Constructing Essential Laminations in Some 3-Manifolds

Thesis by
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Abstract

In trying to understand 3-manifolds (with the hope of eventually classifying them as with 2-manifolds), one approach that has turned out to be fruitful is to study objects of codimension one in them, more specifically, incompressible surfaces, taut foliations (or foliations without Reeb components), and essential laminations (loosely speaking, a lamination in a manifold is a foliation of a closed subset of that manifold).

For a 3-manifold M containing an incompressible surface (with some extra hypotheses), Waldhausen proved great theorems such as: M has infinite fundamental group, the universal cover of M is $\mathbf{R}^3$, homotopic homeomorphisms of M are isotopic, and $\pi_1(M)$ determines M up to homeomorphism.

Similar theorems for manifolds containing taut foliations were proven by Novikov, Haefliger, Rosenberg, and others.

The essential lamination was developed comparatively recently (late 1980's) as a generalization of the incompressible surface and the taut foliation, which themselves qualify as essential laminations. In fact they are just extreme cases of essential laminations: at one end we have surfaces, which are properly embedded, and at the other end we have foliations, which fill up the manifold (empty complement). A typical lamination is in general somewhere in between; it is nowhere dense as in the case of surfaces, but has non-compact leaves as in foliations.

Analogues of some of the theorems of Waldhausen have been proven ([GO]) for closed manifolds admitting essential laminations: they are irreducible, have infinite fundamental group, and are covered by $\mathbf{R}^3$. Other questions such as whether homotopic homeomorphisms are isotopic are being worked on.

An advantage of the essential lamination over its ancestors, the incompressible surface and the taut foliation, is that it is much more common and easier to find. It is conjectured (or hoped) that "most" closed 3-manifolds admit essential laminations (and the results of this thesis are in support of this conjecture).

So believing that the essential lamination is indeed a useful tool in the study of 3-manifolds, and hoping that it is common in them, an important question that arises is: *Which 3-manifolds admit essential laminations?*

In this thesis we answer this question for those manifolds obtained by surgery on 2-bridge knots in the 3-sphere:

Theorem 1 *Surgery on a 2-bridge torus knot* $T_{2,q}$*, with coefficient* $\in (-\infty, q-2)$ *yields a manifold which admits essential laminations.*

Theorem 2 *Nontrivial surgery on a non-torus 2-bridge knot yields a manifold which admits essential laminations.*

This gives as a corollary, for example, that property P is true for 2-bridge knots.

The main method used for 2-bridge knots was to start with a given branched surface or lamination on a given knot and go through some Kirby Calculus to see what this lamination looks like on a different knot for which we cannot get all the desired laminations, and then to try to generalize this "newly found" construction. This is quite general and could potentially be as fruitful for all knots and links.

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Introduction

In trying to understand 3-manifolds (with the hope of eventually classifying them as with 2-manifolds), one approach that has turned out to be fruitful is to study objects of codimension one in them, more specifically, incompressible surfaces, taut foliations (or foliations without Reeb components), and essential laminations (loosely speaking, a lamination in a manifold is a foliation of a closed subset of that manifold).

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Analogues of some of the theorems of Waldhausen have been proven ([GO]) for closed manifolds admitting essential laminations: they are irreducible, have infinite fundamental group, and are covered by $\mathbf{R}^3$. Other questions such as whether homotopic homeomorphisms are isotopic are being worked on.

An advantage of the essential lamination over its ancestors, the incompressible surface and the taut foliation, is that it is much more common and easier to find. It is conjectured (or hoped) that "most" closed 3-manifolds admit essential laminations (and the results of this thesis are in support of this conjecture).

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Theorem 1 *Surgery on a 2-bridge torus knot $T_{2,q}$, with coefficient $\in (-\infty, q-2)$ yields a manifold which admits essential laminations.*

Theorem 2 *Nontrivial surgery on a non-torus 2-bridge knot yields a manifold which admits essential laminations.*

This gives as a corollary, for example, a new and simpler proof that property P is true for 2-bridge knots ([T] gives a proof of this which is over 100 pages); existence of an essential lamination in M implies $\pi_1(M)$ is infinite (Theorem 1 of [GO]). So nontrivial surgery on a 2-bridge knot never yields a homotopy sphere.

The main method used for 2-bridge knots was to start with a given branched surface or lamination on a given knot and go through some Kirby Calculus to see what this lamination looks like on a different knot for which we cannot get all the desired laminations, and then to try to generalize this "newly found" construction. This is quite general and could potentially be as fruitful for all knots and links.

