The Seiberg-Witten Equations on 3-manifolds with Boundary

Thesis by

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To My Parents

... and numerous who helped

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Abstract

The Seiberg-Witten equations have proved to be quite powerful in studying smooth 4-manifolds since their landing in 1994. The corresponding Seiberg-Witten theory on closed 3-manifolds can either be obtained by a dimension reduction from the four-dimensional theory, or by following Floer's approach. Here we investigate the theory on 3-manifolds with boundary. The solutions to the Seiberg-Witten equations are identified with critical points to the Chern-Simons-Dirac functional, regarded as a section of the $U(1)$ bundle over the quotient $\mathcal{B}$ of the configuration space. An infinite tube $[0, \infty) \times \Sigma$ is added to the compact manifold and the asymptotic behavior of the solutions on the cylindrical end are studied. The moduli spaces of solutions under gauge group action are finite dimensional, compact and generically smooth. For a generic perturbation h, the moduli space $\mathcal{M}_h$ can be related to the moduli space $\mathcal{M}_L$ of the Kähler-Vortex equations on the boundary surface Σ, via a limiting map r, which is a Lagrangian immersion with respect to a canonical symplectic structure on $\mathcal{M}_L$. Moreover, for a family of admissible perturbations, the moduli spaces for the perturbed Seiberg-Witten equations are mutually Legendrian cobordant.

Contents

List of Notations

$\mathcal{YM}, \mathcal{E}$	energy functional on 4-manifolds
$\mathcal{CSD}$	Chern-Simons-Dirac functional on 3-manifolds
$\mathcal{F}$	Dirac functional on Riemann surfaces
$Cl(V, g)$	Clifford module on vector space V with inner product g
σ, τ	quadratic-like maps sending spinors to forms
ξ	nonvanishing 1-form
∇_A	spin connection associated with connection A on W
D_A	Dirac operator associated with connection A
D_A^Σ	Dirac operator on Riemann surface Σ
$F_{\hat{A}}$	curvature of connection $\hat{A}$ on determinant line bundle
$\mathcal{A}_\Sigma, \mathcal{A}_Y$	configuration space
Ω	symplectic form on $\mathcal{A}_\Sigma$
$\mathcal{G}_\Sigma, \mathcal{G}_Y$	the gauge group
$\mathcal{B}_\Sigma, \mathcal{B}_Y$	quotient space $\mathcal{B} = \mathcal{A}/\mathcal{G}$
$\mathcal{M}_L, \mathcal{M}_Y, \mathcal{M}_h$	various moduli spaces
$\mathcal{L}$	$U(1)$ bundle on $\mathcal{B}$
ω	connection on $\mathcal{L}, d\omega = i\Omega$
$\mathcal{H}$	set of admissible perturbations
ϕ, ψ	collection of embedding of thickened circles and cylinders
$\mathcal{H}_{\phi,\psi}$	component of perturbations corresponding to ϕ, ψ
s, s_h	section of $\mathcal{L}$
$\mathcal{V}, \mathcal{V}_h$	gradient vector field of s, s_h
μ_h	connection part of ∇h
ν_h	spinor part of ∇h
$L^2_{k,w}$	weighted norm on the cylinder $\mathbb{R} \times \Sigma$
$\hat{L}^2$	extended L^2
λ_{-1}	absolute value of first negative eigenvalue
hol_γ	holonomy around the loop γ

Chapter 1 Introduction

A central problem of low-dimensional topology and geometry is to find invariants (topological, differential, piece-wise linear, etc.) of manifolds. This thesis is an attempt to set up the Seiberg-Witten invariants on 3-manifolds with boundary. As a step toward this, we analyze the structure of moduli space to the Seiberg-Witten equations, which is related with topological invariants on the boundary Riemann surface. Furthermore, a family of admissible perturbations of the Seiberg-Witten equations is considered and a Legendrian Cobordism Theorem of the perturbed moduli spaces is proved.

1.1 History and Background

1.1.1 Why Three and Four Dimensions

Over the decades there has been long and sustained interest in low-dimensional topology. There is a considerable incentive from the theoretical physics, especially from the theory of relativity and string theory, to understand the three and four-dimensional manifolds with appropriate metrics imposed. These theories usually involve a field on a certain manifold, or, in other words, a section of a certain vector bundle on the manifold.

From the mathematics' point of view, low-dimensional topology has its own charm as well. To start with, smooth 4-manifolds have resisted classification. In dimension five or higher, smooth structures on a manifold correspond to the reductions of tangent microbundle [35]. On four-dimension, however, there is a completely different picture.

For any $4n$-dimensional manifold X, define

$$\omega(a,b) = \langle a \smile b, [X] \rangle \tag{1.1}$$

where $a, b \in H^{2n}(X;\mathbb{Z})$, $\smile$ is the cup product on cohomology group, and $[X]$ is the volume form. This gives a symmetric bilinear form on $H^{2n}(X;\mathbb{Z})$, and is referred as the intersection form.

A bilinear form ω on a vector space V is called even if $\omega(v,v)$ is even for all $v \in V$. It is called odd if otherwise.

For topological manifolds, M. Freedman showed [23] in 1982

Theorem 1.1 *(Freedman) Given any integral unimodular quadratic form ω, there is an oriented simply connected four-dimensional manifold M realizing ω as its intersection form. Furthermore, if ω is even, then any two such manifolds are homeomorphic. If ω is odd, then there are exactly two homeomorphism classes of 4-manifolds realizing it.*

For smooth manifolds, however, V. Rohlin proved in 1952 that [55]

Theorem 1.2 *(Rohlin) If a smooth, simply connected compact 4-manifold has even intersection form ω, then $\sigma(\omega)$ is divisible by* 16.

This hinted some peculiar behavior of smooth 4-manifolds.

In around 1982, S. Donaldson [15] found gauge theory offer a particular powerful tool to study the four-dimensional geometry. From there on this area thrived remarkably.

1.1.2 The Gauge Theory Approach

Donaldson's idea [15] is to study the Anti-Self-Dual (ASD) connections on certain complex bundles over a closed 4-manifold X under the gauge group action. It turns out, for generic metrics, the moduli spaces are finite dimensional, compact, oriented manifolds, smooth except possibly at finite points, which correspond to the reducible connections. Such moduli spaces carry much of the information of the original 4-manifold in which we are interested.

From another point of view, the ASD connections are the critical points of certain functional, invariant under the gauge group action, on the configuration space, and the functional generally plays the role of an energy function. For example, on an $SU(2)$ bundle V over X, the Yang-Mills functional, for a connection A, is [22]

$$\mathcal{YM}(A) = \int_X |F_A^+|^2 + |F_A^-|^2 \tag{1.2}$$

where F_A^+ and F_A^- are the self-dual and anti-self-dual part of the curvature F_A. Notice we have

$$-8\pi^2\langle c_2(V), [X]\rangle = \int_X |F_A^+|^2 - |F_A^-|^2 \tag{1.3}$$

where $c_2(V)$ is the second Chern class of the $SU(2)$ bundle V. Therefore,

$$\mathcal{YM} \geq 8\pi^2|\langle c_2(V), [X]\rangle| \tag{1.4}$$

and the equality only holds on ASD or SD connections respectively, depending on the sign of the second Chern class, evaluated on fundamental class $[X]$.

Later Donaldson [17] defined the polynomial invariants for a smooth 4-manifold, using the structure of the moduli spaces. On a complex surface, the Donaldson polynomial invariants are nonvanishing and can be related to effective divisors on certain complex bundles on the surface. On a general 4-manifold, however, the computation of the invariants remained elusive and a challenge.

By studying embedded surfaces in the 4-manifolds, P.B. Kronheimer and T.S. Mrowka [39] [41] [42] were able to understand the general structure of the Donaldson polynomial invariants for the so-called simple-type 4-manifolds.

1.1.3 The Seiberg-Witten Equations

1994 marks another year of breakthrough in low-dimensional topology. The discovery of Seiberg-Witten equations [69] from theoretical physics [56] [57] saw more insights to the subject and solutions to many long standing problems in four-dimensional topology. To name a few, P.B. Kronheimer and T.S. Mrowka [40], J. Morgan, Z.

Szabo and C.H. Taubes [52], and R. Fintushel and R. Stern [20] proved the Thom conjecture independently using the Seiberg-Witten invariants. C.H. Taubes [63] found some new constraints on symplectic 4-manifolds from the perspectives of Seiberg-Witten invariants. And there was a strong hint on the connections between the Donaldson's polynomial invariants and the Seiberg-Witten invariants, with the latter much simpler to handle.

On a closed 4-manifold X the Seiberg-Witten invariants are obtained by studying the moduli spaces of solutions to the Seiberg-Witten equations

$$\begin{aligned} &F_{\hat{A}}^{+} + i\sigma(\Phi, \Phi) = 0 \\ &D_A \Phi = 0 \end{aligned} \tag{1.5}$$

where Φ is a section of the positive spinor bundle W^{+}, A is a connection on W^{+} and $\hat{A}$ is the corresponding connection on the determinant line bundle $\det W^{+}$. σ is the map that sends the traceless part of the endomorphism of $SU(2)$ bundle to the self-dual 2-forms (see next chapter). And D_A is the Dirac operator associated with connection A.

The solutions to the above equations are the critical points for the following functional on X

$$\mathcal{E}_X(A, \Phi) = \int_X |\nabla_A \Phi|^2 + \frac{1}{8}(s + |\Phi|^2)^2 + |F_{\hat{A}}^{-}|^2. \tag{1.6}$$

$\mathcal{E}$ can be rewritten as

$$\mathcal{E}_X(A, \Phi) = \int_X |D_A^{+}\Phi|^2 + \frac{1}{4}|\gamma(F_{\hat{A}}^{+}) - (\Phi \otimes \overline{\Phi}^{*})_0|^2 - 4\pi^2 \langle c_1^2(W), [X] \rangle + \frac{1}{8}s^2. \tag{1.7}$$

So again this $\mathcal{E}$ behaves as an energy function on the configuration space and achieves minimum on the solution set to the Seiberg-Witten equations. The moduli spaces are finite-dimensional smooth compact manifolds for generic metrics and are empty for all but finitely many $spin_c$ structures.

1.1.4 Gauge Theory on 3-manifolds

The three-dimensional manifolds have been studied extensively by W.P. Thurston and many others, mainly using geometric tools. For example [24] [26] [66]. There is a close relation between foliations on three manifolds and the Seiberg-Witten equations.

As soon as the gauge theory was developed for 4-manifolds, it was applied to 3-manifolds as well. Since anti-self-duality is something special for 4-manifolds, the ASD perspective is not available for 3-manifolds. However, one may still adopt the energy functional view and consider the critical point set of a certain functional. In this case the Chern-Simons functional on a closed 3-manifold, for connections A on a $SU(2)$ bundle over the 3-manifold Y, fits the picture well:

$$\mathcal{CS}(A) = \frac{1}{4\pi}\int_Y tr(A \wedge dA + \frac{2}{3}A \wedge A \wedge A). \tag{1.8}$$

This functional is not invariant under gauge transformation. But the gauge transformation will only change the functional by 2π times an integer. Therefore, we may view it as an S^1 valued functional. The critical points of this functional correspond to flat connections on Y. One is lead to try to define analogues of the Morse theory in this setting and study the moduli space when varying a 1-parameter family of metrics. The moduli spaces are 0-dimensional and smooth generically. By the compactness and the orientation, they are of finite points with sign. The number of points counted with sign is determined by the spectral flow of the Chern-Simons functional. This is Floer's approach. [11] [21]

Similarly one can set up the Seiberg-Witten invariants on a closed 3-manifold this way. The three-dimensional analogue of the Seiberg-Witten equations are

$$\begin{aligned} &F_A + \frac{i}{2}\tau(\Phi, \Phi) = 0 \\ &D_A \Phi = 0 \end{aligned} \tag{1.9}$$

and the solutions to these equations are the critical points to the Chern-Simons-Dirac functional

$$\mathcal{CSD}(A,\Phi) = \frac{1}{4\pi}\int (A - A_0)\wedge(F_A + F_{A_0}) + (\Phi, D_A\Phi). \tag{1.10}$$

Three-dimensional Seiberg-Witten invariants can also be obtained by dimension reduction from the four-dimensional equations, that is, by considering translationally invariant solutions on $S^1 \times Y$. The invariants derived this way are closely connected to other classical invariants, such as Casson invariant, Milnor torsion and the Alexander polynomial of knots. [47] [53]

1.1.5 On Manifolds with Boundaries

There is another direction along the line of developments. That is, the study of open manifolds. In 1997 P.B. Kronheimer and T.S. Mrowka defined the moduli spaces for the Seiberg-Witten monopole equations on 4-manifolds with boundary [43]. In their theory a contact structure is imposed on the boundary ∂X. The moduli spaces resemble the ones in closed case and P.B. Kronheimer and T.S. Mrowka were able to use them to prove some important theorems on the contact 3-manifolds ∂X.

The contact structure imposed on the boundary makes it unappealing to try the dimension reduction approach to get invariants on 3-manifolds with boundary. The natural $spin_c$ structure near the boundary of a 3-manifold will be the one arising from the product foliation of $I \times \Sigma$. And it cannot possibly correspond to a contact structure on $S^1 \times \Sigma$. So we need to take some other approach.

Earlier in his thesis C. Herald [30] studied the moduli spaces of flat connections on a 3-manifold with boundary. He considered a flat connection as the critical point of the Chern-Simons functional, regarded as a section of a $U(1)$ bundle over the quotient of the configuration space under gauge group action. The moduli spaces $\mathcal{M}_Y$ are stratified, with dimensions of strata dependent only on the genus of the bounding Riemann surface Σ. Furthermore, there is a restriction map $r : \mathcal{M}_Y \mapsto \mathcal{M}_\Sigma$ where $\mathcal{M}_\Sigma$ is the moduli space of flat connections on Σ. The flat connections on connected Riemann surfaces with genus g are well understood and the moduli space has a canonical symplectic structure. The restriction map turned out to be Lagrangian, and he found, under a suitable class of perturbations, the moduli spaces are Legendrian cobordant to each other with respect to this restriction map.

1.2 Main Results and Organization of Material

1.2.1 Main Results

Here we achieve a similar goal, although following a quite different setup in the technical sense. We study the moduli spaces of solutions to the Seiberg-Witten monopole equations on a 3-manifold Y with boundary Σ. We prove that the moduli spaces are compact, finite-dimensional and generically smooth. There is a restriction map r from $\mathcal{M}_Y$ to $\mathcal{M}_\Sigma$. $\mathcal{M}_\Sigma$ is the moduli space of solutions to the Kähler-Vortex equations on Σ. For the canonical symplectic structure on $\mathcal{M}_\Sigma$, r is Lagrangian. For a suitable class of perturbations, the moduli spaces are Legendrian cobordant to each other.

To set up the theory, we consider critical points of the Chern-Simons-Dirac functional on $\mathcal{A}_{(Y,\Sigma)}$, where the functional is regarded as a section of a $U(1)$ bundle over $\mathcal{B}_{(Y,\Sigma)} = \mathcal{A}_{(Y,\Sigma)}/\mathcal{G}_{(Y,\Sigma)}$. We define a family of admissible perturbations and show, for a generic perturbation, the moduli space is a compact, smooth manifold of half the dimension of moduli space of Kähler-Vortex equations, a counter-part of flat connections in Herald's theory.

There are, however, no good boundary conditions for the Seiberg-Witten equations on (Y, Σ). We get around this difficulty by working in a weighted Sobolev space on the 3-manifold Y^+ obtained by attaching a cylindrical end $[0, \infty) \times \Sigma$ to Y. It turns out, on the cylindrical end the equation has the form $J(\frac{\partial}{\partial t} + Q_t)$ where Q_t is a self-adjoint first order elliptic operator on Σ and J is a bundle automorphism. When $t \to \infty$, $Q_t \to Q_\infty = Q$ which is the linearization of the Kähler-Vortex equations.

For this reason, the setup of our theory is quite different from Herald's, even though the results look somewhat similar.

To start with, we consider the solutions to Seiberg-Witten equations on the cylindrical end $[0, \infty) \times \Sigma$ and turn the equations to a gradient flow equation for the Dirac functional $\mathcal{F}$. Although $\mathcal{F}$ does not resemble the energy functionals, it secretly enjoys their properties. Namely, the difference of $\mathcal{F}$ on the tube controls the energy of the pair (A, Φ) in the configuration space.

Therefore we turn our attention to solutions with a finite $\mathcal{F}$ variation at the end. With the help of the center manifold theory, we are able to understand how a finite energy flow approaches the center manifold, which is modeled on $\ker Q$.

On the other hand, Atiyah-Patodi-Singer [5] [6] [7] had studied the similar situation extensively. We mainly follow them to set up the Fredholm theory and compute the index.

From Atiyah-Patodi-Singer's point of view, it is no miracle the formal dimension of Seiberg-Witten moduli space is precisely half the dimension of the Kähler-Vortex moduli space.

Let $D = \sigma(\frac{\partial}{\partial u} + Q) : C^\infty(Y, E, P) \to C^\infty(Y, F)$, where P is the projection to positive eigenspaces of Q. Then according to Atiyah-Patodi-Singer, $\ker D$ is isomorphic to the space of L^2 solutions of $Df = 0$ on Y^+, and $\ker D^*$ is isomorphic to space of extended L^2 solutions of $D^* f = 0$ on Y^+, that is, in L^2 after modulo an $f_\infty \in \ker Q$.

Our situation is somewhat the reverse. We would like $\ker D$ to contain L^2 solutions after modulo an $f_\infty \in \ker Q$. So we may as well start with its dual $\tilde{D} = D^*$: $C^\infty(Y, F, P) \to C^\infty(Y, E)$. Atiyah-Patodi-Singer asserted that $index \tilde{D} = h(F) - h(E) - h_\infty(E)$. Therefore, the formal dimension, or the original index, is $h(E) - h(F) + h_\infty(E)$.

The bundle isomorphism σ in our case is roughly an almost complex structure J on the bundles involved. That makes $J\frac{\partial}{\partial t}$ self-adjoint and offers a symmetry between E and F under consideration. With $h(E) = h(F)$, $h_\infty(E) = h_\infty(F)$ and $h_\infty(E) + h_\infty(F) = \dim \ker Q$, the half-dimension phenomenon is obvious.

The asymptotic behavior of solutions also implies that the restriction map r : $\mathcal{M}_Y \mapsto \mathcal{M}_\Sigma$ is generically an immersion.

The $U(1)$ bundle obtained from the Chern-Simons-Dirac functional descends from the configuration space $\mathcal{A}_\Sigma$ to $\mathcal{M}_\Sigma$ as well as the symplectic structure Ω. With respect to Ω, we will show that the immersion r is Lagrangian. Moreover, the lift of this restriction map to the $U(1)$ bundle is parallel with respect to a canonical connection ω on the $U(1)$ bundle which is compatible with Ω. ω induces a contact structure on the total space of the $U(1)$ bundle and the restriction map is Legendrian.

To obtain transversality results, instead of perturbing the metrics as in the four-dimensional case, we define a family of perturbations by considering certain integrals on a collection of solid tori and cylinders. Under the perturbation the moduli spaces are cobordant and indeed Legendrian cobordant.

1.2.2 Organization of Material

The material is arranged as follows. Chapter 2 will cover most of the preliminaries. We focus on the immediate needs for the Seiberg-Witten equations and leave readers to the standard texts for more general background. In chapter 3 we study the equations on the cylinder and identify the equations with the gradient flow equation of a functional $\mathcal{F}$ on Σ. The critical points of $\mathcal{F}$ are the solutions to the Kähler-Vortex equations over Σ and we study the moduli spaces. Chapter 4 introduces the Chern-Simons-Dirac functional on a 3-manifold with boundary, and equivalently, a 3-manifold with cylindrical ends. A suitable class of perturbation is introduced and their properties are studied. Chapter 5 explores the energy-like property of Dirac functional $\mathcal{F}$ and show the asymptotic behavior of solutions with finite energy. Chapter 6 consists of all the necessary analytical results to set up the theory: the Fredholm theory and index computation, the compactness, and the transversality. Once these are established, we move on to chapter 7 to the Legendrian cobordism of the perturbed moduli spaces.

More References

[3] [16] [17] [18] [19] [25] [27] [38] [61] [64] [65] [67]

Chapter 2 Preliminaries

This chapter covers the preliminaries to the thesis. We briefly introduce Clifford modules and *spin*$_c$ structures on a Riemannian manifold. Then we discuss the Dirac operators on the spinor bundle, after which we derive the three-dimensional analogue of the Seiberg-Witten equations. Here we put emphasis on what we will need in later chapters and steer our discussion mainly for the three-dimensional case. For more general setting and background, we refer interested readers to the texts at the end of this chapter.

2.1 *Spin*$_c$ Structures

2.1.1 Spin Geometry

Definition 2.1 *If V is a vector space with positive definite inner product g, then the Clifford module $Cl(V,g)$ is*

$$Cl(V,g) = \bigotimes_{i=0}^{\infty} V / \langle v \otimes v + g(v,v) \rangle. \tag{2.1}$$

Here $\bigotimes_{i=0}^{\infty} V$ is the free tensor algebra and $\langle v \otimes v + g(v,v)\rangle$ is the ideal generated by elements of type $v \otimes v + g(v,v)$ for any $v \in V$.

On a Riemannian manifold X

Definition 2.2 *A spin$_c$ structure on a vector bundle $V \to X$ is a unitary vector bundle $S \to X$ together with a map $\gamma : V \to End_{\mathbb{C}}(S)$ satisfying*

$$\gamma(v)\gamma(v) = -|v|^2 \mathbf{Id}_S \tag{2.2}$$

and

$$\gamma : Cl(V) \to End_{\mathbb{C}}(S) \text{ is irreducible.} \tag{2.3}$$

A representation that satisfies equation 2.2 is called a Clifford representation.

There is a natural Clifford representation on the 2^n real dimensional spaces with a metric g:

$$\rho : T^*X \mapsto End(\Lambda^*X)$$

induced by

$$\rho(\theta)(\eta) = \theta \wedge \eta - \iota_\theta \eta \tag{2.4}$$

where ι_θ is the dual of $\theta\wedge$ with respect to g. On a Riemannian manifold (M, g) the $*$-operator induces an involution on Λ^*M. The space of exterior forms splits as the sum of even forms and odd forms

$$\Lambda^*M = \Lambda^{even} \oplus \Lambda^{odd}.$$

If the dimension of the manifold is odd, $*^2 = (-1)^{p(n-p)}\mathbf{Id} = \mathbf{Id}$ and the $*$-operator induces an isometry between Λ^{even} and Λ^{odd}. If we identify these two linear spaces through the isomorphism or, more precisely, if we identify a p-form α with an $(n-p)$-form $*\alpha$, then we cut the dimension of representation space by half.

Lemma 2.3 *For any cotangent θ, the following diagram commutes:*

$$\begin{array}{ccc} \Lambda^*(X) & \xrightarrow{\rho(\theta)} & \Lambda^*(X) \\ \downarrow * & & \downarrow * \\ \Lambda^*(X) & \xrightarrow{\rho(\theta)} & \Lambda^*(X) \end{array} \tag{2.5}$$

Proof: For any form η, we need to show

$$*\rho(\theta)\eta = \rho(\theta)(*\eta). \tag{2.6}$$

By definition, for a p-from η

$$\begin{aligned} *\rho(\theta)\eta &= *(\theta \wedge \eta) - *(\iota_\theta \eta) \\ &= (-1)^{2p-n}\iota_\theta(*\eta) + (-1)^{n-1-2p}\theta \wedge *\eta \\ &= \rho(\theta)(*\eta). \end{aligned}$$

□

2.1.2 $spin_c$ Structures on 3-manifolds

The 2^{n-1} dimensional representation gives a representation of the *spin* group. This representation is not the *spin* representation. That is, this representation does not give rise to a $spin_c$ structure. In three-dimension, however, there is something special. For convenience, instead of considering $\Lambda^0(X) \oplus \Lambda^2(X)$ we consider $\Lambda^0(X) \oplus \Lambda^1(X)$. Locally, this gives a four-dimensional representation of *spin*(3). The Clifford multiplication is given explicitly by

$$\begin{aligned} \rho(\theta)\alpha &= \alpha\theta \\ \rho(\theta)\beta &= - * (\theta \wedge \beta) - (\theta, \beta) \end{aligned} \tag{2.7}$$

for $\alpha \in \Lambda^0(X)$ and $\beta \in \Lambda^1(X)$. It's easy to check that condition 2.2 is satisfied.

