An Abstract Condensation Property

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ABSTRACT

Let $\mathcal{A} = (A, ...)$ be a relational structure. Say that $\mathcal{A}$ *has condensation* if there is an $F : A^{<\omega} \longrightarrow A$ such that for every partial order P, it is forced by P that substructures of P which are closed under F are isomorphic to elements of the ground model. *Condensation holds* if every structure in V, the universe of sets, has condensation. This property, isolated by Woodin, captures part of the content of the condensation lemmas for L, K and other "L-like" models. We present a variety of results having to do with condensation in this abstract sense. Section 1 establishes the absoluteness of condensation and some of its consequences. In particular, we show that if condensation holds in M, then $M \models GCH$ and there are no measurable cardinals or precipitous ideals in M. The results of this section are due to Woodin. Section 2 contains a proof that condensation implies $\diamondsuit_\kappa(E)$ for κ regular and $E \subseteq \kappa$ stationary. This is the main result of this thesis. The argument provides a new proof of the key lemma giving GCH. Section 2 also contains some information about the relationship between condensation and strengthenings of diamond. Section 3 contains partial results having to do with forcing "$Cond(\mathcal{A})$", some further discussion of the relation between condensation and combinatorial principles which hold in L, and an argument that $Cond(G)$ fails in $V[G]$, where G is generic for the partial order adding ω_2 cohen subsets of ω_1.

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Introduction

Let $\mathcal{A}$ be a relational structure. Woodin [Wo] has isolated a property of $\mathcal{A}$ which captures part of the content of the condensation lemma for levels of the constructible hierarchy—*viz.*, that suitable hulls collapse to elements of L no matter where these hulls are taken.

Definition. Let $\mathcal{A} = (|\mathcal{A}|, \dots)$ and $F : |\mathcal{A}|^{<\omega} \longrightarrow |\mathcal{A}|$. $Cond(\mathcal{A}; F)$ is the statement: For every partial order (p.o.) $\mathbb{P}$

$$\mathbb{P} \Vdash \forall X. X \prec_F \mathcal{A} \longrightarrow \exists \mathcal{B} \in V. \mathcal{A} \restriction X \cong \mathcal{B}$$

In other words, for any p.o. $\mathbb{P}$ it is forced by $\mathbb{P}$ that F-closed substructures of $\mathcal{A}$ are isomorphic to structures in V. $Cond(\mathcal{A})$ holds iff $\exists F. Cond(\mathcal{A}; F)$.

We will be concerned with condensation for structures of the form $\mathcal{A} = (X, \in, \dots)$ where X is a transitive set. (Notations for such structures will typically suppress the "$\in$".) $Cond(\mathcal{A}; F)$ will hold for such structures just in case F-closed substructures *collapse* to elements of V.

If $Cond(\mathcal{A})$ holds we say that $\mathcal{A}$ *has condensation*. Let M be a transitive model of ZFC. If $M \models Cond(\mathcal{A})$, $\mathcal{A}$ has *condensation in* M. *Condensation holds in* M iff $M \models \forall \alpha\, Cond(V_\alpha)$. Equivalently, since $M \models AC$, condensation holds in M iff $M \models \forall \kappa \forall A \subseteq \kappa\, Cond(\kappa, A)$. For ordinals α, $Cond(\alpha)$ iff $\forall A \subseteq \alpha. Cond(\alpha, A)$. And if A is a subset of α, $Cond(A)$ iff $Cond(\alpha, A)$ (iff $Cond(\langle \alpha, \in, A \rangle)$).

Any other uses of the term "condensation" should be self-explanatory.

Section 1 establishes the absoluteness of condensation and some of its consequences. In particular, we show that if condensation holds in M, then $M \models GCH$ and there are no measurable cardinals or precipitous ideals in M. The results of this section are due to Woodin [Wo]. Section 2 contains a proof that condensation implies $\diamondsuit_\kappa(E)$ for κ regular and $E \subseteq \kappa$ stationary. This is the main result of this thesis. The argument provides a new proof of the key lemma giving GCH. Section 2 also contains some information about the relationship between condensation and strengthenings of diamond. Section 3 contains partial results having to do with forcing "*Cond*($\mathcal{A}$)", some further discussion of the relation between condensation and combinatorial principles which hold in L, and an argument that $Cond(G)$ fails in $V[G]$, where G is generic for the partial order adding ω_2 cohen subsets of ω_1.

Notations are fairly standard. $[X]^\kappa$ is the collection of subsets of X of size κ. $[X]^{<\kappa}$ is the collection of subsets of X of size less than κ. $X^{<\omega}$ is the collection of finite sequences from X. H_κ is the collection of sets hereditarily of size less than κ. $HC = H_{\omega_1}$. If $F : X^{<\omega} \longrightarrow X$, $C_F = \{U \subseteq X \mid F''U^{<\omega} \subseteq U\}$. (Occasionally, C_F will refer to closed sets meeting some size restriction.) $cl_F(U)$ is the closure of U under F. For any set X, π_X is the collapsing map associated with X. $\subseteq_e$ refers to end-extension. $Coll(\kappa, X)$ is the partial order adding a function $f : \kappa \xrightarrow{onto} X$ with conditions of size $< \kappa$. If t is a set-theoretic expression $(t)^M$ is the result of evaluating t in M. Similarly, if ϕ is a formula $(\phi)^M$ is the restriction of ϕ to M. Thus "$(\phi)^M$" and "$M \models \phi$" have the same meaning. In expository contexts 'V' refers to the universe of sets. In statements of a forcing language 'V' refers to the ground model. Other notations will be handled as they arise.

We make a few preliminary remarks and observations. If $F, G : X^{<\omega} \longrightarrow X$ define $F \leq G$ iff $G \geq F$ iff $C_G \subseteq C_F$.

Lemma 0.1. *If* $Cond(\mathcal{A}; F)$ *and* $F \leq G$ *then* $Cond(\mathcal{A}; G)$. $\dashv$

Say that F is *efficient* if $cl_F(U) = F''U^{<\omega}$. It is sometimes convenient to know that

Lemma 0.2. *For any* F *there is an efficient* $G \geq F$.

Proof. Define $\phi_k : V^{<\omega} \longrightarrow V$ by $\phi_k(s) = s_k$ if $k < \ell(s)$, and $\phi_k(s) = s_0$ if $k \geq \ell(s)$. Let $\mathcal{F}^k$ be the closure under composition of $\{F, \phi_0, \ldots \phi_k\}$ and $\mathcal{F} = \bigcup_k \mathcal{F}^k$. Let $\langle F_n \mid n < \omega \rangle$ list $\mathcal{F}$ so that if $p(n)$ is the least k such that $F_n \in \mathcal{F}^k$, then $p(n) < n$. Define G by $G(s) = F_{\ell(s)}(s)$. It is straightforward to check that $C_G \subseteq C_F$ and $cl_G(U) = G''U^{<\omega}$. $\dashv$

Condensation for sets of size $\leq \omega_1$ is trivial. Thus the structures $\mathcal{A}$ which come up for attention will have domains at least that size when viewed from within a model in which $Cond(\mathcal{A})$ is to be evaluated. Another property common to the domains of structures we consider is closure under $u \mapsto rk(u)$. So in every case we will have $rk(A) = A \cap OR \geq \omega$. Also one may always assume of a potential witness F to $Cond(\mathcal{A})$ that $cl_F(U) \cap \alpha^+ = \alpha$ where $\alpha = min(OR \setminus cl_F(U))$.

Lemma 0.3. *The following are equivalent:*

(1) $Cond(\mathcal{A})$.

(2) *Let* $\alpha = rk(\mathcal{A})$. *There is a* u *such that for every* $\theta \geq \alpha$ *and p.o.* $\mathbb{P}$, *letting* $N = V_\theta$, $\mathbb{P} \Vdash \forall X.\, u, \mathcal{A} \in X \prec N \Longrightarrow \pi_X(\mathcal{A}) \in N$.

(3) *There is a* u *and a* $\theta > \alpha$ *such that for every p.o.* $\mathbb{P}$, *letting* $N = V_\theta$, $\mathbb{P} \Vdash \forall X.\, u, \mathcal{A} \in X \prec N \Longrightarrow \pi_X(\mathcal{A}) \in N$.

Proof. Let $\mathcal{A} = (A, \in, R, \dots)$.

(1) $\Longrightarrow$ (2). Let $u = F$.

(2) $\Longrightarrow$ (3). Trivial.

(3) $\Longrightarrow$ (1). Pick u and let F be an efficient Skolem function for $(V_\theta, \{\mathcal{A}, u\})$. Let $G : A^{<\omega} \longrightarrow A$ be defined by $G(s) = F(s)$ if $F(s) \in A$ and $G(s) = \varnothing$ otherwise. If $X \in C_G$ then $A \cap cl_F(X) = X$. So $\pi_X(\mathcal{A}) = \pi_{cl_F(X)}(\mathcal{A})$, where $\pi_X(\mathcal{A}) = (\pi_X{}''A, \pi_X{}'' \in \cap A^2, \pi_X{}''R, \dots)$.

Thus $Cond(\mathcal{A}; G)$ holds. ⊣

Lemma 0.4. *If* A *is a transitive set,* $Cond(A)$ *holds and* $\mathcal{B} \in A$, *then* $Cond(\mathcal{B})$.

Proof. Take u as in 0.3(2) and let $\bar{u} = \{u, A\}$. If $\bar{u}, \mathcal{B} \in X \prec V_\theta$ then $u, A \in X$. And $\pi_X(\mathcal{B}) \in \{\pi_X(v) \mid v \in A \cap X\} = \pi_X(A)$. So $Cond(\mathcal{B})$ holds by 0.3. ⊣

Lemma 0.5. *The following are equivalent:*

(1) *For all* α, $Cond(V_\alpha)$.

(2) $\{\alpha \mid Cond(V_\alpha)\}$ *is unbounded in* OR.

(3) *For all* κ, *for all* $A \subseteq \kappa$, $Cond(\kappa, A)$.

Proof. Clear. ⊣

It should be noted in connection with Lemma 0.5 that, conceivably, condensation may hold for some set yet fail for every set of ordinals which code that set. We touch upon this point again briefly in section 1.

Finally,

Lemma 0.6. *Let A be a set of ordinals and $\mathcal{B} \in L[A]$. Then $Cond(A)$ implies $Cond(\mathcal{B})$.*

Proof. Choose $\theta > sup(\alpha)$ so that $\mathcal{B} \in L_\theta[A]$. If $A, \mathcal{B} \in X \prec V_\theta$ then $\pi_X(\mathcal{B}) \in L[\pi_X(A)]$. Apply lemma 0.3(3) with this θ and $u = \{F, A\}$, with F a witness to $Cond(A)$. ⊣

Section 1

§1.1 Absoluteness

Lemma 1.1. *Let M be a transitive model of ZFC, $F : \kappa^{<\omega} \longrightarrow \kappa$, $A \subseteq \kappa$ with $F, A \in M$. Suppose that there is an $X \in C_F$ such that $(X, X \cap A)$ collapses to $(\lambda, \overline{A}) \notin M$. Then there is such an X in $M[G]$ where G is M-generic for $P = Coll(\omega, \mathcal{P}(\kappa)^M)$.*

Proof. Work in a model in which $\mathcal{P}(P) \cap M$ is countable. Let G be P-generic over M. And let $(*)$ be the statement: $\exists X \,.\, X \in C_F$ and $\pi_X{}''A \notin \mathcal{P}(\kappa)^M$. In $M[G]$, let x be a real coding $(A, F, \mathcal{P}(\kappa)^M)$ in such a way that it is $\Delta^1_1(x)$ to decide whether $X \in C_F$ and $\pi_X{}''A \in \mathcal{P}(\kappa)^M$. Then, $(*)$ holds and is $\Sigma^1_1(x)$, hence holds in $M[G]$ by absoluteness of Σ^1_1 relations. Thus $(P \Vdash (*))^M$. $\dashv$

Corollary 1.2. *If $Cond(A)$ fails in M, then for any $F \in M$ there is a counterexample to $Cond(A; F)$ in $M[G]$, where G is $Coll(\omega, \mathcal{P}(\kappa)^M)$-generic over M.*

Proof. Assume $Cond(A)$, $A \subseteq \kappa$ fails in M. Let $F : \kappa^{<\omega} \longrightarrow \kappa \in M$. Pick $Q \in M$ such that $(Q \Vdash \exists X. X \in C_F \;\&\; \pi_X{}''A \notin V)^M$. Let H be Q-generic over M and apply lemma 1.1 in $M[H]$. $\dashv$

Remark: In Lemma 1.1 it is not necessary that M satisfy all of ZFC since the facts about forcing, and Σ^1_1-absoluteness used in the proof hold in a sufficiently strong finite fragment of ZFC.

It may be worth drawing the analogy with condensation in L at this point. Σ_1-substructures of limit levels of L, wherever they are found, collapse to elements of L. Similarly if $M \models Cond(A; F)$ then an F-closed substructure of A in *any* end-extension of M collapses to an element of M. $Cond(A; F)$ asserts that this is true for generic extensions. With this restriction the preceding property of L is first-order expressible. Lemma 1.1 shows that this restriction is only apparent for suitable M.

By lemma 1.1 we have that if $M \models Cond(A; F)$ then $Cond(A; F)$ holds. The converse is also true.

Lemma 1.3. *Cond*($A; F$) *iff* $L[A, F] \models$ *Cond*($A; F$).

Proof. The "if" direction follows from lemma 1.1: if there is a counterexample to *Cond*($A; F$) which is V-generic, then there is one which is $L[A, F]$-generic.