Eisenbud, Hirsch, Jankins, and Neumann [EHN], [JN], proved Theorem 1 using different methods. (Actually, they showed existence of foliations transverse to Seifert fibers, which can easily be shown to be essential.) The construction given here will be used to get Theorem 2. Hatcher [H] constructed a branched surface for $T_{2,q}$ which carried laminations (with a linear transverse structure) for surgery coefficients $\in (-\infty, 0]$. The proof of Theorem 1 will consist of showing that this branched surface in fact carries laminations (with a piecewise—infinitely many pieces— linear transverse structure) for surgery coefficients $\in (-\infty, q-2)$. Brittenham [B] and [JN] show there exist no essential laminations for surgery coefficients $\in [q-1, +\infty)$. [JN] conjectures non-existence for $[q-2, q-1)$, which is still open (except for the trefoil where non-existence is proved). The branched surfaces of Theorem 1, however, can be shown in support of the conjecture not to carry any laminations for $[q-2, +\infty)$.

Hatcher [H] already constructed essential laminations for "most" 2-bridge knots (see section 2.0); to get Theorem 2, we construct essential laminations for the rest of the 2-bridge knots.

Charles Delman has also independently proven (Thesis, Cornell University, 1992) Theorem 2 using a different construction.

Chapter 1

Definitions, Notation, and Terminology

A closed subset L of an n-manifold M is called a *codimension-m lamination of M* if it comes equipped with a family of charts $\phi_\alpha : U_\alpha \to I^n = I^m \times I^{n-m}$ where I denotes the open unit interval $(0, 1)$, $\{U_\alpha\}$ is an open cover of M, and $\{\phi_\alpha\}$ are homeomorphisms, such that $\forall \alpha$, $\phi_\alpha(U_\alpha \cap L)$ is a union of level hyperplanes $* \times I^{n-m} \subset I^m \times I^{n-m}$ which are respected by the transition maps.

More precisely, let $p : I^m \times I^{n-m} \to I^m$ be the projection map onto the first m coordinates. Let $C_\alpha = p(\phi_\alpha(U_\alpha \cap L))$. Then $\forall \alpha$, $\phi_\alpha(U_\alpha \cap L) = \bigcup_{x \in C_\alpha} (x \times I^{n-m})$. Furthermore, for each α, β such that $U_\alpha \cap U_\beta$ is nonempty, let $A = p(\phi_\alpha(U_\alpha \cap U_\beta \cap L))$, $B = p(\phi_\beta(U_\alpha \cap U_\beta \cap L))$; then we have: $\forall a \in A$, $\exists b \in B$ such that $\phi_\beta(\phi_\alpha^{-1}(a \times I^{n-m})) = b \times I^{n-m}$.

Given a subgroup $G \subset \text{Homeo}(\mathbf{R}^m)$ of homeomorphisms of $\mathbf{R}^m$ onto itself, L is said to have a *transverse G-structure* if every map $(\phi_\beta \circ \phi_\alpha^{-1})_* : A \to B$ is equal to $g|_A : A \to B$ for some $g \in G$, where $(\phi_\beta \circ \phi_\alpha^{-1})_*$ is the obvious map induced by the transition map $\phi_\beta \circ \phi_\alpha^{-1}$. The induced map is well-defined since the transition maps respect the level hyperplanes. L is called a *measured lamination* if G is the group of isometries of $\mathbf{R}^m$.

The lamination L is called a *foliation* if $L = M$. Given a point $x \in L$, the *leaf* l_x of L containing x is defined to be the union of all points $y \in L$ which can be connected to x by a path γ which lies entirely in level hyperplanes, i.e. $\forall \alpha$, $p(\phi_\alpha(\gamma \cap U_\alpha)) =$ constant. Note that the charts ϕ_α give each leaf l of L an $(n-m)$-manifold structure; in general, however, l need not be a submanifold of M (in the subspace topology).

Examples:

Let $V = D^2 \times S^1$ be a solid torus embedded in S^3 with $T = \partial V$ its torus boundary. Let m and l be the meridian and longitude of V, i.e. the (up to homotopy) unique simple closed curves which are homologically trivial in V and $S^3 - V$ respectively. A simple curve (closed or not) on T is said to have *slope* s if it is homotopic to the image of the line of slope s through the origin in $\mathbf{R}^2$ under the "standard" universal covering map. Thus m has slope $1/0 = \pm\infty$ and l has slope $0/1 = 0$. Clearly l is a codimension-1 lamination in T. In fact, for any closed subset $C \subset m = S^1 \times * \subset S^1 \times S^1 = T$, $C \times l$ is a codimension-1 lamination of T. Now let $C = [a, b] \subset S^1$ be homeomorphic to a closed interval in S^1, and let $h : C \to C$ be a homeomorphism which fixes only the end points a and b of C. Then the suspension of h gives us a lamination $L = C \times [0,1]/(x,0) \sim (h(x),1)$ with only two closed leaves $a \times S^1$, $b \times S^1$, and all interior leaves non-compact, limiting on the two closed leaves (figure 1.1). Thus, we say that the two closed leaves have nontrivial holonomy. If h was piecewise linear, say, then we would say L has a piecewise linear transverse structure.

