A Structured Approach to Physically-Based Modeling for Computer Graphics

Thesis by

Ronen Barzel

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Abstract

This thesis presents a framework for the design of physically-based computer graphics models. The framework includes a paradigm for the structure of physically-based models, techniques for "structured" mathematical modeling, and a specification of a computer program structure in which to implement the models. The framework is based on known principles and methodologies of structured programming and mathematical modeling. Because the framework emphasizes the structure and organization of models, we refer to it as "Structured Modeling."

The Structured Modeling framework focuses on clarity and "correctness" of models, emphasizing explicit statement of assumptions, goals, and techniques. In particular, we partition physically-based models, separating them into *conceptual* and *mathematical* models, and *posed problems*. We control complexity of models by designing in a modular manner, piecing models together from smaller components.

The framework places a particular emphasis on defining a complete formal statement of a model's mathematical equations, before attempting to simulate the model. To manage the complexity of these equations, we define a collection of mathematical constructs, notation, and terminology, that allow mathematical models to be created in a structured and modular manner.

We construct a computer programming environment that directly supports the implementation of models designed using the above techniques. The environment is geared to a tool-oriented approach, in which models are built from an extensible collection of software objects, that correspond to elements and tasks of a "blackboard" design of models.

A substantial portion of this thesis is devoted to developing a library of physically-based model "modules," including rigid-body kinematics, rigid-body dynamics, and dynamic constraints, all built with the Structured Modeling framework. These modules are intended to serve both as examples of the framework, and as potentially useful tools for the computer graphics community. Each module includes statements of goals and assumptions, explicit mathematical models and problem statements, and descriptions of software objects that support them. We illustrate the use of the library to build some sample models, and include discussion of various possible additions and extensions to the library.

Structured Modeling is an experiment in modeling: an exploration of designing via strict adherence to a dogma of structure, modularity, and mathematical formality. It does not stress issues such as particular numerical simulation techniques or efficiency of computer execution time or memory usage, all of which are important practical considerations in modeling. However, at least so far as the work carried on in this thesis, Structured Modeling has proven to be a useful aid in the design and understanding of complex physically-based models.

Contents

Index of Figures

Notation

New Notation

SCOPE :: name	"name" in namespace SCOPE	Notn. 3.1
$\mathsf{A} \overset{f}{\sqsubset} \mathsf{B}$	A is a specialization of B with injection f	Notn. 3.5
$\{\text{Things}\}_{\text{IDs}}$	The space of indexes of Things	Notn. 3.11
S_i	The element in index S labeled by ID i	Notn. 3.11
Things [$a \mapsto$ A, $b \mapsto$ B]	Defines state space Things	Notn. 3.22
x_s	$x(s)$, where s is in a state space w/ aspect x	Notn. 3.27
$x_s(t)$	$x(s(t))$, where $s(t)$ is in a state space w/ aspect x	Notn. 3.28
$\begin{bmatrix} a \leftarrow\!\shortmid & x \\ b \leftarrow\!\shortmid & y \end{bmatrix}$	A tuple describing an element of a state space	Notn. 3.24
${}^{f}x$	Coordinate representation of x in frame f	Notn. 6.6

Standard Notation

$f\colon \mathsf{A} \rightarrow \mathsf{B}$	"function named f, which maps from space A to space B"
	f The function as a whole
	$f(x)$ The value of the function acting on $x \in \mathsf{A}$
iff	If and only if
$\Re$	The real numbers
	$\Re^n$ 1-d real arrays having dimension n
	$\Re^{m\times n}$ 2-d real arrays (matrices) w/ dimensions m and n
$\emptyset$	The empty set.
$\left\{x \;\middle|\; \textit{condition}\right\}$	The set of all x such that *condition* holds.
v^*	Antisymmetric dual $\begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$ of a 3-D vector $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Chapter 1

Overview

As the computer graphics field matures, there is an increasing demand for complex, physically-based models. However, little attention has yet been focused on design methodologies for such models. Hence, models have often been ad-hoc, special purpose, abstruse, and/or hard to extend. This thesis develops a framework for physically-based modeling, in order to alleviate these problems.

This chapter gives some background motivation for the creation of a design framework, and elaborates on the goals for the framework. It additionally contains a brief overview of physically-based modeling, and an outline of the remainder of the thesis.

1.1 The Need for a Design Framework

Creating new types of computer graphics models is to a large extent a programming task (at the current state of the art). Creating physically-based models requires the translation of mathematical expressions of behavior into computer programs. Thus, a practitioner of physically-based modeling typically creates a model by jotting down some equations, then writing a program that embodies the model.

In our experience as practitioners of physically-based modeling, to make our programs (and hence the models) be robust, we would try of course to write the programs "well", i.e., pay attention to program structure and modularity, take advantage of object-oriented techniques and languages, and so forth. However, we found that careful program design didn't eliminate many types of stumbling blocks encountered when modeling. For example:

- Even with a well-structured program, it was difficult to extend a model, to add new mathematical functionality and behavior—changing the equations often required substantial re-writing the program.
- It was difficult to merge different models/programs written by different people. For example, in our lab, we wanted to merge the constraint mechanism of [Barzel,Barr88] (for of rigid bodies) with that of [Platt,Barr88] (for flexible bodies)—but the independently designed programs didn't "mesh" directly.
- A program is only as robust as the model it embodies; careful program design doesn't ensure that the underlying mathematical approach is robust. For example, a rigid-body program that implements the penalty method to meet constraints in dynamic models will generally not perform as well as one using analytic or inverse-dynamics techniques (see, e.g., [Baraff89] and [Barzel88]).
- Programming a model doesn't help us to communicate the model; rather than describe a model directly, research papers often describe the computer program that embodies the model (e.g., [Barzel,Barr88], [Girard,Maciejewski85], [Isaacs,Cohen87], [Bruderlin,Calvert89], [Chadwick,Haumann,Parent89]).

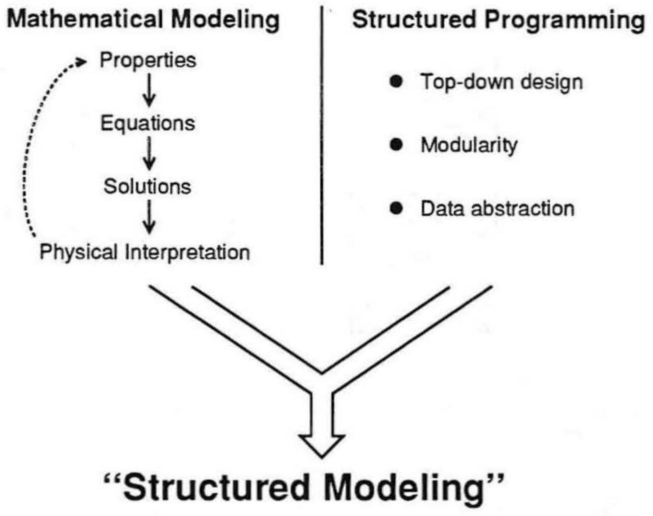


Figure 1.1: Structured Modeling draws from principles of mathematical modeling and structured programming, to derive a framework for physically-based computer graphics modeling. □

- More generally, programming doesn't help us to understand a model. It can be hard to separate the fundamental principles of the behavior of a model from the details of program construction, and to distinguish minor tweaking/debugging of a program from changes to the underlying model.

As a result of the above experience, we are led to the conclusion that it is important to have a common, general-purpose strategy and framework to design models "on blackboards," before implementing them as programs. This work describes such a strategy/framework, that was created by applying structured programming and mathematical modeling principles to the domain of physically-based computer graphics modeling (Fig. 1.1). Given the framework for "blackboard" design of models, we then specify a corresponding computer program framework, that can directly and naturally support the models we design.

1.2 Goals for the Framework

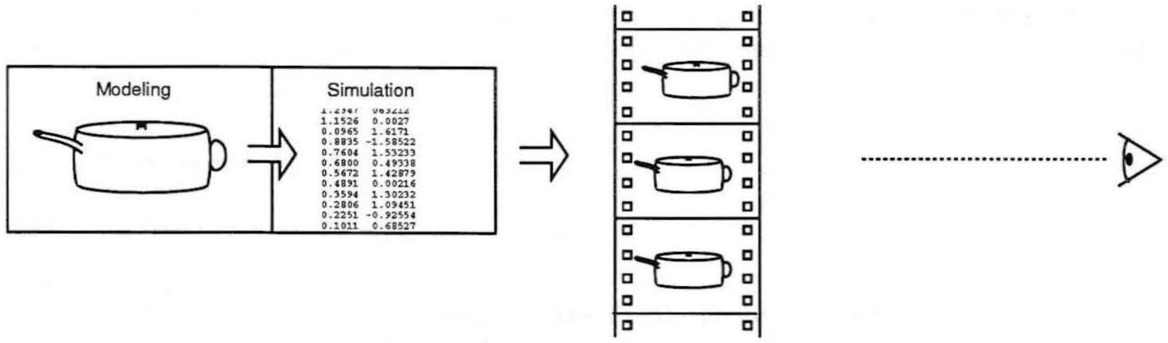
Sec. 1.1 discussed our desire to have a strategy/framework for the design of physically-based models. Here, we enumerate some goals for the framework:

- *To facilitate the understanding and communication of models,* both in discussions with colleagues and in written articles and reports. We want to be able to create complete, well-defined "blackboard" models, independent of the programs that implement them.

- *To facilitate the creation of models with high degrees of complexity.* A variety of factors contribute to the complexity of models: size, in terms of numbers of objects and possible interactions between them; the desire to model real-world phenomena with increasing accuracy; mathematical complexity; intricacy of numerical computational techniques; and so on. We are particularly interested in "non-homogeneous" models, which embody several behaviors that differ qualitatively from each other or over time.

- *To facilitate the reuse of models, techniques, and ideas,* so as to allow new models to "stand upon the shoulders" of previous ones, and to help us merge models that are designed separately.

- *To facilitate the extension of models,* so that newly developed techniques and methods can be used to enhance existing capabilities, without having to "start over."



```
[illegible]
1.1526  0.0027
0.0965  1.6171
0.8835 -1.58522
0.7604  1.53233
0.6800  0.49338
0.5672  1.42879
0.4891  0.00216
0.3594  1.30232
0.2806  1.09451
0.2251 -0.92554
0.1011  0.68527
```

Figure 1.2: There are two phases to the overall process of physically-based modeling: "Modeling" defines the formal behavior and equations of the physical system. Given a definition of behavior, "Simulation" is performed to numerically solve the equations. Since the output shown to the user is the immediate result of the simulation, the user may not see the "behind-the-scenes" modeling. □

- *To facilitate the creation of models that are "correct."* We want each model to achieve its particular goals. We want to avoid *ad hoc* techniques, so that models will be useable (and re-usable) in a wide variety of circumstances.
- *To facilitate the translation of models into programs.* We want the program layout and structure to correspond as closely as possible with the blackboard description of a model.

We will emphasize a basic low-level framework, intended for "hands-dirty" model makers, rather than attempt to define a final high-level specification or environment for end-users. The low-level framework, however, is intended to be a base upon which higher-level constructions can be built.

1.3 Overview of Physically-Based Modeling

Physically-based modeling is modeling that incorporates physical characteristics into models, allowing numerical simulation of their behavior. It has become somewhat of a catchall term for a variety of techniques, that all share the approach of defining physical principles of behavior for their models, then having the computer compute the details of the behavior. Physically-based modeling is a relatively new branch of computer graphics.[1] A brief survey of the field can be found in [Foley et al.90-Ch.20].

Common elements in physically-based modeling are: classical dynamics (motion based on forces, mass, inertia, etc) with rigid or flexible bodies; inter-body interaction; constraint-based control. Physically-based models are founded on mathematical equations, and often involve numerically intensive computation to simulate their behavior.

Physically-based models most often focus on how bodies move and change shape over time. But also, sophisticated rendering techniques can be tied in with descriptions of the models, since rendering can be viewed as simulating a model of the interaction of light with matter. Thus in general, physically-based modeling blurs the traditional distinction between modeling, rendering, and animation in computer graphics (see [Foley et al.90-p.606]).

Notice that we make a distinction between *simulation*, which emphasizes computation of behavior given a model, and *modeling* which focuses on the creation of the models—this thesis will address modeling rather than simulation. Note, however, that since we ultimately simulate our models to determine their behavior and create images or animations, the simulation is what is most directly visible to an observer; the modeling work itself is often behind-the-scenes, a matter of defining *what* to simulate (Fig. 1.2).

Finally, we observe that classically, a formalism of a physical system or behaviors is typically just a collection of equations. Each equation can be thought of as a "model fragment"—but for our goal of creating

[1] The field was first named in a course in the 1987 ACM SIGGRAPH (The Association for Computing Machinery's Special Interest Group on Computer Graphics) conference, "Topics in Physically-Based Modeling" [Barr87].

a complete, well-defined model, we must additionally define the "glue" that connects these fragments. The mathematical modeling techniques presented in Ch. 3 will address formal models that include the "glue" between fragments.

1.4 Outline of Thesis

Beyond this chapter, the thesis is organized as follows:

- The design framework
  - Ch. 2: Framework for "blackboard" design of physically-based models
  - Ch. 3: "Structured" mathematical modeling techniques
  - Ch. 4: Computer program framework
- A library, designed using the framework.
  - Ch. 5: Overview of the library
  - Ch. 6–9: Library modules
  - Ch. 12: Extensions to the library
- Conclusion, Ch. 13
- Miscellaneous mathematical objects, Appendix A
- Details of prototype implementation, Appendix B
- A technique for solving piecewise-continuous ODE's, Appendix C

1.5 Related Work

[Brooks91] addresses the same question as we do: how to manage complexity of computer-graphics models. Like us, he adapts ideas that are familiar to programmers; he addresses issues such as debugging, versioning, documentation, etc. Thus his focus is more macroscopic than ours: his effort might be called "model engineering" (from software engineering), as compared to our "structured modeling" (from structured programming). There is of course overlap between the two.

[Booch91] is similar in spirit to our work—design techniques to manage complexity—but in the domain of object-oriented programming rather than physically-based modeling. Our work does have some object-oriented elements however, in mathematical modeling approach and in program design.

Much of our work is based on structured programming and software engineering; when not directly in techniques, then in the ethic of abstraction and modularity as a basic principles of design. [Dahl,Dijkstra, Hoare72] and [Dijkstra76] provide the foundations of structured programming. ([Booch91] contains an extensive bibliography on software engineering.)

We are not aware of other work that addresses physically-based modeling qua modeling. Most of the works in the field are descriptions of techniques to implement various types of effects and behavior. We list a smattering of works: [Badler et al.91] gives techniques for modeling and simulating articulated figures (figures with limbs, such as people and animals); [Raibert,Hodgins91] discusses walking and running (this work has a scientific modeling perspective; in addition to making computer animation of locomotion, the authors build robots that implement their techniques); [Terzopoulos,Fleischer88] models inelastic flexible bodies; [Reynolds87] models flocking behavior of animals; [Fournier,Reeves86] models ocean waves.

A common and fertile approach in physically-based modeling is to use constraint-based techniques to control models. To name just a few works: [Witkin,Kass88] describes a formation of animation as a constrained multi-point boundary-value problem, thus allowing the user to specify intermediate and final configurations as well as the initial; [Barzel,Barr88] describes how to constrain physical models to follow user-specified paths; [Isaacs,Cohen87] and [Schröder,Zeltzer90] combine constrained dynamics with kinematic control; [Platt,Barr88] uses constraint techniques for flexible bodies; and [Kalra90] discusses a framework to combine various constraint methods.

[Zeleznik et al.91] presents an interactive modeling system which, although not specifically focused on physically-based modeling, similarly blurs the traditional modeling/rendering/animation distinction: "We wish to expand the definition of 'modeling' to include the realms of simulation, animation, rendering, and user interaction."

There is a large body of literature on simulation and numerical techniques; see, e.g., [Ralston, Rabinowitz78] for an introduction to numerical analysis, and [Press et al.86] for a collection of numerical subroutines.

Chapter 2

A Framework for Physically-Based Models

This chapter presents a design framework for physically-based models; the emphasis is on the high-level design or specification of physically-based models, as opposed to their implementation. Our approach is to develop and discuss a model on a blackboard, i.e., create a *blackboard model*, before implementing it in a computer program. The framework developed here, however, we will carry over directly to the design of programs, in Ch. 4.

This chapter might seem somewhat theoretical; to see the methods "in use," the reader is encouraged to skim the models described in Ch. 6–9. (Note however that the techniques and notation used for the mathematical subparts of the models in Ch. 6–9 will not be defined until Ch. 3.)

2.1 Overview

A physically-based model includes physical characteristics of the "thing being modeled" and specifies its behavior via mathematical equations. Thus a large component of physically-based modeling is *mathematical modeling*. We want to organize models into well-defined canonical parts, including a mathematical modeling slant and an emphasis on modularity.

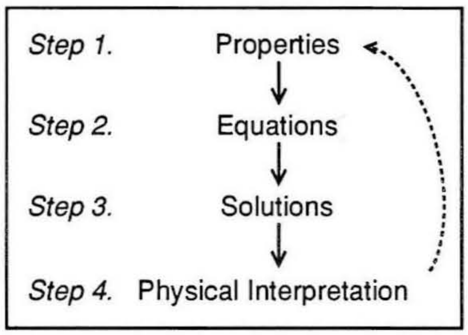
The underlying structure of a physically-based model is independent of its computer implementation. We will therefore talk mostly about the design of *blackboard models*, i.e., models that are worked out on paper or on blackboards,[1] without being tied to programming details. Thus for this chapter, "the model" includes the high-level plans, but we de-emphasize the implementation details.

We are ultimately interested in computer simulation of our models, so the ideas developed here are intended to be practical and implementable—indeed, the design of computer modeling programs discussed in Ch. 4 follows directly from the structure in this chapter. Moreover, Ch. 4 assumes that a blackboard design of a model has been worked out before the model is transferred to a computer; the models described in Ch. 6–9 are designed in such a manner.[2]

Note that the discussion in this chapter will not make a heavy distinction between a *model* (such as a teapot or space shuttle) and a *modeling technique* (such as a particular constraint method). In particular, for the approach we will be discussing, models will be assembled from building blocks, or "modules," which can in turn be built from smaller modules. All the modules in this hierarchy will be designed in the same basic style, be they low-level tools or the top-level model.

[1] We use the term "blackboard models" rather than "paper models" so that the models are not thought of as paper tigers or as origami.

[2] Of course, we don't expect a blackboard model to work perfectly the first time it is implemented; the blackboard model will typically be modified and updated as the implementation is developed.



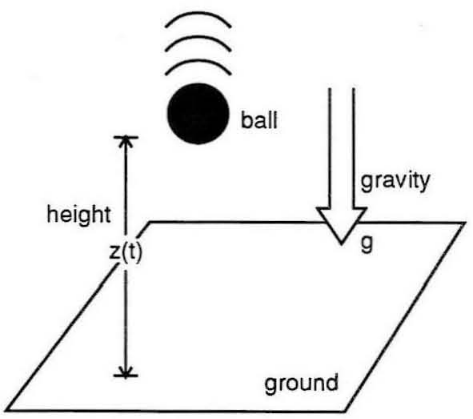


Figure 2.1: (Left) Methodology of applied mathematical modeling. These steps are discussed in the text. □

Figure 2.2: (Right) A falling ball, used in the text to illustrate the applied mathematical modeling procedure of Fig. 2.1. □

2.2 Background: Applied Mathematical Modeling

We present here a brief discussion of mathematical modeling as it is done by applied mathematicians, in order to provide background for the physically-based modeling strategy of Sec. 2.3. This discussion is distilled from the ideas in [Boyce81].

Fig. 2.1 shows the sequence of steps that make up mathematical modeling. To illustrate the steps, we will consider the following simple (trivial) problem: "How long does it take for a falling ball to hit the ground?". We will make a model of the physical system, and use that model to predict an answer to this question. See Fig. 2.2.

Step 1. Choose properties to include in the model. Doing so typically involves making simplifying assumptions. For our example, we assume: horizontal motion is irrelevant; gravity provides a constant downward acceleration g; friction is irrelevant; and the ball starts at rest. We will examine the height of the ball as a function of time $z(t)$.

Step 2. Create equations for the behavior. These are the *mathematical model.* For our example, we have the differential equation:

$$\begin{aligned} \frac{d^2}{dt^2}z(t) + g &= 0 \\ \frac{d}{dt}z(t)\Big|_{t=0} &= 0 \end{aligned}$$

The equations must be well-defined; we don't want there to be any ambiguity or contradictions. The equations can be manipulated, transformed, subject to proofs, and so forth.

Step 3. Solve the equations. The above differential equation yields the following closed form expression: $z(t) = z(0) - gt^2/2$. Given an initial value of $z(0) = h$, we find that $z(t) = 0$ when $t = \sqrt{2h/g}$.

Step 4. Determine the physical interpretation of the solution. How do the results relate to the thing being modeled? What is the mathematics telling us? We can make qualitative or numerical predictions. In the example, we see that balls that start higher will take longer to hit, and if gravity were stronger, balls would hit sooner; quantitatively, if the initial height is 3 meters and gravitational acceleration is 9.8 meters/sec^2, then the ball will hit in approximately .78 seconds.

(Repeat.) If the results aren't sufficient for our needs, we might start again based on what we have learned. For example, if measurements show that bigger balls take longer to hit, we might choose to include some properties such as mass of the ball and air-resistance that depends size of the ball, then update the equations, and so on.

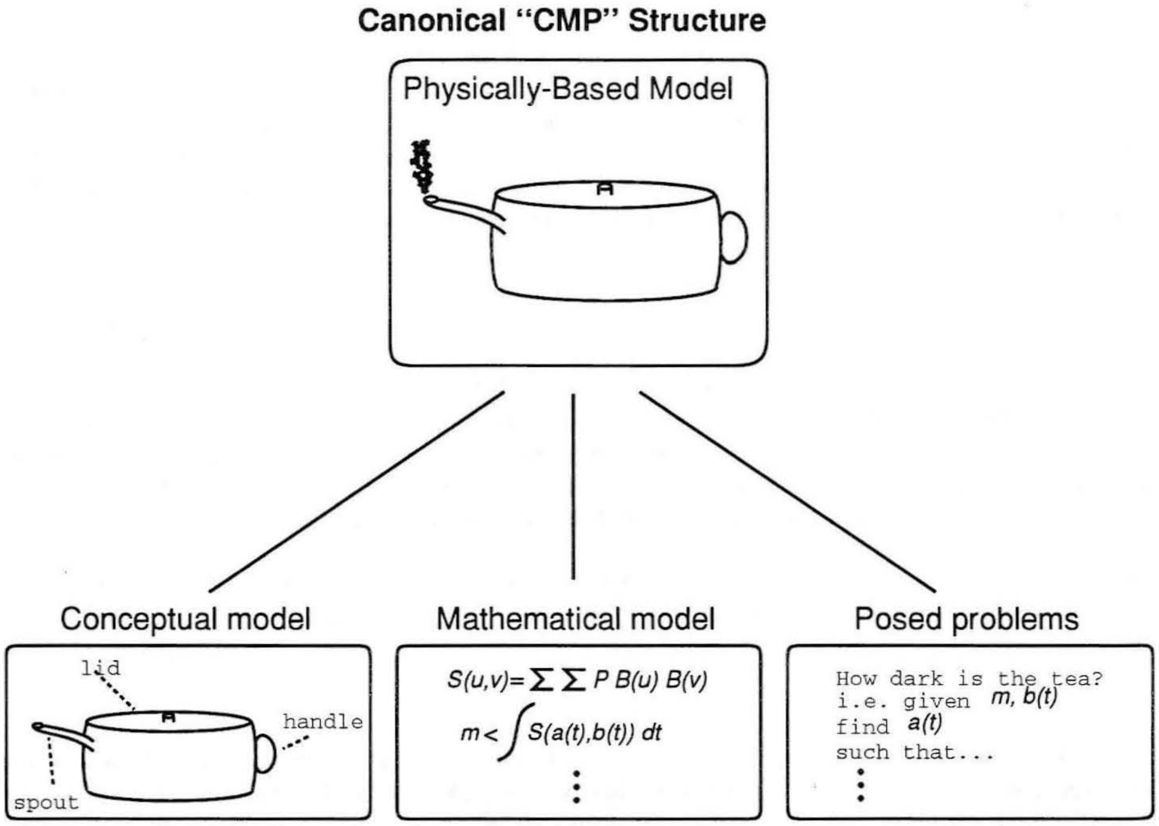


Figure 2.3: Canonical Structure of a Physically-Based Model. The conceptual model describes the physical properties of the thing being modeled. The mathematical model is a collection of equations for the behavior of the model. The posed problems include conceptual statements of tasks to be performed, as well as the corresponding mathematical problems. We call this partition the *CMP structure.* □

The power of this methodology lies in the well-defined mathematical model. Once the equations have been defined, we can draw on the vast body of mathematical knowledge to manipulate them and to solve problems. The equations stand by themselves, irrespective of what the particular problem is. For example, the equation $\frac{d^2}{dt^2}z(t) + g = 0$ in step 2 above could perhaps have been derived for a traffic flow model instead of a mechanics problem—it makes no mathematical difference.

2.3 Canonical "CMP" Structure of a Physically-Based Model

To a large extent, physically-based modeling is an application of mathematical modeling. Because of the power of the applied mathematical modeling methodology, we strive to build that methodology into our idea of a physically-based model. We thus partition a physically-based model into three distinct parts:

- the *Conceptual model,*
- the *Mathematical model,* and
- the *Posed problems,*

where the parts correspond with the steps in Sec. 2.2. We call this partition the conceptual/mathematical/posed-problem structure, or *CMP* for short (Fig. 2.3).

2.3.1 The Conceptual Model ("C")

The *conceptual model* is a description of the properties, features, characteristics, etc. of the thing being modeled, as per step 1 of Sec. 2.2.

In addition to properties that underly the mathematics, the conceptual model may also contain other information about the thing being modeled. For example, a conceptual model of a dynamic rigid body for computer graphics, in addition to terms such as mass and momentum that enter into a mathematical description of classical rigid-body motion, would have terms such as surface color and specularity that do not enter into the equations of motion.

2.3.2 The Mathematical Model ("M")

The *mathematical model* that is part of a CMP structure is directly adopted from mathematical modeling, the result of step 2 in Sec. 2.2. It is a collection of mathematical equations that describe the behavior of the model. The equations are "context free"—purely mathematical expressions that are complete without needing properties or definitions from the conceptual model. There is of course a correspondence with the conceptual model, but that correspondence is in the mind of the designer of the model, rather than inherent in the mathematics.

Note that we distinguish between a mathematical model and a "math problem": the equations in the model are not "problems" but rather predicates, i.e., statements of relationships between entities in the model. For example, the expression

$$\frac{d}{dt}y(t) = f(y(t), t)$$

simply states that the quantity on the left, the rate of change of $y(t)$, is equal to the quantity on the right, the value of $f(\ldots)$. Various problems *could* be posed around the statement—such as an initial-value problem, a boundary-value problem, or testing measured data for agreement—but we don't consider those to be part of the mathematical model (see Fig. 2.4).[3] Phrased from a computer-science point of view, the mathematical model is *declarative* rather than *procedural* (this will be discussed in Sec. 3.5.2).

In addition to equations, a mathematical model includes definitions of the terms that go into the equations. For the above example, we would define y to be a real-valued function of one real argument, f to be a real-valued function of two real arguments, and t to be a real number. It is often convenient to define named properties, e.g., "a function $T(x)$ is said to be *frumious* when $|T(x)| < x^2 \ldots$"

Creating a mathematical model is not necessarily easy. Note that we don't try to define models that are "minimal"; a model may include any or all definitions and equations that are convenient. For physically-based modeling, we have a set of goals, strategies, and techniques that are relevant, as we will discuss in Ch. 3.

2.3.3 The Posed Problems ("P")

We typically want a physically-based model to *do* something, such as make a prediction or simulate some behavior. Mathematically, this translates into solving some problem, as per step 3 of Sec. 2.2. Thus the third part of a physically-based model, once we've defined the conceptual and mathematical models, is the collection of *posed problems*.

Each posed problem includes both a conceptual notion of a task to perform, and the corresponding mathematical problem. The mathematical problems list which terms from the mathematical model have known values, which are unknown, which equations from the mathematical model come into play, and so forth. Sometimes, the mathematical problems may be solved analytically, but in general for physically-based modeling, we will pose problems to be solved numerically, via computer.

[3] Since we distinguish equations from problems, from our point of view the common phrase *to solve an equation* (such as we used in step 3 of Sec. 2.2) is unrigorous. We take it to be slang for *to solve the problem that is implied by an equation*, presupposing that only a single problem is implied in the current context.

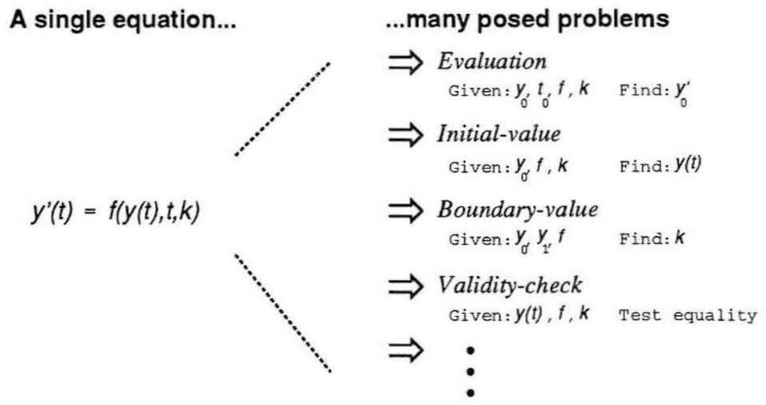


Figure 2.4: A single mathematical equation or model can lend itself to many posed problems. In this case, a differential equation describes a relationship between various quantities; there are many combinations of knowns and unknowns, which lead to various canonical problems. □

The problems can often be stated in a standard canonical form, e.g., as an initial-value ODE problem: "Find $Y(t)$ where $Y' = f(Y,t)$, given f and the initial value Y_0 at t_0." Notice that, unlike the mathematical model, posed problems are often procedural, e.g., "given x, express y as a function of it, then find z such that. . . ." In general, the problems should be mathematically well-posed.[4]

2.3.4 Implementation & Physical Interpretation

To implement a physically-based model, i.e., to make predictions or simulations, we need to solve the posed problems. But how to solve them? We don't offer any specific insights here—that's what the fields of applied mathematics and numerical analysis are about (see, e.g., [Lin,Segel74], [Zwillinger89], [Ralston, Rabinowitz78]). There is also a large body of knowledge and software for solving numerical problems on computers (e.g., [Press et al.86], [NAG]). Computer tools exist as well, to help with mathematical manipulation and numerical problem solution (e.g., [Wolfram91]).

Most often, the designer of a model has some ideas about how to solve the posed problems; notes along these lines can accompany a description of CMP structure, as part of a blackboard model. Given a blackboard model and solution ideas, one can write a special-purpose program that numerically solves the posed problems. Ch. 4 will discuss a more general program framework in which to embed the entire CMP structure.

Once a model has been implemented, and has produced results, we need to interpret those results, as per step 4 of Sec. 2.2. The posed problems describe the conceptual tasks that corresponds with the mathematical problems, helping one to relate the numerical solutions back to the conceptual model. But ultimately, physical interpretation—how the results relate to the thing being modeled—is a process that is undertaken by the designer or user of the model.

2.4 Discussion of the CMP Structure

We have found that separating the conceptual, mathematical, and posed-problem sections of modeling techniques makes it easier to understand them. When we read a new article that discusses a model or modeling technique, or when we come up with an idea of our own, it is instructive to ask four questions:

- "What is the model or technique trying to do?" (conceptual model)
- "What are the underlying equations?" (mathematical model)

[4] That is, each problem has a solution that exists, is unique, and depends smoothly on the known values ([Nihon Sugakkai77]).

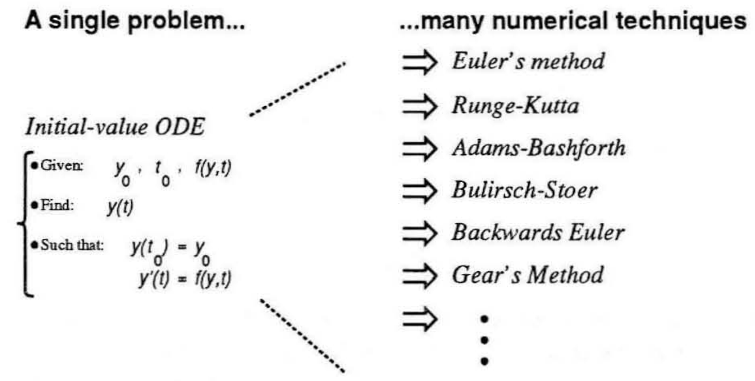


Figure 2.5: A single problem can often be solved by many different of numerical techniques. In this case, there are a variety of solvers for initial-value ODE problems; the choice depends on issues such as accuracy, stiffness, speed, ease of implementation, and so forth. The availability of such techniques, and the ability to suit a numerical technique to the particulars of a given problem, are primary reasons not to "hardwire" a specific technique into one's physically-based model. □

- "What are the knowns and unknowns?" (posed problems)
- "What are the solution techniques?" (implementation)

2.4.1 Why Separate The Numerical Techniques?

The CMP structure doesn't include numerical techniques; they are left as implementation details. But sometime numerical manipulation is presented as an inherent part of a model or method. For example, sometimes modeling techniques are described in ways such as:

"For each frame,[5] update the body positions by adding the velocities. . . ,"

This description has some appeal: the physical interpretation is immediate; one can intuit why the results are plausible; and it is straightforward to implement. Notice that it mingles the numerical manipulation with the conceptual and mathematical models, as well as with the posed problems: The properties of the body and the animation, a differential equation of motion, and Euler's method for solving an initial-value problem are all stated at once.

Despite the appeals of such mingling, we feel it is best to make a clear separation between the statement of a model and the numerical techniques used to solve the posed mathematical problems.

The most pragmatic reason for the separation is that stating a numerical problem in canonical form allows one to take advantage of existing tools and knowledge: there are many numerical routines, libraries, books, and lore upon which one can draw, e.g., [Press et al.86], [Ralston,Rabinowitz78], [NAG]. There are often tools available that are more robust than the intuitive techniques; for example Euler's method is known to be less accurate than others, and unstable for stiff equations ([Press et al.86-Ch.15]).

Furthermore, if the particular numerical solution technique is not hard-wired into the model, one can choose the technique that is appropriate to the particular circumstances of the problem. For example: (see also Fig. 2.5)

"For up to moderately-sized problems, [we use] a Choleski-type matrix factorization procedure. . . For large problems [we use] iterative methods such as successive over-relaxation. . . Multi-grid methods. . . have served well in the largest of our simulations" [Terzopoulos,Fleischer88-p.275]

Finally, keeping the numerical technique separate helps insure that implementation details such as step sizes, error tolerances, and so forth, don't have significant effects on the conceptual behavior. For example,

[5] An animation *frame* is a single image from the sequence that forms the animation.

while we can accept that higher error tolerances may produce less accurate simulations, if we decide to animate at twice the frame rate, we wouldn't want to get qualitatively different behavior (see also Fig. 2.6).

Separation of numerical solution techniques from the body of the model is not unique to our CMP structure. In practice it is done quite frequently (hence the existence of the field of numerical analysis). The CMP structure itself, however, is not typically emphasized; the next sections will expand on the reasons for the partition.

2.4.2 Why Separate Problems From Equations?

The distinction between equations and problems, which we emphasize in Sec. 2.3.2, may seem like hairsplitting or sophistry. Perhaps interesting to academics, but where is the practical benefit?

The primary benefit is in extensibility and reusability of models. As mentioned earlier, a single mathematical model can imply many numerical problems; if the numerical problems aren't hardwired into the model, we can have the opportunity later to pose modified or new problems.

For example, in our own earlier work [Barzel,Barr88], we hardwired a specific numerical problem into our model: we described the motion of rigid bodies as an initial-value ODE expressed in terms of linear and angular momenta (among other things), with the linear and angular velocities were defined as "auxiliary" variables to be computed once the momenta have been determined. After a small amount of experience, we realized that it would be easier for the user if we could specify initial conditions in terms of the velocities rather than the momenta. Unfortunately, because the model (and hence the computer program) had the momenta hardwired as the "more primitive" representation, it required some inelegant patching of the computer program to insert appropriate velocity-to-momentum conversion routines. The mathematical rigid-body model in Ch. 8 of this work, on the other hand, merely describes the relationship between the velocities and momenta, without preferring one to the other; the resulting computer implementation naturally supports user specification of either.

Another reason for the separation of the mathematical model from posed problems is the declarative/procedural distinction between them. Being purely declarative helps make the mathematical model more robust, as will be discussed in Ch. 3. And the declarative/procedural distinction will be significant to the design of programs to implement physically-based models, in Ch. 4.

Note that there is often mathematical manipulation as part of solving a given problem. It is thus possible for there to be ambiguity between an equation that "should" be part of the mathematical model, and one that is "merely" an intermediate result in an analytic solution. For example, many analysis problems are solved via eigenvalues; should the eigenvalue-eigenvector equation be included in the mathematical model? Such questions require judgement calls on the part of the designer of the model. Our general tendency, however, is to include all such equations in the mathematical model; if the equations arise for one posed problem, they may also arise for some later problem, thus we may as well state them up front. Additionally, these equations often have interesting physical interpretations, and thus it can help our understanding of the overall model to include them explicitly. If the intermediate equations are only approximations, however, we are more hesitant to include them in the model (Fig. 2.6).

2.4.3 Why Separate Concepts From Mathematics?

Having equations that stand alone—without conceptual context—is a source of the power of applied mathematical modeling, as discussed in Sec. 2.2. This is thus our primary reason for making an explicit distinction between the conceptual and mathematical models.

There are additional benefits incurred by the separation, however. As mentioned in Sec. 2.3.1, there are properties of the thing being modeled that are important to our overall model, but that don't fit into the mathematical model; such properties have a home in the conceptual model.[6] The conceptual model, since

[6] In some cases, too, the mathematical model has terms or degrees of freedom that don't map directly back to anything in the conceptual model, e.g., as gauge transformations in classical electromagnetic field theory ([Landau,Lifshitz75]).

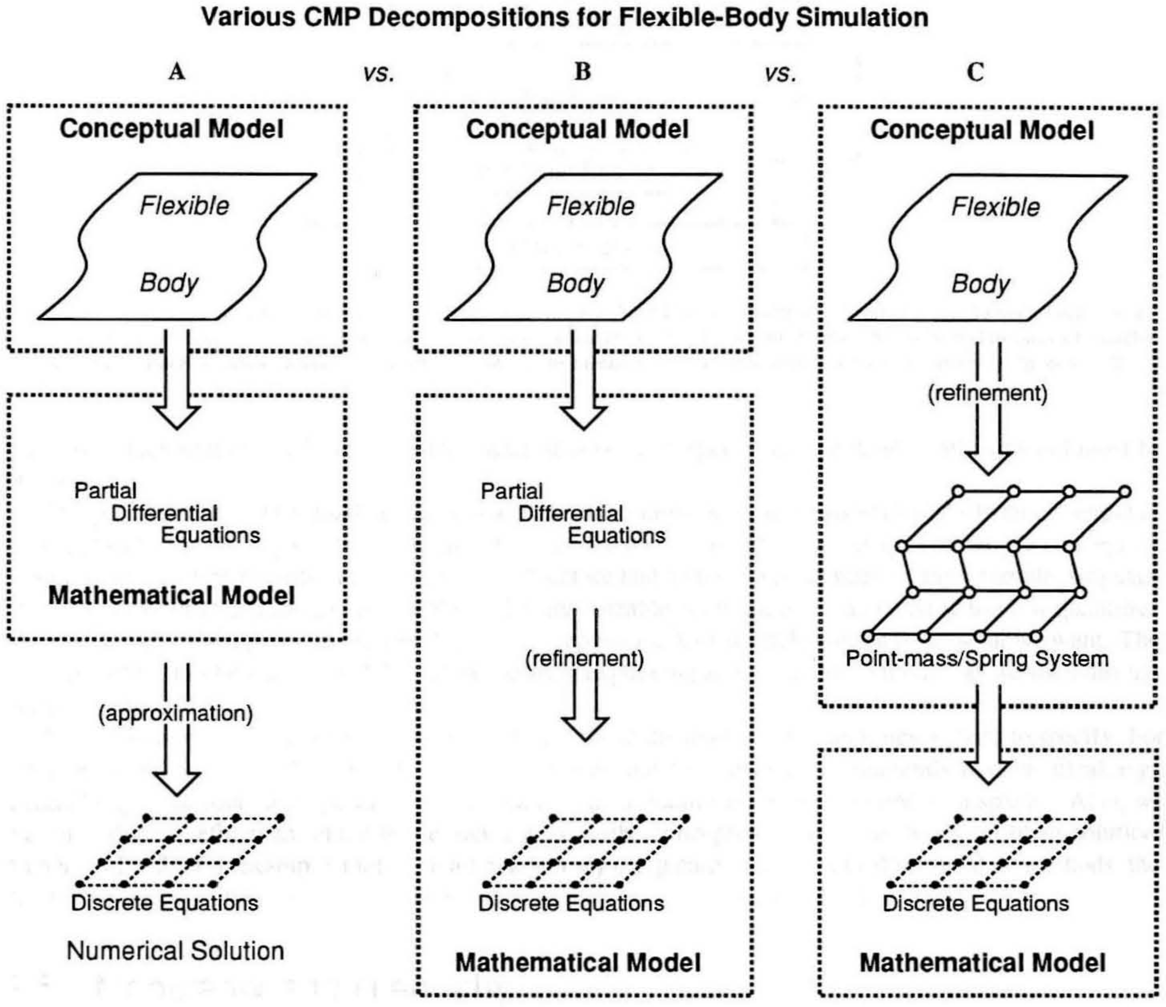


Figure 2.6: Choices in the CMP structure. Example Problem: Simulating the motion of a flexible body. A flexible body can be described mathematically as a continuous surface governed by partial differential equations (PDE's). PDE solutions are typically approximated by discretizing, yielding numerically soluble equations. We illustrate three ways of organizing the CMP structure:

A. The conceptual model is a flexible body, the mathematical model is a continuous surface, the solution technique requires discretization.

B. The mathematical model includes a refinement to discrete equations.

C. The conceptual model includes a refinement to a mass-point/spring system.

Version *A* is the "cleanest": the numerical solver can automatically/adaptively adjust parameters such as discretization, transparently to the high levels of the model, or other solvers can be used. At the other extreme, version *C* has discretization artifacts "hardwired" into the model (often mixed with other details such as polygonalization for rendering), thus to change solution parameters requires altering the high-level model. Version *C*'s main appeals are expedience (it can be simple to implement on top of existing rigid-body simulation systems) and mathematical simplicity (there are no PDE's). Flexible bodies are discussed further in Sec. 12.5. □

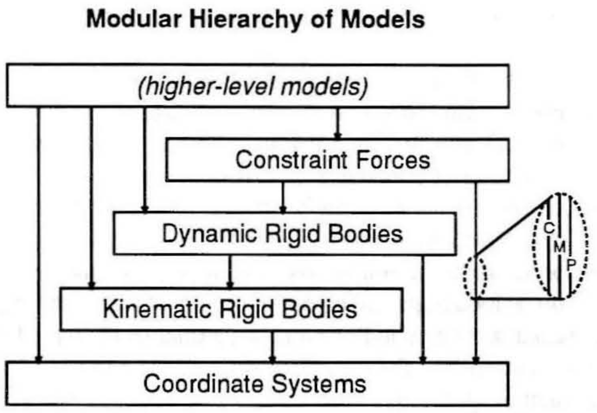


Figure 2.7: Modularity in Physically-based Models. Individual "modules" are defined for parts of the overall model domain. Each module may make use of or build on lower-level models, for concepts, mathematics, and/or posed problems. The diagram illustrates a hierarchy of modules that describe various aspects of rigid-body motion (the hierarchy corresponds with the prototype library in Ch. 6–9). □

it is just a statement in words, can include vague or fuzzy concepts, while the mathematical model must be precise.

The precise nature of the mathematical model lets us examine it for inconsistency, insufficiency, singularities, and so forth. Finding such irregularities in the mathematics can often give us insight into the conceptual model, and help identify parts of the overall model that we had neglected conceptually. For example, singularities in a constraint equation can imply physically unrealizable configurations, and having too few quantities to create a well-posed problem implies that our conceptual model isn't rich enough to do what we want. The precision of the mathematical model similarly helps in explaining and debugging a model, as we shall discuss in Sec. 2.6.4.

Note that the separation between the concepts and the mathematics can sometimes be hard to specify. For some applications, especially scientific or technical ones, the concepts can be inherently mathematical, e.g., a rigid body's conceptual shape may have a mathematical definition, such as a cone or a sphere. Also, we may introduce a refinement into a model that can be at the conceptual, mathematical, or problem-solution/numerical levels. For example, motion of a flexible body is typically simulated by discretization methods; the discretization might be conceptual, mathematical or numerical, as illustrated in Fig. 2.6.

2.5 Modularity and Hierarchy

The CMP structure helps us to organize our thoughts about what's what in a model, helps us to understand the mathematical and numerical behavior, and so forth. But CMP doesn't directly address the question of how to handle big models. We need a way to manage their complexity, even within the CMP structure: That's where modularity and hierarchy come in.

Modularity means that we build separate, loosely connected components, with well-defined boundaries and interfaces. *Hierarchy* means that each module can be built by putting together or invoking simpler modules. These techniques let us work on complex problems by a "divide-and-conquer" approach, breaking down large problem into simpler ones that are easy to comprehend. At the higher levels, one can focus on the interaction between the modular parts, without worrying about the details within them. Furthermore, the modular parts can be reusable—commonly used parts can be designed "once and for all" and kept in public libraries, so that their usage can be consistent, and so that designers won't have to repeatedly reinvent them.

The above is well known from programming, but it applies equally to blackboard physically-based models.[7] Fig. 2.7 illustrates a decomposition of rigid-body modeling into separate modules; dynamic rigid

[7] If we design our blackboard models modularly, it then will be straightforward to make modular program implementations. This will

bodies, e.g., are built from kinematic bodies, but with additional properties such as mass, and their motion is prescribed by Newton's laws.

So how do we make and include modules using the CMP framework? For blackboard models, it's easy: mostly, just say it's modular, and it's modular—we have no formal mechanisms to deal with. That is, the conceptual model, being just words, can just refer in words to concepts from another module, e.g., a high-level model in Fig. 2.7 can say "the swingarm moves as a dynamic rigid body. . . "; the mathematical model of one module can define its quantities and equations in terms of those from another module; [8] and a posed problem of one module can require solving a subproblem from another module.

Notice that there's typically "parallel" modularity and hierarchy in the conceptual/mathematical/posed-problem parts. For example, if one refers to concepts of dynamic rigid bodies, one will typically need to refer also to the corresponding mathematical equations, and one will often pose standard problems. Thus we group all three parts of a CMP partition into a single package or module. Ultimately, we'd like to define libraries of standard, general-purpose CMP modules that can be used repeatedly in many complex modeling tasks; Ch. 6–9 defines a simple prototype library for the domain of rigid-body modeling.

Modularity is a simple concept—there's not really very much that we have to say about it here. It isn't hard to design and build models in a modular manner. What's mostly needed is to *decide* to design models modularly; then, while doing the designs, one pays attention to issues such as generality, ease of use by others, and so forth, that are familiar to most structured programmers.

2.6 Designing a Model

This section discusses some issues for designing a model, or CMP module. Note that we don't make much of a distinction between "top-level models" and "low-level modules"; we like to design top-level models in the same way as modules intended for inclusion in a library. It's not any harder to design a top-level model as a module, and doing so can help make the model easier to extend or combine with other models later on; furthermore, given a rich library of support modules, the top-level module often doesn't need to do very much.

2.6.1 Separating C from M from P

The most important part of designing a model, we feel, is keeping the concepts separate from the mathematics separate from the posed problems. Various reasons for this separation were expounded in Sec. 2.4, but they can all be distilled as:

> *A well-defined CMP partition helps a model be robust, reusable and extensible, and helps us to understand, build, and debug the model.*

There may be choices about what goes into which section (e.g., as per Fig. 2.6), but we insist upon explicitly choosing.

We also want to make sure that implementation details—or things that *ought* to be implementation details—aren't mixed in with the model. The CMP separation helps with this somewhat, but one might still mix numerical details with the conceptual task statement in a posed problem. Watch out for statements like "choose an acceleration to bring the body to rest in one time step," that mix conceptual properties ("acceleration") with solution parameters ("time step"). Sometimes, however, implementation details can affect our choice of model, as will be discussed in the next section.

2.6.2 Top-Down vs. Bottom-Up vs. . . .

How do we go about designing a model? Do we design the conceptual model first, then the mathematical, and finally pose some problems, or do we start with problem tasks, then build the conceptual and mathematical

be discussed in Ch. 4.

[8] Ch. 3 will discuss techniques to support modularity of mathematical models.

models around them? Do we start with the existing library modules and build upwards, or start with the goals and build downwards?

For the CMP structure, a top-down approach—conceptual, then mathematical, then posed problems—corresponds with the mathematical modeling steps of Sec. 2.2. Notice, however, that mathematical modeling includes a *repeat* step, in which we go back and adjust the top level based upon our experience, and which is consistent with the "annealing" design strategy.

Often, practical ability to solve the posed problems limits the effectiveness of top-down design. Some difficulties that can arise for numerical problems are:

- The solution techniques are unacceptably slow.
- Available subroutine libraries don't have the appropriate solvers.
- It's unknown how to solve a problem, e.g., Fermat's last theorem.
- A problem is probably unsolvable, e.g., the halting problem, or is NP-complete,[9] e.g., the traveling salesman problem.

If we encounter a circumstance such as these, we may need to change the mathematical or conceptual models, to produce a more tractable problem. And sometimes, our top-level goals are inherently intractable, and cannot be met as stated.

Note, that there can be circumstances in which computational details affect the choice of conceptual model without impinging upon our goals. For example, if the falling ball of Fig. 2.2 were to bounce, the traditional linear restitution model for rigid-body collision ([Fox67]) leads to a sort of computational Zeno's paradox, in which the ball takes infinitely many ever-smaller bounces, but never reaches continuous contact with the ground. Switching to a quasilinear restitution model,[10] however, puts a finite bound on the number of bounces. Because the traditional model is merely an axiomatic approximation to empirical evidence, switching to a slightly different model produces equally acceptable simulation of behavior. For another example, [Baraff91] shows that the principle of constraints for mechanical systems leads to a problem that is NP-complete; and, while abandoning that principle yields different behavior in indeterminate configurations, the resulting behavior is still consistent with a rigid-body model, and moreover, is computationally tractable.

2.6.3 Standard Outline for a Model

To help maintain the structure of a model, we can fill in the blanks of a standard outline: (Fig. 2.8)

Goals. What we are trying to achieve; the high-level problem.

Conceptual Model. Includes a list of the physical properties of the thing being modeled, and their corresponding mathematical model terms. Can refer to other modules.

Mathematical Model. A collection of definitions and equations; May refer to definitions and equations of other modules. (The mathematical model will be discussed at length in Ch. 3.)

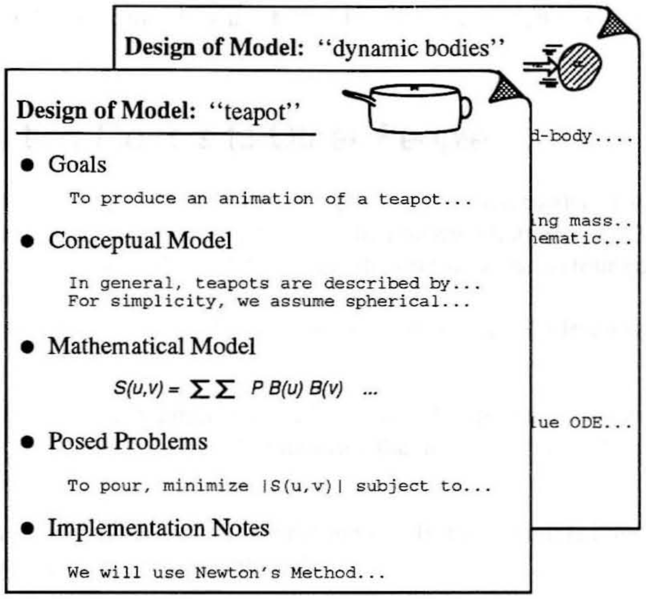
Posed Problems. Statements of the conceptual tasks to perform, and the corresponding mathematical problems, listing the knowns and unknowns for each. A problem may be decomposed into a series of smaller problems, that may be posed in other modules.

Implementation Notes. Details such as what numerical solvers are used for each problem, numerical parameter settings that produce acceptable results, and so forth.

For top-down design, we try to fill in the sections in the listed order. For annealing design, we fill in and update the sections as we learn about them. Each module has its own such outline.

[9] For practical purposes, NP-complete problems are computationally intractable; see [Lewis,Papadimitriou81].

[10] The traditional equation relating velocity before and after a bounce is $v^+ = -ev^-$, where e is the *coefficient of restitution*. Instead, we use $v^+ = \max(-e\, v^- - \kappa,\ 0)$, where κ is a constant velocity-loss term and the final velocity is clamped to 0.



Design of Model: "dynamic bodies"

d-body....

ng mass..
ematic...

ue ODE...

Design of Model: "teapot"

- Goals

  `To produce an animation of a teapot...`

- Conceptual Model

  ```
  In general, teapots are described by...
  For simplicity, we assume spherical...
  ```

- Mathematical Model

  $S(u,v) = \sum\sum \; P \, B(u) \, B(v) \; \ldots$

- Posed Problems

  `To pour, minimize |S(u,v)| subject to...`

- Implementation Notes

  `We will use Newton's Method...`

Figure 2.8: Outline of model design. Each module includes a statement of its goals, conceptual model, mathematical model, and posed problems, as well as implementation and other notes. □

2.6.4 Debugging a Model

Suppose we've implemented a model, i.e., solved some numerical problems, and we have understood the physical interpretation of the results. And, unfortunately, the results are wrong: they aren't what we want or expect. What's going wrong? For complex models, it can be hard to figure out the source of the misbehavior; it is often easy to get caught up in wild goose chases, e.g., trying to fix a numerical solver that isn't in fact broken, and so forth.

Having a CMP structure helps us locate and isolate bugs. There are typically three cases to consider when the simulated behavior is apparently broken:

1. *The problems are not being solved correctly.* Check to see if the numerical routines are really solving the equations (e.g., verify correctness by substituting solutions back into the original equation, or compare nominal derivative values against actual behavior, etc.). If they aren't solving the equations, there's a bug in the numerical techniques or in how we are using them.

2. *They're the wrong problems.* If the numerical solvers are correctly doing what they're told, then we must be telling them the wrong things. The problem has not been properly posed: we don't have the desired correspondence between the conceptual task and the mathematical model. There may be a bug in the mathematical model.

3. *It's what the model predicts.* If the mathematical model does correspond with the conceptual model, and if we are correctly solving the equations, then we are correctly simulating our conceptual model.

If we have identified case 3, the bug isn't "inside" the model. We have a conflict between the behavior that we want or intuitively expect, and what the model produces. This can be resolved in two ways:

Change the model. The conceptual model doesn't include the properties that are necessary to elicit the behavior we want. We must re-think our basic abstraction of the thing being modeled.

Change our expectation. Maybe we are wrong, and the model is right! A robust model might make surprising predictions that turn out to actually correspond with how the real thing actually behaves.[11]

[11] I once made simulations of a jumprope. I had trouble finding even a single frame that had a nice smooth loop of rope sliding along

Debugging will be discussed more concretely in Ch. 4, when we discuss computer implementations of CMP models.

2.7 Communicating Models to Other People

Unsurprisingly, we use the CMP structure to describe a physically-based model. This is of course easiest if the model was designed using the CMP paradigm—then the framework in Sec. 2.6.3 (Fig. 2.8) serves as an outline for a presentation of the model. The CMP structure directly answers the four questions that were listed in Sec. 2.4, which we have found aids in comprehending a model.

To clearly present a physically-based model, we can flesh out Sec. 2.6.3's framework with some explanatory material:

- *Background/Overview.* Before beginning to explain a model, especially to an audience unfamiliar with the problem domain, it is generally useful to describe the domain in general, and provide a few words about what the model will do.
- *Goals.* What are the motivations for defining the model? If it is a module intended to be included in a library, what are some anticipated applications?
- *Conceptual model.* What does the model do? Not merely a list of properties, but an overall introduction to the model: an overview of the basic abstraction, descriptions of the expected or common behaviors, and so forth. Diagrams can be helpful.
- *Mathematical model.* Mathematically, the equations should be "context free"—but to help comprehend them, it helps to explain their physical interpretations alongside them. Ch. 3 will have more to say about presenting mathematical models.
- *Posed Problems.* In addition to what the problems are, conceptually and mathematically, describe why they are interesting—what are they used for? Also, under what circumstances can and can't the problems be solved, i.e., where are the singularities and so forth, and what are their physical interpretations?
- *Implementation Notes.* What solvers are appropriate, what are the numerical issues and parameter settings, etc. Sometimes it may be more convenient to list the numerical techniques for each posed problem directly alongside it, rather than here in a separate section—but be careful not to confuse a particular choice of technique with the statement of the problem.

The modules in Ch. 6–9 are all presented using the above framework. (The reader can judge how effective this approach is!)

For spoken presentations, when fielding questions, it can be helpful to identify which part of the framework the question addresses, in the same way as debugging (Sec. 2.6.4). A common type of question is a "what if. . . " question, i.e., asking what the model will do in an unusual circumstance. Here too, the CMP structure is helpful. Sometimes, the answer is "it's not part of our conceptual model" (e.g., for a question "what if we move the light source close to the jello—does it get hot and melt?). Othertimes, the question can be mapped into the mathematical model, in which case there's typically one of three answers:

1. "There is no solution to the equations"
2. "There's exactly one solution, which is. . . "
3. "There are many solutions"

When there's not exactly one solution, the resulting behavior then depends on the particulars of the numerical solution technique.

the floor; it was wiggling and jangling every which way. This was very frustrating until I came across a stroboscopic multi-exposure of a girl jumping rope. . .

2.8 Summary

We have presented a canonical *CMP* (conceptual/mathematical/posed-problem) structure for physically-based models, which, based on applied mathematical modeling technique, emphasizes a well-defined mathematical model that is separate from its solution and physical interpretation. We differ from "straight" applied mathematics in that: we take a modular, re-usable approach to design of models; we have a heavier emphasis on the separation of the parts, in particular the distinction between an equation and a problem; and our conceptual model can include properties that don't enter into the mathematical model. Because of the importance of the mathematical part of a model, Ch. 3 will address a structured approach to mathematical modeling, which will fit in to our CMP framework.

We have found that the CMP structure enhances our ability to discuss, analyze, and understand modeling techniques, helps us to build and debug models, and produces models that are robust, reusable, and extensible. Furthermore, as we shall see in Ch. 4, it maps well onto programming; the CMP structure can be designed into computer programs.

Despite the presentations in this chapter, there is no cookbook for designing, debugging, and communicating models—those remain creative human processes. However, we do stress the CMP structure for physically-based models. We have used it successfully, for example, for the models in Ch. 6–9. And even when we don't formally write down a CMP structure in the precise form described in this chapter, we at least *think* about the models in that way.

2.9 Related Work

The mathematical modeling methodology we describe in Sec. 2.2, illustrated in [Boyce81], is reasonably widespread in applied mathematics. Its application to modeling physical systems is discussed, e.g., by [Zeidler88-p.viii], which draws a diagram similar to our Fig. 2.1.

Modularity and hierarchy are basic elements of structured programming (see, e.g., [Dahl,Dijkstra, Hoare72]) and object-oriented programming ([Booch91]).

The framework and design methodology that we have discussed is analogous to those found in other fields. For example, engineering design is often separated into task clarification/conceptual design/embodiment design/detail design; see, e.g., [Pahl,Beitz88], [French85], or [Cross89].

Chapter 3

Structured Mathematical Modeling

This entire chapter is directed towards solving a single problem: To be able to write down complete explicit mathematical models of complex systems. The task may seem innocuous at first glance, but it is in fact challenging enough to have spawned a chapter that is fairly dense. The chapter has a fair amount of "talk"—philosophy and other discussion—and also defines a few specific mathematical mechanisms that help structure complex models (see Fig. 3.1).

This chapter will draw some analogies between mathematical modeling and computer programming, to help us take advantage of principles of software design. However, the mathematical techniques we present do not depend on a knowledge of software.

The mathematical exposition does not strictly demand more than a basic familiarity with set theory and functional analysis.[1] The definitions we present are basic, thus somewhat abstract; however the underlying concepts are quite simple, and we have found that the techniques and notation seem natural after modest exposure. The list of notation, p. xvii, may be helpful.

The principles and techniques discussed in this chapter are used for the models in Ch. 6–9 . The reader is encouraged to skim Ch. 6–9 in order to see the principles and techniques "in action."

3.1 Overview

What is a mathematical model? For our purposes, a mathematical model is a collection of mathematical definitions of terms and equations. For physically-based modeling, we expect a mathematical model to be defined in the context of a CMP (conceptual/mathematical/posed-problem) partition as per Ch. 2. Thus, although problem statements ("given x, find y such that. . . ") and physical interpretation ("z is the width. . . ") are closely related to the mathematical model, we consider them to be separate from it. Ch. 2 details reasons for the separation; in this chapter we will take it as a given.

The principles and techniques we will discuss in this chapter are "administrative" in nature, directed towards structuring and managing complexity of mathematical models. For most of our applications, we will be content to re-cast classical equations—"model fragments" as per Sec. 1.3—into our structured form. In order to define such fragments *ab initio,* we refer the reader to [Lin,Segel74].

Since this chapter focuses on mathematical modeling, an unqualified use of "model" can be taken to mean "mathematical model." Similarly, since this chapter focuses on issues significant to structured modeling, an unqualified use of "mathematical modeling" can be taken to mean ". . . from our point of view."

[1] "A good treatise on the theory of functions of a real variable does not strictly require of its readers any previous acquaintance of the subject. . . yet a student armed with no more than a naked, virgin mind is unlikely to survive the first few pages. In the same way, although this book does not call upon any previous knowledge. . . " [Truesdell91-p.xvii]

Outline of Chapter 3

Figure 3.1: Outline of this chapter. Secs. 3.1–3.5 describe our philosophy and attitudes towards mathematical modeling. Secs. 3.6–3.10 present various mathematical mechanisms—definitions, notations, and techniques—that are useful for structured mathematical modeling; Secs. 3.11–3.12 contain further discussion and summary. □

3.2 Motivation for Structured Mathematical Modeling

We adhere to the premise that if we wish to do a simulation, we must first create a well-defined mathematical model; and if a mathematical model is well-defined, we ought to be able to write it down. Our primary goal for mathematical modeling is therefore:

> *Goal:* To be able to write down complete, explicit equations for complex physically-based models.

The difficulty here lies in the words "complete," "explicit," and "complex." Imagine writing complete, explicit equations for a bicycle drive train: the behavior of each and every link, their interactions with each other and with the gears and derailleurs, etc. Clearly, writing them out longhand in full detail, without the aid of special tricks or techniques, is too cumbersome and ugly to consider seriously.

We notice that computer programming has inherent in it the same difficult components as our goal for mathematical modeling: A computer program must be explicit, must be complete, and can be very complex. Over the past twenty years, techniques for structuring and engineering computer programs have been developed to mitigate these difficulties. We will apply many of these techniques to mathematical modeling.

The "Structured Mathematical Modeling" techniques in this chapter are intended to help meet our primary goal—complete, explicit written equations for complex physically-based models. But before developing any specific techniques, we will elaborate on various aspects of the goal.

3.2.1 Complex Models

The primary sources of complexity in our models are size and heterogeneity. The mathematical models in [Boyce81] have perhaps a few dozen variables; our computer graphics models can easily have hundreds or thousands, with intricate relationships or interconnections between them.[2] Moreover, we wish to mingle various behavior modes into a single model, where the differing behaviors can be due to different components of a model as well as from individual components that change over time.

[2] Having merely "thousands" of variables is itself humble, compared to applications such as weather/climate modeling. We emphasize applications, however, in which the many variables cannot be treated statistically or regularly, in that they may have irregular behaviors or interrelationships.

Thus, although we adopt the basic mathematical modeling philosophy of [Boyce81] (as discused in Sec. 2.2), we also need to develop techniques to handle our unusual complexity requirements.

3.2.2 Complete Models

As discussed in Sec. 1.3, most physical models are expressed as "fragments", i.e., a collection of equations that describe individual components of the model. For example, in presenting techniques for animating dynamic legged locomotion, [Raibert,Hodgins91] contains various equations (taken here out of context):

> "Both actuator models have the form $f = k(x - x_r) + b\dot{x}$, where f is...
> The control system computes the desired foot position as: $x_{fh,d} = \ldots$
> The posture control torques are generated by a linear servo: $\tau = \ldots$"

The article describes how and when the various fragments come into play—but this description is in words: there is no mathematical expression for the model as a whole.[3]

Lacking mathematical tools that express models completely, it is hard to communicate, analyze, and so forth. Furthermore, when we ultimately want to simulate a model on a computer, it becomes the programmer's job to ascertain or intuit how the various fragments piece together, and build that structure into a computer program. Thus we need to develop techniques to "glue" together separate model fragments.

3.2.3 Explicit Models

Going hand-in-hand with wanting our models to be complete, we also want them to be explicit. That is, we want to minimize ambiguity, that might lead to errors occurring through (accidental) inappropriate use of equations. We emphasize two ways of doing this: minimizing tacit dependencies, and minimizing tacit assumptions.

In writing equations, minimizing dependencies really means being careful always to write terms as functions that include all their parameters. In addition to minimizing errors in doing mathematical derivations, making all the dependencies explicit greatly eases the ultimate transition to a computer program. Being able to write explicit dependencies requires (and motivates the development of) complexity- and fragment- management techniques as discussed above. For example, in a rigid-body model, the force on a body can depend in principle on the state of all the other bodies at each instant; rather than merely writing "the force f", we want to write "the force function $f(\textit{state-of-all-bodies})$"—but using mathematically precise constructs.

Minimizing tacit assumptions is important in the early stages of overall model design as discussed in Sec. 2.6. But also, as we write our mathematical equations, we want to be careful always to list what our assumptions are, what conditions are required, and so forth. In this way, each equation can be "globally valid". For example, rather than writing an equation "$x = \ldots$," one would write write "in such-and-such circumstances, $x = \ldots$." This helps in understanding the equations, and again can ease the transition to computer programs.[4] The idea of globally valid equations ties in with modularity (Sec. 3.4.2) and declarative models (Sec. 3.5.2).

Unlike our goals of complexity and completeness, for which we identified a need for new techniques, the goal of explicit models to a large extent requires a discipline on the part of the model designer. That is, given techniques that make writing explicit equations feasible, it remains up to the model designer to invest the appropriate effort.

[3] No disparagement is intended. We have chosen to cite [Raibert,Hodgins91] precisely because it is an excellent article, describing superb work; thus it illustrates to us not a lack of discipline on the part of the authors, but rather a need for enhanced expressiveness in mathematical models and notation.

[4] All equations in this chapter in Ch. 6–9 are careful to include their "givens" and notes, in order to help make each be correct and understandable by itself.

3.2.4 Practical Utility

There is one additional goal that we have not yet mentioned: practical utility. We are not interested in purely academic or theoretical techniques: Not only do we want to develop techniques, we want to be able to use them—and moreover, we want to be better off for having used them.

That is, we want our mathematical modeling techniques to make it easier to create models. We want the techniques, as well as the models that we create thereby, to be writable and readable. We want to show models to colleagues, who should be able to understand them. We want to scribble models on whiteboards and napkins.

As such, we try to keep to familiar, existing notation, definitions, and terminology as much as possible to avoid abstruseness. We will not be attempting to fabricate entirely new mathematical languages, but rather will try to augment the existing language with a few simple constructs that have proven to be useful.

Note that it is particularly important that models be understandable to others: Because of structured modeling's modular approach, we will be designing "library" models (or components of models) that are intended to be used by others, often for unforeseen applications. Thus there is a concern for clear and orderly "packaging" and "interfaces" that is familiar to those who design software libraries, but is of lesser significance when one designs a mathematical model purely for one's own one-time use.

Finally, in addition to the above "ergonomic" issues, the techniques we use should be rigorous and robust enough to support derivations, proofs, and so forth.

3.3 Aesthetics & Design Decisions

A balance often must be struck between the goals discussed in Sec. 3.2—in particular, between the goal of practical usefulness and all the goals. For instance, if there are too many parameters and parentheses, putting in all the explicit dependencies will make the equations too cumbersome to read and error-prone to write. Similarly, defining new terms or constructs can clean up an equation, but a plethora of gratuitous definitions can be hard to keep straight and will be particularly inaccessible to newcomers. Or, there can be a tradeoff between full generality (a single model that can do everything) and ease of understanding.

There's no pat answer for what the proper balance is, nor how to strike it. Often, case-by-case decisions will have to be made based on personal aesthetics, individual goals, the levels of experience of the audience and of the designer, and so forth. Thus doing mathematical modeling involves "design decisions" that are similar in spirit to those made in programming, engineering and other constructive tasks—they are an inherent part of being a designer of mathematical models.

There is one area about which we remain resolute: Even if there's shorthand or whatnot being defined for clarity, we will always be careful to make sure that there is an underlying mathematical framework that is solid. Remember the premise that we are trying to make a good/correct mathematical model.

One final trade-off that needs mentioning is between design effort up front vs. ease of use later on. The goal of writing down a complete model before simulating it of course leans heavily towards investing effort in the design, in order to create a better product. But we recognize that there can be diminishing returns from excessive time spent designing "blindly," and that often the best way to design is to experiment in order to gain experience and understanding.

3.4 Borrowing from Programming

We observed earlier (Sec. 3.2) that our approach to mathematical modeling bears similarity to programming: both require complete, explicit descriptions of complex systems. Structured programming, and more recently, object-oriented programming, have been developed to help manage the complexity of computer programs; we can apply much of the their philosophy and many of their techniques to mathematical modeling as well. (Fig. 3.2)

Programming concepts

Ideas we embrace for mathematical modeling:	**Ideas we eschew for mathematical modeling:**
✓ Top-down design	✗ Internal state
✓ Modularity	✗ Procedural definitions
✓ Naming strategies	✗ Machine-readable syntax
✓ Data abstraction	✗ Polymorphism

Figure 3.2: Adopting ideas from programming. Many goals and difficulties that we have for mathematical modeling are similar to those of computer programming. As such, we find that several known principles from computer programming are well worth using for mathematical modeling (Sec. 3.4). Conversely, however, the differences between programming and mathematics compel us to steer clear of some techniques (Sec. 3.5). □

It would be a shame to clutter up mathematics, which has a traditional ethic/aesthetic of austerity, with excessive or unattractive baggage from the world of programming. As such, although borrowing philosophy of programming is probably harmless, when it comes to specific techniques we will try to use a minimal approach, only bringing in methods that are particularly useful. We'd like to keep our mathematics as familiar, readable, and usable as possible, as per Sec. 3.2.4.

3.4.1 Top-Down Design

The top-down design strategy calls for first creating a specification, then creating the high levels of a mechanism that meets the specification, and so on, with the final details of implementation saved for last. For mathematical modeling, the top-down approach mostly enters *before* the design of the mathematical model, i.e., in the specification of goals and physical context for the model, as per the CMP decomposition discussed in Ch. 2.

It is possible, to some extent, to design the mathematical model itself in a top-down manner. The methods of modularity and data abstraction (that will be discussed below) can support specification of the overall structure and relationships within a model, where the details are filled in later.

3.4.2 Modularity

Modularity in programming means that programs are built as a (hierarchical) collection of independent components having well-defined interfaces. It helps us to decompose complex problems into simpler ones that are easier to understand and implement; often, the modular subparts are standardized and kept in libraries. The benefits of modularity apply as well to mathematical modeling.

Mathematics has some modular aspects inherent in it, in that it is based on independent, well-defined axioms and theorems; constructs are defined in terms of more primitive constructs, e.g., a field is defined in terms of sets and a derivative in terms of limits; and often, entire branches of mathematics can be based on earlier branches, e.g., differential geometry is built from linear algebra and calculus ([Millman,Parker77]).

For our purposes, however, modularity in mathematical models will generally coincide with the modular design of physically-based models, as per Sec. 2.5. For example, in Ch. 6–9, we will have a mathematical module for kinematic rigid-body motion, one for dynamic rigid-body motion, one for force mechanisms, and so forth. Each module defines a consistent set of terms and equations for its model, designed to be useful for a variety of applications, rather than for a specific "end-model." The modules can make use of equations and terms from other modules, e.g., dynamic rigid-body motion is built on top of kinematic.

In order for a module to be widely useful, especially in conjunction with other modules, each module must be designed with a care towards not "stepping on the toes" of others. For example, tacit assumptions should be

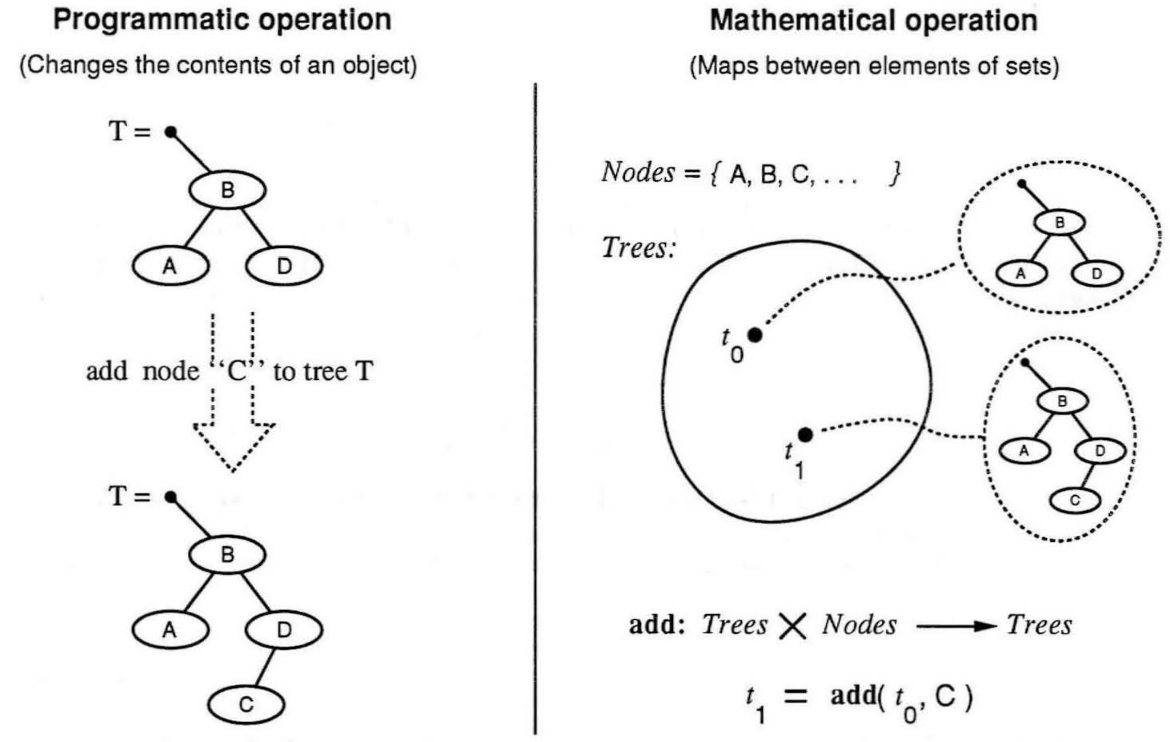

Programmatic operation
(Changes the contents of an object)
T =
B
A
D
add node "C" to tree T
T =
B
A
D
C
Mathematical operation
(Maps between elements of sets)
Nodes = { A, B, C, . . . }
Trees:
t_0
t_1
add: Trees × Nodes → Trees
t_1 = add(t_0, C)

Figure 3.3: No "internal state" in mathematical objects. In most computer programming, it is common for objects to have internal states. For example, adding a node to a binary tree object changes the internal state, but it is still considered the "same" object (where objects are typically identified by their memory addresses). In mathematics, however, we will avoid the notion of internal state: a binary tree is an element of the set *Trees* containing all binary trees, whose nodes are elements of the set *Nodes*. We can define a function **add** that, given any tree in *Trees* and any node in *Nodes*, maps onto some other tree in *Trees*, that has the appropriate nodes in it. Lack of internal state is a characteristic also of functional programming (see Sec. 3.5.1). □

avoided—they may conflict with assumptions in other modules—but instead assumptions should be listed as explicit preconditions for equations, as per (see Sec. 3.2.3). Naming strategies and data abstraction, discussed below, also come into play.

