Definable Equivalence Relations On Polish Spaces

Thesis by
Achim Ditzen

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Contents

Abstract

In the first part of this work we deal with the classification of definable equivalence relations on Polish spaces, where we take definable to mean inside some model of determinacy: We work in ZF+DC+$AD_{\mathcal{R}}$. The classification is up to bireducibility (denoted by $E \sim F$), that is if E and F are equivalence relations on the Baire space $\mathcal{N}$, then $E \sim F$, if there is a mapping $f : \mathcal{N} \to \mathcal{N}$ with $\forall x, y \in \mathcal{N}\ (xEy \Leftrightarrow f(x)Ff(y))$, called a reduction of E into F, and vice versa.

As two equivalence relations on Polish spaces are bireducible just in case there is a bijection between their quotient spaces, our results apply to definable cardinality theory, too. We show that up to bireducibility there are only four infinite hypersmooth equivalence relations: equality on the integers, equality on the Baire space, E_0 on the Cantor space 2^ω given by

$$\alpha E_0 \beta \ \Leftrightarrow\ \exists n \in \omega \forall m > n\ (\alpha(m) = \beta(m)),$$

and E_1 on the countable product of Cantor space $(2^\omega)^\omega$ given by

$$\bar{\alpha} E_0 \bar{\beta} \ \Leftrightarrow\ \exists n \in \omega \forall m > n\ (\alpha_m = \beta_m).$$

Even though we only develop the theory for the context of $AD_{\mathcal{R}}$, it is clear from the proofs that our results apply to a variety of other settings, such as the one encountered in the second part.

In the second part of the thesis we deal with countable Borel equivalence relations E on Polish spaces X, that is with equivalence relations which have countable classes and Borel graphs. The space $\mathcal{M}$ of probability measures on these spaces is again Polish. Of special interest are invariant measures (i.e. those which are preserved under bijections $f : X \to X$ with $f(x)Ex$, so called automorphisms), quasiinvariant measures (i.e. those whose measure class is preserved under automorphisms), and ergodic measures (i.e. those which assign full or null measure to E-invariant Borel sets).

We show that the collections of ergodic measures and of ergodic quasiinvariant measures are Borel. We also classify the complexity of the σ-ideal of nullsets with respect to all invariant measures, showing that this ideal is $\mathbf{\Pi}^1_1$ in the codes of $\mathbf{\Delta}^1_1$ and $\mathbf{\Sigma}^1_1$ sets, and that the σ-ideal of compact nullsets with respect to all invariant measures is $\mathbf{\Pi}^0_2$ if the collection of invariant ergodic measures is at most countable, and $\mathbf{\Pi}^1_1$-complete otherwise.

Chapter 1

Dichotomy Theorems

In this chapter we prove two dichotomy theorems about the reducibility and embeddability relations of equivalence relations. Let us first consider an example: Consider the space X of normal multiplicity free operators on a separable Hilbert space and the equivalence relation E of unitary equivalence. It is well known that two such operators are unitarily equivalent iff their spectral measures are in the same measure class, i.e., have the same null sets. Thus the map f on X into the space Y of measures satisfies the condition

$$xEx' \Leftrightarrow f(x)Ff(x') \text{ for all } x, x' \in X,$$

where F is the relation of being in the same measure class. We call a map satisfying the above condition a **reduction** of E into F, and an **embedding** if it is in addition injective.

Under the axiom of choice reducibility and embeddability are trivial notions which depend only on the cardinalities of the quotient spaces, and on the cardinality of the equivalence classes for embeddability.

We are interested here in these notions for "definable" objects on Polish spaces, i.e., where the spaces are Polish spaces and the equivalence relations and maps are "definable". In this context we want to classify "definable" equivalence relations up to "definable" reducibility and embeddability. "Definable" means in some pointclass such as projective, inductive, inside $L(\mathcal{R})$ or some inner model, etc.

We can also apply our theory to the study of "definable" cardinality theory. Here one investigates the cardinalities of sets I which are "definable" surjective images of the reals, or equivalently, which are "definable" quotient spaces of the reals. Such sets, say I and J, are compared via "definable" injections and bijections:

$$\begin{aligned} I \leq_D J &\Leftrightarrow \text{ there is a "definable" injection from } I \text{ into } J \\ I \sim_D J &\Leftrightarrow \text{ there is a "definable" bijection from } I \text{ onto } J \\ &\Leftrightarrow I \leq_D J \text{ and } J \leq_D I. \end{aligned}$$

An appropriate context for this theory is to take all definable objects to be inside an inner model of ZF+DC+$\mathrm{AD}_{\mathcal{R}}$, the axiom of dependent choice plus determinacy of games of reals. In such models one has full uniformization; thus injections between quotient spaces of the reals correspond exactly to reductions of the equivalence relations giving the quotient spaces.

Let us work from now on in an inner model of ZF+DC+$\mathrm{AD}_{\mathcal{R}}$. It will be clear that our discussion also applies to the other cases mentioned above, provided we have the appropriate level of determinacy.

Our results concern the equivalence relations E_0 on 2^ω, the space of infinite $\{0,1\}$-sequences, and E_1 on $(2^\omega)^\omega$, the space of sequences of such sequences. Since 2^ω is homeomorphic to the Cantor set, we may also think of $(2^\omega)^\omega$ as sequences of elements of the Cantor set.

Two elements $\alpha, \alpha' \in 2^\omega$ are said to be E_0-equivalent iff

$$\exists k \in \omega \forall n \geq k(\alpha(n) = \alpha'(n)),$$

i.e., if they eventually agree. The importance of E_0 stems from the fact that it has two properties: E_0 is not λ**-smooth** for any $\lambda < \Theta$ (i.e., the supremum of the lengths of prewellorderings of the reals), i.e., there is no reduction of E_0 to equality on 2^λ. Also call E **smooth** if it is ω-smooth and Θ**-smooth** if E is λ-smooth for some $\lambda < \Theta$. Thus in this terminology E_0 is not Θ-smooth. On the other hand, E_0 is the increasing union of equivalence relations with finite classes, namely, those equivalence relations on 2^ω which relate elements which agree after a fixed point $k \in \omega$. We call equivalence relations with finite equivalence classes **finite**, and the increasing union of finite equivalence relations **hyperfinite**. There are several equivalent definitions of this notion:

1. E is hyperfinite, i.e., the increasing union of finite equivalence relations.

2. E is **countable** and **hypersmooth**, i.e., the equivalence classes are countable, and E is the increasing union of smooth equivalence relations.

3. There is a bijection T of X with itself such that E is the orbit equivalence relation of T.

We now have the following generalization of a result for the Borel context by Harrington-Kechris-Louveau [90]. We have recently learned that Foreman-Magidor (unpublished) have found the result below independently.

Theorem 1 *(ZF+DC+$AD_{\mathbb{R}}$) Let E be an equivalence relation on a Polish space. Then either E is Θ-smooth or E_0 embeds into E via a continuous function.*

The equivalence relation E_1 on $(2^\omega)^\omega$ is given by

$$\bar{\alpha} E_1 \bar{\alpha}' \Leftrightarrow \exists k \in \omega \forall n \geq k(\alpha_n = \alpha'_n)$$

for all $\bar{\alpha}, \bar{\alpha}' \in (2^\omega)^\omega$. Like E_0, E_1 is hypersmooth, but it is not countable; in fact, it is not reducible to any countable equivalence relation, in particular, not to any hyperfinite equivalence relation. We have the following well known

Fact 2 *E is hypersmooth if and only if E is embeddable into E_1 if and only if E is reducible to E_1.*

Proof: For the forward direction let $E = \bigcup_n F_n$ be a hypersmooth equivalence relation on X, where F_n is smooth and increasing, and let f_n reduce F_n to equality on 2^ω. Then define $f : X \to (2^\omega)^\omega$ by

$$f(\alpha)_{n+1} = f_n(\alpha)$$

and

$$f(\alpha)_0 = \alpha.$$

Clearly, f embeds E into E_1.

For the backward direction assume that $f : X \to (2^\omega)^\omega$ reduces E to E_1. Let F_k be given on $(2^\omega)^\omega$ by

$$\bar{\alpha} F_k \bar{\alpha}' \Leftrightarrow \forall n \geq k(\alpha_n = \alpha'_n).$$

Let $\tilde{F}_k$ be the pullback of F_k via f; i.e., for $x, x' \in X$ set

$$x \tilde{F}_k x' \Leftrightarrow f(x) F_k f(x').$$

Since $E_1 = \bigcup_k F_k$, we have $E = \bigcup_k \tilde{F}_k$, and both are increasing unions of smooth equivalence relations. Thus E is hypersmooth. □

We now have a generalization of a result by Kechris-Louveau [a] for the Borel context:

Theorem 3 *($ZF+DC+AD_{\mathcal{R}}$) Let E be an equivalence relation on a Polish space. If E is hypersmooth, then either E is reducible to E_0, or E_1 is embeddable into E via a continuous function.*

Thus up to reducibility, E_1 is the only non-hyperfinite hypersmooth equivalence relation, and this depends on whether or not E is reducible to a countable equivalence relation.

Recall now the following unpublished

Theorem 4 (Dougherty-Jackson-Kechris) *($ZF+DC+AD_{\mathcal{R}}$) Let E be a countable Θ-smooth equivalence relation on a Polish space. Then E is smooth.*

Then we have the following picture of equivalence relations on Polish spaces, where $<$ indicates that the former relation is reducible into the latter but not vice versa:

$$(=,\omega) < (=,2^\omega) < (E_0, 2^\omega) < (E_1, (2^\omega)^\omega).$$

And there are no other equivalence relations between these. This picture translates immediately into the following CH-type result for definable cardinality theory, where η_0 and η_1 are the cardinalities of the quotient spaces of E_0 and E_1, respectively:

$$\aleph_0 < 2^{\aleph_0} < \eta_0 < \eta_1,$$

with no other cardinalities in between.

Let us now summarize the organization of the rest of the chapter. In Section 2 we give some facts about hyperfinite equivalence relations, which we need later. In Section 3 we construct the pointclasses in which the constructions take place. We also summarize the properties of the pointclasses, which we need later. All constructions will use the combinatorial concept of a tree structure, which we introduce in Section 4. We prove theorem 1 in Section 5 and theorem 3 in Section 6. Finally, in Section 7 we give a generalization of a result by Harrington-Sami [79] to n-ary relations. There are also some applications of this result.

1.1 Hyperfinite Equivalence Relations

We give here some results about hyperfinite equivalence relations, which we need later, and which may be found with references in Dougherty-Jackson-Kechris [a]:

Theorem 5 *(ZF+DC+$AD_{\mathcal{R}}$) Assume that E is a countable equivalence relation on a Polish space X, and $E = \bigcup_n E_n$, where $E_n \subseteq E_{n+1}$ are smooth equivalence relations. Then E embeds into E_0.*

We will show (lemma 6) that such E are hyperfinite, i.e., the increasing union of finite equivalence relations. This will imply (lemma 7) that E is induced by an action of $\mathcal{Z}$. All equivalence relations E induced by $\mathcal{Z}$-actions embed into $E(\mathcal{Z}, {}^{\omega}2) \subseteq ({}^{\omega}2)^{\mathcal{Z}}$, where

$$\bar{\alpha} E(\mathcal{Z}, {}^{\omega}2) \bar{\beta} \Leftrightarrow \exists n \in \mathcal{Z} \forall m \in \mathcal{Z} (\alpha_{m+n} = \beta_m),$$

by the following map $f : X \to ({}^{\omega}2)^{\mathcal{Z}}$: Let $\{U_i : i \in \omega\}$ be a family of subsets of X separating points. For $x \in X$ and $m \in \mathcal{Z}$, let $m.x$ be the action of m on x. Then

$$f(x)(m)(i) = 1 \Leftrightarrow m.x \in U_i.$$

Since $E(\mathcal{Z}, {}^{\omega}2) \sqsubseteq E_0$ by theorem 7.1. of Dougherty-Jackson-Kechris [a], all that remains is to show the above-mentioned lemmas.

Lemma 6 *($AD_{\mathcal{R}}$) Assume that E is a countable equivalence relation on a Polish space X. Then there is a countable group G and a group action of G on X such that $E = E_G$.*

Proof: Let $E = \bigcup_n F_n$, where F_n is the graph of a total function. (To find the F_n, let $R \subset X \times {}^{\omega}X$ be given by $(x, \bar{y}) \in R \Leftrightarrow \{y_n : n \in \omega\} = [x]_E$, let R^* uniformize R and let $(x, y) \in F_n \Leftrightarrow \exists \bar{y}((x, \bar{y}) \in R^* \wedge \bar{y}_n = y)$.) Let $\{R_k : k \in \omega\}$ be a sequence of rectangles, where $R_k = I_k \times J_k$ with I_k and J_k disjoint, such that $\bigcup_k R_k = X^2 - \{(x, x) : x \in X\}$. Let $\hat{F}_n$ be given by $(x, y) \in \hat{F}_n \Leftrightarrow (y, x) \in F_n$. Let $G_{n,m,k} = F_n \cap \hat{F}_m \cap R_k$. Note that $G_{n,m,k}$ is the graph of a partial E-invariant function $g'_{n,m,k}$with disjoint domain and range. Furthermore, $\bigcup_{n,m,k} G_{n,m,k} = E$. Define $g_{n,m,k} : X \to X$ by

$$g_{n,m,k}(x) = \begin{cases} g'_{n,m,k}(x) & \text{if } x \text{ is in the domain of } g'_{n,m,k}, \\ g'^{-1}_{n,m,k}(x) & \text{if } x \text{ is in the range of } g'_{n,m,k}, \\ x & \text{otherwise.} \end{cases}$$

Let G be the group generated by the $g_{n,m,k}$. Since the union of the graphs of the $g_{n,m,k}$ is E, $E_G = E$. □

Lemma 7 *($AD_{\mathcal{R}}$) Assume that $E = \bigcup_n E_n$ is a countable equivalence relation which is the increasing union of smooth equivalence relations E_n. Then E is hyperfinite; i.e., E is the union of an increasing sequence of finite equivalence relations.*

Proof: Let s_n be a selector (i.e., for all x and y, $s_n(x) E_n x$ and $s_n(x) = s_n(y)$ whenever $x E_n y$) of E_n, $G_n = \{g_n^k : k \in \omega\}$ be a group inducing E_n with g_n^0 the identity map, and define F_n by

$$\begin{aligned} xF_ny \;\Leftrightarrow\; & \exists m \leq n[xE_my \\ & \wedge \exists k_0, \ldots, k_m \leq n(x = g_0^{k_0} s_0 g_1^{k_1} \ldots g_m^{k_m} s_m(x)) \\ & \wedge \exists l_0, \ldots, l_m \leq n(y = g_0^{l_0} s_0 g_1^{l_1} \ldots g_m^{l_m} s_m(y))]. \end{aligned}$$

We will show that

1. $F_n \subseteq F_{n+1}$,
2. $F_n \subseteq E_n$,
3. $E \subseteq \bigcup_n F_n$,
4. F_n is an equivalence relation,
5. F_n is finite.

1. and 2. are immediate. For 3., note that if xEy, then there are m and $k_0, \ldots, k_m$ and $l_0, \ldots, l_m$ such that xE_my and $x = g_i^{k_i} s_i(x)$ and $y = g_i^{l_i} s_i$ for $0 \leq i \leq m$. Let $n = \max\{m, k_0, \ldots, k_m, l_0, \ldots, l_m\}$. Then xF_ny.

For 4.: Clearly, F_n is symmetric and reflexive. Let us check that F_n is transitive. Say that xF_ny and yF_nz. Let m, $i_0, \ldots, i_m$, $j_0, \ldots, j_m$, m', $k_0 \ldots, k_{m'}$, $l_0, \ldots, l_{m'} \leq n$ be such that

$$xF_my \;\wedge\; x = g_0^{i_0} s_0 g_1^{i_1} \ldots g_m^{i_m} s_m(x) \;\wedge\; y = g_0^{j_0} s_0 g_1^{j_1} \ldots g_m^{j_m} s_m(y)$$

and

$$yF_{m'}z \;\wedge\; y = g_0^{k_0} s_0 g_1^{k_1} \ldots g_{m'}^{k_{m'}} s_{m'}(y) \;\wedge\; z = g_0^{l_0} s_0 g_1^{l_1} \ldots g_{m'}^{l_{m'}} s_{m'}(z).$$

If $m = m'$, there is nothing to show. Assume without loss of generality that $m < m'$. Let $y' = g_{m+1}^{k_{m+1}} s_{m+1} \dots g_{m'}^{k_{m'}} s_{m'}(y)$. Since $y = g_0^{k_0} s_0 g_1^{k_1} \dots g_{m'}^{k_{m'}} s_{m'}(y)$- $= g_0^{k_0} s_0 g_1^{k_1} \dots g_m^{k_m} s_m(y')$, we have $y' F_m y F_m x$. Thus $s_m(y') = s_m(x)$ and $s_{m'}(y) = s_{m'}(x)$. Thus we have

$$\begin{aligned}
x &= g_0^{i_0} s_0 g_1^{i_1} \dots g_m^{i_m} s_m(x) \\
&= g_0^{i_0} s_0 g_1^{i_1} \dots g_m^{i_m} s_m(y') \\
&= g_0^{i_0} s_0 g_1^{i_1} \dots g_m^{i_m} s_m g_{m+1}^{k_{m+1}} s_{m+1} \dots g_{m'}^{k_{m'}} s_{m'}(y) \\
&= g_0^{i_0} s_0 g_1^{i_1} \dots g_m^{i_m} s_m g_{m+1}^{k_{m+1}} s_{m+1} \dots g_{m'}^{k_{m'}} s_{m'}(x).
\end{aligned}$$

This proves that xF_nz, and thus transitivity.

For 5.: Fix x. Assume that yF_nx and that $m, k_0, \dots, k_m, l_0, \dots, l_m$ witness this fact. Then $s_m(x) = s_m(y)$, so that $y = g_0^{l_0} s_0 g_1^{l_1} \dots g_m^{l_m} s_m(x)$; i.e., y is completely determined by $m, l_0, \dots, l_m$. Since all are bound by n, the F_n-equivalence class of x is finite. □

Lemma 8 *($AD_{\mathbb{R}}$) If E is hyperfinite, then there is a $\mathbb{Z}$-action inducing E.*

Proof: Assume without loss of generality that E is an equivalence relation on $X = {}^{\omega}2$. Let us first see that it suffices to find a relation $R \subseteq E$ such that for each E-equivalence class C, $R \cap C^2$ is a linear ordering of C of either ordertype $\mathbb{Z}$ or of finite ordertype. For assume that we have such an R, then we can define $T : X \to X$ as follows:

$$T(x) = y \Leftrightarrow [(xRy \wedge \forall z \neg(xRzRy)) \vee (yRx \wedge \forall z \neg(zRx \vee yRz)];$$

i.e., $T(x)$ is the successor of x in $R \cap [x]_E^2$ or if x is the last element, then $T(x)$ is the first element. Now let i act on x by T^i. Clearly, this action induces E.

Let E be the increasing union of finite equivalence relations E_n. In order to find R, we will find $\{R_n : n \in \omega\}$ such that

1. $R_n \subseteq R_{n+1} \subseteq E_{n+1}$,

2. For every E_n-equivalence class C, $R_n \cap C^2$ is a linear ordering of C,

3. If xR_ny and $\neg(zE_nx)$, then for all m, $\neg(xR_mzR_my)$.

Let us first see that this suffices. Let $R' = \bigcup_n R_n$. Clearly, $R' \subseteq E$ and $R \cap C^2$ is a linear ordering of C for every E-equivalence class C. Note that 3. implies that if $xR'y$, then there are at most finitely many elements between x and y. The only linear orderings (up to isomorphism) which satisfy this property are finite orderings, $\mathcal{Z}$, ω, and ω^*, the reverse ordering of ω. Thus let us define R by the following rules:

1. If $[x]_E$ has ordertype $\mathcal{Z}$ or is finite, then let R agree with R' on $[x]_E$.
2. If $[x]_E$ has ordertype ω, let for $y, z \in [x]_E$ yRz iff y, z both have an even number of R'-predecessors and $yR'z$ or y, z both have an odd number of R'-predecessors and $zR'y$ or y has an odd number of R'-predecessors and z has an even number of R'-predecessors.
3. If $[x]_E$ has ordertype ω^*, then apply the above definition to the reverse order of $R' \cap [x]_E^2$.

Clearly, R is as desired.

We construct R_n by induction on n: For R_0 put xR_0y iff xE_0y and $x < y$ in the lexicographical ordering. Put $xR_{n+1}y$ iff xR_ny or $xE_{n+1}y$ and not xE_ny and the R_n-least element in $[x]_{E_n}$ is lexicographically less than the R_n-least element in $[y]_{E_n}$.

It is clear that the R_n satisfy 1. - 3.. □

1.2 Auxiliary Pointclasses

We introduce here some pointclasses for later use. The approach is well-known; see e.g., Harrington-Sami [79]. Assume that $\lambda < \theta$ is a cardinal and $\bar{A} = \{A_i : i \in i_0\}$ is a finite sequence of λ-Suslin subsets of the spaces $\mathcal{N}^{n_i}$, $n_i \in \omega$, for $i \in i_0$. Then we can find for each $i \in i_0$ a tree T^i on $\omega^{n_i} \times \lambda$ such that A_i is the projection of the collection of paths through T^i; i.e., $A_i = p[T^i] = \{\bar{\alpha} \in \mathcal{N}^{n_i} : \exists \beta \in \lambda^\omega \forall k \in \omega((\bar{\alpha}|k, \beta|k) \in T^i)\}$. Let $\kappa > \lambda$ be the least ordinal such that $L_\kappa(\mathcal{N} \cup \{T^i : i \in i_0\})$ is admissible. We call the class Γ of Σ_1-definable subsets in $\bigcup_{n\in\omega} P((\mathcal{N} \cup \lambda)^n)$ over $L_\kappa(\mathcal{N} \cup \{T^i : i \in i_0\})$ with parameters in $\lambda \cup \{\lambda, \mathcal{N}\} \cup \{T^i : i \in i_0\}$ the **auxiliary class** for $\bar{A}$. We let $\check{\Gamma}$ denote its dual class, i.e., the class of complements of sets in Γ, and $\Delta = \Gamma \cap \check{\Gamma}$ the ambiguous class of Γ. The classes have the following properties:

1. Γ is closed under $\vee$, $\wedge$, $\exists^\omega$, $\forall^\omega$, $\exists^\lambda$, $\forall^\lambda$, $\exists^{\mathcal{N}}$, $\forall^{\mathcal{N}}$.

2. Γ is closed under substitution of elements from λ, and under permutation, identification and addition of variables.

3. $A_i \in \check{\Gamma}$ for all $i \in i_0$, and in fact for all $i \in i_0$, for all $s \in {}^n\lambda$, and for all $t \in {}^n(\omega^{n_i})$ $p[T^i_{s,t}] \in \check{\Gamma}$. Here $T^i_{s,t} = \{(s', t') \in T^i : s$ and s' are compatible and t and t' are compatible$\}$.

4. There is a pairing function $\lambda \times \lambda \to \lambda$ in Δ. *Proof:* The canonical well-ordering of Ord$\times$Ord given by $(\alpha, \beta) < (\gamma, \delta)$ iff $\max\{\alpha, \beta\} < \max\{\gamma, \delta\}$ or $\max\{\alpha, \beta\} = \max\{\gamma, \delta\}$ and $(\alpha, \beta) < (\gamma, \delta)$ in the lexicographical ordering has a Δ_0 definition. Now define $g : \text{Ord}\times\text{Ord} \to \text{Ord}$ by $g(\gamma, \delta) =$ the ordertype of $\{(\alpha, \beta) \in \text{Ord} \times \text{Ord} : (\alpha, \beta) < (\gamma, \delta)\}$, which has a Σ_1-definition. Since λ was least such that A was λ-Suslin, λ is a cardinal and thus $g(\alpha, \beta) < \lambda$ for all $\alpha, \beta < \lambda$. □

5. Γ is λ-parameterized. *Proof:* Enumerate the Σ_1-formulas with constant symbols for elements from $\lambda \cup \{\lambda, \mathcal{N}, T\}$ effectively. □

6. Γ is normed with norms into κ. *Proof:* Let $B \in \Gamma$ be defined by $\exists x \phi(x, y)$, where ϕ is Δ_0. Let $\psi : B \to \kappa$ be given by $\psi(b) =$ the least ξ such that $L_\kappa(\lambda \cup \{\lambda, \mathcal{N}, T\}) \models \exists x \in L_\xi(\lambda \cup \{\lambda, \mathcal{N}, T\})\phi(x, b)$. This works.□

7. Γ has the reduction property and $\check{\Gamma}$ has the separation property. *Proof:* This follows from Moschovakis [80], p. 204, 4B.10 and 4B.11.□

8. There is a Γ-coding of Δ-sets: There are $C \subseteq \lambda$, $D, \check{D} \subseteq \lambda \times \mathcal{N}$, $C, D \in \Gamma$, $\check{D} \in \check{\Gamma}$, such that

   (a) If $\xi \in C$, then $D_\xi = \check{D}_\xi$.

   (b) For every $B \in \Delta$, $B \subseteq \mathcal{N}$, there is $\xi \in C$ with $B = D_\xi$.

