Carleman Inequalities with Convex Weights

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Abstract

In this thesis we show that if $n \geq 2$, and ϕ is a convex function on the bounded convex domain Ω, then there is a constant $A = A(n, p, q, |\Omega|)$ such that

$$\|e^{\phi} f\|_{L^q(\Omega)} \leq A \|e^{\phi} \Delta f\|_{L^p(\Omega)}$$

holds for all $f \in C_0^{\infty}(\Omega)$, and for the following values of p and q: $p = n/2$ and $q < 2n/(n-3)$ when $n \geq 3$, and $p > 1$ and $q < \infty$ when $n = 2$.

For the one parameter family of weights $\{e^{t\phi}\}_{t \geq 1}$, where ϕ is essentially uniformly convex on a bounded domain Ω, we prove an $L^p(\Omega) \longrightarrow L^q(\Omega)$ inequality for $1/p - 1/q \leq 2/n$ and $2n/(n+3) < p \leq q < 2n/(n-3)$, $n \geq 3$, ($1 < p \leq q < \infty$ for $n = 2$).

For the family of radial weights $e^{|x|^{\rho}}$, $1 < \rho < \infty$, we obtain an $L^p(\mathbf{R}^n) \longrightarrow L^q(\mathbf{R}^n)$ inequality for $1/p - 1/q = 2/n$ and $2n/(n+3) < p \leq q < 2n/(n-3)$, $n \geq 3$. For $2 \leq \rho < \infty$, this can be improved to $1/p - 1/q \leq 2/n$ and $2n/(n-3) < p \leq q < 2n/(n-3)$ when $n \geq 3$. If $n = 2$, the valid range is $1 < p \leq q < \infty$.

Finally, if ϕ is any convex function on $\mathbf{R}$, we obtain an $L^p(\mathbf{R}^n) \longrightarrow L^q(\mathbf{R}^n)$ Carleman inequality for the family of one–dimensional weights $e^{\phi(x_n)}$, for $n \geq 3$, and when $1/p - 1/q = 2/n$ and $2n/(n+3) < p < q < 2n/(n-3)$.

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0.1 Introduction

A fundamental problem in partial differential equations is to determine the extent to which solutions are unique. For example, suppose $p(D)$ is a differential operator and $p(D)u = 0$ in some domain $\Omega \subseteq \mathbf{R}^n$. If u vanishes on an open subset of Ω, does it follow that u is identically zero in Ω? In other words, does the differential operator $p(D)$ have the unique continuation property? The link between weighted Sobolev inequalities and unique continuation was established in the 1930's by Carleman [2], and thereafter this approach to unique continuation became known as the Carleman method. There is a large literature on the use of Carleman inequalities in proving unique continuation theorems. See [8], or [13] for more recent results. If $n \geq 3$, and $V(x) \in L^{n/2}_{loc}(\mathbf{R}^n)$, unique continuation for the Schrödinger operator $-\triangle + V$ is a consequence of the following Carleman inequality, due to Kenig, Ruiz, and Sogge [9].

Theorem 0.1 *Let $n \geq 3$, and let p and p' satisfy $1/p+1/p' = 1$, $1/p-1/p' = 2/n$. If $k \in \mathbf{R}^n$, there is a constant $A = A_n$ such that*

$$\|e^{k\cdot x} f\|_{L^{p'}(\mathbf{R}^n)} \leq A_n \|e^{k\cdot x} \triangle f\|_{L^p(\mathbf{R}^n)}$$

holds for all $f \in C_0^\infty(\mathbf{R}^n)$.

It is natural to ask whether the family of linear weights $k\cdot x$ can be replaced by a larger class of functions in this Carleman inequality. In seeking an appropriate generalization, if the class is to be closed under multiplication by positive constants, then a simple necessary condition is that the class satisfy some type of maximum principle. For instance, suppose ϕ is a continuous function on a domain $\Omega \subseteq \mathbf{R}^n$, and there is a point $x_0 \in \Omega$, and an open set

U containing x_0, with compact closure in Ω, such that

$$(i) \quad \phi(y) \le \phi(x_0) \quad \forall y \in U$$

$$(ii) \quad \phi(y) < \phi(x_0) \quad \forall y \in \partial U$$

Then there does not exist a constant A, independent of t, such that

$$\|e^{t\phi} f\|_{L^q(\Omega)} \le A \|e^{t\phi} \Delta f\|_{L^p(\Omega)}$$

holds for all $t \ge 1$, and all $f \in C_0^\infty(\Omega)$. Although this fact is elementary and well known, we will prove it since we do not have an exact reference.

We may suppose $0 \in \Omega$, $\phi(0) = 0$. Then , for $\delta > 0$ sufficiently small, $B_\delta = \{x \in U : \phi(x) > -\delta\}$ has compact closure in U. Thus there is an $f \in C_0^\infty(U)$ with $f = 1$ on B_δ. Then

$$\|e^{t\phi} f\|_{L^q(\Omega)} \ge \|e^{t\phi}\|_{L^q(B_{\delta/2})},$$

and

$$\begin{aligned} \|e^{t\phi} \Delta f\|_{L^p(\Omega)} &\le C \|e^{t\phi}\|_{L^p(U \setminus B_\delta)} \\ &\le C|U| e^{-\delta t} \\ &= C_1 e^{-\delta t}. \end{aligned}$$

Thus in order for such an inequality to hold, we must have

$$\int_{B_{\delta/2}} e^{qt\phi(x)} dx \le (CA)^q e^{-q\delta t}$$

for all $t \ge 1$. Equivalently,

$$\int_{B_{\delta/2}} e^{qt(\phi(x)+\delta)} dx \le (CA)^q,$$

but since $\phi(x) + \delta > \delta/2$ for $x \in B_{\delta/2}$, this implies

$$e^{qt\delta/2}|B_{\delta/2}| \leq (CA)^q$$

for all $t \geq 1$. This is impossible, since the nonempty open set $B_{\delta/2}$ has positive measure.

There is a positive result in dimension 2 for subharmonic weights due to D. Jerison (private communication with T Wolff).

Theorem 0.2 *Let D be the unit disc in $\mathbf{R}^2$. Then if w is subharmonic in D, there is a constant C such that*

$$\|e^w f\|_{L^2(D)} \leq C\|e^w \Delta f\|_{L^2(D)}$$

holds for all $f \in C_0^\infty(D)$.

Jerison has also pointed out that a similar inequality, (which he proved in dimension 2), in higher dimensions would answer the following question, originally posed by L. Bers. See [15] for a discussion, and related results.

Conjecture 0.1 *Let $n \geq 2$, and suppose u is a harmonic function on the upper half space $\mathbf{R}^n_+ \subset \mathbf{R}^n$, which is C^2 up to the boundary, and whose gradient vanishes on a boundary set of positive measure. Then u is constant.*

The purpose of this thesis is to present results concerning the smaller family of convex weights. In this situation, there are natural osculation arguments using linear weights. Similar osculation arguments appear in [6] and [13]. We now state the main result of this thesis.

Theorem 0.3 *Let Ω be a bounded convex domain in $\mathbf{R}^n$, $n \geq 3$. Then if ϕ is a convex function on Ω, and $q < 2n/(n-3)$, there is a constant*

$C = C(n, q, |\Omega|)$ such that

$$\|e^{\phi} f\|_{L^q(\Omega)} \leq C \|e^{\phi} \triangle f\|_{L^{n/2}(\Omega)}$$

holds for all $f \in C_0^{\infty}(\Omega)$.

Let Ω be a bounded convex domain in $\mathbf{R}^2$, and suppose $1 < p \leq q < \infty$. Then if ϕ is convex on Ω, there is a constant $C = C(p, q, |\Omega|)$ such that

$$\|e^{\phi} f\|_{L^q(\Omega)} \leq C \|e^{\phi} \triangle f\|_{L^p(\Omega)}$$

holds for all $f \in C_0^{\infty}(\Omega)$.

The effect of osculation is to localize matters to the sets where a convex function is close to its linear part. Specifically, we make local estimates on the sets $S_a^{\phi}(C) = \{x \in \Omega : \phi(x) - \phi(a) - \nabla\phi(a) \cdot (x - a) < C\}$. The ability to add up these local estimates amounts to proving a covering lemma for the sets $S_a^{\phi}(C)$. For an arbitrary convex function, the $S_a^{\phi}(C)$ satisfy an L^1 type covering lemma.

Theorem 0.4 (Covering Lemma) *Let Ω be a bounded convex domain in $\mathbf{R}^n$. Suppose ϕ is a convex function on Ω. Then there is a constant D_n and a covering $\Omega = \bigcup_{a \in J} S_a^{\phi}(C)$ such that*

$$\sum_{a \in J} |S_a^{\phi}(C)| \leq D_n |\Omega|.$$

The gap condition in theorem 0.3 approaches the expected value of $2/n$ only in low dimensions. To prove Carleman inequalities for the correct gap, we restrict our attention to certain subsets of the convex functions, allowing us to prove stronger L^{∞} type covering lemmas.

One result in this direction concerns essentially uniformly convex weights on bounded domains, (defined in section 3.2). Hormander [6] proved an

$L^2(\Omega) \longrightarrow L^2(\Omega)$ Carleman inequality for these weights in connection to unique continuation. We show that this can be extended to $L^p \longrightarrow L^q$ estimates for $1/p - 1/q \leq 2/n$. Specifically, we have the following result.

Theorem 0.5 *Suppose Ω is a bounded domain in $\mathbf{R}^n$, $n \geq 2$. Let $\phi \in C^2(\overline{\Omega})$ be essentially uniformly convex, with k linear directions. When $n \geq 3$, let (p, q) satisfy $1/p - 1/q \leq 2/n$, and $2n/(n+3) < p \leq q < 2n/(n-3)$. When $n = 2$, let (p, q) satisfy $1 < p \leq q < \infty$. Then there is a constant $A = A(n, p, q, \phi, \Omega)$ such that for $t \geq 1$*

$$\|e^{t\phi} f\|_{L^q(\Omega)} \leq A t^{-((n-k)/2)(2/n+1/q-1/p)} \|e^{t\phi} \triangle f\|_{L^p(\Omega)}$$

holds for all $f \in C_0^\infty(\Omega)$.

By restricting the range of p and q somewhat, we obtain inequalities for other second order operators $p(D)$ with the correct gap condition. This is described in chapter 3.

If the weights are sufficiently convex, the restriction to bounded domains is not necessary, and the osculation argument can be used to prove estimates on $\mathbf{R}^n$. We demonstrate this for the weights $\phi(x) = |x|^\rho$, for $\rho > 1$. Stromberg [12] proved a surprising $L^2(\mathbf{R}^n) \longrightarrow L^2(\mathbf{R}^n)$ result for these weights as part of a general study of L^2 Carleman inequalities. We present an estimate for the natural $2/n$ gap.

Theorem 0.6 *Let $n \geq 3$, and $\rho > 1$. Let (p, q) satisfy $1/p - 1/q = 2/n$ and $|1/p - 2n/(n+2)| < 1/2n$. Then there is a constant $C = C(n, p, \rho)$, such that*

$$\|e^{|x|^\rho} f\|_{L^q(\mathbf{R}^n)} \leq C \|e^{|x|^\rho} \triangle f\|_{L^p(\mathbf{R}^n)}$$

holds for all $f \in C_0^\infty(\mathbf{R}^n)$.

When $\rho \geq 2$, the extra convexity allows us to prove estimates for smaller gaps as well. Specifically, if $\rho \geq 2$ the estimate in theorem 0.6 holds for the following values of p and q. When $n \geq 3$, the valid range is $1/p - 1/q \leq 2/n$, and $2n/(n+3) < p \leq q < 2n/(n-3)$. For $n = 2$ it is $1 < p \leq q < \infty$. In particular, for $n = 2$ or 3, we have $L^p(\mathbf{R}^n) \longrightarrow L^p(\mathbf{R}^n)$ estimates for $1 < p < \infty$.