3.4.3 Naming Strategies

In mathematics, as in programming, we assign names to things—the names should be mnemonic and unambiguous. We must be careful to avoid naming conflicts, especially when models are defined modularly and independently.

We will adopt several naming strategies from programming for use in mathematical modeling: full-word names, function-name overloading, and namespace scoping. These will be discussed in Sec. 3.6.

3.4.4 Data Abstraction

Data abstraction in programming is analogous to modularity, but for data elements: a collection of related data elements is bundled together as a single logical entity, often called a *data structure* or *abstract data type* (see [Liskov,Guttag86]). Data abstraction provides *encapsulation*, or *information hiding*, in which an abstract data object can be manipulated as a single entity without reference to details of its contents or implementation, and *hierarchy*, in that an abstract data type may be defined in terms of other abstract data types. These capabilities can greatly help to manage complexity.

Object-oriented programming (OOP) takes data abstraction a step further: An abstract data type (or *class*) is specified along with a well-defined set of the operations that may be performed upon that type of object. Thus the particulars of the elements that make up the data type are implementation details irrelevant to its use, and can be changed without affecting the data type's functionality. This separation of implementation from function is another powerful structural tool, that aids in maintenance and extensibility of programs.

The field of mathematics has some data abstraction inherent in it. As discussed in Sec. 3.4.2, types of mathematical objects are often defined hierarchically. Additionally, mathematical "data types"—such as sets, fields, or vectors—are often defined abstractly, via properties and operations, without having any "innards."

For mathematical modeling, we will emphasize the use of data abstraction, in the form of *abstract spaces*; this will be discussed at length in Sec. 3.7. We will also present, in Sec. 3.9, a mathematical mechanism for defining *state spaces*, that encapsulate collections of elements/operations into mathematical entities.[5]

3.5 Distinctions from Programming

Lest we get carried away with the parallels between mathematical modeling and software design, we examine differences between them. In particular, there are some ideas from programming that we make a point of *not* borrowing; this section lists several features of mathematical modeling that conflict with common programming methodology (Fig. 3.2).

3.5.1 No Internal State

One aspect of programming that we find unsuited to mathematical modeling is the idea of "internal state" of an object. As illustrated in Fig. 3.3, a computer-programming object can have variable internal state—in fact, the concept of objects having internal state, with operations to access and modify that state, is the *sine qua non* of object-oriented programming.

In the realm of mathematics, on the other hand, we prefer objects to have no internal state. An operation is merely a map between elements of the domain and elements of the range; "performing" an operation on a given element of the domain just means finding the corresponding element of the range. For example, if we apply the operation "add one" to the number "3," we get "4"—but "3" isn't modified.[6]

Lacking internal state, activities that would be programmatically defined as state changes, mathematically are defined instead as mappings between elements of (abstract) state spaces, as illustrated in Fig. 3.3. We will discuss abstract spaces in general in Sec. 3.7, and a mechanism for state spaces in particular in Sec. 3.9.

Why not define some mathematical mechanism that supports internal state? Because the lack of internal state is not a drawback but rather a feature: Without internal state, the result of an operation on a given object is single-valued, and we can thus define equations that always apply, without fear of the objects involved being in the wrong state. That is, the lack of internal state is central to the use of globally valid equations as per Sec. 3.2.3.

Finally, note that the idea of "globally valid" objects, whose internal state is never modified does in fact arise in some programming paradigms. For example, for parallel programming, [Chandy,Taylor92] uses *definition variables*, whose values don't change, to insure correctness of otherwise *mutable* variables across parallel composition of program blocks. Another example is functional programming, based on a mathematical function approach in which values are computed but no changes are stored (see, e.g., [Henson87]). Note however that we have not explicitly adopted any techniques or terminology from those realms.

[5] Our use of abstract spaces is similar to the formal/algebraic treatment of abstract data types in computer science (see [Horebeek, Lewi89],[Cleaveland86]); but note that we have opposite goals: the study of abstract data types brings mathematical rigor to computer programming constructs, while our use of abstract spaces brings computer programming principles to mathematical modeling.

[6] One is reminded of the applied-mathematicians' joke: "$2 + 2 = 5$, for large values of 2 and small values of 5." Note also the well-known bug in FORTRAN programs: passing the constant 3 to a routine `addone(x)` might in fact change the value of "3" for the remainder of the program.

3.5.2 No Procedural Definitions

Traditional programming paradigms are *procedural* (or *imperative*): The programmer specifies a sequence of instructions for the computer to carry out. Our mathematical models, on the other hand, contain no posed problems or instructions for solving problems, but merely state equations (Sec. 2.3.2). Thus the mathematical models are *declarative.*

Notice that the lack of internal state (Sec. 3.5.1) goes hand-in-hand with lack of procedural specification: Not having any state, there can be no notion of the "current instruction." Once again, we have a similarity with the functional programming paradigm:

> "One of the great benefits of [functional programming] is that the programs may be considered *declaratively.* It is *not* necessary to attempt a mental execution of the program in order to understand what it 'does'... We wish to adopt a more static approach in which it is more appropriate to ask what a program 'means' rather than 'does.'" [Henson87-p.66]

If we replace "programs" with "equations," and "execution" with "solution," the above argument applies as well to mathematical modeling.[7]

In the real world, and hence in our physically-based models, there are often sequential changes of state. Thus we need to be able to express "when *this* happens, do *that*" in a mathematical model—without a procedural specification. In Sec. 3.10, we will discuss "segmented functions," that address this need.

3.5.3 No Machine-Readable Syntax Necessary

If we're not careful as we wear our "programmer's hats" when approaching the design of mathematical models, we might end up defining mathematical mechanisms more suited for computers than for people. The ergonomic goals of Sec. 3.2.4 must be brought into play.

For one thing, the mathematical syntax shouldn't be so subtle or so intricate that it takes a compiler to figure out the intent. That is, important distinctions shouldn't depend on easily-confused notation (e.g., tiny subscripted marks), a single symbol shouldn't imply an overly complex or non-intuitive series of operations, and so forth.

Additionally, we don't want to define syntax so rigid or cumbersome that it's inconvenient to use. Sometimes, flexibility can be enhanced by allowing several alternative notations; as in

$$y = \sin(x)$$
$$y = \sin x$$

Furthermore, when scribbling equations and derivations, one commonly uses shorthand, elisions, and so forth; it should be possible to do so yet still be able to make use of the corresponding mathematical mechanism. On the other side of this same coin, we don't want to invent syntax that is ungainly enough to encourage its elision.

With flexible syntax comes the potential of ambiguity and lack of rigor. If one uses shorthand, leaves off parenthetical arguments, and so forth, one relies on the reader knowing what is meant rather than what is said. This is another "design decision" as per Sec. 3.3. We generally try to mitigate such ambiguity by appropriate annotation in the accompanying text; Eqn. 9.21, for example, has a note "For clarity, the parameters have been left off $\mathcal{M}_{pq}(Y, t)$..."

3.5.4 No Polymorphism

One of the most powerful ideas of object-oriented programming is *polymorphism*: a given name may refer to an object of any one of a number of related classes, that may respond to a given operation in different ways. This allows a single program to be applied to any of the related classes, even ones that weren't defined when the program was originally written.

[7] The similarity between mathematical modeling and functional programming is unsurprising: functional programming is designed to allow formal, mathematical manipulation and analysis of programs.

We said in Sec. 3.4.4 that we would incorporate the idea of data abstraction—i.e., the definition of classes—into mathematical modeling. Thus we might expect or desire to take advantage of polymorphism as well. Mathematics contains some simple uses of polymorphism: for example the expression $a + b$ has slightly different meanings for real numbers, complex numbers, functions, and so forth.

However, as a rule, polymorphism is not suited to mathematical modeling, because it tends to increases ambiguity. That is, when working with polymorphic objects, their behavior isn't completely well-defined upon simple inspection of an expression, but will depend on what type of objects are eventually used. For mathematical modeling, we have no compiler or run-time support to identify or disambiguate objects—we just look at the equations, so their meanings should be evident without added context, as discussed in Sec. 3.2.3.

This is the end of the Principles *part of chapter 3. The next part,* Techniques, *presents some mathematical mechanisms that support the structured mathematical modeling approach we have been discussing.*

3.6 Naming Strategies

In mathematical equations, letters are used for names of parameters, functions, and so forth: for example, the letters f, k, t, and x in the equation "$x = f_k(y, t)$." But as more and more equations and terms are written, it's easy to run out of letters.[8] To squeeze more life out of the letters, one can take advantage of various cases and fonts, e.g., r vs. R vs. $\mathbf{R}$ vs. $\mathcal{R}$ vs. $\Re$ etc. One can also turn to other alphabets, e.g., ξ, Ψ, $\aleph$, etc. But still, for mathematical models of the complexity we're interested in, running out of letters for names has the potential to be a serious problem. In particular, since we design models as separate modules, we run the risk of having naming conflicts when we combine the modules together. Furthermore, we want names to make sense; with too many letters chosen arbitrarily, it can be hard to remember what's what.

Our concern with mathematical names is analogous to computer programming's concern with variable and function names. We present here three naming strategies from computer programming that can be used as well for mathematical modeling.

[8] This is "another example of a growing problem with mathematical notation: There aren't enough squiggles to go around" [Blinn92-p.88]

3.6.1 Full-word Names

Instead of using only a single letter for mathematical names one can use a whole word, or at least a multi-letter name. This is familiar for many fundamental mathematical functions and operations, such as

$$\begin{gathered}\sin(x) \\ \arccos(x) \\ \lim_{n\to\infty} x = 0\end{gathered}$$

and so forth, but it is rarer to see multi-letter terms in other contexts. ([Misner,Thorne,Wheeler73] is an example of a work that does use a full-word approach, defining, e.g., tensors named Einstein and Riemann.) In addition to allowing us more names than single letters, multi-letter names of course have the advantage of being mnemonic.

As with good programming style, one should generally use longer names for global, important, or infrequently-used terms, and use shorter or single-letter names for local and frequently-used terms. We will use full-word names for most abstract spaces we define (see discussion of abstract spaces in Sec. 3.7), such as Locations in Ch. 6 and States in Ch. 7, and even occasional compound-word names, such as StatePaths in Ch. 7. However, we will maintain standard or historical names where they are familiar, such as m for mass and p for linear momentum in rigid body dynamics, Ch. 8.

With multiple-letter names, one has to be careful that when two names are written next to each other, they don't textually merge into another name. For example, to multiply a term y by a term *cat*, we might write

$$x = y\,cat$$

but the right-hand side might be mistaken for a single term *ycat*, which might or might not exist. [9] Confusion can be mitigated through use of spacing, different fonts, parentheses or other symbols, and so forth. For example:

$$\begin{aligned} x &= y\ \mathit{cat} \\ x &= y\ \mathrm{cat} \\ x &= (y)(\mathit{cat}) \\ x &= y * \mathit{cat} \end{aligned}$$

3.6.2 Scoping, Namespaces

How do we avoid "name collisions" when mathematical models are designed as a collection of separate modules? For example, the kinematic rigid-body model in Ch. 7 and the dynamic rigid-body model in Ch. 8 each need to define an abstract space, and in both cases we would like to use the name "States". Can they both do so? If yes, and we want to mix kinematic and dynamic models, how can we tell which States is being referred to in a given equation? Furthermore, can one model refer to the other's name? We describe here a mechanism inspired by that of Common LISP ([Steele90]).

The basic idea is simple: each model has its own *scope*, or *namespace*; a name x is used in one scope is not the same name as x used in another scope. Thus in each model we are free to use whatever names we want, and by fiat they do not collide with names from other models.

In order to be able to unambiguously use names from different scopes, we give each scope its own *scope name*. For example, the kinematic rigid-body model has scope name KINEMATIC.[10] We can think of the scoping mechanism as follows: When we define terms within a particular model, they are all implicitly "actually" defined with to have *full*, or *scoped, names* that include the model's scope name. We use the following notation:

[9] This is why we like to emphasize full-word names, rather than arbitrary mnemonic abbreviations: "*ycat*" is less likely to be accidentally thought of as a defined term if there are no, or few, other non-word names. Of course, one is never entirely safe: consider $x = n\,ever$ vs. $x = never$. As well, in practice full words can be unwieldy, so shorter, mnemonic names are necessary. See also [Kernighan,Plauger78] for guidelines for mnemonic names.

[10] We will consistently use small caps font for scope names.

Notation. Double-colon "::" for scoped names.

(3.1)

If a name, e.g., "States" is defined within a namespace having a scope name, e.g., "KINEMATIC," its full name is written using a double-colon, as: KINEMATIC :: States

Notn. 3.1: We write a scoped name by prefixing the short name with the name of the scope, with two colons between them. The colons serve to set the names apart, and to emphasize that the name-scoping mechanism is being used. □

When working on a particular model, we generally assume that all names are defined within the scope of that model, and there's no need to write out the full names. If we want to use a name from a different model, however, we can use its full name. The dynamic rigid-body model, for example, uses just "States" to refer to its own space and "KINEMATIC :: States" to refer to the kinematic space.

Sometimes, we can avoid conflicts, and save writing, by just choosing which namespace is the default for a given name. For example, Ch. 9 starts by stating that "Systems" will be assumed to mean "RIGID :: Systems" (rather than "KINEMATIC :: Systems").

Of course, when there is no conflict, there's no need to use full names. For example, the model named COORDS (Ch. 6) defines several terms (Vectors, Locations, $\mathcal{L}ab$, etc.) that are used frequently and widely in other models, and are not defined elsewhere—so we do not use the full names. Still, to aid the reader, it is good to start each model by listing where all such "imported" names are defined (this also avoids future ambiguity if a later model provides a different definition of one of these names).

Given that our goal as per Sec. 3.5.3 is clarity and non-ambiguity for the readers, rather than formal syntax, we are not overly strict in our use of namespaces and scoped names. In particular, we find no need to go the programmatic extremes of Common LISP's mechanism, with its internal vs. external names, exporting and shadowing of names, and so forth.

Finally, note that each model must of course be given a unique scope name, or we are back to our original name conflicts.[11]

3.6.3 Function Name "Overloading"

We generally want any given name to have only one meaning (within a given scope), so as to avoid ambiguity, in keeping with the explicitness goal of Sec. 3.2.3. For example, if we define a function $f(t)$, we don't want to also have a parameter named f as well—especially if we will be in the habit of dropping the parentheses and parameters from functions.[12]

However, sometimes there may be a family of related operators or functions that perform analogous operations, but on different types of parameters. For example, in Ch. 6 we define a family of operators, *Rep*, that yield the coordinate representation of a geometric object; one *Rep* acts on elements of an abstract space Locations to yield an element of $\Re^3$, and a different *Rep* acts on elements of an abstract space Orientations to yield an element of $\Re^{3\times3}$.

To eliminate ambiguity, we would give a different name to each operator. Continuing with the the earlier example, we might name one operator *RepLoc* and another *RepOrient.* However, doing so can be cumbersome, and it can even be confusing, if we tend to think of all operators in the *Rep* family as essentially the same.

Thus, as an exception to our general policy of non-ambiguity, we are willing to allow families of functions to share a single name. We will adopt the term "overloading" from programming, where this idea is familiar (in particular *operator overloading* in programming languages—e.g., defining "+" appropriately for various different argument types such as integers, floating-point numbers, or complex numbers—is well-known

[11] One can imagine introducing hierarchy into namespaces, so that one might have a fully scoped name such as "CALTECH:: KINEMATIC:: Systems;" we have not (yet?) found this to be necessary, however.

[12] In fact, it is not uncommon to see expositions in which a single name is used as a parameter and/or as function and/or as a value of the function. In fluid mechanics (see, e.g., [Lin,Segel74-Ch.13]), x is commonly used as an independent world-space variable as well as a material-to-world coordinate mapping $x(a)$. Although there is some benefit—anything named x is always a world-space location—for our purposes we feel that eliminating ambiguity is a strong enough concern that this practice should be discouraged.

([Booch91])). We can give a formal definition of an overloaded mathematical function:

Definition. **Overloaded functions**

(3.2) *An* **overloaded** *function is a function whose domain is the union of the domains of a family of functions (those domains must be disparate), whose range is the union of the ranges of those functions, and whose value for any argument equals that of the member of the family whose domain contains that argument.*

Defn. 3.2: An overloaded function is a single function that "chooses" the appropriate function from a family, based on its argument type. See example, Eqn. 3.3 □

And an example:

Given three functions: fa: $\mathsf{A} \rightarrow \mathsf{X}$*, fb:* $\mathsf{B} \rightarrow \mathsf{Y}$*, and fc:* $\mathsf{C} \rightarrow \mathsf{Z}$*, we can define an overloaded function* $f\colon \mathsf{A} \cup \mathsf{B} \cup \mathsf{C} \rightarrow \mathsf{X} \cup \mathsf{Y} \cup \mathsf{Z}$ *by:*

(3.3)
$$f(x) = \begin{cases} \mathit{fa}(x), & \mathit{if}\ x \in \mathsf{A} \\ \mathit{fb}(x), & \mathit{if}\ x \in \mathsf{B} \\ \mathit{fc}(x), & \mathit{if}\ x \in \mathsf{C} \end{cases}$$

Eqn. 3.3: Example of an overloaded function. Note that the individual functions don't have to act on only a single parameter or even the same number of parameters. For example, if *fb* were to take two real-number arguments, its domain would be $\mathsf{B} = \Re \times \Re$; we can consider $f(x, y)$ for real x and y to be acting on the pair $(x, y) \in \mathsf{B}$ thus $f(x, y) = \mathit{fb}(x, y)$; but for $z \in \mathsf{A}$, we would have $f(z) = \mathit{fa}(z)$. □

In practice, we define only the overloaded function name, and leave implicit the names of the individual functions in the family (for example, see the definition of *Rep*, Defn. 6.5). We have not found it necessary to invent a formal mechanism for naming and referring to the individual functions; as long as we are aware of the underlying unambiguous mechanism, it is sufficient define the overloaded name and leave it at that.

It is important to remember that overloading is a source of ambiguity, and should be used sparingly. One should be careful to define overloaded functions only when it will be obvious which underlying function is implied—that is, we must know to which domain the arguments belong. For example, the function f of Eqn. 3.3 is used unambiguously in the expression:

$$a + 1 = f(a), \mathit{where}\ a \in \mathsf{A}$$

Note that if we expect to occasionally leave off function arguments in our expressions, the overloaded functions will be ambiguous. Note also that we do not overload based on the range spaces of the individual functions; that could too easily lead to unresolvable ambiguity.

Function overloading is commonly used in the mathematical world, without the fanfare that we have put forth. We prefer, however, that some care be put into the decision of whether to overload a particular function; this is a "design decision," as per Sec. 3.3, where the model designer tries to balance clarity and economy of expression against ambiguous meaning. To help eliminate ambiguity, whenever an overloaded function is defined, the reader should be warned in accompanying text "caution: overloading being done here."

The mathematical function-overloading mechanism we have defined is inspired by the programmatic function-overloading mechanism of the C++ language ([Ellis,Stroustrup90]); in particular we adopt C++'s choice to overload functions based on argument type, but not on return type, and the implicit creation of individual functions with uniques names for each domain.

3.7 Abstract Spaces

This section describes a basic element of our approach to mathematical modeling: Prolific definition of abstract spaces. This leads to the mathematical equivalent of data abstraction, encapsulation, hierarchy, etc., familiar from programming, and is thus a powerful tool for the management of complexity.

3.7.1 Defining Abstract Spaces

A primary motivation for the definition of abstract spaces is to have unambiguous meaning. That is, when we define a new abstract space, we are free to assign to it exactly those operators and properties that we need, and no others.[13] By creating abstract spaces tailored for particular uses, and by stating explicitly what their characteristics are, we attempt to eliminate (or at least mitigate) hidden assumptions or effects.

How does one go about defining an abstract space? At its most basic level, it's easy: One merely says, for example,

> *Let* Things *be an abstract space.*[14]

One would then go on from there, and describe various properties of and operations on (elements of) the space. This approach is used for the various geometric objects in Ch. 6.

Often, we can take advantage of existing standard types of abstract spaces. For example, we define a new abstract space to be a vector space, or a Banach algebra. Later in this chapter we will define some types of spaces that are particularly useful for our structured modeling purposes: "indexes" in Sec. 3.8, and "state spaces" in Sec. 3.9. By using these types of spaces we will ultimately be able to, e.g., declare mathematically merely that "x is a rigid-body system" to describe well-defined behavior, properties, etc. for a named collection of rigid bodies that move dynamically.

It may occasionally happen that we will define some new space, and it turns out to be (provably) equivalent to some other, well-known space. For example, in Sec. 3.9.2 we will discuss a space describing rectangles; the space defined there is in fact equivalent to $\Re^2$. If such an equivalence does occur. . . great! We can apply whatever properties are known about that existing space to our own space; we've gained an extra body of knowledge about our space "for free."

One might argue that if it is the case that a new space is equivalent to an existing, well-known space, making our own definition merely confuses things. There is a twofold response:

- With a good name for the space, one can convey a desired connotation: Saying "x is in the space of rectangles" has more intuitive meaning than "x is in $\Re^2$."
- If the model gets refined later on, one can change the properties of the new space, and not need to change definitions of other terms. For example, we may later decide to consider orientation in the plane as a property of a rectangle. The expression "x is in the space of rectangles" could still stand, but otherwise we would have to change it to "x is in $\Re^2 \times SO(1)$."

However, there is of course merit to the original argument; too many gratuitous definitions can be obfuscatory. Whether to define a new space is often an important "design decision," as discussed in Sec. 3.3.

Pedantic interlude

"Abstract space" can mean the universal set of a particular set theory ([Nihon Sugakkai77]); in such usage there can only be a single abstract space under discussion. Our use of "abstract space" is closer to the definition in [James,James76]: "A formal mathematical system consisting of undefined objects and axioms of a geometric nature." However, although most of our applications will in fact be geometric, we don't emphasize the geometric nature. Our working definition is thus "a set of undefined objects"; where we invoke naive set theory and the axiom of comprehension ([Nihon Sugakkai77]) to be able to define such sets by fiat, and the axioms of pairing, power set, union, and so forth to manipulate the sets.

[13] " 'When *I* use a word,' Humpty Dumpty said,. . . 'it means just what I choose it to mean—neither more nor less.' " In [Carroll60], Martin Gardner discusses Carroll's mathematical philosophy that lies behind "Humpty Dumpty's whimsical discourse on semantics."

[14] We will consistently use sans-serif font for abstract spaces. Additionally, we will give each space we define a plural name; this makes it easy to read, e.g., "$t \in$ Things" as "t is an element of the set of Things," i.e. "t is a Thing."

In our use of abstract spaces, we are taking a decidedly non-constructivist approach; constructivist dogma says: "A set is not an entity which has an ideal existence: a set exists only when... we prescribe... what we must do in order to construct an element of the set... " [Bishop,Bridges85-p.5]. Our motivation for defining abstract spaces is, as discussed above, to have precise statements of the properties of mathematical objects we will manipulate; in order to encourage such statements, we leave ourselves free to state them without the attendant formal constructivist mechanism. Since the problems we will be dealing with focus on real-world applications, we are less likely to follow chains of inquiry that lead to the sorts of mathematical monsters feared by constructivists. Ultimately, when we implement our models on computer, we will necessarily be constructivist: a computer program always prescribes how its objects are constructed. But we have admittedly left a possible gap in our methodology: We might in principle define models for which constructive formulations can't be created, i.e., that we can not directly implement on computers. It is intriguing to consider following strict constructivist doctrine in creating mathematical models, leading perhaps to guarantees of computability and so forth. We have not pursued this line of inquiry very far.

Finally, note that we are perhaps a bit optimistic in our notion that precise statements of properties will lead to "unambiguous meaning." As discussed by [Lakatos76], one always assumes a familiarity with the terms and concepts used in the definitions; to whatever extent those concepts are fluid, ambiguities can arise. Still, in practice, reasonable attempts at precision of meaning can help one go a long way—in particular, a lot farther than one would go without *any* attempts to be precise.[15]

3.7.2 Specializations

Occasionally, we will define (or will be given) an abstract space, but will wish to focus on a subset of it that is particularly interesting, defining operators, properties and so forth that apply only to the subset but not the the space as a whole. To formalize this notion, we introduce the idea of a "specialization" (Fig. 3.4):

Definition. **Specialization**

(3.4) *A* **specialization** *of a given abstract space (called the* **general space***) is another abstract space, that is in one-to-one correspondence with some subset of the general space.*

Defn. 3.4: Since the elements of a specialization correspond one-to-one to those of a subset of the general space, properties and operators of the specialization's elements can be thought of as implicitly applying to those of the subset. □

and some attendant notation:

Notation. Subset-like Symbol "$\sqsubset$" for Specializations

(3.5) *If we are given, e.g., spaces named* Specs *and* Gens*, and an injection (one-to-one function)* f: Specs $\rightarrow$ Gens*, then we denote the specialization relationship using the symbol "*$\sqsubset$*", as:*

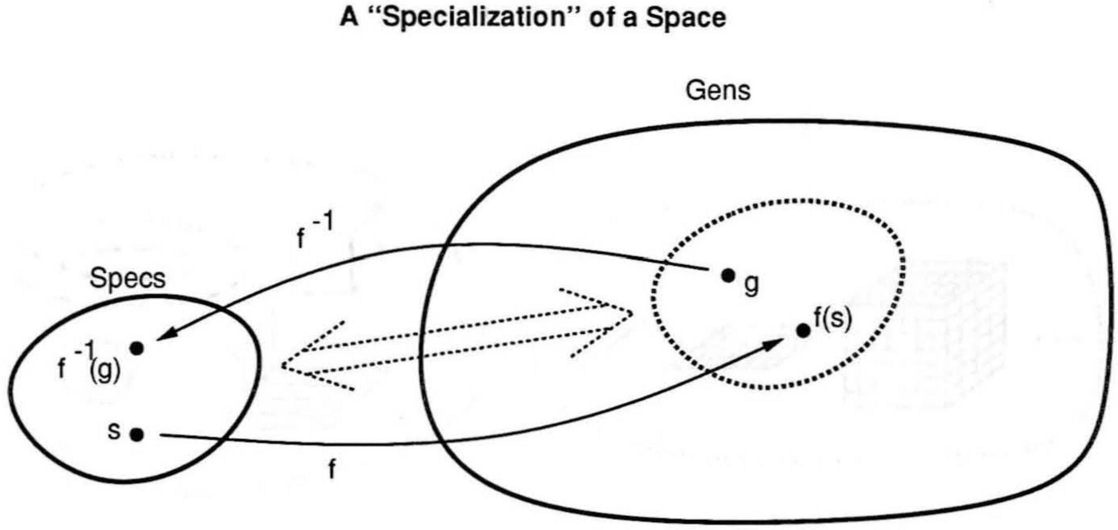
$$\mathsf{Specs} \overset{f}{\sqsubset} \mathsf{Gens} \quad \equiv \quad \text{“Specs is a specialization of Gens”}$$

$$f(\mathsf{Specs}) \quad \equiv \quad \textit{The subset of } \mathsf{Gens} \textit{ corresponding with } \mathsf{Specs}$$

Notn. 3.5: Notation for specializations. If we do not need to name the injection, we will write simply Specs $\sqsubset$ Gens. □

For an element of a specialization, $s \in \mathsf{Specs} \overset{f}{\sqsubset} \mathsf{Gens}$, the corresponding element of the general space is given by $f(s)$. Conversely, for an element $g \in f(\mathsf{Specs}) \subseteq \mathsf{Gens}$, the corresponding element of the specialization is given by $f^{-1}(g)$. Note that $f^{-1}(g)$ is not defined for all general elements $g \in \mathsf{Gens}$, only for those elements that actually correspond with elements of Specs.

[15] We can claim some kinship with Lakatos, however, in the domain of formalism vs. methodology ([Lakatos76-pp.1–5]). Our approach towards mathematical complexity is not geared towards a formal analysis of complexity, but rather methods or heuristics to manage it. For discussion of the complexity of numerical computation, we refer the reader to [Traub,Wasilkowski,Woźniakowski88].



Specs is a specialization of Gens:

$$\text{Specs} \overset{f}{\sqsubset} \text{Gens}$$

Figure 3.4: A specialization of an abstract space. Space Specs corresponds one-to-one with a subset of space Gens ; the function f: Specs $\rightarrow$ Gens is an injection (one-to-one function) that determines the subset and the correspondence. □

Since every element of a specialization has exactly one corresponding general element, we commonly (or implicitly) overload all operators on General spaces to act on the specializations:

(3.6) *Given a specialization* $\text{Specs} \overset{f}{\sqsubset} \text{Gens}$, *and an operator* $\text{OP}: \text{Gens} \rightarrow \text{A}$, *we overload* $\text{OP}: \text{Specs} \rightarrow \text{A}$:

$$\text{OP}(s) = \text{OP}(f(s)), \textit{ for all } s \in \text{Specs}$$

Eqn. 3.6: All general-space operators are overloaded onto the specialization. Thus an element of a specialization can act as the corresponding element of the general space. □

In Sec. 3.9.5 we will give a mechanism for specifying new operations and properties, just for the specialization.

Our notion of a specialization mechanism is of course patterned after class derivation in object-oriented programming. In principle, we might consider overloading operators so that OP (s) would be different from $\text{OP}(f(s))$, thus achieving a mathematical analog of polymorphism. However, as discussed in Sec. 3.5.4, we think that would introduce excessive ambiguity—our idea behind specializations is to assign extra properties to interesting subsets of a spaces, not to change any existing properties. As an aside, C++ programmers might note that evaluating $f(s)$ is analogous to performing "upcast" from a derived class to a base class, while $f^{-1}(g)$ is analogous to a "downcast" and is only "type-safe" when $g \in f(\text{Specs})$ [Ellis,Stroustrup90].

3.7.3 Disparate Unions

Generally, when one thinks of a particular space, even an abstract one, the fact that all its elements belong to that space leads one to think of all those elements as being essentially similar in some conceptual way. While

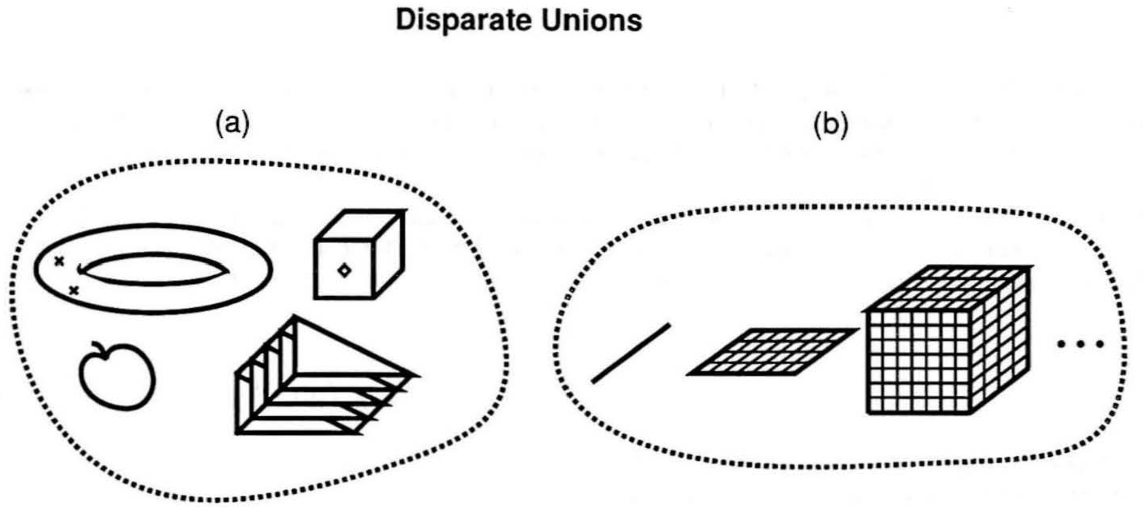


Figure 3.5: Disparate unions. We will occasionally define abstract spaces whose elements may be of various dissimilar types—they needn't even have the same dimensionality or degrees of freedom. Two elements that are of the same type said to be "agnates." (a) A disparate union including points on the surface of a torus, points within the body of a cube, and others. The points marked × are agnates (both on the torus), but neither are agnates of the point marked ◇ (in the cube). (b) A disparate union whose elements have some similarity—all are points from the unit cubes in various dimensions—but that nonetheless have distinct dimensionality. For this space, arithmetic operations are defined between agnates, and we can always construct smooth paths between agnates; but we do not have arithmetic or continuous paths between points that are not agnates. □

this is tautologically true, it will also occasionally be the case that there is a marked *dis*similarity between objects that we have chosen to group together into a single set—in particular, we may group together objects that have different dimensionalities or degrees of freedom (see Fig. 3.5). Thus we define:

Definition. **Disparate Union**

(3.7) *A* **disparate union** *is an abstract space that can be partitioned into a collection of disparate sets; each of those sets is called a* **disparate component** *of the union. Two elements of a disparate union are* **agnates** *if they belong to the same disparate component.*

Defn. 3.7: A disparate union is a collection of different types of elements. Thus if A is an abstract set, two elements $x, y \in$ A might have no or few common operators or properties—they may be "like apples and oranges." Disparate unions, and paths through disparate unions, will be useful for models of things whose state and makeup can vary qualitatively; see the discussion of segmented functions, Sec. 3.10 □

A disparate union may be explicitly defined as the union of various different spaces, for example, Defn. 6.3 defines a space GeomObjs whose elements include locations, scalars, vectors, and others. Or, a space defined in some other manner may nonetheless be a disparate union, for example the set of all "indexes" of a given space, as will be discussed in Sec. 3.8.2. Trivially, any homogeneous space is a disparate union, comprised of a single disparate component; thus all elements of a homogeneous space are agnates.

We will assume that given an element of a disparate union, we can always know to which disparate component it belongs, either by inspection or by construction—i.e., given any two elements, we can determine if they are agnates. C programmers may notice a similarity with the `union` construct: a variable of such a type may in fact be of any of the component types, and it is up to the programmer to know which component type corresponds to any particular value of the union ([Kernighan,Ritchie88]).

More pedantry

We are not familiar with existing mathematical mechanisms that emphasize the manipulation of sets containing elements with varying dimensionality. A related concept is that of a directed systems of sets (see [Spanier66]); that idea, however focuses on ordering and maps between the component sets, particularly in the definition of the direct limit, which defines an equivalence between elements of each set. We are interested, however, in the fact that the elements of the component sets *differ*, not in how they can be equated.

For those who are made uneasy by our assumption that we can always determine from which component set an element originates, we can fall back on [Spanier66]'s definition of *disjoint union* (or *set sum*), in which each element of the union is a pair containing both an element of a component set, and essentially a tag that indicates which component set.

3.8 Identifiers (ID's) and Indexes

Often a model will need to handle a collection of objects, such as a collection of rigid bodies. This section presents a mechanism that allows us to define a collection elements, and manipulate it as a single entity, yet also be able to mathematically "access" individual elements within it. We start, in Sec. 3.8.1, by defining the *identifiers (ID's)* that will be used to name objects. Sec. 3.8.2 defines and gives notation for *indexes*, which are collections of named objects. Sec. 3.8.3 discusses various operations on indexes.

The mechanism described here is used extensively in the models in Ch. 6–9; see, e.g., the index Systems in Defn. 8.14, and its use in Eqn. 8.32.

3.8.1 Definition of ID's

The set of identifiers is trivial:

Definition. IDs

$$\boxed{\mathsf{IDs} \equiv \left(\begin{array}{c} \textit{a discrete space, with an equivalence relation} \\ \textit{to distinguish between the elements} \end{array} \right)} \tag{3.8}$$

Defn. 3.8: A space of identifiers (ID's). We will use the elements of IDs to label various things. We can think of each ID as a name, like "hansel" or "grendel" or "obj193b." □

Having an equivalence relation simply means that we can tell ID's apart: Given $a, b \in \mathsf{IDs}$, we can tell if $a = b$ or if $a \neq b$. Note that we have no other relations defined for IDs. In particular, we do not define an ordering.[16]

We will use ID's from the one space IDs to name many different types of things. One might be worried about confusion—given a particular ID, what type of thing is it naming? One could partition IDs into disjoint subsets, where each subset corresponds to a different type of thing being named. In practice, we have not found this to be necessary. Note also that IDs is infinite; we assume that we can always grab a new ID if we need one.

We will occasionally use sets of ID's. The following definition is convenient:

Definition. IDsets

$$\boxed{\mathsf{IDsets} \equiv \textit{the space } \left\{ \textit{sets of } \mathsf{IDs} \right\}} \tag{3.9}$$

Defn. 3.9: Sets of identifiers. If $s \in \mathsf{IDsets}$, then s is a set of ID's; we can speak of the size of the set ($||s||$) and whether a given identifier $i \in \mathsf{IDs}$ is in the set ($i \in s$) or not ($i \notin s$). □

[16] Why go to all this effort? Why not just label objects with integers? The reason is that we don't want to always have an implied ordering between labeled objects, if there is no natural ordering. A particular collection of objects may need an ordering (or several orderings); in such a case the orderings would be defined explicitly.

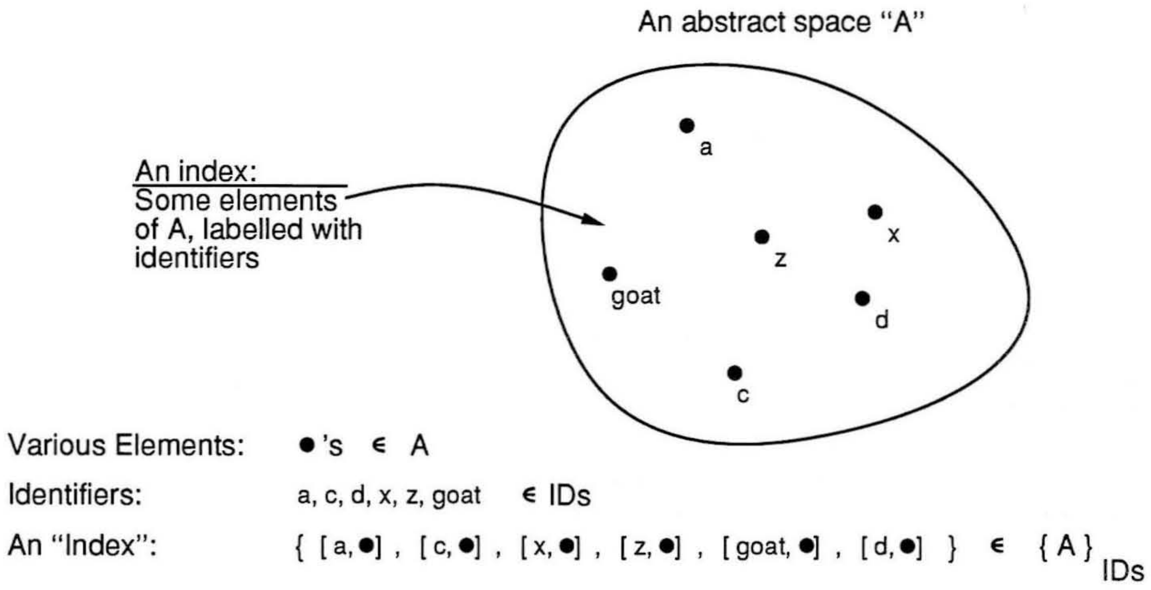

An abstract space "A"
An index:
Some elements of A, labelled with identifiers
a
x
z
goat
d
c
Various Elements: ●'s ∈ A
Identifiers: a, c, d, x, z, goat ∈ IDs
An "Index": { [a,●] , [c,●] , [x,●] , [z,●] , [goat,●] , [d,●] } ∈ {A}IDs

Figure 3.6: An index is a collection of elements of some space, each labeled by an identifier; given an index, we can look up elements based on their identifiers. In this illustration, the •'s are various elements of a space A. It is allowable for two of the •'s to be equal—two different identifiers may label the same element—but no single identifier can be used to label two different elements. □

3.8.2 Indexes

We define an index[17] to be a collection of elements from some space, with each element labeled by a unique ID: (Fig. 3.6)

Definition. **Indexes**

(3.10)

An **index** *of elements of a space* A *is a set of pairs* $[i \in \mathsf{IDs}, a \in \mathsf{A}]$ *such that no two pairs share an ID. That is, if T is an index*

$$\textit{For any two pairs } [i,a],\ [j,b] \in T,\quad i = j \implies a = b$$

Defn. 3.10: An index is a collection of paired ID's and values. A given ID can only occur once within the collection; thus each ID uniquely labels its corresponding value. However, any given value may occur several times, labeled by different ID's, i.e., $a = b \not\Rightarrow i = j$. □

We use the following notation for indexes:

[17] The word "index" is sometimes used to mean a pointer or indicator that selects a single element from a collection; for instance, in the C programming language expression `a[i]`, the variable `i` is called an "array index." Our use of "index" is, instead, in the sense of a table; e.g., the index in the back of a book.

Notation. Assorted Notation for Indexes

(3.11)
$$
\begin{array}{rcl}
\{\mathsf{A}\}_{\mathsf{IDs}} & \equiv & \textit{The set of all indexes of elements of space } \mathsf{A} \\
T \in \{\mathsf{A}\}_{\mathsf{IDs}} & \equiv & \text{“}T \text{ is an index of elements of space } \mathsf{A}\text{”} \\
\mathit{Ids}(T) & \equiv & \textit{The set of ID's used as labels in } T\textit{, i.e.} \\
 & & \left\{ i \in \mathsf{IDs} \;\middle|\; \exists a \in \mathsf{A} \textit{ such that } [i,a] \in T \right\} \\
\mathit{Elts}(T) & \equiv & \textit{The set of elements of } \mathsf{A} \textit{ that are in } T\textit{, i.e.} \\
 & & \left\{ a \in \mathsf{A} \;\middle|\; \exists i \in \mathsf{IDs} \textit{ such that } [i,a] \in T \right\} \\
T_i & \equiv & \textit{The element of } T \textit{ labeled by } i \in \mathit{Ids}(T)\textit{, i.e.} \\
 & & a \in \mathsf{A} \textit{ such that } [i,a] \in T
\end{array}
$$

Notn. 3.11: Notation for indexes. A can be any space. T without subscripts refers to the index as a whole; T with a subscript ID, e.g., T_i, refers to a single element. Note that if an ID is not used as a label in T, that is if $i \notin \mathit{Ids}(T)$, then the subscript notation T_i is invalid. □

Sometimes it is convenient to be able to define an index for which we can be careless, and look up an element by ID, without worrying about whether the ID is actually used in the index—if it isn't, we will accept 0 as its element. We call these "indexes with 0," and extend the notation:

Notation. Indexes with 0

(3.12)
$$
\begin{array}{rcl}
 & & \textit{If space } \mathsf{A} \textit{ has a zero element, } 0\textit{:} \\
T \in \{\mathsf{A}\}^{\circ}_{\mathsf{IDs}} & \equiv & \text{“}T \text{ is an index with zero”} \\
T_i & \equiv & \begin{cases} a \in \mathsf{A} \textit{ such that } [i,a] \in T, & \textit{if } i \in \mathit{Ids}(T) \\ 0, & \textit{if } i \notin \mathit{Ids}(T) \end{cases}
\end{array}
$$

Notn. 3.12: Indexes with 0. If T is defined to be an index with zero, then the subscript notation is not restricted to ID's that are used as labels in T; for ID's that do not label any element, 0 is used instead. $\mathit{Ids}(T)$ and $\mathit{Elts}(T)$ are the same as in Notn. 3.11. □

The remaining discussion of indexes will apply also to indexes with 0's.

Note that two indexes $S, T \in \{\mathsf{A}\}_{\mathsf{IDs}}$ do not necessarily have the same number of entries, or, they may have the same number of entries but may use different ID's. Thus the set $\{\mathsf{A}\}_{\mathsf{IDs}}$ is a disparate union as per Sec. 3.7.3:

(3.13)
$$
\textit{Two indexes } S, T \in \{\mathsf{A}\}_{\mathsf{IDs}} \textit{ are agnates if}
$$
$$
\mathit{Ids}(S) = \mathit{Ids}(T)
$$

Eqn. 3.13: The set of all indexes of A can be partitioned into disparate components so that the indexes in each component have the same ID's. See Defn. 3.7. □

Eqn. 3.35 will illustrate the meaning of *continuity* for index-valued functions.

There is a standard mathematical mechanism similar to our indexes: A mapping from a set Λ to a set A can be called a *family of elements of* A *indexed by* Λ ([Nihon Sugakkai77]). The mapping is commonly written as $\{a_\lambda\}_{\lambda\in\Lambda}$ or just $\{a_\lambda\}$, and Λ is called the *indexing set*. For our purposes, we express the mapping as a set of pairs in order to emphasize its nature as a manipulable mathematical object. Further, we limit our indexes to use IDs as the "indexing set"—but any given index need only be defined over a subset of IDs. LISP programmers will find our indexes to be reminiscent of key-value *association lists* ([Steele90]).

3.8.3 Operations on Indexes

The usual set operations can be applied to indexes, except that we can only perform a union on indexes that don't have any conflicting entries. Several equalities hold:

Given any two indexes $S, T \in \{\mathsf{A}\}_{\mathsf{IDs}}$ of some space A

(3.14)
$$\begin{aligned}
\mathit{Ids}(S \cap T) &= \left\{ i \in \mathit{Ids}(S) \cap \mathit{Ids}(T) \;\middle|\; S_i = T_i \right\} \\
(S \cap T)_i &= S_i,\ \textit{for } i \in \mathit{Ids}(S \cap T) \\
(S \cap T)_i &= T_i,\ \textit{for } i \in \mathit{Ids}(S \cap T) \\
\mathit{Ids}(S - T) &= \mathit{Ids}(S) - \mathit{Ids}(S \cap T) \\
(S - T)_i &= S_i,\ \textit{for } i \in \mathit{Ids}(S - T)
\end{aligned}$$

If $S_i = T_i$ for all $i \in (\mathit{Ids}(S) \cap \mathit{Ids}(T))$,

$$\begin{aligned}
\mathit{Ids}(S \cup T) &= \mathit{Ids}(S) \cup \mathit{Ids}(T) \\
(S \cup T)_i &= \begin{cases} S_i, & \textit{if } i \in \mathit{Ids}(S) \\ T_i, & \textit{if } i \in \mathit{Ids}(T) \end{cases}
\end{aligned}$$

Eqn. 3.14: The intersection of two indexes includes only elements having both the same label and the same value in each index. To subtract from an index, we remove all elements that are in the intersection with the subtrahend. We can only perform a union on two indexes if any ID's that they share are used to label the same value in both. □

We define notation to add and delete elements of an index:[18]

Notation. Index $+$ and $-$

For index $S \in \{\mathsf{A}\}_{\mathsf{IDs}}$ of some space A, *$i \in$* IDs, *$x \in$* A, *and $I \in$* IDsets

(3.15)
$$\begin{aligned}
S + [i, x] &\equiv S \cup \{[i, x]\}, & i &\notin \mathit{Ids}(S) \\
S - i &\equiv S - \{[i, S_i]\}, & i &\in \mathit{Ids}(S) \\
S - I &\equiv S - \left\{ [i, S_i] \;\middle|\; i \in I \right\}, & I &\subseteq \mathit{Ids}(S)
\end{aligned}$$

Notn. 3.15: Adding and deleting entries from an index. Note that to delete an element, we need only specify the ID. We define "subtraction" for an individual ID or for a set of ID's. □

Often we will be given an index of elements of A, and an operator that acts on individual elements of A; we will want to apply the operator to the entire index at once, constructing the index of results.[19] Thus, formally:

Definition. Operators on indexes

Given an operator $\mathrm{OP}: \mathsf{A} \to \mathsf{B}$ *for some spaces* A *and* B, *we overload the operator* $\mathrm{OP}: \{\mathsf{A}\}_{\mathsf{IDs}} \to \{\mathsf{B}\}_{\mathsf{IDs}}$ *as follows: Let $a \in \{\mathsf{A}\}_{\mathsf{IDs}}$, $b \in \{\mathsf{B}\}_{\mathsf{IDs}}$ be indexes, then*

(3.16)
$$\mathrm{OP}(a) = b \iff \begin{cases} b_i = \mathrm{OP}(a_i), \ \textit{all } i \in \mathit{Ids}(a) \\ \mathit{Ids}(b) = \mathit{Ids}(a) \end{cases}$$

Defn. 3.16: If we apply an operator to an index of elements, we get the index of results of applying the operator to each element. Note that we are implicitly overloading every operator on a space to act on indexes of that space. □

If we have an index of functions,[20] we can always define a corresponding function that returns an index:

[18] We are using sloppy language here; as per Sec. 3.5.1, we can't "change" an index by adding or deleting elements. More precisely, these are operations that describe a new index given the operands.

[19] In LISP parlance, we are performing a "map" of the operator over the index ([Steele90]).

[20] "Index of functions"— is that kosher? Sure, if one considers each function to be an element of a space of functions. We will actually use this frequently in Ch. 6–9.

Definition. **Implied function** of an index of functions

(3.17) *Given an index F whose elements are functions into some space* A, *i.e.,* $F \in \{\textit{functions}: \Re \to \mathsf{A}\}_{\mathsf{IDs}}$, *the* **implied function** *is the unique index-valued function* $D: \Re \to \{\mathsf{A}\}_{\mathsf{IDs}}$ *such that:*

$$\begin{aligned} \mathit{Ids}(D(x)) &= \mathit{Ids}(F) && \textit{independent of } x \\ D(x)_i &= F_i(x) && \textit{for all } i \in \mathit{Ids}(F),\ x \in \Re \end{aligned}$$

Defn. 3.17: A implied by an index of functions. Given an index F, each of whose elements is a function, there is a unique implied function whose value is the index of results of evaluating each element of F. The domain of the functions doesn't have to be $\Re$, but can be any space. □

Thus we can "evaluate" an index of functions as if it were a function:

Notation. Evaluating an implied function using parentheses "( )"

(3.18) *Given an index of functions into some space* A, *e.g.,* $F \in \{\textit{functions}: \Re \to \mathsf{A}\}_{\mathsf{IDs}}$, *parentheses implicitly refer to F's implied function:*

$$F(x) \equiv D(x), \textit{where } D: \Re \to \{\mathsf{A}\}_{\mathsf{IDs}} \textit{ is implied by } F$$

Notn. 3.18: We can treat an index of functions as if it were the corresponding implied function of indexes. This notation can of course be used for functions other than of reals or of more than one variable; we would write, e.g., $F(x, y, z)$. □

3.9 State Spaces

Modeling often involves the manipulation of the *state* of an object: its location, configuration, momentum, and the like, as appropriate—all that describes its condition of existence. To accomplish this, it is often useful to define an abstract space such that a single abstract element corresponds to a unique and complete depiction of the state. This gives us "encapsulation" of information, as discussed in Sec. 3.4.4—we can manipulate a single entity, rather than having to handle the collection of separate parameter values that make up the state.

Since we will frequently want to define and use abstract "state spaces," this section provides a mechanism—definitions and notation—to aid in the process. Readers with a background in computer programming will observe that the state space mechanism we describe here is reminiscent of data structures, or, more strongly, of object oriented programming. As discussed in Sec. 3.4, we have intentionally borrowed the object-and-operator paradigm of object oriented programming for use in mathematical modeling.

3.9.1 Basic Definition & Notation

We want to use abstract spaces whose elements correspond with, or encapsulate, the states of objects. Such spaces will need operators to perform the mapping between the abstract elements and the various parameters and other aspects of the corresponding state. Thus we define:

Definition. **State space, aspect operators**

(3.19) *A* **state space** *is an abstract space that has an associated collection of named operators into other spaces. These operators are called* **aspect operators** *of the state space.*

Defn. 3.19: An element of a state space can be thought of as representing a collection of mathematical values of various types. The "aspect" operators tell us what those values are for any given element. □

An example of a state space might be:

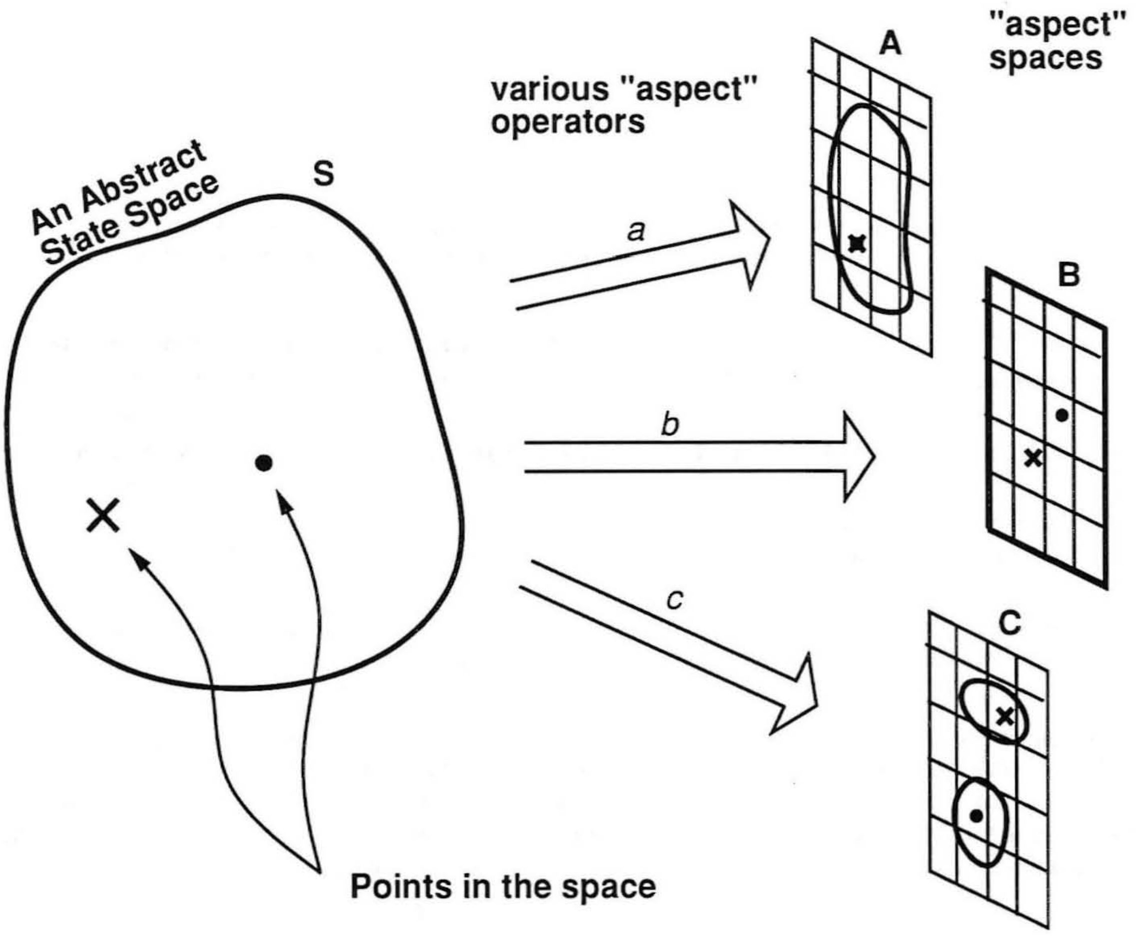


Figure 3.7: A state space is an abstract space, with a collection of named "aspect" operators that project into various aspect spaces; here, we illustrate Eqn. 3.20. A point in the state space is determined uniquely by its entire collection of aspect values. Note that any given aspect operator may be many-to-one, and the projection of the state space may be a subset of the aspect space. Additionally, aspect values may be non-independent, e.g., for all points in $s \in \mathsf{S}$, the values of the aspects might always satisfy $a(s) = 1 + 2b(s)$. □

$$\begin{array}{l} \textit{an abstract space } \mathsf{S} \\ \textit{operator } a\colon \mathsf{S} \rightarrow \mathsf{A} \\ \textit{operator } b\colon \mathsf{S} \rightarrow \mathsf{B} \\ \textit{operator } c\colon \mathsf{S} \rightarrow \mathsf{C} \end{array} \tag{3.20}$$

Eqn. 3.20: A sample state space S, having three aspect operators, a, b, and c. (Fig. 3.7) □

We impose the restriction that an element of a state space can be identified by its complete collection of aspect values; all "measurable" or "distinguishing" properties of an element of a state space must have aspect operators defined for them. Formally:

(3.21)
$$\textit{Given a state space } \mathsf{S} \textit{ with aspects } a, b, \ldots, \textit{ and } x, y \in \mathsf{S}$$
$$x = y \iff \begin{cases} a(x) = a(y) \\ b(x) = b(y) \\ \vdots \end{cases}$$

Eqn. 3.21: An element in a state space is uniquely identified by its complete collection of aspect values. There will never be two different elements that "look alike", i.e., have exactly the same aspect values. □

It is convenient to standardize some terminology:

state space An abstract space, as per Defn. 3.19.

aspect operator An operator on elements of a state space. (In Eqn. 3.20, a, b, and c are aspect operators of S.)

aspect space The target space of an aspect operator. (In Eqn. 3.20, A, B, and C are aspect spaces.)

aspect value The result of applying an aspect operator to a particular element of a state space. (In Eqn. 3.20, if $s \in \mathsf{S}$ is an element of the state space S, $a(s)$, $b(s)$ and $c(s)$ are its aspect values.)

point in a space We use "point in a space" interchangeably with "element of a space". However, "point" tends to have a connotation that the space is continuous, while "element" has a connotation that the space is discrete.

For a more tangible example of a state space, suppose we wish to have a space of geometric arcs, Arcs. Every element in the space Arcs corresponds to a particular geometric arc. We use operators on the space Arcs to examine the properties of the arcs; each operator specifies a geometric property of the arc that corresponds with a given element of Arcs. Thus, we might have operators to tell us the radius and subtended angle of each arc. Rather than list the space and name the operators in long form as in Eqn. 3.20, we use a shorthand:

Notation. Brackets "[ ]" and Arrows "↦" for a State Space

(3.22)

Arcs [*r* ↦ $\Re$ *angle* ↦ $\Re$]	≡	*an abstract space* Arcs *operator* r: Arcs → $\Re$ *operator* *angle*: Arcs → $\Re$

Notn. 3.22: The notation on the left is shorthand for the collection of statements on the right, to define a state space. Note that if we define several state spaces that share aspect names, we are implicitly overloading the aspect operators (see Sec. 3.6.3). □

Generally, we think of and visualize elements of state spaces as points or dots in some nebulous region, as illustrated in Fig. 3.7. However, it is also possible to think of concrete values: by Eqn. 3.21, each element of a state space S has a unique tuple of aspect values. Thus a state space S can be embedded in the Cartesian product of its aspect spaces:

(3.23)
$$\textit{Given a state space } \mathsf{S} \textit{ with aspect operators } a, b, \ldots \textit{ and corresponding aspect spaces } \mathsf{A}, \mathsf{B}, \ldots,$$
$$\textit{where}: \mathsf{S} \textit{ is isomorphic to a set } C \subseteq \mathsf{A} \times \mathsf{B} \times \ldots$$
$$C = \left\{ [a(s), b(s), \ldots] \;\middle|\; s \in \mathsf{S} \right\}$$

Eqn. 3.23: An embedding of a state space. The state space S is isomorphic to the set C comprised of the tuples of aspect values. The isomorphism of the spaces S and C comes directly from the definition of identity of elements of S, Eqn. 3.21. □

That is, when it is more convenient, we can think of an element as its equivalent tuple of aspect values (hence Notn. 3.22 looks like a tuple written vertically). If we wish to define an element of a state space by specifying its tuple of aspect values, we use the following notation:

Notation. Brackets "[ ]" and Backarrows "$\hookleftarrow$" for a Tuple

$$\boxed{\begin{bmatrix} r & \hookleftarrow & r \\ angle & \hookleftarrow & a \end{bmatrix}} \equiv \boxed{\begin{array}{rcl} \multicolumn{3}{c}{c \in \mathsf{Arcs}\ \textit{such that}} \\ r(c) & = & r, \\ angle(c) & = & a \end{array}} \tag{3.24}$$

Notn. 3.24: Shorthand notation to specify an element of a state space by a tuple of aspect values. Note that this notation leaves implicit to which space the element belongs—it needs to be clear from context. The notation is not well-defined if there is not a unique element corresponding with the tuple. It is not always necessary to include values for all the aspect operators, however; this will be discussed in Sec. 3.9.2. □

3.9.2 Internal Properties of a State Space

For a particular state space it may be true that only certain combinations of aspect values are possible. Aspects may be redundant (e.g., two aspects, one describes a length in yards, the other describes the same length in meters); more generally, there may be arbitrary relationships between possible aspect values, reflecting the inherent structure and "topology" of the aspect space. Using the terms of Eqn. 3.23, C would be a proper subset of the embedding space $A \times B \times \ldots$; often, C is a low-dimensionality manifold in the higher-dimensionality embedding space.

We define the *internal properties* of a space to be the aspect value relationships that are guaranteed to hold for all elements of the space. It is convenient to note the internal properties explicitly, at once when the space is defined. Thus we extend Notn. 3.22:

Notation. Internal Properties Below a Line "—"

$$\boxed{\begin{array}{l} \mathsf{Rectangles} \\ [\\ \quad length \mapsto \Re \\ \quad width \mapsto \Re \\ \quad area \mapsto \Re \\ — \\ \quad length \geq 0 \\ \quad width \geq 0 \\ \quad area = length \times width \\ ] \end{array}} \equiv \boxed{\begin{array}{l} \mathsf{Rectangles} \\ [\\ length \mapsto \Re \\ width \mapsto \Re \\ area \mapsto \Re \\ ] \\ \forall r \in \mathsf{Rectangles}, \\ \quad length(r) \geq 0 \\ \quad width(r) \geq 0 \\ \quad area(r) = length(r) \times width(r) \end{array}} \tag{3.25}$$

Notn. 3.25: The notation on the left is shorthand for the statements on the right, that define a state space (Notn. 3.22) and list its collection of internal properties. Note the shorthand leaves implicit the "for all" qualifier—internal properties are by definition true for every element of the space. □

The internal properties defined for a state space are not restrictions that describe acceptable or interesting elements, or those that solve some problem. Rather, as we said above, the properties are descriptive of the internal "topology" of the space.[21]

Eqn. 3.21 says that in general, one needs the complete collection of aspect values to identify an element of a state space. For spaces with internal properties, however, it is often true that the value of only a few aspect operators—or even a single aspect operator—is in fact sufficient to uniquely identify a point in the space. For example, in Notn. 3.25, the values of any two of $length(r)$, $width(r)$, and $area(r)$ are sufficient to identify an element $r \in \mathsf{Rectangles}$. We call a collection of only some aspect values a *sub-tuple* for an element (and for clarity we will occasionally refer to the complete tuple of values as a *full-tuple*), and define:

[21] However, one might imagine a computer program that numerically computes elements of a state space, representing elements by tuples of values; such a program may in fact use the internal properties to restrict the solution to lie on the proper manifold in the embedding space.

Definition. **Identifying Sub-tuple**

(3.26) *A sub-tuple of aspect values that is sufficient to identify an element of a state space is called an* **identifying sub-tuple**. *If eliminating any value guarantees that the remaining values will be insufficient to identify an element, it is called a* **minimal identifying sub-tuple**

Defn. 3.26: An identifying sub-tuple means that there is only a single element in the space that has those particular aspect values; The remaining values of the element's full-tuple can be determined by the internal properties of the space. □

When defining a state space, it is often useful to describe the minimal identifying sub-tuples. Minimal identifying sub-tuples are typically the easiest way to specify particular elements of a space. Note that Notn. 3.24 allows us to describe an element using an identifying sub-tuple, leaving out the unnecessary aspect values; since the notation includes aspect names, there's no ambiguity about which aspect operator values are part of the sub-tuple.

3.9.3 Subscript Notation for Aspect Operators

An aspect operator is just an operator, and we can use ordinary parenthesis notation to evaluate it, as for example in Eqn. 3.21. However, sometimes parentheses can get cumbersome, when equations get complex or in particular when an aspect value is itself a function. We find it convenient to define an alternative notation:

Notation. Subscripts for Aspect Values

(3.27) *Given a state space* S *with an aspect operator* a, *and an element* $s \in \mathsf{S}$, *the following notations are equivalent:*

$a(s)$ *[standard parenthesis notation]*
a_s *[new subscript notation]*

$$a_s \equiv a(s)$$

Notn. 3.27: Subscript notation for aspect operators. In any given expression, we will use whichever notation seems clearer. □

Often, we consider functions into state spaces. Again, to alleviate difficulties with parentheses, we define an alternate subscript notation:

Notation. Subscripts for Aspect-Valued Functions

(3.28) *Given a state space* S *with an aspect operator* a, *and a function* $s: \Re \to \mathsf{S}$, *the following notations are equivalent:*

$a(s(t))$ *[standard parenthesis notation]*
$a_{s(t)}$ *[subscript notation of Notn. 3.27]*
$a_s(t)$ *[new subscript notation]*
$(a \circ s)(t)$ *[function composition notation]*

$$a_s(t) \equiv a_{s(t)} \equiv a(s(t)) \equiv (a \circ s)(t)$$

Notn. 3.28: Subscript notation for functions into state space. The new subscript notation $a_s(t)$ emphasizes the composite result: an aspect value as a function of t. In any given expression, we will use whichever notation seems clearer. This notation can of course be used for functions other than of reals or of more than one variable; we would write, e.g., $a_s(x, y, z) \equiv a(s(x, y, z))$. □

The different notations in Notn. 3.28 have the common feature that from left to right, they all list a then s then t, so even if we get lost among the various parentheses and subscripts, we can get the idea of what's happening.

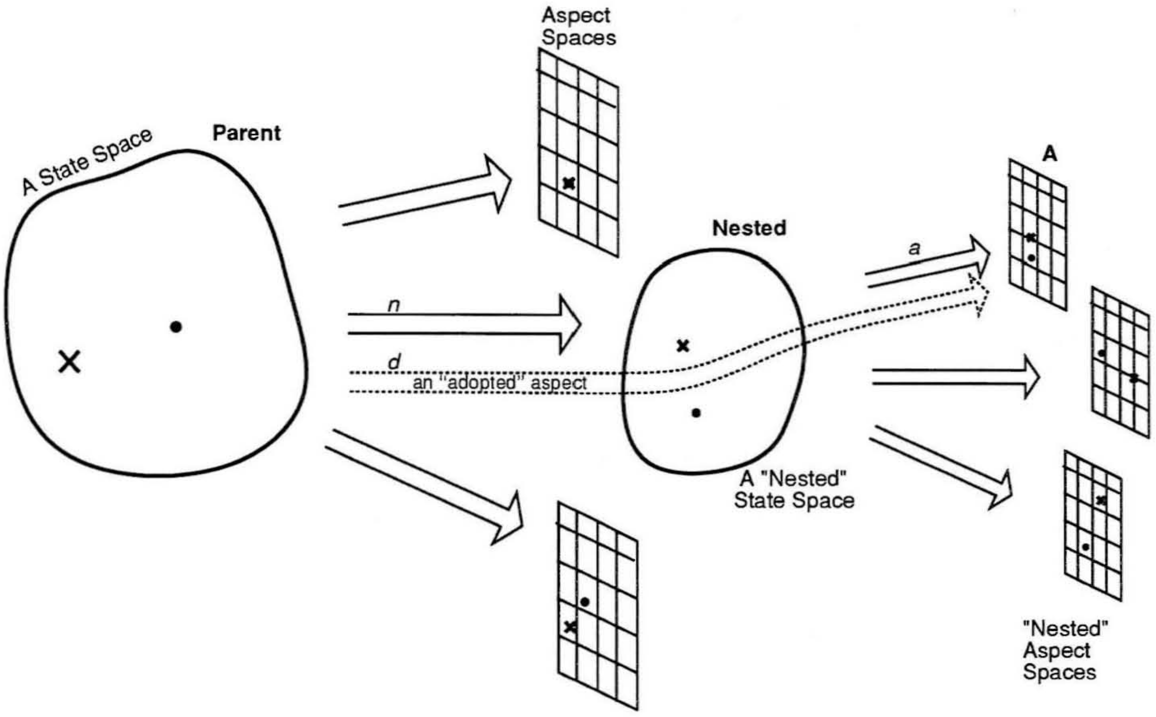


Figure 3.8: Nested state spaces: An aspect space of a state space may be another state space. The aspects of the nested space may be "adopted" as aspects of the parent space through composition of aspect operators; here, we illustrate aspect operator n being adopted into the parent space as aspect operator d . □

3.9.4 Nested State Spaces

An aspect space of a state space may be another state space. That is, one state space may be "nested" inside of another; this is illustrated in Fig. 3.8. The parent space can "adopt" aspects from the nested space to be its own aspects as well:

Definition. **adopt** an aspect

(3.29)

Given a space A*, a state space* Nested *with an aspect operator* a: Nested$\rightarrow$ A*, and a state space* Parent *with aspect operators* n: Parent$\rightarrow$ Nested *and* d: Parent$\rightarrow$ A*,*
Parent **adopts** *a as d iff:*

$$d(p) = a(n(p)),\ \textit{for all}\ p \in \text{Parent}$$

Defn. 3.29: Adopting an aspect from a nested space. The parent space's aspect operator is equivalent to the composition of the nested space's aspect operator with the aspect operator that yields the nested space element. See Fig. 3.8 □

Using the notation of Notn. 3.25, we can re-write Defn. 3.29, perhaps more clearly:

State space Parent *adopts an aspect a of state space* Nested *as its own aspect d if, for some space* A:

(3.30)
$$\begin{array}{ll} \text{Parent} & \text{Nested} \\ [& [\\ \quad n \mapsto \text{Nested} & \quad a \mapsto \text{A} \\ \quad d \mapsto \text{A} &] \\ — & \\ \quad d = a(n) & \\ ] & \end{array}$$

Eqn. 3.30: Adopting an aspect from a nested space. The adoption equivalence in Defn. 3.29 is expressed as a property of the parent space Parent. □

Frequently, we choose to keep the same name for the adopted aspect in the parent space as it has in the nested space. We extend Notn. 3.25 to allow easy definition of adopted aspects:

Notation. Ellipsis "..." for an Adopted Aspect

(3.31)
$$\boxed{\begin{array}{l} \text{Bricks} \\ [\\ \quad \textit{height} \mapsto \Re \\ \quad \textit{base} \mapsto \text{Rectangles} \\ \quad \textit{depth} \qquad \ldots \textit{length} \mapsto \Re \\ \quad \textit{width} \qquad \ldots \textit{width} \mapsto \Re \\ \quad \textit{volume} \mapsto \Re \\ — \\ \quad \textit{height} > 0 \\ \quad \textit{volume} = \textit{height} \cdot \textit{area}(\textit{base}) \\ ] \end{array}} \equiv \boxed{\begin{array}{l} \text{Bricks} \\ [\\ \quad \textit{height} \mapsto \Re \\ \quad \textit{base} \mapsto \text{Rectangles} \\ \quad \textit{depth} \mapsto \Re \\ \quad \textit{width} \mapsto \Re \\ \quad \textit{volume} \mapsto \Re \\ — \\ \quad \textit{height} > 0 \\ \quad \textit{volume} = \textit{height} \cdot \textit{area}(\textit{base}) \\ \quad \textit{depth} = \textit{length}(\textit{base}) \\ \quad \textit{width} = \textit{width}(\textit{base}) \\ ] \end{array}}$$

Notn. 3.31: The notation on the left is shorthand for the definition on the right, in which the nested space's *length* and *width* aspects are adopted into the parent space. In the shorthand notation, each adoption is listed directly underneath the nested space, with ellipses leading from the parent aspect name to the adopted aspect name. The names may be the same or different. Rectangles is used here as defined in Notn. 3.25. □

The examples in Notn. 3.25 and Notn. 3.31 let us derive a corollary from the properties of the spaces Bricks and Rectangles:

$$\forall b \in \text{Bricks},\ \textit{volume}_b = \textit{depth}_b \times \textit{width}_b \times \textit{height}_b$$

A final note on nested spaces: we want to avoid circularity—we disallow a parent state space that nests itself, or that nests a space that directly or indirectly nests the parent.

3.9.5 State Space Specializations

In Sec. 3.7.2, we discussed the idea of a *specialization,* i.e., a subset of a general space, that has additional properties and operators. The state space mechanism can be used to define these additional properties.

To define a state space Specs as a specialization of some general space Gens with an injection f: Specs → Gens, we need only list f as an aspect operator of Specs, and include internal properties that insure f is one-to-one. However, we define notation to make the specialization explicitly visible:

Notation. Specialization Symbol "$\sqsubset$" for State Spaces

$$
\boxed{\begin{array}{l}
\mathsf{Specs} \overset{f}{\sqsubset} \mathsf{Gens} \\
[\\
\quad \vdots \\
— \\
\quad (f \textit{ identifies an element}) \\
]
\end{array}}
\equiv
\boxed{\begin{array}{l}
\mathsf{Specs} \\
[\\
\quad f \mapsto \mathsf{Gens} \\
\quad \vdots \\
— \\
\quad (f \textit{ identifies an element}) \\
]
\end{array}}
\tag{3.32}
$$

Notn. 3.32: The definitions on the right hand side define Specs as a specialization of Gens, as per Defn. 3.4/Fig. 3.4; the notation on the left is a shorthand form. Note that since f is one-to-one, each element $s \in$ Specs has a unique value of f_s; thus a value of f comprises an identifying tuple for an element of Specs. □

If the general space is itself a state space, then the specialization implicitly adopts all of its aspects, by Eqn. 3.6 and Defn. 3.29. That is, we have:

Notation. Specialization of a State Space

$$
\boxed{\begin{array}{ll}
\mathsf{Specs} \overset{f}{\sqsubset} \mathsf{Gstates} & \mathsf{Gstates} \\
[& [\\
\quad x \mapsto \Re & \quad a \mapsto \Re \\
\quad y \mapsto \Re & \quad b \mapsto \Re \\
— &] \\
\quad x = a + b & \\
\quad b = a + y & \\
] &
\end{array}}
\equiv
\boxed{\begin{array}{lll}
\mathsf{Specs} & & \mathsf{Gstates} \\
[& & [\\
\quad f \mapsto \mathsf{Gstates} & & \quad a \mapsto \Re \\
\quad a & \ldots a \mapsto \Re & \quad b \mapsto \Re \\
\quad b & \ldots b \mapsto \Re &] \\
\quad x \mapsto \Re & & \\
\quad y \mapsto \Re & & \\
— & & \\
\quad x = a + b & & \\
\quad b = a + y & & \\
] & &
\end{array}}
\tag{3.33}
$$

Notn. 3.33: A specialization of a state space implicitly adopts all aspects. Note that in this example the internal properties of Specs are sufficient to determine x and y from the adopted aspect values a and b, thus f is sufficient to identify an element of Specs. If no name is needed for the injection aspect operator, we can write just Specs $\sqsubset$ Gstates. □

Several examples of state space specializations can be found in Ch. 9.

3.10 Segmented Functions

Suppose we want to create a model of a thing whose behavior over time includes discontinuous changes in its configuration or properties. For example, as illustrated in Fig. 3.9, the number of bodies in a model may change as bodies are created or destroyed; [Brockett90] describes a formal language for robotic motion that includes discrete changes in force function; and [Kalra90] allows the behavior of a model to change discretely between nodes of a directed graph.

We would like to define a single function of time that completely describes a model's behavior. But at each discontinuous change, the model may switch between incompatible configurations, having different dimensionality or degrees of freedom—how can we have one function that spans these changes?

The answer is simple: Consider the overall configuration space of the model to be the *disparate union* (Sec. 3.7.3) of the various individual configuration spaces—then define the behavior as a path through that overall configuration space. This section introduces the idea of a *segmented function* to describe such paths. The mechanism we define here is used, for example, in the example "tennis-ball cannon" model of Ch. 11.