   Proof: Let U be Γ-universal for $\mathcal{N}$, $U \subseteq \lambda \times \mathcal{N}$. Let $\phi : \lambda^2 \to \lambda$ be the pairing function, $\phi_1, \phi_2 : \lambda \to \lambda$ be such that (ϕ_1, ϕ_2) is its inverse. Then put

$$V_1(\xi, x) \Leftrightarrow U(\phi_1(\xi), x),$$
$$V_2(\xi, x) \Leftrightarrow U(\phi_2(\xi), x).$$

Let D_1, D_2 reduce V_1, V_2 and put

$$\xi \in C \Leftrightarrow \forall x(V_1(\xi, x) \vee V_2(\xi, x)).$$

Then let $D = D_1$ and $\check{D} = (\lambda \times \mathcal{N}) - D_2$. These work. □

9. The recursion theorem holds: There is a Γ-universal set $U \subseteq \lambda \times (\lambda \cup \mathcal{N})$ such that for any Γ-recursive function $f : \lambda \rightarrow \lambda$ there is $\xi \in \lambda$ such that $U_\xi = U_{f(\xi)}$ and for any $V \subseteq \lambda \times (\lambda \cup \mathcal{N})$ in Γ there is $f : \lambda \rightarrow \lambda$ such that $\forall \xi \in \lambda(V_\xi = U_{f(\xi)})$.

10. The reflection theorem holds: Assume that $A \subseteq P(\mathcal{N})$ is Γ on Γ, i.e., $\{\gamma \in \lambda : U_\gamma \in \Gamma\} \in \Gamma$ for the universal set U. Then if $Y \in A$, $Y \in \Gamma$, there is $X \in A$, $X \in \Delta$, $X \subseteq Y$. *Proof:* Let $U \subseteq \lambda \times (\lambda \cup \mathcal{N})$ be universal such that the recursion theorem holds. Let $\phi : U \rightarrow \kappa$ be a Γ-norm. Let $\alpha, \beta \in \lambda$ be such that $U_\alpha = \{\gamma : U_\gamma \in A\}$ and $U_\beta = Y$. Consider the Γ-set $V \subseteq \lambda \times (\lambda \cup \mathcal{N})$ given by

$$V(\delta, y) \Leftrightarrow \phi(\beta, y) < \phi(\alpha, \delta).$$

Let $f : \lambda \rightarrow \lambda$ be given by the $s-m-n$-theorem such that $\forall \xi \in \lambda(V_\xi = U_{f(\xi)})$. Then find by the recursion theorem δ such that $V_\delta = U_{f(\delta)} = U_\delta$. If $U_\delta \notin A$, then $\delta \notin U_\alpha$; thus $\phi(\alpha, \delta) = \infty$; thus

$$\begin{aligned} V(\delta, y) &\Leftrightarrow \phi(\beta, y) < \infty \\ &\Leftrightarrow (\beta, y) \in U \\ &\Leftrightarrow y \in U_\beta = Y; \end{aligned}$$

thus $U_\delta = V_\delta = Y \in A$, a contradiction.

Thus $U_\delta \in A$; thus $\phi(\alpha, \delta) < \infty$; thus $V_\delta \in \Delta$. Since $V_\delta \subseteq Y$, this completes the proof. □

1.3 Trees And Tree Structures

Definition 9 If T is a directed tree, we denote the vertex set by $V(T)$ and the edge set of T by $E(T)$. If $e \in E(T)$ is an edge of T, then we denote by e_0 and e_1 the source and the target of e, respectively. If Γ is a (lightface) pointclass and $X \in \Gamma$ is nonempty, a **Γ-tree structure on X** is a triple $(T, \mathcal{A}, \mathcal{R})$, where T is a finite directed tree, $\mathcal{A}$ is an assignment $v \mapsto A_v$ of nonempty Γ-subsets of X to the vertices of T, and $\mathcal{R}$ is an assignment of Γ-relations $e \mapsto R_e$ to the edges of T such that we have for all edges $e \in E(T)$

$$A_{e_0} R_e A_{e_1},$$

i.e.

$$\forall x \in A_{e_0} \exists x' \in A_{e_1} xRx' \ \wedge \ \forall x' \in A_{e_1} \exists x \in A_{e_0} xRx'.$$

If $(T, \mathcal{A}, \mathcal{R})$ is a tree structure, then $(T, \mathcal{A}', \mathcal{R}')$ is said to **refine** $(T, \mathcal{A}, \mathcal{R})$ if for each vertex $v \in V(T)$ $A_v \supseteq A'_v$ and for each edge $e \in E(T)$ $R_e \supseteq R'_e$.

We now have the following

Lemma 10 *Let Γ be a pointclass which is closed under finite intersections, finite unions and existential quantification over $\mathcal{N}$. Let $X \in \Gamma$ be a nonempty subset of $\mathcal{N}$ with standard basis $\mathcal{B}$ such that $\mathcal{B} \subseteq \Gamma$. Let $(T, \mathcal{A}, \mathcal{R})$ be a Γ-tree structure on X.*

1. *If $\tilde{v} \in V(T)$, $B \subseteq A_{\tilde{v}}$ is a nonempty Γ-set, then there is a refinement $(T, \mathcal{A}', \mathcal{R})$ with $A'_{\tilde{v}} = B$.*
2. *If $\{x_v : v \in V(T)\}$ is a collection of points and $\{B_v : v \in V(T)\}$ is a collection of Γ-sets such that*

   $$x_v \in B_v \subseteq A_v \text{ for all } v \in V(T)$$

   and

   $$x_{e_0} R_e x_{e_1} \text{ for all } e \in E(T),$$

   then there is a refinement $(T, \mathcal{A}', \mathcal{R})$ of $(T, \mathcal{A}, \mathcal{R})$ with

   $$x_v \in A'_v \subseteq B_v \text{ for all } v \in V(T).$$

3. *There is a collection $\{x_v : v \in V(T)\}$ of points such that*

   $$x_v \in A_v \text{ for all } v \in V(T)$$

   and

   $$x_{e_0} R_e x_{e_1} \text{ for all } e \in E(T).$$

4. *If $\epsilon > 0$, then there is a refinement $(T, \mathcal{A}', \mathcal{R})$ of $(T, \mathcal{A}, \mathcal{R})$ such that A'_v has diameter $< \epsilon$ for all $v \in V(T)$.*

5. *If $\tilde{e} \in E(T)$, $S \subseteq R_{\tilde{e}}$, $B \subseteq A_{\tilde{e}_0}$, $C \subseteq A_{\tilde{e}_1}$ are nonempty Γ-sets with BSC, then there is a tree structure $(T, \mathcal{A}', \mathcal{R}')$ refining $(T, \mathcal{A}, \mathcal{R})$ with $B = A'_{\tilde{e}_0}$, $C = A'_{\tilde{e}_1}$, $R_{\tilde{e}} = S$ and $R'_e = R_e$ for $e \neq \tilde{e}$.*

6. *Assume that $\tilde{e} \in E(T)$, $R_{\tilde{e}}$ is an equivalence relation with $R_e \subset R_{\tilde{e}}$ for all $e \in E(T)$, the path between the vertices $s, t \in V(T)$ in T contains $\tilde{e}$, and $B \subseteq A_s$, $C \subseteq A_t$ are nonempty Γ-sets with $BR_{\tilde{e}}C$. Then there is a refinement $(T, \mathcal{A}', \mathcal{R})$ of $(T, \mathcal{A}, \mathcal{R})$ with $A'_s = B$ and $A'_t = C$.*

Proof:

1. We define A'_v by induction on the distance of v to $\tilde{v}$. Let $e \in E(T)$. If A'_{e0} is already defined, let $A'_{e_1} = \pi_1[(A'_{e0} \times A_{e_1}) \cap R_e]$. If A'_{e_1} is already defined, let $A'_{e0} = \pi_0[(A_{e_0} \times A'_{e_1}) \cap R_e]$. Here π_0 and π_1 are the projections onto the first and second coordinate of the product space, respectively.

2. Fix $\tilde{v} \in V(T)$. We define A'_v by induction on the distance of v to $\tilde{v}$. If A'_{e0} is already defined, let $A'_{e_1} = \pi_1[(A'_{e0} \times B_{e_1}) \cap R_e]$. If A'_{e_1} is already defined, let $A'_{e0} = \pi_0[(B_{e_0} \times A'_{e_1}) \cap R_e]$. In the first case we know that $x_{e_1} \in A'_{e_1}$. Similarly, in the second case.

3. Fix $\tilde{v} \in V(T)$ and $x_{\tilde{v}} \in A_{\tilde{v}}$. Then pick x_v by induction on the distance to $\tilde{v}$.

4. Use 3. to find a collection $\{x_v : v \in V(T)\}$ with the guaranteed properties. Find a collection $\{B_v : v \in V(T)\} \subseteq \mathcal{B}$ of basic open sets with diameter $< \epsilon$ and with $x_v \in B_v$ for all $v \in V(T)$. Then use 2. on the collections $\{x_v : v \in V(T)\}$ and $\{B_v \cap A_v : v \in V(T)\}$ to find $\mathcal{A}'$.

5. Let T_0 and T_1 be the two subtrees of T, which remain if $\tilde{e}$ is removed from $E(T)$. Assume that $\tilde{e}_i \in V(T_i)$. Note that $(T_i, \mathcal{A}|V(T_i), \mathcal{R}|E(T_i))$ for $i = 0, 1$ are tree structures, so find refinements $(T_i, \mathcal{A}'|V(T_i),$-$\mathcal{R}|E(T_i))$ with $A'_{\tilde{e}0} = B$ and $A'_{\tilde{e}_1} = C$. Since BSC, $(T, \mathcal{A}', \mathcal{R}')$ with $\mathcal{R}'$ as in the statement is a tree structure with the desired properties.

6. Let T_0 and T_1 be the two subtrees of T which remain if $\tilde{e}$ is removed from $E(T)$. Assume that $s \in V(T_0)$ and $t \in V(T_1)$. Again we know that $(T_i, \mathcal{A}|V(T_i), \mathcal{R}|E(T_i))$ for $i = 0, 1$ are tree structures, so find refinements $(T_i, \mathcal{A}'|V(T_i), \mathcal{R}|E(T_i))$ with $A'_s = B$ and $A'_t = C$. For all $u, v \in V(T_i)$ we have $A'_v R_{\tilde{e}} A'_u$ by induction on the distance between them, using transitivity of $R_{\tilde{e}}$. Since $A'_s R_{\tilde{e}} A'_t$, we have $A'_u R_{\tilde{e}} A'_v$ for all $u, v \in V(T)$. In particular, $A'_{\tilde{e}_0} R_{\tilde{e}} A'_{\tilde{e}_1}$, so that $(T, \mathcal{A}', \mathcal{R})$ is a tree structure.

□

1.4 The First Dichotomy Theorem

We prove here the first dichotomy theorem, working in ZF+DC+$\text{AD}_{\mathcal{R}}$ throughout. We draw here from ideas of Harrington-Kechris-Louveau [90], Harrington-Sami [79], and Foreman [89]. Let E be an equivalence relation on $\mathcal{N}$. By Woodin [a] let $\lambda < \Theta$ be the least cardinal such that E and $\check{E} = \mathcal{N}^2 - E$ are both λ-Suslin, and let T, $\check{T}$ be trees on $\omega^2 \times \lambda$ which prove this. Let Γ be the auxiliary class for E, $\check{E}$, as guaranteed by Section 1.2.

We can define the following λ-smooth equivalence relation R containing E. It is in fact the smallest such equivalence relation in Γ.

Definition 11 Let $R \subseteq \mathcal{N}^2$ be given by

$$\begin{aligned} xRy \;&\Leftrightarrow\; \forall B \in \Delta[B \text{ } E\text{-invariant} \Rightarrow (x \in B \Leftrightarrow y \in B)] \\ &\Leftrightarrow\; \forall B \in \Delta[(\forall v, w \in \mathcal{N}(v \in B \wedge vEw \Rightarrow w \in B)) \\ &\qquad \Rightarrow (x \in B \Leftrightarrow y \in B)], \end{aligned}$$

and $X \subseteq \mathcal{N}$ by

$$\begin{aligned} x \in X \;&\Leftrightarrow\; R_x \neq E_x \\ &\Leftrightarrow\; \exists y(xRy \wedge x\check{E}y). \end{aligned}$$

Since there is a Γ-coding of Δ-sets and Γ is closed under universal quantification, R and X are in $\check{\Gamma}$. If $X = \emptyset$, then E is λ-smooth and there is nothing to show. Thus assume that $X \neq \emptyset$.

1.4.1 The Embedding

We will construct the continuous embedding f of E_0 into $E|\bar{X}$ by constructing a sequence $\{i_n : n \in \omega\}$ of positive integers and a perfect binary tree $\{A^i_s : i \leq i_n, s \in 2^n, n \in \omega\}$ of $\check{\Gamma}$-subsets of X such that

(A) distinct paths through the tree are disjoint; i.e., $\bar{A}^{i_n}_s \cap \bar{A}^{i_n}_t = \emptyset$ if $s, t \in 2^n$, $s \neq t$, and

(B) the sets along one path are decreasing; i.e., $A^i_s \supseteq A^j_t$ if ($s = t \in 2^n$ and $i < j \leq i_n$) or if ($s \subset t$ and $s \neq t$), and

(C) for $s \in 2^n$, the diameter of A^{i_n} is at most $1/(n+1)$.

We then set $f(\alpha)$ to be the unique element in $\bigcap_{n\in\omega} \bar{A}^{i_n}_{\alpha|n}$. By (B) and (C) f is well defined, by (C) f is continuous, and by (A) f is injective. In order for f to be an embedding, it in fact suffices that it satisfies the two conditions of the following lemma.

Lemma 12 (Embedding Lemma) *If a function $f : 2^\omega \to \mathcal{N}$ satisfies*

1. $\forall \gamma, \gamma' \, (\gamma \not{E}_0 \gamma' \Rightarrow f(\gamma) \not{E} f(\gamma'))$,
2. $\forall \gamma \forall k \, (f(0^k\,\hat{}\,0\,\hat{}\,\gamma) E f(0^k\,\hat{}\,1\,\hat{}\,\gamma))$,

then it is an embedding of E_0 into E.

Proof: We prove the statement

$$\forall s, t \in 2^k \forall \gamma \, (f(s\,\hat{}\,\gamma) E f(t\,\hat{}\,\gamma))$$

by induction on k. For $k = 0$ there is nothing to show. Assume the statement for k and let $s, t \in 2^{k+1}$. If $s(k+1) = t(k+1)$, then the induction hypothesis already implies the claim. Thus assume w.l.o.g. that $s(k+1) = 0$ and $t(k+1) = 1$. Then

$$f(s\,\hat{}\,\gamma) E f(0^k\,\hat{}\,0\,\hat{}\,\gamma) E f(0^k\,\hat{}\,1\,\hat{}\,\gamma) E f(t\,\hat{}\,\gamma).$$

□

1.4.2 The Game G

In order to ensure 1. of the embedding lemma, we will play the following game:

$$\begin{array}{lllllll} G: & \mathrm{I} & A_0, B_0 & & A_2, B_2 & & \ldots \\ & \mathrm{II} & & A_1, B_1 & & A_3, B_3 & \ldots, \end{array}$$

where $A_i, B_i \in \check{\Gamma}$, $\emptyset \neq A_{i+1} \subseteq A_i \subseteq X$, $\emptyset \neq B_{i+1} \subseteq B_i \subseteq X$, $A_i R B_i$, $\operatorname{diam}(A_{2i+1}) < 1/(i+1)$, $\operatorname{diam}(B_{2i+1}) < 1/(i+1)$. Whoever violates these rules first, loses. If both follow these rules, then I wins iff $\bigcap_i \bar{A}_i E \bigcap_i \bar{B}_i$. (Here $\bar{A}$ is the topological closure of A.)

Since we want to apply $\mathrm{AD}_{\mathbb{R}}$ to conclude that II has a winning strategy in this game, we should play a coded version of this game, i.e., one where the players play reals instead of ordinals. For this we should fix a prewellordering of the reals of length λ and should code the ordinals by reals with the appropriate rank with respect to this prewellordering. It is easy to modify the argument below to work with the coded version of the game. For simplicity we continue to pretend to play G and assume that it is determined. In order to show that I does not have a winning strategy, and for later use, we need the following lemma.

Lemma 13 *1. If $A \in \check{\Gamma}$, $A^2 \cap R = A^2 \cap E$, then there is $B \in \Delta$, $A \subseteq B$, with $B^2 \cap R = B^2 \cap E$.*

2. If $A \in \Delta$, $A^2 \cap R = A^2 \cap E$, then there is $B \in \Delta$, $[A]_E \subseteq B$, with $B^2 \cap R = B^2 \cap E$.

3. If $A \in \check{\Gamma}$, $A^2 \cap R = A^2 \cap E$, then $A \cap X = \emptyset$.

4. Let A, B be $\check{\Gamma}$-sets with $(A \times B) \cap R \neq \emptyset$. Then $(A \times B) \cap E \neq \emptyset$.

Proof: Let $C(A)$ denote the statement $A^2 \cap R = A^2 \cap E$.

1. Consider $\mathcal{A} \subseteq \Gamma|\mathcal{N}$ defined by

$$\begin{aligned} \mathcal{A}(A) &\Leftrightarrow (\neg A)^2 \cap R \subseteq E \\ &\Leftrightarrow \forall x \forall y[(x \notin A \wedge y \notin A \wedge xRy) \Rightarrow xEy]. \end{aligned}$$

Note that $\mathcal{A}$ is Γ on Γ. Now let $A \in \check{\Gamma}$ with $C(A)$. Then $\mathcal{A}(\neg A)$; thus by reflection there is $B \in \Delta$, $\neg B \subseteq \neg A$, $\mathcal{A}(\neg B)$. But then $A \subseteq B$ and $C(B)$.

2. Apply 1. to $[A]_E$.

3. By 1. find $B' \in \Delta$, $[A]_E \subseteq B'$, $C(B')$. Then inductively find an effectively Δ-sequence $<B_n : n \in \omega>$ of Δ-sets such that

$$B_0 = B', \ B_{n+1} \supseteq [B_n]_E, \ C(B_n),$$

using the fact that 1. holds uniformly. If $B = \bigcup_n B_n$, then $B \in \Delta$, $A \subseteq B$, B is E-invariant and $C(B)$. Thus $B \cap X = \emptyset$.

4. Assume that $A, B \in \check{\Gamma}$ with $(A \times B) \cap E = \emptyset$. Then find by effective $\check{\Gamma}$-separation a Δ-sequence of Δ-sets C_n such that

   (a) $[A]_E \subseteq C_0 \subseteq \mathcal{N} - [B]_E$,

   (b) $[C_n]_E \subseteq C_{n+1} \subseteq \mathcal{N} - [B]_E$.

   Let $C = \bigcup_n C_n$. Then C separates A and B by (a), is E-invariant by (b), and $C \in \Delta$. Thus C is R-invariant and thus $(A \times B) \cap R = \emptyset$.

□

We now have

Lemma 14 *I has no winning strategy in G.*

Proof: Assume that σ is a strategy for I. We will play two runs of G, call the players I, II, I′ and II′ and their moves A_i, B_i and A'_i, B'_i. I and I' will follow σ. We indicate the moves in a diagram after the description of stage n:

Stage 0: I and I′ play their first moves. They are identical. Thus $S_0 = p[\check{T}] \cap R \cap B_0 \times B'_0 \neq \emptyset$ by lemma 13.3. Let $s_0 = t_0 = u_0 = \emptyset$.

Stage n: Assume that the runs have proceeded to the $(2n-2)$nd move and that sequences $s_i, t_i \in \omega^i$ and $u_i \in \lambda^i$ and sets $S_i \subseteq \not{E} \cap X^2$ have been defined for $i < n$ such that

1. I, I′ have followed σ and no player has lost for trivial reasons.

2. $s_i \subseteq s_{i+1}$, $t_i \subseteq t_{i+1}$, $u_i \subseteq u_{i+1}$

3. $p[T_{s_i,t_i,u_i}] \supseteq S_i$

4. $R \supseteq S_i \supseteq S_{i+1}$

5. $B_{2i} \supseteq \pi_0 S_i$, $B'_{2i} = \pi_1 S_i$

6. $A_{2i-1} \supseteq A_{2i} \supseteq A'_{2i-1} \supseteq A'_{2i} \supseteq A_{2i+1}$

Then find sequences $s_n, t_n \in \omega^n$ and $u_n \in \lambda^n$ extending s_{n-1}, t_{n-1} and u_{n-1}, respectively, such that

$$S'_n = p[\check{T}_{s_n,t_n,u_n}] \cap S_{n-1} \neq \emptyset.$$

Let

$$A = A'_{2n-2}$$

and

$$B = \pi_0 S'_n.$$

If $x \in B$, then using 4. and 5. there is $y \in B'_{2n-2}$ such that $(x,y) \in S_{n-1} \subseteq R$ and $z \in A'_{2n-2} = A$ such that yRz, thus xRz. Thus we can shrink A and B down to A_{2n-1} and B_{2n-1} such that $A_{2n-1} R B_{2n-1}$, and they are both sufficiently small. Let A_{2n} and B_{2n} be given by σ. Then let

$$S''_n = S'_n \cap B_{2n} \times \mathcal{N}$$

and

$$A = A_{2n}$$

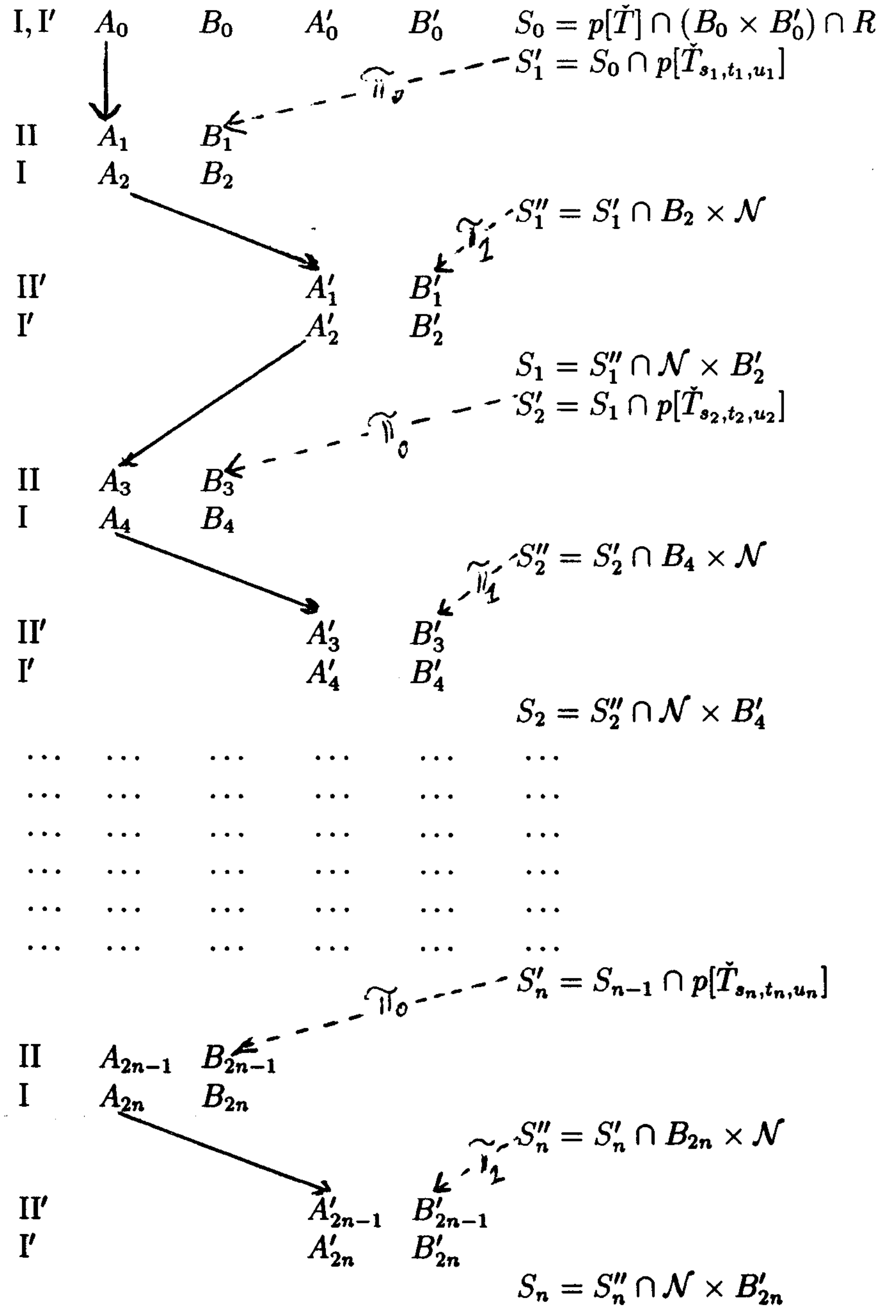
and

$$B = \pi_1 S''_n.$$

Again by transitivity of R we have $(A \times B) \cap R \neq \emptyset$. Thus shrink A and B down to sufficiently small A'_{2n-1} and B'_{2n-1} such that $A'_{2n-1} R B'_{2n-1}$. Let A'_{2n} and B'_{2n} be given by σ and finally set

$$S_n = S''_n \cap \mathcal{N} \times B'_{2n}.$$

This completes stage n.