Finally, for weights which are one dimensional, the relevant covering lemma becomes much simpler and an adaptation of an argument in [13] proves the following.

Theorem 0.7 *Let ϕ be a convex function on $\mathbf{R}$, and for $n \geq 3$, suppose $1/p - 1/q = 2/n$ and $2n/(n+3) < p < q < 2n/(n-3)$. Then there is a constant $A = A_p$ such that, for $\phi = \phi(x_n)$ we have*

$$\|e^{\phi(x_n)} f\|_{L^q(\mathbf{R}^n)} \leq A \|e^{\phi(x_n)} \Delta f\|_{L^p(\mathbf{R}^n)}$$

holds for all $f \in C_0^\infty(\mathbf{R}^n)$.

Chapter 1

Preliminaries

1.1 Some notation

For $p \in [1, \infty)$, and $\Omega \subseteq \mathbf{R}^n$, we denote the usual Lebesque space norms on Ω by $\|g\|_{L^p(\Omega)} = (\int_\Omega |g(x)|^p dx)^{1/p}$. And for such p we let p' denote the exponent conjugate to p. Namely, p and p' satisfy $1/p + 1/p' = 1$.

If E is a measurable subset of $\mathbf{R}^n$, we let χ_E be the characteristic function of E, and $|E|$ be the n–dimensional Lebesque measure of E. In particular, we have $|E| = \int_{\mathbf{R}^n} \chi_E(x)dx$.

The space of smooth functions whose support is compactly contained in Ω is denoted $C_0^\infty(\Omega)$. If $k \geq 1$, then $C^k(\Omega)$ consists of functions whose partial derivatives of order $\leq k$ exist and are continuous functions on Ω. If the partial derivatives of g of order $\leq k$ extend to continous functions on $\overline{\Omega}$, we say that $g \in C^k(\overline{\Omega})$.

We denote the unit sphere in $\mathbf{R}^n$ by S^{n-1}. If a,$b \in \mathbf{R}^n$, we denote their inner product by $a \cdot b$. Also, if $a \in \mathbf{R}^n$, and $f \in S^{n-1}$, we let $L(a, f) = \{a + sf : s \in \mathbf{R}\}$ be the line through a in the direction of f.

1.2 Convex sets and functions

In this section we will set our notation and state some well known facts about convex sets and functions. Througout this thesis, C will denote a constant whose value may change at each occurence.

If $F \subset \mathbf{R}^n$ is compact and convex, then for $f \in S^{n-1}$ we let

$$\gamma_F(f) = sup\{x \cdot f : x \in F\}$$

be the support function of F, and let

$$w_F(f) = \gamma_F(f) + \gamma_F(-f)$$

be the width in the f–direction. If F also has nonempty interior, we let

$$B_F = \frac{1}{|F|}\int_F x dx$$

be the barycenter of F. For such F, and $\delta > 0$, we let δF denote the dilation of F by δ about the barycenter. Sometimes a convex set will be dilated about a point other than the barycenter. When this happens we will explicitly refer to the base point for clarity.

The following five lemmas are standard facts about convex sets. They can be found for instance in [14]. We state them here without proof.

Lemma 1.1 *Suppose F_1, F_2 are compact convex subsets of $\mathbf{R}^n$, with $F_1 \subset F_2$. If $w_{F_1}(f) \leq \epsilon w_{F_2}(f)$ for some $f \in S^{n-1}$, and $\epsilon > 0$, then there is a constant C such that $|F_1| \leq C\epsilon|F_2|$.*

Lemma 1.2 *Suppose F is a compact, convex subset of $\mathbf{R}^n$, and $B_F = 0$. Then there is a constant C such that $w_F(f) \leq C\gamma_F(f)$, for all $f \in S^{n-1}$.*

Lemma 1.3 *Suppose F_1,F_2 are compact , convex subsets of $\mathbf{R}^n$, and $F_1 \bigcap F_2 \neq \emptyset$. Suppose also that for some $A \geq 1$ and for all $f \in S^{n-1}$ we have $w_{F_1}(f) \leq A w_{F_2}(f)$. Then $F_1 \subset \lambda F_2$ with $\lambda \leq CA$.*

Definition 1.1 *A rectangle in $\mathbf{R}^n$ is the image under a rotation of a set $\Pi_{j=1}^n I_j$, where each I_j is a closed interval in $\mathbf{R}$ with nonempty interior.*

Lemma 1.4 *If F is a compact convex set with interior, then there is a rectangle R with the same barycenter as F such that $R \subset F \subset CR$.*

Lemma 1.5 *Suppose F is compact and convex, $a \in F$, and $A \geq 1$. Let $F_A = \{x : x - a = A(y - a)$ for some $y \in F\}$ be the dilation of F by A around a. Then $F_A \subset (CA)F$.*

Corollary 1.1 *If F is a compact convex set with interior, then for $p \in F$ and $f \in S^{n-1}$ we have*

$$\left|L(p,f)\bigcap F\right| \leq C\left|L(B_F,f)\bigcap F\right|.$$

Proof: This is obvious if F is a rectangle, and in that case we may take $C = 1$. For a general F, choose a rectangle R with $R \subset F \subset CR$, with barycenter B_F. Then if $p \in F \subset CR$, and $f \in S^{n-1}$, we have

$$\begin{aligned}
\left|L(p,f)\bigcap F\right| &\leq \left|L(p,f)\bigcap CR\right| \\
&\leq \left|L(B_F,f)\bigcap CR\right| \\
&= C\left|L(B_F,f)\bigcap R\right| \\
&\leq C\left|L(B_F,f)\bigcap F\right|.
\end{aligned}$$

Corollary 1.2 *Suppose F_1, F_2 are compact, convex subsets of $\mathbf{R}^n$ with $F_1 \subset F_2$. Suppose also that $|F_1| \geq \frac{1}{A}|F_2|$ for some $A \geq 1$. Then*

$$\left|L(p,f)\bigcap F_2\right| \leq CA\left|L(B_{F_1},f)\bigcap F_1\right|$$

holds for all $f \in S^{n-1}$, and all $p \in F_2$.

Proof: The hypotheses together with lemma 1.1 imply that if $f \in S^{n-1}$, then

$$w_{F_1}(f) \geq \frac{C}{A} w_{F_2}(f).$$

Then lemma 1.3 implies that $F_2 \subset \lambda F_1$ with $\lambda \leq CA$. Then, using corollary 1.1, for any $p \in F_2$ we have

$$\begin{aligned} \left|L(p,f)\bigcap F_2\right| &\leq \left|L(p,f)\bigcap(\lambda F_1)\right| \\ &\leq C\left|L(B_{F_1},f)\bigcap(\lambda F_1)\right| \\ &= C\lambda\left|L(B_{F_1},f)\bigcap F_1\right|. \end{aligned}$$

This proves the corollary since $\lambda \leq CA$.

We require a few simple properties of convex functions. This material can be found in [3] or [10].

Definition 1.2 *If Ω is a convex domain in $\mathbf{R}^n$, we say a function $\phi : \Omega \longrightarrow R$ is convex if*

$$\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y)$$

holds for all $x, y \in \Omega$ and for all $0 \leq t \leq 1$.

Then it follows that ϕ is differentiable almost everywhere in Ω. Moreover, the restriction of ϕ to any line is convex, and for $f \in S^{n-1}$, $\lambda > 0$, the following monotonicity property of $\nabla\phi$ holds almost everywhere.

$$\nabla\phi(x) \cdot f \leq \frac{\phi(x + \lambda f) - \phi(x)}{\lambda} \tag{1.1}$$

For points $a \in \Omega$ where $\nabla\phi(a)$ exists, we set $T_1^a(x) = \phi(a) + \nabla\phi(a) \cdot (x - a)$. Our local Carleman inequalities will be made on sets where ϕ is close to its linear part. For this we make the following:

Definition 1.3 *For $a \in \Omega$, ϕ convex on Ω, and $t > 0$, we define $S_a^\phi(t)=$ $\{x \in \Omega : \phi(x) - T_1^a(x) < t\}$.*

The following easy lemma will be used in the proof of the proposition below.

Lemma 1.6 *For any $\rho \geq 1$, $S_a^\phi(\rho t) \subseteq \rho S_a^\phi(t)$, where $\rho S_a^\phi(t)$ denotes the dilation by ρ of the set $S_a^\phi(t)$ about the point a. In particular, $|S_a^\phi(\rho t)| \leq \rho^n |S_a^\phi(t)|$.*

Proof: If $f \in S^{n-1}$, we let $L(a, f) = \{a + sf : s \in R\}$ be the line through a parallel to f. It then suffices to show

$$L(a,f) \bigcap S_a^\phi(\rho t) \subseteq \rho \left[L(a,f) \bigcap S_a^\phi(t)\right],$$

where the right hand side refers to the dilation by ρ of the line segment $L(a,f) \bigcap S_a^\phi(t)$ about the point a.

Suppose that z satisfies $\phi(z) - T_1^a(z) = t$. Then (1.1) and the mean value theorem imply that $(\nabla\phi(z) - \nabla\phi(a)) \cdot (z - a) \geq t$. To prove lemma 1.6, we must show that if $\rho \geq 1$, and y is such that $y - a = \rho(z - a)$, then $\phi(y) - T_1^a(y) \geq \rho t$. To establish this, we notice that

$$\begin{aligned}
\phi(y) - T_1^a(y) &= \phi(y) - T_1^z(y) + T_1^z(y) - T_1^a(y) \\
&\geq T_1^z(y) - T_1^a(y) \\
&= \phi(z) - T_1^a(z) + (\nabla\phi(z) - \nabla\phi(a)) \cdot (y - z) \\
&= t + (\rho - 1)(\nabla\phi(z) - \nabla\phi(a)) \cdot (z - a) \\
&\geq t + (\rho - 1)t \\
&= \rho t.
\end{aligned}$$

The following proposition will be used to make Carleman estimates on the sets $S_a^\phi(t)$.

Proposition 1.1 *Let $\Omega \subset \mathbf{R}^n$ be a bounded convex domain with $0 \in \Omega$. Suppose $\psi : \Omega \longrightarrow \mathbf{R}$ is convex and satisfies $\psi(0) = 0$, and $\psi(x) \geq 0$ for all $x \in \Omega$. Then there is a dimensional constant C_n such that*

$$\int_\Omega e^{-\psi(x)} dx \leq C_n |S_0^\psi(1)|.$$

Proof: Since for any $f \geq 0$, $\int_\Omega f(x)dx = \int_0^\infty |\{x \in \Omega : f(x) > t\}|dt$, we have after a change of variable:

$$\int_\Omega e^{-\psi(x)} dx = \int_0^\infty |\{y \in \Omega : \psi(y) < w\}| e^{-w} dw.$$

We break the integral up into two pieces and apply lemma 1.6

$$\int_0^1 |\{y \in \Omega : \psi(y) < w\}| e^{-w} dw \leq |S_0^\psi(1)| \left(\int_0^1 e^{-w} dw \right).$$

For the other term we have, using the lemma,

$$\int_1^\infty |\{y \in \Omega : \psi(y) < w\}| e^{-w} dw \leq |S_0^\psi(1)| \left(\int_1^\infty w^n e^{-w} dw \right).$$

This proves the proposition with $C_n = \left(\int_0^1 e^{-w} dw \right) + \left(\int_1^\infty w^n e^{-w} dw \right)$.

Chapter 2

Convex Weights

This chapter is devoted to proving the following Carleman inequality for convex weights.

