APPLICATIONS OF DESCRIPTIVE SET THEORY TO TOPOLOGY AND ANALYSIS

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CONTENTS

ABSTRACT

In Chapter 1, we prove that for every family I of closed subsets of a Polish space each $\mathbf{\Sigma}^1_1$ set can be covered by countably many members of I or else contains a nonempty $\mathbf{\Pi}^0_2$ set which cannot be covered by countably many members of I. We derive from it the general form of Hurewicz's theorem due to Kechris, Louveau, and Woodin, and a theorem of Feng on the open covering axiom. Also some well-known theorems on finding "big" closed sets inside of "big" $\mathbf{\Sigma}^1_1$ sets are consequeces of our result. Chapter 2 consists of a joint work with A.S. Kechris. We prove that given a σ-ideal I, the possibilty of approximating each $\mathbf{\Sigma}^1_1$ set by a Borel set modulo I is equivalent to a definable form of the countable chain condition. This answers a question of Mauldin. We also characterize the meager ideal on a Polish group G as the only translation invariant, ccc σ-ideal I on G such that each set from I is contained in an F_σ set from I. This partially verifies a conjecture of Kunen. In Chapter 3, we establish a theorem which gives sufficient conditions for a K_σ equivalence relation to continuously embed E_0. As a consequence of this result we show that no indecomposable continuum contains a Borel set which has precisely one point in common with each composant. This solves an old problem in the theory of continua. In Chapter 4, answering a question of A.S. Kechris, we prove that the Topological Vaught Conjecture holds for Polish groups admitting invariant metrics. We also answer a question of R.L. Sami by proving that there exist continuous actions of Polish abelian groups with non-Borel induced orbit eqivalence relations. Actually, we give a fully algebraic characterization of sequences of countable abelian groups (H_n) such that the group $\prod_n H_n$ has a continuous action with non-Borel orbit equivalence relation. In Chapter 5, we give a characterization of local compactness for Polish abelian groups in terms of Haar null sets of Christensen: a Polish abelian group is locally compact iff each family of mutually disjoint closed (or, equivalently, universally measurable) sets which are not Haar null is countable. This completes, in a sense, Dougherty's solution to a problem of Christensen. We also consider the question of the possibility of approximating analytic by Borel sets modulo Haar null sets. Chapter 6 contains two dichotomy theorems for Baire class 1 functions: a Baire class 1 function can be decomposed into countably many continuous functions, or else it contains a function which is as complicated with respect to decompositions into continuous functions as any other Baire class 1 function; an analogous theorem is proved for decompositions into continuous functions with closed domains. These results strengthen a theorem of Jayne and Rogers and answer some questions of Steprāns. Their proofs use effective descriptive set theory as well as infinite games. Some results on decompositions of Borel sets and functions on higher levels are also obtained.

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CHAPTER 1

COVERING ANALYTIC SETS BY FAMILIES OF CLOSED SETS

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Abstract. We prove that for every family I of closed subsets of a Polish space each Σ_1^1 set can be covered by countably many members of I or else contains a nonempty Π_2^0 set which cannot be covered by countably many members of I. We prove an analogous result for κ-Souslin sets and show that if $A^\#$ exists for any $A \subset \omega^\omega$, then the above result is true for Σ_2^1 sets. A theorem of Martin is included stating that this result is also true for weakly homogeneously Souslin sets. As an application of our results we derive from them a general form of Hurewicz's theorem due to Kechris, Louveau, and Woodin and a theorem of Feng on the open covering axiom. Also some well-known theorems on finding "big" closed sets inside of "big" Σ_1^1 and Σ_2^1 sets are consequences of our results.

§1. **Introduction.** Gy. Petruska answering a question of M. Laczkovich proved in [P] that a Σ_1^1 set on $[0, 1]$ either can be covered by countably many closed sets of Lebesgue measure 0 or else it contains a nonempty Π_2^0 set which cannot be covered by countably many closed sets of measure 0. (This is, in fact, an equivalent reformulation, see Remark 2 following the proof of Theorem 1.) It is a trivial observation that the above statement holds if we replace closed sets of Lebesgue measure 0 by closed nowhere dense sets (or equivalently, first category sets). A. Kechris formulated the following general question. Let I be a family of closed subsets of a Polish space X. Put $I_{\text{ext}} = \{Y \subseteq X : \exists \{F_n : n \in \omega\} \subseteq I \; Y \subseteq \bigcup_{n \in \omega} F_n\}$. Find out to what families of closed sets Petruska's theorem can be generalized, i.e., what families of closed sets have the following property: for any Σ_1^1 set A either $A \in I_{\text{ext}}$ or there is a Π_2^0 set $G \subseteq A$ with $G \notin I_{\text{ext}}$. This is a weak form of the covering property which says the same except that the set G is closed rather than merely Π_2^0. The covering property is very restrictive. For example, neither the family of closed sets of Lebesgue measure 0 nor of first category have this property. On the other hand, the families of closed countable sets, of compact sets and of closed sets of extended uniqueness on $[0, 2\pi]$ do have it (Souslin; Kechris [K] and Saint Raymond [SR]; Debs-Saint Raymond [DS], see also [KL, Theorem 5, p. 426]). Surprisingly it turns out that the answer to Kechris' question is affirmative for all families of closed sets. Moreover, assuming that $A^\#$ exists for any $A \subseteq \omega^\omega$ we prove that if $A \in \Sigma_2^1$ then $A \in I_{\text{ext}}$ or there is $G \subset A$ with $G \notin I_{\text{ext}}$ and $G \in \Pi_2^0$. We give several applications of these results. Among them the generalized Hurewicz theorem proved in [KLW] and the theorem of Feng [F] that Σ_1^1 sets fulfil the open covering axiom. Also certain theorems of Kechris, Solovay, and Louveau can be derived from our results.

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Define $I_{\text{perf}} = \{Y \subseteq X: Y \neq \varnothing \text{ and } \forall U \text{ open } U \cap Y \neq \varnothing \Rightarrow U \cap Y \notin I_{\text{ext}}\}$. By $\text{cl}(A)$ we denote the closure of the set A. If $\mathcal{F}$ is a family of subsets of X let $\mathcal{F}^d = \text{cl}(\bigcup \mathcal{F}) \setminus \bigcup\{\text{cl}(F): F \in \mathcal{F}\}$. $\text{MGR}(A)$ denotes the family of all subsets of A which are of first category in A. If $\tau \in \omega^{<\omega}$ then $\text{lh}\,\tau$ is the unique $N \in \omega$ with $\tau \in \omega^N$. By $\tau * \sigma$, $\tau, \sigma \in \omega^{<\omega}$, we denote the concatenation of τ and σ: $\tau * n$, $\tau \in \omega^{<\omega}$, $n \in \omega$, stands for $\tau * \langle 0, n \rangle$. For $x \in \omega^\omega$ or $x \in 2^\omega = \{0, 1\}^\omega$ and $n \in \omega$ by $x|n$ we denote the restriction of x to $n = \{0, \ldots, n-1\}$: in particular $x|0 = \varnothing$. We give ω^ω and 2^ω the product topologies with basic neighborhoods $[\sigma] = \{x: x|\text{lh}\,\sigma = \sigma\}$ for $\sigma \in \omega^{<\omega}$ or $\sigma \in 2^{<\omega}$. If T is a tree on a set X and $u \in X^n$ for some $n \in \omega$, then $T_u = \{v \in T: v \subset u \text{ or } u \subset v\}$. For $T \subset \omega^{<\omega}$ put $[T] = \{x \in \omega^\omega: \forall n\, x|n \in T\}$. If T is a tree on $\omega \times \kappa$ define $p[T] = \{x \in \omega^\omega: \exists y \in \kappa^\omega \forall n\, (x|n, y|n) \in T\}$. For $P \subset Y \times X$ and $y \in Y$ put $P_y = \{x \in X: (y, x) \in P\}$.

§2. Covering $\mathbf{\Sigma}^1_1$ sets by closed sets.

THEOREM 1. *Let I be a family of closed subsets of a Polish space X. Then each $\mathbf{\Sigma}^1_1$ set either is in I_{ext} or contains a $\mathbf{\Pi}^0_2$ subset not in I_{ext}*

This result can also be formulated as follows: Let J be a σ-ideal generated by a family of closed sets in a Polish space. Then any $\mathbf{\Sigma}^1_1$ set not in J contains a $\mathbf{\Pi}^0_2$ subset which is not in J.

We need the following lemma of Petruska. It was proved in [P] in the special case $I =$ the family of measure 0 closed subsets of $[0, 1]$, but the argument works in the general situation as well.

LEMMA ([P, Lemma 4]). *Let $A \subseteq X$ be $\mathbf{\Sigma}^1_1$. Assume $A \notin I_{\text{ext}}$. Then there is a regular Souslin scheme $\{A_\tau: \tau \in \omega^{<\omega}\}$ consisting of closed sets such that*

(i) $A_\varnothing \neq \varnothing$;

(ii) $\bigcup_{x \in \omega^\omega} \bigcap_{n \in \omega} A_{x|n} \subseteq A$;

(iii) *if* $A_\tau \neq \varnothing$, *then* $A \cap A_\tau \in I_{\text{perf}}$ *and is dense in* A_τ;

(iv) $\bigcup\{A_{\tau * n}: n \in \omega\}$ *is dense in* A_τ.

OUTLINE OF THE PROOF OF THE LEMMA. Let A have a representation $A = \bigcup_{x \in \omega^\omega} \bigcap_{n \in \omega} H_{x|n}$ where H_τ, $\tau \in \omega^{<\omega}$, are closed. Put $L_\tau = \bigcup_{x \in \omega^\omega} \bigcap_{n \in \omega} H_{\tau * x|n}$. Then define $A_\tau = \text{cl}(L^*_\tau)$ where $L^*_\tau = L_\tau \setminus \bigcup\{U: U \text{ open and } U \cap L_\tau \in I_{\text{ext}}\}$. □

PROOF OF THEOREM 1. Let $A \subseteq X$ be $\mathbf{\Sigma}^1_1$. Suppose $A \notin I_{\text{ext}}$. Let A_τ, $\tau \in \omega^{<\omega}$, be as in the Lemma.

Case 1. $\exists\, \tau \in \omega^{<\omega} \exists\, U$ open $A_\tau \cap U \neq \varnothing$ and $I_{\text{ext}} \supset \text{MGR}(A_\tau \cap U)$.

Put $A' = A \cap A_\tau \cap U$. Then by (iii) from the Lemma $A' \in I_{\text{perf}}$. Also A' is $\mathbf{\Sigma}^1_1$. Thus A' has the Baire property whence there is a $\mathbf{\Pi}^0_2$ set G such that $G \subseteq A'$ and $A' \setminus G \in \text{MGR}(A_\tau \cap U) \subseteq I_{\text{ext}}$. Thus, $G \notin I_{\text{ext}}$.

Case 2. $\forall\, \tau \in \omega^{<\omega} \forall\, U$ open $A_\tau \cap U = \varnothing$ or $\text{MGR}(A_\tau \cap U) \setminus I_{\text{ext}} \neq \varnothing$.

In the following construction we use an idea from [KLW, Lemma 7] (see also [KL, Theorem 2, p. 425]). Let us fix a complete metric on X. We construct recursively $\phi: \omega^{<\omega} \to \omega^{<\omega}$ and U_τ, $\tau \in \omega^{<\omega}$, with the following properties:

(1) $lh\phi(\tau) = lh\tau$; $\tau \subseteq \rho \Rightarrow \phi(\tau) \subset \phi(\rho)$;

(2) U_τ is open;

(3) $\text{diam}\, U_\tau \leq 1/(lh\tau + 1)$;

(4) $\lim_n \text{diam}\, U_{\tau * n} = 0$;

(5) $\tau \subseteq \rho$ and $\tau \neq \rho \Rightarrow \text{cl}(U_\rho) \subset U_\tau$;
(6) $U_{\tau * n} \cap U_{\tau * m} = \varnothing$ if $n \neq m$;
(7) $U_\tau \cap A_{\phi(\tau)} \neq \varnothing$;
(8) $\{U_{\tau * n} : n \in \omega\}^d \notin I_{\text{ext}}$;
(9) $\{U_{\tau * n} : n \in \omega\}^d \subset U_\tau$.

Put $\phi(\varnothing) = \varnothing$. $U_\varnothing$ any open set with diam $U_\varnothing \leq 1$ and $U_\varnothing \cap A_\varnothing \neq \varnothing$. Assume $\phi(\tau)$, U_τ are constructed for all $\tau \in \omega^N$, $N \in \omega$. Pick $\tau \in \omega^N$. Then $U_\tau \cap A_{\phi(\tau)} \neq \varnothing$ and $\text{MGR}(A_{\phi(\tau)} \cap U_\tau) \backslash I_{\text{ext}} \neq \varnothing$. Thus, we can find $K \subseteq A_{\phi(\tau)} \cap U_\tau$, K closed, nowhere dense in $A_{\phi(\tau)}$ and $K \notin I_{\text{ext}}$. Since K is nowhere dense, we can find a countable discrete set $D \subseteq A_{\phi(\tau)} \cap U_\tau$ with $\text{cl}(D) = K \cup D$ and $D \cap K = \varnothing$. Put $D = \{x_n : n \in \omega\}$ with $x_n \neq x_m$ for $n \neq m$. Let $U_{\tau * n}$ be an open ball centered at x_n with radius $r_n > 0$. By choosing r_n sufficiently small we can arrange that $\text{cl}(U_{\tau * n}) \subseteq U_\tau$, diam $U_{\tau * n} \leq 1/(\text{lh}\tau + 2)$, $\lim_n \text{diam}\, U_{\tau * n} = 0$, $U_{\tau * n} \cap U_{\tau * m} = \varnothing$ if $n \neq m$, and $\{U_{\tau * n} : n \in \omega\}^d = K$. Since $x_n \in A_{\phi(\tau)}$, we also have $U_{\tau * n} \cap A_{\phi(\tau)} \neq \varnothing$. Now for each n we can find a $k \in \omega$ with $U_{\tau * n} \cap A_{\phi(\tau) * k} \neq \varnothing$ by (iv) from the Lemma. Pick such a k and put $\phi(\tau * n) = \phi(\tau) * k$. This finishes the construction.

Now put $G = \bigcap_n \bigcup \{U_\tau : \text{lh}\tau = n\}$. Then G is $\mathbf{\Pi}^0_2$ by (2). We show that $G \subseteq A$ and $G \notin I_{\text{ext}}$. From (5) and (6) it follows that $G = \bigcup_{x \in \omega^\omega} \bigcap_{n \in \omega} U_{x|n}$. Since diam $U_{x|n} \leq 1/(n+1)$, $U_{x|n} \cap A_{\phi(x|n)} \neq \varnothing$ (by (3) and (7)) and $A_{\phi(x|n)}$ are closed, we have $\bigcap_{n\in\omega} U_{x|n} \subseteq \bigcap_{n \in \omega} A_{\phi(x|n)}$. By (1) there is $y \in \omega^\omega$ with $\phi(x|n) = y|n$ for all $n \in \omega$. Then $\bigcap_{n \in \omega} U_{x|n} \subseteq \bigcap_{n\in\omega} A_{y|n} \subseteq A$.

Now we show that $G \notin I_{\text{ext}}$. Note that (3), (5), and (7) guarantee that $\bigcap_{n\in\omega} U_{x|n} \neq \varnothing$ for any $x \in \omega^\omega$. Thus, $U_\tau \cap G \neq \varnothing$ for any $\tau \in \omega^{<\omega}$. Assume there are closed sets $F_n \in I$ with $\bigcup_{n\in\omega} F_n \supset G$. Then, by the Baire Category Theorem applied to $\text{cl}(G)$, there is an open set V and $n_0 \in \omega$ with $V \cap G \neq \varnothing$ and $V \cap \text{cl}(G) \subseteq F_{n_0}$. Now there is a $\tau \in \omega^{<\omega}$ with $U_\tau \subseteq V$ (by (3)). But by (4) and (9) and the fact that $G \cap U_{\tau * n} \neq \varnothing$, $n \in \omega$, we have $\{U_{\tau * n} : n \in \omega\}^d \subseteq V \cap \text{cl}(G) \subseteq F_{n_0}$. But $F_{n_0} \in I$ which contradicts (8) and the proof is complete. □

REMARK. (1) By putting $G' = G \backslash \bigcup \{U : U \text{ open and } U \cap G \in I_{\text{ext}}\}$ we can guarantee that the $\mathbf{\Pi}^0_2$ set produced in Theorem 1 is in I_{perf}.

(2) In the case where I is a σ-ideal of closed sets (i.e., I is a family of closed sets, a subset of an element from I is in I, and if $\{F_n : n \in \omega\} \subseteq I$ and $\bigcup_{n\in\omega} F_n$ is closed, then $\bigcup_{n \in \omega} F_n \in I$) the weak covering property has the following obvious reformulation (this is the original formulation from Petruska's theorem): Let A be $\mathbf{\Sigma}^1_1$. Then either $A \in I_{\text{ext}}$, or there is a nonempty closed set C so that $C \cap A$ contains a $\mathbf{\Pi}^0_2$ set dense in C and for any open set U if $U \cap C \neq \varnothing$, then $\text{cl}(U \cap C) \notin I$. To obtain such a set C from the $\mathbf{\Pi}^0_2$ set G produced in Theorem 1, put $C = \text{cl}(G \backslash \bigcup\{U : U \text{ open and } U \cap G \in I_{\text{ext}}\})$.

We can actually obtain a slightly more accurate conclusion than that in Theorem 1.

COROLLARY 1. *Let I be a family of closed subsets of a Polish space X. Let $A \subset X$ be $\mathbf{\Sigma}^1_1$ and such that $A \subset \bigcup I$. Then either $A \in I_{\text{ext}}$ or there is $G \subset A$ such that G is homeomorphic to ω^ω and $G \in I_{\text{perf}}$.*

PROOF. By Theorem 1 and Remark 1 $A \in I_{\text{ext}}$ or there is $G' \subset A$ which is $\mathbf{\Pi}^0_2$ and $G' \in I_{\text{perf}}$. Since $G' \in I_{\text{perf}}$ and $G' \subset \bigcup I$, G' is dense in itself. Now we find a $\mathbf{\Pi}_2^0$ set $G'' \subseteq G'$ such that G'' is dense in G' and G'' is 0-dimensional. By the Baire Category Theorem $G'' \in I_{\text{perf}}$. If there is no compact set $C \subset G''$ such that $\varnothing \neq U \cap G'' \subset C$ for some open set U, then G'' is homeomorphic to ω^ω. If there is such a compact set C, then $C \notin I_{\text{ext}}$ since $G'' \in I_{\text{perf}}$. We find $C' \subset C$ compact and such that $C' \in I_{\text{perf}}$. Since C' is 0-dimensional, compact, and dense in itself, C' is homeomorphic to 2^ω. Thus, it contains a dense copy of ω^ω, call it G. Again by a Baire category argument $G \in I_{\text{perf}}$. □

§3. **Covering κ-Souslin sets by closed sets.** By $L[A_1, \dots, A_n]$ we denote the smallest inner model M of ZF such that $A_i \cap M \in M$, $i = 1, \dots, n$ (see [J, p. 128]). We say that a family of closed sets I is *hereditary* if closed subsets of elements from I are in I.

THEOREM 2. *Let I be a hereditary family of closed subsets of ω^ω. Let $A \subset \omega^\omega$ be κ-Souslin. Assume that κ-Souslin sets have the Baire property. Then one of the following holds*:

(i) *A can be covered by κ many sets from I*;

(ii) *A contains a $\mathbf{\Pi}_2^0$ set G such that $G \notin I_{\text{ext}}$.*

Moreover if $A = p[T]$ for a tree T on $\omega \times \kappa$, then (i) *can be strengthened to*:

(i)′ *there exist $\lambda < \kappa^+$ and a family of trees on $\omega\{S_\xi : \xi < \lambda\} \in L[T, I]$ such that $A \subset \bigcup\{[S_\xi] : \xi < \lambda\}$ and $\{[S_\xi] : \xi < \lambda\} \subset I$.*

PROOF. For any tree S on $\omega \times \kappa$ define $c(S) = \{s \in \omega^{<\omega} : \{((n_1, \xi_1), \dots, (n_k, \xi_k)) \in S : s \subset (n_1, \dots, n_k)\}$ is not well founded$\}$. We have $[c(S)] = \text{cl}(p[S])$. Now we define recursively

$T^0 = T$;

$T^{\xi+1} = \{u \in T^\xi : [c(T_u^\xi)] \notin I\}$;

$T^\xi = \bigcap_{\zeta<\xi} T^\zeta$ if ξ is a limit ordinal.

There exists $\lambda < \kappa^+$ such that $T^\lambda = T^{\lambda+1}$.

Case 1. $T^\lambda = \varnothing$.

Define $\mathscr{F} = \bigcup_{\xi<\lambda}\{c(T_u^\xi) : u \in T^\xi \setminus T^{\xi+1}\}$. Then $\mathscr{F} \in L[T, I]$ and $\{[S] : S \in \mathscr{F}\} \subset I$. Also, $p[T] \subset \bigcup\{[S] : S \in \mathscr{F}\}$.

Case 2. $T^\lambda \neq \varnothing$.

Subcase 1. $\exists u \in T^\lambda I_{\text{ext}} \supset \text{MGR}(\text{cl}(p[T_u^\lambda]))$. Since by assumption $p[T_u^\lambda]$ has the Baire property, this subcase can be dealt with as in Case 1 in Theorem 1 as long as we show that $p[T_u^\lambda] \notin I_{\text{ext}}$ for $u \in T^\lambda$. But otherwise there exist $u \in T^\lambda$ and $K_n \in I$, $n \in \omega$ such that $p[T_u^\lambda] \subset \bigcup_{n\in\omega} K_n$. Then $[T_u^\lambda] \subset \bigcup_{n\in\omega} p^{-1}(K_n)$. By the Baire Category Theorem there is $n_0 \in \omega$ and $v \in T^\lambda$ with $u \subset v$ and $[T_v^\lambda] \subset p^{-1}(K_{n_0})$, whence $[c(T_v^\lambda)] = \text{cl}(p[T_v^\lambda]) \in I$ which contradicts the definition of λ.

Subcase 2. $\forall\, u \in T^\lambda\ \text{MGR}(\text{cl}(p[T_u^\lambda])) \setminus I_{\text{ext}} \neq \varnothing$. The assumption implies that for any $u \in T^\lambda$ we can find a closed set $K \subset \text{cl}(p[T_u^\lambda])$ which is nowhere dense in $\text{cl}(p[T_u^\lambda])$ and $K \notin I_{\text{ext}}$. By a construction similar to that in Case 2 in Theorem 1 we build a function $\phi : \omega^{<\omega} \to T^\lambda$ so that

(i) $\text{diam}(p[T_{\phi(\sigma)}^\lambda]) \leq 1/(\text{lh}\sigma + 1)$;

(ii) if $\sigma \subset \tau$ and $\sigma \neq \tau$, then $\phi(\sigma) \subset \phi(\tau)$ and $\phi(\sigma) \neq \phi(\tau)$;

(iii) $\{p[T^{\lambda}_{\phi(\sigma * n)}]\colon n \in \omega\}$ is discrete (i.e.. $\mathrm{cl}(p[T^{\lambda}_{\phi(\sigma * k)}]) \cap \mathrm{cl}(\bigcup_{n \neq k} p[T^{\lambda}_{\phi(\sigma * n)}]) = \varnothing$) for any $\sigma \in \omega^{<\omega}$:

(iv) $\{p[T^{\lambda}_{\phi(\sigma * n)}]\colon n \in \omega\}^{d} \notin I_{\mathrm{ext}}$.

The set

$$G = \bigcup_{x \in \omega^{\omega}} \bigcap_{n \in \omega} p[T^{\lambda}_{\phi(x|n)}] = \bigcap_{n \in \omega} \bigcup_{u \in \phi[\omega^{n}]} p[T^{\lambda}_{u}]$$

is contained in A. To show that $G \in \mathbf{\Pi}^0_2$ note that the conditions (i) and (iii) imply the existence of a family of open sets $\{V_\tau\colon \tau \in \omega^{<\omega}\}$ such that $p[T^{\lambda}_{\phi(\tau)}] \subset V_\tau$. if $\rho \subset \tau$ then $V_\tau \subset V_\rho$. if neither $\rho \subset \tau$ nor $\tau \subset \rho$. then $V_\rho \cap V_\tau = \varnothing$. and finally $\operatorname{diam} V_\tau \leq 2/(\mathrm{lh}\tau + 1)$. Since $\bigcap_{n \in \omega} p[T^{\lambda}_{\phi(x|n)}] \neq \varnothing$ for any $x \in \omega^\omega$. we have $\bigcap_{n\in\omega} p[T^{\lambda}_{\phi(x|n)}] = \bigcap_{n \in \omega} V_{x|n}$. It follows that $G = \bigcap_{n\in\omega} \bigcup_{\tau \in \omega^n} V_\tau \in \mathbf{\Pi}^0_2$. As in Case 2 in Theorem 1 we show that $G \notin I_{\mathrm{ext}}$. □

Let I be a family of closed sets of a Polish space X. Call a set $A \subset X$ *I-approximable* if either $A \in I_{\mathrm{ext}}$ or there is a $\mathbf{\Pi}^0_2$ set $G \subset A$ such that $G \notin I_{\mathrm{ext}}$. A is *absolutely approximable* if it is I-approximable for any family I of closed subsets of X. The following simple proposition will prove to be useful.

PROPOSITION 1. *Let X and Y be Polish spaces. Let I be a family of closed subsets of Y. and let $f\colon X \to Y$ be continuous. If $A \subset X$ is I^*-approximable. where $I^* = \{f^{-1}[F]\colon F \in I\}$. then $f[A]$ is I-approximable. In particular. if A is absolutely approximable. then so is $f[A]$.*

PROOF. By assumption either $A \in I^*_{\mathrm{ext}}$, whence $f[A] \in I_{\mathrm{ext}}$, or there is a $\mathbf{\Pi}^0_2$ set $G' \subset A$ such that $G' \notin I^*_{\mathrm{ext}}$. But then $f[G'] \notin I_{\mathrm{ext}}$ and $f[G'] \in \mathbf{\Sigma}^1_1$. Thus. applying Theorem 1 we obtain a $\mathbf{\Pi}^0_2$ set $G \subset f[G'] \subset A$ with $G \notin I_{\mathrm{ext}}$. □

COROLLARY 2. *Assume $\omega_1^{L[x]} < \omega_1$ for any $x \in \omega^\omega$. Let X be a Polish space. Let $A \subset X$ be $\mathbf{\Sigma}^1_2$. and let I be a family of closed sets which is $\mathbf{\Sigma}^1_2$ (in the Effros structure). Then either $A \in I_{\mathrm{ext}}$ or there exists a $\mathbf{\Pi}^0_2$ set $G \subset A$ with $G \notin I_{\mathrm{ext}}$.*

PROOF. First. let us notice that we need to prove the conclusion only for $X = \omega^\omega$. Indeed. let $\phi\colon \omega^\omega \to X$ be a continuous surjection. Put $A' = \phi^{-1}(A)$. $I^* = \{\phi^{-1}(F)\colon F \in I\}$. Note that $A' \in \mathbf{\Sigma}^1_2$ and $I^* \in \mathbf{\Sigma}^1_2$. Now assuming that we have proved the Corollary for $X = \omega^\omega$ A' is I^*-approximable. Then by Proposition 1 A is I-approximable.

Let $X = \omega^\omega$. We obviously can assume that I is hereditary. There exists $x_0 \in \omega^\omega$ such that $A.\ I \in \Sigma^1_2(x_0)$. Thus. by Schoenfield's theorem I is absolute for $L[x_0]$. and there exists a tree T on $\omega \times \omega_1$ such that $T \in L[x_0]$ and $p[T] = A$. Assume A does not contain a $\mathbf{\Pi}^0_2$ set not in I_{ext}. Since by Solovay's theorem $\omega_1^{L[x]} < \omega_1$ for all $x \in \omega^\omega$ implies that ω_1-Souslin sets with trees in L have the Baire property. we conclude from Theorem 2 that there exists a sequence of trees on ω $\{S_\xi\colon \xi < \lambda\} \in L[x_0]$ such that $A \subset \bigcup\{[S_\xi]\colon \xi < \lambda\}$ and $\{[S_\xi]\colon \xi < \lambda\} \subset I$. Since $L[x_0] \cap \omega^\omega$ is countable. $|\{S_\xi\colon \xi < \lambda\}| \leq \omega$. whence $A \in I_{\mathrm{ext}}$. □

A. Kechris pointed out that if one assumes that $A_i^{\#}$ exists for $A_1, \dots, A_n \subset \omega^\omega$ (e.g.. if a Ramsey cardinal exists) then $L[A_1, \dots, A_n]$ contains only countably many reals. Thus. the proof of Theorem 2 and Corollary 4 goes through for an arbitrary family I of closed sets. Therefore. the following corollary holds true.

COROLLARY 3. *Assume $A^{\#}$ exists for any $A \subset \omega^\omega$. Let I be a family of closed subsets of a Polish space X, and let $A \subset X$ be $\mathbf{\Sigma}^1_2$. Then either $A \in I_{\text{ext}}$ or there exists a $\mathbf{\Pi}^0_2$-set $G \subset A$ with $G \notin I_{\text{ext}}$.*

D. A. Martin defined the following game $\Gamma(I, A)$, where I is a family of closed subsets of ω^ω and A is a subset of ω^ω. Player I plays $K_n \in I$, and Player II plays $\sigma_n \in \omega^{<\omega}$, $n \in \omega$, so that $\sigma_n \subset \sigma_{n+1}$, $\sigma_n \neq \sigma_{n-1}$ and $[\sigma_n] \cap K_n = \varnothing$. Player II wins if $\bigcup_{n\in\omega} \sigma_n \in A$. The next theorem and corollary are due to Martin.

THEOREM 3 (Martin). *If Player II has a winning strategy in $\Gamma(I, A)$, then there is a $\mathbf{\Pi}^0_2$ set $G \subset A$ such that $G \notin I_{\text{ext}}$. If Player I has a winning strategy, then $A \in I_{\text{ext}}$.*

PROOF. By Theorem 1 to prove the first part it is enough to show that there is a $\mathbf{\Sigma}^1_1$ subset of A not in I_{ext}. Denote by Σ a winning strategy of Player II. First we construct recursively a countable set $\mathscr{K} \subset I$ such that for any $K_0, \dots, K_{n-1} \in \mathscr{K}$, $K \in I$ there is $K_n \in \mathscr{K}$ such that $\Sigma(K_0, \dots, K_{n-1}, K) = \Sigma(K_0, \dots, K_{n-1}, K_n)$. Now define $B \subset \omega^\omega$ as follows: $x \in B$ iff there exist $K_n \in \mathscr{K}$ and $k_n \in \omega$, $n \in \omega$, such that $k_n < k_{n+1}$ and $(K_0, x|k_0, \dots, K_n, x|k_n)$ agrees with Σ for each $n \in \omega$. Then clearly $B \in \mathbf{\Sigma}^1_1$ and $B \subset A$. Moreover $B \notin I_{\text{ext}}$. Otherwise there are $K_n \in I$, $n \in \omega$ with $B \subset \bigcup_{n\in\omega} K_n$. Now we can define inductively $K'_n \in \mathscr{K}$ so that $\Sigma(K'_0, \dots, K'_{n-1}, K_n) = \Sigma(K'_0, \dots, K'_{n-1}, K'_n)$ for each $n \in \omega$. Put $\sigma_n = \Sigma(K'_0, \dots, K'_n)$. Then $[\sigma_n] \cap K_n = \varnothing$ for every $n \in \omega$ whence $\bigcup_{n\in\omega} \sigma_n \notin \bigcup_{n\in\omega} K_n$. But on the other hand $\bigcup_{n\in\omega} \sigma_n \in B$.

Let Σ be a winning strategy of Player I. Define $\mathscr{F}$ by the condition that $K \in \mathscr{F}$ iff there exist $\sigma_0, \dots, \sigma_n \in \omega^{<\omega}$ and $K_0, \dots, K_n \in I$ such that $(K_0, \sigma_0, \dots, K_n, \sigma_n, K)$ agrees with Σ. It is easy to see that $\mathscr{F}$ is countable. Obviously, $\mathscr{F} \subset I$ and $A \subset \bigcup \mathscr{F}$. Thus, $A \in I_{\text{ext}}$. □

It is easy to see that if $A \subset \omega^\omega$ is homogeneously Souslin, then the outcome of the above game is homogeneously Souslin. Thus, by [MS, Theorem 2.3] the game is determined, i.e., from Theorem 3 homogeneously Souslin subsets of ω^ω are absolutely approximable. Since weakly homogeneously Souslin sets are projections of homogeneously Souslin sets, we get from Proposition 1 that these too are absolutely approximable. In particular, if there exist ω Woodin cardinals and a measurable cardinal above them, then the game is determined for A projective or $A \in L(\mathbf{R}) \cap \text{Power}(\mathbf{R})$ [MS]. Thus, we have the following corollary.

COROLLARY 4 (Martin). *Let I be a family of closed subsets of ω^ω, and let $A \subset \omega^\omega$ be weakly homogeneously Souslin. Then either $A \in I_{\text{ext}}$, or there is a $\mathbf{\Pi}^0_2$ set $G \subset A$ with $G \notin I_{\text{ext}}$. The same holds for A projective or $A \in L(\mathbf{R}) \cap \text{Power}(\mathbf{R})$ if there are ω Woodin cardinals with a measurable cardinal above them.*

§**4. Applications.** In this section we give various applications of the results proved in the §§2 and 3.

A frequently met problem in analysis or descriptive set theory is that of finding a "big" closed set inside of "big" $\mathbf{\Sigma}^1_1$, $\mathbf{\Sigma}^1_2$, or projective sets. The results proved in §§2 and 3 reduce this problem to finding "big" closed sets inside of "big" $\mathbf{\Pi}^0_2$ sets. Let us be more precise. Following [L] call a set $A \subset X$ I-regular if either $A \in I_{\text{ext}}$ or there is a closed set $C \subset X$ with $C \subset A$ and $C \notin I_{\text{ext}}$. (Here again X is a Polish space and I is a family of closed subsets of X.) Note that I has the covering property (see the Introduction) iff every $\mathbf{\Sigma}^1_1$ set is I-regular. From Theorem 1 and Corollary 2 we have the following immediate corollary. (Note that similar conclusions, under appropriate assumptions, can be drawn form Corollaries 3 and 4.)

COROLLARY 5. *Let I be a family of closed subsets of a Polish space X. Every $\mathbf{\Sigma}^1_1$ subset of X is I-regular iff every $\mathbf{\Pi}^0_2$ subset is. If additionally $\omega_1^{L[x]} < \omega_1$ for any $x \in \omega^\omega$ and $I \in \mathbf{\Sigma}^1_2$, then every $\mathbf{\Sigma}^1_2$ subset of X is I-regular iff every $\mathbf{\Pi}^0_2$ subset is.*

This corollary provides a basis for proving I-regularity of $\mathbf{\Sigma}^1_1$ and $\mathbf{\Sigma}^1_2$ sets. For example generalizing results of Kechris [K], Saint Raymond [SR], Solovay [S], and Souslin (the classical perfect set theorem), Louveau in [L] proved that if I is of well-founded type (for definition see [L]), then any $\mathbf{\Sigma}^1_1$ set is I-regular (i.e., I has the covering property) and if $\omega_1^{L[x]} < \omega_1$ for any $x \in \omega^\omega$, then any $\mathbf{\Sigma}^1_2$ set is I-regular. Using Corollary 4 we can obtain both results by simply proving that $\mathbf{\Pi}^0_2$ sets are I-regular (note that $I \in \mathbf{\Pi}^0_2 \subset \mathbf{\Sigma}^1_2$ since I is of well-founded type) and this is not difficult.

Now we will indicate how one can derive from the results proved in §§2 and 3 a theorem of Feng [F, Theorem 1.1, Theorem 2.1, the remark following Theorem 3.4] saying that sets in $\mathbf{\Sigma}^1_1$ (or in $\mathbf{\Sigma}^1_2$ if $\omega_1^{L[x]} < \omega_1$ for all $x \in \omega^\omega$, or in $\bigcup_{n\in\omega} \mathbf{\Sigma}^1_n$ or $L(\mathbf{R}) \cap \mathrm{Power}(\mathbf{R})$ if there are ω Woodin cardinals with a measurable cardinal above them) satisfy the open covering axiom.

COROLLARY 6 (Feng). *Let X be a Polish space, and let $X \times X = K_0 \cup K_1$, where K_1 is closed and symmetric. Assume $A \subset X$ is $\mathbf{\Sigma}^1_1$ ($\mathbf{\Sigma}^1_2$ if $\omega_1^{L[x]} < \omega_1$ for all $x \in \omega^\omega$, or is in $\bigcup_{n\in\omega} \mathbf{\Sigma}^1_n$ or $L(\mathbf{R}) \cap \mathrm{Power}(\mathbf{R})$ if there are ω Woodin cardinals with a measurable cardinal above them). Then either A can be covered by countably many 1-homogeneous sets or contains a perfect compact set which is 0-homogeneous. (A set $S \subset X$ is called i-homogeneous, $i = 1$ or 2 if $(x, y) \in K_i$ for any $x, y \in S$ with $x \neq y$.)*

PROOF. Put $I = \{F \subset X : F \text{ is closed and } F \times F \subset K_1\}$. Notice that I is $\mathbf{\Pi}^0_1$, whence certainly $\mathbf{\Sigma}^1_2$. Then by Theorem 1 or Corollary 2 or Corollary 4 $A \in I_{\text{ext}}$, i.e., A can be covered by countably many 1-homogeneous sets, or there exists a $\mathbf{\Pi}^0_2$ set $G \subset A$ with $G \in I_{\text{perf}}$. This condition means that if $U \subset X$ is open and $U \cap G \neq \varnothing$, then $(U \cap G) \times (U \cap G) \not\subset K_1$. Let us fix a complete metric on G. We are in the following situation: G is complete, $V = (G \times G) \backslash K_1 \subset G \times G$ is open and symmetric and for any nonempty open set $U \subset G$ there are open sets $U_1, U_2 \subset U$ with $U_1 \times U_2 \subset V$, and hence, $U_2 \times U_1 \subset V$. This allows us to construct open sets U_σ, $\sigma \in 2^{<\omega}$, such that

(1) $\sigma \subset \tau$ and $\sigma \neq \tau \Rightarrow \mathrm{cl}(U_\tau) \subset U_\sigma$;

(2) If neither $\sigma \subset \tau$ nor $\tau \subset \sigma$, then $U_\sigma \cap U_\tau = \varnothing$ and $U_\sigma \times U_\tau \subset V$;

(3) $\mathrm{diam}\, U_\sigma \leq 1/\mathrm{lh}(\sigma)$.

Now if we put $C = \bigcap_{n\in\omega} \bigcup_{\mathrm{lh}(\sigma)=n} U_\sigma$, then C is perfect and compact, $C \subset G \subset A$, and also $C \times C \backslash \{(x, x) : x \in G\} \subset V$. This last condition means that C is 0-homogeneous. □

Let us point out that Corollary 6 (for $X = \omega^\omega$) is essentially included (modulo the simple argument presented above) in [L] as it is easy to see that the family I defined in the above proof is of well-founded type.

From this point on we state all results for $\mathbf{\Sigma}^1_1$ sets. But in the proofs we use only the property of $\mathbf{\Sigma}^1_1$ sets established in Theorem 1 (i.e., we actually prove them for absolutely approximable sets). Therefore one can equally well apply Corollaries 3 or 4 and obtain, under appropriately stronger hypothesis, analogous results for sets in $\mathbf{\Sigma}^1_2$, $\mathbf{\Sigma}^1_n$, or $L(\mathbf{R}) \cap \text{Power}(\mathbf{R})$.

One can quite easily deduce from Theorem 1 the strong version of Hurewicz's theorem proved in [KLW, Theorem 4, p. 267] (see also [KL, Theorem 7, p. 419]). The original proof is game theoretic and relies on the fact that games which are Boolean combinations of $\mathbf{\Pi}^0_2$ sets are determined.