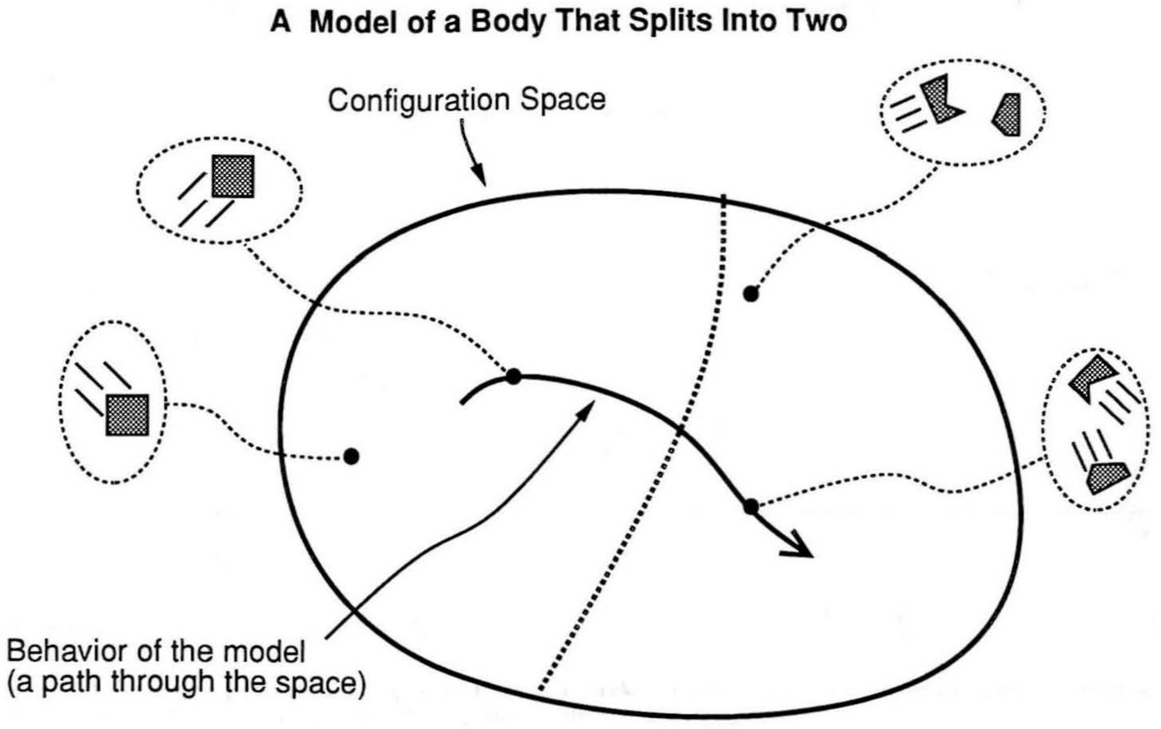

A Model of a Body That Splits Into Two
Configuration Space
Behavior of the model
(a path through the space)

Figure 3.9: Example of a segmented function. Consider a model of a body that splits into two bodies. When there is only one body, we describe the model with 6 degrees of freedom (3-D position and orientation), but when there are two bodies, 12 degrees of freedom are needed. The model's behavior over time is described by a *segmented function:* a piecewise continuous function into a disparate union. □

3.10.1 Definition of a Segmented Function

We'd like a path through a disparate union to be "continuous," but continuity can't be defined in general for disparate unions. A path that stays within a single disparate component of the union can be continuous, however—but to make a transition between components, there must be a discontinuity. Thus we define:

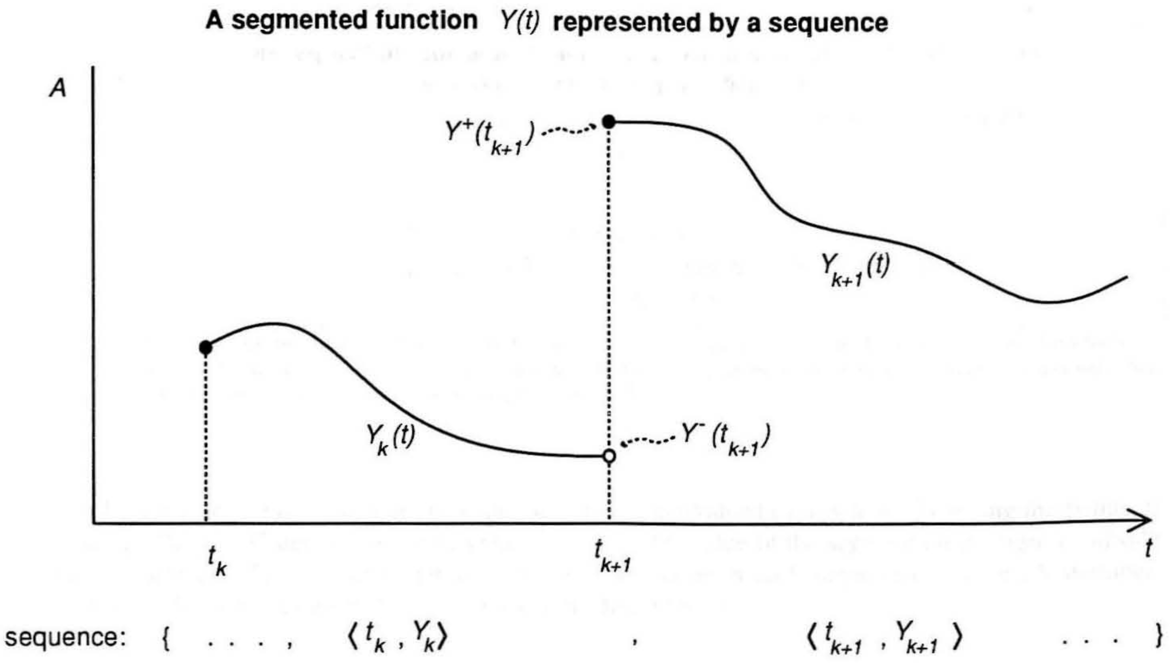
Definition. **Segmented Function**

(3.34) *A* **segmented function** *is a piecewise-continuous function from the real numbers to a disparate union. Each continuous piece is called a* **segment,** *and we refer to each discontinuity as an* **event.**

Defn. 3.34: Each segment of a segmented function stays within a disparate component of the union, i.e., all points along the segment are agnates. The transition from one segment to the next may cross into a different component. Note that we define segmented functions only as paths, i.e., functions from the real numbers, rather than as functions from arbitrary spaces. □

The definition assumes that *continuous* is well-defined within each component of the disparate union. For a space that is discrete, we take *continuous* to mean *constant*. And thus, for a space that is (equivalent to) a Cartesian product of discrete and smooth spaces, a *continuous* means that the discrete parts stay constant and the smooth parts can vary continuously.

A common use of segmented functions is for paths into a space of indexes (Sec. 3.8.2). First, we'll look at what continuity means for index-valued functions:



A segmented function $Y(t)$ represented by a sequence
A
$Y^+(t_{k+1})$
$Y_{k+1}(t)$
$Y_k(t)$
$Y^-(t_{k+1})$
t_k
t_{k+1}
t
sequence: $\{ \;\ldots,\; \langle t_k , Y_k \rangle \;,\; \langle t_{k+1} , Y_{k+1} \rangle \;\ldots\; \}$

Figure 3.10: Sequential representation of a segmented function. Each element in the sequence is a pair containing the corresponding segment's left bound and continuous function. The diagram illustrates two adjacent segments. □

(3.35) *For an index-valued function* $f: \Re \rightarrow \{\mathsf{A}\}_{\mathsf{IDs}}$ *to be continuous, we must have:*

- *the composite function* $(Ids \circ f): \Re \rightarrow \mathsf{IDsets}$ *is constant, and*
- *each composite function* $f_i: \Re \rightarrow \mathsf{A}$ *is continuous, for* $i \in Ids(f(t))$.

Eqn. 3.35: Continuous index-valued functions. For f to be continuous, we must have a given collection of elements, that each vary continuously (we assume that continuity is defined for the space A). □

A segmented function is made from continuous pieces, and thus:

(3.36) *For a segmented index-valued function* $f: \Re \rightarrow \{\mathsf{A}\}_{\mathsf{IDs}}$, *we have:*

- $(Ids \circ f): \Re \rightarrow \mathsf{IDsets}$ *is constant in each segment, and*
- *each* $f_i: \Re \rightarrow \mathsf{A}$ *is continuous in each segment, for* $i \in Ids(f(t))$.

Eqn. 3.36: Segmented index-valued functions. Each segment may have a different set of ID's. If two adjacent segments share an ID i, the value $f(t)_i$ might or might not vary continuously across the event. □

3.10.2 Sequential Representation of a Segmented Function

We'd like to to represent segmented functions concretely, in order to be able to manipulate and evaluate them. Mathematical methods, as well as numerical, are most often designed for continuous functions, and don't robustly handle discontinuities. Thus we isolate the continuous parts of a segmented function, expressing it as a sequence of continuous functions with bounded domains: (see Fig. 3.10)

Definition. **Sequential Representation**

(3.37)

The **sequential representation** *of a segmented function $Y : \Re \to \mathsf{A}$ for some space A is a sequence of real/function pairs:*
$\{\ldots, \langle t_{k-1}, Y_{k-1}\rangle, \langle t_k, Y_k\rangle, \langle t_{k+1}, Y_{k+1}\rangle, \ldots\}$ *such that for each pair* $\langle t_k, Y_k\rangle$,

$t_k \in \Re$	*increasing with k*
$Y_k : [t_k, t_{t+1}] \to \mathsf{A}$	*continuous function on the interval*
$Y(t) = Y_k(t),$	*where* $t_k \le t < t_{k+1}$

Defn. 3.37: Sequential form of a segmented function. The t_k's in the sequence are the values of t for which there are events. Each function Y_k need only be defined on the interval corresponding to its segment, but must be continuous, thus its values must stay within a single disparate component of A. □

In order for a segmented function to be defined and single-valued everywhere, including the points of discontinuity, Defn. 3.37 defines (arbitrarily) that $Y(t)$ take the value of the segment on the right-hand side of each discontinuity. Thus we have not used the rightmost value of each segment's function. Sometimes, however, we will want to examine "both" values at the discontinuity:

Notation. Superscript "−" and "+" for Left and Right Values

(3.38)

Given a segmented function $Y : \Re \to \mathsf{A}$ represented sequentially by a sequence $\{\langle t_k, Y_k\rangle\}$, define:

$$Y^-(t) \equiv \begin{cases} Y(t), & t \neq t_k \text{ for any } k \\ Y_{k-1}(t_k) & t = t_k \text{ for some } k \end{cases}$$
$$Y^+(t) \equiv \begin{cases} Y(t), & t \neq t_k \text{ for any } k \\ Y_k(t_k) & t = t_k \text{ for some } k \end{cases}$$

Notn. 3.38: $Y^+(t_k)$ is the value of the segmented function on the right-hand side of the discontinuity at t_k, and $Y^-(t_k)$ is the value on the left-hand side. For convenience, away from discontinuities, both are defined to be the same as Y. □

Note that Defn. 3.37 and Notn. 3.38 imply/assume an infinite number of segments; often, however, the sequence will be finite on either end: the end segments may extend to $\pm\infty$ if there are no farther events, or we may only be interested in $Y(t)$ for a bounded domain.

3.10.3 Functional Characterization of a Segmented Function

The sequential representation describes a segmented function by enumerating all the segments, i.e., listing the event times t_k and the continuous functions Y_k. We would also like to describe a segmented function "all at once." But, as mentioned in Sec. 3.10.2, we want a mechanism that isolates the discontinuities.

We define here a predicate-based method to characterize segmented functions. In addition to locating the events between segments, this method emphasizes the relationship that links one segment to the next.

Definition. **Functional Characterization**

(3.39)

The **functional characterization** *of a segmented function* $Y : \Re \to \mathsf{A}$ *is a triple of predicates:*

a **body function**	$F : \Re \times \mathsf{A} \to \mathsf{Boolean}$ *that describes the continuous parts of* Y,
an **event function**	$G : \Re \times \mathsf{A} \to \mathsf{Boolean}$ *that isolates the events (discontinuities) in* Y
a **transition relation**	$H : \mathsf{A} \times \Re \times \mathsf{A} \to \mathsf{Boolean}$ *that describes the events in* Y,

such that, for each segment $\langle t_k, Y_k \rangle$,

$$
\begin{array}{ll}
F(t, Y_k(t)) \text{ is true} & \text{for } t_k < t < t_{k+1} \\
G(t, Y^-(t)) \text{ is true} & \text{iff } t = t_k \\
H(Y^-(t_k), t_k, Y^+(t_k)) \text{ is true} & \text{(for each } t_k\text{)}
\end{array}
$$

Defn. 3.39: A segmented function characterized by three predicates. The body function is true [22]along each continuous segment; the event function is true only at the events; and the transition relation is true where the function crosses an event. □

Notice that we refer to Defn. 3.39 as a *characterization*, rather than a representation. The predicates are not necessarily "tight": we have not specified their values for arguments that are off the path of $Y(t)$, and furthermore, for the body function and transition relation, constant-true predicates would always suffice. Thus a single segmented function can be characterized by many predicate triples, and a single predicate triple can characterize many functions.

Still, the functional characterization provides a canonical form that is often useful—and in many circumstances, we can define predicates that *are* tight. For example, Appendix C uses a functional characterization to define segmented functions as solutions to piecewise-continuous ODE's. Notice that the functional characterization gives a declarative way of encapsulating the procedural instruction *"when* x *happens, do* z*":* If we increase t, the value of the event function, $G(t, Y^-(t))$ remains false until an event occurs, i.e., until *"*x *happens";* and in describing the transition from $Y^-(t)$ to $Y^+(t)$ at the event, the transition relation $H(\ldots)$ embodies *"do* z.*"*

This is the end of the Techniques *part of chapter 3. The next part,* Discussion, *contains some notes on designing models as well as an overall summary of the chapter.*

3.11 Designing a Model

Thus far, this chapter has discussed philosophy and goals for modeling, and has presented assorted techniques for mathematical modeling. In this section, we try to give a feel of how these all fit together.

We would like to emphasize again that there is no unique "right" way of defining any particular mathematical model. As discussed in Sec. 3.3, there are design decisions involved that are often not cut-and-dried, but can be based on aesthetics, experience, goals, and so forth. Thus we don't attempt to provide a specific recipe for model design, but just a collection of notes on creating models.

The kinematic rigid-body module in Ch. 7 is small, and serves as a good example of many ideas here; we encourage the reader to refer to it while reading this section. We will use the following notation to point to specific examples:

[§7.3]	= "see Sec. 7.3 for an example"
[Eq7.1]	= "see Eqn. (or Defn. or Notn.) 7.1 for an example"

[22] For predicate calculus purists, writing "$F(\ldots)$ is true" is inelegant; one would just write "$F(\ldots)$". But since we use predicates rarely, we will be explicit.

3.11.1 Writing a Model

A mathematical model is just a collection of equations and definitions written down with some explanatory text. [§7.3]

Context. We write each mathematical model within the context of the overall physically-based model, as per Sec. 2.6.3. That is, the mathematical model is preceded by a description of the *conceptual model* [§7.2], i.e., an explanation of the thing being modeled in terms of the behavior and properties of interest; and is followed by statements of *posed problems* [§7.4], i.e., tasks described in words and as the corresponding mathematical problems.

Exposition. The intention of the written mathematical model is to be understandable by human readers. Thus, although the mathematics of the model should be complete and well-defined without any extra material, we should also include suitable explanatory text or diagrams to help make the mathematics understandable. In particular, we should explain the the physical interpretations of the terms and equations that we write.

Framework. Each mathematical model is intended to be used as a module, and may invoke terms and equations from other modules. This leads to the following framework for writing a model:

> **Names & Notation.** We start each model by stating its scope name,[23] and also list terms from other modules that we will reference, as per Sec. 3.6.2. We may also describe other naming conventions, notation that will be used, and so forth. [§7.3.1]
>
> **Main Body.** The definitions and equations that describe the thing being modeled. This may be divided into several sections, and is often presented in a form paralleling the conceptual model. [§7.3.2–7.3.5]
>
> **Derivations & Proofs.** We put derivations and proofs, if any are needed, in a separate section. Thus the main body of the model contains the "bottom-line" equations that are of practical use for other modules, while this section contains supplementary material that is of interest to "maintainers" of the model. [§6.7]

3.11.2 What is in a Typical Model?

Things that are commonly found in models:

State Spaces. Typically, the thing being modeled, or objects therein, have some sort of configuration, or state. We use the state space mechanism of Sec. 3.9 to define the abstract configuration space [Eq7.1]. Note that we're not minimalist about the choice of aspects in state spaces: we include any quantity that defines or is defined by the configuration [Eq8.4]; in order to be able to specify individual elements, we describe what combinations of aspect values comprise identifying tuples. In many cases, it may make be useful to define a *metric* for a state space ([Hughes92]), i.e., a definition of a distance between two elements. State spaces are also convenient to bundle a collection of quantities into a single entity [Eq7.8].

Indexes. Whenever there are several of a thing—numbers, vectors, functions, sets, etc.—in a model, we will use the index mechanism (Sec. 3.8.2) to be able to manipulate a collection of the things as a single entity. Particularly common is to have an index of states. [Eq7.6]

Predicates. It is often convenient to define a predicate that is true when a particular relationship or property holds for certain values. [Eq7.9]

Functions. We often define abstract spaces, each element of which is a function. Probably the most common functions are those used for the behavior of a thing over time [Eq7.2], but they can be arbitrary [Eq8.16]. Sometimes we define a space that includes all possible functions from one space to another [Eq6.19], and sometimes we define a space containing only functions that we find interesting or useful [Eq7.2].

[23] Generally, the scope name does not need to appear anywhere else within the model. This is like the children's riddle: "*Q.* What belongs to you but is used more by everybody else?" "*A.* Your name."

3.11.3 Things to Do

"Tactical" ideas for designing models.

Modular design. We don't want to define "global" quantities, but rather always encapsulate them in a state space—even for a top-level model. Thus, we wouldn't write *"let y be the number of yaks, and z the number of zebras."* Instead, we define a state space:

$$\begin{array}{ll} \text{ZooStates} & \\ [& \\ y \mapsto \textit{Integers} & \textit{number of yaks} \\ z \mapsto \textit{Integers} & \textit{number of zebras} \\ \vdots & \\ ] & \end{array}$$

then we can write *"for a zoo having state $s \in$ ZooStates . . ., the number of yaks, y_s, is equal to. . ."* [§11.3]

Temporal Behavior. For behavior of objects over time, we don't want to have a "current" configuration. Instead, we define the entire path of a function over time as a single object [Eq7.2]. We often define predicates that describe interesting paths, or paths that are consistent with other items in the model [Eq8.10].

Layered design. We like to take a layered approach, first defining general properties, then building on them to include more details. For example, Defn. 8.10 describes a path of a rigid body that is consistent with respect to arbitrary net force/torque functions; Defn. 8.18 refines this to net force/torque fields; and Defn. 8.25 refines this further to collections of individual force/torque "motives."

Validity tests. When defining a model, it can be helpful to include theorems or redundancies, i.e., "if *those* are true, then *this* must be true as well. For example, the rigid body model includes an energy formulation so that if the bodies' motions are consistent with the applied forces, an energy conservation equation, Eqn. 8.32 must hold. In addition to providing a consistency check on the model, it can be a powerful debugging aid when implementing models, by specifying what to double-check to determine if the solution techniques are working (see Sec. 2.6.4).

3.11.4 Things to Think About

Some things to bear in mind while creating a model.

Remember the context. In designing a model, we want to find a balance between definitions that are intuitive and close to the conceptual model, and those that are well-suited to the particular problems we expect to pose. Sometimes it all might fit together well, but occasionally there is some tension. We have no specific advice other than to try to keep an eye on "both ends" at once.

Declarative model. We have emphasized the importance of a mathematical model being declarative, a statement of relationships rather than a problem to solve. However, it is easy to slip into a problem-posing mode, especially when we expect to program/simulate a particular problem. One might define a model that's technically declarative, but does so by by declarations such as "x = *solution-to-the-problem*." As we write equations that define objects, we try to ask ourselves:

- Does the equation describe *what* the object is?
- Or does it just describe *how* to create one?

There is of course no hard dividing line, but it is something to think about. We have found that as we gain experience, it becomes easier and more natural to create declarative models.

Relationships in the model. There is an aspect of designing a structured mathematical model that is not unlike programming. In order for the expressions to be complete, we want every function to have arguments for everything it depends on—there should be no hidden parameters. This often involves thinking about how to represent and encapsulate relevant information, what the underlying relationships are between entities, how to define functions so that the right information gets to the right place, and so forth. The design process for the "fancy forces" model in Ch. 9 was an exercise in this type of thinking.

3.12 Summary

This chapter was motived by the idea that having written, well-defined mathematical formulations leads to more robust models. Traditional mathematical techniques and methods facilitate the creation of what we call "model fragments" (Sec. 1.3), but do not focus on creating well-defined complex models that include the "glue" between these fragments.

This chapter thus presented techniques and methods that help organize and structure complex mathematical models, in a well-defined, writable way. Many of the ideas were inspired by computer programming methodology, which has been developed to meet similar needs. The techniques fell into five groups:

Naming Strategies (Sec. 3.7). Techniques for naming mathematical entities. Most notably, we introduced *scoped names*, e.g.,

$$\textsc{kinematic}::\mathsf{States}$$

Abstract spaces (Sec. 3.7). Discussion and methods for defining abstract spaces of mathematical entities. Includes the ability to define *specializations*, e.g.,

$$\mathsf{Specs} \sqsubset \mathsf{Gens}$$

Indexes (Sec. 3.8). A mechanism to manipulate collections of named entities, e.g.,

$$\{\mathsf{Things}\}_{\mathsf{IDs}}$$

State Spaces (Sec. 3.9). A mechanism to encapsulate a collection of properties into a single entity, e.g.,

$$\begin{array}{l} \mathsf{Arcs} \\ [\\ \;r \qquad \mapsto \Re \\ \;angle \mapsto \Re \\ ] \end{array}$$

Segmented Functions (Sec. 3.10). A mechanism to describe functions whose values may have discontinuous changes in dimensionality, e.g.,

$$\{\ldots, \langle t_k, Y_k \rangle, \ldots\}$$

We have actually used these mechanisms, informally on whiteboards and on napkins as we developed mathematical models, as well as more formally for the expositions in Ch. 6–9. They were designed (and refined and re-designed) in order to be practical and effective. We have found that with some experience, they are natural and easy to use. Of course, we don't claim that these specific techniques are the best or only ones to meet our goals. Different or additional structuring techniques might well be useful.

3.13 Related Work

We are not familiar with other work that follows our approach of creating "complete, explicit models," with a focus on the relationships and administration of model fragments. For basic applied mathematical modeling exposition, we refer the reader to [Lin,Segel74] and [Boyce81].

Chapter 4

Computer Programming Framework

This chapter puts together the elements of the preceding chapters, to form the basis of a physically-based modeling system:

- Conceptual/mathematical/posed-problem structure and modularity as per Ch. 2.
- Mathematical models as per Ch. 3.

In this chapter, we go from the blackboard to the keyboard: that is, we will assume that the designer has made (a first pass at) a "blackboard" version of the model, as per Ch. 2, and now is ready to sit down at a computer and write some code to implement it. Note that in Ch. 2, numerical solution techniques were largely dismissed as an implementation detail, and were left out of the CMP structure. Here, however, we're focusing on implementation, so the numerical techniques will be on an equal footing.

We will describe a framework for the structure of modeling programs, and how to implement a CMP model within that framework. Several design issues will be discussed, including questions of debugging and efficiency.

The prototype models in Ch. 6–9 include descriptions of their implementations, in accordance with the framework described in this chapter. Appendix B provides a background description of our prototype modeling environment.

4.1 Overview

For computer graphics modeling, we want extensible, reusable modeling tools. This chapter discusses a modeling "system" or "environment" containing a collection of such tools, from which a particular model or modeling program can be built. Note that the tools we build are for implementors—programmers—rather than for end-users.[1]

The modeling system follows a "toolbox" approach. The modules and techniques that comprise the collection of tools are at various levels of representation; they can be and mixed as appropriate for a given application. We emphasize that we are not defining a "standard interface" for physically-based modeling (as, e.g., RenderMan [Upstill90] does for scene description), but rather just an extensible toolbox.

A particular design goal for the system is to decouple the state of a model from the state of the program. A mathematical model as per Ch. 3 can be a single formulation that spans discontinuities and state changes in the thing being modeled; the computer program should do the same. That is, it should be possible to "random

[1] A programmer could use the tools as the "guts" of an end-user modeling workstation/environment, but we are not directly describing such an environment.

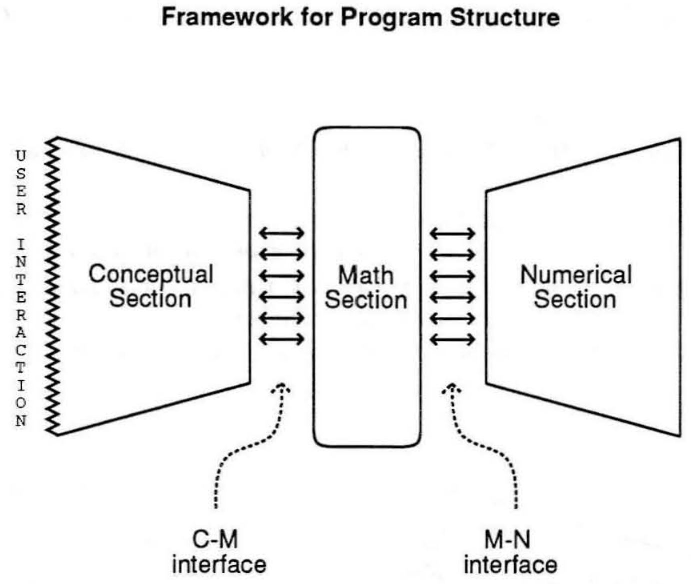


Figure 4.1: Framework for program structure. The structure for a program follows the canonical conceptual/mathematical/posed-problem structure discussed in Ch. 2. The program is divided into three separate sections: The "conceptual section" is the front end, maintaining the data structures that represent the user's conceptual model. The "math section" contains data structures that represent the mathematical model. The "numerics section" contains various numerical problem solvers; the posed problems themselves are embodied in the interfaces between the three sections. □

access" the model over time, with the model's state reflecting the correct configuration for each access. This is discussed further in Sec. 4.7.

A note on terminology: We will use *environment* or *system* to refer to the collection of modules, techniques, and so forth. We will use *program* to refer to a (hypothetical) computer program built using tools from the environment; a program might implement a single model, some small class of models, or a general end-user modeling workstation.

4.2 Framework for Program Structure

The conceptual/mathematical/posed-problem decomposition of a physically-based model that was discussed in Ch. 2 translates into a framework for programs. We divide a program into three distinct sections: (Fig. 4.1)

- the Conceptual Section,
- the Mathematical Section, and
- the Numerics Section.

The *conceptual section* supports the user's conceptual model. It maintains data structures describing the objects being modeled, interacts with the user, and so forth. The *math section* embodies mathematical models as per Ch. 3. It contains various data elements and structures that represent the mathematical objects and equations. The *numerics section* is responsible for numerical solution of specific equations and problems. Notice that the posed problems of the model do not have a section of their own in the program; they are embodied in the interfaces between the three sections.

Putting it all together, a program has the following sequence of functionality:

1. The conceptual section defines a formalized conceptual model, based on user input.
2. The conceptual section creates a mathematical model based on the conceptual model.

3. For posed problems, the appropriate mathematical problems are created based on the mathematical model.

4. For posed problems, the numerics section computes solutions to the problems, which get relayed back to the mathematical model, from there to the conceptual model, and finally back to the user.

Notice that the program maintains two separate data structures that represent the overall model: one for the conceptual model, and one for the mathematical model.[2]

The program is composed of modules that "live" within each section, as well as interface modules that bridge between the sections. The math and numerics sections are mostly just libraries of routines and objects; they have no autonomous operation. The conceptual section, however, administrates the operation of the entire program.

4.2.1 Conceptual Section

The conceptual section maintains the formal version of the user's conceptual model. This is typically a data structure containing objects that the user understands and manipulates, corresponding to entities in the thing being modeled. For example, a rigid-body modeling system might have objects for the primitive body, objects for forces that act on the bodies, and objects for "control points" at which the forces are attached, all grouped hierarchically (see Fig. 4.2). Note that in Ch. 2 the conceptual model was expressed as words; here we formalize the representation as a data structure.

The conceptual model data structure includes sufficient information to construct the corresponding mathematical model data structure (via the C-M interface). In addition, it includes information such as rendering details, user-interaction constructs, hierarchical groupings, and so forth, that are conceptually important, i.e., part of the abstraction, but may be invisible/irrelevant to the mathematical simulation.

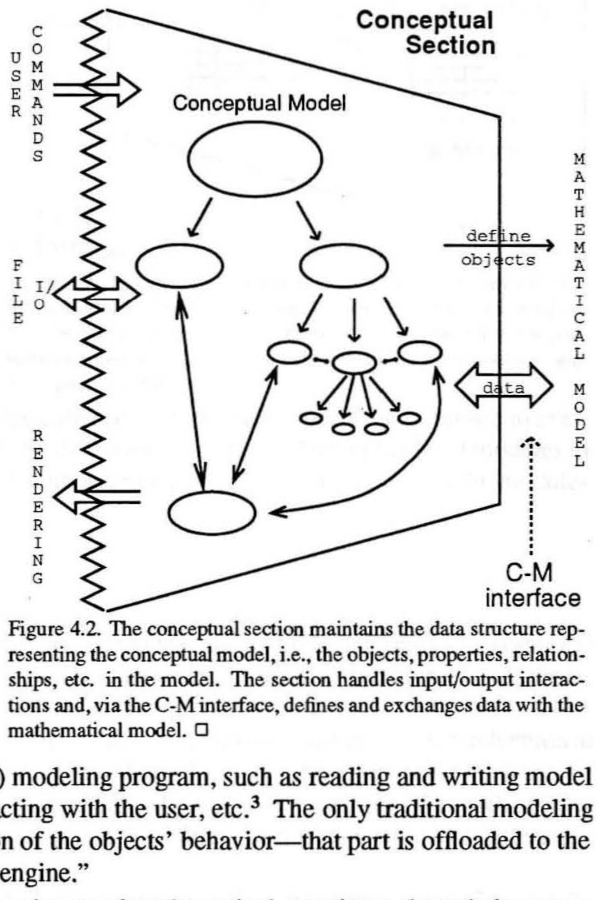


Figure 4.2. The conceptual section maintains the data structure representing the conceptual model, i.e., the objects, properties, relationships, etc. in the model. The section handles input/output interactions and, via the C-M interface, defines and exchanges data with the mathematical model. □

Many tasks performed by the conceptual section are the same as those of a traditional (kinematic) modeling program, such as reading and writing model description files, sending frames to a renderer, interacting with the user, etc.[3] The only traditional modeling program task that is not performed is the computation of the objects' behavior—that part is offloaded to the mathematical section, which is used as an "equation engine."

The conceptual model may actually include some snippets of mathematical equation or knowledge, especially when the high-level abstraction is inherently mathematical (see Sec. 2.4.3). For example, the shape of a body may be defined via a parametric function, as in [Snyder92].

[2] Actually, there's no reason to limit it to two models. One can imagine having separate graphical/interaction/etc. models, as well as the mathematical and conceptual models. For this discussion, however, we will just toss all but the mathematical into the conceptual model; thus the conceptual model we describe is perhaps not as well-structured as it ought to be.

[3] For this discussion, we assume that rendering is purely a conceptual task. In principle, however, a mathematical formulation for rendering could be part of the mathematical model, and the task of producing an image would then be a posed problem.

4.2.2 Numerics Section

The numerics section is the "back end" of a physically-based modeling program. It is a collection of program objects, whose role is straightforward: to solve numerical problems (on request), such as integrating ordinary differential equations (ODE's), finding roots of functions, solving linear systems of equations, and so forth, without regard to any larger context these problems may be embedded in.

The underlying functionality of the numerics section can be provided by numerical software packages, such as [Press et al.86] or [NAG]. However, the subroutines in such packages typically act at a simple level of representation, such as arrays of numbers; and the interface is often procedural, e.g., for an ODE integrator, one steps the solution forward by repeatedly calling a subroutine.

A collection of modules are built on top of the bottom-level subroutines. These perform transformations to higher-level representations, such as from procedural interfaces to functional, from arrays to arbitrary data objects, and so forth. These modules are designed in an object-oriented manner, allowing different solution techniques to be "swapped in" based on circumstances, as discussed in Sec. 2.4.1.

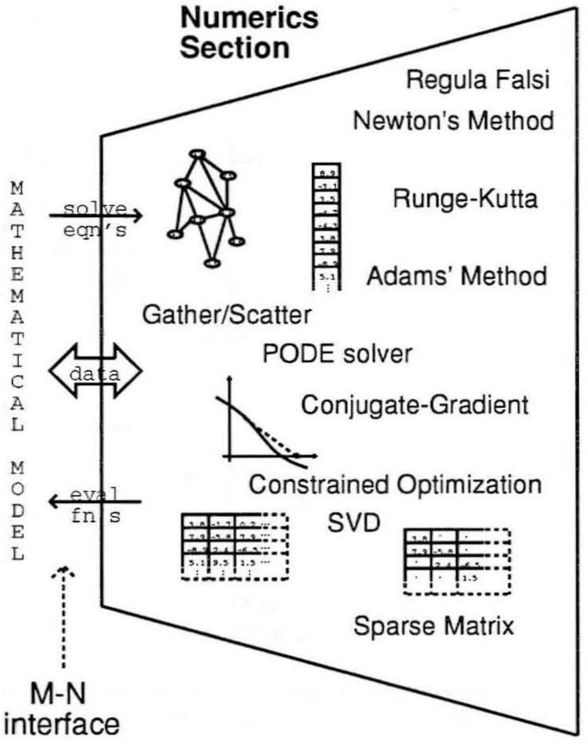


Figure 4.3. The numerics section includes a variety of numerical solution modules. The bottom-level subroutines can be mostly taken from books or libraries. Above those lie various modules that perform changes of representation between low-level subroutines and the higher, interface level. □

Numerical solution subroutines often require that the caller provide problem-specific subroutines to evaluate various functions (such as the derivative function of a differential equation). The higher level modules in the numerics section provide the appropriate items to the underlying numerical subroutines, but the modules must in turn be "fed" routines by the M-N interface.

4.2.3 Math Section

The math section fits between the conceptual and numerics sections. It contains definitions of computer objects that represent elements of mathematical models, such as sets, various kinds of functions, state spaces, ID's and indexes, and so forth (Fig. 4.4). The objects support simple operations that access their values. Note that each object's value is constant[4] over the lifetime of the object—as discussed in Sec. 3.5.1, mathematical objects have no changeable internal state. However, the value of an object may be a function which can be evaluated on different arguments as needed; in order to determine the value of a function, the math section may do simple symbolic/arithmetic computation, or may call on the numerics section (via the M-N interface) to solve the appropriate problems.

Remember that a mathematical model as per Ch. 3 is declarative rather than procedural (Sec. 3.5.2). Translating this to the program, it means that the objects in the math section don't "do" anything—they just "are." That is, they don't manage control flow in the program: at most, they merely evaluate functions on demand.

In addition to definitions of various types of math objects, the math section contains utility routines to help create and manipulate objects. For example, a utility routine might define a function object whose value is the sum of the values of a given set of other function objects.

[4] Ideally. In Sec. 4.8 we will discuss relaxing this requirement, for efficiency reasons

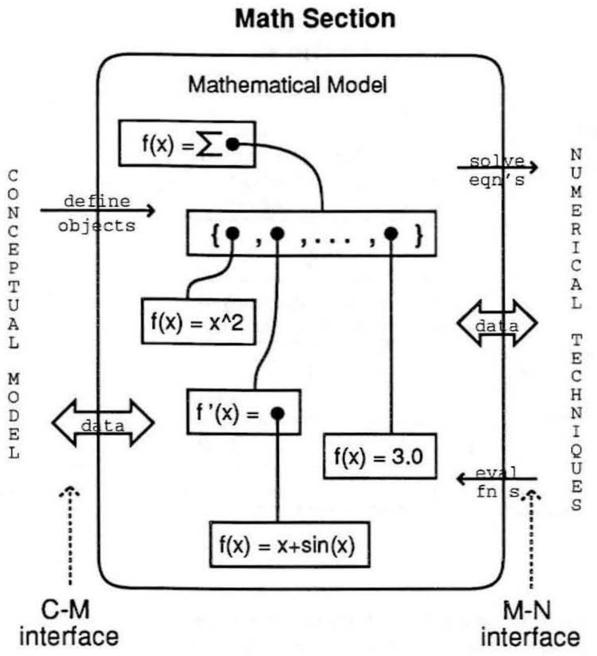


Figure 4.4: The math section contains definitions of objects that support mathematical entities, allowing data structures to be built that represent mathematical models as per Ch. 3. The figure illustrates a function whose value is equal to the sum of the values of a set of other functions. □

The modeling environment can pre-define some math section objects, such as ID's and numbers, as basic primitive elements. However, for the most part, each implementation of a blackboard mathematical model requires definitions of its own abstract spaces, functions, indexes, and so forth.[5] To help in writing new models, the environment can provide math section utilities such as templates and run-time support for these elements; Appendix B discusses the utilities provided in our prototype system.

4.2.4 C-M and M-N Interfaces

Between the Conceptual and Math sections lies the *C-M interface,* and between the *M*ath and *N*umerics sections lies the *M-N interface*. The interfaces are the part of the program that arrange for posed problems to be solved.

C-M Interface

The C-M interface contains the "know-how" to construct a mathematical model from a conceptual model, and how to transfer data between them. Thus, the C-M interface is the high-level instrument by which posed problems are solved: it converts a conceptual task into the appropriate mathematical problem statement, constructing (via the M-N interface) mathematical objects whose values are solutions to the problem. In transferring the resulting data from the mathematical to the conceptual model, the C-M interface implements part of the physical interpretation phase of physically-based modeling (Sec. 2.3.4). For example, Fig. 4.5 illustrates the control flow to draw the conceptual model at a particular instant of time. For models in which objects are created and destroyed over time, the C-M interface will know how to create/destroy or activate/ deactivate conceptual-model objects based on mathematical results.

[5] Thus programmers must be "toolmakers" as well as users of tools. We follow the design philosophy that to write a program, one should write a module library then call it (see, e.g., [Strauss85]).

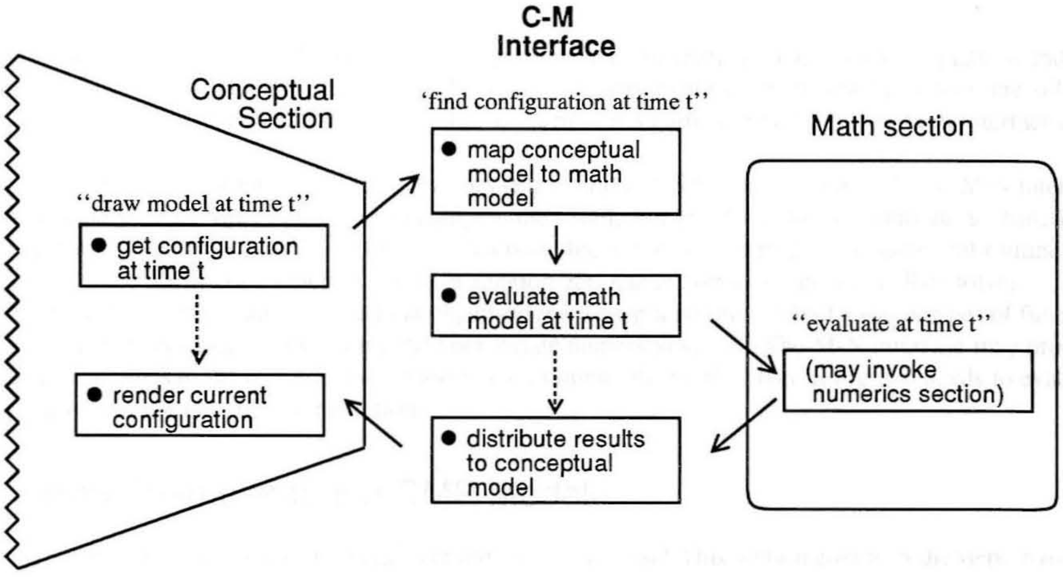

C-M Interface
Conceptual Section
"draw model at time t"
● get configuration at time t
● render current configuration
'find configuration at time t"
● map conceptual model to math model
● evaluate math model at time t
● distribute results to conceptual model
Math section
"evaluate at time t"
(may invoke numerics section)

Figure 4.5: Outline of control flow across the C-M interface, for a sample task. The conceptual section uses the math section as a "computation engine," via the C-M interface. Here, the posed problem is to draw the model at a particular instant of time. If the task will be performed repeatedly with minor variation, such as drawing the model for each frame of an animation, the initial mapping may be performed just once in a "setup" phase. □

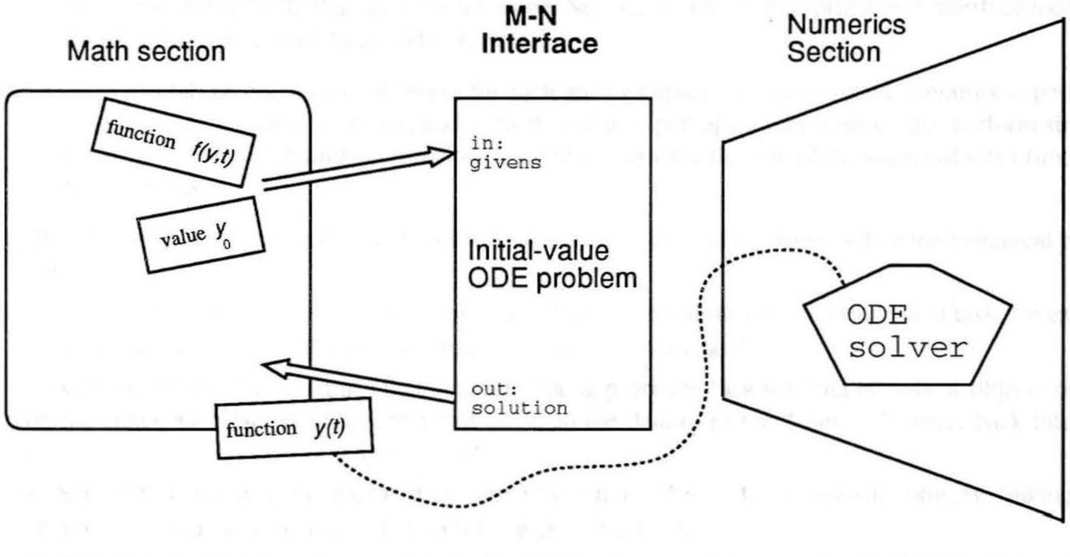


Figure 4.6: Role of the M-N interface, for a sample task. The M-N interface maps between mathematical model objects and numerical solution techniques. Here, an M-N interface routine constructs a function object, $y(t)$, that solves an initial-value ODE, $\frac{d}{dt}y(t) = f(y,t)$, $y(0) = y_0$; the function object is "tethered" to a numerical ODE solver, so that if evaluated, the function will implicitly invoke the numerical routine to compute the result. □

The C-M interface's "know-how" does not have to be completely hard-wired. An obect-oriented design for the conceptual section can allow each type of conceptual object to define methods relating to its corresponding math section elements. The C-M interface may provide "callback" functions to the mathematical section, which evaluate whatever mathematical equations reside in the conceptual model.

M-N Interface

The M-N interace contains the "know-how" to map between mathematical model objects/equations and numerical modules. Thus, the M-N interface is the low-level instrument by which posed problems are solved: given mathematical model objects containing the "knowns" for a particular problem, the M-N interface constructs objects for the solution.

For problems whose solutions are simple values (e.g., "find $x \in \Re$ that minimizes. . . "), the M-N interface can immediately call a numerical solver to compute the result. For problems whose solutions are functions, however, the M-N interface might construct a function object that is "tethered" to a numerical routine; for example, Fig. 4.6 illustrates a function whose evaluation gets handed off to a numerical ODE solver.

The "tethering" can be implemented via object-oriented programming; each of various types of function objects implements evaluation by calling the appropriate numerical solver. The M-N interface may provide "callback" functions to the numerical techniques; for example, the ODE solver in Fig. 4.6 needs to evaluate $f(y, t)$ during the course of its computation.

4.3 How To Implement a CMP Model

Suppose we have a CMP blackboard model worked out—now what? This section goes over the steps involved in creating an implementation of the model.

A CMP module translates into a collection of definitions of data structures, objects, routines, and so forth. Remember that we're not trying to make an end-user program, but to add a collection of tools to our toolbox, as discussed in Sec. 4.1. Here's what to do:

Conceptual Model Define a data structure that represents the conceptual model, including all properties, relationships, and so forth, that are relevant, as per Sec. 4.2.1. Build appropriate user interface tools, to read/write files, send scenes to a renderer, etc.

Mathematical Model Define a class of object for each abstract space, with associated operators to perform evaluations, access elements of sets, and so forth, and also perhaps utility routines that perform simple symbolic tasks, such as defining a function object that computes the sum of the values of other function objects, as per Sec. 4.2.3.

Posed Problems Each posed problem consists of a conceptual task, and a corresponding mathematical problem.

• Define conceptual section routines to provide a high-level interface to the conceptual task; for example, a routine `draw(t)` that draws the model at time `t`, as per Fig. 4.5.

• Define C-M interface routines that construct the appropriate "known" mathematical objects based on the conceptual data structure, and that can map the data from the "solution" object back into the conceptual data structure, as per Sec. 4.2.4.

• Define M-N interface routines that construct "solution" objects from "known" objects, linking the solution to modules in the numerical section, as per Sec. 4.2.4.

• Choose the numerics section modules to use. If necessary, write modules to transform between the high-level representations used in the mathematical objects to lower-level representations needed by the numerical solvers, as per Sec. 4.2.2. If necessary, write new low-level numerical solvers, or incorporate techniques from books or public-domain or commercial packages; these might be specific to the particular CMP module, or (ideally) might be suitable for incorporation into the common numerics section library.

We don't necessarily do all the above in the listed order. Most commonly, we start with the mathematical model definitions, then work bottom-up through the posed problem, and finally put the high-level conceptual interface on top.

Note that if the blackboard CMP model was designed modularly (Sec. 2.5), based on existing CMP modules, we can correspondingly use existing tools in building the new tools. This can be done at any or all levels: math objects that include other math objects; C-M routines that call other C-M routines; and so forth. Additionally, the environment can include basic support for common mathematical constructs (e.g., indexes and state spaces as per Ch. 3) and conceptual tasks (e.g., animation loops that call a `draw(t)` routine for each frame), as well as a rich numerics library; Appendix B describes the tools in our prototype system.

Note also, occasionally not all the above items need to be created. For example, the geometric objects (locations, vectors, scalars, etc.) defined in Ch. 6 need no M-N interface; all their operations can be performed symbolically or arithmetically. For another example, in Fig. 2.6's version C of a flexible-body model, the conceptual model includes a refinement to a collection of rigid bodies; thus given pre-existing rigid-body tools, the flexible-body conceptual tasks could translate directly into rigid-body conceptual tasks, and no new math, numerics, or interface routines would need to be written.

4.4 Procedural Outlook

The math section objects, as we said in Sec. 4.2.3, don't "do" anything. Thus the procedural elements in the program are the two ends: the conceptual section and the numerics section.

Let's look at what happens when the conceptual section performs a task: The C-M interface bundles up the relevant information into a data structure (the mathematical model), and passes it to the M-N interface; then the M-N interface unbundles the information to invoke the numerical solvers, bundles the result back into mathematical data structure objects, and passes it back to the C-M interface; finally, the C-M interface unbundles the results for the conceptual section to use.

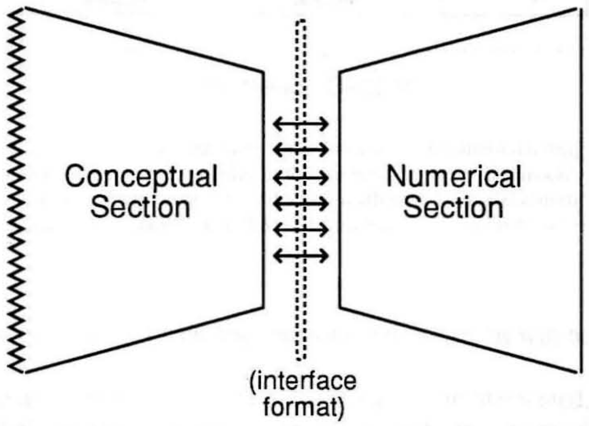


Figure 4.7. From a procedural point of view, the mathematical model is just an intermediate interface format between the conceptual and numerical sections of the program. □

Thus, from a procedural point of view, the mathematical model is essentially just an interface format between the conceptual and numerics sections. (Or even just an interface format between the C-M and M-N interfaces.)

4.5 Why Have a Math Section?

Since, as we discovered in Sec. 4.4, the math section objects is in some sense just define an intricate interface format between the conceptual and numerics sections, one might wonder why we bother to define them. Why not just let the conceptual section call the numerical routines directly? There are several reasons for having a separate mathematical model.

First, having a separate, explicit mathematical model allows the program to directly correspond with the CMP structure discussed in Ch. 2. This helps eliminate "transcription" errors in developing the program from the blackboard model. Conversely, when the program points out flaws in our model, and sends us back to the drawing board, we will have an easier time updating the CMP model.

Furthermore, although the mathematical model is derived from the conceptual model, it is not necessarily a simplification of the conceptual model; it is likely to be organized differently. For example, in the mathematical model, the states of all the bodies that have the same equations of behavior may be grouped into a single index, regardless of the hierarchical relationship of the bodies in the conceptual model. Thus the task of the C-M interface can be complex, and if we were to try to merge the mathematical model organization into the conceptual model, it could end up a tangled mess. Along the same lines, having a separate mathematical

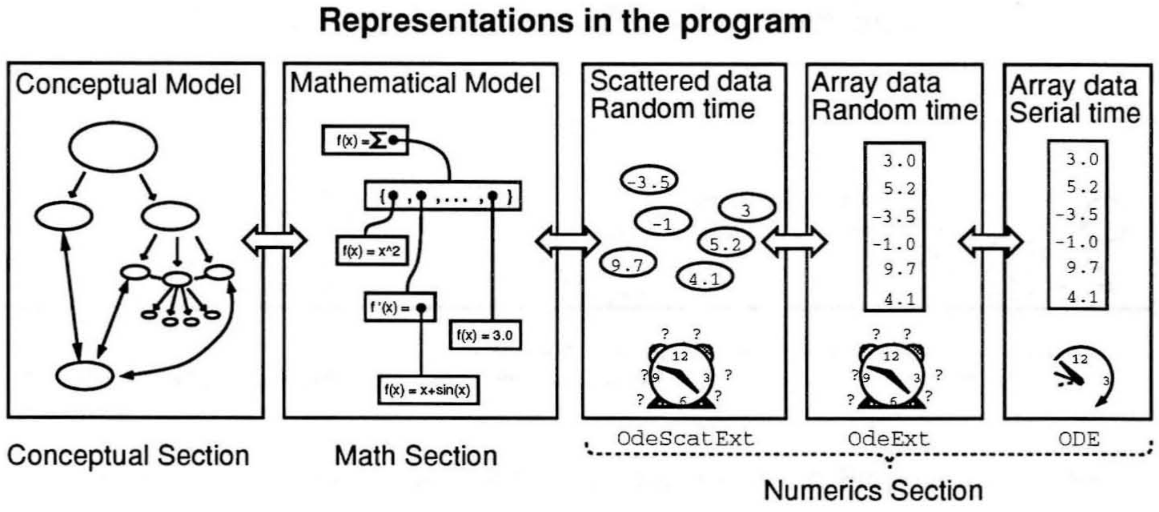


Figure 4.8: A sequence of changes of representation, from the high level that the user interacts with, down to the number-crunching format used by the numerical solverings. The illustration shows several steps within modules in the numerics section: The module `OdeScatExt` evaluates the solution $y(t)$ to an ODE at arbitrary values of t, distributing the results into mathematical model objects; `OdeExt` evaluates $y(t)$ at arbitrary values of t, transferring the result as an array of values; and `ODE` is a traditional step-wise ODE solver that computes array-valued results for increasing values of t. □

model also makes it easier for us to decouple the state of the model from the state of the program, as will be discussed in Sec. 4.7.

Finally, the math section can be sufficiently large and intricate to justify being designed, implemented, tested and debugged independently, or with just the M-N interface. Once it is working, it can be incorporated into the higher levels of the program. (See also the discussion of debugging, Sec. 4.9.)

We can think of the math section as providing a programmatic mathematical manipulation/computation package for the given model. Consider a programmatic package that supports rational numbers, or infinite-precision arithmetic: it would include definitions of the primitive data objects, and operators and subroutines to manipulate the objects and perform numerical computations. The math section for a given model does the same, but the "primitive objects" represent not just numbers, but indexes (Sec. 3.8.2), elements of state spaces (Sec. 3.9), and so forth; and the M-N interface provides the computational support.

4.6 Representational Outlook

A program that provides a high-level model to the user, and also uses "number-crunching" routines to compute simulations, must by necessity transform data between the form of the high-level representation and that of the numerical routines. This transformation can typically be decomposed into a series of separate steps.

We prefer to design programs such that those changes of representation are explicit; hence the separation between the conceptual, math, and numerics sections in the program framework. The C-M and M-N routines are the agents that perform the changes. The changes of representation continue within the numerics section as well. Fig. 4.8 illustrates the changes of representation that we commonly carry out to simulate a model as a function of time. Notice that the whole program can be viewed as a "pipeline," analogous to the well-known rendering pipeline for computer graphics imaging ([Foley et al.90]).[6]

Changing representations is not often considered to be a major functional component of a program: The "meat" of the program might be identified as the number crunching or rendering or user interface; while the

[6] Just as computer hardware is often built to directly support the rendering pipeline, we can imagine hardware support of a modeling pipeline for some classes of posed problems.

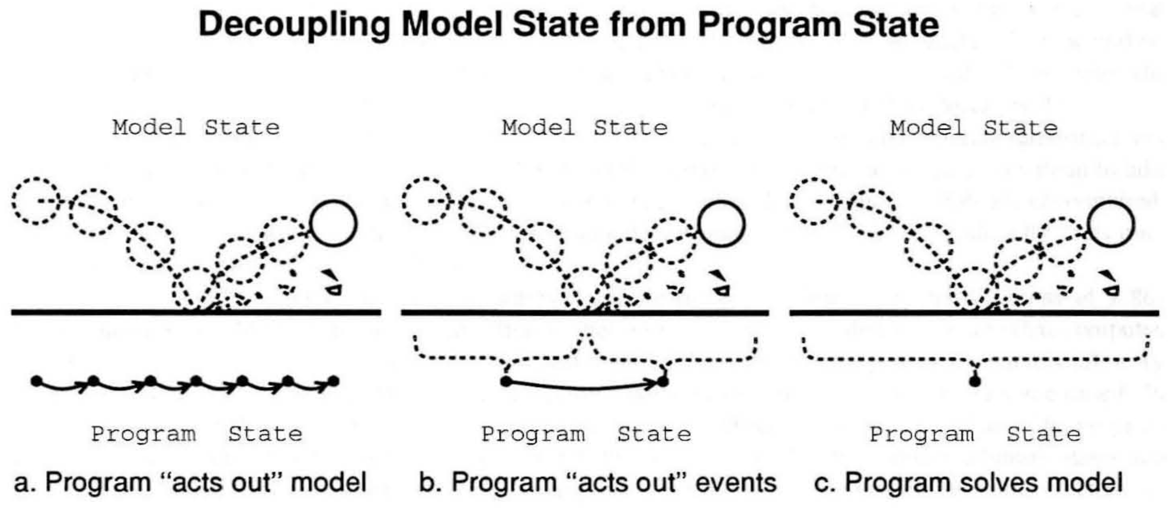

Decoupling Model State from Program State
Model State
Model State
Model State
Program State
Program State
Program State
a. Program "acts out" model
b. Program "acts out" events
c. Program solves model

Figure 4.9: State of the simulated model vs. state of the program. We illustrated three programming paradigms: (a) Each step of program execution moves the model forward in simulation time. (b) The program computes continuous behavior as a mathematical function of simulation time, but if there are changes in the structure of the model, the state of the program must be altered. (c) The program computes the behavior as a mathematical function that spans discontinuities; the state of the program need not be altered to evaluate the model in any of its states. □

code that links various modules together often seems like it's not "doing" anything, merely housekeeping. However, we have found that often, changing representations can be the trickiest part of a program. Housekeeping tasks, such as gathering/scattering together data from a collection of separate objects into a single linear array, are often fraught with bugs.

By identifying representation-change tasks explicitly, we can often separate them out from the "meat," and build separate modules that perform the changes. The changes of representation can thus be designed and debugged independently, and the remaining routines are freed from the need to do the housekeeping. Appendix B describes change-of-representation modules for array gather/scatter, full-to-sparse matrix, and sequential-to-random access of data.

4.7 Decoupling Model State from Program State

For computer graphics modeling, the most commonly posed problem for a model is: *simulate the model's behavior over time*. Mathematically, this means we define a function $Y(t)$ that we evaluate to yield the behavior at any given time t. The most common form for the results of a simulation is a sequence of frames for an animation; that is, we sample the function Y(t) at times $t_0, t_0+\Delta, t_0+2\Delta, \ldots$, rendering and recording an image for each sample. Thus, programs are often designed to evaluate $Y(t)$ only at a sequence of increasing times.

Given that we've expressed our model's behavior mathematically as a function $Y(t)$, however, we don't want to limit ourselves to sampling it sequentially. We would like to "random access" the function, i.e., evaluate it at any arbitrary time values, in arbitrary order. This would allow, for example:

- "Shuttling" back and forth in interactive output;
- Rendering using motion-blur techniques that need arbitrary time samples, e.g., distributed ray tracing ([Cook,Porter,Carpenter84]);
- Solving numerical problems that may span events, such as multi-point boundary-value problems ([Press et al.86]).

Random access of the model is particularly appealing, and difficult, when the model includes events that change its structure, such as adding new bodies or relationships. Mathematically, the function $Y(t)$ would be a *segmented function* as per Sec. 3.10. In the conceptual section of the program, we need to "decouple" the program's state from the state of the model; the next few paragraphs illustrate what we mean by this.

Consider a program that simulates the model's behavior by Euler's method: The program data structures describe the state of the body at some time t_k. The program computes the derivatives, and uses them to take a small step forward, updating the data structures for time t_{k+1}; when discontinuous events are encountered, the program's data structures are modified, and simulation continues. Thus the program directly "acts out" the behavior of the model over time (See Fig. 4.9a).

Sophisticated ODE solution techniques, which are more robust than Euler's method ([Press et al.86-Ch.15]), need to be able to explore the state space: they require a user-provided subroutine that computes derivates based on *hypothetical* states and times, and will call that subroutine repeatedly for many arbitrary values in arbitrary order,[7] before settling on and returning what the model's state *really* is for some time t. In order to use these techniques, one must make a leap in how one thinks about one's program: The program no longer has a "current state" that is always correct, but needs to be able to evaluate arbitrary states and times, that might be off the solution manifold; the program is not acting out the model, but rather calculating a function that describes how the model varies over time.

Using sophisticated numerical solvers thus forces a certain amount of decoupling of program state from model state. However, that decoupling is typically for continuous changes in state only; programs still typically solve forward in time; when an event happens, the program's data structures are changed, and a new continuous problem is solved forward from there. Thus the program is still "acting out" the discrete events (see Fig. 4.9b.)

We go one step further: Solve a mathematical segmented function, which spans discontinuous, state-changing events, and evaluate that solution anywhere. This requires that the conceptual model data structures can be adjusted based on the value of the function: the number of bodies, etc., isn't known a priori. It also requires numerical solvers that can compute them. The example "tennis ball cannon" model in Ch. 11 illustrates a model described by a segmented function, that is solved using the piecewise-continuous ODE solver described in Appendix C.[8]

4.8 Efficiency

The conversion to and from a math section data structure, and the overall emphasis on changes of representation, can give us pause: This approach has some advantages in terms of modularity and so forth, but if it makes our programs too slow or memory-intensive, we won't be able to use it. This section takes a brief look at efficiency.

Start by considering the far extreme: assume that efficiency is not a concern. Since the mathematical model is defined based on the conceptual model, and has no changeable internal state of its own, in principle it doesn't need to be maintained all the time. Whenever the conceptual model needs some numbers, such as the state of the model at a particular instant of time, it can whip up a mathematical model, evaluate it, get the results, and toss it out.

But, since we often solve the same problem over and over again, we can be more frugal than that. Commonly, the solution is a function object, we don't throw it away but rather keep it, and evaluate it repeatedly. This allows the back end to maintain previous values, cache results, take advantage of coherence, and so forth.

In general, the changes of representation involve a structural setup phase, that build data structures, set up memory address pointers, and so forth. The actual transfer of data can just follow those pointers, often

[7] John Platt has coined the apt name "numerics dance" to describe this process [Platt87].

[8] Another approach to random-access of models is to compute sequentially, but save the "history" so that we can play back prior states as needed—this would work even for Euler's method. However, as mentioned earlier, defining the mathematical model as a segmented function can let us solve problems that span events, such as multipoint boundary-value problems: during the "numerics dance," solvers might need to explore possible states of the model across discontinuous events—and whether any individual event *really* occurs won't be known until the solution is complete.

"leapfrogging" past several changes of representation to place the data in its final destination.

Note also that for numerically intensive computations, most of the inner loops are inside of the low-level numerical routines—thus efficiency of the higher levels of the program is not critical. Often, however, there are "callback" routines that must be used in an inner loop, e.g., the routine that computes derivatives for an ODE solver; but these don't necessarily have to go all the way back up the representational pipeline, and also, as before, can be set up once then quickly accessed later.

Our overall approach to dealing with efficiency in our system is as espoused by [Kernighan,Plauger78]: "Make it right before you make it faster." That is, we first design and prototype in an idealized, modular form, written to optimize cleanliness of concept. Once it's written and working, if it turns out to be insufficiently efficient, we can use "profilers" and other diagnostic techniques to determine what needs to be optimized. It's always relatively easy to go back and cut corners to provide speedups (for special cases) when necessary—but if it's originally designed and built with those corners cut, it would be harder to go back and modularize later. Remember that one of our goals is to define general, reusable tools, not optimized special-purpose simulations.[9]

4.9 Debugging

As we implement physically-based models as computer programs, we can encounter two types of bugs:

- *bugs in the program,* and
- *bugs in the model.*

Bugs in the program mean that it doesn't faithfully implement the blackboard model; bugs in the blackboard model means that we goofed in our CMP design. It is important to recognize the difference between these two types of bugs, or we might spend lots of time and effort on wild goose chases: for example, we might pore over trace output, single-step through program execution, and so forth, trying to find out where we've made a "typo" in an expression, but the real problem is a "thinko" in the conceptual model; conversely, we might keep trying to re-derive or re-express a mathematical equation, but the problem turns out to be an `a[i]` that should be an `a[j]`.

Of course, it's not always easy to determine what type of bug we're dealing with—that can be a major part of the debugging task. However, the CMP structure of the model can help us to isolate and identify bugs in the model, as was discussed in Sec. 2.6.4. Similarly, the modular design of the program framework can help us isolate bugs in the program: Are the numerical modules working correctly? Are the results being properly represented as math section objects? and so forth.

In blackboard mathematical models, there are often redundancies, or multiple ways of expressing a quantity; we can take advantage of this to help find bugs in programs. For example, the state of a dynamic rigid body (Ch. 8) includes mass m, momentum p, velocity v, and position x; when we create or use a state object, we can doublecheck to make sure that the relation $p = mv$ holds (i.e., check that the internal properties of the state space are maintained, as per Sec. B.3.6), and for a body whose behavior is computed numerically, we can doublecheck that $v = \frac{d}{dt}x$ (by finite differencing) as expected.[10]

Double-checking can happen at a very broad scope—we can "swap in" additional problems to solve in order double-check our simulation, i.e., the C-M interface can create additional elements in the mathematical model, and request additional solution objects from the M-N interface. The ability to do this "swapping" is a benefit of the modular framework in general, and the separation of the mathematical model in particular.

[9] From Dijkstra's *Notes on Structured Programming:* "My refusal to regard efficiency considerations as the programmer's prime concern is not meant to imply that I disregard them... My point, however, is that we can only afford to optimise (whatever that may be) provided that the program remains sufficiently manageable." [Dijkstra72-p.6]

[10] Because of finite precision, and approximate solutions to problems, we can't expect values that are analytically equivalent to be exactly equal in the program, thus we can only check to see if the values are "close enough" within some tolerance. But how to choose that tolerance? The tolerance doesn't have to be tight—it doesn't affect the accuracy of simulation, it's only a guard against errors. When there's a bug, the results are often **very** wrong, so they will exceed any small tolerance. Also common is buggy behavior that causes the different quantities to diverge as the simulation progresses, so that even if the tolerance is too loose to initially catch the error, after a short time the tolerance will be exceeded.

It can also be useful to put "assertions" in the program, to check for things that the model says can't happen: two disjoint events that happen simultaneously, or singularities in equations, and so forth. When such things do occur, it can often be easy to determine by inspection if they are merely bugs in the program, or if they have uncovered gaps in our conceptual understanding of the problem (see Sec. 2.4.3).

4.10 Summary

This chapter has described an approach to implementing physically-based models that are defined via the CMP structure of Ch. 2. The focus is on an overall framework for the creation of tools in a modular and extensible manner. The tools are intended as basic reusable support for programmers to create end-user applications.

The program framework parallels the CMP structure; in particular it emphasizes that the role of the program is to define and solve mathematical problems, then convey the results back to the user. Thus there is an emphasis on a separate and explicit definition of a programmatic mathematical model and of numerical solution techniques.

The framework also emphasizes the simulation of a model as a change between various representations of the model and data. The changes of representation are performed in a series of well-defined, modularized steps.

A feature of the framework is that it allows a decoupling of the state of a program from the state of the model that is simulated; thus the program can explore the state space of the model to produce solutions and can "random access" the solution in various states—even spanning discontinuities.

4.11 Related Work

[Zeleznik et al.91] presents an interactive modeling system which, although not specifically focused on physically-based modeling, is similarly based on the specification of an extensible, object-oriented framework.

[Kalra90] provides a unified scheme for solving arbitrary constraint problems; in the lingo of our own work, it discusses how to transform a mathematical representation to a numerical representation.

[Blaauw,Brooks91] takes view similar to ours, of computation as a series of changes of representation.

Chapter 5

Overview of Model Library

The preceding chapters presented a structured strategy for designing and implementing physically-based models. The main ideas of the strategy are:

- Decomposition of a model into conceptual/mathematical/posed-problem parts (Ch. 2).
- Modular hierarchy of models (Ch. 2).
- Mathematical modeling techniques and notation (Ch. 3).
- A program framework having separate Conceptual/math/numerics sections (Ch. 4).

We move now from theory and philosophy to practice and applications. This chapter gives an overview of a prototype library for rigid-body modeling, that was designed using the structured strategy, and that is discussed in detail in the upcoming chapters. The prototype library (Ch. 6–9) supports classical dynamic rigid body motion, with geometric constraints. Sample models built using the library are described in Chs. 10,11. The library and models have been implemented as described; implementation notes are included in the upcoming chapters, and Appendix B has an overview of the prototype implementation environment. Some possible extensions to the library are outlined in Ch. 12.

5.1 Goals for the Prototype Library

The prototype library has three major goals:

- **To demonstrate the structured design strategy.**
  The development of the strategy, in Ch. 1–4 included an assortment of philosophy and techniques. Here, we illustrate how we bring them all together, by developing (and using) an extensible library, from initial concept through implementation details.
- **To test the feasibility of the design strategy.**
  The strategy is experimental: an idea that we are putting forth for consideration, but which has not yet passed "the test of time." The prototype library will test the practicality of the strategy, e.g.: Do our models fit into a CMP decomposition? does modularity work for us? are the mathematical techniques practicable? etc. (Ch. 13 includes an evaluation of our experience.)
- **To provide a rigid-body modeling library.**
  This prototype library attempts to serve as a first step towards a general, reusable, and extensible library for rigid-body dynamics, as espoused by the design strategy.

We have attempted to design and present the smallest possible example of a library that would reasonably meet the above goals—our intent is to be illustrative, rather than encyclopedic. However, we try not to cut

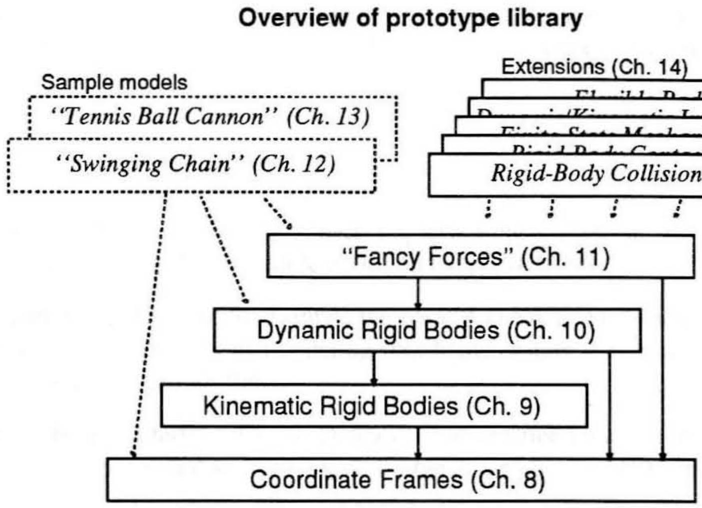

Overview of prototype library
Sample models
"Tennis Ball Cannon" (Ch. 13)
"Swinging Chain" (Ch. 12)
Extensions (Ch. 14)
Rigid-Body Collision
"Fancy Forces" (Ch. 11)
Dynamic Rigid Bodies (Ch. 10)
Kinematic Rigid Bodies (Ch. 9)
Coordinate Frames (Ch. 8)

Figure 5.1: Modules in the prototype library. The Coordinate Frames module provides us with a common framework for working with 3-D coordinate geometry. The Kinematic Rigid Bodies module defines our idea of rigid-body motion. The Dynamic Rigid Bodies module adds classical Newtonian mechanics. The "Fancy Forces" module provides a mechanism to specify forces for the Newtonian model, that supports geometric constraints on bodies. Sample models illustrate the use of the library. We also discuss ideas for extending this library. □

any corners: in order to validly demonstrate and test the strategy, the prototype was developed, structured, implemented, and presented in accordance with the stated design strategy. Thus we've ended up with "just a simple example" that fills more pages than the description of the strategy itself.[1]

Note also that we will present only the result of the design process, rather than chronicle the design process itself. The reader may rest assured that the prototype library was not initially conceived in exactly its final form, but rather we designed, modified, implemented, re-designed, and so forth

5.2 Features of the Library

The library includes the following features for rigid-body modeling:

- Basic Newtonian motion of rigid bodies, in response to forces and torques.
- The ability to measure work done by each force and torque, and balance it against the kinetic energy of the bodies.
- Support for various kinds of forces to apply to bodies, including "dynamic constraint" forces ([Barzel, Barr88]) to allow constraint-based control.[2]
- The ability to handle discontinuities in a model.
- The ability to be extended, both by enhancing the modules we describe, and by adding additional modules.

[1] From Dijkstra's *Notes on Structured Programming:* "I am faced with a basic problem of presentation. What I am really concerned about it the composition of large programs, the text of which may be, say, of the same size as the whole text of this chapter.... For practical reasons, the demonstration programs must be small, many times smaller than the 'lifesize' programs I have in mind." [Dijkstra72-p.1]

[2] We don't intend to be self-serving by describing our own "dynamic constraints" work, nor to imply that it is the primary method of control that should be supported by a rigid-body modeling library. Rather, it comprises a reasonably intricate test case for the design strategy, and has the advantage (to us) that we are experienced with it and thus were able to focus on the structure of the model rather than on getting the technique to work.

To give a feel for how the library could be extended, Ch. 12 will discuss several possible additions: rigid body collision and contact, finite-state control mechanisms, transitions between kinematic and dynamic behavior, and flexible bodies.

5.3 Outline of the Library

The prototype library includes four modules, each implementing a different "sub-model" within the overall domain of rigid-body modeling. Each module builds on the previous modules. (Fig. 5.1)

1. **Coordinate Frames Model** (Ch. 6): Defines the basic 3-D Euclidean world space in which our models exist, and provides support for manipulating coordinate system frames and geometric objects such as orientations, locations, and vectors.

2. **Kinematic Rigid Bodies Model.** (Ch. 7): Defines our abstraction for kinematic motion (i.e., motion without regard to force or inertia) and provides a simple descriptive mechanism.

3. **Dynamic Rigid Bodies Model** (Ch. 8): Provides support for bodies moving under the influence of arbitrary forces and torques. Includes an energy-balance mechanism.

4. **"Fancy Forces" Model** (Ch. 9): Provides a mechanism to define forces and torques, and apply them to arbitrary bodies. Integrated with the "dynamic constraints" force mechanism.

We provide examples of models built using the library:

"Swinging Chains" (Ch. 10): Bodies linked and suspended to form chains swinging in gravity. Illustrates continuous dynamic motion, using the constraints mechanism.

"Tennis Ball Cannon" (Ch. 11): An oscillating cannon fires a stream of balls, that bounce, change size, and disappear. Illustrates discontinuous changes in state and configuration of a model.

The "model fragments" (Sec. 1.3), i.e., the low-level equations of behavior, that are embodied in the library are quite simple, as compared with, say, flexible body mechanics, or fluid dynamics. Our emphasis for this prototype is on modularity in the models, the ability to pose multiple problems from a single model, the interrelationships between the bodies, and other design issues, rather than on particularly complex behaviors.

5.4 Common Mathematical Idioms

Because of our emphasis on explicit statement of assumptions and properties, the expositions in our modules follow a rather axiomatic approach that is more common in pure mathematics than in physics or mechanics: Each module has a series of definitions of abstract spaces and properties, then provides further equations that are derivable from those definitions.[3]

Note that the principles and equations that underlie the various modules in our library are well-known; there will be no fundamentally new or surprising equations or derivations. Thus the mathematical models in the libraries serve mostly to re-cast the known equations into a form convenient for our structured, modular outlook. Because we will be covering well-trodden ground, we will not in general include detailed proofs or derivations, but rather refer interested readers to appropriate references.

The modules include definitions of various state spaces (Sec. 3.9), to describe the configurations of objects. Often, a state space will include a description of the motion as well—in which case the space is a generalization of a physicists' *phase space* ([Marion70]). Note that each point in a state space typically describes the state

[3] In this, we are similar to the field of *Rational Mechanics*: "The traditional approach to mechanics is in no way incorrect, but it fails to satisfy modern standards of criticism and explicitness. Therefore, some parts of the foundations of mechanics heretofore left in the penumbrae of intuition and metaphysics I shall here present in an explicit, compact mathematical style... " [Truesdell91-p.6].

of an object *at an instant*—the behavior of an object over time is described by a path through state space. If the state space includes motion information, we will often explicitly define a *consistent* path to be such that the motion description at each state along the path actually agrees with the trajectory described by the path.

Expositions of physics commonly use a single name as a value or as a function. For example, "x" may be defined as the location of a particle, and later "$x(t)$" would be used to describe the location as a function of time. However, as discussed in Sec. 3.6.3, we prefer a name to have only one meaning. Thus, if we define a value

$$x \in \mathsf{States}$$

for some space States, we will *never* use it as a function; if we are interested in a function, we will always define one explicitly, e.g.,

$$p\colon \Re \rightarrow \mathsf{States},$$

in which case "p" refers to the function as a whole, and "$p(t)$" refers to the value of the function for some $t \in \Re$. We will often define a space of functions, such as

$$\mathsf{StatePaths} \equiv \textit{the set of functions } \{\Re \rightarrow \mathsf{States}\},$$

and we will then use

$$p \in \mathsf{StatePaths}$$

to denote a function p.