Notice the complex dimension of irreducible $spin_c(3)$ representation is 2. If we show the above gives rise to a $spin_c(3)$ representation, then it is necessarily irreducible. To get a $spin_c(3)$ representation we need to impose a Hermitian structure on the real space and prove they are compatible with the Clifford multiplication.

The set of Hermitian structures on $\Lambda^0(X) \oplus \Lambda^1(X)$ is in 1-1 correspondence with the set of unit length 1-forms on X. Given a complex structure J, $J(\mathbf{1})$ is a unit length 1-form where $\mathbf{1}$ is the identity function on X. Conversely, given a unit length 1-form α_0 we can define $J(\mathbf{1}) = \alpha_0$. The metric and orientation on $\alpha_0^\perp$ together define an almost complex structure on the orthogonal complement $\alpha_0^\perp$.

As $\chi(X) = 0$, such a unit length 1-form is always available. Call it e_1 and complete it to a local orthonormal basis $\{e_1, e_2, e_3\}$, compatible with the orientation. The basis for $\Lambda^0(X) \oplus \Lambda^1(X)$ is $\{\mathbf{1}, e_1, e_2, e_3\}$. Regarded as a complex vector space, it is spanned by $\{\mathbf{1}, e_2\}$. By definition our Clifford multiplication gives

$$\begin{aligned}
\rho(e_1)(\mathbf{1}) &= e_1 = J(\mathbf{1}) \\
\rho(e_1)(e_2) &= - * (e_1 \wedge e_2) = -e_3 = -J(e_2) \\
\rho(e_2)(\mathbf{1}) &= e_2 \\
\rho(e_2)(e_2) &= -(e_2, e_2) = -\mathbf{1} \\
\rho(e_3)(\mathbf{1}) &= e_3 = J(e_2) \\
\rho(e_3)(e_2) &= - * (e_3 \wedge e_2) = e_1 = J(\mathbf{1}).
\end{aligned} \tag{2.8}$$

So e_i's correspond with the Pauli matrices

$$e_1 \to \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, e_2 \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, e_3 \to \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

The Pauli matrices all satisfy $e_i^2 = -\mathbf{Id}$ and $e_i e_j = -e_j e_i$ so they give an irreducible representation of the $spin_c(3)$ and the above gives a $spin_c(3)$ structure. We actually proved

Lemma 2.4 *There are always $spin_c(3)$ structures on any compact 3-manifold.* $\square$

Notice that on the spinor bundle W, which is a rank 2 Hermitian bundle in our case, there is a (local) hyperkähler structure. This is compatible with our picture as the three complex structures on $T^0(X) \oplus T^1(X)$, in our case, are exactly given by the 3 basis e_1, e_2, e_3, by letting $J_i(\mathbf{1}) = e_i$ respectively. We may compare this with the four-dimensional case, where the self-dual 2-forms serve more or less similar roles. Except in that case Λ^+ is in general not trivial but, a splitting of $\Lambda^+ = R\omega \oplus \omega^\perp$ would suffice.

2.1.3 The Nonvanishing 1-form ξ

Parallel to the four-dimensional case, we also have

Lemma 2.5 *The pairs* $(\mathfrak{s}, \xi)$*, where* $\mathfrak{s}$ *is a* $spin_c$ *structure and* ξ *is a unit length spinor, are in 1-1 correspondence with the unit length* 1*-forms.*

Proof: Given a pair $(\mathfrak{s}, \xi)$, consider the map

$$\tilde{\rho} : T^0(X) \oplus T^1(X) \mapsto V$$

given by $\tilde{\rho} : \theta \mapsto \rho(\theta)\xi$. $\ker \tilde{\rho}$ is given by pairs (α, β) where $\rho(\alpha + \beta)\xi = 0$. By definition, that is

$$\rho(\beta)\xi + \alpha\xi = 0. \tag{2.9}$$

Acting by Clifford multiplication by β on both sides:

$$-|\beta|^2\xi + \alpha\rho(\beta)\xi = 0. \tag{2.10}$$

Multiplying α on both sides and subtracting 2.9 from 2.10 yields

$$-(|\beta|^2 + |\alpha|^2)\xi = 0. \tag{2.11}$$

Since ξ is of unit length, we conclude $\alpha = 0$ and $\beta = 0$. So $\ker \tilde{\rho} = 0$. By dimensional reasons, $\tilde{\rho}$ must be an isomorphism. Now consider $\tilde{\rho}^{-1}(\mathbb{C}\xi)$, clearly $T^0 \in \tilde{\rho}^{-1}(\mathbb{C}\xi)$. Suppose $\tilde{\rho}^{-1}(\mathbb{C}\xi) = T^0 \oplus M$, then M is a one-dimensional space of 1-forms. So one can choose a θ with unit length.

Another way to look at it is to view the constant function $\mathbf{1}$ as the preimage of ξ and the preimage of $i\xi$ will be a unit length 1-form.

θ acts on ξ by i and on $\xi^\perp$ by $-i$.

Suppose we are now given a unit length 1-form θ, we can recover V by letting $V = \Lambda^0(X) \oplus \Lambda^1(X)$. The complex structure is given by $J(\mathbf{1}) = \theta$ and $J|_{\theta^\perp}$ is determined by the metric and orientation. V inherits a metric from the metric on X and it is Hermitian. So we have a representation $T^* \mapsto End(\mathbb{C}^2)$. To check it gives a $spin_c$ structure, we need to check that $\rho^*(\theta) = -\rho(\theta)$ and $\rho(\theta)^2 = (-|\theta|^2)1_V$. This can be shown readily from the Pauli matrices view we discussed earlier before this Lemma. □

Corollary 2.6 *The following are equivalent:*

1. $(\mathfrak{s}, \xi), |\xi| = 1$.
2. *1-form* θ, $|\theta| = 1$.
3. *oriented 2-plane field.*

Given a 1-form ξ, we also use ξ to denote the 2-plane field associated to it.

2.2 Dirac Operators

2.2.1 The Dirac Operator

By standard theory

Definition 2.7 *Given a $spin_c$ structure (S, γ) on $T^*X \to X$, $\gamma : T^*X \to End(S)$ induces a map $\tilde{\gamma} : T^*X \otimes S \to S$. For a Hermitian connection $\hat{A}$ on the determinant line bundle $\det W \to X$, the Levi-Civita connection on T^*X together with $\hat{A}$ induces a spin connection*

$$\nabla_A : \Gamma(S) \to \Gamma(T^*X \otimes S).$$

The Dirac operator D_A is the composition

$$D_A = \tilde{\gamma} \circ \nabla_A. \tag{2.12}$$

Suppose Φ_0 is the unit length spinor corresponding to the identity function 1. And θ is the unit length 1-form determining the *$spin_c$* structure. There is a unique connection ∇_0 on $\det W$ so that the corresponding spin connection has the property that $\nabla_0^{spin}\Phi_0 \in \Gamma(\theta^\perp)$. Denote the corresponding Dirac operator by D_0. Then we have

Lemma 2.8

$$D_0\Phi_0 = \frac{1}{2} * (\theta \wedge d\theta)\Phi_0 - \frac{i}{2}d^*\theta\Phi_0 - \frac{1}{2}(\iota_\theta d\theta) \tag{2.13}$$

Proof: Notice $\gamma(\theta)\Phi_0 = i\Phi_0$ and θ acts on $\Gamma(\theta^\perp)$ by $-i$. Take the Dirac operator on both sides of the equation and apply

$$\nabla^{spin}(\gamma(\eta))\Phi = \gamma(\nabla^{LC}\eta)\Phi + \gamma(\eta)\nabla^{spin}\Phi. \tag{2.14}$$

We get

$$\begin{aligned} D_0(i\Phi_0) &= D_0(\gamma(\theta)\Phi_0) \\ &= \gamma(d\theta + d^*\theta)\Phi_0 + (-i)D_0\Phi_0 \\ &= -iD_0\Phi_0 + d^*\theta\Phi_0 + i * (\theta \wedge d\theta)\Phi_0 - i\iota_\theta d\theta. \end{aligned}$$

□

The space of connections on a line bundle is an affine space for $i\Omega^1(X)$. For a fixed connection $\hat{A}_0$ and any connection $\hat{A}$, we have $\hat{A} = \hat{A}_0 + a$ where a is an imaginary 1-form. Then $D_A = D_{A_0} + 2\gamma(a)I_W$. Notice the image of $\gamma(\Lambda^1(X))\Phi_0$ is $\Gamma(i\mathbb{R}\Phi_0) \oplus \Lambda^{0,0}(\xi)$ where ξ is the 2-plane field $\theta^\perp$, so we can choose a connection $\hat{A}$ so that $D_A\Phi_0 = \frac{1}{2} * (\theta \wedge d\theta)\Phi_0$. We will call this connection $\hat{A}_0$ and keep in mind

$$D_{A_0}\Phi_0 = \frac{1}{2} * (\theta \wedge d\theta)\Phi_0. \tag{2.15}$$

In particular

Corollary 2.9 $D_A\Phi_0 = 0$ *if* $\theta \wedge d\theta = 0$.

If the 2-plane field defines a contact structure, we can choose θ so that $\theta \wedge d\theta = vol$ then Φ_0 becomes an eigenvector:

Corollary 2.10 $D_A\Phi_0 = \frac{1}{2}\Phi_0$ *if the 2-plane field is a contact structure.*

2.2.2 Calculation with ξ

From another point of view, given a 2-plane field ξ, we can define the *spin$_c$* structure by declaring

$$W = \Lambda^{0,0}(\xi) \oplus \Lambda^{0,1}(\xi).$$

The orientation and the metric together define an almost complex structure on ξ, so the above makes sense. The Clifford multiplication is given by the following rule.

If e_1 is the unit length normal vector to ξ, then e_1 acts on $\Lambda^{0,0}(\xi)$ by multiplication of i and on $\Lambda^{0,1}(\xi)$ by multiplication of $-i$. And for $v \in \xi$, we define

$$\begin{aligned} \gamma(v)\alpha &= \frac{1}{\sqrt{2}}(v + iJv)\alpha \in \Lambda^{0,1}(\xi), \alpha \in \Lambda^{0,0}(\xi) \\ \gamma(v)\beta &= -\frac{1}{\sqrt{2}}\langle v + iJv, \beta\rangle \in \Lambda^{0,0}(\xi), \beta \in \Lambda^{0,1}(\xi). \end{aligned} \tag{2.16}$$

Here J is the almost complex structure on ξ, $\langle\, ,\rangle$ is the Hermitian inner product induced by the Riemannian metric. From above we have

$$\begin{aligned} \gamma(\alpha)\Phi_0 &= (\alpha, 0), \alpha \in \Lambda^{0,0}(\xi) \\ \gamma(\beta)\Phi_0 &= (0, \beta), \beta \in \Lambda^{0,1}(\xi). \end{aligned}$$

Lemma 2.11 *In matrix form, the Dirac operator is*

$$\begin{pmatrix} \frac{\alpha}{2} * \theta \wedge d\theta + i(d\alpha, \theta) + \frac{\alpha i}{2} d^*\theta & \overline{\partial}\alpha + \frac{\alpha}{2}\iota_\theta d\theta \\ \overline{\partial}^*|_\xi \beta - \frac{1}{2}(\iota_\theta d\theta, \beta) & -i\iota_\theta d\beta \end{pmatrix}. \tag{2.17}$$

Proof: For a spinor (α, β) we compute

$$\begin{aligned} D_0(\alpha, 0) &= D_{A_0}(\gamma(\alpha)\Phi_0) \\ &= \gamma(d\alpha)\Phi_0 + \gamma(\alpha)D_{A_0}\Phi_0 \\ &= \gamma(d\alpha)\Phi_0 + \frac{\alpha}{2} * (\theta \wedge d\theta) + \frac{\alpha i}{2} d^*\theta + \frac{\alpha}{2}\iota_\theta d\theta) \\ &= (\frac{\alpha}{2} * (\theta \wedge d\theta) + \frac{\alpha i}{2} d^*\theta + i(d\alpha, \theta), \overline{\partial}\alpha + \alpha\iota_\theta d\theta) \end{aligned}$$

where $\overline{\partial} : \Lambda^{0,0}(\xi) \to \Lambda^{0,1}(\xi)$ is the projection of $d\alpha$ to $\Lambda^{0,1}(\xi)$. Notice we have, for a complex valued function α:

$$d\alpha = (d\alpha, \theta)\theta + \overline{\partial}\alpha + \partial\alpha.$$

On the other hand

$$\begin{aligned} D_0(0,\beta) &= D_{A_0}(\gamma(\beta)\Phi_0) \\ &= \gamma(d\beta + d^*\beta)\Phi_0 + \sum \gamma(e^i)\gamma(\beta)\nabla_i\Phi_0 \\ &= \gamma(\overline{\partial}^*\beta)\Phi_0 + \gamma(d\beta)\Phi_0 - \frac{1}{2}(\iota_\theta d\theta, \beta) \\ &= (\overline{\partial}^*\beta + (*d\beta, \theta) - \frac{1}{2}(\iota_\theta d\theta, \beta), -i\iota_\theta d\beta)). \end{aligned}$$

Notice here we have $\overline{\partial}^* = d^*$ where the adjoint are both taken on Y. We can also consider $\overline{\partial}^*|_\xi$, the adjoint along the 2-plane field. In this case $\overline{\partial}^*|_\xi = \overline{\partial}^* + (*d(), \theta)$. Also notice some of the last terms in 2nd equality above vanish due to the fact that $\gamma(e^i)$ anticommutes with $\gamma(\beta)$ whenever e^i doesn't appear in β and commutes with the part of β that involves e^i. So in matrix form the Dirac operator can be written as

$$\begin{pmatrix} \frac{\alpha}{2} * \theta \wedge d\theta + i(d\alpha, \theta) + \frac{\alpha i}{2} d^*\theta & \overline{\partial}\alpha + \frac{\alpha}{2}\iota_\theta d\theta \\ \overline{\partial}^*|_\xi\beta - \frac{1}{2}(\iota_\theta d\theta, \beta) & -i\iota_\theta d\beta \end{pmatrix}.$$

□

In particular, if the 3-manifold is a product $\mathbb{R} \times \Sigma$ and the 2-plane field is given by $T\Sigma$ then $\theta = dt$, $d\theta = 0$ and $d^*\theta = 0$. The Dirac operator reduces to

$$\begin{pmatrix} i\frac{\partial}{\partial t} & \overline{\partial} \\ \overline{\partial}^* & -i\frac{\partial}{\partial t} \end{pmatrix}.$$

2.3 The Seiberg-Witten Equations

2.3.1 The Map σ

On a four-dimensional manifold X, with a *spin$_c$* structure (W^+, W^-, γ), the representation

$$\gamma : \Gamma(T^*X) \to End(W^+ \oplus W^-)$$

lifts to a map, still called γ:

$$\gamma : \Gamma(\Lambda^*X) \to End(W^+ \oplus W^-)$$

by letting

$$\gamma(\theta_1 \wedge \theta_2) = \gamma(\theta_1)\gamma(\theta_2) - \gamma(\theta_2)\gamma(\theta_1) \tag{2.18}$$

for $\theta_i \in T^*X$ and extended inductively on the degree of forms.

Restricted to $\Gamma^+(X) = \Gamma(\Lambda^+(X))$, γ induces a map, yet again called γ:

$$\gamma : \Gamma^+(X) \to End(W^+)$$

since $\Gamma^+(X)$ acts on W^- trivially. There is a unique map

$$\sigma : \Gamma(W^+) \otimes \Gamma(W^-) \to \Gamma^+(X)$$

characterized by

$$(i\sigma(\varphi, \psi), \theta) = (\gamma(\theta)\varphi, \psi) \tag{2.19}$$

for any $\varphi, \psi \in \Gamma(W^+)$ and $\theta \in \Gamma^+(X)$.

Fibrewise the map σ can be seen as follows. If there is a split $\Lambda^+ = \mathbb{R}\omega \oplus \omega^\perp$, then $W^+ = \Lambda^{0,0} \oplus \Lambda^{0,2}$ for the almost structure determined by the unit-length 2-form ω. The preferred section $\gamma(\omega)\mathbf{1}$ supplies a (local) quarternion structure, together with the orientation, on the $\mathbb{C}^2$ bundle W^+ by declaring $\mathfrak{i} = \gamma(\omega)\imath$. Under this quarternion structure, there is an isomorphism

$$g : \Lambda^+(X) \to Im(\mathbb{H})$$

given by $g(\theta) = \gamma(\theta)\imath$. Then σ is the map

$$\sigma' : \mathbb{H} \times \mathbb{H} \to Im(\mathbb{H})$$

defined by

$$\sigma'(\varphi, \psi) = -Im(\varphi \mathfrak{i} \overline{\psi}) \tag{2.20}$$

composed with g^{-1}.

Remark 2.12 *The map $\varphi \mapsto \sigma'(\varphi, \varphi)$ is the cone on the Hopf map $S^3 \mapsto S^2$.*

2.3.2 The Seiberg-Witten Equations

The Seiberg-Witten equations, for pairs (A, Φ), where A is a connection on W^+ and $\Phi \in \Gamma(W^+)$, are

$$\begin{aligned} F_{\hat{A}}^+ + i\sigma(\Phi, \Phi) &= 0 \\ D_A \Phi &= 0. \end{aligned} \tag{2.21}$$

From this we can derive the three-dimensional analogue by considering the translationally invariant solutions to the above equations on $\mathbb{R} \times Y$.

Given a 3-manifold Y, a $spin_c$ structure (W, γ) induces a $spin_c$ structure $(W, W, \tilde{\gamma})$ on $\mathbb{R} \times Y$, on which the metric is the product metric, by declaring $\tilde{\gamma} = \gamma$ on $\Gamma(Y)$ and $\tilde{\gamma}(dt)$ be the identical map viewed as an isomorphism between the first W and second W.

For a translationally invariant solution (A, Φ) to the Seiberg-Witten equations on

$\mathbb{R} \times Y$, first notice

$$D^+_{\hat{A},\mathbb{R}\times Y} = D_{A,Y}. \tag{2.22}$$

So the second equation is preserved. On the other hand

$$F_{\hat{A},\mathbb{R}\times Y} = F_{\hat{A},Y} \tag{2.23}$$

as the metric on $\mathbb{R}$ is flat. That means

$$F^+_{\hat{A},\mathbb{R}\times Y} = \frac{1}{2}(F_{\hat{A},Y} + dt \wedge *_Y F_{\hat{A},Y}). \tag{2.24}$$

Apply the restriction map to the first equation:

$$\frac{1}{2}F_{\hat{A},Y} + i\sigma(\Phi,\Phi)|_Y = 0 \tag{2.25}$$

or

$$F_{\hat{A},Y} + 2i\sigma(\Phi,\Phi)|_Y = 0. \tag{2.26}$$

Call $\tau(\varphi,\psi) = 2\sigma(\varphi,\psi)|_Y$. Then the three-dimensional Seiberg-Witten equations become

$$\begin{aligned} &F_{\hat{A}} + i\tau(\Phi,\Phi) = 0 \\ &D_A\Phi = 0. \end{aligned} \tag{2.27}$$

In the presence of a preferred unit-length section of W, τ can be computed explicitly.

Given a unit-length 1-form e^1, expand it to a basis $\{e^1, e^2, e^3\}$ (at least locally) preserving the orientation. Let ξ be the global 2-plane field spanned by $\{e^2, e^3\}$, then

$W = \Lambda^{0,0}(\xi) \oplus \Lambda^{0,1}(\xi)$. The correspondence in the quarternionic structure is

$$e^1 \sim \mathfrak{i}, \qquad e^2 \sim \mathfrak{j}, \qquad e^3 \sim \mathfrak{k}. \tag{2.28}$$

In terms of 2-forms it is

$$e^2 \wedge e^3 \sim \mathfrak{i}, \qquad e^3 \wedge e^1 \sim \mathfrak{j}, \qquad e^1 \wedge e^2 \sim \mathfrak{k}. \tag{2.29}$$

Under this identification a pair $(\alpha, \beta) \in \Gamma^{0,0}(\xi) \oplus \Gamma^{0,1}(\xi)$ corresponds to $\alpha + \beta\mathfrak{j}$. So

$$\begin{aligned}
&-(\alpha + \beta\mathfrak{j})\mathfrak{i}(\overline{\alpha} - \mathfrak{j}\overline{\beta}) \\
=&(-\alpha\mathfrak{i} + \beta\mathfrak{k})(\overline{\alpha} - \mathfrak{j}\overline{\beta}) \\
=&-\alpha\mathfrak{i}\overline{\alpha} + \beta\mathfrak{k}\overline{\alpha} + \alpha\mathfrak{i}\mathfrak{j}\overline{\beta} - \beta\mathfrak{k}\mathfrak{j}\overline{\beta} \\
=&(-\alpha\overline{\alpha} + \beta\overline{\beta})\mathfrak{i} + \beta\mathfrak{k}\overline{\alpha} + \alpha\mathfrak{k}\overline{\beta} \\
=&(-|\alpha|^2 + |\beta|^2)\mathfrak{i} - 2\overline{\alpha}\beta\mathfrak{k}.
\end{aligned} \tag{2.30}$$

Notice the correspondence

$$\begin{aligned}
\mathfrak{i} &\xrightarrow{g} \tfrac{1}{2}(dt \wedge e^1 + e^2 \wedge e^3) \xrightarrow{r} \tfrac{1}{2}(e^2 \wedge e^3) \\
\mathfrak{j} &\xrightarrow{g} \tfrac{1}{2}(dt \wedge e^2 + e^3 \wedge e^1) \xrightarrow{r} \tfrac{1}{2}(e^3 \wedge e^1) \\
\mathfrak{k} &\xrightarrow{g} \tfrac{1}{2}(dt \wedge e^3 + e^1 \wedge e^2) \xrightarrow{r} \tfrac{1}{2}(e^1 \wedge e^1).
\end{aligned} \tag{2.31}$$

Here r is the restriction. By the definition of τ:

$$\tau((\alpha, \beta), (\alpha, \beta)) = (-|\alpha|^2 + |\beta|^2, -2\overline{\alpha}\beta). \tag{2.32}$$

Here $(-|\alpha|^2 + |\beta|^2, -2\overline{\alpha}\beta) \in \Gamma(\mathbb{R}\theta) \oplus \Gamma(\mathbb{C}\theta^{\perp})$.

For simplicity of later computation, we take the liberty of changing a constant here and instead consider the equations

$$\begin{aligned} &F_{\hat{A}} + \frac{i}{2}\tau(\Phi, \Phi) = 0 \\ &D_A\Phi = 0. \end{aligned} \tag{2.33}$$

Notice the change of constant in front of τ is non material as it is just a rescaling of Φ in the first equation, and rescaling Φ does not affect the second equation.