So suppose $L[A, F] \models \neg$*Cond*($A; F$). Let $M = L[A, F]$ and κ be such that $A \subseteq \kappa$, $F : \kappa^{<\omega} \longrightarrow \kappa$. Since $M \models \neg$*Cond*($A; F$), there are $P \in M$ and P-names $\sigma, \tau \in M$, such that in M, $P \Vdash (\tau \in C_F \ \& \ \sigma = \pi_\tau{}''A \ \& \ \sigma \notin V)$. If G, H are filters on P which are mutually generic over M, then $\sigma_G \neq \sigma_H$. Let Q be a forcing notion which makes both $\mathcal{P}(P)^M$ and (true) $\mathcal{P}(\kappa)$ countable. If G is Q-generic over V then in $V[G]$, there is a collection $\mathcal{C}$ of size 2^ω consisting of filters on P which are pairwise mutually generic over M. The map $H \mapsto \sigma_H$ is 1-1 on $\mathcal{C}$. Since $|\mathcal{P}(\kappa)^V| = \omega$, there is a filter $H \in \mathcal{C}$ such that $\sigma_H \notin V$. But $\tau_H \in C_F$. Thus it is not forced by Q that images of A by collapses of F-closed sets lie in V. So *Cond*($A; F$) fails. $\dashv$

The next result improves on corollary 1.2.

Lemma 1.4. *Let $A, F \in M \subseteq N$, where M and N are models of (a strong enough fragment of) ZFC. Assume that $M \models \neg$Cond($A; F$) and that there is a real in $N \setminus M$. Then in N there is a counterexample to Cond$(A; F)^M$ — i.e., an $X \in C_F$ with $\pi_X{}''A \notin M$. Furthermore, X can be found with M-countable order-type.*

Proof. Work in M. Let $\theta > \kappa$ be such that V_θ reflects "*Cond*($A; F$)" together with enough of ZFC. Let $P = Coll(\omega, \mathcal{P}(\kappa))$ and find W with $\kappa, A, F, P \in W \prec V_\theta$, $|W| = \omega$. Let $\pi = \pi_W : W \longrightarrow \overline{M}$ be the transitive collapse of W and $\pi(\kappa, A, F, P) = (\bar{\kappa}, \overline{A}, \overline{F}, \overline{P})$. So $\overline{M} \models \neg$*Cond*($\overline{A}; \overline{F}$) and by Lemma 1.1, $\overline{M} \models$ "$\overline{P} \Vdash \exists X \in C_{\overline{F}} \,.\, \pi_X{}''\overline{A} \notin V$." In $\overline{M}$ there are $\overline{P}$-names σ, τ such that $\overline{P} \Vdash (\tau \subseteq \bar{\kappa} \ \& \ \tau \in C_{\overline{F}} \ \& \ \sigma = \pi_\tau{}''\overline{A} \ \& \ \sigma \notin V)$. The names σ, τ can be used to construct a "continuous" 1-1 map $f : {}^\omega 2 \longrightarrow \{(\tau_G, \sigma_G) \mid G$ *is* $\overline{P}$-*generic over* $\overline{M}\}$ so that given $(f(u))_1 = \sigma_{G_u}$, u can be recovered. Then $(f(u))_0 = \tau_{G_u}$ lifts to an F-closed set via π, $\pi^{-1}{}''\tau_{G_u}$, which collapses to σ_{G_u}. Do this as follows.

Define a map $s \mapsto (p_s, \alpha_s)$ from $2^{<\omega}$ into $P \times OR$ so that

(1) $s \perp t \Longrightarrow p_s \perp p_t$,

(2) $s \subseteq t \Longrightarrow p_t \leq p_s$,

(3) $p_{s^\frown 0} \Vdash \alpha_s \notin \sigma$ and $p_{s^\frown 1} \Vdash \alpha_s \in \sigma$ and,

(4) for $s \in {}^\omega 2$, $\langle p_s \mid s \subseteq x \rangle$ is $\overline{M}$-generic (generates a $\overline{P}$-generic filter over $\overline{M}$).

Let $(D_n \mid n \in \omega)$ list the dense subsets of $\overline{P}$ in $\overline{M}$. Given p_s choose α_s and $p_{s^\frown i}$. $i = 0, 1$ as follows. Since $\overline{P} \Vdash \sigma \notin V$ there is an ordinal α and there are conditions $q, r \leq p_s$ such that $q \Vdash \alpha \notin \sigma$ and $r \Vdash \alpha \in \sigma$. (Since p_s does not decide "$\alpha \in \sigma$", $\alpha \neq \alpha_u$ for $u \subsetneq s$.) Now pick $\bar{q} \leq q$ and $\bar{r} \leq r$ in $D_{\ell(s)}$, where $\ell(s)$ = length of s, and let $\alpha_s = \alpha$, $p_{s^\frown 0} = \bar{q}$, and $p_{s^\frown 1} = \bar{r}$.

If $u \in {}^\omega 2$, let $G_u = \{p \in \overline{P} \mid p$ is compatible with p_s for some $s \subseteq u\}$ and let $f(u) = (\tau_{G_u}, \sigma_{G_u})$. G_u is $\overline{M}$-generic. Since $(\overline{P} \Vdash \tau \in C_{\overline{F}})^{\overline{M}}$, τ_{G_u} is $\overline{F}$-closed and $\pi^{-1}{}''\tau_{G_u}$ is F-closed with collapse σ_{G_u}. u can be recovered from σ_{G_u} recursively. Thus if $u \in {}^\omega 2 \cap N \setminus M$ then $X = \tau_{G_u}$ is an F-closed set in N with $\pi_X{}''A = \sigma_{G_u} \notin M$. And the order-type of X is countable in M. $\dashv$

Now as an improvement of corollary 1.2 we have,

Corollary 1.5. *If $Cond(A)$ fails in M, then for any $F \in M$ there is a counterexample to $Cond(A; F)$ in $M[G]$ which is of countable order-type in M where G is $(2^{<\omega}, \subseteq)$-generic over M.* $\dashv$

§1.2 Some consequences of condensation.

Lemma 1.6. *If α is countable, $A \subseteq \mathcal{P}(\alpha)$ and $Cond(A)$, then $|A| \leq \omega_1$.*

We present two arguments for this lemma in this section and a third argument in section 2. Each argument is of interest. The second introduces a fact used again in section 3.

Lemma 1.6 has the following immediate consequence.

Theorem 1.7. *Let κ be an infinite cardinal. $Cond(\mathcal{P}(\kappa))$ implies $2^\kappa = \kappa^+$. Hence condensation implies GCH.*

Proof. Let $P = Coll(\omega, \kappa)$ and G be P-generic. Apply lemma 1.6 in $V[G]$ taking $A = \mathcal{P}(\kappa)^V$. If $V \models |\mathcal{P}(\kappa)| \geq \kappa^{++}$ then $V[G] \models |A| \geq \omega_2$. By lemma 1.6, $Cond(A)$ must fail in $V[G]$, hence in V by absoluteness. $\dashv$

Corollary 1.8. *CH is equivalent to $Cond(\mathcal{P}(\omega))$.* $\dashv$

Proof of 1.6. Choose $F : A^{<\omega} \longrightarrow A$ to witness $Cond(A)$ so that if $X \in C_F$ then X is transitive. This is possible since α is countable. Suppose $A \geq \omega_2$. Let $\langle A_\eta \mid \eta < \theta\rangle$ list A without repetition, $\theta \geq \omega_2$. Let G be generic for Namba forcing ([J], p.289). So $V[G] \models |{\omega_1}^V| = \omega_1$ and $cf({\omega_2}^V) = \omega$. In $V[G]$, let $S \subseteq {\omega_2}^V$ be cofinal with order-type ω. And let $X = cl_F(\{A_\eta \mid \eta \in S\})$. Since $Cond(A; F)$ holds and X is transitive, $X \in V$. In V, $|X| \geq \omega_2$ since $\{\eta \mid A_\eta \in X\}$ is cofinal in ω_2. Thus $V[G] \models |X| \geq |{\omega_2}^V| = \omega_1$. But in $V[G]$, X is the closure of a countable set, hence countable. Contradiction. $\dashv$

The next argument depends on the following fact.

Lemma 1.9. *Let M, N be transitive models of ZFC, $M \subseteq N$, $F: A^{<\omega} \to A$, $F \in M$, $M \models |A| \geq \omega_2$. If there is a real in $N \setminus M$ then $C_F \cap N \setminus M \neq \varnothing$.*

Proof. (This is adapted from [V] where a stronger statement is proved.) Let $h : \delta \longrightarrow A$ be a bijection. Extend h to $\delta^{<\omega}$ by $h(s) = (h(s_0), \ldots h(s_n))$ and let $\widetilde{F} = h^{-1} \circ F \circ h : \delta^{<\omega} \longrightarrow \delta$. Let G be the following two person game of length ω : player I plays intervals $I_n = [\alpha_n, \beta_n] \subseteq \omega_2$ and player II plays ordinals $\gamma_n < \omega_2$ so that $\beta_n < \gamma_n < \alpha_{n+1}$. Player I moves first and wins a play iff $cl_{\widetilde{F}}(\alpha_n \mid n < \omega) \cap \omega_2 \subseteq \cup_n I_n$. This game is open for player II, hence determined, by Gale-Stewart.

Player II cannot have a winning strategy. Let σ be a strategy for player II. Let $\theta > \omega_2$ be large and choose a sequence $X_1 \prec X_2 \prec \cdots \prec V_\theta$ of elementary submodels of V_θ such that $F, h, \sigma \in X_1$, and $X_n \cap \omega_2 = \alpha_n < \omega_2$ is an ordinal with $cf(\alpha_n) = \omega_1$. Let $z = cl_{\widetilde{F}}(\alpha_n \mid n < \omega)$. Then $z \cap \alpha_n$ is bounded in α_n. Let $\beta_n = sup(z \cap \alpha_{n+1})$ and $I_n = [\alpha_n, \beta_n]$ for $n \in \omega$, taking

$\alpha_0 = 0$. And let $\gamma_n = \sigma(I_0, \dots, I_n)$. Since $I_0, \dots, I_n, \sigma \in X_{n+1} \prec V_\theta$, it follows that $\gamma_n < \alpha_{n+1}$, Thus $\langle (I_n, \gamma_n) \mid n < \omega \rangle$ is a run of G which player I wins, since $cl_{\widetilde{F}}(\alpha_n \mid n < \omega) \subseteq \cup_n I_n$. So σ is not winning for II.

So player I has a winning strategy σ . Let $\gamma_0 < \gamma_1 < \cdots < \omega_2$ be the first ω ordinals closed under each of σ,F, and h (hence under $\widetilde{F}$) in the sense that, e.g., $cl_F(\gamma_i) \cap \omega_2 \subseteq \gamma_i$. If $a \subseteq \omega$, let $\widetilde{X}_a = cl_{\widetilde{F}}(\alpha_n \mid n < \omega)$ where $[\alpha_n, \beta_n] = I_n$ is obtained by using σ against II's play $\langle \gamma_n \mid n \in a \rangle$. Let $X_a = h'' \widetilde{X}_a$. Then $n \in a$ iff $\widetilde{X}_a \cap (\gamma_n, \gamma_{n+1}) \neq \varnothing$. So a can be recovered from X_a.

Carrying out this argument in M and using a real $a \in N \setminus M$ yields a set $X_a \in C_F \cap N \setminus M$, hence lemma 1.9. $\dashv$

Proof of 1.6 (2nd). Choose F as before. The result is immediate from lemma 1.9. $\dashv$

Theorem 1.7 is not stated optimally. It is easy to see that $Cond(\mathcal{P}(\kappa))$ implies $Cond(\mathcal{P}(\lambda))$ for $\lambda \leq \kappa$: if π is a collapsing map then $\pi(\mathcal{P}(\lambda)) = \{A \cap \pi(\lambda) \mid A \in \pi(\mathcal{P}(\kappa))\}$. Thus $Cond(\mathcal{P}(\kappa)) \Longrightarrow 2^\lambda = \lambda^+$ for all $\lambda \leq \kappa$.

Definition. Let $\mathcal{A} = (A, \dots)$ with A transitive and let $\alpha \leq A \cap OR$ and $F : A^{<\omega} \longrightarrow A$.

(1) $<\alpha$-$Cond(\mathcal{A}; F)$ iff for all P, $P \Vdash \forall X. X \in C_F \;\;\&\;\; X \cap \alpha \in \alpha \longrightarrow \pi_X(\mathcal{A}) \in V$.

(2) α-$Cond(\mathcal{A}; F)$ iff for all P, $P \Vdash \forall X. X \in C_F \;\;\&\;\; \alpha \subseteq X \longrightarrow \pi_X(\mathcal{A}) \in V$.

α-$Cond(\mathcal{A})$ iff there is an F such that α-$Cond(\mathcal{A}; F)$ and $<\alpha$-$Cond(\mathcal{A})$ there is an F such that $<\alpha$-$Cond(\mathcal{A}; F)$.

$<\kappa^+$-$Cond(\mathcal{A})$ is equivalent to κ-$Cond(\mathcal{A})$. Lemmas 1.1 and 1.3 hold for these notions with no change of argument. We have the following refinement of theorem 1.7.

Theorem 1.7.1.

(1) κ-*Cond*($\mathcal{P}(\lambda)$) *implies* $2^\delta = \delta^+$ *for all* $\delta \in [\kappa, \lambda]$.

(2) κ-*Cond*($\mathcal{P}(\kappa)$) *iff* $2^\kappa = \kappa^+$.