COROLLARY 8 (Kechris-Louveau-Woodin). *Let X be a Polish space. Let $A, B \subseteq X$ be disjoint, and assume that A is $\mathbf{\Sigma}^1_1$. Then either A can be separated from B by a $\mathbf{\Sigma}^0_2$ set, or there is a homeomorphic embedding $\phi: 2^\omega \to X$ such that $\phi[2^\omega] \subset A \cup B$ and $\phi(x) \in B$ iff $x(n) = 0$ for all but finitely many $n \in \omega$.*

PROOF. Apply Theorem 1 to the family $I = \{F : F \text{ is closed and } F \cap B = \varnothing\}$. Then either $A \in I_{\text{ext}}$, i.e., A can be separated from B by a $\mathbf{\Sigma}^0_2$ set, or there is a closed set C as in Remark 2 which in this case means that $A \cap C$ contains a $\mathbf{\Pi}^0_2$ set dense in C and $B \cap C$ is dense in C. Let $G_n \subset C$, $n \in \omega$, be open and dense in C and such that $G_n \supset G_{n+1}$ and $A \supset \bigcap_{n\in\omega} G_n$. Now we recursively construct open in C sets $U_\tau \subset C$ and points $y_\tau \in C$, $\tau \in 2^{<\omega}$, so that

(1) $\tau \subseteq \rho$ and $\tau \neq \rho \Rightarrow U_\tau \supset \text{cl}(U_\rho)$;
(2) $U_{\tau*0} \cap U_{\tau*1} = \varnothing$;
(3) $\text{diam } U_\tau \leq 1/(\text{lh}\tau + 1)$;
(4) $y_\tau \in U_\tau \cap B$;
(5) $y_\tau = y_{\tau*0^n}$ for any $n \in \omega$ where 0^n is a sequence consisting of n 0's;
(6) if $\text{lh}\tau = n + 1$ and $\tau(n) = 1$, then $U_\tau \subset G_n$.

The construction is elementary so we skip its detailed description. Let us only mention that the conditions (4), (5), and (6) can be met since B is dense and the G_n are open and dense in C.

Now define $\phi(x)$ to be the only (by (1) and (3)) point in $\bigcap_{n\in\omega} U_{x|n}$ for $x \in 2^\omega$. Clearly, ϕ is a homeomorphism by (1), (2), and (3). If $x \in 2^\omega$ and $x(n) = 1$ for infinitely many $n \in \omega$, then (6) and the fact that $G_n \supset G_{n+1}$ guarantee that $\phi(x) \in \bigcap_{n\in\omega} G_n \subset A$. If $x(n) = 0$ for $n \geq N$ for some $N \in \omega$, then $\phi(x) = y_{x|N} \in B$ by (4) and (5). Thus, ϕ is as required. □

A. Kechris pointed out to us that the following result proved independently in [K] (for $X = \omega^\omega$) and [SR] follows from the above corollary. Let X be Polish. Let $A \subset X$ be $\mathbf{\Sigma}^1_1$. Then either A can be covered by a K_σ or else contains a closed copy of ω^ω. To see this put $\widetilde{X}$ = a metrizable compactification of X. Then either A can be separated from $\widetilde{X}\setminus X$ by a $\mathbf{\Sigma}^0_2$ set, i.e., A can be covered by a K_σ since $\widetilde{X}$ is compact or else there is a homeomorphic embedding $\phi: 2^\omega \to X$ such that $\phi[2^\omega] \subset (\widetilde{X}\setminus X) \cup A$ and $\phi(x) \in \widetilde{X}\setminus X$ iff $x(n) = 0$ for all but finitely many $n \in \omega$. In this case $\phi[2^\omega] \cap X$ is closed in X and is contained in A. Notice also that $\phi[2^\omega] \cap X$ is homeomorphic to ω^ω.

A special case of the next corollary for $A \in \mathbf{\Pi}^0_2$, as well as its lightface version for $A \in \Sigma^1_1$, $P \subset X \times X$ symmetric Π^1_1 with K_σ sections, is due to Louveau [L1], [L2, Lemma 3.10]. That the following boldface version for $A \in \mathbf{\Sigma}^1_1$ follows from Theorem 1 was pointed out to us by Kechris. (We state it here for $X^k \times X$ instead of $X \times X$ because of one application that we consider below.)

COROLLARY 7. *Let X be Polish and let $P \subset X^{k-1} = X^k \times X$, $k \in \omega$, be such that P has the Baire property in Y^{k-1} for any $Y \subset X$, $Y \in \mathbf{\Pi}^0_2$, and $P_{\overline{x}} \in \mathbf{\Sigma}^0_2$ for any $\overline{x} \in X^k$. Let $A \subset X$ be $\mathbf{\Sigma}^1_1$. Then either there is a sequence $\overline{x}_n \in X^k$, $n \in \omega$, such that $A \subset \bigcup_{n\in\omega} P_{\overline{x}_n}$ or else there is a perfect compact set $C \subset A$ such that $(x_1, \dots, x_{k+1}) \notin P$ for $x_1, \dots, x_{k+1} \in C$ with $x_i \neq x_j$ for $i \neq j$.*

PROOF. Put $I = \{F : F \subset X, F$ is closed, and there is $\overline{x} \in X^k$ with $F \subset P_{\overline{x}}\}$. Then by Theorem 1 and Remark 1 either $A \in I_{\text{ext}}$ which in this case means $A \subset \bigcup_{n\in\omega} P_{\overline{x}_n}$ for some $\overline{x}_n \in X^k$, $n \in \omega$, or else there is a $\mathbf{\Pi}^0_2$ set $G \subset A$ such that $G \in I_{\text{perf}}$. Put $P' = P \cap G^{k+1}$. (We could use now Louveau's theorem for $A \in \mathbf{\Pi}^0_2$, but the argument is short enough to be included here.) Then P' has the Baire property in G^{k+1}. Note also that $P'_{\overline{x}}$ is meager in G for any $\overline{x} \in G^k$. Otherwise, by the Baire Category Theorem, there would exist an open set $U \subset X$ with $\varnothing \neq U \cap G \subset P'_{\overline{x}} \subset P_{\overline{x}}$ which contradicts the fact that $G \in I_{\text{perf}}$. Thus, by the Kuratowski-Ulam theorem (see [O]) P' is meager in G^{k+1}. Since G is Polish in the relative topology, by Mycielski's theorem [M] there exists a compact perfect set $C \subset G \subset A$ with the desired properties. □

This corollary can be used to prove the following result of van Engelen, Kunen, and Miller [EKM]. Let $A \subset \mathbf{R}^2$, $\mathbf{R}$ = the reals, be $\mathbf{\Sigma}^1_1$. Then either A can be covered by countably many lines or it contains a compact perfect set no three points of which are collinear. Simply consider the relation $P \subset (\mathbf{R}^2)^3 = (\mathbf{R}^2)^2 \times \mathbf{R}^2$ defined by: $(x, y, z) \in P$ iff $x, y, z \in \mathbf{R}^2$ are collinear and $x \neq y$ and note that $P \in \mathbf{\Sigma}^1_1$ and $P_{(x,y)}$, for $x, y \in \mathbf{R}^2$, is empty or a line thus, $\mathbf{\Sigma}^0_2$.

Let us mention one more application of Theorem 1. Louveau in [L, Theorem 2.2] proved that in Solovay's model if I is a family of closed subsets of ω^ω of the form $I = \{C_x : x \in \omega^\omega\}$ for some closed set $C \subset \omega^\omega \times \omega^\omega$, then for any set $A \subset \omega^\omega$, $A \notin I_{\text{ext}}$ there exists a $\mathbf{\Sigma}^1_1$ set G such that $G \subset A$ and $G \notin I_{\text{ext}}$. From Theorem 1 it follows that G can be chosen to be $\mathbf{\Pi}^0_2$.

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CHAPTER 2

APPROXIMATION OF ANALYTIC BY BOREL SETS AND DEFINABLE COUNTABLE CHAIN CONDITIONS

BY

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ABSTRACT

Let I be a σ-ideal on a Polish space such that each set from I is contained in a Borel set from I. We say that I fails to fulfil the $\mathbf{\Sigma}^1_1$ countable chain condition if there is a $\mathbf{\Sigma}^1_1$ equivalence relation with uncountably many equivalence classes none of which is in I. Assuming definable determinacy, we show that if the family of Borel sets from I is definable in the codes of Borel sets, then each $\mathbf{\Sigma}^1_1$ set is equal to a Borel set modulo a set from I iff I fulfils the $\mathbf{\Sigma}^1_1$ countable chain condition. Further we characterize the σ-ideals I generated by closed sets that satisfy the countable chain condition or, equivalently in this case, the approximation property for $\mathbf{\Sigma}^1_1$ sets mentioned above. It turns out that they are exactly of the form $MGR(\mathcal{F}) = \{A : \forall F \in \mathcal{F} A \cap F \text{ is meager in } F\}$ for a countable family $\mathcal{F}$ of closed sets. In particular, we verify partially a conjecture of Kunen by showing that the σ-ideal of meager sets is the unique σ-ideal on $\mathbf{R}$, or any Polish group, generated by closed sets which is invariant under translations and satisfies the countable chain condition.

1. Introduction

The main objects of our study will be σ-ideals of subsets of Polish spaces. By a **σ-ideal** on X we mean a family of subsets of X which is closed under taking subsets and countable unions. All σ-ideals considered in this paper are assumed to be **proper**, i.e., they do not contain X, and **uniform**, i.e., they contain all singletons $\{x\}, x \in X$. Here are some other relevant definitions. A σ-ideal

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I is said to be **Borel supported** (**$\mathbf{\Sigma}_2^0$ supported**, resp.) if for any $A \in I$ there is $B \in \mathbf{\Delta}_1^1 \cap I$ ($B \in \mathbf{\Sigma}_2^0 \cap I$, resp.) with $A \subset B$. Note that a σ-ideal is $\mathbf{\Sigma}_2^0$ supported iff it is generated by a family of closed sets. A σ-ideal I has the **approximation property** if for any $A \in \mathbf{\Sigma}_1^1$ there is $B \in \mathbf{\Delta}_1^1$ such that $A \Delta B = (A \setminus B) \cup (B \setminus A) \in I$. Note that, in case I is Borel supported, this is equivalent to saying that if $A \in \mathbf{\Sigma}_1^1$, then there are $B_1, B_2 \in \mathbf{\Delta}_1^1$ such that $B_1 \subset A \subset B_2$ and $B_2 \setminus B_1 \in I$. We say that a σ-ideal I fulfils the **countable chain condition** (the **c.c.c.**) if any family $\mathcal{A}$ of disjoint Borel sets such that $\mathcal{A} \cap I = \emptyset$ is countable. It is well-known that if a Borel supported σ-ideal fulfils the c.c.c., then it has the approximation property (see e.g. the proof of Lemma 5 below). In particular cases, like, e.g., $I =$ the family of meager sets or the family of measure zero sets for some σ-finite Borel measure, this says that analytic sets have the Baire property and are measurable. It also follows from the above fact that, in case I is Borel supported, the members of $\mathcal{A}$ in the definition of the c.c.c. can be assumed to be merely $\mathbf{\Sigma}_1^1$ without changing the meaning of this definition.

Let $\mathcal{A}$ be a family of disjoint sets. One can naturally associate with such a family the equivalence relation $E_{\mathcal{A}}$:

$$(1) \qquad x E_{\mathcal{A}} y \Leftrightarrow (\forall A \in \mathcal{A}\, x \in A \Leftrightarrow y \in A).$$

Thus a Borel supported σ-ideal I does not fulfil the c.c.c. iff there is an equivalence relation E with $|X/E| > \omega$ whose equivalence classes, except for possibly one, are $\mathbf{\Sigma}_1^1$ and do not belong to I. We propose the following definable version of the c.c.c. We say that a Borel supported σ-ideal I fulfils the **$\mathbf{\Sigma}_1^1$ c.c.c.** if there is no $\mathbf{\Sigma}_1^1$ equivalence relation E with $|x/E| > \omega$ whose all, but possibly countably many, equivalence classes are not in I. (We get an equivalent version of this definition if we assume that none of the equivalence classes of E is in I.) The main result of the first part of the present paper is that the $\mathbf{\Sigma}_1^1$ c.c.c. is equivalent with the approximation property (assuming some determinacy and definability of the σ-ideal). This gives an answer to a question of Mauldin [M1]. We also define the pseudo-Borel c.c.c. and prove a version of the above result (the pseudo-Borel c.c.c. replacing the $\mathbf{\Sigma}_1^1$ c.c.c.) without assuming any determinacy hypotheses. As a lemma we prove (see Lemma 4) the following result which seems interesting in its own right: Assume $\mathbf{\Delta}_2^1$-determinacy. If E is a $\mathbf{\Sigma}_1^1$ equivalence relation, then E has countably many equivalence classes iff every E-invariant $\mathbf{\Sigma}_1^1$ set is Borel. (After this paper was written, G. Hjorth showed that $\mathbf{\Delta}_2^1$-determinacy can be replaced in the above statement by the assumption that $x^{\#}$ exists for all $x \in \omega^\omega$, which is equivalent, by results of Harrington and Martin, to $\mathbf{\Sigma}^1_1$-determinacy.)

In the second part we examine which $\mathbf{\Sigma}^0_2$ supported σ-ideals fulfil the $\mathbf{\Sigma}^1_1$ c.c.c. It turns out that the $\mathbf{\Sigma}^1_1$ c.c.c. is equivalent in this case with the c.c.c. Actually we show that $\mathbf{\Sigma}^0_2$ supported σ-ideals fulfilling the c.c.c. are of the form $I = \{A : \forall F \in \mathcal{F}\ A \cap F \text{ is meager in } F\}$ for some countable well-ordered by reverse inclusion family $\mathcal{F}$ of closed sets. On the other hand, if the c.c.c. is violated by a $\mathbf{\Sigma}^0_2$ supported σ-ideal I, then there exists a homeomorphic embedding ϕ: $2^\omega \times \omega^\omega \to X$ such that $\phi[\{\alpha\} \times \omega^\omega] \notin I$ for any $\alpha \in 2^\omega$. This sharpens and generalizes some earlier results of Mauldin [M] and Balcerzak, Baumgartner and Hejduk [BBH]. We use this fact to show that if I is a $\mathbf{\Sigma}^0_2$ supported σ-ideal of subsets of a Polish group which is translation invariant and fulfils the c.c.c., then it is the σ-ideal of meager sets. This gives a partial answer to a question of Kunen [KU].

2. Approximating $\mathbf{\Sigma}^1_1$ sets and the $\mathbf{\Sigma}^1_1$ c.c.c.

It is a well-known fact that if a Borel supported σ-ideal fulfills the c.c.c., then it has the approximation property (see Lemma 5 below). That the reverse implication also holds in certain particular cases was proved in [KLW]. A combination of Theorem 7(ii), Proposition 6(ii) of Section 3 in [KLW] yields the following result: Let I be a Borel supported σ-ideal such that $I \cap \mathbf{\Delta}^1_1$ is $\mathbf{\Pi}^1_1$ in the codes of Borel sets and such that for any $A \in \mathbf{\Delta}^1_1 \setminus I$ there exists a closed set $C \notin I$ with $C \subset A$. Then I has the approximation property iff I fulfills the c.c.c. Also Mauldin [M1] proved, using results from [M], that the σ-ideal of subsets of $[0,1]$ which can be covered by a $\mathbf{\Sigma}^0_2$ set of Lebesgue measure zero (the σ-ideal very strongly violates the c.c.c. as was shown in [M]) does not have the approximation property. Here, using quite different methods and assuming an appropriate amount of determinacy, we are able to prove that the approximation property is actually equivalent to the $\mathbf{\Sigma}^1_1$ c.c.c., for all reasonably definable Borel supported σ-ideals regardless of their other structural properties. This gives an answer to a question of Mauldin [M1], who asked what properties of a σ-ideal are responsible for it having the approximation property.

If E is an equivalence relation on X and $A \subset X$ is E-invariant, we write $|A/E|$ for the cardinality of the family of equivalence classes included in A. If $B \subset X$, then $[B]_E$ denotes the saturation of B with respect to E, i.e., $[B]_E =$

$\{x \in X : \exists y \in B\, xEy\}$. We write $[x]_E$ for $[\{x\}]_E$. If there is no possibility of confusion we will drop the subscript E. If σ and τ are two sequences of elements of a set Y then $\sigma * \tau$ denotes their concatenation. If $y \in Y$, then $\sigma * y = \sigma * (0, y)$. For a definition of $\mathbf{\Pi}^1_1$-rank see [K1, 34B]. Now we define the set $WO \subset 2^\omega$. Let $\langle\ ,\ \rangle : \omega^2 \to \omega$ be a bijection. Put $\alpha \in WO$ iff the relation $\{(n,m) \in \omega^2 : \alpha(\langle n,m\rangle) = 1\}$ well orders ω. WO is $\mathbf{\Pi}^1_1$. Define $|\alpha|$ = the order type of $\{(n,m) \in \omega^2 : \alpha(\langle n,m\rangle) = 1\}$ for $\alpha \in WO$. Then $\alpha \to |\alpha|$ is a $\mathbf{\Pi}^1_1$-rank on WO. For a pointclass Γ, $\mathrm{Det}(\Gamma)$ means that all games in Γ are determined. By $\sigma(\mathbf{\Pi}^1_2)$ we denote the σ-algebra generated by the family of all $\mathbf{\Pi}^1_2$ sets.

THEOREM 1: *Assume* $\mathrm{Det}(\mathbf{\Delta}^1_2)$. *Let I be a Borel supported σ-ideal such that the family $I \cap \mathbf{\Delta}^1_1$ is $\sigma(\mathbf{\Pi}^1_2)$ in the codes of Borel sets. Then I has the approximation property iff I fulfils the $\mathbf{\Sigma}^1_1$ c.c.c.*

The proof of the theorem is split up into several lemmas. The implication $\Rightarrow$ follows from Lemmas 3 and 4 and the implication $\Leftarrow$ follows from Lemmas 5 and 6. Note that the assumption that $I \cap \mathbf{\Delta}^1_1$ is $\sigma(\mathbf{\Pi}^1_2)$ in the codes is used only in the proof of $\Leftarrow$.

The following consequence of Theorem 4 from [KW] will be useful.

LEMMA 1: (Kechris–Woodin) $\mathrm{Det}(\mathbf{\Delta}^1_2)$ *implies* $\mathrm{Det}(\sigma(\mathbf{\Pi}^1_2))$.

We will be also using the following particular case of a theorem due to Solovay. For a proof see [K, Theorem 7.1].

LEMMA 2: (Solovay) *Assume* $\mathrm{Det}(\mathbf{\Delta}^1_2)$. *Let A be a $\mathbf{\Pi}^1_1$ set and ρ a $\mathbf{\Pi}^1_1$-rank on A. Let $B \subset A$ be $\sigma(\mathbf{\Pi}^1_2)$ and such that if $\rho(x) = \rho(y)$ and $x \in B, y \in A$ then $y \in B$. Then $B \in \mathbf{\Pi}^1_1$.*

LEMMA 3: *Let E be a $\mathbf{\Sigma}^1_1$ equivalence relation whose all but countably many classes are not in I. Let A be an E-invariant set. If $A \notin \mathbf{\Delta}^1_1$, then there is no $B \in \mathbf{\Delta}^1_1$ such that $A \Delta B \in I$.*

Proof: Assume otherwise. Since I is Borel supported, we can suppose that there are Borel sets C and D such that $C \cap A = \emptyset, D \subset A$ and $X \setminus (C \cup D) \in I$. Now, $[C]$ and $[D]$ are $\mathbf{\Sigma}^1_1$ and also $[C] \cap A = \emptyset$ and $[D] \subset A$, as A is E-invariant. Let $\{O_n\colon n \in \omega\}$ be the family of all equivalence classes of E which are in I. Each O_n is $\mathbf{\Sigma}^1_1$. If $[C] \cup [D] \cup \bigcup_{n\in\omega} O_n = X$, then, since A is E-invariant, $A = [D] \cup \bigcup_{O_n \subset A} O_n$ and $X \setminus A = [C] \cup \bigcup_{O_n \cap A = \emptyset} O_n$. Now, the Suslin theorem implies that A is Borel which contradicts the assumptions. Thus there exists

$x \in X \setminus ([C] \cup [D] \cup \bigcup_{n\in\omega} O_n)$. Then $[x] \notin I$ and $[x] \subset X \setminus (C \cup D) \in I$, a contradiction. ∎

LEMMA 4: *Assume* $\mathrm{Det}(\mathbf{\Delta}^1_2)$. *Let E be a $\mathbf{\Sigma}^1_1$ equivalence relation. If E has uncountably many equivalence classes, then there exists an E-invariant set $A \in \mathbf{\Sigma}^1_1 \setminus \mathbf{\Delta}^1_1$. (Thus E has countably many equivalence classes iff every E-invariant $\mathbf{\Sigma}^1_1$ set is Borel.)*

Proof: Assume that such an A does not exist. Then $[A] \in \mathbf{\Delta}^1_1$ for any $A \in \mathbf{\Sigma}^1_1$. We claim that either there exists a Borel uncountable set $C \subset X$ such that xEy iff $x = y$ for $x, y \in C$, or there exists an E-invariant set $B \in \mathbf{\Delta}^1_1$ such that $|B/E| > \omega$ and if $B' \subset B$ is $\mathbf{\Delta}^1_1$ and E-invariant then $|B'/E| \le \omega$ or $|(B \setminus B')/E| \le \omega$. (The proof below is related to arguments of Becker [B], Sami and Stern on minimal counterexamples to the Vaught conjecture.) To prove this assume that for any E-invariant $B \in \mathbf{\Delta}^1_1$ there exist E-invariant $\mathbf{\Delta}^1_1$ sets $B_1, B_2 \subset B$ such that $B_1 \cap B_2 = \emptyset$ and $|B_1/E| > \omega$, $|B_2/E| > \omega$. We construct a countable Boolean algebra $\mathcal{A}$ of Borel sets such that:

(i) $\mathcal{A}$ contains a countable topological basis of X;

(ii) if $B \in \mathcal{A}$ and $|[B]/E| > \omega$ then there exist $B_1, B_2 \in \mathcal{A}$ such that $B_1, B_2 \subset B$, $[B_1] \cap [B_2] = \emptyset$, and $|[B_1]/E| > \omega$, $|[B_2]/E| > \omega$;

(iii) the topology generated by $\mathcal{A}$ is Polish.

$\mathcal{A}$ is built recursively starting from a countable topological basis of X. We easily take care of (ii) using the assumption on E. To get (iii), we apply two well-known facts: a topology on a standard Borel space can be extended by Borel sets to obtain a Polish topology (see [K1, Theorem 13.1]), and an increasing union of Polish topologies is Polish (see [K1, Lemma 13.3]).

Now we fix a complete metric d on X which is compatible with the topology generated by $\mathcal{A}$, and do a Cantor-type construction producing open (in this topology) sets $Q_\sigma, \sigma \in 2^{<\omega}$, so that:

(a) $Q_\emptyset = X$;

(b) d-diam$(Q_\sigma) \le 1/(lh\sigma + 1)$;

(c) $|[Q_\sigma]/E| > \omega$;

(d) d-closure$(Q_{\sigma * i}) \subset Q_\sigma$ for $i \in 2$ and $\sigma \in 2^{<\omega}$;

(e) if $\sigma, \tau \in 2^{<\omega}$ are incompatible, then $[Q_\sigma] \cap [Q_\tau] = \emptyset$.

When Q_σ, for some $\sigma \in 2^{<\omega}$, has been constructed, we find by (ii) open (in the topology generated by $\mathcal{A}$) sets $U_0, U_1 \subset Q_\sigma$ such that $|[U_i]/E| > \omega, i = 1, 2$, and

$[U_0] \cap [U_1] = \emptyset$. Now for $i = 1, 2$ find $V_n^i, n \in \omega$, such that V_n^i is open in the topology generated by $\mathcal{A}$, d-closure$(V_n^i) \subset Q_\sigma$, d-diam$(V_n^i) < 1/(lh\sigma + 2)$ and $\bigcup_{n\in\omega} V_n^i = U_i$. Then $|[V_{n_i}^i]/E| > \omega$ for some $n_i \in \omega$. Put $Q_{\sigma * i} = V_{n_i}^i$ for $i = 1, 2$.

Now $C = \bigcap_{n\in\omega} \bigcup_{lh\sigma=n} Q_\sigma$ is an uncountable Borel (in the original topology) set whose distinct elements lie in distinct equivalence classes of E.

If there exists an uncountable Borel set C as above, we can find a $\mathbf{\Sigma}_1^1$ non-Borel set $A \subset C$. Then $[A] \cap C = A$, whence $[A] \notin \mathbf{\Delta}_1^1$, a contradiction.

Thus we can assume, by passing to a Borel invariant subset of X, that $|X/E| > \omega$ and for each $\mathbf{\Sigma}_1^1$ set $A \subset X, |[A]/E| \leq \omega$ or $|(X \setminus [A])/E| \leq \omega$. Using Det$(\mathbf{\Pi}_1^1)$, by Burgess' theorem [Bu], there exists a $\mathbf{\Delta}_2^1$ function $f : X \to WO$ such that $xEy \Leftrightarrow |f(x)| = |f(y)|$. Put $B = \{x \in WO : \exists y \in X\, |f(y)| = |x|\}$. Then $B \in \mathbf{\Sigma}_2^1$ and fulfils the assumptions of Lemma 2 (with $A = WO$ and $\rho(x) = |x|$). Thus $B \in \mathbf{\Pi}_1^1$. Now define

$$B' = \{x \in B : \exists z \in B\,(|z| < |x| \wedge \forall y\,(y \in B \wedge |y| < |x| \Rightarrow |y| \leq |z|))\}.$$

It follows that $B' \in \mathbf{\Sigma}_2^1$. Put $A = f^{-1}(B')$. Then $A \in \mathbf{\Sigma}_2^1$ and is E-invariant. Also A as well as its complement contain uncountably many equivalence classes of E. Thus $A \in \mathbf{\Sigma}_2^1 \setminus \mathbf{\Pi}_1^1$. By Det$(\mathbf{\Delta}_2^1)$ and Lemma 1, each $\mathbf{\Sigma}_1^1$ set is Borel reducible to A. Pick $D \subset 2^\omega$ with $D \in \mathbf{\Sigma}_1^1 \setminus \mathbf{\Delta}_1^1$. Let $\phi : 2^\omega \to X$ be Borel and such that $x \in D \Leftrightarrow \phi(x) \in A$. Since A is E-invariant, $x \in D \Leftrightarrow \phi(x) \in [\phi[D]] \in \mathbf{\Delta}_1^1$. Thus D is $\mathbf{\Delta}_1^1$, a contradiction. ∎

LEMMA 5: *If I does not have the approximation property, then there exists a $\mathbf{\Pi}_1^1$ set A with a $\mathbf{\Pi}_1^1$-rank ρ such that the set $T \subset \omega_1$ defined by $\alpha \in T$ iff $\{x : \rho(x) = \alpha\} \notin I$ is uncountable.*

Proof: Let P be a $\mathbf{\Sigma}_1^1$ set such that there is no $B \in \mathbf{\Delta}_1^1$ with $P \Delta B \in I$. Then the same is true about the $\mathbf{\Pi}_1^1$ set $Q = X \setminus P$. Let ϕ be a Borel mapping from X to the space of all trees on ω such that $\phi(x)$ is well founded iff $x \in Q$. For a tree T on ω and $u \in \omega^{<\omega}$, put $T_u = \{v \in \omega^{<\omega} : u * v \in T\}$. If T is well founded, let $|T|$ denote the rank of T. Suppose $\forall u \in \omega^{<\omega} \exists \xi < \omega_1 \forall \zeta > \xi \{x : \phi(x)_u$ is well founded and $|\phi(x)_u| = \zeta\} \in I$. Then for each $u \in \omega^{<\omega}$ there exists a smallest $\xi = \xi_u < \omega_1$ as above. Put $\bar{\xi} = \sup\{\xi_u : u \in \omega^{<\omega}\} + 1$. Now define $B = \{x \in X : \phi(x)$ is well founded and $|\phi(x)| \leq \bar{\xi}\}$ and $B' = \{x \in X : \exists u \in \omega^{<\omega} \phi(x)_u$ is well founded and $|\phi(x)_u| = \bar{\xi}\}$. Then it is easy to check that $B \subset Q \subset B \cup B'$, $B, B' \in \mathbf{\Delta}_1^1$ and $B' \in I$ which contradicts our assumption on Q. Thus there exists $\bar{u} \in \omega^{<\omega}$ such that $\forall\xi < \omega_1 \exists\zeta > \xi\{x \in X : \phi(x)_{\bar{u}}$ is well founded and $|\phi(x)_{\bar{u}}| = \zeta\} \notin I$. Put $A = \{x \in X : \phi(x)_{\bar{u}}$ is well founded$\}$ and $\rho(x) = |\phi(x)_{\bar{u}}|$. It is easy to verify that these A and ρ work. ∎

Lemma 6: *Assume* $\mathrm{Det}(\mathbf{\Delta}^1_2)$. *Let* $I \cap \mathbf{\Delta}^1_1$ *be* $\sigma(\mathbf{\Pi}^1_2)$ *in the codes. If I does not have the approximation property, then there is a* $\mathbf{\Sigma}^1_1$ *equivalence relation E such that* $|X/E| = \omega_1$ *and all equivalence classes of E, except for perhaps one, are not in I.*

Proof: Take A and ρ as in Lemma 5. Define $A' = \{x \in A : \{y \in A : \rho(y) = \rho(x)\} \notin I\}$. Since $I \cap \mathbf{\Delta}^1_1$ is $\sigma(\mathbf{\Pi}^1_2)$ in the codes, A' is $\sigma(\mathbf{\Pi}^1_2)$. Clearly A' fulfils the assumption of Lemma 2 whence $A' \in \mathbf{\Pi}^1_1$. Then the following equivalence relation is $\mathbf{\Sigma}^1_1$:

$$xEy \Leftrightarrow ((x \in A' \vee y \in A') \Rightarrow (x \in A' \wedge y \in A' \wedge \rho(x) = \rho(y)))\,.$$

Also E has ω_1 equivalence classes and all of them except for perhaps $X \setminus A'$ are not in I. ∎

Assuming more determinacy and using the full strength of Solovay's lemma (see [K, Theorem 7.1]) we obtain the same conclusion (with the same proof) as in Theorem 1 for wider classes of Borel supported σ-ideals or even for all of them if we assume AD. (Note however that, as follows from Lemmas 2 and 3, it is enough to have only $\mathrm{Det}(\mathbf{\Delta}^1_2)$ to prove that the approximation property implies the $\mathbf{\Sigma}^1_1$ c.c.c. for all Borel supported σ-ideals.) For example we have the following result.

Theorem 1': *Assume PD (AD, resp.). Let I be a Borel supported σ-ideal such that* $I \cap \mathbf{\Delta}^1_1$ *is projective in the codes* ($I \cap \mathbf{\Delta}^1_1$ *is arbitrary, resp.). Then the* $\mathbf{\Sigma}^1_1$ *c.c.c. and the approximation property are equivalent.*

We want to make here a few comments on what can be proved without any determinacy hypotheses. We will summarize them in Theroem 1". A family $\mathcal{A}$ of disjoint sets is called **pseudo-Borel** if the relation $E_{\mathcal{A}}$ associated with $\mathcal{A}$ as in (1) in the Introduction is $\mathbf{\Sigma}^1_1$ and there is a $\mathbf{\Pi}^1_1$ equivalence relation F such that

$$x \in \bigcup \mathcal{A} \Rightarrow (\forall y\, xFy \Leftrightarrow xE_{\mathcal{A}}y). \tag{2}$$

Note that if $E_{\mathcal{A}}$ is Borel we can take $F = E_{\mathcal{A}}$. A Borel supported σ-ideal I fulfils the **pseudo-Borel c.c.c.** if every pseudo-Borel family $\mathcal{A}$ of disjoint sets such that $\mathcal{A} \cap I = \emptyset$ is countable. Clearly the c.c.c. implies the $\mathbf{\Sigma}^1_1$ c.c.c., which in turn implies the pseudo-Borel c.c.c.

LEMMA 7: *Assume a Borel supported σ-ideal has the approximation property. Then I fulfils the pseudo-Borel c.c.c.*

Proof: Suppose I does not fulfil the pseudo-Borel c.c.c. Let $\mathcal{A}$ be a pseudo-Borel family of sets witnessing it and let F be a $\mathbf{\Pi}^1_1$ equivalence relation from the definition of pseudo-Borelness. By Lemma 3 applied to $E_{\mathcal{A}}$ it is enough to find an $E_{\mathcal{A}}$-invariant set A such that $A \in \mathbf{\Sigma}^1_1 \setminus \mathbf{\Delta}^1_1$. Since $E_{\mathcal{A}} \in \mathbf{\Sigma}^1_1, X \setminus \bigcup \mathcal{A} \in \mathbf{\Sigma}^1_1$. If $X \setminus \bigcup \mathcal{A} \notin \mathbf{\Delta}^1_1$ we are done. Thus we can assume that $\bigcup \mathcal{A} \in \mathbf{\Delta}^1_1$. But by (2) $\bigcup \mathcal{A}$ is F-invariant and $F | \bigcup \mathcal{A} = E_{\mathcal{A}} | \bigcup \mathcal{A}$. Thus since $F \in \mathbf{\Pi}^1_1$ and $|\bigcup \mathcal{A}/F| = |\mathcal{A}| > \omega$, by Silver's theorem [S], there is a perfect compact set $C \subset \bigcup \mathcal{A}$ such that different elements of C belong to different equivalence classes of $E_{\mathcal{A}}$. Pick $A \subset C$ in $\mathbf{\Sigma}^1_1 \setminus \mathbf{\Delta}^1_1$. Then $[A]_{E_{\mathcal{A}}}$ is $E_{\mathcal{A}}$-invariant and $\mathbf{\Sigma}^1_1$ and, as $[A]_{E_{\mathcal{A}}} \cap C = A, [A]_{E_{\mathcal{A}}} \notin \mathbf{\Delta}^1_1$. ∎

LEMMA 8: *Assume I is a Borel supported σ-ideal such that $I \cap \mathbf{\Delta}^1_1$ is $\mathbf{\Sigma}^1_1$ in the codes of Borel sets. If I fulfils the pseudo-Borel c.c.c., then I has the approximation property.*

Proof: It is enough to prove an analogue of Lemma 6 without the determinacy hypothesis. But since we assume that $I \cap \mathbf{\Delta}^1_1$ is $\mathbf{\Sigma}^1_1$ in the codes, the set A' defined in the proof of Lemma 6 is $\mathbf{\Pi}^1_1$. Put $\mathcal{A} = \{\{x \in A' : \rho(x) = \alpha\} : \alpha < \omega_1\}$. Then $E_{\mathcal{A}}$ is equal to the relation E defined in the proof of Lemma 6 and thus $|X/E_{\mathcal{A}}| > \omega$ and $E_{\mathcal{A}} \in \mathbf{\Sigma}^1_1$. For the $\mathbf{\Pi}^1_1$ equivalence relation F we take

$$xFy \Leftrightarrow (x = y \vee (x \in A' \wedge y \in A' \wedge \rho(x) = \rho(y))).$$

∎

Combining Lemmas 7 and 8 we obtain the following theorem.

THEOREM1": *Let I be a Borel supported σ-ideal such that $I \cap \mathbf{\Delta}^1_1$ is $\mathbf{\Sigma}^1_1$ in the codes of Borel sets. Then I has the approximation property iff I fulfils the pseudo-Borel c.c.c.*

3. $\mathbf{\Sigma}_2^0$ supported σ-ideals

The $\mathbf{\Sigma}_2^0$ supported σ-ideals occur frequently in harmonic analysis and descriptive set theory as σ-ideals generated by families of closed sets. In this section we characterize those $\mathbf{\Sigma}_2^0$ supported σ-ideals which have the approximation property and also give an abstract characterization of the σ-ideal of meager sets. No determinacy assumptions will be used in the sequel.

Let $\mathcal{F}$ be a family of subsets of a Polish space X. Put

$$MGR(\mathcal{F}) = \{B \subset X : \forall A \in \mathcal{F}\, B \cap A \text{ is meager in } A\}.$$

If $A \subset X$, we will write $MGR(A)$ for $MGR(\{A\})$. If I is a σ-ideal and $A \subset X$, we write $I|A = \{B \subset A : B \in I\}$. A family $\mathcal{F}$ of subsets of X is said to be **well-ordered by reverse inclusion** if there is an ordinal α such that $\mathcal{F} = \{A_\xi : \xi < \alpha\}$ and $\xi \leq \zeta < \alpha \Leftrightarrow A_\xi \supset A_\zeta$. By π_X and π_Y we denote the projections from $X \times Y$ onto X and Y, respectively. Also for $A \subset X \times Y$ we write $A_x = \{y \in Y : (x, y) \in A\}$.

LEMMA 9: *Let Y be Polish and let J be a $\mathbf{\Sigma}_2^0$ supported σ-ideal. Assume that for any open set $U \neq \emptyset$ there exists a nowhere dense set $F \subset U$ such that $F \notin J$. Then there is a homeomorphic embedding $\phi : 2^\omega \times \omega^\omega \to Y$ such that $\phi[\{\alpha\} \times \omega^\omega] \notin J$ for any $\alpha \in 2^\omega$.*

Proof: For any family $\mathcal{A}$ of subsets of Y define $\mathcal{A}^d$ to be the set of all points $x \in Y$ such that for any open U with $x \in U$ the set $\{A \in \mathcal{A} : A \cap U \neq \emptyset\}$ is infinite. In the natural way we identify a sequence $\sigma \in (2 \times \omega)^n$ with the sequence $((\sigma)_0, (\sigma)_1) \in 2^n \times \omega^n$. For $\alpha \in \omega^\omega$ by $\alpha|n$ we denote the restriction of α to $n = \{0, \ldots, n-1\}$. We also write $N_\sigma = \{\gamma \in 2^\omega \times \omega^\omega : \pi_{2^\omega}(\gamma)|n = (\sigma)_0, \pi_{\omega^\omega}(\gamma)|n = (\sigma)_1\}$ for $\sigma \in (2 \times \omega)^n, n \in \omega$.

Now we construct recursively open sets $U_\sigma, \sigma \in (2 \times \omega)^{<\omega}$, so that:

(i) $\sigma \subset \tau, \sigma \neq \tau$ implies closure$(U_\tau) \subset U_\sigma$;
(ii) if neither $\sigma \subset \tau$ nor $\tau \subset \sigma$ then $U_\sigma \cap U_\tau = \emptyset$;
(iii) $\operatorname{diam}(U_\sigma) \leq 1/2^{n+(\sigma)_1(n-1)}$, where $n = lh\sigma$;
(iv) $\{U_{\sigma * (i,n)} : n \in \omega\}^d \notin J$ for $i \in 2$;
(v) $U_\sigma \neq \emptyset$.

If U_σ has been defined, find a nowhere dense closed set $F \subset U_\sigma$ with $F \notin J$. Then find two closed sets $F_0, F_1 \subset F$, $F_0, F_1 \notin J$ such that there exist two open sets $V_0, V_1 \subset U_\sigma$ containing F_0 and F_1, respectively, and having disjoint closures.

Since F_i is nowhere dense in $V_i, i = 0, 1$, we can find nonempty pairwise disjoint open sets $W_n^i, n \in \omega$, so that $F_i = \{W_n^i : n \in \omega\}^d, W_n^i \subset V_i$ and $\text{diam}(W_n^i) \leq 1/2^{k+1+n}$, where $k = lh\sigma$. To define W_n^i, first choose $D^i = \{d_n^i : n \in \omega\}$ to be discrete subsets of V_i such that $F_i = \text{closure}(D^i) \setminus D^i$. Then let W_n^i be an appropriately small ball around d_n^i. Put $U_{\sigma*(i,n)} = W_n^i$.

Now define $\phi : 2^\omega \times \omega^\omega \to Y$ by $\phi(\alpha, \beta) =$ the only point in $\bigcap_{n\in\omega} U_{(\alpha|n,\beta|n)}$. It is clear from (i)–(iii) and (v) that ϕ is a homeomorphic embedding. Note also that, by (iii) and (iv), $\{\phi[N_{\sigma*(i,n)}] : n \in \omega\}^d = \{U_{\sigma*(i,n)} : n \in \omega\}^d \notin J$ for any $\sigma \in (2 \times \omega)^{<\omega}$ and $i \in 2$.