In terms of (α, β), the equations can also be written as

$$\begin{aligned} &F_{\hat{A}} = \frac{i}{2}(|\alpha|^2 - |\beta|^2, 2\overline{\alpha}\beta) \\ &D_A(\alpha, \beta) = 0. \end{aligned} \tag{2.34}$$

More References

[4] [14] [29] [32] [36] [37] [44] [49]

Chapter 3 The Equations on a Cylinder

In this chapter we study the equations on a cylinder $\mathbb{R}\times\Sigma$ and relate them to the Dirac functional on Σ. Furthermore we study the critical points to the Dirac functional and show that they solve the Kähler-Vortex equations on Σ. We also study the moduli space of the Kähler-Vortex equations.

3.1 Equations on $\mathbb{R}\times\Sigma$

3.1.1 Simplifying the Equations

Consider a 3-manifold $\mathbb{R}\times\Sigma$ where Σ is an oriented genus g Riemann surface, possibly not path connected. There is a canonical *spin$_c$* structure on Σ, coming from the almost complex structure on Σ, with $W^+ = \Lambda^{0,0}_\Sigma$, $W^- = \Lambda^{0,1}_\Sigma$. The almost complex structure here is given by the metric and orientation. The product foliation of the tangent space $T(\mathbb{R}\times\Sigma)$ gives rise to a *spin$_c$* structure on $\mathbb{R}\times\Sigma$, compatible with the canonical *spin$_c$* structure on Σ induced by the almost complex structure, in the sense that $W_{\mathbb{R}\times\Sigma} = W^+_\Sigma \oplus W^-_\Sigma$ and Clifford multiplication by dt induces an automorphism of W^+ and W^- respectively.

Any *spin$_c$* structure on Σ can be written as $W^+ = \Lambda^{0,0}_\Sigma \otimes L$ and $W^- = \Lambda^{0,1}_\Sigma \otimes L$ where L is a complex line bundle on Σ. Fix a Hermitian connection A on L, it gives rise to a *spin* connection on $W^\pm$. The Dirac operator is given by a matrix

$$\begin{pmatrix} 0 & D_A^+ \\ D_A^- & 0 \end{pmatrix}.$$

On $\mathbb{R} \times \Sigma$, the corresponding Dirac operator is given by

$$\begin{pmatrix} \gamma(dt)\nabla^{spin}_{\frac{\partial}{\partial t}} & D_A^+ \\ D_A^- & \gamma(dt)\nabla^{spin}_{\frac{\partial}{\partial t}} \end{pmatrix}.$$

Notice $\nabla^{spin}_{\frac{\partial}{\partial t}}\Phi = \frac{\partial \Phi}{\partial t}$. Composing with the Clifford multiplication of dt on the spinor bundle, the Dirac operator is represented by the matrix

$$\begin{pmatrix} i\frac{\partial}{\partial t} & D_A^+ \\ D_A^- & -i\frac{\partial}{\partial t} \end{pmatrix}.$$

Consider now a *spin$_c$* structure on $\mathbb{R} \times \Sigma$ which is induced from a line bundle L on Σ. Then, $W = W^+ \oplus W^- = (\Lambda^{0,0}_\Sigma \otimes L) \oplus (\Lambda^{0,1}_\Sigma \otimes L)$, and write $\Phi = (\Phi^+, \Phi^-)$. Here we make no distinction between a bundle on Σ and its pullback to $\mathbb{R} \times \Sigma$.

A connection A on $\pi^*(L)|_{\mathbb{R}\times\Sigma}$ is given by $A = A(t) + u(t)dt$ where $A(t)$ is a connection on $L|_\Sigma$ and $u(t)$ is an imaginary valued function on Σ. We say A is in temporal gauge if $u = 0$. We have

Lemma 3.1 *If A is in temporal gauge, then equations 2.33 can be simplified to*

$$\begin{aligned} \frac{\partial \Phi^+}{\partial t} &= iD_{A(t)}\Phi^- \\ \frac{\partial \Phi^-}{\partial t} &= -iD_{A(t)}\Phi^+ \\ \frac{\partial A(t)}{\partial t} &= -i\overline{\Phi^+}\Phi^- \\ * F_{A(t)} &= \frac{i}{2}(|\Phi^+|^2 - |\Phi^-|^2). \end{aligned} \tag{3.1}$$

Proof: For the Dirac operators we have

$$\begin{aligned} D_A\Phi^\pm &= D_{A(t)+u(t)dt}\Phi^\pm \\ &= D_{A(t)}\Phi^\pm + \gamma(dt)(\frac{\partial \Phi^\pm}{\partial t}) + 2u(t)\gamma(dt)\Phi^\pm \\ &= D_{A(t)}\Phi^\pm \mp i\frac{\partial \Phi^\mp}{\partial t} \pm 2iu(t)\Phi^\pm. \end{aligned}$$

And

$$F_{A(t)+u(t)dt} = F_{A(t)} + \frac{\partial A(t)}{\partial t} \wedge dt + 2du(t) \wedge dt. \tag{3.2}$$

If A is in the temporal gauge, that is, $u(t) = 0$, then

$$\begin{aligned} D_A \Phi^{\pm} &= D^{\pm}_{A(t)} \Phi^{\pm} \mp i \frac{\partial \Phi^{\mp}}{\partial t} \\ F_{A(t)} &= F_{A(t)} + \frac{\partial A(t)}{\partial t} \wedge dt. \end{aligned} \tag{3.3}$$

Plugging into the Seiberg-Witten equations 2.33, we get

$$\begin{aligned} D_{A(t)} \Phi^{+} &= i \frac{\partial \Phi^{-}}{\partial t} \\ D_{A(t)} \Phi^{-} &= -i \frac{\partial \Phi^{+}}{\partial t} \\ * F_{A(t)} &= \frac{1}{2}(|\Phi^{+}|^2 - |\Phi^{-}|^2) \\ \frac{\partial A(t)}{\partial t} &= i \overline{\Phi^{+}} \Phi^{-}. \end{aligned} \tag{3.4}$$

Or

$$\begin{aligned} \frac{\partial \Phi^{+}}{\partial t} &= i D_{A(t)} \Phi^{-} \\ \frac{\partial \Phi^{-}}{\partial t} &= -i D_{A(t)} \Phi^{+} \\ \frac{\partial A(t)}{\partial t} &= i \overline{\Phi^{+}} \Phi^{-} \\ * F_{A(t)} &= \frac{i}{2}(|\Phi^{+}|^2 - |\Phi^{-}|^2). \end{aligned} \tag{3.5}$$

This proves the Lemma. □

3.1.2 The Dirac Functional

Lemma 3.2 *The first three equations of Lemma 3.1 are the gradient flow equations* *of the Dirac functional* $\mathcal{F} : i\Omega^1(\Sigma) \times \Gamma(W^+) \times \Gamma(W^-) \mapsto \mathbb{R}$ *defined by*

$$\mathcal{F}(A, \Phi^+, \Phi^-) = \int_\Sigma (D_A\Phi^+, i\Phi^-) \tag{3.6}$$

where $(,)$ *is the real inner product on* W^-.

Proof: We claim

$$D\mathcal{F}_{(a,\phi^+,\phi^-)} = \int_\Sigma (a, i\overline{\Phi^+}\Phi^-) + (\phi^+, iD_A\Phi^-) + (\phi^-, -iD_A\Phi^+). \tag{3.7}$$

Given the claim, the Lemma follows immediately.

To prove the claim, compute

$$\begin{aligned}
&D\mathcal{F}_{(A,\Phi^+,\Phi^-)}(a, \phi^+, \phi^-)\\
=&\frac{\partial}{\partial t}\{(F(A + ta, \Phi^+ + t\phi^+, \Phi^- + t\phi^-) - F(A, \Phi^+, \Phi^-))\}\Big|_{t=0}\\
=&\frac{\partial}{\partial t}\{(\int_\Sigma t(\gamma(a)\Phi^+, i\Phi^-) + t(D_A\phi^+, i\Phi^-) + t(D_A\Phi^+, i\phi^-))\}\Big|_{t=0}\\
=&\frac{\partial}{\partial t}\{(\int_\Sigma t(\gamma(a)\Phi^+, i\Phi^-) + t(\phi^+, iD_A\Phi^-) + t(-iD_A\Phi^+, \phi^-))\}\Big|_{t=0}\\
=&\int_\Sigma (\gamma(a)\Phi^+, i\Phi^-) + (\phi^+, iD_A\Phi^-) + (\phi^-, -iD_A\Phi^+)\\
=&\int_\Sigma (a, i\overline{\Phi^+}\Phi^-) + (\phi^+, iD_A\Phi^-) + (\phi^-, -iD_A\Phi^+).
\end{aligned}$$

Here we used the fact that $(\gamma(a)\Phi^+, i\Phi^-) = (a, i\overline{\Phi^+}\Phi^-)$. $\Box$

The Hessian of $\mathcal{F}$, at a critical point $(A_0, \Phi_0^+, \Phi_0^-)$, is then given by

$$Hess\mathcal{F}_{(A_0,\Phi_0^+,\Phi_0^-)}((a_1, \varphi_1^+, \varphi_1^-), (a_2, \varphi_2^+, \varphi_2^-)) = \langle (a_1, \varphi_1^+, \varphi_1^-), L(a_2, \varphi_2^+, \varphi_2^-)\rangle \tag{3.8}$$

for any pairs $(a_i, \varphi_i^+, \varphi_i^-), i = 1, 2$. Here L is the self-adjoint operator

$$\begin{aligned}
L : i\Omega^1(\Sigma) \times \Gamma(W^+) \times \Gamma(W^-) \to i\Omega^1(\Sigma) \times \Gamma(W^+) \times \Gamma(W^-)\\
L(a, \varphi^+, \varphi^-) = (i\overline{\varphi^+}\Phi_0^- + i\overline{\Phi_0^+}\varphi^-, iD_{A_0}\varphi^- + i\gamma(a)\Phi_0^-, -iD_{A_0}\varphi^+ - i\gamma(a)\Phi_0^+)
\end{aligned} \tag{3.9}$$

On the configuration space subject to the restriction

$$* F_{A(t)} = \frac{i}{2}(|\Phi^+|^2 - |\Phi^-|^2). \tag{3.10}$$

L is a first order differential operator. That it is of first order on $\Gamma(W^+) \times \Gamma(W^-)$ is clear, to see it is of first order on $i\Omega^1(\Sigma)$ as well, observe that the leading term of $i\Omega^1(\Sigma)$ in $L^*L(a, \varphi^+, \varphi^-)$ is:

$$\overline{-D_{A_0}\varphi^-}\Phi_0^- + \overline{\Phi_0^+}D_{A_0}\varphi^+. \tag{3.11}$$

But

$$2 * d_{A_0}a = i(\langle\Phi_0^+, \varphi^+\rangle - \langle\Phi_0^-, \varphi^-\rangle). \tag{3.12}$$

Apply $\overline{\partial}_{A_0}$ to both sides, and notice that $D_{A_0}\varphi^+ = \overline{\partial}_{A_0}\varphi^+$, and $D_{A_0}\overline{\varphi}^- = *\overline{\partial}^*_{A_0}\varphi^-$. Therefore the leading term is given by $2\overline{\partial}_{A_0} * d_{A_0}a$. In summary

Corollary 3.3 *The linearization L of $\nabla\mathcal{F}$ at a critical point $(A_0, \Phi_0^+, \Phi_0^-)$ is a first order self-adjoint elliptic differential operator. L has real discrete spectrum, unbounded in both directions. Each eigenvalue is of finite multiplicity. In matrix form L can be expressed as*

$$\begin{pmatrix} 0 & i\overline{()}\Phi_0^- & i\overline{\Phi_0^+}() \\ -i\gamma()\Phi_0^- & 0 & iD_{A_0} \\ i\gamma()\Phi_0^+ & -iD_{A_0} & 0 \end{pmatrix}. \tag{3.13}$$

3.2 The Moment Map

3.2.1 The Constraint

The constraint 3.10 is worth more attention. First it is preserved by the gradient flow. More precisely,

Lemma 3.4 *Assume 3.10 holds at $t = t_0$ for the gradient flow from t_0 to t_1, then it also holds at $t = t_1$.*

Proof: From the first two equations we get

$$\frac{\partial}{\partial t}(\frac{i}{2}(|\Phi^+|^2 - |\Phi^-|^2)) = d_\Sigma(\frac{i}{2}(\Phi^+\overline{\Phi^-} + \Phi^-\overline{\Phi^+})). \tag{3.14}$$

And we also know

$$\frac{\partial}{\partial t}(F_{A(t)}) = d_\Sigma(\frac{\partial A}{\partial t}). \tag{3.15}$$

Integrating the above from t_0 to t_1 gives the result. □

We can also interpret these calculations from a more conceptual point of view. Let $\mathcal{A}_\Sigma = \{(A, \Phi^+, \Phi^-)|\Phi^+ \in \Gamma(W^+), \Phi^- \in \Gamma(W^-)\}$ be the configuration space. There is a symplectic structure on $\mathcal{A}_\Sigma$ defined by

$$\Omega((a_1, \varphi_1^+, \varphi_1^-), (a_2, \varphi_2^+, \varphi_2^-)) = \int_\Sigma -a_1 \wedge a_2 + Im(\varphi_1^+\overline{\varphi_2^+} - \varphi_1^-\overline{\varphi_2^-}). \tag{3.16}$$

The symplectic structure is the usual one on the configuration space of Hermitian connections over a $U(1)$ bundle. On the spinor bundle $W^+ \oplus W^-$, it is the symplectic structure on each line bundle summand, with the orientation of W^- reversed.

3.2.2 The Moment Map

Given a symplectic manifold (M, ω) and a Lie group $\mathfrak{G}$ acting on M, the moment map of the action is a map $f : M \mapsto \mathfrak{g}^*$, where $\mathfrak{g}^*$ is the dual of the Lie algebra of $\mathfrak{G}$, satisfying

$$df(\xi) = \omega(X_\xi, .). \tag{3.17}$$

Where ξ is an element of the Lie algebra and X_ξ is the associated left invariant vector field.

Let M be the infinite dimensional configuration space $\mathcal{A}_\Sigma$, and $\mathfrak{G}$ be the gauge group $\mathcal{G} = Map(\Sigma, S^1)$. Then

Lemma 3.5 *The constraint 3.10 is the moment map of the $\mathcal{G}$ action on the symplectic manifold $(\mathcal{A}_\Sigma, \Omega)$.*

Proof: To prove the statement, we split our discussion into the connection part and spinor part. Let us first look at the connection part. The Lie algebra of the group $\mathcal{G}$ is the mapping space $Map(\Sigma, i\mathbb{R})$. Now for $\xi : \Sigma \mapsto i\mathbb{R}$, the left invariant vector field is $X_\xi(A) = d_A\xi \in i\Omega^1(\Sigma)$ as the gauge group element u acts on a connection A by $u(A) = A - u^{-1}du$. For another vector field $\eta \in i\Omega^1(\Sigma)$, we have

$$\begin{aligned}
\omega_A(X_u(A), \eta(A)) &= -\int_\Sigma X_\xi(A) \wedge \eta(A) \\
&= -\int_\Sigma d_A(\xi) \wedge \eta \\
&= \int_\Sigma \xi \wedge d_A\eta \\
&= \int_\Sigma \xi d_A\eta.
\end{aligned}$$

The last equality follows from the fact that ξ is an imaginary function. On the other hand for the map $f : A \mapsto *F_A$, the differential is given by $Df_A(\eta) = *d_A\eta$ so

$$\begin{aligned}
Df_A(\eta)(\xi) &= \int_\Sigma \xi * d_A\eta * vol \\
&= \int_\Sigma \xi d_A\eta.
\end{aligned}$$

This proves the connection part.

For the spinor part, a gauge group element u acts on (Φ^+, Φ^-) by

$$u((\Phi^+, \Phi^-)) = (u\Phi^+, u\Phi^-),$$

hence, for a Lie algebra element ξ, the left invariant vector field is $X_\xi((\Phi^+, \Phi^-)) = (\xi\Phi^+, \xi\Phi^-)$. Notice for function $g : (\Phi^+, \Phi^-) \mapsto \frac{i}{2}(|\Phi^+|^2 - |\Phi^+|^2)$, the differential is given by

$$\begin{aligned} Dg_{(\phi^+,\phi^-)}(\xi)((\Phi^+,\Phi^-)) &= \int_\Sigma \xi(\phi^+\overline{\Phi^+} + \Phi^+\overline{\phi^+} - \phi^-\overline{\Phi^-} - \Phi^-\overline{\phi^-}) \\ &= \int_\Sigma Im(\xi\Phi^+\overline{\phi^+} - \xi\Phi^-\overline{\phi^-}). \end{aligned}$$

Since ξ is purely imaginary. On the other hand

$$\begin{aligned} &\omega((\xi\Phi^+,\xi\Phi^-),(\phi^+,\phi^-)) \\ &= \int_\Sigma Im(\xi\Phi^+\overline{\phi^+} - \xi\Phi^-\overline{\phi^-}). \end{aligned}$$

So this proves the spinor part. $\square$

3.3 Critical Points of $\mathcal{F}$

3.3.1 The Kähler-Vortex Equations

Now we study the critical point set of $\mathcal{F}$ on Σ, with the additional constraint 3.10. This is the set of solutions to the following equations on Σ:

$$\begin{cases} D_A^+\Phi^+ = 0 \\ D_A^-\Phi^- = 0 \\ \Phi^+\overline{\Phi^-} = 0 \\ *F_A = \frac{i}{2}(|\Phi^+|^2 - |\Phi^-|^2) \end{cases} \tag{3.18}$$

Identifying Φ^+ with $\Lambda^{0,0}(L)$ and Φ^- with $\Lambda^{0,1}(L)$, the Dirac operators are $\overline{\partial_A}$ and $\overline{\partial_A}^*$ respectively. So we have:

$$\begin{cases} \overline{\partial_A}\Phi^+ = 0 \\ \overline{\partial_A}^*\Phi^- = 0 \\ \Phi^+\overline{\Phi^-} = 0 \\ *F_A = \frac{i}{2}(|\Phi^+|^2 - |\Phi^-|^2) \end{cases} \tag{3.19}$$

From the first 3 equations and the unique continuation theorem we know either Φ^+ or Φ^- vanishes identically. Integrate the last equation over Σ and we get

$$-2\pi i c_1(\det W) = \int_\Sigma \frac{i}{2}(|\Phi^+|^2 - |\Phi^-|^2). \tag{3.20}$$

If we assume $\Phi^- = 0$ then

$$\begin{aligned} &\overline{\partial_A}\Phi^+ = 0 \\ &* F_A = \frac{i}{2}|\Phi^+|^2. \end{aligned} \tag{3.21}$$

Notice $*F_A = \langle F_A, Vol\rangle = \frac{1}{2}\langle F_A, \omega\rangle$ where ω is the $(1,1)$ symplectic form on Σ. The above can also be written as

$$\begin{aligned} &\overline{\partial_A}\Phi^+ = 0 \\ &\langle F_A, \omega\rangle = i|\Phi^+|^2. \end{aligned} \tag{3.22}$$

These are the Kähler-Vortex equations on Σ, which were studied by many. [12] [33]

3.3.2 The Kazdan-Warner Equation

On a line bundle L over Σ, for a pair (A, s) where $s \in \Gamma(L)$, consider the equations

$$\begin{aligned} &\overline{\partial_A}s = 0 \\ &\langle F_A, \omega\rangle = i|s|^2. \end{aligned} \tag{3.23}$$

Fix the complex structure and change the metric within the conformal structure. This is the same as an action of e^f where f is a real valued function. We have

$$\begin{aligned} e^f(A) &= A + \partial\overline{f} - \overline{\partial} f \\ e^f(s) &= e^f s. \end{aligned} \tag{3.24}$$

Consequently

$$\overline{\partial_{e^f(A)}} = e^f \overline{\partial_A} e^{-f} \tag{3.25}$$

and

$$F_{e^f(A)} = F_A - 2\partial\overline{\partial} f = F_A - \frac{1}{2}\Delta f \omega. \tag{3.26}$$

As an equation for f the second equation of 3.23 becomes

$$-i\langle F_A, \omega\rangle - \Delta f = e^{2f}|s|^2. \tag{3.27}$$

This becomes the Kazdan-Warner equation [34] if both sides are multiplied by -1:

$$\Delta f - k = -e^{2f} h \tag{3.28}$$

where $h = |s|^2$ is a nonnegative, and somewhere positive, function. Furthermore,

$$k = -i\langle F_A, \omega\rangle. \tag{3.29}$$

Equation 3.28 arises naturally when J. L. Kazdan and F. W. Warner tried to prescribe scalar curvature on compact Riemann surfaces. They proved

Theorem 3.6 *(Kazdan-Warner) For any C^∞ function $h \geq 0$ and $h > 0$ somewhere. If $\int k > 0$, then the Kazdan-Warner equation has a unique solution.* $\square$

As a corollary

Corollary 3.7 *When $-i\langle F_A, \omega\rangle > 0$, the pairs (A, s) are in 1-1 correspondence with the holomorphic sections of the line bundle L.* $\square$

Remark 3.8 *Here F_A is the curvature of A, which is on the determinant line bundle of the spin$_c$ structure, with $W = L \oplus (L \otimes K^{-1})$ so*

$$\det W = L^{\otimes 2} \otimes K^{-1}. \tag{3.30}$$

3.3.3 The Moduli Spaces

When $c_1(\det W) \leq 0$, $\Phi^- = 0$. For L to have a section, $c_1(L) \geq 0$. Meanwhile $c_1(\det W) \leq 0$ means $2c_1(L) - c_1(K) \leq 0$. This is

$$0 \leq c_1(L) \leq g - 1. \tag{3.31}$$

After modulo the gauge group action, $[A, \Phi^+]$ correspond to an effective divisor on L. Denote the moduli space of the Kähler-Vortex equations corresponding to $L \oplus (L \otimes K^{-1})$ by $\mathcal{M}_L$. Then $\mathcal{M}_L = \mathcal{S}^{c_1(L)}\Sigma$. Here $\mathcal{S}^n\Sigma = \Sigma^n/S_n$ is the symmetric product of Σ.

A similar argument can be applied to $c_1(\det W) \geq 0$. Here the pairs (A, Φ^-) correspond to holomorphic L valued 1-forms. By Serre duality, they correspond to holomorphic sections of $K \otimes L^{-1}$. In this case the conditions translate to

$$g - 1 \leq c_1(L) \leq 2g - 2. \tag{3.32}$$

The symmetry between W^+ and W^-, and hence, between range of $0 \leq c_1(L) \leq g - 1$ and $g - 1 \leq c_1(L) \leq 2g - 2$, can also be seen through the conjugation map between L and $K \otimes L^{-1}$, which induces isomorphisms between $W = L \oplus (L \otimes K^{-1})$ and $W' = K \otimes L^{-1} \oplus L^{-1}$.

We compile the facts into the following

Theorem 3.9 *Let* $W = L \oplus (L \times K^{-1})$, *then*

$$\text{when } g = 0, \qquad \mathcal{M}_L = \emptyset;$$

$$\text{when } g > 0, \qquad \mathcal{M}_L = \begin{cases} \mathcal{S}^{c_1(L)}\Sigma & \text{if } 0 \leq c_1(L) \leq g-1 \\ \mathcal{S}^{c_1(K \times L^{-1})}\Sigma & \text{if } g-1 \leq c_1(L) \leq 2g-2 \\ 0 & \text{otherwise.} \end{cases} \tag{3.33}$$

Example 3.10 *When* L *is trivial,* $c_1(L) = 0$, *and* $\mathcal{M}_L$ *consists of isolated, whose number counts the number of path connected boundary components. In particular, if* Y *has a single boundary component, then* $\mathcal{M}_L$ *consists of a single point. Same for* $L = K$.