Proof. Clear from foregoing arguments. $\dashv$

Next, a few results concerning the effect of condensation on embeddings.

Lemma 1.10. *Let M be a transitive model of ZFC, $A \in M$, $A \subseteq \kappa$. If $M \models Cond(A)$ and $j : M \longrightarrow N$ is elementary, then $A \in N$.*

Proof. Let $M \models Cond(A; F)$. Then $N \models Cond(j(A); j(F))$. It is easy to check that $j''\kappa$ is closed under $j(F)$. Letting $X = j''\kappa$, $X \cap j(A) = j''A$. So $A = \pi_X{}''j(A) \in N$. $\dashv$

The next two corollaries are immediate.

Corollary 1.11. *If condensation holds in M and $j : M \longrightarrow N$ is elementary, then $M \subseteq N$.* $\dashv$

<u>Remark</u>: This is best possible: one cannot prove in ZFC that $M = N$. For example, assume $V = L[\mu]$ with κ the unique measurable cardinal. We will see shortly that condensation holds in V_κ. Let $X \prec V_\kappa$ be Jónsson. Thus $|X| = \kappa$ and $\kappa \setminus X \neq \varnothing$. The inverse of π_X is a non-trivial elementary embedding $j : M \longrightarrow V_\kappa$ with critical point $\alpha < \kappa$. Condensation holds in M. M is cofinal in V_κ. And $M \neq V_\kappa$ since α is not measurable. Stretch this to a class embedding using the embedding induced by μ.

Corollary 1.12. *If condensation holds then there are no measurable cardinals.* $\dashv$

Corollary 1.13. *$Cond(\omega_2) + 2^{\omega_1} = \omega_2$ implies that there are no precipitious ideals on ω_1.*

Proof. Let I be a countably complete ideal on ω_1 and let $j : V \longrightarrow M = V^{\omega_1}/G$ be the generic embedding. Suppose that M is well-founded. Then ${\omega_2}^V \leq j({\omega_1}^V) < j({\omega_2}^V) < {\omega_3}^V$, since $2^{\omega_1} = \omega_2$. In V, let $A \subseteq \omega_2$ code a well-ordering of length $\alpha \in (j({\omega_2}^V), {\omega_3}^V)$. Since $j({\omega_2}^V)$ is a cardinal in M, by the preceding inequality $A \notin M$. Hence $Cond(A)$ must fail in V. $\dashv$

§1.3 Two examples

The rest of this section is given to discussion of two examples: briefly, condensation in $L[\mu]$, and in more detail, condensation in $HOD^{L(\mathbb{R})}$ under suitable hypotheses.

Example 1. *In $L[\mu]$ the following hold:*

(1) $Cond(V_{\kappa+1})$

(2) $\neg Cond(\mu)$.

$L[\mu] \models \neg Cond(\mu)$ by lemma 1.10 (here regarding $\mu = \mu \cap L[\mu] \in L[\mu]$). Digression: one may take the statement $Cond(\mu)$ to be $Cond(\mathcal{A})$ for any transitive structure $\mathcal{A}$ in $L[\mu]$ such that $L[\mu] = L[\mathcal{A}]$. To be specific, take the simplest. Let $A = \mu \cup \kappa$ and $\mathcal{A} = (A, \in)$. Then $L[\mu] \models \neg Cond(\mathcal{A})$. Since AC holds in $L[\mu]$ and $L[\mu] \models 2^\kappa = \kappa^+$, equivalently $L[\mu] \models \neg Cond(\kappa^+, D)$ for any $D \subseteq \kappa^{+L[\mu]}$ in $L[\mu]$ which codes μ. Although, in general there seems no reason to suppose that the failure of condensation for every set of ordinals coding a set S implies the failure of condensation for S (or for $TC(S)$). Thus one might ask the following:

QUESTION. *Is it consistent that $Cond(\mathcal{P}(\kappa))$ holds and $Cond(A)$ fails for every set of ordinals coding $\mathcal{P}(\kappa)$, for some κ?*

Choice would have to fail in $L(\mathcal{P}(\kappa))$ in a model giving a positive answer.

For $Cond(V_{\kappa+1})$, first argue that $Cond(V_\alpha)$ holds in $L[\mu]$ for $\alpha < \kappa$. This is an easy application of techniques of Kunen [K]. Let $\theta > \kappa$ be large, and $V_\alpha \in X \prec L_\theta[\mu]$, $\alpha < \kappa$. Let $\pi = \pi_X : X \longrightarrow L_{\bar\theta}[\bar\mu]$. $L_{\bar\theta}[\bar\mu]$ is clearly iterable. So let $\delta > \kappa^+$ be regular and iterate $L[\mu]$ and $L_{\bar\theta}[\bar\mu]$ up to $L[F_\delta]$ and $L_\gamma[F_\delta]$ respectively, where F_δ is the club filter on δ. Since $\alpha < \kappa$, $L_{\bar\theta}[\bar\mu] \models |\pi(V_\alpha)| < \pi(\kappa)$. So $\pi(V_\alpha)$ is fixed in the iteration of $L_{\bar\theta}[\bar\mu]$, hence an element of $L_\gamma[F_\delta] \subseteq L[F_\delta] \subseteq L[\mu]$. This argument is independent of setting. In particular it holds in generic extensions of $L[\mu]$. So $L[\mu] \models \forall\alpha < \kappa\, Cond(V_\alpha)$. Now let $j : L[\mu] \longrightarrow L[j(\mu)]$ be the ultrapower embedding. $L[j(\mu)] \models \forall\alpha < j(\kappa)\, Cond(V_\alpha)$. Since $\kappa + 1 < j(\kappa)$, $L[j(\mu)] \models Cond(V_{\kappa+1})$. By absoluteness

$L[\mu] \models Cond(V_{\kappa+1}^{L[j(\mu)]})$. Finally, $Cond(V_{\kappa+1}^{L[\mu]}) = Cond(V_{\kappa+1}^{L[j(\mu)]})$. So $L[\mu] \models Cond(V_{\kappa+1})$.

Applying theorem 1.7.1, the GCH holds for cardinals $\lambda \leq \kappa$ in $L[\mu]$. The usual argument that the GCH holds in $L[\mu]$ for $\lambda \geq \kappa$ establishes that for $\lambda \geq \kappa$, κ-$Cond(\mathcal{P}(\lambda))$ holds. Note the duplication at κ.

Example 2. *Let* $(\mathfrak{T})$ *be the following statement:* $V = L(\mathbb{R}) + Scale(\Sigma_1^2) +$ *There are no uncountable sequences of reals. With* $ZF + DC$ *as background theory,* $(\mathfrak{T})$ *implies that* $HOD \models Cond(HOD_\kappa)$*, where* $\kappa = \omega_1$ *and* $HOD_\kappa = HOD \cap V_\kappa$.

Thus for the rest of this discussion tend to assume $ZF + DC$ is in effect. "$Scale(\Sigma_1^2)$" is the statement that every Σ_1^2 relation on $\mathbb{R}$ has a Σ_1^2-scale. It is a direct consequence of $Scale(\Sigma_1^2)$ that every Σ_1^2 set is the projection of a tree in HOD. That is, if $A \subseteq \mathbb{R}$ is Σ_1^2, then there is a tree $T \subseteq (\omega \times \lambda)^{<\omega}$ for some λ with $T \in HOD$ and $A = p[T] = \{x \in {}^\omega\omega \mid \exists f : \omega \longrightarrow \lambda. \forall n (x \restriction n, f \restriction n) \in T\}$. (And something similar goes for Σ_1^2 relations $A \subseteq \mathbb{R}^n$. We identify $\mathbb{R}$ with ${}^\omega\omega$.) In other words every Σ_1^2 set is Suslin over HOD. (Assuming $V = L(\mathbb{R})$, $Scale(\Sigma_1^2)$ is equivalent to the statement that every Σ_1^2 set is Suslin over HOD.) For the needed background from descriptive set theory see [M], [M-S], [S].

It might be worthwhile to put things somewhat into context before proceeding with the argument. Woodin has proved that $\mathfrak{T}$ is equivalent to $AD^{L(\mathbb{R})}$. Assuming $AD^{L(\mathbb{R})}$, HOD and $L[\mu]$ are alike in that condensation holds up to the least measurable. HOD is the richer model, of course. Woodin has shown that θ is Woodin in HOD, where θ is the supremum of order-types of prewellorderings of $\mathbb{R}$. HOD_{ω_1} is an analogue of L_{ω_1} corresponding to the pointclass Σ_1^2. The fact that HOD_{ω_1} is a model of condensation supports the analogy.

Assume $\mathfrak{T}$ and let $\kappa = \omega_1$. Since there are no uncountable sequences of reals, κ is inaccessible in HOD. $HOD_\kappa \models ZFC$, hence can be identified with its sets of ordinals. Also, if $P \in HOD_\kappa$ then there are filters $G \subseteq P$ generic over HOD below any condition. It is left to the reader to verify that sets of reals claimed to be Σ_1^2 below really are. This is straightforward using reflection in the $L(\mathbb{R})$-hierarchy and section 1 of [S], in particular that $(\Sigma_1^2)^{L(\mathbb{R})} = \Sigma_1(L(\mathbb{R}), \{\mathbb{R}\})$— i.e., in $L(\mathbb{R})$, Σ_1^2 sets of reals are those which are Σ_1 definable with $\mathbb{R}$ as a parameter.

Choose a reasonable coding of countable ordinals and their subsets by reals: Π_1^1 sets W, C coding countable ordinals and their subsets respectively. If $x \in W$ let β_x be the ordinal coded by x. If $x \in C$ let U_x be the set coded by x. Let $H = \{x \in C \mid U_x \in HOD\}$. H is a Σ_1^2 set of reals which codes $HOD \cap [\kappa]^{<\kappa}$ (essentially HOD_κ). Let $<_{OD}$ be the well-ordering of ordinal definable sets given by: $U <_{OD} V$ iff for every (α, β, ϕ) with $V = D(\alpha, \beta, \phi)$ there is $(\alpha', \beta', \phi') <_{lex} (\alpha, \beta, \phi)$ with $U = D(\alpha', \beta', \phi')$ where $D(\alpha, \beta, \phi) = \{x \mid L_\alpha(\mathbb{R}) \models \phi(x, \beta)\}$. (Formulas ϕ are identified with elements of ω in a suitable way.) $<_{OD}$ induces a Σ_1^2 prewellordering on H by $x \leq y$ iff x, $y \in H$, and $U_x \leq_{OD} U_y$. Also, there is a Π_1^2 relation $\leq^*$ such that $y \in H$ implies that $\{x \mid x \leq^* y\} = \{x \mid x \leq y\} \subseteq H$. Thus $\leq$, in turn, yields a Σ_1^2-*norm* on H, $\phi : H \longrightarrow \lambda$, for some λ. (See [M].) Define $A \subseteq \mathbb{R}^2$ by $(x, z) \in A$ iff z codes an initial segment of $\mathcal{P}(\beta_x) \cap HOD$ under $\leq_{OD}$. A is Σ_1^2. To see this let $(z)_n(k) = z(\langle \cdot, \cdot \rangle)$, $\langle \cdot, \cdot \rangle$ a recursive pairing function. $(x, z) \in A$ iff $\forall n\, [U_{(z)_n} \in H \;\; \& \;\; \forall y (U_y \subseteq \beta_x \;\; \& \;\; y \leq^* (z)_n \longrightarrow \exists m U_y = U_{(z)_m})]$. By $Scale(\Sigma_1^2)$ there is a tree $T \subseteq (\omega^2 \times \lambda)^{<\omega}$ for some λ with $A = p[T]$.

Let $\delta > \kappa$ be large and let $\kappa, T \in X \prec HOD_\delta$, $\pi : X \longrightarrow M$, $\pi(\kappa, \lambda, T) = (\bar{\kappa}, \bar{\lambda}, \overline{T})$ and $\overline{M} = \pi(HOD_\kappa)$. $\overline{M}$ is a transitive model of ZFC, hence can be identified with its sets of ordinals. Under this identification:

<u>Claim</u>. $\overline{M}$ is an initial segment of $\bigcup_{\beta<\bar{\kappa}} \mathcal{P}(\beta) \cap HOD$ under $<_{OD}$.

Let $P_\beta = Coll(\omega, \beta)$ and $\Vdash_\beta$ be the associated forcing relation. We have $HOD \models \forall \beta < \kappa, U \subseteq \beta.\ \exists \gamma \Vdash_\gamma (\exists x, z, n.\ \beta = \beta_x \;\; \& \;\; (x, z) \in p[T] \;\; \& \;\; U = U_{(z)_n})$. To see this let $\gamma = (2^\beta)^{HOD}$ and G be P_γ=generic over HOD. In $HOD[G]$ pick x coding β and z coding $\mathcal{P}(\beta) \cap HOD$. Then $(x, z) \in p[T]$. Hence by absoluteness, $HOD[G] \models (x, z) \in p[T]$. ($T(x, z)$ is well-founded iff well-founded in $HOD[G]$. And $(x, z) \in p[T]$ iff $T(x, z)$ is not well-founded. $T(x, z) = \{s \in \lambda^{<\omega} \mid (x \restriction l(s), z \restriction l(s), s) \in T\}$.) Thus $M \models \forall \beta < \bar{\kappa}, U \subseteq$

β. $\exists\gamma \Vdash_\gamma (\exists x, z, n.\ \beta = \beta_x\ \ \&\ \ (x, z) \in p[\overline{T}]\ \ \&\ \ U = U_{(z)_n})$. Let $U \in \overline{M}$, $U \subseteq \beta < \bar{\kappa}$. Pick a suitable $\gamma < \bar{\kappa}$ and let G be P_γ-generic over M. In $M[G]$ pick $(x, z) \in p[\overline{T}]$ with $U = U_{(z)_n}$ for some n. Let $\bar{f} : \omega \longrightarrow \bar{\lambda}$ witness $(x, z) \in p[\overline{T}]$. Define $f : \omega \longrightarrow \lambda$ by $f(n) = \pi^{-1}(\bar{f}(n))$. Applying π^{-1} pointwise for all n, $(x \restriction n, z \restriction n, f \restriction n) \in T$. Thus $(x, z) \in p[T] = A$. So $(z)_n \in H$ and $U = U_{(z)_n} \in HOD$. At this point we have $\overline{M} \subseteq HOD$. Now let $V \subseteq \beta' < \bar{\kappa}$, $V \in HOD$, $V <_{OD} U$. Clearly one can take $\beta > \beta'$ above. Thus for some m, $U_{(z)_m} = V$. So $V \in M[G]$. But this does not depend on the particular generic G chosen. Thus $V \in \overline{M}$ and the claim is established.