Finally, all of the bodies in the prototype models exist in a 3-D Euclidean world; all "vectors" and so forth are thus 3-space objects, as defined in the coordinate frames model, Ch. 6.

5.5 Presentation of Each Module

Each of the upcoming chapters describes a single CMP module, as per Sec. 2.5. We consistently use the following organization for each chapter, combining the CMP framework of Sec. 2.7 with the mathematical modeling framework of Sec. 3.11.1:

Introduction. A small blurb at the front of each chapter, giving an overview of the domain and use of the module in the chapter. More extensive background may be given as an initial section of the chapter, if needed.

Goals. A description of our purpose or desired features for the module.

Conceptual Model. A description of the conceptual model, as per Sec. 2.3.1. Our focus in these prototype modules is on behavior as defined mathematically; thus the conceptual models will include little beyond those elements that will be described by the mathematical model.

Mathematical Model

Names & Notation The scope name (Sec. 3.6.2) for the module, a list of names used from other modules, and a description of any unusual notations that will be used.

Definitions & Equations These will be broken up into sections as appropriate.

Posed Problems As discussed in Sec. 2.4.2, there are potentially many interesting problems that can be posed for a given model. We will typically include only one or two interesting or common problems.

Implementation Notes These describe the prototype implementation; the implementation follows Ch. 4's program framework: separate conceptual, math, and numerics sections, along with C-M and M-N interfaces. Appendix B gives some details of our particular implementation, and discusses the terminology and notation we will use in the notes for each model. Since our focus in this prototype is on the mathematical models, the notes will emphasize the math section and, to a lesser extent, the C-M and M-N interfaces; the conceptual section will simply be sketched at a high level.

Derivations & Proofs We put these at the end (rather than within the mathematical model) so as not to get in the way.

5.6 Related Work

[Fox67] and [Goldstein80] are two classic classical mechanics references. [Marion70] gives a good introduction to the dynamics of particles and rigid bodies. We will refer also to [Craig89] for kinematics and 3-D geometry.

Chapter 6

Coordinate Frames Model

Computer graphics models commonly deal with 3-dimensional geometric objects such as locations, vectors, etc., often working with several different coordinate systems. This module contains some basic definitions and notation for manipulating these objects and representing them in arbitrary orthonormal coordinate frames.

The mathematics for 3-D objects and coordinate systems is well-known; see e.g., [Craig89] and [Foley et al.90]. We assume that the reader has at least a passing familiarity with the ideas of vectors, matrixes, tensors, and so forth; although we will define all our terms axiomatically, this exposition would not suffice as an introductory text in linear algebra.

This module is intended to give us a standard, convenient, and consistent form for using Euclidean 3-D coordinate objects within our modeling environment. It does not address curvilinear coordinate systems, such as cylindrical or spherical coordinates. It also does not address geometry in homogeneous coordinates (which is common in computer graphics [Foley et al.90]), nor in curved spaces.

6.1 Background

Conceptually, the ideas of geometric objects such as coordinate systems, vectors, locations, and so forth are often conveyed via diagrams (e.g., Figs. 6.1,6.2, etc.). Mathematically, however, one needs a formal definition and algebra. There are two common approaches to the definition and use of geometric objects:

- *Numerical Coordinates.* Objects are be defined as collections of coordinate values (also called *components*), along with rules that describe how the values change if one switches coordinate systems. Arithmetic and other operations are defined by matrix operations on the coordinate values.

- *Abstract entities.* Objects are defined as elements of abstract spaces. Arithmetic and other operations are defined in a manner independent of coordinate systems. When coordinate values are needed, mappings are defined from the abstract spaces to any desired coordinate system.

The abstract approach has a simple, compact notation, and can extend to arbitrary manifolds in non-Euclidean spaces. It is thus the approach of choice for studying differential geometry (e.g., [Millman, Parker77]). However, the details of numerical computation are typically hidden or implicit.

The coordinate-based approach, on the other hand, keeps track of numerical values in particular frames of reference. The equations tend to be cumbersome—full of "bookkeeping" of coordinate frames and components—but they apply immediately to numerical computation. It is thus the common approach used for practical applications, e.g., robotics ([Craig89]) or computer graphics ([Foley et al.90]).

For our purposes, we want elegant mathematical models—thus we want the convenience and expressive power of the abstract approach. But we also need to perform numerical computations—thus we want the

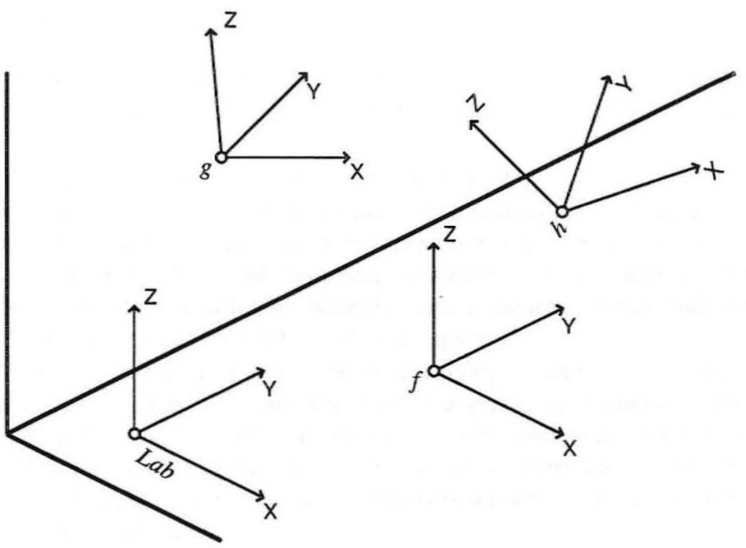


Figure 6.1: Multiple coordinate frames. We illustrate several different right-handed, orthonormal coordinate frames, labeled f, g, h, and $\mathcal{L}ab$, all defined in an absolute, uniform, 3-D world space. $\mathcal{L}ab$ is an arbitrary frame that is chosen to be a fixed standard. □

applicability of coordinate-based formulations.[1] Our approach will therefore be to define the geometric objects as abstract concepts, and define the corresponding abstract mathematical notation, but also provide a notation that lets us express the objects in arbitrary coordinate systems and perform numerical manipulations.

6.2 Goals

We have several goals for this module:

- To have a standard model and notation for geometric objects, that encapsulates coordinate systems and coordinate transformations.
- To support useful geometric objects, be they tensors or non-tensors.[2]
- To be able to use coordinate-free notation and expressions.
- To be able to use coordinate notation and expressions, for arbitrary coordinate systems.
- To support objects and coordinate systems that change as functions of time. A particular special case is an object that has a constant representation in changing coordinate systems, such as a piece of material that is fixed to a rigid body, and is carried with it as the body moves.

The emphasis for the above goals is on basic support for mathematical models and corresponding program implementations, rather than on any *a priori* conceptual model.

[1] "*Pictorial* treatment of geometry... is tied conceptually as closely as possible to the world... *Abstract* differential geometry... is the quickest, simplest mathematical scheme... *Components* [are] indispensible in programming... Today, no one has full power to communicate with others about [geometry] who cannot express himself in all three languages." [Misner,Thorne,Wheeler73-p.199]

[2] A *tensor* is an object that obeys certain transformation rules when we switch between coordinate system (see Eqn. 6.11).

6.3 Conceptual Model

We assume the existence of an absolute 3-dimensional Euclidean space, sometimes called *world space*. The space is homogeneous and isotropic, i.e., has no preferred directions or locations. We also assume that *time* is homogeneous.

We can place x, y, and z axes anywhere we like in space, to define a coordinate system which can be used to represent geometric objects. We will refer to each triple of axes as a *coordinate frame*, or just *frame* (Fig. 6.1). Each frame gives us a coordinate system for measuring objects in the world—i.e., the frame gives us a particular vantage to "look at" the world. We restrict ourselves to frames that are orthonormal, i.e., the x, y, and z axes are perpendicular and all frames have the same scale for distance, and that are right-handed, i.e., if x points to the right and y points forward, z will point upward.

We define several types of abstract geometric objects. Each object can be described in a given frame by a collection of coordinate numbers, which we call the object's *numerical representation*, or just *representation*. We emphasize the distinction between an abstract object and its representations: A single abstract object may have different representations in different frames. A collection of numbers, along with a frame, can uniquely describe a particular type of object; but a collection of numbers without a coordinate frame to go with them doesn't "mean" anything, geometrically.

This module includes several types of geometric objects (of course, these are not all possible geometric objects; we merely define some objects that we have found to be useful for our applications):

Location.[3] An absolute location in space, represented numerically as a 3×1 matrix containing its x, y, and z coordinates (distances from the origin of the frame). Pictorially, we draw a dot (Fig. 6.2).

Orientation. An absolute orientation in space. Can be represented in a variety of ways: pictorially, we draw an orthonormal triple of vectors (Fig. 6.3); numerically we most commonly use a 3×3 rotation matrix, whose columns are the representations of the corresponding vectors.

Scalar. A frame-independent value (0-order tensor). Represented numerically as a single real number.

Vector. A direction with magnitude (1st-order tensor). Represented numerically as a 3×1 matrix containing its x, y, and z coordinates (displacements along each axis). Pictorially, we draw an arrow (Fig. 6.2); the position of the arrow is irrelevant, only its direction and length are significant.

2Tensor. An abstract object (2nd-order tensor), that corresponds with a linear operation on vectors; the operation can be performed by "multiplying" (arithmetic will be discussed in Defn. 6.14) a vector by a 2tensor to yield the new vector. Represented numerically as a 3×3 matrix, where each column represents the result of the operation on an axis. Pictorially, the operation can be illustrated by showing "before" and "after" drawings of a triple of vectors (Fig. 6.3).

Rotations. 2Tensors, whose corresponding operations preserve length and angles.

Basis.[4] A generalization of an orientation; corresponds with a triple of three arbitrary vectors. Represented as the 3×3 matrix whose columns represent the corresponding vectors.

Frame. A coordinate frame is defined by the location of its origin, P, and the orientation of its axes, R. We choose one arbitrary frame, that we call the *lab frame*, to be a fixed standard for reference. The coordinate system defined by the lab frame is called *lab coords* or *world coords*. Pictorially, we draw the frame's orientation situated at the frame's origin.

[3] [Craig89] uses the term *position vector* for these; we prefer to reserve the word *vector* for the 1st-order tensor object only.

[4] Basis objects are not often used conceptually; we include them for mathematical completeness.

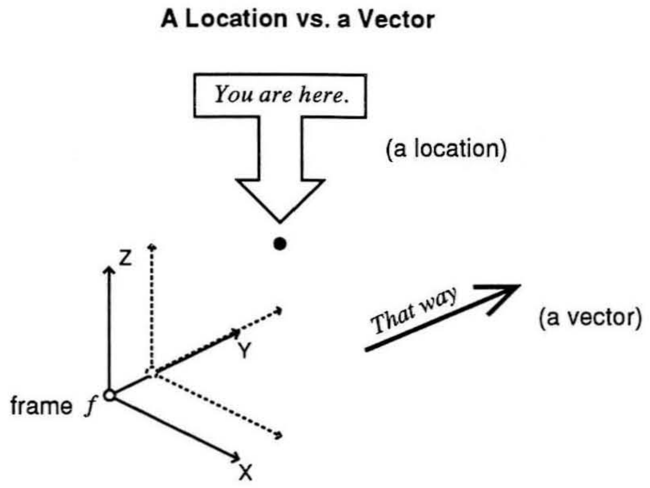


Figure 6.2: We distinguish between a *location*, which describes a position in absolute space, and a *vector*, which describes a direction (and distance). Both are represented by x, y, and z coordinate values in any given frame; but if we translate the frame, the coordinates of a location will vary, but the coordinates of a direction will stay the same. We similarly distinguish between *orientations*, which describe fixed alignments in space, and *rotations*, which describe operations on vectors. □

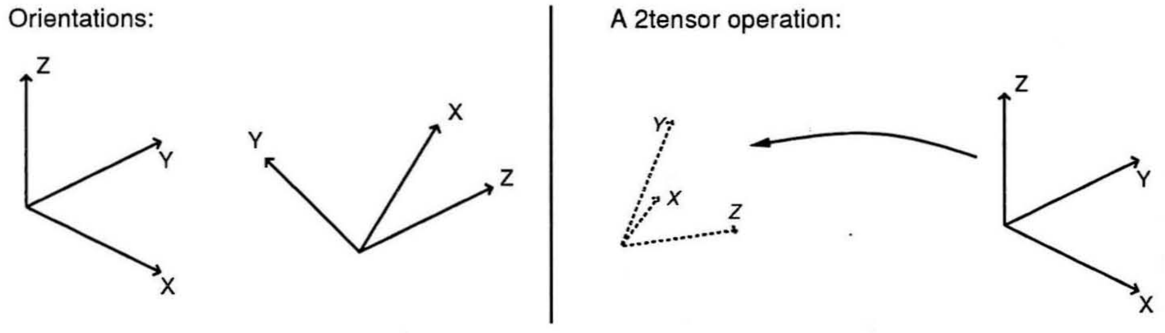


Figure 6.3: We draw an orientation as a triple of vectors, all drawn from the same spot—but the position of that spot is irrelevant. To illustrate a 2tensor, we draw an arc linking an orthonormal triple of vectors to the triple that results when each vector is multiplied by the 2tensor. □

We stress the conceptual distinction between locations and vectors: a location corresponds with a fixed position in absolute space, while a vector corresponds with a magnitude/direction but has no specific position [5] (see Fig. 6.2). Similarly, we distinguish between an absolute orientation in space and a rotation operation that one might use to align to it any particular frame. (In any given frame, however, there is a correspondence between locations and vectors, and between orientations and rotations; see Sec. 6.4.6 and Fig. 6.6.)

Notice that we consider a frame to be a type of geometric object. Thus there is circularity in our definitions: locations and orientations are described by their representations in frames, but a frame is defined by a location/orientation pair. The lab frame serves to break this circularity.

We use the term *moving object* (location, vector, etc.) to refer to one that varies over time. Moving objects can have associated velocity objects; in particular, we define an *instantaneous frame* to be a frame with associated linear and angular velocity vectors. A moving object may be *fixed in a moving frame*, i.e., it follows the frame's motion so that its coordinates never change as seen from that frame.

[5] Differential-geometers may comment that a "direction" (tangent vector) for an arbitrary surface or manifold is an element of the space tangent to the manifold *at a particular point* ([Millman,Parker77-pp.93,213]). For our Euclidean formalism, however, all tangent spaces are isomorphic to each other, thus we are safe in our use of tangent vectors without associated positions in space.

A note on units and dimension:

The conceptual objects described here often have physical *dimension* associated with them, such as length, mass, and time, and are measured in terms of *units*, such as centimeters, grams, and seconds. The terms in the mathematical model, however, are dimensionless, however—just numbers. [Lin,Segel74-Ch.6] discusses the *nondimensionalization* process, which is beyond our scope; we just point out the need for (at least) consistency of units within a given model.

6.4 Mathematical Model

6.4.1 Names & Notation

The scope name for this module is

COORDS

We don't use definitions from any other module. In this module, we will define and extensively use a prefix-superscript notation:

$${}^{f}x \qquad \text{(Notn. 6.6)}$$

6.4.2 Definitions

We start by defining abstract spaces for various types of objects; we refer to these as the *primitive* objects:

Definition. (Primitive Geometric Spaces)

$$\begin{aligned}
\mathsf{Scalars} &\equiv \textit{the set}\left\{\textbf{scalar}\ \textit{objects}\right\}\\
\mathsf{Vectors} &\equiv \textit{the set}\left\{\textbf{vector}\ \textit{objects}\right\}\\
\mathsf{2Tensors} &\equiv \textit{the set}\left\{\textbf{2tensor}\ \textit{objects}\right\}\\
\mathsf{Rotations} &\equiv \textit{the set}\left\{\textbf{rotation}\ \textit{objects}\right\} \subset \mathsf{2Tensors}\\
\mathsf{Locations} &\equiv \textit{the set}\left\{\textbf{location}\ \textit{objects}\right\}\\
\mathsf{Bases} &\equiv \textit{the set}\left\{\textbf{basis}\ \textit{objects}\right\}\\
\mathsf{Orientations} &\equiv \textit{the set}\left\{\textbf{orientation}\ \textit{objects}\right\} \subset \mathsf{Bases}
\end{aligned} \tag{6.1}$$

Defn. 6.1: We define an abstract space for each type of geometric objects. Note that orientations are types of bases whose corresponding vectors are orthonormal. □

The space of coordinate frames is defined as a state space, using the mechanism of Sec. 3.9:

Definition. Frames

$$\begin{array}{ll}
\mathsf{Frames} & \\
[& \\
\ P \mapsto \mathsf{Locations} & \textit{Location of a frame's origin}\\
\ R \mapsto \mathsf{Orientations} & \textit{Orientation of a frame's axes}\\
] &
\end{array} \tag{6.2}$$

Defn. 6.2: Space of coordinate frames. Each frame $f \in \mathsf{Frames}$ has an origin P_f at some location, and an orientation R_f. (The use of subscripted aspect values is as per Notn. 3.27.) □

It will be convenient to define a single set for the primitive geometric objects:

Definition. GeomObjs

$$\boxed{\begin{aligned}\mathsf{GeomObjs} \;\equiv\; & \mathsf{Locations} \cup \mathsf{Bases} \cup \mathsf{Scalars} \\ & \cup \mathsf{Vectors} \cup \mathsf{2Tensors}\end{aligned}} \tag{6.3}$$

Defn. 6.3: All primitive geometric objects. A geometric object $x \in \mathsf{GeomObjs}$ can be a location, basis, scalar, vector, or 2tensor. Thus the space GeomObjs is a disparate union as per Sec. 3.7.3, where two objects are *agnates* if they are both of the same primitive geometric type. □

We define a lab frame, to give us a standard set of coordinates, as per Sec. 6.3.

Definition. $\mathcal{Lab}$

$$\boxed{\mathcal{Lab} \in \mathsf{Frames}} \tag{6.4}$$

Defn. 6.4: The fixed lab frame $\mathcal{Lab}$ is a unique element of Frames. It has no special properties other than being agreed upon by everybody. □

6.4.3 Representation & Notation

An element from one of the above spaces is an abstract object; to make it seem concrete, we can produce a *representation* of the object. The representation depends, of course, on a choice of coordinate frame. Thus we define the representation operator *Rep*:

Definition. Rep

(6.5)

We define an overloaded operator
$Rep\colon \mathsf{Frames} \times \mathsf{GeomObjs} \to \Re \cup \Re^3 \cup \Re^{3\times3}$ *that yields the coordinates of an object in a given frame.*

$$\begin{array}{ll} Rep\colon \mathsf{Frames} \times \mathsf{Scalars} \to \Re & \textit{a single number} \\ Rep\colon \mathsf{Frames} \times \mathsf{Vectors} \to \Re^3 & a\ 3 \times 1 \textit{ matrix} \\ Rep\colon \mathsf{Frames} \times \mathsf{2Tensors} \to \Re^{3\times3} & a\ 3 \times 3 \textit{ matrix} \\ Rep\colon \mathsf{Frames} \times \mathsf{Locations} \to \Re^3 & a\ 3 \times 1 \textit{ matrix} \\ Rep\colon \mathsf{Frames} \times \mathsf{Bases} \to \Re^{3\times3} & a\ 3 \times 3 \textit{ matrix} \end{array}$$

Defn. 6.5: Representation operators convert from an abstract geometric object to a collection of real numbers, given a choice of frame. Note that *Rep* is overloaded (Sec. 3.6.3) to act on elements of the various different spaces. □

Because we commonly work with representations of objects, we define a shorthand notation for representations:

Notation. Prefix-superscripts for Representation in a Frame

(6.6)

For any $f \in \mathsf{Frames}$ and any $x \in \mathsf{GeomObjs}$,

$${}^{f}x \;\equiv\; Rep(f, x)$$

Notn. 6.6: Representation of a geometric object in a given frame. The above terms can be read as "x represented numerically in frame f" or as just "x in f." We most commonly use the prefix-superscript notation. □

Note that, while x is an abstract geometric object, ${}^{f}x$ is always a just collection of numbers. The representation functions are one-to-one, and thus can be used to determine identity:

(6.7)

For any objects $x, y \in \mathsf{GeomObjs}$ that are agnates (i.e., of the same primitive type), and any $f \in \mathsf{Frames}$

$${}^{f}x = {}^{f}y \iff x = y$$

Eqn. 6.7: Given a fixed frame, the representation of a geometric element of a given type is sufficient to identify the element. □

Representation of a frame

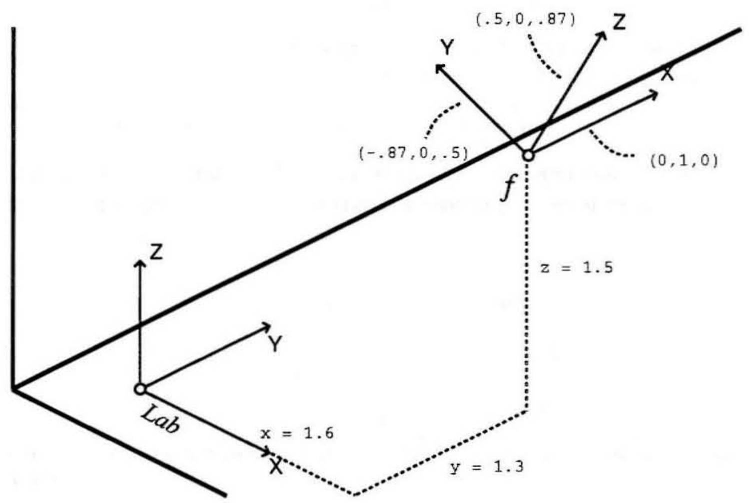


Figure 6.4: Any frame f is defined by its location P_f and orientation R_f, which can in turn be represented in any frame. Here, frame f is at location ${}^{\mathcal{L}ab}P_f = \begin{bmatrix} 1.6 \\ 1.3 \\ 1.5 \end{bmatrix}$ in the lab, and its orientation in the lab is ${}^{\mathcal{L}ab}R_f = \begin{bmatrix} 0 & -.87 & .5 \\ 1 & 0 & 0 \\ 0 & .5 & .87 \end{bmatrix}$. Represented in itself, f is at location ${}^{f}P_f = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ with orientation ${}^{f}R_f = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ as per Eqn. 6.10 □

Thus, for any given frame f, there is an equivalence between geometric objects and their numerical representations. Also, since locations and vectors share a representational space, for a given frame there is a natural correspondence between them; and similarly for orientations and rotations (see Sec. 6.4.6).

The *Rep* operators for orientations and rotations yield orthonormal rotation matrices. (An orientation's matrix corresponds with the rotation that aligns the frame axes to it.) Rotations and orientations may be represented in other ways as well, such as by Euler angles, in angle-axis form, or as unit quaternions (see [Craig89-Ch.2]). We give a definition for the latter:[6]

Definition. Repq

(6.8)

The quaternion representation of orientations and rotations is given by:

$$\textit{Repq}: \text{Frames} \times \text{Orientations} \rightarrow \Re^4$$
$$\textit{Repq}: \text{Frames} \times \text{Rotations} \rightarrow \Re^4$$

Defn. 6.8: Quaternion representation, overloaded for orientations and rotations. For any $r \in$ Orientations $\cup$ Rotations and any $f \in$ Frames, we have $|\textit{Repq}\,(f, r)| = 1$, i.e., the representation is as a unit quaternion. The quaternion representation can be computed from the matrix representation $\textit{Rep}\ (f, r)$ as described, e.g., in [Shoemake85].[7] □

We don't have a direct representation for frames; however, the location and orientation of a frame can of course be represented numerically in any frame (see Fig. 6.4). Combining P_f and R_f, defined in Defn. 6.2, with Notn. 6.6, gives us

[6] Quaternions can themselves be defined as abstract objects. For the purposes of this module, however, we treat them simply as 4-component arrays having the appropriate arithmetic rules.

[7] [Shoemake85] uses left-handed quaternion rotations; the matrix conversion therein describes the transpose of the right-handed matrix we would use.

(6.9)
$$\textit{For } f, g \in \textsf{Frames}$$
$$\begin{aligned} {}^g P_f &= \left(\begin{array}{c} \textit{Frame } f\textit{'s location, represented} \\ \textit{numerically in frame } g \end{array} \right) \\ {}^g R_f &= \left(\begin{array}{c} \textit{Frame } f\textit{'s orientation, represented} \\ \textit{numerically in frame } g \end{array} \right) \end{aligned}$$

Eqn. 6.9: The position and orientation of frame f, as "seen from" frame g. □

Since the location of a frame is defined to be the location of its origin, and since the orientation of a frame is defined to be the orientation of its axes (Sec. 6.3), a frame's representation of its own location and orientation is always trivial:

(6.10)
$$\textit{For any } f \in \textsf{Frames}:$$
$${}^f P_f = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad {}^f R_f = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Eqn. 6.10: A frame represented in terms of itself. The frame's origin is always at its own coordinates (0,0,0), and its orientation representation is the identity. □

6.4.4 Transforming Representations Between Frames

Often, we know the representation of an object in one frame, and we'd like to determine the representation in some other frame. We define the transformation rules:

(6.11)
$$\textit{For } f, g \in \textsf{Frames}$$
$$\begin{array}{ll} {}^g s = {}^f s & s \in \textsf{Scalars} \\ {}^g v = ({}^g R_f)\, {}^f v & v \in \textsf{Vectors} \\ {}^g a = ({}^g R_f)\, {}^f a\, ({}^g R_f)^T & a \in \textsf{2Tensors} \\ {}^g p = {}^g P_f + ({}^g R_f)\, {}^f p & p \in \textsf{Locations} \\ {}^g b = ({}^g R_f)\, {}^f b & r \in \textsf{Bases} \end{array}$$

Eqn. 6.11: Transformation rules. We emphasize that these rules do not transform geometric objects—rather, they change between two different representations *of the same object*. All operations in the above are performed via matrix arithmetic. □

In Eqn. 6.11, the tensor objects—scalars, vectors, and 2tensors— employ the standard tensor transformation rules. Locations, however, take into account the origin of the frame in which they are represented; and bases transform like vectors.

Notice that if we transform from a frame to itself, we can use Eqn. 6.10 to reduce each equality in Eqn. 6.11 to the trivial identity:

(6.12)
$$\textit{Given any } f, g \in \textsf{Frames}, \textit{ and any } x \in \textsf{GeomObjs}$$
$$f = g \implies {}^f x = {}^g x$$

Eqn. 6.12: The transformation rules are consistent. If we "transform" a representation between a frame and the same frame, the representation doesn't change. Note that the converse isn't true, i.e., ${}^f x = {}^g x \not\Rightarrow f = g$; for example, a vector will have the same representation in two frames that have the same orientation but different origin locations. □

We occasionally want to consider the representation of a frame's location and orientation in a second frame, as compared with the second frame's representation in the first. We we have the following equalities:

For any $f, g \in$ Frames

(6.13)
$$\begin{aligned} {}^{g}P_{f} &= -{}^{g}R_{f}\,{}^{f}P_{g} \\ {}^{g}R_{f} &= ({}^{f}R_{g})^{T} \end{aligned}$$

Eqn. 6.13: The relationship between two frame's representations. If we "look at" frame f from within frame g, we get "opposite" representations from those we get if we look at g from within f. These equalities derive from Eqn. 6.10 and Eqn. 6.11 (see Sec. 6.7). □

6.4.5 Arithmetic Operations

We can perform arithmetic operations on the representations of objects, using standard matrix arithmetic, as in Eqn. 6.11 and Eqn. 6.13. However, doing so requires picking a choice of frame, and can lead to cumbersome equations.

We want to define arithmetic operations directly on the abstract geometric objects; the abstract operations should correspond with the matrix operations on the objects' representations. A purist's approach might be to define the arithmetic operations abstractly, then prove that (with proper choice of representation) the representations follow the corresponding matrix arithmetic—in any frame. For our purposes, however, we will define the abstract operations "through the back door," i.e., define them such that they agree with the representations:

Definition. **Arithmetic Operations**

(6.14)

For any binary matrix arithmetic operation $\star$ *or unary matrix operation* $\diamond$, *we define the corresponding abstract geometric operations, for any* $x, y \in$ GeomObjs, *by:*

$$\left.\begin{aligned} {}^{f}(x \star y) &\equiv ({}^{f}x) \star ({}^{f}y) \\ {}^{f}(\diamond y) &\equiv \diamond({}^{f}y) \end{aligned}\right\} \text{ When independent of the choice of } f \in \text{Frames}$$

Defn. 6.14: Arithmetic operations between geometric objects, $x \star y$, are defined by the corresponding matrix arithmetic operations between their representations ${}^{f}x$ and ${}^{f}y$—but only where those operations yield frame-invariant results. Note that x and y are not necessarily agnates, e.g., a matrix may be multiplied by a vector. □

Fig. 6.5 lists the arithmetic operations. Note in particular that there is no meaning to the sum of two locations, or to the negation of a location—but a vector can be added to a location, and the difference between two locations can be found. We list some of the usual arithmetic properties:

For all $a \in$ 2Tensors, $r \in$ Rotations, $n \in$ Orientations, *and* $v, w \in$ Vectors, *we have the following properties:*

(6.15)
$$\begin{aligned} a^{-1}\,a &= a\,a^{-1} = 1 \\ r^{-1} &= r^{T} \\ r\,n &\in \text{Orientations} \\ v^{*}\,w &= v \times w \end{aligned}$$

Eqn. 6.15: The inverse of a 2tensor is both a right- and left-inverse. The inverse of a rotation is its transpose. Orientations are closed under left-multiplication by rotations. The antisymmetric dual,[8] v^{*}, of a vector v performs a cross product on another vector. □

The following "special" objects are defined, with the usual properties:

[8] The antisymmetric dual v^{*} of a vector v may be unfamiliar: The dual of $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is $\begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$.

Arithmetic Operations on Abstract Objects

Binary operations:

addition	Scalars	$+$	Scalars	$\to$	Scalars
subtraction	Scalars	$-$	Scalars	$\to$	Scalars
multiplication	Scalars	$*$	Scalars	$\to$	Scalars
scale	Scalars	$*$	Vectors	$\to$	Vectors
scale	Scalars	$*$	2Tensors	$\to$	2Tensors
scale	Scalars	$*$	Bases	$\to$	Bases
scale	Vectors	$*$	Scalars	$\to$	Vectors
addition	Vectors	$+$	Vectors	$\to$	Vectors
subtraction	Vectors	$-$	Vectors	$\to$	Vectors
inner product	Vectors	$\cdot$	Vectors	$\to$	Scalars
cross product	Vectors	$\times$	Vectors	$\to$	Vectors
outer product	Vectors	$*$	Vectors	$\to$	2Tensors
displacement	Vectors	$+$	Locations	$\to$	Locations
cross product	Vectors	$\times$	2Tensors	$\to$	2Tensors
cross product	Vectors	$\times$	Bases	$\to$	Bases
scale	2Tensors	$*$	Scalars	$\to$	2Tensors
product	2Tensors	$\cdot$	Vectors	$\to$	Vectors
addition	2Tensors	$+$	2Tensors	$\to$	2Tensors
subtraction	2Tensors	$-$	2Tensors	$\to$	2Tensors
product	2Tensors	$\cdot$	2Tensors	$\to$	2Tensors
product	2Tensors	$\cdot$	Bases	$\to$	Bases
displacement	Locations	$+$	Vectors	$\to$	Locations
difference	Locations	$-$	Locations	$\to$	Vectors
scale	Bases	$*$	Scalars	$\to$	Bases
cross product	Bases	$*$	Vectors	$\to$	Bases
addition	Bases	$+$	Bases	$\to$	Bases
substraction	Bases	$-$	Bases	$\to$	Bases

Unary operations:

inverse	Scalars^{-1}	$\to$	Scalars
negation	$-$Scalars	$\to$	Scalars
antisymmetric dual	Vectors*	$\to$	2Tensors
negation	$-$Vectors	$\to$	Vectors
transpose	2Tensors^{T}	$\to$	2Tensors
inverse	2Tensors^{-1}	$\to$	2Tensors
negation	$-$2Tensors	$\to$	2Tensors
negation	$-$Bases	$\to$	Bases

Figure 6.5: The arithmetic operations on abstract geometric objects, in accordance with Defn. 6.14. All the usual operations on tensors are defined. Note, however, that locations may only be subtracted from each other or added to vectors, and bases may be left-multiplied by 2tensors, but not right-multiplied. □

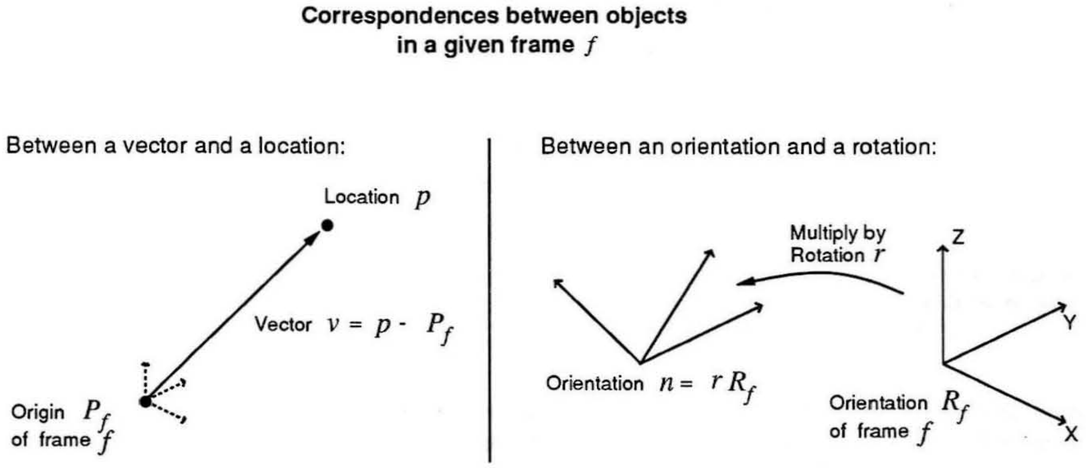

Correspondences between objects in a given frame f
Between a vector and a location:
Location p
Vector $v = p - P_f$
Origin P_f of frame f
Between an orientation and a rotation:
Multiply by Rotation r
Orientation $n = r R_f$
Orientation R_f of frame f
Z
Y
X

Figure 6.6: For a given frame f, there is a natural correspondence between a location $p \in$ Locations and the vector $v \in$ Vectors from the frame's origin to p; both are represented by the same three coordinate values in frame f. Similarly, there is a natural correspondence between a rotation $r \in$ Rotations and the orientation $n \in$ Orientations that results when multiplying the frame's orientation by r. □

(6.16)

Object	*Name*	*Rep.*	*Properties*
$1 \in$ Scalars	*one*	1	$1 * x = x$
$0 \in$ Scalars	*zero*	0	$0 + x = x, 0 * x = 0$
$0 \in$ Vectors	*zero vector*	$\begin{bmatrix}0\\0\\0\end{bmatrix}$	$0 + x = x$
$0 \in$ 2Tensors	*zero 2tensor*	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$0 * x = 0$
$1 \in$ 2Tensors	*identity*	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$	$1 * x = x$
$0 \in$ Bases	*zero basis*	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$x * 0 = 0$

For any $f, g \in$ Frames: ${}^f\!0 = {}^g\!0$, *and* ${}^f\!1 = {}^g\!1$

Eqn. 6.16: Objects with special properties. The representations of these objects are the same in all frames, as can be shown by substituting into Eqn. 6.11. □

6.4.6 Correspondence Between Objects

Given a choice of frame, there is a natural correspondence between locations and vectors, and between orientations and rotations, based on equality of representation: (Fig. 6.6)

For any $f \in \mathsf{Frames}$, $p \in \mathsf{Locations}$, $v \in \mathsf{Vectors}$, $r \in \mathsf{Rotations}$, $n \in \mathsf{Orientations}$, $a \in \mathsf{2Tensors}$, $b \in \mathsf{Bases}$, *we have the following correspondences:*

(6.17)
$$\begin{aligned} v = p - P_f \quad &\iff \quad {}^{f}v = {}^{f}p \\ n = r\,R_f \quad &\iff \quad {}^{f}n = {}^{f}r \\ b = a\,R_f \quad &\iff \quad {}^{f}b = {}^{f}a \end{aligned}$$

Eqn. 6.17: In any given frame, the representation of a location is the same as the representation of the vector from the origin to that location. Similarly, the representation of an orientation corresponds with the rotation that aligns the frame with that orientation. More generally, given a frame, there is a correspondence between bases and 2tensors. □

Due to this correspondence (and perhaps also because the common emphasis is on tensor objects only), many expositions don't define locations and orientations as separate entities. However, we prefer to define them separately, because the correspondence is only valid for a single frame; e.g., if a vector and location correspond in some frame f, they won't necessarily correspond to each other in some other frame g, because of their different transformation rules in Eqn. 6.11.

The correspondence is useful when switching between numerical and abstract forms of an equation; this is illustrated in Sec. 6.7 (for the derivation of Eqn. 6.30).

6.4.7 Using Scalars as Real Numbers

Since the representation of a scalar is independent of the choice of frame (Eqn. 6.11), there is a natural isomorphism between scalars and real numbers. Thus, to make things simple we eliminate the distinction between the two spaces:[9]

(6.18)
$$\begin{aligned} &\mathsf{Scalars} = \Re \\ &{}^{f}s = s \quad \begin{cases} \textit{for all } s \in \mathsf{Scalars}, \\ \textit{independent of } f \in \mathsf{Frames} \end{cases} \end{aligned}$$

Eqn. 6.18: We can use Scalars and $\Re$ interchangeably. The representation operator is trivial: a scalar is represented as itself. We can freely add or drop frame-superscripts on scalars. □

We will continue to use Scalars and the superscript notation in this module, for consistency with the other primitive geometric objects. In general, however, there is little reason to define equations in terms of Scalars rather than reals.

6.4.8 Moving Objects

What we think of conceptually as a *moving object* is mathematically just a function from the reals, i.e., time or path parameters, onto the geometric objects (see Fig. 6.7). Thus we define the following spaces:

[9] The specific space COORDS :: Scalars that we define in this module is interchangeable with $\Re$, but a more general notion of scalars would encompass quantities such as complex numbers.

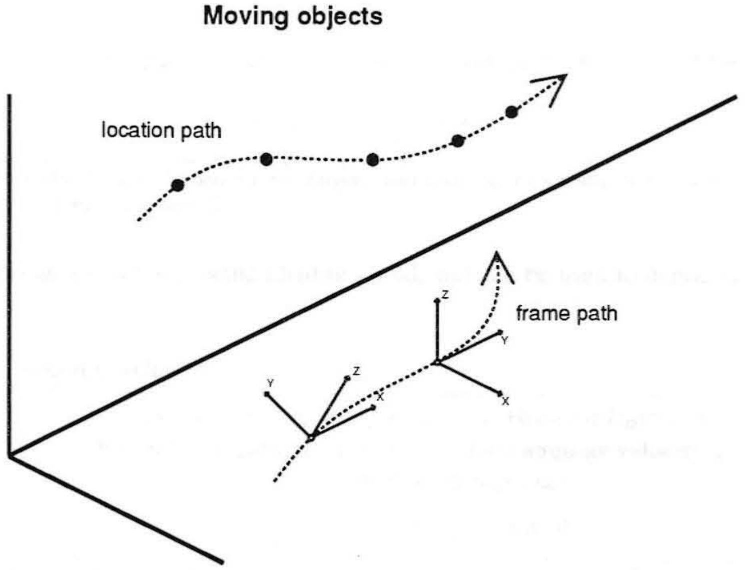


Figure 6.7: A "moving object" is described by a path, i.e., a function from reals to the object's space. We illustrate a location path and a frame path, but one can define paths for any geometric object. □

Definition. ScalarPaths, VectorPaths, etc.

(6.19)
$$\begin{aligned}
\text{ScalarPaths} &\equiv \textit{the set of functions } \{\Re \to \text{Scalars}\} \\
\text{VectorPaths} &\equiv \textit{the set of functions } \{\Re \to \text{Vectors}\} \\
\text{2TensorPaths} &\equiv \textit{the set of functions } \{\Re \to \text{2Tensors}\} \\
\text{RotationPaths} &\equiv \textit{the set of functions } \{\Re \to \text{Rotations}\} \\
\text{LocationPaths} &\equiv \textit{the set of functions } \{\Re \to \text{Locations}\} \\
\text{BasisPaths} &\equiv \textit{the set of functions } \{\Re \to \text{Bases}\} \\
\text{OrientationPaths} &\equiv \textit{the set of functions } \{\Re \to \text{Orientations}\} \\
\text{GeomPaths} &\equiv \text{LocationPaths} \cup \text{BasisPaths} \\
&\quad \cup \text{ScalarPaths} \cup \text{VectorPaths} \\
&\quad \cup \text{2TensorPaths}
\end{aligned}$$

Defn. 6.19: Spaces of functions, for moving objects. Note that RotationPaths ⊂ 2TensorPaths, and OrientationPaths ⊂ BasisPaths. We define GeomPaths analogously to GeomObjs, Defn. 6.3. □

We can define derivatives for these paths by extension from the arithmetic operations (Defn. 6.14). Note that because the difference of two locations is a vector, the derivative of a location path is a vector path. Note also that the derivative of a location is a vector, and that in general, the derivative of a rotation is a 2tensor (not a rotation) and the derivative of an orientation is a basis (not an orientation). We have the following identity:

Definition. $\frac{d}{dt}$ (derivatives of paths)

(6.20)

For a path $x \in$ GeomPaths, *and constant frame* $f \in$ Frames,

$$^{f}\!\left(\tfrac{d}{dt}x(t)\right) = \tfrac{d}{dt}\left({}^{f}x(t)\right), \quad t \in \Re$$

Defn. 6.20: For a fixed frame, the derivative and representation operators commute. Note that if f varies as a function of t, the operators no longer commute. □

For rotations and orientations, the following identities hold, and can be used to define an angular velocity:

Definition. **Angular velocity**

(6.21)

For any differentiable function $r \in$ RotationPaths *or* $r \in$ OrientationPaths, *there exists a unique* **angular velocity** *function* $\omega \in$ VectorPaths *such that*

$$\tfrac{d}{dt}r(t) = \omega^*(t)r(t), \quad t \in \Re$$

Defn. 6.21: The derivative of rotations and orientations always obey these identities. The angular velocity, ω, is a vector function; at each instant its value is a vector that lies along the instantaneous axis of rotation, and whose magnitude is the instantaneous rate of rotation (radians per unit time). □

An analogous equation holds for quaternion representations (Defn. 6.8):

(6.22)

For any differentiable function $r \in$ RotationPaths *or* $r \in$ OrientationPaths, *if* $\omega \in$ VectorPaths *is the angular velocity function, then for any constant* $f \in$ Frames,

$$\tfrac{d}{dt}Repq(f, r(t)) = \tfrac{1}{2}\,{}^{f}\omega(t)Repq(f, r(t)), \quad t \in \Re$$

Eqn. 6.22: The angular velocity equation in Defn. 6.21 has an analog in quaternion representations, using quaternion arithmetic. □

If a frame "moves," it will have associated with it linear and angular velocities. We define a space that associates two velocity vectors with a frame:

Definition. InstFrames

(6.23)

```
InstFrames
[
F  ↦ Frames                      a frame
P      ...P ↦ Locations
R      ...R ↦ Orientations
V  ↦ Vectors                     a linear velocity
ω  ↦ Vectors                     an angular velocity
]
```

Defn. 6.23: Each "instantaneous frame" is a frame along with a linear velocity vector and an angular velocity vector. Note that InstFrames is not a specialization of Frames; it is possible, to have two instantaneous frames that have the same frame but different velocities. □

If we consider paths in the space InstFrames, we are most interested in ones whose velocity aspects actually agree with their motion, as per Defn. 6.21. Thus we define:

Definition. **consistent** *frame function*

(6.24)

Given a differentiable function $f: \Re \rightarrow \text{InstFrames}$, *we say* f *is* **consistent** *iff:*

$$\begin{array}{ll} \frac{d}{dt}P_f(t) = V_f(t), & t \in \Re \\ \frac{d}{dt}R_f(t) = \omega_f^*(t)R_f(t), & t \in \Re \end{array}$$

Defn. 6.24: A path through instantaneous frame space is consistent if at each point on the path, the velocity vectors at that point correspond with the actual derivatives of the function. (The use of subscripted aspects of functions is as per Notn. 3.28.) □

Notice that we can trivially construct an ordinary frame from an instantaneous frame, by projecting the F aspect operator of the instantaneous frame:

(6.25)

Given $j \in \text{InstFrames}$ *define* $f \in \text{Frames}$ *by*

$$f = F_j$$

Eqn. 6.25: Given an instantaneous frame, the corresponding ordinary frame is trivially available. □

Thus we will implicitly overload all functions and notations that expect frames to accept instantaneous frames as well. In particular, the representation operator $Rep(f, x)$ and notation $\overset{f}{x}$ are defined for $f \in \text{InstFrames}$.

Similarly, extending Eqn. 6.25, we can be careless and blur the distinction between differentiable functions to frames and consistent functions to instantaneous frames, since one can trivially be constructed from the other:

(6.26)

For differentiable $f: \Re \rightarrow \text{Frames}$ *and consistent* $j: \Re \rightarrow \text{InstFrames}$

Given f, *define* j *by:*

$$\begin{array}{lcll} F_j(t) & = & f(t), & t \in \Re \\ V_f(t) & = & \frac{d}{dt}P_f(t), & t \in \Re \\ \omega_f^*(t) & = & (\frac{d}{dt}R_f(t))(R_f(t)),^T & t \in \Re \end{array}$$

Or, given j, *define* f *by:*

$$\begin{array}{lcll} f(t) & = & F_j(t), & t \in \Re \end{array}$$

Eqn. 6.26: Switching between differentiable frame functions and consistent instantaneous frame functions. Because one can be constructed from the other, we will blur the distinction between them. □

We define a space of consistent functions:

Definition. FramePaths

(6.27)

$$\text{FramePaths} \equiv \textit{the set of functions} \left\{ f: \Re \rightarrow \text{InstFrames } \textit{such that } f \textit{ is consistent} \right\}$$

Defn. 6.27: Space of moving frames. We restrict the space to consistent (Defn. 6.24) functions. As per Eqn. 6.26 we can interchange consistent instantaneous and ordinary frame functions, so for brevity we name this space "FramePaths " rather than "InstFramePaths." □

A moving frame "carries" its coordinates system along with it:

(6.28)
$$\textit{Given a moving frame } f \in \mathsf{FramePaths}$$
$$\begin{array}{ll} {}^{f(t)}P_f(t) = 0 & \frac{d}{dt}\,{}^{f(t)}P_f(t) = 0 \textit{ (constant)} \\ {}^{f(t)}R_f(t) = 1 & \frac{d}{dt}\,{}^{f(t)}R_f(t) = 0 \textit{ (constant)} \\ {}^{f(t)}V_f(t) = \textit{not constant in general} & \\ {}^{f(t)}\omega_f(t) = \textit{not constant in general} & \end{array}$$

Eqn. 6.28: Represented in itself, a moving frame's origin is always at (0,0,0) and its orientation is the identity, as per Eqn. 6.10. The velocities can have any values, however. □

But notice that if we equate velocities with derivatives in Eqn. 6.28, we get the following "paradox," i.e., a seeming violation of Defn. 6.24:

(6.29)
$$\textit{Given a moving frame } f \in \mathsf{FramePaths}\textit{, in general:}$$
$$\begin{array}{rcl} {}^{f(t)}V_f(t) & \neq & \frac{d}{dt}({}^{f(t)}P_f(t)) \\ {}^{f(t)}\omega^*_f(t) & \neq & \frac{d}{dt}({}^{f(t)}R_f(t))({}^{f(t)}R_f(t))^T \end{array}$$

Eqn. 6.29: Moving frame "paradox." A moving frame can have non-0 velocity coordinates, but the origin is always at 0 as per Eqn. 6.28—thus the velocity is apparently not the derivative of the location. However, the offending inequalities are not equivalent to Defn. 6.24: differentiation and representation do not commute for a non-constant frame (Defn. 6.20). □

6.4.9 Paths & Derivatives

Suppose that we have a constant object, but we represent it in a moving frame. Thus though the object itself does not change, the representation of it might. The derivatives for the various types of objects, represented in a moving frame:

(6.30)
$$\begin{array}{lll} \multicolumn{3}{l}{\textit{For moving frame } f \in \mathsf{FramePaths} \textit{ and constant objects}} \\ s \in \mathsf{Scalars} & v \in \mathsf{Vectors} & a \in \mathsf{2Tensors} \\ p \in \mathsf{Locations} & n \in \mathsf{Orientations} & \end{array}$$
$$\begin{array}{l} \frac{d}{dt}({}^{f(t)}s) = 0 \\ \frac{d}{dt}({}^{f(t)}v) = -\,{}^{f(t)}[\omega_f(t) \times v] \\ \frac{d}{dt}({}^{f(t)}a) = -\,{}^{f(t)}[\omega^*_f(t)\, a + a\, \omega^{*T}_f(t)] \\ \frac{d}{dt}({}^{f(t)}p) = -\,{}^{f(t)}[V_f(t) + \omega_f(t) \times (p - P_f(t))] \\ \frac{d}{dt}({}^{f(t)}n) = -\,{}^{f(t)}[\omega^*_f(t)\, n] \end{array}$$

Eqn. 6.30: These equations describe behavior of the representation of an object, if the object is constant, but the frame we represent it in changes. (See derivation in Sec. 6.7.) □

Suppose now that we have an object that is moving, and we represent it in a moving frame. This is a more general case of Eqn. 6.30:

For moving frame $f \in$ FramePaths *and moving objects*
$s \in$ ScalarPaths $v \in$ VectorPaths $a \in$ 2TensorPaths
$p \in$ LocationPaths $n \in$ OrientationPaths

(6.31)
$$\begin{aligned}
\tfrac{d}{dt}({}^{f(t)}s(t)) &= {}^{f(t)}[\tfrac{d}{dt}s(t)] \\
\tfrac{d}{dt}({}^{f(t)}v(t)) &= {}^{f(t)}[\tfrac{d}{dt}v(t) - \omega_f(t) \times v(t)] \\
\tfrac{d}{dt}({}^{f(t)}a(t)) &= {}^{f(t)}[\tfrac{d}{dt}a(t) - \omega_f^*(t)\, a(t) - a(t)\, \omega_f^{*T}(t)] \\
\tfrac{d}{dt}({}^{f(t)}p(t)) &= {}^{f(t)}[\tfrac{d}{dt}p(t) - V_f(t) - \omega_f(t) \times (p(t) - P_f(t))] \\
\tfrac{d}{dt}({}^{f(t)}n(t)) &= {}^{f(t)}[\tfrac{d}{dt}n(t) - \omega_f^*(t)\, n(t)]
\end{aligned}$$

Eqn. 6.31: These equations describe the behavior of the representation of an object, if both the object and the frame to represent it in are changing. □

Now consider an object that is moving, and that we represent in a moving frame—but its representation in that frame is constant. Thus the object "moves with" or "is carried by" or just "is fixed in" that moving frame; we define:

Definition. **Fixed in a moving frame**

(6.32)
A moving object $x\colon \Re \rightarrow$ GeomObjs
is **fixed in a moving frame** $f \in$ FramePaths *iff:*

$$\tfrac{d}{dt}\, {}^{f(t)}x(t) = 0, \textit{ for all } t$$

Defn. 6.32: A moving object is fixed in a moving frame if its representation in that frame is constant, or, equivalently, the derivative of its representation is 0. □

The representations of fixed objects in other frames are given directly by Eqn. 6.11. The derivatives can be defined as abstract objects:

For moving objects fixed in moving frame $f \in$ FramePaths
$s \in$ ScalarPaths $v \in$ VectorPaths $a \in$ 2TensorPaths
$p \in$ LocationPaths $n \in$ OrientationPaths

(6.33)
$$\begin{aligned}
\tfrac{d}{dt}s(t) &= 0 \\
\tfrac{d}{dt}v(t) &= \omega_f(t) \times v(t) \\
\tfrac{d}{dt}a(t) &= \omega_f^*(t)\, a(t) + a(t)\, \omega_f^{*T}(t) \\
\tfrac{d}{dt}p(t) &= V_f(t) + \omega_f(t) \times (p(t) - P_f(t)) \\
\tfrac{d}{dt}n(t) &= \omega_f^*(t)\, n(t)
\end{aligned}$$

Eqn. 6.33: These equations describe the behavior of an object that is fixed in a moving frame. The derivatives are computed using abstract geometric operations on the various quantities. □

Note that the latter above implies that an orientation that is fixed in a moving frame f has an angular velocity ω that is the same as f's angular velocity ω_f (see Defn. 6.21).

6.5 Posed Problems

The conceptual and mathematical models that we have defined do not immediately imply any complicated numerical problems. However, there are many simple "evaluation" tasks that may often need to be performed. To list a few:

- Define an object given its representation in a particular frame.

- Given an object, evaluate its representation in a particular frame.
- Find the motion of an object fixed in a moving frame.
- Given a location path, find its velocity (also a path)
- Given an orientation path, find its angular velocity (also a path).

6.6 Implementation Notes[†]

The implementation of this module lies entirely in the math section of the program—in the conceptual section, we typically manipulate arrays of numbers, corresponding to the representations in conceptually natural frames, thus no intricate C-M interface is needed; and since we have defined no complicated numerical problems, there is no M-N interface. The math section has scope name `MCO` ("Math COordinates"). Fig. 6.8 lists the definitions for the module.

Primitive objects. The implementation maintains the distinction between abstract geometric objects and their representations: an object is not tied to a particular frame. For each class of object a method, `rep(Frame)`, yields the representation in any specified frame (which is specified via an instance of class `Frame`, described below). Thus:

```
class Vector :
  constructors:  (Frame, double[3])     construct from coords in given frame
  methods:       rep(Frame) : double[3]   represent as coords in given frame

class Orientation :
  constructors:  (Frame, double[3][3])   construct from matrix in given frame
                 (Frame, double[4])      construct from quaternion in given frame
  methods:       rep(Frame)  : double[3][3]   represent as matrix in given frame
                 repq(Frame) : double[4]      represent as quaternion in given frame
```

To construct an instance of a primitive object, one provides a frame along with the representation in that frame, as per Eqn. 6.7. Notice that an `Orientation` can be constructed/represented via matrixes or quaternions (Defn. 6.8); the same is true for the `Rotation` class. Classes `2tensor` and `Location` are defined similarly to `Vector`. `Scalar` is implemented simply as an alias for `double`, as per Eqn. 6.18. The various arithmetic operators, `+`, `*`, `cross`, etc., are defined for these classes as appropriate (Fig. 6.5).

Internally, our implementation stores these objects as their lab-frame representations: when an instance is constructed or represented, the appropriate transformation (Eqn. 6.11) is performed; and arithmetic is performed directly in lab coordinates. The implementation could perhaps be optimized in various ways, however; for example, the representation in the most recently requested frame might be stored, to avoid repeatedly performing the transformation calculations to the same frame.

Frames. The classes for state spaces Frames (Defn. 6.2) and InstFrames (Defn. 6.23) are defined in the standard manner (Sec. B.3.6):

```
class Frame :
  constructors:  (Location P, Orientation R)
                 (InstFrame)                    trivial, as per Eqn. 6.25
  members:       P : Location
                 R : Orientation

class InstFrame :
  constructors:  (Location P, Orientation R, Vector v)
                 (Frame f, Vector v)
  members:       f : Frame
                 P : Location
                 R : Orientation
                 v : Vector
                 w : Vector
```

[†]See Appendix B for discussion of the terminology, notation, and overall approach used here.

Program definitions in scope MCO:

class name	*abstract space*	
`2tensor`	2Tensors	*(Defn. 6.1)*
`2tensorPath`	2TensorPaths	*(Defn. 6.19)*
`Frame`	Frames	*(Defn. 6.2)*
`FramePath`	FramePaths	*(Defn. 6.27)*
`InstFrame`	InstFrames	*(Defn. 6.23)*
`Location`	Locations	*(Defn. 6.1)*
`LocationPath`	LocationPaths	*(Defn. 6.19)*
`Orientation`	Locations	*(Defn. 6.1)*
`OrientationPath`	OrientationPaths	*(Defn. 6.19)*
`Rotation`	Rotations	*(Defn. 6.1)*
`RotationPath`	RotationPaths	*(Defn. 6.19)*
`Scalar`	Scalars	*(Defn. 6.1)*
`ScalarIdx`	$\{\text{Scalars}\}_{\text{IDs}}$	*index of scalars (Notn. 3.11)*
`ScalarPath`	ScalarPaths	*(Defn. 6.19)*
`ScalarPathIdx`	$\{\text{ScalarPaths}\}_{\text{IDs}}$	*index of scalar paths (Notn. 3.11)*
`Vector`	Vectors	*(Defn. 6.1)*
`VectorIdx`	$\{\text{Vectors}\}_{\text{IDs}}$	*index of vectors (Notn. 3.11)*
`VectorPath`	VectorPaths	*(Defn. 6.19)*

global constant	*mathematical object*	
`Lab_Frame`	$\mathcal{L}ab \in$ Frames	*(Defn. 6.4)*
`Zero_Vector`	$0 \in$ Vectors	*(Eqn. 6.16)*
`Zero_2tensor`	$0 \in$ 2Tensors	*(Eqn. 6.16)*
`Identity_2tensor`	$1 \in$ 2Tensors	*(Eqn. 6.16)*

Figure 6.8: Math section definitions in the prototype implementation. In addition to classes for the abstract spaces that we use, we define a few constant global variables. □

Notice the circularity discussed in Sec. 6.3: We need a location and an orientation to construct a frame, but we need a frame to construct a location or orientation. The predefined constant frame `Lab_Frame` provides a starting point.

Paths. Classes for object paths (Defn. 6.19) in the standard manner (Sec. B.3.7):

```
class LocationPath :
  constructors:  (Location)                  constant path
                 (FramePath, double[3])      fixed in given moving frame
  methods:       eval(double t) : Location
                 velocity()     : VectorPath
```

The classes `2tensorPath`, `OrientationPath`, `RotationPath`, `ScalarPath`, and `VectorPath` are defined similarly. In addition to constant paths and fixed paths as shown, there is support for paths that are algebraic combinations of given paths, and for paths that are evaluated by calling arbitrary user-supplied subroutines. Some paths support the `velocity` method, which returns a path whose value is the derivative of the given path, e.g., as per Eqn. 6.33. We additionally define the class `FramePath`, the standard manner for paths into state spaces (Sec. B.3.7).

6.7 Derivations

Since the mathematics of geometric objects is well-known, there are no surprises in this chapter. Hence, we refer the reader to [Craig89] and [Foley et al.90] for further discussion of 3-D geometry and linear algebra, or the reader may simply "take our word for it." To illustrate the use of our notation, however, we provide derivations of Eqn. 6.13 and one equality in Eqn. 6.30.

• Eqn. 6.13

Given: two frames $f, g \in$ Frames.
For any location $p \in$ Locations, we have

$$\begin{aligned} {}^{g}p &= {}^{g}P_f + ({}^{g}R_f)\,{}^{f}p && \textit{(Eqn. 6.11)} \end{aligned}$$

substitute P_g (the origin of frame g) for p to get

$$\begin{aligned} {}^{g}P_g &= {}^{g}P_f + ({}^{g}R_f)\,{}^{f}P_g \\ 0 &= {}^{g}P_f + ({}^{g}R_f)\,{}^{f}P_g && \textit{(via Eqn. 6.10)} \\ {}^{g}P_f &= -\,{}^{g}R_f\,{}^{f}P_g \end{aligned}$$

For any orienation $n \in$ Orientations, we have

$$\begin{aligned} {}^{g}n &= ({}^{g}R_f)\,{}^{f}n && \textit{(Eqn. 6.11)} \end{aligned}$$

substitute R_g (the orientation of frame g) for n to get

$$\begin{aligned} {}^{g}R_g &= ({}^{g}R_f)({}^{f}R_g) \\ 1 &= ({}^{g}R_f)({}^{f}R_g) && \textit{(via Eqn. 6.10)} \\ {}^{g}R_f &= ({}^{f}R_g)^T && \textit{(via Eqn. 6.15)} \end{aligned}$$

• Eqn. 6.30

Given: constant location $p \in$ Locations and moving frame $f \in$ FramePaths.
For any constant frame $g \in$ Frames, we have

$$\begin{aligned}
{}^{g}p &= {}^{g}P_{f(t)} + ({}^{g}R_{f(t)})\,{}^{f(t)}p && \textit{(Eqn. 6.11)} \\
\tfrac{d}{dt}({}^{g}p) &= \tfrac{d}{dt}({}^{g}P_{f(t)} + ({}^{g}R_{f(t)})\,{}^{f(t)}p) \\
{}^{g}(\tfrac{d}{dt}p) &= && \textit{(via Defn. 6.20)} \\
0 &= && \textit{(p is constant)} \\
&= \tfrac{d}{dt}({}^{g}P_{f(t)}) + [\tfrac{d}{dt}({}^{g}R_{f(t)})]\,{}^{f(t)}p + {}^{g}R_{f(t)}\tfrac{d}{dt}({}^{f(t)}p) && \textit{(product rule)} \\
-\,{}^{g}R_{f(t)}\tfrac{d}{dt}({}^{f(t)}p) &= \tfrac{d}{dt}({}^{g}P_{f(t)}) + [\tfrac{d}{dt}({}^{g}R_{f(t)})]\,{}^{f(t)}p \\
&= {}^{g}(\tfrac{d}{dt}P_{f(t)}) + {}^{g}(\tfrac{d}{dt}R_{f(t)})\,{}^{f(t)}p && \textit{(via Defn. 6.20)} \\
&= {}^{g}V_{f(t)} + {}^{g}\omega^*_{f(t)}\,{}^{g}R_{f(t)}\,{}^{f(t)}p && \textit{(via Defn. 6.24)} \\
\tfrac{d}{dt}({}^{f(t)}p) &= -({}^{g}R_{f(t)})^T\,{}^{g}V_{f(t)} - ({}^{g}R_{f(t)})^T\,{}^{g}\omega^*_{f(t)}\,{}^{g}R_{f(t)}\,{}^{f(t)}p && \textit{(mult. by } -R^T\textit{)} \\
&= -\,{}^{f(t)}R_g\,{}^{g}V_{f(t)} - {}^{f(t)}R_g\,{}^{g}\omega^*_{f(t)}({}^{f(t)}R_g)^T\,{}^{f(t)}p && \textit{(via Eqn. 6.13)} \\
\dagger \qquad &= -\,{}^{f(t)}V_{f(t)} - {}^{f(t)}\omega^*_{f(t)}\,{}^{f(t)}p && \textit{(via Eqn. 6.11)} \\
&= -\,{}^{f(t)}V_{f(t)} - {}^{f(t)}\omega^*_{f(t)}({}^{f(t)}p - {}^{f(t)}P_{f(t)}) && \textit{(via Eqn. 6.10)} \\
\ddagger \qquad &= -\,{}^{f(t)}[V_{f(t)} + \omega^*_{f(t)}(p - P_{f(t)})] && \textit{(via Defn. 6.14)} \\
&= -\,{}^{f(t)}[V_{f(t)} + \omega_{f(t)} \times (p - P_{f(t)})] && \textit{(via Eqn. 6.15)}
\end{aligned}$$

Notice in line † that ${}^{f}\omega^*_f\;{}^{f}p$ is a valid matrix product, but there is no corresponding abstract operation—we can't multiply a 2tensor, ω^*_f, by a location, p. But, by subtracting zero in the form ${}^{f}P_f$, we essentially replace p with its corresponding vector in frame f (Eqn. 6.17). Since a product between a 2tensor, ω^*_f, and a vector, $(p - P_f)$, *is* a valid abstract operation, we can make the transition to line ‡.

Chapter 7

Kinematic Rigid Bodies Model

"Kinematic" motion is motion without considerations of mass and force. This includes, e.g., motion described by keyframe animations systems, geometric constraints, and direct user manipulation.

This module provides a basic expression of the kinematic motion of rigid bodies. It provides a simple structure for describing collections of bodies, and a mechanism for describing bodies constrained in fixed hierarchies. The module is administrative rather than technical; we do not give any specific methods or techniques for manipulating bodies, we merely provide a framework within which to express such techniques.

The "traditional" computer graphics animation techniques, as discussed in Ch. 1.2, are kinematic; see, e.g., [Magnenat-Thalmann,Thalmann85]. [Craig89] has an extensive discussion of kinematics in the context of robotics.

7.1 Goals

We have a few simple goals for this module:

- To provide a basic model & notation for rigidly moving bodies.
- To describe kinematic relationships between bodies, e.g., as a hierarchy.
- To describe behavior of points fixed on a body.

The intent is not to define any particular techniques for kinematic rigid-body motion, but to give a common framework and terminology that can be built upon by other, higher-level models and techniques, such as dynamics or articulated key-frame animation.

7.2 Conceptual Model

Our conceptual model for kinematic rigid bodies is quite simple. We describe our basic abstraction of a *body*, as well as two other useful notions, that of a *body point*, and a *hierarchical configuration* of bodies. All bodies exist in the fixed, Euclidean 3-D world space of Ch. 6.

Bodies:

A rigid body doesn't change its shape—that is what we mean by "rigid"—thus a body's motion can be described by its position and orientation. In other words, we think of a rigid body as "carrying" with it a

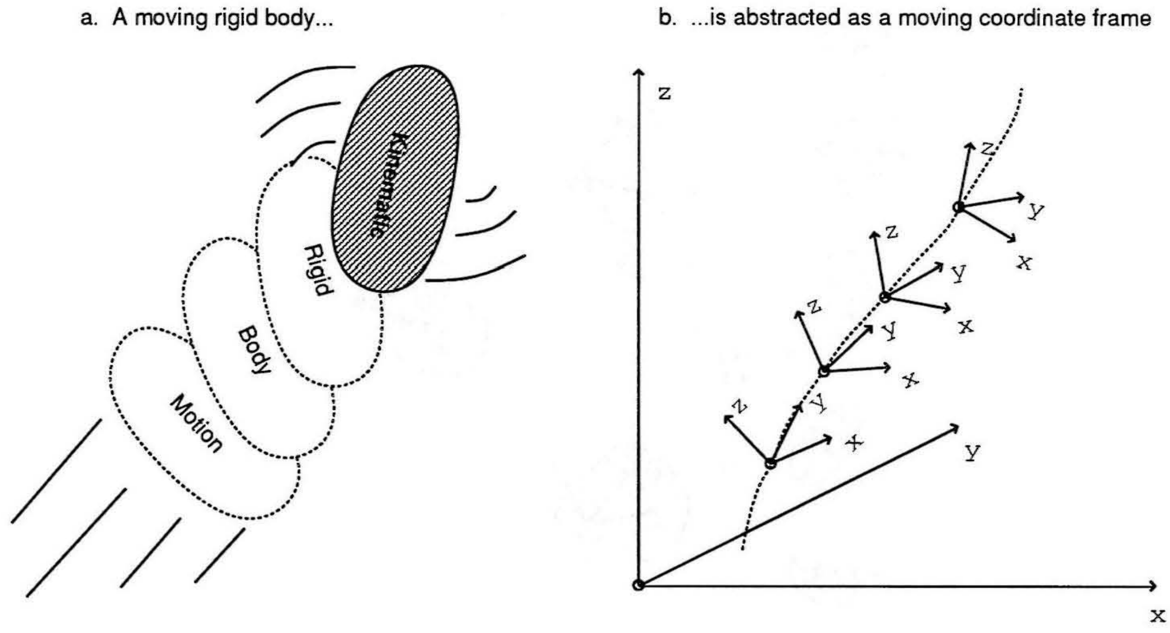

a. A moving rigid body...
b. ...is abstracted as a moving coordinate frame
Kinematic
Rigid
Body
Motion
z
x
y
z
y
x

Figure 7.1: The motion of a rigid body is described by the motion of an orthonormal coordinate frame that is fixed relative to the body. Each body's frame is called its "body coordinates." □

coordinate frame, as per Sec. 6.3. We call that frame the *body frame* (Fig. 7.1). Note that a moving body has a velocity and angular velocity at each instant of time.

The body frame is fixed relative to the body, and serves to define *body coordinates* that can be used in particular to describe the shape or configuration of the body. In this module, we make no restrictions on the body shape, other than that it be constant in body coords; nor do we provide any mechanism for describing the shape.[1] Presumably, in most cases, the origin of body coordinates is at some "natural" spot relative to the body shape (e.g., the center or the corner or the apex, depending on the shape), and the body coordinate axes are aligned with the axes of symmetry of the shape—but this is not required.

Each body in a model is given a unique "name" of some sort, so that we can identify and distinguish the bodies.

Body points:

Frequently, in addition to defining a primitive body, we are interested in specifying "interesting" points on the body. For example, we might specify a point at which to attach a constraint. We define a *body point* to be a location that is is fixed in the body frame, as per Sec. 6.3, i.e., its body coordinates are constant. Thus, as the body moves, it "carries" the body point with it.

For the most part, we think of a body point as being the location of a specific piece of "material" within the body or on its surface. But we don't require this; one could define a body point in the middle of a donut's hole, for example. The origin of the body frame is a body point, at coordinates $(0, 0, 0)$.

Hierarchical Configurations:

We're often interested in describing kinematic relationships between groups of bodies; for example, it is common to model humans, animals, and robot manipulators as *articulatedfigures,* i.e., collections of segments connected at joints (see [Badler et al.91], [Craig89]).

[1] The description of shape is a fundamental part of computer graphics; we refer readers to [Foley et al.90] and [Snyder92].

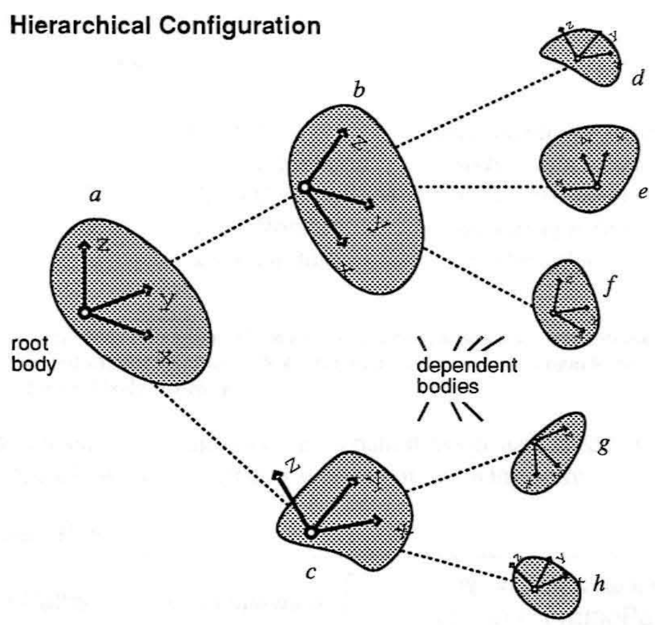


Figure 7.2: A *hierarchical configuration* is a collection of bodies organized into a hierarchical structure. Each body's configuration is specified in the body coordinates of its parent; thus if, e.g., body b is moved or rotated, bodies d, e, and f will be carried with it, and if the root body a is moved, the entire collection moves with it. A common type of hierarchical configuration is an *articulated figure,* in which primitive body segments are connected at joints. □

This module defines a *hierarchical configuration,* i.e., a collection of bodies organized into a tree hierarchy, with the position and orientation of each body's frame described in the coordinates of its parent (Fig. 7.2). An articulated figure can be described by a hierarchical configuration, in which each body's origin is fixed in its parent's frame, but the body's orientation can be adjusted.

7.3 Mathematical Model

7.3.1 Names & Notation

The scope name for this module is

KINEMATIC

We make use of the following definitions from other modules:

IDforests	(Defn. A.1)	Locations	(Defn. 6.1)
IDs	(Defn. 3.8)	LocationPaths	(Defn. 6.19)
FramePaths	(Defn. 6.27)	Orientations	(Defn. 6.1)
InstFrames	(Defn. 6.23)	Vectors	(Defn. 6.1)

We will also use superscript frame-representation notation:

$$ {}^{f}x \quad \text{(Notn. 6.6)} $$

7.3.2 Body State

We define the space of possible configurations of a rigidly-moving kinematic body, States. Note that the space includes only coordinate-frame information; we are not including any shape description.

Definition. States

$$
\begin{array}{lll}
\multicolumn{3}{l}{\mathsf{States}} \\
\multicolumn{3}{l}{[} \\
f \mapsto \mathsf{InstFrames} & & \textit{body coordinate frame} \\
x & \ldots P \mapsto \mathsf{Locations} & \textit{body origin location} \\
\mathbf{R} & \ldots R \mapsto \mathsf{Orientations} & \textit{body orientation} \\
v & \ldots V \mapsto \mathsf{Vectors} & \textit{velocity of origin} \\
\omega & \ldots \omega \mapsto \mathsf{Vectors} & \textit{angular velocity} \\
\multicolumn{3}{l}{]}
\end{array}
\tag{7.1}
$$

Defn. 7.1: Kinematic rigid-body state space. The state of a rigid body at an instant is just its instantaneous coordinate frame (which includes linear and angular velocities). Note that we have renamed the origin to be x and velocity v, in keeping with common usage for rigid body modeling. □

A moving body can be described by its trajectory, i.e., a path through state space. Based on Sec. 6.4.8, we define a space StatePaths, each of whose elements is a path of a moving body:

Definition. StatePaths

$$
\mathsf{StatePaths} \equiv \textit{the set of functions} \left\{ \begin{array}{c} s\colon \Re \rightarrow \mathsf{States} \textit{ such that} \\ (f \circ s) \in \mathsf{FramePaths} \end{array} \right\}
\tag{7.2}
$$

Defn. 7.2: The space of paths through state space. For a path $s \in$ StatePaths, the composite function $(f \circ s)$, that describes the body frame's motion, must be an element of FramePaths; this means that it is consistent as per Defn. 6.24, i.e., that the velocity vectors at each instant agree with the trajectory of the body. □

The moving-frame "paradox," Eqn. 6.29, applies directly to moving bodies:

$$
\begin{array}{rcl}
\multicolumn{3}{c}{\textit{For } s \in \mathsf{StatePaths}} \\
\frac{d}{dt} x_s(t) & = & v_s(t) \\
0 \doteq \frac{d}{dt}\, {}^{f_s(t)}x_s(t) & \neq & {}^{f_s(t)}v_s(t)
\end{array}
\tag{7.3}
$$

Eqn. 7.3: The moving-body paradox. The velocity vector can be non-0 in body coordinates, even the the origin is always at (0,0,0). The velocity vector describes the motion of the body relative to the fixed world space; thus the body "knows" that it is moving, unlike special-relativistic formulations of motion, in which there is no local way to distinguish a frame at rest from one moving with constant velocity. □

7.3.3 Body Points

We define the space of states of body points, Bodypts, analogous to the states of bodies, to define the configuration of a body point.