Example 3.11 *At the midpoint when* $L = K^{\frac{1}{2}}$ *and* $c_1(L) = g - 1$, *both* Φ^+ *and* Φ^- *vanish identically. And the connections are flat. The dimension of moduli space for flat connections is* $2g - 2$, *compatible here with* $\dim \mathcal{M}_L = 2(g-1)$.

More References

[3] [9] [12] [13] [28] [31] [33] [48] [52] [68]

Chapter 4 The Chern-Simons-Dirac Functional

In this chapter we introduce the Chern-Simons-Dirac functional on the 3-manifolds with boundary. While it is not gauge invariant, even in the S^1 sense, it defines a section of a $U(1)$ bundle on the quotient space $\mathcal{B}$. We also introduce a family of admissible perturbations and study their properties.

4.1 The Chern-Simons-Dirac Functional on a 3-manifold with Boundary

4.1.1 The Chern-Simons-Dirac Functional

For a compact 3-manifold the Chern-Simons-Dirac functional is defined by

$$\mathcal{CSD}(A,\Phi) = -\frac{1}{4\pi}\int_Y (A - A_0)\wedge(F_A + F_{A_0}) - \langle\Phi, D_A\Phi\rangle \tag{4.1}$$

for a pair (A,Φ) where A is a connection on the determinant line bundle of the $spin_c$ structure and Φ is a section of the Hermitian bundle W. Here A_0 is a reference connection and changing A_0 will only change the $\mathcal{CSD}(A,\Phi)$ by a constant for every A.

On a closed 3-manifold, acting by a gauge transformation changes the value by 2π times an integer. For a $g : Y \mapsto U(1)$, let $[g] \in H^1(Y;\mathbb{Z})$ be the pullback of the generator of $H^1(U(1);\mathbb{Z})$ through g, then

Lemma 4.1 $\mathcal{CSD}(g(A), g(\Phi)) - \mathcal{CSD}(A,\Phi) = 2\pi[g]\cup c_1(W)$.

Proof: Notice our choice of the gauge transformation $g(A) = A - g^{-1}dg$ makes it so that $\langle \Phi, D_A\Phi\rangle$ is invariant under the gauge transformation since

$$\begin{aligned}
\langle g(\Phi), D_{g(A)}g(\Phi)\rangle &= \langle g\Phi, D_{A-g^{-1}dg}g(\Phi)\rangle \\
&= \langle g\Phi, D_A g(\Phi) - \gamma(g^{-1}dg)g\Phi\rangle \\
&= \langle g\Phi, gD_A\Phi + \gamma(dg)\Phi - \gamma(dg)\Phi\rangle \\
&= \langle g\Phi, gD_A\Phi\rangle \\
&= \langle \Phi, D_A\Phi\rangle.
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
&\int_Y (g(A) - A_0)\wedge(F_{g(A)} + F_{A_0}) \\
&= \int_Y (A - g^{-1}dg - A_0)\wedge(F_{A-g^{-1}dg} + F_{A_0}) \\
&= \int_Y (A - A_0)\wedge(F_A + F_{A_0}) - g^{-1}dg\wedge(F_A - d(g^{-1}dg) \\
&\quad + F_{A_0}) - (A - g^{-1}dg - A_0)\wedge d(g^{-1}dg) \\
&= \int_Y (A - A_0)\wedge(F_A + F_{A_0}) - g^{-1}dg\wedge(F_A + F_{A_0}) + d(A - g^{-1}dg - A_0)\wedge g^{-1}dg \\
&= \int_Y (A - A_0)\wedge(F_A + F_{A_0}) - 2g^{-1}dg\wedge F_A.
\end{aligned}$$

So

$$\mathcal{CSD}(g(A), g(\Phi)) - \mathcal{CSD}(A,\Phi) = \frac{1}{2\pi}\int_Y g^{-1}dg\wedge F_A = 2\pi[g]\cup c_1(W) \tag{4.2}$$

as wanted. □

When Y has a boundary Σ the above computation brings in an extra boundary integral $\frac{1}{4\pi}\int_\Sigma (A - A_0)\wedge g^{-1}dg$ by integration by parts on the third equation above.

4.1.2 The $U(1)$ Bundle $\mathcal{L}_\Sigma$

Let

$$\mathcal{A}_\Sigma = \{(A, \Phi)\big| A \in L^2_{k;\Sigma}(i\Lambda^1(\Sigma)), \Phi \in L^2_{k;\Sigma}(W)\}$$
$$\mathcal{G}_\Sigma = \{g\Big| g \in L^2_{k+1;\Sigma}(Map(\Sigma, U(1)))\}$$

be the configuration space and gauge group for $k \geq 2$. $\mathcal{G}_\Sigma$ acts on $\mathcal{A}_\Sigma$ smoothly. We let $\mathcal{B}_\Sigma = \mathcal{A}_\Sigma/\mathcal{G}_\Sigma$ be the quotient space.

We shall use the correction term in last section to define a $U(1)$ bundle $\mathcal{L}$ over $\mathcal{A}_\Sigma$ and show it descends to $\mathcal{B}_\Sigma$.

Define a map $\Theta : \mathcal{A}_\Sigma \times \mathcal{G}_\Sigma \mapsto U(1)$ by

$$\Theta(A, \Phi, g) = exp(i(\mathcal{CSD}(\tilde{g}(\tilde{A}), \tilde{g}(\tilde{\Phi})) - \mathcal{CSD}(\tilde{A}, \tilde{\Phi}))). \tag{4.3}$$

Here $\tilde{A}$, $\tilde{\Phi}$ and $\tilde{g}$ are global extensions of the corresponding data from Σ to Y. The choices here are immaterial since explicitly we have

$$\Theta(A, \Phi, g) = exp(\frac{i}{4\pi}\int_\Sigma (A - A_0) \wedge g^{-1}dg). \tag{4.4}$$

The formula suggests we write $\Theta(A, \Phi, g) = \Theta(A, g)$. From the above it is also evident that Θ is a cocycle. That is,

$$\Theta(A, g)\Theta(g(A), h) = \Theta(A, gh). \tag{4.5}$$

Θ induces a line bundle $\mathcal{L}_\Sigma$ over $\mathcal{B}_\Sigma$:

$$\mathcal{L}_\Sigma = \mathcal{A}_\Sigma \times_\Theta U(1) = \mathcal{A}_\Sigma \times U(1)/ \sim$$

where the equivalence relation is given by

$$(A, \Phi, u) \sim (g(A), g(\Phi), \Theta(A, g)u).$$

From equation 4.4 and the Sobolev multiplication theorem we conclude

Lemma 4.2 Θ *is smooth.* $\square$

4.1.3 The Connection ω

We can define a connection on $\mathcal{L}_\Sigma$ as follows. First we define a global 1-form, using the trivilization $\mathcal{A}_\Sigma \times \{1\}$, by the formula

$$\omega(a, \varphi^+, \varphi^-) = \frac{i}{4\pi} \int_\Sigma -(A - A_0) \wedge a + Im(\Phi^+ \overline{\varphi^+}^* - \Phi^- \overline{\varphi^-}^*) \tag{4.6}$$

for $(a, \varphi^+, \varphi^-) \in T_{(A,\Phi)} A_\Sigma$ and then extend the connection $U(1)$ equivariantly.

The curvature of the connection ω, evaluated on two vectors in $T_{(A,\Phi)} A_\Sigma$, is given by

$$\begin{aligned} & d\omega((a_1, \varphi_1^+, \varphi_1^-), (a_2, \varphi_2^+, \varphi_2^-)) \\ =& \frac{i}{2\pi} \int_\Sigma -a_1 \wedge a_2 + Im(\varphi_1^+ \overline{\varphi_2^+}^* - \varphi_1^- \overline{\varphi_2^-}^*). \end{aligned} \tag{4.7}$$

This 2-form, without the factor of i, is the symplectic form Ω on $\mathcal{A}_\Sigma$.

For a $U(1)$ bundle U over a symplectic manifold (M, Ω), if there is a connection ω so that $F = d\omega = i\Omega$, then such a connection defines a contact structure on the total space of U.

$\mathcal{G}_\Sigma$ acts on the configuration space $\mathcal{A}_\Sigma$. With the help of Θ, $\mathcal{G}_\Sigma$ acts on $\mathcal{L}_\Sigma$ by sending (A, Φ, u) to $(g(A), g(\Phi), u\Theta(A, g))$.

Lemma 4.3 *The $\mathcal{G}_\Sigma$ action on $\mathcal{L}_\Sigma$ is a contactomorphism, that is, $\mathcal{G}_\Sigma$ preserves the connection ω.*

Proof: Since

$$g(A, \Phi, u) = (A - g^{-1}dg, g\Phi, u\Theta(A, g)) \tag{4.8}$$

we have

$$g_*(\delta A, \delta\Phi, \delta u) = (g^{-1}\delta A g, g\delta\Phi, \delta u\Theta(A,g) + u\delta\Theta(A,g)). \quad (4.9)$$

So

$$\begin{aligned}
& g^*\omega_{(A,\Phi,u)}(\delta A, \delta\Phi, \delta u) \\
=& \omega_{(A-g^{-1}dg, g\Phi, u\Theta(A,g))}(g_*(\delta A, \delta\Phi, \delta u)) \\
=& \frac{i}{4\pi}\int_\Sigma -(A - A_0 - g^{-1}dg)\wedge\delta A + Im(g(\Phi^+)\overline{g\delta(\Phi^+)}^* - g(\Phi^-)\overline{g\delta(\Phi^-)}^*) \\
& + u^{-1}\Theta^{-1}(\delta u\Theta + u\delta\Theta) \\
=& \frac{i}{4\pi}\int_\Sigma -(A - A_0)\wedge\delta A + g^{-1}dg\wedge\delta A + Im(\Phi^+\overline{\delta(\Phi^+)}^* - \Phi^-\overline{\delta(\Phi^-)}^*) \\
& + u^{-1}\delta u + \Theta^{-1}\delta\Theta \qquad (4.10) \\
=& \omega_{(A,\Phi,u)}(\delta A, \delta\Phi, \delta u) + \frac{i}{4\pi}\int_\Sigma g^{-1}dg\wedge\delta A + \Theta^{-1}\delta\Theta \\
=& \omega_{(A,\Phi,u)}(\delta A, \delta\Phi, \delta u) + \frac{i}{4\pi}\int_\Sigma g^{-1}dg\wedge\delta A + \frac{i}{4\pi}\int_\Sigma \delta A\wedge g^{-1}dg \\
=& \omega_{(A,\Phi,u)}(\delta A, \delta\Phi, \delta u).
\end{aligned}$$

This proves the Lemma. □.

4.1.4 The $\mathcal{CSD}$ Functional as a Section

The $U(1)$ bundle $\mathcal{L}_\Sigma$ over $\mathcal{B}_\Sigma$ pulls back to a $U(1)$ bundle $\mathcal{L}_Y$ on $\mathcal{B}_Y$, and inherit the connection by pullback. We next study the $\mathcal{CSD}$ functional as a section of the bundle and understand its critical point set.

Consider the section $s : \mathcal{A} \mapsto \mathcal{A}\times U(1)$ given by

$$s(A,\Phi) = (A, \Phi, exp(i\mathcal{CSD}(A,\Phi))). \quad (4.11)$$

s is Θ equivariant thus descends to a continuous section of the quotient bundle $\mathcal{L}_Y$.

First let us prove a Lemma. If we write $\varphi_i = (\varphi_i^+, \varphi_i^-)$ then

Lemma 4.4

$$\int_Y (D_A\varphi_2, \varphi_1) - (D_A\varphi_1, \varphi_2) = \int_\Sigma (\varphi_1^+, i\varphi_2^+) - (\varphi_1^-, i\varphi_2^-). \tag{4.12}$$

Proof: Let $\mathbf{n}$ be the normal vector of boundary Σ, then by [44] equation 5.7

$$\int_Y (D_A\varphi_2, \varphi_1) - (D_A\varphi_1, \varphi_2) = \int_\Sigma (\varphi_1, \gamma(\mathfrak{n})\varphi_2). \tag{4.13}$$

In our case, $\mathfrak{n}$ acts on W^+ by multiplication by i and on W^- by multiplication by $-i$, and the Lemma follows immediately. □

Given the Lemma, we have

Lemma 4.5 *The gradient vector field of the section s is*

$$\mathcal{V} = \nabla_\omega s(A, \Phi) = \frac{1}{2\pi}(*(F_A + F_{A_0} + \frac{i}{2}\tau(\Phi, \Phi)), D_A\Phi). \tag{4.14}$$

Proof: Given $(a, \varphi) \in T_{(A,\Phi)}A$, we compute

$$\begin{aligned}
Ds(A, \Phi)(a, \varphi) =& s^{-1}ds(a, \varphi) + \omega(a, \varphi) \\
=& -\frac{i}{4\pi}\int_Y a \wedge (F_A + F_{A_0}) + (A - A_0) \wedge da \\
&+ (\Phi, \gamma(a)\Phi) + (\varphi, D_A\Phi) + (\Phi, D_A\varphi)) + \omega(a, \varphi) \\
=& -\frac{i}{4\pi}\int_Y 2a \wedge (F_A + F_{A_0}) + a \wedge i\tau(\Phi, \Phi) + 2(\varphi, D_A\Phi) \\
&+ \frac{i}{4\pi}\int_\Sigma (A - A_0) \wedge a - (\Phi^+, i\varphi^+) + (\Phi^-, i\varphi^-) + \omega(a, \varphi) \\
=& -\frac{i}{4\pi}\int_Y 2a \wedge (F_A + F_{A_0}) + a \wedge i\tau(\Phi, \Phi) + 2(\varphi, D_A\Phi) \\
=& -\frac{i}{2\pi}\int_Y a \wedge (F_A + F_{A_0} + \frac{i}{2}\tau(\Phi, \Phi)) + (\varphi, D_A\Phi).
\end{aligned}$$

The Lemma then follows. □

The critical points for the gradient satisfy the following equations:

$$\begin{aligned} &F_A + F_{A_0} + \frac{i}{2}\tau(\Phi, \Phi) = 0 \\ &D_A\Phi = 0. \end{aligned} \tag{4.15}$$

We define the moduli space to be $\mathcal{M} = \mathcal{V}^{-1}(0)/\mathcal{G}_Y$.

4.2 Perturbations of $\mathcal{CSD}$ Functional

4.2.1 Admissible Perturbations

In [30] a thicken link in Y is chosen and the perturbation is obtained by computing the integral of holonomies of the connection along the loops. Here we take a similar approach.

More precisely, let $\phi = \{\gamma_i\}_{i=1}^k$ be a finite collection of embeddings of solid tori $\gamma_i : S^1 \times D^2 \to Y$ with $\gamma_i(1, 0) = x_0$ where x_0 is a fixed point on Y. A corresponding collection of functions $\tilde{h}_i \in C^2(S^1, \mathbb{R})$ is chosen. We denote the set of such functions by $\tilde{\mathcal{H}}$ and give it the compact-open C^2 topology. Furthermore, let $\psi = \{\delta_j\}_{j=1}^l$ be a finite collection of embeddings of solid cylinders $\delta_j : [0, 1] \times D^2 \to Y$ with $\delta_j(0, 0) = x_0$. We also choose a corresponding collection of functions $\hat{h}_j \in C^2(\mathbb{R}^+, \mathbb{R})$. The set of such functions will be denoted by $\hat{\mathcal{H}}$ while the topology is also given by compact-open C^2 topology.

We define a function $h : \mathcal{A} \to \mathbb{R}$ from the above choices by:

$$h(A, \Phi) = \sum_{i=1}^{k} \int_{D^2} \tilde{h}_i(hol_{\gamma_i}(\gamma_i(1, x), A))\eta(x) + \frac{1}{2}\sum_{j=1}^{l} \int_{D_2} (\int_{I\times\{x\}} \hat{h}_j(|\Phi|^2))\eta(x)$$

here $\eta(x)$ is a fixed radially symmetric cut-off function on the unit disc. The function h coming from the above choices is called an admissible function and we denote the space of admissible functions by $\mathcal{H}$.

The gauge invariance of the admissible perturbation is immediate from the expression. Let

$$s_h(A,\Phi) = s(A,\Phi)exp(ih(A,\Phi)). \tag{4.16}$$

Then s_h defines a section on the $U(1)$ bundle $\mathcal{L}$. We denote the gradient vector field of s_h by $\mathcal{V}_h$ and call $\mathcal{M}_h = \mathcal{V}_h^{-1}(0)/\mathcal{G}_Y$ the perturbed moduli space corresponding to h.

For a given admissible perturbation h, there are two ways to vary it:

1. Fix the embeddings and vary functions $\tilde{h}_i$'s and $\hat{h}_j$'s;

2. Vary the embeddings.

For most of our purposes it suffices to consider (1). For this reason we use $\mathcal{H}_{\phi,\psi}$ to denote the set of admissible perturbations coming from the embeddings (ϕ,ψ).

There is a partial ordering in the collection of sets $\mathcal{H}_{\phi,\psi}$. The order is given by the inclusion relation of the embeddings of ϕ and ψ. More precisely, we say $\mathcal{H}_{\phi,\psi} \lessdot \mathcal{H}_{\phi',\psi'}$ if and only if $\phi \subset \phi'$ and $\psi \subset \psi'$. In this case we may regard $\mathcal{H}_{\phi,\psi}$ as a subset of $\mathcal{H}_{\phi',\psi'}$ by putting

$$\tilde{h}_i, \hat{h}_j = 0, \text{for } \gamma_i \notin \phi, \delta_j \notin \psi.$$

For two collections of embeddings ϕ_1, ψ_1 and ϕ_2, ψ_2, there is an obvious common upper bound for $\mathcal{H}_{\phi,\psi}$ and $\mathcal{H}_{\phi',\psi'}$, that is, $\mathcal{H}_{\phi_1\cup\phi_2,\psi_1\cup\psi_2}$. The set $\mathcal{H}$ can be viewed as the limit set under this partial ordering. Notice the topology on $\mathcal{H}_{\phi,\psi}$ is compatible with the partial ordering $\lessdot$; therefore, there is a unique (roughest) topology on the limit set, and we take this topology as the one on $\mathcal{H}$. The topology is characterized as the roughest of all on which so that when restricted to individual $\mathcal{H}_{\phi,\psi}$ the two are the same.

Under this topology the space $\mathcal{H}$ is path connected. For any two admissible perturbations h^0 and h^1, a path connecting the two can be seen as follows. Denote the embeddings of thicken tori and interval by ϕ_i, ψ_i, $i = 0, 1$, respectively. Consider the component $\mathcal{H}_{\phi_0\cup\phi_1,\psi_0\cup\psi_1}$. h^0 and h^1 can be regarded as its elements by extending the $\tilde{h}_i$'s, $\hat{h}_j$'s to be zero outside the original tori and intervals. Therefore, we may define the path by $h^t = (1-t)h^0 + th^1$.

4.2.2 The Gradient of h

The gradient of the function h, denoted by $\nabla h(A, \Phi)$, is a vector field on the configuration space which is characterized by

$$Dh(A,\Phi)(a,\phi) = \langle \nabla h(A,\Phi), (a,\phi)\rangle_{L^2}. \tag{4.17}$$

From the definition it is immediate that $\mathcal{V}_h = \mathcal{V} + \nabla h$.

Lemma 4.6 *Let h be an admissible function, then*

$$\begin{aligned}\nabla h(A,\Phi)(x) = (&\sum_{i=1}^{k} \tilde{h}_i'(hol_{\gamma_i}(\gamma_i(1,x),A))\eta(P_2\gamma_i^{-1}(x))(\gamma_i^{-1})^*(d\theta),\\ &\sum_{j=1}^{l} \hat{h}_j'(|\Phi|^2)\Phi\eta(Q_2\delta_j^{-1}(x))).\end{aligned} \tag{4.18}$$

Remark 4.7 *Here x is a point on Y, P_2 is the projection of $S^1 \times D^2$ to D^2 and Q_2 is the projection of $I \times D^2$ to D^2. And θ is the coordinate on S^1 in $S^1 \times D^2$. Since η is a bump function on the disc, the above makes sense although γ_i^{-1}'s and δ_j^{-1}'s cannot be extended to a global diffeomorphism.*

Given the Lemma, let

$$\mu_h(A) = \sum_{i=1}^{k} \tilde{h}_i'(hol_{\gamma_i}(\gamma_i(1,x),A))\eta(P_2\gamma_i^{-1}(x)) * (\gamma_i^{-1})^*(d\theta) \tag{4.19}$$

and

$$\nu_h(\Phi) = \sum_{j=1}^{l} \hat{h}_j'(|\Phi|^2)\Phi\eta(Q_2\delta_j^{-1}(x))), \tag{4.20}$$

then the perturbed equations for $\mathcal{V}_h$ are

$$\begin{aligned} &F_A + F_{A_0} + \frac{i}{2}\tau(\Phi,\Phi) + \mu_h(A) = 0 \\ &D_A\Phi + \nu_h(\Phi) = 0. \end{aligned} \tag{4.21}$$

Proof of Lemma: We again break the proof into the connection part and the spinor part. First let us consider the connection part. For $t \in (-\epsilon, \epsilon)$ and $a \in i\Lambda^1(Y)$ we have

$$hol_{\gamma_i}(\gamma_i(1,x), A + ta) = hol_{\gamma_i}(\gamma_i(1,x), A) exp(t \int_{\gamma_i} \langle \gamma_i^*(a), d\theta\rangle d\theta), \tag{4.22}$$

where θ is the coordinate on S^1. So

$$\begin{aligned} &\frac{\partial}{\partial t}\Big\{ \sum_{i=1}^{k} \int_{D^2} \tilde{h}_i(hol_{\gamma_i}(\gamma_i(1,x), A+ta))\eta(x)\Big\}\Big|_{t=0} \\ =&\frac{\partial}{\partial t}\Big\{ \sum_{i=1}^{k} \int_{D^2} \tilde{h}_i(hol_{\gamma_i}(\gamma_i(1,x), A) exp(t\int_{S^1} \langle \gamma_i^*(a), d\theta\rangle d\theta))\eta(x)\Big\}\Big|_{t=0} \\ =&\sum_{i=1}^{k} \int_{D^2} \tilde{h}_i'(hol_{\gamma_i}(\gamma_i(1,x), A)) \int_{\gamma_i(1,x)} \langle a, (\gamma_i^{-1})^* d\theta\rangle (\gamma_i^{-1})^* d\theta \eta(x) \\ =&\sum_{i=1}^{k} \int_{D^2} \int_{\gamma_i(1,x)} \tilde{h}_i'(hol_{\gamma_i}(\gamma_i(1,x), A)) \langle a, (\gamma_i^{-1})^* d\theta\rangle (\gamma_i^{-1})^* d\theta \eta(x). \end{aligned}$$

Here we notice the fact that we are considering $U(1)$ connections so they are Abelian. It follows from above that

$$\begin{aligned} Dh(A,\Phi)(a,0) &= \sum_{i=1}^{k} \int_{D^2}\int_{\gamma_i(1,x)} \tilde{h}_i'(hol_{\gamma_i}(\gamma_i(1,x), A)) a\eta(x) \\ &= \langle \sum_{i=1}^{k} \int_{D^2}\int_{\gamma_i(1,x)} \tilde{h}_i'(hol_{\gamma_i}(\gamma_i(1,x), A))\eta(x)(\gamma_i^{-1})^*(d\theta), a\rangle. \end{aligned}$$