Suppose that there is $\alpha \in 2^\omega$ such that $\phi[\{\alpha\} \times \omega^\omega] \in J$. Then there exist $F_n \in J \cap \mathbf{\Pi}_1^0, n \in \omega$, such that $\phi[\{\alpha\} \times \omega^\omega] \subset \bigcup_{n\in\omega} F_n$. By the Baire Category Theorem there is $\tau \in \omega^k$, for some $k \in \omega$, and $n_0 \in \omega$ such that $\phi[N_{(\alpha|k,\tau)}] \subset F_{n_0}$. But then $\{\phi[N_{\alpha|(k+1),\tau*(n)}] : n \in \omega\}^d \subset F_{n_0} \in J$, a contradiction. ∎

The following theorem generalizes and strengthens some results proved in [M] and [BBH]. It was shown in [BBH, Theorem 2.3] that (ii) holds for the σ-ideal of all subsets of 2^ω which can be covered by $\mathbf{\Sigma}_2^0$ sets of Lebesgue measure zero. A bit weaker result for the same σ-ideal was proved earlier in [M, Theorem 1] and this weaker result was generalized in [BBH, Theorem 1.5] to a slightly wider class of $\mathbf{\Sigma}_2^0$ supported σ-ideals.

THEOREM 2: *Let I be a $\mathbf{\Sigma}_2^0$ supported σ-ideal. Then precisely one of the following possibilities holds:*
(i) *$I = MGR(\mathcal{F})$ for a countable family $\mathcal{F}$ of closed subsets of X, which can be assumed to be well-ordered by reverse inclusion;*
(ii) *there is a homeomorphic embedding $\phi : 2^\omega \times \omega^\omega \to X$ such that $\phi[\{\alpha\} \times \omega^\omega] \notin I$ for any $\alpha \in 2^\omega$.*

Proof: For $F \subset X$ closed put $F' = F \setminus \bigcup\{U : U$ is open, $U \cap F \neq \emptyset$ and $I|(U \cap F) = MGR(U \cap F)\}$ and $F^* = F \setminus \bigcup\{U : U$ is open and $U \cap F \in I\}$. Now define by transfinite recursion:
$F_0 = X^*$;
$F_\lambda = (\bigcap_{\gamma<\lambda} F_\gamma)^*$ if λ is limit;
$F_{\gamma+1} = F'_\gamma$.

Claim: Let $U \subset X$ be open. Assume $F_{\gamma+1} \cap U = F_\gamma \cap U$. Then $F_\xi \cap U = F_\gamma \cap U$ for any $\xi > \gamma$.

Proof of the Claim: First we prove that if $W \cap F_\gamma \in I$ for an open set W, then $W \cap F_\gamma = \emptyset$. This is clear if γ is limit or 0. Assume γ is a successor. Let λ be the biggest limit ordinal $\leq \gamma$ or $\lambda = 0$. Then $W \cap F_\gamma$ must be meager in F_λ. So there exists a biggest $\theta < \gamma$ with $W \cap F_\gamma$ meager in F_θ. It follows that there exists an open set V such that $\emptyset \neq V \cap F_{\theta+1} \subset W \cap F_\gamma$. We thus have $V \cap F_{\theta+1} \in MGR(F_\theta)$ and $V \cap F_{\theta+1} \in I$, whence $V \cap F_{\theta+1} = \emptyset$, a contradiction.

Now, if $U \cap F_\gamma \subset F_{\gamma+1}$, we show by induction on $\xi > \gamma$ that $U \cap F_\gamma \subset F_\xi$. For ξ limit it is a consequence of the observation from the previous paragraph. For successors it follows directly from the inductive hypothesis and the inclusion $U \cap F_\gamma \subset F_{\gamma+1}$. This finishes the proof of the Claim.

There exists a smallest $\alpha < \omega_1$ such that $F_\alpha = F_{\alpha+1}$.

CASE 1: $F_\alpha = \emptyset$.

Put $\mathcal{F} = \{F_\gamma : \gamma < \alpha\}$. First notice that $F_{\gamma+1}$ is nowhere dense in F_γ for $\gamma < \alpha$. Otherwise there is an open set U such that $F_{\gamma+1} \supset F_\gamma \cap U \neq \emptyset$. Then by the Claim $F_\xi \supset F_\gamma \cap U$ for all $\xi > \gamma$. In particular, $F_\alpha \supset F_\gamma \cap U \neq \emptyset$ which contradicts our assumption on F_α.

Now we show that $I = MGR(\mathcal{F})$. Let $A \in I$. Then $A \cap (F_\gamma \setminus F_{\gamma+1}) \in MGR(F_\gamma \setminus F_{\gamma+1})$ for $\gamma < \alpha$. But since $F_{\gamma+1} \in MGR(F_\gamma)$, we have $A \in MGR(F_\gamma)$. For the opposite direction assume that $A \cap F_\gamma \in MGR(F_\gamma)$. Since $F_{\gamma+1}$ is closed, $A \cap (F_\gamma \setminus F_{\gamma+1}) \in MGR(F_\gamma \setminus F_{\gamma+1})$. Thus $A \cap (F_\gamma \setminus F_{\gamma+1}) \in I$ for $\gamma < \alpha$. Also clearly $X \setminus F_0 \in I$ and $\bigcap_{\gamma<\lambda} F_\gamma \setminus F_\lambda \in I$ for λ limit. Since I is a σ-ideal,

$$A = A \cap (X \setminus F_0) \cup \bigcup_{\lambda<\alpha,\lambda \ limit} A \cap (\bigcap_{\gamma<\lambda} F_\gamma \setminus F_\lambda) \cup \bigcup_{\gamma<\alpha} A \cap (F_\gamma \setminus F_{\gamma+1}) \in I.$$

CASE 2: $F_\alpha \neq \emptyset$.

By the Claim $F_\alpha = F_\xi$ for all $\xi > \alpha$. Thus $F'_\alpha = F_\alpha$ and $F^*_\alpha = F_\alpha$. This easily implies that the assumptions of Lemma 9 are fulfilled for $Y = F_\alpha$ and $J = I|F_\alpha$. Thus we obtain (ii). ∎

Note that (i) implies that I fulfils the c.c.c. Thus it follows from Theorem 2 that if a $\mathbf{\Sigma}^0_2$ supported σ-ideal does not fulfil the c.c.c., then there exists a "perfect" family of G_δ's outside of I, i.e., (ii) holds. A similar fact was proved for a different class of σ-ideals in [KLW]. Namely by Theorem 2 of Section 3 in

[KLW], if I is a Borel supported σ-ideal such that $I \cap \mathbf{\Delta}^1_1$ is $\mathbf{\Pi}^1_1$ in the codes and for any $A \in \mathbf{\Delta}^1_1 \setminus I$ there is a closed set $C \notin I$ with $C \subset A$, then if I does not fulfil the c.c.c., then there is a "perfect" family of closed sets not in I. In particular, in this case, as well as in the case of $\mathbf{\Sigma}^0_2$ supported σ-ideals, the c.c.c., the $\mathbf{\Sigma}^1_1$ c.c.c., and the pseudo-Borel c.c.c. are equivalent.

The next theorem lists a few characterizations of the σ-ideals of the form $MGR(\mathcal{F})$ for a countable, well-ordered by reverse inclusion family $\mathcal{F}$ of closed sets.

THEOREM 3: *Let I be a $\mathbf{\Sigma}^0_2$ supported σ-ideal. Then the following are equivalent.*
(i) *I is of the form $MGR(\mathcal{F})$ for a countable family $\mathcal{F}$ of closed subsets of X well-ordered by reverse inclusion;*
(ii) *I fulfils the c.c.c.;*
(iii) *I fulfils the pseudo-Borel c.c.c.;*
(iv) *$I \cap \mathbf{\Delta}^1_1$ is $\mathbf{\Delta}^1_1$ in the codes of Borel sets;*
(v) *$I \cap \mathbf{\Delta}^1_1$ is $\mathbf{\Sigma}^1_1$ in the codes of Borel sets;*
(vi) *I has the approximation property.*

Proof: (i) $\Rightarrow$ (ii). Let $\mathcal{A}$ be an uncountable family of disjoint $\mathbf{\Sigma}^1_1$ sets with $\mathcal{A} \cap I = \emptyset$. Then, since $\mathcal{F}$ is countable, there is $F \in \mathcal{F}$ and an uncountable family $\mathcal{A}' \subset \mathcal{A}$ such that $A \cap F$ is not meager in F for any $A \in \mathcal{A}'$. This yields a contradiction, since $MGR(F)$ fulfils the c.c.c.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i). Suppose (i) does not hold. Let ϕ be as in Theorem 2(ii). Put $\mathcal{A} = \{\phi[\{\alpha\} \times \omega^\omega] : \alpha \in 2^\omega\}$. Then $E_{\mathcal{A}}$ is Borel. Indeed, notice that since ϕ is a homeomorphic embedding $\phi[2^\omega \times \omega^\omega]$ is $\mathbf{\Pi}^0_2$. Put $B = \phi[2^\omega \times \omega^\omega]$. Then

$$\begin{aligned} xE_{\mathcal{A}}y &\Leftrightarrow ((x \notin B \wedge y \notin B) \vee (\exists \alpha \in 2^\omega x, y \in \phi[\{\alpha\} \times \omega^\omega])) \\ &\Leftrightarrow ((x \notin B \wedge y \notin B) \vee (\exists! \alpha \in 2^\omega x, y \in \phi[\{\alpha\} \times \omega^\omega])). \end{aligned}$$

Since $E_{\mathcal{A}}$ is Borel, $\mathcal{A}$ is a pseudo-Borel family.

(i) $\Rightarrow$ (iv). By a standard calculation, see e.g. [K, 16.1].

(iv) $\Rightarrow$ (v) is obvious.

(v) $\Rightarrow$ (i). Suppose that I is not of the required form. Let ϕ be as in Theorem 2(ii). Let $B \subset \omega^\omega \times 2^\omega$ be such that $B \in \mathbf{\Delta}^1_1$ and $\pi_{\omega^\omega}[B] \notin \mathbf{\Pi}^1_1$. Define $B' \subset \omega^\omega \times X$ by $(\alpha, x) \in B' \Leftrightarrow x \in \phi[2^\omega \times \omega^\omega] \wedge (\alpha, \pi_{2^\omega}(\phi^{-1}(x))) \in B$. Clearly $B' \in \mathbf{\Delta}^1_1$. It is easy to check that $B'_\alpha \notin I$ or $B'_\alpha = \emptyset$ for any $\alpha \in \omega^\omega$ and

$\{\alpha \in \omega^\omega : B'_\alpha \notin I\} = \{\alpha \in \omega^\omega : B_\alpha \neq \emptyset\} = \pi_{\omega^\omega}[B] \notin \mathbf{\Pi}^1_1$. Thus $\{\alpha \in 2^\omega : B'_\alpha \in I\} \notin \mathbf{\Sigma}^1_1$ which gives a contradiction since if $I \cap \mathbf{\Delta}^1_1$ is $\mathbf{\Sigma}^1_1$ in the codes, then $\{\alpha \in 2^\omega : A_\alpha \in I\}$ is $\mathbf{\Sigma}^1_1$ for any Borel set $A \subset 2^\omega \times X$.

(vi) $\Rightarrow$ (iii) is simply Lemma 7.

((iii) $\wedge$ (v)) $\Rightarrow$ (vi) is Lemma 8. ∎

Consider now 2^ω as a group with the coordinatewise addition modulo 2. Kunen [Ku, 1.27] asked if all Borel supported σ-ideals on 2^ω which are translation invariant and fulfil the c.c.c. are: the family of meager sets, the family of Lebesgue measure zero sets or the intersection of the two. The following corollary provides a partial answer to this question.

Corollary: *Let X be a Polish space and let H be a group of homeomorphisms of X such that $\bigcup_{h \in H} h[U] = X$ for any open nonempty set $U \subset X$. Let I be a $\mathbf{\Sigma}^0_2$ supported σ-ideal on X. If I fulfils the c.c.c. and is such that $h[A] \in I$ if $A \in I$, then I is the σ-ideal of meager sets. In particular, if G is a Polish group and I is a $\mathbf{\Sigma}^0_2$ supported translation invariant σ-ideal on G which fulfils the c.c.c., then I is the σ-ideal of meager sets.*

Proof: First notice that, by invariance under homeomorphisms from H, I cannot contain a nonempty open set. By Theorem 3 there is a well-ordered by reverse inclusion countable family $\mathcal{F}$ of closed subsets of X such that $I = MGR(\mathcal{F})$. Let $F_0 \in \mathcal{F}$ be such that $F' \subset F_0$ for any $F' \in \mathcal{F}$. Then $X \setminus F_0$ is open and $X \setminus F_0 \in I$. Thus $X \setminus F_0 = \emptyset$, i.e., $F_0 = X$. If $\mathcal{F} \neq \{F_0\}$, let $F_1 \in \mathcal{F}$ be such that $F' \subset F_1$ for any $F' \in \mathcal{F} \setminus \{F_0\}$. If $\mathcal{F} = \{F_0\}$, put $F_1 = \emptyset$. It follows that $MGR(X \setminus F_1) \subset I$. Since $X \setminus F_1$ is nonempty and open, we get $MGR(X) \subset I$ by invariance of $MGR(X)$ and I under homeomorphisms from H. If there is a set $A \in I \setminus MGR(X)$, then, since I is $\mathbf{\Sigma}^0_2$ supported, we can find $A \in \mathbf{\Sigma}^0_2$, $A \in I \setminus MGR(X)$. Now the Baire Category Theorem implies that there is an open set in I which is impossible. ∎

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CHAPTER 3

K_σ EQUIVALENCE RELATIONS AND INDECOMPOSABLE CONTINUA

3.1. K_σ equivalence relations

Let E_0 be the equivalence relation on 2^ω defined by xE_0y iff $\exists N \in \omega \forall n > N\ x(n) = y(n)$, for $x, y \in 2^\omega$. If E is a Borel equivalence relation on a Polish space, we say that E_0 continuously embeds in E, $E_0 \sqsubseteq_c E$, if there is a continuous injection $\phi : 2^\omega \to X$ such that xE_0y iff $\phi(x)E\phi(y)$ for $x, y \in 2^\omega$.

We prove below a theorem which gives a sufficient condition for a K_σ equivalence relation to continuously embed E_0. A corollary of this result solves an old problem in the theory of indecomposable continua concerning the existence of a Borel set having precisely one point in common with each composant. Theorem 3.1 is related to and was inspired by the Glimm-Effros theorem on continuous actions of Polish groups discovered in the study of C^*-algebras [G], [E] and its generalization to actions of arbitrary groups of homeomorphisms due to Becker and Kechris [BK].

Theorem 1. *Let X be Polish, and let F be a K_σ equivalence relation on X. Assume that $\{x \in X : [x]_F$ is not locally closed at $x\}$ is not meager. Then $E_0 \sqsubseteq_c F$.*

Proof. Since F is K_σ and contains the diagonal of $X \times X$, X is K_σ. Hence, there exist open U_n, $n \in \omega$, such that $\bigcup_n U_n$ is dense and for each n, $\overline{U}_n$ is compact. Thus $\{x \in U_{n_0} : [x]_F$ is not locally closed at $x\}$ is not meager for some n_0. So, restricting F to $\overline{U}_{n_0}$, we can assume that X is compact.

Now, we can find $F_k \subset F$, $k \in \omega$, such that F_k is compact, symmetric (i.e., $(x, y) \in F_k$ implies $(y, x) \in F_k$), $\{(x, x) : x \in X\} \subset F_k$, and $F_k^{2k+3} \subset F_{k+1}$. ($F_k^n$ is defined recursively: $F_k^1 = F_k$, and $F_k^{n+1} = \{(x, y) : \exists z\ (x, z) \in F_k^n$ and $(z, y) \in F_k\}$.) We write $A \perp_k B$ for $A, B \subset X$ if $(A \times B) \cap F_k = \emptyset$.

Claim 1. There exists an open nonempty set $U \subset X$ such that given $k \in \omega$ and $\emptyset \neq W \subset U$ open there are nonempty compact $C_0, C_1 \subset W$ and $n \in \omega$ such that

(i) $C_0 \perp_k C_1$;

(ii) $C_0 \subset [C_1]_{F_n}$ and $C_1 \subset [C_0]_{F_n}$;

(iii) $C_0 = \overline{V}$ for some open V.

Proof of Claim 1. Let $\{V_m : m \in \omega\}$ be an open basis for X with each V_m nonempty. Put

$$A_{m,p} = ([V_m]_F \cap V_p) \cup (V_m \cap [V_p]_F).$$

Note that $A_{m,p}$ are F_σ. Put

$$B_k^r = \bigcup\{A_{m,p} : V_m \perp_k V_p \text{ and } V_m, V_p \subset V_r\},\ k, r \in \omega.$$

First, we show that if $x \notin \bigcap_k \bigcap_r (B_k^r \cup (X \setminus V_r))$, then $[x]_F$ is locally closed at x. If x is as above, then $x \in V_r$ and $x \notin B_k^r$ for some $k, r \in \omega$. Let $y \in V_r$. Then $(x, y) \in F$ iff $(x, y) \in F_k$. Since $F_k \subset F$, it is enough to show that $(x, y) \notin F_k$ implies $(x, y) \notin F$. But if $(x, y) \notin F_k$, then there are $V_m, V_p \subset V_r$ such that $x \in V_m$, $y \in V_p$, and $V_m \perp_k V_p$. Since $x \notin B_k^r$ and $x \in V_m$, $x \notin [V_p]_F$ whence $(x, y) \notin F$. It follows that $[x]_F \cap V_r = [x]_{F_k} \cap V_r$ whence $[x]_F$ is locally closed at x.

By assumption, $\bigcap_k \bigcap_r (B_k^r \cup (X \setminus V_r))$ is not meager. Since $B_k^r \cup (X \setminus V_r)$ is F_σ, there exists a nonempty open set U such that for all r, k, $\text{int}(B_k^r \cup (X \setminus V_r))$ is dense in U. Let $\emptyset \neq W \subset U$ be open. If $V_r \subset W$, then for all k, $\text{int}(B_k^r)$ is dense in V_r, whence for any k there are $V_p, V_m \subset V_r$ such that $V_p \perp_k V_m$ and $\text{int}([V_p]_F \cap V_m) \neq \emptyset$. Now, we can find $l, n \in \omega$ such that $\overline{V}_l \subset V_m$ and $\overline{V}_l \subset [V_p]_{F_n}$. Put $C_0 = \overline{V}_l$ and $C_1 = \overline{V}_p \cap [\overline{V}_l]_{F_n}$. Then C_0 and C_1 are as required, which finishes the proof of Claim 1.

We construct recursively nonempty compact sets C_s, $s \in 2^{<\omega}$ (as usual $C_s \subset C_t$ if $s \supset t$, and $\text{diam}(C_s) \leq 1/(lh(s) + 1)$) along with a sequence of natural numbers $n_0 < n_1 < n_2 < \cdots$ so that to some pairs (C_s, C_t), $s, t \in 2^k$, an n_i with $i < k$ will be assigned in which case, we write $C_s \overset{n_i}{\longleftrightarrow} C_t$. The following additional conditions will be fulfilled. (By 0^k we denote the sequence consisting of k 0's.)

(1) $C_{0^k} = \overline{U}_k$ where U_k is open;

(2) if $C_s \overset{n_k}{\longleftrightarrow} C_t$, then $C_t \subset [C_s]_{F_{n_k}}$ and $C_s \subset [C_t]_{F_{n_k}}$;

(3) $C_{0^{k+1}} \overset{n_k}{\longleftrightarrow} C_{0^k * i}$ for $i = 0, 1$;

(4) $C_{0^{k+1}} \perp_{n_{k-1}+2} C_{0^k * 1}$;

(5) if $C_s \overset{n_k}{\longleftrightarrow} C_t$, then $C_{s*i} \overset{n_k}{\longleftrightarrow} C_{t*i}$ for $i = 0, 1$.

Assume the construction has been carried out.

Claim 2. $E_0 \sqsubseteq_c F$.

Proof of Claim 2. Call $s, t \in 2^{<\omega}$ k-close if $lh(s) = lh(t)$, there is $p \leq k$ such that $s|(p+1) = 0^{p+1}$, $t|(p+1) = 0^p * 1$ or vice versa, and for any m with $p+1 \leq m < lh(s)$, $s(m) = t(m)$. Immediately from (3) and (5), we get that if $s, t \in 2^{<\omega}$ are k-close, then $[C_s]_{F_{n_k}} \supset C_t$ and $[C_t]_{F_{n_k}} \supset C_s$. Also, it is clear that if $s, t \in 2^{<\omega}$, $lh(s) = lh(t)$, and $s(i) = t(i)$ for all $i \geq k+1$, then there is a sequence $s_0, s_1, \ldots, s_m$ such that $m \leq 2k$, $s_0 = s$, $s_m = \tau$, and s_i, s_{i+1} are k-close for $i < m$. Thus, if $s, t \in 2^{<\omega}$ are as above, then $[C_s]_{F^{2k}_{n_k}} \supset C_t$ and $[C_t]_{F^{2k}_{n_k}} \supset C_s$. Since $F^{2k}_{n_k} \subset F_{n_k+1}$, we obtain the following conclusion.

(i) Let $s, t \in 2^{<\omega}$, $lh(s) = lh(t)$, and $s(i) = t(i)$ for $i \geq k+1$. Then $[C_s]_{F_{n_k+1}} \supset C_t$ and $[C_t]_{F_{n_k+1}} \supset C_s$.

Also we have the following fact.

(ii) Let $s, t \in 2^{<\omega}$, $lh(s) = lh(t)$. Assume $s(k) \neq t(k)$, $k \geq 1$. Then $C_s \perp_{n_{k-1}} C_t$. If $s(0) \neq t(0)$, then clearly $C_s \perp_0 C_t$.

To see this, assume $s(k) = 0$, $t(k) = 1$, and put $s' = s|(k+1)$, $t' = t|(k+1)$. By (i), $C_{s'} \subset [C_{0^{k+1}}]_{F_{n_{k-1}+1}}$ and $C_{t'} \subset [C_{0^k*1}]_{F_{n_{k-1}+1}}$. Now if $C_s \not\perp_{n_{k-1}} C_t$, then there are $x \in C_s$, $y \in C_t$ with $xF_{n_{k-1}}y$. Since $C_s \subset C_{s'}$ and $C_t \subset C_{t'}$, we get $z_0 \in C_{0^{k+1}}$ and $z_1 \in C_{0^k*1}$ with $(x, z_0) \in F_{n_{k-1}+1}$ and $(yz_1) \in F_{n_{k-1}+1}$. Thus $(z_0, z_1) \in F^3_{n_{k-1}+1} \subset F_{n_{k-1}+2}$ which contradicts (4).

Define $\phi : 2^\omega \to X$ by letting $\phi(\alpha)$ be the unique element in $\bigcap_n C_{\alpha|n}$ for $\alpha \in 2^\omega$. Since $\{(x,x) : x \in X\} \subset F_k$ for all k, from (ii) we get that if $lh(s) = lh(t)$ and $s \neq t$, then $C_s \cap C_t = \emptyset$. Thus, ϕ is 1-to-1 and continuous. If $\alpha, \beta \in 2^\omega$ and $(\alpha, \beta) \notin E_0$, that is, $\alpha(k) \neq \beta(k)$ for infinitely many $k \in \omega$, then by (ii) and the fact that $F_k \subset F_{k+1}$ for all k, we have $(\phi(\alpha), \phi(\beta)) \notin F_k$ for all k whence $(\phi(\alpha), \phi(\beta)) \notin F$. If $\alpha, \beta \in 2^\omega$ and $(\alpha, \beta) \in E_0$, then $\alpha(k) = \beta(k)$ for $k \geq N$ and some $N \in \omega$. By (ii), $[C_{\alpha|m}]_{F_{n_N+1}} \supset C_{\beta|m}$ and $[C_{\beta|m}]_{F_{n_N+1}} \supset C_{\alpha|m}$ for all k. Hence $[C_{\alpha|m}]_{F_{n_N+1}} \ni \phi(\beta)$ and $[C_{\beta|m}]_{F_{n_N+1}} \ni \phi(\alpha)$. This allows us to pick sequences $y_m \to \phi(\alpha)$ and $z_m \to \phi(\beta)$ with $y_m \in [\phi(\beta)]_{F_{n_N}+1}$ and $z_m \in [\phi(\alpha)]_{F_{n_N}+1}$ Since $[\phi(\alpha)]_{F_{n_N}+1}$ and $[\phi(\beta)]_{F_{n_N}+1}$ are closed, $\phi(\alpha) \in [\phi(\beta)]_{F_{n_N}+1}$ and $\phi(\beta) \in [\phi(\alpha)]_{F_{n_N}+1}$, whence $\phi(\alpha)F\phi(\beta)$, and Claim 2 is proved.

Thus, to finish the proof of the theorem, it is enough to construct $\{C_s : s \in 2^{<\omega}\}$. The construction is recursive on the length of $s \in 2^{<\omega}$. To avoid cluttering pages with notation, we will describe only the first three steps of the construction. Let U be as in Claim 1. Put $C_\emptyset = \overline{U}_0$ where U_0 is a nonempty, open set with $\mathrm{diam}(U_0) < 1$ and $\overline{U}_0 \subset U$. Find n_0 and D_0, D_1 as in Claim 1 for $W = U_0$ and $k = 0$. Let V be an open set with $D_0 = \overline{V}$. Let $D^1, \ldots, D^m$ be compact sets with diameter $< 1/2$ and whose union is D_1. Then for some i_0, $[D^{i_0}]_{F_{n_0}} \cap D_0$ has a nonempty interior. Let U_1 be open with diameter $< 1/2$ and such that $\overline{U}_1 \subset \mathrm{int}([D^{i_0}]_{F_{n_0}} \cap D_0)$. Finally, put $C_{\langle 0 \rangle} = \overline{U}_1$ and $C_{\langle 1 \rangle} = D^{i_0} \cap [C_0]_{F_{n_0}}$. Now, we define C_s for s with $lh(s) = 2$. Let n_1, D_{00}, D_{01} be as in Claim 1 for $W = U_1$ and $k = n_0 + 2$. Let V be open with $D_{00} = \overline{V}$. Put $D_{10} = C_1 \cap [D_{00}]_{F_{n_0}}$ and $D_{11} = C_1 \cap [D_{01}]_{F_{n_0}}$. We could define $C_{\langle i,j \rangle}$ to be D_{ij} except that their diameters may be too big, so in the remainder of the proof, we modify them appropriately. First, find $D^1_{11} \subset D_{11}$ compact with diameter $< 1/3$ and such that the interior of $\left[[D^1_{11}]_{F_{n_0}} \cap D_{01}\right]_{F_{n_1}} \cap U_1$ is nonempty. Next, find $D^1_{01} \subset [D^1_{11}]_{F_{n_0}} \cap D_{01}$ compact with diameter $< 1/3$ and such that the interior of $[D^1_{01}]_{F_{n_1}} \cap U_1$ is nonempty. Find D^1_{10} compact with diameter $< 1/3$ and such that the interior of $[D^1_{10}]_{F_{n_0}} \cap [D^1_{01}]_{F_{n_1}} \cap U_1$ is nonempty. Let U_2 be an open set such that $\mathrm{diam}(U_2) < 1/3$ and $\overline{U}_2 \subset [D^1_{10}]_{F_{n_0}} \cap [D^1_{01}]_{F_{n_1}} \cap U_1$. Put finally $C_{\langle 0,0 \rangle} = \overline{U}_2$, $C_{\langle 1,0 \rangle} = [C_{\langle 0,0 \rangle}]_{F_{n_0}} \cap D^1_{10}$, $C_{\langle 0,1 \rangle} = [C_{\langle 0,0 \rangle}]_{F_{n_1}} \cap D^1_{01}$, and $C_{\langle 1,1 \rangle} = [C_{\langle 0,1 \rangle}]_{F_{n_0}} \cap D^1_{11}$. This finishes the proof of the theorem.

Corollary 2. *Let X be a Polish space. Let F be a K_σ equivalence relation on X each equivalence class of which is dense. If F has at least two equivalence classes, then $E_0 \sqsubseteq_c F$.*

Proof. By Theorem 1 it is enough to show that for any $x \in X$, $[x]_F$ is not locally closed at x. But if it were, then, since $[x]_F$ is dense, there would exist an open set U with $x \in U \subset [x]_F$. But then no equivalence class different from $[x]_F$ could be dense.

3.2. Application to indecomposable continua

A continuum is a metric compact connected space. A continuum is called indecomposable if it is not the union of two proper subcontinua. Indecomposable continua, first constructed by Brouwer in 1910, occur naturally in dynamical systems and also have their own extensive literature. A composant of a continuum C is a maximal set any two points of which lie in a proper subcontinuum of C. Each indecomposable continuum is partitioned into disjoint composants. We will call the equivalence relation E_C induced by this partition the composant equivalence relation, i.e., xE_Cy iff x and y lie in a proper subcontinuum of C. The study of composants is crucial in understanding the structure of indecomposable continua. Mazurkiewicz [Ma] proved that there is a perfect closed set $P \subset C$ which has at most one point in common with each composant. (An immediate consequence of it is that there are $2^{\aleph_0}$ composants.) A natural question to ask is whether there is a Borel set $T \subset C$ which has precisely one point in common with each composant. Such a set T is called a Borel transversal. (This question is formulated explicitly in Mauldin's [M] but was considered earlier by continuum theorists.) A partial answer was obtained by Cook [C] who proved that a Borel transversal cannot be F_σ. (More general facts about F_σ transversals were obtained in the recent paper by Dębski and Tymchatyn [DT].) By an argument of Mauldin [M], some other partial results can be deduced from the work of Emeryk [Em] and Krasinkiewicz [K]. Rogers in [R] noticed the relation of the question of the existence of a Borel transversal to the Glimm-Effros theorem. He applied the Glimm-Effros theorem to prove that certain indecomposable continua (solenoids and the Knaster continuum) carry a Borel probability measure μ which is ergodic in the sense that it assigns to each composant measure 0 and for any Borel $X \subset C$ if each composant of C is either contained in X or disjoint from it, then $\mu(X) = 0$ or $\mu(X) = 1$. Such continua do not have Borel transversals.

Below, we answer the question of the existence of Borel transversals in the negative for all indecomposable continua. The following corollary will imply that each indecomposable continuum carries an ergodic (in the sense described above) probability measure (see Corollary 4).

Corollary 3. *Let C be an indecomposable continuum. Then $E_0 \sqsubseteq_c E_C$ where E_C is the composant equivalence relation.*

Proof. By [R, Theorem 3.3], E_C is K_σ. It is well known, see for example [Ku, Ch.5, §48, VI, Theorems 2 and 7], that each composant is dense and that there are at least two composants, that is, $[x]_{E_C}$ is dense for each $x \in C$ and E_C has at least two equivalence classes. Thus, Corollary 3 follows from Corollary 2.

To state the next corollary, we need the following definition. Let E be a Borel equivalence relation on a Polish space Y. A Borel probability measure μ on Y is called E-ergodic if $\mu([x]_E) = 0$ for any $x \in Y$ and $\mu(X) = 0$ or 1 if $X \subset Y$ is Borel and is the union of a family of E-equivalence classes. The next corollary follows from Corollary 3 by, by now, standard arguments (see [E]).

Corollary 4. *Let C be an indecomposable continuum with the composant equivalence relation E_C.*

(i) There exists an E_C-ergodic Borel probability measure on C.

(ii) There does not exist a Borel set which has precisely one point in common with each composant.

The following theorem, improving on a result of Rogers [R, Theorem 3.3], gives an important structural property of the composant equivalence relation. It shows that the composant equivalence relation is hypersmooth, see [KL].

Theorem 5. *The composant equivalence relation on an indecomposable continuum is the increasing union of a sequence of compact equivalence relations.*

Proof. Let C be an indecomposable continuum with the composant equivalence relation E_C. Let $\{U_n : n \in \omega\}$, with $U_{n+1} \subset U_n$, be an open basis at $x_0 \in C$. Let C_n, $n \in \omega$, be proper subcontinua of C such that $x_0 \in C_n$, $C_n \subset C_{n+1}$, and $\bigcup_n C_n = [x_0]_{E_C} =$ the composant of x_0. Define for $x, y \in C$ and $n \in \omega$

$$xE_ny \text{ iff } x = y \text{ or } x, y \in K \text{ for some subcontinuum } K \subset (C \setminus U_n) \cup C_n.$$

One checks easily that each E_n is an equivalence relation and that $E_n \subset E_{n+1}$. To see that E_n is closed, let $x_kE_ny_k$, $k \in \omega$, and $x_k \to x$, $y_k \to y$. We can assume that $x_k \neq y_k$ for all k. Let K_k be a continuum witnessing $x_kE_ny_k$. Then $K = \lim_k K_k$ is a continuum, $x, y \in K$, and $K \subset (C \setminus U_n) \cup C_n$ since $(C \setminus U_n) \cup C_n$ is closed. Thus, xE_ny.

Since $U_n \setminus C_n \neq \emptyset$ for all n (as C_n is nowhere dense, see [Ku]), each subcontinuum $K \subset (C \setminus U_n) \cup C_n$ is proper, whence $E_n \subset E_C$ for all $n \in \omega$. To see that $\bigcup_n E_n = E_C$, let xE_Cy. If xE_Cx_0, we can find an $n \in \omega$ such that $x, y \in C_n$. But then xE_ny. If $\neg(xE_Cx_0)$, let K be a proper subcontinuum of C with $x, y \in K$. There is n such that $K \cap U_n = \emptyset$, as $x_0 \notin K$. Then xE_ny. This finishes the proof.

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CHAPTER 4
POLISH GROUP ACTIONS

4.1. The Topological Vaught Conjecture for Polish groups with an invariant metric

Let G be a Polish group acting on a set X. Put for $x, y \in X$

$$xE_G^X y \Leftrightarrow \exists g \in G \; gx = y.$$

Then $E_G^X \subset X \times X$ is an equivalence relation and is called the *equivalence relation induced by the action of G on X*. (Sometimes, if there is no possibility of confusion, we drop the superscript X.) If X is Polish and the action of G is continuous, then E_G is analytic. The Topological Vaught Conjecture (TVC) says that either E_G has countably many equivalence classes, or there exists a perfect set which has at most one point in common with each equivalence class. It is a generalization of the famous Vaught conjecture from model theory and was first formulated by Miller. The TVC is still open, so it seems interesting to ask for what classes of Polish groups it holds. If E_G is Borel, the TVC follows from Silver's theorem. And indeed, in case G is locally compact, E_G turns out to be Borel. R.L. Sami in [S] proved that the TVC holds for abelian Polish groups. A.S. Kechris asked if the TVC holds for Polish groups admitting an invariant metric. (By invariant metric we mean a two-sided invariant metric.) Each abelian Polish group admits an invariant metric. There exist, however, groups admitting invariant metrics which are very far from being abelian, for instance, they may contain the free group with $2^{\aleph_0}$ generators. Below, we show that the TVC does hold for Polish groups with invariant metric. We actually prove a much stronger dichotomy theorem reminiscent of the Glimm-Effros theorem (see [E], [G]). Recently, G. Hjorth established analogous results for Polish nilpotent groups and for Polish groups whose quotient by the center admits an invariant metric.

For the definition of E_0 and $\sqsubseteq_c$ see 3.1.

Theorem 1. *Let G be a Polish group admitting an invariant metric. Let X be a Polish G-space. Then either $E_0 \sqsubseteq_c E_G$ or E_G is G_δ.*

Corollary 2. *The Topological Vaught Conjecture holds for Polish groups admitting an invariant metric.*

Proof of the corollary. By the thorem we have two cases: either $E_0 \sqsubseteq_c E_G$ or E_G is G_δ. It is easy to see that there is a perfect set which has at most one point in common with each equivalence class of E_0. So, if $E_0 \sqsubseteq_c E_G$, then the same is true of E_G. If E_G is G_δ, then the function $f: X \to 2^\omega$ defined by $f(x) = \{n \in \omega : x \in [V_n]_G\}$ is Borel ($\{V_n : n \in \omega\}$ a topological basis of X) and has the property that $xE_G y$ iff $f(x) = f(y)$. From this it follows by standard methods that the TVC holds for E_G.

In the sequel, I will use the following known facts.

(Effros) Let X be a Polish G-space, G a Polish group. Let $x \in X$ be such that $[x]_G$ is nonmeager. Then $[x]_G$ is G_δ, and the mapping $g \to gx$, $G \to [x]_G$, is open.

(Becker-Kechris) Let X and G be as above. Assume the action has a dense orbit and $E_0 \not\sqsubseteq_c E_G$. Then there is $x \in X$ with $[x]_G$ nonmeager.

Lemma 3. *Let G be a Polish group, and let X be a Polish G-space. If $[y]_G$ is nonmeager, $y \in X$, and $x \notin [y]_G$, then there is $V \subset G$ open such that $e \in V$ and $\overline{Vx} \cap [y]_G = \emptyset$.*

Proof. Since $[y]_G$ is G_δ, $X \setminus [y]_G = \bigcup_n F_n$, F_n closed. Put $F'_n = \{g \in G : gx \in F_n\}$. Then F'_n are closed, and $\bigcup_n F'_n = G$. Thus, there is n_0 and an open set $U \neq \emptyset$ with $U \subset F'_{n_0}$. Let $g \in G$ be such that $e \in gU$. Put $V = gU$. Then $\overline{Vx} = \overline{gUx} = g\overline{Ux} \subset gF_{n_0}$, and clearly $gF_{n_0} \cap [y]_G = \emptyset$, whence $\overline{Vx} \cap [y]_g = \emptyset$.

Lemma 4. *Let G be a Polish group admitting an invariant metric, and let X be a Polish G-space. If $[y]_G$ is nonmeager and $[x]_G$ is dense, $x, y \in X$, then for any nonempty open set $V \subset G$, $\overline{Vx} \cap [y]_G \neq \emptyset$.*

Proof. Let $\emptyset \neq V \subset G$ be open. We show that $\overline{Vx} \cap [y]_G \neq \emptyset$. Since G admits an invariant metric, we can assume that $e \in V$ and $gVg^{-1} = V$ for any $g \in G$. Let W be open, symmetric and such that $e \in W$, $W^3 \subset V$, $gWg^{-1} = W$ for any $g \in G$. Since Wy is open in $[y]_G$, $\overline{Wy}$ has a nonempty interior, whence $[x]_G \cap \overline{Wy} \neq \emptyset$. Thus, there are $h \in G$ and $h_k \in W$ with $h_k y \to hx$. If we show that $\overline{Vhx} \cap [y]_G \neq \emptyset$, then, since $\overline{Vhx} \cap [y]_G = \overline{hVx} \cap [y]_G = h\overline{Vx} \cap [y]_G$,

$h\overline{Vx} \cap [y]_G \neq \emptyset$, that is, $\overline{Vx} \cap [y]_G \neq \emptyset$. Thus, we can assume that $hx = x$, that is, $h_k x \to x$. Since $[x]_G$ is dense, we can find $g_n \in G$ such that $g_n x \to y$. Fix n. For k large enough, $d(g_n h_k y, g_n x) < 1/(n+1)$, d a metric on X. Thus, we can find a subsequence (h_{k_n}) such that $d(g_n h_{k_n} y, g_n x) < 1/(n+1)$. Call this subsequence (h_n). Since $g_n x \to y$, $g_n h_n y \to y$. Since $g \to gy$ is open, for n large enough we have $g_n h_n \in WG_y$.