Now let $\alpha, \beta, \alpha', \beta'$ be the reals produced in the two runs. Then by 3. we see that $\beta \not E \beta'$, by 6. $\alpha = \alpha'$. Thus we must either have $\beta \not E \alpha$ or $\beta' \not E \alpha'$; thus σ was not a winning strategy. □

Thus fix a winning strategy τ for II.

1.4.3 The Game G'

In order to ensure 2. of the embedding lemma, it will be convenient to play the following game:

$$G': \begin{array}{llllllll} \mathrm{I} & A_0, B_0 & & A_2, B_2 & & \dots & \\ \mathrm{II} & & A_1, B_1, R_1 & & A_3, B_3, R_3 & & \dots, \end{array}$$

where players I and II take turns playing pairs of nonempty decreasing (i.e., $A_i \supseteq A_{i+1}$, $B_i \supseteq B_{i+1}$) $\check{\Gamma}$-subsets A_i, B_i of X, and II plays in addition nonempty decreasing binary $\check{\Gamma}$-relations R_{2i+2} such that

$$A_0 R B_0 \ \wedge\ A_{2i+1} R_{2i+1} B_{2i+1} \ \wedge\ A_{2i+2} R_{2i+1} B_{2i+2}.$$

If either player violates these rules, the first to do so loses. If both players play according to the rules, player II wins if $\bigcap_n \bar{A}_n E \bigcap_n \bar{B}_n$.

Using lemma 13 and the tree T proving that E is λ-Suslin, we have

Lemma 15 *Player II has a winning strategy in G'.*

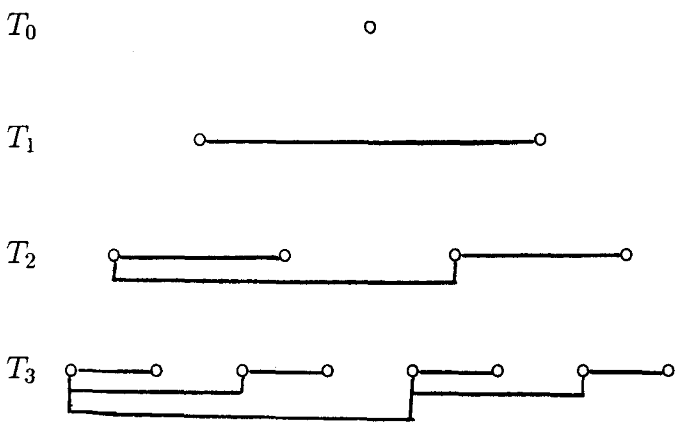
Proof: We will describe the strategy of II, which is winning. Assume that player I plays nonempty $\check{\Gamma}$-subsets A_0, B_0 of X with $A_0 R B_0$. Let $S_0 = (A_0 \times B_0) \cap E$, which is nonempty by lemma 13. Since $E = p[T]$, player II can pick $s_1, t_1 \in \omega^1$ and $u_1 \in \lambda^1$ such that $R_1 = p[T_{(s_1,t_1,u_1)}]$ intersects $A_0 \times B_0$. Player II plays $A_1 = \pi_0[(A_0 \times B_0) \cap R_1]$ and $B_1 = \pi_1[(A_0 \times B_0) \cap R_1]$ and R_1. Assume that player I responds with legal moves A_2, B_2. Since $A_2 R_1 B_2$, player II can find $s_3, t_3 \in \omega^3$ extending s_1 and t_1, respectively, and $u_3 \in \lambda^3$ extending u_1 such that $R_3 = p[T_{(s_3,t_3,u_3)}]$ intersects $A_2 \times B_2$. Player II plays $A_3 = \pi_0[(A_2 \times B_2) \cap R_3]$ and $B_3 = \pi_1[(A_2 \times B_2) \cap R_3]$ and R_3. He continues in the same manner. At the end of the run II assured that $\bigcap_n \bar{A}_n = \{\bigcup_n s_n\}$ and $\bigcap_n \bar{B}_n = \{\bigcup_n t_n\}$ and $(\bigcup_n s_n, \bigcup_n t_n, \bigcup_n u_n) \in [T]$, so that indeed $\bigcap_n \bar{A}_n E \bigcap_n \bar{B}_n$, and II wins the run. □

1.4.4 Trees on 2^n

Before we can give the construction of the complete binary tree of $\check{\Gamma}$-subsets $\{A^i_s\}$ of X, we will need to construct a directed tree T^n on each 2^n. If $s,t \in 2^n$ are linked by an edge in T^n, then this will indicate that there will have to be one round of a run of G' among the A^i_s, A^i_t. But let us construct the trees first. T_0 has just one vertex and no edges. Assume that T^n is given. We obtain T^{n+1} by taking two copies of T_n and joining their zeros:

$$\forall i,j \in \{0,1\} \forall s,t \in 2^n [(s\hat{\ }{<}i{>}, t\hat{\ }{<}j{>}) \in E(T^{n+1}) \Leftrightarrow$$
$$((i=j \wedge (s,t) \in E(T^n)) \vee (i=0 \wedge j=1 \wedge s=t=0^n))].$$

We have:



In fact, we can also give a direct definition of T^n, though it is not immediate from it that T^n is indeed a directed tree:

$$V(T^n) = 2^n$$
$$E(T^n) = \{(s,t) \in 2^n \times 2^n : \exists k < n \exists u \in 2^{n-(k+1)} (s = 0^k\hat{\ }0\hat{\ }u \wedge t = 0^k\hat{\ }1\hat{\ }u)\}.$$

1.4.5 Construction of the binary tree

We are now ready to construct the $\{A^i_s\}$ and $\check{\Gamma}$-relations $\{R^i_e : i \le i_n, e \in E(T^n), n \in \omega\}$ such that in addition to (A)-(C) of the above, we have:

(D) There is a sequence $\{j_{s,t} : \exists n \in \omega (s,t \in 2^n \wedge s(n-1) \neq t(n-1))\}$ of indices such that for each pair $\gamma, \gamma' \in 2^\omega$ with $\gamma \not\!E_0 \gamma'$, if $\{n_k : k \in \omega\}$ is the increasing enumeration of $\{n : \gamma(n) \neq \gamma'(n)\}$, then

$$\begin{array}{lllll} I & A^{j_0}_{\gamma|n_0}, A^{j_0}_{\gamma'|n_0} & & A^{j_1}_{\gamma|n_1}, A^{j_1}_{\gamma'|n_1} & & \ldots \\ II & & A^{j_0+1}_{\gamma|n_0}, A^{j_0+1}_{\gamma'|n_0} & & A^{j_1+1}_{\gamma|n_1}, A^{j_1+1}_{\gamma'|n_1} & \end{array}$$

is a run of G where II follows τ, and none of the players loses for trivial reasons, where $j_i = j_{\gamma|n_i, \gamma'|n_i}$ for $i \in \omega$.

(E) Note that for each $\gamma \in 2^\omega$ and each $n \in \omega$ $(0^n{}^\frown 0^\frown \gamma|k, 0^k{}^\frown 1^\frown \gamma|k)$ is an edge of T^{n+1+k}. Thus $R^i_{(0^n{}^\frown 0^\frown \gamma|k, 0^k{}^\frown 1^\frown \gamma|k)}$ exists. There are indices $\{j^n_s : n \in \omega, s \in 2^{<\omega}\}$ such that for each $\gamma \in 2^\omega$ and each $n \in \omega$

$$\begin{array}{ll} I & II \\ A^{j_0}_{0^n{}^\frown 0}, A^{j_0}_{0^n{}^\frown 1} & \\ & A^{j_0+1}_{0^n{}^\frown 0}, A^{j_0+1}_{0^n{}^\frown 1}, R^{j_0+1}_{(0^n{}^\frown 0, 0^n{}^\frown 1)} \\ A^{j_1}_{0^n{}^\frown 0^\frown \gamma|1}, A^{j_1}_{0^n{}^\frown 1^\frown \gamma|1} & \\ & A^{j_1+1}_{0^n{}^\frown 0^\frown \gamma|1}, A^{j_1+1}_{0^n{}^\frown 1^\frown \gamma|1}, R^{j_1+1}_{(0^n{}^\frown 0^\frown \gamma|1, 0^k{}^\frown 1^\frown \gamma|1)} \\ \ldots & \\ & \ldots \end{array}$$

is a run of G' where II follows τ', and none of the players loses for trivial reasons, where $j_i = j^n_{\gamma|i}$ for all $i \in \omega$.

(F) $(T^n, \mathcal{A}^i_n, \mathcal{R}^i_n)$ is a tree structure for each $n \in \omega$ and each $i \leq i_n$. $R^0_{0^n{}^\frown 0, 0^n{}^\frown 1} = R$ for all $n \in \omega$, where $\mathcal{A}^i_n = \{A^i_s : s \in 2^n\}$ and $\mathcal{R}^i_n = \{R^i_e : e \in E(T^n)\}$.

Condition (D) guarantees 1. of the embedding lemma and condition (E) guarantees 2. of the embedding lemma. Condition (F) is used to ensure that the construction can be carried on. Let us construct at stage n the A^i_s and R_0 and i_n for $s \in 2^n$ and $i \leq i_n$.

1.4.6 Stage 0

Let $A^0_\emptyset = X$, $j^0_\emptyset = i_0 = 0$.

1.4.7 Stage 1

In order to find two disjoint, R-related $\check{\Gamma}$-subsets of X, we show

Lemma 16 (Splitting Lemma) *Let $A, B \subseteq X$ be in $\check{\Gamma}$, A, B nonempty, ARB. Then there are nonempty, disjoint $C \subseteq A$, $D \subseteq B$ in $\check{\Gamma}$ with CRD and $\bar{C} \cap \bar{D} = \emptyset$.*

Proof: Assume first then $(A \times B) \cap R \not\subseteq \{(x, x) : x \in \mathcal{N}\}$. Then find two distinct, R-related points $a \in A$, $b \in B$, choose disjoint neighborhoods V, W about them and let $C = \pi_0[R \cap ((A \cap V) \times (B \cap W))]$ and $D = \pi_1[R \cap ((A \cap V) \times (B \cap W))]$.

But we cannot have $A \times B \cap R \subseteq \{(x, x) : x \in \mathcal{N}\}$. Otherwise we have $A = B$, and thus $A^2 \cap R = A^2 \cap E$; thus $A \cap X = \emptyset$, a contradiction. □

Thus we can find by the splitting lemma two disjoint nonempty $\check{\Gamma}$-subsets A^0_0, A^0_1 of X which are R-related. Let $R^0_{0,1} = R$. Let A^1_0, A^1_1 be the answer according to τ in the situation

$$\begin{array}{ll} I & A^0_0, A^0_1 \\ II & \end{array}$$

of G and $R^1_{0,1} = R$. Let $j_{0,1} = 0$. This satisfies (D). Let $A^2_0, A^2_1, R^2_{0,1}$ be the answer according to τ' in the situation

$$\begin{array}{ll} I & A^1_0, A^1_1 \\ II & \end{array}$$

of G'. Let $j^0_\emptyset = 1$. This satisfies (E). We played such that (F) is satisfied till now. Use the lemma 10.4 of Section 1.3 to shrink the A^2_s to have sufficiently small diameter in order to satisfy (C) and such that (F) remains satisfied. This completes stage 1; thus $i_1 = 3$.

1.4.8 Stage $n + 1$

Set $R^0_{s^\frown i, t^\frown i} = R^{i_n}_e$, $R^0_{0^n {}^\frown 0, 0^n {}^\frown 1} = R$ for $s \in 2^n$ and $i \in 2$; set $A^0_{s^\frown i} = A^{i_n}_s$ for $e = (s, t) \in E(T^n)$ and $i \in 2$. Thus we have the tree structure $(T^{n+1}, \mathcal{A}^0_{n+1}, \mathcal{R}^0_{n+1})$ with $\mathcal{A}^0_{n+1} = \{A^0_s : s \in 2^{n+1}\}$ and $\mathcal{R}^0_{n+1} = \{R^0_e : e \in E(T^{n+1})\}$.

Let $\{s_i : i < 2^n\}$ enumerate 2^n. We set out to shrink the A^i_s to ensure (A), i.e., that the closures of the $A^{i_{n+1}}_s$ are pairwise disjoint. For $s, t \in 2^n$

with $s \neq t$ and $i, j \in 2$ we have $\bar{A}^0_{s^\frown i} \cap \bar{A}^0_{t^\frown j} = \bar{A}^{i_n}_s \cap \bar{A}^{i_n}_t = \emptyset$. We will proceed by induction on $i \leq 2^n$ such that $\bar{A}^{i+1}_{s_i{}^\frown 0} \cap \bar{A}^{i+1}_{s_i{}^\frown 1} = \emptyset$ and such that $(T^{n+1}, \mathcal{A}^{i+1}_{n+1}, \mathcal{R}^{i+1}_{n+1})$ is a tree structure. Assume that we have defined $\mathcal{A}^i_{n+1}$. Since all the R^i_e are contained in R, and $(T^{n+1}, \mathcal{A}^i_{n+1}, \mathcal{R}^i_{n+1})$ is a tree structure, we have $A^i_s R A^i_t$ for all $s, t \in 2^{n+1}$. By the above splitting lemma, find subsets $B_j \subset A^i_{s_i{}^\frown j}$ such that $\bar{B}_0 \cap \bar{B}_1 = \emptyset$ and $B_0 R B_1$. Then use lemma 10 to find a tree structure $(T^{n+1}, \mathcal{A}^{i+1}_{n+1}, \mathcal{R}^{i+1}_{n+1})$ refining $(T^{n+1}, \mathcal{A}^i_{n+1}, \mathcal{R}^i_{n+1})$ with $B_j = A^{i+1}_{s_i{}^\frown j}$. Thus we can guarantee (A), and at the end of this induction we have constructed $\mathcal{A}^i_{n+1}$ and $\mathcal{R}^i_{n+1}$ for $i \leq 2^n$.

In order to guarantee (C), just use lemma 10 on each of the vertices of T^{n+1} successively. This takes 2^{n+1} steps, so that we have constructed $\mathcal{A}^i_{n+1}$ and $\mathcal{R}^i_{n+1}$ for $i \leq 2^n + 2^{n+1}$.

We now set out to guarantee (D). Let $\{(s_m, t_m) : m \in 2^{2n}\}$ be an enumeration of $2^n \times 2^n$. Assume that $\mathcal{A}^i_{n+1}$ and $\mathcal{R}^i_{n+1}$ have been defined for $i = 2^n + 2^{n+1} + m$. Let $\{l_0 \ldots l_k\}$ enumerate $\{l < n : s_m(l) \neq t_m(l)\}$ in increasing order. Note that

I	II
$A^{j_0}_{s_m\|l_0}, A^{j_0}_{t_m\|l_0}$	
	$A^{j_0+1}_{s_m\|l_0}, A^{j_0+1}_{t_m\|l_0}$
$A^{j_1}_{s_m\|l_1}, A^{j_1}_{t_m\|l_1}$	
	$A^{j_1+1}_{s_m\|l_1}, A^{j_1+1}_{t_m\|l_1}$
$\ldots$	
	$\ldots$
$\ldots$	
	$A^{j_k+1}_{s_m\|l_k}, A^{j_k+1}_{t_m\|l_k}$

with $j_i = j_{s_m|l_i, t_m|l_i}$ is a partial run of G, where neither player lost for trivial reasons and II followed his winning strategy τ. Set $j_{s^\frown 0, t^\frown 1} = 2^n + 2^{2n} + m = i$ and note that $A^i_{s_m{}^\frown 0}, A^i_{t_m{}^\frown 1}$ is a legal move of I. Let B_0, B_1 be the answer of τ to this move and use lemma 10 to find a tree structure $(T^{n+1}, \mathcal{A}^{i+1}_{n+1}, \mathcal{A}^{i+1}_{n+1})$ refining $(T^{n+1}, \{A^i_s\}, \{R^i_e\})$ with $A^{i+1}_{s_m{}^\frown 0} = B_0$ and $A^{i+1}_{t_m{}^\frown 1} = B_1$ and $R^{i+1}_e = R^i_e$ for all $e \in E(T^{n+1})$. After 2^{2n} of these steps, we have defined $\mathcal{A}^i_{n+1}$ and $\mathcal{R}^i_{n+1}$ and $\{j_{s,t}\}$ for $i \leq 2^n + 2^{n+1} + 2^{2n}$, and we ensured (D) for this stage.

We are left with ensuring (E). Enumerate the edgeset $E(T^{n+1}) = \{e^m : m < m_0\}$. Assume that $\mathcal{A}^i_{n+1}$ and $\mathcal{R}^i_{n+1}$ have been defined for $i = 2^n + 2^{n+1} +$

$2^{2n}+m$. Recall that e_0^m and e_1^m denote the ends of e^m. Let l_0 be such that $e^m=(0^{l_0-1}{}^\frown 0^\frown u, 0^{l_0-1}{}^\frown 1^\frown u)$ for some $u\in 2^{n+1-l_0}$. Note that $A^i_{e_0^m}, A^i_{e_1^m}$ is a legal move for I in the situation

$$
\begin{array}{ll}
I & II \\
A^{j_0}_{e_0^m|l_0}, A^{j_0}_{e_1^m|l_0} & \\
 & A^{j_0+1}_{e_0^m|l_0}, A^{j_0+1}_{e_1^m|l_0}, R^{j_0+1}_{e^m|l_0} \\
A^{j_1}_{s_m|l_1}, A^{j_1}_{t_m|l_1} & \\
 & A^{j_1+1}_{s_m|l_1}, A^{j_1+1}_{t_m|l_1} \\
\ldots & \\
 & \ldots \\
\ldots & \\
 & A^{j_{n-l_0}}_{e_0^m|n}, A^{j_k+1}_{e_1^m|n}, R^{j_0+1}_{e^m|n}
\end{array}
$$

of G', where $j_l=j^n_{u|l}$. Let $j^n_u=i$, and B_0, B_1, R^0 be the response to this move of I by II, who is following τ'. Then use lemma 10 to find a tree structure $(T^{n+1}, \mathcal{A}^{i+1}_{n+1}, \mathcal{A}^{i+1}_{n+1})$ refining $(T^{n+1}, \{A^i_s\}, \{R^i_e\})$ with $A^{i+1}_{s_m{}^\frown 0}=B_0$ and $A^{i+1}_{t_m{}^\frown 1}=B_1$ and $R^{i+1}_{e^m}=R^0$ and $R^{i+1}_e=R^i_e$ for $e\neq e^m$. We are done after m_0 steps in ensuring (E), and we set $i_n=2^n+2^{n+1}+2^{2n}+m_0$. This completes the construction of the $n+1$st stage and thus the proof.

1.5 The Second Dichotomy Theorem

We prove here the second dichotomy theorem. First we need

Lemma 17 *Let E be an equivalence relation on a Polish space X. Then E is hypersmooth iff $E\leq E_1$ iff $E\sqsubseteq E_1$.*

Proof: Assume that $E\leq E_1$. Let $F_n=\{(x,y)\in(2^\omega)^\omega\times(2^\omega)^\omega : \forall k\geq n\ (x_n=y_n)\}$. Let $f: X\to(2^\omega)^\omega$ reduce E to E_1. Let $\tilde{F}_n=(f\times f)^{-1}[F_n]=\{(x,y)\in X^2 : (f(x),f(y))\in F_n\}$. Since $\tilde{F}_n\leq F_n$ and F_n is smooth, so is $\tilde{F}_n$. Clearly, E is the increasing union of the $\tilde{F}_n$, so that E is hypersmooth.

Assume on the other hand that E is hypersmooth, and let E be the increasing union of an increasing sequence $\{\tilde{F}_n : n\in\omega\}$ of smooth equivalence relations, and let $f_{n+1}: X\to 2^\omega$ be a reduction of $\tilde{F}_n$ to equality. Let

$f_0 : X \to 2^\omega$ be an injection. Then $f : X \to (2^\omega)^\omega$ given by $f(x)_n = f_n(x)$ is an embedding of E into E_1. □

Since any hypersmooth equivalence relation E is isomorphic to $E_1|A$ for some $A \subseteq (2^\omega)^\omega$, it suffices to consider $E_1|A$ for $A \subseteq (2^\omega)^\omega$. By Woodin [a] let $\lambda < \theta$ be the least ordinal such that A is λ-Suslin and let T be a tree on $2 \times \lambda$ such that $A = p[T]$, where we identify $(2^\omega)^\omega$ with 2^ω via the recursive isomorphism $\bar{\alpha} \mapsto \alpha$ and $\alpha(<n,m>) = \bar{\alpha}(n)(m)$ with $<n,m> = 1/2(m+n)(m+n+1)+n$. Let Γ be an auxiliary pointclass for $\{A\}$ given by Section 1.2.

Let for $m > n$

$$\begin{aligned} Y_{n,m} &= \bigcup\{B \in \check{\Gamma} : B^2 \cap F_m \subseteq F_n\}, \\ X_{n,m} &= A - Y_{n,m}, \\ X_n &= \bigcup_{m>n} X_{n,m}, \\ X &= \bigcap_n X_n. \end{aligned}$$

By a reflection argument we have

$$Y_{n,m} = \bigcup\{B \in \Gamma \cap \check{\Gamma} : B^2 \cap F_m \subseteq F_n\};$$

thus

$$\begin{aligned} y \in Y_{n,m} &\Leftrightarrow \exists B \in \Gamma \cap \check{\Gamma}(y \in B \wedge B^2 \cap F_m \subseteq F_n) \\ &\Leftrightarrow \exists \xi \in C(y \in D_\xi \wedge \forall x \forall x'((x \in \check{D}_\xi \\ &\qquad \wedge x' \in \check{D}_\xi \wedge x F_m x') \Rightarrow x F_m x')), \end{aligned}$$

which is in Γ. Thus $X_{n,m} \in \check{\Gamma}$, and thus so are X_n and X.

We will show that $X = \emptyset$ implies that $E_1|A$ is reducible to E_0 and that $X \neq \emptyset$ implies that E_1 is continuously embeddable into $E_1|A$.

1.5.1 Case I: $X = \emptyset$

Note that $X = A - \bigcup_n \bigcap_{m>n} Y_{n,m}$; thus $Y = \bigcup_n \bigcap_{m>n} Y_{n,m}$ contains A. Since $Y \in \Gamma$, we can find by the separation property an $A' \in \Gamma \cap \check{\Gamma}$ with $A \subseteq A' \subseteq Y$. Since $A' = \bigcup_n (A' \cap \bigcap_{m>n} Y_{n,m})$, we can find by the effective reduction property a uniformly Γ collection $\{A_n : n \in \omega\}$ such that $A' = \bigsqcup_n A_n$

and $A_n \subseteq A' \cap \bigcap_{m>n} Y_{n,m}$. Thus $\{A_n : n \in \omega\}$ is uniformly $\Gamma \cap \check{\Gamma}$, since $A_n = A' - \bigcup_{n \neq m} A_m$.

Now define F'_n on A' by

$$xF'_n y \Leftrightarrow [(x, y \in \bigcup_{m \leq n} A_m \wedge xF_n y) \vee \exists m > n(x, y \in A_m \wedge xF_m y)].$$

Note that $F'_n \in \Gamma \cap \check{\Gamma}$ uniformly and that $E_1|A' = \bigcup_n F_n|A' = \bigcup_n F'_n|A'$. Also note that F'_n is uniformly smooth; i.e., we have uniformly $\Gamma \cap \check{\Gamma}$-recursive reductions ϕ_n of F'_n to equality on 2^ω. Let $A'' = \phi_0[A']$. Thus $A'' \in \check{\Gamma}$. Let $\{B^n_k\}_{k,n\in\omega}$ be a uniformly $\Gamma \cap \check{\Gamma}$ family such that $\{B^n_k\}_{k\in\omega}$ is a separating family for F'_n. Define equivalence relations F''_n on A'' and $C^n_k \subseteq A''$ by

$$\begin{aligned}
\alpha F''_n \beta &\Leftrightarrow \exists x \exists y[\phi_0(x) = \alpha \wedge \phi_0(y) = \beta \wedge xF'_n y] \\
&\Leftrightarrow \forall x \forall y[(\phi_0(x) = \alpha \wedge \phi_0(y) = \beta) \Rightarrow xF'_n y], \\
\alpha \in C^n_k &\Leftrightarrow \exists x[\phi_0(x) = \alpha \wedge x \in B^n_k] \\
&\Leftrightarrow \forall x[\phi_0(x) = \alpha \Rightarrow x \in B^n_k].
\end{aligned}$$

Let $E'' = \bigcup_n F''_n$. We also have the following:

Lemma 18 *F''_k is countable.*

Proof: Let $\alpha \in A''$ and $x \in A'$ such that $\phi_0(x) = \alpha$. Let n be such that $x \in A_n$. Note that since A_n and ϕ_0 are F'_0-invariant, n does not depend on x, but only on α. If $n \geq k$, let $g(\alpha) = (n, 0)$. If $n < k$, then $x \in A_n \subseteq Y_n = \bigcap_{m<n} Y_{n,m} \subseteq Y_{n,k}$. Let ξ be least such that

$$x \in [D_\xi]_{F_n} \wedge D^2_\xi \cap F_k \subseteq F_n.$$

Let $g(\alpha) = (n, \xi)$.

We will show that $g|[\alpha]_{F''_k}$ is injective. Let $\beta, \beta' \in [\alpha]_{F''_k}$ with $g(\beta) = g(\beta')$. Let $x, x' \in A'$ with $\phi_0(x) = \beta$ and $\phi_0(x') = \beta'$. Let n be such that $x, x' \in A_n$.