On the other hand for the spinor part,

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left\{\frac{1}{2}\sum_{j=1}^{l}\int_{D_2}(\int_I \hat{h}_j(\langle\Phi+t\phi,\Phi+t\phi\rangle)\eta(x)\right\}\Bigg|_{t=0}\\
=&\frac{1}{2}\frac{\partial}{\partial t}\left\{\sum_{j=1}^{l}\int_{D_2}(\int_I \hat{h}_j(|\Phi|^2+2t\langle\Phi,\phi\rangle+O(t^2)))\eta(x)\right\}\Bigg|_{t=0}\\
=&\sum_{j=1}^{l}\int_{D_2}(\int_I \hat{h}_j'(|\Phi|^2)\langle\Phi,\phi\rangle\eta(x).
\end{aligned}
$$

Hence

$$
\begin{aligned}
Dh(A,\Phi)(0,\phi) &= \sum_{j=1}^{l}\int_{D_2}(\int_I \hat{h}_j'(|\Phi|^2)\langle\Phi,\phi\rangle\eta(x)\\
&= \langle\sum_{j=1}^{l}\int_{D_2}(\int_I \hat{h}_j'(|\Phi|^2)\Phi)\eta(x),\phi\rangle
\end{aligned}
$$

and this proves the lemma. □

Remark 4.8 *The gradient of $h \in \mathcal{H}_{\phi,\psi}$ vanishes identically outside the support of ϕ and ψ. More precisely, μ_h vanishes outside support of ϕ and ν_h vanishes outside support of ψ.*

4.2.3 Bound for the Perturbation

Given Lemma 4.6, the same computation will yield, again by applying the Abelian property of the connections,

Lemma 4.9 *Let h be an admissible function, then*

$$
\begin{aligned}
&Hessh(A,\Phi)((a_1,\phi_1),(a_2,\phi_2))\\
=&\sum_{i=1}^{k}\tilde{h}_i''(hol_{\gamma_i}(\gamma_i(1,x),A))\langle a_1,(\gamma_i^{-1})^*d\theta\rangle\langle a_2,(\gamma_i^{-1})^*d\theta\rangle\eta(P_2\gamma_i^{-1}(x))\\
&+\sum_{j=1}^{l}(2\hat{h}_j''(|\Phi|^2)\langle\Phi,\phi_1\rangle\langle\Phi,\Phi_2\rangle+\hat{h}_j'(|\Phi|^2)\langle\phi_1,\phi_2\rangle)\eta(Q_2\delta_j^{-1}(x)).
\end{aligned}
\tag{4.23}
$$

□

A corollary of the above is

Corollary 4.10 *Let h be an admissible perturbation, then*

1. *The map $A \mapsto \nabla h(A)$ is smooth from $\mathcal{A}$ to $L^2_2(\Omega^1(Y) \times \Gamma(W))$.*
2. *$Hessh(A) : L^2_2 \mapsto L^2_1$ is a compact operator.*

Proof: (1) is obvious from 4.9 and the fact that

$$\tilde{h}_i'(hol_{\gamma_i}(\gamma_i(1,x), A)) \le C_1 \tag{4.24}$$

for a constant independent of A and only dependent of h, since $\tilde{h}_i \in C^2$. Furthermore,

$$\tilde{h}_i''(hol_{\gamma_i}(\gamma_i(1,x), A)) \le C_2 \tag{4.25}$$

also from the assumption that $\tilde{h}_i \in C^2$. This proves (2) by 4.9 along with the Rellich Lemma. □

From the expression of ν_h in equation 4.20 we have

$$D_A\nu_h(\Phi) = \sum_{j=1}^{l} \gamma(\hat{h}_j(|\Phi|^2))D_A\Phi\eta + \gamma(d[\hat{h}_j(|\Phi|^2)\eta])\Phi. \tag{4.26}$$

For $\Phi \in L^2_k$ the coefficients depend on Φ but since $\hat{h}_j \in C^2$ they are all uniformly bounded.

We restate the above facts in the following

Corollary 4.11 *Let h be an admissible perturbation. Then there is a constant C depending only on h so that $\|\nabla h(A)\|_{L^2_2} \le C$ for each $A \in \mathcal{V}_h^{-1}(0)$.*

4.2.4 A Unique Continuation Theorem

Definition 4.12 *A solution (A, Φ) to the perturbed equations is degenerate if $\Phi \equiv 0$. It is nondegenerate if otherwise.*

For a degenerate solution to the unperturbed equation, $F_A = 0$ and we recover the flat connection. In our study we will consider only the nondegenerate solutions.

In 1957 N. Aronszajn [1] proved the following unique continuation theorem for a second order differential operator:

Theorem 4.13 *If u solves*

$$|\mathbb{A}u|^2 \leq M\{\sum_{i=1}^{n} |\frac{\partial u}{\partial x^i}|^2 + |u|^2\} \tag{4.27}$$

in a domain $V \subset \mathbb{R}^n$ and if at some point $x_0 \in V$,

$$\int_{B(x_0,r)} |u|dx = O(r^{\alpha+n}) \tag{4.28}$$

for any $\alpha > 0$, then u vanishes identically on V.

Here $\mathbb{A}$ is a second order differential operator, possibly with variable coefficients, and the leading coefficients $a_{ij} \in C^{2,1}$ (second derivative Lipshitzian).

We can apply the theorem to the perturbed equation

$$D_A\Phi + \nu_h(\Phi) = 0. \tag{4.29}$$

Let $\mathbb{A} = D_A^2$, then $\mathbb{A}$ is a second order differential operator. If Φ solves 4.29 then Φ satisfies

$$\mathbb{A}\Phi + D_A\nu_h(\Phi) = 0. \tag{4.30}$$

$D_A\nu_h$ is a nonlinear operator, with coefficients depending on Φ. But by equation 4.26 all coefficients are bounded for L_k^2 solutions. So there exists a constant $M > 0$ so that

$$|\mathbb{A}\Phi|^2 \leq M\{\sum_{i=1}^{n} |\frac{\partial \Phi}{\partial x^i}|^2 + |\Phi|^2\} \tag{4.31}$$

in local coordinates. By Aronszajn's unique continuation theorem:

Corollary 4.14 *If Φ solves 4.29 and around a point $y_0 \in Y^+$,*

$$\int_{B(y_0,r)} |\Phi| dx = O(r^{\alpha+3}) \tag{4.32}$$

for any $\alpha > 0$, then $\Phi \equiv 0$ identically.

More References

[32] [68]

Chapter 5 Asymptotic Behavior

In this chapter we discuss the asymptotic behavior of solutions to the Seiberg-Witten equations on the cylindrical end. We first show that on the cylinder a solution with finite $\mathcal{F}$ variation converges to a solution to the Kähler-Vortex equations on Σ slice-wise. Using the theory of center manifold, we describe a finite dimensional model for the L^2 moduli space. Furthermore, the convergence of a Seiberg-Witten equation to the Kähler-Vortex solution is exponentially fast.

5.1 Finite $\mathcal{F}$ Variation

5.1.1 $\mathcal{F}$ as an Energy Functional

Definition 5.1 *$(A(t), \Phi(t))$ on the cylindrical end is of finite $\mathcal{F}$ variation if for any $t \in [t_0, \infty)$*

$$|\mathcal{F}(A(t), \Phi(t)) - \mathcal{F}(A(t_0), \Phi(t_0))| < \infty. \tag{5.1}$$

For simplicity, we denote $\mathcal{F}(A(t), \Phi(t))$ by $\mathcal{F}_t$.

The following Lemma explores the similarity between $\mathcal{F}$ and other energy functionals.

Lemma 5.2

$$\mathcal{F}_{t_1} - \mathcal{F}_{t_0} = \int_{t_0}^{t_1} \int_{\Sigma} |D_A^+\Phi^+|^2 + |D_A^-\Phi^-|^2 + |\Phi^+\Phi^-|^2. \tag{5.2}$$

Proof:

$$\mathcal{F}_t = \int_{\Sigma} (D_A\Phi^+, i\Phi^-) \tag{5.3}$$

so

$$\begin{aligned}
\frac{\partial}{\partial t}\mathcal{F}_t &= \int_\Sigma \frac{\partial}{\partial t}(D_A\Phi^+, i\Phi^-) \\
&= \int_\Sigma (\frac{\partial}{\partial t}(D_A\Phi^+), i\Phi^-) + (D_A\Phi^+, \frac{\partial}{\partial t}(i\Phi^-)) \\
&= \int_\Sigma (D_A(\frac{\partial\Phi^+}{\partial t}) + \gamma(\frac{\partial A}{\partial t})\Phi^+, i\Phi^-) + (D_A\Phi^+, i\frac{\partial\Phi^-}{\partial t}) \\
&= \int_\Sigma (D_A(iD_A\Phi^-), i\Phi^-) + (i\overline{\Phi^+}\Phi^+\Phi^-, i\Phi^-) + (D_A\Phi^+, D_A\Phi^+) \\
&= \int_\Sigma |D_A\Phi^-|^2 + |D_A\Phi^+|^2 + |\Phi^+\Phi^-|^2.
\end{aligned}$$

□

If a Seiberg-Witten solution (A, Φ) is put in temporal gauge, then it satisfies 3.5. Substituting into the above Lemma, we get

$$\mathcal{F}_{t_1} - \mathcal{F}_{t_0} = \int_{t_0}^{t_1}\int_\Sigma \left|\frac{\partial\Phi^+}{\partial t}\right|^2 + \left|\frac{\partial\Phi^+}{\partial t}\right|^2 + \left|\frac{\partial A}{\partial t}\right|^2. \tag{5.4}$$

Then the variation will bound the L^2 norm of $\frac{\partial(A,\Phi)}{\partial t}$, which in turn bounds the L^2 distance between $(A,\Phi)_{t_0}$ and $(A,\Phi)_{t_1}$ for arbitrarily large t_0, t_1. And $(A,\Phi)_t$ will converge when $t \to \infty$.

5.1.2 L^2 Convergence

The real picture is less fortunate. In reality we have

$$\frac{\partial A}{\partial t} = i\overline{\Phi}^+\Phi^- - 2du. \tag{5.5}$$

For an imaginary function u. As we may not always be able to extend u to a globally defined function, there might be no gauge transformation to neutralize it. Hence the above argument is not applicable.

The situation, however, had been extensively studied by S. Lojasiewicz [46] for finite dimensions and by L. Simon [58] for infinite dimensions. Simon's theory can actually imply more but it does not lie in the center of our theory and we will only state the relevant consequences without proof.

For bounds induced by energy,

Lemma 5.3 *Let (B, Ψ) be a C^∞ solution to the Kähler-Vortex equations. Let U be a neighborhood around (B, Ψ). Then there is a neighborhood $V \subset U$ of (B, Ψ), and constants $0 < \beta \leq 1$, $0 < \theta \leq \frac{1}{2}$, so that if $(C, \Theta) \in V$ then there is a C^∞ Kähler-Vortex solution (B', Φ') so that*

$$\begin{aligned} &\|(C, \Theta) - (B', \Phi')\|_{L^2} \leq (\|\nabla_{(C,\Theta)}\mathcal{F}\|_{L^2})^\beta \\ &|\mathcal{F}(C, \Theta)|^{1-\theta} \leq \|\nabla_{(C,\Theta)}\mathcal{F}\|_{L^2}. \end{aligned} \tag{5.6}$$

□

And for path length:

Lemma 5.4 *Let (B, Ψ), U as in last Lemma. Then there is a neighborhood $V \subset U$ and a constant $\theta = \theta(B, \Psi)$ with $0 < \theta \leq \frac{1}{2}$, so that the following holds. If (A, Φ) solves the Seiberg-Witten equations on cylinder, then*

$$\int_{t_0}^{t_1} \|\frac{\partial(A, \Phi)}{\partial t}\|_{L^2} \leq \frac{2}{\theta}|\mathcal{F}_{t_1} - \mathcal{F}_{t_0}|^\theta. \tag{5.7}$$

□

These, combined with the standard arguments, imply

Theorem 5.5 *There exists a constant ϵ_0, so that if an $L^2_{k,loc}$ solution (A, Φ) satisfies*

$$|\mathcal{F}_t| < \epsilon \tag{5.8}$$

for $t > T_0$, then $(A(t), \Phi(t))$ converges to a C^∞ solution to the Kähler-Vortex equations. □

5.2 The Center Manifold

5.2.1 The General Picture

In chapter 3 we showed that our equations are the gradient flow equations for a Dirac functional $\mathcal{F}$ with constraint 3.10. The constraint is preserved by a flow line. Furthermore, the Hessian of $\mathcal{F}$ is self adjoint with real and discrete spectrum. To understand our position let us first quote a general theorem [51]

Theorem 5.6 *Let H be a Hilbert space. Suppose that*

$$H = H^+ \oplus H^0 \oplus H^-$$

is a decomposition of H. Let

$$L^+ : H^+ \to H^+$$
$$L^- : H^- \to H^-$$

be densely defined, closed, unbounded Fredholm linear operators. Suppose that for each $t \geq 0$ the maps

$$e^{-L^+ t} : H^+ \to H^+$$
$$e^{L^- t} : H^- \to H^-$$

are bounded linear maps, and that these define semi-groups of endomorphisms of H^+ and H^- which vary continuously with t in the strong operator topology. Suppose that there exist positive constants λ_+, λ_-, D such that

$$\sup_{t \geq 0} \max(e^{\lambda_+ t}\|e^{-L^+ t}\|, e^{\lambda_- t}\|e^{L^- t}\|) \leq D \tag{5.9}$$

where the norms are the operator norms on bounded linear operators on $H^\pm$. Let $L : H \to H$ be the densely defined linear operator given by $L(x^+, x^0, x^-) = L^+(x^+) + L^-(x^-)$. Let $U \subset H$ be a neighborhood of $\{0\}$ and let $n : U \mapsto H$ be a smooth function *which vanishes to second order at $\{0\} \in H$. Consider the densely defined vector field $\mathcal{V}$ on U given by*

$$\mathcal{V}(x) = L(x) + n(x).$$

Then for any $m \geq 1$ there is a connected neighborhood $\mathcal{N} \subset H^0$ of $\{0\}$ and a C^m map $\xi : \mathcal{N} \to H^+ \oplus H^-$ whose graph $\mathcal{P}$ satisfies:

1. *$\mathcal{P} \subset U$;*

2. *$\mathcal{P}$ is tangent to H^0 at $\{0\}$;*

3. *$\mathcal{P}$ is contained in the domain of the vector field $\mathcal{V}$;*

4. *$\mathcal{V}$ is everywhere tangent to $\mathcal{P}$;*

5. *Every critical point of $\mathcal{V}$ sufficiently close to $\{0\}$ is contained in $\mathcal{P}$.*

□

5.2.2 Our Case

Now we shall show that this theorem applies to our context. In our case L is the linearization of $\nabla\mathcal{F}$ as given in equation 3.9. Let H^+, H^-, and H^0 be the closures of the positive, negative, and zero eigenspaces of L respectively. By Corollary 3.3 L has real discrete spectrum. So $H = H^+ \oplus H^0 \oplus H^-$ gives an L^2-orthogonal decomposition as required.

Furthermore e^{-L^+t} and e^{L^-t} define semi-groups of bounded endomorphisms of H^+ and H^- respectively, which vary continuously with respect to t in the strong operator norm. To see this, let us consider e^{-L^+t} on H^+. Any element in H^+ can be written as an infinite linear combination $\sum_\lambda \alpha_\lambda$ where λ ranges over positive eigenvalues. Moreover, such an infinite sum defines an element of $H \subset L^2$ if and only if

$$\sum_\lambda \|\alpha_\lambda\|^2 < \infty. \tag{5.10}$$

By definition

$$e^{-L^+t}(\alpha_\lambda) = e^{-\lambda t}\alpha_\lambda. \tag{5.11}$$

This clearly defines a semigroup endomorphism on the span of the eigenspaces of L^+. To show that e^{-L^+t} actually defines a semi-group of endomorphisms of H^+ we need only see that for each $t \geq 0$,

$$\sum_\lambda e^{-\lambda t}\|\alpha_\lambda\|^2 < C(t)\sum_\lambda \|\alpha_\lambda\|^2 \tag{5.12}$$

for some constant $C(t)$ only dependent of t. Since all the λ's are bounded below by the first positive eigenvalue λ_1 this is manifest. It also shows that these operators vary continuously with t in the strong operator topology.

The same argument can be applied to show that e^{L^-t} defines a semigroup of bounded endomorphisms on H^-, continuous in the strong operator norm. The above computation also indicates that the condition

$$\sup_{t\geq 0} \max(e^{\lambda_+ t}\|e^{-L^+t}\|, e^{\lambda_- t}\|e^{L^-t}\|) \leq D \tag{5.13}$$

is satisfied for $\lambda_+ = \lambda_1$, and $\lambda_- = \lambda_{-1}$.

For a coordinate patch $U_{(B,\Psi)}$ centered at (B, Ψ), we have a function $n : U \cap L^2_k \mapsto H$ given by

$$n(a, \varphi^+, \varphi^-) = (i\overline{\varphi}^+\varphi^-, i\gamma(a)\varphi^-, -i\gamma(a)\varphi^+) \tag{5.14}$$

which is smooth and vanishes to the 2nd order at $\{0\}$ with $\nabla\mathcal{F} = L + n$. Therefore applying Theorem 5.6 we have

Corollary 5.7 *Let (B, Ψ) be a solution to the Kähler-Vortex equations. Let $U_{(B,\Psi)}$ be a coordinate patch centered at (B, Ψ). Let $H^\perp$ be the L^2-orthogonal complement of $H^0 = kerL$. Then there is a neighborhood $\mathcal{N} \subset H^0$ of $\{0\}$ and a C^2 function $\xi : \mathcal{N} \mapsto H^\perp$ whose graph is a C^2-center for the densely defined vector field $\nabla\mathcal{F}$ on*

U. □

5.2.3 Main Theorem

For the local decomposition for the coordinate patch at (B, Ψ):

$$\mathcal{K}_{(B,\Psi)} = H^0_{(B,\Psi)} \times H^{\perp}_{(B,\Psi)} \tag{5.15}$$

we denote by π^0 and $\pi^{\perp}$ the L^2-orthogonal projections of $\mathcal{K}_{(B,\Psi)}$ onto $H^0_{(B,\Psi)}$ and $H^{\perp}_{(B,\Psi)}$. The main theorem of this chapter can be accounted as following:

Theorem 5.8 *Let (B, Ψ) be a solution to the Kähler-Vortex equations on Σ. Let $\mathcal{N}$ be a neighborhood of $\{0\}$ in $H^0_{(B,\Psi)}$, and let $\xi : \mathcal{N} \mapsto H^{\perp}$ be a C^2 function whose graph $\mathcal{P} \subset \mathcal{N} \times H^{\perp}_{(B,\Psi)} \subset \mathcal{K}_{(B,\Psi)}$ is a C^2-center manifold. Fix a $\delta > 0$. For any C^3 solution (A, Φ) to the monopole equations, satisfying*

$$\mathcal{F}(A(t_0), \Phi(t_0)) - \mathcal{F}(A(\infty), \Phi(\infty)) < \delta \tag{5.16}$$

then there is a unique path $p : [t_0, \infty) \to H^0_{(B,\Psi)}$ so that

$$\frac{\partial p(t)}{\partial t} = \nabla_{p(t)} \mathcal{F}. \tag{5.17}$$

Furthermore, if we let $(A', \Phi') = p(t) + \Theta(p(t))dt \in \mathcal{A}_{Y^+}$, for $t_0 \leq t < \infty$, there is a constant κ so that

$$\|(A, \Phi) - (A', \Phi')\|_{L^2_3([t+\frac{1}{2}, t+\frac{3}{2}] \times \Sigma)} < \kappa e^{-\lambda_{-1} t/2}. \tag{5.18}$$

□

We will not devote all our efforts to the proof of every point of the Theorem; instead, we prove it from two aspects that we are most interested: approximate solutions in the center manifold and L^2-exponential approach to the center manifold. We suggest interested readers to look into [51] chapter 5 for a technically complete treatment of a similar subject.

5.3 Exponential Approach

5.3.1 Approximate Solutions in the Center Manifold

This Lemma shows that an approximate solution to the gradient flow equation is not far from a genuine solution:

Lemma 5.9 *Fix a $C > 0$ and let $b : [t_0, \infty) \to H^0_{(B,\Psi)}$ be a C^2-path such that $\lim_{t\to\infty} b(t)$ exists and*

$$\|\dot{b}(t) - \nabla_{b(t)}\mathcal{F}\|_{L^2} \leq Ce^{-\frac{\lambda_{-1}t}{2}} \tag{5.19}$$

then there is a unique C^2-path $z : [t_0, \infty) \to H_{(B,\Psi)}$ with $\dot{z}(t) = \nabla_{z(t)}\mathcal{F}$ and

$$\|z(t) - b(t)\|_{L^2} \leq \frac{4}{\lambda_{-1}}Ce^{-\frac{\lambda_{-1}t}{2}}. \tag{5.20}$$

Proof: The proof is standard. Define the Banach space $\tilde{B}$ to be the space of continuous paths $\gamma : [t_0, \infty) \to H_{(B,\Psi)}$ such that

$$\sup_{t\in[t_0,\infty)} e^{\frac{\lambda_{-1}t}{2}}\|\gamma(t)\|_{L^2} < \infty. \tag{5.21}$$

For $\gamma \in \tilde{B}$ define:

$$\|\gamma\|_{\tilde{B}} = \sup_{t\in[t_0,\infty)} e^{\frac{\lambda_{-1}t}{2}}\|\gamma(t)\|_{L^2} \tag{5.22}$$

For a path $\bar{b} : [t_0, \infty) \to H_{(B,\Psi)}$ with $\lim_{t\to\infty} \bar{b}(t) = \bar{b}(\infty)$ we define a map $J_{\bar{b}} : \tilde{B} \mapsto \tilde{B}$ by

$$J_{\bar{b}}(\gamma)[t] = -\bar{b}(t) + \bar{b}(\infty) - \int_t^\infty \Xi(\bar{b}(s) + \gamma(s))ds. \tag{5.23}$$

Here we still use Ξ to denote the pullback of vector field $\Xi = \nabla\mathcal{F}$ to $H_{(B,\Psi)}$.

