$$

$$S_W = \{x + (b, bs, bs^2, bs^3) : b \in (-2, 2), \gamma_4(s) \in W\},$$

and any black δ-tube is the δ neighborhood of a line

$$l(t_0, z) = \{z + (c, ct_0, ct_0^2, ct_0^3), |c| < 2\},$$

where t_0 is a point such that $\gamma_4(t_0) \in B$.

i) It is easy to check that the intersection of S_W and $l(t_0, z)$ consists of at most 2 points. The claim follows from the observations that the tangent plane $T(b, s)$ of S_W at the point corresponding to the parameter values (b, s) is spanned by the vectors $e_1 = (0, 1, 2s, 3s^2)$ and $e_2 = (1, s, s^2, s^3)$, and the angle between $l(t_0, z)$ and $T(b, s)$ is greater than a fixed constant depending on the distance between W and B. We omit the details.

ii) It is easy to check that for fixed x, the intersection of S_W and S_B consists of $\lesssim 1$ points for y in a dense subset of $\mathbb{R}^4$. Therefore, by changing y slightly if necessary and replacing δ with 2δ, we can assume that $S_W \cap S_B$ consists of $\lesssim 1$ points.

Note that if E and F are subsets of a metric space, then

$$\mathcal{N}(E, \delta) \cap \mathcal{N}(F, \delta) \subseteq \mathcal{N}(E \cap \mathcal{N}(F, 2\delta), \delta);$$

hence, it suffices to prove that the induced Lebesgue measure of the set of points on $S_B \cap Q_1$, that are in the 4δ neighborhood of $S_W \backslash Q_x$, is $\lesssim \delta^{-C\varepsilon}\delta^2$.

Let $A_\lambda = \{z : |z - y| \in [\lambda, 2\lambda]\}$. We prove that for all $\lambda \in (0, 1/2)$, the measure of the set of points on $S_B \cap A_\lambda \cap Q_1$ that are in the 4δ neighborhood of $S_W \backslash Q_x$ is $\lesssim \delta^{-C\varepsilon}\delta^2$. This yields the claim since $S_B \cap Q_1$ can be covered by $\lesssim \log(\delta^{-1})$ A_λ's.

Note that the area element on the surface S_B is

$$\mathrm{d}A = f(a,t)a\,\mathrm{d}a\mathrm{d}t, \tag{2.5}$$

where f is a bounded function. Hence, the measure of a subset of $S_B \cap A_\lambda$ of the form $\{x + (a, at, at^2, at^3) : |a - a_0| < \alpha, |t - t_0| < \alpha/\lambda\}$ is $\lesssim \alpha^2$. Therefore, by using part (i) of the lemma, we only need to show that the measure of the set

$$S_t := \{t \in [-1,1] : \exists a, b, s \text{ such that } |F(a,b,t,s)| \leq 4\delta\}$$

is $\lesssim \delta^{-\varepsilon}\delta/\lambda$, where $F : \mathbb{R}^4 \to \mathbb{R}^4$ is the function defined via

$$F(a,b,t,s) = x - y + (a - b, at - bs, at^2 - bs^2, at^3 - bs^3).$$

Note that any derivative of F of order less than two is bounded by C and

$$JF = \det \begin{pmatrix} 1 & t & t^2 & t^3 \\ -1 & -s & -s^2 & -s^3 \\ 0 & a & 2at & 3at^2 \\ 0 & -b & -2bs & -3bs^2 \end{pmatrix} = ab(t-s)^4 \gtrsim \lambda\delta^\varepsilon C.$$

Hence, a quantitative version of the inverse function theorem, for example the one in [6], implies that $F^{-1}(B(0, 4\delta))$ is contained in $\lesssim 1$ balls of diameter

$\lesssim \delta^{-\varepsilon}\delta/\lambda$. This shows that the measure of the set S_t is $\lesssim \delta^{-\varepsilon}\delta/\lambda$. $\square$

Proof of Lemma 2.3.1. We prove part i) only.

Let j_0 be the smallest integer so that $m/2^{j_0} \leq \mu/2$. By Lemma 2.2.1, we have $\Phi_{\mathcal{W}_g^{j_0}} \leq \mu/2$. Note that $\tilde{\Phi}_{\mathcal{W}} \leq \Phi_{\mathcal{W}_g^{j_0}} + \tilde{\Phi}_{\mathcal{W}_b^{j_0}}$. Therefore, $\{\tilde{\Phi}_{\mathcal{W}} \geq \mu\} \subset \{\tilde{\Phi}_{\mathcal{W}_b^{j_0}} \geq \mu/2\}$ and it is enough to prove part i) with $\tilde{\Phi}_{\mathcal{W}_b^{j_0}}$ instead of $\tilde{\Phi}_{\mathcal{W}}$. Also by Lemma 2.2.1, $\mathcal{W}_b^{j_0}$ is a union of $\lesssim 2^{j_0} \lesssim m/\mu$ bushes. Similarly, let k_0 be the smallest integer so that $n/2^{k_0} \leq \nu/2$. Note that $\Phi_{\mathcal{B}} \leq \Phi_{\mathcal{B}_g^{k_0}} + \Phi_{\mathcal{B}_b^{k_0}}$; hence, by the same reasoning, it is enough to prove part i) with $\Phi_{\mathcal{B}_b^{k_0}}$ instead of $\Phi_{\mathcal{B}}$ and $\mathcal{B}_b^{k_0}$ is a union of $\lesssim 2^{k_0} \lesssim n/\nu$ bushes.

Denote the bushes in $\mathcal{W}_b^{j_0}$ (resp. $\mathcal{B}_b^{k_0}$) by β_w (resp. β_b). We have

$$\int \Phi_{\mathcal{B}_b^{k_0}} \tilde{\Phi}_{\mathcal{W}_b^{j_0}} = \sum_{\beta_b} \sum_{\beta_w} \int_{Q_1 \setminus 2Q} \Phi_{\beta_b} \Phi_{\beta_w}, \tag{2.6}$$

where Q is the δ^{ε}-cube containing the base of β_w.

Now, we divide each black bush into $\approx \log(\delta^{-1})$ disjoint segments β_b^k. The segment β_b^0 consists of the parts of the tubes that are in the δ neighborhood of the basepoint, and for $k > 0$, β_b^k consists of the parts of the tubes whose distance to the basepoint is between $2^{k-1}\delta$ and $2^k\delta$. We have

$$(2.6) \lesssim \sum_{k=0}^{\log(\delta^{-1})} \sum_{\beta_b^k} \sum_{\beta_w} \int_{Q_1 \setminus 2Q} \Phi_{\beta_b^k} \Phi_{\beta_w}. \tag{2.7}$$

We need the following lemma to estimate the right-hand side of the inequality (2.7).

Lemma 2.3.3. Fix a black bush segment β_b^k.

i) There are $\lesssim 2^{2k}\delta^{-1}$ white tubes that intersect β_b^k.

ii) For any white bush β_w that intersects β_b^k, we have

$$\int_{Q_1\setminus 2Q} \Phi_{\beta_b^k}\Phi_{\beta_w} \lesssim \delta^{3-C\epsilon}2^{-k},$$

where Q is the δ^ε-cube containing the basepoint of the white bush β_w.

iii) For any white tube w that intersects β_b^k, we have

$$\int_{Q_1} \Phi_{\beta_b^k}\chi_w \lesssim \delta^3 2^{-k}.$$

Proof. i) Note that there are at most δ^{-1} tubes through a given point, and (2.5) implies that the maximum possible cardinality of a δ-separated set of points on β_b^k is $\lesssim 2^{2k}$. Hence, there are at most $2^{2k}\delta^{-1}$ white tubes that intersect β_b^k.

ii) Part ii) of Lemma 2.3.2 shows that the measure of the set of points that belong to both β_b^k and β_w is $\lesssim \delta^{4-C\varepsilon}$. The claim follows from the following pointwise inequalities:

$$\Phi_{\beta_b^k} \lesssim 2^{-k}\delta^{-1}, \tag{2.8}$$

$$\Phi_{\beta_w}\chi_{Q_1\setminus 2Q} \lesssim \delta^{-\varepsilon}. \tag{2.9}$$

To prove (2.8), note that the angle between the axis of the adjacent tubes is $\gtrsim \delta$. Also note that the distance between the points on β_b^k and the basepoint of β_b is at least $2^k\delta$. These show that at most $2^{-k}\delta^{-1}$ many tubes passes through a given point on β_b^k.

Proof of (2.9) is similar, since the points in the complement of $2Q$ are at least at a distance δ^ε to the basepoint of the bush.

iii) This follows from part i) of Lemma 2.3.3 and (2.8). $\square$

We continue the proof of part i) of Lemma 2.3.1. Fix a black bush segment β_b^k. Using part ii) of Lemma 2.3.3, and remembering that there are at most m/μ white bushes, we obtain

$$\sum_{\beta_w} \int_{Q_1 \setminus 2Q} \Phi_{\beta_b^k} \Phi_{\beta_w} \lesssim \frac{m}{\mu} \delta^{3-C\epsilon} 2^{-k}. \tag{2.10}$$

On the other hand, parts i) and iii) of Lemma 2.3.3 imply that

$$\sum_{\beta_w} \int_{Q_1 \setminus 2Q} \Phi_{\beta_b^k} \Phi_{\beta_w} \lesssim 2^{2k} \delta^{-1} \delta^3 2^{-k}. \tag{2.11}$$

Using (2.10) and (2.11) in (2.7), and remembering that there are at most n/ν black bushes, we obtain

$$(2.7) \lesssim \sum_{k=0}^{\log(\delta^{-1})} \frac{n}{\nu} \min(\frac{m}{\mu}, 2^{2k} \delta^{-1}) \delta^{3-C\epsilon} 2^{-k} \lesssim \log(\delta^{-1}) \frac{n}{\nu} (\frac{m}{\mu})^{1/2} \delta^{5/2-C\varepsilon},$$

which yields the claim of part i) using Tschebyshev's inequality. $\square$

2.4 Main Lemma in $\mathbb{R}^5$; Bushfields

The following lemma is the main lemma for the proof in $\mathbb{R}^5$. Let $m = |\mathcal{W}|$, $n = |\mathcal{B}|$.

Lemma 2.4.1. Let $d = 5$. With the notation in Section 2.2, for any μ and ν we have

i) $|\{x \in Q_1 : \tilde{\Phi}_{\mathcal{W}}(x) \geq \mu, \Phi_{\mathcal{B}} \geq \nu\}| \lesssim \delta^{7/2-C\varepsilon} \frac{nm^{1/4}}{\nu^2 \mu^{3/4}}$,

ii) $|\{x \in Q_1 : \Phi_{\mathcal{W}}(x) \geq \mu, \tilde{\Phi}_{\mathcal{B}} \geq \nu\}| \lesssim \delta^{7/2-C\varepsilon} \frac{n^{1/4} m}{\nu^{3/4} \mu^2}$.

In the proof of the lemma, we use a geometric construction called bushfield. A bushfield is a set of tubes intersecting a common tube τ; we call τ the basetube of the bushfield. We call a bushfield consisting of white (resp. black) tubes a white (resp. black) bushfield. We begin the proof with the following lemma about the geometric properties of the bushfields.

Lemma 2.4.2. Let *bf* be a bushfield of white δ-tubes with basetube w and β be a bush of black δ-tubes with basepoint p. Let A_λ be the cylinder $A_\lambda = \{y \in \mathbb{R}^5 : \text{dist}(w, y) \in [\lambda, 2\lambda]\}$. Then

i) If $y \in A_\lambda$, then $\Phi_{bf}(y) \lesssim \lambda^{-1}$.

ii) $|bf \cap A_\lambda| \lesssim \lambda^2 \delta^2$; hence, there are at most $\lambda^2 \delta^{-4}$ δ-separated tubes intersecting $bf \cap A_\lambda$.

iii) $|(bf \cap A_\lambda) \cap (b \backslash \mathcal{N}(p, \delta^\varepsilon))| \lesssim \delta^{-C\varepsilon} \delta^5$.

Proof. Using the maps T_N^t that are defined before Lemma 2.6.2, it is easy to see that for all s and t in $[-1, 1]$, there exists a linear map T_s^t, that takes the curve

γ_5 to itself and in particular takes $\gamma_5(s)$ to $\gamma_5(t)$, such that the entries in the matrix representation of T_s^t and its inverse are bounded by a fixed constant. Because of this and translation invariance, it is enough to prove the lemma by assuming that W is a segment around $\gamma_5(0)$, and the basetube of bf is the δ neighborhood of the line $l(\gamma_5(0), 0)$.