If $n \geq k$, then the definition of F''_k implies that $xF'_n x'$; thus $xF'_0 x'$; thus $\beta = \phi_0(x) = \phi_0(x') = \beta'$.

If $n < k$, let ξ be such that $x, x' \in [D_\xi]_{F_n}$ and $D^2_\xi \cap F_k \subseteq F_n$. Let $y, y' \in D_\xi$ be such that $xF_n y$ and $x'F_n y'$. Since β, β' are in the same F''_k-equivalence class, we have $xF'_k x'$; thus by transitivity $yF'_k y'$. But then we have $yF'_n y'$, since $y, y' \in D_\xi$. Thus by transitivity $xF'_n x'$; thus $xF'_0 x'$; thus $\beta = \phi_0(x) = \phi_0(x') = \beta'$.

Thus g maps every F''_k-equivalence class injectively into λ; thus each class has to be countable. □

Furthermore $\{C^n_k\}_{k\in\omega}$ is a separating family for F''_n; thus F''_n is smooth via the map $\phi_n : A'' \to 2^\omega$ given by

$$\phi_n(x) = \{k \in \omega : x \in C^n_k\}.$$

Thus $E'' = \bigcup_n F''_n \sqsubseteq E_0$ by theorem 5 of Section 1.1. Since $E_1|A$ is reducible to E'', it is also reducible to E_0.

1.5.2 Case II: $X \neq \emptyset$

Similarly to the recursive isomorphism between 2^ω and $(2^\omega)^\omega$, we get an injection $2^{<\omega} \to (2^{<\omega})^\omega$, $s \mapsto \bar{s} = <s_m : m \in \omega>$, given for $s \in 2^p$ by

$$s_m(k) = \begin{cases} s(<m,k>) & \text{if } <m,k> < p \\ \uparrow & \text{if } <m,k> \geq p; \end{cases}$$

i.e., s codes a finite number of finite sequences followed by empty sequences. For $p \in \omega$ let $L(p)$ be the least m such that $s_m = \emptyset$ for any (or equivalently for all) $s \in 2^p$. Let $\sim_j$ be the following equivalence relation on $2^{<\omega}$:

$$s \sim_j t \Leftrightarrow \forall m \geq j(s_m = t_m).$$

Note that with the above identification $2^\omega \to {}^\omega({}^\omega 2)$, we have

$$\alpha F_j \beta \Leftrightarrow \forall p(\alpha|p \sim_j \beta|p)$$

(i.e., we consider F_j as living on 2^ω pulled back via the bijection above. Similarly for A and E_1.) Furthermore, it is clear that for any p we have

$$\sim_0 |2^p \subseteq \sim_1 |2^p \subseteq \ \dots \ \subseteq \sim_{L(p)} |2^p,$$

and $\sim_0 |2^p$ is equality and $\sim_{L(p)} |2^p$ is $2^p \times 2^p$.

In order to find the continuous embedding f of E_1 into $E_1|A$, we will construct a strictly increasing function $M : \omega \to \omega$ and a collection $\mathcal{U} = \{U_s\}_{s\in 2^{<\omega}}$ of nonempty $\check{\Gamma}$-subsets of X such that

(a) $\forall s \in 2^{<\omega}[\emptyset \neq U_s \subseteq X \ \wedge \ (U_s \supseteq U_{s^\frown 0} \cup U_{s^\frown 1}) \ \wedge \ \mathrm{diam}(U_s) \leq 2^{-\mathrm{lh}(s)} \ \wedge$ $(\bar{U}_{s^\frown 0} \cap \bar{U}_{s^\frown 1}) = \emptyset]$,

(b) $\forall \alpha \in 2^\omega (\bigcap_p \bar{U}_{\alpha|p} \subseteq A)$,

(c) $\forall p \in \omega \forall s, t \in 2^p \forall j \leq L(p)(s \sim_j t \Rightarrow U_s F_{M(j)} U_t)$,

(d) $\forall p \in \omega \forall s, t \in 2^p \forall j \leq L(p)[\neg(s \sim_j t) \Rightarrow (\bar{U}_s \times \bar{U}_t) \cap F_j = \emptyset]$.

Assume that this can be done. Define $f : 2^\omega \to A$ as follows: For $\alpha \in 2^\omega$ (a) implies that $\{\bar{U}_{\alpha|p} : p \in \omega\}$ is a decreasing sequence of closed sets whose diameter tends to 0; thus $\bigcap_p \bar{U}_{\alpha|p}$ is a singleton $\{f(\alpha)\}$. Clearly, f is continuous, and by (b) it is into A.

If $\alpha, \beta \in 2^\omega$, $\neg(\alpha E_1 \beta)$, then for infinitely many j there is p such that $\neg(\alpha|p \sim_j \beta|p)$; thus $\neg(f(\alpha) F_j f(\beta))$ for infinitely many j; thus $\neg(f(\alpha) E_1 f(\beta))$.

If $\alpha, \beta \in 2^\omega$, $\alpha E_1 \beta$, then there is $j > 0$ such that for all $p \in \omega$, $\alpha|p \sim_j \beta|p$; thus $f(\alpha) F_{M(j)} f(\beta)$ by (c) and by the fact that $F_{M(j)}$ is closed.

Thus we are done, once we show that we can construct U and M satisfying (a) - (d).

In order to ensure (b), it is convenient to consider the following game G and to use the fact that A is λ-Suslin to show that II has a winning strategy. Alternatively, one could use the argument given in the proof below directly in the construction.

$$\begin{array}{llllll} G: & I & B_1 & & B_2 & & \dots \\ & II & & B_1' & & B_2' & & \dots \end{array}$$

In a run of G, players I and II take turns playing nonempty $\check{\Gamma}$-sets as indicated above such that $X \supseteq B_i \supseteq B_i' \supseteq B_{i+1}$. Player II wins the run iff $\bigcap_i \bar{B}_i \subseteq A$.

We will now describe II's winning strategy τ. Let T be a tree on $2 \times \lambda$ such that $p[T] = A$ and for all $n \in \omega$, $s \in 2^n$ and $t \in \lambda^n$ $p[T_{s,t}] \in \check{\Gamma}$.

Assume that I plays $B_1 \subseteq X \subseteq A$. Let $x_1 \in B_1$ and $t_1 \in \lambda^1$ such that $x_1 \in p[T_{x_1|1,t_1}] \cap B_1 = B_1'$. Let II play B_1' and I answer with B_2. Let $x_2 \in B_2$ and $t_2 \in \lambda^2$ such that $t_2 \supseteq t_1$ and $x_2 \in p[T_{x_2|2,t_2}] \cap B_2 = B_2'$. Continue in the same fashion.

Let $x = \bigcup_i x_i|i = \lim_i x_i$. Since $x_i \in B_j$ for $j > i$, we have $x \in \bar{B}_j$ for all j. By construction the diameter of B_i' is $\leq 2^{-i}$. Thus $\bigcap_i \bar{B}_i = \{x\}$. But $(x|i, t_i) \in T$ for all i; thus $x \in A$. Thus τ is winning.

To ensure (b) we will also find collections $\mathcal{V} = \{V_s : s \in 2^\omega\}$ and $\mathcal{V}' = \{V_s' : s \in 2^\omega\}$ such that

(b1) For all $\alpha \in 2^\omega$,

$$\begin{array}{lcccccc} I & V_{\alpha|0} & & V_{\alpha|1} & & \dots & \\ II & & V'_{\alpha|0} & & V'_{\alpha|1} & & \dots \end{array}$$

is a run of G where II followed τ.

(b2) $\forall p \in \omega \forall s \in 2^p \forall i \in 2 (U_s \supseteq V_{s^\frown i} \supseteq V'_{s^\frown i} \supseteq U_{s^\frown i})$.

Definition 19 A **labeled tree** is a pair (T, l_T), where T is a directed tree and $l_T : E(T) \to \omega$ is a labeling of the edges of T. If (T, l_T) is a labeled tree, we say that vertices $s, t \in V(T)$ are **n-linked** if there is an edge with label n between them, and we denote this by $s \overset{n}{—} t$. If all edges in the path between s and t have labels $\leq n$, we say that s and t are **n-connected** and write $s\text{-}^n\text{-}t$.

We have

Lemma 20 *Let S be a finite nonempty set and $\sim_0 \subseteq \sim_1 \subseteq \dots \subseteq \sim_k$ equivalence relations on S such that $\sim_0$ is equality and $\sim_k$ is $S \times S$. Then there is a finite tree T with vertex set S such that for all $s, t \in S$ and $0 < i \leq k$*

$$s \sim_i t \Leftrightarrow s\text{-}^i\text{-}t.$$

Proof: We proceed by induction on k. For $k = 1$ take T to be any tree on S and set all labels of T to 1. Assume that the lemma is known for k and that $\sim_0 \subseteq \sim_1 \subseteq \dots \subseteq \sim_{k+1}$ satisfy the assumptions of the lemma. Let $C_0, \dots, C_l$ be the $\sim_k$-equivalence classes of S. We can apply the assumption to each of the C_j and the equivalence relations $\sim_0 \subseteq \sim_1 \subseteq \dots \subseteq \sim_k$ restricted to C_j to obtain a tree T_j for which the lemma holds. Now pick a vertex $c_j \in V_{C_j}$ for each $j \in 0, \dots, l$, let T' be a tree on $\{c_j : 0 \leq j \leq l\}$ and label all edges of T' with $k + 1$.

Let T be the following labeled tree on S: If $s, t \in C_j$, then let s, t be n-linked in T iff s, t are n-linked in T_j. If $s \in C_i$, $t \in C_j$ and $i \neq j$, then let s, t be n-linked iff $n = k + 1$ and $s = c_i$ and $t = c_j$. It is easy to check that T works. □

Since $\{\sim_j |2^p : j \leq L(p)\}$ satisfies the prerequisites of the previous lemma, we fix for each p a labeled tree T_p on 2^p guaranteed by the lemma.

Using the fact that $M : \omega \to \omega$ will be strictly increasing and that the F_n are increasing equivalence relations, we can rewrite (c) as (c′) $(T_p, \mathcal{U}|2^p, \mathcal{F}_p)$ is a tree structure for all $p \in \omega$, where $\mathcal{F}_p = \{F_{m(l_{T_p}(e))}\}$.

We will construct M, U, V, and $\tilde{V}$ in stages. At the end of stage p we will have constructed $M|(L(p)+1)$, $\mathcal{U}|2^{\leq p}$, $\mathcal{V}|2^{\leq p}$, and $\tilde{\mathcal{V}}|2^{\leq p}$. But conditions (a),(b1),(b2),(c′),(d) are not strong enough to ensure the induction step. Thus we impose in addition conditions

(e1) $\forall B \in \check{\Gamma}(\emptyset \neq B \subseteq U_\emptyset \Rightarrow (B^2 \cap E_{M(0)}) \not\subseteq E_0)$,

(e2) $\forall p \in \omega \forall j \leq L(p+1) \forall B \in \check{\Gamma}(\emptyset \neq B \subseteq \bigcup_{s \in 2^p} U_s \Rightarrow (B^2 \cap F_{M(j)}) \not\subseteq F_j)$.

We now consider the following 3 cases separately:

(A) The construction of $V_\emptyset$, $\tilde{V}_\emptyset$, $U_\emptyset$, and $M(0)$.

(B) The construction of $\mathcal{V}|2^{p+1}$, $\tilde{\mathcal{V}}|2^{p+1}$, $\mathcal{U}|2^{p+1}$, and $M(L(p+1))$, given $\mathcal{V}|2^{<p+1}$, $\tilde{\mathcal{V}}|2^{<p+1}$, $\mathcal{U}|2^{<p+1}$, and $M|(L(p)+1)$, in the case that $L(p+1) > L(p)$.

(C) The construction of $\mathcal{V}|2^{p+1}$, $\tilde{\mathcal{V}}|2^{p+1}$, and $\mathcal{U}|2^{p+1}$, given $\mathcal{V}|2^{<p+1}$, $\tilde{\mathcal{V}}|2^{<p+1}$, $\mathcal{U}|2^{<p+1}$, and $M|(L(p)+1)$, in the case that $L(p+1) = L(p)$.

Case (A):

Lemma 21 *Let $B \in \check{\Gamma}|X$, $x_1, \ldots, x_k \in B$. Let $n \in \omega$. Then there is $m > n$ and a nonempty $B' \in \check{\Gamma}|B$ such that*

$$\forall C \in \check{\Gamma}|B'(C \neq \emptyset \Rightarrow C^2 \cap F_m \not\subseteq F_n)$$

and $x_1, \ldots, x_k \in B'$.

Proof: Since $X = \bigcap_n \bigcup_{m>n} X_{m,n}$, $B \subseteq \bigcup_{m>n} X_{m,n}$. Since the $X_{m,n}$ are increasing in m, find m such that $x_1, \ldots, x_k \in X_{m,n}$. Let $B' = B \cap X_{m,n}$. Then B' is as desired. □

Apply the lemma to X and $n = 0$ and some $x \in X$. Thus there is $M(0)$ and $V_\emptyset \in \check{\Gamma}$, $\emptyset \neq V_\emptyset \subseteq X$ such that (e1) holds for $V_\emptyset$ is place of $U_\emptyset$. Let $\tilde{V}_\emptyset$ be the reply of II to $V_\emptyset$ according to τ. Let $U_\emptyset \subseteq \tilde{V}_\emptyset$ be a nonempty $\check{\Gamma}$-set with diameter ≤ 1. This completes stage 0.

Case (B): Let $B = \bigcup_{s \in 2^p} U_s$ and pick by lemma 10 for each $s \in 2^p$ $x_s \in U_s$ such that

$$\forall j \in (L(p)+1) \forall s \in 2^p \forall t \in 2^p (s \sim_j t \Rightarrow x_s F_{M(j)} x_t).$$

By lemma 21 we can find $B' \in \check{\Gamma}$, $B' \subseteq B$, which contains all the x_s and $m = M(L(p+1)) > M(L(p))$ such that

$$\forall C \in \check{\Gamma} | B' \; [C \neq \emptyset \Rightarrow C^2 \cap F_m \subseteq F_{L(p+1)}].$$

Then use lemma 10 to find a refinement $(T, \mathcal{U}'|2^p, \mathcal{F}_p)$ of $(T, \mathcal{U}|2^p, \mathcal{F}_p)$ such that $\bigcup_{s \in 2^p} U'_s \subseteq B'$. By lemma 10 we may assume that (a) is satisfied. (c) and (d) are satisfied by construction. We have

Lemma 22 *There are collections* $\mathcal{U}^i = \{U^i_s : s \in 2^p\}$ *for* $i = 0,\ 1$ *of* Γ*-sets such that*

(1) $\forall s, t \in 2^p \forall i, i' \in 2 \; (U^i_s E_m U^{i'}_t)$,

(2) $(T, \mathcal{U}^i, \mathcal{F}_p)$ *are tree structures refining* $(T, \mathcal{U}'|2^p, \mathcal{F}_p)$,

(3) $\forall s, t \in 2^p (\bar{U}^0_s \times \bar{U}^1_t) \cap F_{L(p+1)} = \emptyset$.

We first show

Lemma 23 *Assume that* $\emptyset \neq B \in \check{\Gamma}$, $n < m \in \omega$ *such that*

$$\forall C \in \check{\Gamma} (\emptyset \neq C \subseteq B \Rightarrow C^2 \cap F_m \not\subseteq F_n).$$

Assume that $\emptyset \neq C, D \subseteq A$, $C, D \in \check{\Gamma}$ *such that* $C F_m D$. *Then there are nonempty* $C', D' \in \check{\Gamma}$ *with* $C' \subseteq C$, $D' \subseteq D$, $C' F_m D'$, *and* $(\bar{C}' \times \bar{D}') \cap F_n = \emptyset$.

Proof: First note that

$$(C \times D) \cap F_m \subseteq F_n \Rightarrow C^2 \cap F_m \subseteq F_n.$$

To see this, let $c, c' \in C$ with $c F_m c'$. Since $C F_m D$, let $d \in D$ with $c F_m d$. By transitivity $c' F_m d$, and then by assumption, $c F_n d$ and $c' F_n d$. Thus $c F_n c'$.

Thus we have $(C \times D) \cap F_m \not\subseteq F_n$. Let $c \in C$ and $d \in D$ with cF_md and $\neg(cF_nd)$. Then there must be $k \in [n,m)$ and $l \in \omega$ such that $c_{<k,l>} \neq d_{<k,l>}$. Let

$$\tilde{C} = C \cap \{x : x_{<k,l>} = c_{<k,l>}\},$$
$$\tilde{D} = D \cap \{x : x_{<k,l>} = d_{<k,l>}\}.$$

Clearly, $\bar{\tilde{C}} \times \bar{\tilde{D}} \cap F_n = \emptyset$ and $(c,d) \in (\tilde{C} \times \tilde{D}) \cap F_m$. Then $C' = \tilde{C} \cap [\tilde{D}]_{F_m}$ and $D' = \tilde{D} \cap [\tilde{C}]_{F_m}$ are as desired. □

Proof: (of lemma 22) Let $\{(s_j, t_j) : j = 1, \dots, j_0\}$ be an enumeration of $2^p \times 2^p$. We will find collections $\mathcal{U}^{i,j} = \{U_s^{i,j} : s \in 2^p\}$ of $\check{\Gamma}$-sets such that

(i) $(T_p, \mathcal{U}^{i,j}, \mathcal{F}_p)$ is a good tree structure,

(ii) $U_s = U_s^{0,0} = U_s^{1,0}$ for all $s \in 2^p$,

(iii) $(T_p, \mathcal{U}^{i,j+1}, \mathcal{F}_p)$ refines $(T_p, \mathcal{U}^{i,j}, \mathcal{F}_p)$,

(iv) for $j > 0$ $(\bar{U}_{s_j}^{0,j} \times \bar{U}_{t_j}^{1,j}) \cap F_{L(p+1)} = \emptyset$,

(v) for $j > 0$ $(U_{s_j}^{0,j} F_m U_{t_j}^{1,j})$.

Assume that this can be done. Let $U^i = U^{i,j_0}$. If $i = i'$, then (1) holds since $(T_p, \{U_s^i : s \in 2^p\}, \mathcal{F}_p)$ is a tree structure. We also have $U_{s_p}^0 F_m U_{t_p}^1$ by (v). Since $m > M(L(p))$, transitivity yields (1); (2) holds by (i); (3) holds by (iv) since $U_{s_j}^0 \times U_{t_j}^1 = U_{s_j}^{0,j_0} \times U_{t_j}^{1,j_0} \subseteq U_{s_j}^{0,j} \times U_{t_j}^{1,j}$, which is disjoint from $F_{L(p+1)}$. Thus we are left with constructing the $U^{i,j}$.

$U^{i,0}$ is given by (ii). Assume that $U^{i,j}$ are given. Note that the above argument shows that

$$\forall s,t \in 2^p \forall i,i' \in 2\ (U_s^{i,j} F_m U_t^{i',j}).$$

In particular, $U_{s_{j+1}}^{0,j} F_m U_{t_{j+1}}^{1,j}$. By lemma 23 find $C \subseteq U_{s_{j+1}}^{0,j}$ and $D \subseteq U_{t_{j+1}}^{1,j}$ such that

$$C F_m D \wedge (\bar{C} \times \bar{D}) \cap F_{L(p+1)} = \emptyset.$$

Then use lemma 10 to find $\mathcal{U}^{i,j+1}$ such that $(T_p, \mathcal{U}^{i,j+1}, \mathcal{F}_p)$ is a tree structure refining $(T_p, \mathcal{U}^{i,j}, \mathcal{F}_p)$ with $C = U_{s_{j+1}}^{0,j+1}$ and $D = U_{t_{j+1}}^{1,j+1}$. □

Now apply lemma 22 to $\{U' : s \in 2^p\}$ to obtain $\{U_s^0 : s \in 2^p\}$ and $\{U_s^1 : s \in 2^p\}$ as guaranteed. Put $\tilde{U}_s = U_{s|p}^{s(p)}$ for $s \in 2^{p+1}$. Then $(T_{p+1}, \tilde{\mathcal{U}}|2^{p+1}, \mathcal{F}_{p+1})$ is a tree structure to which the following lemma applies:

Lemma 24 *Let $M : L(p+1) \to \omega$ be increasing. Assume that $\mathcal{U} = \{\tilde{U}_s : s \in 2^{p+1}\}$, $\mathcal{V} = \{V_s : s \in 2^{\leq p}\}$, and $\mathcal{V}' = \{V'_s : s \in 2^{\leq p}\}$ are collections of nonempty $\check{\Gamma}$-sets such that*

(i) *$(T_{p+1}, \tilde{\mathcal{U}}|2^{p+1}, \mathcal{F}_{p+1})$ is a tree structure.*

(ii) *For all $s \in 2^p$,*

$$\begin{array}{lcccccccc} I & V_{s|0} & & V_{s|2} & & \cdots & & V_s & \\ II & & V'_{s|0} & & V'_{s|2} & & \cdots & & V'_s \end{array}$$

is a partial run of G where II followed τ,

(iii) *$\forall s \in 2^{p+1}(\tilde{U}_s \subseteq V'_{s|p})$.*

Then there are collections $\mathcal{U}' = \{U'_s : s \in 2^{p+1}\}$, $\mathcal{V} = \{V_s : s \in 2^{p+1}\}$, and $\mathcal{V}' = \{V'_s : s \in 2^{p+1}\}$ of nonempty $\check{\Gamma}$-sets such that

(i) *$(T_{p+1}, \mathcal{U}'|2^{p+1}, \mathcal{F}_{p+1})$ is a tree structure refining $(T_{p+1}, \tilde{\mathcal{U}}|2^{p+1}, \mathcal{F}_{p+1})$.*

(ii) *For all $s \in 2^{p+1}$,*

$$\begin{array}{lcccccccc} I & V_{s|0} & & V_{s|2} & & \cdots & & V_s & \\ II & & V'_{s|0} & & V'_{s|2} & & \cdots & & V'_s \end{array}$$

is a partial run of G where II followed τ.

(iii) *$\forall s \in 2^{p+1}(U'_s \subseteq V'_s \subseteq V_s \subseteq \tilde{U}_s)$.*

Proof: Let $\{s_j : 0 \leq j \leq 2^{p+1}\}$ enumerate 2^{p+1}. We will construct families $\mathcal{U}^j = \{U^j_s : s \in 2^{p+1}\}$ for $0 \leq j \leq 2^{p+1}$ such that

(1) $(T_{p+1}, \mathcal{U}^{j+1}, \mathcal{F}_{p+1})$ is a tree structure refining $(T_{p+1}, \mathcal{U}^j, \mathcal{F}_{p+1})$,

(2)

$$\begin{array}{lcccccccc} I & V_{s_j|0} & & \cdots & & V_{s_j|p-1} & & U^j_{s_j} & \\ II & & V'_{s_j|0} & & \cdots & & V'_{s_j|p-1} & & U^{j+1}_{s_j} \end{array}$$

is a partial run of G where II followed τ,

(3) $U^0_s = \tilde{U}_s$ for all $s \in 2^{p+1}$.

If this can be done, then we can set $U'_s = U_s^{2^{p+1}}$, $V_{s_j} = U^j_{s_j}$, and $V'_{s_j} = U^{j+1}_{s_j}$. Thus we are left with the construction of the $\mathcal{U}^j$, which we produce by induction on j. Assume that $\mathcal{U}^j$ is given. Let $U^{j+1}_{s_j}$ be the move of II according to τ in the situation

I	$V_{s_j\|0}$		$V_{s_j\|2}$		$\ldots$		$V_{s_j\|p-1}$		$U^j_{s_j}$
II		$V'_{s_j\|0}$		$V'_{s_j\|2}$		$\ldots$		$V'_{s_j\|p-1}$	

Then find U^{j+1}_s for $s \neq s_j$ by lemma 10. □

Apply lemma 22 to $m = M(L(p+1))$ and $\mathcal{U}'|2^{p+1}$ to obtain $\mathcal{U}|2^{p+1}$, $\{V_s : s \in 2^{p+1}\}$, and $\{V'_s : s \in 2^{p+1}\}$ such that (b1) and (b2) are satisfied and (a),(c),(d), and (e2) remain satisfied. Thus we are done with case (B).

Case (C): The construction is similar to case (B).

Lemma 25 *Let $(T_p, \mathcal{U}|2^p, \mathcal{F}_{p+1})$ be a tree structure, $n < L(p)$. Assume that*

$$\forall B \in \check{\Gamma}(\emptyset \neq B \subseteq \bigcup_{s \in 2^p} U_s \Rightarrow (B^2 \cap F_{M(n+1)}) \not\subseteq F_{n+1}).$$

Then there are two collections $\mathcal{U}^0 = \{U^0_s : s \in 2^p\}$ and $\mathcal{U}^1 = \{U^1_s : s \in 2^p\}$ of $\check{\Gamma}$-sets such that

(1) *$(T_p, \mathcal{U}^i, \mathcal{F}_{p+1})$ is a tree structure refining $(T_p, \mathcal{U}|2^p, \mathcal{F}_{p+1})$ for $i = 0,\ 1$,*

(2) $\forall s \in 2^p\ (U^0_s F_{M(n+1)} U^1_s)$,

(3) $\forall s, t \in 2^p\ (s \text{-}^n\text{-} t \Rightarrow (\bar{U}^0_s \times \bar{U}^1_t) \cap F_{n+1} = \emptyset$.