It is immediate from the claim that $\overline{M} \in HOD$. That $HOD \models Cond(HOD_\kappa)$ follows using lemma 1.4, since $\mathbb{R} \setminus HOD \neq \varnothing$.

Section 2

In this section we present the main result of this thesis, theorem 2.2, that condensation implies $\diamond$. The argument supplies a new proof of lemma 1.6. Also, we discuss the relationship between condensation and $\diamond^*$, and present a gap-2 version of $\diamond^*$ which implies $Cond(\omega_2)$.

If κ is a regular cardinal and $E \subseteq \kappa$ is stationary, recall that $\diamond_\kappa(E)$ asserts that there is a sequence $\langle S_\alpha \mid \alpha < \kappa \rangle$ such that (1) for all α, $S_\alpha \subseteq \alpha$ and (2) for every $A \subseteq \kappa$, $\{\alpha \in E \mid A \cap \alpha = S_\alpha\}$ is stationary in κ. $\diamond_\kappa = \diamond_\kappa(\kappa)$. If $\kappa = \omega_1$ the subscript is supressed. $\diamond_\kappa(E)$ is equivalent to the assertion that there is a sequence $\langle S_\alpha \mid \alpha < \kappa \rangle$ such that (1) for all α, $S_\alpha \subseteq \mathcal{P}(\alpha)$ and $|S_\alpha| \leq \alpha$ and (2) for every $A \subseteq \kappa$, $\{\alpha \in E \mid A \cap \alpha \in S_\alpha\}$ is stationary in κ [D].

Let $\mathcal{A} = (A, \in, \ldots)$ be a structure with A transitive. For $\mathcal{A}, F \in X$ let $\pi_X : X \longrightarrow M_X$ be the transitive collapse of X, $\mathcal{A}_X = \pi_X(\mathcal{A})$, $F_X = \pi_X(F)$. The next lemma is just a special case of lemma 1.1.

Lemma 2.1. *Assume $Cond(\mathcal{A}; F)$. Let θ be such that $V_\theta \models Cond(\mathcal{A}; F) + T$ with T a sufficiently strong fragment of ZFC. If $\mathcal{A}, F \in X \subseteq Y$ and $X, Y \prec V_\theta$ then $\mathcal{A}_X \in M_Y$.*

Proof. Let $\overline{X} = \pi_Y{}''X$. $M_Y \models Cond(\mathcal{A}_Y; F_Y) + T$ and $\overline{X} \cap A_Y$ is closed under F_Y. Applying lemma 1.1 (and the remark following its proof) with $M = M_Y$ we must have $\mathcal{A}_X = \pi_{\overline{X}}(\mathcal{A}_Y) \in M_Y$. $\dashv$

Theorem 2.2. *Let $\lhd$ be a well-ordering of H_{ω_2} of minimal order-type and $\mathcal{A} = (H_{\omega_2}, \lhd)$. Assume $Cond(\mathcal{A})$. Then for every stationary $E \subseteq \omega_1$, $\diamond(E)$ holds.*

Proof. By theorem 1.7, $2^{\omega_1} = \omega_2$. So $|H_{\omega_2}| = \omega_2$ and $\lhd$ has order-type ω_2. Every initial segment of $\lhd$ is an element of H_{ω_2}.

Let $F : H_{\omega_2}^{<\omega} \longrightarrow H_{\omega_2}$ witness $Cond(\mathcal{A})$ and take V_θ as in lemma 2.1, with $cf(\theta) > \omega_1$. Let $H : V_\theta^{<\omega} \longrightarrow V_\theta$ be a skolem function for $(V_\theta, \{\mathcal{A}, F\})$. So if $X \in C_H$ then $\mathcal{A}, F \in X \prec V_\theta$. H can be chosen so that for any $X \subseteq V_\theta$,

$cl_H(X) = H''X^{<\omega}$. Let $C = \{X \in [V_\theta]^\omega \mid X \prec (V_\theta, H)\}$. C is club in $[V_\theta]^\omega$. If $X \in C$ then $X \in C_H$, $X \cap H_{\omega_2} \in C_F \cap [H_{\omega_2}]^\omega$ and $X \cap \omega_1$ is an ordinal $\alpha_X < \omega_1$.

For $X \prec V_\theta$ let $X^+ = \{f(\alpha_X) \mid f \in {}^{\omega_1}V \cap X\}$. If $X \in C$ then $X^+ = cl_H(X \cup \{\alpha_X\})$. $X^+ \subseteq cl_H(X \cup \{\alpha_X\})$ is clear. For the reverse inclusion let $p \in [V_\theta]^{<\omega}$ and define $f_p(\alpha) = cl_H(p \cup \{\alpha\})$ for $\alpha < \omega_1$. $f_p \in V_\theta$ since $cf(\theta) > \omega_1$. If $p \in X \in C$ then $f_p \in X$. Now let $u \in cl_H(X \cup \{\alpha_X\}) = H''(X \cup \{\alpha_X\})^{<\omega}$. Then $u = H(s)$ for some $s \in (p \cup \{\alpha_X\})^{<\omega}$ with $p \in X$. Thus $u \in f_p(\alpha_X) \in X^+$. Furthermore, each $f_p(\alpha)$ is countable. So there is a $g \in X$ such that for all $\alpha < \omega_1$, $g(\alpha) : \omega \xrightarrow{onto} f_p(\alpha)$. Then $g(\alpha_X) \in X^+$. Also $\omega \subseteq X \subseteq X^+$. Thus $f_p(\alpha_X) \subseteq X^+$ and $u \in X^+$.

It follows that if $X \in C$ then $\mathcal{A}, F \in X^+ \prec V_\theta$. Hence by lemma 2.1, for $X \in C$, $\mathcal{A}_X \in M_{X^+}$. Thus $\mathcal{A}_X = \pi_{X^+}(f(\alpha_X))$ for some $f \in X$. Let $\alpha = \alpha_X$, $\mathcal{A}_X = (A_X, \lhd_X)$. By elementarity $f(\alpha) = (A_\alpha, \lhd_\alpha)$ where $A_\alpha \models \alpha = \omega_1 +$ "*I am* H_{ω_2}". $\lhd_\alpha$ well-orders A_α and every initial segment of $\lhd_\alpha$ is an element of A_α. Also α is countable in X^+. So π_{X^+} fixes every element of $A_\alpha \cap X^+$. So $A_\alpha \cap X^+$ is a $\lhd_\alpha$-initial segment of A_α and $\mathcal{A}_X = \pi_{X^+}(f(\alpha)) = f(\alpha) | X^+ = (A_\alpha \cap X^+, \lhd_\alpha \mid \alpha_{X^+})$ where $\lhd_\alpha \mid \beta$ is the length β initial segment of $\lhd_\alpha$. (Note: possibly A_α is countable, in which case $\pi_{X^+}(f(\alpha)) = f(\alpha) = \mathcal{A}_X$. We cannot, at this point, prove that A_α is countable, however. It is clear that $\ell(\lhd_\alpha) \leq \omega_1$. Whether $\ell(\lhd_\alpha) < \omega_1$ relates to the question of whether condensation implies $\diamondsuit^*$. See theorem 2.4).

Let $E \subseteq \omega_1$ be stationary. The set $\mathcal{S}_o = \{X \in C \mid \alpha_X \in E\}$ is stationary in $[V_\theta]^\omega$. By Fodor's theorem there is a single f and a stationary $\mathcal{S} \subseteq \mathcal{S}_o$ such that for $X \in \mathcal{S}$, $\mathcal{A}_X = \pi_{X^+}(f(\alpha_X))$. Define a $\diamondsuit(E)$ sequence using this f in the usual way: given $S \restriction \alpha$ let (S_α, C_α) be the $\lhd_\alpha$-least pair of subsets of α such that C_α is club in α and $\beta \in C_\alpha \cap E$ implies that $S_\alpha \cap \beta \neq S_\beta$, if such exists, and $\varnothing$ otherwise. Let $S = \langle S_\alpha \mid \alpha < \omega_1 \rangle$. Suppose that S fails to be a $\diamondsuit(E)$ sequence. Let (U, D) be the $\lhd$-least counterexample. Take $X \in \mathcal{S}$ with $S, U, D, E \in X$. Then $X \models$ "(U, D) is the $\lhd$-least pair showing that S is not $\diamondsuit(E)$." So $M_X \models$ "$(U \cap \alpha, D \cap \alpha)$ is the $\lhd_X$-least pair showing that $S \restriction \alpha$ is not $\diamond(E \cap \alpha)$," where $\alpha = \alpha_X$. Since $X \in \mathcal{S}$, $\lhd_X$ is an initial segment of $\lhd_\alpha$. Thus $(U \cap \alpha, D \cap \alpha)$ satisfies the definition of (S_α, C_α). So $U \cap \alpha = S_\alpha$. On the other hand, since $D \in X \in \mathcal{S}$, $\alpha \in D \cap E$, contradicting the choice of U and D. So S is a $\diamond(E)$ sequence. $\dashv$

Remark: One can avoid the use of X^+ above (and hence the need to assume that $cf(\theta) > \omega_1$) as follows. Let $X^* = cl_H(X \cup \{\alpha_X\})$. Then $X \subseteq X^*$ and $\mathcal{A}, F \in X^* \prec V_\theta$. By lemma 2.1, $\mathcal{A}_X = \pi_{X^*}(H(s))$ for some $s \in (X \cup \{\alpha_X\})^{<\omega}$. For some $p \in X$, $s \in (p \cup \{\alpha_X\})^{<\omega}$. Thus, although $s \notin X$, there is a parameter matix $m \in X$ such that $s = m(\alpha_X)$. Fix this m on a stationary subset of $\mathcal{S}_0$, and define f on ω_1 by $f(\alpha) = H(m(\alpha))$. Then f can be used as above to define a $\diamond(E)$ sequence.

Theorem 2.2(b). *Let κ be a regular cardinal and $\lhd$ be a well-ordering of H_{κ^+} of minimal order-type. Let $\mathcal{A} = (H_{\kappa^+}, \lhd)$ and assume $< \kappa$-$Cond(\mathcal{A})$ holds. Then for every stationary $E \subseteq \kappa$, $\diamond_\kappa(E)$ holds.*

Proof. The proof, with the obvious changes, is identical. In choosing C one must stipulate that $X \cap \kappa$ is an ordinal. $\dashv$

Corollary 2.3. *Assume $Cond(\mathcal{P}(\lambda))$. Then for every regular $\kappa < \lambda$ and stationary $E \subseteq \kappa$, $\diamond_\kappa(E)$ holds.* $\dashv$

The proof of 2.2 contains a proof that $Cond(H_{\omega_2}, \lhd) + 2^{\omega_1} = \omega_2$ implies CH. This by itself is no improvement on theorem 1.7 which assumes only $Cond(\mathcal{P}(\omega))$. But the argument is of a different type. In fact the method of argument supplies another proof of lemma 1.6 (hence, that $Cond(\mathcal{P}(\omega))$ is sufficient for CH).

Proof of lemma 1.6 (3rd). Let $\bar{\alpha} < \omega_1$, $A \subseteq \mathcal{P}(\bar{\alpha})$ and assume $Cond(A)$. The only new observation needed is that if R is a relation on A, then also $Cond(R)$ holds. (This is not necessarily true if $\bar{\alpha} \geq \omega_1$. E.g., if there is an inner model with a measurable cardinal then the measure in $L[\mu]$ gives a counterexample. The large cardinal assumption is probably irrelevant.) To see this let $R \subseteq A^n$ and choose F witnessing $Cond(A)$ so that if $X \in C_F$ then X is transitive. Let $R, A, F \in X \prec V_\theta$. Since π_X is the identity on

$\mathcal{P}(\bar{\alpha})$ we have $\pi_X(R) = \{\boldsymbol{a} \mid \boldsymbol{a} \in R \cap X\} = \{\boldsymbol{a} \mid \boldsymbol{a} \in R \cap (A^n \cap X)\} = \{\boldsymbol{a} \mid \boldsymbol{a} \in R \cap (A \cap X)^n\}$. Thus $\pi_X(R)$ is definable from R and $A \cap X = \pi_X(A)$. So $Cond(R)$ holds.