Now suppose instead of gluing $[a, b] \times 0$ to $[a, b] \times 1$ by h, we do the following. Pick $a < c < b$, "split" $[a, b] \times 0$ at $c \times 0$, give the $[a, c] \times 0$ portion a right hand twist along the meridian, then join it to the $[c, b] \times 0$ portion again, but on the "b side" (i.e. l_a joins l_b) and then glue to $[a, b] \times 1$ by identity (figure 1.2).

To split at c means remove the leaf l_c containing c, and replace it by two parallel leaves l_1 and l_2 with nothing in between. Similarly when l_a and l_b seem to join after the twist, they are actually only getting very close to each other, but remain disjoint.

For simplicity, assume $[a, b]$ is actually the unit interval $[0, 1]$. If c is rational, say p/q, then we see that all the leaves of L will be simple closed curves of slope p/q. On the other hand, if c is irrational, then every leaf of L will be non-compact, dense in L, and every arc transverse to L intersects it in a Cantor set.

This idea of "splitting and joining" will be used often later on in one dimension higher (i.e. 2-dimensional laminations in 3-manifolds) without any further explanation. We often use phrases such as "split off a batch of leaves of thickness ϵ" or "slice off ϵ leaves". For example, in figure 1.2, we start with a batch of curves $(b - a)$ thick and slice off $(c - a)$ curves. Also, for the sake of simplicity, we usually think of the lamination as intersecting transverse arcs in intervals, while in reality the intersection is often a Cantor set.

A *train track* τ in a surface F is a space locally modeled on figure 1.3(a) with neighborhood $N(\tau)$ vertically fibered by intervals, and *horizontal* and *vertical boundary* $\partial_h N(\tau)$, $\partial_v N(\tau)$ as shown in figure 1.3(b).

More precisely, τ is a 1-complex (graph) of valence 3, i.e. every 0-cell (vertex) belongs locally to exactly three 1-cells (edges) (only locally because the two endpoints of an edge may coincide), with the following property: the edges are smoothly embedded in the ambient surface F in such a way that at each vertex there is a unique edge whose union with each of the other two edges (again locally) forms a smooth 1-submanifold of F, i.e. they are tangent. The vertices are called *branch points.*

A codimension-1 lamination L in the surface F is said to be *carried* by the train track τ if it lies (or equivalently, can be isotoped to lie) in a neighborhood $N(\tau)$ of τ such that L is transverse to the vertical fibers of $N(\tau)$. L is *fully carried* by τ if it intersects every fiber of $N(\tau)$.

For example, the lamination of figure 1.2 is carried by the train track of figure 1.4(a). The lamination could be specified by assigning "weights" to the train track, indicating the thickness of the lamination at each part, as in figure 1.4(b).

We have similar definitions in one dimension higher:

A *branched surface* B in a 3-manifold M is a space locally modeled on figure 1.5(a), with neighborhood $N(B)$ vertically fibered by intervals, and *horizontal* and *vertical boundary* $\partial_h N(B)$, $\partial_v N(B)$ as shown in figure 1.5(b).

The main difference to notice is that in a train track, the branch locus consists of a discrete set of double points, whereas in a branched surface it consists of 1-manifolds intersecting transversely in a discrete set of triple points. The definition of when a lamination is (fully) carried by a branched surface is the same as with train tracks.

Note: we always require that if ∂M is nonempty, then B is *properly embedded* in M and is transverse to ∂M (so $B \cap \partial M = \partial B$ is a train track $\subset \partial M$).

Let $B, B' \subset M$ be branched surfaces in a 3-manifold, or train tracks in a surface. Then B' is a *splitting* of B, or equivalently B is a *pinching* of B' if there is an I-bundle J in M such that $N(B) = N(B') \cup J$, $J \cap N(B') \subset \partial J$, $\partial_h J \subset \partial_h N(B')$, and $\partial_v J \cap N(B') \subset \partial_v N(B')$ has finitely many components whose fibers are fibers of $\partial_v N(B')$.

For example, in figure 1.6 the train track in (a) can be split three different ways into (b), (c), and (d). Note that any lamination carried by (a) is also carried by either (b), or (c), or (d).