We claim that $J_{\bar{b}}$ is a $\frac{1}{2}$-contraction and the unique fixed point w for the map satisfies

$$\|w(t)\|_{L^2} < \frac{4C}{\lambda_{-1}} e^{-\frac{\lambda_{-1}t}{2}}. \tag{5.24}$$

To show $J_{\bar{b}}$ is a $\frac{1}{2}$-contraction we observe that for γ_1, γ_2:

$$\begin{aligned}
&\|J_{\bar{b}}(\gamma_1) - J_{\bar{b}}(\gamma_2)\|_{\tilde{B}} \\
=&\|(-\bar{b}(t) + \bar{b}(\infty) - \int_t^\infty \Xi(\bar{b}(s) + \gamma_1(s))ds) - \\
&\quad (-\bar{b}(t) + \bar{b}(\infty) - \int_t^\infty \Xi(\bar{b}(s) + \gamma_2(s))ds)\|_{\tilde{B}} \\
=&\|\int_t^\infty \Xi(\bar{b}(s) + \gamma_2(s))ds - \Xi(\bar{b}(s) + \gamma_2(s))ds\|_{\tilde{B}} \\
=& \sup_{t\in[t_0,\infty)} e^{\frac{\lambda_{-1}t}{2}} |\int_t^\infty (\Xi(\bar{b}(s) + \gamma_2(s)) - \Xi(\bar{b}(s) + \gamma_2(s)))ds| \\
\leq& \sup_{t\in[t_0,\infty)} e^{\frac{\lambda_{-1}t}{2}} \int_t^\infty |\nabla_{\bar{b}(s)+\gamma_2(s)}\mathcal{F} - \nabla_{\bar{b}(s)+\gamma_1(s)}\mathcal{F}| \\
\leq& \sup_{t\in[t_0,\infty)} e^{\frac{\lambda_{-1}t}{2}} \int_t^\infty \frac{\lambda_{-1}}{4} \|\pi^0(\bar{b}(s) + \gamma_2(s)) - \pi^0(\bar{b}(s) + \gamma_1(s))\|_{L^2} ds \\
\leq& \frac{\lambda_{-1}}{4} \sup_{t\in[t_0,\infty)} e^{\frac{\lambda_{-1}t}{2}} \int_t^\infty \|\gamma_2(s) - \gamma_1(s)\|_{L^2} ds \\
\leq& \frac{\lambda_{-1}}{4} e^{\frac{\lambda_{-1}t_0}{2}} \int_{t_0}^\infty \|\gamma_1(s) - \gamma_2(s)\|_{L^2} ds \\
\leq& \frac{\lambda_{-1}}{4} e^{\frac{\lambda_{-1}t_0}{2}} \int_{t_0}^\infty \|\gamma_1 - \gamma_2\|_{\tilde{B}} e^{-\frac{\lambda_{-1}s}{2}} ds \\
=& \frac{\lambda_{-1}}{4} \|\gamma_1 - \gamma_2\|_{\tilde{B}} \int_{t_0}^\infty e^{-\frac{\lambda_{-1}(s-t_0)}{2}} ds \\
=& \frac{1}{2} \|\gamma_1(s) - \gamma_2(s)\|_{\tilde{B}}
\end{aligned}$$

To estimate the norm of fixed point w, we only have to estimate the norm of $J_{\bar{b}}(0)$. But

$$J_{\bar{b}}(0) = \int_0^\infty (\dot{b}(t) - \nabla_{b(t)}\mathcal{F}). \tag{5.25}$$

So by integral over the assumption

$$\|\dot{b}(t) - \nabla_{b(t)}\mathcal{F}\|_{L^2} \leq Ce^{-\frac{\lambda_{-1}t}{2}} \tag{5.26}$$

we have $\|J_{\bar{b}}(0)\|_{\tilde{B}} \leq \frac{2C}{\lambda_{-1}}$ thus $\|w\|_{\tilde{B}} \leq \frac{4C}{\lambda_{-1}}$ and the L^2 estimates depending on t holds readily from the definition of $\|\cdot\|_{\tilde{B}}$.

Now we define $z(t) = b(t)+w(t)$. Since $J_{\bar{b}(t)}(w) = w$ by differentiating the equation defining $J_{\bar{b}(t)}(w)$ we obtain

$$\frac{\partial z}{\partial t} = \Xi(z(t)) \tag{5.27}$$

or $\nabla_{z(t)}\mathcal{F} = \frac{\partial z}{\partial t}$. As vector field $\nabla\mathcal{F}$ is in C^1, $z(t)$ is in C^2, and we have

$$\|z(t) - b(t)\|_{L^2} \leq \frac{4C}{\lambda_{-1}}e^{-\frac{\lambda_{-1}t}{2}} \tag{5.28}$$

and that proves the Lemma. □

From the uniqueness, we immediately have

Corollary 5.10 *Let $z_1, z_2 : [T_0, \infty) \to \mathcal{H}_{(B,\Psi)}$ be flow lines for $\nabla\mathcal{F}$. If there is a constant C so that*

$$\|z_1(t) - z_2(t)\|_{L^2} < Ce^{-\frac{\lambda_{-1}}{2}(t-T_0)} \tag{5.29}$$

for all $t \geq T_0$, then $z_1(t) = z_2(t)$. □

The Corollary says that distinct flow lines in the center manifold cannot approach each other exponentially fast.

5.3.2 L^2-exponential Approach to the Center Manifold

This Lemma shows that the approach to the center manifold is exponentially fast, with the speed controlled by $e^{-\frac{\lambda_{-1}t}{2}}$.

Lemma 5.11 *Fix the notation and hypotheses as in the statement of Theorem 5.8. We decompose* $(A(t), \Phi(t)) = (B, \Psi) + b(t) + c(t)$ *where* $b(t) \in H^0_{(B,\Psi)}$ *and* $c(t) \in H^{\perp}_{(B,\Psi)}$. *There are constants* $E_0 > 0$, *such that if there is a solution to the Kähler-Vortex equation* $(B, \Psi) + b_\infty \in U_{(B,\Psi)}$ *so that*

$$\sup_{t \in [t_0, \infty)} \|b(t) - b_\infty\|_{L^2_2} + \|c(t)\|_{L^2_2} = E \leq E_0 \tag{5.30}$$

then for all $t \geq t_0$,

$$\|c(t)\|_{L^2} \leq E e^{-\frac{\lambda_{-1} t}{2}}. \tag{5.31}$$

Proof: We first derive the equations $b(t)$ and $c(t)$ satisfy. Since

$$\frac{\partial (b + c)}{\partial t} = \nabla_{(B,\Psi)+b+c} \mathcal{F} \tag{5.32}$$

and $\nabla_{(B,\Psi)+b}$ is tangent to H^0, we have

$$\frac{\partial b}{\partial t} = L(b) + n(b) = n(b) \tag{5.33}$$

and

$$\frac{\partial c}{\partial t} = L(c) + n(b + c) - n(b). \tag{5.34}$$

n is quadratic so

$$\begin{aligned} \|n(b + c) - n(b)\|_{L^2} \leq & C(\|b + c\|_{L^2_2} + \|c\|_{L^2_2}) \|b + c - b\|_{L^2} \\ \leq & 2C \cdot E \cdot \|c\|_{L^2} \\ \leq & 2E_0 C \|c\|_{L^2}. \end{aligned} \tag{5.35}$$

The Sobolev constant C is independent of b, c, so we can choose $E_0 < \frac{\min(\lambda_1, \lambda_{-1})}{8C}$, then

$$\|n(b + c) - n(b)\|_{L^2} \leq \frac{\min(\lambda_1, \lambda_{-1})}{4} \|c\|_{L^2}. \tag{5.36}$$

Taking inner product with c we have

$$|\langle n(b+c)-n(b),c\rangle| \leq \frac{\min(\lambda_1,\lambda_{-1})}{4}\|c\|_{L^2}^2. \tag{5.37}$$

Let $c=c^+ + c^-$, where $c^\pm$ denote the $H^\pm$ component of c. We estimate, for $c^+ \neq 0$,

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\|c^+(t)\|_{L^2}^2 =&\langle c^+(t), \frac{\partial}{\partial t}c^+(t)\rangle \\ =&\langle c^+(t), \frac{\partial}{\partial t}c(t)\rangle \\ =&\langle c^+(t), L(c)+n(b+c)-n(b)\rangle \\ \geq&\lambda_1\|c^+\|_{L^2}^2 - \frac{\min(\lambda_1,\lambda_{-1})}{4}\|c^+\|_{L^2}\|c\|_{L^2}.\end{aligned} \tag{5.38}$$

The inequality is a closed condition, so it can be extended to the closure of $c^+ \neq 0$. For an open set where $c^+ = 0$, the inequality is obvious. Divide out a factor of $\|c^+\|_{L^2}$ we get

$$\frac{\partial}{\partial t}\|c^+(t)\|_{L^2} \geq \lambda_1\|c^+\|_{L^2} - \frac{\min(\lambda_1,\lambda_{-1})}{4}\|c\|_{L^2}. \tag{5.39}$$

Similarly we can prove

$$\frac{\partial}{\partial t}\|c^-(t)\|_{L^2} \leq -\lambda_{-1}\|c^-\|_{L^2} + \frac{\min(\lambda_1,\lambda_{-1})}{4}\|c\|_{L^2}. \tag{5.40}$$

Subtracting the two inequality:

$$\begin{aligned}&\frac{\partial}{\partial t}(\|c^+(t)\|_{L^2} - \|c^-(t)\|_{L^2}) \\ \geq&\lambda_1\|c^+\|_{L^2} + \lambda_{-1}\|c^-\|_{L^2} - \frac{\min(\lambda_1,\lambda_{-1})}{2}\|c\|_{L^2} \\ \geq&\min(\lambda_1,\lambda_{-1})(\|c^+\|_{L^2}+\|c^-\|_{L^2}) - \frac{\min(\lambda_1,\lambda_{-1})}{2}\|c\|_{L^2} \\ \geq&\frac{\min(\lambda_1,\lambda_{-1})}{2}\|c\|_{L^2}\end{aligned} \tag{5.41}$$

Therefore $\|c^+(t)\|_{L^2} - \|c^-(t)\|_{L^2}$ is nondecreasing. If $\|c^+(t)\|_{L^2} > \|c^-(t)\|_{L^2}$, then $\|c\|_{L^2} < 2\|c^+(t)\|_{L^2}$ and

$$\begin{aligned}\frac{\partial}{\partial t}\|c^+(t)\|_{L^2} \geq& \lambda_1\|c^+\|_{L^2} - \frac{\min(\lambda_1, \lambda_{-1})}{4}\|c\|_{L^2} \\ \geq& \lambda_1\|c^+\|_{L^2} - \frac{\min(\lambda_1, \lambda_{-1})}{4}2\|c^+\|_{L^2} \\ \geq& \frac{\lambda_1}{2}\|c^+\|_{L^2}.\end{aligned} \tag{5.42}$$

That means $\frac{\partial}{\partial t}(log\|c^+\|_{L^2}) \geq \frac{\lambda_1}{2}$, so $\|c^+\|_{L^2}$ is exponentially increasing with speed $e^{\frac{\lambda_1}{2}t}$. That contradicts the assumption that it is bounded by E. So $\|c^+(t)\|_{L^2} \leq \|c^-(t)\|_{L^2}$. That implies $\|c\|_{L^2} < 2\|c^-(t)\|_{L^2}$ and

$$\begin{aligned}\frac{\partial}{\partial t}\|c^-(t)\|_{L^2} \leq& -\lambda_{-1}\|c^-\|_{L^2} + \frac{\min(\lambda_1, \lambda_{-1})}{4}\|c\|_{L^2} \\ \leq& -\lambda_{-1}\|c^-\|_{L^2} + \frac{\min(\lambda_1, \lambda_{-1})}{4}2\|c^+\|_{L^2} \\ \leq& -\frac{\lambda_{-1}}{2}\|c^-\|_{L^2}.\end{aligned} \tag{5.43}$$

Integrating the inequality form T_0 to t, we get

$$\begin{aligned}\|c^-(t)\|_{L^2} \leq& \|c^-(T_0)\|_{L^2} e^{-\frac{\lambda_{-1}}{2}(t-T_0)} \\ \leq& E e^{-\frac{\lambda_{-1}}{2}(t-T_0)}.\end{aligned} \tag{5.44}$$

□

More References

[53] [60] [62]

Chapter 6 Analytical Results

We set up the analytical theory in this chapter. First we discuss the Fredholm theory for L^2_k solutions and compute the index. Then we show the compactness of the perturbed moduli spaces. Last we prove the transversality. As a result of these, a generic moduli space is a compact smooth manifold, with dimension $\frac{1}{2}\mathcal{M}_L$.

6.1 Fredholm Theory

6.1.1 Linearization Map

Consider the maps

$$\begin{aligned}
\delta_1 &: Map(Y, i\mathbb{R}) \to \Gamma(i\Lambda^1) \oplus \Gamma(W) \\
\delta_2 &: \Gamma(i\Lambda^1) \oplus \Gamma(W) \to \Gamma(i\Lambda^2) \oplus \Gamma(W)
\end{aligned}$$

defined by

$$\begin{aligned}
\delta_1(f) &= (-df, f\Phi) \\
\delta_2(a, \phi) &= (da + i\tau(\Phi, \phi) + (D\mu_h)_A(a), D_A\phi + \gamma(a)\Phi + (D\nu_h)_\Phi(\phi))
\end{aligned} \tag{6.1}$$

at (A, Φ), where

$$(D\mu_h)_A(a) = \sum_{i=1}^{k} \int_{D^2 \times S^1} \tilde{h}_i''(hol_{\gamma_i}(\gamma_i(1, x), A))\eta(P_2\gamma_i^{-1}(x))\langle a, (\gamma_i^{-1})^* d\theta\rangle * (\gamma_i^{-1})^*(d\theta) \tag{6.2}$$

and

$$(D\nu_h)_\Phi(\phi) = \sum_{j=1}^{l} \int_{[0,1]\times D^2} (2\hat{h}_j''(|\Phi|^2)\langle \Phi, \phi\rangle \Phi + \hat{h}_j'(|\Phi|^2)\phi)\eta(Q_2\delta_j^{-1}(x)). \tag{6.3}$$

The first map δ_1 is the linearization of the gauge fixing map

$$g(A, \Phi) = (A - g^{-1}dg, g\Phi) \tag{6.4}$$

for $g \in Map(Y, S^1)$. The second map δ_2 is the linearization of the perturbed Seiberg-Witten equations 4.21. The maps are purely local and do not depend on the global geometry of the underlying manifolds. Now consider the noncompact manifold $Y^+ = Y \cup_\Sigma \Sigma \times [0, \infty)$. The metric is given by the product metric at the end.

To set up the Fredholm theory and for purpose of index computation, for simplicity, we may as well consider the unperturbed equations and solutions. Still using notation δ_1, δ_2, consider

$$\begin{aligned}
\delta_1 :& L_k^2(\Gamma(i\mathbb{R})) \to L_{k-1}^2(\Gamma(i\Lambda^1) \oplus \Gamma(W)) \\
\delta_2 :& L_{k-1}^2(\Gamma(i\Lambda^1) \oplus \Gamma(W)) \to L_{k-2}^2(\Gamma(i\Lambda^2) \oplus \Gamma(W)) \\
&\delta_1(f) = (-df, f\Phi) \\
&\delta_2(a, \phi) = (da + i\tau(\Phi, \phi), D_A\phi + \gamma(a)\Phi)
\end{aligned}$$

and

$$0 \longrightarrow L_k^2(\Gamma(i\mathbb{R})) \xrightarrow{\delta_1} L_{k-1}^2(\Gamma(i\Lambda^1) \oplus \Gamma(W)) \xrightarrow{\delta_2} L_{k-2}^2(\Gamma(i\Lambda^2) \oplus \Gamma(W)) \longrightarrow 0. \tag{6.5}$$

Lemma 6.1 *At any solution (A, Φ) to the Seiberg-Witten equations, the above is a complex.*

Proof: To show that $\delta_2 \circ \delta_1 = 0$, we compute

$$\begin{aligned}
&\delta_2 \circ \delta_1(f) \\
=&\delta_2(-df, f\Phi) \\
=&(d(-df) + i\tau(\Phi, f\Phi), D_A(f\Phi) + \gamma(-df)\Phi).
\end{aligned}$$

$d(-df) = 0$. On the other hand τ is locally modeled on

$$\tau : (\varphi, \psi) \mapsto Im(\varphi i \overline{\psi}). \tag{6.6}$$

In particular, if $\psi = f\varphi$ for a purely imaginary function f, then

$$\begin{aligned} \tau(\varphi, \psi) =& Im((\varphi i \overline{f\varphi})) \\ =& |f| Im((\varphi i)(\overline{\varphi i})) \\ =& 0. \end{aligned} \tag{6.7}$$

So $i\tau(\Phi, f\Phi) = 0$ and the first component vanishes. For the second component, notice

$$D_A(f\Phi) = \gamma(df)\Phi + f D_A \Phi. \tag{6.8}$$

And the vanishing of second component follows from $D_A\Phi = 0$. □

At any nondegenerate solution (A, Φ), the map δ_1 is injective so the zeroth cohomology of the complex is trivial. The formal tangent space to the moduli space $\mathcal{M}$ at $[A, \Phi]$, called the Zariski tangent space, is the first cohomology of the above complex. The condition that $\mathcal{M}$ is smooth at $[A, \Phi]$ is the vanishing of the second cohomology, called the obstruction space.

6.1.2 Weighted Norms

In the above we worked on the space of L^2 solutions. This choice can easily be seen to be improper. In last chapter we showed, for a finite $\mathcal{F}$ variation solution, there is a limit at the cylindrical end to a Kähler-Vortex solution. If a corresponding Kähler-Vortex solution is nontrivial, then the Seiberg-Witten solution cannot be possibly in L^2.

To overcome this difficulty, let us fix a Kähler-Vortex solution (B, Ψ). On the cylindrical end, suppose the $spin_c$ structure is induced by the product foliation, that is, $W = \pi^*(W_\Sigma^+) \oplus \pi^*(W_\Sigma^-)$. Consider a pair $(A, \Phi) + \pi^*(B, \Psi)$. For simplicity we will not distinguish (B, Ψ) from its pullback $\pi^*(B, \Psi)$. Substitute into the Seiberg-Witten equations 4.15 and we get

$$\begin{aligned} &dA + F_{A_0} + i\tau(\Phi, \Psi) + \frac{i}{2}\tau(\Phi, \Phi) = 0 \\ &D_B\Phi + \gamma(A)(\Phi + \Psi) = 0. \end{aligned} \tag{6.9}$$

The linearization for pair (a, φ), then, is

$$\delta_2'(a, \varphi) = (da + i\tau(\varphi, \Psi) + i\tau(\varphi, \Phi), D_B\varphi + \gamma(a)(\Phi + \Psi)). \tag{6.10}$$

Now we can allow (a, φ) to vary on proper Sobolev spaces.

Let $\tilde{t}$ be a global extension to Y^+ of the coordinate projection map $t : \mathbb{R}^+ \times \Sigma \to \mathbb{R}^+$. For any $w \in \mathbb{R}$ and a fixed A_0 we define

$$\begin{aligned} \|A\|_{L^2_{k,w,A_0}} &= \|e^{w\tilde{t}}A\|_{L^2_{k,A_0}} \\ \|\Phi\|_{L^2_{k,w,A_0}} &= \|e^{w\tilde{t}}\Phi\|_{L^2_{k,A_0}}. \end{aligned} \tag{6.11}$$

We denote L^2_{k,w,A_0} to be the completion of $C_c^\infty(Y^+)$ under the above norm. It is a standard fact that for different A_0's all the norms are equivalent, so we will omit the index A_0 from now on.

In last chapter, we showed that the approach to the center manifold is exponentially fast, with speed faster than $e^{-\frac{\lambda_{-1}}{2}t}$. Therefore, for a $w < \frac{\lambda_{-1}}{2}$ the following complex makes sense:

$$\begin{aligned} 0 \longrightarrow \quad & L^2_{k,w}(\Gamma(i\mathbb{R})) \quad \xrightarrow{\delta_1} L^2_{k-1,w}(\Gamma(i\Lambda^1) \oplus \Gamma(W)) \\ \xrightarrow{\delta_2'} \; & L^2_{k-2}(\Gamma(i\Lambda^2) \oplus \Gamma(W)) \longrightarrow \quad 0 \end{aligned} \tag{6.12}$$

From Lemma 6.1,

Corollary 6.2 *6.12 is a complex.*

6.1.3 Operators at the End

To understand the Fredholm property of complex 6.12, we wrap it up to form a single operator, by taking adjoint with respect to L^2-inner product after conjugation. Define

$$\begin{aligned} &\mathcal{D}'_w = (e^{-w\tilde{t}}\delta_1^* e^{w\tilde{t}}, \delta'_2) : \\ &L^2_{k,w}(\Gamma(i\Lambda^1) \oplus \Gamma(W)) \to L^2_{k-1,w}(\Gamma(i\mathbb{R}) \oplus \Gamma(i\Lambda^2) \oplus \Gamma(W)). \end{aligned} \tag{6.13}$$

Identifying $i\Lambda^2$ with $i\Lambda^1$ with $*$, we get

$$\mathcal{D}_w : L^2_{k,w}(\Gamma(i\Lambda^1) \oplus \Gamma(W)) \to L^2_{k-1,w}(\Gamma(i\mathbb{R}) \oplus \Gamma(i\Lambda^1) \oplus \Gamma(W)). \tag{6.14}$$

$\delta_1^* : L^2_k(\Gamma(i\Lambda^1) \oplus \Gamma(W)) \to L^2_{k-1}(\Gamma(i\mathbb{R}))$ is characterized by

$$\begin{aligned} \delta_1^*(a, \varphi) &= ((a, \varphi), \delta_1(f)) \\ &= ((a, \varphi), (-df, f(\Phi + \Psi))) \\ &= (a, -df) + (\varphi, f(\Phi + \Psi)) \\ &= (-d^*a, f) + (-i(\varphi, i(\Phi + \Psi)), f) \\ &= (-d^*a - i(\varphi, i(\Psi + \Phi)), f). \end{aligned}$$

So

$$\delta_1^*(a, \varphi) = -d^*a - i(\varphi, i(\Phi + \Psi)) \tag{6.15}$$

and

$$e^{-w\tilde{t}}\delta_1^* e^{w\tilde{t}}(a, \varphi) = -d^*a - i(\varphi, i(\Phi + \Psi)) - w(dt, a). \tag{6.16}$$

If we write $a = b + cdt$ where $b \in i\Lambda^1(\Sigma)$ and $c \in i\Lambda^0(\Sigma)$ then the above is

$$e^{-w\tilde{t}}\delta_1^* e^{w\tilde{t}}(c, b, \varphi) = -d_\Sigma^* b + \frac{\partial c}{\partial t} + wc - i(\varphi, i(\Phi + \Psi)). \tag{6.17}$$

On the other hand, if in addition we express $\varphi = (\varphi^+, \varphi^-)$, then

$$
\begin{aligned}
*\delta_2'(c, b, \varphi^+, \varphi^-) = (\; & *_\Sigma\, d_\Sigma b + ip_1\tau(\varphi, \Psi + \Phi), \\
& *_\Sigma \frac{\partial b}{\partial t} + *_\Sigma d_\Sigma c + ip_2\tau(\varphi, \Psi + \Phi), \\
& i\frac{\partial \varphi^+}{\partial t} + D_B^-\varphi^- + (\gamma(b + cdt)(\Psi + \Phi))^+, \\
& - i\frac{\partial \varphi^-}{\partial t} + D_B^+\varphi^+ + (\gamma(b + cdt)(\Psi + \Phi))^-).
\end{aligned} \tag{6.18}
$$

Here p_1, p_2 are projections to proper components. In summary we proved

Lemma 6.3

$$
\begin{aligned}
\mathcal{D}_w(c, b, \varphi^+, \varphi^-) = (\; & - d_\Sigma^* b - \frac{\partial c}{\partial t} - wc - i(\varphi, i(\Psi + \Phi)) + *_\Sigma d_\Sigma b + ip_1\tau(\varphi, \Psi + \Phi), \\
& *_\Sigma \frac{\partial b}{\partial t} + *_\Sigma d_\Sigma c + ip_2\tau(\varphi, \Psi + \Phi), \\
& i\frac{\partial \varphi^+}{\partial t} + D_B^-\varphi^- + (\gamma(b + cdt)(\Psi + \Phi))^+, \\
& - i\frac{\partial \varphi^-}{\partial t} + D_B^+\varphi^+ + (\gamma(b + cdt)(\Psi + \Phi))^-).
\end{aligned} \tag{6.19}
$$

6.1.4 Operator in Matrix Form

We can express $\mathcal{D}_w$ in matrix form as $D_{w,\infty} + O$ where

$$
D_{w,\infty} = \begin{pmatrix} -\frac{\partial}{\partial t} - w & - *_\Sigma\, d_\Sigma & i(\cdot, i\Psi^+) & -i(\cdot, i\Psi^-) \\ - *_\Sigma\, d_\Sigma & *_\Sigma \frac{\partial}{\partial t} & \cdot\Psi^- & \overline{\Psi}^+ \cdot \\ i\gamma(\cdot)\Psi^+ & \gamma(\cdot)\Psi^- & i\frac{\partial}{\partial t} & D_B \\ -i\gamma(\cdot)\Psi^- & \gamma(\cdot)\Psi^+ & D_B & -i\frac{\partial}{\partial t} \end{pmatrix} \tag{6.20}
$$

and

$$
O = \begin{pmatrix} 0 & -d_\Sigma^* & i(\cdot, i\Phi^+) & -i(\cdot, i\Phi^-) \\ 0 & 0 & \cdot\Phi^- & \overline{\Phi}^+ \cdot \\ i\gamma(\cdot)\Phi^+ & \gamma(\cdot)\Phi^- & 0 & 0 \\ -i\gamma(\cdot)\Phi^- & \gamma(\cdot)\Phi^+ & 0 & 0 \end{pmatrix}. \tag{6.21}
$$

Let J be the involution on $\Gamma(i\Lambda(\Sigma))\oplus\Gamma(W^+\oplus W^-)$. Explicitly $J = *$ on $\Gamma(i\Lambda^1(\Sigma))$ and $J = \pm i$ on $W^\pm$. Clearly $J^2 = -1$ and can be regarded as an almost complex structure on the bundles involved.