Theorem 2.1 *Let $n \geq 3$. Let Ω be a bounded convex domain in $\mathbf{R}^n$, and $\phi : \Omega \longrightarrow \mathbf{R}$ a convex function on Ω. Then if $q < 2n/(n-3)$, there is a constant $A = A(n, q, |\Omega|)$ such that*

$$||e^{\phi} f||_{L^q(\Omega)} \leq A||e^{\phi} \triangle f||_{L^{n/2}(\Omega)}$$

holds for all $f \in C_0^\infty(\Omega)$.

Let $n = 2$. Then if $p > 1$, and $q < \infty$, there is a constant $A = A(p, q, |\Omega|)$ such that

$$||e^{\phi} f||_{L^q(\Omega)} \leq A||e^{\phi} \triangle f||_{L^p(\Omega)}$$

holds for all $f \in C_0^\infty(\Omega)$.

2.1 Osculation by linear weights

The local Carleman estimates we will make are based on the following.

Theorem 2.2 (Kenig–Ruiz–Sogge, [9]) *Suppose $n \geq 3$, and $\frac{1}{r} - \frac{1}{r'} = \frac{2}{n}$. Let the pair of exponents (p, q) satisfy*

$$\frac{1}{p} - \frac{1}{q} = \frac{2}{n} \tag{2.1}$$

$$\left|\frac{1}{p} - \frac{1}{r}\right| < \frac{1}{2n}. \tag{2.2}$$

Then if $k \in \mathbf{R}^n$, there is a constant $A = A(n, p)$ such that

$$||e^{k \cdot x} f||_{L^q(\mathbf{R}^n)} \leq A||e^{k \cdot x} \Delta f||_{L^p(\mathbf{R}^n)}$$

holds for all $f \in C_0^\infty(\mathbf{R}^n)$.

Suppose $n = 2$, and $\frac{1}{p} - \frac{1}{q} < 1$. If $\Omega \subset \mathbf{R}^2$ is bounded, and $k \in \mathbf{R}^2$, there is a constant $A = A(p, q, |\Omega|)$ such that

$$||e^{k \cdot x} f||_{L^q(\Omega)} \leq A||e^{k \cdot x} \Delta f||_{L^p(\Omega)}$$

holds for all $f \in C_0^\infty(\Omega)$.

Also, when $n \geq 3$ and $p = r$, we may replace Δf by $p(D)f$, where $p(D)$ is any second order constant coefficient differential operator with principal part $Q(\xi) = -\xi_1^2 - \cdots - \xi_j^2 + \xi_{j+1}^2 + \cdots + \xi_n^2$, for some $0 \leq j \leq n$. This substitution is also valid when $n = 2$, and $1/p - 1/q < 1$.

We remark that condition (2.1) is a necessary scaling condition, and that (2.2) involves an interval around r because the adjoint of the operator under consideration is another operator of the same type. Also notice that (2.1) and (2.2) imply that $q < 2n/(n-3)$, which is the condition on q appearing in theorem 2.1.

We begin with the local osculation estimate. This is done by replacing ϕ by its linear part on $S_a^\phi(t)$ and applying theorem (2.2). The result is a local inequality with a factor proportional to the measure of $S_a^\phi(t)$. We then cover Ω with the sets $S_a^\phi(t)$ and add up the estimates. This can be done in several ways, each reducing the Carleman inequality to a certain covering lemma for the sets $S_a^\phi(t)$. For the case of general convex weights the relevant estimate is given below.

Proposition 2.1 *Let $n \geq 3$, and $t \geq 1$. If $\Omega = \bigcup_{a\in J} S_a^\phi(t)$ is any covering of Ω by the sets $S_a^\phi(t)$, and $q < 2n/(n-3)$, there is a constant $A = A(n,q)$ such that*

$$||e^\phi f||_q \leq Ae^t \left(\sum_{a\in J} |S_a^\phi(t)|\right)^{1/q} ||e^\phi \triangle f||_{n/2}$$

holds for all $f \in C_0^\infty(\Omega)$.

When $n = 2$, and $1/p - 1/q < 1$, there is an $A = A(p,q)$ such that

$$||e^\phi f||_q \leq Ae^t \left(\sum_{a\in J} |S_a^\phi(t)|\right)^{1/q} ||e^\phi \triangle f||_p$$

holds for all $f \in C_0^\infty(\Omega)$.

Proof: We give the proof for $n \geq 3$. Notice first that $q < 2n/(n-3)$, as in theorem 2.2. For any such q, let p be the corresponding exponent in that theorem. That is, $1/p - 1/q = 2/n$. Then since $e^{\phi(x)} \approx e^{T_1^a(x)}$ on $S_a^\phi(t)$ we have

$$\int_{S_a^\phi(t)} e^{q\phi}|f|^q dx \leq e^{qt} \int_{S_a^\phi(t)} e^{qT_1^a(x)}|f|^q dx.$$

Applying theorem 2.2 to the right hand side gives

$$\int_{S_a^\phi(t)} e^{q\phi}|f|^q dx \leq C^q e^{qt} ||e^{T_1^a(x)} \triangle f||_p^q.$$

Now applying Holder's inequality (since $1/p - 1/q = 2/n$) to the right hand side gives

$$\int_{S_a^\phi(t)} e^{q\phi}|f|^q dx \leq Ce^{qt} ||e^{-(\phi - T_1^a(x))}||_q^q ||e^\phi \triangle f||_{n/2}^q.$$

Since $\phi(x) - T_1^a(x)$ is a nonnegative convex function on Ω, an application of proposition 1.1 then implies

$$\int_{S_a^\phi(t)} e^{q\phi}|f|^q dx \leq \left(CC_n e^t\right)^q |S_a^\phi(t)| \left(||e^\phi \Delta f||_{n/2}^q\right).$$

Now since $\Omega = \bigcup_{a\in J} S_a^\phi(t)$, the integral over Ω is majorized by the sum of the integrals over the sets $S_a^\phi(t)$. Thus,

$$||e^\phi f||_q \leq CC_n e^t \left(\sum_{a\in J} |S_a^\phi(t)|\right)^{1/q} ||e^\phi \Delta f||_{n/2}.$$

This completes the proof of the proposition.

2.2 Covering estimates

We state and prove a covering lemma for the sets $S_a^\phi(t)$, which combined with proposition 2.1 proves theorem 2.1.

Theorem 2.3 (Covering Lemma) *Let Ω be a bounded domain in $\mathbf{R}^n$, $n \geq 2$. Suppose ϕ is a convex function on Ω. Then there is a constant D_n and a covering $\Omega = \bigcup_{a\in J} S_a^\phi(t)$ such that*

$$\sum_{a\in J} |S_a^\phi(t)| \leq D_n |\Omega|.$$

This covering property significantly restricts the manner in which these sets may intersect. While it is impossible to bound the number of sets $S_a^\phi(t)$ to which a given point may belong, (as can be seen by considering the one parameter family $\phi(x) = t|x|$ for $t > 0$), it does turn out that these sets can always be made disjoint by performing dilations by a fixed factor. This property reflects the specific structure of these sets, and is not merely a consequence of the convexity of the $S_a^\phi(t)$. Specifically, taking $n = 2$ it is easy to see theorem 2.3 is false for coverings by general convex sets. More precisely, we mean that given any constant D_n, there is a bounded domain Ω and a covering $\Omega = \bigcup_j F_j$ of Ω by convex sets F_j, such that if $\Omega = \bigcup F_{k(j)}$ is any subcover then $\sum_{k(j)} |F_{k(j)}| > D_n|\Omega|$. It is also not difficult to show it remains false if we only consider coverings by squares. This is somewhat surprising given that squares have extremely nice covering properies. The problem here of course is that we are requiring the sets to cover all of Ω, and not merely a certain fraction. However, even if we relax the statement of theorem 2.3 in this manner, it can still be shown to be false for coverings by arbitrary rectangles. This can be shown by a Kakeya set construction as in [5].

The proof of the Covering Lemma will be broken into several stages. We begin by analyzing the intersection properties of the sets $S_a^\phi(t)$ in one dimension, and hence along lines in $\mathbf{R}^n$, and proceed to deduce some covering properties of the $S_a^\phi(t)$.

2.2.1 Intersection properties

Suppose Ω is a bounded convex domain in $\mathbf{R}^n$, and $\psi : \Omega \to \mathbf{R}$ is convex. Then $inf\{\psi(y) : y \in \Omega\}$ exists and is finite. We will denote this infimum by $inf_{y\in\Omega}\psi(y)$. For such ψ, set

$$V_\psi = \{x \in \Omega : \psi(x) < inf_{y\in J}\psi(y) + 1\}.$$

The convexity of ψ implies that V_ψ is a convex open subset of Ω. Our first result is an intersection estimate when $n = 1$ for the sets $V_{\psi(x)-kx}$.

Proposition 2.2 *If ψ is convex on an interval $J = (J_1, J_2)$, then for $k, l \in \mathbf{R}$ we have*

$$|V_{\psi(x)-kx} \bigcap V_{\psi(x)-lx}| \leq 2/|k - l|.$$

Proof: Replacing ψ by the (convex) function $\psi - l$ shows we may assume $l = 0$. Now suppose $b \in V_\psi$. We claim that if $hk \geq 2$ then $b - h \notin V_{(\psi(x)-kx)}$. To prove this for such an h, we may assume $b - h \in J$, for otherwise it is obvious. If $b - h \in J$, then since $b \in V_\psi$

$$\psi(b-h) - k(b-h) - inf_{w\in J}(\psi(w) - kw) \geq$$

$$\psi(b-h) - k(b-h) - inf_{w\in J}(\psi(w) - kw) + [\psi(b) - inf_{y\in J}\psi(y) - 1] =$$

$$[\psi(b-h) - inf_{y\in J}\psi(y)] + [\psi(b) - kb - inf_{w\in J}(\psi(w) - kw)] + kh - 1.$$

Since the first two terms are nonnegative, the preceding line is $\geq kh - 1$, and so if $kh \geq 2$ we obtain

$$\psi(b-h) - k(b-h) - inf_{w\in J}(\psi(w) - kw) \geq 1$$

and so $b - h \notin V_{(\psi - k)}$. The proposition now follows by taking b to be the right endpoint of V_ψ when $k > 0$, and the left endpoint when $k < 0$.

Observe that if $\phi : J \to \mathbf{R}$ is convex, then

$$S_a^\phi(t) = \{x \in J : \phi(x) - T_1^a(x) < t\} = V_{t^{-1}(\phi(x) - \phi'(a)x)}. \qquad (2.3)$$

The intersection estimate in proposition 2.2 immediately implies an estimate for the intersection of the sets $S_a^\phi(t)$ along lines in $\mathbf{R}^n$. As before, if $p \in \Omega$, and $f \in S^{n-1}$, then $L(p, f)$ denotes the line through p in the direction f.

Proposition 2.3 *Suppose the convex set $S_a^\phi(t) \bigcap S_b^\phi(t)$ is nonempty, and let B denote its barycenter. Then*

$$\left|S_a^\phi(t) \bigcap S_b^\phi(t) \bigcap L(B, f)\right| \leq \frac{2t}{|(\nabla\phi(b) - \nabla\phi(a)) \cdot f|}$$

holds for all $f \in S^{n-1}$.