Note that bf is contained in the 2δ neighborhood of the set

$$S_{bf} = \{(u,0,0,0,0) + a(1,t,t^2,t^3,t^4) | u \in (-1,1), a \in (-2,2), \gamma_5(t) \in W\} \quad (2.12)$$

It is easy to see that S_{bf} can also be parametrized as

$$S_{bf} = \{(u, a, at, at^2, at^3) : u \in (-2,2), a \in (-2,2), \gamma_5(t) \in W\}. \quad (2.13)$$

Using this parametrization, we see that $bf \cap A_\lambda$ is contained in the 2δ neighborhood of

$$S_{bf}^{\lambda} = \{(u, a, at, at^2, at^3) : u \in (-2,2), |a| \in [\lambda/2, 2\lambda], \gamma_5(t) \in W\}. \quad (2.14)$$

Also as before, we define

$$S_\beta = \{(b, bs, bs^2, bs^3, bs^4) : |b| \in (\delta^{-\varepsilon}, 2), \gamma_5(s) \in B\}. \quad (2.15)$$

Note that $\beta \backslash \mathcal{N}(p, \delta^\varepsilon)$ is contained in the 2δ neighborhood of the set $p + S_\beta$.

i) Let bf_i be the set of tubes in bf whose direction is $\gamma_5(t)$ for some $t \in [i\delta, (i+$

$1)\delta]$. Note that because of δ-separatedness every point in $\mathbb{R}^5$ is contained in $\lesssim 1$ of the tubes in bf_i. Let P_i^δ be the 2δ neighborhood of the 2-plane P_i through the origin that is spanned by the vectors $\gamma_5(0)$ and $\gamma_5(i\delta)$. Note that all of the tubes in bf_i are contained in P_i^δ. Also note that the angle between the planes P_i and P_j is

$$\begin{aligned}\measuredangle(P_i, P_j) &\approx \measuredangle(\gamma_5(i\delta) - \gamma_5(0), \gamma_5(j\delta) - \gamma_5(0)) \\ &\approx \measuredangle((0,1,i\delta,(i\delta)^2,(i\delta)^3),(0,1,j\delta,(j\delta)^2,(j\delta)^3)) \approx |i-j|\delta.\end{aligned}$$

This and the observation that the distance between the points in A_λ and the basetube is approximately λ show that any point in A_λ is contained in $\lesssim 1/\lambda$ P_i^δ's, which is the claim of part i).

ii) Note that the volume element on S_{bf} with respect to parametrization (2.13) is

$$dW = f(a,t)a\, dudadt, \tag{2.16}$$

where f is a bounded function. This and (2.14) prove the first part. The second part follows from the observations that there are at most $\lambda^2\delta^{-3}$ δ-separated points on $bf \cap A_\lambda$, and at most δ^{-1} δ-separated tubes pass through a given point.

iii) This is similar to the proof of Lemma 2.3.2. Using parametrization (2.12), it is easy to check that for p in a dense subset of $\mathbb{R}^5$, the intersection of

S_{bf} and S_β consists of $\lesssim 1$ points. Hence, by changing p slightly if necessary and replacing δ with 2δ, we can assume that $S_{bf} \cap S_\beta$ consists of $\lesssim 1$ points.

As in the proof of Lemma 2.3.2 ii), it suffices to prove that the induced Lebesgue measure of the set of points on $S_{bf} \cap A_\lambda$, that are in the 4δ neighborhood of $S_\beta \backslash Q_x$, is $\lesssim \delta^{-C\varepsilon}\delta^3$.

Formula (2.16) implies that the measure of a subset of $S_{bf} \cap A_\lambda$ of the form $\{(u, a, at, at^2, at^3) : |u - u_0| < \alpha, |a - a_0| < \alpha, |t - t_0| < \alpha/\lambda\}$ is $\lesssim \alpha^3$. Also note that for fixed t, the intersection of the 2-plane $\{(u, a, at, at^2, at^3) : |u| < 2, |a| < 2\}$ with the 4δ neighborhood of S_β is of measure $\lesssim \delta^2$. This is because of the transversality as in the proof of Lemma 2.3.2 i). Hence, it suffices to prove that the measure of the set

$$S_t = \{t : \exists u, a, b, s \text{ such that } |F(u, a, b, t, s)| \leq 4\delta\}$$

is $\lesssim \delta^{-\varepsilon}\delta/\lambda$, where $F : \mathbb{R}^5 \to \mathbb{R}^5$ is the function defined via

$$F(u, a, b, t, s) = p + (u - b, a - bs, at - bs^2, at^2 - bs^3, at^3 - bs^4).$$

Note that any derivative of F of order less than two is bounded by C and

$$JF = \det \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & t & t^2 & t^3 \\ -1 & -s & -s^2 & -s^3 & -s^4 \\ 0 & 0 & a & 2at & 3at^2 \\ 0 & -b & -2bs & -3bs^2 & -4bs^3 \end{pmatrix} = abC_{W,B} \gtrsim \lambda\delta^{\varepsilon},$$

where $C_{W,B}$ is a constant that depends on the distance between W and B only. Hence, $F^{-1}(B(0,4\delta))$ is contained in $\lesssim 1$ balls of diameter $\lesssim \delta^{-\varepsilon}\delta/\lambda$. This shows that the measure of the set S_t is $\lesssim \delta^{-\varepsilon}\delta/\lambda$. □

Proof of Lemma 2.4.1. We prove part i) only.

Let j_0 be the smallest integer so that $m/2^{j_0} \leq \mu/2$. Using Lemma 2.2.1 as in the proof of Lemma 2.3.1, we note that it is enough to prove part i) with $\tilde{\Phi}_{\mathcal{W}_b^{j_0}}$ instead of $\tilde{\Phi}_{\mathcal{W}}$, and $\mathcal{W}_b^{j_0}$ is a union of $\lesssim 2^{j_0} \lesssim m/\mu$ bushes.

Now, we decompose the black tubes into bushfields. Let Ω be the set $\{\Phi_{\mathcal{B}} > \nu/2\}$. Fix a number $\eta \in (0,1)$ which is determined later. We need the following lemmas.

Lemma 2.4.3. Let τ be a black tube. If $|\tau \cap \Omega| \geq \eta|\tau| \approx \eta\delta^4$, then τ intersects $\gtrsim \eta\nu^2$ tubes from $\mathcal{B}$.

Proof. Without loss of generality, we can assume that τ is the δ-tube with axis $l(\gamma_5(0), 0)$. Note that the set $\tau \cap \Omega$ is covered by the black tubes $\approx \nu/2$ times.

Since we are trying to find a lower bound for the number of tubes required to cover $\tau \cap \Omega$ $\nu/2$ times, we can assume that all the tubes that intersect τ in a small angle are in the covering.

Let $\mathcal{B}_i$ be the set of tubes b in $\mathcal{B}$ that intersect τ and such that the direction of b is $\gamma_5(t)$ for some $t \in [i\delta, (i+1)\delta]$. Note that using the tubes in $\mathcal{B}_i$, one can cover the set $\tau \cap \Omega$ at most once. This is because of δ-separatedness.

The angle between τ and the tubes in $\mathcal{B}_i$ is approximately $i\delta$. This shows that $|\tau \cap b| \lesssim \delta^4/(|i|+1)$; hence, to cover the set $\tau \cap \Omega$ with the tubes in $\mathcal{B}_i$, we need at least $\eta(|i|+1)$ tubes from $\mathcal{B}_i$. This yields the claim of the lemma, since we have to cover the set $\tau \cap \Omega$ approximately $\nu/2$ times and $\sum_{i=0}^{\nu/2} \eta(i+1) \approx \eta\nu^2$. $\square$

Lemma 2.4.4. Given $\eta > 0$ we can decompose $\mathcal{B}$ as

$$\mathcal{B} = \mathcal{B}_r \cup \mathcal{B}_s,$$

where each tube b in $\mathcal{B}_r$ satisfies $|b \cap \Omega| \leq \eta|b|$, and $\mathcal{B}_s$ is a union of $\lesssim n/(\eta\nu^2)\log(\delta^{-1})$ bushfields.

Proof. Let A be a large enough constant. Choose $An/(\eta\nu^2)\log(\delta^{-1})$ tubes from $\mathcal{B}$ randomly. The following claim yields the lemma.

Claim. With high probability all the tubes b in $\mathcal{B}$ with $|b \cap \omega| \geq \eta|b|$ intersect at least one of the tubes from the random sample.

Proof of the claim. Lemma 2.4.3 implies that b intersects at least $\eta\nu^2$ tubes;

hence, b intersects none of the tubes from the random sample with probability $\lesssim (1-\eta\nu^2/n)^{An/(\eta\nu^2)\log(\delta^{-1})} \approx \delta^A$. This shows that the above-mentioned probability is $\geq 1 - Cn\delta^A$, which is $\geq 1/2$ if A is large enough. □

Choose such a sample. Let $\mathcal{B}_s$ be the set of tubes that intersect one of the tubes in the sample and $\mathcal{B}_r$ be the set of remaining tubes. Obviously, $\mathcal{B}_s$ is a union of $\lesssim n/(\eta\nu^2)\log(\delta^{-1})$ bushfields, and any tube $b \in \mathcal{B}_r$ satisfies $|b \cap \Omega| \leq \eta|b|$. □

We continue the proof of Lemma 2.4.1. Note that

$$\{\tilde{\Phi}_{\mathcal{W}_b^{j_0}} \geq \mu, \Phi_{\mathcal{B}} \geq \nu\} \subseteq \{\tilde{\Phi}_{\mathcal{W}_b^{j_0}} \geq \mu, \Phi_{\mathcal{B}_s} \geq \frac{\nu}{2}\} \cup \{\Phi_{\mathcal{B}_r} \geq \frac{\nu}{2}\}. \tag{2.17}$$

Using Lemma 2.4.4, we obtain

$$\|\Phi_{\mathcal{B}_r}\chi_{\{\Phi_{\mathcal{B}_r}\geq\frac{\nu}{2}\}}\|_1 \lesssim \sum_{b\in\mathcal{B}_r} |b \cap \{\Phi_{\mathcal{B}_r} \geq \frac{\nu}{2}\}| \lesssim \eta\delta^4 n.$$

Thus,

$$|\{\Phi_{\mathcal{B}_r} \geq \frac{\nu}{2}\}| \lesssim \frac{\eta n}{\nu}\delta^4. \tag{2.18}$$

Now, we estimate the measure of the set $\{\tilde{\Phi}_{\mathcal{W}_b^{j_0}} \geq \mu, \Phi_{\mathcal{B}_s} \geq \nu/2\}$ as in the proof of Lemma 2.3.1. Denote the bushfields in $\mathcal{B}_s$ by bf and white bushes by

β_w. We have

$$\int \Phi_{\mathcal{B}_s} \tilde{\Phi}_{\mathcal{W}_b^{j_0}} = \sum_{bf} \sum_{\beta_w} \int_{Q_1 \backslash 2Q} \Phi_{bf} \Phi_{\beta_w}, \tag{2.19}$$

where Q is the δ^ε-cube containing the base of β_w.