Definition. Bodypts

$$
\begin{array}{ll}
\multicolumn{2}{l}{\mathsf{Bodypts}} \\
\multicolumn{2}{l}{[} \\
x \mapsto \mathsf{Locations} & \textit{location of body point} \\
v \mapsto \mathsf{Vectors} & \textit{velocity of body point} \\
\multicolumn{2}{l}{]}
\end{array}
\tag{7.4}
$$

Defn. 7.4: Body point state space. A body point has a location x and a velocity v. The velocity v is the point's velocity in space. □

We can specify a body point state, given its body-frame coordinates and the state of the body it belongs to:

Definition. Bodypt

(7.5)
$$\text{Define a function } Bodypt\text{: States} \times \Re^{3\times 3} \rightarrow \text{Bodypts such that, for all } s \in \text{States and } coords \in \Re^{3\times 3}\text{:}$$
$$^{f(s)}x(Bodypt(s, coords)) = coords$$
$$v(Bodypt(s, coords)) = v_s + \omega_s \times (x(Bodypt(s, coords)) - x_s)$$

Defn. 7.5: A function that yields the state of a body point, given its body-frame coordinates. The expression for the point's velocity comes from Eqn. 6.33, that gives derivates of objects fixed in moving frames. □

7.3.4 Collections of Bodies

The definitions of Sec. 7.3.2 are generic, without the concept of *which* body. Now, we will provide definitions allowing us to name and identify bodies. We define a *system* to be a collection of names (ID's) of bodies, and their corresponding states:

Definition. Systems

(7.6)
$$\text{Systems} \equiv \{\text{States}\}_{\text{IDs}}$$

Defn. 7.6: Each system is an index of states (as per Sec. 3.8.2). That is, it is a collection of body states elements, each labeled with a different "name," or ID. Given a system $Y \in$ Systems, the state of a body labeled with $b \in$ IDs is given by Y_b. □

The above describes the state of a collection of bodies at a single instant of time. For a collection of bodies that are moving over time, we define a *system path:*

Definition. SysPaths

(7.7)
$$\text{SysPaths} \equiv \{\text{StatePaths}\}_{\text{IDs}}$$

Defn. 7.7: Each system is an index of state paths. Note that each element in the index is a function. That is, for a system path $\mathcal{Y} \in$ SysPaths, the state of body b at an instant of time t is given by $\mathcal{Y}_b(t)$. □

To construct a system $Y \in$ Systems given a system $\mathcal{Y} \in$ SysPaths, we evaluate all the paths in $\mathcal{Y}$ at a common time t. This is written as $Y = \mathcal{Y}(t)$, as per Notn. 3.18.

7.3.5 Hierarchical Configurations

A hierarchical model consists of the tree hierarchy that organizes the bodies, along with the specification of each child's state relative to that of its parent. Sec. A.1 defines the IDforests mechanism for hierarchies of names. All that remains for us is to define a mechanism to specify each body's frame relative to its parent.

We start by defining a space that encapsulates representations of frames.[2]

Definition. FrameReps

(7.8)
FrameReps
[
$x \mapsto \Re^3$ *represents a location*
$\mathbf{R} \mapsto \Re^{3\times 3}$ *represents an orientation*
]

Defn. 7.8: Each "frame rep" will be used to specify the representation of a frame in some other frame. Note that that x and $\mathbf{R}$ are numbers—they will be representations of abstract geometric objects, rather than the abstract objects themselves. □

[2] This is the most general way of describing a frame relative to some coordinate system. But for specific applications, other representations might be more convenient. For example, articulated figures are typically described in terms of joint angles. Such descriptions could be defined as specializations (Sec. 3.7.2) of FrameReps.

We have a notion of two bodies that are "aligned" by a given frame representation:

Definition. **Aligned bodies,** given a frame representation

(7.9) *A body state $b \in$* States *is* **aligned** *to a body state $a \in$* States *given a frame representation $c \in$* FrameReps *iff:*

$$\begin{aligned} {}^{f(a)}x(b) &= x(c) \\ {}^{f(a)}\mathbf{R}(b) &= \mathbf{R}(c) \end{aligned}$$

Defn. 7.9: The numbers in frame rep $c \in$ FrameReps give the coordinate values for body b, when "looked at" from body a. Note that given any two of a, b, or c, the third is uniquely determined. □

A hierarchical configuration bundles together a collection of frame-representation constraints along with a forest (collection of trees) of body names:

Definition. HierConfigs

(7.10)

```
HierConfigs
[
  forest    ↦ IDforests                    the ID hierarchy
  roots       ...roots ↦ IDsets              independent bodies
  nonroots    ...nonroots ↦ IDsets           dependent bodies
  leaves      ...leaves ↦ IDsets             bodies having no children
  parent      ...parent ↦ {IDs}_IDs          parent of each dependent
  config    ↦ {FrameReps}_IDs              the frame representations
—
  Ids(config) = nonroots
]
```

Defn. 7.10: A hierarchical model. *forest* is a forest of body ID's. For each body $c \in$ *nonroots*, we will use frame rep $config_c$ to align body c in its parent's coordinates. The internal property *Ids*(*config*) = *nonroots* guarantees that every non-root body has a corresponding frame rep. □

Putting together Defns. 7.9,7.10, we tell if a particular collection of body states satisfies a given hierarchical configuration:

Definition. **Satisfies**

(7.11) *A collection of body states $Y \in$* Systems **satisfies** *a hierarchical configuration $m \in$* HierConfigs *iff, For all $c \in nonroots(m)$,*

$$Y_c(t) \textit{ is aligned to } Y_{parent(m)_c} \textit{ given } config(m)_c$$

Defn. 7.11: A given system satisfies a model if each child body is properly aligned to its parent. Note that no top-level restrictions are placed on the roots. □

7.4 Posed Problems

A well-known problem in modeling is the *forward kinematics problem*: Given a static hierarchical configuration, and given the states at the roots, determine the states of all the bodies (see [Craig89-Ch.3]). This can be expressed using our constructs as:

$$
\begin{array}{rl}
\textit{given:} & B \in \text{Systems } \textit{and} \\
 & H \in \text{HierConfigs}, \\
 & \textit{such that } \mathit{Ids}(B) = \mathit{roots}(H) \\
\textit{find:} & Y \in \text{Systems} \\
\textit{such that:} & \left\{ \begin{array}{l} Y_b = B_b \textit{ for all } b \in \mathit{Ids}(B) \\ Y \textit{ satisfies } H \end{array} \right.
\end{array}
\tag{7.12}
$$

Eqn. 7.12: The forward-kinematics problem. We are given B, the states of the roots of the hierarchy H. We want to determine the states of all the children as well. The solution system Y includes the states of all the bodies, roots (directly as given) and nonroots (determined from H). □

Solving this problem is straightforward. One proceeds top-down from the roots, at each node constructing the frame that is aligned with its parent.

The complementary problem to the above is an *inverse kinematics problem:* Given the states of the roots and leaves of the hierarchy, determine the relationships between the bodies:

$$
\begin{array}{rl}
\textit{given:} & B \in \text{Systems } \textit{and} \\
 & F \in \text{IDforests}, \\
 & \textit{such that } \mathit{Ids}(B) = \mathit{roots}(F) \cup \mathit{leaves}(F) \\
\textit{find:} & Y \in \text{Systems } \textit{and} \\
 & H \in \text{HierConfigs}, \\
\textit{such that:} & \left\{ \begin{array}{l} Y_b = B_b \textit{ for all } b \in \mathit{Ids}(B) \\ \mathit{forest}(H) = F \\ Y \textit{ satisfies } H \end{array} \right.
\end{array}
\tag{7.13}
$$

Eqn. 7.13: An inverse-kinematics problem. We are given the hierarchical structure of body names F, but no configuration specifications, and we are given B, the states of the roots and leaves. We want to determine the states of all the intermediate bodies as well. The solution includes the system Y and the complete hierarchical configuration H. For specific types of models we may be given additional input or restrictions, such as given segment lengths for articulated figures. □

Unlike the forward kinematics problem, solving inverse problems can be difficult. There is not necessarily a unique solution; there may be none or many. [Craig89-Ch.4] discusses this problem in detail.

7.5 Implementation Notes†

7.5.1 Conceptual Section Constructs

The conceptual bodies can be defined using an object-oriented method: A base class `Body` includes the body's name, position and orientation data members; various derived classes, e.g. `Sphere`, `Cylinder`, `Banana`, etc., implement their own `draw` methods to draw the body at its current position and orientation. The base class can also maintain a list of body points associated with the body.

7.5.2 Math Section Constructs

The scope name for this module's math section is `MKIN`; Fig. 7.3 lists the class definitions. The classes for the state spaces States, FrameReps, and HierConfigs are defined in the standard manner (Sec. B.3.6):

```
class State :
  constructors:  (MCO::InstFrame ff)
  members:       f : MCO::InstFrame
                 x : MCO::Location
                 R : MCO::Orientation
                 v : MCO::Vector
                 w : MCO::Vector
```

†See Appendix B for discussion of the terminology, notation, and overall approach used here.

Program definitions in scope **MKIN**:

class name	*abstract space*	
`FrameRep`	FrameReps	*(Defn. 7.8)*
`FrameRepIdx`	$\{\text{FrameReps}\}_{\text{IDs}}$	*index of frame reps (Notn. 3.11)*
`HierConfig`	HierConfigs	*(Defn. 7.10)*
`State`	States	*(Defn. 7.1)*
`StatePath`	StatePaths	*(Defn. 7.2)*
`SysPath`	SysPaths	*(Defn. 7.7)*
`System`	Systems	*(Defn. 7.6)*

Figure 7.3: Math section definitions in the prototype implementation. □

```
class FrameRep :
  constructors:  (double[3], double[4])
  members:       x : double[3]
                 r : double[4]
```

```
class HierConfig :
  constructors:  (MMISC::Forest, FrameRepIdx)
  members:       forest   : MMISC::Forest
                 config   : FrameRepIdx
                 roots    : MM::IdSet
                 nonroots : MM::IdSet
```

The class for StatePaths is defined in the standard manner for paths (Sec. B.3.7):

```
class StatePath :
  constructors:  (MCO::FramePath ff)
  members:       f : MCO::FramePath
                 x : MCO::LocationPath
                 R : MCO::OrientationPath
                 v : MCO::VectorPath
                 w : MCO::VectorPath
```

7.5.3 M-N Interface

The scope name for the M-N interface **NKIN**. We define a routine to compute the system that solves the forward-kinematics problem. It is implemented using the routine `NMISC::IdForestPreorder(...)` of Sec. A.3.1, to traverse the hierarchy.

```
SolveForward(System, HierConfig) : System
```

7.5.4 C-M Interface

The C-M interface routines must be able to do the following:

- Map between conceptual body names and math ID's.
- Map between `MKIN::State` objects and conceptual body objects.
- Map between `MKIN::System` objects and the conceptual data structures.
- Construct a `MKIN::HierConfig` from the conceptual data structures.

Chapter 8

Dynamic Rigid-Bodies Model

This model describes classical dynamics of rigid bodies, i.e., motion of rigid bodies based on inertia and the influence of applied forces and torques, in accordance with Newton's laws. For collections of bodies, each body's motion is due directly only to the forces and torques acting on it; interaction between bodies is mediated by forces.

The force-based paradigm of motion is simple, reasonably intuitive, and uniform across applications. We don't address other paradigms, such as Lagrangian or Hamiltonian, that are based upon energy and generalized or canonical coordinates. Generalized coordinates are useful for deriving equations of motion for specific problems with constrained motion and limited degrees of freedom; using the force-based paradigm, we constrain the motion through explicit introduction of forces, as will be discussed in Ch. 9.

The presentation in this module, while axiomatic, assumes that the concepts of mass, force, and so forth are familiar to the reader. Detailed discussion and analysis can be found in [Fox67] and [Goldstein80].

8.1 Goals

Our goals for this model are:

- To describe bodies that move under the influence of forces and torques.
- To keep track of energy expenditures, for analysis and debugging.
- To provide support for higher-level modules that implement collisions, constraints, force mechanisms, etc.

At this level, we will not specify provide any mechanism for describing how individual forces and torques arise; we take forces and torques as "givens" from a higher level module.

8.2 Conceptual Model

The dynamic rigid bodies model is built on the kinematic rigid body model of Ch. 7, set in the fixed 3-D world space of Ch. 6. To the kinematic model, we add the notion of *dynamics,* i.e., motion in accordance with forces and inertia, as per Newton's laws and the classical paradigm.[1]

[1] Actually, the classical rigid-body paradigm, and Newton's laws, can be derived from more basic postluates known as Euler's laws; see [Fox67].

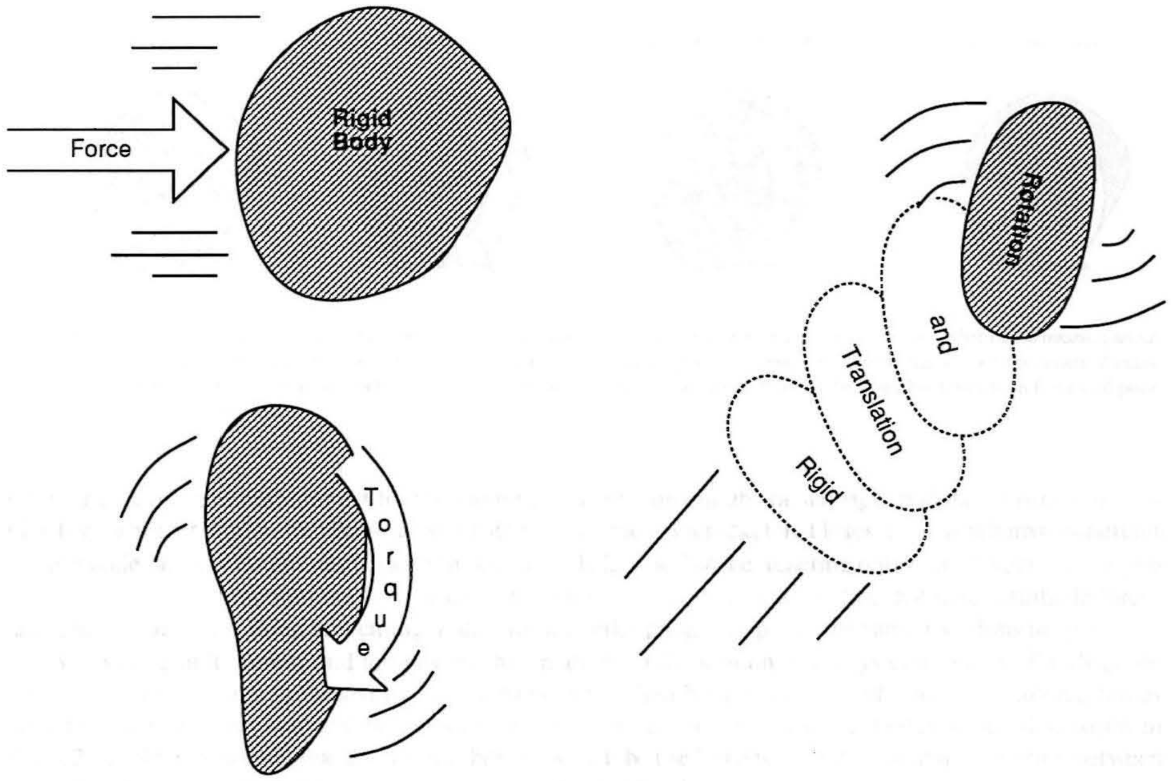


Figure 8.1: Rigid-body motion. A body moves under the influence of forces and torques, in accordance with Newton's laws of motion. The motion of a body can be separated into the linear velocity of the center of mass of the body, and the angular velocity of the body's rotation about the center of mass. The body features are encapsulated as a constant net mass and an inertia tensor that is fixed in the body frame. □

8.2.1 Bodies

We assume that each body is completely rigid, i.e., does not flex or deform in any circumstances, and that each body has a constant mass that is distributed throughout the body. This abstraction is not unreasonable for many real-world objects, so long as they are subject to relatively mild pressures and accelerations.

A dynamic body is a kinematic body as per Ch. 7—i.e., is uniquely identifiable and has a *body frame* associated with it—along with a *mass distribution.* We don't need to know the details of the mass distribution; it is encapsulated into an *inertia tensor,* and a location of the *center of mass.* Note that the center of mass does not have to be at the origin of the body frame. Since the body is rigid, the mass distribution is fixed in the body frame.

A body moves in response to *forces* and *torques* acting on it, in accordance with Newton's laws (Fig. 8.1). Motion is separable into translational motion of the center of mass (linear velocity), and rotation of the body about the center of mass (angular velocity). In the absence of any applied forces or torques, the center of mass moves in a straight line—but the body frame origin may rotate about it if the body is spinning. Forces and torques will be discussed in greater detail below (Sec. 8.2.2).

As in the kinematic model, this model includes no explicit description of the shape of a body (the body doesn't even need to be contiguous). The inertia tensor depends on the shape, but not uniquely: bodies with different shapes may share the same inertia tensor. In practice a body's behavior often does depend on details of its shape or mass distribution, e.g., wind resistance or gravitational field variation; We don't disallow such effects in this model, but merely observe that for our purposes, they are indirect, and can be encapsulated

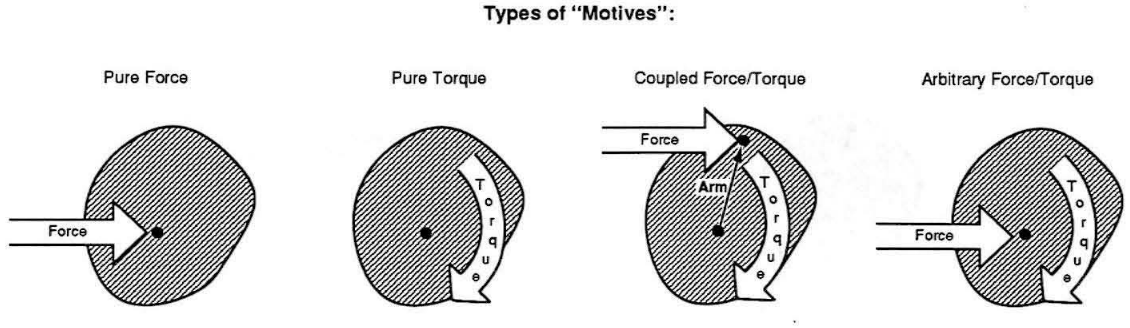


Figure 8.2: There are four types of motives: Pure forces, which cause translation but not rotation; Pure torques, that cause rotation about the center of mass, but not translation; Coupled force/torque, in which a force applied to a body at some distance from the center of mass yields a corresponding torque about the center of mass; And arbitrary force/torque pairs, that can be thought of as a pure force and pure torque acting simultaneously. □

(by higher-level modules) into net forces and torques acting on our abstracted rigid bodies. This model does disallow, however, bodies that don't have constant mass, such as a rocket that loses mass as it burns propellant.

For collections of bodies, each moves independently in world space, reacting only to the forces and torques applied to it. All interactions between bodies are thus mediated by forces. Notice that unless suitable forces are applied, there is nothing preventing bodies from overlapping, occupying the same locations in space.

We assume finite forces and torques for this module, so the motion is always continuous. For discontinuities in velocity, such as caused by collisions between rigid bodies, the idea of *impulses*—infinite forces applied instantaneously—would be introduced; we leave this for higher-level modules (see discussion in Sec. 12.1). Note, however, that the this module could still be used to describe the continuous motion between discontinuities, as in the "tennis ball cannon" example of Ch. 11.

8.2.2 "Motives"—Force/Torque Objects

We assume the existence of forces and torques, which are primitive objects that can be described by vectors, and which, when applied to bodies, cause them to move.[2]

We find it convenient to bundle a single force and/or a single torque into an object that we call a *motive*. The most common type of motive is a *coupled force/torque*, in which a force applied away from the center of mass of a body induces a torque on the body (the vector from the center of mass to the point of application is called the *moment arm*). We can also have a *pure force* or *pure torque*, which can be thought of as applied to the center of mass, and for generality, an arbitrary *force/torque pair* (Fig. 8.2).

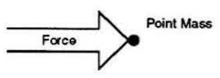
Motives exist in themselves as abstract objects, but to be used, a motive must be applied to some specific body: an *applied motive* is a motive along with (the name of) the body it acts on. If the motive is a coupled force/torque, its moment arm describes the point of application on the body. We typically give each applied motive a unique name in order to identify it.

Any number of motives can be applied to a body simultaneously. The forces and torques from each can be added, to produce a single *net force* and *net torque* on the body. The behavior of the body under the influence of the collection of motives is the same as if only the net force and torque were applied to it.

Motives are typically described at any instant by a force/torque *field*, i.e., a function of the configuration of the model.[3] For example, a Hooke's-law spring applies a force that is proportional to how much the spring is stretched. The fields are defined over hypothetical configurations—i.e., they describe what the motive would

[2] It is not strictly necessary to define torques as primitive objects—the effect of any torque can be mimicked by a suitably chosen pair of forces—but are convenient abstractions.

[3] This use of the term "field" is in the geometric sense "function over some manifold," rather than the abstract algebraic sense "set having addition and multiplication operations."



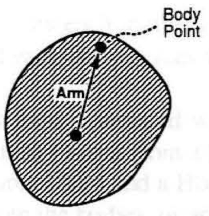


Figure 8.3: (Left) A point mass is a body that has all its mass at a single point. Point masses move under the influence of forces, but not torques. □

Figure 8.4: (Right) A body point is fixed in body coordinates. Each body point has an associated moment arm, which is the vector from the center of mass. □

be, *if* the model were in any given state—not just for the path taken by a particular simulation.[4] Note that a single field can depend on the configuration of the entire model—thus the force on a given body can depend on the configurations of the other bodies in the model, giving us an avenue to create interactions between bodies.

8.2.3 Point Masses

A *point mass* is a hypothetical object: a body that has all of its mass concentrated at a single mathematical point (Fig. 8.3). A point mass has a location and velocity, and will move under the influence of forces in the same manner as an ordinary body. Point masses have a zero inertia tensor, and have no associated orientation or angular velocity. We do not define the application of a torque to a point mass.

Point masses provide a simple abstraction of rigid body motion when orientation is not an issue. They are often used as experimental "test particles." Many flexible-body models are comprised of networks of point masses and interconnecting forces. In addition, the center of mass of any arbitrary deforming (non-rigid) body moves in as a point mass experiencing the sum of all forces acting on the body.

8.2.4 Body Points

A dynamic body point is essentially the same as a kinematic body point (Sec. 7.2), i.e., a location that moves with a body, fixed in the body's coordinate system (Fig. 8.4). As with kinematic body points, since we have no explicit description of the shape of a body, we do not care whether a body point is "in" or "on" the body.

Body points provide convenient "handles" at which to apply forces. Since a force applied at a body point yields a coupled torque, we associate a *moment arm* with each body point, as per Sec. 8.2.2. Note that the moment arm is the vector from the center of mass, not from the body origin.

8.2.5 Energy

It is a fundamental property of classical mechanics that energy is always conserved, or at least accounted for. Thus, although we are using a force-based paradigm rather than energy-based (i.e., Newtonian rather than Lagrangian), we can still keep track of energy in our models.

[4] It is particularly important that fields be defined off the solution path for a particular simulation, because most numerical solution must explore the configuration space in order to determine that path.

Each body has an associated *kinetic energy*, based on its linear and angular motion. When a body is acted upon by motives, its kinetic energy can change. The *work* done by each motive is the kinetic energy it adds to the body (negative work means that the force removes kinetic energy, i.e., slows the body).[5]

We can analyze the behavior of a model by examining the work done by each motive. In addition, we can test for consistency of the model by checking that the kinetic energy of each body agrees with the work done by all motives acting on it.

Often, a motive field is *conservative*, i.e., has a *potential energy* field associated with it such that the potential energy and work done by the motive add up to a constant—energy taken from a body is "stored" in the potential energy field, and vice versa. Examples include a gravitational field and a Hooke's-law spring.

Non-conservative motives are *dissipative* if they remove energy from the bodies, or are *active* if they add energy. The work done by these motives corresponds with a conversion between mechanical (kinetic) energy and other forms of energy, such as heat, electromagnetic, or chemical, that are out of the ken of this model.

8.3 Mathematical Model

8.3.1 Names & Notation

The scope name (see Sec. 3.6.2) for this module is

RIGID

We use the following terms from other modules:

IDs	(Defn. 3.8)	*Lab*	(Defn. 6.4)
InstFrames	(Defn. 6.23)	Locations	(Defn. 6.1)
KINEMATIC::*Bodypt*	(Defn. 7.5)	2Tensors	(Defn. 6.1)
KINEMATIC::Bodypts	(Defn. 7.4)	Orientations	(Defn. 6.1)
KINEMATIC::StatePaths	(Defn. 7.2)	Scalars	(Defn. 6.1)
KINEMATIC::States	(Defn. 7.1)	Vectors	(Defn. 6.1)

We also extensively use subscript notation for state spaces, and prefix-superscript frame representation notation:

x_s	(Notn. 3.27)
$x_s(t)$	(Notn. 3.28)
${}^{f}x$	(Notn. 6.6)

8.3.2 Mass Distributions

If we know the shape of a body and the mass density everywhere within it, we can compute the mass distribution values.

If a body occupying volume V has mass density $\rho\colon \Re^3 \to \Re$, we compute the mass $\mathtt{m} \in \Re$, center of mass $\mathtt{xc} \in \Re^3$, and inertia tensor matrix $\mathtt{I} \in \Re^{3\times 3}$ by:

(8.1)
$$\begin{aligned}
\mathtt{m} &= \int_V \rho(r)\, d^3r \\
\mathtt{xc}_i &= \frac{1}{\mathtt{m}} \int_V r_i\, \rho(r)\, d^3r \\
\mathtt{I}_{ij} &= \int_V (|r|^2 \mathbf{1}_{ij} - r_i\, r_j)\, \rho(r)\, d^3r \;-\; \mathtt{m}\,(|\mathtt{xc}|^2 \mathbf{1}_{ij} - \mathtt{xc}_i\, \mathtt{xc}_j)
\end{aligned}$$

Eqn. 8.1: Mass distribution equations. The variable of integration, $r \in \Re^3$, is a point in body coordinates, within the volume V. Note that we transfer the the inertia tensor to be about the center of mass, rather than about the origin of body coordinates. Full discussion of these equations is beyond our scope; see, e.g., [Fox67-App.B]. □

[5] Although the motive may arise from a field over possible configurations (Sec. 8.2.2), the actual work done by a motive depends on the particular path taken by the body.

Because our model does not include a description of the shape of a body, we generally take the mass distribution values to be primitive values that are given to us "pre-computed," by a higher-level model.[6] We encapsulate the rigid mass distribution properties of a body into a single space MassDists, so that each element of MassDists corresponds with a particular mass distribution:

Definition. MassDists

$$\begin{array}{lll} \multicolumn{3}{l}{\text{MassDists}} \\ [& & \\ \mathtt{m} & \mapsto \Re & \textit{mass} \\ \mathtt{I} & \mapsto \Re^{3x3} & \textit{inertia tensor matrix} \\ \mathtt{xc} & \mapsto \Re^{3} & \textit{center of mass coordinates} \\ ] & & \end{array} \tag{8.2}$$

Defn. 8.2: A space describing mass distributions of rigid bodies. $\mathtt{xc}$ gives the offset of the center of mass from the origin of the body's coordinate system. $\mathtt{I}$ is the inertia tensor about the center of mass. Note that each element is a collection of numbers, not geometric objects (we use typewriter font to remind us of this). These numbers describe the mass distribution of a body in its own coordinate system. □

A point mass (Sec. 8.2.3) has all its mass concentrated at its origin. We can think of it as the end result of a limiting process in which we keep the mass of a body constant while shrinking the shape to a zero volume located at the origin. We see in Eqn. 8.1 that both $\mathtt{xc}$ and $\mathtt{I}$ go to zero. Thus we define the set of point mass distributions to be a subset of all mass distributions:

Definition. PtMassDists

$$\text{PtMassDists} \equiv \left\{ \mathrm{M} \in \text{MassDists} \;\middle|\; \begin{array}{l} \mathtt{I}(\mathrm{M}) = 0 \\ \mathtt{xc}(\mathrm{M}) = 0 \end{array} \right\} \tag{8.3}$$

Defn. 8.3: The set of point masses. A point mass has all of its mass concentrated at its origin. Its inertia tensor $\mathtt{I}$ is thus 0, and its center of mass $\mathtt{xc}$ is at its origin. □

8.3.3 State of a Single Body

The instantaneous dynamic state of a body is completely determined by its mass distribution and its kinematic state. However, there are assorted useful geometric, momentum, and energy terms that also describe various properties of the body's dynamic state. We group all the terms into a single state space, States, that has internal properties which ensure the appropriate relationships hold amongst the various terms. Thus each element of States encapsulates a complete dynamic state:

[6] [Fox67] includes a table giving the mass properties for a variety of homogeneous bodies in canonical shapes; [Lien,Kajiya84] computes integral properties of arbitrary nonconvex polyhedra; [Snyder92] computes the mass properties for homogeneous bodies described by parametric surfaces. The net mass properties of a rigid body formed as a compound of simpler bodies is described in [Fox67-App.B].

Definition. States

(8.4)

States

[

M	$\mapsto$ MassDists	*mass distribution*
m	$\ldots \mathtt{m} \mapsto \Re$	*mass*
$\mathbf{I}$	$\mapsto$ 2Tensors	*inertia tensor*
k	$\mapsto$ KINEMATIC :: States	*kinematic state*
f	$\ldots f \mapsto$ InstFrames	*body coord frame*
x	$\ldots x \mapsto$ Locations	*location of body coords origin*
$\mathbf{R}$	$\ldots \mathbf{R} \mapsto$ Orientations	*body coords orientation*
v	$\ldots v \mapsto$ Vectors	*body velocity*
ω	$\ldots \omega \mapsto$ Vectors	*body angular velocity*
arm	$\mapsto$ Vectors	*moment arm of body origin*
xc	$\mapsto$ Locations	*location of center of mass*
vc	$\mapsto$ Vectors	*velocity of center of mass*
p	$\mapsto$ Vectors	*linear momentum*
L	$\mapsto$ Vectors	*angular momentum*
KEv	$\mapsto \Re$	*linear kinetic energy*
$KE\omega$	$\mapsto \Re$	*angular kinetic energy*
KE	$\mapsto \Re$	*net kinetic energy*

—

$$\begin{aligned} {}^{f}\mathbf{I} &= \mathtt{I}(M) & arm &= x - xc \\ -({}^{f}arm) = {}^{f}xc &= \mathtt{xc}(M) & v &= vc + \omega \times arm \\ KEv &= \tfrac{1}{2} m\, vc \cdot vc & p &= m\, vc \\ KE\omega &= \tfrac{1}{2} \omega \cdot \mathbf{I}\omega & L &= \mathbf{I}\,\omega \\ KEv + KE\omega &= KE & & \\ M \in \text{PtMassDists} &\implies \begin{cases} \mathbf{R} = R_{lab} \\ \omega = 0 \end{cases} & & \end{aligned}$$

]

Defn. 8.4: Instantaneous dynamic state of a rigid body. Note that a body has separate position and velocity information for its body frame origin (x and v) vs. for its center of mass (xc and vc); The vector arm is the displacement of the origin relative to the center of mass; it is fixed in body coordinates, given by the mass distribution. Identifying tuples for the space include: [M, k]—mass distribution and kinematic state; [M, xc, $\mathbf{R}$, p, L]—mass distribution and dynamic state; [M, f, xc, vc]—mass distribution and mixed state. □

Notice that we use the single space States to encompass both bodies and point masses; since a point mass has no intrinsic notion of orientation, we (arbitrarily) choose to define that a point mass has the lab's orientation $R = R_{lab}$, and zero angular velocity, $\omega = 0$. From the above and from Defn. 8.3 we have the following corollary properties of a point mass:

Given $p \in$ States *with* $M_p \in$ PtMassDists

(8.5)
$$\begin{aligned} \mathbf{I}_p &= 0, & \mathbf{R}_p &= Lab, \\ \omega_p &= 0, & xc_p &= x_p, \\ arm_p &= 0, & vc_p &= v_p, \\ L_p &= 0, & KE_p &= KEv_p, \\ KE\omega_p &= 0 & & \end{aligned}$$

Eqn. 8.5: Various properties of point masses. □

It may seem like overkill to use States for point masses: mightn't it be "cleaner" to have a separate, simpler space just for point masses? However, by having a single space, the remaining discussion will apply uniformly both to bodies and to point masses.

8.3.4 Motion of a Single Body

A moving body varies its state over time; that is, a moving body is described by a path through States space. Thus we define the space:

Definition. StatePaths

(8.6)
$$\text{StatePaths} \;\equiv\; \textit{the set of functions} \left\{ \begin{array}{c} s: \Re \to \text{States} \textit{ such that} \\ (k \circ s) \in \text{KINEMATIC} :: \text{StatePaths}, \textit{and } \mathrm{M}_s(t) \textit{ is} \\ \textit{constant} \end{array} \right\}$$

Defn. 8.6: The set of paths through state space. We only consider functions s for which the resulting kinematic state function $k_s(t)$ is a kinematic state path (Defn. 7.2). Additionally, we require that the mass distribution $\mathrm{M}_s(t)$ doesn't change. □

Since we require that the mass distribution of a state path be constant, we can drop the path parameter when using the mass distribution aspects:

Notation. Dropping the parameter (t) for a state path

(8.7)
For a state path $s \in$ StatePaths
$$\begin{aligned} \mathrm{M}_s &\equiv \mathrm{M}_s(t) \\ m_s &\equiv m_s(t) \end{aligned}$$

Notn. 8.7: Since, by definition, the mass distribution of a state path doesn't depend on the path parameter, for clarity we usually leave it off. □

From Defn. 8.4 and Defn. 8.6, we have:

(8.8)
Given a state path $s \in$ StatePaths
$$\begin{aligned} {}^{f_s(t)}\mathbf{I}_s(t) &= \mathtt{I}(\mathrm{M}_s), \; \textit{constant} \\ {}^{f_s(t)}xc_s(t) &= m_s, \; \textit{constant} \end{aligned}$$

Eqn. 8.8: For a state path s, the inertia tensor $\mathbf{I}_s(t)$ and location of center of mass $xc_s(t)$ are **not** constant—they move with the body, changing over time. But their representations in the body's moving coordinate frame are constant, given by the constant mass distribution. Using the parlance of Sec. 6.4.9, they are "fixed" in the moving body frame. □

Defn. 8.6 requires that the motion be continuous, and that the mass distribution in the body be constant. That, along with Defn. 7.2 and Defn. 6.24 ensure that, as expected:

(8.9)
Given a state path $s \in$ StatePaths
$$\begin{aligned} \tfrac{d}{dt}x_s(t) &= v_s(t) \\ \tfrac{d}{dt}xc_s(t) &= vc_s(t) \\ \tfrac{d}{dt}\mathbf{R}_s(t) &= \omega_s^*(t)\mathbf{R}_s(t) \end{aligned}$$

Eqn. 8.9: Velocities of a state path agree with the derivatives as expected. Note that the "moving-body paradox," Eqn. 7.3, applies to dynamic rigid bodies' origins, orientations, and centers of mass. □

In addition to moving continuously and rigidly, we want a body to move dynamically under the influence of forces and torques. Thus we define: (Fig. 8.5)

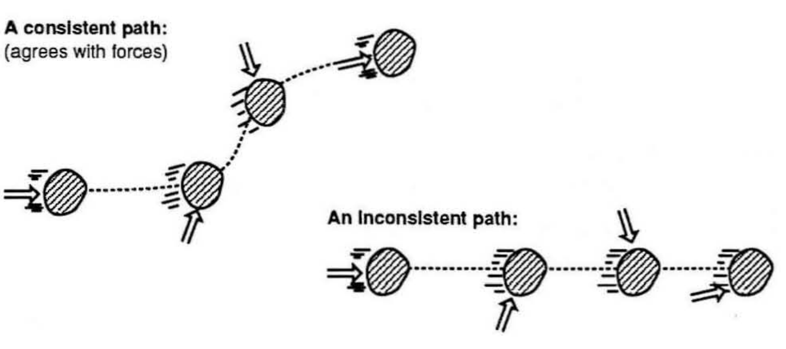


Figure 8.5: Consistent vs. inconsistent paths. A state path is an arbitrary function that describes physically realizable motion of a rigid body. If we have a net force function and a net torque function (torque not illustrated), we say that the path and functions are *consistent* if the motion of the body agrees with the force and torque, as per "$F = m\,a$." □

Definition. **Consistent** (path, net force, net torque)

A state path $s \in \mathsf{StatePaths}$ *is* **consistent** *with net force and torque functions* $F, T\colon \Re \rightarrow \mathsf{Vectors}$ *iff:*

$$\frac{d}{dt}p_s(t) = F(t) \quad \textit{(force equals change in momentum)}$$
$$\frac{d}{dt}L_s(t) = T(t) \quad \textit{(torque equals change in angular momentum)}$$
$$\mathrm{M}_s \in \mathsf{PtMassDists} \implies T(t) = 0 \tag{8.10}$$

Defn. 8.10: Dynamic behavior of a rigid body. At each instant of time, the change in momenta is due to the *net force* $F(t)$ and *net torque* $T(t)$. As discussed in Sec. 8.2.3, we don't apply torques to point masses. □

By Defn. 8.4 and Eqn. 8.9 we have, for a path $s \in \mathsf{StatePaths}$: $\frac{d}{dt}p_s(t) = \frac{d}{dt}m_s\, vc_s(t) = m\frac{d^2}{dt^2}x_s(t)$. Thus Defn. 8.10 is equivalent to the common "$F = ma$" 2nd-order Newtonian equation of motion.

8.3.5 Body Points

A dynamic body point is similar to a kinematic body point, but we find it convenient to include the point's moment arm as part of its instantaneous state.

Definition. Bodypts

(8.11)

```
Bodypts
[
k     ↦ KINEMATIC :: Bodypts   kinematic state of body point
x         ... x ↦ Locations        location
v         ... v ↦ Vectors          velocity
arm ↦ Vectors                  displacement from body c.m.
]
```

Defn. 8.11: Body point state space. The state of a body point includes its kinematic description as well as a moment arm, i.e., a vector from the center of mass of the body the point location. Note, however, that there is no explicit indication of *which* body. □

We can specify a body point state by giving its body-frame coordinates, and the state of the body it belongs to:

Definition. Bodypt

(8.12)

$$\boxed{\begin{array}{c}\textit{Define a function Bodypt: } \mathsf{States} \times \Re^{3\times3} \rightarrow \mathsf{Bodypts} \textit{ such that, for all} \\ s \in \mathsf{States} \textit{ and } coords \in \Re^{3\times3}\textit{:} \\ k(Bodypt(s, coords)) = \text{KINEMATIC}::Bodypt(k_s, coords) \\ arm(Bodypt(s, coords)) = x(Bodypt(s, coords)) - xc_s \end{array}}$$

Defn. 8.12: A function that yields the state of a body point, given its body-frame coordinates. The kinematic state of the body point is specified via KINEMATIC :: *Bodypt* (Defn. 7.5). The moment arm is the vector from the body center of mass to the body point. □

Some properties of a body point:

Given a body state $s \in$ States and body point $p \in$ Bodypts, where $p = Bodypt(s, coords)$ for some $coords \in \Re^{3\times3}$, we have:

(8.13)

$$\begin{aligned} {}^{f(s)}x_p &= coords \\ {}^{f(s)}arm_p &= coords + {}^{f(s)}arm_s \\ &= coords + \mathbf{xc}(\mathrm{M}_s) \\ v_p &= v_s \ + \omega_s \times (x_p - x_p) \\ &= vc_s + \omega_s \times arm_p \end{aligned}$$

Eqn. 8.13: The location and moment arm of a body point defined via *Bodypt* (Defn. 8.12) can easily be expressed in body coordinates. The velocity can be expressed geometrically in terms of either body origin or body center of mass. These equalities are derived from from Defn. 7.5, Defn. 8.12, and the properties in Defn. 8.4. □

8.3.6 A Collection of Bodies

For a collection of continuously moving dynamic bodies, the interaction between the bodies is mediated by forces, as discussed in Sec. 8.2.1. That is, each body doesn't directly "see" any other bodies, it just responds to the forces it "feels", in accordance with Defn. 8.10. However, we are free to define whatever force functions we like—in particular, the force function for any given body may at any instant be determined by the states of all the bodies. In this way, the motion of a body may be determined (indirectly) by interaction with other bodies.

In order to manage collections of bodies, we proceed analogously to the scheme used for kinematic bodies (Sec. 7.3.4). We define an instantaneous *system*:

Definition. Systems

(8.14)

$$\boxed{\mathsf{Systems} \equiv \{\mathsf{States}\}_{\mathsf{IDs}}}$$

Defn. 8.14: Each instantaneous system is an index of states (as per Sec. 3.8.2). That is, it is a collection of body state elements, each labeled with a different "name," or ID. Given a system $Y \in$ Systems , the state of a body labeled with $b \in$ IDs is given by Y_b □

Each point in the space Systems describes the state of a collection of bodies, at an instant of time. For moving bodies, we analogously define a *system path* to be a labeled collection of paths:

Definition. SysPaths

(8.15)

$$\boxed{\mathsf{SysPaths} \equiv \{\mathsf{StatePaths}\}_{\mathsf{IDs}}}$$

Defn. 8.15: Each system path is an index of state paths. Note that each element in the index is a function. That is, for a system path $\mathcal{Y} \in$ SysPaths, the state of body b at an instant of time t is given by $\mathcal{Y}_b(t)$ □

Given $\mathcal{Y} \in$ SysPaths, the instantaneous state of the system at a time t is given by evaluating all paths in $\mathcal{Y}$ at t; this is written as $\mathcal{Y}(t)$ as per Notn. 3.18.

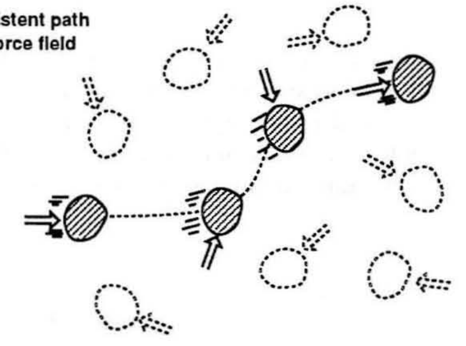


Figure 8.6: A consistent path in a force field. A force field defines what the force would be on a body for any possible state it may be in. A path is consistent with a force field and a torque field (not shown) if it is consistent with the forces traced out as the body path moves through the field. Note that if there are several bodies in a system, the force/torque on any one of them can depend on the state of all of them; thus the consistency of each individual path can depend on all the other paths. □

We define a few types of *fields*, i.e., functions over instantaneous systems:

Definition. (fields)

(8.16)
$$\begin{aligned}\mathsf{SysScalarFields} &\equiv \left\{\begin{array}{c}\textit{the set of functions}\\ \mathsf{Systems}\times\Re\rightarrow\mathsf{Scalars}\end{array}\right\}\\ \mathsf{SysVectorFields} &\equiv \left\{\begin{array}{c}\textit{the set of functions}\\ \mathsf{Systems}\times\Re\rightarrow\mathsf{Vectors}\end{array}\right\}\\ \mathsf{SysLocationFields} &\equiv \left\{\begin{array}{c}\textit{the set of functions}\\ \mathsf{Systems}\times\Re\rightarrow\mathsf{Locations}\end{array}\right\}\end{aligned}$$

Defn. 8.16: Scalar, vector, and location fields are functions on instantaneous systems and time. Since an instantaneous system $Y \in$ Systems labels each state with the ID naming its body, an individual field function $F(Y,t)$ can depend on particular named bodies. □

As an example of a field, we can define a vector field *velA* that yields the velocity of the body named *bodyA*:

(8.17)
Given an identifier bodyA $\in$ IDs, *we can define a vector field velA* $\in$ SysVectorFields *by:*

$$velA(Y,t) = v(Y_{bodyA})$$

Eqn. 8.17: Sample vector field, that gives the velocity of a body named *bodyA*. Note that the field is only well-defined for systems that include this body, i.e., for $bodyA \in Ids(Y)$. Note also that the field is independent of any bodies other than *bodyA* that may be in the system, as well as the time t. □

Having made the above definitions, we can extend Defn. 8.10 to apply to a collection of bodies acting under the influence of corresponding labeled collections of force and torque fields: (Fig. 8.6)

Definition. **Consistent** (system path, net force index, net torque index)

(8.18)
A system path $\mathcal{Y} \in$ SysPaths *is* **consistent** *with*
$F, T \in \{\mathsf{SysVectorFields}\}_{\mathsf{IDs}}$ *iff:*

$$Ids(\mathcal{Y}) = Ids(F) = Ids(T)$$
$$\mathcal{Y}_b \textit{ is consistent with } (F_b \circ \mathcal{Y}) \textit{ and } (T_b \circ \mathcal{Y}), \textit{ for all } b \in Ids(\mathcal{Y})$$

Defn. 8.18: A system path is consistent if each individual path is consistent (Defn. 8.10) with its corresponding composite net force and torque functions. For each ID b, the net force and torque acting on body b are given by $F_b(\mathcal{Y}(t),t)$ and $T_b(\mathcal{Y}(t),t)$. Notice the force/torque on each body can depend on the state of all the other bodies. □

8.3.7 Motives

The description of rigid body motion in Defn. 8.18 assumes the existence of fields giving the net force and torque on each body in a system. Here, we will use the idea of *motives*—force/torque objects—as per Sec. 8.2.2, and allow a collection of separate motives to apply to each body: we will derive the net force and torque fields on each body by adding the contributions from a collection of motive fields.

We start by defining a space of motives. Each element in the space is a value of a force and/or torque, along with the corresponding moment arm:

Definition. Motives

(8.19)
$$
\begin{array}{l}
\mathsf{Motives} \\
\big[\\
\quad \begin{array}{lll}
F & \mapsto \mathsf{Vectors} & \textit{force} \\
T & \mapsto \mathsf{Vectors} & \textit{torque} \\
\mathit{arm} & \mapsto \mathsf{Vectors} & \textit{moment arm}
\end{array} \\
\quad \text{—} \\
\quad T = \mathit{arm} \times F \qquad \textit{or} \qquad \mathit{arm} = 0 \\
\big]
\end{array}
$$

Defn. 8.19: The space of motives. For a *coupled force/torque,* we have a non-zero force and moment arm, yielding the corresponding torque $T = \mathit{arm} \times F$. A *pure force* has force but no torque ($F \neq 0$, $\mathit{arm} = T = 0$). A *pure torque* has torque but no force ($F = \mathit{arm} = 0$, $T \neq 0$). An arbitrary *force/torque pair* has ($F \neq 0$, $T \neq 0$, $\mathit{arm} = 0$). (See Fig. 8.2.) □

A pure force is of course equivalent to a force applied at the center of mass ($\mathit{arm} = 0$). Since we have here no notion of body shape, there is no requirement that the moment arm actually be "in" or "on" the body.

An *applied motive* describes a motive that is applied to a particular body name. Thus we define:

Definition. AppliedMotives

(8.20)
$$
\begin{array}{l}
\mathsf{AppliedMotives} \\
\big[\\
\begin{array}{lll}
\mathit{motive} \mapsto \mathsf{Motives} & & \textit{motive} \\
F & \ldots F \mapsto \mathsf{Vectors} & \quad \textit{force} \\
T & \ldots T \mapsto \mathsf{Vectors} & \quad \textit{torque} \\
\mathit{arm} & \ldots \mathit{arm} \mapsto \mathsf{Vectors} & \quad \textit{moment arm} \\
\mathit{body} & \mapsto \mathsf{IDs} & \textit{body that the motive is applied to}
\end{array} \\
\big]
\end{array}
$$

Defn. 8.20: Motives applied to bodies. Each element defines a particular motive and the ID of a body at which it is applied. □

We define fields of motives and applied motives, analogous to the fields in Defn. 8.16:

Definition. (motive fields)

(8.21)
$$
\begin{array}{rl}
\mathsf{SysMotiveFields} \equiv & \textit{the set of functions} \\
& \{\mathsf{Systems} \times \Re \rightarrow \mathsf{Motives}\} \\
\mathsf{AppliedMotiveFields} \equiv & \textit{the set of functions} \\
& \left\{ \begin{array}{c} a\colon \mathsf{Systems} \times \Re \rightarrow \mathsf{AppliedMotives} \textit{ such that} \\ \mathit{body}(a(Y,t)) \textit{ is constant} \end{array} \right\}
\end{array}
$$

Defn. 8.21: Motive and applied motive fields. For applied motive fields, we restrict the definition to those functions whose body ID doesn't vary, no matter what parameters its given. □

We require that each applied motive field corresponds with the application of a motive field to a single body. Thus the *body* aspect of an applied motive field is independent of the parameters, and we typically drop them:

Notation. Dropping parameters (Y, t) for applied motive field body

(8.22)
$$\text{For } a \in \mathsf{AppliedMotiveFields}$$
$$\mathit{body}(a) \equiv \mathit{body}(a(Y,t))$$

Notn. 8.22: Since, by definition, the body of an applied motive field doesn't depend on the parameters, for clarity we usually leave them off. □

A common occurrence in models is a force that is applied to a body at a particular body point. For example, we might attach a spring force to a body point at one end of a cylinder; as the cylinder moves, the spring follows it, always acting on the given point of the body. In terms of motives, this means:

(8.23)
If an applied motive field $a \in \mathsf{AppliedMotiveFields}$ *describes a force applied to a body named* $\mathit{body}(a) \in \mathsf{IDs}$ *in system* $\mathcal{Y} \in \mathsf{SysPaths}$ *at body coordinates* $coords \in \Re^{3\times 3}$*, we have:*

$$\mathit{arm}_a(\mathcal{Y}(t), t) = \mathit{arm}(\mathit{Bodypt}(\mathcal{Y}_{\mathit{body}(a)}(t), coords)), \quad \text{for all } t \in \Re$$

Eqn. 8.23: For us to say that a force is "applied to a body point," the applied motive field's *arm* aspect must always agree with the body point's *arm* aspect as the body moves over time. □

Typically, a model will contain many applied motive fields acting on the various in the model. Since each applied motive field specifies the body it's applied to, we don't need to explicitly organize the various motive fields by body. We group all the applied motive fields into an index; this will let us refer to them by ID later, e.g., for the energy expressions in Sec. 8.3.8.

The following functions extract the net force and torque on each body given an index of applied motives.

Definition. Fnet, Tnet

(8.24)
We define the net force and torque functions,
$$\mathit{Fnet}\colon \{\mathsf{AppliedMotives}\}_{\mathsf{IDs}} \to \{\mathsf{Vectors}\}^{\circ}_{\mathsf{IDs}}$$
$$\mathit{Tnet}\colon \{\mathsf{AppliedMotives}\}_{\mathsf{IDs}} \to \{\mathsf{Vectors}\}^{\circ}_{\mathsf{IDs}}$$ *such that, for all* $A \in \{\mathsf{AppliedMotives}\}_{\mathsf{IDs}}$,

$$\mathit{Fnet}(A)_b \equiv \sum_{\substack{a \in \mathit{Elts}(A) \\ \mathit{body}(a)=b}} F_a$$

$$\mathit{Tnet}(A)_b \equiv \sum_{\substack{a \in \mathit{Elts}(A) \\ \mathit{body}(a)=b}} T_a$$

$$\mathit{Ids}(\mathit{Fnet}(A)) \equiv \mathit{Ids}(\mathit{Tnet}(A)) \equiv \left\{ \mathit{body}(a) \;\middle|\; a \in \mathit{Elts}(A) \right\}$$

Defn. 8.24: Net force and torque, given an index of applied motives. For each body b, *Fnet* $(A)_b$ yields the sum of all the individual motive forces that are applied to body b. Note that *Fnet* (A) is defined to be an index with zero, so that if A doesn't apply any forces to a particular body, the body will be assigned a net force of 0. Note that we don't need to refer to the elements of A by their ID labels. (Similarly for *Tnet*.) □

Using the above definition, we can extend Defn. 8.18's specification of consistent system paths to handle a collection of applied motives; this is the "bottom line" specification of rigid body motion in our model:

Definition. **Consistent** (system path, applied motive index)

(8.25)

A system path $\mathcal{Y} \in \mathsf{SysPaths}$ *is* **consistent** *with* $\mathcal{A} \in \{\mathsf{AppliedMotiveFields}\}_{\mathsf{IDs}}$ *iff:*

$\mathcal{Y}$ *is consistent with* $(Fnet \circ \mathcal{A}), (Tnet \circ \mathcal{A})$

Defn. 8.25: A system is consistent with an index of applied motives if it is consistent (Defn. 8.18) with the net force and torque implied by the motives. Here, as in Defn. 8.24, the ID's that label elements of the index of applied motive fields, $\mathcal{A}$, are not neeed. □

8.3.8 Energy

The following expression describes the rate of change of energy of a body, in relation to the force and torque applied to it:

(8.26)

For any state path $s \in \mathsf{StatePaths}$ *consistent with net force and torque functions* $F, T: \Re \to \mathsf{Vectors}$,

$$\tfrac{d}{dt} KE_s(t) = F(t) \cdot vc_s(t) + T(t) \cdot \omega_s(t)$$

Eqn. 8.26: Energy balance. The rate of change of the body's kinetic energy is equal to the power applied by the net force and torque acting on it, for a consistent system. This equation can be derived from Defn. 8.4 and Defn. 8.10 □

Thus we describe the instantaneous *power* being applied by a force and torque:

(8.27)

Given a body with state $b \in \mathsf{States}$ *that is acted on by a force and torque* $F, T \in \mathsf{Vectors}$, *the power* $P \in \Re$ *being applied is given by:*

$$P = F_m \cdot vc_b + T_m \cdot \omega_b$$

Eqn. 8.27: The instantaneous power applied to a body by an applied force and torque. □

To measure the energy imparted to a body by a motive, we need to keep track of the power and work associated with the motive as the body moves along its path. We start by defining a *power motive* (or just *pmotive*), that extends a motive to include power-related terms:

Definition. Pmotives, PmotivePaths

(8.28)

```
Pmotives
[
  M    ↦ Motives                 a motive
  F        ...F ↦ Vectors
  T        ...T ↦ Vectors
  arm      ...arm ↦ Vectors
  v    ↦ Vectors                 a body velocity
  ω    ↦ Vectors                 a body angular velocity
  P    ↦ ℜ                       power applied by motive
  W    ↦ ℜ                       work done
—
  P ≡ F·v + T·ω
]
PmotivePaths ≡ the set of functions {ℜ → Pmotives}
```

Defn. 8.28: Each pmotive element of Pmotives describes with the instantaneous application of a motive M to a body with velocities v and ω. P is the power exerted. W is an additional parameter, that will be used to accumulate the work done by the motive over time. We also define the set of pmotive-valued paths, PmotivePaths, without any *a priori* restrictions. □

For a given system path, we can associate a pmotive path with an applied motive field; this pmotive path keeps track of the work done by that applied motive field on its body in the system:

Definition. **Consistent** (pmotive path, applied motive field, system path)

(8.29)

A power motive path $p \in \mathsf{PmotivePaths}$ *is* **consistent** *with* $a \in \mathsf{AppliedMotiveFields}$ *and* $\mathcal{Y} \in \mathsf{SysPaths}$ *iff*

$$\begin{aligned} M_p(t) &= motive_a(\mathcal{Y}(t), t) \\ v_p(t) &= vc(\mathcal{Y}_{body(a)}(t)) \\ \omega_p(t) &= \omega(\mathcal{Y}_{body(a)}(t)) \\ \tfrac{d}{dt} W_p(t) &= P_p(t) \end{aligned}$$

Defn. 8.29: We describe a power motive path p that corresponds with an applied motive field, for a given system. The first three equations signify that the power motive path p continually describes the actual application of the motive, i.e., the motive and velocity aspects agree. The final equation further restricts the path p to the one in which the rate of change of work is equal at each instant to the applied power (this is analogous to Defn. 6.24's restricting frame paths to those with a velocity matching the change in location). Thus $W_p(t)$ is the total work done by a in system $\mathcal{Y}$, up to time t. □

The above definition relates a single applied motive field to a single corresponding pmotive path. For an index of applied motive fields, we can create a corresponding index of pmotive paths, where the label of each applied motive field is used to label its corresponding pmotive path. Thus we extend the above definition:

Definition. **Consistent** (pmotive paths, appl. motive fields, system path)

(8.30)

An index of pmotive paths $\mathcal{P} \in \{\mathsf{PmotivePaths}\}_{\mathsf{IDs}}$ *is* **consistent** *with* $\mathcal{A} \in \{\mathsf{AppliedMotiveFields}\}_{\mathsf{IDs}}$ *and* $\mathcal{Y} \in \mathsf{SysPaths}$ *iff:*

$$Ids(\mathcal{P}) = Ids(\mathcal{A}),$$
$$\mathcal{Y} \textit{ is consistent with } \mathcal{A},$$
$$P_f \textit{ is consistent with } \mathcal{A}_f \textit{ and } \mathcal{Y}, \textit{for all } f \in Ids(\mathcal{A})$$

Defn. 8.30: For an index of motive paths $\mathcal{P}$ to be consistent with an index of applied motive fields $\mathcal{A}$, for a given system of bodies $\mathcal{Y}$: First, each path in $\mathcal{P}$ must be have the same ID as a field in $\mathcal{A}$; Second, the $\mathcal{Y}$ and $\mathcal{A}$ must themselves be consistent, as per Defn. 8.25. Finally, each pmotive path in $\mathcal{P}$ must be consistent with its corresponding field in $\mathcal{A}$, as per Defn. 8.29. □

Putting together Eqn. 8.26 through Defn. 8.30, we have balance of energy for an entire system of bodies:

(8.31)

Given $\mathcal{P} \in \{\mathsf{PmotivePaths}\}_{\mathsf{IDs}}$ *consistent with* $\mathcal{A} \in \{\mathsf{AppliedMotiveFields}\}_{\mathsf{IDs}}$ *and* $\mathcal{Y} \in \mathsf{SysPaths}$, *then for all* $b \in Ids(\mathcal{Y})$:

$$KE(\mathcal{Y}_b(t)) - \sum_{\substack{f \in Ids(\mathcal{A}) \\ body(\mathcal{A}_f) = b}} W(\mathcal{P}_f(t)) \textit{ is constant}$$

Eqn. 8.31: Energy balance. For each body in a consistent system, the total kinetic energy of the body is balanced by the sum of the work done by all the motives acting on it. Note that we don't have a precise value (e.g., 0, as one might expect) for this equation, because W is only specified up to a constant offset, by the differential equation in Defn. 8.29. □

Note that Eqn. 8.31 is merely Eqn. 8.26 applied independently and simultaneously to each body in the system. That is, no energy is transferred directly between bodies. Rather, the energy added to or removed from each body is due only to the forces acting on it. However, we can sum them together to make a single equation for entire system:

(8.32)

Given $\mathcal{P} \in \{\mathsf{PmotivePaths}\}_{\mathsf{IDs}}$ *consistent with* $\mathcal{A} \in \{\mathsf{AppliedMotiveFields}\}_{\mathsf{IDs}}$ *and* $\mathcal{Y} \in \mathsf{SysPaths}$, *we have:*

$$\sum_{a \in Ids(\mathcal{Y})} KE(\mathcal{Y}_a(t)) - \sum_{f \in Ids(\mathcal{A})} W(\mathcal{P}_f(t)) \textit{ is constant}$$

Eqn. 8.32: Net energy balance for a system. The total kinetic energy of the bodies is balanced by the total work done by the motives. □

As discussed in Sec. 8.2.2, some motive fields are *conservative*, and we know a potential energy associated with the motive. In such a case, the potential energy must balance the work done by the motive:

$$\begin{gathered}\textit{Given } \mathcal{P} \in \{\mathsf{PmotivePaths}\}_{\mathsf{IDs}} \textit{ consistent with} \\ \mathcal{A} \in \{\mathsf{AppliedMotiveFields}\}_{\mathsf{IDs}}, \textit{ and } \mathcal{Y} \in \mathsf{SysPaths}, \textit{ If we have a} \\ \textit{potential energy field } U_f \in \mathsf{SysScalarFields} \textit{ for applied motive field } \mathcal{A}_f \\ \\ U_f(\mathcal{Y}(t), t) + W(P_f(t)) \textit{ is constant}\end{gathered} \tag{8.33}$$

Eqn. 8.33: If a motive field $\mathcal{A}_f$ has a potential energy U_f associated with it, then as a system evolves over time, any lost potential energy is gained as work done by the motive, and vice versa. □

Furthermore, for a conservative motive field, the force/torque is given by the gradient of the potential energy field.

8.4 Posed Problems

Forward Dynamics

The most common problem we solve is to determine the behavior of a collection of bodies that start out in some configuration, and are acted on by some collection of force fields. This is the *forward dynamics problem.*

$$\begin{array}{rl} \textit{given:} & t_0 \in \Re, \\ & Y_0 \in \mathsf{Systems}, \\ & \mathcal{A} \in \{\mathsf{AppliedMotiveFields}\}_{\mathsf{IDs}}, \\ \textit{find:} & \mathcal{Y} \in \mathsf{SysPaths} \\ \textit{such that:} & \left\{ \begin{array}{l} \mathcal{Y}(t_0) = Y_0 \\ \mathcal{Y} \textit{ is consistent with} A \end{array} \right. \end{array} \tag{8.34}$$

Eqn. 8.34: The forward dynamics problem. Given an initial condition of a collection of bodies (at time t_0 the state is Y_0), and a collection of motive fields applied to the bodies, determine the behavior of the bodies for other values of t. The resulting system $\mathcal{Y}$ must match the initial conditions, and must be consistent with the applied motives as per Defn. 8.25. □

The definition of *consistent* (Defn. 8.25) can be expanded to yield a first-order ordinary differential equation in canonical, numerical form:

$$\begin{gathered}\textit{For a system path } \mathcal{Y} \in \mathsf{SysPaths} \textit{ consistent with} \\ \mathcal{A} \in \{\mathsf{AppliedMotiveFields}\}_{\mathsf{IDs}}, \textit{ if } Ids(\mathcal{Y}) = \{a, b, \ldots\}, \textit{ then:} \\ \begin{array}{rcl} \frac{d}{dt}\,{}^{\mathcal{L}ab}xc(\mathcal{Y}_a(t)) & = & {}^{\mathcal{L}ab}vc(\mathcal{Y}_a(t)) \\ \frac{d}{dt}\,{}^{\mathcal{L}ab}\mathbf{R}(\mathcal{Y}_a(t)) & = & {}^{\mathcal{L}ab}\omega^*(\mathcal{Y}_a(t))\mathbf{R}(\mathcal{Y}_a(t)) \\ \frac{d}{dt}\,{}^{\mathcal{L}ab}p(\mathcal{Y}_a(t)) & = & {}^{\mathcal{L}ab}Fnet(\mathcal{A}(\mathcal{Y}(t), t))_a \\ \frac{d}{dt}\,{}^{\mathcal{L}ab}L(\mathcal{Y}_a(t)) & = & {}^{\mathcal{L}ab}Tnet(\mathcal{A}(\mathcal{Y}(t), t))_a \\ \frac{d}{dt}\,{}^{\mathcal{L}ab}xc(\mathcal{Y}_b(t)) & = & {}^{\mathcal{L}ab}vc(\mathcal{Y}_b(t)) \\ & \vdots & \\ \textit{i.e.,} \quad \mathtt{Y}' & = & \mathtt{F}_{\mathcal{A}}(\mathtt{Y}, \mathtt{t}) \end{array}\end{gathered} \tag{8.35}$$

Eqn. 8.35: Canonical numerical ODE form for rigid-body motion. The dynamic state of all the bodies can be expressed in lab coordinates, and collected into a single linear array "Y". The derivative of Y is a function of Y and time t—the function can be constructed from $\mathcal{A}$, along with the bodies' mass distributions and the rigid-body equations of motion. Note that, as per Defn. 8.4, the values of xc, $\mathbf{R}$, p and L are sufficient to identify the state of a body, given that we know its (constant) mass distribution. □

To solve the forward-dynamics problem, we start with an initial value for Y, then numerically integrate the above ODE, using any convenient numerical integrator. For any body in the system that is a point mass, we can make Y smaller by eliminating the **R** and L entries. Note that if Eqn. 8.35 is integrated as written, numerical inaccuracies will cause ${}^{\mathcal{L}ab}\mathbf{R}(\mathcal{Y}_a(t))$ to quickly diverge from a rotation matrix; instead, we commonly use the quaternion representation of the rotation, Eqn. 6.22; inaccuracies can still creep into the solution, but they are less significant.

Forward Dynamics and Work

A corollary problem to forward dynamics is to compute the amount of work done by each motive in the model; this means finding the consistent power motive paths: That is, we extend Eqn. 8.34:

$$
\begin{aligned}
\textit{given:}\quad & t_0 \in \Re, \\
& Y_0 \in \mathsf{Systems}, \\
& \mathcal{A} \in \{\mathsf{AppliedMotiveFields}\}_{\mathsf{IDs}}, \\
\textit{find:}\quad & \mathcal{Y} \in \mathsf{SysPaths} \\
& \mathcal{P} \in \{\mathsf{PmotivePaths}\}_{\mathsf{IDs}} \\
\textit{such that:}\quad & \begin{cases} \mathcal{Y}(t_0) = Y_0 \\ \mathcal{P} \textit{ is consistent with } \mathcal{A} \textit{ and } \mathcal{Y} \end{cases}
\end{aligned}
\tag{8.36}
$$

Eqn. 8.36: An augmented forward dynamics problem. Solve the forward dynamics problem (Eqn. 8.34), but also find the corresponding consistent (Defn. 8.30) index of pmotive paths. Then the total work done by an applied motive labeled $i \in \mathit{Ids}(\mathcal{A})$ up to time t is given by $W(\mathcal{P}_i(t))$. □

Expanding the definition of consistent for pmotive paths (Defn. 8.30) gives:

For a pmotive path index $\mathcal{Y} \in \{\mathsf{PmotivePaths}\}_{\mathsf{IDs}}$ *consistent with* $\mathcal{A} \in \{\mathsf{AppliedMotiveFields}\}_{\mathsf{IDs}}$ *and* $\mathcal{Y} \in \mathsf{SysPaths}$ *, if* $\mathit{Ids}(\mathcal{A}) = \{j, k, \ldots\}$*, then we augment Eqn. 8.35*

$$
\begin{aligned}
\tfrac{d}{dt} W(\mathcal{P}_j(t)) = P(\mathcal{P}_j(t)) \quad &= F(\mathcal{A}_j(\mathcal{Y}(t),t) \cdot vc(\mathcal{Y}_{body(\mathcal{A}_j)}(t)) \\
&\quad + T(\mathcal{A}_j(\mathcal{Y}(t),t) \cdot \omega(\mathcal{Y}_{body(\mathcal{A}_j)}(t)) \\
\tfrac{d}{dt} W(\mathcal{P}_k(t)) = P(\mathcal{P}_k(t)) \quad &= F(\mathcal{A}_k(\mathcal{Y}(t),t) \cdot vc(\mathcal{Y}_{body(\mathcal{A}_k)}(t)) \\
&\quad + T(\mathcal{A}_k(\mathcal{Y}(t),t) \cdot \omega(\mathcal{Y}_{body(\mathcal{A}_k)}(t)) \\
&\vdots \\
\textit{i.e.,}\quad \mathtt{Y}' &= \mathtt{F}_{\mathcal{A}}(\mathtt{Y}, \mathtt{t}) \\
\mathtt{P}' &= \mathtt{G}_{\mathcal{A}}(\mathtt{P}, \mathtt{Y}, \mathtt{t})
\end{aligned}
\tag{8.37}
$$

Eqn. 8.37: Canonical numerical ODE form for work done by motives. We append equations for the work to the rigid-body motion ODE of Eqn. 8.35. □

As a "sanity check" for our simulations, we can plug the solution results into the energy-balance equations, Eqn. 8.31 and Eqn. 8.32. For conservative motives, we can check against the potential energy as per Eqn. 8.33. Note that initial values for the work terms can be chosen arbitrarily, since the energy balance equations are specified only up to a constant. We will commonly use an initial work value of 0, or, for conservative forces, $-U_f(Y_0, t_0)$, so that Eqn. 8.33 will always sum to 0.

Piecewise-continuous Forward Dynamics

The forward dynamics problem as expressed above is continuous (for finite motives). However, a model that includes discontinuities can use the above ODE's to describe the continuous parts of the behavior; for example, the "tennis ball cannon" example in Ch. 11 sets up a forward dynamics problem as an initial-value piecewise-continuous ODE problem.

Program definitions in scope `MRIG`:

class name	*abstract space*	
`ApplMotive`	AppliedMotives	*(Defn. 8.20)*
`ApplMotiveField`	AppliedMotiveFields	*(Defn. 8.21)*
`ApplMotiveFieldIdx`	$\{\text{AppliedMotiveFields}\}_{\text{IDs}}$	*index of fields*
`ApplMotiveIdx`	$\{\text{AppliedMotives}\}_{\text{IDs}}$	*index of applied motives*
`ApplMotiveIdxField`	$\text{Systems} \times \Re \to \{\text{AppliedMotives}\}_{\text{IDs}}$	*field that yields an index*
`MassDist`	MassDists	*(Defn. 8.2)*
`Motive`	Motives	*(Defn. 8.19)*
`Pmotive`	Pmotives	*(Defn. 8.28)*
`PmotivePath`	PmotivePaths	*(Defn. 8.28)*
`PmotivePathIdx`	$\{\text{PmotivePaths}\}_{\text{IDs}}$	*index of pmotive paths*
`State`	States	*(Defn. 8.4)*
`StatePath`	StatePaths	*(Defn. 8.6)*
`SysLocationField`	SysLocationFields	*(Defn. 8.16)*
`SysLocationFieldIdx`	$\{\text{SysLocationFields}\}_{\text{IDs}}$	*index of location fields*
`SysMotiveField`	SysMotiveFields	*(Defn. 8.16)*
`SysPath`	SysPaths	*(Defn. 8.15)*
`SysScalarField`	SysScalarFields	*(Defn. 8.16)*
`SysScalarFieldIdx`	$\{\text{SysScalarFields}\}_{\text{IDs}}$	*index of location fields*
`SysVectorField`	SysVectorFields	*(Defn. 8.16)*
`SysVectorFieldIdx`	$\{\text{SysVectorFields}\}_{\text{IDs}}$	*index of location fields*
`System`	Systems	*(Defn. 8.14)*

Figure 8.7: Math section definitions in the prototype implementation. □

Other Posed Problems

Other problems can be posed as well. *Inverse dynamics problems* compute forces and torques that yield given desired behaviors or constraints on behavior. Often, the resulting forces and torques are then used in a forward dynamics formulation, to yield the complete behavior, as in [Isaacs,Cohen87] or [Barzel,Barr88] (the latter of which we incorporate into the "fancy forces" module, Ch. 9).

[Witkin,Kass88] finds behavior that optimizes objectives such as energy, for motion that can be constrained at multiple points. They pose a constrained optimization problem, in which the dynamics of body motion (Defn. 8.10 in our formulation) is treated as an additional constraint, on discretized force and motion functions; force function parameters are found that optimize the given objective functions.

8.5 Implementation Notes†

8.5.1 Conceptual Section Constructs

In the conceptual section of a program, a conceptual body object can be built from kinematic conceptual bodies (Sec. 7.5.1), which maintain the position and orientation and "know" how to draw the body, augmented with mass distribution values.

This module supports a few simple motives; classes are defined for Hookean springs, constant fields, viscous damping, and so forth. Each class defines a routine to compute the associated field function (or a pair of routines that compute equal-and-opposite fields, e.g., for a spring linking two bodies). Motive classes also support a "`draw`" method, that can draw arrows or otherwise illustrate the force.

8.5.2 Math Section Constructs

The math section for this module has scope name `MRIG` ("Math RIGid-body dynamics"); Fig. 8.7 lists the classes that are defined. All are defined straightforwardly, as described in Sec. B.3. We will give a few notes.

†See Appendix B for discussion of the terminology, notation, and overall approach used here.

• Class definition for MassDists (Defn. 8.2):

```
class MassDist :
  constructors:  (double m,I[3][3], double cm[3])  construct arbitrary body
                 (double m)                         construct point mass
  members:       m          : double         net mass
                 minv       : double         inverse of the mass, i.e. 1/m
                 I          : double[3][3]   inertia tensor matrix
                 Iinv       : double[3][3]   inverse of the inertia matrix, i.e. I^-1
                 cm         : double[3]      body coords of center of mass
                 is_ptmass  : int            true if is a point mass, Defn. 8.3
```

The state space class `MassDist`, defined the standard manner (Sec. B.3.6), has two constructors: the constructor for point masses sets `I` and `cm` to 0; conversely, the constructor for arbitrary bodies signals an error if 0 is given for `I`. Both constructors precompute the inverses of `m` and `I`, so that they won't need to be repeatedly computed during execution. Notice that we add an extra member that lets us easily distinguish point masses.

• Class definition for States (Defn. 8.4):

```
class State :
  constructors:  (MassDist, MKIN::State)
                 (MassDist, MCO::Location xc, MCO::Orientation R, MCO::Vector p,L)
                 (MassDist, MCO::Location xc, MCO::Vector p)
                 (MassDist, MCO::Frame, MCO::Vector vc,w)
                 (MassDist, MKIN::State, MCO::Vector p,L)
  members:       mdist : MassDist
                 m     : double
                 minv  : double
                 I     : MCO::2tensor
                 Iinv  : MCO::2tensor
                 k     : MKIN::State
                 f     : MCO::InstFrame
                 x     : MCO::Location
                 R     : MCO::Orientation
                 v     : MCO::Vector
                 w     : MCO::Vector
                 arm   : MCO::Vector
                 xc    : MCO::Location
                 vc    : MCO::Vector
                 p     : MCO::Vector
                 L     : MCO::Vector
                 KEv   : double
                 KEw   : double
                 KE    : double
```

The state space class `State` has several constructors that define elements based on the various identifying tuples discussed in Defn. 8.4. In addition, there is a constructor that takes a non-minimal tuple—a mass distribution, a kinematic state, and momentum vectors p and L—and checks that the given values are consistent with the internal properties of the space ($p = m\,vc$, etc.).

• Class definitions for fields (Defn. 8.16):

```
class SysVectorField :
  constructors:  (MCO::Vector v)    constant
  methods:       eval(System, double t) : MCO::Vector
```

(The classes `SysScalarField` and `SysVectorField` are similar.) The first constructor above creates a constant field. The second, a field that extracts a vector property of an individual body, where the `Fieldcode` parameter chooses which property; for example, the *velA* example field in Eqn. 8.17 would be constructed via the parameter pair (`BODY_VEL`, *bodyA*). In addition to these, there is support for fields that are algebraic combinations of other fields, for fields that are evaluated by calling arbitrary user-supplied subroutines, and so forth.

8.5.3 M-N Interface

The scope name for the M-N interface is `NRIG`. We define a routine to solve the forward dynamics problem, Eqn. 8.34, given initial conditions and a field that computes an applied motive index.

```
SolveForward(double t0, System Y0, ApplMotiveIdxField A) : SysPath
```

This function returns a `SysPath` object—when can then be evaluated at arbitrary values of time t. The object is set up internally so that the ODE solver `NUM::OdeScatExt` (Sec. B.4.6) is invoked to integrate Eqn. 8.35 as necessary to perform the evaluation. Notice that this routine does *not* take an index of applied motive fields, i.e., $\mathcal{A} \in \{\text{AppliedMotiveFields}\}_{\text{IDs}}$ of Eqn. 8.35, but rather the implied function (Defn. 3.17) that returns an index; an instance of the implied function's class, `ApplMotiveIdxField`, can be created trivially from the index of applied motive fields, as per Sec. B.3.7. (Using the implied function allows the numerical solver to perform a single evaluation to compute all the motives; this is convenient when the motives must be computed simultaneously, as in the fancy force mechanism of Ch. 9.)

A similar routine

```
SolveForwardEnergy(...) : PmotivePathIdx
```

returns the a `PmotivePathIdx` object that solves the augmented forward dynamics problem, Eqn. 8.36.

Both the above routines can also be used to define the continuous behavior of an initial-value piecewise-continuous ODE problem; the returned paths invoke the `NUM::podeScatExt` solver (Sec. B.4.9).

8.5.4 C-M Interface

To solve a forward dynamics problem, the C-M interface performs the following initial setup:

- Construct a `System` instance, `Y0`, from the initial conceptual model state. (This defines a map from conceptual body objects to math ID's.)
- Construct a `ApplMotiveFieldIdx` instance, `A`, from the conceptual motive objects. (This defines a map from conceptual force objects to math ID's.)
- Construct the solution object: `Y = NRIG::SolveForward(t0, Y0, A)`.
- For energy computation, construct the solution object: `P = NRIG::SolveForwardEnergy(...)`.

To set the conceptual model state for any time `t`, the C-M interface does:

- Compute the `System` instance `Yt = Y(t)`.
- For each ID `i` in `Yt`, set the state of the corresponding conceptual body object based on `Yt[i]`.
- For energy computation, similarly set the state of each force object based on `P(t)`.

The conceptual section can then draw the bodies, print the energy values, check energy balance, etc.