As another example, if $\tau \subset \partial V$ is a train track on the boundary of a solid torus, which carries a lamination consisting of closed curves of slope $p/q \in \mathbf{Q}$, then τ can be split into a finite number of simple closed curves of the same slope, the splitting being "guided" by the lamination.

There are definitions for when a lamination (codimension one in a 3-manifold) is called essential, and also when a branched surface is called essential. In [GO] it is proven that if L is a lamination fully carried by B, then L is essential if and only if B is. In fact the laminations constructed in this thesis are shown to be essential by showing that they are carried by essential branched surfaces. So here we first define the essential branched surface, and then define a lamination to be *essential* if it is carried by an essential branched surface. (The "original" definition of an essential lamination is pretty similar to that of an essential branched surface; the interested reader may find it in [GO].)

First some auxiliary definitions (figure 1.7(a-e)). Let B be a branched surface in a 3-manifold M. Then:

A *disc of contact* for B is a disc D which lies in the neighborhood $N(B)$ transverse to the fibers, such that $\partial D \subset \text{int}(\partial_v N(B))$ ($\text{int}(X)$ means topological interior of X, also denoted $\mathring{X}$).

A *half disc of contact* for B is a disc D in $N(B)$ transverse to the fibers such that $D \cap \partial M = D \cap \partial N(B) =$ an arc $\subset \partial D$, and the complementary arc in ∂D is in $\text{int}(\partial_v N(B)\ )$.

A *monogon* for B is a disc $D \subset M - \overset{\circ}{N}(B)$ such that $D \cap N(B) = \partial D$ is a union of two arcs, one in $\partial_h N(B)$, the other a single fiber in $\partial_v N(B)$.

B contains a *Reeb branched surface* if:

1) there exists a solid torus $R \subset M$ such that $C = R \cap B$ is a branched surface which carries a sublamination of a Reeb foliation of R, which contains ∂R as the compact leaf, and at least one non-compact leaf, or,

2) (when M has nonempty boundary) there exists a properly embedded annulus $A \subset M$ (i.e. $A \cap \partial M = \partial A$) which is boundary parallel, i.e. $M - A$ has two components one of which is a solid torus R with $\partial R = A \cup A'$, $A' \subset \partial M$ an annulus, such that $C = R \cap B$ is a branched surface which carries a sublamination of a "half" Reeb foliation of R transverse to A', containing A as the compact leaf, and at least one non-compact leaf.

Now, B is said to be *essential* in M if the following five conditions hold:

i) B has no discs or half discs of contact.

ii) $\partial_h N(B)$ is *essential* in $M - \overset{\circ}{N}(B)$, i.e. $\partial_h N(B)$ is incompressible and ∂-incompressible (defined below) in $M - \overset{\circ}{N}(B)$, B has no monogons, and no component of $\partial_h N(B)$ is a sphere or disc properly embedded in M.

iii) $M - \overset{\circ}{N}(B)$ is irreducible, and $\partial M - \overset{\circ}{N}(B)$ is incompressible in it.

iv) B contains no Reeb branched surfaces.

v) B fully carries a lamination.

A surface F is *incompressible* in a 3-manifold M if every simple closed curve on F which bounds a disc in $M - F$ also bounds a disc in F. Often 'F incompressible in M' is also defined as '$\pi_1(F)$ injects into $\pi_1(M)$'. For

2-sided surfaces these two definitions are equivalent. In general, however, the latter implies the former, but not vice versa.

A surface F properly embedded in M is *∂-incompressible* if every properly embedded arc in F that can be homotoped in M rel endpoints to an arc in ∂M can also be homotoped so in F.

Let $K \subset S^3$ be a knot, i.e. an embedded S^1. Let $p/q \in \mathbf{Q} \cup \{\infty\}$ be a possibly infinite rational number (∞ represents $p = 1$, $q = 0$) with p, q relatively prime. Then *p/q Dehn surgery* on K consists of removing a solid torus neighborhood $V = N(K)$ of K and gluing it back to $S^3 - \mathring{N}(K)$ by a homeomorphism $h : \partial V \to \partial(S^3 - \mathring{N}(K))$ such that h(meridian)= p(meridians) + q(longitudes). Although this does not uniquely determine h (in fact there are infinitely many such h's), it does uniquely determine the resulting manifold (every such h yields the same manifold). Closed 3-manifolds are often specified this way, i.e. by surgery description, and we sometimes use Kirby Calculus to find different surgery descriptions for the same manifold (see Rolfsen [R]).