Now, we divide each black bushfield into $\approx \log(\delta^{-1})$ disjoint segments bf^k. The segment bf^0 consists of the parts of the tubes that are in the δ neighborhood of the basetube, and for $k > 0$, bf^k consists of the parts of the tubes whose distance to the basetube is between $2^{k-1}\delta$ and $2^k\delta$. We have

$$(2.19) \lesssim \sum_{k=0}^{\log(\delta^{-1})} \sum_{bf^k} \sum_{\beta_w} \int_{Q_1 \backslash 2Q} \Phi_{bf^k} \Phi_{\beta_w}. \tag{2.20}$$

Fix a black bushfield segment bf^k. Note that, as in the case $d = 4$, $\chi_{Q_1 \backslash 2Q} \Phi_{\beta_w} \lesssim \delta^{-\varepsilon}$. Using this and parts i) and iii) of Lemma 2.4.2, and remembering that there are at most m/μ white bushes, we obtain

$$\sum_{\beta_w} \int_{Q_1 \backslash 2Q} \Phi_{bf^k} \Phi_{\beta_w} \lesssim \frac{m}{\mu} \delta^{4-C\epsilon} 2^{-k}. \tag{2.21}$$

On the other hand, part ii) of Lemma 2.4.2 shows that there are at most $2^{2k}\delta^{-2}$ white tubes that intersect bf^k. Using this and parts i) and iii) of Lemma 2.4.2, we obtain

$$\sum_{\beta_w} \int_{Q_1 \backslash 2Q} \Phi_{bf^k} \Phi_{\beta_w} \lesssim 2^k \delta^{2-C\varepsilon}. \tag{2.22}$$

Using (2.21) and (2.22) in (2.20), and remembering that there are at most $n/(\eta\nu^2)(\log(\delta^{-1}))^2$ black bushfields, we obtain

$$
\begin{aligned}
(2.20) \quad &\lesssim \sum_{k=0}^{\log(\delta^{-1})} \frac{n}{\eta\nu^2}(\log(\delta^{-1}))^2 \min(\frac{m}{\mu}, 2^{2k}\delta^{-2})\delta^{4-C\epsilon}2^{-k} \\
&\lesssim (\log(\delta^{-1}))^3 \frac{n}{\eta\nu^2}(\frac{m}{\mu})^{1/2}\delta^{3-C\varepsilon}.
\end{aligned}
$$

Thus, using Tschebyshev's inequality, we obtain

$$
|\{\tilde{\Phi}_{\mathcal{W}_b^{j_0}} \geq \mu, \Phi_{\mathcal{B}_s} \geq \frac{\nu}{2}\}| \lesssim \frac{nm^{1/2}}{\eta\nu^3\mu^{3/2}}\delta^{3-C\varepsilon}. \tag{2.23}
$$

Using (2.18) and (2.23) in (2.17), we obtain

$$
|\{\tilde{\Phi}_{\mathcal{W}_b^{j_0}} \geq \mu, \Phi_{\mathcal{B}} \geq \nu\}| \lesssim \frac{\eta n}{\nu}\delta^4 + \frac{nm^{1/2}}{\eta\nu^3\mu^{3/2}}\delta^{3-C\varepsilon}. \tag{2.24}
$$

Minimizing the right-hand side of the inequality (2.24) by choosing a suitable η yields the claim of the lemma. □

2.5 Bilinear Estimate

In this section, we estimate the $L^{p'}$ norm of the function $\min(\Phi_{\mathcal{W}}, \Phi_{\mathcal{B}})$. We need the following numerical inequalities. For proofs see [35]. Let $\theta \in [1/2, 1]$ and a_j, b_k, a, b, x and y be nonnegative real numbers. Let

$$f(x,y) := \min(x,y)^\theta \max(x,y)^{1-\theta}.$$

Then

$$f(ax, by) \leq f(x,y)f(a,b), \tag{2.25}$$

$$f(\sum_j a_j, \sum_k b_k) \leq \sum_{j,k} f(a_j, b_k). \tag{2.26}$$

The following inequality is an immediate corollary of (2.26). Let a, b, c and d be nonnegative real numbers. Then

$$\min(a+b, c+d)^\theta \max(a+b, c+d)^{1-\theta} \lesssim a^{1-\theta}(b+c)^\theta + c^{1-\theta}(c+d)^{1-\theta}. \tag{2.27}$$

For technical reasons, we work with the function Ψ_θ defined below instead of $\min(\Phi_{\mathcal{W}}, \Phi_{\mathcal{B}})$. This is because of the asymmetry of the bounds in Lemmas 2.3.1 and 2.4.1. Here, θ is a dimension dependent parameter in $[1/2, 1]$.

Definition.

$$\Psi_\theta := \chi_{Q_1} \cdot \min(\Phi_{\mathcal{W}}, \Phi_{\mathcal{B}})^\theta \max(\Phi_{\mathcal{W}}, \Phi_{\mathcal{B}})^{1-\theta},$$

$$S_\theta := \chi_{Q_1}.\min(\Phi^*_{\mathcal{W}}, \Phi^*_{\mathcal{B}})^\theta \max(\Phi^*_{\mathcal{W}}, \Phi^*_{\mathcal{B}})^{1-\theta},$$

$$T_\theta := \chi_{Q_1}.(\tilde{\Phi}^{1-\theta}_{\mathcal{W}} \Phi^\theta_{\mathcal{B}} + \Phi^\theta_{\mathcal{W}} \tilde{\Phi}^{1-\theta}_{\mathcal{B}}).$$

Note that the inequality (2.27) implies that

$$\Psi_\theta \leq S_\theta + T_\theta. \tag{2.28}$$

By using the estimates in Lemmas 2.3.1 and 2.4.1, we obtain an estimate for T_θ, and using the rescaling and induction arguments from [35], we prove the same estimate for Ψ_θ. In some sense, the estimates in Lemmas 2.3.1 and 2.4.1 are stronger than the estimates we need; in the following lemma we bring them into the relevant form using trivial estimates.

Lemma 2.5.1. Let $\theta = 1/2$ for $d = 4$ and $\theta = 4/7$ for $d = 5$. Let $p = (d+2)/d$ and $1/p + 1/p' = 1$. Then

$$\|\delta T_\theta\|_{p'}^{p'} \lesssim \delta^{-C\varepsilon}(\delta^d \max(|\mathcal{B}|, |\mathcal{W}|))^{d/(d-1)}. \tag{2.29}$$

Proof. First note that there are $\lesssim \delta^{-1}$ same colored tubes containing a given point. Hence,

$$\|\delta T_\theta\|_\infty \lesssim 1. \tag{2.30}$$

Also note that $\|\Phi_{\mathcal{W}}\|_1 \lesssim |\mathcal{W}|\delta^{d-1}$. This and the similar estimate for $\Phi_{\mathcal{B}}$ imply via Tschebyshev's inequality that

$$\{\Phi_{\mathcal{W}} \geq \mu, \Phi_{\mathcal{B}} \geq \nu\} \lesssim \delta^{d-1} \min(\frac{|\mathcal{W}|}{\mu}, \frac{|\mathcal{B}|}{\nu}). \tag{2.31}$$

i) The case $d = 4$: Let

$$Y(\mu,\nu) = |\{x \in Q_1 : \tilde{\Phi}_{\mathcal{W}}(x) \geq \mu, \Phi_{\mathcal{B}} \geq \nu\}|$$

and $m = |\mathcal{W}|$, $n = |\mathcal{B}|$. Using part i) of Lemma 2.3.1, we obtain

$$\begin{aligned} Y(\mu,\nu) &\lesssim \delta^{5/2-C\varepsilon} \min(\frac{nm^{1/2}}{\nu^2\mu^{3/2}}, \frac{m}{\mu}\delta^{1/2}) \\ &\leq \delta^{5/2-C\varepsilon}(\frac{nm^{1/2}}{\nu^2\mu^{3/2}})^{2/3}(\frac{m}{\mu}\delta^{1/2})^{1/3} \\ &= \delta^{-C\varepsilon}\frac{(\delta^4 n\delta^4 m)^{2/3}}{(\nu\mu)^{4/3}}\delta^{-8/3}. \end{aligned} \tag{2.32}$$

Summing over the dyadic values of μ and ν between 1 and δ^{-1} gives

$$\|\chi_{Q_1}\sqrt{\tilde{\Phi}_{\mathcal{W}}\Phi_{\mathcal{B}}}\|_{8/3}^{8/3} \lesssim \delta^{-C\varepsilon}(\delta^4 n\delta^4 m)^{2/3}\delta^{-8/3}.$$

Estimating $\|\chi_{Q_1}\sqrt{\tilde{\Phi}_{\mathcal{B}}\Phi_{\mathcal{W}}}\|_{8/3}$ in the same way gives

$$\|\delta T_{1/2}\|_{8/3} \lesssim \delta^{-C\varepsilon}(\delta^4 n\delta^4 m)^{1/4}. \tag{2.33}$$

Interpolating (2.33) with (2.30) yields the claim of the lemma for $d = 4$.

ii) The case $d = 5$:

Define $Y(\mu,\nu)$ in the same way. Using part i) of Lemma 2.4.1, we obtain

$$Y(\mu,\nu) \lesssim \delta^{-C\varepsilon} \frac{nm^{1/4}}{\nu^2 \mu^{3/4}} \delta^{7/2}.$$

Using $\mu \lesssim \delta^{-1}$, we obtain

$$Y(\mu,\nu) \lesssim \delta^{-C\varepsilon} \frac{nm^{1/4}}{\nu^2 \mu^{3/2}} \delta^{11/4} \lesssim \delta^{-C\varepsilon} \frac{(\delta^5 \max(m,n))^{5/4}}{\nu^2 \mu^{3/2}} \delta^{-7/2}.$$

Summing over the dyadic values of μ and ν between 1 and δ^{-1} gives

$$\|\chi_{Q_1} \tilde{\Phi}_{\mathcal{W}}^{3/7} \Phi_{\mathcal{B}}^{4/7}\|_{7/2}^{7/2} \lesssim \delta^{-C\varepsilon} (\delta^5 \max(m,n))^{5/4} \delta^{-7/2}.$$

Estimating $\|\chi_{Q_1} \tilde{\Phi}_{\mathcal{B}}^{3/7} \Phi_{\mathcal{W}}^{4/7}\|_{7/2}$ in the same way gives

$$\|\delta T_{4/7}\|_{7/2}^{7/2} \lesssim \delta^{-C\varepsilon} (\delta^5 \max(m,n))^{5/4}.$$

□

Lemma 2.5.2. Let $\theta = 1/2$ for $d = 4$ and $\theta = 4/7$ for $d = 5$. Let $p = (d+2)/d$ and $1/p + 1/p' = 1$. Fix two disjoint segments W and B of γ_d. For any $\delta > 0$, we have that, for any δ-separated $\mathcal{W}$ and $\mathcal{B}$ the following inequality is valid:

$$\|\delta \Psi_\theta\|_{p'}^{p'} \lesssim \delta^{-C\varepsilon} (\delta^d \max(|\mathcal{B}|, |\mathcal{W}|))^{d/(d-1)}. \tag{2.34}$$

Proof. We begin with the following rescaling lemma.

Lemma 2.5.3. Fix $\delta \in (0, \delta_0)$ (δ_0 is determined in the proof) and assume that the claim of Lemma 2.5.2 has been proved for $\delta^{1-\varepsilon}$. Then, we have

$$\|\delta \Psi_\theta\|_{L^{p'}(Q)}^{p'} \leq A_\varepsilon \delta^{C\varepsilon^2/2 - C\varepsilon}((\delta^d \max(|\mathcal{B}|, |\mathcal{W}|))^{d/(d-1)}, \tag{2.35}$$

where Q is a δ^ε-cube and where $\mathcal{W}$ and $\mathcal{B}$ are δ-separated sets of tubes.

Proof. Fix a δ^ε-cube Q. For each $w \in \mathcal{W}$, let $k(w)$ be the cardinality of the set of white tubes w_1 such that $w_1 \cap Q$ is contained in the double of w. Let $\mathcal{W}_\mu$ be the set of white tubes w with $k(w) \in [\mu, 2\mu]$. Define $k(b)$ and $\mathcal{B}_\nu$ analogously. Note that $k(w)$ and $k(b)$ are restricted to values between 1 and $\delta^{-\varepsilon}$. Let

$$\Psi_\theta^{\mu\nu} := \chi_Q . \min(\Phi_{\mathcal{W}_\mu}, \Phi_{\mathcal{B}_\nu})^\theta \max(\Phi_{\mathcal{W}_\mu}, \Phi_{\mathcal{B}_\nu})^{1-\theta}.$$

Note that (2.26) implies that

$$\Psi_\theta \leq \sum_{\mu\nu} \Psi_\theta^{\mu\nu}, \tag{2.36}$$

pointwise on Q, where the sum is over the dyadic values of μ and ν. We estimate the $L^{p'}$ norm of the functions $\Psi_\theta^{\mu\nu}$. We can assume that $\mu \geq \nu$.