Now, we show that $\forall U \ni e$ open$\exists N \forall n, m \geq N$ $G_{h_n y} \subset UG_{h_m y}$. Put $y_n = h_n y$. Let $e \in U_0$ be open, symmetric, $U_0^4 \subset U$, and $gU_0 g^{-1} = U_0$ for all $g \in G$. Since $U_0 y$ is open in $[y]_G$, we can find $O \subset X$ open with $\emptyset \neq O \cap [y]_G \subset U_0 y$. There is $g \in G$ with $gx \in O$. It follows that $gy_n \to gx$, so, $gy_n \in O$ for $n \geq N$ some $N \in \omega$, whence $gy_n \in U_0 y$. Thus, $gy_n \in U_0^2 gy_m$ for $n, m \geq N$, so $y_n \in U_0^2 y_m$. It follows that

$$G_{y_n} \subset U_0^2 G_{y_m} U_0^2 \subset U_0^4 G_{y_m} \subset UG_{y_m}.$$

Put

$$H = \{g \in G : \exists n_k \to \infty \exists p_k \in G_{h_{n_k} y}\, p_k \to g\}.$$

First, note that $H \subset G_x$. Indeed, if $g \in H$, then $x \leftarrow h_{n_k} y = p_k h_{n_k} y \to gx$, so $g \in G_x$. Next, we show that $g_n \in W^3 H$ for n large enough. Since $g_n h_n \in WG_y$, $g_n \in Wh_n^{-1}(h_n G_y h_n^{-1}) \subset W^2 G_{h_n y}$. Again put $y_n = h_n y$. Let W_i, $i \in \omega$, be open with $e \in W_i$, $\bigcup_{i \geq k} W_k \cdots W_i \to e$ as $k \to \infty$, and $\overline{\bigcup_i W_0 W_1 \cdots W_i} \subset W$. Let N be such that $G_{y_n} \subset W_0 G_{y_m}$ for $n, m \geq N$. Let $n \geq N$. Pick $n = n_0 < n_1 < n_2 < \cdots$ so that $G_{y_k} \subset W_i G_{y_m}$ for $k, m \geq n_i$. Let $g \in G_{y_{n_0}} = G_{y_n}$. Pick $g_i \in G_{y_{n_i}}$, $i \geq 1$, so that $gg_1^{-1} \in W_0$ and $g_i g_{i+1}^{-1} \in W_i$ for $i \geq 1$. This is possible since $G_{y_{n_i}} \subset W_i G_{y_{n_{i+1}}}$. It is easy to check that (g_i) is Cauchy, so we can put $h = \lim_i g_i$. Clearly $h \in H$, and

$$gh^{-1} = \lim_i gg_i^{-1} = \lim_i (gg_1^{-1})(g_1 g_2^{-1}) \cdots (g_{i-1} g_i^{-1}) \in \overline{\bigcup_i W_0 W_1 \cdots W_i} \subset W.$$

So, $g \in Wh \subset WH$. Thus, $G_{y_n} \subset WH$ for $n \geq N$, whence $g_n \in W^2 WH = W^3 H$ for $n \geq N$.

Combining $g_n \in W^3 H$ for large n and $H \subset G_x$, we get $g_n \in W^3 G_x$ for large n. Since $W^3 \subset V$, there are $c_n \in G_x$ such that $g_n c_n \in V$ for large n. But then $g_n c_n x = g_n x \to y$, so $\overline{Vx} \cap [y]_G \neq \emptyset$, and the lemma is proved.

We list two corollaries to Lemmas 3 and 4.

Corollary 5. *Let G be a Polish group admitting an invariant metric, and let X be a Polish G-space. If $[y]_G$ is nonmeager and $[x]_G$ is dense, $x, y \in X$, then $[x]_G = [y]_G$.*

Corollary 6. *Let G and X be as above. Assume all orbits are dense. Then either $E_0 \sqsubseteq_c E_G$ or there is only one orbit.*

Proof. If $E_0 \not\sqsubseteq_c E_G$, then there is a nonmeager orbit. Since all orbits are dense, this is the only orbit by Corollary 5.

Proof of the theorem. Define the equivalence relation xFy iff $\overline{[x]_G} = \overline{[y]_G}$. Then F is G_δ since xFy iff $\forall n\ [x]_G \cap V_n \neq \emptyset \Leftrightarrow [y]_G \cap V_n \neq \emptyset$ iff $\forall n\ x \in [V_n]_G \Leftrightarrow y \in [V_n]_G$ for an open basis $\{V_n : n \in \omega\}$ for the topology on X. Clearly $E_G \subset F$. If $E_G = F$, E_G is G_δ and we are done. If $E_G \neq F$, then for some $x \in X$ the invariant set $[x]_F$ contains at least two orbits. Since $[x]_F$ is G_δ and each orbit contained in it is dense in it, $E_0 \sqsubseteq_c E_G|[x]_F$ by Corollary 6; thus, $E_0 \sqsubseteq_c E_G$.

4.2. Complexity of equivalence relations induced by Polish group actions

4.2.1. Introduction. As mentioned above, it was proved by R. L. Sami [S, Theorem 2.1] that the topological Vaught conjecture holds for Borel actions of abelian Polish groups. The proof, however, was different from the one in the locally compact case; in particular, it did not show that E_G^X was Borel for G Polish abelian. The natural question was raised by Sami (see [S, p.339]) whether E_G^X is Borel for all Borel (or, equivalently, continuous if X is a Polish space, see [BK]) actions of Polish abelian groups on standard Borel spaces. We answer this question in the negative. We consider groups of the form $H_0 \times H_1 \times H_2 \times \cdots$ where the H_n's are countable. Such groups are equipped with the product topology (each H_n carring the discrete topology) which is Polish and compatible with the group structure. We fully characterize those sequences (H_n) of countable abelian groups for which all Borel actions of $H_0 \times H_1 \times H_2 \times \cdots$ induce Borel equivalence relations. This happens precisely when all but finitely many of the H_n's are torsion and, for each prime p, for all but finitely many n's the p-component of H_n is of the form $F \times \mathbb{Z}(p^\infty)^m$, where F is a finite p-group, $\mathbb{Z}(p^\infty)$ is the quasicyclic p-group (i.e., $\mathbb{Z}(p^\infty) \simeq \{z \in \mathbb{C} : \exists n\ z^{p^n} = 1\}$), and $m \in \omega$. In particular, if $H_n = H$, $n \in \omega$, and H is countable abelian, then all Borel actions of $H \times H \times H \times \cdots$ induce Borel equivalence relations iff $H \simeq \bigoplus_p (F_p \times \mathbb{Z}(p^\infty)^{n_p})$, where F_p is afinite abelian p-group, $n_p \in \omega$, and p varies over the set of all primes. Thus, e.g., the group $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \cdots$ is abelian, Polish, and has a Borel action which induces non-Borel equivalence relation. This answers Sami's question. On the other hand, $\mathbb{Z}(2^\infty) \times \mathbb{Z}(2^\infty) \times \mathbb{Z}(2^\infty) \times \cdots$ provides an interesting example of a Polish abelian group which is not locally compact but whose Borel actions induce only Borel equivalence relations. This shows that the implication "G locally compact $\Rightarrow E_G^X$ Borel" cannot be reversed. Some results for non-abelian H_n's are also obtained.

Now, we state some definitions and establish notation. By ω we denote the set of all natural numbers $\{0, 1, 2, \ldots\}$. Ordinal numbers are identified with the set of their predecessors, in particular $n = \{0, 1, \ldots, n-1\}$, for $n \in \omega$. By $\mathbb{Z}$, $\mathbb{Z}(p)$, $\mathbb{Z}(p^\infty)$, p a prime, we denote the group of integers, the cyclic group with p elements, and the quasicyclic p-group, respectively. By e we denote the identity element of a group and by $\langle X \rangle$, for a subset X of a group, the subgroup generated by X. We write $\langle h \rangle$ for $\langle \{h\} \rangle$. If H is a group, $\bigoplus_\omega H$ stands for the direct sum of countably many copies of H. A group H is called *p-compact* if for any decreasing sequence of groups $G_k < \mathbb{Z}(p) \times H$ with $\pi[G_k] = \mathbb{Z}(p)$, for each $k \in \omega$, we have $\pi[\bigcap_{k\in\omega} G_k] = \mathbb{Z}(p)$ where $\pi : \mathbb{Z}(p) \times H \to \mathbb{Z}(p)$ is the projection. If H is an abelian group and p is a prime, by the *p-component of H* we mean the maximal p-subgroup of H.

For a sequence of sets (H_n), $n \in \omega$, we write

$$H^n = H_0 \times \cdots \times H_{n-1},\ H^{<\omega} = \bigcup_{n\in\omega} H^n, \text{ and } H^\omega = H_0 \times H_1 \times \cdots.$$

We also write A^ω for the product of infinitely many copies of A. If $x \in H^\omega$, put $lhx = \omega$; if $\sigma \in H^n$, some $n \in \omega$, put $lh\sigma = n$. For $\sigma \in H^{<\omega}$ and $x \in H^{<\omega} \cup H^\omega$, we write $\sigma * x$ for the concatenation of σ and x. If $x \in H^{<\omega} \cup H^\omega$ and $X \subset \omega$, we write $x|X$ for the unique element $y \in H^{<\omega} \cup H^\omega$ such that the domain of y is ω, if $X \cap lhx$ is infinite, and n, if $X \cap lhx$ is finite and has n elements, and $y(i) = x(\text{the (i+1)'th element of } X)$. A set $S \subset H^{<\omega}$ is called a *tree on* (H_n) if $\sigma \in S$ implies $\sigma|n \in S$ for any $n < lh(\sigma)$. If S is a tree on (H_n) and $\sigma \in H^{<\omega}$, put $S_\sigma = \{\tau \in H^{<\omega} : \sigma * \tau \in S\}$. For a tree S on (H_n), H_n countable, define $S' = \{\sigma \in S : \exists \tau \in S\ \sigma \subset \tau, \sigma \neq \tau\}$. By transfinite induction define, for $\beta \in \omega_1$,

$S^0 = S$ and $S^\beta = (S^\gamma)'$ if $\beta = \gamma + 1$, and $S^\beta = \bigcap_{\gamma<\beta} S^\gamma$ if β is limit. Put $ht(S) = \min\{\beta : S^\beta = S^{\beta+1}\}$. For $\sigma \in H^{<\omega}$, put $r_S(\sigma) = \min\{\beta \in \omega_1 : \sigma \notin S^\beta\}$ if there exists $\beta < \omega_1$ with $\sigma \notin S^\beta$, and $r_S(\sigma) = \omega_1$ otherwise. If there is no danger of confusion, we will omit the subscript in r_S. A tree on (H_n) is *well-founded* if there is no sequence $\sigma_i \in S$, $i \in \omega$, such that $\sigma_i \subset \sigma_{i+1}$ and $lh(\sigma_i) \to \infty$ as $i \to \infty$. Now, assume that the H_n's are groups. The identity element $(e, e, \dots)$ of H^ω is denoted by $\vec{e}$. A tree S on (H_n) is called a *coset tree* if $S \cap H^n$ is a left coset of a subgroup of H^n for any $n \in \omega$, i.e., if $\sigma_1, \sigma_2, \sigma_3 \in S \cap H^n$, then $\sigma_1\sigma_2^{-1}\sigma_3 \in S$. A coset tree S is called a *group tree* if $S \cap H^n$ is a subgroup of H^n for any $n \in \omega$. The notion of a group tree was introduced by Makkai in [M] and rediscovered by the author. We say that (H_n) *admits group (coset) trees of arbitrary height* if for any $\beta < \omega_1$, there is a group (coset) tree T on (H_n) with $ht(T) > \beta$. Let S be a coset tree on a sequence of groups (H_n). Then for each $n \in \omega$ there is a unique subgroup G_n of H^n which $S \cap H^n$ is a coset of. We actually have $G_n = \sigma^{-1}(S \cap H^n)$ for any $\sigma \in S \cap H^n$. Define

$$\alpha(S) = \bigcup_{n\in\omega} G_n.$$

Thus $\alpha(S) = \bigcup_{n\in\omega} \sigma_n^{-1}(S \cap H^n)$ where $\sigma_n \in S \cap H^n$ if $S \cap H^n \neq \emptyset$ and $\sigma_n = e$ otherwise. It is easy to see that $\alpha(S)$ is a group tree.

4.2.2. Main results.

Theorem 7. *Let (H_n) be a sequence of countable abelian groups. Then the equivalence relation induced by any Borel action of H^ω is Borel iff for all but finitely many n, H_n is torsion, and for all primes p for all but finitely many n the p-component of H_n is of the form $F \times \mathbb{Z}(p^\infty)^k$, where $k \in \omega$ and F is a finite abelian p-group.*

If H is countable, abelian, and torsion, then $H = \bigoplus_p H_p$, where p ranges over the set of all primes, and H_p is the p-component of H (see [F]). Thus we get the following corollary.

Corollary 8. *Let H be an abelian countable group. Then the equivalence relations induced by Borel actions of H^ω are Borel iff H is isomorphic to $\bigoplus_p(F_p \times \mathbb{Z}(p^\infty)^{n_p})$, where p ranges over the set of all primes, $n_p \in \omega$, and F_p is a finite abelian p-group.*

For not necessarily abelian countable groups, we have the following version of one implication from Theorem 7.

Theorem 9. *Let (H_n) be a sequence of countable groups. If for each prime p, for all but finitely many n, H_n is p-compact, then the equivalence relations induced by Borel actions of H^ω are Borel.*

It is an open question whether the converse of Theorem 9 holds. This would be a natural extension of Theorem 7, since, as we show in Lemma 16, a countable abelian group is p-compact iff it is torsion and its p-component has the form as in Theorem 1.

Some of the ingredients of the proofs are: the theorem of Becker and Kechris [BK] on the existence of universal actions, the structure theory for countable abelian groups, and a construction of group trees of arbitrary height. It turns out that both conditions in Theorem 7 are equivalent to (H_n) not admitting group trees of arbitrary height (Lemma 21). This generalizes the known results that the sequence (H_n), $H_n = \mathbb{Z}$ for each $n \in \omega$, admits group trees of arbitrary height (Makkai [M, Lemma 2.6]), and that the sequence (H_n), $H_n = \bigoplus_\omega \mathbb{Z}(2)$ for each n, admits group trees of arbitrary height (Shelah [M, Appendix]). (See also [L, p. 979] for a proof of the latter result and its generalizations to groups which are direct sums of κ many copies of $\mathbb{Z}(2)$ for certain cardinals κ.) The known proofs in the above two cases—$\mathbb{Z}$ and $\bigoplus_\omega \mathbb{Z}(2)$—were different from each other, and Makkai's construction for $\mathbb{Z}$ rested on Dirichlet's theorem on primes in arithmetic progressions. We present a construction (Lemma 19) that encompasses both these cases and is purely combinatorial.

Here is how Theorems 7 and 9 follow from the lemmas in Sections 4.2.3-4.2.5. In Section 4.2.3, we prove that all Borel actions of H^ω, (H_n) a sequence of countable groups, induce Borel equivalence relations iff (H_n) does not admit well-founded coset trees of arbitrary height (Lemma 11). In Section 4.2.4, we show that (H_n) does not admit well-founded coset trees of arbitrary height iff it does not admit group trees of arbitrary height (Lemma 15). Then, in Section 4.4.5, we show that if for each prime p, for all but finitely many n, H_n is p-compact, then (H_n) does not admit group trees of arbitrary height (Lemma 17). This proves Theorem 9. Next, we prove that if (H_n) is a sequence of abelian groups, then

(H_n) does not admit group trees of arbitrary height iff for all but finitely many n, H_n is torsion and, for all primes p, for all but finitely many n, the p-component of H_n has the form as in Theorem 7 (Lemma 21). This proves Theorem 7.

4.2.3. Group actions and coset trees. The following construction is from [BK]. Let G be a Polish group. Consider $\mathcal{F}(G)$ the space of all closed subsets of G with the Effros Borel structure, i.e., the Borel structure generated by sets of the form $\{F \in \mathcal{F}(G) : F \cap V \neq \emptyset\}$ for $V \subset G$ open. Put $\mathcal{U}_G = \mathcal{F}(G)^\omega$, and define the following G-action on $\mathcal{U}_G$: $(g, (F_n)) \to (gF_n)$.

Theorem. (Becker-Kechris [BK]) *$\mathcal{U}_G$ with the above G-action is a universal Borel G space, i.e., if X is a standard Borel space on which G acts by Borel automorphisms, then there is a Borel injection $\pi : X \to \mathcal{U}_G$ such that $\pi(gx) = g\pi(x)$ for $g \in G$ and $x \in X$.*

Let X be a standard Borel G-space. Let $\pi : X \to \mathcal{U}_G$ be a Borel injection whose existence is guaranteed by the above theorem. Then, for $x, y \in X$, we have

$$xE_G^X y \Leftrightarrow \pi(x)E_G^{\mathcal{U}_G}\pi(y).$$

This shows that the following corollary to the theorem above is true.

Lemma 10. *Let G be a Polish group. The relation induced by any Borel G-action is Borel iff the relation induced by the G-action on $\mathcal{U}_G$ is Borel.*

Lemma 11. *Let (H_n) be a sequence of countable groups. The equivalence relation induced by any Borel H^ω-action is Borel iff (H_n) does not admit well-founded coset trees of arbitrary height.*

Proof. Let $\mathcal{T}$ be the family of all trees on (H_n). The set $\mathcal{T}$ is a Polish space with the topology generated by sets of the form $\{T \in \mathcal{T} : \sigma \in T\}$ and $\{T \in \mathcal{T} : \sigma \notin T\}$ for $\sigma \in H^{<\omega}$.

($\Leftarrow$) By Lemma 10, it is enough to prove that the H^ω-action on $\mathcal{U}_{H^\omega}$ induces a Borel relation. Let $\mathcal{T}_p$ be the family of all pruned trees on (H_n), i.e., trees with no finite branches, with the topology inherited from $\mathcal{T}$. This topology makes $\mathcal{T}_p$ a Polish space. The mapping $\phi : \mathcal{T}_p \to \mathcal{F}(H^\omega)$ given by $\phi(T) = \{x \in H^\omega : \forall n \in$

$\omega\ x|n \in T\}$ is a Borel isomorphism. For $x \in H^\omega$ and $T \in \mathcal{T}_p$ define

$$xT = \{\sigma \in H^{<\omega} : \ \sigma \in x|m(T \cap H^m) \quad \text{where} \quad m = lh(\sigma)\}.$$

Then easily $xT \in \mathcal{T}_p$. Also $\phi(xT) = x\phi(T)$. Thus it is enough to check that the following action of H^ω on $\mathcal{T}_p^\omega$ induces a Borel equivalence relation: $\big(x, (T_n)\big) \to (xT_n)$, for $x \in H^\omega, (T_n) \in \mathcal{T}_p^\omega$.

Now define $\Phi : \mathcal{T}_p \times \mathcal{T}_p \to \mathcal{T}$ by

$$\Phi(T, S) = \{\sigma \in H^{<\omega} : \ T \cap H^m = \sigma(S \cap H^m) \quad \text{where} \quad m = lh(\sigma)\}.$$

Easily $\Phi(T, S)$ is a coset tree. Define the mapping $\Psi : \mathcal{T}_p^\omega \times \mathcal{T}_p^\omega \to \mathcal{T}$ by

$$\Psi\big((T_n), (S_n)\big) = \bigcap_{n \in \omega} \Phi(T_n, S_n).$$

Note that the intersection of a family of coset trees is a coset tree. Thus, for any $(T_n), (S_n) \in \mathcal{T}_p^\omega$, $\Psi\big((T_n), (S_n)\big)$ is a coset tree. Also note that

$$(T_n) E_{H^\omega}^{\mathcal{T}_p^\omega} (S_n) \ \Leftrightarrow \ \Psi\big((T_n), (S_n)\big) \quad \text{is not well-founded.}$$

Indeed, if $\sigma_0 \subset \sigma_1 \subset \cdots$, $lh(\sigma_i) \to \infty$, and $\sigma_i \in \Psi\big((T_n), (S_n)\big)$, then $xS_n = T_n$ for each $n \in \omega$ where $x = \bigcup_{i \in \omega} \sigma_i$. If $xS_n = T_n$ for all $n \in \omega$ and some $x \in H^\omega$, then $x|i \in \Psi\big((T_n), (S_n)\big)$ and $\{x|i : i \in \omega\}$ witnesses that $\Psi\big((T_n), (S_n)\big)$ is not well-founded. Clearly Ψ is a Borel mapping. Thus, if we assume that there is $\beta \in \omega_1$ such that any well-founded coset tree on (H_n) has height $< \beta$, we get

$$(\mathcal{T}_p \times \mathcal{T}_p) \setminus E_{H^\omega}^{\mathcal{T}_p^\omega} = \Psi^{-1}\big(\{T \in \mathcal{T} : T \quad \text{well-founded and} \quad ht(T) < \beta\}\big).$$

But $\{T \in \mathcal{T} : T$ well-founded and $ht(T) < \beta\}$ is Borel, whence $E_{H^\omega}^{\mathcal{T}_p^\omega}$ is Borel.

($\Rightarrow$) Assume (H_n) admits well-founded coset trees of arbitrary height. Define the following continuous action of H^ω on $\mathcal{T}$:

$$(x, T) \to xT = \{\sigma \in H^{<\omega} : \ \sigma \in x|m(T \cap H^m) \quad \text{where} \quad m = lh(\sigma)\}.$$

Define a Borel function $\Phi_1 : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ by

$$\Phi_1(T, S) = \{\sigma \in H^{<\omega} : \ \forall m \leq lh(\sigma)\ T \cap H^m = \sigma|m(S \cap H^m)\}.$$

Now, if $E^{\mathcal{T}}_{H^\omega}$ is Borel, $\Phi_1\big[(\mathcal{T}\times\mathcal{T})\setminus E^{\mathcal{T}}_{H^\omega}\big]$ is $\mathbf{\Sigma}^1_1$. Also $\Phi_1\big[(\mathcal{T}\times\mathcal{T})\setminus E^{\mathcal{T}}_{H^\omega}\big]\subset\{T\in\mathcal{T}: T \text{ is well-founded}\}$. Since $\{T\in\mathcal{T}: T \text{ is well-founded}\}$ is a $\mathbf{\Pi}^1_1$ set and $T\to ht(T)$ is a $\mathbf{\Pi}^1_1$-norm on it, by the boundedness principal, there is $\beta\in\omega_1$ such that, for any $T,S\in\mathcal{T}$, if $(T,S)\notin E^{\mathcal{T}}_{H^\omega}$, then $ht\big(\Phi_1(T,S)\big)<\beta$. But note that if T is a coset tree, then $\Phi_1\big(T,\alpha(T)\big)=T$. Thus, for any well-founded coset tree T on (H_n), $ht(T)=ht\big(\Phi(T,\alpha(T))\big)<\beta$, a contradiction.

4.2.4. Coset and group trees. The next several lemmas lead to a proof that the existence of well-founded coset trees of arbitrary height is equivalent to the existence of group trees of arbitrary height (Lemma 15). We will use a few times the easy fact that $\{r(\sigma):\sigma\in T\}\supset ht(T)$ for any tree T on (H_n).

Lemma 12. *Let S be a coset tree. Then:*
(i) $\alpha(S')=\alpha(S)'$;
(ii) if $S^\xi\cap H^k\neq\emptyset$ for each $k\in\omega$, then $\alpha(S^\xi)=\alpha(S)^\xi$.
Proof. To show (i), let $\sigma\in H^n$. Then $\sigma\in\alpha(S')$ implies that there are $\tau_1,\tau_2\in S'$ such that $\sigma=\tau_1^{-1}\tau_2$. Now we can find $g,h\in H_n$ with $\tau_1*g,\tau_2*h\in S$. But then $\sigma*(g^{-1}h)=(\tau_1*g)^{-1}(\tau_2*h)\in\alpha(S)$. Thus $\sigma\in\alpha(S)'$. On the other hand, if $\sigma\in\alpha(S)'$, then there are $g\in H_n$ and $\tau_1,\tau_2\in S$ with $\tau_1^{-1}\tau_2=\sigma*g$. But then $\sigma=(\tau_1|n)^{-1}(\tau_2|n)$ and $\tau_1|n,\tau_2|n\in S'$, whence $\sigma\in\alpha(S')$.

Notice that if $S_n\supset S_{n+1}$, $n\in\omega$, are coset trees, and, for some $k\in\omega$, $\bigcap_{n\in\omega}(S_n\cap H^k)\neq\emptyset$, then $\alpha(\bigcap_{n\in\omega}S_n)\cap H^k=\bigcap_{n\in\omega}\alpha(S_n)\cap H^k$. To see this, pick $\sigma\in\bigcap_{n\in\omega}S_n\cap H^k$. Then

$$\alpha(\bigcap_{n\in\omega}S_n)\cap H^k=\sigma^{-1}(\bigcap_{n\in\omega}S_n\cap H^k)=\bigcap_{n\in\omega}\sigma^{-1}(S_n\cap H^k)=\bigcap_{n\in\omega}\alpha(S_n)\cap H^k.$$

Using (i) and the above observation, we get (ii) by transfinite induction.

Lemma 13. *Let T be a group tree. Let $\sigma_n\in H^n$, $n\in\omega$, be such that $(\sigma_{n+1}|n)^{-1}\sigma_n\in T^\beta$ for some $\beta\in\omega_1$. Put $S=\bigcup_{n\in\omega}\sigma_n(T\cap H^n)$. Then S is a coset tree, and for any $\xi\leq\beta$ we have $S^\xi=\bigcup_{n\in\omega}\sigma_n(T^\xi\cap H^n)$.*
Proof. For $\xi\leq\beta$, define

$$S^{(\xi)}=\bigcup_{n\in\omega}\sigma_n(T^\xi\cap H^n).$$

In particular, $S^{(0)} = S$. First note that each $S^{(\xi)}$ is a coset tree. Indeed, if $m < n$, then $(\sigma_n|m)^{-1}\sigma_m \in T^\xi$. This follows easily by induction from our assumptions that it holds for $n = m+1$ and the fact that T^ξ is a group tree. To check that $S^{(\xi)}$ is a tree, let $\tau \in T^\xi \cap H^n$. Then, for $m < n$, $(\sigma_n\tau)|m = (\sigma_n|m)(\tau|m) = \sigma_m\big(\sigma_m^{-1}(\sigma|m)(\tau|m)\big) \in S^{(\xi)} \cap H^m$ since $\big(\sigma_m^{-1}(\sigma_n|m)\big)(\tau|m) \in T^\xi \cap H^m$. Thus $S^{(\xi)}$ is a tree, and because of the way it was defined, it is a coset tree. It is obvious that $\alpha(S^{(\xi)}) = T^\xi$ and that $\sigma_n \in S^{(\xi)}$ for any $n \in \omega$, $\xi \leq \beta$.

Now, we show by induction that, for $\xi \leq \beta$, $\alpha(S^\xi) = T^\xi$ and $\sigma_n \in S^\xi$ for each $n \in \omega$. Both statements are true for $\xi = 0$. If ξ is limit and $\sigma_n \in S^\zeta$ for all $\zeta < \xi$, then clearly $\sigma_n \in S^\xi$. By Lemma 12(ii), we also have $\alpha(S^\xi) = \alpha(S)^\xi = T^\xi$. If ξ is a succesor, say $\xi = \zeta + 1$, then, by Lemma 12(i) and the induction hypothesis, we get $\alpha(S^\xi) = \alpha(S^\zeta)' = (T^\zeta)' = T^\xi$. Since $\sigma_{n+1} \in S^\zeta$, $\sigma_{n+1}|n \in S^\xi$. Since $(\sigma_{n+1}|n)^{-1}\sigma_n \in T^\beta \subset T^\xi$, we have $\sigma_n = (\sigma_{n+1}|n)\big((\sigma_{n+1}|n)^{-1}\sigma_n\big) \in S^\xi$.

Thus $\alpha(S^{(\xi)}) = T^\xi = \alpha(S^\xi)$, i.e., for each $n \in \omega$, $S^{(\xi)} \cap H^n$ and $S^\xi \cap H^n$ are left cosets of the same subgroup of H^n. Also $(S^{(\xi)} \cap H^n) \cap (S^\xi \cap H^n) \neq \emptyset$, as σ_n belongs to the intersection. Thus we get $S^{(\xi)} \cap H^n = S^\xi \cap H^n$ for each $n \in \omega$, i.e., $S^{(\xi)} = S^\xi$.

Lemma 14. *Let T be a group tree with $ht(T) > \omega$. Then there exist $\sigma_n \in H^n$ such that :*

(i) $(\sigma_{n+1}|n)^{-1}\sigma_n \in T$*;*

(ii) $\bigcup_{n\in\omega} \sigma_n(T \cap H^n)$ *is a well-founded tree of height* $< \omega \cdot 2$.

Proof. We start with the following observation. Let K be a countable group and let K_n, $n \in \omega$, be a strictly decreasing sequence of subgroups of K. Then there exist $g_n \in K$, $n \in \omega$, such that $g_n^{-1}g_{n+1} \in K_n$ and $\bigcap_{n\in\omega} g_nK_n = \emptyset$. To see that this is true, enumerate $K = \{k_n : n \in \omega\}$ and pick $g_n \in K$ recursively so that $g_{n+1}K_{n+1} \subset g_nK_n$ and $k_n \notin g_{n+1}K_{n+1}$.

Now, assume that T is a group tree and $ht(T) > \omega$. Let σ_0 be such that $r(\sigma_0) = \omega$. Put $k_0 = lh(\sigma_0) + 1$. Then $\{r(\sigma) : \sigma \in T \cap H^{k_0}\} \cap \omega$ is cofinal in ω. Let $p_n : H^n \to H^{k_0}$, $n > k_0$, denote the projection on the first k_0 coordinates. Since $\{\sigma \in H^{k_0} : r(\sigma) \geq m\} = p_{k_0+m}[T \cap H^{k_0+m}]$, there is an increasing sequence $k_0 < m_0 < m_1 < m_2 < \cdots$ such that $p_{m_{n+1}}[T \cap H^{m_{n+1}}] \neq p_{m_n}[T \cap H^{m_n}]$ and, obviously, $p_{m_{n+1}}[T \cap H^{m_{n+1}}] \subset p_{m_n}[T \cap H^{m_n}]$. Pick $\tau_n \in H^{k_0}$, $n \in \omega$, as in the preceding paragraph for $K_n = p_{m_n}[T \cap H^{m_n}]$, i.e.,

$$\tau_{n+1}^{-1}\tau_n \in p_{m_n}[T \cap H^{m_n}] \quad \text{and} \quad \bigcap_{n\in\omega} \tau_n\big(p_{m_n}[T \cap H^{m_n}]\big) = \emptyset.$$

We recursively construct $\sigma_n \in H^n$, $n \in \omega$, so that

$$\sigma_{m_n}|k_0 = \tau_n \text{ and } (\sigma_{n+1}|n)^{-1}\sigma_n \in T.$$

First, find $\rho_n \in H^{m_n}$ with $\rho_n|k_0 = \tau_n$ and $(\rho_{n+1}|m_n)^{-1}\rho_n \in T$. For ρ_0 take any extension of τ_0 in H^{m_0}. Now assume ρ_n has been constructed. Then $\tau_{n+1}^{-1}(\rho_n|k_0) = \tau_{n+1}^{-1}\tau_n \in p_{m_n}[T \cap H^{m_n}]$. Let $\sigma \in T \cap H^{m_n}$ be such that $\tau_{n+1}^{-1}(\rho_n|k_0) = \sigma|k_0$. Note that $(\rho_n\sigma^{-1})|k_0 = \tau_{n+1}$, and let ρ_{n+1} be an arbitrary extension of $\rho_n\sigma^{-1}$ in $H^{m_{n+1}}$. Now, put $\sigma_n = \rho_l|n$ if $0 \le n \le m_0$ and $l = 0$ or if $m_{l-1} < n \le m_l$ and $l > 0$.

We have $\sigma_{m_n}|k_0 = \rho_n|k_0 = \tau_n$. Also $(\sigma_{n+1}|n)^{-1}\sigma_n \in T$, i.e., (i), is easy to see. Put $S = \bigcup_{n\in\omega}\sigma_n(T \cap H^n)$. To check (ii), let $\sigma \in S \cap H^{k_0}$. Pick the unique $k \in \omega$ such that $\sigma \in \tau_k\big(p_{m_k}[T \cap H^{m_k}]\big) \setminus \tau_{k+1}\big(p_{m_{k+1}}[T \cap H^{m_{k+1}}]\big)$. Then for any $\sigma' \in S$ with $\sigma' \supset \sigma$, we have $lh\sigma' < m_{k+1}$. Otherwise, $\sigma' \in \sigma_n(T \cap H^n)$ for some $n \ge m_{k+1}$, whence $\sigma = p_n(\sigma') \in \tau_n(p_n[T \cap H^n])$, a contradiction. Thus $r_S(\sigma) < \omega$ for any $\sigma \in S \cap H^{k_0}$. It follows that S is well-founded and $ht(S) \le \omega + k_0$.

Lemma 15. *Let (H_n) be a sequence of countable groups. Then the following conditions are equivalent:*
(i) (H_n) admits well-founded coset trees of arbitrary height;
(ii) (H_n) admits coset trees of arbitrary height;
(iii) (H_n) admits group trees of arbitrary height.
Proof. (i)⇒(ii) is obvious.

(ii)⇒(iii). Note that if $S \subset T$ are coset trees and $S \neq T$, then $\alpha(S) \subset \alpha(T)$ and $\alpha(S) \neq \alpha(T)$. To see this, pick $k \in \omega$ such that $S \cap H^k \neq T \cap H^k$ and $\sigma \in S \cap H^k$. Then $\alpha(S) \cap H^k = \sigma^{-1}(S \cap H^k) \neq \sigma^{-1}(T \cap H^k) = \alpha(T) \cap H^k$.

Now, let S be a given coset tree. Define $\gamma = \min\{\min\{\xi : \exists k\ S^\xi \cap H^k = \emptyset\}, ht(S)\}$. Then, by Lemma 12(ii) and the above observation, we have $\alpha(S)^\xi = \alpha(S^\xi) \neq \alpha(S^\zeta) = \alpha(S)^\zeta$ for $\xi < \zeta < \gamma$, whence $ht(\alpha(S)) \ge \gamma$. But it is easy to see that $ht(S) < \gamma + \omega$. Thus (ii)⇒(iii) is proved.

(iii)⇒(i). Let T be a group tree of height $> \beta + \omega$. We show that there is a well-founded coset tree of height $\ge \beta$. To this end consider T^β. Then $ht(T^\beta) > \omega$.

Apply Lemma 14 to T^β to find $\sigma_n \in H^n$, $n \in \omega$, as in Lemma 14(i) and (ii). Put $S = \bigcup_{n\in\omega} \sigma_n(T \cap H^n)$. Then, by Lemma 13, S is a coset tree and $S^\beta = \bigcup_{n\in\omega} \sigma_n(T^\beta \cap H^n) \neq \emptyset$. By Lemma 14(ii), $S^{\beta+\omega\cdot 2} = \left(\bigcup_{n\in\omega} \sigma_n(T^\beta \cap H^n)\right)^{\omega\cdot 2} = \emptyset$. Thus S is a well-founded tree with $ht(S) \geq \beta$.

4.2.5. Group trees and algebraic properties of groups.

Lemma 16. *Let H be a countable group. If H is not torsion, it is not p-compact for any prime p.*

Proof. Clearly, if a subgroup of H is not p-compact, neither is H. Thus it is enough to show that $\mathbb{Z}$ is not p-compact. This is witnessed by the following sequence of subgroups of $\mathbb{Z}(p) \times \mathbb{Z}$:

$$G_k = \{\left(m(p+1)^k \text{ mod } p,\ m(p+1)^k\right) : m \in \mathbb{Z}\},\ \ k \in \omega.$$

Lemma 17. *Let (H_n) be a sequence of countable groups. If (H_n) admits group trees of arbitrary height, then there exist a prime p and infinitely many $n \in \omega$ such that H_n is not p-compact.*

Proof. If for infinitely many $n \in \omega$ H_n is not torsion, we are done by Lemma 16. Also, if (H_n) admits group trees of arbitrary height, so does $(H_n)_{n\geq N}$ for any $N \in \omega$. This follows from Lemma 15 as soon as we notice that if S is a coset tree on (H_n) and $\sigma \in H^N$, then S_σ is a coset tree on $(H_n)_{n\geq N}$, and that, given $\beta < \omega_1$, if $ht(S)$ is large enough, then $ht(S_\sigma) > \beta$ for some $\sigma \in H^N$. Thus, we can assume that H_n is torsion for each n, and that there exists a group tree on (H_n) of height $> \omega^2$.

Let T be a group tree on (H_n). Let p be a prime. Assume $\sigma \in T \cap H^n$, $r(\sigma) < \omega_1$, and the order of σ is a power of p. Let $\beta < r(\sigma)$. Then there is $\tau \supset \sigma$ such that $r(\tau) = \beta$ and the order of τ is a power of p. To see this, let $\tau' \supset \sigma$, $\tau' \neq \sigma$ and $r(\tau') \geq \beta$. Let $l \in \omega$ be such that p does not divide it and the order of $l\tau'$ is a power of p. Since the order of σ is a power of p, there is $l' \in \omega$ such that $l'l\sigma = \sigma$. Put $\tau_1 = l'l\tau'$. Note that $\tau_1 \supset \sigma$ and $\tau_1 \neq \sigma$. Since, for any $\gamma \in \omega_1$ and $m \in \omega$, $\{\tau \in T \cap H^m : r(\tau) \geq \gamma\}$ is a subgroup of H^m (this follows easily from the facts that $\{\tau \in T \cap H^m : r(\tau) \geq \gamma\} = T^\gamma \cap H^m$ and that T^γ is a group tree), $r(\tau_1) = r(l'l\tau') \geq r(\tau') \geq \beta$. If $r(\tau_1) = \beta$, we are done. If $r(\tau_1) > \beta$, we repeat the above construction and get $\tau_2 \supset \tau_1$, $\tau_2 \neq \tau_1$, whose order is a power of p and

$r(\tau_2) \geq \beta$. Again, if $r(\tau_2) = \beta$, we are done; otherwise we repeat the construction. Note that we cannot do it indefinitely, since then we would produce a sequence $\sigma \subset \tau_1 \subset \tau_2 \subset \cdots$, $\tau_m \neq \tau_{m+1}$, whence $r(\sigma) = \omega_1$, a contradiction. Thus we must obtain $\tau_m \supset \sigma$ such that $r(\tau_m) = \beta$ and the order of τ_m is a power of p.

Next, notice that if $\tau \in T \cap H^n$, $r(\tau)$ is a limit, and the order of τ is a power of p, p a prime, then H_n is not p-compact. Indeed, let γ_k, $k \in \omega$, be a strictly increasing sequence of ordinals tending to $r(\tau)$. Put $G_k = \{\sigma \in T \cap H^{n+1} : r(\sigma) \geq \gamma_k\}$. Let $\pi : H^{n+1} \to H^n$ be the projection. Notice that (G_k) is a decreasing sequence of subgroups of H^{n+1} and $\tau \in \bigcap_{k\in\omega} \pi[G_k] \setminus \pi[\bigcap_{k\in\omega} G_k]$. Let $C = \langle\tau\rangle$. Then $C < H^n$ and $C \simeq \mathbb{Z}(p^m)$ for some $m \in \omega$. Put $G'_k = G_k \cap (C \times H_n)$. Let $\phi : C \to \mathbb{Z}(p)$ be a surjective homomorphism. Let $\Phi = \phi \times id : C \times H_n \to \mathbb{Z}(p) \times H_n$. Since Φ is finite-to-1, $\Phi[\bigcap_{k\in\omega} G'_k] = \bigcap_{k\in\omega} \Phi[G'_k]$. Note also that $\pi' \circ \Phi = \phi \circ \pi$ where $\pi' : \mathbb{Z}(p) \times H_n \to \mathbb{Z}(p)$ is the projection. Thus

$$\phi\big[\pi[\bigcap_{k\in\omega} G'_k]\big] = \pi'\big[\bigcap_{k\in\omega} \Phi[G'_k]\big].$$

But $\pi[\bigcap_{k\in\omega} G'_k] \neq C$ whence $\pi[\bigcap_{k\in\omega} G'_k] \subset ker(\phi)$. Thus $\phi\big[\pi[\bigcap_{k\in\omega} G'_k]\big] = \{0\}$ and finally

$$\pi'\big[\bigcap_{k\in\omega} \Phi[G'_k]\big] = \{0\}.$$

On the other hand,

$$\bigcap_{k\in\omega} \pi'\big[\Phi[G'_k]\big] = \phi\big[\bigcap_{k\in\omega} \pi[G'_k]\big] = \mathbb{Z}(p).$$

Thus the decreasing sequence of groups $\Phi[G'_k]$, $k \in \omega$, witnesses that H_n is not p-compact.