It can be convenient to allow the conceptual model to associate dimensional units with the various quantities. For example, mass may be specified in grams or kilograms, and distance in meters or inches. The mathematical model, as discussed in Sec. 6.3, must be expressed using uniform units. Thus the C-M interface should perform the proper scaling. Note in particular that the inertia tensor is in units of mass-length ^{2} thus varies quadratically with changes in length scale.

Chapter 9

"Fancy Forces" Model

The rigid body model in Ch. 8 is force-based: by assumption, all interactions between bodies and all environmental effects on a body are mediated by the forces (and torques) applied to the bodies. In particular, if we—as creators and users of a model—wish to influence the behavior of the bodies, we must do so through application of forces. Ch. 8 assumed the existence of forces and force fields, but did not discuss how to create them.

This chapter presents a mechanism to create forces and apply them to bodies. It supports forces due explicitly to things in the model, such as springs and gravity, as well as *constraint forces* that can be introduced in order to connect bodies together, specify their motion, and so forth. All the forces fit into a common mathematical model.

The constraint forces in this model use the "dynamic constraints" method described in [Barzel,Barr88]. Indeed, this entire chapter essentially rephrases that work, to use the structured design framework. In particular, the original work defined mathematical "model fragments" for the individual constraints and relied on a word/pseudocode description to combine them—here, we create a complete mathematical model that includes the "glue" between the fragments.

For an example of the "fancy forces" mechanism in use, see the "swinging chain" model in Ch. 10. Although the material in this chapter is self-contained, we refer readers to [Barzel,Barr88] for fuller discussion; and more extensive treatment of the topic can be found in [Barzel88].

9.1 Goals

Our goals for this model are:

- To pre-define various types of force objects, which can later be applied to arbitrary bodies.
- To allow a single force object to generate related forces on several bodies; e.g., an elastic spring would exert equal-and-opposite forces on two bodies.
- To support geometric constraints (via the "dynamic constraints" mechanism of [Barzel,Barr88]).
- To have a uniform mathematical formulation, that can accomodate the various types of forces.

9.2 Conceptual Model

This module describes a mechanism for creating force and torque objects, for use in conjunction with the rigid-body dynamics model of Ch. 8. Note that Ch. 8 defined a *motive* to be a force and/or torque vector, and

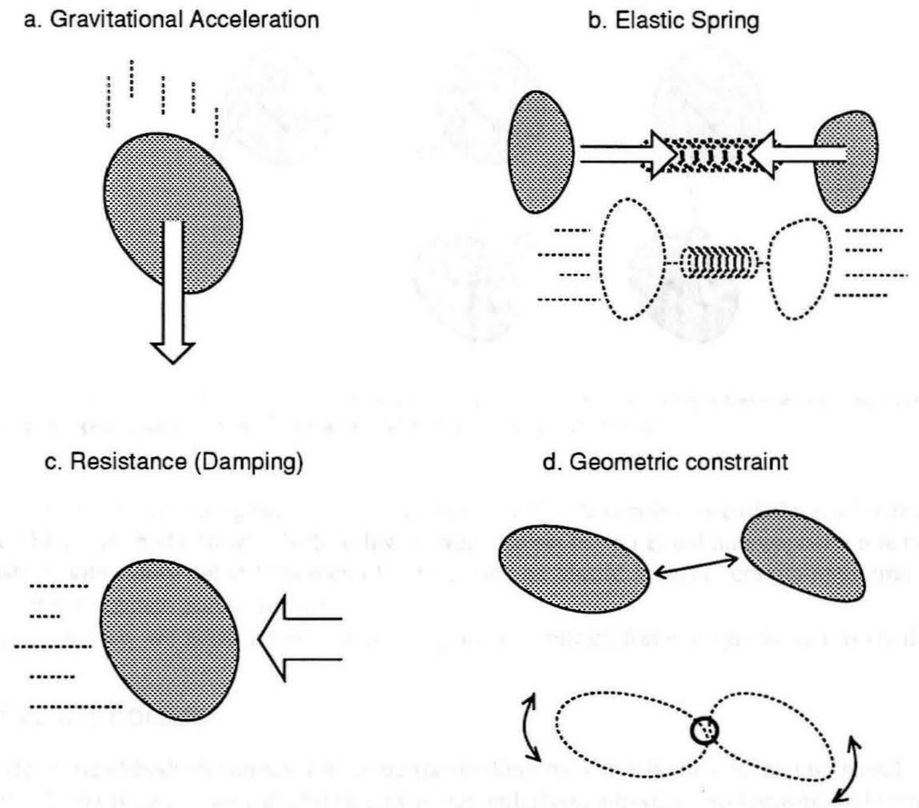


Figure 9.1: Various types of force objects. (a) Gravitational acceleration yields a constant force on a body, proportional to its mass. (b) An elastic spring yields equal-and-opposite forces on a pair of bodies based on their separation distance. (c) Damping yields a force negatively proportional to the body velocity, thus resisting motion. (d) Geometric constraints (a "point-to-point" constraint is illustrated) induce forces whose values are not known explicitly, but instead adapt so as to maintain the constraints no matter what other motions and forces affect the bodies. □

motive field to be a function that yields a single motive given the configuration of a collection of bodies. Here, we return to the word *force*, but with a more general connotation:

- A *force object* is a conceptual entity that is a source of motives.

A force object may be defined to act on one, two, or more bodies. When a force object is applied to a specific set of bodies (at a specified point on each body), it determines a motive field on each body. Fig. 9.1 illustrates several example.

9.2.1 Force Pairs

If a force object acts on one body, it will typically apply a force along with the corresponding coupled torque (due to the moment arm of the point of application; see Sec. 8.2.2). A force object that acts on several bodies will apply a separate coupled force/torque to each.

Commonly, a force object that acts on two bodies will apply forces to the bodies that are equal-and-opposite. Such a *force pair* adds no net linear momentum to the system as a whole—the position of the aggregate center of mass isn't affected—though it may add angular momentum, unless the points of application on the two bodies are at the same location in space, i.e., the bodies are touching.

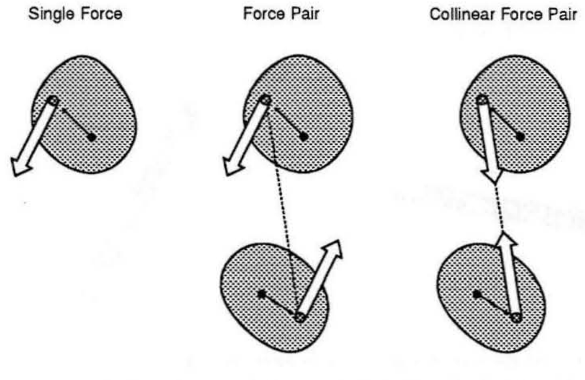


Figure 9.2: A force object acting on several bodies applies a different force to each. Commonly, a force object acting on two bodies will apply an equal-and-opposite pair of forces. (The corresponding torques are not illustrated.) □

A force pair that points along the line connecting the application points is called a *collinear force pair*—these force objects embody Newton's third law ("every action has an equal and opposite reaction") and are thus consistent with many natural sources of force, such as elastic springs, gravitational and electrostatic attraction, surface contact, and so forth.[1]

In the general case, we allow a force object to apply an arbitrary force/torque to each body that it acts on.

9.2.2 Explicit Forces

For our model of rigid-body dynamics, forces are the medium by which bodies interact with each other and the environment. In many cases, we can abstract away the underlying physical mechanism, and simply express a force as an explicit function of the configuration of the model. We give a few examples:

Gravitational Acceleration (Fig. 9.1a) For laboratory-scale environments on the surface of the earth, we can express the earth's gravitational attraction on a body as a constant downward acceleration, i.e., a force proportional to the body mass, acting at the center of mass. At larger scales, we would need to use Newton's radial, inverse-square law of gravitation.

Damping (Fig. 9.1c) Newton's first law tells us that objects in motion tend to stay in motion. However, in practice, bodies are almost always affected by friction and air resistance. When we create models, it is often convenient to gloss over the details, and essentially use the pre-Newtonian law "a body in motion tends to come to rest". To do so, we introduce a damping force that is opposite to body's velocity (and similarly, a damping torque that is opposite to the body's angular velocity). Damping forces are *dissipative* (Sec. 8.2.5)—they always remove energy from the bodies they act on.

Damping that is linearly proportional to body speed is called *viscous;* it can quickly bring a body to rest, resisting motion as if the body were slogging through goo. Quadratic damping—proportional to the square of the speed—is a rough approximation for air resistance: it opposes motion at high speeds, but is negligible at low speeds. Reasonable results can often be achieved by combining small amounts of viscous and quadratic damping.[2]

Elastic Spring (Fig. 9.1b) It is often convenient to model the effect of attaching two bodies by a spring (or a rubber band). We assume that the mass and motion of the spring itself is unimportant—thus we don't introduce the spring into the model as a body, but merely as a force object that acts on the bodies it

[1] If a collinear force pair acts on the bodies' centers of mass, it is called a *central* force ([Goldstein80]).

[2] We must stress that applying a motion-resisting damping force to a body is a very simple *ad hoc* approximation to real-world effects. A more studied approximation to air resistance would take into account the shape of the body and the profile that it offers to the air; and, of course, the resulting disturbance of the air is significant as well.

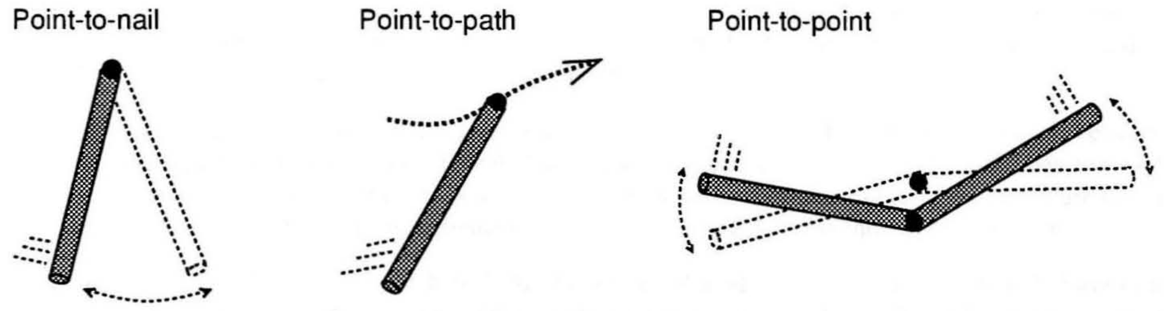


Figure 9.3: Some geometric constraints. The *point-to-nail* constraint requires that a body point be at a constant location in space; the body is free swing and rotate about that point. More generally, the *point-to-path* constraint specifies a kinematic path for a body point to follow. Two bodies may be assembled via a *point-to-point* constraint; the specified body points must stay attached, but the bodies may otherwise move freely. □

attaches. A spring always pulls or pushes along its own length, thus yielding a collinear force pair (Sec. 9.2.1). The strength is usually based on Hooke's law, i.e., is proportional to the displacement of the spring from its rest length.

Often, there is friction or other dissipation in the spring; we can add a damping term that opposes change in the length of the spring, to yield a *spring-and-dashpot* (analogous to an automotive shock absorber). Damping helps to limit oscillations that springs induce in a model.

9.2.3 Geometric Constraints

A powerful way to control models to specify constraints on the geometric configuration of the bodies. Fig. 9.3 illustrates three basic constraints on the locations body points: *point-to-nail, point-to-path,* and *point-to-point.* Other constraints might involve the orientations or surfaces of bodies.

In order to effect the constraints, forces must be applied to the bodies. Thus a geometric constraint is a type of force object; the resulting forces are called *constraint forces.* Unlike explicit force objects (Sec. 9.2.2), we can't know in advance precisely what the forces should be—the constraint forces need to take into account the influences of other forces that act on the bodies.

But what is the "meaning," or physical interpretation, of the constraint objects? There are two outlooks:

- *An abstraction of mechanical mechanisms.* For example, a point-to-point constraint can be thought of as an idealized ball-and-socket joint: the resulting constraint forces are equipollent to the forces that a frictionless physical joint would exert, but we don't need to model the details of the joint.
- *"The hand of god."* Sometimes, we don't have or need an underlying physical mechanism. For example, we may perform an experiment in which we presuppose that one point on a body follows a given path, in order to determine the dynamic response of the rest of the system.

Note that it is possible to *overconstrain* a model, and describe a physically unrealizable assemblage of a collection of bodies. As a trivial example, one might constrain the opposite ends of a rigid body to be at the same location in space. We would like our implementation to inform us when a model encounters an unrealizable set of constraints—but we place no restrictions on the resulting behavior.

9.3 Mathematical Model

The mathematical model for this module has a few features that bear mentioning:

- *Generality*. We have opted for generality in the design of the mathematical model, to support not just the specific examples of force objects listed in Sec. 9.2, but a wide class of explicit or implicit (constraint) forces. The resulting model is thus somewhat abstract.
- *Housekeeping*. The mathematical model follows the philosophy of Ch. 3: All dependencies between quantities in the model are made explicit. This requires a fair amount of "housekeeping," because each force object can apply to arbitrary bodies, and furthermore because the values of the constraint forces can depend on all the other forces in the system. The resulting model is thus somewhat arcane.
- *Infrastructure*. Unlike those of Ch. 6–8, the mathematical model here does not immediately follow the conceptual model: instead, we first define a mathematical infrastructure. That infrastructure is then used to define the mathematical elements that correspond with the conceptual objects of Sec. 9.2.

We include an overview of the mathematical model in Secs. 9.3.1–9.3.3, before going on to the exposition in the remaining sections.

9.3.1 Overview: Connection with Rigid Body Model

The link between this model and the dynamic rigid body model (Ch. 8) is via Defn. 8.25, which defines the behavior of a collection of bodies via an applied motive index

$$\mathcal{A} \in \{\mathsf{AppliedMotiveFields}\}_{\mathsf{IDs}},$$

i.e., a function which, given the state of a system, $Y \in \mathsf{Systems}$ at time $t \in \Re$ yields an index of applied motives[3] $A \in \{\mathsf{AppliedMotives}\}_{\mathsf{IDs}}$:

$$A = \mathcal{A}(Y,t).$$

Recall that each element of A is an applied motive, i.e., a motive and the name of the body it applies to.

The role of the "fancy forces" model is thus to define a function $\mathcal{A}$—i.e., to relate a state Y and time t with a collection A of motives that are applied to bodies—consistent with any given collection of force objects acting on bodies in a model. We will provide an abstract definition of $\mathcal{A}$ in Defn. 9.15; and the equation that ultimately relates A, Y, and t is given in Eqn. 9.21.

9.3.2 Overview: Contrast with Previous Formulation

The exposition of the "dynamic constraints" method in [Barzel88] (and, more compactly, in [Barzel,Barr88]) starts with a statement of an inverse dynamics problem—determine the unknown constraint forces given requirements for behavior—and proceeds to derive a linear equation for the constraint forces.

Here, we take the opposite approach. We axiomatically define a series of abstract constructs and mechanisms for defining and constraining motives, and refine and build on them—until we have constructed the essentially same linear constraint equation as the original work.

The most significant feature of the current mathematical model is that it is complete and explicit, in keeping with the goals and philosophy discussed in Sec. 3.2. The original work, in contrast, described the various "model fragments" in isolation, and described the resulting linear constraint equation, but the details of which constraints applied to which bodies were left implicit.

9.3.3 Overview: Decomposition of Force Objects

We decompose a force object into three parts: A *proto-motive*, which is an array of "knobs" that can be tweaked; a *motive-generator function* that creates motives based on the settings of those knobs; and a *constraint function* that examines the results to see if they are appropriate: (Fig. 9.4)

[3] Formally, $\mathcal{A}$ is an index of functions, and we are discussing its *implied function* as per Defn. 3.17.

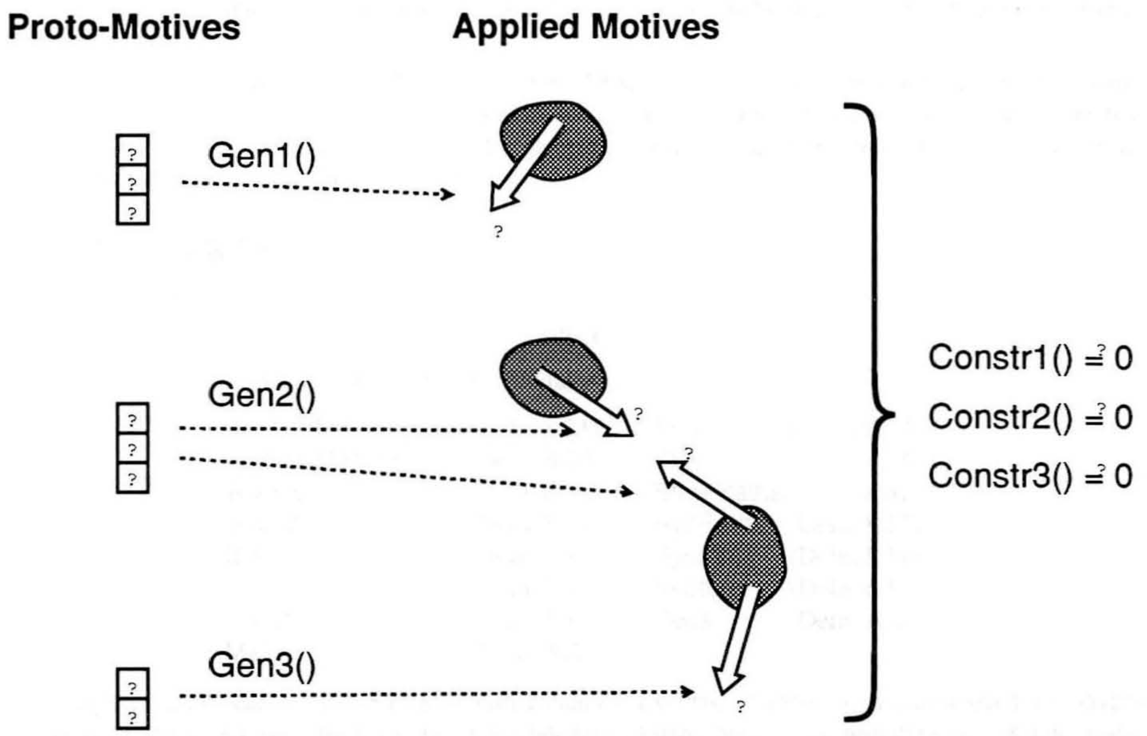

Proto-Motives
Applied Motives
Gen1()
Gen2()
Gen3()
?
Constr1() $\overset{?}{=}$ 0
Constr2() $\overset{?}{=}$ 0
Constr3() $\overset{?}{=}$ 0

Figure 9.4: Outline of proto-motive mechanism. Each *proto-motive* is a real-number array of some size. Each *motive-generator function* (labeled "Gen1," etc.) converts a proto-motive into one or more applied motives. *Constraint functions* ("Constr1," etc.) examine the entire state of the system of bodies and collection of applied motives. A system is consistent if the values of the proto-motives are such that all the constraint functions evaluate to 0. □

Proto-motive. The forces and torques that are applied by a force object are often interrelated, and have limited degrees of freedom (d.o.f.). For example: the torque on each body is often due to the force acting at a moment arm (3 d.o.f. per body); or, the forces on two bodies may be a collinear equal-and-opposite pair (1 d.o.f. for the two forces). The proto-motive is an array of real numbers (a "vec" as per Sec. A.2) having as many dimensions as there are degrees of freedom.

Motive-generator function. Given the names of the bodies, and the points of application on each, the motive-generator function maps from proto-motive values to a collection of applied motives. For example, a single force on a single body has 3 d.o.f., and the 3 proto-motive values can be used directly as the coordinates of the force vector. A force pair on two bodies also has 3 d.o.f.: the 3 proto-motive values can be used as the coordinates of the force on one body, and their negation as the coordinates of the force on the other body. (In both cases the torque follows from the moment arm, so adds no d.o.f.)

Defn. 9.6 will define $\mathsf{ProtoGens}[n, k]$ to be the space of motive-generator functions that act on n bodies, and have k degrees of freedom. Ultimately, in Sec. 9.3.8, we will define several common motive-generator functions: a single pure force, a single pure torque, a single coupled force/torque, a force pair, and a collinear force pair.

Constraint function. The proto-motives provide us with "knobs" to tweak, in order to adjust the forces and torques—but we need to choose the settings for those knobs. Thus we define a *constraint function,* that examines a collection of applied motives, and the state of the system, and tells us whether they are acceptable—the function yields an array of reals, that are all equal to 0 if the motives are OK. A constraint function may examine all the motives and bodies in the system in order to decide whether they are acceptable. Thus the value of a constraint function may be influenced by the settings of other force objects.

Defn. 9.7 will define ProtoConstrs$[n, k]$ to be the space of constraint functions that check n bodies and yield k values. Secs. 9.3.9,9.3.10 will define several constraint functions for the force objects in Sec. 9.2: for gravitational acceleration, for damping, for an elastic spring, and for point-to-nail, point-to-path, and point-to-point constraints.

9.3.4 Names & Notation

The scope name for this module is:

FORCES

We make use of the following terms from other modules:

AppliedMotiveFields	(Defn. 8.21)	*Fnet*	(Defn. 8.24)
AppliedMotives	(Defn. 8.20)	*Tnet*	(Defn. 8.24)
Bodypt	(Defn. 8.12)	StatePaths	(Defn. 8.6)
Bodypts	(Defn. 8.12)	SysPaths	(Defn. 8.15)
IDs	(Defn. 3.8)	Systems	(Defn. 8.14)
$\mathcal{L}ab$	(Defn. 6.4)	Vectors	(Defn. 6.1)
Locations	(Defn. 6.1)	Vecs	(Defn. A.2)
Mats	(Defn. A.2)		

We use Systems to mean RIGID :: Systems rather than KINEMATIC :: Systems, and similarly for Bodypts, etc. We also extensively use subscript notation for indexes and state spaces, and specializations of state spaces:

$\mathcal{X}_i$	(Notn. 3.11, for an index $\mathcal{X}$ and ID i)
x_s	(Notn. 3.27, for aspect x of state space element s)
$\mathsf{A} \sqsubset \mathsf{B}$	(Notn. 3.32 and Notn. 3.33)

9.3.5 Application Information

In order to specify which bodies a particular force object will be applied to, and what the IDs of the resulting motives should be, we define a state space containing *application information* elements:

Definition. ApplicInfos

(9.1)

```
ApplicInfos
[
  a  ↦ IDs   ID of an applied motive
  b  ↦ IDs   ID of a body
  pt ↦ ℜ³    body coordinates of application point
]
```

Defn. 9.1: Each application info $p \in$ ApplicInfos contains the ID of an applied motive, the ID of a body that the motive acts on, and the body coordinates of the point at which the force should be applied (for coupled force/torque motives). □

Because a force object may act on several bodies, we define *application sets* of information elements:

Definition. ApplicSets

$$
\begin{aligned}
\text{ApplicSets} &\equiv \textit{the space } \left\{\textit{ordered sets of } \text{ApplicInfos}\right\} \\
\text{ApplicSets}[n] &\equiv \left\{s \in \text{ApplicSets} \;\middle|\; \|s\| = n\right\}
\end{aligned}
\tag{9.2}
$$

Defn. 9.2: An application set $s \in$ ApplicSets$[n]$ has size $s = n$, i.e. contains n elements; since the set is ordered, we can examine its elements by number: $s_0, s_1, \ldots s_{n-1}$. □

We would like to identify when the motives listed in an application set are to be found in a given index of applied motives. Thus we define:

Definition. **Compatible** (motive index, application set)

(9.3)

An applied motive index $A \in \{\text{AppliedMotives}\}_{\text{IDs}}$ *and an application set* $s \in \text{ApplicSets}$ *are* **compatible** *iff:*

$$\begin{array}{rcll} a(r) & \in & \mathit{Ids}(A), & \text{all } r \in s \\ b(r) & = & \mathit{body}(A_{a(r)}), & \text{all } r \in s \end{array}$$

Defn. 9.3: An application set s is compatible with an index of applied motives A if each application info $r \in s$ corresponds with an applied motive in A. That is, the ID $a(r)$ must be the label of an element of A; that element must be applied to body ID $b(r)$, where a and b are as per Defn. 9.1. Note that A may also have additional elements that are not specified by any pair in s. □

Furthermore, we would like to identify when a collection of application sets account for all the motives in an index:

Definition. **Compatible** (motive index, application set index)

(9.4)

An applied motive index $A \in \{\text{AppliedMotives}\}_{\text{IDs}}$ *and an application set index* $\mathcal{S} \in \{\text{ApplicSets}\}_{\text{IDs}}$ *are* **compatible** *iff:*

- *A is compatible with s, for all $s \in \mathit{Elts}(\mathcal{S})$*
- $||A|| = \sum_{s \in \mathit{Elts}(\mathcal{S})} ||s||$

Defn. 9.4: Compatible index of applied motives and index of application sets. The applied motive index must be compatible (Defn. 9.3) with each application set. The total number of applied motives must agree with the total number of application infos—this implies that each application info and each application set is unique. □

It will be convenient to have some notation for various quantities relating to the point of application of a body force, as specified by application info:

Notation. Point of application

(9.5)

Given application info $r \in \text{ApplicInfos}$ *and system state* $Y \in \text{Systems}$, *we write:*

with $s = Y_{b(r)}$ *and* $d = \mathit{Bodypt}(s, pt(r))$,

$$\begin{array}{rcll} x(Y,r) & \equiv & {}^{\mathcal{L}ab}x_d & \textit{location} \\ v(Y,r) & \equiv & {}^{\mathcal{L}ab}v_d & \textit{velocity} \\ \mathit{arm}(Y,r) & \equiv & {}^{\mathcal{L}ab}\mathit{arm}_d & \textit{moment arm} \\ \mathit{acc}_0(Y,r) & \equiv & {}^{\mathcal{L}ab}(\mathit{arm}_d^* \mathbf{I}^{-1}(\omega_s \times L_s) + \omega_s \times (\omega_s \times \mathit{arm}_d)) & \\ & & & \textit{free acceleration} \\ \mathit{acc}_F(Y,r) & \equiv & \frac{1}{m_s} & \textit{acceleration force-dependence} \\ \mathit{acc}_T(Y,r) & \equiv & -{}^{\mathcal{L}ab}\mathit{arm}_d^* \mathbf{I}^{-1} & \textit{acceleration torque-dependence} \end{array}$$

Notn. 9.5: Quantities for a point of application. The acceleration acc_0 is due to the rotation of the body; acc_F and acc_T are coefficients that yield acceleration due to force and torque. *Bodypt* maps from body point coordinates to a dynamic body point state element (Defn. 8.12). Derivations for the acceleration terms are given in Sec. 9.6.1. □

9.3.6 Proto-Motive Mechanism

This section defines the proto-motive mechanism that is outlined in Sec. 9.3.3. First, we define the class of *motive-generator functions* (Sec. 9.3.3), which map from "vec"-valued *proto-motives* to applied motives:

Definition. ProtoGens

$$\text{ProtoGens}[n,k] \equiv \textit{the set of functions} \left\{ f\colon \begin{pmatrix} \text{Systems} \times \Re \times \\ \text{ApplicSets}[n] \times \\ \text{Vecs}[k] \end{pmatrix} \to \{\text{AppliedMotives}\}_{\text{IDs}} \atop \textit{such that } f(Y,t,s,x) \textit{ and } s \textit{ are compatible} \right\} \tag{9.6}$$

Defn. 9.6: Motive-generator functions for n motives, using k degrees of freedom. Given a system $Y \in$ at time $t \in \Re$, and an application set $s \in$ ApplicSets$[n]$ naming the n motives to generate and the bodies to apply them to, a motive-generator function maps from a proto-motive value $x \in$ Vecs $[k]$ to an index of applied motives. □

And next, the class of *constraint functions,* which yield constraint residues, given an index of applied motives:

Definition. ProtoConstrs

$$\text{ProtoConstrs}[n,k] = \textit{set of functions} \left\{ \begin{pmatrix} \text{Systems} \times \Re \times \\ \text{ApplicSets}[n] \times \\ \{\text{AppliedMotives}\}_{\text{IDs}} \end{pmatrix} \to \text{Vecs}[k] \right\} \tag{9.7}$$

Defn. 9.7: Constraint functions for n motives, yielding k residues. Given a system $Y \in$ at time $t \in \Re$, and an application set $s \in$ ApplicSets$[n]$ naming the n motives and bodies of interest, a constraint function maps from an index of applied motives $A \in \{\text{AppliedMotives}\}_{\text{IDs}}$ to a size-k vec of residues that are 0 if the constraint is met. □

We bundle together a constraint function and motive-generator function into a *proto-specifier,* that describes a *force object* as per Sec. 9.2:

Definition. ProtoSpecs

(9.8)

ProtoSpecs
[

n	$\mapsto$ *Integers*	*number of motives*
cz	$\mapsto$ *Integers*	*size of the constraint*
pz	$\mapsto$ *Integers*	*size of the proto-motive*
Constr	$\mapsto$ ProtoConstrs$[n, cz]$	*constraint function*
Gen	$\mapsto$ ProtoGens$[n, pz]$	*motive-generator function*

]

Defn. 9.8: Each proto-specifier $p \in$ ProtoSpecs has a constraint function and a motive-generator function. [4] Commonly, we will have $cz_p = pz_p$, i.e. the number of degrees of freedom agrees with the number of constraint equations, but we do not require this. □

A proto-specifier takes a proto-motive as input, and yields a constraint residue as output, with an index of applied motives as the intermediate result (Fig. 9.4). Note that all proto-specifiers in a model share a single applied motive index.

Do a given proto-specifier index and application set index jibe? We define:

Definition. **Compatible** (proto-specifier index, application set index)

(9.9)

A proto-specifier index $\mathcal{P} \in \{\text{ProtoSpecs}\}_{\text{IDs}}$ *and an application set index* $\mathcal{S} \in \{\text{ApplicSets}\}_{\text{IDs}}$ *are* **compatible** *iff:*

$$\begin{aligned} \mathit{Ids}(\mathcal{P}) &= \mathit{Ids}(\mathcal{S}) \\ \|\mathcal{S}_i\| &= n(\mathcal{P}_i),\ \textit{all } i \in \mathit{Ids}(\mathcal{P}) \end{aligned}$$

Defn. 9.9: Compatible proto-specifier and application set indexes. Each application set has the number of elements that is appropriate for its corresponding proto-specifier. □

[4] Notice that we have slightly extended the state-space notation of Sec. 3.9, by specifying the sizes of the functions, n, cz , and pz, "inline" rather than via separate explicit properties.

Similarly, does a proto-specifier index agree with a proto-motive index?

Definition. **Compatible** (proto-specifier index, proto-motive index)

(9.10)

A proto-specifier index $\mathcal{P} \in \{\text{ProtoSpecs}\}_{\text{IDs}}$ *and a proto-motive index* $\mathcal{X} \in \{\text{Vecs}\}_{\text{IDs}}$ *are* **compatible** *iff:*

$$\begin{aligned} \mathit{Ids}(\mathcal{P}) &= \mathit{Ids}(\mathcal{X}) \\ \mathit{sz}(\mathcal{X}_i) &= \mathit{pz}(\mathcal{P}_i),\ \mathit{all}\ i \in \mathit{Ids}(\mathcal{P}) \end{aligned}$$

Defn. 9.10: Compatible proto-specifier and proto-motive indexes. Each proto-motive is the appropriate size for its corresponding proto-specifier. □

Putting together the various types of compatibility, we define:

Definition. **Compatible ensemble**

(9.11)

A group of indexes $\mathcal{P} \in \{\text{ProtoSpecs}\}_{\text{IDs}}$, $\mathcal{X} \in \{\text{Vecs}\}_{\text{IDs}}$, $\mathcal{S} \in \{\text{ApplicSets}\}_{\text{IDs}}$, *and* $A \in \{\text{AppliedMotives}\}_{\text{IDs}}$ *form a* **compatible ensemble** *iff:*

- *$\mathcal{P}$ is compatible (Defn. 9.10) with $\mathcal{X}$,*
- *$\mathcal{P}$ is compatible (Defn. 9.9) with $\mathcal{S}$, and*
- *$\mathcal{S}$ is compatible (Defn. 9.4) with A*

Defn. 9.11: Structurally compatible ensemble. Each proto-specifier $\mathcal{P}_i$ maps its corresponding proto-motive $\mathcal{X}_i$ to the applied motives listed in its corresponding application set $\mathcal{S}_i$. All the resulting applied motives are gathered in A. □

From Defn. 9.4, we know that the elements of $\mathcal{S}$ in a compatible ensemble are unique. Thus we can use them as subscripts, instead of ID's.

Notation. Application Sets as Subscripts

(9.12)

Given a compatible ensemble $\mathcal{P} \in \{\text{ProtoSpecs}\}_{\text{IDs}}$, $\mathcal{X} \in \{\text{Vecs}\}_{\text{IDs}}$, $\mathcal{S} \in \{\text{ApplicSets}\}_{\text{IDs}}$,, *and* $A \in \{\text{AppliedMotives}\}_{\text{IDs}}$: *if* $s \in \text{ApplicSets}$ *and* $i \in \mathit{Ids}(\mathcal{S})$ *are such that* $s = \mathcal{S}_i$, *we write:*

$$\begin{aligned} \mathcal{X}_s &\equiv \mathcal{X}_i \\ \mathcal{P}_s &\equiv \mathcal{P}_i \\ \mathit{cz}_s &\equiv \mathit{cz}(\mathcal{P}_i) \\ \mathit{pz}_s &\equiv \mathit{pz}(\mathcal{P}_i) \\ \mathit{Constr}_s &\equiv \mathit{Constr}(\mathcal{P}_i) \\ \mathit{Gen}_s &\equiv \mathit{Gen}(\mathcal{P}_i) \end{aligned}$$

Notn. 9.12: Subscript notation for compatible sets. In a compatible ensemble, the elements $\mathcal{S}$ are unique (Defn. 9.4), so we can them as subscripts to refer to the the corresponding elements of $\mathcal{X}$ and $\mathcal{P}$. For further shorthand, we also use those subscripts directly for the aspects of elements of $\mathcal{P}$. □

The components of a compatible ensemble (Defn. 9.11) have the proper structure—all the right sizes, the right ID's in the right places, etc.—but we also want to have the proper values:

Definition. **Consistent**

(9.13)

A compatible ensemble $\mathcal{P} \in \{\text{ProtoSpecs}\}_{\text{IDs}}$, $\mathcal{X} \in \{\text{Vecs}\}_{\text{IDs}}$, $\mathcal{S} \in \{\text{ApplicSets}\}_{\text{IDs}}$, *and* $A \in \{\text{AppliedMotives}\}_{\text{IDs}}$ *is* **consistent** *with* $Y \in \text{Systems}$ *and* $t \in \Re$ *iff:*

For all $s \in \mathit{Elts}(\mathcal{S})$:
- $\mathit{Constr}_s(Y, t, s, A) = 0$
- $\mathit{Gen}_s(Y, t, s, \mathcal{X}_s) \subseteq A$

Defn. 9.13: A compatible ensemble is consistent with an instantaneous state if all the constraints are met, and if all the applied motives agree with the motive-generator functions. □

Note, from Defn. 9.4 and Defn. 9.6, the latter equation above is equivalent to:

(9.14)
$$\begin{gathered}\textit{[given same as Defn. 9.13], for all } s \in \mathit{Elts}(\mathcal{S}) \textit{ and all} \\ i \in \mathit{Ids}(\mathit{Gen}_s(Y,t,s,\mathcal{X}_s)) \\ \mathit{Gen}_s(Y,t,s,\mathcal{X}_s)_i = A_i\end{gathered}$$

Eqn. 9.14: Each motive-generator function Gen_s yields an index of applied motives; and each applied motive in that index is labeled by the same ID in the overall applied motive index A. □

We are now in a position to define an index of applied motive fields (as per the discussion in Sec. 9.3.1), based on an index of proto-specifiers and an index of application sets:

Definition. **Consistent** (applied motive field, proto-specifier index)

(9.15)
$$\boxed{\begin{gathered}\textit{An index of applied motive fields } \mathcal{A} \in \{\mathsf{AppliedMotiveFields}\}_{\mathsf{IDs}} \textit{ is} \\ \textbf{consistent} \textit{ with } \mathcal{P} \in \{\mathsf{ProtoSpecs}\}_{\mathsf{IDs}} \textit{ and } \mathcal{S} \in \{\mathsf{ApplicSets}\}_{\mathsf{IDs}} \textit{ iff:} \\ \\ \textit{for any } Y \in \mathsf{Systems} \textit{ and } t \in \Re, \\ \textit{there exists } \mathcal{X} \in \{\mathsf{Vecs}\}_{\mathsf{IDs}}, \\ \textit{such that the ensemble } \mathcal{P}, \mathcal{X}, \mathcal{S}, \mathcal{A}(Y,t) \textit{ is consistent with } Y \textit{ and } t\end{gathered}}$$

Defn. 9.15: The field A is consistent with $\mathcal{P}$ and $\mathcal{S}$ if at each point in space/time it maps to a collection of motives that are consistent with $\mathcal{P}$ and $\mathcal{S}$. □

The definitions in this section provide us with a useful framework, but they are too general to perform computations. Next section will look at a restricted class of proto-specifiers, whose functions are linear.

9.3.7 Linear Proto-Motive Mechanism

Sec. 9.3.6 defined the structural relationships between the components for arbitrary proto-specifiers. Here, we examine proto-specifiers whose constraint and motive-generator functions are linear; these will be sufficient to handle both explicit force objects (Sec. 9.2.2) and "dynamic constraint" force objects (Sec. 9.2.3).

The presentation in this section defines canonical forms for linear proto-motive and constraint functions. These forms are inserted into the definition of a consistent ensemble, Defn. 9.13, resulting in a constraint equation, Eqn. 9.21, that is linear in the proto-motives.

We start by defining fields analogous to the location, vector, and scalar fields of Defn. 8.16, except that these fields also take application sets as parameters:

Definition. (fields)

(9.16)

$$
\begin{aligned}
\mathsf{SysVecFields}[n,k] &\equiv \textit{the set of functions} \\
&\quad \left\{ \begin{pmatrix} \mathsf{Systems} \times \Re \\ \times \mathsf{ApplicSets}[n] \end{pmatrix} \to \mathsf{Vecs}[k] \right\} \\
\mathsf{SysMatidxFields}[n,j,k] &\equiv \textit{the set of functions} \\
&\quad \left\{ \begin{pmatrix} \mathsf{Systems} \times \Re \\ \times \mathsf{ApplicSets}[n] \end{pmatrix} \to \{\mathsf{Mats}[j,k]\}^{\circ}_{\mathsf{IDs}} \right\} \\
\mathsf{SysMatidxAFields}[n,j,k] &\equiv \textit{the set of functions} \\
&\quad \left\{ \begin{array}{l} f \in \mathsf{SysMatidxFields}[n,j,k] \textit{ such that} \\ \mathit{Ids}(f(Y,t,s)) = \left\{ a(r) \;\middle|\; r \in s \right\} \end{array} \right\} \\
\mathsf{SysMatidxBFields}[n,j,k] &\equiv \textit{the set of functions} \\
&\quad \left\{ \begin{array}{l} f \in \mathsf{SysMatidxFields}[n,j,k] \textit{ such that} \\ \mathit{Ids}(f(Y,t,s)) = \left\{ b(r) \;\middle|\; r \in s \right\} \end{array} \right\}
\end{aligned}
$$

Defn. 9.16: Fields over instantaneous state, that are additionally parametrized by applications sets. Note that elements of SysMatidxFields$[n, j, k]$ are functions that yield indexes with 0. The spaces SysMatidxAFields $[n, j, k]$ and SysMatidxBFields$[n, j, k]$ are subsets of SysMatidxFields$[n, j, k]$, that yield indexes whose ID's are specified by the application set parameter $s \in$ ApplicSets; the former uses the applied motive ID's listed in s, while the latter uses the body ID's. □

We define the set of motive-generator functions that are linear in the proto-motives—this is a subset of the general class of motive-generator functions (Defn. 9.6). Notice the use of a state space specialization, as per Notn. 3.32, to define aspect operators on the subset.

Definition. Lgens

(9.17)

$$
\begin{array}{l}
\mathsf{Lgens}[n,k] \overset{gen}{\sqsubset} \mathsf{ProtoGens}[n,k] \\
[\\
\quad G \mapsto \mathsf{SysMatidxAFields}[n,3,k] \quad \textit{force coefficients} \\
\quad H \mapsto \mathsf{SysMatidxAFields}[n,3,k] \quad \textit{torque coefficients} \\
\quad J \mapsto \mathsf{SysMatidxAFields}[n,3,1] \quad \textit{motive arms} \\
\text{—} \\
\quad \textit{For } Y \in \mathsf{Systems},\, t \in \Re,\, s \in \mathsf{ApplicSets}[n] \textit{ and } X \in \mathsf{Vecs}[k], \\
\qquad {}^{\mathcal{cab}}\!\mathcal{F}(gen(Y,t,s,X)_{a(r)}) = G(Y,t,s)_{a(r)}\, X, \quad \textit{for all } r \in s \\
\qquad {}^{\mathcal{cab}}\!\mathcal{T}(gen(Y,t,s,X)_{a(r)}) = H(Y,t,s)_{a(r)}\, X, \quad \textit{for all } r \in s \\
\qquad {}^{\mathcal{cab}}\!\mathit{arm}(gen(Y,t,s,X)_{a(r)}) = J(Y,t,s)_{a(r)} \qquad \textit{for all } r \in s \\
]
\end{array}
$$

Defn. 9.17: Linear motive-generator functions. Each generated force vector is the product of a matrix in G and the proto-motive X (when represented in lab coords); similarly for the torque vectors. The motive arms are independent of X. Note that we must have either $J(Y,t,s)_{a(r)} = 0$ or $H(Y,t,s)_{a(r)} = J(Y,t,s)^{*}_{a(r)} G(Y,t,s)_{a(r)}$, by Defn. 8.19. □

Similarly, we define the subset of constraint functions (Defn. 9.7) that depend linearly on the forces and torques in the applied motive index:

Definition. Lconstrs

$$
\begin{array}{l}
\mathsf{Lconstrs}[n,k] \overset{constr}{\sqsubset} \mathsf{ProtoConstrs}[n,k] \\
[\\
\quad \begin{array}{lll}
K & \mapsto \mathsf{SysVecFields}[n,k] & \textit{independent term} \\
\Gamma & \mapsto \mathsf{SysMatidxBFields}[n,k,3] & \textit{force-dependency coefficients} \\
\Lambda & \mapsto \mathsf{SysMatidxBFields}[n,k,3] & \textit{torque-dependency coefficients} \\
\mathit{all} & \mapsto \{0,1\} & \textit{all-or-some selector}
\end{array} \\
\text{—} \\
\mathit{For}\ Y \in \mathsf{Systems}, t \in \Re, s \in \mathsf{ApplicSets}[n], A \in \{\mathsf{AppliedMotives}\}_{\mathsf{IDs}} \\
\quad constr(Y,t,s,A,\mathit{all}) = \\
\quad K(Y,t,s) + \begin{cases}
\displaystyle\sum_{a \in \mathit{Elts}(A)} \begin{pmatrix} \Gamma(Y,t,s)_{\mathit{body}(a)}\, \overset{cab}{F}_a + \\ \Lambda(Y,t,s)_{\mathit{body}(a)}\, \overset{cab}{T}_a \end{pmatrix}, & \mathit{all} = 1 \\
\displaystyle\sum_{r \in s} \begin{pmatrix} \Gamma(Y,t,s)_{b(r)}\, \overset{cab}{F}(A_{a(r)}) + \\ \Lambda(Y,t,s)_{b(r)}\, \overset{cab}{T}(A_{a(r)})) \end{pmatrix}, & \mathit{all} = 0
\end{cases} \\
]
\end{array} \tag{9.18}
$$

Defn. 9.18: Linear constraint functions. The constraint is the sum of a force/torque-independent term K and products of coefficients with forces and torques. The *all* selector chooses whether to perform the summation over all applied motives in a given index (A), or only over those motives specified in the given application set (s). □

Notice that by Defn. 9.16, $\Gamma(Y,t,s)_b$ and $\Lambda(Y,t,s)_b$ will be 0 for body ID's b that are not listed in s. Thus the $\mathit{all} = 1$ and $\mathit{all} = 0$ cases are similar in structure: both examine only the bodies listed in s, accumulating $\Gamma(Y,t,s)_b$ times a force and $\Lambda(Y,t,s)_b$ times a torque for each body b—but in the $\mathit{all} = 0$ case the force and torque are from the motive listed in s, while in the $\mathit{all} = 1$ case, they are the *net force* and *net torque* on the body.

We define the set of linear proto-specifiers (Defn. 9.8):

Definition. LprotoSpecs

$$
\begin{array}{l}
\mathsf{LprotoSpecs} \overset{P}{\sqsubset} \mathsf{ProtoSpecs} \\
[\\
\quad \begin{array}{lll}
\mathit{Lconstr} & \mapsto \mathsf{Lconstrs}[n, cz] & \textit{linear constraint function} \\
K & \quad \ldots K \mapsto \mathsf{SysVecFields}[n, cz] & \\
\Gamma & \quad \ldots \Gamma \mapsto \mathsf{SysMatidxBFields}[n, cz, 3] & \\
\Lambda & \quad \ldots \Lambda \mapsto \mathsf{SysMatidxBFields}[n, cz, 3] & \\
\mathit{all} & \quad \ldots \mathit{all} \mapsto \{0,1\} & \\
\mathit{Lgen} & \mapsto \mathsf{Lgens}[n, pz] & \textit{linear generator function} \\
G & \quad \ldots G \mapsto \mathsf{SysMatidxAFields}[n, 3, pz] & \\
H & \quad \ldots H \mapsto \mathsf{SysMatidxAFields}[n, 3, pz] &
\end{array} \\
\text{—} \\
\quad \mathit{Constr} = constr(\mathit{Lconstr}) \\
\quad \mathit{Gen} = gen(\mathit{Lgen}) \\
]
\end{array} \tag{9.19}
$$

Defn. 9.19: A Linear proto-specifier is a special case of a proto-specifier, having functions that are linear as per Defns. 9.17,9.18. □

Continuing with the shorthand notation defined in Notn. 9.12, we will use application sets as subscripts when we have a compatible ensemble:

Notation. Application sets as subscripts

[given same as Notn. 9.12], if $s \in$ ApplicSets *and* $p \in$ LprotoSpecs *are such that* $s = \mathcal{S}_i$ *and* $P(p) = \mathcal{P}_i$*, we write:*

(9.20)
$$\begin{aligned} K_s &\equiv K(p) \\ \Gamma_s &\equiv \Gamma(p) \\ \Lambda_s &\equiv \Lambda(p) \\ G_s &\equiv G(p) \\ H_s &\equiv H(p) \\ \mathit{all}_s &\equiv \mathit{all}(p) \end{aligned}$$

Notn. 9.20: For a compatible ensemble, we use subscripts to refer to the aspect values of any linear proto-specs. □

Defn. 9.13 and Defns. 9.17,9.18 lead to a linear equation for compatible proto-specifiers and motives (written using Notn. 9.20). Sec. 9.6.2 contains the derivation of the following:

If a compatible ensemble $\mathcal{P} \in \{\text{LprotoSpecs}\}_{\text{IDs}}$, $\mathcal{X} \in \{\text{Vecs}\}_{\text{IDs}}$, $\mathcal{S} \in \{\text{ApplicSets}\}_{\text{IDs}}$, $A \in \{\text{AppliedMotives}\}_{\text{IDs}}$ *is consistent with* $Y \in$ Systems *and* $t \in \Re$*, then*

(9.21)
$$\forall p \in \mathit{Elts}(\mathcal{S}), \qquad K_p(Y,t,p) + \sum_{q \,\in\, \mathit{Elts}(\mathcal{S})} \mathcal{M}_{pq}(Y,t)\,\mathcal{X}_q = 0$$

where

$$\begin{aligned} \mathcal{M}_{pq} &\in \text{Mats}[cz_p, pz_q] \\ &= \begin{cases} 0, & \mathit{all}_p = 0 \;\&\; p \neq q \\ \sum_{r \in q} ([\Gamma_p]_{b_r}[G_q]_{a_r} + [\Lambda_p]_{b_r}[H_q]_{a_r}), & \mathit{otherwise} \end{cases} \end{aligned}$$

Eqn. 9.21: A consistent collection of linear proto-motives is based on an array equation, linear in the proto-motives. For clarity the parameters have been left off of $\mathcal{M}_{pq}(Y,t)$, $\Gamma_p(Y,t,p)$, $\Lambda_p(Y,t,p)$, $G_q(Y,t,q)$, and $H_q(Y,t,q)$. See derivation in Sec. 9.6.2. □

Eqn. 9.21 is the *linear constraint-force equation* of [Barzel,Barr88]. We will discuss it further in Sec. 9.4.

The definitions in this section provide canonical forms for motive-generator functions, constraint functions, and proto-specifiers. In the coming sections, we fill in those forms, for the various types of force objects discussed in Sec. 9.2.

9.3.8 Motive-Generator Functions

This section defines several commonly useful motive-generator functions. These are all linear functions as per Defn. 9.17, thus each is defined by a pair of functions $G, H \in$ SysMatidxAFields$[n,3,k]$, where k is the number of degrees of freedom in the resulting motives. Each function yields an index containing one entry per motive. We will define motive-generator functions for one and for two motives.

For one arbitrary pure force:

Definition. $\mathcal{G}pure\mathcal{F}$

Define $\mathcal{G}pure\mathcal{F} \in$ Lgens$[1,3]$ *by:*

(9.22)
$$\begin{aligned} G_{\mathcal{G}pure\mathcal{F}}(Y,t,s)_{a(s_0)} &\equiv 1 \\ H_{\mathcal{G}pure\mathcal{F}}(Y,t,s)_{a(s_0)} &\equiv 0 \end{aligned}$$

Defn. 9.22: Motive generator for a pure force. The three values of a proto-motive are used directly as the components of the force; the torque and motive arm are 0. Note that $G(Y,t,s)$ and $H(Y,t,s)$ are independent of Y and t. $J(Y,t,s)$ must be zero, by Defn. 9.17. □

For one arbitrary pure torque:

Definition. $\mathcal{G}pure\mathcal{T}$

(9.23)

Define $\mathcal{G}pure\mathcal{T} \in \mathsf{Lgens}[1,3]$ *by:*

$$\begin{aligned} G_{\mathcal{G}pure\mathcal{T}}(Y,t,s)_{a(s_0)} &\equiv 0 \\ H_{\mathcal{G}pure\mathcal{T}}(Y,t,s)_{a(s_0)} &\equiv 1 \end{aligned}$$

Defn. 9.23: Motive generator for a pure torque. Analogous to Defn. 9.22 but with opposite values for G and H. □

Most often, we will generate a motive consisting of a coupled force/torque, where the force that is applied at a specified point in body coordinates:

Definition. $\mathcal{G}coupled$

(9.24)

Define $\mathcal{G}coupled \in \mathsf{Lgens}[1,3]$ *by:*

$$\begin{aligned} G_{\mathcal{G}coupled}(Y,t,s)_{a(s_0)} &\equiv 1 \\ J_{\mathcal{G}pure\mathcal{F}}(Y,t,s)_{a(s_0)} &\equiv arm(Y,s_0) \end{aligned}$$

Defn. 9.24: Motive generator for a coupled force/torque. The three values of the proto-motive are used directly as the components of the force; the moment arm is as described specified by s_0, and the torque is coupled. □

In the most general case for one motive, we may wish to generate an arbitrary force and torque; this requires six degrees of freedom:

Definition. $\mathcal{G}arbit$

(9.25)

Define $\mathcal{G}arbit \in \mathsf{Lgens}[1,6]$ *by:*

$$\begin{aligned} G_{\mathcal{G}arbit}(Y,t,s)_{a(s_0)} &\equiv \begin{bmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \end{bmatrix} \\ H_{\mathcal{G}arbit}(Y,t,s)_{a(s_0)} &\equiv \begin{bmatrix} 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{bmatrix} \\ J_{\mathcal{G}pure\mathcal{F}}(Y,t,s)_{a(s_0)} &\equiv 0 \end{aligned}$$

Defn. 9.25: Motive generator for an arbitrary force/torque. This acts on a proto-motive vec with six values: The first three values of the proto-motive are used for the components of the force, and the last three are used for the components of the torque. □

For a force object that acts on two bodies, we most commonly generate an equal-and-opposite force pair (Sec. 9.2.1); this requires three degrees of freedom:

Definition. $\mathcal{G}pair$

(9.26)

Define $\mathcal{G}pair \in \mathsf{Lgens}[2,3]$ *by:*

$$\begin{aligned} G_{\mathcal{G}pair}(Y,t,s)_{a(s_0)} &\equiv 1 \\ G_{\mathcal{G}pair}(Y,t,s)_{a(s_1)} &\equiv -1 \\ J_{\mathcal{G}pure\mathcal{F}}(Y,t,s)_{a(s_0)} &\equiv arm(Y,s_0) \\ J_{\mathcal{G}pure\mathcal{F}}(Y,t,s)_{a(s_1)} &\equiv arm(Y,s_1) \end{aligned}$$

Defn. 9.26: Motive generator for a force pair. The proto-motive values are used directly as the components of the force for application s_0, and the negation is used for s_1. The moment arms are as specified by s, and the torques are coupled. □

For a collinear force pair, in which the paired forces lie on the line connecting the application points (Sec. 9.2.1), only one degree of freedom is needed:

Definition. $\mathcal{G}collinear$

Define $\mathcal{G}collinear$ $\in$ Lgens$[2,1]$ *by:*

$$\begin{aligned} G_{\mathcal{G}collinear}(Y,t,s)_{a(s_0)} &\equiv x(Y,s_0) - x(Y,s_1) \\ G_{\mathcal{G}collinear}(Y,t,s)_{a(s_1)} &\equiv -G_{\mathcal{G}collinear}(Y,t,s)_{a(s_0)} \\ J_{\mathcal{G}pure\mathcal{F}}(Y,t,s)_{a(s_0)} &\equiv arm(Y,s_0) \\ J_{\mathcal{G}pure\mathcal{F}}(Y,t,s)_{a(s_1)} &\equiv arm(Y,s_1) \end{aligned} \tag{9.27}$$

Defn. 9.27: Motive generator for a collinear force pair. There is only one degree of freedom: the force is determined by scaling the line segment that separates the application points. The moment arms are as specified by s, and the torques are coupled to the forces. □

9.3.9 Constraint Functions for Explicit Force Objects

This section discuss explicit force objects (Sec. 9.2.2); we will discuss geometric constraint objects (Sec. 9.2.3) in Sec. 9.3.10.

Constraint functions for explicit force objects (Sec. 9.2.2) are expressed directly in terms of Defn. 9.18, by specifying values for $K \in$ SysVecFields$[n,k]$ and $\Gamma, \Lambda \in$ SysMatidxBFields$[n,k,3]$. We will define constraint functions for the objects described in Sec. 9.2.2; functions for other objects can be defined analogously.

For a gravitational acceleration force:

Definition. $\mathcal{C}grav$

Define the family of functions $\mathcal{C}grav[g]$ $\in$ Lconstrs$[1,3]$ *for $g \in \Re$ by:*

$$\begin{aligned} all_{\mathcal{C}grav} &\equiv 0 \\ \Gamma_{\mathcal{C}grav}(Y,t,s)_{b(s_0)} &\equiv -1 \\ \Lambda_{\mathcal{C}grav}(Y,t,s)_{b(s_0)} &\equiv 0 \\ K_{\mathcal{C}grav}(Y,t,s) &\equiv \begin{bmatrix} 0 \\ 0 \\ -g\, m(Y_b(s_0)) \end{bmatrix} \end{aligned} \tag{9.28}$$

Defn. 9.28: Constraint for gravitational acceleration. The values of Γ and Λ are such that Defn. 9.13 and Defn. 9.18 imply that the lab coordinates of the force are given directly by K, when this constraint is met. K itself yields a downward value proportional to the body mass and the gravitational acceleration parameter. Here, we assume that the lab frame z-axis points "up." □

For a damping force and torque:

Definition. $\mathcal{C}damp$

Define the family of functions $\mathcal{C}damp[\mu_v, \mu_w, \nu_v, \nu_w]$ $\in$ Lconstrs$[1,6]$ *for $\mu_v, \mu_\omega, \nu_v, \nu_\omega \in \Re$ by:*

$$\begin{aligned} all_{\mathcal{C}damp} &\equiv 0 \\ \Gamma_{\mathcal{C}damp}(Y,t,s)_{b(s_0)} &\equiv \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \\ \Lambda_{\mathcal{C}damp}(Y,t,s)_{b(s_0)} &\equiv \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \\ K_{\mathcal{C}damp}(Y,t,s) &\equiv \begin{bmatrix} -(\mu_v + \nu_v \| v\!c(Y_{b(s_0)})\|)\ {}^{\mathcal{L}ab}v\!c(Y_{b(s_0)}) \\ -(\mu_\omega + \nu_\omega \| \omega(Y_{b(s_0)})\|)\ {}^{\mathcal{L}ab}\omega(Y_{b(s_0)}) \end{bmatrix} \end{aligned} \tag{9.29}$$

Defn. 9.29: Constraint for viscous and quadratic damping. μ_v and ν_v are the coefficients of viscous and quadratic damping for the linear velocity, and μ_ω and ν_ω are for the rotation. The first three values of K give the force, and the last three give the torque, in a manner similar to Defn. 9.28. □

For a Hooke's law spring, with viscous damping:

Definition. Cspring

(9.30)

Define the family of functions $\mathcal{C}spring[l_0, k, \mu] \in$ Lconstrs[2, 3] *for $l_0, k, \mu \in \Re$ by:*

$$\begin{aligned}
all_{cspring} &\equiv 0\\
\Gamma_{cspring}(Y,t,s)_{b(s_0)} &\equiv -1\\
\Gamma_{cspring}(Y,t,s)_{b(s_1)} &\equiv 0\\
\Lambda_{cspring}(Y,t,s)_{b(s_0)} &\equiv 0\\
\Lambda_{cspring}(Y,t,s)_{b(s_1)} &\equiv 0\\
K_{cspring}(Y,t,s) &\equiv (k\,(l-l_0) + \mu\, v \cdot \hat{r})\,\hat{r},\\
\textit{where}\ \ & r = x(Y,s_1) - x(Y,s_0)\\
& v = v(Y,s_1) - v(Y,s_0)\\
& l = \|r\|\\
& \hat{r} = r/l
\end{aligned}$$

Defn. 9.30: Constraint for a damped spring force. The value of K constrains the force of application s_0 to point towards the application point of s_1, based on rest length l_0, spring constant k, and damping coefficient μ. $\hat{r}$ is the unit vector from application point s_0 to s_1, and $v \cdot \hat{r}$ is the rate of change of separation between the bodies. This function is not well-defined if the two application points coincide in space, unless $l_0 = \mu = 0$. (No constraint is placed on application s_1; presumably an opposite force will be generated for it.) □

9.3.10 Constraint Functions for Geometric Constraints

We model geometric constraints via the "dynamic constraints" method described in [Barzel,Barr88]. We will recap that method here, before defining some specific constraint functions.

For any geometric constraint, we define a "deviation measure" that equals 0 when the constraint is met (Fig. 9.5). The deviation measure is function of the state of the system, and the bodies the constraint acts on; i.e., it is a vec field $D \in$ SysVecFields$[n, k]$, as per Defn. 9.16. We allow a deviation measure to be applied to any number of bodies, n, and have any fixed size k. In general D may have explicit dependence on time (such as for a point-to-path constraint), but in many cases D is purely geometric.

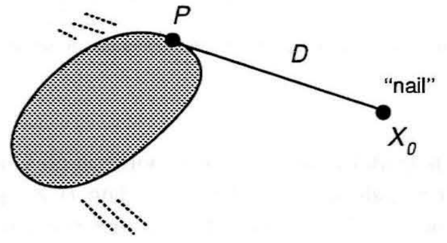


Figure 9.5. The "deviation measure" for a geometric constraint is a function D whose value is 0 when the constraint is met. Here, we show a point-to-nail constraint, whose deviation is given by $\mathcal{L}ab(X_p - X_0)$ for body point p constrained to a nail at X_0. □

To express the requirement that a deviation measure equal zero, we use:

(9.31)

Given a deviation measure $D \in$ SysVecFields$[n,k]$, *an application set $s \in$* ApplicSets$[n]$, *a system path $\mathcal{Y} \in$* SysPaths, *and a constant $\tau \in \Re$ we require:*

$$\tfrac{d^2}{dt^2} D(\mathcal{Y}(t),t,s) + \tfrac{2}{\tau}\tfrac{d}{dt} D(\mathcal{Y}(t),t,s) + \tfrac{1}{\tau^2} D(\mathcal{Y}(t),t,s) = 0$$

Eqn. 9.31: Required behavior for the value of deviation measure D as a model moves over time. We have picked a second-order differential equation that describes critical damping: from any initial condition, the value decays smoothly down to zero, "assembling" the model.[5] The *time constant*, τ, controls the rate of assembly; τ must be positive. (See, e.g., [Boyce,DePrima77] for discussion of critically damped differential equations.) □

[5] Analytically, the value of D asymptotically approaches 0, but doesn't ever reach 0. When implemented numerically, however, it soon reaches zero within error tolerances.

Equations other than Eqn. 9.31 could perhaps be used to describe the constraint; we choose Eqn. 9.31 because it describes "assembly" of a model from an initial condition in which the constraint isn't met, and also because, it results in an expression for the constraint as an equation that is linear in the forces and torques on the bodies, which is numerically tractable when we implement the model.

To convert Eqn. 9.31 into a form having explicit dependence on motives, we create auxiliary functions that describe the behavior of the deviation measure over time, for a dynamic system of bodies:

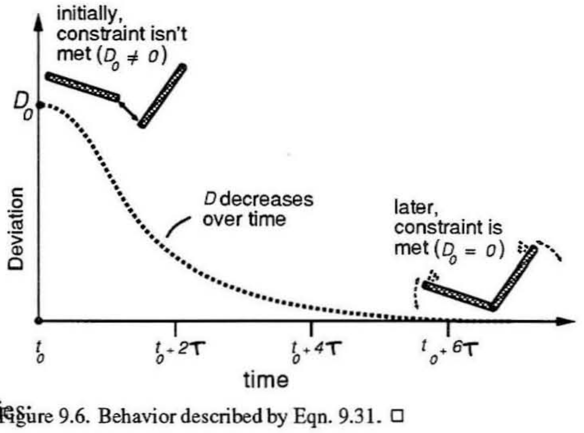


Figure 9.6. Behavior described by Eqn. 9.31. □

(9.32)

Given a deviation measure $D \in \mathsf{SysVecFields}[n,k]$, *an application set* $s \in \mathsf{ApplicSets}[n]$, *and a system path* $\mathcal{Y} \in \mathsf{SysPaths}$ *consistent with applied motive fields* $\mathcal{A} \in \{\mathsf{AppliedMotiveFields}\}_{\mathsf{IDs}}$, *we create auxiliary functions* $D^{(1)} \in \mathsf{SysVecFields}[n,k]$ *and* $D^{(2)} \in \mathsf{ProtoConstrs}[n,k]$ *such that:*

$$\begin{aligned} D^{(1)}(\mathcal{Y}(t),t,s) &\equiv \tfrac{d}{dt}D(\mathcal{Y}(t),t,s) \\ D^{(2)}(\mathcal{Y}(t),t,s,\mathcal{A}(t)) &\equiv \tfrac{d^2}{dt^2}D(\mathcal{Y}(t),t,s) \end{aligned}$$

Eqn. 9.32: Auxiliary deviation measure functions. $D^{(1)}$ yields the instantaneous rate of change of the deviation; $D^{(2)}$ yields the instantaneous acceleration. □

Notive in Eqn. 9.32 that since D is geometric, it depends only on positions and orientation information in $\mathcal{Y}(t)$. But $D^{(1)}$ will depend also on the velocities in $\mathcal{Y}(t)$ (Eqn. 8.9), and $D^{(2)}$ will depend further on the net force and torque applied to each body (Defn. 8.10), since $\mathcal{Y}$ is consistent with $\mathcal{A}$ (Defn. 8.25). By the chain rule for differentiation, the dependency of $D^{(2)}$ on the net forces and torques is linear. Thus we can expand $D^{(2)}$:

(9.33)

Given $D^{(2)}$ *of Eqn. 9.32, we create auxiliary functions* $\Gamma, \Lambda \in \mathsf{SysMatidxBFields}[n,k,3]$ *and* $\beta \in \mathsf{SysVecFields}[n,k]$ *such that, then for any* $Y \in \mathsf{Systems}$, $t \in \Re$, $s \in \mathsf{ApplicSets}[n]$, *and* $A \in \{\mathsf{AppliedMotives}\}_{\mathsf{IDs}}$,

$$\begin{aligned} D^{(2)}(Y,t,s,A) &= \beta(Y,t,s) \\ &+ \sum_{r\in s}\Gamma(Y,t,s)_{b(r)}\mathit{Fnet}(A)_{b(r)} + \sum_{r\in s}\Lambda(Y,t,s)_{b(r)}\mathit{Tnet}(A)_{b(r)} \end{aligned}$$

Eqn. 9.33: Auxiliary deviation measure functions. $D^{(2)}$, which describes the acceleration of the deviation measure, depends linearly on the net force and net torque on each body; Γ gives the coefficients for the forces, Λ gives the coefficients for the torques, and β gives the part of $D^{(2)}$ that is independent of the forces and torques. *Fnet* and *Tnet* are as per Defn. 8.24. □

The auxiliary definitions of Eqn. 9.32 and Eqn. 9.33 are in a form allowing us to define a specialization of linear constraint functions to express Eqn. 9.31:

Definition. GeomConstrs$[n,k]$

(9.34)

$$\begin{array}{lll}
\multicolumn{3}{l}{\mathsf{GeomConstrs}[n,k] \sqsubset \mathsf{Lconstrs}[n,k]} \\
[& & \\
\quad \tau & \mapsto \Re & \textit{time constant} \\
\quad D & \mapsto \mathsf{SysVecFields}[n,k] & \textit{deviation measure} \\
\quad D^{(1)} & \mapsto \mathsf{SysVecFields}[n,k] & \textit{rate of change} \\
\quad \beta & \mapsto \mathsf{SysVecFields}[n,k] & \textit{acceleration w/o forces} \\
— & & \\
\multicolumn{3}{l}{\quad all = 1} \\
\multicolumn{3}{l}{\quad \tau > 0} \\
\multicolumn{3}{l}{\quad D,\ D^(1),\ \beta,\ \Gamma,\ \textit{and}\ \Lambda\ \textit{relate by Eqns. 9.32,9.33}} \\
\multicolumn{3}{l}{\quad K(Y,t,s) = \beta(Y,t,s) + \tfrac{2}{\tau}D^{(1)}(Y,t,s) + \tfrac{1}{\tau^2}D(Y,t,s)} \\
] & &
\end{array}$$

Defn. 9.34: Constraint functions for geometric constraints, using the "dynamic constraints" method. For a constraint function $f \in$ GeomConstrs$[n,k]$ the constraint requirement $f(Y,t,s,A) = 0$ (Defn. 9.13) is met when Eqn. 9.31 holds. □

Now, we can define various constraint functions that are elements of GeomConstrs$[n,k]$; in each case, we will define a family of functions that is parameterized by the time constant τ. We will define functions for the three body point constraints described in Sec. 9.2.3; functions for other objects can be defined analogously.

For a point-to-nail constraint:

Definition. Cptnail

(9.35)

Define the family of functions $\mathit{Cptnail}[\tau, X_0] \in \mathsf{GeomConstrs}[1,3]$ *for* $\tau \in \Re$ *and* $X_0 \in$ Locations *by:*

$$\begin{array}{rcl}
D_{cptnail}(Y,t,s) & = & x(Y,s_0) - {}^{cab}X_0 \\
D^{(1)}_{cptnail}(Y,t,s) & = & v(Y,s_0) \\
\beta_{cptnail}(Y,t,s) & = & acc_0(Y,s_0) \\
\Gamma_{cptnail}(Y,t,s)_{b(s_0)} & = & acc_F(Y,s_0) \\
\Lambda_{cptnail}(Y,t,s)_{b(s_0)} & = & acc_T(Y,s_0)
\end{array}$$

Defn. 9.35: Point-to-nail constraint function. The deviation measure is thus simply the difference between the point location and the nail location X_0. The rate of change and acceleration of the deviation are just those of the point (as per Notn. 9.5). □

For a point-to-point constraint:

Definition. Cptpt

(9.36)

Define the family of functions $\mathit{Cptpt}[\tau] \in \mathsf{GeomConstrs}[2,3]$ *for* $\tau \in \Re$ *by:*

$$\begin{array}{rcl}
D_{cptpt}(Y,t,s) & = & x(Y,s_0) - x(Y,s_1) \\
D^{(1)}_{cptpt}(Y,t,s) & = & v(Y,s_0) - v(Y,s_1) \\
\beta_{cptpt}(Y,t,s) & = & acc_0(Y,s_0) - acc_0(Y,s_1) \\
\Gamma_{cptpt}(Y,t,s)_{b(s_0)} & = & acc_F(Y,s_0) \\
\Gamma_{cptpt}(Y,t,s)_{b(s_1)} & = & -acc_F(Y,s_1) \\
\Lambda_{cptpt}(Y,t,s)_{b(s_0)} & = & acc_T(Y,s_0) \\
\Lambda_{cptpt}(Y,t,s)_{b(s_1)} & = & -acc_T(Y,s_1)
\end{array}$$

Defn. 9.36: Point-to-point constraint function. The deviation measure is the difference between the locations of the two points of application. The remainder of the terms are analogous to Defn. 9.35. Each body point's force- and torque-dependence for acceleration, acc_F and acc_T, result in a separate entry in Γ and Λ. □

For a point-to-path constraint:

Definition. Cptpath

Define the family of functions $\mathcal{C}ptpath[\tau, P] \in \mathsf{GeomConstrs}[1,3]$ *for* $\tau \in \Re$ *and path* $P : \Re \rightarrow \mathsf{Locations}$ *by:*

$$\begin{aligned} D_{cptpath}(Y,t,s) &= x(Y,s_0) - {}^{cab}P(t) \\ D^{(1)}_{cptnail}(Y,t,s) &= v(Y,s_0) - \tfrac{d}{dt}\,{}^{cab}P(t) \\ \beta_{cptnail}(Y,t,s) &= acc_0(Y,s_0) - \tfrac{d^2}{dt^2}\,{}^{cab}P(t) \\ \Gamma_{cptnail}(Y,t,s)_{b(s_0)} &= acc_F(Y,s_0) \\ \Lambda_{cptnail}(Y,t,s)_{b(s_0)} &= acc_T(Y,s_0) \end{aligned} \tag{9.37}$$

Defn. 9.37: Point-to-path constraint function. A generalization of the point-to-nail constraint, Defn. 9.35, but for a "moving nail." □

9.3.11 Proto-Specifiers

Given the definitions in the previous sections, all that remains in order to specify a force object is to match up a motive generator function of Sec. 9.3.8 with a constraint function of Sec. 9.3.9 or Sec. 9.3.10, to form a proto-specifier as per Defn. 9.8 and Defn. 9.19. We define proto-specifiers for the conceptual force objects described in Sec. 9.2:

Definition. (various proto-specifiers)

We define several families of proto-specifiers, all elements of $\mathsf{LprotoSpecs}$*:*

(9.38)

p		n_p	pz_p	cz_p	$Lgen_p$	$Lconstr_p$
$\mathcal{G}ravity[g]$	$=$	1	3	3	$\mathcal{G}pure\mathcal{F}$	$\mathcal{C}grav[g]$
$\mathcal{D}amp[\mu_v, \mu_w, \nu_v, \nu_w]$	$=$	1	6	6	$\mathcal{G}arbit$	$\mathcal{C}damp[\mu_v, \mu_w, \nu_v, \nu_w]$
$\mathcal{S}pring[l_0, k, \mu]$	$=$	2	1	3	$\mathcal{G}collinear$	$\mathcal{C}spring[l_0, k, \mu]$
$\mathcal{P}ointToNail[\tau, X_0]$	$=$	1	3	3	$\mathcal{G}coupled$	$\mathcal{C}ptnail[\tau, X_0]$
$\mathcal{P}ointToPoint[\tau]$	$=$	2	3	3	$\mathcal{G}pair$	$\mathcal{C}ptpt[\tau]$
$\mathcal{P}ointToPath[\tau, P]$	$=$	1	3	3	$\mathcal{G}coupled$	$\mathcal{C}ptpath[\tau, P]$

Defn. 9.38: Various proto-specifiers. Each is defined by specifying its motive-generator function *Lgen* and constraint function, *Lconstr*. In most cases, the number of degrees of freedom (*pz*) is equal to the number of constraint terms (*cz*); see discussion in Sec. 9.4. □

Usually, a constraint function has a natural or intended motive generator function that goes with it. For example, the point-to-point constraint function *Cptpt* is paired with the equal-and-opposite force pair motive generator *Gpair* to form *PointToPoint*. But we can also mix and match. For example, if we pair constraint *Cptpt* with generator *Gcoupled*—which applies a force to only one body—then we have a constraint in which a body "follows" or "shadows" another without affecting its motion.[6] Or, we could apply a pure force to meet a point-to-path constraint, thus the body will translate but not rotate in order to follow the path.

9.4 Posed Problems

We have one prominent posed problem for this model: to create the forces and torques that are described by a given set of linear proto-specifiers, i.e., that are due to the motive generators and that satisfy the constraints.

[6] Strictly speaking, since $\mathcal{G}coupled(Y,t,s,X)$ requires an application set $s \in \mathsf{ApplicSets}[1]$ while $\mathcal{C}ptpt(Y,t,s,A)$ requires an application set in $s \in \mathsf{ApplicSets}[2]$, they can't be used together, as per Defn. 9.19. However, we can trivially extend $\mathcal{G}coupled$ to act on elements $s \in \mathsf{ApplicSets}[2]$ by ignoring s_1.

That is, as discussed in Sec. 9.3.1,

$$\begin{array}{rl} \textit{given:} & \mathcal{P} \in \{\mathsf{LprotoSpecs}\}_{\mathsf{IDs}}, \textit{ compatible with} \\ & \mathcal{S} \in \{\mathsf{ApplicSets}\}_{\mathsf{IDs}}, \\ \textit{define:} & \mathcal{A} \in \{\mathsf{AppliedMotiveFields}\}_{\mathsf{IDs}} \\ & \textit{where } \mathcal{A} \textit{ is consistent with } \mathcal{P} \textit{ and } \mathcal{S} \\ \textit{evaluate:} & \mathcal{A}(Y,t) \\ & \textit{for any } Y \in \mathsf{Systems} \textit{ and } t \in \Re \end{array} \tag{9.39}$$

Eqn. 9.39: Motive evaluation. The indexes $\mathcal{P}$ and $\mathcal{S}$ describe the force objects in the model and how they are applied to the bodies. They determine a motive field index $\mathcal{A}$ (Defn. 9.13). During the course of a simulation, we will want to evaluate $\mathcal{A}$ for many different values of Y and t. □

Because we are using linear proto-specifiers, to evaluate $\mathcal{A}(Y,t)$ as per Defn. 9.13 we use Eqn. 9.21, which we paraphrase:

$$\begin{array}{rl} \forall p, & K_p(Y,t,p) + \sum_q \mathcal{M}_{pq}(Y,t)\,\mathcal{X}_q = 0 \\ \textit{where} & \mathcal{M}_{pq} \in \mathsf{Mats}[cz_p, pz_q] \\ & K_p \in \mathsf{Vecs}[cz_p] \\ & \mathcal{X}_q \in \mathsf{Vecs}[pz_q] \end{array}$$

We can express the above in matrix form, if we arbitrarily assign an order to the elements of the proto-motive ensemble:

$$\begin{bmatrix} \mathcal{M}_{11}(Y,t) & \mathcal{M}_{12}(Y,t) & \dots & \mathcal{M}_{1n}(Y,t) \\ \mathcal{M}_{21}(Y,t) & \mathcal{M}_{22}(Y,t) & \dots & \mathcal{M}_{2n}(Y,t) \\ & \vdots & & \\ \mathcal{M}_{n1}(Y,t) & \mathcal{M}_{n2}(Y,t) & \dots & \mathcal{M}_{nn}(Y,t) \end{bmatrix} \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \\ \vdots \\ \mathcal{X}_n \end{bmatrix} + \begin{bmatrix} K_1(Y,t,p_1) \\ K_2(Y,t,p_2) \\ \vdots \\ K_n(Y,t,p_n) \end{bmatrix} = 0$$

Each element in the above is an array; we can consider the above to be a block matrix form[7] of a $(\sum_p cz_p) \times (\sum_q pz_q)$ matrix equation:

$$\mathbf{M}(Y,t)\,\mathbf{X} + \mathbf{K}(Y,t) = 0$$

For any Y and t, we can compute $\mathbf{M}(Y,t)$ and $\mathbf{K}(Y,t)$, and solve for $\mathbf{X}$ using one of many standard numerical techniques (see [Press et al.86]). We note some characteristics of $\mathbf{M}$ to be taken into account:

- $\mathbf{M}$ is generally sparse.
- $\mathbf{M}$ is not necessarily square; a proto-specifier p may have $pz_p \neq cz_p$. If there are fewer degrees of freedom than constraints ($pz_p < cz_p$), there won't in general be a solution unless the constraints are redundant (as is true for the proto-specifier *Spring*).
- The explicit force objects ($\mathit{all}_p = 0$) have only diagonal entries; the corresponding values $\mathcal{X}_p$ can be found first, and the remaining matrix can be reduced.[8]
- $\mathbf{M}$ can often be partitioned into independent blocks, which can be solved separately; each block corresponds with a group of bodies that may be constrained with respect to each other, but there are no constraints between the blocks.
- The structure of $\mathbf{M}$ is due to the indexes $\mathcal{P}$ and $\mathcal{S}$, and is independent of Y and t. Thus the partitioning and so forth can be performed once per collection of force objects, and then used for each evaluation of $\mathcal{A}(Y,t)$. Furthermore, many of the functions Γ, Λ, G, and H defined in Secs. 9.3.8–9.3.10 are constants, which can be folded once per collection of force objects, to minimize repeated computation.