Now assume $|A| \geq \omega_2$. Let $\lhd$ be a well-ordering of a subset of A of length ω_2. By the preceding $Cond(\lhd)$ holds. As in the proof of 2.2 find $C \subseteq [V_\theta]^\omega$ such that $X \in C$ implies that $\lhd, F \in X^+ \prec V_\theta$ so that $\lhd_X = \pi_X(\lhd) \in M_{X^+}$. So there is an $f \in X$ such that $\lhd_X = \pi_{X^+}(f(\alpha_X)) = f(\alpha_X) \cap X^+$. Since $M_X \models ot(\lhd_X) = \omega_2$ and α_X is countable in M_{X^+}, $M_{X^+} \models ot(\lhd_X) \leq \omega_1$. Thus by elementarity $f(\alpha_X) = \lhd_{\alpha_X}$ orders a subset of $\mathcal{P}(\bar{\alpha})$ and $ot(f(\alpha_X)) \leq \omega_1$. Now fix f on a stationary set $\mathcal{S}$. W.l.o.g., for all $\alpha \leq \omega_1$, $|f(\alpha)| \leq \omega_1$. Let $a \in fld(\lhd)$. Take $X \in \mathcal{S}$ with $a \in X$. Then $a \in fld(\lhd_{\alpha_X})$. So $fld(\lhd) \subseteq \bigcup\{fld(\lhd_\alpha) \mid \alpha < \omega_1\}$. This implies that $|\lhd| = \omega_1$ contradicting the choice of $\lhd$. Thus $|A| \leq \omega_1$. $\dashv$

Let κ be a regular cardinal, $\mathcal{F}$ a normal filter on κ. $E \subseteq \kappa$ is *$\mathcal{F}$-positive* if $\kappa \setminus E \notin \mathcal{F}$. $\diamond^*(\mathcal{F})$ is the statement: there is a sequence $\langle S_\alpha \mid \alpha < \kappa\rangle$ such that (1) for all $\alpha < \kappa, S_\alpha \subseteq \mathcal{P}(\alpha)$ and $|S_\alpha| \leq \alpha$, and (2) for all $A \subseteq \kappa$, $\{\alpha \mid A \cap \alpha \in S_\alpha\} \in \mathcal{F}$. $\diamond^*_\kappa = \diamond^*(Cub_\kappa)$ where Cub_κ is the club filter on κ. $wCC(\kappa)$ is the statement that for every $\mathcal{A} = (A, \dots)$ with $\kappa^+ \subseteq A$ there is an α such that $sup\{ot(X \cap \kappa^+) \mid X \prec \mathcal{A}\ \&\ X \cap \kappa = \alpha\} \geq \alpha^+$. $wCC(\kappa)$ is the weak Chang conjecture for κ. wCC is $wCC(\omega_1)$.

It is easy to see that for every normal filter $\mathcal{F}$ on κ and $\mathcal{F}$-positive E, $\diamond^*_\kappa \implies \diamond^*(\mathcal{F}) \implies \diamond_\kappa(E) \implies \diamond_\kappa$. Here are some facts relating $\diamond^*_\kappa$ and $wCC(\kappa)$. (See [B], [D-K], [D-L].)

(1) $wCC(\kappa)$ implies $\neg\diamond^*_\kappa$.

(2) Assuming $V = L$, $wCC(\kappa)$ iff $\neg\diamond^*_\kappa$ iff κ is ineffable.

(3) If $\kappa \geq \omega_2$ is a successor cardinal and $wCC(\kappa)$ holds, then $0^\dagger$ exists.

As a corollary to the proof of 2.2 we have:

Theorem 2.4. *Let $\mathcal{A}$ be as in 2.2. Assume $Cond(\mathcal{A})$ and $\neg wCC$. Then for every stationary $E \subseteq \omega_1$ there is a normal filter $\mathcal{F}$ such that $E \in \mathcal{F}$ and $\diamond^*(\mathcal{F})$ holds.*

Proof. Take f as in the proof of 2.2. So on a stationary set $\mathcal{S} \subseteq \mathcal{S}_o = \{X \mid \alpha_X \in E\}$, $\mathcal{A}_X = \pi_{X^+}(f(\alpha_X))$. Let $\mathcal{E} = \{p(C_A \cap \mathcal{S}) \mid A \subseteq \omega_1\}$ where $C_A = \{X \mid A \in X\}$ and $p(U) = \{\alpha_X \mid X \in U\}$. Then $\mathcal{E}$ generates a non-trivial normal filter $\mathcal{F}$ on ω_1 with $E \in \mathcal{F}$. $p(C_A \cap \mathcal{S}) \subseteq p(\mathcal{S}) \subseteq p(\mathcal{S}_o) = E$. So $E \in \mathcal{F}$. To check that $\mathcal{F}$ is normal let $B_\alpha \in \mathcal{F}$ for $\alpha < \omega_1$ and $B = \triangle_\alpha B_\alpha = \{\beta \mid \beta \in \bigcap_{\alpha<\beta} B_\alpha\}$. Let $p(C_{A_\alpha} \cap \mathcal{S}) \subseteq B_\alpha$. And let A code $\langle A_\alpha \mid \alpha < \omega_1\rangle$ so that on a club $C \subseteq [V_\theta]^\omega$, $A \in X \in C$ implies that $\langle A_\alpha \mid \alpha < \omega_1\rangle \in X$. If $X \in C_A \cap \mathcal{S}$ then for $\alpha < \alpha_X$, $A_\alpha \in X$ so that $\alpha_X \in p(C_{A_\alpha} \cap \mathcal{S}) \in B_\alpha$. Thus $\alpha_X \in B$ and $p(C_A \cap \mathcal{S}) \subseteq B$. This gives $B \in \mathcal{F}$. $\mathcal{F}$ is clearly nontrivial.

Since wCC fails, by shrinking $\mathcal{S}$ if necessary one can, for each $\alpha < \omega_1$, bound the order-type of $X \cap \omega_2$ for $X \in \mathcal{S}$ with $\alpha_X = \alpha$ independently of X. Let $b(\alpha)$ give this bound. One can assume also that $b \in X$ so that $b(\alpha_X) < \alpha_{X^+}$. So for $X \in \mathcal{S}$, $\mathcal{A}_X = \pi_{X^+}(f(\alpha_X)) = f(\alpha_X)$. f can be chosen so that $f(\alpha)$ is countable for all α. It is now easy to check that $S_\alpha = \mathcal{P}(\alpha) \cap f(\alpha)$ defines a $\diamondsuit^*(\mathcal{F})$ sequence. $\dashv$

Again nothing in this argument depends on the fact that ω_1 is the least uncountable cardinal. Thus more generally:

Theorem 2.4(b). *Let $\mathcal{A} = (H_\kappa, \lhd)$ where κ is regular and $\lhd$ is a well-ordering of H_κ of minimal type. Assume $<\kappa$-$Cond(\mathcal{A})$ and $\neg wCC(\kappa)$. Then for every stationary $E \subseteq \kappa$ there is a normal filter $\mathcal{F}$ such that $E \in \mathcal{F}$ and $\diamondsuit^*(\mathcal{F})$ holds.*

This is a partial converse of (1). By (2) the full converse holds in L. Theorem 2.4 itself is a little odd. $wCC(\kappa)$ is a large cardinal property in L. By (3), in K if $wCC(\kappa)$ holds then κ is at least inaccessible. And $\diamondsuit^*$ holds (hence wCC fails) in the model of example 2 in section 1. So it seems likely that condensation simply refutes wCC and $wCC(\kappa^+)$ more generally.

$\diamondsuit^*$ has the following equivalent formulation: there is a sequence $S = \langle S_\alpha \mid \alpha < \omega_1\rangle$ such that (1) $|S_\alpha| = \omega$ and (2) for every $A \subseteq \omega_1$, $\{X \in [H_{\omega_2}]^\omega \mid \pi_X(A) \in S_{\alpha_X}\}$ is club. Boost this a bit to get a "gap-2" strong diamond principle $\diamondsuit^2$.

Definition. $\diamondsuit^2$ is the statement: there is a sequence $S = \langle S_\alpha \mid \alpha < \omega_1 \rangle$ such that (1) $|S_\alpha| = \omega$ and (2) for every $A \subseteq \omega_2$, $\{X \in [H_{\omega_3}]^\omega \mid \pi_X(A) \in S_{\alpha_X}\}$ is club.

Theorem 2.5.

(a) L, $L[\mu] \models \diamondsuit^2$.

(b) $\diamondsuit^2$ *implies* $Cond(\omega_2)$.

Proof. (a) In L, let $S_\alpha = L_\beta$ where β is least such that $L_\beta \models \alpha$ is countable. It is easy to verify that this defines a $\diamondsuit^2$ sequence. So turn to $L[\mu]$.

Working in $L[\mu]$, let κ be the measurable. Let $F = Cub_{\omega_1}$. Note that $\mathbb{R} \subseteq L[F]$. So $\omega_1{}^{L[F]} = \omega_1$. For $\alpha < \omega_1$ let η_α be the least η such that $L_\eta[F] \models |\alpha| = \omega$, $\gamma_\alpha = \omega_1{}^{L_{\eta_\alpha}[F]}$ and let β_α be the least β such that $L_\beta[F] \models |\gamma_\alpha| = \omega$. Take $S_\alpha = (HC)^{L_{\beta_\alpha}[F]}$. This will do.

We show that S_α is countable. Let $a \subseteq \omega$ code $\tau \geq \alpha$. Let $\theta > \kappa$ be large and $a \in X \prec L_\theta[\mu]$, with $|X| = \omega$. $\pi_X : X \longrightarrow L_{\bar\theta}[\bar\mu]$ with $\bar\kappa = \pi_X(\kappa)$. Thus $\alpha < \omega_1{}^{L_{\bar\theta}[\bar\mu]} < \bar\kappa < \omega_1$ and $L_{\bar\theta}[\bar\mu]$ is iterable. Iterate $L_{\bar\theta}[\bar\mu]$ up to $L_\delta[F]$. Then $\delta \geq \eta_\alpha$ and $\gamma_\alpha \leq \omega_1{}^{L_\delta[F]} = \omega_1{}^{L_{\bar\theta}[\bar\mu]}$. So γ_α is countable and we may assume that $\tau \geq \gamma_\alpha$. So $\delta \geq \beta_\alpha$. Now $S_\alpha = (HC)^{L_{\beta_\alpha}[F]} \subseteq (HC)^{L_\delta[F]} = (HC)^{L_{\bar\theta}[\bar\mu]}$ is countable.

Let $A \subseteq \omega_2$ and $C = \{Y \cap H_{\omega_3} \mid A \in Y \prec L_\theta[\mu]\}$. C is club in $[H_{\omega_3}]^\omega$. Let $X \in C$ and $X = Y \cap H_{\omega_3}$. $A_X = \pi_X(A) = \pi_Y(A)$. Let $\alpha = \alpha_X = \alpha_Y$. $\pi_Y : Y \longrightarrow L_{\bar\theta}[\bar\mu] \models \alpha = \omega_1$. Iterate this model to $L_\delta[F]$. Also $L_\delta[F] \models \alpha = \omega_1$. Let $\nu = \omega_2{}^{L_\delta[F]} = \omega_2{}^{L_{\bar\theta}[\bar\mu]}$. Then $A_X \subseteq \nu \leq \omega_1{}^{L_{\eta_\alpha}[F]} = \gamma_\alpha$. Since $\nu < \bar\kappa$, ν and A_X are fixed in the iteration of $L_{\bar\theta}[\bar\mu]$. So $A_X \in L_\delta[F] \subseteq L_{\beta_\alpha}[F] \models |\nu| = \omega$. It follows that $A_X \in S_\alpha$. So $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ is $\diamondsuit^2$.

(b) Let $A \subseteq \omega_2$ and assume $Cond(A)$ fails. A review of the proof of lemma 1.4 shows that for any $F : \omega_2{}^{<\omega} \longrightarrow \omega_2$ there is an α (hence a stationary set of α) such that there is a "perfect set" of distinct images of A by $X \in C_F$ with $\alpha_X = \alpha$. So there can be no $\diamondsuit^2$ sequence. $\dashv$

Section 3

This section addresses the problem of obtaining $Cond(A)$ by forcing. One would like, assuming GCH, to add $Cond(A)$ with set forcing, without collapsing cardinals. The immediate motivation for this is corollary 1.13. A small forcing notion which adds $Cond(\omega_2) + 2^{\omega_1} = \omega_2$ shows that the existence of a precipitous ideal is not a consequence of large cardinal axioms. This problem has been attacked from more than one direction now. And to our knowledge is still open. Thus we speculate that the difficulty of this problem is related to the difficulty of forcing $Cond(A)$. But the matter of forcing $Cond(A)$ retains its interest in spite of this question about precipitous ideals. It was raised in a slightly different guise in Lee Stanley's thesis [Stn]. Stanley shows how to force the existence of higher-gap morasses, noting that these forced morasses lack some of the properties of the "natural" morasses which can be constructed in L, K and higher core models. (See [W].) These properties are precisely the ones needed to verify that the forced morasses have condensation. This is notable since a gap-2 morass at ω_1 which has condensation kills all precipitous ideals on ω_1.