Now, let T be a group tree on (H_n) with $ht(T) > \omega^2$. There exists a prime p and $\sigma \in T$ such that the order of σ is a power of p and $\omega^2 \leq r(\sigma) < \omega_1$. To show this, first find $\tau \in T$ with $r(\tau) = \omega^2$. The group $G = \langle\tau\rangle$ is cyclic and finite. Thus there are $\sigma_1, \sigma_2, \ldots, \sigma_m \in T \cap H^n$, $n = lh(\tau)$, which commute with each other, their orders are powers of distinct primes and $\tau = \sigma_0 \cdots \sigma_m$. Note that for each $0 \leq i \leq m$ there is $k \in \omega$ with $k\tau = \sigma_i$. Thus, since $\{\sigma \in T \cap H^n : r(\sigma) \geq \omega^2\}$ is a subgroup of H^n, $r(\sigma_i) \geq \omega^2$ for all $0 \leq i \leq m$. Also $\{\sigma \in T \cap H^n : r(\sigma) \geq \omega_1\}$ is a subgroup of H^n, thus there is i such that $r(\sigma_i) < \omega_1$, and we are done.

Now, fix the prime p and $\sigma \in T$ as above. Let $N \in \omega$. We show that there are more than N numbers n such that H_n is not p-compact. Indeed, we can recursively produce $\tau_0, \tau_1, \dots, \tau_N \in T$ so that $\sigma \subset \tau_0$ and $r(\tau_0) = \omega^2$, $\tau_i \subset \tau_{i+1}$, the order of each τ_i is a power of p, and $r(\tau_i) = \omega \cdot (N + 1 - i)$, $1 \leq i \leq N$. But then if we put $n_i = lh(\tau_i)$, we get $n_0 < n_1 < \cdots < n_N$ and H_{n_i} is not p-compact since ω^2 and $\omega \cdot (N+1-i)$, $1 \leq i \leq N$, are limit.

In the following lemma, we essentially find all abelian countable groups which are p-compact.

Lemma 18. *Let H be an abelian countable group. Let p be a prime. Then the following conditions are equivalent:*

(i) H is p-compact;

(ii) H is torsion, and the p-component of H is of the form $F \times \mathbb{Z}(p^\infty)^n$ where F is a finite p-group and $n \in \omega$;

(iii) H is torsion, and there is no surjective homomorphism mapping a subgroup of H onto $\bigoplus_\omega \mathbb{Z}(p)$.

Proof. (ii)⇒(i). Let $G_k < \mathbb{Z}(p) \times H$, $k \in \omega$, $G_{k+1} < G_k$, and $\pi[G_k] = \mathbb{Z}(p)$ where $\pi : \mathbb{Z}(p) \times H \to \mathbb{Z}(p)$ is the projection. Now, $H = H_p \times H'$ and $G_k = (G_k)_p \times G'_k$ where H_p and $(G_k)_p$ are the p-components of H and G_k, respectively, and the order of any element of H' or G'_k is not divisible by p [F, Thm. 8.4]. Clearly we have $(G_k)_p < \mathbb{Z}(p) \times H_p$. We say that a group fulfils the minimum condition if each strictly decreasing sequence of subgroups if finite. Since, as one can easily see, $\mathbb{Z}(p^\infty)$ and finite groups fulfil the minimum condition, and the property of fulfilling the minimum condition is preserved under taking finite products, $\mathbb{Z}(p) \times H_p$ fulfils the minimum condition. Thus there is $k_0 \in \omega$ such that $(G_k)_p = (G_{k_0})_p$ for $k \geq k_0$. But then

$$\begin{aligned} \pi[\bigcap_{k \in \omega} G_k] &= \pi[\bigcap_{k \in \omega} (G_k)_p \times \bigcap_{k \in \omega} G'_k] \supset \pi[(G_{k_0})_p \times \{0\}] \\ &= \pi[(G_{k_0})_p \times G'_{k_0}] = \pi[G_{k_0}] = \mathbb{Z}(p). \end{aligned}$$

(i)⇒(iii). By Lemma 16, H is torsion. Note that if F_1 can be mapped by a homomorphism onto F_2, F_1, F_2 groups, and F_2 is not p-compact, then F_1 is not p-compact either. Indeed, let $\phi : F_1 \to F_2$ be a surjective homomorphism, and let the sequence (G_k) of subgroups of $\mathbb{Z}(p) \times F_2$ witness that F_2 is not p-compact, then

$$G'_k = \{(m, g) \in \mathbb{Z}(p) \times F_1 : (m, \phi(g)) \in G_k\}$$

witness that F_1 is not p-compact. Thus to prove that H is not p-compact, assuming (iii) fails, it is enough to show that $\bigoplus_\omega \mathbb{Z}(p)$ is not p-compact. Let $\{e_i : i \in \omega\}$ be an independent set generating $\bigoplus_\omega \mathbb{Z}(p)$. Let us fix a sequence of sets $X_k \subset \omega$, $k \in \omega$, such that $X_{k+1} \subset X_k$ and $\bigcap_{k\in\omega} X_k = \emptyset$. Define $G_k < \mathbb{Z}(p) \times \bigoplus_\omega \mathbb{Z}(p)$ by

$$G_k = \langle\{(m, me_i) : i \in X_k, m \in \mathbb{Z}(p)\}\rangle.$$

Then (G_k) witnesses that $\bigoplus_\omega \mathbb{Z}(p)$ is not p-compact.

(iii)⇒(ii). Assume (iii). Let H_p the p-component of H. Let $H_p^1 = \bigcap_{n\in\omega} nH_p$ be its first Ulm group. If H_p/H_p^1 is infinite, then $H_p/H_p^1 \simeq \bigoplus_{m\in\omega} \mathbb{Z}(p^{n_m})$ for a sequence $n_m \in \omega \setminus \{0\}$ [F, Thm. 17.2 and remarks on p. 155]. Thus H_p/H_p^1, and hence H_p, can be mapped homomorphicaly onto $\bigoplus_\omega \mathbb{Z}(p)$. Therefore H_p/H_p^1 is finite. Put $F = H_p/H_p^1$. But then H_p^1 is divisible [F, Lemma 37.2] and $H_p \simeq F \times H_p^1$ [F, Thm. 21.2]. Now, by [F, Thm. 23.1], either $H_p^1 \simeq \mathbb{Z}(p^\infty)^n$, for some $n \in \omega$, and we are done, or $H_p^1 \simeq \bigoplus_\omega \mathbb{Z}(p^\infty)$. But in the latter case H_p^1, and hence H, contains an isomorphic copy of $\bigoplus_\omega \mathbb{Z}(p)$, a contradiction.

Remark. (In this remark the notation and terminology follow [F].) One can give other characterizations of p-compactness among countable torsion abelian groups. For example p-compactness of H is equivalent to the following conditions:
(iv) the p-component of H fulfils the minimum condition;
(v) for any finite p-group $F < H$ the p-rank of H/F is finite.
Obviously (ii)⇒(iv), and (iv)⇒(i) as in the proof of (ii)⇒(i). Now, assuming (iv) and noticing that a homomorphic image of a group fulfilling the minimum condition fulfils the minimum condition, we get that the p-component of H/F, $F < H$ finite, fulfils the minimum condition. This obviously implies that its p-rank is finite. Thus (iv)⇒(v). To see (v)⇒(ii), let H_p be the p-component of H. Let τ be its Ulm type. First note that if $\tau = \gamma + 1$, for some γ, then H_p^γ/H_p^τ is finite. Otherwise, $r_p(H_p^\gamma/H_p^\tau) = \infty$, and since $H_p \simeq H_p/H_p^\tau \times H_p^\tau$, we get $r_p(H_p) = \infty$. Now, we claim, that τ is neither a limit ordinal nor a succesor of a limit ordinal. Otherwise, using the above observation there is a sequence of groups $G_n < H_p/H_p^\tau$, $n \in \omega$, such that $G_{n+1} < G_n$, $G_{n+1} \neq G_n$ and

$\bigcap_{n\in\omega} G_n$ is finite. Put $G = \bigcap_{n\in\omega} G_n$. Then we can pick recursively $g_k \in H_p/H_p^\tau$ so that $pg_k \in G$ and for each k there is an n with $g_k \in G_n$ and $g_i \notin G_n$ for $i < k$. Then clearly the image of $\{g_k : k \in \omega\}$ under the natural homomorphism $H_p/H_p^\tau \to (H_p/H_p^\tau)/G$ is infinite independent. Again, since $H_p \simeq H_p/H_p^\tau \times H_p^\tau$, $r_p(H/G') = r_p(H_p/G') = \infty$ for some finite p-group G'. Next, notice that τ is not of the form $\gamma \dot{+} 2$ because in this case $H_p^{\gamma+1}/H_p^\tau$ is finite and $r_p(H_p^\gamma/H_p^{\gamma+1}) = \infty$ whence $r_p\big((H_p^\gamma/H_p^\tau)/(H_p^{\gamma+1}/H_p^\tau)\big) = \infty$. And as before $r_p(H/G') = \infty$ for some finite p-group G'. Thus $\tau \leq 1$, and if $\tau = 1$, then H_p/H_p^1 is finite. If $\tau = 0$, H_p is divisible, and since $r_p(H_p) < \infty$, there is $n \in \omega$ with $H_p \simeq \mathbb{Z}(p^\infty)^n$. If $\tau = 1$, put $F = H_p/H_p^1$. Then $H_p \simeq F \times H_p^1$, F finite, H_p^1 divisible. Since $r_p(H_p^1) < \infty$, there is $n \in \omega$ with $H_p^1 \simeq \mathbb{Z}(p^\infty)^n$.

Now, we make a technical definition useful in proving the existence of group trees of arbitrary height. An abelian countable group H is called *managable* if there exist two decreasing sequences of subgroups (G_n^0), (G_n^1) with $\bigcap_{n\in\omega} G_n^i = \{e\}$, for $i = 0, 1$, and a homomorphism $\phi:\ H \times H \to H$ such that $\phi[G_n^0 \times G_n^1] = H$ for any $n \in \omega$.

Lemma 19. *Let H be a countable abelian group. If H is managable, then (H_n), where $H_n = H$ for each $n \in \omega$, admits group trees of arbitrary height.*

Proof. Fix two decreasing sequences of subgroups (G_n^0) and (G_n^1) and a homomorphism ϕ as in the definition of managability. For each ordinal $\beta < \omega_1$, we produce a group tree T_β such that:

– if $\beta = \gamma + 1$, then $T_\beta^\gamma \cap H = H$ and $\forall h \in H\ (h \neq e \Rightarrow (T_\beta)_h$ is well-founded);

– if β is limit, then $\forall \gamma < \beta\ \exists n \in \omega\ (T_\beta^\gamma \cap H^2 \supset G_n^0 \times G_n^1)$ and $\forall \sigma \in H^2\ (\sigma \neq (e,e) \Rightarrow (T_\beta)_\sigma$ is well-founded).

Then clearly $\omega_1 > r_{T_\beta}(h) \geq \beta$ for any $h \in H \setminus \{e\}$ in the first case, and for any $\gamma < \beta$, $\omega_1 > r_{T_\beta}(\sigma) \geq \gamma$ for some $\sigma \in H^2 \setminus \{(e,e)\}$ in the latter. Thus $ht(T_\beta) \geq \beta$ for any $\beta \in \omega_1$.

Put $T_0 = \{\vec{e}\}$ and $T_1 = H \cup \{\vec{e}\}$. Assume T_γ has been defined for all $\gamma < \beta$. If $\beta = \gamma + 1$ and γ is a successor, put

$$T_\beta = \{\emptyset\} \cup H \cup \{\sigma(0) * \sigma :\ \sigma \in T_\gamma, lh\sigma \geq 1\}.$$

If $\beta = \gamma + 1$ and γ is a limit, put

$$T_\beta = \text{ the tree generated by } \{\phi(\sigma(0), \sigma(1)) * \sigma : \ \sigma \in T_\gamma, lh\sigma \geq 2\}.$$

Checking that the T_β's work is straightforward. Now, assume β is a limit ordinal. Note that it is enough to construct two group trees S_0 and S_1 such that there is an increasing sequence $\gamma_n \to \beta$ with $S_0^{\gamma_n} \cap H \supset G_n^0$ and $S_1^{\gamma_n} \cap H \supset G_n^1$ and $\forall h \in H \ (h \neq e \Rightarrow (S_0)_h$ and $(S_1)_h$ are well-founded). If S_0 and S_1 are defined, let

$$T_\beta = \{\sigma \in H^{<\omega} : \ \sigma|\{2k : \ k \in \omega\} \in S_0 \text{ and } \sigma|\{2k+1 : \ k \in \omega\} \in S_1\}.$$

We will define a group tree $S = S_0$ as above; the construction of S_1 is analogous. Put $G_n^0 = G_n$. Fix an increasing sequence of successors $\gamma_n \to \beta$, $n \in \omega$. Find pairwise disjoint infinite sets X_n, $n \in \omega$, with $\bigcup_{n\in\omega} X_n = \omega$. Let

$$R_n = \{\emptyset\} \cup \{h * \sigma : h \in G_n, \sigma|X_n \in T_{\gamma_n}, \sigma|(\omega \setminus X_n) \subset \vec{e}, \text{ and}$$
$$\text{if } lh\sigma > \min X_n, \text{ then } h = (\sigma|X_n)(0)\}.$$

Note that each R_n is a group tree. Define

$$S = \bigcup_{k\in\omega} \langle H^k \cap \bigcup_{n\in\omega} R_n \rangle.$$

Easily S is a group tree. To see $S^{\gamma_n} \cap H \supset G_n$, just notice that, for each $h \in G_n$, $r_{T_{\gamma_n}}(h) \geq \gamma_n$, and there is a monotone 1-to-1 mapping $\psi : \ (T_{\gamma_n})_h \to S$ defined by $\psi(\sigma) = h*\tau$, where $\tau \in H^{<\omega}$ is maximal such that $\tau|X_n = h*\sigma$ and $\tau|(\omega\setminus X_n) \subset \vec{e}$. To show that $(S)_h$ is well-founded for $h \in H \setminus \{e\}$, fix $h \in H$ with $h \neq e$, and assume towards a contradiction that $h * x$ is an infinite branch through S for some $x \in H^\omega$. Find $n \in \omega$ with $h \notin G_n$. Let $k \in \omega$ be such that $k \cap X_i \neq \emptyset$ for $i \in n$. Put $\tau = x|k$ and $n_i = \min X_i$ for $i \in n$. If $\tau(n_{i_0}) \neq e$ for some $i_0 \in n$, notice that $x|X_{i_0}$ is an infinite branch through $T_{\gamma_{i_0}}$ with $(x|X_{i_0})(0) \neq e$ which contradicts the inductive assumption. Thus we can assume that $\tau(n_i) = e$ for all $i \in n$. Then, since the R_i's are group trees, $h * \tau = \sigma \cdot \prod_{i\in n}(h_i * \tau_i)$ for some $\sigma \in G_n \times H^k$ with $\sigma(n_i) = e$ and some $h_i * \tau_i \in R_i \cap H^{k+1}$. By the definition of R_i, $h_i = \tau_i(n_i) = \tau(n_i) = e$. Thus $h = \sigma(0) \in G_n$, a contradiction.

Lemma 20. *Let (H_n) be a sequence of countable groups. Then (H_n) admits group trees of arbitrary height if either of the following conditions holds.*

(i) There exists a sequence $n_0 < n_1 < \cdots$ *such that* (H_{n_k}) *admits group trees of arbitrary height.*

(ii) For each n, G_n *is a homomorphic image of a subgroup of* H_n, *and* (G_n) *admits group trees of arbitrary height.*

Proof. (i) Let T be a group tree on (H_{n_k}). Define $\bar{T}$ a group tree on (H_n) as follows

$$\sigma \in \bar{T} \quad \text{iff} \quad \sigma|X \in T \text{ and } \sigma|(\omega \setminus X) = \vec{e}|(\omega \setminus X)$$

where $X = \{n_k : k \in \omega\}$. Then $ht(\bar{T}) \geq ht(T)$.

(ii) Fix $H'_n < H_n$ and surjective homomorphisms $\phi_n : H'_n \to G_n$. Let T be a group tree on (G_n). Define $\bar{T}$ a group tree on (H_n) as follows

$$\sigma \in \bar{T} \quad \text{iff} \quad \forall k < lh\sigma \ \big(\sigma(k) \in H'_n \text{ and } \big(\phi_0(\sigma(0)), \cdots, \phi_k(\sigma(k))\big) \in T\big).$$

Then $ht(\bar{T}) \geq ht(T)$.

Lemma 21. *Let* (H_n) *be a sequence of countable abelian groups. Then* (H_n) *does not admit group trees of arbitrary height iff* H_n *is torsion for all but finitely many* n*, and for each prime* p*, for all but finitely many* n *the* p*-component of* H_n *is of the form* $F \times \mathbb{Z}(p^\infty)^k$*, where* F *is a finite* p*-group,* $k \in \omega$.

Proof. The implication $\Leftarrow$ follows from Lemmas 8 and 9. To see $\Rightarrow$, assume the conclusion does not hold. Then either there exist infinitely many n such that H_n contains an isomorphic copy of $\mathbb{Z}$ or, by Lemma 18, there exist a prime p and infinitely many n such that a subgroup of H_n can be mapped homomorphically onto $\bigoplus_\omega \mathbb{Z}(p)$. Thus, by Lemma 20, it is enough to show that (H_n), where $H_n = \mathbb{Z}$ for each n or $H_n = \bigoplus_\omega \mathbb{Z}(p)$ for each n, admits group trees of arbitrary height.

By Lemma 19, it suffices to prove that $\mathbb{Z}$ and $\bigoplus_\omega \mathbb{Z}(p)$ are managable. For $\mathbb{Z}$, put $G^0_n = \langle 2^n \rangle$, $G^1_n = \langle 3^n \rangle$. Define $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ by $\phi(m, l) = m + l$. For $\bigoplus_\omega \mathbb{Z}(p)$, fix an infinite independent set $\{e_i : i \in \omega\}$ generating $\bigoplus_\omega \mathbb{Z}(p)$. Find a decreasing sequence of nonempty sets $X_n \subset \omega$, $n \in \omega$, such that $\bigcap_{n \in \omega} X_n = \emptyset$. Put $G^0_n = \langle \{e_i : i \in X_n\} \rangle$ and $G^1_n = \{e\}$. Fix a function $f : \omega \to \omega$ so that, for any $n, m \in \omega$, $f^{-1}(m) \cap X_n \neq \emptyset$. Define $\phi' : \bigoplus_\omega \mathbb{Z}(p) \to \bigoplus_\omega \mathbb{Z}(p)$ to be the unique homomorphism extending $\phi'(e_i) = e_{f(i)}$. Let $\phi : \bigoplus_\omega \mathbb{Z}(p) \times \bigoplus_\omega \mathbb{Z}(p) \to \bigoplus_\omega \mathbb{Z}(p)$ be the composition of the projection to the first coordinate with ϕ'.

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CHAPTER 5
ON HAAR NULL SETS

Let G be a Polish abelian group. Christensen [C] calls a universally measurable set $A \subset G$ *Haar null* if there exists a probability Borel measure μ on G such that $\mu(g + A) = 0$ for all $g \in G$. It was proved in [C] that in case G is locally compact a universally measurable set is Haar null iff it is of Haar measure zero. Also, the union of a countable family of Haar null sets is Haar null, i.e., Haar null sets constitute a σ-ideal. One of the first questions asked by Christensen in [C] was whether any family of mutually disjoint, universally measurable sets which are not Haar null is countable, as is the case when the group is Polish locally compact. This was answered in the negative by Dougherty [D] who constructed such uncountable families, for example, in all infinite dimensional Banach spaces. (Haar null sets are called "shy" in [D] following the terminology of [HSY].) This gives rise to the question whether the existence of such uncountable families characterizes non-locally-compact, Polish, abelian groups. We prove that this is indeed the case, i.e., a Polish, abelian group is not locally compact iff there exists an uncountable family of universally measurable or, equivalently, closed, pairwise disjoint sets which are not Haar null. We also consider the problem of approximating sets modulo Haar null sets. We show that in each non-locally-compact, Polish, abelian group there exists an analytic set A such that $A \triangle B$ is not Haar null for any co-analytic set B; but each analytic Haar null set is contained in a Borel Haar null set. (This last statement answers a question of Dougherty [D, p.86].) Additionally, we prove that for any $\alpha < \omega_1$ there exists $A \in \mathbf{\Sigma}^0_\alpha$ such that $A \triangle B$ is not Haar null for any $B \in \mathbf{\Pi}^0_\alpha$.

The definition of Haar null sets was extended by Topsøe and Hoffmann-Jørgensen [TH-J] and Mycielski to all Polish groups. A universally measurable set $A \subset G$ is said to be Haar null if there exists a Borel probability measure μ such that $\mu(gAh) = 0$ for all $g, h \in G$. Haar null sets are still closed under countable unions and coincide with Haar measure zero sets in locally compact groups. We prove all our results for Polish groups which admit an invariant metric. (A metric d on G is *invariant* if $d(g_1hg_2, g_1kg_2) = d(h,k)$ for any $g_1, g_2, h, k \in G$.) This class of groups contains properly all Polish, abelian groups, since each metric group G admits a left-invariant metric which, obviously, is invariant when G is abelian. Any invariant metric on a Polish group is automatically complete.

By $cl(A)$ we denote the closure of A. $\mathbb{N}$ stands for the set of all natural numbers (and $0 \in \mathbb{N}$) and $2^{\mathbb{N}}$ for the countable infinite product of $\{0,1\}$ with the product topology. By $\mathbb{N}^n$ or 2^n, for $n \in \mathbb{N}$, we denote the set of all sequences of elements of $\mathbb{N}$ or $\{0,1\}$, respectively, of length n indexed by $\{0,\dots,n-1\}$, and by $\mathbb{N}^{\mathbb{N}}$ the set of all infinite sequences of elements of $\mathbb{N}$. Put also $\mathbb{N}^{<\mathbb{N}} = \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$. If α is a sequence, by $\alpha|n$, for some $n \in \mathbb{N}$, we denote the sequence $(\alpha(0),\dots,\alpha(n-1))$; in particular, $\alpha|0 = \emptyset$. If $\sigma \in \mathbb{N}^n$, $m \in \mathbb{N}$, $\sigma * m$ denotes the unique $\tau \in \mathbb{N}^{n+1}$ such that $\tau|n = \sigma$ and $\tau(n) = m$.

First, we prove the following purely topological theorem.

Theorem 1. *Assume G is a Polish, non-locally-compact group admitting an invariant metric. Then there exists a closed set $F \subset G$ and a continuous function $\phi : F \to 2^{\mathbb{N}}$ such that for any $x \in 2^{\mathbb{N}}$ and any compact set $K \subset G$ there is $g \in G$ with $gK \subset \phi^{-1}(x)$.*

Proof. A family $\mathcal{A}$ of subsets of G is called *discrete* if each $g \in G$ has an open neighborhood intersecting at most one member of $\mathcal{A}$. Let d be an invariant metric on G; d is complete.

Claim. *Let $U \subset G$ be open and nonempty. There exist $g_n \in U$ and open U_n, $n \in \mathbb{N}$, such that $U_n \subset U_{n+1}$, $\bigcup_n U_n = G$, and the family $\{g_nU_n : n \in \mathbb{N}\}$ is discrete.*

Fix an increasing sequence of finite sets (Q_n) such that $\bigcup_n Q_n$ is dense in G. Find $\delta > 0$ and an infinite set $D \subset U$ whose points are at distance at least δ from each other. For any finite sets $A, B \subset G$ there is $g \in D$ such that $d(gA,\ B) \geq \delta/2$. If not, then for any $g \in D$ there are $a \in A$ and $b \in B$ with $d(ga, b) < \delta/2$. But then there exist distinct $g, g' \in D$ with the same pair a, b; hence $d(g,g') = d(ga, g'a) \leq d(ga,b) + d(b, g'a) < \delta$, contradicting $d(g,g') \geq \delta$. Thus, we can inductively choose $g_n \in D$ so that $d(g_nQ_n,\ \bigcup_{i<n} g_iQ_i) \geq \delta/2$. Let $W = \{g \in G :\ d(e,g) < \delta/5\}$. Put $U_n = Q_nW$. Then $\bigcup_n U_n = (\bigcup_n Q_n)W = G$ since $\bigcup_n Q_n$ is dense, and, by invariance of d, $\{g_nU_n : n \in \mathbb{N}\}$ is discrete, which finishes the proof of the claim.

For $\sigma \in \mathbb{N}^{<\mathbb{N}}$, define $g_\sigma \in G$ and $V_\sigma \subset G$ open so that:

(i) $V_\emptyset = G$;

(ii) if $m < n$, then $V_{\sigma * m} \subset V_{\sigma * n}$;

(iii) $\bigcup_m V_{\sigma * m} = G$;

(iv) $\{g_{\sigma * m} V_{\sigma * m} : m \in \mathbb{N}\}$ is discrete;

(v) if $\sigma \in \mathbb{N}^n$, $n \geq 1$, then $d(g_{\sigma * m}, e) < 2^{-\sigma(n-1)}$ for each $m \in \mathbb{N}$.

Put $g_\emptyset = e$ and $V_\emptyset = G$. For $\sigma \in \mathbb{N}^n$, let $U = \{g : d(g, e) < 2^{-\sigma(n-1)}\}$ if $n \geq 1$ and $U = G$ if $n = 0$. Find $g_m \in U$ and $U_m \subset G$ open with the properties as in the Claim. Put $g_{\sigma * m} = g_m$ and $V_{\sigma * m} = U_m$.

Let $W_\sigma = \bigcap_{k \leq n} g_{\sigma|k} V_{\sigma|k}$ for $\sigma \in \mathbb{N}^n$. Put

$$F = \bigcap_n \bigcup_{\sigma \in \mathbb{N}^n} cl(W_\sigma).$$

Notice that for each n the family $\{W_\sigma : \sigma \in \mathbb{N}^n\}$ is discrete, whence

$$\bigcup_{\sigma \in \mathbb{N}^n} cl(W_\sigma) = cl(\bigcup_{\sigma \in \mathbb{N}^n} W_\sigma);$$

therefore, F is closed. For $x \in 2^{\mathbb{N}}$, put

$$F_x = \bigcap_n \bigcup \{cl(W_\sigma) : \sigma \in \mathbb{N}^n \text{ and } \sigma(i) \text{ is even iff } x(i) = 1 \text{ for } i < n\}.$$

Then $F_{x_0} \cap F_{x_1} = \emptyset$ if $x_0 \neq x_1$. Indeed, assume $x_0(n) = 0$ and $x_1(n) = 1$ for some n. Since $\{W_\sigma : \sigma \in \mathbb{N}^{n+1}\}$ is discrete, $cl(W_\sigma) \cap cl(W_{\sigma'}) = \emptyset$ if $\sigma, \sigma' \in \mathbb{N}^{n+1}$, $\sigma \neq \sigma'$. Thus

$$\bigcup \{cl(W_\sigma) : \sigma \in \mathbb{N}^{n+1} \text{ and } \sigma(n) \text{ is even}\} \cap \bigcup \{cl(W_\sigma) : \sigma \in \mathbb{N}^{n+1} \text{ and } \sigma(n) \text{ is odd}\} = \emptyset.$$

But

$$F_{x_0} \subset \bigcup \{cl(W_\sigma) : \sigma \in \mathbb{N}^{n+1} \text{ and } \sigma(n) \text{ is odd}\},$$

while

$$F_{x_1} \subset \bigcup \{cl(W_\sigma) : \sigma \in \mathbb{N}^{n+1} \text{ and } \sigma(n) \text{ is even}\}.$$

Note also that $\bigcup_{x\in 2^{\mathbb{N}}} F_x = F$. Now define $\phi : F \to 2^{\mathbb{N}}$ by letting $\phi(g)$ be equal to the unique x with $g \in F_x$. To prove that ϕ is continuous, it is enough to see that preimages of basic clopen subsets of $2^{\mathbb{N}}$ are closed. But for $\tau \in 2^n$, $n \in \mathbb{N}$, we have

$$\phi^{-1}(\{x \in 2^{\mathbb{N}} : x|n = \tau\})$$
$$= F \cap \bigcup\{cl(W_\sigma) : \sigma \in \mathbb{N}^n \text{ and } \sigma(i) \text{ even iff } \tau(i) = 1 \text{ for } i < n\}.$$

And again, since $\{W_\sigma : \sigma \in \mathbb{N}^n\}$ is discrete,

$$\bigcup\{cl(W_\sigma) : \sigma \in \mathbb{N}^n \text{ and } \sigma(i) \text{ is even iff } \tau(i) = 1 \text{ for } i < n\}$$

is closed.

Let $K \subset G$ be compact. We want to show that for any $x \in 2^{\mathbb{N}}$ there is $g \in G$ with $gK \subset F_x$. For simplicity of notation we will only find $g \in G$ such that $gK \subset F$. It will be clear from the proof that the same argument applies to each F_x in place of F. We will produce $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that for each $n \in \mathbb{N}$

(a) $g_{\alpha|n} g_{\alpha|n-1} \cdots g_\emptyset K \subset V_{\alpha|n+1}$;

(b) $\sum_{i \geq n+2} d(g_{\alpha|i}, e) < d(g_{\alpha|n} g_{\alpha|n-1} \cdots g_\emptyset K,\ G \setminus V_{\alpha|n+1})$.

Then by (b), $\prod_i g_{\alpha|i}$ exists, since d is complete. By (a), (b), and the invariance of d, $(\prod_i g_{\alpha|i})K \subset g_{\alpha|n+1} V_{\alpha|n+1}$ for each n. Thus, since $V_\emptyset = G$, $(\prod_i g_{\alpha|i})K \subset \bigcap_n g_{\alpha|n} V_{\alpha|n} \subset F$, and we are done.

Assume $\alpha|n$ has been defined. By (ii) and (iii), there is $\delta > 0$ such that for all m large enough

$$(*) \qquad g_{\alpha|n} g_{\alpha|n-1} \cdots g_\emptyset K \subset V_{\alpha|n*m} \text{ and } d(g_{\alpha|n} g_{\alpha|n-1} \cdots g_\emptyset K,\ G \setminus V_{\alpha|n*m}) > \delta.$$

Also, by (v), given $\epsilon > 0$ for m large enough we have $d(g_{\alpha|n*m*k}, e) < \epsilon$ for all k. Thus, we can pick an m so that $(*)$ holds, and for each k

$$d(g_{\alpha|n*m*k}, e) < (1/2^{n+1}) \min\{\delta,\ \min\{d(g_{\alpha|i} g_{\alpha|i-1} \cdots g_\emptyset K,\ G \setminus V_{\alpha|i+1}) : i \leq n-1\}\}$$

with the convention $\min \emptyset = \infty$. Put $\alpha|n+1 = \alpha|n * m$. This finishes the proof of the theorem.

Corollary 2. *Let G be a Polish group admitting an invariant metric. Then each family of universally measurable or, equivalently, closed, pairwise disjoint sets which are not Haar null is countable iff G is locally compact.*

Proof. ($\Leftarrow$) If G is locally compact, Haar null sets coincide with sets of Haar measure zero, see [C] and [TH-J]. Since G is Polish, Haar measure is σ-finite.

($\Rightarrow$) Assume G is not locally compact. Since for any Borel probability measure on G there is a compact set of positive measure, it follows that the sets $\phi^{-1}(x)$, $x \in 2^{\mathbb{N}}$, from the Theorem are not Haar null.

Proposition 3. *Let G be a Polish group.*
(i) If $A \subset G$ is analytic and Haar null, then there exists a Borel set $B \subset G$ which is Haar null and $A \subset B$.
(ii) Assume that G is not locally compact and admits an invariant metric. Then there exists an analytic set A such that for no co-analytic set B $A \triangle B$ is Haar null. For any $\alpha < \omega_1$ there exists $A \in \mathbf{\Sigma}^0_\alpha$ such that for no $B \in \mathbf{\Pi}^0_\alpha$ $A \triangle B$ is Haar null.

Proof. If $Z \subset X \times Y$, then, as usual, $Z_x = \{y \in Y : (x, y) \in Z\}$ for $x \in X$.

(i) Let A be analytic and Haar null. Let μ be a probability Borel measure witnessing it. Then the family of sets

$$\Phi = \{X \subset G : X \in \mathbf{\Sigma}^1_1 \text{ and } \forall g_1, g_2 \in G \ \mu(g_1 X g_2) = 0\}$$

is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, i.e., for any $\mathbf{\Sigma}^1_1$ set $P \subset Y \times G$, Y a Polish space, the set $\{y \in Y : P_y \in \Phi\}$ is $\mathbf{\Pi}^1_1$. To check this, let $P \subset Y \times G$ be $\mathbf{\Sigma}^1_1$, Y Polish. Define $\widetilde{P} \subset G \times G \times Y \times G$ by $(g_1, g_2, y, g) \in \widetilde{P}$ iff $g \in g_1 P_y g_2$. Then $\widetilde{P} \in \mathbf{\Sigma}^1_1$. It follows from [K, Theorem 29.26] that $\{(g_1, g_2, y) : \mu(\widetilde{P}_{(g_1,g_2,y)}) = 0\}$ is $\mathbf{\Pi}^1_1$, whence so is

$$\{y \in Y : \forall g_1, g_2 \in G \ \mu(\widetilde{P}_{(g_1,g_2,y)}) = 0\} = \{y \in Y : P_y \in \Phi\}.$$

Now, since $A \in \Phi$, by (the dual form of) the First Reflection Theorem, see [K, Theorem 35.10 and the remarks following it], there exists a Borel set B with $B \supset A$ and $B \in \Phi$, so B is as required.

(ii) Let F and $\phi : F \to 2^{\mathbb{N}}$ be as in the Theorem. The argument below is essentially the same as Balcerzak's argument in the proof of Lemma 2.1 from [B]. Let $\Lambda =$ co-analytic sets or $\Lambda = \mathbf{\Pi}^0_\alpha$ for some $\alpha < \omega_1$. Let $U \subset 2^{\mathbb{N}} \times G$ be universal for $\Lambda | G$, i.e., $U \in \Lambda$ and $\{B \subset G : B \in \Lambda\} = \{U_x : x \in 2^{\mathbb{N}}\}$. Put $A = (G \setminus F) \cup \bigcup_{x \in 2^{\mathbb{N}}} (\phi^{-1}(x) \setminus U_x)$. Note that $A = (G \setminus F) \cup \{g \in F : (\phi(g), g) \notin U\}$ whence, since ϕ is continuous and F is closed, $G \setminus A \in \Lambda$. Also, for any $x \in 2^{\mathbb{N}}$, we have $A \triangle U_x \supset \phi^{-1}(x)$. Thus, $A \triangle B$ is not Haar null for any $B \in \Lambda$.

Remark. Proposition (i) can also be deduced from a theorem of Dellacherie. If μ witnesses that an analytic set A is Haar null, put $\tilde{\mu}(X) = \sup\{\mu^*(gXh) : g, h \in G\}$, where $X \subset G$ and μ^* is the outer measure induced by μ. Then it is easy to check that $\tilde{\mu}$ is what is called in [De] a caliber. Thus, since $\tilde{\mu}(A) = 0$, by [De, Theorem 2.4], there exists a Borel set $B \supset A$ with $\tilde{\mu}(B) = 0$, i.e., $\mu(gBh) = 0$ for any $g, h \in G$.

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CHAPTER 6

DECOMPOSING BOREL SETS AND FUNCTIONS AND THE STRUCTURE OF BAIRE CLASS 1 FUNCTIONS

6.1. Introduction

All spaces considered are metric separable and are denoted usually by the letters X, Y, or Z. If a metric separable space is additionally complete, we call it Polish; if it is a continuous image of ω^ω or, equivalently, of a Polish space, it is called Souslin.

In the first part of the paper our main concern is to determine how difficult it is to represent a Borel set as a union of simpler Borel sets or the graph of a Borel function as a union of the graphs of simpler Borel functions. Using Effective Descriptive Set Theory, in particular Louveau's theorem, we show that if $A \subset X$, X Polish, is Borel, then $A \in \mathbf{\Sigma}^0_\alpha$ or there is a continuous injection $\phi : \omega^\omega \to A$ such that $\phi^{-1}(B)$ is meager for any $B \subset A$ which is $\mathbf{\Sigma}^0_\alpha$. This gives a new proof of J. Stern's result that if a Borel set A is the union of $< \text{cov}(\mathcal{M})$ sets in $\mathbf{\Sigma}^0_\alpha$, then A is itself $\mathbf{\Sigma}^0_\alpha$. ($\text{cov}(\mathcal{M})$ is the smallest cardinality of a family of meager sets covering $\mathbb{R}$.) We prove similar results for functions. Put, for $f: X \to Y$ and a family of functions $\mathcal{G}$,

$$dec(f, \mathcal{G}) = \min\{|\mathcal{F}| : \bigcup \mathcal{F} = X,\ \forall Z \in \mathcal{F}\ f|Z \in \mathcal{G}\}.$$

Let $\mathbf{B}_\alpha$ stand for the family of functions on the α's level of the Baire hierarchy. We show, e.g., that given $f: X \to Y$ Borel, X Polish, either $dec(f, \mathbf{B}_\alpha) \leq \omega$ or there is a continuous injection $\phi : \omega^\omega \to X$ such that $\phi^{-1}(A)$ is meager for any $A \subset X$ with $f|A \in \mathbf{B}_\alpha$; thus $dec(f, \mathbf{B}_\alpha) \geq \text{cov}(\mathcal{M})$. These results imply that the decomposition coefficients defined in [CMPS] and proved there to be $> \omega$ are actually $\geq \text{cov}(\mathcal{M})$.

In the second part, we apply some of the ideas of the first part to study Baire class 1 functions. The structure of Baire class 1 functions was recently extensively studied in a number of papers, see e.g. [KL], [R]. We prove two dichotomy results of the following form: a Baire class 1 function "decomposes" into countably many continuous functions or "contains" a very complicated function. Two kinds of decompositions will be considered: decomposition into continuous functions with closed domains (considered first by Jayne and Rogers [JR]) and into continuous functions with arbitrary domains (first considered by Lusin); thus, a function $f: X \to Y$ will be regarded as simple in the first sense if $X = \bigcup_n X_n$, $n \in \omega$, each X_n is closed and $f|X_n$ is continuous, and it will be simple in the latter sense if $X = \bigcup_n X_n$, $n \in \omega$, and $f|X_n$ is continuous for each n. To define containment between functions, put for $g: X_1 \to Y_1$ and $f: X_2 \to Y_2$

$$g \sqsubseteq f \text{ iff } \exists \phi: X_1 \to X_2,\ \psi: g[X_1] \to Y_2 \text{ embeddings with } \psi \circ g = f \circ \phi.$$

Now, we identify the functions which will be contained in each complicated with respect to a decomposition Baire class 1 function. For the decomposition into continuous functions with closed domains the functions are modeled on the well-known Lebesgue's example of an increasing function on $[0,1]$ which is continuous exactly at all irrational points; for the decomposition into continuous functions with arbitrary domains the function is the so-called Pawlikowski's function defined in [CMPS]. Here are the precise definitions.

Definition of Lebesgue's functions L and L_1. Let Q be the set of all points in 2^ω which are eventually equal to 1. For each $x \in Q$ fix a number $a_x > 0$ so that
1) if $x, y \in Q$, $x \neq y$, then $a_x \neq a_y$;
2) $a_x < 1/3^{n_0}$, where n_0 is the smallest natural number such that $x(n) = 1$ for $n \geq n_0$.
Let $H: 2^\omega \to [0,1]$ be the well-known embedding $H(x) = \sum_{n=0}^{\infty} x(n)/3^{n+1}$. Let L, $L_1: 2^\omega \to \mathbb{R}$ be defined by

$$L(x) = \begin{cases} H(x), & \text{if } x \notin Q; \\ H(x) + a_x, & \text{if } x \in Q; \end{cases}$$

and

$$L_1(x) = \begin{cases} 0, & \text{if } x \notin Q; \\ a_x, & \text{if } x \in Q. \end{cases}$$

Definition of Pawlikowski's function P. Let $\omega + 1$ have the natural, order topology. Let $P: (\omega+1)^\omega \to \omega^\omega$ be defined by $P(\eta) = \gamma$, $\eta \in (\omega+1)^\omega$, where for $n \in \omega$

$$\gamma(n) = \begin{cases} 0, & \text{if } \eta(n) = \omega; \\ \eta(n) + 1, & \text{if } \eta(n) < \omega. \end{cases}$$

Finally, we can formulate the results. Let $f: X \to Y$ be Baire class 1, X Souslin. Then either $X = \bigcup_n X_n$, $n \in \omega$, X_n closed and $f|X_n$ continuous, or $L \sqsubseteq f$ or $L_1 \sqsubseteq f$; also, either $X = \bigcup_n X_n$, $n \in \omega$, and $f|X_n$ continuous, or $P \sqsubseteq f$. The first part of the above sentence sharpens a result of Jayne and Rogers from [JR]. An interesting feature of the second part is that its proof uses Effective Desriptive Set Theory even though its statement mentions only functions on the first level of Baire hierarchy.