Let $\bar{\mathcal{W}}_\mu$ be a maximal subset of $\mathcal{W}_\mu$ that satisfies the property:

(*): *If $w_1, w_2 \in \bar{\mathcal{W}}_\mu$, then $w_1 \cap Q$ is not contained in the double of w_2.*

Define $\bar{\mathcal{B}}_\nu$ analogously. Replace the tubes in $\bar{\mathcal{W}}_\mu$ (resp. $\bar{\mathcal{B}}_\nu$) with their doubles and let

$$\bar{\Psi}_\theta^{\mu\nu} := \chi_Q . \min(\Phi_{\bar{\mathcal{W}}_\mu}, \Phi_{\bar{\mathcal{B}}_\nu})^\theta \max(\Phi_{\bar{\mathcal{W}}_\mu}, \Phi_{\bar{\mathcal{B}}_\nu})^{1-\theta}.$$

Note that the maximality of $\bar{\mathcal{W}}_\mu$ (resp. $\bar{\mathcal{B}}_\nu$) implies that $\Phi_{\mathcal{W}_\mu} \lesssim \mu \Phi_{\bar{\mathcal{W}}_\mu}$ (resp. $\Phi_{\mathcal{B}_\nu} \lesssim \nu \Phi_{\bar{\mathcal{B}}_\nu}$), which implies via (2.25) that

$$\Psi_\theta^{\mu\nu} \lesssim \mu^\theta \nu^{1-\theta} \bar{\Psi}_\theta^{\mu\nu}.$$

Taking the $L^{p'}$ norms, we obtain

$$\|\Psi_\theta^{\mu\nu}\|_{p'} \lesssim \mu^\theta \nu^{1-\theta} \|\bar{\Psi}_\theta^{\mu\nu}\|_{p'}. \tag{2.37}$$

Finally, note that property (*) implies that

$$|\bar{\mathcal{W}}_\mu| \lesssim \mu^{-1} |\mathcal{W}|, \quad |\bar{\mathcal{B}}_\nu| \lesssim \nu^{-1} |\mathcal{B}|. \tag{2.38}$$

Dilating the cube Q by $\delta^{-\varepsilon}$, we obtain a cube Q' of side 1, and we obtain $\delta^{1-\varepsilon}$-separated sets $\bar{\mathcal{W}}_\mu$, $\bar{\mathcal{B}}_\nu$ of $2\delta^{1-\varepsilon}$-tubes. Hence, we can apply the hypothesis to obtain

$$\|\delta^{1-\varepsilon} \bar{\Psi}_\theta^{\mu\nu}(\delta^\varepsilon x)\|_{L^{p'}(Q')}^{p'} \leq A_\varepsilon \delta^{-(1-\varepsilon)C\varepsilon} (\delta^{(1-\varepsilon)d} \max(|\bar{\mathcal{W}}_\mu|, |\bar{\mathcal{B}}_\nu|))^{d/(d-1)}.$$

Making the change of variables $x \to \delta^{\varepsilon} x$, we obtain

$$\delta^{-p'\varepsilon}\delta^{-d\varepsilon}\|\delta\bar{\Psi}_{\theta}^{\mu\nu}\|_{L^{p'}(Q)}^{p'} \leq A_{\varepsilon}\delta^{-C\varepsilon}(\delta^{d}\max(|\bar{\mathcal{W}}_{\mu}|,|\bar{\mathcal{B}}_{\nu}|))^{d/(d-1)}\delta^{C\varepsilon^2}\delta^{-d^2\varepsilon/(d-1)} \tag{2.39}$$

Using estimates in (2.38), we have

$$\begin{aligned}\max(|\bar{\mathcal{W}}_{\mu}|,|\bar{\mathcal{B}}_{\nu}|) &\lesssim \max(\frac{|\mathcal{W}|}{\mu},\frac{|\mathcal{B}|}{\nu})\\ &\leq \frac{1}{\nu}\max(|\mathcal{W}|,|\mathcal{B}|).\end{aligned} \tag{2.40}$$

Using estimate (2.39) and then estimate (2.40) in (2.37) and making the necessary cancelations, we get

$$\begin{aligned}\|\delta\Psi_{\theta}^{\mu\nu}\|_{L^{p'}(Q)}^{p'} &\lesssim \mu^{\theta p'}\nu^{(1-\theta)p'}\|\delta\bar{\Psi}_{\theta}^{\mu\nu}\|_{L^{p'}(Q)}^{p'}\\ &\lesssim A_{\varepsilon}\delta^{-C\varepsilon}(\delta^{d}\max(|\bar{\mathcal{W}}_{\mu}|,|\bar{\mathcal{B}}_{\nu}|))^{d/(d-1)}\delta^{C\varepsilon^2}\mu^{\theta p'}\nu^{(1-\theta)p'}\delta^{p'\varepsilon}\delta^{-d/(d-1)\varepsilon}\\ &\lesssim A_{\varepsilon}\delta^{-C\varepsilon}(\delta^{d}\max(|\mathcal{W}|,|\mathcal{B}|))^{d/(d-1)}\delta^{C\varepsilon^2}\mu^{\theta p'}\nu^{(1-\theta)p'-d/(d-1)}\delta^{p'\varepsilon}\delta^{-d\varepsilon/(d-1)}\\ &\lesssim A_{\varepsilon}\delta^{-C\varepsilon}(\delta^{d}\max(|\mathcal{W}|,|\mathcal{B}|))^{d/(d-1)}\delta^{C\varepsilon^2};\end{aligned} \tag{2.41}$$

the last inequality follows from the fact that $\mu,\nu \lesssim \delta^{-\varepsilon}$ when we note that

$$(1-\theta)p' > \frac{d}{d-1}.$$

Using (2.41) in (2.36), we have

$$\begin{aligned}\|\delta\Psi_\theta\|_{p'} &\le \sum_{\mu\nu}\|\delta\Psi_\theta^{\mu\nu}\|_{p'}\\ &\lesssim \sum_{\mu\nu} A_\varepsilon\delta^{-C\varepsilon}(\delta^d\max(|\mathcal{W}|,|\mathcal{B}|))^{d/(d-1)}\delta^{C\varepsilon^2}\\ &\lesssim A_\varepsilon\delta^{-C\varepsilon}(\delta^d\max(|\mathcal{W}|,|\mathcal{B}|))^{d/(d-1)}\delta^{C\varepsilon^2}\log(\delta^{-1})^2,\end{aligned}$$

since there are $\lesssim \log(\delta^{-1})^2$ terms in the summation. This yields the claim of the lemma given that δ_0 is small enough. □

We continue the proof of Lemma 2.5.2. Note that the lemma is obvious for $\delta \ge \delta_0$, and we prove the lemma for the values of δ such that $\delta^{1-\varepsilon} > \delta_0$. An obvious induction argument yields the claim of the lemma.

We estimate T_θ using Lemma 2.5.1 and estimate S_θ using Lemma 2.5.3 in the following way. For each δ^ε-cube Q, applying Lemma 2.5.3 to the sets

$$n_{\mathcal{W}}(Q) := \{w \in \mathcal{W} : w \sim Q\}, \quad n_{\mathcal{B}}(Q) := \{b \in \mathcal{B} : b \sim Q\},$$

we obtain

$$\|\delta S_\theta\|_{L^{p'}(Q)}^{p'} \le \delta^{C\varepsilon^2/2 - C\varepsilon} A_\varepsilon \left(\delta^d \max(n_{\mathcal{W}}(Q), n_{\mathcal{B}}(Q))\right)^{d/(d-1)}.$$

Summing over Q, we obtain

$$\begin{aligned}
\|\delta S_\theta\|_{p'}^{p'} &\le \delta^{C\varepsilon^2/2-C\varepsilon} A_\varepsilon \sum_Q \left(\delta^d \max(n_{\mathcal{W}}(Q), n_{\mathcal{B}}(Q))\right)^{d/(d-1)} \\
&\le \delta^{C\varepsilon^2/2-C\varepsilon} A_\varepsilon \left(\sum_Q \delta^d \max(n_{\mathcal{W}}(Q), n_{\mathcal{B}}(Q))\right)^{d/(d-1)} \\
&\lesssim \delta^{C\varepsilon^2/2-C\varepsilon} A_\varepsilon \left(\delta^d \max(\sum_Q n_{\mathcal{W}}(Q), \sum_Q n_{\mathcal{B}}(Q))\right)^{d/(d-1)} . \qquad (2.42)
\end{aligned}$$

Note that $\sum_Q n_{\mathcal{W}}(Q) \lesssim |\mathcal{W}|$ and $\sum_Q n_{\mathcal{B}}(Q) \lesssim |\mathcal{B}|$. This is because by Lemma 2.2.1 each tube belongs to at most one bush; hence, each tube is related to $\lesssim 1$ δ^ε-cubes. Using these bounds in (2.42), we obtain

$$\|\delta S_\theta\|_{p'}^{p'} \lesssim \delta^{C\varepsilon^2/2} A_\varepsilon \delta^{-C\varepsilon} (\delta^d \max(|\mathcal{W}|, |\mathcal{B}|))^{d/(d-1)},$$

which yields the claim of the lemma. □

2.6 Proof of Theorem 2.1.3

We do the rest of the proof in general dimensions. Lemma 2.5.2 and the following theorem yield the claim of Theorem 2.1.3.

Theorem 2.6.1. *Let $d \geq 3$ and $\varepsilon > 0$. Assume that p, q and r satisfy the inequalities (2.2), (2.3) and (2.4), and $r \geq q \geq p$. Let W and B be disjoint segments of γ_d. Assume that for any $\delta > 0$, and for any δ-separated $\mathcal{W}$ and $\mathcal{B}$, we have*

$$\|\delta \min(\Phi_{\mathcal{W}}, \Phi_{\mathcal{B}})\|_{L^{p'}(Q_1)}^{p'} \lesssim \delta^{-\varepsilon}(\delta^d \max(|\mathcal{W}|, |\mathcal{B}|))^{p'/r'},$$

where $1/p + 1/p' = 1$ and $1/r + 1/r' = 1$. Then the restricted X-ray transform X is bounded from the Sobolev space $W^{p,C\varepsilon}(Q_1)$ to $L^q(L^r)$.

In the proof of Theorem 2.6.1, we work with the operator

$$X_\delta f(l) = \frac{1}{\delta^{d-1}} \int_{l_\delta} f(x)dx,$$

where l_δ is the δ neighborhood of l in $\mathbb{R}^d$. X_δ is simply the operator X thickened by δ. It is easy to see that the adjoint map X_δ^* of X_δ which takes functions defined on G_d to functions defined on $\mathbb{R}^d$ is defined via

$$X_\delta^* f(u) = \int_{-1}^{1} \int_{H_t} \chi_{l_\delta(t,x)}(u) f(l(t,x))dxdt.$$

The hypothesis of Theorem 2.6.1 is essentially a bilinear estimate for X_δ^*; in the proof, we convert it to a linear estimate. The argument is quite standard, and we omit some details; the proof below is a variation of the one in [35].

We need the following rescaling map for the curve γ_d: Fix a point $\gamma_d(t_0)$ and consider the basis $\{\gamma_d(t_0), \gamma_d'(t_0), ..., \gamma_d^{(d-1)}(t_0)\}$ for $\mathbb{R}^d$. Define $T_N^{t_0}$ via $T_N^{t_0}(\gamma_d^{(j)}(t_0)) = N^j \gamma_d^{(j)}(t_0)$, $j = 0, 1, 2, ..., d-1$.

Lemma 2.6.2. i) $T_N^{t_0}$ takes the curve γ_d to itself, thus taking the γ_d-rays to γ_d-rays. Moreover, we have the following formula:

$$T_N^{t_0}(\gamma_d(t)) = \gamma_d(N(t - t_0) + t_0).$$

ii) $T_N^{t_0}$ takes a segment of length N^{-1} centered at $\gamma_d(t_0)$ of the curve γ_d to a segment of length ≈ 1.

Proof. We prove that $T_N^{t_0}(\gamma_d(t)) = \gamma_d(N(t - t_0) + t_0)$, which yields the claims of the lemma. Let A be the $d \times d$ matrix whose ith column is $\gamma_d^{i-1}(t_0)$, i.e., $A = [\gamma_d(t_0)\ \gamma_d'(t_0)\ ...\ \gamma_d^{(d-1)}(t_0)]$ and $B = \text{diag}(1, N, ..., N^{d-1})$. Note that

$$T_N^{t_0}(\gamma_d(t)) = ABA^{-1}[1, t, t^2, ..., t^{d-1}]^T. \tag{2.43}$$

Let $f(t) = t^j$. Using the equality

$$f(t) = f(t_0) + f'(t_0)(t - t_0) + ... + f^{(j)}(t_0)\frac{(t - t_0)^j}{j!}, \tag{2.44}$$

and the definition of $\gamma_d(t)$, we have

$$[1, t, ..., t^{d-1}]^T = A[1, t - t_0, \frac{(t-t_0)^2}{2!}, ..., \frac{(t-t_0)^{d-1}}{(d-1)!}]^T. \quad (2.45)$$

Using (2.44) and (2.45) in (2.43), we have

$$\begin{aligned} T_N^{t_0}(\gamma_d(t)) &= AB[1, t-t_0, \frac{(t-t_0)^2}{2!}, ..., \frac{(t-t_0)^{d-1}}{(d-1)!}]^T \\ &= A[1, N(t-t_0), \frac{(N(t-t_0))^2}{2!}, ..., \frac{(N(t-t_0))^{d-1}}{(d-1)!}]^T. \quad (2.46) \end{aligned}$$

Using (2.45) by replacing t with $N(t-t_0)+t_0$, we obtain

$$(2.46) = [1, N(t-t_0)+t_0, ..., (N(t-t_0)+t_0)^{d-1}] = \gamma_d(N(t-t_0)+t_0).$$

□

Let s be a segment of the curve γ_d of length N^{-1} centered at $\gamma_d(t_0)$. We denote $T_N^{t_0}$ by T_s, and we denote the subset of G_d consisting of all lines whose directions are in s by G_d^s. Since T_s takes γ_d-rays to γ_d-rays, there is an action $T_s : G_d \to G_d$. We give some more definitions:

Definition. Let Y be a subset of a metric space. We denote the characteristic function of $\mathcal{N}(Y, \eta)$ by $\chi_{Y,\eta}$.