Proof: We prove the statement of the lemma for subtrees T of T_p by induction on the number of -^{n}--equivalence classes of $V(T)$. The case $V(T)$ having one -^{n}--equivalence class has been shown in lemma 22. Assume T is a subtree such that $V(T)$ has $k+1$ -^{n}--equivalence classes $C_0, \ldots, C_k$. Let T' be a labeled tree on $\{C_i : i \leq k\}$ given by $C_i \text{-}^k\text{-} C_j$ iff there are vertices $c_i \in C_i$ and $c_j \in C_j$ with $c_i \text{-}^k\text{-} c_j$ in T. Assume without loss of generality that C_k is an endnode of T' and that C_{k-1} is m-linked to C_k. Let e be the unique edge in T between C_{k-1} and C_k. We may assume without loss of generality that $e_0 \in C_{k-1}$ and $e_1 \in C_k$. We have $m = M(l_{T_p}(e)) > n$. Let T'' be the subtree of T with vertexset $C_0 \cup \ \ldots \ \cup C_{k-1}$ and T''' the subtree of T with vertexset C_k. Since T'' has k -^{n}-equivalence classes, we can apply the lemma to $(T'', U|V(T''), \mathcal{F}_{p+1}|E(T'')\})$ to obtain $\tilde{U}^i|V(T'')$ satisfying (1),(2),(3) for T''. Let $\tilde{U}^0_{e_1} = \tilde{U}^1_{e_1} = \pi_1[(\tilde{U}^0_{e_0} \times U_{e_1}) \cap F_{M(m)}]$. Then we have $\tilde{U}^0_{e_0} F_{M(m)} \tilde{U}^0_{e_1}$ and $\tilde{U}^1_{e_0} F_{M(m)} \tilde{U}^1_{e_1}$ by transitivity and (2). Now apply lemma 22 to obtain tree structures $(T''', \mathcal{U}^i|C_k, \mathcal{F}_{p+1}|E(T'''))$ with $\tilde{U}^0_s = \tilde{U}^1_s$ for $s \in C_k$. Then apply lemma 22 to $\tilde{U}^0|C_k$ to obtain $\{U^i_s : s \in C_k\}$ satisfying (1) - (3) for T''' and refining $\tilde{U}^0|C_k$. Put

$$U^0_{e_0} = \pi_0[(\tilde{U}^0_{e_0} \times U^0_{e_1}) \cap F_m]$$

and

$$U^1_{e_0} = \pi_0[(\tilde{U}^1_{e_0} \times U^1_{e_1}) \cap F_m]$$

and then find $\mathcal{U}^i|V(T'')$ such that $(T'', \mathcal{U}^i|V(T''), \mathcal{F}_{p+1}|E(T''))$ is a refinement of $(T'', \tilde{\mathcal{U}}^i|V(T''), \mathcal{F}_{p+1}|E(T''))$ by lemma 10. This completes the induction step and the proof of the lemma. □ Now we proceed as in case (B).

1.6 A Perfect Set Theorem for n-ary Relations

In Harrington-Sami [79] it was shown under the axiom of determinacy $AD_{\mathcal{R}}$ that every equivalence relation on a Polish space either has a perfect set of pairwise inequivalent elements or its equivalence classes are wellorderable. We generalize this theorem to n-ary relations.

Theorem 26 *(ZF+DC+$AD_{\mathcal{R}}$) Let $\lambda < \Theta$. Let A be a λ-Suslin subset of $\mathcal{N}$ and $R \subseteq R'$ be co-λ-Suslin relations on $\mathcal{N}^n$ such that*

$$\forall \bar{x} \in [\mathcal{N}]^{n-1} \forall \bar{y} \in \mathcal{N}^n [(\forall i < (n-1) R(\bar{x}, y_i)) \Rightarrow R'(\bar{y})],$$

and R is closed under permutation of arguments. Then either there is a sequence $\{A_\xi : \xi < \lambda\}$ of λ-Suslin subsets of A with $A^n_\xi \subseteq R'$ or there is a perfect set $P \subseteq A$ such that $[P]^n \cap R = \emptyset$.

Here are some applications (which extend corresponding results of H. Friedman and K. Kunen - A. Miller for the Borel and analytic case respectively):

Corollary 27 *Let* $d : X \times X \longrightarrow \mathcal{R}_0^+$ *be a distance function on a Polish space* X*. Then either the metric space induced by d is separable or there is some* $\epsilon > 0$ *and a perfect set* $A \subseteq X$ *of points such that* $\forall x, x' \in A\ (x \neq x' \Rightarrow d(x, x') > \epsilon)$.

Proof: Take the relations R_n on X the collection of pairs which have d-distance at least 2^{-n}. Then apply the perfect set theorem to each pair R_n and R_{n+1}. □

Corollary 28 *Let* $R \subseteq \mathcal{R}^2 \times \mathcal{R}^2 \times \mathcal{R}^2$ *be the relation of colinearity, i.e., say* $R(x, y, z)$ *iff* x, y, z *are contained in a single line. Then any subset of the plane is either contained in countably many lines or contains a perfect set of points no 3 of which are colinear.*

Proof: Apply the perfect set theorem. □

1.6.1 Proof of the Perfect Set Theorem

For simplicity of notation we assume that $n = 3$ and $A = \mathcal{N}$. There are no additional difficulties encountered in the general case.

Let T and T' be trees on $\lambda \times \omega^3$ such that $\mathcal{N}^3 - R = p[T]$ and $\mathcal{N}^3 - R' = p[T']$. Let Γ be an auxiliary class for $\mathcal{N}^3 - R$ and $\mathcal{N}^3 - R'$. We denote by Δ its ambiguous class of Γ and set

$$
\begin{aligned}
X &= \{x \in \mathcal{N} : \exists A \in \Delta(x \in A \wedge A^3 \subseteq R')\}, && (1.1)\\
Y &= \mathcal{N} - X. && (1.2)
\end{aligned}
$$

Note that $X \in \Gamma$ and $Y \in \check{\Gamma}$. If $Y = \emptyset$, then we are done. Thus assume that $Y \neq \emptyset$.

In order to ensure that every triple of our perfect set is not R-related, we play the following game G:

$$
\begin{array}{llll}
\text{I} & A_0^0, A_0^1, A_0^2 & & A_2^0, A_2^1, A_2^2 \\
\text{II} & & A_1^0, A_1^1, A_1^2 & \quad \ldots,
\end{array}
$$

where I and II play sets in $\check{\Gamma}$ satisfying $\emptyset \neq A_j^i \subseteq Y$, $A_{j+1}^i \subseteq A_j^i$ and $\mathrm{diam}(A_{2j+1}^i) < 1/(j+1)$. The first player who violates these rules loses. If both players play within the rules, then

$$\text{I wins iff } R(\bar{\alpha}), \text{ where } \{\alpha_i\} = \bigcap_j \bar{A}^i_j.$$

Since we want to use determinacy to conclude that II has a winning strategy, we should play a coded version of G: We should fix a surjection $\phi : \mathcal{R} \to \lambda$, and the players should play reals which via ϕ and the $\check{\Gamma}$-universal set code $\check{\Gamma}$-sets. For simplicity of notation we continue to use G and assume that it is determined. It is easy to modify the argument below to work with the coded version of the game. Thus by the following lemma, II has a winning strategy τ.

Lemma 29 *I does not have a winning strategy in G.*

We first show:

Lemma 30 (Splitting Lemma) *If $A \in \check{\Gamma}$, $\emptyset \neq A \subseteq Y$, then $A^3 \cap \neg R' \neq \emptyset$.*

Proof: Let $\mathcal{A} = \{A \in \Gamma : (\neg A)^3 \subseteq R'\}$ and note that

$$\mathcal{A}(A) \Leftrightarrow \forall \bar{x}[(\forall i \in 3(x_i \notin A)) \Rightarrow R'(\bar{x})]$$

is Γ on Γ. Now if $A \in \check{\Gamma}$, $A \subseteq Y$, $A^3 \subseteq R'$, then $\mathcal{A}(\neg A)$; thus by Γ-reflection there is a $\neg B \subseteq \neg A$, $\neg B \in \Delta$, $\mathcal{A}(\neg B)$. But then $B \supseteq A$, $B \in \Delta$, $B^3 \subseteq R'$. Thus $B \subseteq X$; thus $B \cap Y = \emptyset$, thus $A = \emptyset$. □

Proof: (of lemma 29) Assume otherwise and let σ be a winning strategy for I. We will play 3 runs of G. Call the players in the ith run I^i and II^i. I^i will follow σ. Call the reals produced in the ith run $(\alpha^i, \beta^i, \gamma^i)$. The games will be played in such a manner that $\alpha^i = \alpha^{i'}$, $\beta^i = \beta^{i'}$, and $\neg R(\gamma^1, \gamma^2, \gamma^3)$. Since I^i followed σ, we have $\forall i\ R(\alpha^i, \beta^i, \gamma^i)$, and we get a contradiction to the premises of the theorem.

We will play the games in stages. In stage 0 we will play the 0th moves of the games, in stage k the $2k-1$st and the $2k$th moves of the games. We will also at stage k find nonempty S_k and S^i_k $(i < 3)$ in $\check{\Gamma}|\mathcal{N}^3$ and sequences $s_k, t_k, u_k \in \omega^k$ and $v_k \in \lambda^k$, such that $S_k \subseteq p[T'_{s_k,t_k,u_k,v_k}] \subseteq (\mathcal{N}^3 - R')$.

We will denote the kth move in the ith game by A^i_k, B^i_k, C^i_k. The reader may want to refer to the diagram of the run of the game, which is given after the description of the moves:

Stage 0: Players $\mathrm{I}^0, \mathrm{I}^1, \mathrm{I}^2$ make their first moves A_0^i, B_0^i, C_0^i. They are identical. Let $s_0 = t_0 = u_0 = v_0 = \emptyset$,

$$S_0 = (\mathcal{N}^3 - R') \cap \prod_i C_0^i.$$

Note that $S_0 \neq \emptyset$ by the Splitting Lemma, since $C_0^i = C_0^{i'}$.

Stage k: Find $s_k \subseteq s_{k-1}$, $t_k \subseteq t_{k-1}$, $u_k \subseteq u_{k-1}$, and $v_k \supseteq v_{k-1}$ such that

$$S_k^0 = S_{k-1} \cap p[T'_{s_k,t_k,u_k,v_k}] \neq \emptyset.$$

Let A_{2k-1}^0, B_{2k-1}^0 be obtained as a subset of A_{2k-2}^2, B_{2k-2}^2 by some standard procedure such that the diameter of A_{2k-1}^0 and B_{2k-1}^0 is at most $1/2$ the diameter of A_{2k-2}^2 and B_{2k-2}^2. Let $C_{2k-1}^0 = \pi_0[T'_{s_k,t_k,u_k,v_k}]$. Find the $A_{2k}^0, B_{2k}^0, C_{2k}^0$ via σ.

Let

$$\begin{aligned} S_k^1 &= S_k^0 \cap (C_{2k}^0 \times \mathcal{N}^2), \\ A_{2k-1}^1 &= A_{2k}^0, \\ B_{2k-1}^1 &= B_{2k}^0, \\ C_{2k-1}^1 &= \pi_1[S_k^1]. \end{aligned}$$

Then find $A_{2k}^1, B_{2k}^1, C_{2k}^1$ via σ.

Let

$$\begin{aligned} S_k^2 &= S_k^1 \cap (\mathcal{N} \times C_{2k}^1 \times \mathcal{N}), \\ A_{2k-1}^2 &= A_{2k}^1, \\ B_{2k-1}^2 &= B_{2k}^1, \\ C_{2k-1}^2 &= \pi_2[S_k^2]. \end{aligned}$$

Then find $A_{2k}^2, B_{2k}^2, C_{2k}^2$ via σ. Finally, let $S_k = S_k^2 \cap (\mathcal{N}^2 \times C_{2k}^2)$. This completes stage k.

Note that the construction is such that

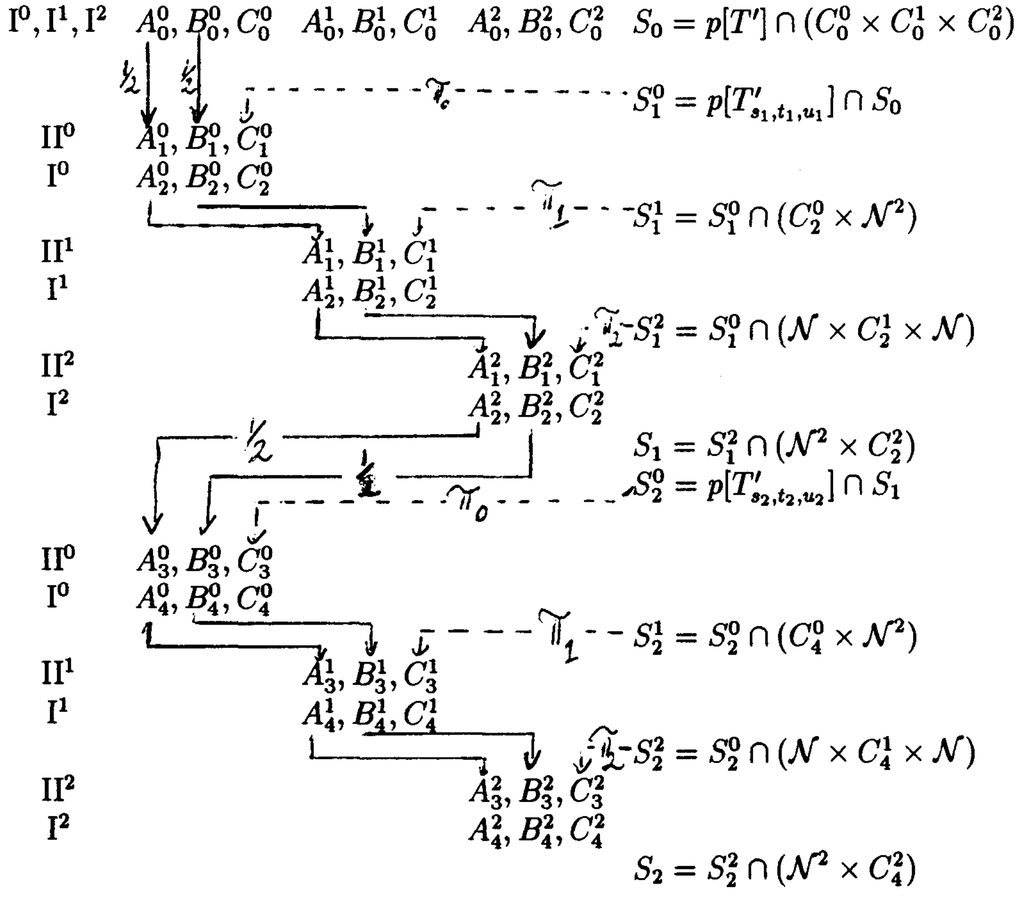
$$A_{2k-2}^2 \supseteq A_{2k-1}^0 \supseteq A_{2k}^0 \supseteq A_{2k-1}^1 \supseteq A_{2k}^1 \supseteq A_{2k-1}^2 \supseteq A_{2k}^2$$

and

$$B^2_{2k-2} \supseteq B^0_{2k-1} \supseteq B^0_{2k} \supseteq B^1_{2k-1} \supseteq B^1_{2k} \supseteq B^2_{2k-1} \supseteq B^2_{2k},$$

Thus $\alpha^0 = \alpha^1 = \alpha^2$ and $\beta^0 = \beta^1 = \beta^2$.

Also, the construction of the S_k guaranties that $\neg R'(\gamma^0, \gamma^1, \gamma^2)$, since $\gamma^0 = \bigcup_k s_k$, $\gamma^1 = \bigcup_k t_k$, $\gamma^2 = \bigcup_k u_k$ and the v_k witness that $(\gamma^0, \gamma^1, \gamma^2) \in p[T'] = \neg R$. Thus $\neg R(\alpha^i, \beta^i, \gamma^i)$ for some $i = 0, 1, 2$. Thus σ was not winning.



Here a solid arrow indicates that the set at the head of the arrow has been copied and equals the set at the tail of the arrow. A solid arrow with 1/2 inserted in the middle indicates that the set at the head is a nonempty subset of the set at the tail with at most half the diameter. A broken arrow indicates that the set at the head is a projection of the set at the tail. We show which projection was used. □

Let H_k be the collection of all lexicographically strictly increasing 3-sequences of 2^k. Let $h_k : H_k \to l_k$ be a bijection, where l_k is the cardinality of H_k. Let $h = \bigcup_k h_k$. Let $l : 2^{<\omega} \to \omega$, $s \mapsto l_k$, where k is the length of s.

We will find collections $\{A^i_s : s \in (2^{<\omega} - 2^{<2}), i < l(s)\}$ and $\{B^i_s : s \in (2^{<\omega} - 2^{<2}), i < l(s)\}$ of $\check{\Gamma}$-sets contained in Y such that

1. $B^i_s \subseteq A^i_s \subseteq B^j_t \subseteq A^j_t$ for $s \supseteq t$.

2. $B^i_s \subseteq A^i_s \subseteq B^j_s \subseteq A^j_s$ for $i > j$.

3. If $\gamma_0, \gamma_1, \gamma_2$ is a lexicographically strictly increasing sequence in 2^ω and k_0 is the least integer k such that $\gamma_0|k, \gamma_1|k, \gamma_2|k$ are pairwise distinct, then $\{(A^{h(\bar{\gamma}|k)}_{\gamma_0|k}, A^{h(\bar{\gamma}|k)}_{\gamma_1|k}, A^{h(\bar{\gamma}|k)}_{\gamma_2|k}) : k \geq k_0\}$ are the moves of I and $\{(B^{h(\bar{\gamma}|k)}_{\gamma_0|k}, B^{h(\bar{\gamma}|k)}_{\gamma_1|k}, B^{h(\bar{\gamma}|k)}_{\gamma_2|k}) : k \geq k_0\}$ are the moves of II in a run of G where II followed τ.

4. $\forall k > 2\ \forall s, t \in 2^k\ (s \neq t \Rightarrow A^0_s \cap A^0_t = \emptyset)$.

Assuming that this can be done, we find an injective continuous map $f : 2^\omega \to \mathcal{N}$ given by $\gamma \mapsto \bigcap_k \bar{A}^0_{\gamma|k}$, which satisfies

$$\forall \bar{\gamma} \in [2^\omega]^3 \neg R(f(\gamma_0), f(\gamma_1), f(\gamma_2))$$

by 3. Thus we have the desired perfect set, once we have constructed the A^l_s and the B^l_s.

The inductive construction of the collection $\{A^l_s : s \in 2^{<\omega}, l < l(s)\}$: and $\{B^l_s : s \in 2^{<\omega}, l < l(s)\}$: Let A^0_s for $s \in 2^2$ be disjoint, nonempty subsets of Y.

Assume that A^0_s has been defined for $s \in 2^n$ with $n \geq 2$. We will construct A^i_s, B^i_s for $s \in 2^n$ and A^0_s for $s \in 2^{n+1}$. The A^i_s, B^i_s are defined by induction on i. Let $h_n^{-1}(i) = (s_0, s_1, s_2)$. Let l be the least integer such that $s_0|l, s_1|l, s_2|l$ are pairwise incompatible. Consider $A^i_{s_0}, A^i_{s_1}, A^i_{s_2}$ as the next move of I in the run

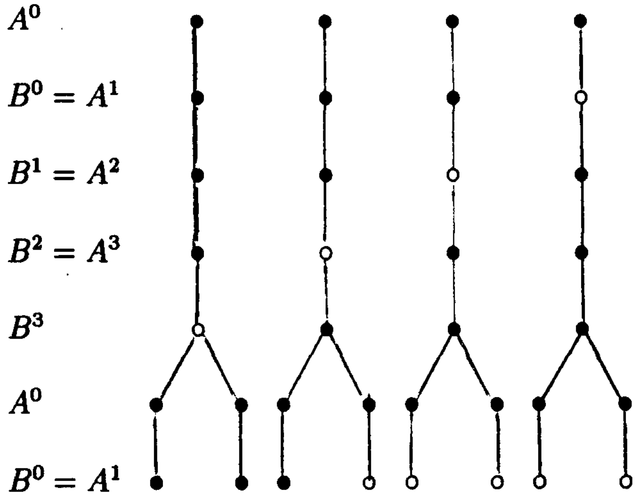
I	II
$A^{h(\bar{s}\|l)}_{s_0\|l}, A^{h(\bar{s}\|l)}_{s_1\|l} A^{h(\bar{s}\|l)}_{s_2\|l}$	
	$B^{h(\bar{s}\|l)}_{s_0\|l}, B^{h(\bar{s}\|l)}_{s_1\|l} B^{h(\bar{s}\|l)}_{s_2\|l}$
$A^{h(\bar{s}\|l+1)}_{s_0\|l+1}, A^{h(\bar{s}\|l+1)}_{s_1\|l+1} A^{h(\bar{s}\|l+1)}_{s_2\|l+1}$	
	$B^{h(\bar{s}\|l+1)}_{s_0\|l+1}, B^{h(\bar{s}\|l+1)}_{s_1\|l+1} B^{h(\bar{s}\|l+1)}_{s_2\|l+1}$
$\dots$	
	$\dots$
$\dots$	
	$\dots$
$A^i_{s_0}, A^i_{s_1}, A^i_{s_2}$	

and let B_0, B_1, B_2 be the next move of II in this run according to τ. Then define

$$B^i_s = \begin{cases} A^i_s & \text{if } s \neq s_0, s_1, s_2 \\ B_i & \text{if } s = s_i \end{cases}$$
$$A^{i+1}_s = B^i_s.$$

Assume finally that all the A^i_s, B^i_s for $s \in 2^n$ and $i < l_n$ have been found. For each $s \in 2^n$ and let $A^0_{s^\frown 0}$ and $A^0_{s^\frown 1}$ be nonempty $\check{\Gamma}$ subsets of $B^{l_n - 1}_s$.

The first few levels of the construction:



The solid dots indicate that the sets have been obtained from the previous ones by shrinkage; the others have just been copied. This completes the construction of the A_s^l and the B_s^l and thus the proof of the theorem.

Chapter 2

Measures for Countable Borel Equivalence Relations

2.1 Ergodic Measures for Countable Borel Equivalence Relations

We will discuss here countable Borel equivalence relations E on a standard Borel space X, i.e., relations with countable equivalence classes which are Borel subsets of the product space $X \times X$.

Consider a countable group G and a Borel action of G on X, i.e., a homomorphism from G into the group $\mathrm{Aut}(X)$ of Borel automorphisms (i.e., Borel bijections) of X. Then the orbit equivalence relation $E = E_G$ on X is Borel. Feldman and Moore showed that every countable Borel equivalence relation is of this form:

Theorem 1 (Feldman-Moore [77]) *Let E be a countable Borel equivalence relation on a standard Borel space X. Then there is a countable Borel group action such that E is the orbit equivalence relation of that action.*

Thus assume that E is the orbit equivalence relation of a Borel group action of a group G. Let $\mathcal{M} = \mathcal{M}(X)$ denote the space of probability measures on X. Let $\mathcal{B}$ denote the Borel σ-algebra of X. $\mathcal{M}$ becomes a standard Borel space with the smallest σ-algebra $\mathcal{A}$ which makes all the functionals $\mu \mapsto \mu(B)$ for $B \in \mathcal{B}$ measurable. A measure $\mu \in \mathcal{M}$ is called **non-atomic** if $\mu(\{x\}) = 0$ for all $x \in X$, **E-invariant** if $g\mu = \mu$ for every

$g \in G$, and E-**quasi-invariant** if $g\mu \sim \mu$ for every $g \in G$. (It can be easily shown that these notions are independent of G; see e.g., Dougherty-Jackson-Kechris [a].) A measure μ is called E-**ergodic** if $\mu(A) = 0$ or 1, for every Borel E-invariant (i.e., A is the union of E-equivalence classes) set $A \subsetneq X$. As usual, we omit mentioning E when it is clear from the context. It can be shown (see e.g. Dougherty-Jackson-Kechris [a]) that the sets $\mathcal{I}$ and $\mathcal{EI}$ of E-invariant and E-ergodic E-invariant measures, respectively, are Borel in $\mathcal{M}(X)$, and so is $\mathcal{Q}$, the set of E-quasi-invariant measures.

We show:

Theorem 2 *Let E be a countable Borel equivalence relation on a standard Borel space X. Then the set $\mathcal{E}$ of ergodic probability measures is Borel in the space of probability measures $\mathcal{M}$ on X. In particular, the set of quasi-invariant ergodic probability measures $\mathcal{EQ} = \mathcal{E} \cap \mathcal{Q}$ is Borel.*

This improves on a result by Krieger [71, p.187], who computed that the set of quasi-invariant, ergodic probability measures is $\mathbf{\Pi}^1_1$.