[7] We refer readers to [Horn,Johnson85] and [Golub,Van Loan85] for discussion of matrices.

[8] In [Barzel,Barr88], the explicit forces are treated as a special case *a priori*, and are not entered into the matrix. Here, we prefer a more uniform mathematical treatment, and leave it to the implementation to special-case the computation of the explicit forces.

- **M** may be singular or ill-conditioned, implying the lack of a unique solution. If there are multiple solutions, any solution is acceptable; this may occur through extraneous degrees of freedom in the proto-specifiers, through redundancies in the constraints specified in the model, or through the existence of equipollent sets of forces that meet the constraints. If there are no solutions, it can indicate improperly constructed proto-specifiers (e.g., an orientation-based constraint function paired with a pure force motive generator), or an overconstrained system as discussed in Sec. 9.2.3. Additionally, if using the constraints to "assemble" models as discussed in Sec. 9.3.10, the path implied by Eqn. 9.31 may be physically unrealizable; see discussion in [Barzel,Barr88]. A least-squares solution is often practicable.

Once we've computed **X**, and hence the proto-motives, $\mathcal{X}$, we can determine the actual motives, i.e., the forces and torques, by evaluating all the motive-generator functions, as per Eqn. 9.14:

$$\mathcal{A}(Y,t) = \bigcup_{s\in\mathcal{S}} Gen_s(Y,t,s,\mathcal{X}_s)$$

The generator functions are evaluated by multiplying each $\mathcal{X}_s$ by the values of $G_s(Y,t,s)$ and $H_s(Y,t,s)$, as per Defn. 9.17; these values were already computed to construct **M**, and can be reused.

9.5 Implementation Notes[†]

9.5.1 Conceptual Section Constructs

The conceptual section of a program supports various types of force objects. Each instance of a force object includes the name(s) of the body (bodies) that it acts on and the body points of application, along with the time constant τ for geometric constraints and other type-specific parameters.

Each type of force object knows how to draw itself, for illustration or debugging, e.g., a helix between points attached by a spring, or the curve of a point-to-path constraint.

Paths for point-to-path constraints can be defined symbolically or via a curve editor, as described in [Snyder92].

9.5.2 Math Section Constructs

The math section for this module has scope name `MFRC` ("Math fancy FoRCes"). Fig. 9.7 lists the objects that are defined. Classes for the various state spaces, sets, and indexes are defined as per Sec. B.3. We will give a few additional notes.

- Class definitions for ProtoGens$[n,k]$ and Lgens$[n,k]$:

```
class ProtoGen :
  constructors:  (int n,k)
  members:       n : int
                 k : int
  methods:       eval(MRIG::System Y, double t, ApplicSet S, MMISC::Vec x)
                      : MRIG::ApplMotiveIdx

class Lgen (derived from ProtoGen) :
  constructors:  (int n,k, SysMatIdxField G,H,J)
  members:       G : SysMatIdxAField
                 H : SysMatIdxAField
                 J : SysMatIdxAField
  methods:       eval(MRIG::System Y, double t, ApplicSet S, MMISC::Vec x)
                      : MRIG::ApplMotiveIdx
```

[†]See Appendix B for discussion of the terminology, notation, and overall approach used here.

Program definitions in scope `MFRC`:

class name	*abstract space*	
`ApplicInfo`	ApplicInfos	*(Defn. 9.1)*
`ApplicSet`	ApplicSets	*(Defn. 9.2)*
`ApplicSetIdx`	$\{\text{ApplicSets}\}_{\text{IDs}}$	*index of application sets*
`GeomConstr`	GeomConstrs	*(Defn. 9.34)*
`Lconstr`	Lconstrs	*(Defn. 9.18)*
`Lgen`	Lgens	*(Defn. 9.17)*
`LprotoSpec`	LprotoSpecs	*(Defn. 9.19)*
`LprotoSpecIdx`	$\{\text{LprotoSpecs}\}_{\text{IDs}}$	*index of linear proto-specs*
`ProtoConstr`	ProtoConstrs	*(Defn. 9.7)*
`ProtoGen`	ProtoGens	*(Defn. 9.6)*
`ProtoSpec`	ProtoSpecs	*(Defn. 9.8)*
`SysMatField`	MatFields	*(Defn. 9.16)*
`SysMatIdxAField`	SysMatidxAFields	*(Defn. 9.16)*
`SysMatIdxBField`	SysMatidxBFields	*(Defn. 9.16)*
`SysMatIdxField`	SysMatidxFields	*(Defn. 9.16)*
`SysVecField`	SysVecFields	*(Defn. 9.16)*

global constant	*function*	
`Garbit`	$\mathcal{G}\mathit{arbit}$	*(Defn. 9.25)*
`Gcollinear`	$\mathcal{G}\mathit{collinear}$	*(Defn. 9.27)*
`Gcoupled`	$\mathcal{G}\mathit{coupled}$	*(Defn. 9.24)*
`Gpair`	$\mathcal{G}\mathit{pair}$	*(Defn. 9.26)*
`GpureF`	$\mathcal{G}\mathit{pure}\mathcal{F}$	*(Defn. 9.22)*
`GpureT`	$\mathcal{G}\mathit{pure}\mathcal{T}$	*(Defn. 9.23)*

class name	*function family*	
`Cdamp`	*Cdamp*	*(Defn. 9.29)*
`Cgrav`	*Cgrav*	*(Defn. 9.28)*
`Cptnail`	*Cptnail*	*(Defn. 9.35)*
`Cptpath`	*Cptpath*	*(Defn. 9.37)*
`Cptpt`	*Cptpt*	*(Defn. 9.36)*
`Cspring`	*Cspring*	*(Defn. 9.30)*
`Damp`	$\mathcal{D}\mathit{amp}$	*(Defn. 9.38)*
`Gravity`	$\mathcal{G}\mathit{ravity}$	*(Defn. 9.38)*
`PointToNail`	$\mathcal{P}\mathit{oint}\mathcal{T}\mathit{o}\mathcal{N}\mathit{ail}$	*(Defn. 9.38)*
`PointToPath`	$\mathcal{P}\mathit{oint}\mathcal{T}\mathit{o}\mathcal{P}\mathit{ath}$	*(Defn. 9.38)*
`PointToPoint`	$\mathcal{P}\mathit{oint}\mathcal{T}\mathit{o}\mathcal{P}\mathit{oint}$	*(Defn. 9.38)*
`Spring`	$\mathcal{S}\mathit{pring}$	*(Defn. 9.38)*

Figure 9.7: Math section definitions in the prototype implementation. □

The class `ProtoGen` is an abstract class, that declares a generic `eval` method. The class `Lgen` is derived from `ProtoGen`; it has members for the G, H and J fields, and defines `eval` to compute an applied motive index, as per Defn. 9.17. The classes `ProtoConstr` and `Lconstr` are defined similarly. These classes all include checks to make sure that n and k are used consistently.

• Constant instances for motive-generator functions: In Sec. 9.3.8, we defined several specific motive-generator functions, $\mathcal{G}\mathit{arbit}$, $\mathcal{G}\mathit{pair}$, and so forth. For each of these functions, we define an instance of class `Lgen`, with the appropriate values for members `G`, `H`, and `J`. Each instance is defined once, as a global constant, to be used or referenced by any proto-specifier. For example:

```
GpureF = Lgen(n=1, k=3, G=1, H=0, J=0)
```

• Class definitions for explicit constraint functions: In Sec. 9.3.9, we defined several parametrized families of constraint functions. For each family, we derive a class from `Lconstr`. For example:

```
class Cspring (derived from Lconstr) :
  constructors:  (double l0,k,mu)
  members:       l0 : double    rest length
                 k  : double    spring constant
                 mu : double    damping coefficient
```

The members for Γ, Λ, and K compute their values based on the member variables l_0, k and μ, as per Defn. 9.30; for any given triple of parameters, l_0, k, and μ, a specific instance can be created.

• Class definitions for geometric constraint functions (Sec. 9.3.10: We define a class derived from *Lconstr*:

```
class GeomConstr (derived from Lconstr) :
  constructors:  (int n,k, double tau, SysVecField D,D1,B, SysMatIdxField Ga,La)
  members:       tau  : double          time constant τ
                 D    : SysVecField
                 D1   : SysVecField
                 Beta : SysVecField
```

It defines the member for K to compute its value as per Defn. 9.34. We derive more specific classes for the various types of constraints. For example:

```
class PointToPoint (derived from GeomConstr) :
  constructors:  (double tau)


class PointToNail (derived from GeomConstr) :
  constructors:  (double tau, MCO::Location X0)
  members:       X0 : MCO::Location
```

For any given time constant τ, along with constraint-specific parameters (such as X_0 above), an instance of a geometric constraint function can be created.

9.5.3 M-N Interface

The scope name for the M-N interface is `NFRC`. We define a routine that solves the posed problem (Eqn. 9.39): define an applied motive index field that is consistent with given proto-specifier and application set indexes.

```
LsolveProto(LprotoSpecIdx P, ApplicSetIdx S) : MRIG::ApplMotiveIdxField
```

This routine returns an object that can be evaluated for arbitrary values of Y and t. The object is set up internally to follow the solution procedure outlined in Sec. 9.4:

1. Given values of Y and t.
2. Compute the various terms K, Γ, G, etc.
3. Compute the $\mathcal{M}_{pq}$ matrixes.
4. Gather the $\mathcal{M}_{pq}$ and K_p values into $\mathbf{M}$ and $\mathbf{K}$, using `NUM::GatScat2` (Sec. B.4.2).
5. Solve the equation $\mathbf{M}\,\mathbf{X} + \mathbf{K} = 0$ to get $\mathbf{X}$, using `NUM::LinSys` (Sec. B.4.3).
6. Scatter $\mathbf{X}$ into $\mathcal{X}_q$ values, using `NUM::GatScat2`.
7. Compute an index of motives A, based on the $\mathcal{X}_q$ values and the precomputed G, H, and J.
8. return A

As an optimization, at the time the object is created we partition the matrix into independent blocks, based on the connectivity information in `S`; then the single large linear system solution becomes a series of smaller ones.

9.5.4 C-M Interface

The C-M Interface constructs indexes $\mathcal{P} \in \{\textsf{LprotoSpecs}\}_{\textsf{IDs}}$ and $\mathcal{S} \in \{\textsf{ApplicSets}\}_{\textsf{IDs}}$ based on the conceptual force objects. For each force object, the interface must:

- Choose a label $i \in \textsf{IDs}$ for the force object
- Choose or create the appropriate motive-generator and constraint function instances for P_i.
- Create an application set $s = S_i$. For each $r \in s$, we must:
  - Map from each conceptual body name to its mathematical model ID, for $b(r)$
  - Create an (arbitrary) unique ID for $a(r)$
  - Set $pt(r)$ to the body coordinates of the application point.

Having done the above, the C-M interface can call `NFRC::LsolveProto` to get an `MRIG::ApplMotiveIdxField`, which can be used as a parameter to `NRIG::SolveForward` of Sec. 8.5.3, in order to simulate the resulting behavior.

9.6 Derivations

9.6.1 Acceleration of a Body Point

We derive an expression for the acceleration of a body point, for use in Notn. 9.5. Given:

dynamic body path	$s \in \textsf{StatePaths}$	*consistent (Defn. 8.10) with*
net force, torque functions	$F, T: \Re \rightarrow \textsf{Vectors},$	*and*
body point path	$p: \Re \rightarrow \textsf{Bodypts}$	*such that*
	$p(t) \equiv \mathit{Bodypt}(s(t), \mathit{coords})$	
for constant	$\mathit{coords} \in \Re^{3\times 3}$	

First, we express the derivative of the body's inverse inertia tensor. Since the inertia tensor is fixed in the body frame (Eqn. 8.8), Eqn. 6.33 gives (after replacing ω^{*T} with the equivalent $-\omega^*$):

$$\frac{d}{dt}\mathbf{I}_s^{-1}(t) = \omega_s^*(t)\mathbf{I}_s^{-1}(t) - \mathbf{I}_s^{-1}(t)\omega_s^*(t)$$

Using the above, we express the derivatives of the body's angular velocity vector (for clarity, we drop the (t) parameters after the first occurrence):

$$\begin{aligned}
\omega_s(t) &= \mathbf{I}_s^{-1}(t)\, L_s(t) \\
\tfrac{d}{dt}\omega_s &= (\tfrac{d}{dt}\mathbf{I}_s^{-1})\, L_s + \mathbf{I}_s^{-1}\, (\tfrac{d}{dt}L_s) \\
&= (\omega_s^*\mathbf{I}_s^{-1} - \mathbf{I}_s^{-1}\,\omega_s^*)\, L_s + \mathbf{I}_s^{-1}T(t) \\
&= \omega_s^*\mathbf{I}_s^{-1}L_s - \mathbf{I}_s^{-1}\,\omega_s^* L_s + \mathbf{I}_s^{-1}T \\
&= \omega_s^*\omega_s - \mathbf{I}_s^{-1}(\omega_s \times L_s) + \mathbf{I}_s^{-1}T \\
&= -\mathbf{I}_s^{-1}(\omega_s \times L_s) + \mathbf{I}_s^{-1}T
\end{aligned}$$

We start with the velocity of a body point, from Eqn. 8.13, and differentiate to get the acceleration (again, we drop the (t) parameters):

$$\begin{aligned}
v_p(t) &= vc_s(t) + \omega_s(t) \times \mathit{arm}_p(t) \\
\tfrac{d}{dt}v_p &= \tfrac{d}{dt}vc_s + (\tfrac{d}{dt}\omega_s) \times \mathit{arm}_p + \omega_s \times (\tfrac{d}{dt}\mathit{arm}_p) \\
&= \tfrac{d}{dt}vc_s - \mathit{arm}_p \times (\tfrac{d}{dt}\omega_s) + \omega_s \times (\tfrac{d}{dt}\mathit{arm}_p) \\
&= \tfrac{1}{m_s}F(t) - \mathit{arm}_p^*(-\mathbf{I}_s^{-1}(\omega_s \times L_s) + \mathbf{I}_s^{-1}T) + \omega_s \times (\omega_s \times \mathit{arm}_p) \\
&= \tfrac{1}{m_s}F - \mathit{arm}_p^*\mathbf{I}_s^{-1}T + \mathit{arm}_p^*\mathbf{I}_s^{-1}(\omega_s \times L_s) + \omega_s \times (\omega_s \times \mathit{arm}_p)
\end{aligned}$$

The first term above gives acc_F in Notn. 9.5, the second term gives acc_T, and the remaining terms give acc_0.

9.6.2 Derivation of the Linear Constraint Equation

We derive Eqn. 9.21. Given:

$$\begin{aligned}
&\textit{compatible}\\
&\quad \mathcal{P} \in \{\text{LprotoSpecs}\}_{\text{IDs}},\\
&\quad A \in \{\text{AppliedMotives}\}_{\text{IDs}}\\
&\quad \mathcal{S} \in \{\text{ApplicSets}\}_{\text{IDs}}\\
&\quad \mathcal{X} \in \{\text{Vecs}\}_{\text{IDs}}\\
&\textit{consistent with}\\
&\quad Y \in \text{Systems}\\
&\quad t \in \Re
\end{aligned}$$

We plug into the constraint equation, Defn. 9.13. For notational clarity, after the first occurrence we will leave the (Y,t,p) parameters off K, Γ, Λ, and the (Y,t,q) off G, and H.

$$0 = \mathit{Constr}_p(Y,t,p,A) \qquad \textit{(Defn. 9.13)}$$

If $\mathit{all}_p = 1$, we have:

$$\begin{aligned}
0 &= K_p(Y,t,p) + \sum_{j\in \mathit{Ids}(A)} \begin{pmatrix} \Gamma_p(Y,t,p)_{\mathit{body}(A_j)}\ {}^{\mathit{cab}}\mathcal{F}(A_j) + \\ \Lambda_p(Y,t,p)_{\mathit{body}(A_j)}\ {}^{\mathit{cab}}\mathcal{T}(A_j) \end{pmatrix} && \textit{(Defn. 9.18)}\\
&= K_p + \sum_{q\in \mathit{Elts}(\mathcal{S})} \sum_{r\in q} \begin{pmatrix} \Gamma_p(Y,t,p)_{b(r)}\ {}^{\mathit{cab}}\mathcal{F}(A_{a(r)}) + \\ \Lambda_p(Y,t,p)_{b(r)}\ {}^{\mathit{cab}}\mathcal{T}(A_{a(r)}) \end{pmatrix} && \textit{(Defn. 9.11)}\\
&= K_p + \sum_{q\in \mathit{Elts}(\mathcal{S})} \sum_{r\in q} \begin{pmatrix} [\Gamma_p]_{b(r)}\ {}^{\mathit{cab}}\mathcal{F}(\mathit{Gen}_q(Y,t,q,\mathcal{X}_q)_{a(r)}) + \\ [\Lambda_p]_{b(r)}\ {}^{\mathit{cab}}\mathcal{T}(\mathit{Gen}_q(Y,t,q,\mathcal{X}_q)_{a(r)}) \end{pmatrix} && \textit{(Eqn. 9.14)}\\
&= K_p + \sum_{q\in \mathit{Elts}(\mathcal{S})} \sum_{r\in q} \begin{pmatrix} [\Gamma_p]_{b(r)} G_q(Y,t,q)_{a(r)} \mathcal{X}_q + \\ [\Lambda_p]_{b(r)} H_q(Y,t,q)_{a(r)} \mathcal{X}_q \end{pmatrix} && \textit{(Defn. 9.17)}\\
&= K_p + \sum_{q\in \mathit{Elts}(\mathcal{S})} \left(\sum_{r\in q} ([\Gamma_p]_{b(r)}[G_q]_{a(r)} + [\Lambda_p]_{b(r)}[H_q]_{a(r)}) \right) \mathcal{X}_q
\end{aligned}$$

And if $\mathit{all}_p = 0$, we have:

$$\begin{aligned}
0 &= K_p(Y,t,p) + \sum_{r\in p} \begin{pmatrix} \Gamma_p(Y,t,p)_{b(r)}\ {}^{\mathit{cab}}\mathcal{F}(A_{a(r)}) + \\ \Lambda_p(Y,t,p)_{b(r)}\ {}^{\mathit{cab}}\mathcal{T}(A_{a(r)}) \end{pmatrix} && \textit{(Defn. 9.18)}\\
&= K_p + \sum_{r\in p} \begin{pmatrix} \Gamma_p(Y,t,p)_{b(r)}\ {}^{\mathit{cab}}\mathcal{F}(\mathit{Gen}_s(Y,t,p,\mathcal{X}_p)_{a(r)}) + \\ \Lambda_p(Y,t,p)_{b(r)}\ {}^{\mathit{cab}}\mathcal{T}(\mathit{Gen}_q(Y,t,p,\mathcal{X}_p)_{a(r)}) \end{pmatrix} && \textit{(Eqn. 9.14)}\\
&= K_p + \sum_{r\in p} \begin{pmatrix} [\Gamma_p]_{b(r)} G_p(Y,t,r)_{a(r)} \mathcal{X}_p + \\ [\Lambda_p]_{b(r)} H_p(Y,t,r)_{a(r)} \mathcal{X}_p \end{pmatrix} && \textit{(Defn. 9.17)}\\
&= K_p + \sum_{r\in p} \left([\Gamma_p]_{b(r)}[G_p]_{a(r)} + [\Lambda_p]_{b(r)}[H_p]_{a(r)} \right) \mathcal{X}_p\\
&= K_p + \sum_{q\in \mathit{Elts}(\mathcal{S})} \left(\delta_{p=q} \sum_{r\in q} [\Gamma_p]_{b(r)}[G_q]_{a(r)} + [\Lambda_p]_{b(r)}[H_q]_{a(r)} \right) \mathcal{X}_q
\end{aligned}$$

This leads directly to Eqn. 9.21.

Chapter 10

Swinging Chain Model

This chapter describes a "swinging chain" model: A collection of cylinders linked end-to-end to form a chain. The two ends of the chain are fixed in space, and the chain dangles between them.

This model demonstrates how we define a top-level physically-based model, illustrating the use of the library of modules defined in Ch. 6–9. In particular, the model makes use of the geometric constraints supported by the "fancy forces" mechanism of Ch. 9.

The functionality for this model is directly supported by the library routines, thus there's little that needs to be defined in this chapter—just a description of the conceptual model, in terms of the conceptual models of Ch. 6–9 . However, we illustrate some aspects of the mathematical model that is implied.

10.1 Goals

The purpose of this model is to demonstrate how we define a top-level model, and to test the prototype modeling library. In particular, we:

- Test and illustrate rigid-body dynamics (Ch. 8)
- Test and illustrate constraints and other "fancy forces" (Ch. 9).

10.2 Conceptual Model

This model has only one high-level conceptual object in it: A linked chain. We fix the two endpoints in space, and let the chain dangle and swing under the influence of gravity, with some damping so that the chain will come to rest in a catenary shape (Fig. 10.1). The chain is made from a collection of 6 separate links, each of which is a rigid body as per Ch. 8.

Links. Each link is a cylinder of length 20cm and radius 2cm. with spherical endcaps. The body frame has its origin at the base of the cylinder, and the cylinder extends along the positive z-axis (Fig. 10.2). The mass of the cylinder is 0.5kg, distributed homogeneously (except in the endcaps), giving an inertia tensor of $\begin{bmatrix} 17.2 & 0 & 0 \\ 0 & 17.2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ kg-cm^2 in body coordinates,[1] with the center of mass at z =10cm.

[1] From Eqn. 8.1 we can derive the formula for the inertia tensor of a homogeneous cylinder aligned with the z axis, having mass m, radius r and length l (or we can just look it up in [Fox67-Table II]):

$$\begin{bmatrix} \frac{1}{12}m(3r^2+l^2) & 0 & 0 \\ 0 & \frac{1}{12}m(3r^2+l^2) & 0 \\ 0 & 0 & \frac{1}{2}m\,r^2 \end{bmatrix}$$

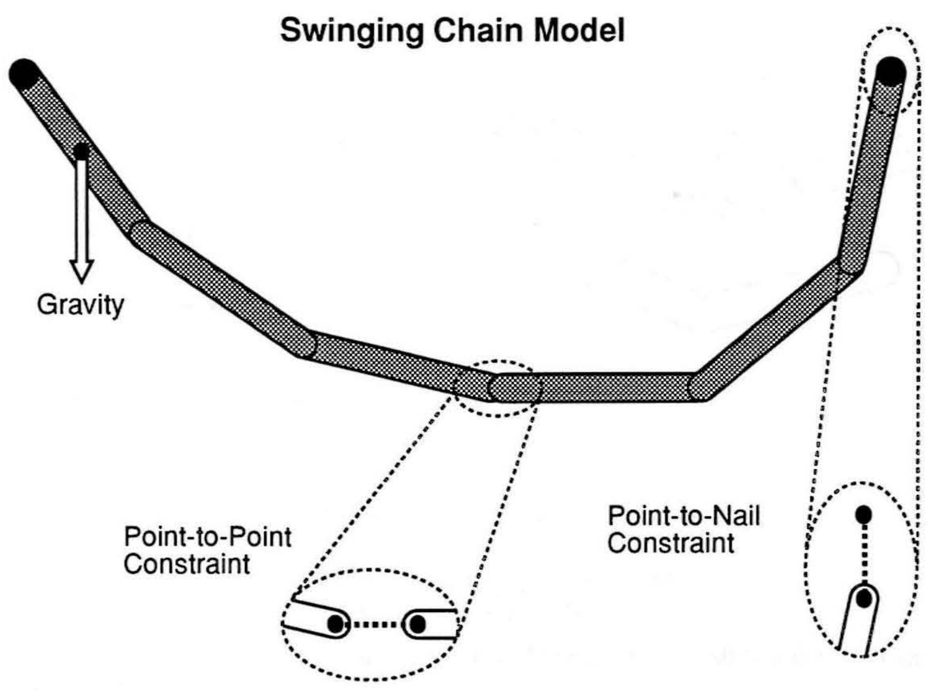


Figure 10.1: Elements in the "swinging chain" sample model. The chain is formed from a collection of cylindrical links. Each link is acted on by a constant downward gravity force, and by a viscous damping force/torque (not illustrated). Each adjacent pair of links is connected with a "point-to-point" constraint, and each end link has its endpoints fixed in space with a "point-to-nail" constraint. After an initial excitation, the chain will swing freely, gradually losing energy due to the damping, until it comes to rest in a catenary shape hanging between the fixed points. □

Joints. The joints between each link are formed using point-to-point constraints (Sec. 9.2.3). We connect the tip of one link to the origin of the adjacent link. Notice that the bodies interpenetrate in order to meet the constraint—but because of the spherical endcaps, no seams are formed at the joints. (Fig. 10.3)

Ends. The tip of the "topmost" and origin of the "bottommost" link are held fixed at locations $(50, 0, 110)$cm and $(-50, 0, 110)cm$ in lab coordinates, via point-to-nail constraints.

Other forces. Each body feels a downward gravitational acceleration of 9.8m/sec^2. We apply a viscous damping force (Sec. 9.2.2) of 0.1dyne/(cm/sec) and torque of 0.1cm-dyne/(radians/sec) to each body.

Initial assembly. We initially place the bodies in an arbitrary configuration, and let the constraint forces assemble the chain and pull it into place. A value of .2sec for the time constant τ for the constraints (see Sec. 9.3.10) causes the model to assemble in roughly one second. In their "rush" to assemble, the bodies pick up a fair amount of kinetic energy from the action of the constraint forces. In some circumstances that energy might be an unwanted by-product, but for this model, it serves to start the chain off jangling.

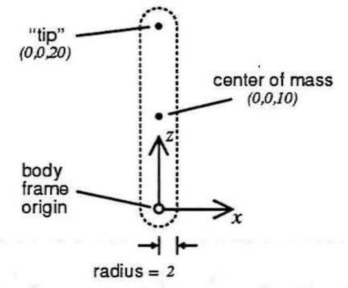
10.3 Mathematical Model

No new mathematical definitions are required for this model, but we will illustrate the "proto-specifier ensemble" (Defn. 9.11) that describes the various forces. (We are essentially hand-simulating some of the function of the C-M interface of the fancy forces implementation.)

First, we assign ID's to the bodies in the model: Let

$$\mathit{cyl0},\ \mathit{cyl1},\ \mathit{cyl2},\ \mathit{cyl3},\ \mathit{cyl4},\ \mathit{cyl5} \in \mathsf{IDs}$$

correspond with the cylinders, in left-to-right order as in Fig. 10.1.



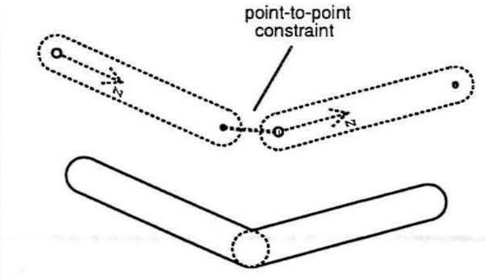


Figure 10.2: (Left) Detail of a link, in body coordinates (y-axis goes into the page). □

Figure 10.3: (Right) Detail of a joint between links. □

The application of the force objects to the model is described by an index of proto-specifiers, and an index of application sets:

$$\begin{aligned} \mathcal{P} &\in \{\mathsf{ProtoSpecs}\}_{\mathsf{IDs}} \quad \textit{(Defn. 9.8)} \\ \mathcal{S} &\in \{\mathsf{ApplicSets}\}_{\mathsf{IDs}} \quad \textit{(Defn. 9.2)} \end{aligned}$$

We construct the entries in these indexes according to the following table (recall that the two indexes share the same ID's, Defn. 9.9):

$i \in$ IDs	$\mathcal{P}_i$		$\mathcal{S}_i$		
	[*Constr*,	*Gen*]	[a	, b	, arm]
joint01	[*Cptpt* ,	$\mathcal{G}$*pair*]	[*jnt01a*,	*cyl0*,	$(0,0,20)$]
			[*jnt01b*,	*cyl1*,	$(0,0,0)$]
joint12	[*Cptpt* ,	$\mathcal{G}$*pair*]	[*jnt12a*,	*cyl1*,	$(0,0,20)$]
			[*jnt12b*,	*cyl2*,	$(0,0,0)$]
		⋮			
joint45	[*Cptpt* ,	$\mathcal{G}$*pair*]	[*jnt45a*,	*cyl4*,	$(0,0,20)$]
			[*jnt45b*,	*cyl5*,	$(0,0,0)$]
end0	[*Cptnail*,	$\mathcal{G}$*coupled*]	[*end0* ,	*cyl0*,	$(0,0,0)$]
end5	[*Cptnail*,	$\mathcal{G}$*coupled*]	[*end5* ,	*cyl5*,	$(0,0,20)$]
grav0	[*Cgrav* ,	$\mathcal{G}$*coupled*]	[*grav0* ,	*cyl0*,	$(0,0,10)$]
grav1	[*Cgrav* ,	$\mathcal{G}$*coupled*]	[*grav1* ,	*cyl1*,	$(0,0,10)$]
		⋮			
grav5	[*Cgrav* ,	$\mathcal{G}$*coupled*]	[*grav5* ,	*cyl5*,	$(0,0,10)$]
damp0	[*Cdamp*,	$\mathcal{G}$*arbit*]	[*damp0*,	*cyl0*,	]
damp1	[*Cdamp*,	$\mathcal{G}$*arbit*]	[*damp1*,	*cyl1*,	]
		⋮			
damp5	[*Cdamp*,	$\mathcal{G}$*arbit*]	[*damp5*,	*cyl5*,	]

A few things to notice in the above table:

- Each joint acts on two bodies, thus has two application info's in its application set.
- Each damping object generates an arbitrary force/torque; its generator function doesn't examine the arm aspect value of the application set, so we have left that entry blank.
- The a column lists the ID's of all the motives in the model. Some of these ID's happen to be the same as used in column i, but that's irrelevant.

Chapter 11

"Tennis Ball Cannon"

This chapter describes a "tennis ball cannon" model: A series of balls is shot from a gun; each ball flies in an arc, and bounces when it hits the ground. The model contains discontinuous behaviors—the firing of the gun, the bouncing of the balls, and others.

This model illustrates the *segmented function* formulation: we define the behavior of the model as a function $\mathcal{C}$ that encapsulates the discontinuous events; the value of the solution, $\mathcal{C}(t)$, includes information such as which balls are part of the model at any time t.

Unlike the "swinging chain" (Ch. 10), this model isn't already directly supported by the library modules in Ch. 6–9. Thus, we build on the library models, but we define extra special-purpose mathematical constructs for this application. (Some of the special-purpose mechanisms that are defined here can be generalized, as discussed in the library extensions, Ch. 12.)

11.1 Goals

The purpose of this model is to illustrate the segmented function mechanism described in Sec. 3.10, and to test the piecewise-continuous ODE solution utilities described in Sec. B.4.7. In order to fully exercise these tools, we include a few somewhat contrived features in the model: (Fig. 11.1)

- discontinuities regularly in time (cannon fires)
- discontinuities based on state (balls bounce)
- increase of dimensionality (balls created when cannon fires)
- decrease in dimensionality (balls removed when too small)
- change in properties (ball radius changes)
- change in continuous behavior (wind resistance changes)
- discontinuities in motion (balls bounce)

11.2 Conceptual Model

We describe the abstractions of the various elements in the model. Note that we are not trying to accurately model a real thing, but rather have contrived an assortment of features to meet our goals.

Environment. In lab coordinates (Sec. 6.3), the positive z axis points "up." The ground is an infinite flat surface in the x-y plane at $z = 0$.

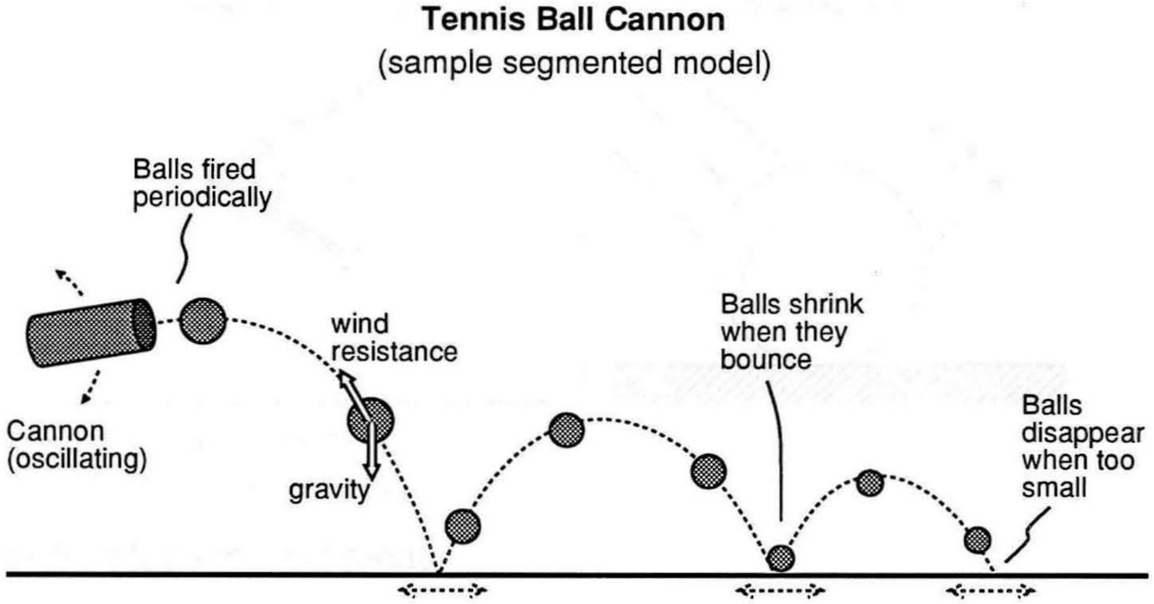


Figure 11.1: Model of a cannon firing a series of balls; illustrates discontinuities in a model. The cannon oscillates up and down, firing a stream of balls. Each ball experiences gravity, and wind resistance based on radius. When a ball hits the ground it bounces, but also instantaneously shrinks by a fixed factor (thus will experience less wind resistance after the bounce). When a ball shrinks below a minimum size, it is removed from the model. The contact points where the balls bounce shift back and forth due to the oscillation of the cannon. The bounces are completely elastic, but because of the wind resistance energy is lost, so each rebound is lower than the previous. □

Cannon. The cannon is just a barrel: a cylinder, with length 0.8 meters and radius 0.22 meters. The back end is fixed at coordinates $(-5, 0, 5)$ in the world; the muzzle points towards the positive x direction, but oscillating ± 0.4 radians from horizontal, with a period of 2 seconds per cycle. (In the body frame, the back end is at the origin, and the muzzle lies on the x-axis.)

Firing. To "fire," the cannon spontaneously creates a ball at the muzzle; the ball has an initial velocity of 15 meters/second, outward in the barrel direction. (We create the ball at the muzzle, so it doesn't need to travel down the barrel.) The cannon has an unlimited supply of balls, and fires with a period of 0.2 seconds between shots.

Balls. Each ball is a rigid sphere, with radius 0.2 meters (initially) and mass 1 kilogram.[1] We include no rotation effects—the motion of the ball is that of a point mass (Sec. 8.2.3) at the center of the sphere.

Bouncing. When a ball hits the ground ($z = 0$), it instantaneously shrinks—the new radius is $3/4$ the previous. The location of the center of the ball is instantaneously dropped so that the bottom stays on the ground (Fig. 11.3). The velocity is instantaneously negated in the z direction, but left alone in x and y directions,[2] i.e., the ball bounces elastically. The mass is not altered.

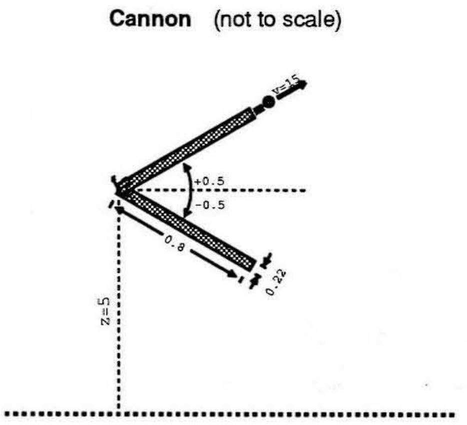
Disappearing. If, when hitting the ground, the ball shrinks so that its radius is less than 0.09 meters, the ball is "too small" and is instantaneously removed from the model.[3] With the stated parameters, this happens on the third contact, after two bounces.

Forces. Each ball is acted on by two forces: a constant downward gravitational acceleration of 9.8 meters/second^2; and a air resistance (drag) force that is opposite to and quadratic in the velocity, scaled by $.5$ kilogram/meter^3 times the cross sectional area (πr^2). For simplicity, we directly hardwire the net force on each ball to be the sum of the two contributions, rather than using the "fancy forces" mechanism, Ch. 9.

[1] Perhaps more a bocce ball than a tennis ball!

[2] Actually, because of the configuration of the gun and lack of crosswise forces, the velocity in the y direction is always 0.

[3] Think of it as falling through cracks in the floor?



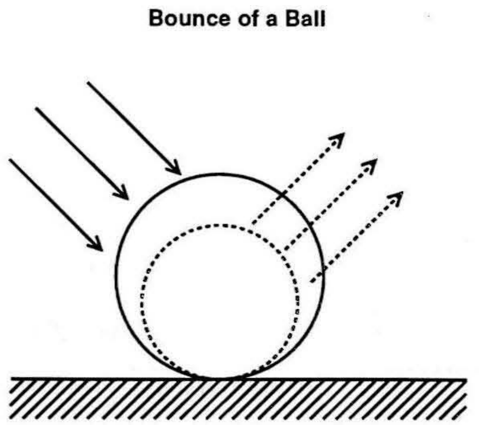


Figure 11.2: (Left) Detail of cannon. □

Figure 11.3: (Right) Detail of a ball bouncing. □

11.3 Mathematical Model

11.3.1 Names & Notation

The scope name for this model is:

TSEG ("Test of a SEGmented model")

We use of the following terms from other modules:

AppliedMotiveFields	(Defn. 8.21)	*Rep*	(Defn. 6.5)
IDs	(Defn. 3.8)	*Repq*	(Defn. 6.8)
IDsets	(Defn. 3.9)	RIGID :: States	(Defn. 8.4)
FramePaths	(Defn. 6.27)	Systems	(Defn. 8.14)
$\mathcal{L}ab$	(Defn. 6.4)	Vectors	(Defn. 6.1)
Locations	(Defn. 6.1)		

We use Systems to mean RIGID :: Systems rather than KINEMATIC :: Systems. For convenience, we define the constant parameters for the model:

Definition. (constant parameters)

(11.1)

$loc_c \in$ Locations	${}^{\mathcal{L}ab}loc_c = (-5,0,5)$	*cannon position*
$len_c \in \Re$	$\theta_c = 0.8$	*cannon length*
$\tau_c \in \Re$	$\tau_c = 2$	*cannon oscillation period*
$\theta_c \in \Re$	$\theta_c = 0.5$	*cannon oscillation amplitude*
$\tau_b \in \Re$	$\tau_b = 0.2$	*cannon firing period*
$v_b \in \Re$	$v_b = 15$	*ball initial speed*
$g \in \Re$	$g = 9.8$	*gravitational acceleration*
$w \in \Re$	$w = .5$	*air resistance coefficient*
$m_0 \in \Re$	$m_0 = 1$	*ball mass*
$r_0 \in \Re$	$r_0 = 0.2$	*initial ball radius*
$rmin \in \Re$	$rmin = 0.09$	*minimum ball radius*
$\gamma \in \Re$	$\gamma = 0.75$	*radius shrink factor*
$\hat{Z} \in$ Vectors	${}^{\mathcal{L}ab}\hat{Z} = (0,0,1)$	*"up" vector*

Defn. 11.1: Constant parameters for the model, as described in Sec. 11.2. □

11.3.2 Definitions

It is convenient to pre-define a sequence of ID's that we will use in the model:

Definition. idq

We define a non-repeating sequence of ID's:

$$idq \equiv \left\{ idq_0, idq_1, \ldots \;\middle|\; idq_i \in \mathsf{IDs} \text{ and } idq_i = idq_j \Rightarrow i = j \right\} \tag{11.2}$$

Defn. 11.2: A sequences of ID's, that we will use to label bodies and motives in the model. □

The motion of the cannon's body frame is describe by an explicit function:

Definition. fcannon

We define a function fcannon $\in$ FramePaths *by:*

$$\begin{aligned} P_{fcannon}(t) &= loc_c \\ Repq(\mathcal{L}ab, R_{fcannon}(t)) &= \begin{bmatrix} \cos(\frac{1}{2}\theta_c \sin(\frac{2\pi t}{\tau_c})) \\ 0 \\ -\sin(\frac{1}{2}\theta_c \sin(\frac{2\pi t}{\tau_c})) \\ 0 \end{bmatrix} \end{aligned} \tag{11.3}$$

Defn. 11.3: Cannon body frame function, *fcannon*. We use a quaternion to describe a rotation of $-\theta_c \, sin(\frac{2\pi t}{\tau_c})$ radians about the lab frame's y-axis, so the cannon muzzle's angle from the horizontal oscillates with a period τ_c and amplitude θ_c. □

The instantaneous frame of a ball at the instant it is fired can described in turn:

Definition. bshot

We define a function bshot: $\Re \rightarrow$ RIGID :: States *by:*

$$\begin{aligned} Rep(fcannon(t), x_{bshot}(t)) &= (len_c, 0, 0) \\ Rep(fcannon(t), v_{bshot}(t)) &= (v_b, 0, 0) \\ \mathbf{R}_{bshot}(t) &= R_{\mathcal{L}ab} \\ \omega_{bshot}(t) &= 0 \\ m_{bshot}(t) &= m_0 \\ \mathbf{I}_{bshot}(t) &= 0 \end{aligned} \tag{11.4}$$

Defn. 11.4: Initial state of a ball. *bshot*(t) describes the instantaneous state of a ball fired at time t. The position and velocity are fixed in the frame of the gun. The orientation, angular velocity, and inertia tensor are 0, since the balls are point masses. Note that *bshot* is *not* a state path as per Defn. 8.6—its velocity vectors don't describe its own motion—rather, it is a mapping between a time that a shot might occur, and the initial state of the shot ball. □

The model as a whole is described at an instant by the following space:

Definition. States

(11.5)

$$\begin{array}{lll}
\text{States} & & \\
[& & \\
\quad ids & \mapsto \text{IDsets} & \textit{balls' IDs} \\
\quad R & \mapsto \{\Re\}_{\text{IDs}} & \textit{radius of each ball} \\
\quad Y & \mapsto \text{Systems} & \textit{state of each ball} \\
\quad M & \mapsto \{\text{AppliedMotiveFields}\}_{\text{IDs}} & \textit{motives on each ball} \\
\quad tfire & \mapsto \Re & \textit{time of most recent shot} \\
\quad seq & \mapsto Integers & \textit{ball ID sequence number} \\
— & & \\
\quad ids = Ids(Y) = Ids(R) = Ids(M) & & \\
\quad \forall i \in ids,\ body(M_i) = i & & \\
] & &
\end{array}$$

Defn. 11.5: State of the model. For each ball named $i \in ids$, its radius is given by R_i, its dynamic state is given by Y_i, and M_i is the applied motive that acts on it. *tfire* is the time of the shot most recently fired by the cannon. *seq* is the sequence number of the ID for the ball to be fired next. Since we apply only one motive per ball, we simply label each motive with the ball's ID. □

The motive field that is applied to a ball with a given radius and id is given by:

Definition. netf

(11.6)

Define a function netf: $\Re \times \text{IDs} \rightarrow \text{AppliedMotiveFields}$ *such that, for any* $r \in \Re$ *and* $i \in \text{IDs}$:

$$\mathcal{M} = netf(r, i) \quad \Longrightarrow$$
$$\begin{cases} F(\mathcal{M}(Y,t)) = -m(Y_{body(M)}\, g)\, \hat{Z} - \pi\, r^2 w\, \| v(Y_{body(M)})\|\; v(Y_{body(M)}) \\ T(\mathcal{M}(Y,t)) = 0 \end{cases}$$

Defn. 11.6: Motive applied to a ball. The net force we apply to a ball has two parts: a constant gravitational acceleration times the mass of the ball, pointing downward ($-\hat{Z}$), and a drag force that is quadratic in and opposite to the ball's velocity ($-v(Y_{body(mf)})$). □

We define a function that returns the post-bounce state of a bouncing ball:

Definition. bbounce

(11.7)

We define a function bbounce: $\text{States} \times \text{IDs} \rightarrow \text{RIGID} :: \text{States}$ *by:*

$$bbounce(s, i) \equiv \begin{bmatrix} \text{M} \leftarrow & \text{M}(Y(s)_i) \\ x \leftarrow & x(Y(s)_i) - (1-\gamma)R(s)_i\hat{Z} \\ v \leftarrow & v(Y(s)_i) - 2(\hat{Z} \cdot v(Y(s)_i))\hat{Z} \end{bmatrix}$$

Defn. 11.7: Bounce of a ball. Negates the z component of the ball's velocity, and lowers the center by the change in radius. Since the ball is a point mass, the mass distribution, position and velocity form an identifying tuple. (The use of tuple notation is as per Notn. 3.24.) □

Finally, several auxiliary functions that will be convenient later on:

Definition. height, hit, big

(11.8)

We define several functions

height:	States × IDs → $\Re$	*height of a ball*
hit:	States	*balls hitting the ground*
big:	States	*balls big enough to survive*

$$\begin{aligned} height(s,i) &\equiv x(Y(s)_i)\cdot\hat{Z} - R(s)_i \\ hit(s) &\equiv \left\{ i \in ids(s) \;\middle|\; height(s,i) = 0 \right\} \\ big(s) &\equiv \left\{ i \in hit(s) \;\middle|\; \gamma\, R(s)_i \ge rmin \right\} \end{aligned}$$

Defn. 11.8: Auxiliary functions. For a model state $s \in$ States, the height above the ground of ball $i \in$ IDs is given by $height(s,i)$, and the set of balls that are hitting the ground is given by $hit(s)$.[4] Of the balls that hit the ground, $big(s)$ yields those that are large enough to "survive" the bounce—i.e., shrinking them won't reduce them below the minimum radius. □

11.3.3 Behavior of the Model

The behavior of the model over time is that of a segmented function, as per Sec. 3.10. We will describe the behavior using the functional characterization described in Sec. 3.10.3; but as per Sec. B.4.9 we have several numerical event functions, each with a corresponding transition function.

For continuous motion, we define a *consistent* path through state space:

Definition. **consistent**

(11.9)

A state function $s\colon \Re \to$ States *is* **consistent** *iff:*

$$\begin{aligned} ids_s(t) &\quad \textit{is constant} \\ R_s(t) &\quad \textit{is constant} \\ M_s(t) &\quad \textit{is constant} \\ tfire_s(t) &\quad \textit{is constant} \\ seq_s(t) &\quad \textit{is constant} \\ Y_s &\quad \textit{is consistent with } M_s \end{aligned}$$

Defn. 11.9: A continuous, consistent, path through state space. The set of ball ID's doesn't change, the radius of each ball doesn't change, the motive field that is applied to each ball doesn't change, and the gun never fires. But the balls can move, as long as their motion is consistent (Defn. 8.25) with the motive fields. □

To describe the cannon firing, we have an event function, that "goes off" when the cannon fires, and a transition function that adds a new ball:

[4] Because of numerical inaccuracies, in practice we use

$$hit(s) \equiv \left\{ i \in ids(s) \;\middle|\; height(s,i) < \epsilon \text{ and } v(Y(s)_i)\cdot\hat{Z} < 0 \right\}$$

for some small tolerance ϵ. Thus $hit(s)$ is the set of balls that are numerically close to hitting the ground, where we take care not to include balls that have already bounced and are moving away.

Definition. Gfire, Hfire

Define event function Gfire: $\Re \times \mathsf{States} \rightarrow \Re$ *and transition function Hfire*: $\Re \times \mathsf{States} \rightarrow \mathsf{States}$ *by:*

$$Gfire(t,s) \equiv tfire_s + \tau_b - t$$

(11.10)

$$Hfire(t,s) \equiv \begin{bmatrix} R & \leftarrow\!\!\dashv & R_s + [idq_{seq(s)}, r_0] \\ Y & \leftarrow\!\!\dashv & Y_s + [idq_{seq(s)}, bshot(t)] \\ M & \leftarrow\!\!\dashv & M_s + [idq_{seq(s)}, netf(r_0, idq_{seq(s)})] \\ tfire & \leftarrow\!\!\dashv & t \\ seq & \leftarrow\!\!\dashv & seq(s)+1 \end{bmatrix}$$

Defn. 11.10: Event and transition functions to fire cannon. For a model in state $s \in \mathsf{States}$, the last time the cannon fired was $tfire_s$, since the period is τ_b (Defn. 11.1), the cannon will fire next when $t = tfire_s + \tau_b$. To make a transition to the initial state of the next segment, the function *Hfire* adds an entry for a new ball to the various indexes, sets the time of firing, and increments the sequence number. (The use of $+$ to add elements to an index is as per Notn. 3.15, and the use of the tuple notation is as per Notn. 3.24.) □

To describe a ball (or balls) bouncing, we have an event function that "goes off" when any ball contacts the ground, and a transition function that adjusts the radius, state, and motives of the bouncing balls:

Definition. Gbounce, Hbounce

Define event function Gbounce: $\Re \times \mathsf{States} \rightarrow \Re$ *and transition function Hbounce*: $\Re \times \mathsf{States} \rightarrow \mathsf{States}$ *by:*

$$Gbounce(t,s) \equiv \min_{i \in ids(s)} height(s,i)$$

(11.11)

$$Hbounce(t,s) \equiv \begin{bmatrix} R & \leftarrow\!\!\dashv & R_s - hit(s) \bigcup_{i \in big(s)} \{[i, \gamma\, R(s)_i]\} \\ Y & \leftarrow\!\!\dashv & Y_s - hit(s) \bigcup_{i \in big(s)} \{[i, bbounce(s,i)]\} \\ M & \leftarrow\!\!\dashv & M_s - hit(s) \bigcup_{i \in big(s)} \{[i, netf(\gamma\, R(s)_i, i)]\} \\ tfire & \leftarrow\!\!\dashv & tfire_s \\ seq & \leftarrow\!\!\dashv & seq_s \end{bmatrix}$$

Defn. 11.11: Event and transition functions for balls bouncing. For a model in state $s \in \mathsf{States}$, the value $Gbounce(t,s)$ is the minimum height of any ball, thus is 0 when any ball hits the ground. To make the transition to the post-bounce segment, $Hbounce(t,s)$ removes all index entries for balls that are colliding with the ground ($hit(s)$), and adds new entries for all colliding balls large enough to survive the transition ($big(s)$); the cannon-firing aspect values are not altered. Both $Gbounce(t,s)$ and $Hbounce(t,s)$ are independent of t. □

Note that transition functions *Hfire* and *Hbounce* are commutative, so if both types of events happen simultaneously, the two transition functions can be applied sequentially in either order.

11.4 Posed Problems

For this sample model, we have just a single posed problem: determine the behavior of the model over time. If we start with no balls, we just specify the time that the cannon first fires:

Program definitions in scope `MTSEG`:

class name	*mathematical object*	
`State`	States	*(Defn. 11.5)*
`StatePath`	$\Re \rightarrow$ States	*(segmented) state function*
`idq`	*idq*	*(Defn. 11.2)*

routine		*function*	
`fcannon(double)`	: `MCO::instFrame`	*fcannon*	*(Defn. 11.3)*
`bshot(double)`	: `MRIG::State`	*bshot*	*(Defn. 11.4)*
`netf(double,MM::Id)`	: `MRIG::ApplMotiveField`	*netf*	*(Defn. 11.6)*
`bbounce(State,MM::Id)`	: `MRIG::State`	*bbounce*	*(Defn. 11.7)*
`height(State,MM::Id)`	: `double`	*height*	*(Defn. 11.8)*
`hit(State)`	: `MM::IdSet`	*hit*	*(Defn. 11.8)*
`big(State)`	: `MM::IdSet`	*big*	*(Defn. 11.8)*
`gfire(double,State)`	: `double`	*Gfire*	*(Defn. 11.10)*
`hfire(double,State)`	: `State`	*Hfire*	*(Defn. 11.10)*
`gbounce(double,State)`	: `double`	*Gbounce*	*(Defn. 11.11)*
`hbounce(double,State)`	: `State`	*Hbounce*	*(Defn. 11.11)*

Figure 11.4: Math section definitions for the sample segmented model. □

$$
\begin{array}{rl}
\textit{given:} & \textit{parameters of Defn. 11.1} \\
 & \textit{initial firing time } t_0 \\
\textit{find:} & \textit{segmented function } C\colon \Re \rightarrow \text{States} \\
\textit{such that:} & \left\{ \begin{array}{l} \textit{Continuously, } C \textit{ is consistent, and discontinuities in } C \textit{ are} \\ \textit{described by event/transition function pairs Gfire, Hfire} \\ \textit{and Gbounce, Hbounce.} \end{array} \right.
\end{array}
\tag{11.12}
$$

Eqn. 11.12: Behavior of the tennis ball cannon model. We start with an "empty" model, i.e., no balls. The first ball is shot at time t_0, and the cannon fires regularly from then on. □

The above is an initial-value piecewise-continuous ODE problem, that can be solved as described in Appendix C. The continuous behavior is an ODE as per the rigid body forward dynamics problem, Sec. 8.4. The initial state $C_0 \in$ States for the problem can be constructed as:

$$
C_0 = \begin{bmatrix} ids & \leftarrow\!\shortmid & \emptyset \\ R & \leftarrow\!\shortmid & \emptyset \\ Y & \leftarrow\!\shortmid & \emptyset \\ M & \leftarrow\!\shortmid & \emptyset \\ tfire & \leftarrow\!\shortmid & t_0 - \tau_b \\ seq & \leftarrow\!\shortmid & 0 \end{bmatrix},
$$

which will trigger a "fire" event at $t = t_0$, and add the first ball to the model. (Or, if this gives results in boundary-condition difficulties, we can evaluate $\mathit{Hfire}(t_0, C_0)$, to yield the initial state of the first non-empty segment.)

11.5 Implementation Notes[†]

The implementation of the "tennis ball cannon" follows straightforwardly from this chapter's description of the model.

11.5.1 Math Section Constructs

The math section for this module has scope name `MTSEG` ("Math Test of a SEGmented model"); the definitions are listed in Fig. 11.4.

[†]See Appendix B for discussion of the terminology, notation, and overall approach used here.

```
class State :
  constructors: (MCO::ScalarIdx R, MRIG::System Y, MRIG::ApplMotiveFieldIdx M, double tfire, int seq)
  members:      ids   : MM::IdSet
                R     : MCO::ScalarIdx
                Y     : MRIG::System
                M     : MRIG::ApplMotiveFieldIdx
                tfire : double
                seq   : int
```

The class for state space States is defined as per Sec. B.3.6. The constructor is given the various indexes as parameters; it checks to see that the sets of ID's are mutually consistent, and that each motive is labeled with the ID of the body that it is applied to, as per Defn. 11.5. The class for state paths, `StatePath` is defined in the standard manner as per Sec. B.3.7.

```
class Idq :
  methods: seq(int i) : MM::Id
```

The class `idq` implements an infinite sequence of ID's, *idq* (Defn. 11.2), by a "generator" mechanism: the sequence is maintained as a data structure whose elements are created on demand; when a sequence element is requested that isn't in the data structure, a new, unique ID is defined and added.

The routines listed in Fig. 11.4 are implemented by directly transcribing the corresponding mathematical functions. `netf` constructs an `ApplMotiveField` instance whose force is given by an algebraic combination of `SysVectorField` instances, and whose torque is a constant 0 `SysVectorField`, as per Sec. 8.5.2.

11.5.2 M-N Interface

The M-N interface for this module has scope name `NTSEG`

```
SolveForward(double t0) : StatePath
```

The state path instance returned by this function embodies the solution function C for the behavior of the model (Sec. 11.4). In order to evaluate $C(t)$, the piecewise-continuous ODE solver `NUM::PodeScatExt` (Sec. B.4.9) is called.

The continuous part of the solution is constructed via `MRIG::SolveForward` (Sec. 8.5.3). To handle the discontinuities, we set up the numerical solver to invoke the routines `gfire` and `gbounce`—events are signaled when the values they compute cross zero.[5] When an event is found, the solver returns a code telling us which of the two events it is, so that we can invoke `hfire` or `hbounce` as appropriate. The solver may determine that, to within numerical tolerances, *both* events happen simultaneously, in which case it returns both codes; for this model we can safely call both `hfire` and `hbounce` in either order.

11.5.3 Conceptual Section

The conceptual section can use the definitions of the dynamic rigid-body module, Sec. 8.5.1.

11.5.4 C-M Interface

The state of the solution for any time value includes the set of ID's of balls that are in the model at that time. The C-M interface can dynamically adjust the conceptual section state, adding or removing "`sphere`" objects to conform with the solution for any time value. The various balls can be distinguished by the position of their ID's in the sequence `idq`, if, e.g., we want to cycle through a series of colors for the balls that are created.

For more general models, in which many different types of events may result in the mathematical model's generating new ID's, we may want the C-M interface to be able to distinguish between ID's based on the

[5] Notice that both functions are set up to be positive before an event, zero at the event, and negative past the event, as discussed in Sec. C.1.1.

events that caused them. For example a single model may create a new ball when a cannon is fired, and create new glowing embers when fireworks explode—the C-M interface needs to know whether a given ID in the system belongs to a ball or to an ember. We suggest two methods to achieve this:

- Define separate sequences similar to *idq*. Each event would choose ID's from a separate sequence, and the C-M interface could determine to which sequence any given ID belongs..
- More generally, partition the space IDs into subsets, and require different parts of the model to use ID's from different subsets. This could be implemented as subclasses of the class `MM::Id`, with runtime tagging to be able to determine which subclass a given instance is a member of.

Chapter 12

Extensions to the Prototype Library

The library and sample models that were presented in Ch. 6–11 serve as a small, prototype example and test case. However, one of our goals for the structured modeling approach is to support extensibility, i.e., addition of new capabilities that are built on and integrated with an existing library.

This chapter, therefore, discusses how the prototype library could be extended and enhanced to support a handful of new capabilities: collision and contact, finite-state control mechanisms, interchanging kinematic and dynamic motion, and flexible bodies. Note that unlike Ch. 6–11, which were all implemented as described, the discussion in this chapter is speculative, put forth to suggest ways in which the given library might be extended. Thus we will describe some conceptual ideas, and informally sketch some mathematical equations, but will not give a formal description of modules or implementation details.

12.1 Rigid-Body Collision

We'd like our rigid bodies to bounce when they collide with each other. The rigid-body dynamics module in Ch. 8 doesn't notice when bodies collide—it doesn't "know" the extents of the bodies, and freely lets them pass through each other. Here, we'll talk briefly about collisions.

First, we observe that conceptually, the rigid-body abstraction is at its weakest when addressing collisions between bodies: for continuous free motion, with accelerations that are small compared to the rigidity of the material, the rigid-body abstraction works well; but when bodies collide, there is a sudden extreme change in their velocities—thus there are deformations, shock waves, and so forth in even the most rigid of materials. Still, it is often useful to abstract a collision as a discontinuous change in the velocity/momentum of a perfectly rigid bodies. (Fig. 12.1)

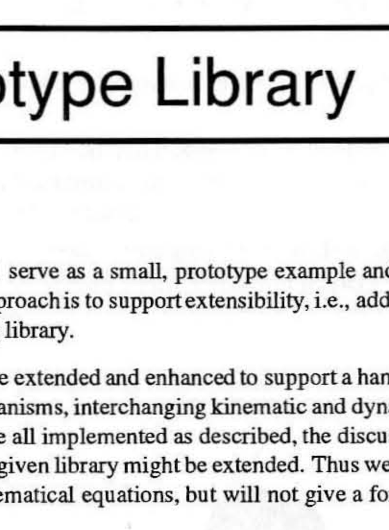


Figure 12.1. Rigid-body collision. When rigid bodies collide, they experience a discontinuous change in velocity (both linear and angular). □

Discontinuous changes in momentum are described via *impulsive forces* (or just *impulses*). The common empirical model for collisions is that the points of contact emerge from an impulsive collision with some fraction e of the relative perpendicular velocity that they had going in:

$$\Delta V_{\perp}^{+} = -e\,\Delta V_{\perp}^{-}$$

where e is called the *coefficient of restitution*, and is determined experimentally for different pairs of materials. If $e = 1$, the bodies leave with the same relative velocity that they had going in, whereas if $e = 0$, they will remain in contact.[1] For more complete discussion of rigid-body collision, see [Fox67-Ch10.8], [Baraff89], [Moore,Wilhelms88].

Rigid-body collisions can be incorporated into the prototype library, by generalizing the method used in the "tennis ball cannon" model in Ch. 11. That is, we mathematically characterize the behavior as a segmented function by defining two functions that describe the collisions:

- *An event function* g such that $g(t, Y(t))$ is the minimum separation between any bodies, where negative values indicate penetration.
- *A transition function* h such that $h(t, Y(t))$ computes the effects of impulses on the colliding bodies.

Of course, we need to include the shapes of the bodies into our conceptual and mathematical models, in order to define the event function. In particular, we need algorithms to determine the separation between bodies; see [Baraff90], [Von Herzen,Barr,Zatz90], [Moore,Wilhelms88], or [Snyder92].

When several bodies mutually collide, or when collisions involve bodies that are constrained (as per Ch. 9) or in contact (as per Sec. 12.2), the impulsive collision calculation expands to involve a simultaneous system of equations; see [Baraff89], [Moore,Wilhelms88].

12.2 Rigid-Body Contact

Rigid bodies are often in continuous contact, e.g., a collection of bodies in a pile. In general, we can let rigid bodies touch, or let them move apart, but we don't want them to interpenetrate. Mathematically, this translates into a *non-holonomic* constraint, i.e., an inequality relation: The separation between bodies must be greater than or equal to 0. (Fig. 12.2)

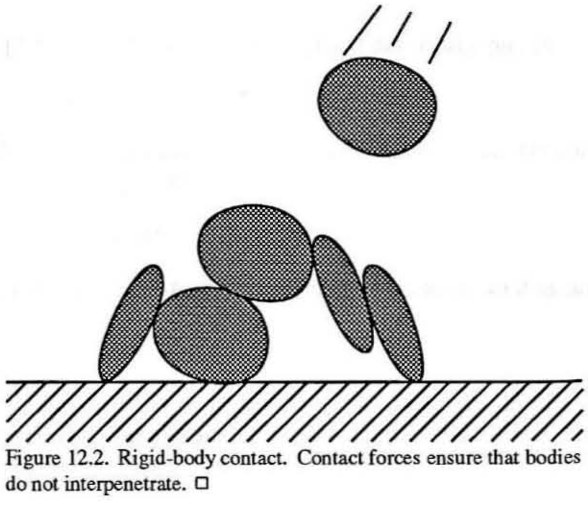


Figure 12.2. Rigid-body contact. Contact forces ensure that bodies do not interpenetrate. □

As long as two bodies maintain contact, they mutually apply *contact forces* to keep from interpenetrating; these forces are essentially the same as those of the geometric constraints described in Ch. 9. The contact forces are "one way," however, in that they push bodies apart, but don't pull to hold bodies together.

Determining when to introduce and remove contact forces is a complex task, especially in the presence of multiple-body contact, surface friction, impulsive collisions, and so forth. A treatment of the subject is beyond our scope; we refer readers to the series of articles [Baraff89], [Baraff90], and [Baraff91].

For our current discussion, we observe that a description of rigid-body contact can fit into the segmented-model formulation. Extending the functions described in Sec. 12.1, we have:

- *Event function:* Determines when bodies collide, and, for bodies in contact, determines when the contact forces "let go."
- *Transition function:* Computes impulsive behavior, activates contact constraints to keep bodies from interpenetrating, deactivates contact constraints to let bodies separate.

[1] As discussed in Sec. 2.6.2, we actually prefer a quasilinear restitution model.

12.3 Finite-State Control

Often, a model may have several different modes of behavior, that it switches between based on special events; control mechanisms to describe these changes of behavior are often defined as finite-state machines (FSM's). These are graphs or tables that have a collection of state *nodes*, and, for each node, a list of interesting events and the corresponding transitions (*arcs*) to new nodes.

For example, gait control for human or animal locomotion is commonly described as a finite state machine—when a foot leaves the ground, the model enters a state in which the leg is brought forward, when the leg contacts the ground, weight is shifted onto it, and so forth (Fig. 12.3); see [Girard, Maciejewski85], [Raibert,Hodgins91]. The "tennis ball cannon" model in Ch. 11 can be described via a simple machine that has one state node, with two event arcs that loop back to it, one followed periodically for the "fire" event, and the other followed when a ball bounces. [Brockett90] discusses a finite-state model for a formal robotics language. [Kalra90] gives a general formulation for finite-state control over models (from which we take the notation we use below).

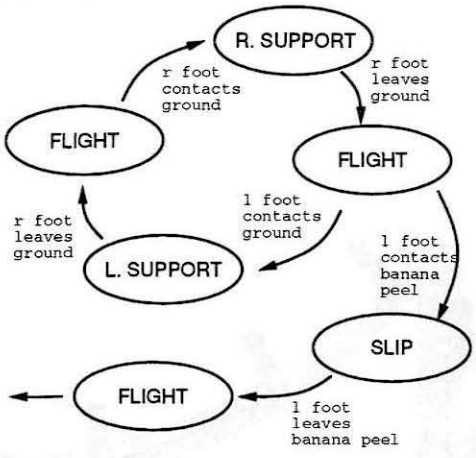


Figure 12.3. Finite-state machine. A finite-state machine can be used as a control mechanism for models. □

Mathematically, we can describe a finite-state graph via a collection of triples, each describing an arc:

$$(B_i, L, B_j),$$

where B_i names a node, L an event, and B_j the node at the terminus of the arc, and where we use ID's to name events and nodes. The space of all possible finite-state graphs would then be:

$$\mathsf{FSMs} \equiv \textit{sets of triples}\ \{(B_i, L, B_j), \ldots\}$$

The state of a model controlled by an FSM would include an aspect for the current state node, as well as an aspect containing the FSM itself.

$$\begin{array}{lll} \multicolumn{3}{l}{\mathsf{ModelStates}} \\ [& & \\ \ \textit{fsm} & \mapsto \mathsf{FSMs} & \textit{The controller for the model} \\ \ B & \mapsto \mathsf{IDs} & \textit{The current state node} \\ \ \vdots & & \\ ] & & \end{array}$$

Consider a segmented function through the above space. Since the aspect spaces for B and *fsm* are discrete, they must stay constant within each segment of the function. However, when an event occurs, a transition function can modify B:

$$h(t, s, L) \equiv \begin{bmatrix} B \leftarrow & B_j \textit{ such that } (B_s, L, B_j) \in \textit{fsm}_s \\ \vdots & \end{bmatrix}$$

Notice that in addition to time t and model state $s \in$ ModelStates, the transition function takes a parameter $L \in$ IDs, which is the name of the event that occurred, as discussed in Sec. B.4.9.[2] In addition to updating B,

[2] If multiple events can occur simultaneously, and if the transition function can't be evaluated sequentially for each event, the FSM would need to include arcs to follow for each combination of events.

the transition function can adjust the model state appropriately (based on s and L), in the same style as *Hfire* (Defn. 11.10) and *Hbounce* (Defn. 11.11) of the "tennis ball cannon." We can even modify *fsm*—the model can change its control program!

12.4 Mixed Dynamic/Kinematic Motion

Often, a model has some parts that can most easily be described kinematically, while other parts have dynamic motion. The "tennis ball cannon" in Ch. 11, for example, has kinematic motion for the cannon barrel, and dynamic motion for the balls. More interestingly, individual bodies may switch between kinematic and dynamic motion, as in Fig. 12.4.

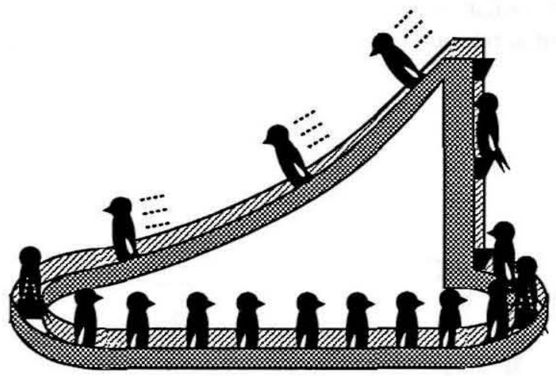


Figure 12.4. Interchanging dynamic and kinematic behavior. Here, the penguins slide down the incline dynamically, but make the return trip via a kinematic conveyor belt. □

Any desired kinematic motion of a dynamic body can in principle be achieved by introducing appropriate time-varying geometric constraints, such as described in Defn. 9.37—but doing so may be inconvenient, and moreover, requires extra work in the numerical simulation of the dynamics.

Instead, we can directly change the rules of behavior for any given body. For example, we define a model whose state includes both kinematic and dynamic elements:

ModelStates

[

$K \mapsto \text{KINEMATIC} :: \mathsf{Systems}$ *kinematic body states*

$D \mapsto \text{RIGID} :: \mathsf{Systems}$ *dynamic body states*

$M \mapsto \{\text{RIGID} :: \mathsf{MassDists}\}_{\mathsf{IDs}}$ *mass properties of each body*

$\vdots$

—

$Ids(K) \cup Ids(D) = Ids(M)$ *M records mass of* all *the bodies*

$\forall i \in Ids(D),\ \text{M}(D_i) = M_i$ *dynamic bodies agree with M*

]

The motion of the bodies in K is governed by some kinematic rule, while the motion of the bodies in D is determined by a set of applied motive fields. For a body named $i \in \mathsf{IDs}$ to go from dynamic to kinematic, the transition function would perform:

$$\begin{array}{lll} D & \leftarrow\!\dashv & D - i \\ K & \leftarrow\!\dashv & K + [i, k(D_i)] \end{array}$$
remove from dynamic index / *add kinematic state to index*

and for kinematic to dynamic:

$$\begin{array}{lll} D & \leftarrow\!\dashv & D + [i, \begin{bmatrix} \text{M} \leftarrow\!\dashv & M_i \\ k \leftarrow\!\dashv & K_i \end{bmatrix}] \\ K & \leftarrow\!\dashv & K - i \end{array}$$
add dynamic state to index / *remove from kinematic index*

We maintain the body frame (and velocity) across the events, so the transitions appear seamless. The transitions can be controlled via the finite-state mechanism described in Sec. 12.3.

On a related note, consider a model of 100,000 dominoes arranged on end, ready to be knocked over. To perform a dynamic simulation of all the dominoes would be infeasible. But we notice that there are only a few dominoes moving at any one time—the rest are static. (Fig. 12.5)

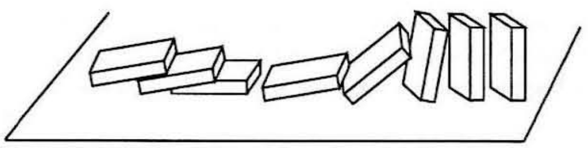


Figure 12.5. Mixed static and dynamic parts. The dominoes are static before and after falling. □

Thus we would design a model with separate "active" and "static" body indexes. When something collides with a static body, we switch it to active, with an initial condition based on the collision impulse (Sec. 12.1). When an active body is at rest and in contact only with static bodies, we switch it to static. Computationally, we need only compute the behavior of the active bodies, and need only test for collisions amongst active bodies and between active bodies and neighboring static bodies.

12.5 Flexible Bodies

We would like to extend our library to support dynamic modeling of flexible bodies, integrated with the rigid bodies (Fig. 12.6). It is common in computer graphics to model flexible bodies as a grid of point masses, with some some sorts of springs linking them together. This approach has the advantage that it's easy to implement, especially given an existing rigid-body modeling system.[3] However, we feel that this approach introduces excessive implementation detail into the conceptual model, as discussed in Sec. 2.4. Instead, we perform much the same computation, but with a different outlook (choice *A* of Fig. 2.6, rather than the mass-point/spring model, choice *C*).

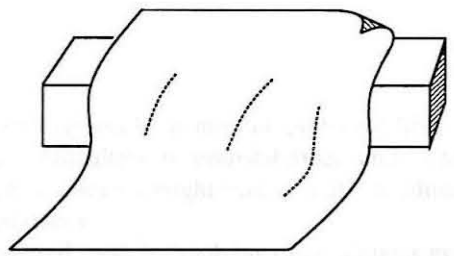


Figure 12.6. A flexible body, interacting with a rigid body. □

Our basic conceptual model of a flexible body is that it is a smooth, continuous surface. For the mathematical model, we follow the development in [Terzopoulos et al.87] and [Platt89]: start with an expression for the strain energy of the model; taking a variatonal derivative yields a partial differential equation (PDE) describing the continuous surface that minimizes the energy. In order to numerically solve the PDE, it is discretized using a finite element or finite difference approximation. As discussed in [Platt89], a finite difference approximation can be made that is mathematically equivalent to a point-mass-and-spring system.

Programmatically, once we numerically solve the (discretized) equations, we propogate the result back to the mathematical section, and thence to the conceptual section—this means reconstructing a continuous function from the discretized results, by interpolation appropriate to the discretization method.

There are several advantages to creating a smooth surface at the mathematical and conceptual levels, rather than using the mass-point/spring model:

- The body can be rendered without regard to the numerical discretization. Many rendering methods will resample the smooth surface (adaptively) to create high-quality graphics; this resampling is not restricted to the numerical discretization grid points.
- The interaction with the other objects in the model is not restricted to grid points. Penetration tests, e.g., can be performed on the smooth surface, without having to worry if small objects will "slip between the cracks."
- The grid density can be chosen automatically by the numerical solver, based on problem-specific data, or can vary adaptively over time or across the body. The user doesn't need to guess a grid density in advance that will suffice throughout the simulation.

[3] In fact, we've done so: we've succesfully simulated models of flexible 3-D bodies as roughly 1000 point masses on sparse grids, with roughly 10,000 springs between them, using our implementation of the prototype library modules of Ch. 6–8 .