Proof: We reduce to the situation of proposition 2.2. Specifically, set $h(x) = t^{-1}(\phi(x) - \nabla\phi(a) \cdot x)$, and let $h_{Bf}(s) = h(B + sf)$. Then h_{Bf} is a convex function on the bounded open interval $J = \{s \in \mathbf{R} : B + sf \in \Omega\}$. Now using (2.1) we have

$$S_a^\phi(t) \bigcap L(B,f) = V_h \bigcap L(B,f)$$

$$= \{B + sf \in \Omega : h(b + sf) < inf_{y\in\Omega} h(y) + 1\}$$

$$\subseteq \{B + sf \in \Omega : h(B + sf) < inf_{y=B+sf, s\in J} h(y) + 1\}$$

$$= V_{h_{Bf}},$$

but as $h_{Bf}(s) = t^{-1}\left(\phi_{Bf}(s) - (\nabla\phi(a) \cdot f)\, s\right) - t^{-1}\nabla\phi(a) \cdot B$, this implies

$$\begin{aligned} S_a^\phi(t) \bigcap L(B,f) &\subseteq V_{t^{-1}(\phi_{Bf}(s) - \nabla\phi(a)\cdot fs) - t^{-1}\nabla\phi(a)\cdot B} \\ &= V_{t^{-1}(\phi_{Bf}(s) - \nabla\phi(a)\cdot fs)}. \end{aligned}$$

Of course we also have

$$S_b^\phi(t) \bigcap L(B,f) \subseteq V_{t^{-1}(\phi_{Bf} - \nabla\phi(b)\cdot f)}.$$

The proposition now follows from proposition 2.2. An important consequence of this is the following.

Corollary 2.1 *For any $A \geq 1$, if $\left|S_a^\phi(t) \cap S_b^\phi(t)\right| \geq \frac{1}{A}|S_a^\phi(t)|$ then there exists an $\alpha = \alpha(A)$ such that*

$$|(\nabla\phi(b) - \nabla\phi(a)) \cdot (x - a)| \leq t\alpha(A)$$

holds for all $x \in S_a^\phi(t)$.

Proof: We may assume $x \neq a$. Taking $f = (x-a)/|x-a| \in S^{n-1}$, we see from proposition 2.3 that

$$|(\nabla\phi(b) - \nabla\phi(a)) \cdot (x-a)| \leq \frac{2t|x-a|}{\left|S_a^\phi(t) \bigcap S_b^\phi(t) \bigcap L(B,(x-a)/|x-a|)\right|}.$$

The corollary now follows from corollary 1.2 by taking $F_1 = S_a^\phi(t) \bigcap S_b^\phi(t)$, $F_2 = S_a^\phi(t)$, and observing that for $x \in S_a^\phi(t) = F_2$ we have

$$|x-a| \leq \left|L(a, \frac{x-a}{|x-a|}) \bigcap F_2\right|.$$

In the next proposition we consider the situation when two of the sets $S_a^\phi(t)$ have "large"overlap. We'll show that when this happens their union is contained in a set of the form $S_a^\phi(At)$, for some dimensional constant A. Since a local Carleman estimate can be made (using theorem 2.2) on such a set, this allows us to basically throw out one of the two sets in this situation. We will see that this will reduce matters to consideration of the case when two such sets have "small" overlap.

Proposition 2.4 *For any* $A \geq 1$, *if* $\left|S_a^\phi(t) \bigcap S_b^\phi(t)\right| \geq \frac{1}{A}|S_a^\phi(t)|$, *then there is a* $C = C(A)$ *such that* $S_a^\phi(t) \subseteq S_b^\phi(tC(A))$.

Proof: Suppose $x \in S_a^\phi(t)$ and $y \in S_a^\phi(t) \bigcap S_b^\phi(t)$. Then

$$\begin{aligned}
&\phi(x) - T_1^b(x) = \\
&\quad [\phi(x) - T_1^a(x)] + [(\nabla\phi(b) - \nabla\phi(a)) \cdot (y-a)] + [T_1^a(y) - T_1^b(y)] \\
&\quad -[(\nabla\phi(b) - \nabla\phi(a)) \cdot (x-a)].
\end{aligned}$$

Taking absolute values and applying the triangle inequality, we see it suffices to show that each of the four terms on the RHS has absolute value $\leq C(A)t$. The first term is nonnegative and less than t, as $x \in S_a^\phi(t)$. The second and fourth terms have absolute value less than $t\alpha(A)$ by corollary 2.1. And the third term has absolute value at most $2t$, as $y \in S_a^\phi(t) \bigcap S_b^\phi(t)$. Hence

$$\phi(x) - T_1^b(x) < tC(A)$$

and so $x \in S_b^\phi(tC(A))$. This proves the proposition.

2.2.2 Covering properties

Our first proposition in this section will tell us that we may select a covering $\Omega = \bigcup_{a \in J} S_a^\phi(t)$ such that the sets $S_a^\phi(t)$ don't overlap too much. This preliminary statement will be refined to give the Covering Lemma (theorem 2.3).

First, there is a set $\mathcal{J} \subseteq \Omega$, with $|\mathcal{J}| = |\Omega|$, and such that if $a \in \mathcal{J}$, then $\nabla\phi(a)$ exists. Then since $\mathcal{J}$ has full measure in Ω, we have $\Omega = \bigcup_{a \in \mathcal{J}} S_a^\phi(t)$. We next pick a countable subcover

$$\Omega = \bigcup_{k=1}^{\infty} S_{a_k}^\phi(t)$$

with the property

$$|S_{a_j}^\phi(t)| \geq \frac{1}{2} sup_{k \geq j} |S_{a_k}^\phi(t)|. \tag{2.4}$$

.

Proposition 2.5 *Suppose $A \geq 1$, and let $C = C(A)$ be as in proposition 2.4. Then given the cover $\Omega = \bigcup_{k=1}^{\infty} S_{a_k}^\phi(t)$ with property (2.4) above, we can obtain a new cover $\Omega = \bigcup S_{b_j}^\phi(tC)$ with $\{b_j : j = 1, 2, \ldots\} \subseteq \{a_k : k = 1, 2, \ldots\}$ and the property*

$$\left|S_{b_i}^\phi(t) \bigcap S_{b_j}^\phi(t)\right| \leq \frac{2}{A} min\left(|S_{b_i}^\phi(t)|, |S_{b_j}^\phi(t)|\right) \tag{2.5}$$

holds whenever $b_i \neq b_j$.

Proof: We begin by selecting $S^{\phi}_{a_1}(t)$ and for $j \geq 2$, if $|S^{\phi}_{a_1}(t) \bigcap S^{\phi}_{a_j}(t)| \geq (1/A)|S^{\phi}_{a_j}(t)|$, we throw out $S^{\phi}_{a_j}(t)$ and notice that $S^{\phi}_{a_j}(t) \subseteq S^{\phi}_{a_1}(Ct)$ by proposition 2.4. We thus obtain a new covering of Ω by taking all those sets we did not throw out and replacing $S^{\phi}_{a_1}(t)$ with $S^{\phi}_{a_1}(Ct)$. Namely

$$\Omega = S^{\phi}_{a_1}(Ct) \bigcup \left(\bigcup_{\alpha(j)} S^{\phi}_{a_{\alpha(j)}}(t) \right)$$

with the property that

$$\begin{aligned} \left| S^{\phi}_{a_1}(t) \bigcap S^{\phi}_{a_{\alpha(j)}}(t) \right| &\leq (1/A)|S^{\phi}_{a_{\alpha(j)}}(t)| \\ &\leq (2/A) min \left[|S^{\phi}_{a_1}(t)|, |S^{\phi}_{a_{\alpha(j)}}(t)| \right] \end{aligned}$$

holds for all j.

Next, we choose $S^{\phi}_{a_{\alpha(1)}}(t)$, recalling that $|S^{\phi}_{a_{\alpha(1)}}(t)| \geq \frac{1}{2} sup_{j \geq 1} |S^{\phi}_{a_{\alpha(j)}}(t)|$. Then for $j \geq 2$, we throw out $S^{\phi}_{a_{\alpha(j)}}(t)$ if $|S^{\phi}_{a_{\alpha(1)}}(t) \bigcap S^{\phi}_{a_{\alpha(j)}}(t)| \geq \frac{1}{A}|S^{\phi}_{a_{\alpha(j)}}(t)|$. We then obtain a new cover

$$\Omega = S^{\phi}_{a_1}(Ct) \bigcup S^{\phi}_{a_{\alpha(1)}}(Ct) \bigcup \left(\bigcup_{j=1}^{\infty} S^{\phi}_{a_{\beta(j)}}(t) \right).$$

This cover has the property

$$\left| S^{\phi}_{a_1}(t) \bigcap S^{\phi}_{b}(t) \right| \leq \frac{2}{A} min \left[|S^{\phi}_{a_1}(t)|, |S^{\phi}_{b}(t)| \right]$$

for $b \in \{\alpha(1), \beta(j)\}$, and

$$\left| S^{\phi}_{a_{\alpha(1)}}(t) \bigcap S^{\phi}_{d}(t) \right| \leq \frac{2}{A} min \left[|S^{\phi}_{a_{\alpha(1)}}(t)|, |S^{\phi}_{d}(t)| \right]$$

holds for $d \in \{a_1, \beta(j)\}$. This construction can be iterated, obtaining a cover with the desired property.

2.2.3 Proof of the covering lemma

We have obtained a preliminary cover by the sets $S_a^\phi(Ct)$ in proposition 2.5. When A is large, the sets in this cover have small relative intersection. This covering property is not sufficient to prove our covering lemma, even when we restrict our consideration to coverings by convex sets. This fact can be demonstrated by a Kakeya set construction, as in [5].

At this point we have not fully exploited the special structure of the sets $S_a^\phi(t)$, which is necessary to gain the full strength of the covering lemma. The crucial fact is that when the sets $S_a^\phi(t)$ have a small relative intersection as in proposition 2.5, they can be shrunk around their barycenters by a fixed factor and be made disjoint. Since dilations have a known effect on volume, the estimate in the covering lemma follows easily. We begin with the main proposition, and then deduce the covering lemma. In what follows, $\epsilon > 0$ should be considered small, but otherwise fixed.

Proposition 2.6 *If* $0 < |S_a^\phi(t) \bigcap S_b^\phi(t)| < \epsilon^n min\{|S_a^\phi(t)|, |S_b^\phi(t)|\}$ *then*

$$\left(\epsilon S_a^\phi(t)\right) \bigcap \left(\epsilon S_b^\phi(t)\right) = \emptyset,$$

where the dilations by ϵ *are taken with respect to the barycenters.*

Proof: We consider two cases. Let C be the constant appearing in lemma 1.2.

Case 1:

$$S_a^\phi(t) \subseteq \left(\frac{\epsilon^{-1}}{8C} S_b^\phi(t)\right). \tag{2.6}$$

The proof in this case relies only on the convexity of the sets $S_a^\phi(t)$. We show that if $\left(\epsilon S_a^\phi(t)\right) \bigcap \left(\epsilon S_b^\phi(t)\right) \neq \emptyset$ then $\left(\epsilon S_a^\phi(t)\right) \subseteq S_b^\phi(t)$. Then

$$\left|S_a^\phi(t) \bigcap S_b^\phi(t)\right| \geq |\epsilon S_a^\phi(t)| = \epsilon^n |S_a^\phi(t)|$$

which is a contradiction.