Because every closed 3-manifold M can be obtained by surgery on a knot or link $K \subset S^3$ (Theorem of Lickorish and Wallace in [R]), in trying to construct essential laminations in M, we first try to do so in $S^3 - \mathring{N}(K)$, and then hope that the lamination extends to an essential lamination in M after the Dehn filling (i.e. gluing back the solid torus or tori).

More precisely, if M is obtained by p/q surgery on a knot $K \subset S^3$, then we either want 1) essential laminations in $S^3 - \mathring{N}(K)$ which are disjoint from the boundary torus and remain essential after the Dehn filling, or 2) essential laminations in $S^3 - \mathring{N}(K)$ which intersect the boundary torus transversely in simple closed curves of slope p/q, and which remain essential after Dehn filling.

Of course the reason we want simple closed curves of slope p/q on the boundary torus is so that we can cap them off with the meridional discs in the solid torus of the Dehn filling. Making these boundary curves closed is crucial and sometimes difficult.

A knot $K \subset S^3$ is called an *n-bridge* knot if it can be drawn (isotoped) such that it has only n local maxima (or equivalently, n local minima), with n being minimal. A local maximum is measured in the sense of height, i.e. the z-coordinate, with the knot thought of as sitting in $\mathbf{R}^3$.

For example, figure 1.8 shows a 2-bridge representation of the trefoil knot; that 2 is minimal follows from the fact that only the unknot has a 1-bridge representation.

It is an easy exercise to show that any 2-bridge knot can be isotoped to look like figure 1.9, which is denoted $[b_1, b_2, \ldots, b_n]$, where b_i is the number of half-twists in the "i^{th} sequence of twists"; b_i is positive or negative according to whether the twists are right or left handed.

In order to describe a specific lamination in a given knot complement in S^3, we first describe a branched surface that carries it, and then designate weights on the branched surface, indicating the thickness of the lamination at each part of it (see figure 1.4).

We describe the branched surface by first isotopying it so that the height function on it is Morse, i.e. the branched surface is transverse to horizontal planes in $\mathbf{R}^3$ (which represent level 2-spheres in S^3) except at isolated singularities of the Morse height function, and then drawing the intersection of the branched surface with these horizontal planes in successive stages.

Example: Figure 1.10(a) shows a surface (a branched surface with no branch points) whose boundary (thick lines) is the trefoil. The surface is described on the right, 1.10(b), by its intersection with horizontal planes, with the four dots in each diagram representing the knot's intersection with the planes. The dotted lines represent the saddles.

In a branched surface with nonempty branch locus, the intersections with (almost all) horizontal planes are train tracks; figure 1.11 shows an example. A lamination carried by this branched surface would be described as: we start with two vertical batches of sheets; the left batch splits into two parts (batches), one part continuing vertically down, the other joining the right batch in a saddle.

As another example, figure 2.12 shows a branched surface in the trefoil complement which looks similar to the surface of figure 1.10(a). The latter is in fact a subset of the former; it also qualifies as a lamination carried by the branched surface of figure 2.12.

Chapter 2

Main Results and Constructions

2.0 Outline

Theorem 1 *Surgery on a 2-bridge torus knot $T_{2,q}$, with coefficient* $\in (-\infty, q-2)$ *yields a manifold which admits essential laminations.*

Theorem 2 *Nontrivial surgery on a non-torus 2-bridge knot yields a manifold which admits essential laminations.*

Corollary *Property P is true for 2-bridge knots.*

> *Proof:* For torus knots, see Moser [M]. For non-torus 2-bridge knots, Theorem 2 above, together with Theorem 1 of [GO] imply infinite fundamental group for the manifold obtained by surgery. □

In this chapter we describe the constructions for the laminations of Theorems 1 and 2. That these laminations are essential will be proven in the next chapter.

In [H], Hatcher constructs branched surfaces which for "most" 2-bridge knots carry laminations of all boundary slopes. This construction is explained in section 2.1.1 below. The knots for which not all boundary slopes were obtained consisted of three classes (m, n, and p are positive integers):

(a) Torus knots $T_{2,q}$; slopes $(-\infty, 0]$ were obtained.

(b) Knots of the form $[2m+1, -2n]$ (odd right-handed twists, even left-handed); slopes $(-\infty, -4n] \cup [0, \infty)$ were obtained.

(c) Knots of the form $[2m+1, -(2n+1), 2p+1]$; slopes $(-\infty, -2a] \cup [0, \infty)$ were obtained, where $a = \min(2m+1, 2p+1)$.

We show in section 2.1.1 that the branched surface of (a) also carries the extra laminations claimed to exist in Theorem 1.

For Theorem 2, we find new branched surfaces in section 2.1.2 which for knots of (b) and (c) either carry laminations of all boundary slopes or are disjoint from the knot and remain essential after Dehn filling.