- The solution method is independent of the higher levels. The numerics section doesn't have to always use the finite-difference method if in some configurations, e.g., a modal analysis (see [Pentland, Williams89]) or deforming rigid body method is appropriate. The various methods can be swapped in without affecting the higher level of the program.[4]

It might seem expensive to make and use the interpolated solution, rather than directly using the numerical data. However, we feel that it will buy us much in the way of cleanliness, robustness, and applicability. Moreover, the eventual use of "smart" or adaptive solvers will allow us to simulate systems that require fine sampling for only a few extreme configurations; if we were to have only a pre-assigned sampling density, we would need to globally choose the finest sampling, thus making the overall computation too slow to be feasible. Thus, ultimately, this approach may be faster.

In order to integrate rigid-body and flexible-body dynamics, we must have a compatible mathematical formalism. For rigid bodies, in Ch. 8 we chose a force-based, Newtonian formalism; but flexible bodies are typically expressed using an energy-based, Lagrangian formalism. We may need to extend Ch. 8 to include the energy formulation, or, conversely, we may wish to express flexible bodies using a force-based formulation. Or both.

12.6 Summary

This chapter has speculated on how several techniques and features could fit into the prototype library of Ch. 5–9. In doing so, we have doubtlessly glossed over many difficulties, incompatabilities, and so forth. Our intent is to try to convey a feeling of where the modeling methodology might lead, and also to illustrate how we use the methodology in thinking about as-yet-unsolved issues.

Our choice of topics in this chapter has deliberately emphasized models involving discontinuous events and changes of state. We feel that a primary benefit of the structured approach towards modeling will ultimately be in the ability to create models that are hetereogeneous—i.e., that incorporate various different behaviors mixed together—both across time and across the elements of the model.

[4] This is analogous to the linear system solver `NUM::linsys`, discussed in Sec. B.4.3, that chooses between a sparse method or singular-value decomposition.

Chapter 13

Concluding Remarks

This chapter offers retrospective evaluation and comments on aspects of the design framework. We will elaborate on a small assortment of issues, based in particular on our experience with the prototype modeling library of Ch. 5–12, and conclude with thoughts about where the ideas we have presented may lead.

13.1 Did We Meet Our Goals?

Sec. 1.2 enumerated various goals for a design framework. Here, we list the goals again here, accompanied by discussion of how well the design strategy that we have presented meets those goals, based on our experience developing and implementing the library described in Ch. 5–12.

- *To facilitate the understanding and communication of models:* We have found that the CMP structure (Ch. 2) helps us to understand models, by providing a powerful and convenient partitioning of the major parts of a model. Having a common framework and terminology has proven helpful for discussing with colleagues models that are under development, and for presenting final versions of models such as in Ch. 6–9.

- *To facilitate the creation of models with high degrees of complexity:* The use of modularity in the design of models (Sec. 2.5) helps manage complexity. The use of structured mathematical modeling techniques (Ch. 3) helps us to isolate and write well-defined mathematical equations for complex models; in particular, the ability to defined the behavior of a complex model as a segmented function (Sec. 3.10, Sec. 4.7) lends itself to very "clean" organization for models and programs.

- *To facilitate the reuse of models, techniques, and ideas.* Modularity design as per Sec. 2.5 is helpful in the re-use of models at the blackboard level. The toolbox-oriented program framework lets us re-use models and program code at the implementation level; in particular, the "structured numerics" library (Sec. B.4) has proven to be usable in a wide variety of applications.

- *To facilitate the extension of models.* In our sample library, the "fancy forces" module (Ch. 9) was successfully designed and implemented as an extension to the earlier rigid-body modeling modules; it did of course engender some minor alterations to the earlier models, but the modular design and implementation framework kept those changes localized. Similarly, as per our discussion in Ch. 12, future extensions to the library seem relatively straightforward.

- *To facilitate the creation of models that are "correct."* The emphasis on explicit statement of the goals and conceptual properties of the model (Ch. 2) helps make sure that we are "on track" as we design models. The emphasis on standalone mathematical models (Sec. 2.4, Ch. 3) helps us to verify mathematical consistency. The overall framework helps us to debug a model by identifying and localizing various types of bugs, as discussed in Sec. 2.6.4 and Sec. 4.9.

- *To facilitate the translation of models into programs.* By using object-oriented, functional, and procedural programming styles in the various separate sections of a program, rather than choosing a single style overall, we have been able to best match the programming methods to the local programming tasks. And, since the program framework (Ch. 4) follows the blackboard framework closely, it has proven to be straightforward to implement models; in particular, the final version of the "fancy forces" module was implemented without much difficulty, by following the description in Ch. 9. The program framework lends itself to debugging, as discussed in Sec. 4.9.

13.2 Notes on the Design Framework

13.2.1 Structure of Physically-based Models

As per the discussion of applied mathematics in Ch. 2, we think that a primary part of physically-based modeling is:

- Identification of a well-defined mathematical model, that has no "conceptual" or numerical solution influences mixed into it.

The separation of equations from problems (as per Sec. 2.4.2) is perhaps less basic: for given goals, there are often specific problems to be posed, and we may be best off posing them directly. But when we are interested in reusable general-purpose models, in which different problems may be posed for different applications, the separation becomes significant.

13.2.2 Mathematical Models

We have found the mathematical techniques in Ch. 3 to be convenient and helpful in designing complete mathematical models. But can these techniques (or future ones) keep a rein on mathematical models that can grow arbitrarily complex? It would be nice if mathematical models were always tractable by hand, via the use of modularity, structuring techniques, and so forth. In our experience with the "fancy forces" mathematical model, Ch. 9, however, we have found that we construct a long chain of definitions that can be tricky to use correctly.

If our goal of having well-defined mathematical models *written on paper* proves to be unattainable or impractical, we may need to modify it: we'd rather not give up well-defined mathematical models, but we may give up writing them on paper—that is, we may turn to computer-aided mathematical modeling, as discussed in Sec. 13.4.

13.2.3 Program Framework

Our experience using the program framework that we defined in Ch. 4, with its conceptual/math/numerics separation, has been favorable. Nevertheless, given the volatile nature of computer program technology, we're skeptical about *any* specific program structure passing the test of time.

We have found that the approach of identifying and supporting changes of representation, as discussed in Sec. 4.6, has been particularly valuable in defining a high-level, structured numerics library. The numerics library described in Sec. B.4 has proven useful not just for our own prototype physically-based modeling library, but for other projects as well.

In defining "math section" objects and classes for the modules of Ch. 6–9, we essentially implemented a limited, special-purpose symbolic mathematics mechanism, directly in the C++ programming language. This was a straightforward task, given well-defined "blackboard" mathematical models. Nevertheless, it would perhaps be more elegant to define these in a mathematical modeling language; see Sec. 13.4.

13.3 Have We Made Modeling Easy?

We can't claim that we've "magically" made modeling easy. It can be a lot of work to completely specify a model in the "structured modeling" manner we have described—for example, if nothing else, the prototype modules in Ch. 6–9 have a lot of bulk. In fact, for a small, one-time-use model, it would probably be easier to just express and implement the model from scratch.

So what have we done? Our goal has been to maximize correctness, modularity, and reusability, so that we can build models more intricate than we could otherwise. Thus we:

- Invest more effort in the small—i.e., carefully describe the details of individual modules,

to yield

- Better results in the large—i.e., complex models that are robust and flexible.

Even our simple prototype library has provided us with models and modeling programs that are more flexible than were previously available to us. For example, we now construct programs that support the optional calculation of work done by each force object—this capability is achieved "for free" given the pre-existing modular library.

But what about models that are *very* large? If we look at programming, we see that modularity and structure are crucial elements for the design of large programs, but as projects get increasingly large, structured programming is in itself insufficient: high-level languages are used, and for large enough projects, we enter the realm of software engineering (which addresses issues such as revision control, metrics, management techniques, and so forth). Similarly, we feel that our "structured modeling" approach includes some fundamental elements for the design of large models, but for large models, higher level mechanisms than we have described will need to be created, and for sufficiently large projects, an approach to "model engineering" will be needed; this need is discussed by [Brooks91].

13.4 Computer-Assisted Mathematical Modeling

Computer tools can potentially aid us to define and use mathematical models. We identify three areas in which computer tools can be of use; there is of course some overlap between them.

CAD for Mathematical Models. Remember that for our purposes, a mathematical model is a collection of definitions and equations, as opposed to a posed problem. Thus we want computer-aided design (CAD) tools that help us to *construct* equations and definitions, rather than tools to solve equations. A mathematical modeling CAD tool would include such features as:[1]

- Declarative definitions (rather than procedural).
- Modularity, including support for libraries, name scoping, etc.
- Data abstraction, i.e., definition of new abstract spaces.
- Extensible notation.
- Type-checking of operators and domains.
- Interface with programming languages and problem-solution tools.
- Ability to arrange and annotate models for clarity of presentation.

Note that the emphasis is not on automated generation of equations and definitions, but rather on a utility that helps us to manually generate the equations and definitions that we're interested in.

[1] Our hypothetical CAD tool is similar in many respects to a "Smart Paper" proposal of [Barr86].

Solving Posed Problems. Symbolic mathematics programs, such as Maple ([Char et al.91]) and Mathematica ([Wolfram91]), are powerful tools for problem-solving.[2] These programs incorporate the ability to manipulate expressions symbolically with built-in numerical evaluation and solution capability. On a different slant, [Abelson et al.89] describes techniques for automated construction, execution, and analysis of numerical problems from high-level models.

Program Construction. Numerical problem-solving subroutines often require that the user provide "call-back" subroutines; e.g., a subroutine that computes a derivative for an ODE solver, or an objective function for an optimizer. These subroutine are derived from mathematical model constructs, and thus can potentially be created automatically by symbolic manipulation. The subroutines can be constructed at runtime, based on end-user specified model configurations, resulting in special-purpose routines that efficiently evaluate the necessary components for a given problem; this approach was used in [Witkin, Kass88].

13.5 Future Directions

We hope that we have provided a stepping stone upon which others may build, to create new generations of modeling methodologies. We can imagine the existence of rich modeling environments, full of building blocks and modules (rigid bodies, constraints, flexible bodies, fluids, walking and running, quantum mechanics, weather, aerodynamics, molecular modeling, etc.) that can all be interconnected and expanded, so that users and researchers can easily, build highly complex models. Future work for such environments might entail:

- Specification languages for physically-based models.
- Standards for exchanging physically-based models.
- Automated techniques to link together different model components.
- Interactive modeling/simulation workstations.
- Specialized hardware to support key aspects of the modeling/simulation process.

Clearly, the ideas that we have presented don't directly provide the above capabilities—but we have tried to identify and examine some of the concepts that may underlie them.

[2] Note that these programs are not quite suited to our idea of CAD for mathematical modeling. They have procedural approaches; they emphasize automation in order to minimize the user's work; and finally, they don't emphasize emphasize modularity or data abstraction, both of which are fundamental to our mathematical modeling strategy.

Appendix A

Miscellaneous Mathematical Constructs

This appendix contains a few mathematical constructs that are used by some of the models in Ch. 6–9 . Notice that these are purely mathematical utilities, rather than physically-based models. Thus they do not include conceptual models, goal statements, and so forth, just some mathematical definitions.

The mathematical constructs defined here are not as fundamental as the various constructs of Ch. 3 (such as ID's and state spaces); thus we have chosen not to include them in that chapter. On the other hand, these constructs seem too general-purpose to be defined as auxiliary constructs of some specific physically-based model.

Therefore, to give such constructs a home, we have defined here a small library of miscellaneous mathematical definitions. This library includes program implementations of the constructs, in the math section, numerics section, and M-N interface parts of the program framework (Ch. 4).

A.1 Trees

The index mechanism of Sec. 3.8.2 provides a basic structuring capability for mathematical models: The ability to manipulate groups of elements by name. Here, we define an additional structuring capability, familiar from computer science: Hierarchical "tree" relationships between mathematical entities.

We define a state space, such that each element of the space is a forest:

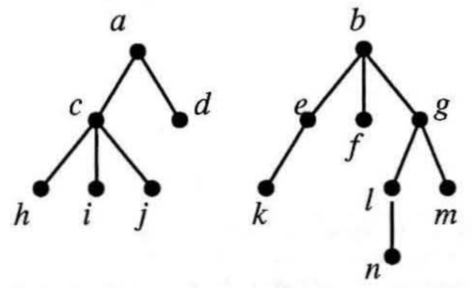


Figure A.1. A sample forest. c's *parent* is a, h's parent is c, and so forth. a's *children* are $\{c, d\}$, and so forth. $\{a, b\}$ are the *roots*. $\{d, f, h, i, j, k, m, n\}$ are the *leaves*. c's *descendants* are $\{h, i, j\}$, a's descendants are $\{c, d, h, i, j\}$, and so forth. □

Definition. IDforests

(A.1)

$$
\begin{array}{l}
\textsf{IDforests} \\
\Bigl[\\
\quad \begin{array}{lll}
\textit{ids} & \mapsto \textsf{IDsets} & \textit{All ID's in the forest} \\
\textit{roots} & \mapsto \textsf{IDsets} & \textit{The ID's at the roots} \\
\textit{nonroots} & \mapsto \textsf{IDsets} & \textit{The ID's not at the roots} \\
\textit{leaves} & \mapsto \textsf{IDsets} & \textit{The ID's at the leaves} \\
\textit{parent} & \mapsto \{\textsf{IDs}\}_{\textsf{IDs}} & \textit{Parent of an ID} \\
\textit{children} & \mapsto \{\textsf{IDsets}\}_{\textsf{IDs}} & \textit{Set of children of each ID} \\
\textit{descendants} & \mapsto \{\textsf{IDsets}\}_{\textsf{IDs}} & \textit{Set of descendants of each ID}
\end{array} \\
— \\
\quad 1.\ \textit{Ids}(\textit{parent}) \subseteq \textit{ids} \\
\quad 2.\ \textit{Ids}(\textit{children}) = \textit{Ids}(\textit{descendants}) = \textit{ids} \\
\quad 3.\ \textit{roots} \cup \textit{nonroots} = \textit{ids},\ \textit{roots} \cap \textit{nonroots} = \emptyset \\
\quad 4.\ \textit{nonroots} = \textit{Ids}(\textit{parent}) \\
\quad 5.\ \textit{leaves} = \left\{ i \in \textit{ids} \;\middle|\; \textit{children}_i = \emptyset \right\} \\
\quad 6.\ \forall i \in \textsf{IDs},\quad j \in \textit{children}_i \Leftrightarrow \textit{parent}_j = i \\
\quad 7.\ \forall i \in \textsf{IDs},\quad \textit{descendants}_i = \textit{children}_i \bigcup_{j \in \textit{children}_i} \textit{descendants}_j \\
\quad 8.\ \forall i \in \textsf{IDs},\quad i \notin \textit{descendants}_i \\
\Bigr]
\end{array}
$$

Defn. A.1: The space of forests of trees of ID's. Each element of IDforests is a forest, which may have several trees. (1) Not all ID's need have parents. (2) All ID's have a set of children and a set of descendants (3) Every ID is a root or a nonroot. (4) The nonroots are those ID's that have parents. (5) The leaves are those ID's that have no children. (6) Every ID is the parent of its children. (7) The descendants of an ID are its children and its children's descendants. (8) No ID is its own descendant. □

This space is used for the kinematic hierarchical configurations, in Ch. 7.

A.2 Arrays

It is convenient to define some constructs for arrays. We define a *vec* to be a 1-D array of real numbers, of arbitrary size, and a *mat* to be a 2-D array of real numbers, of arbitrary size:[1]

Definition. Vecs, Mats

(A.2)

$$
\begin{array}{rcl}
\textsf{Vecs} & \equiv & \left\{ x \in \Re^k \;\middle|\; \textit{positive integer } k \right\} \\
\textsf{Mats} & \equiv & \left\{ x \in \Re^{j \times k} \;\middle|\; \textit{positive integers } j, k \right\}
\end{array}
$$

Defn. A.2: Vecs is the set of all 1-D arrays of real numbers, no matter what their size; for example, if $a \in$ Vecs and $b \in$ Vecs are two vecs, it may be true that $a \in \Re^2$ while $b \in \Re^3$. Similarly, Mats is the set of all 2-D arrays of reals. Thus, Vecs and Mats are disparate unions as per Sec. 3.7.3; two vecs or two mats are *agnates* if they have the same sizes. □

In order to tell us the size of a mat or vec, we define operators:

[1] We chose the names "vec" and "mat" to be reminiscent of "vector" and "matrix"; we chose not use the latter names directly, so as to avoid potential confusion with the geometric objects in Ch. 6.

Program definitions in scope `MMISC`:

class name	*abstract space*	
`Forest`	IDforests	*(Defn. A.1)*
`Mat`	Mats	*(Defn. A.2)*
`MatIdx`	$\{\text{Mats}\}_{\text{IDs}}$	*index of mats*
`MatIdx0`	$\{\text{Mats}\}^{\circ}_{\text{IDs}}$	*index of mats, with 0*
`Vec`	Vecs	*(Defn. A.2)*
`VecIdx`	$\{\text{Vecs}\}_{\text{IDs}}$	*index of vecs*

Figure A.2: Math section definitions for the miscellaneous mathematical constructs. □

Definition. *lsz*, *Rsz*, *Sz*

$$\text{For } m \in \text{Mats},\ v \in \text{Vecs}$$

$$\begin{aligned} lsz(m) &\equiv j,\ where\ m \in \Re^{j\times k} \\ rsz(m) &\equiv k,\ where\ m \in \Re^{j\times k} \\ sz(v) &\equiv k,\ where\ m \in \Re^{k} \end{aligned} \tag{A.3}$$

Defn. A.3: Operators for the "left size" (*lsz*) and "right size" (*rsz*) of a mat, and for the size (*sz*) of a vec. □

It is convenient to give ourselves shorthand notation for arrays of known sizes:

Notation. Vecs and Mats

$$\begin{aligned} \text{Mats}[j,k] &\equiv \left\{ m \in \text{Mats} \;\middle|\; lsz(m) = j,\ rsz(m) = k \right\} \\ \text{Mats}[j,*] &\equiv \left\{ m \in \text{Mats} \;\middle|\; lsz(m) = j \right\} \\ \text{Mats}[*,k] &\equiv \left\{ m \in \text{Mats} \;\middle|\; rsz(m) = k \right\} \\ \text{Vecs}[k] &\equiv \left\{ v \in \text{Vecs} \;\middle|\; sz(m) = k \right\} \end{aligned} \tag{A.4}$$

Notn. A.4: Arrays of known sizes. We use the bracket notation on the left as shorthand for the subsets on the right. For example, Mats[4, 3] is the space of 4 × 3 arrays of reals, Vecs[3] is the space of all arrays of 3 reals. Mats[2, *] has its size partially-specified—the space includes,e.g., 2 × 2 as well as 2 × 3 arrays. □

We use will use standard matrix arithmetic for mats and vecs.

A.3 Implementation Notes[†]

A.3.1 Implementation of Trees

The class `MMISC::IdForest` that implements the space of forests could in principle be implemented in our standard manner for state spaces (Sec. B.3.6), i.e., explicitly store all the aspect values. However, that can quickly grow unwieldy, especially for the *descendants* aspect. Instead, a more compact mechanism is used, and routines are defined to compute the various aspect values on demand.

For numerical computations involving trees, one often needs to perform an operation on each ID in a tree. The routines `NMISC::IdForestPreorder(...)` and `NMISC::IdForestPostorder(...)` call a user-provided subroutine once for each ID in a forest, the former visiting a parent before visiting its children, and the latter, after.

A.3.2 Implementation of Arrays

The classes `MMISC::Vec` and `MMISC::Mat` are straightforward interfaces to a standard array package.

[†]See Appendix B for discussion of the terminology, notation, and overall approach used here.

Appendix B

Prototype Implementation

This appendix discusses a prototype implementation of a structured modeling environment as per Ch. 4 and used by the modules in Ch. 6–9. We discuss here the fundamental support routines and structure for the conceptual/math/numerics framework discussed in Ch. 4; this structure is not specific to the rigid-body modeling library in Ch. 6–9.

The appendix is in four parts. First, we give an overview of the presentation style that we use to describe the implementation, both here and in the models in Ch. 6–9. This is followed by a brief discussion of the conceptual section. Next, we describe the math section—focusing in particular on how to implement a mathematical model such as in Ch. 6–9, using the support that is provided by the environment. Finally, we discuss the numerics section, which contains a modular interface between low-level numerical subroutines and the higher levels of the programming environment.

The discussion is at the level of functional specification, intending to provide a starting point for the design of future implementations, rather than a prescription of syntactic details of the prototype. Note, however, that the specification is taken from existing, working code—it is not speculation.

B.1 Overview of the Presentation Style

In describing the prototype implementation, both here and in the modules in Ch. 6–9 , we have tried to convey the essentials of the structure and methodology, so that readers interested in designing and implementing their own modeling environments would be able to draw on our experience.

We have chosen not to include listings (or even snippets) of our working C++ source code or class definitions—we feel that the syntactic intricacies of the C++ language, compounded by the programming idiosyncrasies of the author, would obfuscate the mostly simple mechanisms that are at work. Instead, we give functional specifications of modules and classes. But we try to sprinkle in enough "grit" to keep the description anchored to the implementation, rather than drifting off into hyperbole.

Object-oriented programming today does not have a widely accepted uniform terminology. We list the terms that we use, and refer the reader to [Booch91] for further discussion:

- *Class.* An abstract data type, in an object-oriented language.
- *Instance.* An object that belongs to some class.
- *Data Member* (or, just *member*). A named data element that contains part of the state of an instance.
- *Method.* A named operation on an instance.
- *Operator.* A method invoked via special syntax.
- *Constructor.* A method that is invoked to initialize a new object.

We will describe a class using the following form:

```
class Classname :
  constructors:  (...parameters...)     a constructor for the class
                 (...other parms...)    we may have several constructors
  members:       gamma : double     data member named "gamma" of type double
                 width : integer    we may have several data members
  methods:       eval(...parms...) : integer      method "eval" returns an integer
                 rep(...parms...)  : double[3]    returns an array of 3 doubles
```

We will leave various details out of our descriptions: implementation details such as private or hidden data members (any interesting implementation issues will be discussed in the accompanying text); optimization details such as pass-by-reference or inline definitions; class-mechanism administratrivia such as destructors or assignment operators; debugging tools such as instance names and tracing methods; numerical details such as tolerance parameters.

Our program descriptions will follow the LISP-like namespace style that is discussed in Sec. 3.6.2, in which we specify a "current scope" for a module; all non-primitive terms presumed to be in that scope unless explicitly prefixed with a scope name. Since C++ scoping doesn't work in that way, the actual implementation explicitly prefixes all symbols with their scope names. To keep the program definitions from being too unwieldy, we use abbreviated scope names:

scope name	*defined*	*program section*	*description*
`MCO`	Sec. 6.6	Math	COordinate frames
`MKIN`	Sec. 7.5.2	Math	KINematic rigid bodies
`NKIN`	Sec. 7.5.3	m-N interface	KINematic rigid bodies
`MRIG`	Sec. 8.5.2	Math	dynamic RIGid bodies
`NRIG`	Sec. 8.5.3	m-N interface	dynamic RIGid bodies
`MFRC`	Sec. 9.5.2	Math	fancy FoRCes
`NFRC`	Sec. 9.5.3	m-N interface	fancy FoRCes
`MTSEG`	Sec. 11.5.1	Math	Test SEGmented model
`NTSEG`	Sec. 11.5.1	m-N interface	Test SEGmented model
`MMISC`	Sec. A.3	Math	MISCellaneous mathematics
`NMISC`	Sec. A.3	Numerics	MISCellaneous mathematics
`MM`	Sec. B.3	Math	support Modules
`NUM`	Sec. B.4	Numerics	structured nUMerics library

B.2 The Conceptual Section

The conceptual section maintains a data structure containing objects that correspond to things in the model. The data structure objects are updated based on user interaction, simulation results, and so forth.

As discussed in Sec. 4.2.1, the conceptual section of the program performs many of the same tasks as a traditional (kinematic) modeling program, except that the behavioral computation is offloaded to a mathematical/numerical computation "engine," via the C-M interface. We have focused on the construction of that engine as the most novel part of the program framework, and have given short shrift to the conceptual section. Our prototype implementation provides some basic support interfaces:

- to [Snyder92] for shape models and their mass-distribution properties,
- to rendering software and hardware,
- to animation/control and recording software and hardware,

There is still much room for work at the conceptual level. In particular, there is a need for high-level physically-based modeling description languages and data formats.

B.3 The Math Section

The math section of a program, as discussed in Sec. 4.2 and Sec. 4.4, is primarily a collection of definitions of data types—i.e., classes—that support the various entities in the blackboard mathematical models. Here, we discuss how these definitions and support are implemented.

B.3.1 Overview: Classes and Abstract Spaces

Our mathematical models are based on abstract spaces, as discused in Sec. 3.7 (and evidenced in the models of Ch. 6–9). This maps well into object-oriented programming—our basic procedure to implement a given model is:

- *For each abstract space of interest in the mathematical model, we define a corresponding class in the program.*

For example, some classes that we define, and their corresponding spaces, are:

class name	*abstract space*	
`MM::Id`	IDs	(Defn. 3.8)
`MM::IdSet`	IDsets	(Defn. 3.9)
`MCO::Vector`	Vectors	(Defn. 6.1)
`MCO::VectorPath`	VectorPaths	(Defn. 6.19)
`MRIG::ApplMotive`	AppliedMotives	(Defn. 8.20)
`MRIG::ApplMotiveIdx`	$\{\text{AppliedMotives}\}_{\text{IDs}}$	*(an index, Notn. 3.11)*

Notice that we use singular names for the classes, in keeping with the programming style that the name is that of the data type. Notice also that we often explicitly create classes, such as `ApplMotiveIdx`, for commonly used abstract spaces that are mathematically defined implicitly or indirectly, such as the space of all indexes of applied motives, $\{\text{AppliedMotives}\}_{\text{IDs}}$; we will discuss indexes in Sec. B.3.5. Finally, notice that we create classes even for abstract spaces whose elements are functions, such as VectorPaths whose elements are functions from reals to vectors; we will discuss functions in Sec. B.3.7.

Each class defines various methods that support mathematical operators and functions defined over the abstract space. For example, as discussed in Sec. 6.6, the `MCO::Vector` class has a method `rep(Frame)` that supports the mathematical operator *Rep* (Defn. 6.5), returning the representation of a vector in a given coordinate frame.

B.3.2 Immutable Objects

In Sec. 3.5.1 we pointed out that mathematical model entities have no changeable "internal state"—as opposed to program objects, which typically do. However, we would like to closely follow the mathematical model in our implementation. Thus we adopt a restricted style for the math section objects:

- *The state of a math section object may not be changed after the object is constructed.*

This can be supported in object-oriented programming by defining no "modifier" methods—only "accessors"—and by declaring all the member data to be constant/read-only. The constructor method for the class must correctly initialize the contents of the object, since it will not be changed later on; thus the arguments to the constructor must be sufficient to completely specify an instance.[1]

We think of an instance of a math section object as a being a primitive abstract value, rather than a storage area whose contents can be modified.[2] We will use math section objects as if they were primitive data types

[1] If it is too cumbersome to require all necessary construction parameters to be available at once, we can take a more procedural approach: Define a class to have a flag that indicates "under construction," to have modifier methods that may only be used if the flag is set, to have accessor methods that may *not* be used if the flag is set, and to have a special "wrap" method that resets the flag.

[2] Note the contrast with conceptual section objects: each instance of a conceptual object corresponds with a thing in the model—it is a storage location that holds various parameters that describe the configuration of the thing, and that are updated and modified as we manipulate or simulate the thing—and it may not be destroyed during the lifetime of the thing.

like integers or floating-point numbers. If we have two instances that represent the same value, they can be used interchangeably.

In addition to being close to the mathematical model, this approach lends a certain ease of use to the math section: An instance of a math object is always correct, by construction. We manipulate and copy entire instances rather than pointers, so that pointer corruption and data aliasing are not major issues. We write functions or operators that take math object instances as arguments and construct new instances for the result. (To minimize the runtime cost of copying arguments and return values, the arguments can be passed by reference, and the user can specify the destination into which the return value should be constructed; see Sec. 4.8.)

The strict adherence to immutability works well for primitive, fixed-size objects—even rather large ones, such as `MRIG::State` (Sec. 8.5.2). But for compound, variable-size objects such as sets and indexes, it can be impractical as the objects grow large: we often perform operations such as $x \leftarrow x \cup y$ in which we will replace the value of variable `x` with a new value; but constructing a new value from scratch, only to immediately throw away the old value can be prohibitively expensive. But since we know that the old value will no longer be needed, we can provide an "increment" operator $\leftarrow\cup$, so that $x \leftarrow\cup\, y$ efficiently updates variable `x` to have the new value. Thus we relax our immutability requirement, to support arithmetic increment operators for variable-size objects.

B.3.3 ID's

We define a class `Id` to implement the abstract space IDs (Defn. 3.8). The only operations that the class supports are constructing a new unique ID, and testing to see if two instances are equal. Internally, an ID is simply represented as a 32-bit integer code; to create a new ID, a global counter is incremented.

B.3.4 ID sets

We define a class `IdSet` to implement IDsets, the space of sets of ID's (Defn. 3.9). The class is implemented using a standard "container" mechanism, including support for iterating through all elements of a set, and testing to see if a given ID is an element of the set.

B.3.5 Indexes

We provide templates for defining indexes (Defn. 3.10) and indexes with zero (Notn. 3.12). The template for indexes provides:

```
class Indexed<Thing> :
  members:    ids : IdSet     the set of ID's used as labels in the index
  operators:  [i] : Thing     the element labeled by ID i
```

The subscripting operator `x[i]` for an index `x` generates a runtime error if `i` is not in `ids`. The template additionally provides set and arithmetic operators (Notn. 3.15), both normally and in "increment" form as discussed in Sec. B.3.2. Internally, the class is implemented using a standard "container" mechanism that keys on the ID's internal integer code. The template for indexes with zero is the same except that it requires an argument indicating the "0" element for the class that is indexed; that element is returned by `x[i]` if `i` is not in `ids`.

B.3.6 State Spaces

State spaces are implemented by directly using the object-oriented mechanism. Consider the example space Rectangles of Notn. 3.25:

Rectangles
[
$$\begin{aligned} &\mathit{length} \mapsto \Re \\ &\mathit{width} \mapsto \Re \\ &\mathit{area} \mapsto \Re \end{aligned}$$
—
$$\begin{aligned} &\mathit{length} \geq 0 \\ &\mathit{width} \geq 0 \\ &\mathit{area} = \mathit{length} \times \mathit{width} \end{aligned}$$
]

We support each aspect directly as a data member—thus we are representing an element of the state space by the full tuple of aspect values:

```
class Rectangle :
  constructors:  (double length,width)        minimal identifying tuple
                 (double length,area)         minimal identifying tuple
                 (double width,area)          minimal identifying tuple
                 (double length,width,area)   full tuple
  members:       length : double
                 width  : double
                 area   : double
```

All the data members are declared to be constant, as per Sec. B.3.2. As discussed in Sec. 3.9.2, for this space, any pair of the aspect values is sufficient to identify an element; we define three constructors accordingly, each of which computes the third aspect value from the given two. We also define a constructor that accepts a full tuple. All the constructors verify that the arguments are non-negative and that the area property holds; a runtime error is generated if this is not the case.

In general, we don't implement constructors for all possible identifying tuples—merely for those that we find to be useful. But all constructors must be careful to construct an instance that satisfies all the internal properties of the space (or else to generate a runtime error). As discussed in Sec. B.3.2, we want to guarantee that all math object instances are valid, by construction.

We do generally implement state spaces by providing members for all aspect values, as described above, because of its simplicity. However, if memory space is a concern, or if we don't want to spend the time computing all aspect values at construction time, we can use a more compact representation. For example, `Rectangle` could store just the length and width as members, and could provide an accessor method to compute the area.

Nested state spaces (Sec. 3.9.4) are implemented as ordinary data members. Inherited aspects can simply duplicate the data of the nested space, or can be references to the nested space's aspect values.

B.3.7 Paths and Other Functions

The mathematical models often define abstract spaces whose elements are paths (functions from the reals) or more general functions. For example, we have KINEMATIC :: StatePaths (Defn. 7.2), SysVectorFields(Defn. 8.16), and ProtoGens(Defn. 9.6).

Like state spaces, we don't provide any direct support for function spaces, but they are implemented readily using the object oriented mechanism. Consider the space of Rectangles (Sec. B.3.6) as a function of time:

$$\text{RectanglePaths} \equiv \textit{the set of functions } \{\Re \rightarrow \text{Rectangles}\}$$

It is implemented simply as:

```
class RectanglePath :
  methods:  eval(double t) : Rectangle    evaluate the function at time t
```

The class doesn't provide an implementation of the `eval` method. Instead, it is a base class, and we use the object-oriented mechanism to define subclasses that implement various specific functions, based on their constructor arguments. For example, one subclass might define `eval` to return a given constant value. Another might define `eval` to compute a rectangle that oscillates as some explicit function of time.

Most importantly, we can define a subclass that invokes a numerical routine to solve a posed mathematical problem—this is the primary "hook" through which we access the numerics section of the program. For example, Sec. 8.5.3 describes a subclass of `MRIG::SysPath` that invokes a differential equation solver to evaluate the function, and Sec. 9.5.3 describes a subclass of `MRIG::ApplMotiveIdxField` that sets ups and solves a linear system of equations.

For paths into state spaces, we can often compose paths for each aspect operator. For example:

```
class RectanglePath :
  members:  length : doublePath
            width  : doublePath
            area   : doublePath
  methods:  eval(double t) : Rectangle
```

Each aspect path is set to a subclass that implements its `eval` by calling `RectanglePath`'s eval, and returns the proper aspect value of the result.

Every index of functions has an implied function that returns an index, by simply evaluating all the element functions and making an index of the results (Defn. 3.17). Correspondingly, we may have a class `ThingFuncIdx`, being indexes of functions, and another class `ThingIdxFunc`, being functions that return indexes. The latter could support a subclass that is constructed from a specific index of functions instance, and whose `eval` method simply evaluates all the element functions and constructs an index to hold the results.

B.3.8 Discussion

We have used the methods that we have outlined, to implement the mathematical models of the modules in Ch. 6–9. We have found that the mathematical modeling method fits well with an object-oriented implementation. Implementing a mathematical model from a given blackboard specification is not hard—simply a matter of typing in the appropriate class definitions and so forth. There is a great deal of rote in the task, especially in the minutiae of the object-oriented class specifications. We can imagine embedded or special-purpose mathematical modeling languages that more directly support our mathematical modeling constructs. See discussion in Secs. 13.2.3, 13.4.

B.4 The Numerics Section

The numerics section of a program, as discussed in Ch. 4, is responsible for computing numerical solutions to the posed problems. There is a large body of knowledge and software for numerical computation, that we can take advantage of for physically-based modeling; see, e.g., [Press et al.86], [Ralston,Rabinowitz78], [Golub,Van Loan85], [NAG].

We can assume that we have a rich library of numerical subroutines readily available to us. So what more needs to be done? The answer is that we need to join the "back end" numerical subroutines with the higher levels of abstraction in a modeling program. This section discusses the structure/mechanism that we build on top of an existing numerical library.

B.4.1 Overview of the "Structured Numerics" Library

The numerics section bridges between the numerical subroutines and the mechanisms that are used by the math section and M-N interface of a program (Sec. 4.2.4), in order to overcome some basic design differences (Fig. B.1). In bridging between them, we perform changes of representation as discussed in Sec. 4.6.

Thus we build a *structured numerics* library: A modular, object-oriented collection of interfaces to numerical techniques. The library is designed to provide access to numerical techniques in a form convenient for the high-level abstraction and goals, rather than driven by the low-level design and implementation goals.

Numerical subroutine design	*vs.*	**Program requirements**
Data formats:		
• *Arrays of numbers.* Numerical subroutines most commonly operate on arrays of numbers.		• *Conceptually separate objects.* We may group together separate objects for the purpose of solving a particular problem. E.g., an index of dynamic body states is grouped into an array for Eqn. 8.35
Access methods:		
• *Procedural, arcane.* Special-purpose routines or sequences must often be followed to provide data or access solutions. E.g., ODE solvers typically require repeated calls to a subroutine to advance the solution by steps; a different routine can interpolate within the most recent step.		• *Uniform interface.* For any given problem, we would like to specify the problem conveniently, and access the solution freely. E.g., we want to evaluate a solution function $Y(t)$ for arbitrary values of time t, as discussed in Sec. 4.7.
Choice of algorithm:		
• *One algorithm per subroutine.* Each subroutine and algorithm is typically suitable to to a specific regime of the problem. E.g., a routine may solve a sparse linear least squares problem using a Lanczos algorithm.		• *One interface for various algorithms.* A problem may need to be solved in different regimes. E.g., a linear system of equations may be sparse or dense, ill- or well-conditioned.
Programming paradigm:		
• *Subroutine paradigm.* Numerical routines often assume or are geared towards use by a program that solves a single problem; they are often not re-entrant.[4]		• *Object-oriented paradigm.* Many separate objects can coexist, each of which solves a particular problem; solution evaluations may be intermingled.

Figure B.1: This table lists various differences in design between numerical "back end" subroutines and higher levels of the program. The design of numerical routines is low-level, driven by algorithm and implementation issues, whereas for the higher level of the program we have explicitly tried to be driven by abstraction and high-level goals. The *structured numerics* library is designed to bridge between the low and high levels. □

Fig. B.2 illustrates the hierarchy of modules in our prototype structured numerics library. As per Sec. 4.1, we provide a collection of tools at various levels of representation, so that for any particular application, the programmer may choose the most convenient form. We are not attempting to provide a single interface or paradigm to span all problems.[3]

Despite our highfalutin talk of "abstraction," "changes of representation," and so forth, the structured numerics library is very mundane: it deals with programming details such as data format and subroutine interfaces, and the bulk of its implementation is involved with housekeeping issues such as storage allocation, cacheing intermediate results, and so forth. Note in particular that little or no numerical computation takes places within our modules—ultimately, they are merely interfaces to existing numerical subroutine libraries.

The next several sections discuss various modules in our prototype library (those that are used by the models in Ch. 6–9). We will simply outline the structure and interface to the modules, to "give a feel" for the library—complete implementation specifications are beyond the scope of this book. For the underlying solution techniques, we refer readers to [Press et al.86].

[3] In general, numerical computation today is a job for "hobbyists": to most effectively get meaningful results, one must have an understanding of issues such as accuracy and tolerance, a feel for the character of the particular problem being solved, a familiarity with the strengths and weaknesses of various known solution techniques, and so forth. This limits our ability to provide "universal" high-level interfaces that allow the user to ignore implementation and solution details. Still, well-packaged interfaces can be designed that apply to a reasonably broad range of problems. And, by designing our library as a collection of tools at various levels, we allow users to "reach down" and access lower-level routines as necessary without breaking the spirit or function of the library.

[4] A *re-entrant* technique is one that can be used simultaneously or in an interleaved manner by different applications. For example, we might want to advance one ODE solution, then solve a different ODE, then continue advancing the original. FORTRAN numerical libraries often store information about a problem in shared "common blocks," so that starting a second problem solution precludes the ability to continue the first.

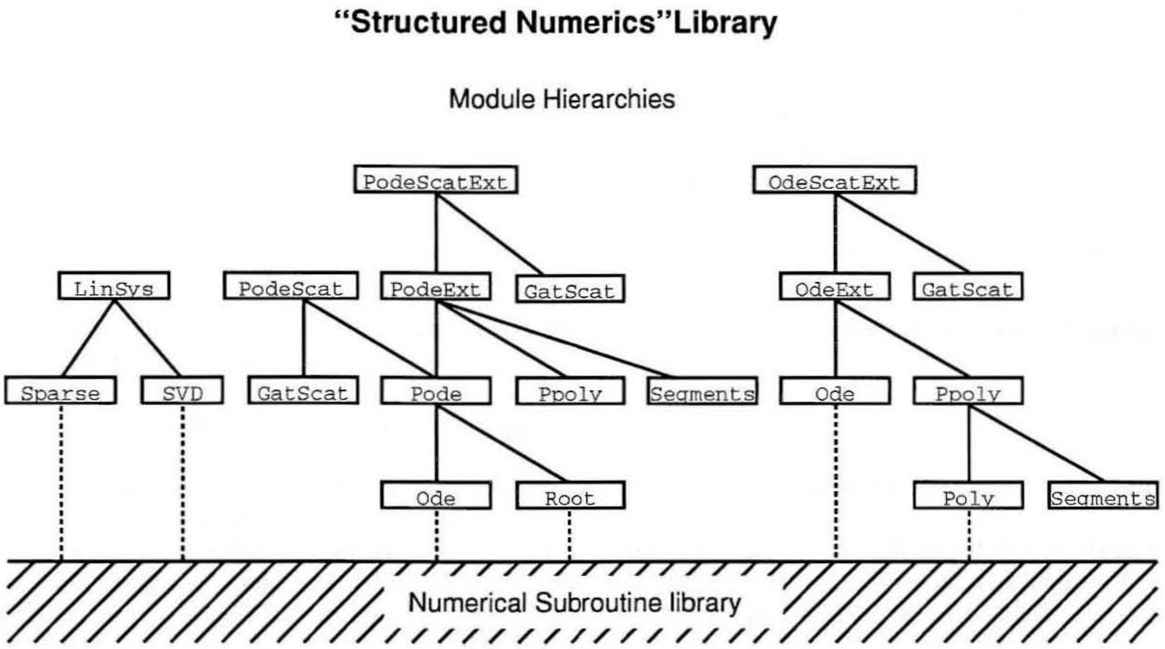


Figure B.2: Schematic diagram of some module hierarchies in our prototype "structured numerics" library, built on top of a standard numerical subroutine library. The library is intended to be used as a toolbox, with modules at various levels of representation available for different applications. For example, `Ode`, `OdeExt`, and `OdeScatExt` provide different interfaces to ordinary differential equation initial-value problems; the "best" one to use depends on the application. Secs. B.4.2–B.4.11 describe the modules. □

B.4.2 `GatScat`: Array Gather/Scatter

This is a utility class, that does the housekeeping to map between a collection of separate elements, each having local data, and a single linear array of data. The following methods are supported:

`add(...)`	Add an element to the collection—an element is be specified via subroutines to read and write its local data, or simply by the memory location of the local data. Each element can be a different size.
`gather(array)`	Gather each element's local data into an array.
`scatter(array)`	Scatter an array of data into each element's local data.

After adding each element once, the gather and scatter methods can be used repeatedly with different data sets. Other methods map between elements and their array indexes, return the total size of the array, and so forth. `GatScat` is used by `OdeScatExt` (Sec. B.4.6) and `PodeScatExt` (Sec. B.4.9).

The class `GatScat2` is similar, but maps between separate elements and a 2-d block matrix of data. Each element specifies its row and column in the matrix; not all entries in the matrix need to be specified. The `gather` and `scatter` methods support both sparse and full arrays. Additionally, `GatScat2` can partition the matrix into independent blocks, based on which entries have elements associated with them. `GatScat2` is used by the "fancy forces" model, `NFRC::LsolveProto` (Sec. 9.5.3), and is compatible with `LinSys` (Sec. B.4.3).

B.4.3 `LinSys`: Linear Systems of Equations

This is a class that solves matrix equations of the form $M\,X + B = 0$, computing values of matrix X given values of M and B. If the matrix M is singular or ill-conditioned, the least-squares solution is computed.

For our applications, M is commonly sparse, though not always, and M may sometimes be ill-conditioned or singular. We can't always tell in advance—for example, a system may generally be well-conditioned, but may occasionally pass through an ill-conditioned "zone" during the course of a simulation.

LinSys is a "black box" that lets the user specify the system using ordinary 2-d array data format, and that runs as fast as possible in the general case, yet is guaranteed to work (albeit more slowly) in the special cases. Its features are:

- It partitions M into independent blocks, and solves each one separately.
- Within each block, if the matrix is sparse it constructs the appropriate sparse representation, and uses Sparse (an interface to a Lanczos method) to solve the system. Sparse may report failure if the matrix is too ill-conditioned.
- If the matrix is not sparse, or if the sparse solution fails, singular-value decomposition is used instead, which is slow but has essentially guaranteed success.

LinSys is compatible with GatScat2 (Sec. B.4.2), which can be used to pre-compute the partition and sparsity of the matrix.

B.4.4 Ode: Ordinary Differential Equation

This is a class that solves ordinary differential equation (ODE) initial-value problems: it computes $y(t)$ where $\frac{d}{dt}y = f(y,t)$, given a value $y_0 = y(t_0)$.

Ode provides an object-oriented interface to various techniques in the numerical library; that is, an abstract base class is defined, and assorted derived classes implement Runge-Kutta, Adams, and other techniques. Ode performs no changes of representation, supporting array data format and sequential access methods:

step()	Advance the solution to some larger value of t.
interp(t)	Interpolate the solution within the last step.
solve(t)	Advance until and return the solution at time t.

Unlike LinSys (Sec. B.4.3), our current implementation of Ode does not choose solution techniques automatically.

B.4.5 OdeExt: Extruded ODE

This is a class that solves ordinary differential equation (ODE) initial-value problems, as does Ode (Sec. B.4.4). Unlike Ode, however, this class doesn't require sequential access to the solution $y(t)$. Instead, the solution is "extruded," allowing the user to evaluate $y(t)$ for any values of t, in any order. Its primary interface method is thus:

eval(t)	Evaluate the solution for any time $t > t_0$.

OdeExt uses array data format, the same as Ode.

Internally, OdeExt works by maintaining the solution as a piecewise-polynomial function, using Ppoly (Sec. B.4.10). If evaluation is requested at a time beyond the bounds of the stored solution, Ode is used to advance the solution as needed; the result of each step along the way is added the the stored solution.[5]

For many problems, we only need to evaluate $y(t)$ for monotonically or almost-monotonically increasing values of t. Thus, to save memory, OdeExt can maintain a "sliding window" that moves forward automatically as the solution is advanced—solution values prior to the start of the window are thrown away.

B.4.6 OdeScatExt: Scattered, Extruded ODE

This class combines the extruded ODE-solving functionality of OdeExt (Sec. B.4.5) with the array gather/scatter functionality of GatScat (Sec. B.4.2). This is used, for example, by the rigid-body dynamics model routine NRIG::SolveForward (Sec. 8.5.3) to construct an object that computes an index of rigid body states, for any value of time t.

[5] Most ODE-solving techniques internally compute a polynomial to describe the value of the function in the most recent step. Often, the polynomial itself is not made directly available—but its order k is known, thus we can accurately resample the solution to construct an explicit polynomial function.

B.4.7 `Pode`: Piecewise-Continuous ODE

This class is analogous to `Ode` (Sec. B.4.4), but supports piecewise-continuous ODE's, using the solution technique discussed in Appendix C.

B.4.8 `PodeExt`: Extruded PODE

This class is analogous to `OdeExt` (Sec. B.4.5), but supports piecewise-continuous ODE's. The solution $y(t)$ is a segmented function as described in Sec. 3.10, whose size may change at the discontinuous events. `PodeExt` allows the transition function that starts a segment (see Appendix C) to specify arbitrary data for the solver to associate with that segment; The `eval(t)` method returns the associated data for the segment that spans time t, and the size of the solution within that segment, as well as the array containing the solution at time t.

The `Segments` utility (Sec. B.4.11) is used to keep track of the separate segments of the solution; within each segment the solution is accumulated as a piecewise-polynomial function, in the same manner as `OdeExt`.

B.4.9 `PodeScatExt`: Scattered, Extruded PODE

Analogous to `OdeScatExt` (Sec. B.4.6), this class combines the functionality of `PodeExt` (Sec. B.4.8) with the array gather/scatter functionality of `GatScat` (Sec. B.4.2). It is used by the "tennis ball cannon" model (Sec. 11.5.2).

In addition to gathering and scattering the solution values, `PodeScatExt` allows the user to specify separate event functions (see Appendix C), and will gather their values into an array for `PodeExt`. When an event is detected, i.e., a root of an event functions is discovered, the user's transition function is passed a code indicating which of the events was found (see Sec. 11.5.2).

We also provide a class `PodeScat`, that supports scattered data, but only sequential access to the solution. Functionally it is equivalent to using `PodeScatExt` with a "sliding window" of zero width, so that no prior values are saved—but the interface is simpler.

B.4.10 `Ppoly`: Piecewise Polynomial Functions

This class supports piecewise polynomial functions. It uses the `Poly` class (which implements Chebychev polynomials) for each piece, and keeps track of them via the `Segments` utility (`segments`). It supports the following methods:

`addl(...)`	Add a polynomial segment on the left end.
`addr(...)`	Add a polynomial segment on the right end.
`eval(t)`	Evaluate the solution for any time t within the endpoints of the stored function. This method uses `Segments` to determine which polynomial spans time t, then evaluates that polynomial.

The class also provides miscellaneous support, such as methods to "trim" the function throwing away polynomials outside a given range.

`Ppoly` is used by `OdeExt` (Sec. B.4.5) and `PodeExt` (Sec. B.4.8).

B.4.11 `Segments`: Partition of the Real Number Line

This is a utility class that does the housekeeping to maintain a collection of adjacent segments of the real number line, maintaining arbitrary data for each segment. Starting with an initial endpoint for the structure, it supports the following methods:

`addl(t, data)`	Add a segment on the left end, and store the provided data.
`addr(t, data)`	Add a segment on the right end, and store the provided data.
`data(t)`	Look up and return the data of the segment containing t.

The class also provides miscellaneous support, such as methods to iterate through all the segments.
`Segments` is used by `PodeExt` (Sec. B.4.8) and `Ppoly` (Sec. B.4.10).

B.4.12 Discussion

We have presented the "structured numerics" library as part of the overall program framework for physically based modeling, that we described in Ch. 4. However, it is not specific to that framework.

Our prototype structured numerics library has been used (and extended) by colleagues for applications other than physically-based modeling: a modular, object-oriented numerics library that supports high-level representations is useful even for projects that don't include the rest of the "structured modeling" mechanism.

Currently, numerical subroutine libraries are as we have described—low-level, array-oriented, and so forth—typically written in FORTRAN. However, with the advent and popularity of object-oriented programming, it is likely that in the near future, the bottom-level numerical routines available from vendors will be object-oriented and modular, and will support high-level representations such as we described. If so, our comment at the start of Sec. B.4:

> "We can assume that we have a rich library of numerical subroutines readily available... So what more needs to be done?"

will need to be updated:

> "We can assume that we have a rich library of object-oriented numerical modules readily available...Nothing more needs to be done."

Appendix C

Solving Piecewise-Continuous ODE's

This appendix describes an algorithm to solve initial-value problems for Piecewise-Continuous Ordinary Differential Equations (PODE's). The solution to a PODE is a "segmented function" as discussed in Sec. 3.10; we will use the mechanism developed there to specify a PODE to be solved.

The solution technique makes use of common ODE-solving and root-finding techniques as "black boxes," allowing easy tailoring of the solution mechanism to specific problems, by appropriate choice of "boxes" and their parameters.

C.1 Formalism for Piecewise-Continuous ODE's (PODE's)

C.1.1 Definition

We define a *piecewise continuous ordinary differential equation* (PODE): an ordinary differential equation [1]

$$\text{(C.1)} \qquad \begin{gathered} \textit{For } y\colon \Re \to \Re^n \textit{ and } \mathcal{F}\colon \Re^n \times \Re \to \Re^n, \\ \tfrac{d}{dt} y = \mathcal{F}(y,t) \end{gathered}$$

Eqn. C.1: Canonical ordinary differential equation. For a piecewise-continuous ODE, the function $\mathcal{F}(y(t),t)$ is continuous and has a bounded derivative, except at isolated values of t. The solution function $y(t)$ to a PODE is C^1-continuous except at those values of t. □

The solution to a PODE is a *segmented function,* as per Sec. 3.10, i.e., is piecewise continuous. Note that in general we are interested in solving for functions

$$y\colon t \to \bigcup_{i=1,2,\ldots} \Re^i,$$

but for the present discussion we will assume that the solution stays in $\Re^n$ for constant n; Sec. C.2 will re-introduce arbitrary dimensionality in the solution.

[1] The PODE method described in this chapter will work in exactly the same manner with piecewise-continuous differential-algebraic equations (DAE's), e.g., of the form $\mathcal{F}(y, dy/dt, t) = 0$ [Petzold82]. However, because of their greater familiarity, we present the discussion in terms of ODE's.

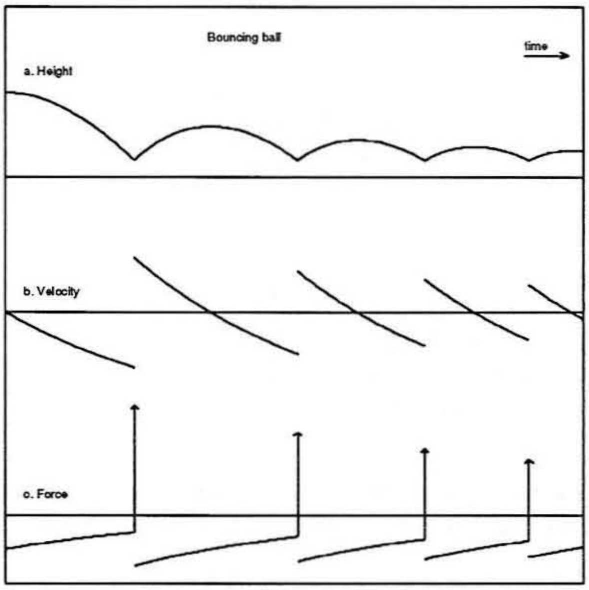


Figure C.1: Example of a PODE: A bouncing ball, with damping. (a) The height of the ball. (b) The velocity of the ball. (c) The force on the ball; at each bounce, an infinite force is required to instantaneously change the velocity. □

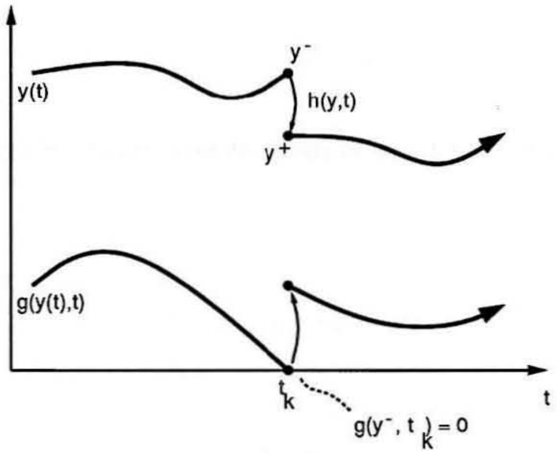


Figure C.2: A PODE. $y(t)$ is the continuous solution of $\frac{d}{dt}y = f(y,t)$ [not shown], until $g(y(t),t) = 0$. Then $y(t)$ is changed discontinuously by adding $h(y,t)$, and the solution progresses continuously again. □

We will use a "functional characterization" to describe a PODE, as discussed in Sec. 3.10.3. That is, we define three functions:

(C.2)	*"body function"*	$f\colon \Re^n \times \Re \rightarrow \Re^n$
	"event function"	$g\colon \Re^n \times \Re \rightarrow \Re$
	"transition function"	$h\colon \Re^n \times \Re \rightarrow \Re^n$

Eqn. C.2: Functional characterization of a PODE. We describe a PODE by three functions, in the manner of Defn. 3.39. Note however that here, we define the functions numerically, rather than in predicate form. The functions are discussed in the text below. □

$f(y,t)$: $f(y,t) \equiv \mathcal{F}(y,t)$ wherever $\mathcal{F}(y,t)$ is continuous and has a bounded derivative. $f(y,t)$ need not be defined where $\mathcal{F}(y,t)$ is discontinuous or singular.

$g(y,t)$: A "event" function, that locates discontinuities by defining legal states: as long as the solution $y(t)$ is in a legal state, $f(y,t)$ must be continuous and well-behaved; if the solution attempts to enter an illegal state, a discontinuity occurs at the boundary. Thus we define:

$$g(y,t)\begin{cases} > 0: & \textit{y is a legal state at t} \\ < 0: & \textit{y is an illegal state at t} \\ = 0: & \textit{Boundary state: } \mathcal{F}(y,t) \textit{ may be discontinuous or singular; } y(t) \textit{ may be discontinuous} \end{cases} \tag{C.3}$$

$g(y,t) = 0$ occurs only at the discontinuities or singularities in $\mathcal{F}(y,t)$.[2] The solution $y(t)$ always has the property that $g(y(t),t) \geq 0$. $g(y,t)$ itself must be continuous where it is non-0.[3,4]

$h(y,t)$: A "transition" function, that bridges discontinuities by describing the change in the solution $y(t)$: at each discontinuity, $y(t)$ is offset by adding $h(y,t)$ (see Fig. C.2). That is, given a t_k such that $y^- = y(t_k)$ and $g(y^-, t_k) = 0$, we have

$$y^+ = y^- + h(y^-, t_k) \tag{C.4}$$

In effect $h(y,t)$ integrates $f(y,t)$ across the discontinuity; thus if $f(y,t)$ contains a delta-function, then $h(y,t)$ is non-0. i.e.,

$$\begin{aligned} &\textit{if} \\ &\mathcal{F}(y,t) = \mathcal{H}(y,t)\delta(t-t_k) + \left\{ \begin{array}{c} \textit{terms w/o} \\ \delta\textit{-functions} \end{array} \right\}, \\ &\textit{then} \\ &h(y,t) = \int_{t_k^-}^{t_k^+} \mathcal{F}(y,t)dt \\ &\qquad\quad = \mathcal{H}(y(t_k), t_k). \end{aligned} \tag{C.5}$$

C.1.2 Continuous ODE Segments

As an alternative to Eqn. C.2, a PODE can be conveniently specified by a sequence of continuous ODE segments (see Fig. C.3):

[2]For some applications, there may be several conditions, any of which might signal a discontinuity. Thus $g(y,t)$ may be a vector, and the comparisons $g(y,t) > 0$, < 0, and $= 0$ refer to its minimum component. The implementation can support vectors; this allows us to support multiple independent events, as discussed in Sec. B.4.9

[3]That is, we may have: $g(y(t),t)$ continuous for $t \neq t_k$, with $\lim_{t \to t_k^-} g(y(t),t) = 0$ and $\lim_{t \to t_k^+} g(y(t),t) \neq 0$, as in Fig. C.2.

[4]We could also just have $g \neq 0$ is the continuous state, and $g = 0$ signals a discontinuity. But having legal/illegal states is convenient for debugging, and is typically not a hardship for the designer of the event function (just negate it if you want it to be legal but it's negative).

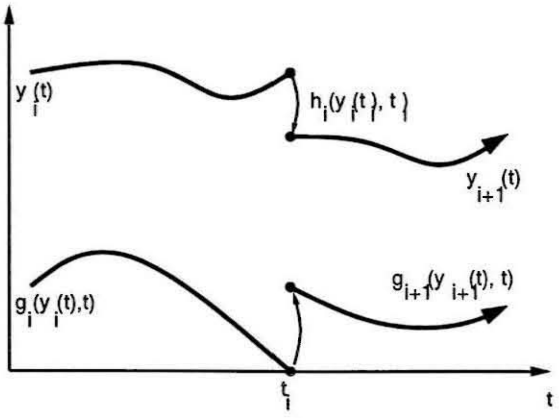


Figure C.3: A PODE considered as a sequence of continuous ODE segments. $y_i(t)$ is the continuous solution of the ith segment $\frac{d}{dt}y_i = f_i(y,t)$, until $g_i(y_i(t),t) = 0$ at $t = t_i$. $h_i(y_i(t_i),t_i)$ yields the initial conditions for segment $i+1$. □

(C.6)
$$\begin{array}{l} \frac{d}{dt}y_1 = f_1(y_1,t) \textit{ while } g_1(y_1(t),t) > 0 \\ \frac{d}{dt}y_2 = f_2(y_2,t) \textit{ while } g_2(y_2(t),t) > 0 \\ \vdots \\ \frac{d}{dt}y_i = f_i(y_i,t) \textit{ while } g_i(y_i(t),t) > 0 \\ \vdots \end{array}$$

where each of the f_i's and g_i's are continuous. t_i is the time of the end of the ith segment; that is, we define t_i to be the earliest $t \geq t_{i-1}$ such that:

(C.7)
$$g_i(y_i(t_i),t_i) = 0$$

Notice that we may have $t_i = t_{i-1}$; in such a case, we refer to segment i as having *zero-length*.

At t_i, we switch from the ith to the $(i+1)$th ODE. The initial conditions for the new ODE are given by:

(C.8)
$$y_{i+1}(t_i) = y_i(t_i) + h_i(y_i(t_i),t_i)$$

The functions of Eqn. C.2 are produced by:

(C.9)
$$\begin{array}{l} f(y,t)=f_i(y,t), \textit{ where } t_{i-1} < t < t_i \\ g(y,t)=g_i(y,t), \textit{ where } t_{i-1} < t \leq t_i \\ h(y,t)=h_i(y,t), \textit{ where } t = t_i, \end{array}$$

and the solution is[5]

(C.10)
$$y(t) = y_i(y,t), \textit{ where } t_{i-1} \leq t \leq t_i$$

[5]Notice that at each t_i, the solution $y(t_i)$ has two values: the value before and after the discontinuity. The solution technique described in Sec. C.2 finds and returns both these values.

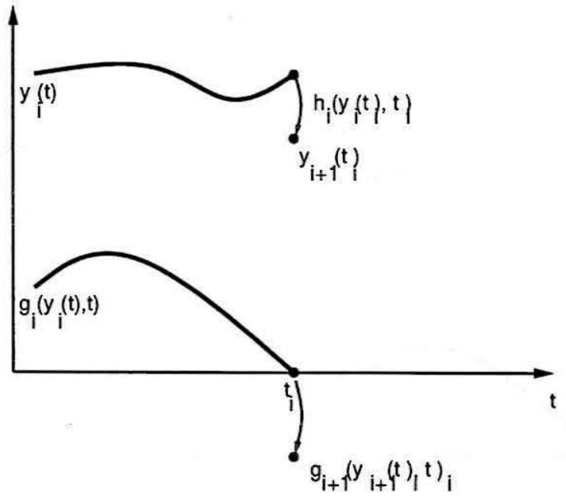


Figure C.4: Unsolvable PODE: The initial conditions of segment $i+1$ are such that $g_{i+1}(y,t)$ is negative at the start of the segment. □

C.1.3 Errors

There are two errors that are endemic to PODE's defined as per the previous section:

- If $g_{i+1}(y_{i+1}(t_i), t_i) < 0$, the initial conditions for segment $i+1$ specify an illegal state; the PODE can not be solved (see Fig. C.4)
- If $t_i = t_{i-1}$ for all i after some i_0, i.e. if all segments after segment i_0 have zero length, the PODE cannot progress beyond t_{i_0}. Such an occurrence is often an indication of an an attempt to simulate a continuous process by repeated application of the jump function $h(y,t)$

The solution technique described in Sec. C.2 can detect both these errors.

C.2 Solving a PODE

Note: For notational convenience, we define the following three functions:

$$\begin{aligned} F_i(t) &\equiv f_i(y_i(t), t) \\ G_i(t) &\equiv g_i(y_i(t), t) \\ H_i(t) &\equiv h_i(y_i(t), t) \end{aligned} \tag{C.11}$$

Overview

A PODE described as a series of continuous ODE's (as per Eqn. C.6) can be solved easily: Each segment can be solved in turn using a standard ODE solver;[6] the end of a segment is marked by a root of the composite function $G_i(t)$.

Thus the algorithm for solving a PODE, illustrated in Fig. C.5, is:

1. Solve segment i. Use a standard ODE solver to determine the function $y_i(t)$.

2. Monitor $G_i(t)$. Watch to see if the value crosses below zero.

[6] Each segment is an ODE initial-value problem, and can be solved using previously mentioned techniques.

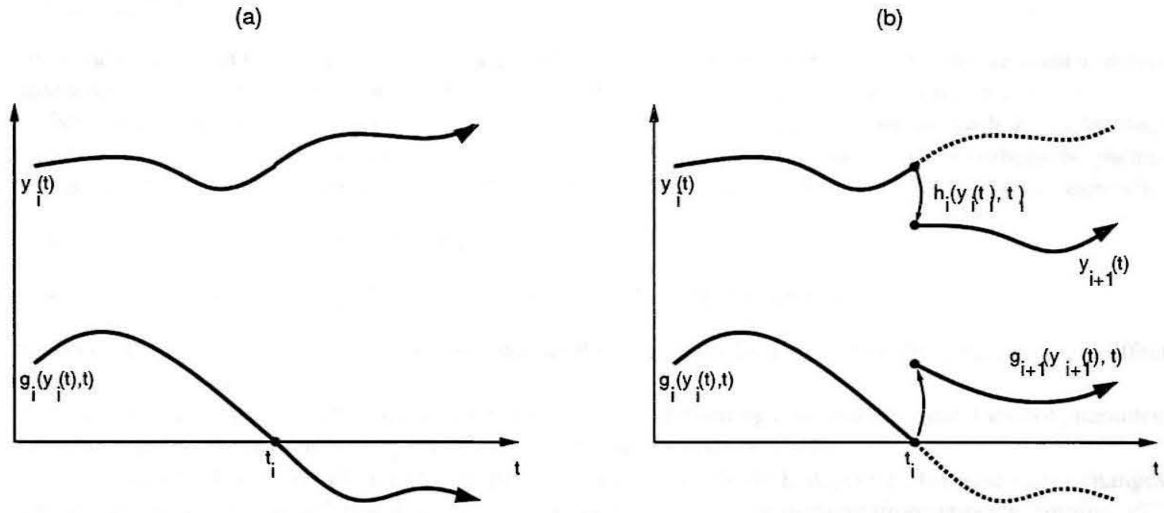


Figure C.5: Solving a PODE: (a) $y_i(t)$ is determined using a standard ODE-solver. An ODE solution step might extend past the end of the segment; $g_i(y_i(t),t)$ is sampled to see if it goes below zero; if it does the root t_i is found, by using a standard root-finder with the interpolation of $y_i(t)$ over the last ODE step. (b) Once the end of segment i has been determined, at $t = t_i$, $h_i(y_i(t_i),t_i)$ is added to $y_i(t_i)$ to yield the initial-conditions for segment $i+1$. The solution of segment i for $t > t_i$ is discarded. □

3. **Find root t_i.** If a zero-crossing is encountered, use a standard root-finder to find the time t_i, as per Eqn. C.7. This marks the end of segment i.

4. **Switch to segment $i+1$.** $H_i(t_i)$ is added to $y_i(t_i)$, to yield the initial conditions of segment $i+1$ (Eqn. C.8).

Note that the algorithm works by solving $y_i(t)$ some distance past t_i, and by observing a negative value of $G_i(t)$. Thus there are extra restrictions on the f_i's and g_i's:

- $f_i(y,t)$ and $g_i(y,t)$ must extend continuously some amount past t_i
- The algorithm finds zero-crossings, rather than zeros, of $G_i(t)$; the composite function must be negative for some $t > t_i$

To minimize the amount that $y_i(t)$ is solved past the end of its segment, we alternate (1) and (2) above. That is, we have the ODE solver solve $y_i(t)$ for one step; then we verify that $G_i(t)$ remains positive; then we take another ODE step, and repeat.

We discuss the four parts of the algorithm in greater detail in the following sections.

C.2.1 Solving Continuous Segment i

The standard ODE-solving technique may be used as a "black-box" to produce the continuous function $y_i(t)$ that is the solution to $\frac{d}{dt}y_i = f_i(y_i,t)$. The initial conditions are determined from the final state of the previous segment (for the first segment, the initial conditions are provided by the user). We assume that the ODE-solver takes steps, and reports back a local solution function after each step.[7]

Note that, at each step, we are interested in a continuous local solution function $y_i(t)$, rather than just the value of y_i at the end of the step. Thus our choice of ODE-solver is restricted to techniques that support interpolation of the solution in the last step; Adams-based methods work well. Extrapolation-based techniques (e.g., Bulirsch-Stoer) are not well-suited to our PODE method. (See, e.g., [Ralston,Rabinowitz78] or [Press et al.86] for discussion of Adams and extrapolation methods.)

[7] The ODE-solver is free to use whatever step-size it determines to be appropriate; the step-size does not have to be constant.

C.2.2 Sampling $G_i(t)$

Given the result of an ODE step, i.e., given a continuous function $y_i(t)$ over some interval, we want to determine whether $G_i(t)$ is always positive. We attempt to do so by sampling $G_i(t)$ at several values of t.

We can't guarantee to find all zero-crossings of $G_i(t)$ without extra information, such as a Lipschitz bound on $G_i(t)$.[8] Rather than require the user to provide explicit Lipschitz information, or perhaps the partial derivatives of $g_i(y, t)$ with respect to y and t, the implementation described here takes the following approach:

- Sample at the end of every ODE step
- Sample at least every τ_g, where τ_g is a user-adjustable sampling interval.

To allow adaptive control over the sampling interval, the user may adjust τ_g "on the fly", e.g., as a side-effect of evaluating $g_i(y, t)$.

Again, we emphasize that this approach is chosen for programming convenience, and does not guarantee correctness unless the user adjusts τ_g based on the Lipschitz bounds of $G_i(t)$.

For some PODE's, it is sufficient to sample $G_i(t)$ only at each ODE step (e.g., because $G_i(t)$ changes slowly compared to $y_i(t)$, or because $G_i(t) < 0$ for all $t > t_i$). The implementation supports "turning off" the minimum sampling interval.

C.2.3 Finding t_i, the Root of $G_i(t)$

If a sample of $G_i(t)$ is negative, we have detected a zero-crossing. The previous (positive) sample and the new (negative) sample bracket a root of the composite function $G_i(t)$. A standard root-finder can be used to determine the root value t_i at which $G_i(t_i) = 0$. (Since $G_i(t)$ is continuous, and since we start with a bracketing interval, techniques such as Regula Falsi are guaranteed to find the root. See, e.g., [Press et al.86].)

Because $G_i(t)$ is sampled at the end of every ODE step, we know that the root bracket lies entirely within the current step. Thus the ODE-solver's local solution function $y_i(t)$ can be used; no new ODE-solving steps need to be taken during the root-finding process.

C.2.4 Switching to Segment $i + 1$

Given that we have found a discontinuity at $t = t_i$, the following occurs:

- We compute the initial conditions for the next segment: $y_{i+1}(t_i) = y_i(t_i) + H_i(t_i)$
- i is (implicitly) incremented; i.e., the user switches to the next set of functions, f_{i+1}, g_{i+1}, and h_{i+1}.
- We reset the ODE solver, to have it discard history and start integrating the new segment from the new initial conditions.

$G_{i+1}(t_i)$ should be evaluated, to ensure that the initial conditions of the new segment describe a legal state. If $G_{i+1}(t_i) < 0$, the PODE is unsolvable (see Fig. C.4).

C.2.5 Zero-Length Segments

A zero-length segment occurs if two conditions hold (see Fig. C.6):

- $G_i(t_{i-1}) = 0$; i.e., the initial conditions of the segment are on a boundary of the legal state, and
- $G_i(t) < 0$ for $t > t_{i-1}$; i,.e. the solution immediately attempts to enter an illegal state.

No special care need be taken to detect zero-length segments; we can merely check if $t_i = t_{i-1}$.

[8] A Lipschitz bound is a value L such that $|G_i(t_2) - G_i(t_1)| \leq L|t_2 - t_1|$ for all t_1, t_2 in a region of interest. See [Kalra,Barr89].

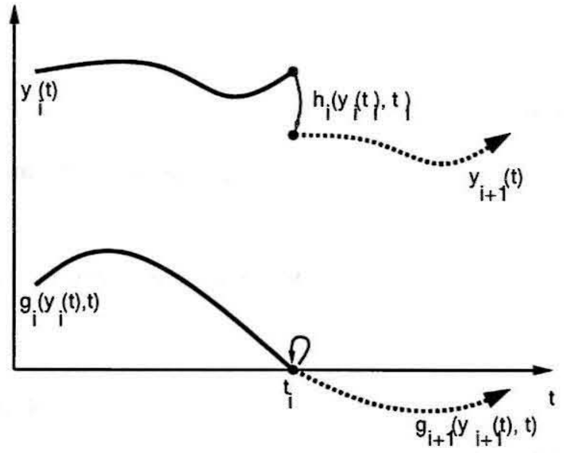


Figure C.6: Zero-length segments: $g_{i+1}(y, t) = 0$ at the start of segment $i + 1$ at $t = t_i$, and $g_{i+1}(y_{i+1}(t), t) < 0$ for $t > t_i$. Thus the end of the segment is the same as the start: $t_{i+1} = t_i$. □

C.2.6 Numerical Issues

There are several numerical issues that arise in the implementation of the PODE solution algorithm:

- As discussed earlier, without Lipschitz bounds on $g(y_i(t), t)$, it is possible to sample too sparsely, and miss zero-crossings.

- The test of $g > 0$ should be tolerant to within some bounds; proper error-analysis should be done to determine the tolerance. A simplifying approach is to put the burden on the user, in the calculation of $g(y, t)$, to add the appropriate absolute tolerance terms, so that comparison against zero is valid.

- The root-finder typically doesn't find the exact 0, but instead brackets the root with a t^{pos} and a t^{neg}, such that $G_i(t^{\text{pos}}) > 0$ and $G_i(t^{\text{neg}}) < 0$, where t^{pos} and t^{neg} differ by some small tolerance. A handy trick is to choose t^{pos} as the root, thus ensuring that $G_i(t)$ is never negative.

- If $G_i(t) = 0$ over a finite region before becoming negative, the root-finder is likely to choose an arbitrary point within the region as the root. If $G_i(t) = 0$ and then becomes positive, no zero-crossing will be detected, hence no root will be looked for.

- As discussed in Sec. C.2.3, the initial root bracket lies within a single ODE step. The positive g-value at the left side of the bracket, however, may have been determined by the value of $y_i(t)$ at the end of the previous step. Because of numerical imprecision, interpolating the current step back to the end of the previous step yields a different value for $y_i(t)$; thus the composite function $G_i(t)$ may have a discontinuity at the left endpoint of the bracket. Occasionally, the discontinuity is such that we have a positive value at the left endpoint, and negative values in the rest of the region; in such a case, we take the left endpoint to be the root.

- In adaptively adjusting τ_g, the user must avoid continually decreasing τ_g in a converging series, or the solver will not be able to make any forward progress. (This might happen, e.g., if at each sample of $g(t)$, the user estimates that the next discontinuity will occur at some t_0, and conservatively sets $\tau_g = \frac{1}{2}(t_0 - t)$).

C.3 Computational Costs

The PODE algorithm doesn't directly do any numerical computation; rather it serves mainly to control the execution of the underlying ODE-solver and root-finder. The computational cost thus depends heavily on the choice of those routines.

We break down the costs into three major parts:

- "Startup overhead" for the ODE-solver, at each discontinuity.
- Cost of solving and testing the continuous segments.
- Cost of determining t_i for each discontinuity.

C.3.1 Startup Overhead

The ODE "startup overhead," if any, at a discontinuity depends on the solution method. For example, adaptive step-size methods may start with very small steps; multi-step methods may use slower methods to generate initial samples; and implicit methods may need to numerically determine the initial Jacobian matrix.

For equations with a large number of discontinuities, the startup overhead may be a significant factor to be considered when choosing the ODE-solver.

C.3.2 Continuous Segments

To solve each continuous ODE-segment, the only cost added by the PODE method over the underlying ODE-solver is the cost of sampling $g_i(y_i(t), t)$ to ensure that the end of the segment hasn't been reached.

For globally continuous equations, the PODE method may be used by providing a constant positive function, e.g. $g(y, t) = 1$, with τ_g large. The cost of using PODE over using the underlying ODE-solver is then minimal. (But, of course, for this case PODE provides no advantage over using the underlying ODE-solver directly.)

C.3.3 Determining t_i

It is important to note that once a discontinuity is detected, no ODE-solving is performed to determine t_i. Rather, $y_i(t)$ is interpolated in the last ODE step.

Thus, the cost of determining t_i is the cost of doing the root-finding. This depends on the root-finding technique, which will require evaluation of $g_i(y_i(t), t)$ some number of times.

Each evaluation of the composite function incurs the cost of doing the interpolation to evaluate $y_i(t)$; this depends on the ODE-solver, but is typically a relatively fast operation.

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