Further, it turns out that L, L_1, and P are as complicated as any other Baire class 1 function with respect to the decomposition into continuous functions with closed domains, in case of L and L_1, and with arbitrary domains, in case of P; thus, the above dichotomy results are in a sense best possible. Put

$$dec_c(f) = \min\{|\mathcal{F}| : \bigcup \mathcal{F} = X,\ \forall Z \in \mathcal{F}\ Z \text{ is closed and } f|Z \text{ is continuous}\}$$

and

$$dec(f) = \min\{|\mathcal{F}| : \bigcup \mathcal{F} = X,\ \forall Z \in \mathcal{F}\ f|Z \text{ is continuous}\},$$

i.e., $dec(f) = dec(f, \mathbf{B}_0)$. Note that if $g \sqsubseteq f$, then clearly $dec_c(g) \leq dec_c(f)$ and $dec(g) \leq dec(f)$. By a result of Cichoń and Morayne [CM],

$$\sup\{dec_c(f) : f : X \to Y,\ X \text{ Souslin},\ f \text{ Baire class 1}\} \leq \mathbf{d},$$

where $\mathbf{d}$ is the smallest cardinality of a dominating subset of ω^ω. We prove that $dec_c(L) = dec_c(L_1) = \mathbf{d}$. Thus indeed L and L_1 are as complicated as any other Baire class 1 function as far as decomposing into continuous functions with closed domains is concerned, i.e., $dec_c(L) = dec_c(L_1) \geq dec_c(f)$ for any Baire class 1 function f. We prove an analogous result for P. Put

$$\mathbf{dec} = \sup\{dec(f) : f : X \to Y,\ X \text{ Souslin},\ f \text{ Baire class 1}\}.$$

We show that $dec(P) = \mathbf{dec}$. (This answers two questions of Steprāns [St, Q.7.1 and Q.7.2].) Thus combinig the above results, we get that for any $f: X \to Y$ Baire class 1, X Polish, we have $dec_c(f) \leq \omega$ or $dec_c(f) = \mathbf{d}$, and $dec(f) \leq \omega$ or $dec(f) = \mathbf{dec}$. The equality $dec(P) = \mathbf{dec}$ also gives, via the work of Steprāns, an interesing characterization of $\mathbf{dec}$ as the covering coefficient of a certain combinatorially defined σ-ideal on ω^ω. (It is known that $\text{cov}(\mathcal{M}) \leq \mathbf{dec} \leq \mathbf{d}$, [CMPS], and that it is consistent that $\text{cov}(\mathcal{M}) < \mathbf{dec}$, Steprāns [St], and $\mathbf{dec} < \mathbf{d}$, Shelah-Steprāns [SS].)

In order to prove $dec(P) = \mathbf{dec}$, we define and study complete semicontinuous functions. A lower semicontinuous (lsc) function $F : X \to [0,1]$ is called *lsc complete* if each lsc function $f : 2^\omega \to [0,1]$ can be obtained as $F \circ \phi$ for some continuous $\phi : 2^\omega \to X$. Using a Wadge-type game, we give an internal characterization of lsc complete functions as those lsc functions $F : X \to [0,1]$ for which there is a $\mathbf{\Pi}^0_2$ set $D \subset X$ such that $0 \in F[D]$ and for any open set U, $F[U \cap D]$ is of the form $\{y \in [0,1] : y \geq y_0\}$ or $\{y \in [0,1] : y > y_0\}$ for some $y_0 \in [0,1]$. Also, we prove the existence of "minimal" lsc complete functions. We give a new proof of the inequality $\mathbf{dec} \geq \text{cov}(\mathcal{M})$, first established in [CMPS], by showing that $dec(f) \geq \text{cov}(\mathcal{M})$ for any lsc complete f.

If X is a compact, metric space, let $K(X)$ denote the space of all closed subsets of X with the Hausdorff metric. A particular attention has been devoted to the fact that the restriction of the Lebesgue measure to $K([0,1])$ provides a natural example of a complicated usc function [JM, vMP]. We apply some of the results mentioned above to Borel measures on compact metric spaces X viewed as usc functions on $K(X)$. Using the characterization of complete lsc functions, we show that any Borel, probability, nonatomic measure on a compact metric space is usc complete. In fact, we prove a more general version of this result for capacities. This generalizes van Mill and Pol's result for the Lebesgue measure [vMP]. Also, we use the theorem that $\mathbf{dec} = dec(P)$ to characterize probability, Borel measures μ on a compact metric space X for which $dec(\mu) = \mathbf{dec}$, e.g., if X does not have isolated points, then $dec(\mu) = \mathbf{dec}$ unless μ is a finite, convex combination of Dirac measures. This generalizes the result of Jackson and Mauldin that $dec(\lambda) > \omega$, where λ is the Lebesgue measure [JM].

6.2. Decomposing Borel sets and functions into simpler Borel sets and functions

By $\mathbf{B}_\alpha$, $\alpha < \omega_1$, we denote the αth class of the Baire hierarchy of real functions, i.e., for $f : X \to Y$, $f \in \mathbf{B}_\alpha$ if for any $U \subset Y$ open $f^{-1}(U)$ is $\mathbf{\Sigma}^0_{1+\alpha}$ in X. In particular, $\mathbf{B}_0$ is the class of continuous functions. (Note that the enumeration of the $\mathbf{B}_\alpha$'s starts with $\alpha = 0$ while that of the $\mathbf{\Sigma}^0_\alpha$'s with $\alpha = 1$.) Also define $f : X \to \mathbb{R}$ to be in $\mathbf{L}_\alpha$ ($\mathbf{U}_\alpha$, respectively) if $f^{-1}((r, \infty))$ ($f^{-1}((-\infty, r))$, respectively) is $\mathbf{\Sigma}^0_{1+\alpha}$ in X for all $r \in \mathbb{R}$. Thus $\mathbf{L}_0$, $\mathbf{U}_0$ are the classes of lower and upper semicontinuous functions, respectively. By a classical theorem of Lebesgue and Hausdorff a real function is in $\mathbf{B}_\beta$ iff it is a pointwise limit of a sequence of functions from $\bigcup_{\alpha<\beta} \mathbf{B}_\alpha$, and it is in $\mathbf{L}_\beta$ ($\mathbf{U}_\beta$, respectively) iff it is a pointwise limit of an increasing (decreasing, respectively) sequence of functions from $\mathbf{B}_\beta$. Let $\mathrm{cov}(\mathcal{M})$ be the smallest cardinality of a family of meager sets covering $\mathbb{R}$. Recall that the Gandy-Harrington topology on a recursively presented Polish space is the topology generated by all Σ^1_1 sets and that it is strong Choquet. (See [HKL] for some background on the Gandy-Harrington topology.) We will refer to the Gandy-Harrington topology as the G-H topology and sets open with respect to it will be called G-H open.

We say that a set D *separates* A *and* B if $A \subset D$ and $D \cap B = \emptyset$. We will use the following theorem due to Louveau (see [L1]):

Let A_0, A_1 be Σ^1_1 sets such that for some $D \in \mathbf{\Pi}^0_\alpha$, $1 \leq \alpha < \omega_1^{CK}$, $A_0 \subset D$ and $A_1 \cap D = \emptyset$ modulo sets meager in the Gandy-Harrington topology. Then A_0 and A_1 can be separated by a set from $\Pi^0_\alpha(\Delta^1_1)$.

Let $\mathcal{A}$ be a family of subsets of a Polish space X. Let $C \subset X$, and let $\alpha < \omega_1$. We say that

– $\mathcal{A}$ is $\mathbf{\Pi}^0_\alpha$ *on* C iff $\forall A \in \mathcal{A}\ \exists D \in \mathbf{\Pi}^0_\alpha\ A \cap C \subset D \subset A$;

– $\mathcal{A}$ is *relatively* $\mathbf{\Pi}^0_\alpha$ *on* C iff $\forall A \in \mathcal{A}\ \exists D \in \mathbf{\Pi}^0_\alpha\ A \cap C \subset D \cap C \subset A$ (i.e., $A \cap C = D \cap C$).

Lemma 2.1. *Let $\mathcal{A}_1$, $\mathcal{A}_2$ be countable families of Borel subsets of a Polish space X, and let $1 < \beta < \omega_1$. Then precisely one of the following two possibilities holds.*
(i) $X = \bigcup_n C_n$ and, for each $n \in \omega$, $\mathcal{A}_1$ or $\mathcal{A}_2$ is $\mathbf{\Pi}^0_\alpha$ (relatively $\mathbf{\Pi}^0_\alpha$, respectively) on C_n for some $\alpha < \beta$;
(ii) There is a continuous injection $\phi : \omega^\omega \to \bigcup \mathcal{A}_1 \cap \bigcup \mathcal{A}_2$ such that if $\mathcal{A}_1$ or $\mathcal{A}_2$ is $\mathbf{\Pi}^0_\alpha$ (relatively $\mathbf{\Pi}^0_\alpha$, respectively) on C for some $\alpha < \beta$, then $\phi^{-1}(C)$ is meager.

Proof. We will prove the statements for "$\mathbf{\Pi}^0_\alpha$ on C" and "relatively $\mathbf{\Pi}^0_\alpha$ on C" simultaneously. Let $\mathcal{A}$ be a countable family of Borel subsets of X. Fix $A \subset X \times \omega$ such that

(1) $$\mathcal{A} = \{\{x \in X : (x, n) \in A\} : n \in \omega\}.$$

Since the argument below relativizes, we can assume that X is a recursively presented Polish space, $\beta < \omega_1^{CK}$, and $A \in \Delta_1^1$. Note that for $C \subset X$

$$\mathcal{A} \text{ is } \mathbf{\Pi}^0_\alpha \text{ on } C \text{ iff } (C \times \omega) \cap A \text{ and } (X \times \omega) \setminus A \text{ can be separated by a } \mathbf{\Pi}^0_\alpha \text{ set,}$$

and also

$$\mathcal{A} \text{ is relatively } \mathbf{\Pi}^0_\alpha \text{ on } C \text{ iff}$$
$$(C \times \omega) \cap A \text{ and } (C \times \omega) \setminus A \text{ can be separated by a } \mathbf{\Pi}^0_\alpha \text{ set.}$$

Let $\Phi : \mathcal{P}(X) \to \mathcal{P}(X)$ denote either the identity function, or the constant function $\Phi(C) = X$ for all $C \in \mathcal{P}(X)$. Put

$$P^\Phi = \{C \subset X : C \in \Sigma^1_1 \text{ and}$$
(2) $$(C \times \omega) \cap A \text{ and } (\Phi(C) \times \omega) \setminus A \text{ can be separated by a set in } \bigcup_{\alpha<\beta} \mathbf{\Pi}^0_\alpha\}.$$

Claim 1. *$\bigcup P^\Phi$ is Π^1_1.*

If $C \in P^\Phi$, then, by Louveau's theorem, there is $D \in \Pi^0_\alpha(\Delta^1_1)$, for some $\alpha < \beta$, which separates the Σ^1_1 sets $(C \times \omega) \cap A$ and $(\Phi(C) \times \omega) \setminus A$. Put

$$C' = \{x \in X : \forall n \, (x, n) \in (D \cap A) \cup ((X \times \omega) \setminus (D \cup A))\}.$$

Then $C \subset C'$, $C' \in \Sigma^1_1$, and, as is easy to see, D separates $(C' \times \omega) \cap A$ and $(\Phi(C') \times \omega) \setminus A$, i.e., $C' \in P^\Phi$. Thus

$$\bigcup P^\Phi = \{x \in X : \exists D \in \bigcup_{\alpha<\beta} \Pi^0_\alpha(\Delta^1_1) \, \forall n \, (x, n) \in (D \cap A) \cup ((X \times \omega) \setminus (D \cup A))\}$$

which is Π^1_1.

Below in this proof all topological notions—meager, G_δ, etc.—refer to the Gandy-Harrington topology.

Claim 2. *Let $C \subset X$ be such that $(C \times \omega) \cap A$ and $(\Phi(C) \times \omega) \setminus A$ can be separated by a set from $\bigcup_{\alpha<\beta} \mathbf{\Pi}^0_\alpha$. Then there are $C_n \in P^\Phi$, $n \in \omega$, such that $C \setminus \bigcup_n C_n$ is meager.*

There exist $C_n \in \Sigma^1_1$, $n \in \omega$, such that $C_n \setminus C$ does not contain a nonmeager set with the Baire property, for each n, and $C \setminus \bigcup_n C_n$ is meager. Let $D \in \bigcup_{\alpha<\beta} \mathbf{\Pi}^0_\alpha$ separate $(C \times \omega) \cap A$ and $(\Phi(C) \times \omega) \setminus A$. Note that $(C_n \times \omega) \cap A \subset D$ and

$((\Phi(C_n) \times \omega) \setminus A) \cap D = \emptyset$ modulo meager sets. Thus, by Louveau's theorem, for each $n \in \omega$ there is a set in $\bigcup_{\alpha<\beta} \Pi^0_\alpha(\Delta^1_1)$ which separates $(C_n \times \omega) \cap A$ and $(\Phi(C_n) \times \omega) \setminus A$. Therefore $C_n \in P^\Phi$.

Let A_1, A_2 and P_1^Φ, P_2^Φ be defined as in (1) and (2) for $\mathcal{A} = \mathcal{A}_1$ and $\mathcal{A} = \mathcal{A}_2$, respectively. If $\bigcup P_1^\Phi \cup \bigcup P_2^\Phi \supset \bigcup \mathcal{A}_1 \cap \bigcup \mathcal{A}_2$, then actually $\bigcup P_1^\Phi \cup \bigcup P_2^\Phi = X$, as $X \setminus \bigcup \mathcal{A}_i = X \setminus \{x \in X : \exists n\, (x,n) \in A_i\} \in P_i^\Phi$, $i = 1, 2$, whence (i) holds. If not, put

$$E_1 = \bigcup \mathcal{A}_1 \cap \bigcup \mathcal{A}_2 \setminus \bigcup P_1^\Phi \cup \bigcup P_2^\Phi.$$

By Claim 1, E_1 is a nonempty Σ^1_1 set. If $C \subset X$ is such that $(C \times \omega) \cap A_i$ and $(\Phi(C) \times \omega) \setminus A_i$ can be separated by a $\mathbf{\Pi}^0_\alpha$ set, for some $\alpha < \beta$, then, by Claim 2, $C \cap E_1$ is meager. Note that $(\{x\} \times \omega) \cap A_i$ and $(\Phi(\{x\}) \times \omega) \setminus A_i$ can be separated by a $\mathbf{\Pi}^0_1$ set for any $x \in X$; thus E_1 does not have isolated points. Let $\{B_n : n \in \omega\}$ be a countable basis of E_1. Put $E_2 = E_1 \setminus \bigcup_n (\overline{B}_n \setminus B_n)$. Then E_2 is a dense G_δ in E_1, whence it is strong Choquet (see [HKL, Proposition 2.1(iii)]). Since it is clearly regular and has countable basis, it is Polish by Choquet's theorem. Moreover, since E_2 does not have isolated points, we can find a dense G_δ subset of E_2 homeomorphic to ω^ω. This finishes the proof of the lemma.

Theorem 2.2. *Let X be a Polish space, and let $1 < \beta < \omega_1$. Let $A \subset X$ be Borel. Then either $A \in \mathbf{\Sigma}^0_\beta$, or there is a continuous injection $\phi : \omega^\omega \to A$ such that for any $C \subset A$, $C \in \mathbf{\Sigma}^0_\beta$, $\phi^{-1}(C)$ is meager in ω^ω.*

Proof. Let $\mathcal{A}_1 = \mathcal{A}_2 = \{A\}$. If $X = \bigcup_n C_n$ and, for each $n \in \omega$, $\mathcal{A}_1$ is $\mathbf{\Pi}^0_\alpha$ on C_n for some $\alpha < \beta$, then A is $\mathbf{\Sigma}^0_\beta$. Otherwise, from Lemma 2.1, we get a continuous injection ϕ as required.

I was informed by A. Miller that the following corollary was proved by Stern [Sr, Theorem 3.2]. Stern's proof is different from the one presented here and uses Steel's forcing. Also, [BD, Theorem 2] contains a similar but weaker result. The corollary immediately follows from Theorem 2.2 if $\alpha > 1$ and is trivial if $\alpha = 1$.

Corollary 2.3. *Let A be a Borel set in a Polish space and let $1 \leq \beta < \omega_1$. Assume A is the union of $<$ cov($\mathcal{M}$) sets in $\mathbf{\Sigma}^0_\beta$. Then $A \in \mathbf{\Sigma}^0_\beta$.*

Theorem 2.4. *Let X be a Polish space, and let $1 \leq \beta < \omega_1$. Let $\mathcal{G}$ be one of the* *following*

$$\bigcup_{\alpha<\beta} \mathbf{B}_\alpha, \ \bigcup_{\alpha<\beta} \mathbf{L}_\alpha, \ \bigcup_{\alpha<\beta} \mathbf{U}_\alpha, \ \bigcup_{\alpha<\beta} \mathbf{L}_\alpha \cup \mathbf{U}_\alpha.$$

Let $f : X \to \mathbb{R}$ be a Borel function. Then either $dec(f, \mathcal{G}) \leq \omega$, or there is a continuous injection $\phi : \omega^\omega \to X$ such that if $f|C \in \mathcal{G}$, then $\phi^{-1}(C)$ is meager, so $dec(f, \mathcal{G}) \geq cov(\mathcal{M})$.

Proof. Put $\mathcal{A}_1 = \mathcal{A}_2 = \{f^{-1}(\mathbb{R} \setminus V_n) : n \in \omega\}$, where $\{V_n : n \in \omega\}$ is a countable topological basis of $\mathbb{R}$. Note, that $\mathcal{A}_1$ is relatively $\mathbf{\Pi}^0_{1+\alpha}$ on $C \subset X$ iff $f|C \in \mathbf{B}_\alpha$. Thus, an application of Lemma 2.1 similar to the one in Theorem 2.2 gives the conclusion for $\mathcal{G} = \bigcup_{\alpha<\beta} \mathbf{B}_\alpha$. To obtain it for $\mathcal{G} = \bigcup_{\alpha<\beta} \mathbf{L}_\alpha$, $\mathcal{G} = \bigcup_{\alpha<\beta} \mathbf{U}_\alpha$, and $\mathcal{G} = \bigcup_{\alpha<\beta} \mathbf{L}_\alpha \cup \mathbf{U}_\alpha$ apply a similar argument respectively to the families $\mathcal{A}_1 = \mathcal{A}_2 = \{f^{-1}((-\infty, q]) : q \in \mathbb{Q}\}$, $\mathcal{A}_1 = \mathcal{A}_2 = \{f^{-1}([q, \infty)) : q \in \mathbb{Q}\}$, and $\mathcal{A}_1 = \{f^{-1}((-\infty, q]) : q \in \mathbb{Q}\}$, $\mathcal{A}_2 = \{f^{-1}([q, \infty)) : q \in \mathbb{Q}\}$.

It was proved in [CMPS, Corollary 3.3] that $dec(f, \bigcup_{\alpha<\beta} \mathbf{L}_\alpha \cup \mathbf{U}_\alpha) > \omega$, for some $f \in \mathbf{B}_\beta$ and also [CMPS, Theorem 5.7] that $dec(f, \mathbf{L}_0 \cup \mathbf{U}_0) \geq \text{cov}(\mathcal{M})$ for some $f \in \mathbf{B}_1$. Laczkovich showed that for any $\beta < \omega_1$ there is $f \in \mathbf{L}_\beta$ with $dec(f, \mathbf{B}_\beta) > \omega$ (see [CM] for a proof); and by [CMPS, Theorem 5.6] there is $f \in \mathbf{L}_0$ with $dec(f, \mathbf{B}_0) \geq \text{cov}(\mathcal{M})$. The next corollary improves on these results. Let me first mention, however, that Steprāns established in [St] the consistency with ZFC of the existence of $f \in \mathbf{L}_0$ such that $dec(f, \mathbf{B}_0) > \text{cov}(\mathcal{M})$.

Corollary 2.5. *Let X be Polish uncountable.*
(i) For each $1 \leq \beta < \omega_1$ there exists $f : X \to \mathbb{R}$, $f \in \mathbf{B}_\beta$, with $dec(f, \bigcup_{\alpha<\beta} \mathbf{L}_\alpha \cup \mathbf{U}_\alpha) \geq cov(\mathcal{M})$.
(ii) For each $\beta < \omega_1$ there exists $f : X \to \mathbb{R}$, $f \in \mathbf{L}_\beta$, such that $dec(f, \mathbf{B}_\beta) \geq cov(\mathcal{M})$.

Proof. By [CMPS, Corollary 3.3], there exists $f : X \to \mathbb{R}$, $f \in \mathbf{B}_\beta$ such that $dec(f, \bigcup_{\alpha<\beta} \mathbf{L}_\alpha \cup \mathbf{U}_\alpha) > \omega$. Thus (i) follows from Theorem 2.4. To prove (ii), use the fact that there is $f : X \to \mathbb{R}$, $f \in \mathbf{L}_\beta$ such that $dec(f, \mathbf{B}_\beta) > \omega$ [CM, Corollary 3.4] and apply Theorem 2.4.

Remarks. 1. By the proof of Theorem 4.8 from [CMPS], for $\beta < \omega_1$ and any

$f \in \mathbf{B}_{\beta+1}$ there is $g \in \mathbf{L}_\beta$ such that $dec(g, \mathbf{B}_\beta) \geq dec(f, \mathbf{L}_\beta \cup \mathbf{U}_\beta)$. Thus (ii) in our Corollary 2.5 actually follows from (i).

2. I do not know whether the method employed here can be used to show that the more subtle decomposition coefficients studied by Morayne in [M] are also $\geq \mathrm{cov}(\mathcal{M})$. Perhaps the refined version of Louveau's theorem from [L2] can be of some help.

6.3. Decomposing Baire class 1 functions into continuous functions with closed domains

In [JR, Theorem 1] Jayne and Rogers proved that for any function $f: X \to Y$, X Souslin, either there are closed sets $X_n \subset X$, $n \in \omega$, such that $\bigcup_n X_n = X$ and $f|X_n$ is continuous, or there is an F_σ set $A \subset Y$ such that $f^{-1}(A)$ is not F_σ. The next result—the first dichotomy theorem for Baire class 1 functions—sharpens Jayne and Rogers's theorem and, perhaps, provides an explanation why it is true. (For a derivation of [JR, Theorem 1] from Theorem 3.1 see the remark following the proof of Theorem 3.1.)

Theorem 3.1. *Let $f: X \to Y$ be Baire class 1, X Souslin. Then precisely one of the following holds.*
(i) There are closed sets $X_n \subset X$, $n \in \omega$, such that $\bigcup_n X_n = X$ and $f|X_n$ is continuous.
(ii) $L \sqsubseteq f$ or $L_1 \sqsubseteq f$.

We will need a few auxiliary notions. For a sequence of sets $A_k \subset X$, $k \in \omega$, and $x \in X$, we write $A_k \to x$ if each A_k is nonempty and for any $\epsilon > 0$ $A_k \subset B(x, \epsilon)$ for k large enough. A function $f: X \to Y$ is *strongly discontinuous at* $x \in X$ if there exist a sequence of open sets $V_k \subset X$ and an open set $U \subset Y$ such that $V_k \to x$, $f(x) \in U$ and $f[V_k] \cap U = \emptyset$. A point $x \in X$ is *f-isolated* if there is an open set $U \subset Y$ such that $f^{-1}(U) = \{x\}$.

First, we give characterizations of L and L_1.

Lemma 3.2. *Let $g: 2^\omega \to Y$. Assume each $x \in Q$ is g-isolated, g is continuous at each $x \in 2^\omega \setminus Q$, and given $\epsilon > 0$ $osc(g, x) < \epsilon$ for all but finitely many points in Q.*
(i) If $g|(2^\omega \setminus Q)$ is an embedding, then $\psi_0 : L[2^\omega] \to Y$ given by $\psi_0(L(x)) = g(x)$

is a well-defined embedding, and $\psi_0 \circ L = g$.

(ii) If $g|(2^\omega \setminus Q)$ *is constant, then* $\psi_1 : L_1[2^\omega] \to Y$ *given by* $\psi_1(L_1(x)) = g(x)$ *is a well-defined embedding, and* $\psi_1 \circ L_1 = g$.

Proof. (i) Since $g|(2^\omega \setminus Q)$ is 1-to-1 and each $x \in Q$ is g-isolated, g is 1-to-1. Also L is 1-to-1, thus ψ_0 is well-defined and 1-to-1. Let $L(x_n) \to L(x)$ and $L(x_n) \neq L(x)$. Clearly $x \in 2^\omega \setminus Q$ and $x_n \to x$. Since x is a continuity point of g, $g(x_n) \to g(x)$. Thus, $\psi_0(L(x_n)) \to \psi_0(L(x))$; whence ψ_0 is continuous. Assume $g(x_n) \to g(x)$. Since each $x \in Q$ is g-isolated, $x \in 2^\omega \setminus Q$. Since $\mathrm{osc}(g, x_n) \to 0$, we can find $z_n \in 2^\omega \setminus Q$ such that $d(z_n, x_n) \to 0$ and $d(g(z_n), g(x_n)) \to 0$. Thus $g(z_n) \to g(x)$. Since $g|2^\omega \setminus Q$ is an embedding, $z_n \to x$, whence $x_n \to x$. Thus $L(x_n) \to L(x)$, i.e., $\psi_0^{-1}(g(x_n)) \to \psi^{-1}(g(x))$; whence ψ is an embedding.

(ii) If $L_1(x) = L_1(y)$, then $x, y \in 2^\omega \setminus Q$ or $x = y$, so $g(x) = g(y)$. Thus, ψ_1 is well-defined. Note that if $L_1(x) \neq L_1(y)$, then $x \neq y$ and $x \in Q$ or $y \in Q$. Since each element of Q is g-isolated, $g(x) \neq g(y)$. Thus ψ_1 is 1-to-1. Let $L_1(x_n) \to L_1(x)$ and $L_1(x_n) \neq L_1(x)$. Then clearly $x_n \in Q$ and $x \in 2^\omega \setminus Q$. Since $\mathrm{osc}(g, x_n) \to 0$, there are $z_n \in 2^\omega \setminus Q$ with $d(g(x_n), g(z_n)) \to 0$. But $g(z_n) = g(x)$. Thus $g(x_n) \to g(x)$, so $\psi(L_1(x_n)) \to \psi(L_1(x))$. So ψ is continuous. Since $L_1[2^\omega]$ is compact, ψ is an embedding.

Lemma 3.3. *Let* $f : X \to Y$. *Assume the sets of all continuity and of all discontinuity points of* f *are both dense. For* $\epsilon > 0$ *let* S_ϵ *be the set of all strong discontinuity points at which the oscilation of* f *is* $< \epsilon$. *Then for any* $\emptyset \neq U \subset X$ *open* $f[U \cap S_\epsilon]$ *is infinite.*

Proof. Let S be the set of all strong discontinuity points of f. Note that if there is a sequence $x_n \to x$, x_n are continuity points of f and $f(x_n) \not\to f(x)$, then $x \in S$. To see this, find first a subsequence (x_{n_k}) of (x_n) and an open set $V \subset Y$ such that $f(x) \in V$ and $f(x_{n_k}) \notin V$. Since each x_{n_k} is a continuity point, we can find open sets $W_k \ni x_{n_k}$ and an open set $V' \subset V$ such that $f(x) \in V'$ and $f[W_k] \cap V' = \emptyset$. By making W_k small in diameter, we ensure that $W_k \to x$.

Now, we show that S is dense. Let $\emptyset \neq U \subset X$ be open. Let $x \in U$ be a discontinuity point of f. Let $x_n \in U$ and $V \subset Y$ open be such that $x_n \to x$, $f(x) \in V$, and $f(x_n) \notin V$. Let y_k^n, $n, k \in \omega$, be continuity points of f such that

$y^n_k \in U$ and $y^n_k \to x_n$. If for some n $f(y^n_k) \not\to f(x_n)$, then $x_n \in S$. If for all n $f(y^n_k) \to f(x_n)$, then we can choose a "diagonal" sequence $y^n_{k_n}$ so that $y^n_{k_n} \to x$ and $f(y^n_{k_n}) \not\to f(x)$, so $x \in S$. In any case, $S \cap U \neq \emptyset$.

Let $\emptyset \neq U \subset X$ be open. We construct by induction a sequence $x_n \in S \cap U$ such that $f(x_n) \neq f(x_m)$ if $n \neq m$. Let $x_0 \in S \cap U$. Since $x_0 \in S$, there is $\emptyset \neq V_0 \subset U$ with $f(x_0) \notin f[V_0]$. Let $x_1 \in S \cap V_0$. Find $\emptyset \neq V_1 \subset V_0$ open so that $f(x_1) \notin f[V_1]$. Let $x_2 \in S \cap V_1$. Continuing this procedure, we obtain a sequence (x_n) as required. Thus, for any $\emptyset \neq U \subset X$ open $f[S \cap U]$ is infinite. Since $S_\epsilon = S \cap \{x \in X : \operatorname{osc}(f, x) < \epsilon\}$ and $\{x \in X : \operatorname{osc}(f, x) < \epsilon\}$ is dense, as it contains all continuity points of f, and obviously open, we also have that $f[S_\epsilon \cap U]$ is infinite.

Lemma 3.4. *Let $f : X \to Y$, X Polish, be Baire class 1. Assume that the set of all discontinuity points of f is dense. Then there is a compact perfect set $K \subset X$ and a countable set $D \subset K$ such that*

(i) D is dense in K;

(ii) each $x \in D$ is $f|K$-isolated;

(iii) given $\epsilon > 0$ $\operatorname{osc}(f|K, x) < \epsilon$ for all but finitely many points in D.

Proof. Fix $\phi : \omega \to \omega$ such that $\phi(n) \leq n$ and $\forall n \exists^\infty k \; n = \phi(k)$. We construct sequences $F_n \subset X$ closed and $q_n \in X$ so that

1) $F_{n+1} \subset F_n$;

2) $\{q_k : k \leq n\} \subset F_n$;

3) $\forall x \in F_n \exists k \leq n \; d(x, q_k) \leq 1/(n+1)$;

4) $d(q_{n+1}, q_{\phi(n)}) \leq 1/(n+1)$;

5) q_n is $f|F_n$-isolated;

6) $\operatorname{osc}(f|F_n, q_n) < 1/(n+1)$;

7) $\operatorname{int}(F_n)$ is dense in F_n.

We will put $K = \bigcap_n F_n$ and $D = \{q_n : n \in \omega\}$. K is clearly closed and by 3) totally bounded, whence compact. By 2), $D \subset K$, and by 3) D is dense in K. By 4), as $\forall n \exists^\infty k \; n = \phi(k)$, D is dense-in-itself; thus K is perfect. Since $K \subset F_n$, each q_n is $f|K$-isolated by 5), and $\operatorname{osc}(f|K, q_n) < 1/(n+1)$ by 6).

Since f is Baire class 1, continuity points of f are dense in X; thus, we can apply Lemma 3.3. Let $q_0 \in S_1$. There is $V \subset Y$ open and a sequence of open sets $V_k \subset X$ such that $V_k \to q_0$, $V_k \subset B(q_0, 1)$, $f(q_0) \in V$, and $f[V_k] \cap V = \emptyset$. Put $F_0 = \{q_0\} \cup \bigcup_k \overline{V}_k$. Assume F_n and q_k, $k \leq n$, has been defined. Let $\emptyset \neq U \subset B(q_{\phi(n)}, 1/(n+2)) \cap F_n$ be open. (This is possible by 7).) By Lemma 3.3, find $p_0, \ldots, p_{n+1} \in U \cap S_{1/(n+1)}$ so that $f(p_i) \neq f(p_j)$ if $i \neq j$. Let $W_i \subset Y$ be open such that $f(p_i) \in W_i$ and $\overline{W}_i \cap \overline{W}_j = \emptyset$ if $i \neq j$. For each $k \leq n$ there is at most one $i \leq n+1$ such that $f^{-1}(\overline{W}_i)$ is comeager in $W \cap F_n$ for some open $W \ni q_k$. Thus, by the pigeonhole principle, there is $i_0 \leq n+1$ such that for each $k \leq n$ $X \setminus f^{-1}(\overline{W}_{i_0})$ is not meager in any neighborhood of q_k in F_n. But $X \setminus f^{-1}(\overline{W}_{i_0})$ is F_σ, so using 7), we can find $V_m^k \subset (X \setminus f^{-1}(\overline{W}_{i_0})) \cap F_n$, $n \in \omega$, open and such that $V_m^k \subset B(q_k, 1/(n+2))$, $V_m^k \to q_k$. By the choice of the p_i's and by making W_{i_0} smaller if neccessary, we can find $V_m \subset B(p_{i_0}, 1/(n+2)) \cap (X \setminus f^{-1}(W_{i_0})) \cap F_n$ open with $V_m \to p_{i_0}$. Put $q_{n+1} = p_{i_0}$ and

$$F_{n+1} = \{q_k : k \leq n+1\} \cup \bigcup_{k \leq n} \bigcup_m \overline{V_m^k} \cup \bigcup_m \overline{V}_m.$$

All the requirements 1)-7) are easy to check.

The following lemma is certainly well-known.

Lemma 3.5. *Let $f : \omega^\omega \to Y$ be continuous. Then there is a closed, non-σ-bounded set $H_1 \subset \omega^\omega$ such that $f|H_1$ is constant, or there exists a closed, non-σ-bounded set $H_2 \subset \omega^\omega$ such that $f|H_2$ is an embedding.*

Proof. Case 1. $\exists U \subset \omega^\omega$ open, nonempty and such that $f[U]$ is finite.
Then, since f is continuous, there is $\emptyset \neq V \subset U$ open and such that $f|V$ is constant. Put $H_1 = \overline{V'}$ for some open nonempty V' with $\overline{V'} \subset V$.

Case 2. $\forall U \subset \omega^\omega$ open, nonempty, $f[U]$ is infinite.
Define recursively $\sigma_s \in \omega^{<\omega}$, $s \in \omega^{<\omega}$, so that
1) $s \subset t \Rightarrow \sigma_s \subset \sigma_t$ and $s \perp t \Rightarrow \sigma_s \perp \sigma_t$;
2) $\{f[N_{\sigma_{s*n}}] : n \in \omega\}$ is a discrete family;
3) $diam(f[N_{\sigma_s}]) \leq 1/(lh(s)+1)$;
4) $\{\sigma_{s*n}(l) : n \in \omega\}$ is infinite, where $l = lh(\sigma_s)$.
Assume σ_s is defined. Let $l = lh(\sigma_s)$. Since for each $p \in \omega$ $f[N_{\sigma_s * p}]$ is infinite, we can find a sequence $x_n \in N_{\sigma_s}$, $n \in \omega$, such that $x_{n_1}(l) \neq x_{n_2}(l)$ and $f(x_{n_1}) \neq f(x_{n_2})$ if $n_1 \neq n_2$. We can assume that $\{f(x_n) : n \in \omega\}$ is a discrete set.

Now using continuity of f, we easily find σ_{s*n}, $n \in \omega$, so that $x_n \in N_{\sigma_{s*n}}$, $diam(f[N_{\sigma_{s*n}}]) \leq 1/(l+2)$, $lh(\sigma_{s*n}) > l$, and $\{f[N_{\sigma_{s*n}}] : n \in \omega\}$ is discrete.

Put $H_2 = \{x \in \omega^\omega : \exists^\infty s \; x|lh(\sigma_s) = \sigma_s\}$.

Proof of Theorem 3.1. Let $\mathcal{F}$ be the family of all closed sets $F \subset X$ such that $f|F$ is continuous. It follows from [S, Theorem 1] that either X can be covered by countably many members of $\mathcal{F}$, i.e., we get (i), or there is $X' \subset X$ which is Polish in the relative topology and X' cannot be covered by countably many sets from $\mathcal{F}$. Thus, we can assume that X is Polish and that (i) fails.

By a transfinite derivation process, we produce an ordinal $\alpha < \omega_1$ and a descending transfinite sequence of closed sets F_ξ, $\xi < \alpha$, so that

1) $f|(\bigcap_{\gamma<\xi} F_\gamma \setminus F_\xi)$ is continuous for all $\xi < \alpha$;

2) the set of discontinuity points of $f|F$ is dense in F, where $F = \bigcap_{\xi<\alpha} F_\xi$.

Case 1. $F = \emptyset$

Then since $\bigcap_{\gamma<\xi} F_\gamma \setminus F_\xi$ is F_σ, we can easily find countably many closd sets X_n, $n \in \omega$, so that $\bigcup_n X_n = X$ and $f|X_n$ is continuous which contradicts our assumption.

Case 2. $F \neq \emptyset$

Let $K \subset F$ and $D \subset K$ be as in Lemma 3.4. (We apply it to $f|F$.) We can assume that $X = K$. Since continuity points of f constitute a dense G_δ and no point in D is a continuity point of f, by Hurewicz's theorem, we can find an embedding $\phi_1 : 2^\omega \to K$ so that $x \in Q \Rightarrow \phi_1(x) \in D$ and $x \notin Q \Rightarrow \phi_1(x)$ is a continuity point of f. Consider $g = f \circ \phi_1|(2^\omega \setminus Q)$; g is continuous. We identify $2^\omega \setminus Q$ with ω^ω. Then $H \subset \omega^\omega$ is non-σ-bounded iff there is no G_δ set G such that $G \cap H = \emptyset$ and $Q \subset G$. Let $H \subset \omega^\omega$ be closed, non-σ-bouned such that either $g|H$ is constant or $g|H$ is an embedding (Lemma 3.5). Again by Hurewicz's theorem, there is an embedding $\phi_2 : 2^\omega \to 2^\omega$ such that $x \in Q \Rightarrow \phi_2(x) \in Q$ and $x \notin Q \Rightarrow \phi_2(x) \in H$. Put $\phi = \phi_1 \circ \phi_2$. Then clearly

a) $x \in Q \Rightarrow x$ is $f \circ \phi$-isolated;

b) $x \notin Q \Rightarrow x$ is a continuity point of $f \circ \phi$;

c) given $\epsilon > 0$ $\text{osc}(f \circ \phi, x) < \epsilon$ for all but finitely many $x \in Q$.

We now have two subcases.

Subcase 1. $f \circ \phi|(2^\omega \setminus Q)$ is constant.

Subcase 2. $f \circ \phi|(2^\omega \setminus Q)$ is an embedding.

An application of Lemma 3.2 in each of these subcases finishes the proof.

Remark. To derive [JR, Theorem 1] from Theorem 3.1 combine the following three obvious facts: (1) if f is not Baire class 1, then there is an open, so F_σ, set $A \subset Y$ with $f^{-1}(A)$ not F_σ; (2) if $g \sqsubseteq f$, $g: X_1 \to Y_1$, and there is an F_σ set $B \subset Y_1$ with $g^{-1}(B)$ not F_σ, then there is an F_σ set $A \subset Y$ with $f^{-1}(A)$ not F_σ; (3) $L^{-1}(\{\sum_{n=0}^{\infty} x(n)/3^{n+1} : x \in 2^\omega\})$ and $L_1^{-1}(\{0\})$ are not F_σ.

The first part of the following proposition is due to Cichoń and Morayne. We include its proof here for the sake of completenss. It was also known to Morayne that there is a Baire class 1 function f with $dec_c(f) = \mathbf{d}$.