2.1 Construction for Theorem 1

2.1.1 Special Case: Trefoil

We will first do this for the case of the trefoil $T_{2,3}$, which contains the main idea; then we will do the general case $T_{2,q}$, which is similar. The branched surface in the exterior of $T_{2,3}$ constructed in [H] is shown in figure 2.12. Actually, only part of the branched surface is shown on the left; the jagged edges of the horizontal strips mean that they extend further up and down; the curve diagrams on the right though, which are intersections of level 2-spheres with the branched surface, do fully describe the branched surface. The four dots are $S^2 \cap T_{2,3}$; the dotted lines represent the saddles.

In [H] it is shown that this branched surface carries laminations for all boundary slopes $\in (-\infty, 0]$ (figure 2.13). The numbers represent the "thickness" of the lamination (or the saddles) at the indicated points; for $0 \leq x \leq 1$, $\bar{x}$ denotes $1 - x$. So we start with two batches of sheets of thickness 1, then go through a saddle with thickness 1 (full) on the right and $\epsilon < 1$ (partial) on the left. This saddle is linear, so that in diagram 2, the two horizontal lines are both the linear map from $[0, \epsilon]$ to $[0, 1]$. In fact all saddles in figure 2.13 are linear. (In later constructions we will use nonlinear saddles also.) The saddle in diagram 2 also is 1 (full) on the right and ϵ on the left. If $\bar{\epsilon}$ is an integer multiple of ϵ, then by repeating this saddle $\bar{\epsilon}/\epsilon$ times we will reach diagram 5. If $\bar{\epsilon} = k\epsilon + r$, $0 < r < \epsilon$, however, then the last occurrence of this saddle will instead be r on the left and r/ϵ on the right, preserving the ϵ to 1 ratio.

The significance of this ϵ to 1 ratio and linearity of saddles is that we get the identity map from $[0, 1]$ to $[0, 1]$ for the two vertical lines in diagram

7, and hence also in diagrams 8 and 9. It is important to get this identity map in the last diagram in order for the boundary curves of the lamination on the torus boundary to be closed (i.e. no holonomy), so that we can cap them off with discs after Dehn surgery.

Now, to compute the boundary slope of this lamination as a function of ϵ, first note that for $\epsilon = 1$ we jump from diagram 1 straight to diagram 5, and so the lamination just becomes $S \times [0,1]$ where S is a Seifert surface for $T_{2,3}$ which has boundary slope 0 (figure 1.10). Now for $\epsilon < 1$, we see that in going from diagram 2 to 3, there is a batch of thickness 1 on the top right corner rotating counterclockwise and a batch of thickness ϵ on the top left corner rotating clockwise, so we get a contribution of $-1+\epsilon$ to the total slope. Adding up all the repeats of this move, we get a total addition of $+\bar{\epsilon} - \bar{\epsilon}/\epsilon$ to the slope of the $\epsilon = 1$ lamination (which by above is 0), so the slope is $\bar{\epsilon} - \bar{\epsilon}/\epsilon$. So as ϵ ranges over (0,1], the slope ranges over $(-\infty, 0]$. This was the construction of [H].

Now we claim that this branched surface in fact carries laminations with boundary slopes in $(-\infty, +1)$. To get the extra (0,1) range, the idea is that in diagram 2 (still figure 2.13) where we get a contribution of $+\epsilon - 1$ to the slope, we change the weights on the saddle to $\bar{\epsilon}$ (full) on the left, and $\delta > 0$ (small) on the right (figure 2.14).

This gives a contribution of $+\bar{\epsilon} - \delta$ to the slope. This move cannot be repeated though, as we have already reached diagram 5 of figure 2.13. Thus by making ϵ and δ small, the total slope, $\bar{\epsilon} - \delta$, approaches 1 as desired.

But there is a problem: if we just make the saddles all linear, the boundary curves of the lamination will not be closed (figure 2.15). In fact, because of the holonomy, the slope is really 0.

To avoid this holonomy, we proceed as follows. In figure 2.14, do the saddle in diagram 1 such that the horizontal lines in diagram 2 look as in figure 2.16. We keep the saddle in diagram 2 of figure 2.14 linear, and make δ small enough as in figure 2.16. Now we claim that in figure 2.15, the saddle in diagram 5 can be chosen such that in diagram 6, the maps in the two vertical lines are mirror images, and so in the last diagram, both lines can be made identity.