In fact we can use a result of A. Kechris to extend this result to orbit equivalence relations of locally compact group actions:

Corollary 3 *Let G be a locally compact Polish Group acting in a Borel way on a Polish space X. Let E be the orbit equivalence relation of G on X. Then the set of ergodic probability measures $\mathcal{E}$ is Borel in the space of probability measures $\mathcal{M}$ on X.*

Proof: By Kechris [a] there is a Borel subspace Y of X with $Y \cap [x]_E$ countable for each $x \in X$ and a Borel reduction $f : X \to Y$ reducing E to $E|Y$. Now the mapping $\mu \to f\mu$ is Borel and reduces the set of ergodic measures of E to the set of ergodic measures of $E|Y$. □

Let us first notice that we can reduce the set $\mathcal{E}$ of ergodic probability measures to the set of quasi-invariant, ergodic measures. Consider the mapping which maps μ to $\mu^* = \sum_{i=0}^{\infty} 2^{-(i+1)} g_i\mu$, where $G = \{g_i : i \in \omega\}$ is some enumeration of G. This map is clearly Borel and maps into $\mathcal{Q}$. Furthermore, since μ is translated only by elements of G, $\mu(A) = 0$ iff $\mu^*(A) = 0$ for any G-invariant A. Thus μ is E-ergodic iff μ^* is E-ergodic. Thus for any E, $\mathcal{E}$ is Borel-reducible to $\mathcal{EQ}$, so it suffices to show that $\mathcal{EQ}$ is Borel. For our proof we will use an ergodic-decomposition-theorem patterned after the following well-known result of Varadarajan, which holds in even greater generality:

Theorem 4 (Varadarajan [63], p.208) *Let E be a countable, Borel equivalence relation on a standard Borel space X. Let $\mathcal{EI}$ be the set of E-invariant E-ergodic measures on X. If there is an E-invariant measure on X, then $\mathcal{EI}$ is nonempty Borel, and there is a function $\beta : X \to \mathcal{EI}$, $\beta : x \mapsto \beta_x$ such that*

(i) *β is a Borel measurable map from X onto $\mathcal{EI}$,*

(ii) *β is E-invariant; i.e., $\beta_x = \beta_y$ for all $x, y \in X$ with xEy,*

(iii) *If $X_e = \{x \in X : \beta_x = e\}$ for e in $\mathcal{EI}$, then $e(X_e) = 1$ for all e,*

(iv) *For any E-invariant measure μ,*

$$\mu(A) = \int_X \beta_x(A)\, d\mu(x)$$

for any Borel A.

Furthermore, if β' is another map with the above properties, then $\beta = \beta'$ μ-a.e. for all E-invariant measures μ.

Kifer-Pirogov proved a similar result for quasi-invariant measures which share a common Radon-Nikodym derivative of Borel $\mathcal{Z}$-actions, using results from Dynkin [71]:

Theorem 5 (Kifer-Pirogov [72], p.80) *Let E be a countable Borel equivalence relation on a standard Borel space X induced by a $\mathcal{Z}$-action τ. Assume that $\rho : \mathcal{Z} \times X \to \mathcal{R}^+$ (here $\mathcal{R}^+$ denotes the set of strictly positive reals) is Borel such that*

$$\rho_{t+s} = \rho_t(x)\rho_s(\tau^t x).$$

Let $\mathcal{Q}^\rho$ be the set of quasi-invariant measures μ on X such that $d\tau^{-t}\mu/d\mu = \rho_t$ μ-a.e.. If $\mathcal{Q}^\rho$ is nonempty, then the set $\mathcal{EQ}^\rho$ of ergodic measures in $\mathcal{Q}^\rho$ is nonempty Borel, and there is a function $\beta : X \to \mathcal{EQ}^\rho$, $\beta : x \mapsto \beta_x$ such that

(i) *β is a Borel measurable map from X onto $\mathcal{EQ}^\rho$,*

(ii) *β is E-invariant; i.e., $\beta_x = \beta_y$ for all $x, y \in X$ with xEy,*

(iii) *If $X_e = \{x \in X : \beta_x = e\}$ for e in $\mathcal{EQ}^\rho$, then $e(X_e) = 1$ for all e,*

(iv) *For any measure* $\mu \in \mathcal{Q}^\rho$,

$$\mu(A) = \int_X \beta_x(A)\ d\mu(x)$$

for any Borel A.

Furthermore, if β' *is another map with the above properties, then* $\beta = \beta'$ μ*-a.e. for all measures* $\mu \in \mathcal{Q}^\rho$.

We will combine the methods of Varadarajan and Kifer-Pirogov with descriptive set theory to obtain the following more general version of the ergodic decomposition:

Theorem 6 *Let* E *be a countable, Borel equivalence relation on a standard Borel space* X *induced by a countable group* G *of Borel automorphisms. Let* $\rho : G \times X \to \mathcal{R}^+$ *be Borel such that for all* $x \in X$ *and all* g *and* $h \in G$,

$$\rho(gh, x) = \rho(h, x)\rho(g, hx).$$

Let $\mathcal{Q}^\rho$ *be the Borel set of all quasi-invariant measures* μ *on* X *such that for all* $g \in G$,

$$dg^{-1}\mu/d\mu = \rho_g.$$

If $\mathcal{Q}^\rho$ *is nonempty, then there is a map* $\beta : X \to \mathcal{Q}^\rho$, $x \mapsto \beta^x$ *such that*

(i) β *is a Borel measurable map from* X *onto* $\mathcal{E}\mathcal{Q}^\rho$,

(ii) β *is* E*-invariant; i.e.,* $\beta_x = \beta_y$ *for all* $x, y \in X$ *with* xEy,

(iii) *If* $X_e = \{x \in X : \beta_x = e\}$ *for* e *in* $\mathcal{E}\mathcal{Q}^\rho$, *then* $e(X_e) = 1$ *for all* e,

(iv) *For any measure* $\mu \in \mathcal{Q}^\rho$,

$$\mu(A) = \int_X \beta_x(A)\ d\mu(x)$$

for any Borel A.

Furthermore, if β' *is another map with the above properties, then* $\beta = \beta'$ μ*-a.e. for all measures* $\mu \in \mathcal{Q}^\rho$.

In particular, we have:

Corollary 7 *If Q^ρ is not empty, then it contains some ergodic measures.*

Corollary 8 *For each ρ satisfying the above product rule, the set of quasi-invariant, ergodic measures with ρ as a Radon-Nikodym derivative is Borel.*

Let us mention that, for specific examples, a more careful analysis of the proof yields concrete upper bounds for the Borel complexity of the space of ergodic measures. For the shift of $\mathcal{Z}$ on $2^{\mathcal{Z}}$ one finds that $\mathcal{E}$ is $\mathbf{\Pi}^0_3$, and for the shift of F_2 on 2^{F_2} one finds that $\mathcal{E}$ is $\mathbf{\Pi}^0_{\omega+1}$. Recently A. Kechris showed that in the first case $\mathcal{E}$ is indeed $\mathbf{\Pi}^0_3$-complete. For the action of F_2 it is not known whether $\mathcal{E}$ is $\mathbf{\Pi}^0_{\omega+1}$-complete.

The proof that $\mathcal{E}$ is Borel runs via a reduction to $\mathcal{QE}$. We would like to know:

Question 9 Is there a direct proof showing that $\mathcal{E}$ is Borel?

Since any two uncountable, standard Borel spaces X and Y are Borel-isomorphic via a Borel measurable bijection f, we may and do assume in the rest of the paper that we are dealing with perfect Polish spaces. Such an f also induces a Borel isomorphism between $\mathcal{M}(X)$ and $\mathcal{M}(Y)$. We will at times even assume that we are in a particular Polish space, when this seems desirable. An important Polish space is the Baire space $\mathcal{N}$ of all functions from ω to ω, with the product topology, taking ω to have the discrete topology.

2.1.1 Uniformities

In order to state precisely what we mean by a theorem to hold uniformly, we need codings: A **coding** of a set A is a pair (C, π), where C is a subset of a Polish space, and $\pi : C \to A$ is a surjection onto A. Let A, B be sets with codings (C, π) and (C', π'), respectively, where $C \subseteq Y$ and $C' \subseteq Y'$, X, X' are Polish spaces. We say that $f : X \times A \to X'$ and $g : X \times A \to B$ are **Borel in the codes** if there are Borel functions $\hat{f} : X \times Y \to X'$ and $\hat{g} : X \times Y \to Y'$ such that

$$\forall x \in X \forall y \in C \ (f(x, \pi(y)) = \hat{f}(x, y))$$

and

$$\forall x \in X \forall y \in C\ (g(x, \pi(y)) = \pi' \hat{g}(x, y)).$$

In a Polish space X a set is $\mathbf{\Sigma}^1_1$ if it is the image of a closed subset of some Polish space under a continuous function. It is $\mathbf{\Pi}^1_1$ if it is the complement of a $\mathbf{\Sigma}^1_1$ set. It is well known that a subset of a Polish space is Borel *iff* it is $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$. This gives rise to a standard way of coding Borel subsets in a Polish space X: We can find $\mathbf{\Pi}^1_1$ sets $C \subseteq \mathcal{N}$ and $D, \check{D} \subseteq \mathcal{N} \times X$ such that

1. for all $\alpha \in C$, $D_\alpha = X - \check{D}_\alpha$, and
2. for every Borel set $A \subseteq X$, there is an $\alpha \in C$ such that $A = D_\alpha$.

This yields a coding $C \to \mathcal{B}$, $\alpha \mapsto D_\alpha$. C is called the set of Borel codes for X. We may choose $(C, D, \check{D})$ in such a manner that:

1. $A, B \mapsto A \cap B$, $A, B \mapsto A \cup B$, $A \mapsto X - A$, $A \subseteq \omega \times X \mapsto \bigcup_i A_i$, $A \subseteq \omega \times X \mapsto \bigcap_i A_i$ are Borel in the codes, by Moschovakis [80, Section 7.B].
2. For each Borel $A \subset Y \times X$, where Y is Polish, the function $y \mapsto A_y$ is Borel in the codes, by Moschovakis [80, Section 7.B].
3. Let $\mathcal{B}$ denote the collection of Borel subsets of X. It is well known that $\mathcal{M} \times \mathcal{B} \to \mathcal{R}^+$ given by $(\mu, A) \mapsto \mu(A)$ is Borel in the codes.
4. If we code $\mathcal{F}_b(X, \mathcal{R})$, the set of bounded, real-valued Borel functions on X by $C' = \{\alpha \in C : D_\alpha$ is the graph of a bounded real valued Borel function on $X\}$, then it is well known that $\mathcal{M} \times \mathcal{F}_b(X, \mathcal{R}) \to \mathcal{R}$ given by $(\mu, f) \mapsto \mu(f)$ is Borel in the codes.
5. The collections of Borel in the codes functions are closed under composition.

2.1.2· Proof of Theorem 2

Fix now a countable group G acting in a Borel way on a perfect space X, inducing an equivalence relation E. Call a Borel function $\rho : g \times X \to \mathcal{R}^+$ a **strict cocycle** if for all $x \in X$ and all $g, h \in G$

$$\rho(gh, x) = \rho(h, x)\rho(g, hx).$$

We code strict cocycles by the Borel codes of their graphs and denote the collection of strict cocycles by $\mathcal{C}$. If ρ is a strict cocycle, let $\mathcal{Q}^\rho$ be the collection of all quasi-invariant measures μ such that for all $g \in G$ $dg^{-1}\mu/d\mu = \rho_g$ μ-a.e. Let $\mathcal{E}\mathcal{Q}^\rho$ denote the collection of ergodic measures in $\mathcal{Q}^\rho$. We will actually show the following stronger effective version of theorem 5:

Theorem 10 (Effective Ergodic-Decomposition Theorem) *There is a Borel in the codes function $D : \mathcal{Q} \times \mathcal{C} \times X \to \mathcal{Q}$ such that for any $\mu \in \mathcal{Q}$ and any $\rho \in \mathcal{C}$ with $\mu \in \mathcal{Q}^\rho$, we have for $D_{\mu,\rho} : x \mapsto \mu^x$:*

1. $\forall x \in X \forall g \in G \ (\mu^x = \mu^{gx} \in \mathcal{Q}^\rho)$,
2. $\forall \nu \in \mathcal{Q}^\rho \ [(\nu(X_\nu) = 1 \Leftrightarrow \nu \in \mathcal{E})$, *where* $X_\nu = \{x \in X : \mu^x = \nu\}]$,
3. $\forall \nu \in \mathcal{Q}^\rho \ (\nu(\{x \in X : \mu^x \in \mathcal{E}\}) = 1)$,
4. $\forall \nu \in \mathcal{Q}^\rho \forall A \in \mathcal{B}$
$$\nu(A) = \int \mu^x(A) d\nu(x).$$

From this we will deduce theorem 2, using the following lemma:

Lemma 11 *There is a Borel in the codes function $\mathcal{Q} \to \mathcal{C}$, $\mu \mapsto \rho^\mu$, such that $\forall \mu \in \mathcal{Q} \ (\mu \in \mathcal{Q}^{\rho^\mu})$.*

Proof: (of lemma 11) Assume without loss of generality that $X = [0,1]$. Let us first show how to find a strict cocycle for non-atomic measures: Let $\mathcal{A}$ denote the collection of atomic measures, $\mathcal{NA}$ be the collection of non-atomic measures. Let $f : (\mathcal{NA} \times X) \to [0,1]$ be given by $f^\mu : x \mapsto \mu([0,x))$. Then

1. f is Borel,
2. f^μ is continuous,
3. f^μ is increasing,
4. $f^\mu \mu = m$, the Lebesgue measure,
5. $\mu \sim \nu \Rightarrow f^\mu \mu \sim f^\mu \nu$,
6. $\mu \sim \nu \Rightarrow d\nu/d\mu = (df^\mu \nu / df^\mu \mu) \circ f^\mu$ μ-a.e.

Consider now the function $D : \mathcal{M}([0,1]) \times [0,1] \to \mathcal{R}^+$ given by

$$D(\nu, x) = \begin{cases} \lim_{n\to\infty} \frac{\nu(B_{1/n}(x))}{m(B_{1/n}(x))} & \text{if it exists,} \\ 1 & \text{otherwise,} \end{cases}$$

where $B_r(x) = (x - r, x + r)$, and m is the Lebesgue measure on $[0.1]$. The function $(\nu, x, n) \mapsto \frac{\nu(B_{1/n}(x))}{m(B_{1/n}(x))}$ is clearly Borel, and so D is Borel.

Using the results of the previous section, we see now that $\tilde{\rho} : (\mathcal{M} \times G \times [0,1]) \to \mathcal{R}^+$ given by

$$\tilde{\rho}^{\mu}(g, x) = D(f^{\mu}(g^{-1}\mu))(x),$$

is Borel. By Rudin [87, Chapter 7] $D(\nu, x)$ is the symmetric derivative of ν at x, which is equal, for ν absolutely continuous with respect to m, to $d\nu/dm$ m-a.e. Thus if we let $\bar{\rho} : \mathcal{M} \times G \times X \to \mathcal{R}^+$ be given by $\bar{\rho}^{\mu}(g, x) = \tilde{\rho}^{\mu}(g, f^{\mu}(x))$, and $A \subseteq \mathcal{M} \times X$ be given by $A_{\mu} = \{x \in X : \exists g \in G \exists h \in G\ (\bar{\rho}^{\mu}(gh, x) \neq \bar{\rho}^{\mu}(h, x)\bar{\rho}^{\mu}(g, hx))\}$, then $\bar{\rho}$ and A are Borel, A_{μ} is μ-null and $\bar{\rho}^{\mu}_g = dg^{-1}\mu/d\mu$ μ-a.e. Thus if we set

$$\rho^{\mu}(g, x) = \begin{cases} \bar{\rho}^{\mu}(g, x) & \text{if } x \notin [A_{\mu}]_E \\ 1 & \text{if } x \in [A_{\mu}]_E \end{cases},$$

then $\mu \mapsto \rho^{\mu}$ is as desired for non-atomic measures μ.

In order to deal with general measures, we will decompose them effectively into atomic and non-atomic parts. In fact, there is a Borel function $\mathcal{Q} \to (\mathcal{Q} \cap \mathcal{A}) \times (\mathcal{Q} \cap \mathcal{NA})$, $\mu \mapsto (\mu', \mu'')$ such that $\mu = (1 - \lambda)\mu' + \lambda\mu$, where $\lambda = \sum_{x \in X} \mu(\{x\})$. In order to see this, note that $\mu \mapsto A_{\mu}$, where $A_{\mu} = \{x \in X : \mu(\{x\}) > 0\}$, is Borel in the codes. Let μ'' be the unique atomic measure with $\mu''(\{x\}) = \lambda^{-1}\mu(\{x\})$ for $x \in A_m u$. Let $\mu' = (1 - \lambda)^{-1}(\mu - \lambda\mu'')$.

There is easily a Borel in the codes function $\mathcal{Q} \cap \mathcal{A} \to \mathcal{C}$, $\mu \mapsto \rho^{\mu}$, such that $\mu \in \mathcal{Q}^{\rho^{\mu}}$. We then can set for any $\mu \in \mathcal{Q}$

$$\rho^{\mu}(g, x) = \begin{cases} \rho^{\mu'}(g, x) & \text{if } \mu(\{x\}) = 0, \\ \rho^{\mu''}(g, x) & \text{if } \mu(\{x\}) > 0. \end{cases}$$

Here μ' and μ'' are the non-atomic and atomic parts of μ as given by the above. This clearly works. □

Theorem 2 easily follows:

Proof: (of theorem 2) We assume theorem 10. Then we have

$$\mu \in \mathcal{EQ} \Leftrightarrow (\mu(\{x : D_{\mu,\rho^\mu}(x) = \mu\}) = 1),$$

which is Borel. □

It remains to prove theorem 10.

2.1.3 Some Ergodic Theory

We collect here three results from probability and ergodic theory. Let $(X, \mathcal{B}, \mu)$ be a measure space, i.e., X a set, $\mathcal{B}$ a σ-algebra on X, and μ a measure on $\mathcal{B}$. Let $\mathcal{A} \subseteq \mathcal{B}$ be a σ-algebra and $f \in L^1(\mu)$. A function $\hat{f} \in L^1(\mu)$ is called the **conditional expectation** of f with respect to $\mathcal{A}$, denoted by $\mu(f|\mathcal{A})$, if $\hat{f}$ is $\mathcal{A}$-measurable, and for any $A \in \mathcal{A}$ $\int_A \hat{f}\, d\mu = \int_A f\, d\mu$. The conditional expectation is determined uniquely μ-a.e.. An operator $T : L^1(\mu) \to L^1(\mu)$ is a **conditional-expectation operator** if $T(f) = \mu(f|\mathcal{A})$ for all f and some fixed $\mathcal{A}$. We have the following results about limits of conditional-expectation operators:

Theorem 12 (Billingsley [86, p.493]) *Let $(X, \mathcal{B}, \mu)$ be a measure space and $f \in L^1(\mu)$. Assume that $\{\mathcal{A}_n : n \in \omega\}$ is a decreasing sequence of σ-algebras such that $\bigcap_n \mathcal{A}_n = \mathcal{A}$. Then $\mu(f|\mathcal{A}_n)$ converges to $\mu(f|\mathcal{A})$ μ-a.e.*

Theorem 13 (Burkholder-Chow [61, p.494]) *Let $(X, \mathcal{B}, \mu)$ be a probability measure space and T and T' be conditional-expectation operators on $L^1(\mu)$ associated with the σ-algebras $\mathcal{A}$ and $\mathcal{A}'$. Let S_n be given by $S_0 = T$, $S_{2n+1} = T'S_{2n}$ and $S_{2n+2} = TS_{2n+1}$. Then for any $f \in L^2(\mu)$, $\lim_n S_n f = \mu(f|\mathcal{A} \cap \mathcal{A}')$ μ-a.e. and in the $L^2(\mu)$-norm.*

We can use the Hurewicz Ergodic Theorem and the Hopf Decomposition Theorem to compute the conditional expectation with respect to the σ-algebra of invariant sets with respect to a single Borel automorphism. The following is well known, but we could not find a convenient reference.

Theorem 14 *Let T be a Borel transformation of a standard Borel space X with a quasi-invariant probability measure μ, $\rho^n : X \to \mathcal{R}^+$ be Borel such that for $n, m \in \mathcal{Z}$*

$$\rho^{n+m} = \rho^m(\rho^n \circ T^m)$$

and

$$\rho^n = \frac{dT^{-n}\mu}{d\mu} \quad \mu\text{-a.e.}$$

Let $[\mathcal{B}]_T$ *denote the* σ*-algebra of* T*-invariant Borel sets. Then for any* $f \in L^1(\mu)$,

$$\bar{f}(x) = \lim_{n\to\infty} \frac{\sum_{i=-n}^{n} f(T^i(x))\rho^i(x)}{\sum_{i=-n}^{n} \rho^i(x)}$$

is $\mu(f|[\mathcal{B}]_T)$.

Proof: In order to prove this theorem, let X, T, μ, ρ satisfy the assumptions and let E be the orbit equivalence relation of T on X. Call a Borel set $A \subseteq X$ a **partial transversal** if it intersects each E-equivalence class in at most 1 point. Say that $A \subseteq X$ is **smooth** (or **dissipative**) if it is the T-closure of a partial transversal. Hurewicz calls a set $A \subseteq X$ **conservative** if it contains no partial transversals of positive measure. We now have by a standard exhaustion argument:

Lemma 15 *Let* μ *be a probability measure on* X *which is quasi-invariant with respect to* T. *Then there is a unique* μ*-a.e. decomposition of* X *into* T*-invariant Borel sets* C_T *and* D_T *such that* T *is conservative on* C_T *and dissipative on* D_T.

The Hopf Decomposition Theorem will allow us to compute C_T and D_T effectively:

Theorem 16 (Hopf, see Petersen [83], p.125) *Let* $\tilde{T}$ *be a positive contraction of* $L^1(\mu)$, $u \in L^1(\mu)$ *be strictly positive and*

$$C = \{x \in X : \sum_{i=0}^{\infty} \tilde{T}^i u(x) = \infty\}.$$

Then

1. C *is independent of* u*; i.e., if* u' *satisfies the assumptions of the theorem, and* C' *is defined by replacing* u *by* u' *in the definition of* C, *then* $C = C'$ μ*-a.e.*

2. For all nonnegative $u \in L^1(\mu)$,

$$\sum_{i=0}^{\infty} \tilde{T}^i u(x) < \infty \quad \mu\text{-a.e. on } X - C,$$

Write $C_{\tilde{T}}$ for C and $D_{\tilde{T}}$ for $X - C$.

Since the Hopf Decomposition Theorem applies to positive contractions of $L^1(\mu)$, we associate with T the positive isometry $\tilde{T} : L^1(\mu) \to L^1(\mu)$ given by

$$\tilde{T}(f)(x) \;=\; f(Tx)\,\rho^1(x).$$

We have the following fact about the relationship between the conservative and dissipative parts of T, $\tilde{T}$, and $\tilde{T}^{-1}$. Equalities and inclusions below are μ-a.e.:

Lemma 17

$$\begin{aligned} C_T &= C_{\tilde{T}} \cup C_{\tilde{T}^{-1}} \\ &= \{x \in X : \sum_{i=0}^{\infty} \rho^i(x) = \infty\} \cup \{x \in X : \sum_{i=0}^{-\infty} \rho^i(x) = \infty\}, \\ D_T &= D_{\tilde{T}} \cap D_{\tilde{T}^{-1}} \\ &= \{x \in X : \sum_{i=\infty}^{\infty} \rho^i(x) < \infty\}. \end{aligned}$$

Proof: (of lemma 17) Note that if S is a partial transversal, then

$$0 < \sum_{i=-\infty}^{\infty} \tilde{T}^i \chi_S < \infty$$

on $\bigcup_{i=-\infty}^{\infty} T^i[S]$. Thus $D_T \cap (C_{\tilde{T}} \cup C_{\tilde{T}^{-1}}) = \emptyset$ and $D_T \subseteq D_{\tilde{T}} \cap D_{\tilde{T}^{-1}}$ by the Hopf Decomposition Theorem. In order to see $D_T \supseteq D_{\tilde{T}} \cap D_{\tilde{T}^{-1}}$, assume that this is false and that $A = (D_{\tilde{T}} \cap D_{\tilde{T}^{-1}}) - D_T$ has positive measure. By shrinking A we may assume without loss of generality that $\sum_{i=\infty}^{\infty} \rho^i(x) < M$ on A for some $M > 0$. But then we have

$$\infty \;>\; \int_A \sum_{i=-\infty}^{\infty} \rho^i d\mu \;=\; \sum_{i=-\infty}^{\infty} \int_A \rho^i(x) d\mu(x)$$

$$= \sum_{i=-\infty}^{\infty} \int_{T^{-i}[A]} d\mu(x)$$
$$= \sum_{i=-\infty}^{\infty} \mu(T^{-i}[A]).$$

Since the integral is finite, B, given by

$$B = \{x \in X : x \in T^{-i}[A] \text{ for infinitely many } i \in \mathcal{Z}\},$$

must have measure 0, and we may shrink A to ensure $B = \emptyset$. For $x \in A$, let i_x be the greatest integer i such that $T^{i_x}x \in A$, and put

$$S = \{T^{i_x}x : x \in A\}.$$

Since the $T^i[S]$, $i \in \mathcal{Z}$, cover A, S has positive measure. S is clearly a subset of A. Assume that $x \in S$ and $T^i x \in S$ with $i \neq 0$. Assume that $i > 0$; otherwise let $z = T^i x \in S$ with $T^{-i}z = x \in S$. By the definition of S there is $y \in A$ with $T^{i_y}y = x$. But then $T^{i_y+i}y = T^i x \in S \subseteq A$, a contradiction to the maximality of i_y. Thus S is a partial transversal of positive measure. But this contradicts $A \cap D_T = \emptyset$. □

For the conservative part of T, the Hurewicz Ergodic Theorem yields theorem 12:

Theorem 18 (Hurewicz [44], p.195) *Let T be a transformation of X with quasi-invariant measure μ and $\rho^n : X \to \mathcal{R}^+$ be Borel such that*

$$\rho^n = dT^{-n}\mu/d\mu \quad \mu\text{-a.e.},$$

and

$$\rho^{n+m} = (\rho^n \circ T^m)\, \rho^m.$$

Let $f \in L^1(\mu)$. Then:

1.

$$\bar{f} = \lim_{n\to\infty} \frac{\sum_{i=0}^n f^i(T^i x)\rho^i(x)}{\sum_{i=0}^n \rho^i(x)}$$

exists and is finite μ-a.e. on C_T, the conservative part of T.