Let Y be a subset of G_d^s, then we have

(i) $\|\chi_{T_sY}\|_{p,r} \approx N^{\frac{1}{p}+\frac{d(d-1)}{2r}} \|\chi_Y\|_{p,r}$,

(ii) $\|\chi_{T_sY,\delta}\|_{p,r} \lesssim N^{\frac{1}{p}+\frac{d(d-1)}{2r}} \|\chi_{Y,\delta/N}\|_{p,r}$,

(iii) $X_\delta^*\chi_Y(x) \approx N^{-1}X_\delta^*\chi_{T_sY}(T_sx)$.

To prove (i), note that T_s expands s by a factor $\approx N$ by Lemma 2.6.2, and for any $\gamma_d(t) \in s$, T_s expands volumes in H_t by $\approx N^{d(d-1)/2}$. This follows from the observations that

$$\det(T_s) \approx N.N^2...N^{d-1} = N^{d(d-1)/2}, \tag{2.47}$$

and T_s essentially preserves the lengths in $\gamma_d(t)$ direction. Inequality (ii) follows from (i) and the observation that $\mathcal{N}(T_sY, \delta) \subseteq T_s\mathcal{N}(Y, C\delta/N)$. Finally, inequality (iii) follows from the fact that T_s expands s by a factor $\approx N$.

Lemma 2.6.3. Fix a large constant C. Let $\varepsilon > 0$, $d \geq 3$ and p, q, r be as in Theorem 2.6.1. Let $Z \subset G_d$ and R be a subset of $\mathbb{R}^d$ such that for any γ_d-ray l, $\mathcal{N}(l, \delta) \cap R$ is contained in a cube of side 1. Let S be a subset of R satisfying: If $x \in S$, then there are two segments s_1 and s_2 of γ_d such that

i) s_1 and s_2 are of length C^{-1},

ii) The distance between s_1 and s_2 is at least C^{-1},

iii) $\min(X_\delta^*(\chi_{Z\cap G_{s_1}}), X_\delta^*(\chi_{Z\cap G_{s_2}})) \geq \eta$.

Then,

$$|S| \lesssim \delta^{-\varepsilon}\eta^{-p'}||\chi_{Z,\delta}||_{q',r'}^{p'}.$$

Proof. First note that it suffices to prove the lemma with R replaced with Q_1. To see this, assume that we have proved the lemma for cubes of side 1. Tile R by cubes of side 1, $R = \cup_i Q^i$ say. Let Z^i be the δ neighborhood of the set

$\{l \in Z : \mathcal{N}(l, \delta) \cap Q^i \neq \emptyset\}$. Note that

$$|S| \lesssim \sum_i |S \cap Q^i| \lesssim \delta^{-\varepsilon}\eta^{-p'} \sum_i \|\chi_{Z^i}\|^{p'}_{q',r'}$$
$$\lesssim \delta^{-\varepsilon}\| \sum_i \chi_{Z^i}\|^{p'}_{q',r'} \lesssim \delta^{-\varepsilon}\|\chi_{Z,\delta}\|^{p'}_{q',r'};$$

the third inequality follows from the fact that $p' \geq q' \geq r'$, and the last inequality can be obtained by noting that for any γ_d-ray l, $\mathcal{N}(l, \delta)$ intersects $\lesssim 1$ of the cubes Q^i.

Also note that γ_d can be covered with $\lesssim 1$ segments of length slightly larger than C^{-1} so that any segment of length C^{-1} is contained in one of the segments in the covering. The set $\mathcal{C}$ of pairs of the segments in the covering has $\lesssim 1$ members and for any pair of segments s_1 and s_2 as in the lemma there is a pair $(c_1, c_2) \in \mathcal{C}$ so that $s_i \subset c_i$, i=1,2. Hence, it suffices to prove the lemma assuming that the segments s_1 and s_2 are independent of x.

Let $Z_i = Z \cap G_{s_i}$, $i = 1, 2$. Let $\mathcal{W}$ (resp. $\mathcal{B}$) be δ-separated subsets of Z_1 (resp. Z_2). Denote the characteristic function of the δ neighborhood of $w \in \mathcal{W}$ in G_d by D_w and the characteristic function of the $C\delta$-tube whose axis is w by χ_w. Note that $X^*_\delta D_w \lesssim \delta\chi_w$. Hence, $X^*_\delta Z_1 \lesssim \delta \sum_w \chi_w = \delta\Phi_{\mathcal{W}}$. Similarly, we have $X^*_\delta Z_2 \lesssim \delta\Phi_{\mathcal{B}}$. Using these and the hypothesis of Theorem 2.6.1, we obtain

$$\begin{aligned}\|\min(X_\delta^*\chi_{Z_1}, X_\delta^*\chi_{Z_2})\|_{L^{p'}(Q_1)}^{p'} &\lesssim \|\delta\min(\Phi_{\mathcal{W}}, \Phi_{\mathcal{B}})\|_{L^{p'}(Q_1)}^{p'} \\ &\lesssim \delta^{-\varepsilon}(\delta^d \max(|\mathcal{W}|, |\mathcal{B}|))^{p'/r'} \\ &\lesssim \delta^{-\varepsilon}\max(|\mathcal{N}(Z_1,\delta)|, |\mathcal{N}(Z_2,\delta)|)^{p'/r'} \\ &\lesssim \delta^{-\varepsilon}\|\chi_{Z,\delta}\|_{r',r'}^{p'} \lesssim \delta^{-\varepsilon}\|\chi_{Z,\delta}\|_{q',r'}^{p'};\end{aligned}$$

we used the fact that $q' \geq r'$ in the last inequality. This yields the claim of the lemma using Tschebyshev's inequality. □

Lemma 2.6.4. With the hypothesis of Theorem 2.6.1, we have

$$\|X_\delta^*\chi_Y\|_{L^{p'}(Q_1)} \lesssim \delta^{-C\varepsilon}\|\chi_{Y,\delta}\|_{q',r'},$$

for any $Y \subset G_d$.

Proof. Below, we prove that

$$|\{x \in Q_1 : X_\delta^*\chi_Y(x) \geq \lambda\}| \lesssim \delta^{-C\varepsilon}\lambda^{-p'}\|\chi_{Y,\delta}\|_{q',r'}^{p'}. \tag{2.48}$$

This yields the claim of the lemma as in the proof of Lemma 2.5.1. Note that (2.48) is obvious for $\lambda < \delta^B$, where B is a large enough constant. The reason for this is that the left-hand side is bounded by 1 and the right-hand side is $\gtrsim 1$ if λ is small and Y is non-empty. Therefore, we assume that $\lambda > \delta^B$.

Now, we prove (2.48). Fix a sufficiently large constant C that depends on

ε and B. Let $A = \{x \in Q_1 : X_\delta^* \chi_Y(x) \geq \lambda\}$ and A_σ be the set of all points $x \in Q_1$ such that

i) there are two segments s_1 and s_2 of length σ of γ_d,

ii) the distance between s_1 and s_2 is between σ and $C\sigma$,

iii) $X_\delta^*(\chi_{Y\cap G_{s_i}}) \geq C^{-1}\delta^\varepsilon\lambda$ for $i = 1, 2$.

We claim that $\cup_\sigma A_\sigma \supseteq A$, where the union is over dyadic $\sigma > \delta^K$, where K is a constant which depends on B.

Let $x \in A$. Let σ be the smallest number such that $X_\delta^*(\chi_{Y\cap G_s})(x) \geq (C\sigma)^{\varepsilon/K}\lambda$ for some segment s of length $C\sigma$. Note that the lower bound for λ implies that $\sigma \geq \delta^K$. Divide s into $\approx C$ segments s_i of length σ. Since σ is minimal, for any segment s_i, $X_\delta^*(\chi_{Y\cap G_{s_i}})(x) < \sigma^{\varepsilon/K}\lambda$. On the other hand, $\sum_i X_\delta^*(\chi_{Y\cap G_{s_i}})(x) \geq X_\delta^*(\chi_{Y\cap G_s})(x) \geq (C\sigma)^{\varepsilon/K}\lambda$. Hence, there should be at least 3 segments s_i such that $X_\delta^*(\chi_{Y\cap G_{s_i}}) \geq C^{-1}\sigma^{\varepsilon/K}\lambda$, which proves the claim since $\sigma > \delta^K$.

By pigeonholing, there is a σ such that $|A_\sigma| \gtrsim \delta^\varepsilon |A|$. Using the rescaling maps and Lemma 2.6.3, we find a bound for $|A_\sigma|$, which is independent of σ.

To do this, consider a covering of γ_d with $C\sigma$-segments s_i with bounded overlap. Let A_σ^i be the set of points $x \in A_\sigma$ such that the two σ-segments in the definition of A_σ are contained in s_i. Note that $A_\sigma = \cup_i A_\sigma^i$.

Fix one of the s_i's. Note that the sets $R = T_{s_i}(Q_1)$, $Z = T_{s_i}(Y \cap G_{s_i})$ and $S = T_{s_i}(A_\sigma^i)$ satisfy the hypothesis of Lemma 2.6.3 with $\eta = \delta^{-1}\lambda$. R satisfies the hypothesis since T_{s_i} essentially preserves distances in γ_d direction. Thus, using Lemma 2.6.3, we obtain

$$|T_{s_i} A^i_\sigma| \lesssim \delta^{-\varepsilon}(\sigma^{-1}\lambda)^{-p'} \|\chi_{T_{s_i}(Y\cap G_{s_i}),\delta}\|^{p'}_{q',r'}. \tag{2.49}$$

Using property (ii) of the map T_{s_i}, we have

$$(2.49) \lesssim \delta^{-\varepsilon}(\sigma^{-1}\lambda)^{-p'}\sigma^{-p'(\frac{1}{q'}+\frac{d(d-1)}{2r'})}\|\chi_{Y\cap G_{s_i},\sigma\delta}\|^{p'}_{q',r'}.$$

Using (2.47), we have

$$|A^i_\sigma| \lesssim \delta^{-\varepsilon}\sigma^{\frac{d(d-1)}{2}}(\sigma^{-1}\lambda)^{-p'}\sigma^{-p'(\frac{1}{q'}+\frac{d(d-1)}{2r'})}\|\chi_{Y\cap G_{s_i},\sigma\delta}\|^{p'}_{q',r'}. \tag{2.50}$$

Equation (2.3) implies that $\sigma^{\frac{d(d-1)}{2}}\sigma^{p'}\sigma^{-p'(\frac{1}{q'}+\frac{d(d-1)}{2r'})} \lesssim 1$. Using this in (2.50), we obtain

$$|A^i_\sigma| \lesssim \delta^{-\varepsilon}\lambda^{-p'}\|\chi_{Y\cap G_{s_i},\sigma\delta}\|^{p'}_{q',r'}. \tag{2.51}$$

Now, note that the sets $\mathcal{N}(Y \cap G_{s_i}, \sigma\delta)$ have bounded overlap. Thus, using (2.51), we get

$$\begin{aligned} |A_\sigma| &\leq \sum_{s_i} |A^i_\sigma| \\ &\lesssim \delta^{-\varepsilon}\lambda^{-p'}\sum_{s_i}\|\chi_{Y\cap G_{s_i},\sigma\delta}\|^{p'}_{q',r'} \\ &\lesssim \delta^{-\varepsilon}\lambda^{-p'}\|\chi_{Y,\sigma\delta}\|^{p'}_{q',r'} \lesssim \delta^{-\varepsilon}\lambda^{-p'}\|\chi_{Y,\delta}\|^{p'}_{q',r'}; \end{aligned}$$

the last inequality follows from the observation that $\mathcal{N}(Y,\sigma\delta)\subset\mathcal{N}(Y,\delta)$. □

Proof of Theorem 2.6.1.

Using duality, Lemma 2.6.4 implies that

$$\|X_\delta f\|_{L^q(L^r)} \lesssim \delta^{-\varepsilon}\|f\|_{L^p(Q_1)}. \tag{2.52}$$

Now, we trade ε derivatives for the $\delta^{-\varepsilon}$ factors. This argument is standard; we follow [33] and omit the details. We can assume that $\|f\|_{W^{p,\varepsilon}}=1$. Using a suitable partition of unity (see, e.g., [33], p. 597), one can find functions f_j, $j=1,2,...$ with Fourier support in $\{\xi : |\xi|\approx 2^j\}$ such that $\sum_j 2^{\eta j}\|f_j\|_p \lesssim \|f\|_{W^{p,\varepsilon}}=1$ for small η and

$$|Xf| \lesssim 1+\sum_j |X_{2^{-j}}f_j|. \tag{2.53}$$

Using (2.53) and (2.52) with $\varepsilon=\eta$, we have

$$\|Xf\|_{q,r} \lesssim 1+\sum_j \|X_{2^{-j}}f_j\|_{q,r} \lesssim 1+\sum_j 2^{\eta j}\|f_j\|_p \lesssim 1,$$

which is the claim of Theorem 2.6.1. □

Chapter 3 Elliptic Maximal Function

3.1 Overview and General Discussion

In 1986, Bourgain [1] proved that the circular maximal function

$$\mathcal{M}_C f(x) = \sup_{t>0} \int_{S^1} f(x+ts)\, d\sigma(s)$$

is bounded on $L^p(\mathbb{R}^2)$ if $p > 2$. Different proofs were given in [22] and [26].