Jensen's coding theorem shows that condensation can be added globally without collapsing cardinals, assuming GCH. The argument is essentially top down or "Easton style" using a class partial order to produce a model of $V = L[a]$, $a \subseteq \omega$. The basic strategy is to code $A \subseteq \kappa^+$ by $B \subseteq \kappa$, in the sense that $A \in L[B]$. The distributivity arguments involved in adding this B require the assumption $V = L[A]$, hence the success of coding from ∞ down to κ^+. Some such assumption is evidently necessary. For example, if λ is measurable, $\kappa < \lambda$ and $A \subseteq \kappa^+$ is such that $(\kappa^+)^{L[A]} = \kappa^+$, then there is no $P \in V_\lambda$ which adds $B \subseteq \kappa$ with $A \in L[B]$ without collapsing κ^+. The reason is that $B^\sharp$ exists after forcing with P. So $(\kappa^+)^V = (\kappa^+)^{L[A]} \leq (\kappa^+)^{L[B]} < (\kappa^+)^{V[G]}$. Thus one cannot in general apply the preceding strategy locally to add $Cond(A)$. On the other hand if there is an object C with condensation which computes that κ^+ is "accessible", one might try to add $Cond(A)$ for $A \subseteq \kappa^+$ by reworking the coding relative to C.

§3.1 Coding to add *Cond*(*A*).

Recall almost disjoint (adj) coding. Let $\boldsymbol{b} = \langle b_\nu \mid \nu < \kappa^+ \rangle$ be an almost disjoint sequence of subsets of κ: if $\nu \neq \mu$ then $|b_\nu \cap b_\mu| < \kappa$. Let P consist of pairs (b, u) with $b \subseteq \kappa$, $u \subseteq \kappa^+$ and $|b|, |u| < \kappa$. And let $(b, u) \leq_A (\bar{b}, \bar{u})$ if $\bar{b} \subseteq_e b, \bar{u} \subseteq u$ and for $\nu \in \bar{u} \cap A$, $(b \backslash \bar{b}) \cap b_\nu = \varnothing$. Let $P_A = P(\boldsymbol{b}, A) = (P, \leq_A)$. P_A is κ-closed and, if $\kappa^{<\kappa} = \kappa$, satisfies the κ^+-c.c. P_A adds a set $B \subseteq \kappa$ such that $\nu \in A$ iff $|B \cap b_\nu| < \kappa$. If this last condition is fulfilled, say that the pair $(\boldsymbol{b}, B)$ *codes* A.

Lemma 3.1. *If $(\boldsymbol{b}, B)$ has condensation and codes A, then A has condensation.*

Proof. $A \in L[\boldsymbol{b}, B]$. ⊣

Lemma 3.2. *(GCH) Let $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n$ be adj sequences with condensation, $\boldsymbol{b}_k = \langle b^k_\nu \mid \nu < \omega_{k+1} \rangle$, $b^k_\nu \subseteq \omega_k$. Let $A \subseteq \omega_{n+1}$. Taking $B_{n+1} = A$, suppose that B_k is $P(\boldsymbol{b}_k, B_{k+1})$-generic over $V[B_n, \ldots, B_{k+1}]$, $1 \leq \kappa \leq n$. then A has condensation in $V[B_n, \ldots, B_1]$.*

Proof. By induction working in $V[B_n, \ldots, B_1]$. $B_1 \subseteq \omega_1$ has condensation. Assume B_k has condensation. $(\boldsymbol{b}_k, B_k)$ codes B_{k+1}. So B_{k+1} has condensation by 3.1. ⊣

Lemma 3.3. *(GCH) Assume that $\omega_2, \ldots, \omega_{n+1}$ are accessible in $L[C]$ for some C such that Cond(C) holds. Then for $A \subseteq \omega_{n+1}$ there is a p.o. P such that if G is P-generic over V, then $V[G] \models$ Cond(A) and P preserves cardinals.*

Proof. Let $\omega_{i+1} = (\delta_i^+)^{L[C]}$, $1 \leq i \leq n$. Choose $a \subseteq \omega_n$ such that for $1 \leq i \leq n$, $|\delta_i|^{L[a \cap \omega_i]} = \omega_i$ and let $a_i = a \cap \omega_i$. $L[C, a_i]$ correctly computes ω_{i+1}. So there is an adj sequence $\boldsymbol{b}_i \in L[C, a_i]$ for coding subsets of ω_{i+1} by subsets of ω_i. Note *Cond*($\boldsymbol{b}_1$) holds. At this point one could iterate lemma 3.2 getting *Cond*(a_i) hence *Cond*($\boldsymbol{b}_i$) successively. But this is inefficient. Instead just code from ω_{n+1} to ω_1 using the $\boldsymbol{b}_i$'s. Code A by $B_n \subseteq \omega_n$ using $\boldsymbol{b}_n \in L[C, a_n]$. Then code the pair (a_{k+1}, B_{k+1}) by B_k using $\boldsymbol{b}_k \in L[C, a_k]$ for $1 \leq k < n$. After n steps one has $B_1 \subseteq \omega_1$ such that $A \in L[C, a_1, B_1]$. Since

C has condensation and $a_1, B_1 \subseteq \omega_1$, $\langle C, a_1, B_1\rangle$ has condensation. Thus $Cond(A)$ holds. The required P is the finite iteration which accomplishes this. $\dashv$

Lemma 3.3 can be improved upon considerably in two ways. The coding can be pushed past ω_ω. And the condition "κ^+ is accessible in $L[C]$" can be replaced by the condition: for some B,

$$(*)^{C,B}_\kappa: \quad \forall\delta \in [\kappa, \kappa^+)\ L[C, B \cap \delta] \models |\delta| = \kappa.$$

The next two lemmas give examples.

Lemma 3.4. *(GCH) Assume $Cond(C)$ and that for $n < \omega$, ω_n is accessible in $L[C]$. Then for any $A \subseteq \omega_\omega$, there is a partial order P which adds $Cond(A)$ and preserves cardinals.*

Proof. One can assume that $L[A \cap \omega_n]$ correctly computes ω_n and that $L[A \cap \omega_n] \models |\delta_n| = \omega_n$, where $\omega_{n+1} = (\delta_n^+)^{L[C]}$. Let $A_n = A \cap \omega_n$. So $L[C, A_n]$ correctly computes ω_{n+1}. And there is an adj sequence $\boldsymbol{b}_n = \langle b^n_\nu \mid \nu < \omega_{n+1}\rangle \in L[C, A_n]$ for coding subsets of ω_{n+1} by subsets of ω_n. Let $\boldsymbol{b}_n$ be the $<_{L[C,A_n]}$-least such sequence.

Define P as follows. A condition is $p = (p_n \mid 1 \leq n < \omega)$, $p_n = (b_n, u_n)$ with $b_n \subseteq \omega_n$, $u_n \subseteq \omega_{n+1}$ and $|b_n|, |u_n| < \omega_n$. $p \leq \bar{p}$ iff for all n, $p_n \leq_{\langle \bar{b}_{n+1}, A_{n+1}\rangle} \bar{p}_n$—i.e., for all n,

(1) $\bar{b}_n \subseteq_e b_n, \bar{u}_n \subseteq u_n$

(2) If $\nu \in \bar{u}_n \cap \langle \bar{b}_{n+1}, A_{n+1}\rangle$ then $b_n \setminus \bar{b}_n \cap b^n_\nu = \varnothing$

where $\langle S, T\rangle = \{2\eta \mid \eta \in S\} \cup \{2\eta + 1 \mid \eta \in T\}$. P is as required.

The following sets are dense. For $\nu < \omega_{n+1}, E^\nu_n = \{p \mid \nu \in u_n\}$. For $\eta < \omega_n, \nu < \omega_{n+1}, D^{\eta,\nu}_n = \{p \mid sup\, b_n > \eta \text{ and if } \nu \notin \langle b_{n+1}, A_{n+1}\rangle \text{ then } b_n \cap b^n_\nu \setminus \eta \neq \varnothing\}$.

If $p, \bar{p}$ agree in their first components they are compatible. Thus E^ν_n is dense.

Let $\bar{p} \in P, \eta < \omega_n, \nu < \omega_{n+1}$. Let $p_k = \bar{p}_k$ if $k \neq n$ and let $u_n = \bar{u}_n$. If $\nu \notin \langle \bar{b}_{n+1}, A_{n+1}\rangle$, let $\tau = \nu$. Otherwise let $\tau \in \omega_{n+1} \setminus \langle \bar{b}_{n+1}, A_{n+1}\rangle$ be any ordinal. b^n_τ is adj from each $b^n_\nu, \nu \neq \tau$. Let $\eta_\nu = sup(b^n_\tau \cap b^n_\nu)$ and

$\bar{\eta} = sup\{\eta_\nu \mid \nu \in \bar{u}_n \cap \langle \bar{b}_{n+1}, A_{n+1}\rangle\}$. Since $|u_n| < \omega_n, \bar{\eta} < \omega_n$. Now take $\xi \in b^n_\tau, \xi > \eta, \bar{\eta}$ and let $b_n = \bar{b}_n \cup \{\xi\}$. Then $p \leq \bar{p}$ and $p \in D^{\eta,\nu}_n$.

Let G be P-generic, $B_n = \bigcup\{(p_n)_0 \mid p \in G\}$. Using the density of E^ν_n and $D^{\eta,\nu}_n$ it is easy to check that for all n, $(\boldsymbol{b}_n, B_n)$ codes $\langle B_{n+1}, A_{n+1}\rangle$. If $\nu \in \langle B_{n+1}, A_{n+1}\rangle$ pick $\bar{p} \in G$ with $\nu \in \langle \bar{b}_{n+1}, A_{n+1}\rangle$ and $p' \leq p, p' \in G \cap E^\nu_n$. If $p \leq p'$ then $(b_n, u_n) \leq_{\langle b'_{n+1}, A_{n+1}\rangle} (b'_n, u'_n)$. So since $\nu \in u_n \cap \langle b'_{n+1}, A_{n+1}\rangle$, $b_n \cap b^n_\nu \subseteq sup\, b'_n$. Thus $B_n \cap b^n_\nu \subseteq b'_n$ and $|B_n \cap b^n_\nu| < \omega_n$. If $\nu \notin \langle B_{n+1}, A_{n+1}\rangle$, let $\eta < \omega_n$ and take $p \in G \cap D^{\eta,\nu}_n$. Then $b_n \cap b^n_\nu \setminus \eta \neq \varnothing$. Thus $B_n \cap b^n_\nu$ is unbounded in ω_n.

$A \in L[C, A_1, B_1]$. To see this, working in $L[C, A_1, B_1]$ recursively decode the sequence $(\langle B_n, A_n\rangle \mid 1 \leq n < \omega)$ using the fact that $\boldsymbol{b}_n$ is $<_{L[C,A_n]}$-least, and that $(\boldsymbol{b}_n, B_n)$ codes $\langle B_{n+1}, A_{n+1}\rangle$. So the sequence is in $L[C, A_1, B_1]$. Then obviously $\eta \in A$ iff $2\eta + 1 \in \langle B_n, A_n\rangle$ whenever $\eta < \omega_n$. As before, since C has condensation and $A_1, B_1 \subseteq \omega_1$, $\langle C, A_1, B_1\rangle$ has condensation. Since $A \in L[C, A_1, B_1]$, $Cond(A)$ holds.

It remains to show that P preserves cardinals. Let $P_n = \{p \restriction [n,\omega) \mid p \in P\}$ and $\leq_n = \leq\restriction P_n$. For $D \subseteq \omega_n$, let $P^n_D = \{p \restriction n \mid p \in P\}$ and define $\leq^n_D$ on P^n_D by $p \leq^n_D \bar{p}$ iff (1) for $1 \leq i < n-1, (b_i, u_i) \leq_{\langle \bar{b}_{i+1}, A_{i+1}\rangle} (\bar{b}_i, \bar{u}_i)$ and (2) $(b_{n-1}, u_{n-1}) \leq_{\langle D, A_n\rangle} (\bar{b}_{n-1}, \bar{u}_{n-1})$. Now let $P^{(n)}$ consist of those $p \in P$ with $sup\, u_{n-1} < sup\langle b_n, \varnothing\rangle$. $P^{(n)}$ is dense in P. To see this, let $p \in P$. $c = \omega_n \setminus \bigcup\{b^n_\nu \mid \nu \in u_n\}$ is unbounded in ω_n since $|u_n| < \omega_n$ and $\langle b^n_\nu \mid \nu < \omega_{n+1}\rangle$ is adj. Let δ be least in $c \setminus sup\, u_{n-1}$, $b'_n = b_n \cup \{\delta\}$, $p'_k = p_k$ if $k \neq n$, and $p'_n = \langle b'_n, u_n\rangle$. Then $p' \in P^{(n)}$. For $\nu \in u_n$, $b'_n \setminus b_n \cap b^n_\nu = \{\delta\} \cap b^n_\nu = \varnothing$. So $p'_n \leq_{\langle b_{n+1}, A_{n+1}\rangle} p_n$. Since p' differs from p only in the n-th coordinate, $p' \leq p$. So $P^{(n)}$ is dense. <u>Claim</u>: for $p, \bar{p} \in P^{(n)}$,

(1) $p \restriction [n,\omega) \Vdash_{P_n} p \restriction n \in P^n_{B_n}$, and

(2) $p \leq \bar{p}$ iff $p \restriction [n,\omega) \leq_n \bar{p} \restriction [n,\omega)$ and $p \restriction [n,\omega) \Vdash_{P_n} p \restriction n \leq^n_{B_n} \bar{p} \restriction n$.