In proving this, we may assume the barycenter of $S_b^\phi(t)$ is the origin. For convenience we let B_a denote the barycenter of $S_a^\phi(t)$. Now we suppose $p \in \epsilon S_a^\phi(t)$. We need to show that $p \in S_b^\phi(t)$. By definition, there is a point $p' \in S_a^\phi(t)$ such that

$$p = \epsilon p' + (1-\epsilon)B_a.$$

Now (2.6) implies that $p' = \frac{\epsilon^{-1}b_1}{8C}$ for some $b_1 \in S_b^\phi(t)$. So $p = (b_1/8C) + (1-\epsilon)B_a$. As $\epsilon S_a^\phi(t) \bigcap \epsilon S_b^\phi(t) \neq \emptyset$, there are $a_2 \in S_a^\phi(t)$ and $b_2 \in S_b^\phi(t)$ such that

$$\epsilon a_2 + (1-\epsilon)B_a = \epsilon b_2.$$

Another application of (2.6) implies that

$$a_2 = \frac{\epsilon^{-1}b_3}{8C}$$

holds for some $b_3 \in S_b^\phi(t)$. Combining the preceeding two lines shows that $(1-\epsilon)B_a = \epsilon b_2 - (b_3/8C)$. Now lemma 1.2 implies that $(-b_3/C) \in S_b^\phi(t)$, and so we set $b_4 = (-b_3/C)$. Thus

$$p = (b_1/8C) + \epsilon b_2 + (b_4/8). \tag{2.7}$$

Since $S_b^\phi(t)$ is a convex set with its barycenter at the origin, (2.7) implies that $p \in S_b^\phi(t)$. This proves case 1.

Case 2: Here we have

$$S_a^\phi(t) \not\subseteq \frac{\epsilon^{-1}}{8C} S_b^\phi(t). \tag{2.8}$$

As before, we assume that $\epsilon S_a^\phi(t) \bigcap \epsilon S_b^\phi(t) \neq \emptyset$ and reach a contradiction. This case is more difficult, and it is here that the structure of the sets $S_a^\phi(t)$ will be used.

By lemma 1.4, we may choose a rectangle R_a with $R_a \subset S_a^\phi(t) \subset CR_a$, with the same barycenter as $S_a^\phi(t)$. We may assume this barycenter is the origin. Now (2.8) and lemma 1.3 imply

$$w_{S_a^\phi(t)}(f) \geq C\epsilon^{-1} w_{S_b^\phi(t)}(f) \tag{2.9}$$

holds for some $f \in S^{n-1}$. We may assume that R_a has a side parallel to f. Let $p \in \left(\epsilon S_a^\phi(t)\right) \bigcap \left(\epsilon S_b^\phi(t)\right)$. Then (2.9) implies

$$\begin{aligned} \gamma_{S_b^\phi(t)}(f) &\leq (p \cdot f) + w_{S_b^\phi(t)}(f) \\ &\leq (p \cdot f) + C\epsilon w_{S_a^\phi(t)}(f) \\ &\approx (p \cdot f) + C\epsilon \gamma_{S_a^\phi(t)}(f) \\ &\leq C\epsilon \gamma_{S_a^\phi(t)}(f) + C\epsilon \gamma_{S_a^\phi(t)}(f). \end{aligned}$$

We have used lemma 1.2 in stating $w_{S_a^\phi(t)}(f) \approx \gamma_{S_a^\phi(t)}(f)$. Thus we have

$$\gamma_{S_b^\phi(t)}(f) \leq C\epsilon \gamma_{S_a^\phi(t)}(f), \tag{2.10}$$

and similarly we have

$$\gamma_{S_b^\phi(t)}(-f) \leq C\epsilon \gamma_{S_a^\phi(t)}(-f). \tag{2.11}$$

We claim that there exists a point $p_1 \in L(p, f) \bigcap S_a^\phi(t)$ with

$$p_1 \cdot f \geq (1/2C_n) sup_{x \in S_a^\phi(t)}(x \cdot f) \geq C\epsilon^{-1} \gamma_{S_b^\phi(t)}(f).$$

The existence of such a point p_1 can be seen as follows. (2.10) implies $\exists x_0 \in R_a$ with $x_0 \cdot f \geq C\epsilon^{-1}\gamma_{S_b^\phi(t)}(f)$. Set $p_1 = p + [(x_0 - p) \cdot f]f$. Then $p_1 \in R_a \subset S_a^\phi(t)$ as R_a has a side parallel to f, and $p_1 \cdot f = x_0 \cdot f \geq C\epsilon^{-1}\gamma_{S_b^\phi(t)}(f)$.

Since $p, p_1 \in S_a^\phi(t)$, the convexity of $S_a^\phi(t)$ implies the line segment connecting p and p_1 lies in $S_a^\phi(t)$. Let $p_2 \in S_a^\phi(t)$ denote the midpoint of this line segment. So

$$|p - p_2| = (1/2)|p - p_1| = |p_1 - p_2|,$$

and

$$p_2 \cdot f \geq C\epsilon^{-1}\gamma_{S_b^\phi(t)}(f).$$

Set

$$g(x) = \phi(x) - T_1^b(x) \quad \text{and} \quad h(y) = \phi(y) - T_1^a(y).$$

Claim: $g(p_2) \geq C\epsilon^{-1}t$.

To prove this claim, first observe that if $y \in S_b^\phi(t)$, then

$$\left|\frac{p_2 \cdot f}{y \cdot f}\right| \geq \frac{p_2 \cdot f}{\gamma_{S_b^\phi(t)}(f)} \geq C\epsilon^{-1}.$$

Let $q = \lambda p_2 + (1 - \lambda)b$, for some $\lambda > 0$. If λ is sufficiently small, then $q \in S_b^\phi(t)$. Choose λ so that $q \in S_b^\phi(t)$ with $g(q) = (t/2)$. We can now show that $\lambda \leq C\epsilon$. In fact,

$$\frac{|p_2 - b|}{|q - b|} = \frac{|p_2 - b|}{\lambda|p_2 - b|} = \lambda^{-1}.$$

But then since $q \in S_b^\phi(t)$,

$$\begin{aligned}
\epsilon^{-1} &\leq C\left(\frac{p_2 \cdot f}{q \cdot f}\right) \\
&= \frac{C(p_2 \cdot f)}{\lambda(p_2 \cdot f) + (1 - \lambda)(b \cdot f)} \\
&= \frac{C}{\lambda + (1 - \lambda)\frac{(b \cdot f)}{(p_2 \cdot f)}} \\
&= \frac{C}{\lambda + (1 - \lambda)(O(\epsilon))},
\end{aligned}$$

Thus $\lambda \leq C\epsilon$. Then, since g is convex and $g(b) = 0$, we have

$$g(p_2) \geq \frac{|p_2 - b|}{|q - b|}(g(q) - g(b)) \geq C\epsilon^{-1}t.$$

This proves the claim, and in particular implies that if ϵ is small, then $g(p_2) \geq (10)t$. Also, since $p \in S_b^\phi(t)$, $g(p) < t$, and so $g(p_2) - g(p) \geq 9t$. The mean value theorem implies that

$$g(p_2) - g(p) = \nabla g(c) \cdot (p_2 - p)$$

for some $c = \alpha p + (1 - \alpha)p_2 \in S_a^\phi(t)$. Since $(p_2 - p) = |p_2 - p|f$, we have

$$\nabla g(c) \cdot f \geq \frac{9t}{|p_2 - p|}. \tag{2.12}$$

Now since $h(y) = \phi(y) - T_1^a(y)$ is convex, then (using (1.1))

$$\begin{aligned} \nabla h(c) \cdot f &\leq \frac{h(c + |p_2 - p|f) - h(c)}{|p_2 - p|} \\ &\leq \frac{t}{|p_2 - p|} \end{aligned} \tag{2.13}$$

since both c and $c + |p_2 - p|f$ lie in $S_a^\phi(t)$. (Notice that the latter lies on the line segment connecting p_1 and p_2 in $S_a^\phi(t)$.) Subtracting (2.13) from (2.12) and using the definitions of g and h gives

$$[\nabla\phi(b) - \nabla\phi(a)] \cdot f \geq \frac{8t}{|p_2 - p|} > 0.$$

Running the exact same argument, using (2.11) in place of (2.10) will show we also have the same estimate for $-f$, namely

$$[\nabla\phi(b) - \nabla\phi(a)] \cdot (-f) > 0.$$

This is of course impossible, and we have reached the desired contradiction. This proves case 2, and completes the proof of the proposition.

We now have all the ingredients to prove the covering lemma.

Proof of Theorem 2.3 (Covering Lemma): We begin by fixing an $A \gg 1$, and let

$$\Omega = \bigcup_{j=1}^{\infty} S_{b_j}^{\phi}(C)$$

be the covering in proposition 2.5. (We are taking $t = 1$). We must show there is a constant D_n such that $\sum_{j=1}^{\infty} |S_{b_j}^{\phi}(C)| \leq D_n |\Omega|$. First notice that lemma 1.6 implies

$$\sum_j |S_{b_j}^{\phi}(C)| \leq C^n \sum_j |S_{b_j}^{\phi}(1)|$$

Then the covering property (2.3) shows that the sets $S_{b_j}^{\phi}(1)$ satisfy the hypothesis of proposition 2.6 when A is large. Since the sets $\left(\sqrt[n]{2/A}\right) S_{b_j}^{\phi}(1)$ are disjoint,

$$\sum_j \left| \left(\sqrt[n]{2/A}\right) S_{b_j}^{\phi}(1) \right| \leq |\Omega|$$

Thus, using the effect of dilations on volumes, we have

$$(2/A) \sum_j |S_{b_j}^{\phi}(1)| \leq |\Omega|$$

Putting these estimates together yields

$$\sum_j |S_{b_j}^{\phi}(C)| \leq (AC^n/2)|\Omega|$$

This completes the proof of the Covering Lemma.

In order to complete the proof of theorem 2.1, we need only combine the Covering Lemma with proposition 2.1.

Chapter 3

Uniform Convexity and Related Results

The strength of theorem 2.2 is that it holds for the general class of convex weights. Unfortunately, it is a low dimensional result in the sense that it is only in low dimensions that the exponents have nearly the expected gap of $2/n$. The purpose of this chapter is to demonstrate how the method of chapter 2 can be applied to various subclasses of the convex weights to yield Carleman inequalities with the correct gap.

3.1 Osculation estimates

In chapter 2 we performed the osculation in a manner that would yield an L^1 type covering estimate for the sets $S_a^\phi(t)$. In order to obtain Carleman inequalities with the gap $2/n$, it is necessary to prove L^∞ type covering lemmas for the $S_a^\phi(t)$. We do not have a covering property of this type for general convex weights, so we must restrict our attention to certain subclasses to obtain the sharp gap. The analogue of proposition 2.1 is the following.

Proposition 3.1 *Let $n \geq 2$, and let the pair of exponents (p,q) be as in theorem 2.2. Let $\Omega \subseteq \mathbf{R}^n$ be convex, and ϕ be convex on Ω. Let $\Omega = \bigcup_{a\in J} S_a^\phi(t)$ be any covering of Ω by the sets $S_a^\phi(t)$. Then there is a constant $A = A(n,p,q)$ such that*

$$\|e^\phi f\|_{L^q(\Omega)} \leq Ae^t \left\| \sum_{a\in J} e^{-p(\phi(x)-T_1^a(x))} \right\|_{L^\infty(\Omega)}^{1/p} \left\| e^\phi \triangle f \right\|_{L^p(\Omega)}$$

holds for all $f \in C_0^\infty(\Omega)$.

Proof: This follows the same pattern as proposition 2.1. As before, we give the proof for $n \geq 3$. We begin with the local estimate

$$\int_{S_a^\phi(t)} e^{q\phi}|f|^q dx \leq (Ae^t)^q \left(\int_\Omega e^{-p(\phi(x)-T_1^a(x))} e^{p\phi(x)} |\triangle f|^p dx \right)^{q/p}$$

which follows from theorem 2.2. Then since the $S_a^\phi(t)$ cover Ω, we have

$$\int_\Omega e^{q\phi}|f|^q dx \leq (Ae^t)^q \sum_{a\in J} \left(\int_\Omega e^{-p(\phi(x)-T_1^a(x))} e^{p\phi(x)} |\triangle f|^p dx \right)^{q/p}.$$

Now as $(q/p) > 1$ we may bring the sum inside the integral and obtain

$$\int_\Omega e^{q\phi}|f|^q dx \leq (Ae^t)^q \left(\int_\Omega \sum_{a\in J} e^{-p(\phi(x)-T_1^a(x))} e^{p\phi(x)} |\triangle f|^p dx \right)^{q/p}.$$

The proposition now follows by taking qth roots and applying Holder's inequality to the right hand side.