In [24], Schlag generalized this result and obtained sharp $L^p \to L^q$ estimates for $\mathcal{M}_C$.

In this part of the thesis, we attempt to generalize Bourgain's theorem in a different direction; we consider a natural generalization of the circular maximal function by taking maximal averages over ellipses instead of circles.

More explicitly, let $\mathcal{E}$ be the set of all ellipses in $\mathbb{R}^2$ centered at the origin with axial lengths in $[1/2, 2]$. Note that we do not restrict ourselves to the ellipses whose axes are parallel to the co-ordinate axes. The *elliptic maximal function*, M, is defined in the following way: Let $f : \mathbb{R}^2 \to \mathbb{R}$, then

$$Mf(x) = \sup_{E \in \mathcal{E}} \frac{1}{|E|} \int_E f(x+s) d\sigma(s), \quad x \in \mathbb{R}^2,$$

where $d\sigma$ is the arclength measure on E and $|E|$ is the length of E.

We are interested in the L^p mapping properties of M.

Proposition 3.1.1. M is not bounded in L^p for $p \leq 4$.

Proof. It is easy to see that M is not bounded in L^p for $p < 4$ by applying it to the characteristic function, f_δ, of the δ-neighborhood of the unit circle and taking the limit $\delta \to 0$. A simple calculation shows that for all $x \in B(0,1)$, we have $Mf_\delta(x) \gtrsim \delta^{1/4}$. Therefore, $\|M\|_p \gtrsim \delta^{1/4}$, whereas $\|f_\delta\|_p \approx \delta^{1/p}$.

To prove that M is not bounded in L^4, consider the function

$$g_\delta(x) = (|1 - |x|| + \delta)^{-1/4} \chi_{B(0,2)\setminus B(0,1)}. \tag{3.1}$$

Note that $\|g_\delta\|_4 \approx \log(1/\delta)^{1/4}$. On the other hand, we have $Mg_\delta(x) \gtrsim \log(1/\delta)$ for all $x \in B(0,1)$ and hence $\|Mg_\delta\|_4 \gtrsim \log(1/\delta)$ (see [24] for the details). $\square$

In light of Proposition 3.1.1, one may conjecture that M is bounded in L^p for $p > 4$. We are far from proving this conjecture. However, we obtain some non-trivial estimates for M. We will state our results for the key exponent $p = 4$.

The setup is the following; we work with the family of maximal functions:

$$\mathcal{M}_\delta f(x) = \sup_{E \in \mathcal{E}} \frac{1}{|E^\delta|} \int_{x+E^\delta} f(u)\, du, \tag{3.2}$$

where E^δ is the δ neighborhood of the ellipse E and $|E^\delta|$ is the two- dimensional Lebesgue measure of E^δ. We investigate the L^4 mapping properties of $\mathcal{M}_\delta$.

Applying $\mathcal{M}_\delta$ to the functions in (3.1), we see that the inequality

$$\|\mathcal{M}_\delta f\|_4 \lesssim A(\delta)\,\|f\|_4, \qquad \delta > 0 \tag{3.3}$$

can not hold if $A(\delta) = o(\log(1/\delta)^{3/4})$. On the other hand, estimating the right-hand side of (3.2) by $\delta^{-1}\|f\|_1$ implies that $\|M_\delta f\|_1 \lesssim \delta^{-1}\|f\|_1$ and estimating it by $\|f\|_\infty$ implies that $\|M_\delta f\|_\infty \leq \|f\|_\infty$. By interpolating these bounds, we see that (3.3) holds for $A(\delta) \gtrsim \delta^{-1/4}$.

Let E^δ denote the δ-neighborhood of the ellipse E. We have the following basic property of the elliptic annuli.

Lemma 3.1.2. Let E_1 and E_2 be ellipses such that the distance Δ between their centers is $\gtrsim \delta^{2/5}$. Then

$$|E_1^\delta \cap E_2^\delta| \lesssim \frac{\delta^{5/4}}{\Delta^{1/4}}.$$

We prove this lemma in Section 3.2 (Theorem 3.2.1(i)). Now, using this lemma and Cordoba's L^2 Kakeya argument [8], we prove the simple fact that (3.3) holds for $A(\delta) \gtrsim \delta^{-3/16}$.

Lemma 3.1.3.

$$\|\mathcal{M}_\delta f\|_4 \lesssim \delta^{-3/16}\|f\|_4, \qquad \delta > 0.$$

Proof. The lemma follows by interpolating the trivial L^∞ bound with the following restricted weak type estimate:

$$\|\mathcal{M}_\delta f\|_{2,\infty} \lesssim \delta^{-3/8}\|f\|_{2,1}. \tag{3.4}$$

Fix a set A in $B(0,1)$ and $\lambda \in [0,1]$. Let $\Omega = \{x : M_\delta(\chi_A) > \lambda\}$. Take a δ-separated set $\{x_1, ..., x_m\}$ in Ω. We have

$$|\Omega| \lesssim m\delta^2. \tag{3.5}$$

For each x_j, choose an ellipse E_j such that $|E_j^\delta \cap A| > \lambda\delta$. Using Cauchy-Schwarz inequality, we have

$$\begin{aligned} m\delta\lambda &\leq \sum_{j=1}^{m} |E_j^\delta \cap A| = \int_A \sum_j \chi_{E_j^\delta} \\ &\leq |A|^{1/2} \|\sum_j \chi_{E_j^\delta}\|_2 \\ &= |A|^{1/2} \left(\sum_{j,k} |E_j^\delta \cap E_k^\delta| \right)^{1/2}. \end{aligned} \tag{3.6}$$

Now, we estimate the sum $\sum_{j,k} |E_j^\delta \cap E_k^\delta|$ using Lemma 3.1.2. We have $|E_j^\delta \cap E_k^\delta| \lesssim \frac{\delta^{5/4}}{|x_j - x_k|^{1/4}}$ given that $|x_j - x_k| \gtrsim \delta^{2/5}$. Using this, we obtain for fixed j

$$\begin{aligned} \sum_k |E_j^\delta \cap E_k^\delta| &\lesssim \delta^{-2} \int_{1 \gtrsim |x_j - x| \gtrsim \delta^{2/5}} \frac{\delta^{5/4}}{|x_j - x|^{1/4}} dx + \delta^{-2} \int_{|x_j - x| \lesssim \delta^{2/5}} \delta dx \\ &\lesssim \delta^{-3/4}. \end{aligned}$$

Thus,

$$\sum_{j,k} |E_j^\delta \cap E_k^\delta| \lesssim m\delta^{-3/4}. \tag{3.7}$$

Using (3.7) in (3.6), we have

$$m\delta\lambda \lesssim |A|^{1/2}(m\delta^{-3/4})^{1/2}.$$

Hence

$$|\Omega| \lesssim m\delta^2 \lesssim \left(\delta^{-3/8}\frac{|A|^{1/2}}{\lambda}\right)^2,$$

which proves (3.4). □

We have the following improvement:

Theorem 3.1.4. *For all $\varepsilon > 0$, inequality (3.3) holds with $A(\delta) = \delta^{-1/6}\delta^{-\varepsilon}$.*

Theorem 3.1.4 is a corollary of the following stronger theorem, which is the main result of this chapter.

Theorem 3.1.5. $\|\mathcal{M}_\delta f\|_{24/7,\infty} \lesssim \delta^{-1/3}|\log(\delta)|^{5/4}\|f\|_{2,1}$.

In the proof of Theorem 3.1.5, we use a combinatorial method of Kolasa and Wolff [18], [34].

Note that in the proof of Lemma 3.1.3, we assumed that any two ellipses can be third order tangent to each other in a given set of ellipses. However, in Section 3.3 (see Theorem 3.3.1), we obtain a Marstrand's three circle lemma

[20] type result for ellipses and use it together with the combinatorial method to obtain a bound for the third order tangencies. This is the main ingredient of the proof of Theorem 3.1.5.

This technique was also used in [24], [26], [25] and [21].

3.2 Intersection of Elliptic Annuli

The proof of Theorem 3.1.5 utilizes an analysis of the intersection properties of elliptic annuli. In this section and the next one, we obtain the necessary geometric facts about ellipses. The main result of this section is Theorem 3.2.1 below. It gives a bound for the measure of the intersection of two elliptic annuli. Let $E_z^{e,f}$ denote the ellipse $\{x \in \mathbb{R}^2 : (\frac{x_1-z_1}{e})^2 + (\frac{x_2-z_2}{f})^2 = 1\}$, and let $E_z^{e,f,\theta}$ denote the ellipse $E_z^{e,f}$ rotated counter-clockwise by an angle θ around its center. Let $\mathcal{N}(A,\delta)$ denote the δ neighborhood of the set A. Also let $d(x,y)$ denote the distance between the points $x, y \in \mathbb{R}^2$.

Theorem 3.2.1. *Let $d(z,y) = \Delta \gtrsim \delta^{2/5}$. Then*

i) the measure of the set $\mathcal{N}(E_z^{e,f,\theta},\delta) \cap \mathcal{N}(E_y^{a,b},\delta)$ is $\lesssim \delta(\delta/\Delta)^{1/4}$,

ii) if the measure of the set $\mathcal{N}(E_z^{e,f,\theta},\delta) \cap \mathcal{N}(E_y^{a,b},\delta)$ is $\gtrsim \delta(\delta/(u\Delta))^{1/4}, 1 \lesssim u << (\Delta/\delta)^{1/3}$, then we have

$$\min_{\pm}(|(fe)^{2/3} - (ab)^{2/3}(1 \pm d_{a,b}(z,y))|) \lesssim \min((u\Delta)^{3/4}\delta^{1/4}, u^{3/2}(\delta/\Delta)^{1/2}),$$

where $d_{a,b}((p_1,p_2),(q_1,q_2)) = ((p_1-q_1)^2/a^2 + (q_1-q_2)^2/b^2)^{1/2}$.

In the rest of this section, we prove Theorem 3.2.1.

Let S^1 denote the unit circle. First, we find a relationship between the parameters z_1, z_2, e and f of an ellipse $E_z^{e,f}$ and the measure of the set $\mathcal{N}(E_z^{e,f},\delta) \cap \mathcal{N}(S^1,\delta)$. We begin with the following basic lemma.

Lemma 3.2.2. Let N be a positive integer. There exist constants K_1 and K_2

such that for all $\alpha > 0$ and for all $\delta > 0$, we have

$$\sum_{i=0}^{N} |a_i|\alpha^i > \delta \quad \Longrightarrow \quad \exists x_1 \in (0, K_1\alpha) \text{ and } x_2 \in (-K_1\alpha, 0)$$
$$\text{such that } |\sum_{i=0}^{N} a_i x_j^i| > K_2\delta, \;\; j = 1, 2.$$

Proof. The statement is trivial if $\alpha = 1$, and the general case follows from this by the change of variable $y = x\alpha$. □

Let S_1^1 be $S^1 \cap \{x \in \mathbb{R}^2 : x_2 > 0, |x_1| < 2/3\}$. In the proof of the following theorem, we use without mentioning equalities of the form

$$|e - f| \approx |e^s - f^s|, \;\; 0 < c_1 \le e, f \le c_2 < \infty, \;\; s \in \mathbb{R}\backslash\{0\},$$

where the implicit constants depend on s, c_1, c_2.