(1) is trivial. For (2), let $p \leq \bar{p}$. $p \restriction [n,\omega) \leq_n \bar{p} \restriction [n,\omega)$ is clear. $p \restriction [n,\omega) \Vdash_{P_n} p \restriction n \leq^n_{B_n} \bar{p} \restriction n$ will hold if for all $q \leq_n p \restriction [n,\omega)$, letting $b^q_n = (q_n)_0$, $p_{n-1} \leq_{\langle b^q_n, A_n\rangle} \bar{p}_{n-1}$. Note that if $\bar{p} \in P^{(n)}$ and $B \supseteq_e \bar{b}_n$, then $\bar{u}_{n-1} \cap \langle B, A_n\rangle = \bar{u}_{n-1} \cap \langle \bar{b}_n, A_n\rangle$. Thus for any $p \in P$, $p_{n-1} \leq_{\langle B, A_n\rangle} \bar{p}_{n-1}$ iff $p_{n-1} \leq_{\langle \bar{b}_n, A_n\rangle} \bar{p}_{n-1}$. But if $q \leq_n p \restriction [n,\omega)$, then $b^q_n \supseteq_e \bar{b}_n$. So

$p_{n-1} \leq_{\langle b_n^q, A_n \rangle} \bar{p}_{n-1}$. For the converse, $p \restriction [n, \omega) \Vdash_{P_n} p \restriction n \leq^n_{B_n} \bar{p} \restriction n$ implies that $p_{n-1} \leq_{\langle b_n, A_n \rangle} \bar{p}_{n-1}$, hence that $p_{n-1} \leq_{\langle \bar{b}_n, A_n \rangle} \bar{p}_{n-1}$. Adding the condition $p \restriction [n, \omega) \leq_n \bar{p} \restriction [n, \omega)$ will ensure that $p \leq \bar{p}$. The claim follows.

So for every n, forcing with P is equivalent to forcing with the two-step iteration $P_n * P^n_{B_n}$. P_n is ω_n-closed. And since conditions in P^n_D are compatible if their first components are, $P_n \Vdash P^n_{B_n}$ has the ω_n-c.c. So ω_n is not collapsed. Since for all $n < \omega$, ω_n is not collapsed, neither is ω_ω. And P has the $\omega_{\omega+1}$-c.c. So cardinals $> \omega_\omega$ are preserved. $\dashv$

Lemma 3.5. *(GCH) Assume Cond(C). Let $\kappa = \omega_2$ and assume that for some $B \subseteq \kappa^+$, $(*)^{C,B}_\kappa$ holds. Then for any $A \subseteq \omega_3$ there is a P which adds Cond(A) and preserves cardinals.*

Proof. One can assume that $A \in L[C, B]$ and that $H_\kappa = L_\kappa[B \cap \kappa]$. The first step is to add $D \subseteq \kappa$ such that $B \in L[C, D]$. Then by lemma 3.6 below one can add $Cond(D)$. In the resulting model $Cond(A)$ will hold.

Add D exactly as in [B-J-W] (p.9ff). In $L[C, B \cap \kappa]$ let $h : H_\kappa \longrightarrow \kappa$ be a bijection. For any $b \subseteq \kappa$ let $S(b) = \{h(b \cap \delta) \mid \delta < \kappa\}$. If $b \neq b'$, then $S(b) \cap S(b')$ is bounded in κ. Define a sequence $(b_\delta \mid \delta < \kappa^+), b_\delta \subseteq \kappa$, as follows. Given $(b_\nu \mid \nu < \delta)$, let b_δ be the $<_{L[C, B \cap \delta]}$-least $b \subseteq \kappa$ such that $b \neq b_\nu$ for $\nu < \delta$. b_δ exists since $b \restriction \delta$ is uniformly definable from C and $B \cap \delta$ and $L[C, B \cap \delta] \models |\delta| = \kappa$. Now let D code B relative to the sequence $(S(b_\delta) \mid \delta < \kappa)$. Then $B \in L[C, B \cap \kappa, D]$. To see this, working in $L[C, B \cap \kappa, D], B \cap \kappa$ is available. Given $B \cap \delta$, $\delta \in B$ iff $D \cap S(b_\delta)$ is bounded. And b_δ is uniformly defined from $B \cap \delta$ and C. So $B \cap \delta + 1$ is decided. Since the definitions are uniform, the sequence $(B \cap \delta \mid \delta < \kappa^+)$ is in $L[C, B \cap \kappa, D]$. Thus $B \in L[C, B \cap \kappa, D]$. Now code $B \cap \kappa$ into D to get the required subset of κ. $\dashv$

The last two arguments are just direct adaptations from [B-J-W]. The point here is that under certain conditions these arguments can be localized to add condensation. Here is a guess at a more general fact of this sort:

Conjecture. *(GCH) Assume Cond(C). Let λ be a cardinal. Suppose that there is a B such that for every cardinal $\kappa < \lambda$, $(*)^{C,B}_\kappa$ holds. Then for* *any $A \subseteq \lambda$ there is a partial order P which adds $Cond(A)$ and preserves cardinals.*

"Proof". It is the need to ensure $(*)_\kappa^{\emptyset,A}$ in order to code from κ^+ to κ which is responsible for introducing the assumption $V = L[A]$ in [B-J-W].

We haven't seen any reason to develop this argument. Our immediate goal has been to force $Cond(A)$ for some $A \subseteq \omega_3$ which computes ω_3 correctly thereby killing all precipitous ideals on ω_1. So we are more interested in the question: when does C exist? Or, are there large cardinal assumptions which imply that the conditions of lemma 3.5 cannot be satisfied?

§3.2 Condensation and morasses.

Given some condensation one can add more. For $A \subseteq \omega_2$ condensation can be added outright. An $(\omega_1, 1)$-morass has condensation. Only "coarse" properties of the morass are needed for this. The partial order used in the following argument is distilled from the one Jensen used to add an $(\omega_1, 1)$-morass [Stn]. It adds what we will call an $(\omega_1, 1, A)$-weak-morass, for the purpose of discussion later.

Lemma 3.6. *(CH) For any $A \subseteq \omega_2$ there is a partial order P such that if G is P-generic then $V[G] \models Cond(A)$, and P preserves cardinals.*

Proof. Fix $A \subseteq \omega_2$. Let $F : H_{\omega_2}^{<\omega} \longrightarrow H_{\omega_2}$ be a skolem function for H_{ω_2} and $C_F = \{X \cap \omega_2 \mid |X| = \omega \text{ and } X \text{ is closed under } F\}$. For $\alpha < \omega_1$, let $C_F^\alpha = \{X \in C_F \mid \alpha_X = \alpha\}$ where, as before, $\alpha_X = X \cap \omega_1$. Let $A_X = \pi_X{}'' A$, $Q^{A,F} = Q = \{A_X \mid X \in C_F\}$, and $Q_\alpha = \{A_X \mid X \in C_F^\alpha\}$. For $\alpha < \omega_1$ and $\nu \in (\omega_1, \omega_2)$, let $X_\alpha^\nu = cl_F(\alpha \cup \{\nu\})$, $D_\nu = \{\alpha \mid X_\alpha^\nu \cap \omega_1 = \alpha\}$, $\pi_\alpha^\nu = \pi_{X_\alpha^\nu}$, and $A_\alpha^\nu = \pi_\alpha^\nu{}'' A$. Note that $\subseteq$ and $\subseteq_e$ agree on C_F^α, and that $(C_F^\alpha, \subseteq)$ is a tree. Also, for each α, $(Q_\alpha, \subseteq_e)$ is a tree with height $\leq \omega_1$.

Define P as follows. Conditions are pairs $p = (s, U)$ satisfying:

(1) $s : d \longrightarrow Q$ where d is a closed and bounded subset of ω_1, and for $\alpha \in d$, $s(\alpha)$ is a branch in Q_α of length $< \omega_1$. Let $\alpha_p = max(d)$.

(2) $U \subseteq \bigcup\{D_\nu \times \{\nu\} \mid \nu \in (\omega_1, \omega_2)\}$.

(3) $|s| = |U| = \omega$.

(4) If $(\alpha, \nu) \in U$ then $A^\nu_\alpha \in s(\alpha)$.

(5) If $(\alpha, \nu) \in U$ then $(\alpha_p, \nu) \in U$. And if $\alpha \leq \beta \leq \alpha_p$ and $\beta \in d \cap D_\nu$ then $(\beta, \nu) \in U$.

$(s, U) \leq (t, V)$ if $t \subseteq s$ and $V \subseteq U$. Any two conditions (s, U) and (s, V) are compatible. So P has the ω_2-c.c. since CH holds. Also P is ω_1-dense. So cardinals are preserved. In fact P has an ω_1-closed dense subset P^*. The conditions in P^* satisfy the additional constraint:

(6) If $\nu, \nu' \in U_1$ and $\nu < \nu'$ then $\nu \in X^{\nu'}_{\alpha_p}$,

where $U_1 = \{\nu \mid \exists\alpha\, (\alpha, \nu) \in U\}$. This ensures that if $p_0 \geq p_1 \geq \ldots$, $\alpha = sup_n \alpha_{p_n}$ and $\nu, \nu' \in (\bigcup_n U_n)$, with $\nu < \nu'$, then $\nu \in X^{\nu'}_\alpha$. Hence $A^\nu_\alpha \subseteq_e A^{\nu'}_\alpha$. And $\bigcup_n s_n$ can be extended to a condition with $\alpha \in dom(s)$.

Let G be P-generic, $S = \bigcup\{s \mid (s, U) \in G \text{ for some } U\}$, $\mathcal{U} = \bigcup\{U \mid (s, U) \in G \text{ for some } s\}$. Then $S : D \longrightarrow Q$ where $D \subseteq \omega_1$ is club and for $\alpha < \omega_1$, $S(\alpha)$ is a branch in Q_α of length $< \omega_1$. The key property of S which entails $Cond(A)$ is this: letting $A_\alpha = \bigcup S(\alpha)$,

$(*)$ for all $\nu \in (\omega_1, \omega_2)$, $\{\alpha \mid A^\nu_\alpha \subseteq_e A_\alpha\}$ contains a club.

To see that $(*)$ implies $Cond(A)$, let $B_\nu = \{\alpha \mid A^\nu_\alpha \subseteq_e A_\alpha\}$, $B = \langle B_\nu \mid \nu \in (\omega_1, \omega_2)\rangle$ and let $F, A, S, B \in X \prec V_\theta$. Let $\alpha = \alpha_X$. If $\nu \in X \cap \omega_2$ then $\alpha \in B_\nu$ and $A^\nu_\alpha \subseteq_e A_\alpha$. Thus $A_X = \bigcup\{A^\nu_\alpha \mid \nu \in X \cap \omega_2\} \subseteq_e A_\alpha$. To verify that $(*)$ holds in $V[G]$, note that if $p = (s, U)$ and $(\alpha, \nu) \in U$ then $p \Vdash (D \cap D_\nu) \setminus \alpha \subseteq B_\nu$. And for each ν the set of p with $(\alpha, \nu) \in U$ for some α is dense. $\dashv$

It goes without saying by now that lemma 3.6 has the obvious "gap-1" generalization:

Lemma 3.6(b). *Suppose that κ is regular and $\kappa^{<\kappa} = \kappa$. For any $A \subseteq \kappa^+$ there is a partial order P such that if G is P-generic then $V[G] \models <\kappa$-$Cond(A)$, and P preserves cardinals.*

<u>Remarks</u>: (1) It is a consequence of work of Shelah and Stanley [S-Stn]

that, in the absence of $0^\sharp$, one can force to add $B \subseteq \omega_3$ such that $(*)^{\emptyset,B}_{\omega_2}$ holds without changing cofinalities. Thus assuming $\neg 0^\sharp$, by lemmas 3.5 and 3.6, one can force to add $Cond(A)$ for any $A \subseteq \omega_3$ without collapsing cardinals. This can be improved somewhat, but again apparently short of what is needed to kill all precipitous ideals on ω_1 with set forcing.

(2) Let $A_\alpha = \bigcup S(\alpha)$ as in the foregoing argument. Note that in $V[G]$, the following holds:

$$\nu \in A \quad \text{iff} \quad \exists h : \omega_1 \xrightarrow{onto} \nu. \{\alpha \mid ot(h''\alpha) \in A_\alpha\} \text{ contains a club.}$$
$$\text{iff} \quad \forall h : \omega_1 \xrightarrow{onto} \nu. \{\alpha \mid ot(h''\alpha) \in A_\alpha\} \text{ contains a club}$$

Thus A is Δ_1-definable over H_{ω_2} in the parameter S. (Solovay observed that this holds for the top of a morass.) So one cannot simply iterate the partial order of lemma 3.6 to add $Cond(\omega_2)$ without violating $2^{\omega_1} = \omega_2$.

Instead one might try to add $Cond(\omega_2)$ by adding $Cond(A)$ for some $A \subseteq \omega_3$ which codes A. A natural object to consider for accomplishing this is a gap-2 morass at ω_1 which codes A. Unfortunately, as mentioned, generic gap-2 morasses lack the needed condensation properties. (This point is observed in [Stn], p.75f.) Short of introducing the rather elaborate definition of a gap-2 morass, we can indicate the problem by attempting to step-up lemma 3.6 as follows: given $A \subseteq [\omega_2, \omega_3)$, use 3.6(b) to obtain an $(\omega_2, 1, A)$-weak-morass. This can be coded by a set $A^* \subseteq \omega_2$. Apply 3.6 again to obtain an $(\omega_1, 1, A^*)$-weak-morass. (Though more ideas are involved, this is, roughly speaking, the basic strategy for adding a gap-2 morass at ω_1: add an $(\omega_2, 1)$-morass M, then an $(\omega_1, 1)$-morass which codes the bottom part of M.) One can extract from this a family $\mathcal{M} = \langle A_{\alpha,\nu} \mid \alpha \leq \omega_1, \nu < \omega_2 \rangle$ satisfying (1) for $\mu \in [\omega_2, \omega_3)$,

$$(*)^2_\mu \quad \{\nu \mid A^\mu_\nu \subseteq_e A_{\omega_1,\nu}\} = B^2_\mu \text{ contains a club,}$$

and (2) for $\nu \in [\omega_1, \omega_2)$,

$$(*)^1_\nu \quad \{\alpha \mid (A_{\omega_1,\nu})^\nu_\alpha \subseteq_e A_{\alpha,\bar{\nu}}\} = B^1_\nu \text{ contains a club,}$$

where $\bar{\nu} = \pi^\nu_\alpha(\nu)$, $S^p_U = \pi^p_U{}''S$, and π^p_U is the collapse of $X^p_U = cl_F(U \cup p)$, with p a finite set of ordinals, for some suitable F. If $p = \{\nu_0, \ldots, \nu_k\}$, $X^{\nu_0,\ldots,\nu_k}_U = X^p_U$.