Proof of claim: The saddle of diagram 5 of figure 2.15 will be done in several stages (or pieces). First note that $\delta\epsilon$ is now $\delta\alpha$ because of the $\alpha : 1$ ratio in figure 2.16. First we slice off $\bar{\epsilon}$ sheets from the right side of the bottom line and $\delta\alpha$ sheets from the right side of the top line. This yields figure 2.17(a). Then we slice off x_1 from the right side of the bottom line and y_1 from the right side of the top line, getting figure 2.17(b).

Note that the two vertical lines are starting to look the same, and in the limit will be mirror images. In figure 2.16, we pick α, x_i, and y_i so small that not only this process converges, but there will be enough of the two horizontal lines remaining that we will get figure 2.18.

The top part of $h : \bar{y} \longrightarrow \bar{y}$ is not linear, but only piecewise linear (2 pieces in fact) because of the saddle in diagram 2 of figure 2.14. The important point however is that both g and h have no interior fixed points and have opposite sense: g moves points "down", h "up". So there exists a homeomorphism $f_1 : \bar{x} \longrightarrow \bar{y}$ such that $f_1^{-1} h f_1 = g$, so $f_1 = h^{-1} f_1 g$, so we get figure 2.19, which proves the claim.

Remark 2.1.1 These laminations have a piecewise linear structure since all the maps in all the diagrams are piecewise linear. (Even f_1 is piecewise linear, since g and h are.) Similarly, the laminations constructed in the next section for the general case $T_{2,q}$ have a piecewise linear structure.

2.1.2 General Case: $T_{2,q}$

For the general case $T_{2,q}$, avoiding holonomy needs a little more care, though basically it is the same idea. We do it for $T_{2,5}$; everything is the same for $T_{2,q}$, $q \geq 5$.

First, as with the trefoil, we draw $T_{2,5}$ as the 2-bridge knot $[-2, -2-2, -2]$, i.e. with four pairs of left-handed half-twists, as in figure 2.20. (For $T_{2,q}$, we have $(q-1)/2$ pairs of half-twists.) Then we construct the branched surface described on the right in figure 2.20. According to Theorem 1, our aim is to show this branched surface carries laminations with boundary slopes up to (but not including) 3 ($q - 2 = 5 - 2 = 3$).

As with the trefoil, we use the saddles of diagrams 2, 5, and 8 to get slopes of up to 3, and try to kill the holonomy by appropriately picking the saddles of diagrams 10 and 12.

In order to have no holonomy, i.e. identity maps for the horizontal lines in the last diagram, we want the maps of diagram 11 to be mirror images. So we would like diagram 10 to qualitatively look like figure 2.16, so that we can carry out the same procedure as in figures 2.17, 2.18, and 2.19.

To do this we use induction, the induction step being 'going from diagram 2 to 5' (or 5 to 8, or ...). More precisely, we start with diagram n of figure 2.21 and want to reach the same position qualitatively (but rotated 90 degrees) in diagram $n+3$. (Because we are aiming for slopes close to the maximum of 3, all batches are either very thin or very thick, denoted by S for small, ~ 0, or L for large, ~ 1; the S's are not necessarily equal, nor are the L's.)

This is achieved by doing the saddle in diagram n in such a way that the vertical line of diagram $n+2$ looks as shown. This is the main difference with the case of the trefoil, where this saddle was just linear (diagram 2 of figure 2.14). □ (Theorem 1)

Remark 2.1.2 This construction for Theorem 1 reduces the missing slopes of (a) from $(0, \infty)$ to $[q-2, \infty)$. The same construction works for (b) and (c), reducing the range of the missing slopes roughly by half. (More precisely, for (b) we can get slopes $(-\infty, -4n+1) \cup (-2n+1, \infty)$, and for (c), $(-\infty, -a) \cup (-2n-1, \infty)$.)

2.2 Construction for Theorem 2

This will be done in two parts by dividing knots of (b) ∪ (c) into:

(I) Knots of the form[1] $[2m, 2n]$.

(II) The rest (i.e. (b) ∪ (c) - (I)).

Lemma 2.2.1 *For every 2-bridge knot K of type (I) (i.e. of the form $[2m, 2n]$), $S^3 - \mathring{N}(K)$ admits essential laminations which are disjoint from the boundary, and which remain essential for all nontrivial Dehn fillings.*

Proof: For knots of type (I) we have figure 2.22. The knot $[2, 2n]$ with surgery coefficient $-1/m$ in the last diagram of figure 2.22 is still of type (I). However, $[2, 2n] = [-2, 2n-1]$, so by remark 2.1.2, laminations of boundary slope $-1/m$ are realizable on it. Furthermore, the unknot with coefficient $r/(r+s)$ (last diagram) is disjoint from, and isotopic to, a leaf of the lamination (figure 2.23). Therefore, by [G1] (operation 2.4.2), for all surgeries but one[2] on the unknot, the lamination (on $[2, 2n]$) remains essential. So all surgeries but one (trivial surgery) on the original $[2m, 2n]$ knot yield manifolds which admit essential laminations. □ (Lemma 2.2.1)

Note: We never stated explicitly in any theorem that the laminations of remark 2.1.2 are essential. However, they are proven to be essential along with the laminations of Theorem 1; see remark 3.2.1.