Proposition 3.6. *(i) [CM] Let $f: X \to Y$ be Baire class 1, X Souslin. Then $dec_c(f) \leq \mathbf{d}$.*
(ii) $dec_c(L_1) = dec_c(L) = \mathbf{d}$.

Proof. (i)[CM] Let $\pi: X \times Y \to X$ be the projection. The graph of f is Souslin, so there is $\phi: \omega^\omega \to f \subset X \times Y$ continuous and onto. For any $x \in \omega^\omega$, $K_x = \{y \in \omega^\omega : \forall n\, y(n) \leq x(n)\}$ is compact, whence so is $\phi[K_x]$. Thus $\phi[K_x]$ is a graph of a continuous function defined on $\pi[\phi[K_x]]$ which is also compact whence closed in X. Also, clearly $X = \bigcup_{x \in D} \pi[\phi[K_x]]$ for any dominating set $D \subset \omega^\omega$.

(ii) The inequality $\leq$ follows from (i). To see $\geq$, note that if $L|F$ is continuous, $F \subset 2^\omega$ closed, then each point in $Q \cap F$ is isolated in F. Thus $F \setminus Q$ is still closed in 2^ω, whence it is compact. Thus if $\bigcup \mathcal{F} = 2^\omega$ and for any $F \in \mathcal{F}$ F is closed and $L|F$ is continuous, then $\bigcup\{F \setminus Q : F \in \mathcal{F}\} = 2^\omega \setminus Q$ and each $F \setminus Q$ is compact. Since $2^\omega \setminus Q$ is homeomorphic to ω^ω and any compact subset of ω^ω is bounded, we get $|\mathcal{F}| \geq \mathbf{d}$. The proof for L_1 is similar.

Corollary 3.7. *Let $f: X \to Y$ be Baire class 1, X Souslin. Then $dec_c(f) \leq \omega$ or $dec_c(f) = \mathbf{d}$.*

Proof. If (i) of Theorem 3.1 holds, then $dec_c(f) \leq \omega$. If (ii) holds, then $dec_c(f) \geq dec_c(L)$ or $dec_c(f) \geq dec_c(L_1)$; thus $dec_c(f) = \mathbf{d}$ by Proposition 3.6.

6.4. Decomposing Baire class 1 functions into continuous functions

with arbitrary domains

In this section, we prove the second dichotomy theorem for Baire class 1 functions.

Theorem 4.1. *Let $f : X \to Y$ be Baire class 1, X Souslin. Then either there are $X_n \subset X$, $n \in \omega$, such that $\bigcup_n X_n = X$ and $f|X_n$ are continuous (i.e., $dec(f) \leq \omega$), or $P \sqsubseteq f$.*

Most of the proof of Theorem 4.1 consists of showing preparatory results to establish two main lemmas: 4.6 and 4.7. Lemma 4.6 shows that if $dec(f) > \omega$, then the restriction of f to a subset Z of X has three characteristic properties of P. (It is not difficult to check that P satisfies properties (i)-(iii) from Lemma 4.6.) Lemma 4.7 then shows that P is contained in $f|Z$.

Let $\widetilde{X}, \widetilde{Y}$ be Polish with $X \subset \widetilde{X}$ and $Y \subset \widetilde{Y}$. It is well known that f can be extended to a Borel function $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$. Assume in the rest of this section that $\widetilde{X}$ and $\widetilde{Y}$ are recursively presented Polish spaces $X \in \Sigma^1_1$ and $\widetilde{f} \in \Delta^1_1$.

Lemma 4.2. *Either $dec(f) \leq \omega$, or there is a Σ^1_1 set $\emptyset \neq A \subset X$ such that $f|B$ is not continuous for any Σ^1_1 set $\emptyset \neq B \subset A$.*

Proof. This lemma is, in a sense, a first level analog of Lemma 2.1; its original proof was a simplified version of that of Lemma 2.1. The usage of reflection was suggested to me by G. Hjorth. Let $P = \{C \subset \widetilde{X} : C \in \Sigma^1_1 \text{ and } f|C \text{ is continuous}\}$. By reflection, for $C \in \Sigma^1_1$ with $f|C$ continuous, there is $C' \in \Delta^1_1$ such that $C \subset C'$ and $f|C'$ is continuous. Thus, $x \in \bigcup P$ iff $\exists C' \in \Delta^1_1$ and $f|C'$ continuous. Thorefore, $\bigcup P \in \Pi^1_1$. If $X \subset \bigcup P$, clearly $dec(f) \leq \omega$. If $X \not\subset \bigcup P$, put $A = X \setminus \bigcup P$.

A set $Z \subset X$ is called *singular* if there is an open set $U \subset Y$ such that $f^{-1}(U) \cap Z$ is nonempty, closed, and nowhere dense in Z.

Lemma 4.3. *Let $f : X \to Y$ be Baire class 1. Let $\emptyset \neq A \subset X$ be Σ^1_1. Then either*
(i) $\exists B \subset A$ $B \in \Sigma^1_1$, $B \neq \emptyset$, and $f|B$ continuous, or
(ii) $\forall B \subset A, \emptyset \neq B \in \Sigma^1_1$, $\exists C \subset B$ C singular and Σ^1_1.

Proof. Assume that for some $\emptyset \neq B \subset A$, $B \in \Sigma^1_1$, the following holds:

$$(*) \qquad \forall C \subset B, C \in \Sigma^1_1\ \forall V \subset Y \text{ open } f^{-1}(V) \cap C \neq \emptyset \Rightarrow \mathrm{int}_C(f^{-1}(V) \cap C) \neq \emptyset.$$

We prove that $(*)$ implies that $f|B$ is continuous. Let $V \subset Y$ be basic open. Put

$$C = \{x \in B : \forall W \subset X \text{ basic open } (x \notin W \text{ or } \exists z \in W\ z \in B \setminus f^{-1}(V))\}.$$

Then $C \in \Sigma^1_1$ and $C = B \setminus \mathrm{int}_B(f^{-1}(V) \cap B)$. If $f^{-1}(V) \cap C = \emptyset$, $f^{-1}(V) \cap B$ is open in B. So assume $f^{-1}(V) \cap C \neq \emptyset$. Then by $(*)$ there is $W \subset X$ basic open such that $\emptyset \neq W \cap C \subset f^{-1}(V) \cap C$. But then $W \cap B \subset \mathrm{int}_B(f^{-1}(V) \cap B)$, whence $W \cap C = \emptyset$, a contradiction.

Now, assume that for all $\emptyset \neq B \subset A$, $B \in \Sigma^1_1$, we have $\neg(*)$. We show that (ii) holds. Thus, let $\emptyset \neq B \subset A$, $B \in \Sigma^1_1$. Pick $C_1 \subset B$, $C_1 \in \Sigma^1_1$, and $V \subset Y$ basic open such that $f^{-1}(V) \cap C_1 \neq \emptyset$ and $\mathrm{int}_{C_1}(f^{-1}(V) \cap C_1) = \emptyset$. Note that $f^{-1}(V)$ is an F_σ and $f^{-1}(V) \cap C_1 \in \Sigma^1_1$. Since $f^{-1}(V) \cap C_1$ with the Gandy-Harrington topology is a Baire space, there is $\emptyset \neq C_2 \subset f^{-1}(V) \cap C_1$, $C_2 \in \Sigma^1_1$, and $\overline{C}_2 \subset f^{-1}(V)$. Thus, C_2 is closed and nowhere dense in $C = C_2 \cup (C_1 \setminus f^{-1}(V))$. Also, $f^{-1}(V) \cap C = C_2$. Thus C is singular and Σ^1_1.

Lemma 4.4. *Let $\emptyset \neq D_n \subset \cdots \subset D_1 \subset A \subset X$ be all G-H open with D_1 closed nowhere dense in A. Assume S_1, $S_2 \subset A \setminus D_1$ are disjoint and such that $A \setminus S_1$ and $A \setminus S_2$ are G-H open. Then there are $i_0 \in \{1,2\}$ and a G-H open set $A' \subset A$ such that*

(i) $A' \cap D_n \neq \emptyset$;

(ii) $A' \cap D_1$ is nowhere dense in A';

(iii) if D_{i+1} is nowhere dense in D_i, then $A' \cap D_{i+1}$ is nowhere dense in $A' \cap D_i$;

(iv) $A' \cap S_{i_0} = \emptyset$.

Proof. Claim. *There are $i_0 \in \{1,2\}$ and relatively open sets $\emptyset \neq W_i \subset D_i$, $1 \leq i \leq n$, such that*

(i) $\bigcup_{1 \leq i \leq n} W_i \subset \overline{A \setminus (S_{i_0} \cup D_1)}$;

(ii) $\forall 1 \leq j \leq n$ $(\bigcup_{1 \leq i \leq j} W_i) \cap D_j$ is dense in $(\bigcup_{1 \leq i \leq n} W_i) \cap D_j$.

Assuming the claim has been proved, put

$$A' = \bigcup_{1 \leq i \leq n} W_i \cup (A \setminus (D_1 \cup S_{i_0})).$$

It is clear that (i), (ii), and (iv) hold. To see (iii), note that for any $1 \leq j \leq n$, $(\bigcup_{1\leq i\leq j} W_i)\cap D_j$ is relatively open in D_j and dense in $A'\cap D_j = (\bigcup_{1\leq i\leq n} W_i)\cap D_j$. Now, if D_{j+1} is nowhere dense in D_j, then there is a set $W \subset ((\bigcup_{1\leq i\leq j} W_i)\cap D_j)\setminus D_{j+1}$ relatively open in D_j and dense in $A'\cap D_j$. But then $W\cap(A'\cap D_{j+1}) = \emptyset$, whence $A'\cap D_{j+1}$ is nowhere dense in $A'\cap D_j$.

Thus, it is enough to prove the claim. Put $Z_i = A\setminus(D_1\cup S_i)$, $i = 1, 2$. The claim will follow, if we show that there are $i_0 \in \{1,2\}$ and relatively open sets $\emptyset \neq W_i \subset D_i$, $1 \leq i \leq n$, $\emptyset \neq W_0 \subset A$ such that

(i) $Z_{i_0}\cap W_0$ is dense in $\bigcup_{i\leq n} W_i$;

(ii) $\forall 1 \leq j \leq n$ $(\bigcup_{1\leq i\leq j} W_i)\cap D_j$ is dense in $(\bigcup_{1\leq i\leq n} W_i)\cap D_j$.

For $Z \subset A$ put $Z^0 = \mathrm{int}_A(\overline{Z})$ and $Z^{j+1} = \mathrm{int}_{D_{j+1}}\overline{Z^j}$, $j \leq n$. Note first that $Z_1^j\cup Z_2^j$ is dense in D_j for any $j \leq n$. This is proved by induction: since $\overline{Z}_1\cup\overline{Z}_2 = \overline{Z_1\cup Z_2} \supset A$, $Z_1^0\cup Z_2^0 = \mathrm{int}_A\overline{Z}_1\cup\mathrm{int}_A\overline{Z}_2$ is dense in A. A similar argument shows that denseness of $Z_1^j\cup Z_2^j$ in D_j implies denseness of $Z_1^{j+1}\cup Z_2^{j+1}$ in D_{j+1}. Note also that if $Z_i^j = \emptyset$, then $Z_i^{j+1} = \emptyset$. Since $Z_1^n\cup Z_2^n$ is dense in D_n, there is $i_0 \in \{1,2\}$ such that $Z_{i_0}^n \neq \emptyset$; thus $Z_{i_0}^j \neq \emptyset$ for any $j \leq n$. Put $W_j = Z_{i_0}^j$. Obviously W_j is relatively open in D_j. It is also clear that $W_0 = Z_{i_0}^0$ is dense in $\bigcup_{i\leq n} W_i$ and $Z_{i_0}\cap Z_{i_0}^0$ is dense in $Z_{i_0}^0$; whence $Z_{i_0}\cap W_0$ is dense in $\bigcup_{i\leq n} W_i$. To see (ii), note that $W_j \subset D_j$ and W_j is dense in $\bigcup_{j\leq i\leq n} W_i$. Since obviously $(\bigcup_{1\leq i<j} W_i)\cap D_j$ is dense in $(\bigcup_{1\leq i<j} W_i)\cap D_j$, we get (ii) which finishes the proof of the claim.

Lemma 4.5. *Assume f is not constant on any G-H open set. Let $Y \supset U_1 \supset U_2 \cdots \supset U_n$ be basic open. Assume that $f^{-1}(U_n)\cap A \neq \emptyset$ and that $f^{-1}(U_1)\cap A$ is closed and nowhere dense in A. For $i \leq m$, let $V^i \subset A$ be relatively open, and let $V_i \subset Y$ be open with $V^i\cap f^{-1}(V_i) \neq \emptyset$ and $V_i\cap U_1 = \emptyset$. Then there are basic open sets $O_i \subset V_i$, $i \leq m$, and a G-H open set $A' \subset A$ such that*

(i) $\overline{O}_i\cap\overline{O}_j = \emptyset$ if $i \neq j$;

(ii) $\emptyset \neq f^{-1}(O_i)\cap A' \subset V^i$;

(iii) $f^{-1}(U_n)\cap A' \neq \emptyset$;

(iv) $f^{-1}(U_1)\cap A'$ is closed and nowhere dense in A';

(v) if $f^{-1}(U_{i+1})\cap A$ is nowhere dense in $f^{-1}(U_i)\cap A$, then $f^{-1}(U_{i+1})\cap A'$ is nowhere dense in $f^{-1}(U_i)\cap A'$.

Proof. First, note that since f is not constant on any G-H open set, $f|(V^i \cap f^{-1}(V_i))$, $i \leq m$, attains infinitely many values. Thus by shrinking the V_i's, we can assure that $\overline{V}_i \cap \overline{V}_j = \emptyset$ if $i \neq j$ but still $V^i \cap f^{-1}(V_i) \neq \emptyset$. Thus (i) will be fulfilled automatically as long as $O_i \subset V_i$.

Now by recursion on $i \leq m$, we will find $A'_i \subset A$ G-H open and basic open sets $O_i \subset V_i$ such that (iii)-(v) hold for $A' = A'_i$ and
(vi) $f^{-1}(O_i) \cap V^i \neq \emptyset$,
(vii) $f^{-1}(O_i) \cap A'_i = \emptyset$, and
(viii) $A'_{i+1} \subset A'_i$.
I will just show how to obtain O_0 and A'_0 from A; one gets O_{i+1} and A'_{i+1} from A'_i by the same argument. Since f is not constant on $V^0 \cap f^{-1}(V_0)$, there are $O^1, O^2 \subset V_0$ open and such that $O^1 \cap O^2 = \emptyset$ and $f^{-1}(O^i) \cap V^0 \neq \emptyset$, $i \in \{1,2\}$. Consider the sets

$$A \supset f^{-1}(U_1) \cap A \supset \cdots \supset f^{-1}(U_n) \cap A,$$

and

$$S_1 = f^{-1}(O^1) \cap A \text{ and } S_2 = f^{-1}(O^2) \cap A.$$

Apply Lemma 4.4 to $D_i = f^{-1}(U_i) \cap A$ and S_1, S_2 to obtain $i_0 \in \{1,2\}$ and $A' \subset A$. Put $A'_0 = A'$ and $O_0 = O^{i_0}$. It is clear that (iii)-(viii) are fulfilled by these sets.

Having produced the A'_i's and the O_i's, put

$$A' = A'_m \cup \bigcup_{i \leq m} (f^{-1}(O_i) \cap V^i).$$

Now, it is easy to check that A' along with the O_i's fulfil (i)-(v).

Lemma 4.6. *Assume $A \subset X$ is Σ^1_1 and (ii) of Lemma 4.3 holds. Then there is a set $Z \subset A$ such that*
(i) $f[Z]$ is homeomorphic to ω^ω;
(ii) $f|Z : Z \to f[Z]$ is 1-to-1 and open;
(iii) for any $\emptyset \neq U \subset f[Z]$ relatively open there is $\emptyset \neq V \subset U$ relatively open such that $(f|Z)^{-1}(V)$ is nowhere dense in $(f|Z)^{-1}(U)$.

Proof. Let us fix a winning strategy Σ for $\underline{\alpha}$ in the Choquet game for X with the Gandy-Harrington topology. (See [HKL] for details on the Choquet game for this topology.) Let d be a totally bounded metric on X, and let ρ be a complete metric on Y. We recursively define finite trees $T_n \subset \omega^{<\omega}$, $n \in \omega$, so that

1) $\bigcup_n T_n = \omega^{<\omega}$.
2) $T_n \subset T_{n+1}$;
3) if $\sigma * k \in T_n$, then $\sigma * l \in T_n$ for all $l < k$;

Additionally, we construct $A_n \subset X$ G-H open and $U_\sigma \subset Y$, $\sigma \in T_n$, basic open so that

4) $A_{n+1} \subset A_n$;
5) $\rho - diam(U_\sigma) \leq 1/(lh(\sigma)+1)$;
6) $\sigma \subset \tau \in T_n \Rightarrow U_\tau \subset U_\sigma$;
7) $\sigma, \tau \in T_n$, $\sigma \perp \tau \Rightarrow \overline{U}_\sigma \cap \overline{U}_\tau = \emptyset$;
8) if $\sigma * 0 \in T_n$, then $A_n \cap f^{-1}(U_{\sigma*0})$ is closed and nowhere dense in $A_n \cap f^{-1}(U_\sigma \setminus \bigcup_{\sigma*k \in T_n, k\geq 1} \overline{U}_{\sigma*k})$;
9) $d - diam(A_n \cap f^{-1}(U_\sigma)) \leq 1/(lh(\sigma)+1)$;
10) if $\sigma * k \in T_{n+1} \setminus T_n$ for some $k \geq 1$, then $\forall x \in A_{n+1} \cap f^{-1}(U_{\sigma*0})\ \exists \sigma * m \in T_{n+1}, m \geq 1\ \forall y \in A_{n+1} \cap f^{-1}(U_{\sigma*m})\ d(x,y) < 1/(n+1)$;
11) Let $\sigma \in T_n$ be terminal. Let $\sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_n = \sigma$ be such that σ_i is terminal in T_i, $i \leq n$. Then

$$A_{n+1} \cap f^{-1}(U_\sigma) \subset \Sigma\big(A_0 \cap f^{-1}(U_{\sigma_0}), \ldots, A_n \cap f^{-1}(U_{\sigma_n})\big).$$

Let $\{\sigma_n : n \in \omega\} = \omega^{<\omega}$, and assume that $\forall \sigma \in \omega^{<\omega}\ \exists^\infty n\ \sigma = \sigma_n$. This will guarantee that 1) holds. Assume that A_n, T_n, and U_σ, $\sigma \in T_n$, have been constructed. First, we show that in the construction at the n+1'st stage we have to worry only about conditions 2)-10). Let $\sigma^0, \ldots, \sigma^q$ be the terminal nodes of T_n. For any $i \leq q$ and $j \leq n$, let $\sigma^i_j \subset \sigma^i$ be terminal in T_j. Define

$$B_i = \Sigma\big(A_0 \cap f^{-1}(U_{\sigma^i_0}), \ldots, A_n \cap f^{-1}(U_{\sigma^i_n})\big),$$

and

$$A'_n = (A_n \setminus \bigcup_{i\leq q} f^{-1}(U_{\sigma^i})) \cup \bigcup_{i \leq q} B_i.$$

Note that A'_n, T_n, and U_σ, $\sigma \in T_n$, still fulfil 2)-11). Moreover, if we construct $A_{n+1} \subset A'_n$, T_{n+1}, and U_σ, $\sigma \in T_{n+1}$, with properties 2)-10), they will automatically fulfil 11). Thus having constructed A_n, T_n, U_σ, $\sigma \in T_n$, with 2)-10), it is enough to find A_{n+1}, T_{n+1}, and U_σ, $\sigma \in T_{n+1}$, with 2)-10), and this is what will be done below.

Put $\sigma_n = \sigma$ and $lh(\sigma) = l$.

Case 1. $\exists k < \sigma(l-1)\ \sigma|(l-1) * k \notin T_n$ or $\sigma \in T_n$.

We do not do anything, i.e., $T_{n+1} = T_n$ and $A_{n+1} = A_n$.

Case 2. $\sigma \notin T_n$, $\sigma|(l-1) \in T_n$, and $\sigma(l-1) = 0$.

Put $T_{n+1} = T_n \cup \{\sigma\}$. Let $A \subset A_n \cap f^{-1}(U_{\sigma|(l-1)})$ be Σ^1_1 and singular. Let $V \subset U_{\sigma|(l-1)}$ be open and such that $f^{-1}(V) \cap A$ is nonempty, closed in A, and nowhere dense in A. Let $U_\sigma \subset V$ be basic open such that $\rho - diam(U_\sigma) < 1/(l+1)$, $f^{-1}(U_\sigma) \cap A \neq \emptyset$. Let $\emptyset \neq A' \subset f^{-1}(U_\sigma) \cap A$ be Σ^1_1 such that $d - diam(A') < 1/(l+1)$. Put

$$A_{n+1} = (A_n \setminus f^{-1}(U_{\sigma|(l-1)})) \cup (A \setminus f^{-1}(V)) \cup A'.$$

Case 3. $\sigma \notin T_n$, $\sigma(l-1) > 0$, and $\forall k < \sigma(l-1)\ \sigma|(l-1) * k \in T_n$.

Let $\overline{\sigma} = \sigma|(l-1)$ and $\sigma_0 = \overline{\sigma} * 0$. Find relatively open, nonempty sets $V^0, \ldots, V^m \subset A_n \cap f^{-1}(U_{\overline{\sigma}} \setminus \bigcup_{\overline{\sigma}*k \in T_n} \overline{U}_{\overline{\sigma}*k})$ so that $\overline{V^i} \cap \overline{V^j} = \emptyset$ if $i \neq j$, $d - diam(V^i) \leq 1/(2(n+1))$, and $\forall x \in A_n \cap f^{-1}(U_{\sigma_0})\ \exists i \leq m\ \forall y \in V^i\ d(x,y) < 1/(2(n+1))$. (This is possible by 8).) Additionally, find $V_i \subset U_{\overline{\sigma}} \setminus \bigcup_{\overline{\sigma}*k \in T_n} \overline{U}_{\overline{\sigma}*k}$ open with $f^{-1}(V_i) \cap V^i \neq \emptyset$ and $\rho - diam(V_i) < 1/(l+2)$. Put

$$T_{n+1} = T_n \cup \{\sigma, \overline{\sigma} * (\sigma(l-1)+1), \ldots, \overline{\sigma} * (\sigma(l-1)+m)\}.$$

Let $T = \{\tau :\ \sigma_0 * \tau \in T_n\}$. T is a tree. Let $\tau_0, \ldots, \tau_q$ be all the terminal nodes in T. Let

$$A = A_n \cap f^{-1}(U_{\overline{\sigma}} \setminus \bigcup_{\overline{\sigma}*k \in T_n, k \geq 1} \overline{U}_{\overline{\sigma}*k}).$$

For each τ_j consider the sets $U_{\sigma_0} \supset U_{\sigma_0 * \tau_j|1} \supset \cdots \supset U_{\sigma_0 * \tau_j}$ and the set $A_j = A \setminus \bigcup_{\sigma_0 \subset \sigma,\ \sigma \perp \tau_j} f^{-1}(U_\sigma)$. Applying repeatedly Lemma 4.5, we define recursively on $j \leq q$ basic open sets $O_i^{j+1} \subset O_i^j \subset V_i$ and G-H open sets $A'_j \subset A_j$ so that (i)-(v) of Lemma 4.5 hold for $A = A_j$, $O_i = O_i^j$ and $A' = A'_j$. Finally, put $U_{\overline{\sigma}*(\sigma(l-1)+i)} = O_i^q$, $i \leq m$, and

$$A_{n+1} = \bigcup_{j \leq q} A'_j \cup (A_n \setminus f^{-1}(U_{\overline{\sigma}})) \cup \bigcup_{\overline{\sigma}*k \in T_n,\ k \geq 1} f^{-1}(U_{\overline{\sigma}*k}) \cap A_n.$$

If the T_n's are constructed, let $G = \bigcap_n \bigcup_{\sigma\in\omega^n} U_\sigma = \bigcup_{\eta\in\omega^\omega} \bigcap_n U_{\eta|n}$ and $Z = \bigcap_n A_n \cap f^{-1}(G)$. By 5)-7), G is homeomorphic to ω^ω. Let $\eta \in \omega^\omega$, and let $\sigma_n \subset \eta$ be terminal in T_n. Then, by 11) and 4), the following is a play in the Choquet game for the Gandy-Harrington topology:

$$A_0 \cap f^{-1}(U_{\sigma_0}), \Sigma(A_0 \cap f^{-1}(U_{\sigma_0})),$$
$$A_1 \cap f^{-1}(U_{\sigma_1}), \Sigma(A_0 \cap f^{-1}(U_{\sigma_0}), A_1 \cap f^{-1}(U_{\sigma_1})), \ldots$$

where $\underline{\beta}$ plays first, and $\underline{\alpha}$ responds by its winning strategy Σ. Thus, $\bigcap_n f^{-1}(U_{\sigma_n}) \cap A_n \neq \emptyset$, whence there is $x \in Z$ with $\{f(x)\} = \bigcap_n U_{\eta|n}$. By 9), such an x is unique. Therefore, $f[Z] = G$, and $f|Z$ is 1-to-1. By 9), for any $x \in Z$ and $\epsilon > 0$ there exists $R \subset Z$ such that $x \in R$, $d - diam(R) < \epsilon$, and $f[R]$ is open in G. It follows that $f|Z : Z \to f[Z]$ is open. To see (iii), let $U \subset G$ be relatively open. Let $\sigma \in \omega^{<\omega}$ be such that $U_\sigma \cap G \subset U$. Then by 7) and 9), $f^{-1}(U_{\sigma*0} \cap G)$ is nowhere dense in $f^{-1}(U_\sigma \cap G)$ so also in $f^{-1}(U)$.

Let Z be as in Lemma 4.6. We want to show that $P \sqsubseteq f|Z$. If we put $F = (f|Z)^{-1}$, this will follow from the next lemma.

Lemma 4.7. *Assume $F : \omega^\omega \to Z$ is continuous, 1-to-1, onto, and for any $\emptyset \neq U \subset \omega^\omega$ open there is $\emptyset \neq V \subset U$ open such that $F[V]$ is nowhere dense in $F[U]$. Then, $P \sqsubseteq F^{-1}$.*

We will deduce the above statement from Lemma 4.8. To formulate it we need several definitions. Let d be a totally bounded metric on Z. Let us equip $\mathcal{F}(Z)$, the set of all closed nonempty subsets of Z, with the Hausdorff metric induced by d. We denote this metric also by d. Total boundedness of the metric d on Z implies the following fact which will be used repeatedly in the proofs below: given $\epsilon > 0$ and $K \in \mathcal{F}(Z)$ there is a finite set $A \subset Z$ with $d(A, K) < \epsilon$. In the sequel, $\mathcal{F}(Z)$ is always considered as a topological space with the topology induced by d. Let Ω denote the set of all nonempty open subsets of ω^ω. For $\emptyset \neq U \subset \omega^\omega$ open, let $\Omega(U)$ be the set of all nonempty open subsets of U. Call a nonempty open subset of ω^ω *n-good*, for $n \in \omega$, if it is a finite union of sets of the form N_σ with $\sigma \in \omega^m$, $m \geq n$. A function $\phi : X \to \Omega$ is called *n-good*, $n \in \omega$, if $\phi(x)$ is n-good for any $x \in X$. For $\phi : X \to \Omega$ define $\phi^F : X \to \mathcal{F}(Z)$ by

$$\phi^F(x) = \overline{F[\phi(x)]}.$$

A function $\phi : X \to \Omega$ is called *disjoint* if $\phi^F(x_1) \cap \phi^F(x_2) = \emptyset$ for $x_1 \neq x_2$, $x_1, x_2 \in X$; it is called *continuous* if ϕ^F is continuous.

Lemma 4.8. *There is a sequence of functions $\phi_n : (\omega+1)^n \to \Omega$, $n \in \omega$, such that*
(i) $d - diam(\phi_n^F(\eta)) \leq 1/(n+1)$, $\eta \in (\omega+1)^n$;
(ii) $\phi_{n+1}(\eta) \subset \phi_n(\eta|n)$, $\eta \in (\omega+1)^{n+1}$;
(iii) ϕ_n is n-good;
(iv) ϕ_n is disjoint;
(v) ϕ_n is continuous.

Proof of Lemma 4.7 from Lemma 4.8. Notice the following fact which is a simple consequence of König's lemma:
$(*)$ Assume $U_n \subset \omega^\omega$ is n-good, $n \in \omega$, and $U_{n+1} \subset U_n$; then, $\bigcap_n U_n \neq \emptyset$.
So, in particular, by (ii) and (iii), $\bigcap_n \phi(\eta|n) \neq \emptyset$ for any $\eta \in (\omega+1)^\omega$, and by (i) and the fact that F is 1-to-1,
$(**)$ $\bigcap_n \phi(\eta|n)$ has precisely one element.

Define $\phi : (\omega+1)^\omega \to Z$ by

$$\phi(\eta) = \text{ the unique element of } \bigcap_n \phi_n^F(\eta|n) = F[\bigcap_n \phi_n(\eta|n)].$$

Note that $\phi(\eta)$ is well defined for all $\eta \in (\omega+1)^\omega$ by $(**)$. ϕ is continuous by (v), (i), and (ii) and 1-to-1 by (i) and (iv); thus, since $(\omega+1)^\omega$ is compact, ϕ is an embedding.

Put $G = \bigcap_n \bigcup_{\eta \in (\omega+1)^n} \phi_n(\eta)$, and define $\psi : G \to \omega^\omega$ as follows. Let $x \in G$. By (iv) and (ii), there is a unique $\eta \in (\omega+1)^\omega$ with $x \in \bigcap_n \phi_n(\eta|n)$. Let $\psi(x)$ be the unique element in $\bigcap_n N_{P(\eta)|n}$. We claim that ψ is an embedding, and that it is onto ω^ω. Continuity of ψ is obvious. By $(**)$ and the fact that P is onto, ψ is onto. To show that it is open, we have to find, for any $x \in G$ and N_σ with $x \in N_\sigma$, an $n \in \omega$ such that $\phi_n(\eta|n) \subset N_\sigma$ where η is the unique element of $(\omega+1)^\omega$ with $x \in \bigcap_n \phi_n(\eta|n)$. But if for infinitely many n, $\phi_n(\eta|n) \setminus N_\sigma \neq \emptyset$, then we apply $(*)$ to the family $\phi_n(\eta|n) \setminus N_\sigma$ for $n > lh(\sigma)$, which is legal by (iii), and obtain $y \in \bigcap_n \phi_n(\eta|n) \setminus N_\sigma$. But then, by (i), $F(x) = F(y)$ even though $x \neq y$ contradictng the fact that F is 1-to-1.

Now, we claim that $\phi \circ P^{-1} \circ \psi = F|G$. Note first that for any $x \in G$, $P^{-1} \circ h(x) =$ the unique element in $\bigcap_n \phi_n^F(\eta|n)$. But for any n, $F(x) \in \phi_n^F(\eta|n)$; thus, by (i), $F(x) = \phi \circ P^{-1} \circ \psi(x)$. Now since F, ϕ, P, and ψ are all 1-to-1, and ψ is onto, it follows that $\psi^{-1} \circ P = F^{-1} \circ \phi$. Since ψ^{-1} and ϕ are embeddings, we get $P \sqsubseteq F^{-1}$.

To prove Lemma 4.8, we will need one more auxiliary fact.

Lemma 4.9._(i) Let $U \in \Omega$, and let $\delta > 0$. There is $\phi : \omega + 1 \to \Omega(U)$ disjoint, continuous, and such that $d - diam(\phi^F(\alpha)) < \delta$ for any $\alpha \in \omega + 1$._
(ii) Let $U, V \in \Omega$. Assume $d(\overline{F[U]}, \overline{F[V]}) \leq \epsilon$, $\epsilon > 0$. Let $\phi : \omega + 1 \to \Omega(U)$ be disjoint and continuous. Let $n \in \omega$. Then, there is $\xi : \omega + 1 \to \Omega(V)$ disjoint, continuous, n-good, and such that $d(\phi^F(\alpha), \xi^F(\alpha)) \leq 2\epsilon$ for $\alpha \in \omega + 1$.

Proof. (i) Let $V \subset U$, $V \in \Omega$, be such that $F[V]$ is nowhere dense in $F[U]$. Find $\sigma \in \omega^m$, for some $m \geq n$, such that $N_\sigma \subset V$ and $d - diam(F[N_\sigma]) < \delta$. Put $\phi(\omega) = N_\sigma$. Since $\overline{F[\phi(\omega)]}$ is nowhere dense in $\overline{F[U]}$, there are $W_i \subset \overline{F[U]}$, $i \in \omega$, relatively open and such that $\overline{W_i} \cap \overline{W_j} = \emptyset$ if $i \neq j$, $\overline{W_i} \cap \overline{F[\phi(\omega)]} = \emptyset$, $\overline{W_i} \to \overline{F[\phi(\omega)]}$ and $d - diam(W_i) < \delta$. For each i find an n-good set V_i so that $V_i \subset U \cap F^{-1}(W_i)$ and $d(\overline{F[V_i]}, \overline{W_i}) < 1/(i+1)$. Put $\phi(i) = V_i$.

(ii) This is a refinement of the argument proving (i). Find a finite set $A \subset F[V]$ such that $d(\phi^F(\omega), A) < (3/2)\epsilon$. We find an n-good set $W \subset V$ so that $F[W]$ is nowhere dense in $F[V]$ and $d(A, \overline{F[W]}) < (1/2)\epsilon$. To this end, for any $x \in A$, let $\sigma_x \in \omega^m$, $m \geq n$, be such that $N_{\sigma_x} \subset V$, $d(\{x\}, F[N_{\sigma_x}]) < (1/2)\epsilon$, and $F[N_{\sigma_x}]$ is nowhere dense in $F[V]$. Then put $\xi(\omega) = \bigcup_{x \in A} N_{\sigma_x}$. Let $A_i \subset F[V]$ be finite such that $A_i \to \overline{F[\xi(\omega)]}$ and $A_i \cap \overline{F[\xi(\omega)]} = \emptyset$. This is possible since $\overline{F[\xi(\omega)]}$ is nowhere dense in $\overline{F[V]}$. Since ϕ is continuous, by modifying finitely many of the A_i's, we can assume that $d(A_i, \phi^F(i)) < (3/2)\epsilon$ for all $i \in \omega$. Now, since $\overline{F[\xi(\omega)]}$ is nowhere dense in $\overline{F[V]}$, using a technique similar to that used in constructing $\xi(\omega)$ above we can find sets $W_i \subset V$ which are n-good and such that $\overline{F[W_i]} \cap \overline{F[W_j]} = \emptyset$ if $i \neq j$, $d(A_i, \overline{F[W_i]}) < \epsilon/(2i+2)$, $\overline{F[W_i]} \cap \overline{F[\xi(\omega)]} = \emptyset$. Put $\xi(i) = W_i$ for $i \in \omega$. It is easy to check that ξ is as required.

Proof of Lemma 4.8. For a metric space X, we write X' for the set of all nonisolated points of X. First we observe that the following general claim holds.

Claim 1. *Let X be compact. Let $\psi : X \to \Omega$ be disjoint and continuous, and let $\phi : X' \times (\omega + 1) \to \Omega$ be disjoint, continuous, n-good, $n \in \omega$, with $\phi(x, \alpha) \subset \psi(x)$ and $d - diam(\phi(x, \alpha)) < \delta$, $\delta > 0$, for all $(x, \alpha) \in X' \times (\omega + 1)$. Then there is $\widetilde{\phi} : X \times (\omega+1) \to \Omega$ which extends $\phi$and has all the above mentioned properties of ϕ except that $\widetilde{\phi}(x, \alpha) \subset \psi(x)$ and $d - diam(\widetilde{\phi}(x, \alpha)) < \delta$ hold for all $(x, \alpha) \in X \times (\omega + 1)$.*

Proof of Claim 1. First, we define an extension $\widetilde{\phi}$ which satisfies all the required conditions except perhaps $d - diam(\widetilde{\phi}(x, \alpha)) < \delta$. Let $x \in X \setminus X'$. Find $y_x \in X'$ such that $d(\psi^F(x),\ \psi^F(y_x))$ is minimal among $d(\psi^F(x),\ \psi^F(y))$ for $y \in X'$. Consider $\phi(y_x, \cdot) : \omega + 1 \to \Omega(\psi(y_x))$. By Lemma 4.9(ii), there is $\xi_x : \omega+1 \to \Omega(\psi(x))$ disjoint, continuous, n-good, and such that $d(\phi^F(y, \alpha),\ \xi_x(\alpha)) \leq 2d(\psi^F(x),\ \psi^F(y_x))$. Put

$$\widetilde{\phi}(x, \alpha) = \xi_x(\alpha), \quad \text{for } x \in X \setminus X' \text{ and } \alpha \in \omega + 1.$$

If $x \in X'$, we put $\widetilde{\phi}(x, \alpha) = \phi(x, \alpha)$. It is clear that $\widetilde{\phi}$ is n-good and that $\widetilde{\phi}(x, \alpha) \subset \psi(x)$ for $(x, \alpha) \in X \times (\omega+1)$. Also, is $(x, \alpha) \neq (x', \alpha')$, then $\widetilde{\phi}^F(x, \alpha) \cap \widetilde{\phi}^F(x', \alpha') = \emptyset$. It is enough to check the continuity of $\widetilde{\phi}^F$ on sequences of the form $(x_n, \alpha_n) \to (y, \alpha)$ where $x_n \in X \setminus X'$, $y \in X'$, and $\alpha_n, \alpha \in \omega + 1$. Let y_n be the $y_{x_n} \in X'$ used to define ξ_{x_n}. Then, by definition of y_n,

$$d(\psi^F(x_n),\ \psi^F(y_n)) \leq d(\psi^F(x_n),\ \psi^F(x)) \to 0.$$

Hence, since X is compact, $x_n \to y$, and ψ^F is continuous ans 1-to-1, $y_n \to y$. Thus,

(1) $$\phi^F(y_n, \alpha_n) \to \phi^F(y, \alpha)$$

as ϕ^F is continuous on $X' \times (\omega + 1)$. On the other hand,

(2) $$d(\widetilde{\phi}^F(x_n, \alpha_n),\ \phi^F(y_n, \alpha_n)) \leq 2d(\psi^F(x_n),\ \psi^F(y_n)) \to 0.$$

Thus by (1) and (2), $\widetilde{\phi}^F(x_n, \alpha_n) \to \phi^F(y, \alpha)$. To get $d - diam(\widetilde{\phi}(x, \alpha)) < \delta$, we modify $\widetilde{\phi}$ constructed above as follows. The set

$$\{(x, \alpha) \in X \times (\omega + 1) :\ d - diam(\widetilde{\phi}^F(x, \alpha)) < \delta\}$$

is open and contains $X' \times (\omega + 1)$. Thus,

$$\{(x, \alpha) \in X \times (\omega + 1) : \ d - diam(\widetilde{\phi}^F(x, \alpha)) \geq \delta\}$$

is contained in a set of the form $\{x_1, \dots, x_m\} \times (\omega + 1)$ where each x_i is an isolated point in X. Therefore, it suffices to redefine $\widetilde{\phi}$ on each $\{x_i\} \times (\omega + 1)$ separately so that $\widetilde{\phi}^F(x_i, \alpha) \subset \psi^F(x_i)$ and $d - diam(\widetilde{\phi}^F(x_i, \alpha)) < \delta$, and this can be done by Lemma 4.9(i).