In some situations it is desirable to have an L^∞ type covering estimate with a smaller gap between exponents. For this we record the following generalization of proposition 3.1.

Proposition 3.2 *Let (p,q) be as in proposition 3.1. For $1/q \leq 1/s \leq 1/r \leq 1/p$, there is a constant $A = A(n,r,s)$ such that*

$$\|e^\phi f\|_{L^s(\Omega)} \leq$$

$$Ae^t sup_{a\in J}|S_a^\phi(t)|^{2/n+1/s-1/r}\left\|\sum_{a\in J} e^{-(r/2)(\phi(x)-T_1^a(x))}\right\|_{L^\infty(\Omega)}^{1/r}\left\|e^\phi\triangle f\right\|_{L^r(\Omega)}$$

holds for all $f\in C_0^\infty(\Omega)$.

Proof: The proof of this proposition is really a composite of the proofs of proposition 2.1 and proposition 3.1. First, Holder's inequality and theorem 2.2 imply

$$\int_{S_a^\phi(t)} e^{s\phi}|f|^s dx \leq (Ae^t)^s|S_a^\phi(t)|^{s(1/s-1/q)}\left\|e^{T_1^a(x)}\triangle f\right\|_{L^p(\Omega)}^s.$$

We then write $e^{T_1^a(x)} = e^{(-1/2)(\phi(x)-T_1^a(x))}e^{(-1/2)(\phi(x)-T_1^a(x))}e^{\phi(x)}$ and then another application of Holder's inequality along with proposition 1.1 gives

$$\int_{S_a^\phi(t)} e^{s\phi}|f|^s dx \leq$$

$$(CAe^t)^s|S_a^\phi(t)|^{s(1/s-1/q)+s(1/p-1/r)}\left\|e^{(-1/2)(\phi(x)-T_1^a(x))}e^\phi\triangle f\right\|_{L^r(\Omega)}^s.$$

Since $1/p-1/q=2/n$, we may take the sup over $a\in J$ and obtain

$$\int_{S_a^\phi(t)} e^{s\phi}|f|^s dx \leq$$

$$(CAe^t)^s sup_{a\in J}|S_a^\phi(t)|^{s(2/n+1/s-1/r)}\left\|e^{(-1/2)(\phi(x)-T_1^a(x))}e^{\phi(x)}\triangle f\right\|_{L^r(\Omega)}.$$

Now we proceed as before, dominating the integral over Ω by the sum of the integrals over the $S_a^\phi(t)$. Since $(s/r)\geq 1$, we may bring the sum inside the integral on the right hand side obtaining

$$\|e^\phi f\|_{L^s(\Omega)}^s \leq$$

$$(CAe^t)^s sup_{a\in J}|S_a^\phi(t)|^{s(2/n+1/s-1/r)}\left(\int_\Omega\sum_{a\in J} e^{(-r/2)(\phi-T_1^a)}e^{r\phi}|\triangle f|^r dx\right)^{s/r}.$$

To complete the proof we simply take the $1/s$ power and apply Holder's inequality once more.

3.2 Uniform convexity

Our first application of these osculation estimates is a result concerning essentially uniformly convex weights. Hormander [6] proved an L^2 Carleman inequality for this class of weights in connection to uniqueness for the Cauchy problem. Our definition comes from that paper.

We suppose that Ω is a bounded convex domain in $\mathbf{R}^n$, and that $\phi \in C^2(\overline{\Omega})$. For $x, y \in \Omega$, consider the linear hull of the vectors

$$\nabla\phi(x) - \nabla\phi(y)$$

This is a subspace of $\mathbf{R}^n$, and we may choose coordinates so that this subspace is defined by

$$\{(x_1, x_2, \ldots, x_n) : x_{k+1} = \ldots = x_n = 0\}$$

Observe that $k = 0$ if and only if ϕ is linear. Now if $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi–index, we define the multi–index $\alpha* = (\alpha_1, \ldots, \alpha_k, 0, \ldots, 0)$, and for $y = (y_1, \ldots, y_n) \in \mathbf{R}^n$, we set $|y|^{*2} = y_1^2 + \ldots + y_k^2$.

Definition 3.1 *The function $\phi \in C^2(\overline{\Omega})$ is essentially uniformly convex if the quadratic form $\sum_{j=1}^n \sum_{k=1}^n y^j y^k \partial^2\phi/\partial x_j \partial x_k$ is positive definite in the plane $\{y^{k+1} = \ldots = y^n = 0\}$.*

In this situation, we will say ϕ is linear in the $(x_{k+1}, \ldots, x_n)$ directions. Our aim is to use proposition 3.2 to deduce a Carleman inequality for the one parameter family of weights $e^{t\phi}$, where $t \geq 1$, and ϕ is essentially uniformly convex. The following lemmas estimate the relevant quantities in proposition 3.2.

Lemma 3.1 *Suppose $\phi \in C^2(\overline{\Omega})$ is essentially uniformly convex, and $t \geq 1$. Then there is a constant $C = C(\phi, \Omega)$ such that*

$$|S_a^{t\phi}(1)| \leq Ct^{-k/2}.$$

Proof Since ϕ is linear in the $(x_{k+1}, \ldots, x_n)$ directions, it follows that there is a constant C such that

$$C^{-1}t|x-a|^{*2} \leq t(\phi(x) - T_1^a(x)) \leq Ct|x-a|^{*2}. \tag{3.1}$$

The constant C is uniform as $\phi \in C^2(\overline{\Omega})$. The lemma follows immediately from the left inequality, since the set where the left hand side is < 1 is essentially a cube in $\mathbf{R}^k$ of side $(C/t)^{1/2}$, crossed with a rectangle in the remaining $n-k$ directions whose side lengths are bounded by the diameter of Ω.

Lemma 3.2 *Let R be a rectangle in $\mathbf{R}^n$, centered at 0, and with sides parallel to the axes. Let F be a bounded subset of $\mathbf{R}^n$, and for $a \in F$, let R_a be the translate of R with center at a. Then for $A \geq 1$, there is a constant $C = C(n, A)$ and a covering $F = \bigcup_{j=1}^N R_{a_j}$ such that each point in F lies in at most C of the sets AR_{a_j}.*

Proof: We begin by selecting a covering $F = \bigcup_{a \in J} R_a$ with the property that if x_j is the center of R_{a_j}, then $x_j \notin R_{a_l}$ for $j \neq l$. Then it follows that there is a constant C_n such that

$$C_n^{-1}R_{a_i} \bigcap C_n^{-1}R_{a_j} = \emptyset. \tag{3.2}$$

This follows from the fact that if $\lambda R_a \bigcap \lambda R_b \neq \emptyset$ there is a constant C_n such that

$$\lambda R_a \subseteq \lambda C_n R_b. \tag{3.3}$$

Hence if (3.2) is false, then $C_n^{-1}R_{a_i} \subseteq R_{a_j}$, which contradicts $x_i \notin R_{a_j}$.

Now suppose $y \in F$, lies in M of the sets AR_{a_j}, say $y \in AR_1 \bigcap \cdots \bigcap AR_M$. Then (3.3) implies that

$$\bigcup_{j=1}^{M} AR_j \subseteq C_n AR_1. \tag{3.4}$$

Now (3.2) implies that the sets $(C_nA)^{-1}AR_j$ are disjoint and so

$$\begin{aligned} \left|\bigcup_{j=1}^{M}(C_nA)^{-1}AR_j\right| &= \sum_{j=1}^{M}\left|(C_nA)^{-1}AR_j\right| \\ &= C_n^{-n}M|R_1|. \end{aligned}$$

On the other hand, (3.4) implies that

$$\begin{aligned} \left|\bigcup_{j=1}^{M}(C_nA)^{-1}AR_j\right| &\leq \left|\bigcup_{j=1}^{M}AR_j\right| \\ &\leq |C_nAR_1| \\ &= (C_nA)^n|R_1|. \end{aligned}$$

Combining these two observations yields $M \leq (C_n^2A)^n$. This proves the lemma.

Proposition 3.3 *Suppose $\phi \in C^2(\overline{\Omega})$ is essentially uniformly convex, and $t \geq 1$. Then, for any $\rho > 0$, there is a constant $C = C(\rho, \phi, \Omega)$ and a covering $\Omega = \bigcup_{a \in J} S_a^{t\phi}(1)$ such that*

$$\left\|\sum_{a \in J} e^{(-\rho t)(\phi(x) - T_1^a(x))}\right\|_{L^\infty(\Omega)} \leq C$$

Proof: First observe that (3.1) implies that for some C we have

$$\left\{x : \sum_{j=1}^{k}|x_j - a_j|^2 \leq 1/(Ct)\right\} \subset S_a^{t\phi}(1) \subset \left\{x : \sum_{j=1}^{k}|x_j - a_j|^2 \leq C/t\right\}.$$

We then cover Ω with the sets

$$R_a = \left\{ x : \sum_{j=1}^{k} |x_j - a_j|^2 \leq 1/(Ct) \right\}.$$

We then have

$$\Omega = \bigcup_{a \in J} R_a.$$

Since the R_a are rectangles of the same size with sides parallel to the axes, lemma 3.2 is applicable, and so we may assume that each point in Ω is contained in at most C_n elements of $\bigcup_{a \in J} C^3 R_a$.

Then, as $R_a \subset S_a^{t\phi}(1)$, we also have

$$\Omega = \bigcup_{a \in J} S_a^{t\phi}(1),$$

and each point in Ω lies in at most C_n elements of $\bigcup_{a \in J} S_a^{t\phi}(1)$. This is because

$$S_a^{t\phi}(1) \subset \left\{ x : \sum_{j=1}^{k} |x_j - a_j|^2 \leq C/t \right\} \subset C^3 R_a,$$

where we have used lemmas 1.5 and 1.6.

Now if $x \in \Omega$, we have

$$\sum_{a \in J} e^{(-\rho t)(\phi(x) - T_1^a(x))} \leq C t^{k/2} \int_{\Omega} e^{(-\rho t)(\phi(x) - T_1^a(x))} da.$$

But then (3.1) implies that

$$\sum_{a \in J} e^{(-\rho t)(\phi(x) - T_1^a(x))} \leq C t^{k/2} \int_{\Omega} e^{-C^{-1} \rho t |x-a|^{*2}} da.$$

We may integrate out the last $n - k$ variables with a bound $diam(\Omega)^{(n-k)}$. We are left with

$$\sum_{a \in J} e^{(-\rho t)(\phi(x) - T_1^a(x))} \leq C_{n,\Omega} t^{k/2} \int_{R^k} e^{(-C^{-1} \rho t) |x-a|^{*2}} da_1 \ldots da_k.$$

After a linear change of variables, we obtain

$$\sum_{a\in J} e^{(-\rho t)(\phi(x)-T_1^a(x))} \leq C_{n,\Omega}\rho^{-k}\int_{R^k} e^{-|y|^2}\,dy.$$

Since the RHS is bounded by a constant independent of x, this proves the proposition.

We remark that proposition 3.3 is false if $\phi(x)$ is assumed only to be convex. For example, if $\phi(x) = |x|$, then

$$\|\sum_{a\in J} e^{(-\rho t)(\phi(x)-T_1^a(x))}\|_{L^\infty(\Omega)} \geq C^{-1}t^{(n-1)/2}|\Omega|.$$

We may now put together lemma 3.1 together with propositions 3.2 and 3.3 to prove the following.