2. *$\bar{f}$ is T-invariant μ-a.e. on C_T*

3. *$\bar{f} = \mu(f\|[\mathcal{B}]_T)$ on C_T; i.e., for all Borel T-invariant subsets A of X with $A \subseteq C_T$ μ-a.e., we have $\int_A f d\mu = \int_A \bar{f} d\mu$.*

For the dissipative part of T, we have for any nonnegative $f \in L^1(\mu)$

$$0 < \sum_{i=-\infty}^{\infty} \tilde{T}^i(f)(x) < \infty \quad \mu\text{-a.e. on } D_T;$$

thus

$$\sum_{i=-\infty}^{\infty} f(T^i(x))\rho^i(x) < \infty \quad \mu\text{-a.e. on } D_T,$$

so in particular, $0 < \sum_{i=-\infty}^{\infty} \rho^i(x) < \infty$ μ-a.e. on D_T. Thus for $f \in L^1(\mu)$,

$$\bar{f}(x) = \frac{\sum_{i=-\infty}^{\infty} f^i(T^i x)\rho^i(x)}{\sum_{i=-\infty}^{\infty} \rho^i(x)}$$

exists μ-a.e on D_T, and is a version of $\mu(f\|[B]_T)$ on D_T, i.e., is T-invariant μ-a.e. and $\int_A \bar{f} d\mu = \int_A f d\mu$ for all Borel T-invariant subsets A of D_T: Indeed, on D_T we have μ-a.e.

$$\begin{aligned}
\bar{f}(Tx) &= \frac{\sum_{i=-\infty}^{\infty} f^i(T^i Tx)\rho^i(Tx)}{\sum_{i=-\infty}^{\infty} \rho^i(Tx)} \\
&= \frac{\sum_{i=-\infty}^{\infty} f^i(T^i Tx)\rho^i(Tx)\rho(x)}{\sum_{i=-\infty}^{\infty} \rho^i(Tx)\rho^1(x)} \\
&= \frac{\sum_{i=-\infty}^{\infty} f^i(T^{i+1} x)\rho^{i+1}(x)}{\sum_{i=-\infty}^{\infty} \rho^{i+1}(x)} \\
&= \bar{f}(x)
\end{aligned}$$

and for any Borel T-invariant $A \subseteq D_T$,

$$\begin{aligned}
\int_A \bar{f} d\mu &= \int_A \frac{\sum_{i=-\infty}^{\infty} f(T^i(x))\rho^i(x)}{\sum_{i=-\infty}^{\infty} \rho^i(x)} d\mu(x) \\
&= \sum_{i=-\infty}^{\infty} \int_A \frac{f(T^i(x))\rho^i(x)}{\sum_{j=-\infty}^{\infty} \rho^j(x)} d\mu(x)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i=-\infty}^{\infty} \int_A \frac{f(x)}{\sum_{j=-\infty}^{\infty} \rho^j(T^{-i}x)} d\mu(x) \\
&= \sum_{i=-\infty}^{\infty} \int_A \frac{f(x)\rho^{-i}(x)}{\sum_{j=-\infty}^{\infty} \rho^{j-i}(x)} d\mu(x) \\
&= \int_A f(x) \frac{\sum_{i=-\infty}^{\infty} \rho^i(x)}{\sum_{i=-\infty}^{\infty} \rho^i(x)} d\mu(x) \\
&= \int_A f(x) d\mu(x).
\end{aligned}$$

This completes the proof of theorem 14.

2.1.4 Proof of the Effective Ergodic Decomposition Theorem

We will need the following lemma:

Lemma 19 *There is a Borel in the codes function* $c : \mathcal{C} \times \mathcal{F}_b(X, \mathcal{R}) \to \mathcal{F}_b(x, \mathcal{R})$, *such that for each strict cocycle* ρ, $c_\rho : f \mapsto \hat{f}$ *satisfies*

1. $\hat{f}$ *is G-invariant,* $\hat{f} = f$ *if* f *is G-invariant,* $\widehat{fh} = \hat{f}h$ *if* h *is G-invariant,*
2. $||\hat{f}||_\infty \leq ||f||_\infty$,
3. $\forall \mu \in \mathcal{Q}^\rho[\forall A \in [\mathcal{B}]_G \; (\int_A \hat{f} d\mu = \int_A f d\mu)$, *i.e.,* $\hat{f} = \mu(f | [B]_G))]$.

Proof: In order to prove the lemma, we will first verify it for cyclic subgroups of G and then apply the results of Burkholder-Chow mentioned in the previous section. Let $\{g_n : n \in \omega\}$ be some enumeration of G and let $G_n = <g_i : i \leq n>$.

Lemma 20 *There is a Borel in the codes function* $c' : \omega \times \mathcal{C} \times \mathcal{F}_b(X, \mathcal{R}) \to \mathcal{F}_b(x, \mathcal{R})$, *such that for each strict cocycle* ρ, $c'_{n,\rho} : f \mapsto \hat{f}$ *satisfies*

1. $\hat{f}$ *is* $<g_n>$*-invariant,* $\hat{f} = f$ *if* f *is* $<g_n>$*-invariant,* $\widehat{fh} = \hat{f}h$ *if* h *is* $<g_n>$*-invariant,*
2. $||\hat{f}||_\infty \leq ||f||_\infty$,

3. $\forall \mu \in \mathcal{Q}^\rho [\forall A \in [\mathcal{B}]_{<g_n>} (\int_A \hat{f} d\mu = \int_A f d\mu)$, *i.e.*, $\hat{f} = E(f \| [\mathcal{B}]_{<g_n>})]$.

Proof: To define $c'_{n,\rho} : \mathcal{F}_b \to \mathcal{F}_b$, put for $f \in \mathcal{F}_b$

$$\tilde{f}(x) = \begin{cases} \lim_{m\to\infty} \frac{\sum_{i=-m}^{m} f(g_n^i x)\rho(g_n^i, x)}{\sum_{i=-m}^{m} \rho(g_n^i, x)} & \text{if this limit exists} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\hat{f}(x) = \begin{cases} \tilde{f}(x) & \text{if } \forall i \in \mathcal{Z}(\tilde{f}(g^i(x))) = \tilde{f}(x)) \\ 0 & \text{otherwise.} \end{cases}$$

By theorem 14 $c'_{n,\rho} : f \mapsto \hat{f}$ is as desired. c' is clearly Borel in the codes. □

We will now show that lemma 19 holds for G replaced by G_n, uniformly in n. By induction on n, assume that $\tilde{c}_n : \mathcal{C} \times \mathcal{F}_b \to \mathcal{F}_b$ is given for G_n (the case $n = 0$ is covered by the previous lemma), and c' is given by the previous lemma. For $\rho \in \mathcal{C}$, let $\hat{c}_{n+1,\rho} : \mathcal{F}_b \to \mathcal{F}_b$ be given by

$$\hat{c}_{n+1,\rho}(f)(x) = \begin{cases} \lim_{k\to\infty} \overbrace{\tilde{c}_{n,\rho} \circ c'_{n+1,\rho} \circ \tilde{c}_{n,\rho} \circ \ \dots \ \circ c'_{n+1,\rho}}^{2k \text{ factors}}(f)(x) & \text{if it exists,} \\ 0 & \text{otherwise.} \end{cases}$$

By the theorem of Burkholder-Chow on iterates of conditional-expectation operators mentioned in the previous section,

$$\hat{c}_{n+1,\rho}(f) = \mu(f \| [\mathcal{B}]_{G_{n+1}}) \ \ \mu\text{-a.e.}$$

for every quasi-invariant measure $\mu \in \mathcal{Q}^\rho$. Thus put

$$\tilde{c}_{n+1,\rho}(f)(x) = \begin{cases} \hat{c}_{n+1,\rho}(f)(x) & \text{if } \forall g \in G_{n+1} \ (\tilde{c}_{n+1,\rho}(f)(gx) = \tilde{c}_{n+1,\rho}(f)(x)) \\ 0 & \text{otherwise.} \end{cases}$$

Finally, let $c_\rho : \mathcal{F}_b \to \mathcal{F}_b$ be given by

$$c_\rho(f)(x) = \begin{cases} \lim_{n\to\infty} \tilde{c}_{n,\rho}(f)(x) & \text{if it exists and is the same for each } y \in [x]_G \\ 0 & \text{otherwise.} \end{cases}$$

By the reverse martingale theorem, c is as desired. This completes the proof of lemma 19.

We are now ready to prove the Effective Ergodic Decomposition Theorem 10:

Proof: Assume without loss of generality that $X = {}^\omega 2$. Let W be a countable, Q-linear space such that:

(A) $1 \in W \subseteq C(X, \mathcal{R})$, W separates probability measures.

(B) If $W \subseteq H \subseteq \mathcal{F}_b$ with H is a Q-subspace and closed under uniformly bounded, pointwise limits, then $H = \mathcal{F}_b$.

(C) W is closed under composition with elements of G; i.e., if $f \in W$ and $g \in G$, then $f \circ g \in W$.

Fix $\rho \in \mathcal{C}$ and $\mu \in \mathcal{Q}^\rho$. Let $c_\rho : f \mapsto \hat{f}$. Consider

$$R = \{x \in X : \exists c, c' \in Q \exists f, f' \in W (\widehat{cf + c'f'})(x) \neq c\hat{f}(x) + c'\hat{f}'(x)\},$$

which is $\mathcal{Q}^\rho$-null (i.e., $\mu \in \mathcal{Q}^\rho \Rightarrow \mu(R) = 0$) by the conditional-expectation properties. Thus its G-closure $\bar{R}$ is $\mathcal{Q}^\rho$-null. By setting $\hat{f} = 0$ on $\bar{R}$, we can and do assume that the above equality holds everywhere; i.e., we can assume that c_ρ is Q-linear on W. For $x \in X - \bar{R}$ let μ^x be the unique probability measure in $\mathcal{Q}$ such that

$$\forall f \in W(\mu^x(f) = \hat{f}(x)).$$

For $x \in \bar{R}$, let $\mu^x = \mu$. Let us verify that except for a Q^ρ-null set, μ^x is in Q^ρ. For that notice that $\mu(\mu^x(fh)) = \mu(\mu^x(f)h)$ for h G-invariant, $\mu \in \mathcal{Q}^\rho$, $f \in W$, so $\mu^x(f) = \mu(f \| [\mathcal{B}]_G)$ for all $f \in \mathcal{F}_b$, except on a $\mathcal{Q}^\rho$-null set. Since for $\mu \in \mathcal{Q}^\rho$ $\mu(f \circ g \| [\mathcal{B}]_G) = \mu(\rho_g f \| [\mathcal{B}]_G)$ μ-a.e., we have $\mu^x(f \circ g) = \mu^x(\rho_g f)$ for all $f \in \mathcal{F}_b$, except on a $\mathcal{Q}^\rho$-null set. So $\mu^x \in \mathcal{Q}^\rho$, except on a $\mathcal{Q}^\rho$-null set. We can then change μ^x on an E-invariant $\mathcal{Q}^\rho$-null set μ, to ensure that $\mu^x \in \mathcal{Q}^\rho$ for all $x \in X$.

Since $\bar{R}$ is G-invariant and c_ρ's range is contained in the set of G-invariant functions, 1. (of theorem 10) holds.

In order to verify 2.(of theorem 10), we show that the following are equivalent for $\nu \in \mathcal{Q}^\rho$:

(a) ν is ergodic.

(b) $\nu(\{x : \nu = \mu^x\}) = 1$.

(c) $\forall f \in W$ $(\mu^x(f) = \nu(f)$ ν-a.e.).

The equivalence of (b), (c) is immediate.

To see (a) $\Rightarrow$ (c): Assume that ν is ergodic and let $f \in W$. Then $\mu^x(f)$ is constant ν-a.e., as it is invariant. Thus $\nu(x \mapsto \mu^x(f)) = \mu^x(f)$ ν-a.e. Thus

$$\begin{aligned}\mu^x(f) &= \int \mu^x(f) d\nu(x) \quad \nu-\text{a.e.} \\ &= \int \hat{f} d\nu(x) \quad \nu-\text{a.e.} \\ &= \nu(f) \quad \nu-\text{a.e.}\end{aligned}$$

To see (b) $\Rightarrow$ (a): Let $A \in [\mathcal{B}]_G$. Then $\nu(A) = \mu^x(A) = \nu(A|[\mathcal{B}]_G) = \chi_A$ ν-a.e.; thus $\nu(A) = 0$ or 1.

3. (of theorem 10) follows from 2. and 4.

To see 4. (of theorem 10), note that for any $f \in W$ and any $\nu \in \mathcal{Q}^\rho$,

$$\begin{aligned}\nu(f) &= \int f \; d\nu \\ &= \int \hat{f} \; d\nu \\ &= \int \mu^x(f) \; d\nu(x).\end{aligned}$$

By the Bounded Convergence Theorem, the space H of bounded Borel functions f satisfying

$$\nu(f) = \int \mu^x(f) d\nu(x)$$

is closed under uniformly bounded, pointwise limits; thus by property (B) of W and the above, $H = \mathcal{F}_b$. 4. follows by applying this to χ_A.

The proof that D is Borel in the codes is straightforward. □

2.2 Ideals of Compressible Sets

Let $X = 2^\omega$ and E be a countable, Borel equivalence relation on X; i.e., E has countable equivalence classes and E is a Borel subset of $X \times X$.

Call a function $f : X \to X$ **E-invariant** if $\forall x \in X\ (f(x)Ex)$. A probability measure μ on X is said to be **E-invariant** if for every E-invariant, Borel-measurable bijection $f : X \to X$, we have $f\mu = \mu$. Let $\mathcal{I}$ denote the collection of E-invariant probability measures. We will study the σ-ideals

$$\mathcal{J} = \{A \subseteq X : \forall \mu \in \mathcal{I}(\mu(A) = 0)\},$$

and

$$J = \{A \subseteq X : A \text{ is compact and } A \in \mathcal{J}\}.$$

In our analysis we will obtain similar results as C. Uzcátegui [90] obtained in his Ph.D. thesis for smooth sets. We will also frequently use results from Kechris-Louveau-Woodin [87], who study σ-ideals of compact sets in a general setting.

Our main tool will be the following characterization of Δ^1_1 equivalence relations without invariant measure, which is a direct effectivisation of the corresponding Borel result by Nadkarni [91]:

Theorem 1 *Let E be a countable Δ^1_1 equivalence relation on a recursively presented Polish space Y. Then the following are equivalent:*

1. *E has no invariant probability measures,*
2. *There is an E-invariant Δ^1_1 $f : Y \to Y$, which maps each E-equivalence class into a proper subset of itself; i.e., $\forall x \in X \exists x' \in [x]_E$ ($x \notin$ range of f).*

This result relativizes. Let E be a Δ^1_1 countable equivalence relation X. Call a function $f : A \to A$ an **α-compression** iff $A, f \in \Delta^1_1(\alpha)$ and f maps equivalence classes into proper subsets of themselves; i.e., $\forall x \in A\ (f(x)Ex)$ and $\forall x \in A \exists x' \in ([x]_E \cap A)$ ($x' \notin$ range of f). Call f a **compression** if f is a 0^ω-compression. From the theorem we easily obtain:

Corollary 2 *Let E be a countable equivalence Δ^1_1 relation on X. Then the following are equivalent for $A \in \Delta^1_1(\alpha)$:*

1. *$A \in \mathcal{J}$,*

2. *There is a $\Delta^1_1(\alpha)$-compression of $[A]_E$.*

Proof: 2 $\Rightarrow$ 1 is immediate. To see 1 $\Rightarrow$ 2: If $A \in \Delta^1_1(\alpha)$, then $[A]_E \in \Delta^1_1(\alpha)$, and there is a $\Delta^1_1(\alpha)$ bijection $g : [A]_E \to 2^\omega$. With this bijection we can pull back $E|[A]_E$ to a $\Delta^1_1(\alpha)$ equivalence relation F. Since $[A]_E$ is in $\mathcal{J}$, there is no F-invariant probability measure. Thus we find a $\Delta^1_1(\alpha)$ compression for F, which we can transfer via g to a $\Delta^1_1(\alpha)$ compression of $[A]_E$. □

This enables us to compute the complexity of $\mathcal{J}$:

Corollary 3 *$\mathcal{J}$ is Π^1_1 in the codes on the $\mathbf{\Delta}^1_1$ and $\mathbf{\Sigma}^1_1$ sets.*

Proof: For $\mathbf{\Delta}^1_1$ sets A, we have

$$A \in \mathcal{J} \Leftrightarrow \exists f \in \Delta^1_1(A)\ (f : [A]_E \to [A]_E \text{ is an } A\text{-compression}),$$

which is clearly Π^1_1 in the codes.

In order to compute the complexity of $\mathcal{J}$ for $\mathbf{\Sigma}^1_1$ sets, we use that for any Σ^1_1-set $G \subseteq \mathcal{N} \times X$, the relation $\mu(G_\alpha) > r$ is Σ^1_1 in μ, α, r. Thus we have for any set G $\mathcal{N}$-universal for the $\mathbf{\Sigma}^1_1$-subsets of X

$$G_\alpha \notin \mathcal{J} \Leftrightarrow \exists \mu \in \mathcal{I} \exists r \in (0,1]\ (\mu(G_\alpha) > r).$$

Here $\mathcal{I}$ is the set of E-invariant measures, which is Δ^1_1 in the Polish space of all probability measures on X with the weak*-topology. □

From this we have immediately by Uzcategui [90, theorem 1.1.16].

Corollary 4 *$\mathcal{J}$ has a largest Π^1_1 set; i.e., there is a Π^1_1 set $A \in \mathcal{J}$ which contains every Π^1_1 set in $\mathcal{J}$.*

We now turn to J. Recall that the collection $\mathcal{K}(X)$ of compact subsets of a compact metric space X is again a compact metric space with the Hausdorff topology, i.e., the topology generated by sets of the form

$$\{K \in \mathcal{K}(X) : K \cap U \neq \emptyset\}$$

and

$$\{K \in \mathcal{K}(X) : K \subseteq U\},$$

where U is open.

A σ-ideal I of compact sets is called **thin** if any pairwise, disjoint collection of sets in I is countable. I is called **calibrated** if for all $K, K_n \in \mathcal{K}(X)$ with $K_n \in I$ and $\mathcal{K}(K - \bigcup_{n\in\omega} K_n) \subseteq I$, we have $K \in I$. I is **strongly calibrated** if for any $K \notin I$ and any $P \in \mathbf{\Pi}^0_2|(X \times 2^\omega)$ with $\exists^{2^\omega} P \subseteq K$ (i.e., $\exists\alpha \in 2^\omega P(x,\alpha) \Rightarrow K(x)$), there is a $K' \in \mathcal{K}(P)$ with $\exists^{2^\omega} K' \notin I$. Say that a collection $\mathcal{A}$ of subsets of X is **compatible** with a σ-ideal I of compact subsets, if the smallest σ-ideal $\mathcal{I}$ of subsets of X containing I and $\mathcal{A}$ has no additional compact subsets over I; i.e., $\mathcal{K}(X) \cap \mathcal{I} = I$. A σ-ideal is **controlled** if there is a Σ^1_1 in the codes of $\mathbf{\Pi}^0_2$-sets collection $\mathcal{A}$ of $\mathbf{\Pi}^0_2$-sets compatible with I and with $\emptyset \in \mathcal{A}$.

If $\mathcal{EI}$, the collection of E-ergodic, E-invariant measures on X, is countable, then $J = \bigcap_{\mu\in\mathcal{EI}} J_\mu$, where J_μ is the σ-ideal of nullsets of μ. Each J_μ is $\mathbf{\Pi}^0_2$, thus J is $\mathbf{\Pi}^0_2$. In this case J is thin, calibrated, strongly calibrated, and controlled.

If $\mathcal{EI}$ is uncountable, we still have $J = \bigcap_{\mu\in\mathcal{EI}} J_\mu$, and $\gamma : \mathcal{P}(X) \to [0,1]$ given by $\gamma(A) = \sup\{\mu^* : \mu \in \mathcal{I}\}$ is an analytic submeasure, so that J is $\mathbf{\Pi}^1_1$. But J is not thin, since the ergodic measures have pairwise disjoint support. So J cannot be controlled and thus is not $\mathbf{\Pi}^0_2$. Thus J is truly $\mathbf{\Pi}^1_1$ by the dichotomy theorem for σ-ideals. J is still strongly calibrated and calibrated. $\mathcal{J}$ is not Σ^1_1 on the codes of $\mathbf{\Delta}^1_1$, $\mathbf{\Pi}^1_1$, or $\mathbf{\Sigma}^1_1$-sets, however, since this would imply that J is controlled.

2.2.1 Proof of Theorem 1

We will assume that E is a countable Δ^1_1 equivalence relation on X, which is not compressible. By Feldman-Moore [77], we may assume that we have a countable group G acting in a Δ^1_1-way on X such that E is the orbit equivalence relation of that G-action. We will say that a property P holds almost everywhere if $[\{x \in X : P(x)\}]_E$ is compressible, and we will write $\forall^* x\, P(x)$ in this situation. If P is a property of Δ^1_1-sets and points in X, i.e., $P \subseteq \Delta^1_1|Y_1 \times \ldots \times \Delta^1_1|Y_n \times X$ for each Y_i is ω, X, or $\mathcal{N}$, we will say that P holds uniformly almost everywhere if for each $\bar{A} \in \Delta^1_1|Y_1 \times \ldots \Delta^1_1|Y_n$ we can compute codes of a compressible Δ^1_1-set C and a Δ^1_1 compression $c : C \to C$ recursively from a code of $\bar{A}$ such that $\forall x \notin R\, [P(\bar{A}, x)]$.

The construction of the E-invariant probability measure is similar to the construction of the Haar measure on a compact group. Here we compare the size of Δ^1_1-sets via E-invariant functions; i.e., if between A and each of the $A_1, \ldots A_n$ there is an E-invariant bijection and the A_i are pairwise disjoint, then $\bigcup_{i=1}^n A_i$ will have n times the measure of A. But things get a little messier since our results only hold almost everywhere. We show:

Key Lemma 5 *There is a partial function $m : \Delta^1_1|(X) \times X \to [0, \infty)$, $(A, x) \mapsto m^x(A)$ such that we have uniformly, for every $A \in \Delta^1_1(X)$ and every $B \in \Delta^1_1|(\omega \times X)$ with pairwise disjoint sections,*

$$\forall^* x \; (0 \leq m^x(A) \leq 1), \tag{2.1}$$

$$\forall^* x \; (m^x(\bigcup_{i=0}^{\infty} B_i) = \sum_{i=0}^{\infty} m^x(B_i)), \tag{2.2}$$

$$\forall^* x \; \forall g \in G \; (m^x(A) = m^x(gA)). \tag{2.3}$$

Assuming the key lemma, we apply it to a countable algebra of sets which is uniformly Δ^1_1, given by the following lemma:

Lemma 6 *There is a $C = \{C_n : n \in \omega\} \in \Delta^1_1|X$, a Polish topology τ, a complete metric d on X, and recursive functions $f_1 : \omega^3 \to \omega$ and $f_2 : \omega^2 \to \omega$ such that*

1. *C is a Boolean algebra, $C_0 = X$,*
2. *d induces τ; C is a clopen basis for τ,*
3. *$\forall k, m(C_k = \bigsqcup_l C_{f_1(k,l,m)})$,*
4. *$\forall k, l(g_l^{-1}[C_k] = C_{f_2(k,l)})$,*
5. *$\forall k, l, m$ the d-diameter of $C_{f_1(k,l,m)} < 1/(m+1)$.*

Here $G = \{g_l : l \in \omega\}$ is some Δ^1_1 enumeration of G.

By the key lemma pick $x \in X$ such that

1. $\forall n \; [m^x(C_n) \in [0, 1]]$,
2. $\forall k, m \; [m^x(C_k) = (\sum_{l=0}^{\infty} m^x(C_{f_1(k,l,m)}))]$,
3. $\forall k, l \; [m^x(g_l C_k) = m^x(C_{f_2(k,l)})]$.

This is possible since the set of points $x \in X$ for which one of these fails is compressible. We now apply the following, immediate consequence of theorems 13.2 and 13.8 of Munroe [71] to find the G-invariant probability measure on X easily.