Notice the block

$$\begin{pmatrix} 0 & \cdot\Psi^- & \overline{\Psi}^+\cdot \\ \gamma(\cdot)\Psi^- & 0 & -iD_B \\ \gamma(\cdot)\Psi^+ & iD_B & 0 \end{pmatrix} \tag{6.22}$$

is L in Corollary 3.3, the linearization of the Kähler-Vortex equations, and

$$\begin{pmatrix} -d_\Sigma^* & i(\cdot, i\Phi^+) & -i(\cdot, i\Phi^-) \end{pmatrix} \tag{6.23}$$

expresses the linearization of constraint 3.10.

These illustrated

Lemma 6.4 *The limiting operator $D_{w,\infty}$ is*

$$\begin{pmatrix} -\frac{\partial}{\partial t} - w & v \\ v^* & J(\frac{\partial}{\partial t} + L) \end{pmatrix}. \tag{6.24}$$

Moreover, J anticommutes with L. $\square$

6.1.5 Fredholm Theory

Lemma 6.5 *$\frac{w}{2}$ is not in the spectrum of*

$$Q = \begin{pmatrix} -w & v \\ v^* & L \end{pmatrix} \tag{6.25}$$

if $\lambda(L) > 0$ and $0 < w < \lambda_1$.

Proof: For an eigenvalue $\lambda > 0$ of L, there are a pair of eigenvalues of $\begin{pmatrix} -w & v \\ v^* & L \end{pmatrix}$

given by $-\frac{w-\lambda}{2} \pm \frac{\sqrt{(w+\lambda)^2+4\beta^2}}{2}$. Notice

$$-\frac{w-\lambda}{2} - \frac{\sqrt{(w+\lambda)^2+4\beta^2}}{2} < 0 \tag{6.26}$$

and

$$-\frac{w-\lambda}{2} + \frac{\sqrt{(w+\lambda)^2+4\beta^2}}{2} \geq \lambda \geq \lambda_1. \tag{6.27}$$

So $\frac{w}{2}$ will not be in the spectrum. □

Lemma 6.6 *$\mathcal{D}_{w,\infty}$ is Fredholm for $w < \lambda_1$.*

Proof: Conjugating $\mathcal{D}_{w,\infty}$ by $e^{-\frac{w}{2}t}$ induces an operator $\mathcal{D}_\infty$ on L^2 norm, which equals $\mathcal{D}_{w,\infty} - \frac{w}{2}$ on the cylindrical end. By Lockhart and McOwen [45], $\mathcal{D}_\infty$ is Fredholm if $Q - \frac{w}{2}$ is invertible. Corresponding to an eigenvalue λ and an eigenfunction φ_λ of L, the eigenfunction on cylinder has form $e^{-\lambda t}\varphi_\lambda$. For it to live in L^2, we have to restrict to $\lambda > 0$. The Lemma now follows from Lemma 6.5. □

As a Corollary

Corollary 6.7 *The deformation complex 6.12 is Fredholm.* □

6.2 Index Calculation

6.2.1 Extended L^2

Definition 6.8 *For an operator of form $\frac{\partial}{\partial t} + P$ at the cylindrical end, a pair $(\tilde{A}, \tilde{\Phi})$ is in extended L^2 if $(\tilde{A}, \tilde{\Phi}) \in L^2_{loc}$ and for t large,*

$$(\tilde{A}, \tilde{\Phi}) = (B, \Psi) + (A, \Phi) \tag{6.28}$$

for $(B, \Psi) \in \ker P$ and $(A, \Phi) \in L^2$.

We denote extended L^2 space by $\hat{L}^2$. M. Atiyah, V. Patodi and A. Shapiro proved [5]

Lemma 6.9 *Let $D : C^\infty(Y,E;1-P) \mapsto C^\infty(Y,F)$ be a linear first order elliptic differential operator on Y. Near the boundary $I \times \Sigma$ D takes form $D = \sigma(\frac{\partial}{\partial t} + A)$ for a bundle isomorphism σ and a first order self-adjoint elliptic operator A on Σ. Let $C^\infty(Y,E;1-P)$ denote the space of sections f of E satisfying $(1-P)f(\cdot,0) = 0$ where P is the spectral projection of A corresponding to eigenvalues ≥ 0. Let $D^* : C^\infty(Y,F;P) \mapsto C^\infty(Y,E)$ be the adjoint operator. Then*

(1) $\ker D$ *is isomorphic to the space of L^2 solutions of $Df = 0$ on Y^+.*

(2) $\ker D^*$ *is isomorphic to the space of extended L^2 solutions of $D^* f = 0$ on Y^+.*

In our situation, however, we would like to compute index where kernel is in extended L^2 and cokernel is in L^2. Therefore it is natural to introduce

$$\mathcal{D}_w^* : L^2_{k,w}(\Gamma(i\mathbb{R}) \oplus \Gamma(i\Lambda^1) \oplus \Gamma(W)) \to L^2_{k-1,w}(\Gamma(i\Lambda^1) \oplus \Gamma(W)),$$

the adjoint of $\mathcal{D}_w$ defined by

$$\mathcal{D}_w^* = (\delta_1, e^{-w\tilde{t}}(*\delta_2')^* e^{w\tilde{t}}). \tag{6.29}$$

Then by Atiyah-Patodi-Singer,

Lemma 6.10 *$-ind\mathcal{D}_w^*$ is the formal dimension of complex 6.12.* □

6.2.2 The Atiyah-Patodi-Singer Index Theorem

Atiyah-Patodi-Singer also proved

Lemma 6.11

$$indD = h(E) - h(F) - h_\infty(F) \tag{6.30}$$

where $h(E)$ is the dimension of the space of L^2 solutions of $Df = 0$ on Y^+, $h(F)$ is the corresponding dimension of D^* and $h_\infty(F)$ is the dimension of subspace of $\ker A$ *consisting of limiting values of extended L^2 sections f of F satisfying $D^* f = 0$.*

Let

$$\mathcal{D}^* : L^2_k(\Gamma(i\mathbb{R}) \oplus \Gamma(i\Lambda^1) \oplus \Gamma(W)) \to L^2_{k-1}(\Gamma(i\Lambda^1) \oplus \Gamma(W))$$

be the operator on L^2 spaces. Then by Atiyah-Singer-Patodi

$$ind\mathcal{D}^* = h(E) - h(F) - h_\infty(F) \tag{6.31}$$

where $E = \Gamma(i\mathbb{R}) \oplus \Gamma(i\Lambda^1) \oplus \Gamma(W)$ and $F = \Gamma(\Gamma(i\Lambda^1) \oplus \Gamma(W)$.

Atiyah-Patodi-Singer also showed that

$$h_\infty(E) + h_\infty(F) = \dim\ker Q = \dim \mathcal{M}_L \tag{6.32}$$

On the cylindrical end, the operator takes form $J(\frac{\partial}{\partial t} + Q)$ where $J^2 = -1$ and J anticommutes with Q. The operator is self-adjoint so $h(E) = h(F)$ and $h_\infty(E) = h_\infty(F) = \frac{1}{2}\dim \mathcal{M}_L$. Therefore

Corollary 6.12 *$indD^* = -\frac{1}{2}\dim \mathcal{M}_L$.* □

6.2.3 An Excision Argument

On a compact subset Y_t of Y^+, the L^2 norm and L^2_w norm are commensurate and the operator $\mathcal{D}^*_w$ and $\mathcal{D}^*$ are equivalent. By the excision principal [8], the difference $indD^* - indD^*_w$ only depends on the part $Y^+ \setminus Y_t = [t, \infty) \times \Sigma$. To see the difference on $Y^+ \setminus Y_t$, we take away Y_t and glue back a cylinder $[0, t] \times \Sigma$. Extend both operators to the whole cylinder and consider the difference. While $\mathcal{D}^*_w$ takes form $J(\frac{\partial}{\partial t} + Q)$, $\mathcal{D}^*$ takes form $J(\frac{\partial}{\partial t} + Q - \frac{w}{2})$. Since $\frac{w}{2}$ is not in the spectrum of Q, the index of the two operators are the same. Therefore,

Corollary 6.13 *$ind\mathcal{D}^*_w = ind\mathcal{D}^*$.* □

As a consequence, and combine the previous results,

Theorem 6.14 *The map $\mathcal{D}$ is Fredholm. The formal dimension of the moduli space is $\frac{1}{2}\dim \mathcal{M}_L$.* □

6.2.4 The Orientation

We can orient the moduli spaces in the following way,

Lemma 6.15 *Given an orientation of $H^0(Y)$, $H^1(Y)$ and $H^2(Y)$, there is a corresponding orientation for $\mathcal{M}_h$, for any admissible h.*

Proof: For a family of Fredholm maps $\mathcal{K} : X \mapsto Fred(V, W)$, where V, W are bundles over a manifold M, we can associate an index bundle defined as follows. Over X, we define the virtual bundle

$$Ind\mathcal{K}_x = ker\mathcal{K}(x) - coker\mathcal{K}(x).$$

$\dim ker\mathcal{K}(x)$ may not be continuous on x but $\dim ker\mathcal{K}(x) - \dim coker\mathcal{K}(x)$ is continuous and an orientation is defined to be a section of the real determinant line bundle

$$\begin{aligned} &\det(ker\mathcal{K}(x) - coker\mathcal{K}(x))(x) \\ =&\Lambda^{\dim ker\mathcal{K}(x)}(ker\mathcal{K}(x)) \otimes (\Lambda^{\dim coker\mathcal{K}(x)}(coker\mathcal{K}(x)))^*. \end{aligned}$$

Notice when $\mathcal{K}$ is surjective the usual definition is recovered.

In our case the family of operators $\mathcal{K}$ are

$$\begin{aligned} \mathcal{K} : \mathcal{A}_Y \times \mathcal{H} \mapsto Fred(&L^2_{k,w}(\Omega^1(Y) \oplus \Gamma(W)), \\ &L^2_{k-1,w}(\Omega^0(Y) \oplus \Omega^2(Y) \oplus \Gamma(W))). \end{aligned}$$

The block $\begin{pmatrix} 0 & iD_A \\ -iD_A & 0 \end{pmatrix}$ in L has positive determinant, and so does the corresponding part in Q. So we only have to worry the part

$$Fred(L^2_{k,w}(\Omega^1), L^2_{k-1,w}(\Omega^0 \oplus \Omega^2)).$$

The operators are homotopic to

$$(d^*, d) : \Omega^1(Y) \to \Omega^0(Y) \oplus \Omega^2(Y). \tag{6.33}$$

From Hodge theory

$$(\Lambda^{\max}\Omega^0)^* \otimes (\Lambda^{\max}\Omega^2)^* \otimes \Lambda^{\max}\Omega^1 = (\Lambda^{\max}H^0)^* \otimes (\Lambda^{\max}H^2)^* \otimes \Lambda^{\max}H^1 \tag{6.34}$$

So an orientation of $H^0(Y)$, $H^1(Y)$ and $H^2(Y)$ induces an orientation on $\mathcal{M}_h$. $\square$

6.3 Compactness

6.3.1 A Weitzenböck Formula

On a four-dimensional manifold X, the Weitzenböck formula states

$$D_A^2 = \nabla_A^* \nabla + \frac{s}{4} Id_W + \frac{1}{2}\gamma(F_{\hat{A}}). \tag{6.35}$$

Now given a 3-manifold Y and a $spin_c$ structure, consider $Y \times \mathbb{R}$ with the $spin_c$ structure induced by the given $spin_c$ structure on Y. For translationally invariant spinors the Weitzenböck formula for four-dimension should hold and the four-dimensional Dirac operator is equal to the three-dimensional Dirac operator, similar to discussions in 3.1. Notice here we identify $\omega \in \Omega^2(Y)$ with $\omega \wedge dt + *_Y\omega$ so computing the norm gives a factor of 2. This proves the Weitzenböck formula for 3-manifolds:

$$D_A^2 = \nabla_A^* \nabla + \frac{s}{4} Id_W + \frac{1}{4}\gamma(F_{\hat{A}}) \tag{6.36}$$

For the perturbing terms in equations 4.21, we have the estimates:

Lemma 6.16 *Let (A, Φ) be a solution to the perturbed Seiberg-Witten equation, then there exists constants C_1 and C_2, only dependent of h such that*

$$\begin{aligned}|\langle \gamma(\mu_h(A))\Phi, \Phi\rangle| \leq& C_1|\Phi|^2\\ |\langle D_A\nu_h(\Phi), \Phi\rangle| \leq& C_2|\Phi|^2.\end{aligned}$$

Proof: The first inequality follows immediately from the bound on $\tilde{h_i}'$. For the second inequality, notice

$$\begin{aligned}D_A\nu_h(\Phi) =& D_A\sum_{j=1}^{l}\hat{h}_j'(|\Phi|^2)\Phi\eta(Q_2\delta_j^{-1}(x)))\\ =& \sum_{j=1}^{l}\hat{h}_j'(|\Phi|^2)D_A\Phi\eta(Q_2\delta_j^{-1}(x))\\ &+\gamma(d\hat{h}_j'(|\Phi|^2))\Phi\eta(Q_2\delta_j^{-1}(x))\\ =& \sum_{j=1}^{l}\hat{h}_j'(|\Phi|^2)(-\nu_h(\Phi))\eta(Q_2\delta_j^{-1}(x))\\ &+\gamma(d\hat{h}_j'(|\Phi|^2))\Phi\eta(Q_2\delta_j^{-1}(x))\end{aligned}$$

and the estimate follows. □

6.3.2 Uniform Bounds on Φ

The above estimates enable the pointwise uniform bound of finite $\mathcal{F}$ variational solutions:

Lemma 6.17 *Let (A, Φ) be an $L^2_{k,w}$ solution to the perturbed Seiberg-Witten equations, then there exists constant C, independent of (A, Φ), so that*

$$|\Phi| \leq C.$$

Proof: We show in last chapter that on the cylindrical end the solution converges to a solution to the Kähler-Vortex equations. The moduli space of Kähler-Vortex solutions is compact. So there is a uniform bound for $|\Phi|$ at the cylindrical end. If the maximum of Φ is achieved at a point x_0 then

$$\begin{aligned}
0 \leq & \frac{1}{2}\triangle|\Phi|^2 \\
= & \langle \nabla_A^* \nabla_A \Phi, \Phi \rangle - \langle \nabla_A \Phi, \nabla_A \Phi \rangle \\
\leq & \langle \nabla_A^* \nabla_A \Phi, \Phi \rangle \\
= & \langle D_A^2 \Phi, \Phi \rangle - \frac{1}{2}\langle \gamma(F_A)\Phi, \Phi \rangle - \frac{s}{4}\langle \Phi, \Phi \rangle \\
= & -\frac{s}{4}\langle \Phi, \Phi \rangle - \frac{1}{4}|\Phi|^4 - \langle \gamma(\mu_h(A))\Phi, \Phi \rangle - \langle D\nu_h(\Phi), \Phi \rangle \\
\leq & -\frac{s}{4}\langle \Phi, \Phi \rangle - \frac{1}{4}|\Phi|^4 + C_1|\Phi|^2 + C_2|\Phi|^2
\end{aligned}$$

where C_1 and C_2 are only dependent of h and not of (A, Φ) from the above lemmas. The result then follows. □

We will use the pointwise bound to prove a convergence result on a compact subset of the open manifold Y^+, much as in the closed case. And we will use the asymptotic behavior of the finite $\mathcal{F}$ variational solutions to prove a convergence result on the cylindrical end, in the spirit of [41]. We then patch these two together and prove the global convergence.

6.3.3 Convergence on Compact Subset

Lemma 6.18 *On a compact subset K_1 of Y^+, for a sequence of the solutions (A_i, Φ_i), there is a subsequence $\{i'\} \subset \{i\}$ and gauge transformations $u_{i'}$ so that $u_{i'}(A_{i'}, \Phi_{i'})$ converge to a solution (A, Φ) strongly.*

Proof: The uniform bounds of Φ_i make it transparent and similar to the closed case. Fix a smooth connection A_0 on K_1, for any connection A we can find a gauge transformation g in the identity component of $Map(K_1, S^1)$ so that $g(A) - A_0$ is co-closed and annihilates the normal vectors at the boundary. Furthermore, by choosing g from proper component we can make it so that the harmonic part of $g(A) - A_0$ is also bounded by a constant independent of A, since the torus $H^1(K_1, \mathbb{R})/H^1(K_1, \mathbb{Z})$ is compact. Therefore, by applying proper gauge transformations we may assume the sequences (A_i, Φ_i) are such that

$$\begin{aligned} A_i &= A_0 + a_i \\ d^* a_i &= 0. \end{aligned} \tag{6.37}$$

The perturbed Seiberg-Witten equations for pair (a_i, Φ_i) then become

$$\begin{aligned} da_i + \frac{i}{2}\tau(\Phi_i, \Phi_i) + \mu_h(A_0 + a_i) &= 0 \\ D_{A_0}\Phi_i + \gamma(a_i)\Phi_i + \nu_h(\Phi_i) &= 0. \end{aligned} \tag{6.38}$$

The pointwise uniform bound, combined with the estimates on the admissible perturbations, implies that there is a constant C_1,

$$\begin{aligned} \|\frac{1}{2}\sigma(\Phi_i, \Phi_i)\|_{L^p} &\le C_1 \\ \| * \mu_h(A_0 + a_i)\|_{L^p} &\le C_1 \\ \|\nu_h(\Phi_i)\|_{L^p} &\le C_1 \end{aligned} \tag{6.39}$$

for any $p > 0$. Here L^p means the L^p norm on K_1. Now consider the operator $d + d^*$ and D_{A_0} on K_1. The first has a good boundary condition to make it an elliptic operator: vanishing of the normal vectors at the boundary. The second does not have a good boundary condition. However, the pointwise uniform bound of Φ_i bounds the boundary integral by a constant C_2. Along with the Garding's inequality, we have, for any p:

$$\begin{aligned} \|a_i\|_{L^p_k} &\le C_3(\|(d + d^*)a_i\|_{L^p_{k-1}} + \|a_i\|_{L^p_{k-1}}) \\ \|\Phi_i\|_{L^p_k;U} &\le C_4(\|D_{A_0}\Phi_i\|_{L^p_{k-1;U'}} + \|\Phi_i\|_{L^p_{k-1;U'}} + C_2). \end{aligned}$$

Here U, U' are open sets such that $U \Subset U'$. C_4 depends on U and U'. In the following argument, we omit the U, U' subscript, with the understanding that in each step, we can always choose appropriate open sets.

We start the bootstrapping argument by

$$\begin{aligned}
\|a_i\|_{L^p_1} &\leq C_3(\|(d+d^*)a_i\|_{L^p} + \|a_i\|_{L^p}) \\
&\leq C_3(\|da_i\|_{L^p} + C_1) \\
&\leq C_3(\|\frac{i}{2}\tau(\Phi_i,\Phi_i) + \mu_h(A_0+a_i)\|_{L^p} + C_1) \\
&\leq C_3(C_1 + C_1 + C_1) \\
&\leq C_5
\end{aligned}$$

and

$$\begin{aligned}
\|\Phi_i\|_{L^p_1} &\leq C_4(\|D_{A_0}\Phi_i\|_{L^p} + \|\Phi_i\|_{L^p} + C_2) \\
&\leq C_4(\|\gamma(a_i)\Phi + \nu_h(\Phi_i)\|_{L^p} + C_1 + C_2) \\
&\leq C_4(3C_1 + C_2) \\
&\leq C_6.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\|a_i\|_{L^p_2} &\leq C_3(\|(d+d^*)a_i\|_{L^p_1} + \|a_i\|_{L^p_1}) \\
&\leq C_3(\|da_i\|_{L^p_1} + C_5) \\
&\leq C_3(\|\frac{i}{2}\tau(\Phi_i,\Phi_i) + \mu_h(A_0+a_i)\|_{L^p_1} + C_5) \\
&\leq C_3(\|\frac{i}{2}\tau(\Phi_i,\Phi_i)\|_{L^p_1} + \|\mu_h(A_0+a_i)\|_{L^p_1} + C_5) \\
&\leq C_3(C_1\|\Phi_i\|_{L^p_1} + C_1 + C_5) \\
&\leq C_3(C_1C_5 + C_1 + C_5) \\
&\leq C_7.
\end{aligned}$$

Similar estimates hold for $\|\Phi_i\|_{L^p_2}$. The convergence then follows from Rellich's compactness theorem. $\square$

6.3.4 The Compactness

The convergence on compact subset, along with the connectedness of the critical point set, enables us to prove

Theorem 6.19 *For any sequence (A_i, Φ_i) of $\hat{L}^2_{k,\delta}$ solutions, there is a convergent subsequence, possibly after gauge transformations, in the $\hat{L}^2_{k,\delta}$ topology on Y^+.*

Proof: Let $Y_t = Y \cup_\Sigma [0,t] \times \Sigma$. Consider a sequence (A_i, Φ_i) on Y^+. For each $n > 0$, there is a convergent subsequence, after applying the gauge transformation. By a diagonal arguement, we can pass further to a subsequence, still called (A_i, Φ_i), so that it converges to an (A_0, Φ_0) on Y^+.

Each equation is a closed condition, so (A_0, Φ_0) still solves the Seiberg-Witten equations, and is a critical point for the Chern-Simons-Dirac functional. Furthermore,

$$
\begin{aligned}
&|\mathcal{F}_{t_1}(A_0,\Phi_0) - \mathcal{F}_{t_0}(A_0,\Phi_0)| \\
\leq &|\mathcal{F}_{t_1}(A_0,\Phi_0) - \mathcal{F}_{t_1}(A_i,\Phi_i)| + |\mathcal{F}_{t_1}(A_i,\Phi_i) - \mathcal{F}_{t_0}(A_i,\Phi_i)| + |\mathcal{F}_{t_0}(A_0,\Phi_0) - \mathcal{F}_{t_0}(A_i,\Phi_i)| \\
\leq &C
\end{aligned}
\tag{6.40}
$$

Where the last inequality follows from the continuity of $\mathcal{F}$ for large enough i's. The convengence now follows from the connectedness of the critical point set of finite $\mathcal{F}$ variation. □

As a corollary,

Corollary 6.20 *For any admissible perturbation h, the moduli space $\mathcal{M}_h$ is compact.*
□

6.4 Transversality

6.4.1 The Map $\mathcal{S}$

Consider a solution $(\tilde{A}, \tilde{\Phi})$ to the perturbed Seiberg-Witten equations 4.21 on Y^+. There exists a solution (B, Ψ) to the Kähler-Vortex equations so that on the cylindrical end $(\tilde{A}, \tilde{\Phi})$ asymptotically approaches (B, Ψ). If we write $(\tilde{A}, \tilde{\Phi}) = (A, \Phi) +$

$\pi^*(B, \Psi)$ then (A, Φ) satisfies

$$\begin{aligned} &dA + i\tau(\Phi, \Psi) + \frac{i}{2}\tau(\Phi, \Phi) + \mu_h(A + B) = 0 \\ &D_B\Phi + \gamma(A)(\Phi + \Psi) = 0. \end{aligned} \tag{6.41}$$

This motivates the following definition

Definition 6.21 *Define*

$$\mathcal{S} : \mathcal{H} \times L^2_{k,\delta}(i\Lambda^1) \times L^2_{k,\delta}(W) \times \mathcal{KV} \mapsto L^2_{k-1,\delta}(i\Lambda^2) \times L^2_{k-1,\delta}(W)$$

to be the map

$$\mathcal{S}(h, A, \Phi, B, \Psi) = (dA + i\tau(\Phi, \Psi) + \frac{i}{2}\tau(\Phi, \Phi) + \mu_h(A + B), D_B\Phi + \gamma(A)(\Phi + \Psi)) \tag{6.42}$$

where $\mathcal{KV}$ denotes the solution space for Kähler-Vortex equations.