Theorem 3.2.3. *Let* $d(z, 0) = \Delta \gtrsim \delta^{2/5}$. *Then*

i) the arclength of $E_z^{e,f} \cap \mathcal{N}(S_1^1, \delta)$ *is* $\lesssim (\delta/\Delta)^{1/4}$.

ii) if $|E_z^{e,f} \cap \mathcal{N}(S_1^1, \delta)| \gtrsim (\delta/(u\Delta))^{1/4}$ *for some* $1 \lesssim u << (\Delta/\delta)^{1/3}$, *then we have*

$$|z_1| \lesssim \min(u^{3/2}(\delta\Delta)^{1/2}, u^{9/4}(\delta/\Delta)^{3/4}), \tag{3.8}$$
$$|f - e^2| \lesssim \min((u\Delta)^{3/4}\delta^{1/4}, u^{3/2}(\delta/\Delta)^{1/2}), \tag{3.9}$$
$$|z_2 + f - 1| \lesssim \min((u\Delta)^{3/4}\delta^{1/4}, u^{3/2}(\delta/\Delta)^{1/2}). \tag{3.10}$$

Proof. Consider the function

$$f(x) := z_2 + f(1 - ((x - z_1)/e)^2)^{1/2} - (1 - x^2)^{1/2}.$$

Take a point $t \in (-2/3, 2/3)$ such that $|f(t)| < \delta$. Note that the set $E_z^{e,f} \cap \mathcal{N}(S^1, \delta)$ consists of at most four connected components. Hence, it suffices to prove that there exists $x_1 \in (t - (\delta/\Delta)^{1/4}, t)$ and $x_2 \in (t, t + (\delta/\Delta)^{1/4})$ such that $|f(x_j)| > \delta$ for $j = 1, 2$, and if x_1 or x_2 are not in the $(\delta/(u\Delta))^{1/4}$ neighborhood of t for $1 \lesssim u << (\Delta/\delta)^{1/3}$, then (3.8), (3.9) and (3.10) are valid.

We consider the first five terms of the Taylor expansion of $f(x)$ around t. Let $w := (1 - (t - z_1)^2/e^2)^{-1/2}(1 - t^2)^{1/2}$. We can assume that $w \approx 1$.

$$\begin{aligned}
f(x) &= z_2 + (fw^{-1} - 1)(1 - t^2)^{1/2} \\
&+ \left[(\frac{f}{e^2}(z_1 - t)w + t)(1 - t^2)^{-1/2}\right](x - t) \\
&+ \frac{1}{2}\left[(1 - t^2)^{-3/2}(1 - \frac{f}{e^2}w^3)\right](x - t)^2 \\
&+ \frac{1}{2}\left[(1 - t^2)^{-5/2}(t - w^5\frac{f}{e^4}(t - z_1))\right](x - t)^3 \\
&+ \frac{1}{8}\left[(1 - t^2)^{-7/2}(1 + 4t^2 - w^7\frac{f}{e^4}(1 + 4(t - z_1)^2/e^2))\right](x - t)^4 \\
&+ \frac{1}{24}\left[\eta\frac{3 + 4\eta^2}{(1 - \eta^2)^{9/2}} - \frac{f}{e^8}(\eta - x_1)\frac{3e^2 + 4(\eta - z_1)^2}{(1 - (\eta - z_1)^2/e^2)^{9/2}}\right](\eta - t)^5, \\
&\quad \eta \in (t - |x - t|, t + |x - t|). \\
&=: a_0 + a_1(x - t) + a_2(x - t)^2 + a_3(x - t)^3 + a_4(x - t)^4 + Er.
\end{aligned}$$

Choose u such that

$$\sum_{i=0}^{4} |a_i| \left(\frac{\delta}{u\Delta}\right)^{i/4} = \delta.$$

We have

$$|a_i| \leq (u\Delta)^{i/4}\delta^{1-i/4} \text{ for } i = 0, 1, 2, 3, 4.$$

We consider two cases:

(i) $u \gtrsim (\Delta/\delta)^{1/3}$. Lemma 3.2.2 shows that if we omit the error term Er, then the arclength of the intersection is $\lesssim (\delta/\Delta)^{1/3}$. It is easy to see using the hypothesis $\Delta \gtrsim \delta^{2/5}$ that the error term is not significant.

(ii) $u << (\Delta/\delta)^{1/3}$. Using the definitions of a_0, a_1, a_2 and a_3, we obtain

$$z_2(1-t^2)^{-1/2} + fw^{-1} = 1 + O(\delta), \tag{3.11}$$

$$\frac{f}{e^2}(t - z_1)w = t + O((u\Delta)^{1/4}\delta^{3/4}), \tag{3.12}$$

$$\frac{f}{e^2}w^3 = 1 + O((u\Delta)^{1/2}\delta^{1/2}), \tag{3.13}$$

$$\frac{f}{e^4}(t - z_1)w^5 = t + O((u\Delta)^{3/4}\delta^{1/4}). \tag{3.14}$$

Substituting (3.13) into (3.14), we obtain

$$(ef)^{-2/3}(t - z_1)(1 + O((u\Delta)^{1/2}\delta^{1/2})) = t + O((u\Delta)^{3/4}\delta^{1/4}), \tag{3.15}$$

which implies that

$$(\frac{e^{1/3}}{f^{2/3}} - 1)\frac{t - z_1}{e} + \frac{t - z_1}{e} - t = O((u\Delta)^{3/4}\delta^{1/4}). \tag{3.16}$$

Substituting (3.13) into (3.12), we obtain

$$\frac{f^{2/3}}{e^{4/3}}(t-z_1)(1+O((u\Delta)^{1/2}\delta^{1/2}))=t+O((u\Delta)^{1/4}\delta^{3/4}), \tag{3.17}$$

which implies that

$$t(1-\frac{e^{4/3}}{f^{2/3}})=z_1+O(|z_1-t|(u\Delta)^{1/2}\delta^{1/2}+(u\Delta)^{1/4}\delta^{3/4}). \tag{3.18}$$

Subtracting (3.15) from (3.17), we obtain

$$(t-z_1)(f^{4/3}-e^{2/3}+O((u\Delta)^{1/2}\delta^{1/2}))=O((u\Delta)^{3/4}\delta^{1/4}). \tag{3.19}$$

Substituting (3.13) into (3.11), we obtain

$$z_2(1-t^2)^{-1/2}+(\frac{f^{4/3}}{e^{2/3}}-1)=O((u\Delta)^{1/2}\delta^{1/2}). \tag{3.20}$$

Now, there are two cases $|z_2|\approx\Delta$ or $|z_1|\approx\Delta$.

Case a) Assume $|z_2|\approx\Delta$. Equation (3.20) implies that

$$|e-f^2|\approx\Delta.$$

Using this in (3.19), we obtain

$$t-z_1=O(u^{3/4}(\delta/\Delta)^{1/4}), \tag{3.21}$$

which implies using (3.17) that

$$t = O(u^{3/4}(\delta/\Delta)^{1/4}). \tag{3.22}$$

Using the fact $|e - f^2| \approx \Delta$ and (3.21) in (3.16), we obtain

$$\frac{t - z_1}{e} - t = O((u\Delta)^{3/4}\delta^{1/4}).$$

This and the definition of w implies that

$$w = 1 + O((u\Delta)^{3/4}\delta^{1/4}).$$

On the other hand, using (3.21) and (3.22) in the definition of w, we obtain

$$w = 1 + O(u^{3/2}(\delta/\Delta)^{1/2}).$$

Hence, using (3.13), we have

$$|f - e^2| \lesssim \min((u\Delta)^{3/4}\delta^{1/4}, u^{3/2}(\delta/\Delta)^{1/2}). \tag{3.23}$$

Using (3.21), (3.22) and (3.23) in (3.18), we obtain

$$|z_1| \lesssim \min(u^{3/2}(\delta\Delta)^{1/2}, u^{9/4}(\delta/\Delta)^{3/4}). \tag{3.24}$$

Finally, using (3.22) and the estimates for $|w-1|$ in (3.11), we obtain

$$|z_2+f-1| \lesssim \min((u\Delta)^{3/4}\delta^{1/4}, u^{3/2}(\delta/\Delta)^{1/2}). \tag{3.25}$$

Case b) Assume $|z_1| \approx \Delta$. Using (3.18), we obtain

$$|f-e^2| \approx \Delta, \ |t| \approx \Delta. \tag{3.26}$$

Using (3.13), we obtain

$$(w^2-1)(f/e^2)^{2/3} + (f/e^2)^{2/3} - 1 = O((u\Delta)^{1/2}\delta^{1/2}),$$

which implies using (3.26) that

$$|w^2-1| \approx \Delta. \tag{3.27}$$

Using the definition of w, we obtain

$$w^2 - 1 \approx (t-z_1)^2/e^2 - t^2.$$

Hence (3.27) implies that

$$\left|\frac{t-z_1}{e} - t\right| \approx \Delta. \tag{3.28}$$

Using (3.15), we obtain

$$(\frac{e^{1/3}}{f^{2/3}} - 1)\frac{t - z_1}{e} + \frac{t - z_1}{e} = t + O((u\Delta)^{3/4}\delta^{1/4}),$$

which implies using (3.28) that

$$|e - f^2||t - z_1| \approx \Delta.$$

Hence $|e - f^2| \gtrsim \Delta$ and (3.20) implies that $|z_2| \gtrsim \Delta$. Thus the estimates that we obtained in case a) are valid.

Applying Lemma 3.2.2 (with $K_1\delta$ instead of the δ in the lemma, for a sufficiently large K_1), we see that $|f(x) - Er| > K\delta$, for some $x_1 \in (t - K(\delta/(u\Delta))^{1/4}, 0)$ and $x_2 \in (0, t + K(\delta/(u\Delta))^{1/4})$.

Now, we prove that

$$Er = O(\delta)$$

for $x \in (t - K(\delta/(u\Delta))^{1/4}, t + K(\delta/(u\Delta))^{1/4})$. Note that the estimates that we obtained in part a) imply that

$$|e - 1|, |f - 1| \lesssim \Delta.$$

Let

$$h(\eta) := \frac{\eta(3 + 4\eta^3)}{(1 - \eta)^{9/2}}.$$

We have

$$\begin{aligned}
|Er| &\lesssim (h(\eta) - \frac{f}{e^5} h(\frac{\eta - z_1}{e}))|x-t|^5 \\
&= (h(\eta) - h(\frac{\eta - z_1}{e}) + h(\frac{\eta - z_1}{e})(1 - \frac{f}{e^5}))|x-t|^5 \\
&\lesssim (|\eta - \frac{\eta - z_1}{e}| + \Delta)|x-t|^5 \lesssim (|\eta(e-1)| + |z_1| + \Delta)|x-t|^5 \\
&\lesssim \Delta(\frac{\delta}{u\Delta})^{5/4} \\
&\lesssim \delta.
\end{aligned}$$

Finally, we prove that u can not be $<< 1$. Assume that $u << 1$. Using the definition of a_4 and the estimates we obtained above, we obtain

$$\begin{aligned}
|a_4| &\gtrsim |1 - \frac{f}{e^4}| - \frac{f}{e^4}|1 - w^7| - |t^2 - (\frac{t - z_1}{e})^2| - (\frac{t - z_1}{e})^2 |1 - w^7 \frac{f}{e^4}| \\
&\gtrsim \Delta.
\end{aligned}$$

Hence, u can not be $<< 1$. This yields the upper bound for the arclength of the intersection. □

Let $\min_{\pm}(A \pm B)$ denote $\min(A+B, A-B)$.

Corollary 3.2.4. Let $d(z,0) = \Delta \gtrsim \delta^{2/5}$. Then

i) The arclength of $E_z^{e,f} \cap \mathcal{N}(S^1, \delta)$ is $\lesssim (\delta/\Delta)^{1/4}$,

ii) if it is $\gtrsim (\delta/(u\Delta))^{1/4}, 1 \lesssim u << (\Delta/\delta)^{1/3}$, then we have

$$\min_{\pm}(|(fe)^{2/3} - 1 \pm d(z,0)|) \lesssim \min((u\Delta)^{3/4}\delta^{1/4}, u^{3/2}(\delta/\Delta)^{1/2}).$$

Proof. We divide $\mathcal{N}(S^1,\delta)$ into four segments; $\mathcal{N}(S^1,\delta)=\cup_{i=1}^4\mathcal{N}(S^1_i,\delta)$, where $\mathcal{N}(S^1_1,\delta)$ is as before and $\mathcal{N}(S^1_i,\delta)$ is obtained by rotating $\mathcal{N}(S^1_1,\delta)$ around the origin $i\pi/2$ degrees. Note that if the intersection of the ellipse with $\mathcal{N}(S^1,\delta)$ is large, then its intersection with one of $\mathcal{N}(S^1_i,\delta)$ should be large, too.