Theorem 7 *Let (X, d) be a metric space and G a countable group of bijections of X. Assume that $\mathcal{C} \subseteq P(X)$ and $m : \mathcal{C} \to [0, 1]$ are such that*

1. *For every $\epsilon > 0$ there is a countable cover of X by sets from $\mathcal{C}$, each with d-diameter $< \epsilon$. $\forall A \subseteq X \forall \epsilon > 0 \exists \{C_n : n \in \omega\} \subseteq \mathcal{C}$ $(A \subseteq \bigcup\{C_n : n \in \omega\} \wedge \forall n$ (diameter$(C_n) < \epsilon$)),*
2. *$\mathcal{C}$ is closed under G; i.e., $\forall C \in \mathcal{C} \forall g \in G$ $(g^{-1}[C] \in \mathcal{C})$,*
3. *$\forall \epsilon > 0 \forall \delta > 0 \forall C \in \mathcal{C} \exists \{C_n : n \in \omega\}$ $(C \subseteq \bigcup\{C_n : n \in \omega\} \wedge \sum_n m(C_n) \leq m(C) + \delta \wedge \forall n$ (diameter$(C_n) < \epsilon$)),*
4. *m is G-invariant; i.e., $\forall C \in \mathcal{C} \forall g \in G$ $(m(g^{-1}[C]) = m(C))$.*

Then there is a G-invariant measure μ such that every Borel set is μ-measurable.

Thus we are done, once we prove the lemmas.

2.2.2 An Algebra of Δ^1_1-Sets

Before we prove lemma 6, we need two well-known facts, which we prove since we did not find a convenient reference. Let X be a recursively presented Polish space, E a countable Δ^1_1 equivalence relation induced by a Δ^1_1 group action of a countable group G. Let $\{g_i : i \in \omega\}$ be a Δ^1_1 enumeration of G with g_0 the identity function.

Fact 8 *There is a recursive in the Δ^1_1-codes operation $C \mapsto C'$ such that if $C \subseteq X$, then there is a complete metric d on X and $C' \subseteq \omega \times \omega \times X$ such that*

1. *C is clopen in the metric topology of d,*
2. *$\{C'_{k,m}\}$ is a clopen basis for this topology,*

3. $\forall k \forall m$ ($C'_{k,m}$ *has d-diameter* $\leq 1/(k+1)$),

4. $\forall k$ [($\bigcup_m C'_{m,k} = X$), *and this union is disjoint*],

5. $\forall k$ ($\bigcup_m C'_{k,2m} = C$),

6. $\forall k$ ($\bigcup_m C'_{k,2m+1} = X - C$),

7. *The original topology and the metric topology of d have the same Borel sets.*

Proof: Fix C. Let $\tilde{C} = (\{0\} \times C) \cup (\{1\} \times (X - C))$. Effectively find $G \subseteq \mathcal{N} \times 2 \times X$, $G \in \Pi^0_1$ such that

$$\begin{aligned} \tilde{C}(i,x) &\Leftrightarrow \exists \alpha \in \mathcal{N}\ G(\alpha, i, x) \\ &\Leftrightarrow \exists! \alpha \in \mathcal{N}\ G(\alpha, i, x). \end{aligned}$$

Find $N = \langle N_{k,m} : k, m \in \omega \rangle$ in Δ^0_1 a basis for $\mathcal{N} \times 2 \times X$ such that $\forall k, m(N_{k,m}$ has diameter $< 1/(k+1)$ in the usual bounded metric of $\mathcal{N} \times 2 \times X)$, $\forall k(\bigsqcup_m N_{k,2m} = \mathcal{N} \times \{0\} \times X)$, and $\forall k(\bigsqcup_m N_{k,2m+1} = \mathcal{N} \times \{1\} \times X)$. Let

$$C'_{k,m} \Leftrightarrow \exists i \exists \alpha (G(\alpha, i, x) \wedge N_{k.m}(\alpha, i, x)).$$

Clearly, this operation is effective. If we put the subspace topology on G, then G is complete, $\{N_{k,m} \cap G\}$ is a basis, and the usual metric on $\mathcal{N} \times 2 \times X$ restricted to G is complete. Thus we can transfer this structure by the projection onto X, which is a bijective from G onto X. □

Fact 9 *There is a recursive in the* Δ^1_1*-codes operation* $O_1 : C \mapsto C'$ *and recursive functions* $g : \omega^3 \to \omega$ *and* $h : \omega^2 \to \omega$ *such that if* $C \subseteq \omega \times X$, *then* $C' \subseteq \omega \times X$, *and there is a complete metric d on* X, *inducing a topology* τ *such that*

1. τ *is a Polish topology with the same Borel sets as the original topology on* X,

2. $\{C'_i\}$ *is a clopen basis for* τ,

3. $\forall i, j, k(C'_{g(i,j,k)}$ *and* $C'_{h(j,k)}$ *have d-diameter* $\leq 1/(k+1)$),

4. $\bigsqcup_j C'_{g(i,j,k)} = C_i$,

5. $\bigsqcup_j C'_{h(j,k)} = X$.

Proof: For each i, let $(\tilde{C}^i, d_i, \tau_i)$ be given by the previous fact for C_i. Let $\tilde{C} = <\tilde{C}^i : i \in \omega>$. Let $d = \sup\{d_i/(i+1) : i \in \omega\}$. Then the topology τ induced by d is Polish, has the same Borel structure as the original topology on X, and $\tilde{C}^i = \{\tilde{C}^i_{k,m}\}$ is a subbasis for τ.

Note that the d_i-diameter of $\tilde{C}^i_{k,m}$ is $\leq 1/(k+1)$. Thus the d-diameter of any

$$\tilde{C}_{\bar{m}} = \bigcap_{i=0}^{k-1} \tilde{C}^i_{k,m_i}$$

is at most $1/(k+1)$ for $\bar{m} \in \omega^k$. Furthermore, the collection $\{\tilde{C}_{\bar{m}} : \bar{m} \in \omega^k\}$ is pairwise disjoint and covers X.

Thus let C' enumerate all the $\tilde{C}_{\bar{m}}$, taking care that the enumeration of the sequences $\bar{m}$ is recursive in such a manner that we can guarantee 3.-5. from 3.,5.,6. of the previous fact. □

We are now ready to prove lemma 6.

Proof: Since Δ^1_1-sets are closed uniformly under Boolean operations and taking preimages under Δ^1_1-functions, there is a recursive in the Δ^1_1-codes operation $O_1 : C' \mapsto C''$ such that if C' is a Δ^1_1-sequence of sets, then C'' enumerates the smallest Boolean algebra containing C and is closed under preimages by elements of G. Now put C^0 to be the standard basis for X, and put inductively $C^{n+1} = O_1(O_2(C^n))$. Let d_n be the metric for the topology generated by C^n given by fact 9 and then proceed as there by setting $d = \sup\{d_i/(i+1) : i \in \omega\}$. □

2.2.3 Proof of the Key Lemma

We first fix some notation. We write

$f : A \preceq B$	if $f \in \Delta^1_1$ is injective and preserves E-equivalence classes,
$f : A \prec B$	if $f : A \preceq B$ and $B - f[A]$ is full in B,
$f : A \sim B$	if f is a Δ^1_1-bijection preserving E-equivalence classes,
A^Q	for $A \cap Q$ if Q is E-invariant.

As a first step we show:

Lemma 10 *For every pair A, B of Δ^1_1-subsets of X we can find Δ^1_1-sets $Q \subseteq \omega \times X$ and $C \subset X$ and Δ^1_1-functions $f : A \to B$ and $c : C \to C$ such that*

(a) *C and all Q_k are E-invariant, and these sets partition $[A]_E \cap [B]_E$.*

(b) *c is a compression of C. f and c are E-invariant.*

(c) *$f^{-1}(b)$ has cardinality k or $k+1$ for each $b \in Q_k \cap B$, and there is some $b' \in [b]_E \cap B$ such that $f^{-1}(b')$ has cardinality k.*

Codes for Q, C, f, c can be computed recursively from codes for A and B. Furthermore, if A, B are assigned Q, C, f, c and A', B' are assigned Q', C', f', c'; then we have:

1. *If $P \in \Delta^1_1$ is E-invariant and $A^P = A'^P$ and $B^P = B'^P$, then $Q \cap (\omega \times P) = Q' \cap (\omega \times P)$ and $f|(P \cap A) = f'|(P \cap A')$.*

2. *If $g : A \sim A'$ and $h : B \sim B'$, then $C'' = \bigcup_n (Q_n \Delta Q'_n)$ is compressible via a compression c''. The codes for C'' and c'' can be computed recursively from codes of A, B, A', B'.*

3. *If $P \in \Delta^1_1$ is E-invariant and $A^P = B^P$, then $P \subseteq Q_1$ and $f|A^P$ is the identity.*

This lemma allows us to compare the sizes of sets. On part Q_k, B fits into A k-times but not $k+1$-times.

Proof: Assume w.l.o.g. that $[A]_E \subseteq [B]_E$. First assume that there is a function $\tilde{f} : A \to \omega \times B$ in Δ^1_1 such that

(1) $\tilde{f}$ is injective and $\tilde{f}_2$, the second coordinate function, is E-invariant.

(2) If $(n+1, y)$ is in the range of $\tilde{f}$ and $y' \in [y]_E \cap B$, then (n, y') is also in the range of $\tilde{f}$.

Furthermore, assume that a code for $\tilde{f}$ can be found recursively in the codes of A and B. Let $f = \tilde{f}_2$ and $C = \{x \in A : \forall n \in \omega\ ((n, x)$ is in the range of $\tilde{f})\}$. Then the mapping $c : A^C \prec A^C$ given by

$$c(x) = f^{-1}(f_1(x) + 1, f_2(x))$$

is a compression of A^C, and on the remainder f is as desired. Put

$$Q(n, x) \ \Leftrightarrow\ \forall y' \in ([x]_E \cap B)\ ((n, y') \in \text{range}(\tilde{f})).$$

Then Q is as desired.

In order to show that $\tilde{f}$ can be found effectively, let us show that given $A, B \in \Delta^1_1$, there is a partition of $[A]_E$ into E-invariant Δ^1_1-sets P_1 and P_2 and a partition of A^{P_2} into Δ^1_1-sets A' and A'' such that there are Δ^1_1-functions $h_1 : A^{P_1} \prec B^{P_1}$ and $h_2 : A'' \sim B^{P_2}$. In particular $[A'']_E = P_2$. Furthermore, codes of all these objects can be found recursively in the codes of A and B.

Once this is done, we can set inductively

$$\begin{aligned} A_0 &= A, \\ A_{n+1} &= A'_n, \\ f_1|(A_n - A_{n+1}) &= n \text{ everywhere}, \\ f_2|(A_n - A_{n+1}) &= h_1 \cup h_2 \text{ for the pair } (A_n, B). \end{aligned}$$

Since $[A]_E \subseteq B$, we see by an easy induction that $f|(A - A_{n+1})^{[A_{n+1}]_E}$ has range $(n+1) \times B^{[A_{n+1}]_E}$, so that f indeed satisfies (2). (1) is clear from the definition.

Thus assume that $A, B \in \Delta^1_1$ are given. We have to find P_1, P_2, A', A'', h_1, h_2. By induction, let

$$\begin{aligned} F_0 &= (A \times B) \cap G_0, \\ F_{n+1} &= (A \times B) \cap G_{n+1} - (\pi_0[\bigcup_{i=0}^{n} F_i] \times \pi_1[\bigcup_{i=0}^{n} F_i]), \\ F &= \bigcup_{i \in \omega} F_i. \end{aligned}$$

Recall that G_n was the graph of g_n, where g_n was the n-th function in the enumeration of a group inducing E such that g_0 was the identity map. Note that F is the graph of a partial bijection. If $x \in A$ and $y \in B$ were E-related and if neither x were in the domain of F nor y in the range of F, then $(x, y) \in G_n$ for some n, and thus $(x, y) \in F_n$, a contradiction. Thus for each $x \in [A]_E \cap [B]_E$, either $[x]_E \cap A$ is exhausted by the domain of F or $[x]_E \cap B$ is exhausted by the range of F. Let

$$P_1 = \{x \in ([A]_E \cap [B]_E) : [x]_E \cap B \text{ is not exhausted by the range of } F\},$$

and let h_1 have graph $F \cap (A^{P_1} \times B^{P_1})$. By the definition of P_1 $f : A^{P_1} \prec B^{P_1}$. Let $\check{P}_1 = [A]_E - P_1$. Put $A'' =$ domain of $F \cap (A^{P_1} \cap B^{P_1})$, and let h_2 have graph $F \cap (A'' \times B^{P_1})$. Then $h_2 : A'' \sim B^{P_2}$, since B is exhausted on P_2 by the range of F. Let $A' = A^{P_1} - A''$.

Finally, let us check that 1.-3. are satisfied. 1. is clear, since all the constructions were local; i.e., the construction on a specific equivalence class depended only on the intersection of that equivalence class with A and B. To see 2., look at $Q_m \cap Q'_n$ for $m < n$. The following mapping c is a compression of $A^{Q_m \cap Q'_n}$: For $x \in A^{Q_m \cap Q'_n}$, assume that x is the kth element (in the lexicographical ordering) in the preimage of $f(x)$ under f. Then $k \leq m+1 \leq n$, so since $g \circ f(x)$ has at least size n, let y be the kth element (in the lexicographical ordering) of the preimage $f'^{-1}(g \circ f(x))$. Let $c(x) = g^{-1}(y)$. To see 3. assume by 1. that $A = B$. Then the identity function g_0 is a bijection between A and B; thus $F = G_0 \cap (A \times B)$, and thus $A = A''$ (for the pair A, B), and h_2 is the identity on A. From this it follows that f is the identity. □

Let us introduce the following notation.

Definition 11 For $A, B \in \Delta^1_1$, define

$$[A/B](x) = \begin{cases} i & \text{if } x \in Q_i \\ 0 & \text{if } \forall i (x \notin Q_i), \end{cases}$$

where Q is given by the above lemma.

We observe the following:

Lemma 12 *1. If $A, A', B \in \Delta^1_1$ and $A \cap A' = \emptyset$, then uniformly*

$$\begin{aligned} \forall^* x\ ([A/B](x) + [A'/B](x) &\leq [(A \cup A')/B](x) \\ &\leq ([A/B](x) + [A'/B](x) + 2)). \end{aligned}$$

2. If $A, B, C \in \Delta^1_1$ and $[A]_E \cap [C]_E \subseteq [B]_E$, then

$$\begin{aligned} \forall^* x\ ([A/B](x)[B/C](x) &\leq [A/C](x) \\ &< ([A/B](x) + 1)([B/C](x) + 1)). \end{aligned}$$

Proof: We show that the first inequality of 1. holds almost everywhere. The other inequalities are proved by similar arguments. Fix A, A', B. Let C be the collection of points where the first inequality of 1. fails. Thus we have

$$\forall x \in C\ ([(A \cup A')/B](x) < [A/B](x) + [A'/B](x)).$$

Let Q, f, Q', f', and Q'', f'' be given for A, B, A', B, and $A \cup A', B$ by lemma 10. On $Q_n \cap Q'_m \cap Q''_l$ with $l < n+m$, define $h : A \cup A' \to (l+1) \times B$ by $h(x) = (k, f''(x))$, where x is the $(k+1)$st element in the lexicographical ordering of $f''^{-1}(f''(x))$. Define $g : (n+m) \times B \to A \cup A'$ by

$$g(k,x) = \begin{cases} \text{the } k+1\text{st element in the lex. ord. of } f'^{-1}(x) & \text{if } k < n, \\ \text{the } (k-n)+1\text{st element in the lex. ord. of } f''^{-1}(x) & \text{if } k \geq n. \end{cases}$$

Thus on $Q_n \cap Q'_m \cap Q''l$, h and g are injective and h is not surjective on each equivalence class. Thus on $Q_n \cap Q'_m \cap Q''l$, $c = g \circ h$ is a compression of $A \cup A'$, which can be extended to a compression of $Q_n \cap Q'_m \cap Q''l$ by setting it to be the identity. □

We now construct a decreasing Δ^1_1-sequence $\{F_k : k \in \omega\}$ of almost full sets such that

$$\forall^* x \forall k \in \omega\ ([F_k/F_{k+1}](x) \geq 2).$$

We call these **reference sets**. It clearly suffices to show that for each $A \in \Delta^1_1$ we can find a subset $A' \in \Delta^1_1$ such that A' is almost full in A and

$$\forall^* x \in [A]_E\ ([A/A'](x) \geq 2),$$

and that we can find a code for A', the exceptional set and its compression recursively from a code of A. So fix $A \in \Delta^1_1$ and set

$$S_n(x) \Leftrightarrow [x \in A \wedge \forall y \in ([x]_E \cap A)\ (x|n \leq_{\text{lex}} y|n),$$

where $\leq_{\text{lex}}$ is the lexicographical ordering on $2^{<\omega}$. Then $S_n(x)$ holds for all n iff $x \in A$ is the lexicographically least point of $[x]_E \cap A$. Thus $\bigcap_n S_n$ is a partial transversal, i.e., a set which intersects each equivalence class in at most one point.

Lemma 13 *If $T \in \Delta^1_1$ is a partial transversal, then $[T]_E$ is compressible. A code for $[T]_E$ and its compression may be computed recursively from a code of T.*

Proof: Put an order of ordertype ω on each equivalence class of $[T]_E$ by setting

$$x < y \ \Leftrightarrow\ (xEy \wedge \exists n \in \omega\ (g_n(x) \in T \wedge \forall m \leq n\ g_m(y) \notin T)),$$

where $G = \{g_n\}$ is the group inducing E. Then the map taking each $x \in [T]_E$ to its successor is a compression on $[T]_E$. $\square$

Thus we may assume that $\bigcap_n S_n = \emptyset$. Now put

$$A_1(x) \Leftrightarrow \exists n(S_n(x) \wedge n \text{ is least such that } \exists y \in ([x]_E \cap A) \neg S_n(y))$$

and

$$A_2 = A - A_1.$$

The A_i partition A into full subsets, and we pick for A' in each equivalence class the smaller of the two parts:

$$\begin{aligned} A'(x) \quad \Leftrightarrow \quad & (x \in A_1 \wedge [A_2/A_1](x) \geq 1) \vee, \\ & (x \in A_2 \wedge [A_2/A_1](x) < 1). \end{aligned}$$

Clearly, a code for A' may be computed recursively from a code for A. Put

$$\begin{aligned} C_1 &= \{x \in [A]_E : [A_2/A_1](x) \geq 1\},. \\ C_2 &= \{x \in [A]_E : [A_2/A_1](x) < 1\} \end{aligned}$$

Then we have for almost all $x \in C_1$,

$$\begin{aligned} [A/A'](x) &= [A/A_1](x) \\ &= [(A_1 \cup A_2)/A_1](x) \\ &= [A_1/A_1](x) + [A_2/A_1](x) \\ &\geq 2, \end{aligned}$$

and for almost all $x \in C_2$,

$$\begin{aligned} [A/A'](x) &= [A/A_2](x) \\ &= [(A_1 \cup A_2)/A_2](x) \\ &= [A_1/A_2](x) + [A_2/A_2](x) \\ &\geq 2, \end{aligned}$$

since the set of point in $[A_1]_E \cap [A_2]_E \supseteq C_2$ where $[A_1/A_2](x) = [A_2/A_1](x) = 0$ is clearly compressible, and thus $[A_1/A_2](x) > 0$ for almost all $x \in C_2$. Thus A' is as desired, and we can fix a sequence of reference sets. Now put for $A \in \Delta^1_1|X$ and $x \in X$,

$$m^x(A) = \begin{cases} \lim_{n\to\infty} \frac{[A/F_n](x)}{[X/F_n](x)} & \text{if this limit exists,} \\ 0 & \text{if this limit does not exist.} \end{cases}$$

Since $[A/B]$ is E-invariant, we have $m^x = m^y$ for xEy. Clearly, m^x is nonnegative and almost everywhere $[X/F_n] = [A/F_n] + [(X - A)/F_n] \geq [A/F_n]$, so that $m^x(A) \in [0, 1]$ almost everywhere.

Lemma 14 *Let $A \in \Delta^1_1$. Then*

$$\lim_{n\to\infty} \frac{[A/F_n](x)}{[X/F_n](x)}$$

exists for almost all x, uniformly in A and the sequence of reference sets.

Proof: By lemma 12 and the properties of the reference sets, we have almost everywhere

1. $[A/F_i][F_i/F_{i+j}] \leq [A/F_{i+j}] \leq ([A/F_i] + 1)([F_i/F_{i+j}] + 1)$,
2. $[X/F_i][F_i/F_{i+j}] \leq [X/F_{i+j}] \leq ([X/F_i] + 1)([F_i/F_{i+j}] + 1)$,
3. $[X/F_n] \rightarrow$ pointwise,
4. $[F_n/F_{n+m}] \rightarrow \infty$ pointwise.

Thus we have almost everywhere

$$\frac{[A/F_{i+j}]}{[X/F_{i+j}]} \leq \frac{[A/F_i] + 1}{[X/F_i]} \frac{[F_i/F_{i+j}] + 1}{[F_i/F_{i+j}]},$$

and since $\lim_{j\to\infty}([F_i/F_{i+j}] + 1)/[F_i/F_{i+j}] = 1$, we have

$$\limsup_{j\to\infty} \frac{[A/F_{i+j}]}{[X/F_{i+j}]} \leq \frac{[A/F_i] + 1}{[X/F_i]},$$

and thus

$$\limsup_{j\to\infty} \frac{[A/F_j]}{[X/F_j]} \leq \liminf_{j\to\infty} \frac{[A/F_j] + 1}{[X/F_j]} = \lim_{j\to\infty} \frac{[A/F_j]}{[X/F_j]} = m(A, x).$$

Since the set of points where the limit does not exist is at most the set where lemma 12 fails or where the sequence of reference sets does not have the required properties, we can recursively find a code for this set and for a compression of it from a code of A and the sequence of reference sets. □

We already checked that m^x takes values in $[0,1]$. Since for any $A \in \Delta^1_1$ and any g in the group G inducing E we have $[A/F_n] = [gA/F_n]$ almost everywhere, we have $m^x(A) = m^x(gA)$ almost everywhere by the above lemma. Thus in order to verify that m satisfies the Key Lemma, we are left with showing that m^x is almost everywhere σ-additive. We know from lemma 12 that m^x is almost everywhere finitely additive.

Fix a disjoint Δ^1_1-sequence $\{A_i : i \in \omega\}$. By finite additivity we have

$$\forall^* x \ (m^x(\bigcup_i A_i) \geq \sum_i m^x(A_i)).$$

Put $P(n,x)$ iff $m^x(\bigcup_i A_i) > \sum_i m^x(A_i) + 2^{-n}$. It clearly suffices to show that P_n is uniformly compressible. Note that P_n is E-invariant. Let $A = \bigcup_{i=0}^{\infty} A_i$. By the properties of reference sets we have

$$\forall^* x \ ([X/F_n](x) \geq 2^n)$$

uniformly, and thus uniformly

$$\forall^* x \ (m^x(F_n) \leq 2^n).$$

Thus almost everywhere on P_n we have $m^x(F_n) < m^x(A)$ and so we can apply the following lemma.

Lemma 15 *Assume that $P \in \Delta^1_1$ is E-invariant and $A,\ B \in \Delta^1_1$ with*

$$\forall x \in P(m^x(A) < m^x(B)).$$

Then there is an $f : A^p \preceq B^P$ almost everywhere and furthermore, codes for f, the exceptional set, and its compression can be found recursively in the codes of A, B, and P.

Proof: Let $n : P \to \omega$ be given by $n(x)$ is the least integer m such that $[A/F_m](x) < [B/F_m](x)$. Let f^m, Q^m and $\tilde{f}^m, \tilde{Q}^m$ be given by lemma 10 for A, F_m and B, F_m, respectively. Define $f : A \to B$ by

$$f(x) = \begin{cases} \text{the } k\text{-th element in the lexicographical ordering of} \\ (\tilde{f}^{n(x)})^{-1}(f^{n(x)}(x)), \text{ where } x \text{ is the } k\text{th element in the} \\ \text{lexicographical ordering of } (f^{n(x)})^{-1}(f^{n(x)}(x)). \end{cases}$$

This works □

Thus find a subset A' of A such that $A' \cap P_n \sim F_n \cap P_n$ almost everywhere, and thus

$$\forall^* x \in P\ (m^x(A') = m^x(F_n)).$$

Since $A = A' \cup (A - A')$, we know by finite additivity and the fact that $\forall^* x\ (m^x(F_n) \leq 2^{-n})$ (uniformly), that

$$\forall^* x \in P_n\ (m^x(A - A') > \sum_{i=0}^{\infty} m^x(A_i)).$$

Thus we are done once, we show the following lemma.

Lemma 16 *Assume that A_i is a Δ^1_1-sequence and $B \in \Delta^1_1$, $P \in \Delta^1_1$ is E-invariant. If*

$$\forall^* x \in P\ (m^x(B) > \sum_{i=0}^{\infty} m^x(A_i)),$$

then we can find uniformly a mapping $f : \bigcup_{i=0}^{\infty} A_i \preceq B$.

Proof: Since

$$\forall^* x \in P\ (m^x(B) > m^x(A_0)),$$

find $f_0 : A_0 \preceq B$ by lemma 15. Let $B' = B - f_0[A_0]$. Then

$$\forall^* x \in P\ (m^x(B') > \sum_{i=1}^{\infty} m^x(A_i) \geq m^x(A_1));$$

thus find $f_1 : A_1 \preceq B$ by lemma 15. Continue in the same manner. □

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