Then the linearization of $\mathcal{S}$ is given by

$$\begin{aligned} &D\mathcal{S}(g, a, \varphi, b, \psi) \\ &= (da + i\tau(\varphi, \Psi + \Phi) + i\tau(\Phi, \psi) + D\mu_h(A + B)(g), \\ &\quad D_B\varphi + \gamma(a + b)(\Psi + \Phi) + \gamma(A)(\varphi + \psi) + D\nu_h(\Psi + \Phi)(g)). \end{aligned} \tag{6.43}$$

6.4.2 Surjectivity of $\mathcal{S}$

Theorem 6.22 *$\mathcal{S}$ is smooth and 0 is a regular value.*

Proof: Fix a nondegenerate solution $(A + B, \Phi + \Psi) = (A_0, \Phi_0)$. Rewrite $D\mathcal{S}$ as

$$\begin{aligned} D\mathcal{S}(g, a, \varphi) = (&da + i\tau(\varphi, \Phi_0) + D\mu_h(A_0)(a) + D\mu_h(A_0)(g) + i\tau(\Phi, \psi), \\ &D_{A_0}\varphi + \gamma(a)\Phi_0 + D\nu_h(\Phi_0)(\phi) + D\nu_h(\Phi_0)(g) + \gamma(A)\psi). \end{aligned} \tag{6.44}$$

$D\mathcal{S}$ is of closed range. If $(\alpha, \beta) \in coker D\mathcal{S}$. (α, β) is orthogonal to the image of $D\mathcal{S}$. The surjectivity of $\mathcal{S}$ at 0 follows from the vanishing of (α, β) on Y^+.

For any $x \in Y^+$, choose embeddings $\gamma_j : S^1 \times D^2 \to Y^+, j = 1, 2, 3$, so that $\gamma_j(-1, 0) = x$. Furthermore, we require that $(\gamma_j^{-1})^* d\theta$ be linearly independent at $T_y^* Y$ for $y \in U$, a small neighborhood of x.

Let $\phi' = \phi_0 \cup_i \gamma_i$ then $\phi_0 \subset \phi'$. Let

$$h_1 = \begin{cases} h_0 & \text{on } \phi_0 \\ 0 & \text{on } \gamma_i\text{'s} \end{cases} \tag{6.45}$$

then h_1 is an extension of h_0 to $\mathcal{H}_{\phi_0 \cup_i \gamma_i, \psi}$. And (A_0, Φ_0) solves the perturbed equations corresponding to h_1 as well.

Now we set $a = 0$, $\varphi = 0$, $b = 0$, $\psi = 0$, and change the g, then

$$\langle \sum_i g_i'(hol_{\gamma_i}(\gamma_i(1, x), A_0))\eta(P_2 \gamma_i^{-1}(x))\gamma_{i*}(d\theta), \alpha \rangle = 0. \tag{6.46}$$

$\{\gamma_{i*}(d\theta)\}$ spans the basis for $T_y^* Y$ for $y \in U$, and we have the freedom to change g_i's, so $\alpha \equiv 0$ in U. From the arbitrary choice of x, $\alpha \equiv 0$ on Y^+. Notice here the holonomy inside g' becomes irrelevant, since we have the freedom to change γ_i's as well because we allow intersections of the respective embeddings in the definition of the perturbation.

Now for the β part. For any $x \in Y^+$ where $\Phi_0(x) \neq 0$, in

$$\langle D_{A_0}\varphi + \gamma(a)\Phi_0 + D\nu_{h_0}(\Phi_0)(\varphi) + D\nu_{h_0}(\Phi_0)(g), \beta \rangle = 0 \tag{6.47}$$

set $\varphi = 0$ and $g = 0$ then we get

$$\langle \gamma(a)\Phi_0, \beta \rangle = 0. \tag{6.48}$$

Let $a = 0$, $\varphi = 0$ and we will have

$$\langle D\nu_{h_0}(\Phi_0)(g), \beta \rangle = 0. \tag{6.49}$$

That is

$$\eta\langle g'(|\Phi_0|^2)\Phi_0, \beta\rangle = 0. \tag{6.50}$$

Notice $End(W, W) = \mathbb{R} \oplus \Lambda^1$. If $\beta \neq 0$, then we can find $u_1, u_2 \in \mathbb{R}$ and g, a so that

$$[u_1 g' + u_2\gamma(a)]\Phi_0 = \beta \tag{6.51}$$

since $\Phi_0 \neq 0$ by assumption. That means

$$u_1\eta\langle g'\Phi_0, \beta\rangle + u_2\langle\gamma(a)\Phi_0, \beta\rangle = |\beta|^2 > 0. \tag{6.52}$$

A contradiction. That proves $\beta = 0$ identically by the Unique Continuation Theorem.
□

Combining the results of this chapter and by Sard-Smale theorem [59]

Theorem 6.23 *For any admissible perturbation h the moduli space $\mathcal{M}_h$ is a compact oriented manifold of dimension $\frac{1}{2}\dim\mathcal{M}_L$. The moduli spaces are smooth for generic perturbations.*

More References

[2] [8] [10] [43] [50] [54]

Chapter 7 The Cobordism

7.1 The Limiting Map

We study the limiting map r and show it is a Lagrangian immersion. The composition of r with the section s_h is Legendrian. We define a lift $\tilde{r}$ to construct the Legendrian cobordism between perturbed moduli spaces.

7.1.1 The Map r

The asymptotic behavior of finite $\mathcal{F}$ variational solutions in chapter 5 allows us to define a limiting map from the Seiberg-Witten solution set to the Kähler-Vortex solution set. The map descends to a map

$$r : \mathcal{M}_h(Y, \Sigma) \mapsto \mathcal{M}_L(\Sigma)$$

By Lemma 5.9, Corollary 5.10,

Lemma 7.1 *The map r is continuous and is an immersion.*$\square$

We now proceed to prove the Legendrian property of r. First

Lemma 7.2 *Suppose $\gamma : [0,1] \mapsto \mathcal{A}$ is a smooth path such that $\mathcal{V}_h(\gamma(t)) = 0$ for all $t \in [0,1]$. Then $\tilde{s}_h \circ \gamma$ is a horizontal lift of γ to $\mathcal{A} \times U(1)$. Consequently, $r \circ \tilde{s}_h \circ \gamma : [0,1] \mapsto \mathcal{A}_\Sigma \times U(1)$ is a horizontal lift of $r \circ \gamma$.*

Proof: We need to show $(\tilde{s}_h \circ \gamma)_*(\frac{\partial}{\partial t})$ is horizontal with respect to connection ω. We apply the fact that $\mathcal{V}_h$ is the gradient of the section $\tilde{s}_h$ with respect to ω:

$$\begin{aligned}\omega((\tilde{s}_h \circ \gamma)_*(\frac{\partial}{\partial t})) =& D_\omega \tilde{s}_h(\gamma_*(\frac{\partial}{\partial t}))\\ =&\langle \nabla \tilde{s}_h(\gamma(t)), \gamma_*(\frac{\partial}{\partial t})\rangle\\ =&\langle \mathcal{V}_h(\gamma(t)), \gamma_*(\frac{\partial}{\partial t})\rangle\\ =&0.\end{aligned}$$

□

7.1.2 The Lagrangian Immersion

In chapter 3 we described a symplectic structure Ω on $\mathcal{A}_\Sigma$ defined by 3.16. Ω is compatible on $W^+ \oplus W^-$ with the metric and almost complex structure in the sense

$$\Omega(\phi_1, \phi_2) = g(\phi_1, J\phi_2) \tag{7.1}$$

except that on W^- the orientation is reversed. So the symplectic structure can also be written as

$$\Omega((a_1, \varphi_1^+, \varphi_1^-), (a_2, \varphi_2^+, \varphi_2^-)) = -\int_\Sigma a_1 \wedge a_2 + \int_\Sigma (\varphi_1^+, i\varphi_2^+) - (\varphi_1^-, i\varphi_2^-). \tag{7.2}$$

Let $\gamma_i : [0,1] \to \mathcal{A}_Y$, $i = 1, 2$ be two smooth paths so that $\mathcal{V}_h(\gamma_i(t)) = 0$ for all $t \in [0,1]$. Then

Lemma 7.3 $\Omega((r \circ \gamma_1)_*(\frac{\partial}{\partial t}), (r \circ \gamma_2)_*(\frac{\partial}{\partial t})) = 0.$

Proof: Let

$$(\gamma_i)_*(\frac{\partial}{\partial t}) = (a_i, \varphi_i)$$

on Y and

$$(\gamma_i)_*(\frac{\partial}{\partial t}) = (a_i, \varphi_i^+, \varphi_i^-)$$

on the cylindrical end. For $t \in \mathbb{R}^+$, let $Y_t = Y^+ \setminus [t, \infty) \times \Sigma$. Then

$$\begin{aligned}
&\Omega((r \circ \gamma_1)_*(\frac{\partial}{\partial t}), (r \circ \gamma_2)_*(\frac{\partial}{\partial t})) \\
&= \lim_{t \to \infty} - \int_{\{t\} \times \Sigma} a_1 \wedge a_2 + \int_{\{t\} \times \Sigma} [(\varphi_1^+, i\varphi_2^+) - (\varphi_1^-, i\varphi_2^-)] Vol_\Sigma.
\end{aligned} \tag{7.3}$$

Let ξ be the unit length 1-form corresponding to the *spin*$_c$ structure on Y^+, then the integral can also be written as

$$- \int_{\{t\} \times \Sigma} a_1 \wedge a_2 + \int_{\{t\} \times \Sigma} [(\varphi_1^+, i\varphi_2^+) - (\varphi_1^-, i\varphi_2^-)] * \xi. \tag{7.4}$$

By Stokes theorem on Y_t:

$$\begin{aligned}
&\Omega((r \circ \gamma_1)_*(\frac{\partial}{\partial t}), (r \circ \gamma_2)_*(\frac{\partial}{\partial t})) \\
&= \lim_{t \to \infty} - \int_{Y_t} d(a_1 \wedge a_2) + \int_{Y_t} d\{[(\varphi_1^+, i\varphi_2^+) - (\varphi_1^-, i\varphi_2^-)] * \xi\}.
\end{aligned} \tag{7.5}$$

Since $\mathcal{V}_h(\gamma_i(t)) = 0$, (a_i, φ_i) satisfies

$$\begin{aligned}
&da_i + i\tau(\Phi, \varphi_i) + D\mu_{hA}(\varphi_i) = 0 \\
&D_A \varphi_i + \gamma(a_i)\Phi + D\nu_{h\Phi}(\varphi_i) = 0.
\end{aligned} \tag{7.6}$$

Therefore

$$
\begin{aligned}
&\int_{Y_t} d(a_1 \wedge a_2) \\
&= \int_{Y_t} da_1 \wedge a_2 - a_1 \wedge da_2 \\
&= \int_{Y_t} -(i\tau(\Phi, \varphi_1) + D\mu_{h_A}(a_2)) \wedge a_2 + a_1 \wedge (i\tau(\Phi, \varphi_2)) + D\mu_{h_A}(a_1) \\
&= \int_{Y_t} -i\tau(\Phi, \varphi_1) \wedge a_2 + a_1 \wedge (i\tau(\Phi, \varphi_2)) - Hessh(a_1, a_2) + Hessh(a_2, a_1) \\
&= \int_{Y_t} -i\tau(\Phi, \varphi_1) \wedge a_2 + a_1 \wedge (i\tau(\Phi, \varphi_2)).
\end{aligned}
$$

The map τ is characterized by

$$
i\tau(\varphi, \psi) \wedge \theta = (\gamma(\theta)\varphi, \psi) \tag{7.7}
$$

for $\varphi, \psi \in \Gamma(W)$ and $\theta \in i\Gamma^1(Y)$. Hence

$$
\begin{aligned}
&\int_{Y_t} d(a_1 \wedge a_2) \\
&= -\int_{Y_t} (\gamma(a_2)\Phi, \varphi_1) - (\gamma(a_2)\Phi, \varphi_1) \\
&= \int_{Y_t} (D_A\varphi_2 + D\nu_{h\Phi}(\varphi_2), \varphi_1) - (D_A\varphi_1 + D\nu_{h\Phi}(\varphi_1), \varphi_2) \\
&= \int_{Y_t} (D_A\varphi_2, \varphi_1) - (D_A\varphi_1, \varphi_2) + Hessh(\varphi_2, \varphi_1) - Hessh(\varphi_1, \varphi_2) \\
&= \int_{Y_t} (D_A\varphi_2, \varphi_1) - (D_A\varphi_1, \varphi_2).
\end{aligned}
$$

By equation 4.12

$$
\int_{Y_t} (D_A\varphi_2, \varphi_1) - (D_A\varphi_1, \varphi_2) = \int_{\{t\}\times\Sigma} (\varphi_1^+, i\varphi_2^+) - (\varphi_1^-, i\varphi_2^-). \tag{7.8}
$$

Therefore

$$
-\int_{\{t\}\times\Sigma} d(a_1 \wedge a_2) + \int_{\{t\}\times\Sigma} (\varphi_1^+, i\varphi_2^+) - (\varphi_1^-, i\varphi_2^-) = 0. \tag{7.9}
$$

That means

$$\Omega((r \circ \gamma_1)_*(\frac{\partial}{\partial t}), (r \circ \gamma_2)_*(\frac{\partial}{\partial t})) = 0. \tag{7.10}$$

□

From this Lemma we immediately have

Corollary 7.4 *The map r is a Lagrangian immersion.* □

By definition a Legendrian immersion is a horizontal lift of a Lagrangian immersion. For the contact structure on $\mathcal{L}_\Sigma \to \mathcal{M}_\Sigma$ and from Lemma 7.3 we have

Lemma 7.5 *For generic h, the compositions $s_h \circ r : \mathcal{M}_h \mapsto \mathcal{L}_\Sigma$ are Legendrian.*

7.2 Legendrian Cobordism

7.2.1 The Cobordism

Suppose (M, Ω) is a symplectic manifold and $(\mathcal{L}, \omega)$ is a $U(1)$ bundle with a contact structure over M. The contact structure on $\mathcal{L}$ is obtained from the connection of $\mathcal{L}$ where $d\omega = i\Omega$. There is a canonical contact structure on $\mathcal{L} \times T^*[0,1]$. Let $(u, v) \in [0,1] \times \mathbb{R}$ be the coordinates on $T^*[0,1]$ and π_i, $i = 1,2$ be the projections onto the first and second factors respectively in $\mathcal{L} \times T^*[0,1]$ (and $M \times T^*[0,1]$ as well). On $M \times T^*[0,1]$ there is a canonical symplectic structure

$$\tilde{\Omega} = \pi_1^*\Omega + \pi_2^*(du \wedge dv). \tag{7.11}$$

Now $\mathcal{L} \times T^*[0,1]$ defines a $U(1)$ bundle over $M \times T^*[0,1]$, and the connection given by the 1-form

$$\tilde{\omega} = \pi_1^*\omega - \pi_2^*(vdu) \tag{7.12}$$

defines a compatible contact structure on $\mathcal{L} \times T^*[0,1]$.

Definition 7.6 *For $i = 0, 1$, let $f_i : N_i \mapsto \mathcal{L}$ be immersed Legendrian submanifolds. A Legendrian cobordism between f_0 and f_1 is an immersed Legendrian submanifold $f : N \mapsto \mathcal{L} \times T^*[0,1]$, transversal to $\partial(\mathcal{L} \times T^*[0,1])$ and*

$$p \circ f|_{\partial N} = f_1(N_1) \times \{1\} \cup f_0(N_0) \times \{0\}$$

where $p : T^*[0,1] \mapsto [0,1]$ is the projection. Two oriented Legendrian submanifolds N_0 and N_1 are oriented Legendrian cobordant if there is an oriented Legendrian cobordism (N, f) so that $\partial N = N_1 - N_0$ as oriented manifolds.

The space $\mathcal{H}$ is path connected. Given a path h_t in the space of admissible perturbations, there is a corresponding section s_{h_t} of $\mathcal{L}_Y \times [0,1]$, given by the $\mathcal{G}$ equivariant map

$$s_{h_t}(A, \Phi, t) = e^{i(CS\mathcal{D}(A,\Phi)+h_t(A,\Phi))} : \mathcal{A} \times [0,1] \mapsto U(1). \tag{7.13}$$

Let $\mathcal{M}_{h_t}$ be the moduli space on Y corresponding to the perturbation h_t. For simplicity we will write s_{h_t} as s_t and $\mathcal{M}_{h_t}$ as $\mathcal{M}_t$ unless we want to emphasize the dependence on the perturbation.

For a Fredholm operator $\mathcal{S} : \mathcal{A} \times \mathcal{H} \mapsto \mathcal{A}'$, we can consider the operator from the path space $\mathcal{PH}$ of $\mathcal{H}$:

$$\mathcal{S}_t : \mathcal{A} \times \mathcal{PH} \mapsto \mathcal{A}'.$$

By standard theory the operator $\mathcal{S}_t$ is Fredholm of 1 more index. Regard $\cup_t \mathcal{M}_t$ as sitting inside $\mathcal{B} \times [0,1]$, for a generic path h_t. Apply the same argument and we have

Lemma 7.7 *For generic path h_t, the space $\mathcal{M}_t$ is a smooth, compact manifold of dimension $\frac{1}{2} \dim \mathcal{M}_L + 1$.* □

7.2.2 The Lift $\tilde{r}$

With this understood, we define a lift

$$\tilde{r} : \mathcal{M}_t \mapsto \mathcal{L}_\Sigma \times T^*[0,1]$$

of r. We first declare

$$\overline{r} : \mathcal{A} \times [0,1] \mapsto \mathcal{A}_\Sigma \times U(1) \times T^*[0,1]$$

by

$$\overline{r}(A, \Phi, t) = (r(A, \Phi), \tilde{s}_{h_t}(A, \Phi), t, \frac{\partial}{\partial t} h_t(A, \Phi)). \tag{7.14}$$

This is equivariant with respect to the gauge group action so it induces a map on the quotient. We let $\tilde{r}$ to be the induced map restricted to $\mathcal{M}_t$.

Lemma 7.7 supplies a smooth cobordism between two moduli spaces $\mathcal{M}_h$ for generic h's. By the help of the above lift $\tilde{r}$ the cobordism is indeed a Legendrian cobordism.

Theorem 7.8 *For generic path h_t between $\mathcal{M}_{h_i}$, $i = 0, 1$, $\tilde{r} : \mathcal{M}_t \mapsto \mathcal{L}_\Sigma$ is a Legendrian cobordism.*

Proof: From last section we know $r_t : \mathcal{M}_{h_t} \mapsto \mathcal{M}_\Sigma$ is an immersion for each t. Similarly the map from $\mathcal{M}_t$ to $\mathcal{M}_\Sigma \times [0,1]$ is also an immersion. To prove it gives a Legendrian cobordism, we must show that the lift to $U(1)$ bundle is horizontal. That is, for any vector field $(\delta a, \delta\phi, \delta t) \in T_{(A,\Phi,t)}(\mathcal{A} \times [0,1])$ where (A, Φ) is a solution corresponding to h_t, we need to show $\tilde{r}_*(\delta a, \delta\phi, \delta t)$ is horizontal. Since the connection $\pi_1^*(\omega) + \pi_2^*(vdu)$ is the same as ω when restricted to $\mathcal{A}$, from earlier computation we know $\tilde{r}_*(\delta a, \delta\phi, 0)$ is horizontal. So we only have to show that $\tilde{r}_*(0, 0, \delta t)$ is horizontal.

And we compute

$$\tilde{r}_*(\delta t) = (0, \tilde{s}_{h_{t_0}}(A,\Phi) i \frac{\partial}{\partial t}|_{t=t_0} h_t(A,\Phi)\delta t, \delta t, \frac{\partial^2}{\partial t^2}|_{t=t_0} h_t(A,\Phi)\delta t). \tag{7.15}$$

Consider the part $-\pi_2^*(vdu)$ of the connection on $\mathcal{A} \times T^*[0,1]$, here $v = \frac{\partial}{\partial t} h_t(A,\Phi)$, the last factor of the image of map $\tilde{r}$. Evaluating this part

$$\begin{aligned}(-ivdu)(\tilde{r}_*(\delta t)) =& -\frac{\partial}{\partial t} h_t(A,\Phi)\delta t + \frac{\partial}{\partial t} h_t(A,\Phi)\delta t \\ =&0.\end{aligned} \tag{7.16}$$

So $\tilde{r}_*(\delta t)$ is horizontal and combine above we have

$$\tilde{\omega}(\tilde{r}_*(\delta a, \delta\phi, \delta t)) = 0 \tag{7.17}$$

as wanted. □

Remark 7.9 *Similar to the discussion on the orientation of $\mathcal{M}_h$, given an orientation of $H^0(Y)$, $H^1(Y)$ and $H^2(Y)$, for any path h_t in the space of admissible perturbations, there is also a corresponding orientation for the cobordism moduli space $\mathcal{M}_t$. And this gives the oriented Legendrian cobordism.*

7.3 Concluding Remarks

Our approach here focuses on setup of the theory instead of the geometric implications and applications.

Mainly by studying solutions on 3-manifolds with cylindrical ends, we established that the moduli spaces are compact oriented manifolds of finite dimension and are related to invariants on the boundary. The moduli spaces are smooth for generic perturbations. These should pave the path for further developments.

Analogous to four-dimensional theory, one can derive the Seiberg-Witten invariants through the above data, obtaining a map from $H^2(Y)$ to $\mathbb{Z}$. Or, in light of the

Donaldson polynomial invariants, a formal power series $\sum_i \alpha_i q^i$ where the coefficients α_i's are obtained from moduli spaces of proper dimensions. Since the moduli spaces are generically empty for all but finitely many *spin*$_c$ structures, the above should be a polynomial of degree given by the associated line bundle on the boundary Σ. Furthermore, by the symmetry between W^+ and W^-, if we shift our base point of *spin*$_c$ structure from the canonical one $\mathbb{C} \oplus K^{-1}$ to $K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}$, then we will get a polynomial of $q + q^{-1}$.

It is then natural to ask the relationship between this polynomial with other known invariants.

Another point of interest is the contact structures on Y. Our theory is based on a foliation on Y. Or more precisely, we assume that our *spin*$_c$ structure under consideration arises from the product foliation of $I \times \Sigma$ near the boundary. One may also ask what is the case when the *spin*$_c$ structure arises from a contact structure on Y.

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