Let $|E_z^{e,f}\cap\mathcal{N}(S^1_1,\delta)|>(\delta/(u\Delta))^{1/4}$, for some $1\lesssim u<<(\Delta/\delta)^{1/3}$. Triangle inequality and (3.8) imply that

$$\min_{\pm}(|y_1\pm d(z,0)|)\leq|z_1|\lesssim\min(u^{3/2}(\delta\Delta)^{1/2},u^{9/4}(\delta/\Delta)^{3/4}). \tag{3.29}$$

Equation (3.9) implies that

$$|f-(ef)^{2/3}|\approx|f-e^2|\lesssim\min((u\Delta)^{3/4}\delta^{1/4},u^{3/2}(\frac{\delta}{\Delta})^{1/2}).$$

Hence, we have

$$f-1=(fe)^{2/3}-1+O(\min((u\Delta)^{3/4}\delta^{1/4},u^{3/2}(\frac{\delta}{\Delta})^{1/2})). \tag{3.30}$$

Using (3.29) and (3.30) in (3.10), we obtain

$$\min_{\pm}(|(fe)^{2/3}-1\pm d(z,0)|)\lesssim\min((u\Delta)^{3/4}\delta^{1/4},u^{3/2}(\frac{\delta}{\Delta})^{1/2}).$$

Applying Theorem 3.2.3 (after a rotation) also in the cases where $\mathcal{N}(S^1_1,\delta)$ is replaced with $\mathcal{N}(S^1_i,\delta)$, $i=2,3,4$ yields the claim of the corollary. □

Proof of Theorem 3.2.1. By a dilation, a translation and then a rotation, we can transform $E^{a,b}_{x_0,y_0}$ into S^1 and $E^{e,f,\theta}_z$ into E^{e_1,f_1}_w. We have

$$\begin{aligned} e_1 f_1 &= \frac{ef}{ab} \\ d(w,0) &= d_{a,b}(z,y); \end{aligned}$$

here the first equality holds since the area of the region inside $E^{e,f,\theta}_z$ is equal to the area of the region inside E^{e_1,f_1}_w times ab.

The claim follows by applying Corollary 3.2.4 to E^{e_1,f_1}_w, S^1 and using the fact that for any sets A and B, we have

$$\mathcal{N}(A,\delta) \cap \mathcal{N}(B,\delta) \subset \mathcal{N}(A \cap \mathcal{N}(B, 2\delta), \delta). \qquad \square$$

3.3 Two Ellipses Theorem

Theorem 3.3.1 below is the basic ingredient of the proof of Theorem 3.1.5. It can be considered as a Marstrand's three circles theorem [20] type result for ellipses.

Theorem 3.3.1. *(Two ellipses Theorem) Fix $\Delta \gtrsim \delta^{2/5}$, $d \gtrsim \delta$ and $u \gtrsim 1$. Take any two ellipses E_1 and E_2 such that the distance between their centers (c_1, c_2 respectively) is approximately d. Then the δ-entropy of the set*

$$S := \{x \in \mathbb{R}^2 : |x - c_i| \gtrsim \Delta,\ i = 1, 2,\ \exists \text{ an ellipse } E \text{ centered at } x \text{ such that}$$
$$|E^\delta \cap E_i^\delta| \gtrsim \delta(\delta/u\Delta)^{1/4}, i = 1, 2, \}$$

is $\lesssim \frac{1}{\delta^2 d^{1/2}} |\log(\delta)| u^{3/4} (\delta/\Delta)^{1/4}$.

Proof. By making the suitable translations, rotations and dilations we can assume that $E_2 = S^1$ and $E_1 = E_y^{a,b}$, where $|y| \approx d$. Since the statement of the theorem is void if $u \gtrsim (\Delta/\delta)^{1/3}$, we can further assume that $u << (\Delta/\delta)^{1/3}$.

Denote $u^{3/2}(\delta/\Delta)^{1/2}$ by ξ, and consider the functions

$$F(x) = (|x|^2, d_{a,b}(x, y)^2),$$
$$G(r, s) = \min_{\pm}(| - 1 \pm \sqrt{r} + (ab)^{2/3}(1 \pm \sqrt{s})|).$$

Theorem 3.2.1 implies that the set S is contained in the set

$$\bar{S} := \{x \in \mathbb{R}^2 : |x| \gtrsim \Delta,\ d(x, y) \gtrsim \Delta, G(F(x)) \lesssim \xi\}. \tag{3.31}$$

It is easy to see that the measure of the set $B_\xi := \{(r,s) : G(r,s) \lesssim \xi\}$ is $\lesssim \xi$ (note that $\xi \lesssim 1$).

Below, we prove that the measure of the inverse image of a set of measure ξ under F is at most $(\xi/d)^{1/2}(|\log(\xi/d)|+1)$, which yields the claim of the lemma.

Let B_ξ be a set of measure ξ and A_η be the set where the Jacobian of F, JF, is less than η. Co-area formula (see, e.g., [12] Theorem 3.2.3) implies that

$$|F^{-1}(B_\xi)| \lesssim |A_\eta| + \frac{\xi}{\eta}. \tag{3.32}$$

Claim. $|A_\eta| \lesssim (\eta/d)(|\log(\eta/d)|+1)$.

Proof. Without loss of generality, we can assume that $|y_1| \gtrsim d$. It is easy to calculate that

$$JF \approx \frac{x_1(x_2-y_2)}{b^2} - \frac{x_2(x_1-y_1)}{a^2} = x_1x_2\frac{a^2-b^2}{a^2b^2} - \frac{x_1y_2}{b^2} + \frac{x_2y_1}{a^2}.$$

Hence,

$$A_\eta = \{x \in \mathbb{R}^2 : x_1 \in (-2,2), |x_2 - \frac{x_1y_2a^2}{x_1(a^2-b^2)+y_1b^2}| \lesssim \frac{\eta a^2b^2}{|x_1(a^2-b^2)+y_1b^2|}\}. \tag{3.33}$$

This shows that if $|a^2-b^2| << d$, then $|A_\eta| \lesssim \eta/d$. Now, assume that

$|a^2 - b^2| \gtrsim d$. (3.33) implies that

$$
\begin{aligned}
|A_\eta| &\lesssim \int_{-2}^{2} \min\left(\frac{\eta a^2 b^2}{|x_1(a^2 - b^2) + y_1 b^2|}, 1\right) \mathrm{d}x_1 \\
&\lesssim \frac{\eta}{d}\left(|\log(\frac{\eta}{d})| + 1\right),
\end{aligned}
$$

which proves the claim.

Claim of the theorem follows from (3.32) and the claim above by choosing $\eta = (\xi d)^{1/2}$. □

3.4 Proof of Theorem 3.1.5

Let $A \subset \mathbb{R}^2$, $0 < \lambda \leq 1$ and $\Omega = \{x \in \mathbb{R}^2 : \mathcal{M}_\delta \chi_A(x) > \lambda\}$. We need to prove that

$$|\Omega| \lesssim \left(|\log(\delta)|^{5/4} \delta^{-1/3} \frac{|A|^{1/2}}{\lambda} \right)^{24/7},$$

Without loss of generality, we can assume that $A \subset B(0,1)$. Let $\{x_j\}_{j=1}^{M}$ be a maximally δ separated set in Ω. Note that

$$|\Omega| \lesssim M\delta^2. \tag{3.34}$$

Choose ellipses E_j centered at x_j such that

$$|E_j^\delta \cap A| > \lambda |E_j^\delta| \approx \lambda\delta.$$

We have

$$\begin{aligned} M\delta\lambda &\lesssim \sum_{j=1}^{M} |E_j^\delta \cap A| = \int_A \sum_{j=1}^{M} \chi_{E_j^\delta} \\ &\leq |A|^{1/2} \| \sum_{j=1}^{M} \chi_{E_j^\delta} \|_2 \\ &= |A|^{1/2} \left(\sum_{j,k=1}^{M} |E_j^\delta \cap E_k^\delta| \right)^{1/2}. \end{aligned} \tag{3.35}$$

Let

$$S_{\Delta,u} = \left\{(j,k) : |x_j - x_k| \in (\Delta, 2\Delta), \delta(\frac{\delta}{u\Delta})^{1/4} \leq |E_j^\delta \cap E_k^\delta| \leq \delta(\frac{\delta}{2u\Delta})^{1/4}\right\}.$$

Using this notation, we can estimate $\sum_{j,k=1}^{M} |E_j^\delta \cap E_k^\delta|$ as

$$\begin{aligned}\sum_{j,k=1}^{M} |E_j^\delta \cap E_k^\delta| &\lesssim \sum_{\delta^{2/5} \lesssim \Delta \lesssim 1} \sum_{u} |S_{\Delta,u}| \delta(\frac{\delta}{u\Delta})^{1/4} + \sum_{j=1}^{M} \delta \min(M, \delta^{-6/5}) \\ &\lesssim \sum_{\delta^{2/5} \lesssim \Delta \lesssim 1} \sum_{u} |S_{\Delta,u}| \delta(\frac{\delta}{u\Delta})^{1/4} + M^{17/12} \delta^{3/10}, \qquad (3.36)\end{aligned}$$

where the summations are over the dyadic values of Δ and the dyadic values of $u \in (1, \delta^{-K})$ (since the terms with u greater than a high power of δ^{-1} makes negligible contribution, and Lemma 3.1.2 implies that $S_{\Delta,u}$ is empty if $\Delta > \delta^{2/5}$ and $u << 1$).

Now, we find a bound for the cardinality of the set $S_{\Delta,u}$ using Theorem 3.3.1. Consider the set of triples:

$$\begin{aligned} Q &= \{(j, k_1, k_2) : |x_j - x_{k_i}| \in (\Delta, 2\Delta), \\ &\quad \delta(\frac{\delta}{u\Delta})^{1/4} \leq |E_j^\delta \cap E_{k_i}^\delta| \leq \delta(\frac{\delta}{2u\Delta})^{1/4}, i = 1, 2\}.\end{aligned}$$

We calculate the cardinality of Q in two different ways. Let

$$S_j := |\{k : (j,k) \in S_{\Delta,u}\}|.$$

Note that there are at least S_j^2 triples in Q whose first co-ordinate is j. Hence, we have

$$|S_{\Delta,u}| = \sum_{j=1}^{M} S_j \leq M^{1/2}\left(\sum_{j=1}^{M} S_j^2\right)^{1/2} \lesssim (M|Q|)^{1/2}. \qquad (3.37)$$

On the other hand, we can choose k_1 in M different ways, and for fixed k_1, there are at most $\min(M, d^2/\delta^2)$ indices k_2 such that $|x_{k_2} - x_{k_1}| \in (d, 2d)$. For any such (k_1, k_2), by Theorem 3.3.1 and δ-separatedness, there are at most $\delta^{-2}\min\left(|\log(\delta)|d^{-1/2}u^{3/4}(\delta/\Delta)^{1/4}, \Delta^2\right)$ indices j such that $(j, k_1, k_2) \in Q$. Summing over dyadic $d \in (\delta, 1)$, we obtain

$$\begin{aligned} |Q| &\lesssim M\sum_{d}\min\left(M, \frac{d^2}{\delta^2}\right)\frac{1}{\delta^2}\min\left(|\log(\delta)|d^{-1/2}u^{3/4}(\delta/\Delta)^{1/4}, \Delta^2\right) \\ &\lesssim \frac{M}{\delta^2}|\log(\delta)|\min\left(\frac{(Mu)^{3/4}}{(\delta\Delta)^{1/4}}, M\Delta^2\right). \end{aligned} \qquad (3.38)$$

Using (3.38) in (3.37), we have

$$\begin{aligned} |S_{\Delta,u}| &\lesssim (M|Q|)^{1/2} \\ &\lesssim \frac{M}{\delta}|\log(\delta)|^{1/2}\min\left(\frac{(Mu)^{3/8}}{(\delta\Delta)^{1/8}}, M^{1/2}\Delta\right) \\ &\lesssim \frac{M}{\delta}|\log(\delta)|^{1/2}\left(\frac{(Mu)^{3/8}}{(\delta\Delta)^{1/8}}\right)^{2/3}\left(M^{1/2}\Delta\right)^{1/3} \\ &\lesssim \frac{M^{17/12}}{\delta^{13/12}}|\log(\delta)|^{1/2}(u\Delta)^{1/4}. \end{aligned} \qquad (3.39)$$

Using (3.39) in (3.36) together with the fact that there are at most $\log(\delta)^2$

terms in the summation, we obtain

$$\begin{aligned}\sum_{j,k=1}^{M} |E_j^\delta \cap E_k^\delta| &\lesssim \sum_{\Delta}\sum_{u} \delta \left(\frac{\delta}{u\Delta}\right)^{1/4} \frac{M^{17/12}}{\delta^{13/12}} |\log(\delta)|^{1/2} (u\Delta)^{1/4} + M^{17/12}\delta^{3/10} \\ &\lesssim M^{17/12}\delta^{1/6} |\log(\delta)|^{5/2}. \qquad (3.40)\end{aligned}$$

Using (3.40), (3.35) and (3.34), we have

$$|\Omega| \lesssim M\delta^2 \lesssim \left(|\log(\delta)|^{5/4} \delta^{-1/3} \frac{|A|^{1/2}}{\lambda} \right)^{24/7}, \qquad (3.41)$$

which yields the claim of the theorem.

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