Theorem 3.1 *Suppose Ω is a bounded domain in $\mathbf{R}^n$, $n \geq 2$, and $\phi \in C^2(\overline{\Omega})$ is essentially uniformly convex, and linear in the $(x_{k+1}, \ldots, x_n)$ directions. Also suppose $t \geq 1$, and (p, q) is as in theorem 2.2. Then for $1/q \leq 1/s \leq 1/r \leq 1/p$ there is a constant $A = A(n, r, s, \phi, \Omega)$ such that*

$$\|e^{t\phi}f\|_{L^s} \leq At^{(-k/2)(2/n+1/s-1/r)}\|e^{t\phi}\Delta f\|_{L^r}$$

holds for all $f \in C_0^\infty(\Omega)$.

If $n \geq 3$ and $2n/(n+2) \leq r \leq 2$, (or $n = 2$ and $1 < r \leq 2$), we may replace Δf by $p(D)f$, where $p(D)$ is any second order constant coefficient differential operator with principal part given by $Q(\xi) = -\xi_1^2 - \cdots - \xi_j^2 + \xi_{j+1}^2 + \cdots + \xi_n^2$, for some $0 \leq j \leq n$.

When $r = s = 2$, this is essentially in [6], with the inclusion of an additional estimate on the gradient. In that paper, it is necessary that $\phi \in C^3(\overline{\Omega})$ since ϕ is approximated uniformly by quadratic polynomials rather than linear ones.

3.3 Estimates on $\mathbf{R}^n$

We show here that in some cases where $\phi(x)$ is sufficiently convex, we may obtain Carleman inequalities on all of $\mathbf{R}^n$. The particular example we have in mind is the family $\phi(x) = |x|^\rho$, for $\rho > 1$. There are Carleman inequalities on $L^2(\mathbf{R}^n)$ for these weights due to Stromberg [12]. We will prove $L^p \to L^q$ inequalities for $1/p - 1/q = 2/n$. When $\rho \geq 2$, the extra convexity can be used to prove estimates with smaller gap conditions.

Theorem 3.2 *Let $n \geq 3$, $1 < \rho < \infty$, and suppose (p, q) are as in theorem 2.2. Then there is a constant $A = A(n, p, \rho)$ such that*

$$\|e^{|x|^\rho} f\|_{L^q(\mathbf{R}^n)} \leq A \|e^{|x|^\rho} \triangle f\|_{L^p(\mathbf{R}^n)}$$

holds for all $f \in C_0^\infty(\mathbf{R}^n)$.

Proof: First, observe that

$$\begin{aligned} \|e^{|x|^\rho} f\|_{L^q(\{y:|y|^\rho \leq 100\})} &\leq e^{100\rho} \|f\|_{L^q(\mathbf{R}^n)} \\ &\leq A e^{100\rho} \|\triangle f\|_{L^p(\mathbf{R}^n)} \\ &\leq A e^{100\rho} \|e^{|x|^\rho} \triangle f\|_{L^p(\mathbf{R}^n)}. \end{aligned}$$

Thus in order to prove the theorem it suffices to show $\|e^{|x|^\rho} f\|_{L^q(\{y:|y|^\rho \geq 100\})} \leq C \|e^{|x|^\rho} \triangle f\|_{L^p(\mathbf{R}^n)}$. In view of proposition 3.1, this latter estimate is a consequence of lemma 3.3 below. (Notice that in proposition 3.1, Ω is assumed convex only to insure a direct definition for the convexity of ϕ on Ω. In particular, the proposition is valid on (nonconvex) subsets of Ω.)

Lemma 3.3 *Let $n \geq 2$, and $\rho > 1$. Set $\phi(x) = |x|^\rho$. Then there is a constant $A = A(n, \rho)$, and a covering $\{y : |y|^\rho \geq 100\} = \bigcup_{a \in J} S_a^\phi(\rho - 1)$ such that*

$$\|\sum_{a \in J} e^{-p(\phi(x) - T_1^a(x))}\|_{L^\infty(\mathbf{R}^n)} \leq A.$$

Proof: We begin by showing that the sets $S_a^\phi(\rho-1)$ are essentially balls of radius $|a|^{(2-\rho)/2}$ centered at a. We then obtain a covering $\{y : |y|^\rho \geq 100\} = \bigcup_{a\in J} S_a^\phi(\rho-1)$, with the property that at most $C_{n\rho}$ of the sets $S_a^\phi(\rho-1)$ intersect. Finally, we demonstrate that this covering has the desired property.

Step 1: If $|a|^\rho \geq 100$, there is a C_ρ such that

$$\{x : |a|^{\rho-2}|x-a|^2 \leq C_\rho^{-1}\} \subseteq S_a^\phi(\rho-1) \subseteq \{x : |a|^{\rho-2}|x-a|^2 \leq C_\rho\}. \quad (3.5)$$

To prove the assertion, first note that $x_0 \in S_a^\phi(\rho-1)$ if and only if

$$|x_0|^\rho + (\rho-1)|a|^\rho - \rho|a|^{\rho-2}a\cdot x_0 < \rho - 1.$$

Write $(x_0-a)\cdot a = \mu|a|$, and $\lambda = |x_0 - a - \frac{\mu a}{|a|}|$. Then $|x_0|^\rho = [(|a|+\mu)^2+\lambda^2]^{\rho/2}$ and $a\cdot x_0 = |a|^2 + \mu|a|$. Thus, we have $x_0 \in S_a^\phi(\rho-1)$ iff

$$\left[(|a|+\mu)^2+\lambda^2\right]^{\rho/2} - |a|^\rho - \rho|a|^{\rho-1}\mu < \rho - 1.$$

Factoring $|a|^\rho$ from the left hand side, and setting $x = \mu/|a|$, and $y = \lambda/|a|$, we see the above line is equivalent to

$$\left[(1+x)^2+y^2\right]^{\rho/2} - \rho x - 1 < (\rho-1)|a|^{-\rho} < (\rho-1)/100. \quad (3.6)$$

If we set $f(x,y) = [(1+x)^2+y^2]^{\rho/2}$, then $f(x,y)$ is convex and the left hand side of (3.6) is $f(x,y) - T_1^0(x,y) = H(c)(x,y)\cdot(x,y)$, where $H(c)$ is the 2×2 matrix of second partial derivatives $\left(\frac{\partial^2\phi(c)}{\partial x\partial y}\right)$, and c is a point on the line segment connecting (x,y) to $(0,0)$. The eigenvalues of $H(c)$ are $\lambda_1 = \rho(\rho-1)[(1+c_1)^2+c_2^2]^{(\rho-2)/2}$, and $\lambda_2 = \rho[(1+c_1)^2+c_2^2]^{(\rho-2)/2}$, and since

$$min(\lambda_1,\lambda_2)(x^2+y^2) \leq H(c)(x,y)\cdot(x,y) \leq max(\lambda_1,\lambda_2)(x^2+y^2),$$

we can conclude that

$$C_\rho^{-1}(x^2+y^2) \leq \left[(1+x)^2+y^2\right]^{\rho/2} - \rho x - 1 \leq C_\rho(x^2+y^2),$$

as λ_1 and λ_2 are $\approx C_\rho$ on the set where (3.6) is valid. Recalling now the definitions of x and y, we have shown that $z \in S_a^\phi(\rho - 1)$ implies

$$C_\rho^{-1}|a|^{\rho-2}|z-a|^2 \leq \phi(z) - T_1^a(z) \leq C_\rho |a|^{\rho-2}|z-a|^2.$$

This proves the assertion of Step 1.

Step 2: If $|a|^\rho \geq 100$, and $A \geq 1$, there is a covering

$$\{|a|^\rho \geq 100\} = \bigcup_{a \in J} B\left(a, C_\rho^{-1/2}|a|^{(2-\rho)/2}\right) = \bigcup_{a \in J} B(a, r_a),$$

with the following properties:

$$\leq C \text{ of the sets } B(a, Ar_a) \text{ intersect,} \tag{3.7}$$

$$\text{if } B(a, r_a) \bigcap B(b, r_b) \neq \emptyset \text{ then } |a|/2 < |b| < 2|a|. \tag{3.8}$$

We begin by showing (3.8) holds. If $x \in B(a, r_a) \bigcap B(b, r_b)$, then

$$|a - b| \leq |x - a| + |x - b| \leq r_a + r_b.$$

Then, as $C_\rho \geq 1$, $|a|^\rho \geq 100, |b|^\rho \geq 100$, we have

$$|a| \leq |b| + |a - b| \leq |b| + |a|/10 + |b|/10.$$

Hence, $|a| \leq 11|b|/9$, and similarly $|b| \leq 11|a|/9$. This proves (3.8). We acquire a covering satisfying (3.7) along the lines of lemma 3.2. Begin by choosing some $B(a_1, r_{a_1})$. Suppose $B(a_k, r_{a_k})$ have been chosen for $k = 1, 2, \ldots, n$. If this collection does not cover the set, select $B(a_{n+1}, r_{a_{n+1}})$ with $a_{n+1} \notin \bigcup_{k=1}^n B(a_k, r_{a_k})$. In this way, we obtain a covering of $\{|c|^\rho \geq 100\}$ by the sets $B(a, r_a)$. To show that this cover satisfies (3.7), first note that (3.8) implies that if $B(a_j, r_{a_j}) \bigcap B(a_k, r_{a_k}) \neq \emptyset$ then $B(a_j, r_{a_j}) \subseteq B(a_k, c_\rho r_{a_k})$. This implies that if $k \neq j$,then

$$\frac{1}{c_\rho} B(a_j, r_{a_j}) \bigcap \frac{1}{c_\rho} B(a_k, r_{a_k}) = \emptyset, \tag{3.9}$$

since $a_k \notin B(a_j, r_{a_j})$ if $k > j$. The verification of (3.7) now follows exactly as in lemma 3.2. We omit the details.

Step 3: Let $\{y : |y|^\rho \geq 100\} = \bigcup_{a \in J} B(a, r_a)$ be the covering obtained in Step 2. Then $\{y : |y|^\rho \geq 100\} = \bigcup_{a \in J} S_a^\phi(1)$ as $B(a, r_a) \subseteq S_a^\phi(1)$. We fix an x and show $\sum_{a \in J} e^{-(\phi(x) - T_1^a(x))} \leq C$, where C is independent of x. This estimate is obtained by dividing the sum into 3 regions. Let $J_1 = J \bigcap \{a : |a| \geq \frac{10\rho}{\rho - 1}|x|\}$. We begin by showing $\sum_{a \in J_1} e^{-(\phi(x) - T_1^a(x))} \leq C$.