For $\mu \in [\omega_2, \omega_3)$, let Φ_μ be the directed system $\langle \pi^\mu_{\nu,\nu'} \mid \nu, \nu' \in B^2_\mu, \nu < \nu' \rangle$, with $\pi^\mu_{\nu,\nu'} = \pi^\mu_{\nu'} \circ (\pi^\mu_\nu)^{-1} : A^\mu_\nu \longrightarrow A^\mu_{\nu'}$. The limit of this system is an initial segment of A.

Let $\mathcal{M}, A, F \in Y \prec V_\theta$, Y countable. Let $\alpha = \alpha_Y = Y \cap \omega_1$. The preceding conditions permit the conclusion that for $\mu \in [\omega_2, \omega_3) \cap Y, \pi_Y(\Phi_\mu)$ is a directed system through the sequence $\langle A_{\alpha,\nu} \mid \nu < \nu_\alpha \rangle$, where $\nu_\alpha = sup\{\nu \mid A_{\alpha,\nu} \neq \varnothing\} < \omega_1$. And the limit of $\pi_Y(\Phi_\mu)$ is an initial segment of $\pi_Y(A)$. But this is the extent of the control the conditions exert. One can't conclude that the limit is also an initial segment of some target set. Or from another point of view, the conditions do not seem to control the collapse of B^2_μ. The additional structure of a gap-2 morass does not improve the situation.

What is needed in place of the conditions $(*)^2_\mu$ and $(*)^1_\nu$ to lift the argument of 3.6 is something like this: a map $X \mapsto A_X$ defined on $[\omega_2]^\omega$ such that for $\mu \in [\omega_2, \omega_3)$,

$$(**) \quad \{X \mid A^\mu_X \subseteq_e A_X\} \text{ contains a club in } [\omega_2]^\omega.$$

where, again, $A^\mu_X = \pi^\mu_X{}''A$.

§3.3 $Q^{A,F}$ and an example.

There are several things one can ask or say about the trees $(Q^{A,F}_\alpha, \subseteq_e) = Q_\alpha$ used in the proof of 3.6. For example wCC implies that $\{\alpha \mid ht(Q_\alpha) \geq \omega_1\}$ is stationary. In the extension $V[G]$ of 3.6, F can be chosen so that each Q_α is a countable well-ordering (viz., $S(\alpha)$). But $Cond(A)$ doesn't require this much. In fact, if $2^{\omega_1} = \omega_2$ this case must be far from typical by the observation that it implies that A is $\mathbf{\Delta}_1$-definable over H_{ω_2}. Assuming $Cond(A, F)$ then, what are the possibilities for Q_α? The next lemma is a precursor to lemma 1.4. To some extent it just unpacks the definition of $Cond(A)$. It has as an immediate consequence that for $A \subseteq \omega_2$, if $Cond(A)$ fails then for any F a counterexample to $Cond(A; F)$ appears with any new real. Let $h : X \mapsto A_X = \pi_X{}''A, A \subseteq \omega_2$.

Lemma 3.7. *$\neg Cond(A)$ iff for every $F : {\omega_2}^{<\omega} \longrightarrow \omega_2$ there is an α and a perfect subtree of C_F^α which is preserved by h.*

Proof. Assume C_F^α contains a perfect subtree which is preserved by h. So there is an embedding $e : 2^{<\omega} \longrightarrow C_F^\alpha$ such that $h \circ e : 2^{<\omega} \longrightarrow (Q_\alpha, \subseteq_e)$ is also tree preserving. If $a \in {}^\omega 2 \cap V[G] \setminus V$, then $h(X_a) \notin V$, where $X_a = \bigcup_{s \subseteq a} e(s)$. So $\neg Cond(A; F)$.

For the converse the argument is like that of 1.4. Assume $\neg Cond(A)$. Let $F : {\omega_2}^{<\omega} \longrightarrow \omega_2$. One can assume that if X is closed under F then $X \cap \omega_2$ is an ordinal and if $\omega_1 \subseteq X$ then $X \cap \omega_2$ is an ordinal. Since $\neg Cond(A; F)$ pick $P, p \in P$, and P-names σ and τ such that $p \Vdash \tau \in C_F$ & $\pi_\tau{}''A = \sigma \notin V$ & $\tau \cap \omega_2 = \alpha$. If $\omega_1 \subseteq X$ then $\pi_X{}''A = A \cap sup X$. So $\alpha < \omega_1$. Now using the fact that $p \Vdash \sigma \notin V$, build the embedding $e : 2^{<\omega} \longrightarrow C_F^\alpha$ as in the proof of lemma 1.4. $\dashv$

We conclude this section with an example running in the opposite direction which puts lemma 3.7 to a little use. GCH and $\diamond$ hold in $V[G]$.

Theorem 3.8. *Let P be the forcing notion which adds ω_2 Cohen subsets of ω_1. Then $Cond(G)$ fails in $V[G]$.*

We shall need a slightly stronger version of lemma 1.9. Let $F : {\omega_2}^{<\omega} \longrightarrow \omega_2$. For $\alpha < \omega_1$ let G_α be the following game: player I plays intervals $I_n = [\alpha_n, \beta_n] \leq \omega_2$ and player II plays ordinals γ_n so that $\beta_n < \gamma_n < \alpha_{n+1}$. Player I moves first and wins a play iff $X = cl_F(\alpha \cup \{\alpha_n \mid n < \omega_1\}) \subseteq \bigcup_n I_n$ and $X \cap \omega_1 = \alpha$.

Fact 3.9. *For any $F : {\omega_2}^{<\omega} \longrightarrow \omega_2$. there is an $\alpha < \omega_1$ such that I wins the game G_α.*

Proof. Suppose otherwise. For each α, G_α is determined. Let τ_α be a winning strategy for II in G_α. Note that if τ dominates τ_α, τ is also a winning strategy for II in G_α. So there is a single strategy which is winning for II in G_α for every $\alpha < \omega_1$. Now let σ be a winning strategy for I in the game G used in the proof of 1.9. Play σ against τ to get $I_n = [\alpha_n, \beta_n], n < \omega$. Since

σ is winning, $X = cl(\alpha_n \mid n < \omega) \subseteq \bigcup_n I_n$. Let $\alpha = X \cap \omega_1$. Then σ beats τ in G_α. Contradiction. $\dashv$

Proof of 3.8. The idea is to verify the condition for $\neg Cond(G)$ given in lemma 3.7. (Actually this is not quite what happens.) We can assume CH without loss of generality: add a single Cohen subset of ω_1 getting CH, then ω_2 more. This is the same as just adding ω_2 Cohen subsets of ω_1.

Let $P = \{p : a \longrightarrow 2 \mid a \subseteq \omega_2 \ \ \& \ \ |a| = \omega\}$ and $\dot{F}$ be a name for a potential witness to $Cond(G)$ in $V[G]$. By CH, let $\boldsymbol{w} = \langle w_\nu \mid \nu < \omega_2\rangle$ list $[\omega_2]^\omega$. Let $q \in P, \theta$ be large and $\mathcal{A} = (V_\theta, \{\dot{F}, P, \boldsymbol{w}, q\}, \dots)$. Let $C = \{X \prec \mathcal{A} \mid |X| = \omega\}$, $C^\alpha = \{X \in C \mid X \cap \omega_1 = \alpha\}$, $C \restriction \omega_2 = \{X \cap \omega_2 \mid X \in C\}$ and $C^\alpha \restriction \omega_2 = \{X \cap \omega_2 \mid X \in C^\alpha\}$.

$H \subseteq X \cap P$ is *X-generic* if H consists of compatible conditions and meets every dense set $D \in X$ *in X*—i.e., $H \cap D \cap X \neq \varnothing$ for every $D \in X$ with D dense in P. If M_X is the collapse of X, then $\pi_X''H$ is $\pi_X(P)$-generic over M_X in the usual sense.

If $X \prec \mathcal{A}$, G is V-generic and $X \cap G$ is X-generic, then $X[G]$ is closed under $\dot{F}_G$. Here $X[G] = \{\tau_G \mid \tau \in V^P \cap X\}$, $\tau_G =$ the G-interpretation of τ.

Note the following about the elements of C^α. If $X, Y \in C^\alpha$ and $\eta < \omega_2, \eta \in X \cap Y$ then $[\eta]^\omega \cap X = [\eta]^\omega \cap Y$. If $X \subsetneq Y$ and $\nu = min\,(Y \setminus X)$ then $X \cap \omega_2 = Y \cap \nu$ and $cf(\nu) = \omega_1$ iff $X \cap \omega_2$ is bounded in ν. So if $X \subsetneq Y, \nu = min\,(Y \setminus X)$ and X is bounded in ν, then for every $a \in [\omega_2]^\omega \cap Y, a \cap X \in X$.

Using 3.9 we can get an embedding $e : 2^{<\omega} \longrightarrow C^\alpha \restriction \omega_2$ for some α with the following properties:

(1) $s \subseteq t \Longrightarrow e(s) \subseteq_e e(t)$,

(2) $e(s) \cap e(t) = e(u)$ where $u = s \cap t$,

(3) $e(s)$ is bounded in $\nu = min\,(e(s^\frown i) \setminus e(s))$.

Take $X \in C^\alpha$ such that $e(s) = X \cap \omega_2$. X_s can be chosen minimally. So $s \subseteq t$ implies $X_s \subseteq X_t$. By the preceding, if $\nu = min\,(e(s^\frown i) \setminus e(s))$ then $cf(\nu) = \omega_1$. And if $s \subseteq t$ and $p \in X_t$ then $p \cap X_t \in X_s$.

If $a \in {}^\omega 2$ let $e(a) = \bigcup\{e(s) \mid s \subseteq a\}$, $X_a = \bigcup\{X_s \mid s \subseteq a\}$, and $X = \bigcup\{X_s \mid s \in 2^{<\omega}\}$. For $s \in 2^{<\omega}$ choose $D_s \in X_s$ dense in P so that if $a \in {}^\omega 2$,

$\langle D_s \mid s \subseteq a\rangle$ lists all the dense subsets of $P \in X_a$. Note $X_a \in C^\alpha$.

Now choose conditions $p_s \in X_s \cap D_s$ all compatible so that if $\nu = min\,(X_{s\frown i}, X_s)$ then $p_{s\frown i}(\nu) = i$. Assume this is done. Let $H = \{p_s \mid s \in 2^{<\omega}\}$ and $p = \bigcup H$. $H_a = H \cap X_a = \{p_s \mid s \subseteq a\}$ is X_a-generic, for $a \in {}^\omega 2$. Let G be P-generic with $p \in G$. Then $H \subseteq G$. So $G \cap X_a$ is X_a-generic. Thus $X_a[G]$ is closed under $\dot{F}_G$. Also $X_a[G] \cap \omega_2 = e(a)$. Thus $\pi_{X_a[G]}{}''G = \pi_{e(a)}{}''G$, regarding G as a subset of ω_2: $\nu \in G$ if $\bar{p}(\nu) = 1$ for some $\bar{p} \in G$. To recover a from $\pi_{X_a[G]} = \overline{G}_a$ proceed by induction. Suppose it has been computed that $s \subseteq a$. Let $\nu_i = min\,(e(s\frown i) \setminus e(s))$. Then $\pi_{e(s\frown 0)}(\nu_0) = \pi_{e(s\frown 1)}(\nu_1) = \beta$. Thus $s\frown 0 \subseteq a$ iff $\beta \notin \overline{G}_a$ and $s\frown 1 \subseteq a$ iff $\beta \notin \overline{G}_a$. So $p \Vdash \neg Cond(G;\dot{F})$. And $p \leq q$ since $q \in X$ for any $X \in C$. The set of p which force $\neg Cond(G;\dot{F})$, therefore, is dense. And $P \Vdash \neg Cond(G;\dot{F})$.

For $s, t \in 2^{<\omega}$ let $s \prec t$ if $\ell(s) < \ell(t)$ or $\ell(s) = \ell(t)$ and $s <_{lex} t$. Choose p_s by induction along $\prec$. Fix s. By hypothesis $\{p_t \mid t \prec s\}$ is a set of compatible conditions in X_s. Let $\bar{p} = \bigcup\{p_t \cap X_s \mid t \prec s\}$. Then $\bar{p} \in X_s$. In fact, letting $s = \bar{s}\frown i$, we have $\bar{p} \in X_{\bar{s}}$. So $dom(\bar{p}) \subseteq \nu = min\,(X_s \setminus X_{\bar{s}})$. Let $q = \bar{p} \cup \{\langle \nu, i\rangle\}$. $q \in X_s$ extends to an element of D_s. Let p_s be such a condition. p_s is clearly compatible with each $p_t, t \prec s$. This finishes the argument. $\dashv$

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