Claim 2. *Let $\psi : (\omega + 1)^n \to \Omega$ be disjoint and continuous. Then there exists $\phi : (\omega + 1)^{n+1} \to \Omega$ disjoint, continuous, $(n + 1)$-good with $\phi(\eta) \subset \psi(\eta|n)$ and $d - diam(\phi^F(\eta)) < 1/(n + 2)$ for any $\eta \in (\omega + 1)^{n+1}$.*

Proof of Claim 2. Write $(\omega + 1)^{n+1}$ as $(\omega + 1)^n \times (\omega + 1)$. Let $X = (\omega + 1)^n$. Put $X^{(0)} = X$ and $X^{(k+1)} = (X^{(k)})'$. Then, $X^{(n)} = \{(\omega, \dots, \omega)\}$. Define $\phi : X^{(n)} \times (\omega + 1) \to \Omega(\psi(\omega, \dots, \omega))$ using Lemma 4.9(i) with $\delta = 1/(n + 2)$. Using Claim 1 extend ϕ consecutively to $X^{(n-1)} \times (\omega + 1)$, $X^{(n-2)} \times (\omega + 1)$, ..., and finally to $X^{(0)} \times (\omega + 1) = (\omega + 1)^{n+1}$.

To construct ϕ_n as in the conclusion of Lemma 4.8, let ϕ_0 be defined according to Lemma 4.9(i) with $\delta = 1$. If ϕ_n is defined, we find ϕ_{n+1} by applying Claim 2 to $\psi = \phi_n$.

6.5. Complete semicontinuous functions

In this section, we study complete semicontinuous functions. The results obtained here will be used to prove that $dec(P)$ is highest possible and as a consequence establish an analogue of Corollary 3.7 for the decomposition into functions with arbitrary domains.

It will be convenient to widen the range of applicability of the definition of semicontinuity to certain functions whose image is contained in a compact space equipped with a closed linear order. Let K be a compact, metric space. Let $\preceq \subset K \times K$ be closed. Assume $\preceq$ linearly orders K. A function $f : X \to K$, X a metric, separable space, is called *lower semicontinuous* (lsc) if $f^{-1}(\{y \in K : y_0 \preceq y \text{ and } y_0 \neq y\})$ is open for any $y_0 \in K$. A lsc function $f : X \to K$ will be called *K-lsc complete* if for any $g : 2^\omega \to K$ lsc there is a continuous function $\phi : 2^\omega \to X$ such that $g = f \circ \phi$. If $K = [0, 1]$ and $\preceq = \leq$, we say lsc complete. Since for any compact, metric K and any $\preceq$ closed linear order on K there exists an embedding $h : K \to [0, 1]$ such that $x \preceq y$ iff $h(x) \leq h(y)$, we always implicitely assume that K is embedded in $[0,1]$ and $\preceq = \leq |K$. A *ray* is a subset of K of the form $\{y \in K : y_0 \leq y\}$ or $\{y \in K : y_0 \leq y \text{ and } y_0 \neq y\}$ for some $y_0 \in K$. We adopt the notation $\{y \in K : y_0 \leq y\} = [y_0, \infty)$ and $\{y \in K : y_0 \leq y \text{ and } y_0 \neq y\} = (y_0, \infty)$.

Theorem 5.1. *Let $F : X \to K$ be lsc. Then F is K-lsc complete if, and only if, there exists $D \subset X$ Polish in the relative topology and such that $F[D] = K$ and for any $U \subset D$ relatively open $F[U]$ is a ray.*

Proof. ($\Rightarrow$) It is enough to find $g : 2^\omega \to K$ lsc such that $g[2^\omega] = K$ and for any $U \subset 2^\omega$ open $g[U]$ is a ray, since then there is a continuous function $\phi : 2^\omega \to X$ such that $F \circ \phi = g$, and it is easy to check that $D = \phi[2^\omega]$ works. To define g, fix a nondeacreasing surjection $h : 2^\omega \to K$. Define the function $sup : (2^\omega)^\omega \to 2^\omega$ by $sup((x_n)) = \sup_n x_n$. Finally, put $g = h \circ sup$, and note that $(2^\omega)^\omega$ is homeomorphic to 2^ω.

($\Leftarrow$) Assume we have D as above. First, we show that

$$(1) \qquad \forall y, z \in K \left(y < z \Rightarrow F^{-1}(y) \cap D \subset \overline{F^{-1}(z) \cap D}\right),$$

then that (1) implies

$$(2) \qquad \exists G \subset X \big(G \text{ Polish, zero-dimensional, and } \forall f : G \to K \text{ lsc } f \cap F \neq \emptyset \text{ (i.e., } \exists x \in G\ f(x) = F(x))\big),$$

and, finally, that (2) implies F is K-lsc complete.

If (1) fails for some $y < z$, there is $U \subset D$ relatively open such that $U \cap F^{-1}(y) \neq \emptyset$ and $U \cap F^{-1}(z) = \emptyset$, i.e., $y \in F[U]$ and $z \notin F[U]$, which contradicts the assumption that $F[U]$ is a ray.

To prove (2) from (1), let $Q \subset K$ be countable and such that $\forall y \in K \forall \epsilon > 0 \exists z \in Q\ y - \epsilon < z \leq y$. Note that $\min K \in Q$. For each $y \in Q$, let $Q_y \subset F^{-1}(y) \cap D$ be countable and dense in $F^{-1}(y) \cap D$. Let G be zero-dimensional, *bpi*02 subset of D such that $\bigcup_{y \in Q} Q_y \subset G$. We show that G works. Let $G = \bigcap_n G_n$, G_n open and $G_{n+1} \subset G_n$. Let $f : G \to K$ be lsc. Note that $Q \subset F[G]$ and for any $y, z \in Q$ with $y < z$ we have $F^{-1}(y) \cap G \subset \overline{F^{-1}(z) \cap G}$. This last condition implies that if V is open and $y \in F[V]$, then $z \in F[V]$ for $y, z \in Q$, $y < z$. We recursively construct a sequence of open sets U_n and $z_n \in Q$, $n \in \omega$, such that:

(i) $U_n \subset G_n$;

(ii) $\overline{U}_{n+1} \subset U_n$;

(iii) $diam U_n < 1/(n+1)$;

(iv) $z_n \leq \inf f[U_n]$, and $z_n \in F[U_n]$;

(v) $\inf f[U_n] - 1/(n+1) < \inf F[U_{n+1}] \leq \inf f[U_n]$.

Let U_0 be open such that $diam U_0 < 1$, $U_0 \subset G_0$, and $\min K \in F[U_0]$. If U_n has been defined, find $z_{n+1} \in Q$ with $\inf f[U_n] - 1/(n+1) < z_{n+1} \leq \inf f[U_n]$ and $z_n \leq z_{n+1}$. Such a z_{n+1} exists by the definition of Q and by (iv). Since $z_n \in F[U_n]$, $z_{n+1} \in F[U_n]$. Since F is lsc, there is $V \subset U_n$ open such that $z_{n+1} \in F[V]$ and $\inf f[U_n] - 1/(n+1) < \inf F[V]$. We get U_{n+1} by making V small enough. Now, let x be the only element in $\bigcap_n U_n$. Then $x \in G$, and since f and F are lsc, by (v), we get

$$f(x) = \sup_n \inf f[U_n] = \sup_n \inf F[U_n] = F(x).$$

Now we show that (2) implies that F is K-lsc complete. We can assume that G is a closed subset of ω^ω so that $G =$ the set of all branches of T, for some tree $T \subset \omega^{<\omega}$. Let $f: 2^\omega \to K$ be lsc. We show that there is a continuous function $\phi: 2^\omega \to G$ such that $f = F \circ \phi$. We play the following game: Players I and II play interchangeably; I plays $x_n \in 2$, II plays $y_n \in \omega$ so that $(y_0, \ldots, y_n) \in T$; I wins iff $f((x_n)) \neq F((y_n))$. By Martin's theorem, the game is determined. A winning strategy for I induces a continuous function $\psi: G \to 2^\omega$ such that $f \circ \psi \cap F = \emptyset$, which contradicts (2) since $f \circ \psi$ is lsc. Therefore, II has a winning strategy. It induces a continuous function $\phi: 2^\omega \to G$ such that $f = F \circ \phi$.

Now, we present a construction of a family of Baire class 1 functions. These functions will be used in the proof of the existence of "minimal" lsc complete functions and in the proof that the decomposition coefficient of Pawlikowski's function is highest possible. Let $\preceq_n \subset 2^n$, $n \in \omega$, be partial orders. Assume that for $\sigma, \tau \in 2^{n+1}$

$$\sigma \preceq_{n+1} \tau \Rightarrow \sigma|n \preceq_n \tau|n.$$

Define $T_{(\preceq_n)} \subset \prod_n 2^n$ by

$$x \in T_{(\preceq_n)} \text{ iff } \forall n \, x(n) \preceq_n x(n+1)|n.$$

Let $F_{(\preceq_n)}: T_{(\preceq_n)} \to 2^\omega$ be defined by

$$F_{(\preceq_n)}(x) = \text{ the unique } y \in 2^\omega \text{ with } \forall n \forall^\infty k \; y|n = x(k)|n.$$

Define a partial order $\preceq$ on 2^ω by

(3) $$x \preceq y \text{ iff } \forall n \; x|n \preceq_n y|n.$$

Lemma 5.2. *$F_{(\preceq_n)}$ is Baire class 1 and onto.*

Proof. It is clear that $F_{(\preceq_n)}$ is a pointwise limit of a sequence of continuous functions, whence it is Baire class 1. For $y \in 2^\omega$ define $x \in T_{(\preceq_n)}$ by $x(n) = y|n$, $n \in \omega$. Then $F_{(\preceq_n)}(x) = y$. Thus $F_{(\preceq_n)}$ is onto.

Lemma 5.3. *If $C \subset 2^\omega$ is closed and linearly ordered by $\preceq$, then $F_{(\preceq_n)}|F^{-1}_{(\preceq_n)}(C)$ is C-lsc complete.*

Proof. By Lemma 5.2, $G = F^{-1}_{(\preceq_n)}(C)$ is $\mathbf{\Pi}^0_2$. Therefore, to check that $F_{(\preceq_n)}|G$ is C-lsc complete, it is enough, by Theorem 5.1, to show that $F_{(\preceq_n)}|G$ is lsc and that for any $U \subset G$ relatively open $F_{(\preceq_n)}[U]$ is a ray with respect to $\preceq |C$. To this end, it is enough to see that

(i) if $U \subset T_{(\preceq_n)}$ is open, $y \in F_{(\preceq_n)}[U]$, and $y \preceq z$, then $z \in F_{(\preceq_n)}[U]$, and

(ii) if $x \in T_{(\preceq_n)}$ and $y \preceq F_{(\preceq_n)}(x)$, $y \neq F_{(\preceq_n)}(x)$, then there is an open set $U \subset T_{(\preceq_n)}$ such that $x \in U$ and if $z \preceq y$ then $z \notin F_{(\preceq_n)}[U]$.

To see (i), find $x \in U$ with $F_{(\preceq_n)}(x) = y$. Fix $n \in \omega$ such that if $x'(i) = x(i)$ for $i \leq n$, then $x' \in U$. Define $\bar{x}$ so that $\bar{x}(i) = x(i)$ for $i \leq n$, and $\bar{x}(i) = z|i$ for $i > n$. It is easy to check that $\bar{x} \in T_{(\preceq_n)}$, and clearly $\bar{x} \in U$ and $F_{(\preceq_n)}(\bar{x}) = z$. To see (ii), note that there is $n \in \omega$ such that $x(n) \not\preceq_n y|n$. Then $U = \{x' \in T_{(\preceq_n)} : x'(n) = x(n)\}$ works.

Lemma 5.4 *Assume there is a closed uncountable, linearly ordered by $\preceq$ subset of 2^ω. Then $dec(F_{(\preceq_n)}) \geq dec(f)$ for any lsc $f : 2^\omega \to [0,1]$.*

Proof. Let $C \subset 2^\omega$ be closed, uncountable, linearly ordered by $\preceq$. We can easily find a copy C_0 of 2^ω inside C such that the lexicographic order is equal to $\preceq$ on C_0. Let $\phi : C_0 \to [0,1]$ be an increasing homeomorphism, e.g., the Cantor function. By Lemma 5.3, $F_{(\preceq_n)}[U]$ is a ray in $(C_0, \preceq)$ for any relatively open $U \subset F^{-1}_{(\preceq_n)}(C_0)$. Thus $\phi \circ F_{(\preceq_n)}[U]$ is a ray in $[0,1]$. It follows that $\phi \circ F_{(\preceq_n)}|F^{-1}_{(\preceq_n)}(C_0)$ is lsc complete. Thus, $dec(\phi \circ F_{(\preceq_n)}) \geq dec(f)$ for any $f : 2^\omega \to [0,1]$ lsc. But since ϕ is continuous, $dec(F_{(\preceq_n)}) \geq dec(\phi \circ F_{(\preceq_n)})$.

$\sigma \in 2^n$ is called *splitting* if $\sigma * 0 \preceq_{n+1} \sigma * 1$, or $\sigma * 1 \preceq_{n+1} \sigma * 0$.

Lemma 5.5. *Assume that for each $n \in \omega$ and any $\sigma, \tau \in 2^n$ with $\sigma \preceq_n \tau$ we have*

(4) $$\forall i \in 2\, \exists j \in 2\ \ \sigma * i \preceq_{n+1} \tau * j \text{ and } \forall i \in 2\, \exists j \in 2\ \ \sigma * j \preceq_{n+1} \tau * i.$$

Assume also that for any $\sigma \in 2^{<\omega}$ there is a splitting $\tau \in 2^{<\omega}$ with $\sigma \subset \tau$. Then there is a perfect, closed set linearly ordered by $\preceq$.

Proof. The conclusion will follow easily if we can show that if $\sigma_0, \ldots, \sigma_k \in 2^n$, $\sigma_0 \preceq_n \cdots \preceq_n \sigma_k$, and $i \leq k$, then there are $\tau_0, \ldots, \tau_{k+1} \in 2^m$ for some $m > n$ with $\tau_j|n = \sigma_j$ for $j \leq i$ and $\tau_j|n = \sigma_{j-1}$ for $j > i$, $\tau_0 \preceq_m \cdots \preceq_m \tau_{k+1}$, and $\tau_i \neq \tau_{i+1}$. To see this, let $\tau \supset \sigma_i$ be splitting. Assume $\tau * 0 \preceq_m \tau * 1$ where $m = lh(\tau * 0)$. Put $\tau_i = \tau * 0$ and $\tau_{i+1} = \tau * 1$. By (4), we can extend $\sigma_{i+1}, \ldots, \sigma_k$ one by one to $\tau_{i+2}, \ldots, \tau_{k+1}$, respectively, so that $\tau_{i+1} \preceq_m \cdots \preceq_m \tau_{k+1}$. Similarly, we extend $\sigma_{i-1}, \ldots, \sigma_0$ to $\tau_{i-1}, \ldots, \tau_0$.

Remark. Before we proved Lemma 5.5, J. Pawlikowski pointed out that in case $\sigma \preceq_n \tau$ iff $\forall i < n\ \sigma(i) \leq \tau(i)$, $\sigma, \tau \in 2^n$, one can get a perfect closed set linearly ordered by $\preceq$ by the following simple argument. (Lemmas 5.4 and 5.5 applied to this $\preceq$ will be used in the proof of Theorem 6.1.) Identify ω with the rationls, $\mathbb{Q}$. For any $r \in \mathbb{R}$, let $\alpha_r \in 2^\omega$ be the characteristic function of $\{q \in \mathbb{Q} : q < r\}$. Then $\{\alpha_r : r \in \mathbb{R}\}$ is a Borel uncoutable subset of 2^ω linearly ordered by $\preceq$. Now, any perfect closed subset of $\{\alpha_r : r \in \mathbb{R}\}$ works.

In the next theorem, we prove the existence of complete semicontinuous functions which are in a sense minimal. This result will not be used in the sequel we nevertheless find it interesting.

Now, let $\preceq_n =$ the lexicographic order for each $n \in \omega$. Put $T_l = T_{(\preceq_n)}$. In this case, $\preceq$ is the lexicographic order on 2^ω; it linearly orders 2^ω. Let K be a perfect, compact, metric space linearly ordered by a closed linear order. Fix $\psi : 2^\omega \to K$ a nondecreasing surjection such that

(5) $$\exists D \subset 2^\omega\ D \text{ dense and } \psi|D \text{ 1-to-1}.$$

Put $F_K = \psi \circ F_{(\preceq_n)}$. By Lemma 5.3, F_K is K-lsc complete. We show that it is in a sense a minimal such function.

Theorem 5.5. *Let $f : X \to K$ be K-lsc complete, K compact perfect. Then there is an embedding $\phi : T_l \to X$ such that $F_K = f \circ \phi$.*

Proof. Let $D \subset X$ be as in Theorem 5.1. Without loss of generality we can assume that $D = X$. Define $h : 2^{<\omega} \to K$ by $h(\sigma) = \psi(\sigma * 00\cdots)$. Let $S \subset \bigcup_k \prod_{n\leq k} 2^n$ be the pruned tree with $T_l = [S]$. For $\tau \in S$ we recursively, with respect to $lh(\tau)$, define $U_\tau \subset X$ open and such that:
(i) $diam U_\tau \leq 1/\big(lh(\tau)+1\big)$;
(ii) if $\tau_1 \subset \tau_2$ and $\tau_1 \neq \tau_2$, then $\overline{U}_{\tau_1} \subset U_{\tau_2}$, and if $\tau_1 \perp \tau_2$, then $U_{\tau_1} \cap U_{\tau_2} = \emptyset$;
(iii) $h(\tau(n-1)) \in f[U_\tau] \subset \big(h(\tau(n-1)) - 1/(n+1), \infty\big)$.
Define $U_\emptyset$ to be any open set of diameter < 1 containing an $x \in X$ such that $f(x) = \min K$. This is possible since f is onto. If U_τ is defined, consider the set

$$A = \{\tau' \in S :\ lh(\tau') = lh(\tau) + 1,\ \tau \subset \tau'\}.$$

Enumerate A so that $A = \{\tau_0, \tau_1, \ldots, \tau_m\}$ for some $m \in \omega$, the τ_i's are pairwise different, and $\tau_0(n) \preceq_n \tau_1(n) \preceq_n \cdots \preceq_n \tau_m(n)$, where $n = lh(\tau)$. Note that by (5)

$$h(\tau(n-1)) = h(\tau_0(n)) < h(\tau_1(n)) < \cdots < h(\tau_m(n)).$$

Now, we find recursively U_{τ_i}, $i \leq m$. Let U'_{τ_0} and V_0 be open and such that $\overline{U'}_{\tau_0} \cap \overline{V}_0 = \emptyset$, $\overline{U'}_{\tau_0}, \overline{V}_0 \subset U_\tau$, $h(\tau_0(n)) \in h[U'_{\tau_0}]$, $h(\tau_1(n)) \in h[V_0]$, and $diam U'_{\tau_0} < 1/(n+2)$. Put

$$U_{\tau_0} = U'_{\tau_0} \cap f^{-1}\big(\big(h(\tau_0(n)) - 1/(n+2), \infty\big)\big).$$

Then find U'_{τ_1} and V_1 open and such that $\overline{U'}_{\tau_1} \cap \overline{V}_1 = \emptyset$, $\overline{U'}_{\tau_1}, \overline{V}_1 \subset V_0$, $h(\tau_1(n)) \in h[U'_{\tau_1}]$, $h(\tau_2(n)) \in h[V_1]$, and $diam U'_{\tau_1} < 1/(n+2)$. Put

$$U_{\tau_1} = U'_{\tau_1} \cap f^{-1}\big(\big(h(\tau_1(n)) - 1/(n+2), \infty\big)\big).$$

Repeat this procedure $m+1$ times.

Define ϕ by

$$\phi(x) = \text{ the unique element of } \bigcap_n U_{x|n}.$$

By (i) and (ii), ϕ is continuous, and, by (ii), it is 1-to-1, whence it is an embedding since T_l is compact. Since f and F_K are lsc, (iii) implies that $F_K = f \circ \phi$.

6.6. The value of *dec* for Baire class 1 functions

In [CMPS] it was proved that $dec(P) \geq \text{cov}(\mathcal{M})$, and in [St] that it is consistent that $dec(P) > \text{cov}(\mathcal{M})$. Thus P provides a particularly simple example of a complicated Baire class 1 function. Below we show that $dec(P)$ is actually highest possible. This answers two questions of Steprāns [St, Questions 7.1 and 7.2].

Theorem 6.1. *$dec(P) =$ **dec**, where P is Pawlikowski's function.*

Proof. If Y is a metric separable space, define

$$dec_{1/2}(Y) = \sup\{dec(f) : f : Y \to [0,1], \ f \text{ lsc}\}.$$

Of course, the value of $dec_{1/2}(Y)$ would remain the same if we used usc instead of lsc functions in its definition.

First we show that $\mathbf{dec} = dec_{1/2}(2^\omega)$. The inequality $\geq$ is clear since each lsc is Baire class 1. To see $\leq$, first we show that $dec_{1/2}(Y) \leq dec_{1/2}(2^\omega)$ for any metric separable space Y. By a result due to Smirnov (see [E, Problem 1.8.G.]), $Y = \bigcup_{\alpha<\omega_1} Y_\alpha$, where Y_α, $\alpha < \omega_1$, are zero-dimensional. Each Y_α embeds in 2^ω, and each lsc function on Y_α extends to 2^ω; thus, $dec_{1/2}(Y_\alpha) \leq dec_{1/2}(2^\omega)$. By a result of Adyan and Novikov, $dec_{1/2}(2^\omega) \geq \aleph_1$ (see [JM, Theorem 4]); thus, we get

$$dec_{1/2}(Y) \leq \aleph_1 \sup_{\alpha\in\omega_1} dec_{1/2}(Y_\alpha) \leq dec_{1/2}(2^\omega).$$

Now, let $f : X \to Y$ be Baire class 1. Again, by Smirnov's theorem $Y = \bigcup_{\alpha<\omega_1} Y_\alpha$ and each Y_α is zero-dimensional. Since Y_α embeds in $[0,1]$, we can assume that $f|f^{-1}(Y_\alpha) : f^{-1}(Y_\alpha) \to [0,1]$. By Lindenbaum's theorem (see [CMPS, Theorem 4.4]), any Baire class 1 function $h : Z \to [0,1]$ can be represented as $h = g_2 \circ g_1$ where $g_1 : Z \to [0,1]$ is usc and $g_2 : [0,1] \to [0,1]$ is lsc, so $dec(h) \leq dec(g_2)dec(g_1)$; whence

$$dec(f) \leq \aleph_1 \sup_{\alpha<\omega_1} dec_{1/2}(Y_\alpha)dec_{1/2}([0,1]) \leq dec_{1/2}(2^\omega).$$

Thus $\mathbf{dec} \leq dec_{1/2}(2^\omega)$.

The theorem will be proved, if we can show that $dec_{1/2}(2^\omega) \leq dec(P)$. Let $G : \omega^\omega \to 2^\omega$ be defined by

$$G(x)(n) = \min\{1,\ x(n)\} \text{ for } n \in \omega.$$

Let $\preceq_n$ be the partial order on 2^n defined by

$$\sigma \preceq_n \tau \text{ iff } \forall i < n\ \sigma(i) \leq \tau(i).$$

Let $\preceq$ be the partial order on 2^ω arising from $(\preceq_n)$ by formula (3). Since $(\preceq_n)$ fulfils the assumptions of Lemma 5.5, there is a perfect, closed subset of 2^ω linearly ordered by $\preceq$. Now, it follows from Lemma 5.4 that $dec(F_{(\preceq_n)}) \geq dec_{1/2}(2^\omega)$. Thus, it is enough to show that there is a homeomorphism $\phi : (\omega+1)^\omega \to 2^\omega$ such that $G \circ P = F_{(\preceq_n)} \circ \phi$ since then

$$dec(P) \geq dec(G \circ P) = dec(F_{(\preceq_n)}) \geq dec_{1/2}(2^\omega).$$

Let $\eta \in (\omega+1)^\omega$. Put $\phi(\eta) = x$, where $x = \big(x(n)\big) \in \prod_n 2^n$, and for $i < n \in \omega$ we have

$$x(n)(i) = \begin{cases} 0, & \text{if } \eta(i) \geq n; \\ 1, & \text{if } \eta(i) < n. \end{cases}$$

It is easy to check that ϕ is continuous, 1-to-1, and onto, whence, since $(\omega+1)^\omega$ is compact, ϕ is a homeomorphism. Now, $G \circ P(\eta)(i) = 0$ iff $\eta(i) = \omega$ iff $\forall n > i\ x(n)(i) = 0$ iff $F_{(\preceq_n)}(x)(i) = 0$.

Remark. It follows from Theorem 6.1, via the work of Steprāns [St, Definition 4.1, Proposition 4.1], that **dec** $= \text{cov}(\mathcal{J}_p)$, where $\mathcal{J}_p$ is a σ-ideal on ω^ω. The interesting fact about $\mathcal{J}_p$ is that its definition is purely combinatorial.

The following corollary is analogous to Corollary 3.7.

Corollary 6.2. *Let $f : X \to Y$ be Baire class 1, X Souslin. Then $dec(f) \leq \omega$ or $dec(f) =$* **dec**.

Proof. If (i) of Theorem 4.1 holds, then $dec(f) \leq \omega$. If (ii) holds, then $dec(f) =$ **dec** by Theorem 6.1.

It was proved in [CMPS, Theorem 5.5] that there exists a lsc function f such that $dec(f) \geq \text{cov}(\mathcal{M})$. We strengthen this result below. A function $f : Y \to [0,1]$, Y a metric space, is called *closed-to-1* if for any $y \in [0,1]$ $f^{-1}(y)$ is closed in Y. Obviously, each continuous function is closed-to-1; however there exist plenty of closed-to-1, lsc functions which are not continuous, e.g., Pawlikowski's function P being 1-to-1 is closed-to-1. The method of proof presented here is different from the one in [CMPS].

Theorem 6.3. *Let $F : X \to [0,1]$ be lsc complete. If $\mathcal{F}$ is a family of subsets of X such that $\bigcup \mathcal{F} = X$, and $F|Y$ is closed-to-1 for any $Y \in \mathcal{F}$, then $|\mathcal{F}| \geq cov(\mathcal{M})$.*

Proof. Let $\{V_n : n \in \omega\}$ be a countable topological basis of X. Let D be as in Theorem 5.1. Without loss of generality we can assume that $D = X$. Fix $n \in \omega$ and $Y \in \mathcal{F}$. We claim that there is at most one $y \in [0,1]$ such that $V_n \cap F^{-1}(y) \neq \emptyset$ and Y is dense in $V_n \cap F^{-1}(y)$. If not, let $y_1 < y_2$ be two such y's. Since (1) from the proof of Theorem 5.1 holds, we have

$$Y \cap V_n \cap F^{-1}(y_1) \subset \overline{Y \cap V_n \cap F^{-1}(y_2)},$$

whence, since $Y \in \mathcal{F}$, $F|(Y \cap V_n \cap F^{-1}(y_1)) \equiv y_2$, a contradiction. Thus, we can pick $y_0 \in [0,1]$ such that for any $n \in \omega$ and any $Y \in \mathcal{F}$ either $V_n \cap F^{-1}(y_0) = \emptyset$, or Y is not dense in $V_n \cap F^{-1}(y_0)$. Then, clearly, Y is nowhere dense in $F^{-1}(y_0)$. Since $F^{-1}(y_0)$ is $\mathbf{\Pi}^0_2$, as F is Baire class 1, and $\bigcup \mathcal{F} \supset F^{-1}(y_0)$, we have $|\mathcal{F}| \geq \text{cov}(\mathcal{M})$.

Remark. Below, we prove a result which relates the value of *dec* to the value of an ordinal rank on the family of all Baire class 1 functions. For the definition of the oscilation rank β on Baire class 1 functions we refer the reader to [KL] where it was studied in great detail.
(i) Let $f : X \to \mathbb{R}$, X Polish, be Baire class 1. Assume $\beta(f) < \omega$; then $dec(f) \leq \omega$.
(ii) $\beta(P) = \omega$.
To see (i), put $\beta(f) = n$. Then there exists $\epsilon_0 > 0$ such that $\beta(f, \epsilon) = n$ for $\epsilon < \epsilon_0$. Let $X^1_k = \{x \in X : osc(f,x) < 1/k\}$ and $X^m_k = \{x \in X \setminus \bigcup_{i=1}^{m-1} X^i_k : osc(f|X \setminus \bigcup_{i=1}^{m-1} X^i_k, x) < 1/k$ for $m \leq n$ and for k with $1/k < \epsilon_0$. By our assumption, $\bigcup_{i=0}^{n} X^i_k = X$ for any k. Now define $A_1 = \bigcap_{k=1}^{\infty} X^1_k$ and $A^m_l = (\bigcap_{j=1}^{\infty} \bigcup_{i=1}^{m} X^i_j) \setminus (\bigcap_{j=1}^{l} \bigcup_{i=1}^{m-1} X^i_j)$ for $1 < m \leq n$ and $l < \omega$. Notice that $A_1 \cup \bigcup_{m=2}^{n} \bigcup_{l=1}^{\infty} A^m_l = X$;

thus, it is enough to see that $f|A_1$ and $f|A_l^m$, $l < \omega$, $1 < m \leq n$, are continuous. But we have $A_1 \subset X_k^1$ for all $k < \omega$. Also, if $k > l$, then $\bigcup_{i=1}^{m-1} X_k^i \subset \bigcap_{j=1}^{l} \bigcup_{i=1}^{m-1} X_j^i$ for $m \leq n$. Thus, $A_l^m \subset \bigcup_{i=1}^{m} X_k^i \setminus \bigcup_{i=1}^{m-1} X_k^i = X_k^m$ for all $k > l$. Therefore, $osc(f|A_1, x) = 0$ for $x \in A_1$ and $osc(f|A_l^m, x) = 0$ for $x \in A_m^1$.

We leave proving (ii) to the reader.

6.7. Applications to measures

Let λ be the Lebesque measure on $[0,1]$. Then the restriction of λ to $K([0,1])$ is usc. We denote this restriction by the same letter λ. Van Mill and Pol proved in [vMP, Theorem 3.1] that λ is usc complete. (Actually, they showed that for any compact, metric space X, not only 2^ω, and any usc function $f : X \to [0,1]$ there is a continuous function $\phi : X \to K([0,1])$ such that $f = \lambda \circ \phi$.) Below we are able to generalize this result using the characterization from Theorem 5.1. Let X be a compact, metric space. Recall that a function $c : K(X) \to [0,1]$ is called a *capacity* if

(i) $c(F_1) \leq c(F_2)$ for $F_1, F_2 \in K(X)$ with $F_1 \subset F_2$;

(ii) $c(\bigcap_n F_n) = \inf_n c(F_n)$ for any sequence $F_n \in K(X)$, $n \in \omega$, with $F_{n+1} \subset F_n$;

(iii) if $F \in K(X)$ and $F = \bigcup_n F_n$ for some sequence $F_n \in K(X)$, $n \in \omega$, with $F_n \subset F_{n+1}$, then $c(F) = \sup_n c(F_n)$.

Notice that the restriction of any probability, Borel measure on X to $K(X)$ is a capacity; however, there exist lots of important capacities which cannot be obtained in this way.

Corollary 7.1. *Let X be a compact, metric space. Let $c : K(X) \to [0,1]$ be a capacity. Assume that $c(X) = 1$ and $c(D) = 0$ for any finite set $D \subset X$. Then c is usc complete.*

Proof. First notice that conditions (i) and (ii) guarantee that c is usc. Thus, by Theorem 5.1, it is enough to check that $c[U]$ is a ray for any open set $U \subset K(X)$. Let $F_0 \in U$. We will show that for any real r with $c(F_0) \geq r \geq 0$ there is $F' \in U$ with $c(F') = r$. We can easily find $D \subset F_0$ finite such that for any $F \in K(X)$ if $D \subset F \subset F_0$, then $F \in U$. Let $\mathcal{F}$ be a maximal, linearly ordered by inclusion family of closed subsets F of X such that $D \subset F \subset F_0$ and $c(F) \geq r$. Put $F' = \bigcap \mathcal{F}$. Then $F' \in U$. We can find a decreasing sequence $F_n \in \mathcal{F}$, $n \in \omega$, such that $F' = \bigcap_n F_n$; thus, by (ii), $c(F') \geq r$. If F' is finite, then $r = 0$ and $c(F') = r$. Otherwise, we can find a decreasing sequence of open sets V_n, $n \in \omega$, such that $D \cap V_n = \emptyset$, $F' \cap V_n \neq \emptyset$, and $\bigcap_n V_n = \emptyset$. Put $F_n = F' \setminus V_n$. Then by the definition of F', $c(F_n) < r$. By (iii), $c(F') = \sup_n c(F_n) \leq r$. Thus $c(F') = r$.

Jackson and Mauldin proved in [JM, Theorem 5] that $dec(\lambda) > \omega$, where λ is the restriction to $K([0,1])$ of the Lebesgue measure on $[0,1]$. It follows from van Mill-Pol's result [vMP, Theorem 3.1] mentioned above that $dec(\lambda) = \mathbf{dec}$. In the next corollary, using Corollary 7.1 and Theorem 6.1, we characterize those Borel, probability measures μ on compact, metric spaces for which $dec(\mu) = \mathbf{dec}$. By δ_x we denote the Dirac measure concentrated at x, i.e., $\delta_x(A) = 1$ if $x \in A$ and $\delta_x(A) = 0$ otherwise.

Corollary 7.2. *Let X be a compact, metric space. Let μ be a Borel, probability measure. Let us denote by the same letter the restriction of μ to $K(X)$. Then $dec(\mu) = \mathbf{dec}$ unless $\mu = \sum_{x\in D} \alpha_x \delta_x$ where $\alpha_x > 0$, $\sum_{x\in D} \alpha_x = 1$, and $\{x \in D :$ x is not isolated$\}$ is finite. Moreover, if μ is of the above form, then $dec(\mu) = n+1$ where $n = |\{x \in D : x$ is not isolated$\}|$.*

Proof. If μ is not purely atomic, then there is a closed set $F_0 \subset X$ such that $\mu(F_0) > 0$ and $\mu(\{x\}) = 0$ for any $x \in F_0$. Then by Corollary 7.1, $(1/\mu(F_0))\mu|\{F \in K(X) : F \subset F_0\}$ is usc complete. It follows that $dec(\mu) = \mathbf{dec}$.

Put $N = \{x \in X : x$ is not isolated and $\mu(\{x\}) \neq 0\}$. Assume N is infinite. We will find a continuous function $\phi : (\omega+1)^\omega \to K(X)$ such that if $\eta_k, \eta \in (\omega+1)^\omega$, $\eta_k \to \eta$, then $P(\eta_k) \twoheadrightarrow P(\eta)$ implies $\mu \circ \phi(\eta_k) \twoheadrightarrow \mu \circ \phi(\eta)$. Then, clearly, if $\mu|Y$ is continuous, so is $P|\phi^{-1}(Y)$; thus, $dec(\mu) \geq dec(P)$, and we are done by Theorem 6.1. Find a converging sequence $x^n_\omega \in N$, $n \in \omega$. Put $y = \lim_n x^n_\omega$. Find x^n_k, $k \in \omega$, with $x^n_k \to x^n_\omega$. By choosing subsequences, we can assume that
(i) $\forall k, l \in \omega+1 \, \forall n, m \in \omega \;\; x^n_k \neq x^m_l$ if $k \neq l$ or $n \neq m$;
(ii) $\forall k \in \omega+1 \, \forall n \in \omega \;\; d(x^n_k, y) \leq 1/n$;
(iii) $\forall n \in \omega \;\; \mu(\{x^m_k : n < m \in \omega, \; k \in \omega+1\}) < \mu(\{x^n_\omega\})$.
Define $\phi : (\omega+1)^\omega \to K(X)$ by

$$\phi(\eta) = \{y\} \cup \{x^n_{\eta(n)} : n \in \omega\} \text{ for } \eta \in (\omega+1)^\omega.$$

By (ii), the set on the right hand side is closed. It is routine to check that ϕ is continuous. Let $\eta_k, \eta \in (\omega+1)^\omega$, $\eta_k \to \eta$. Assume $P(\eta_k) \not\to P(\eta)$. Then there is $n \in \omega$ such that $\eta_k(n) \in \omega$ for infinitely many k, $\eta_k(n) \to \omega$, and $\eta(n) = \omega$. Let n_0 be the smallest such n. Without loss of generality we can assume that $\eta_k(n) = \eta(n)$ for all $n < n_0$ and $\eta_k(n_0) \in \omega$ for all $k \in \omega$. Then

$$\begin{aligned}\mu \circ \phi(\eta_k) \leq & \mu(\{x^n_{\eta(n)} : n < n_0\}) + \mu(\{x^{n_0}_{\eta_k(n_0)}\}) \\ & + \mu(\{x^n_l : l \in \omega+1,\ n \in \omega,\ n > n_0\}) + \mu(\{y\}).\end{aligned}$$

Since $\eta_k(n_0) \to \omega$, by (i), $\mu(\{x^{n_0}_{\eta_k(n_0)}\}) \to 0$; thus,

$$\begin{aligned}\limsup_k \mu \circ \phi(\eta_k) \leq & \mu(\{x^n_{\eta(n)} : n < n_0\}) \\ & + \mu(\{x^n_l : l \in \omega+1,\ n \in \omega,\ n > n_0\}) + \mu(\{y\}).\end{aligned}$$

On the other hand, by (iii),

$$\begin{aligned}\mu \circ \phi(\eta) &= \mu(\{x^n_{\eta(n)} : n < n_0\}) + \mu(\{x^{n_0}_\omega\}) + \mu(\{y\}) \\ &> \mu(\{x^n_{\eta(n)} : n < n_0\}) + \mu(\{x^n_l : l \in \omega+1,\ n \in \omega,\ n > n_0\}) + \mu(\{y\}).\end{aligned}$$

Therefore, $\mu \circ \phi(\eta_k) \not\to \mu \circ \phi(\eta)$.

If $|N| = n < \aleph_0$, put

$$X_i = \{F \in K(X) : |F \cap N| = i\}, \text{ for } i \in \{0, \ldots, n\}.$$

It is easy to check that $\mu|X_i$ is continuous, so $dec(\mu) \leq n+1$. To show that $dec(\mu) \geq n+1$, assume towards contradiction that $dec(\mu) \leq n$. Let $Y_0, \ldots, Y_{n-1}$ be such that $\mu|Y_i$ is continuous and $\bigcup_{i=0}^{n-1} Y_i = K(X)$. Now find an open set $U \supset N$ such that

$$\mu(U \setminus N) < \min\{\mu(\{x\}) : x \in N\}. \tag{6}$$

Notice that for any $A \subset N$ the set $\{F \in K(X) : F \subset U,\ F \cap N = A\}$ is $\mathbf{\Pi}^0_2$, so we can apply the Baire Category Theorem, and that if $A \subset A'$ then

$$\{F \in K(X) : F \subset U,\ F \cap N = A'\} \subset \overline{\{F \in K(X) : F \subset U,\ F \cap N = A\}}$$

(this holds since the points in N are not isolated). Using this, we recursively construct $A_j \subset N$ and $Z_j \subset \{F \in K(X) : F \subset U,\ F \cap N = A_j\}$, $j \in \{0, \ldots, n\}$, so that

(iv) $|A_j| = j$;

(v) $A_j \subset A_{j+1}$ for $j < n$;

(vi) $\forall j \leq n \exists i \leq n-1\ Z_j \subset Y_i$;

(vii) $\overline{Z}_j$ contains a nonempty, relatively open subset of $\{F \in K(X) : F \subset U,\ F \cap N = A_j\}$;

(viii) $Z_{j+1} \subset \overline{Z}_j$ for $j < n$.

Using (vi), by the pigeonhole principle, we get $j_1 < j_2 \leq n$ and $i_0 \leq n-1$ with $Z_{j_1}, Z_{j_2} \subset Y_{i_0}$. Let $x_0 \in A_{j_2} \setminus A_{j_1}$. Then

$$\mu|Z_{j_2} \geq \mu(A_{j_2}) \geq \mu(A_{j_1}) + \mu(\{x_0\}). \tag{7}$$

On the other hand,

$$\mu|Z_{j_1} \leq \mu(A_{j_1}) + \mu(U \setminus N),$$

whence, since $\mu|Y_{i_0}$ is continuous,

$$\mu|(Y_{i_0} \cap \overline{Z}_{j_1}) \leq \mu(A_{j_1}) + \mu(U \setminus N). \tag{8}$$

But (7) and (8) contradict (6), since, by (viii), $Z_{j_2} \subset Y_{i_0} \cap \overline{Z}_{j_1}$.

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