For $a \in J_1$, we have $\phi(x) - T_1^a(x) \geq c_\rho |a|^\rho$, and hence

$$\sum_{a \in J_1} e^{-(\phi(x) - T_1^a(x))} \leq \sum_{a \in J_1} e^{-c_\rho |a|^\rho}.$$

We next replace the sum by the appropriate integral. More precisely, we claim

$$\sum_{a \in J_1} e^{-c_\rho |a|^\rho} \leq C \int_{\mathbf{R}^n} e^{-c_\rho |y|^\rho} |y|^{(\rho - 2)n/2} dy.$$

This inequality is justified as follows. We have

$$\begin{aligned}
\frac{1}{|S_a^\phi(1)|} \int_{S_a^\phi(1)} e^{-c_\rho |y|^\rho} dy &= e^{-c_\rho |a|^\rho} |S_a^\phi(1)|^{-1} \int_{S_a^\phi(1)} e^{-c_\rho (|y|^\rho - |a|^\rho)} dy \\
&\geq e^{-c_\rho |a|^\rho} |S_a^\phi(1)|^{-1} \int_{S_a^\phi(1) \bigcap \{|y| \leq |a|\}} e^{-c_\rho (|y|^\rho - |a|^\rho)} dy \\
&\geq d_\rho e^{-c_\rho |a|^\rho} \frac{\left| S_a^\phi(1) \bigcap \{|y| \leq |a|\} \right|}{|S_a^\phi(1)|} \\
&\geq c d_\rho e^{-c_\rho |a|^\rho}.
\end{aligned}$$

Thus we have

$$\sum_{a \in J_1} e^{-c_\rho |a|^\rho} \leq \sum_{a \in J_1} \int_{\mathbf{R}^n} e^{-c_\rho |y|^\rho} \frac{\chi_{S_a^\phi(1)}(y)}{|S_a^\phi(1)|} dy.$$

Bringing the sum inside the integral, and observing that (3.7) and (3.8) imply that

$$\sum_{a \in J_1} \frac{\chi_{S_a^\phi(1)}(y)}{|S_a^\phi(1)|} \leq C |y|^{(\rho - 2)n/2},$$

we see that

$$\sum_{a\in J_1} e^{-c_\rho|a|^\rho} \le C_\rho \int_{\mathbf{R}^n} e^{-c_\rho|y|^\rho}|y|^{(\rho-2)n/2}dy.$$

This proves the claim, and since the integral on the right is clearly bounded by a constant depending only on ρ and n, we have the required estimate for the sum over J_1.

Let $J_2 = J\bigcap\{a : \frac{|x|}{100\rho} \le |a| \le \frac{10\rho}{\rho-1}|x|\}$. Recall also that $a \in J$ implies that $|a|^\rho \ge 100$. Recall from the proof of Step 1 that the estimate $\phi(x) - T_1^a(x) \ge c_{\rho n}|a|^{\rho-2}|x-a|^2$ holds for all x. In particular, for $a \in J_2$ we have $\phi(x) - T_1^a(x) \ge c_{\rho n}|x|^{\rho-2}|x-a|^2$. Hence

$$\begin{aligned}
\sum_{a\in J_2} e^{-(\phi(x)-T_1^a(x))} &\le \sum_{a\in J_2} e^{-c_\rho|x|^{\rho-2}|x-a|^2} \\
&\le C_{\rho n}\int_{\mathbf{R}^n} e^{-c_\rho|x|^{\rho-2}|x-y|^2}|x|^{(\rho-2)n/2}dy \\
&= C_{\rho n}\int_{\mathbf{R}^n} e^{-c_\rho|u|^2}du \\
&\le CC_{n\rho}.
\end{aligned}$$

This provides the desired estimate for the sum over J_2.

Lastly, let $J_3 = J\bigcap\{a : |a| \le \frac{|x|}{100\rho}\}$. If $a \in J_3$, then $\phi(x) - T_1^a(x) \ge |x|^\rho/2$. Hence, we have

$$\begin{aligned}
\sum_{a\in J_3} e^{-(\phi(x)-T_1^a(x))} &\le e^{-|x|^\rho/2}\sum_{a\in J_3} 1 \\
&= e^{-|x|^\rho/2}\sum_{a\in J_3}\frac{1}{|S_a^\phi(1)|}\int_{\mathbf{R}^n}\chi_{S_a^\phi(1)}(y)dy \\
&= e^{-|x|^\rho/2}\int_{\mathbf{R}^n}\left(\sum_{a\in J_3}\frac{\chi_{S_a^\phi(1)}(y)}{|S_a^\phi(1)|}\right)dy \\
&\le Ce^{-|x|^\rho/2}\int_{|y|\le c|x|}|y|^{(\rho-2)n/2}dy \\
&\le C_1e^{-|x|^\rho/2}\int_0^{c|x|} r^{(\rho-2)n/2+n-1}dr
\end{aligned}$$

$$
\begin{aligned}
&\leq C_2|x|^{\rho n/2}e^{-|x|^\rho/2}\\
&\leq C_3.
\end{aligned}
$$

Combining our estimates, we have obtained a cover $\{y : |y|^\rho \geq 100\} = \bigcup_{a\in J} S_a^\phi(1)$ with the property that $\sum_{a\in J} e^{-(\phi(x)-T_1^a(x))} \leq C$. This completes the proof of lemma 3.3.

We finish by presenting a strengthening of theorem 3.2 which is valid when $\rho \geq 2$.

Theorem 3.3 *Let $\rho \geq 2$. For $n = 2$, let $1/p - 1/q < 1$. For $n \geq 3$, let $1/p - 1/q \leq 2/n$ and $2n/(n+3) < p \leq q < 2n/(n-3)$. Then there is a constant $A = A(n,p,\rho)$ such that*

$$\|e^{|x|^\rho} f\|_{L^q(\mathbf{R}^n)} \leq A\|e^{|x|^\rho}\Delta f\|_{L^p(\mathbf{R}^n)}$$

holds for all $f \in C_0^\infty(\mathbf{R}^n)$.

Proof: When $\rho = 2$, the cover obtained in lemma 3.3 extends to a cover of $\mathbf{R}^n$, as the sets $S_a^\phi(t)$ are all balls of constant radius. The theorem for $\rho = 2$ then follows from proposition 3.2. If $\rho > 2$, lemma 3.3 and proposition 3.2 imply that

$$\|e^{|x|^\rho} f\|_{L^q(\{|y|^\rho > 100\})} \leq A\|e^{|x|^\rho}\Delta f\|_{L^p(\mathbf{R}^n)}.$$

On the other hand,

$$
\begin{aligned}
\|e^{|x|^\rho} f\|_{L^q(\{|y|^\rho \leq 100\})} &\leq e^{100}\|f\|_{L^q(\{|y|^\rho\leq 100\})}\\
&\leq e^{100}\|e^{|x|^2} f\|_{L^q(\mathbf{R}^n)}\\
&\leq C\|e^{|x|^2}\Delta f\|_{L^p(\mathbf{R}^n)}\\
&\leq C_1\|e^{|x|^\rho}\Delta f\|_{L^p(\mathbf{R}^n)}.
\end{aligned}
$$

Combining the two estimates proves the theorem.

We conclude by considering the situation of one–dimensional weights. In this instance, the covering lemma is particularly simple to prove, and we obtain $L^p(\mathbf{R}^n \longrightarrow L^q(\mathbf{R}^n)$ estimates for the same range of exponents valid for linear weights. A similar osculation argument using one–dimensional weights appears in [13].

We wish to invoke proposition 3.1 so we first prove the following lemma.

Lemma 3.4 *Suppose ϕ is a convex function on $\mathbf{R}$, and Q is a cube with sides parallel to the coordinate axes in $\mathbf{R}^n$. For $x \in \mathbf{R}^n$, we write $\phi = \phi(x_n)$. Then for $p \geq 1$, there is a covering $Q \subseteq \bigcup_{a\in J} S_a^\phi(1)$, and a constant A such that*

$$\|\sum_{a\in J} e^{-p(\phi(x_n)-T_1^a(x_n))}\|_{L^\infty(Q)} \leq A.$$

Proof: We have $Q = \bigcup_{a\in\mathbf{R}^n} \left(S_a^\phi(1)\bigcap Q\right)$. Using the Besicovitch covering lemma, we may select a subcover $Q = \bigcup_k \left(S_{a_k}^\phi(1)\bigcap Q\right)$ with the property that each point in Q belongs to at most 2 elements of the cover, and that each element of the cover intersects at most 2 other elements. Let $e_n = (0,\ldots,0,1)$. We may assume the sets are ordered so that $k > j$ implies that $\gamma_{S_{a_k}^\phi(1)\bigcap Q}(e_n) \geq \gamma_{S_{a_j}^\phi(1)\bigcap Q}(e_n)$. Then if $(k-j) > 1$, $S_{a_k}^\phi(1)\bigcap S_{a_j}^\phi(1)\bigcap Q = \emptyset$, and $a_k \cdot e_n > a_j \cdot e_n$.

<u>Claim</u> If $x \in S_{a_k}^\phi(1)\bigcap Q$, then

$$\phi(x_n) - T_1^{a_{k+2j}}(x_n) > |j|,$$

and

$$\phi(x_n) - T_1^{a_{k+2j+1}}(x_n) > min\{|j|, |j+1|\},$$

holds for all integers j. We prove the first assertion of the claim for $j \geq 0$. The other cases are similar. If $j = 1$ the claim is just a restatement of the fact that $S^{\phi}_{a_{k+2}}(1) \bigcap S^{\phi}_{a_k}(1) \bigcap Q = \emptyset$. We proceed for $j > 1$ by induction on j. If $x \in S^{\phi}_{a_k}(1) \bigcap Q$, then

$$\begin{aligned} \phi(x_n) - T_1^{a_{k+2j}}(x_n) &= \phi(x_n) - T_1^{a_{k+2j-2}}(x_n) + T_1^{a_{k+2j-2}}(x_n) - T_1^{a_{k+2j}}(x_n) \\ &> (j-1) + T_1^{a_{k+2j-2}}(x_n) - T_1^{a_{k+2j}}(x_n). \end{aligned}$$

The linear function $g(y) = T_1^{a_{k+2j-2}}(y) - T_1^{a_{k+2j}}(y)$ satisfies

$$g'(y) = \phi'(a_{k+2j-2} \cdot e_n) - \phi'(a_{k+2j} \cdot e_n) \leq 0,$$

and so $g(y)$ is decreasing. Hence, since $x_n < a_{k+2j-2} \cdot e_n$, we have

$$\begin{aligned} g(x_n) &\geq g(a_{k+2j-2} \cdot e_n) \\ &= T_1^{a_{k+2j-2}}(a_{k+2j-2} \cdot e_n) - T_1^{a_{k+2j}}(a_{k+2j-2} \cdot e_n) \\ &= \phi(a_{k+2j-2} \cdot e_n) - T_1^{a_{k+2j}}(a_{k+2j-2} \cdot e_n) \\ &> 1, \end{aligned}$$

since $a_{k+2j-2} \notin S^{\phi}_{a_{k+2j}}(1)$. Hence, we have

$$\phi(x_n) - T_1^{a_{k+2j}}(x_n) > j$$

which completes the induction, and proves the claim.

We may show that our covering has the required property by summing a geometric series. If $x \in S^{\phi}_{a_k}(1) \bigcap Q$, then

$$\begin{aligned} \sum_l e^{-p(\phi(x_n) - T_1^{a_l}(x_n))} &= \\ \sum_j e^{-p(\phi(x_n) - T_1^{a_{k+2j}}(x_n))} &+ \sum_j e^{-p(\phi(x_n) - T_1^{a_{k+2j+1}}(x_n))} \\ &\leq 4 \sum_{j=0}^{\infty} e^{-pj} \\ &\leq A. \end{aligned}$$

This proves the lemma.

For a given $f \in C_0^\infty(\mathbf{R}^n)$, we may choose a cube Q, as in lemma 3.4, large enough to contain the support of f. We may then apply lemma 3.4 together with proposition 3.1 to conclude the following.

Theorem 3.4 *Suppose ϕ is convex on $\mathbf{R}$, and for $n \geq 3$, we have $1/p - 1/q = 2/n$ and $2n/(n+3) < p < q < 2n/(n-3)$. Then there is a constant $A = A_p$ such that*

$$\|e^{\phi(x_n)} f\|_{L^q(\mathbf{R}^n)} \leq A \|e^{\phi(x_n)} \Delta f\|_{L^p(\mathbf{R}^n)}$$

holds for all $f \in C_0^\infty(\mathbf{R}^n)$.

We remark that in theorems 3.2, 3.3 and 3.4, we may replace Δf by $p(D)f$ under the conditions provided in theorem 2.2.

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