[1] At first glance, these may seem to be disjoint from (b) and (c), but they are not: in (b), $[2m+1, -2] = [2m, 2]$; in (c), $[2m+1, -1, 2p+1] = [2m+2, 2p+2]$. To check this simply note that the brackets represent the same number (mod$\mathbf{Z}$) as continued fractions. See [HT] for more details.

[2] The unique surgery for which the lamination does not remain essential is represented by the"tangency curve" of the unknot with the leaf it is isotopic to, i.e. surgery 1. But $r/(r+s) = 1$ iff $r/s = 1/0 = \infty$, i.e. trivial surgery on the original $[2m, 2n]$ knot.

Lemma 2.2.2 *For every knot K of type (II) (i.e. of the form $[2m+1,-2n]$, $(m \geq 1,\ n \geq 2)$ or $[2m+1,-2n+1,2p+1]$, $(m,n,p \geq 1)$, but not $[2m,2n]$), $S^3 - \overset{\circ}{N}(K)$ admits branched surfaces which carry laminations of all finite boundary slopes, transverse to the boundary. (These branched surfaces/laminations are proven to be essential in the next chapter.)*

Proof:

Step 1. The 7_3 knot.

The simplest knot of type (II) is [3,-4], which is knot 7_3 in Rolfsen's tables [R]. For this knot, according to remark 2.1.2, we are missing slopes [-7,-3]. It turns out that surgery -7 on the [3,-4] knot gives the same manifold as surgery -7/2 on the [3,-2] knot (5_2 in [R]), as in figure 2.24.

By remark 2.1.2, -7/2 on [3,-2] has an essential lamination. If we *trace back* this lamination, along with the branched surface carrying it, through the surgery and isotopy "moves" (Kirby Calculus) to -7 on [3,-4] (which is a bit of work, but we don't need to do it here), we get the branched surface of figure 2.25 in $S^3 - [3,-4]$.

Showing that this branched surface carries laminations is a bit involved. We construct the lamination so that in the "spiral regions around the knot" (figure 2.26a) it has a triple spiral structure (figure 2.26b): There are exactly two closed curves. (The leaves containing these closed curves though are non-compact elsewhere.) The inner and outer spirals have the same sense; they spiral *in* as they turn clockwise. The middle spiral has the opposite sense; it spirals *out* as it turns clockwise. The "infiniteness" of the inner spiral enables us to realize all boundary slopes.

The purpose of this is to make things work in going from diagram 6 to 7 in figure 2.25, which is the crucial point of the construction. (There are only two other saddles, and they pose no problems. In diagram 3, the saddle is arbitrary, as long as it's full on the right, and partial on the left. In diagram 9, there is a unique saddle which will give identity for the vertical line in diagram 10.) The point is that in diagram 7 a new spiral region is created (which is more visible in 8), and the lamination there must have the same structure as the other spiral (top left in 7) since the two spirals will eventually have to join (diagram 11). It turns out that a triple spiral structure is the minimum amount of structure that will work.

Figure 2.27 shows an enlarged picture of diagram 7 with the details of the saddle. We have to find maps F, G, and H which will give the new spiral the triple spiral structure. Because of the "old spiral" (i.e. the spiral in diagram 6 to which the saddle is joining), F has to be an increasing homeomorphism, while G and H can be any homeomorphisms.

The points labeled $a \cdots f$ occur in pairs, signifying that in diagram 6, before the saddle, they are joined pairwise; the two curves through b and c are (were) the two closed curves of the old spiral.

Let $x = (b+c)/2$, $G(x') = H(x') = x$, $z = (x+c)/2$, and $y = (x+b)/2$, say. Now choose F "increasing enough" such that $F(z) = y$. (There will be other requirements imposed on F later.) The only requirement on G and H, which is easily satisfied, is that y' and z', where $y' = G^{-1}(y)$, $z' = H^{-1}(z)$, lie on the same curve (leaf) which goes all the way around on the right, so that the curve $z' \overset{H}{\longmapsto} z \overset{F}{\longmapsto} y \overset{G^{-1}}{\longmapsto} y' \longmapsto z'$ closes up. Call this curve γ. Now x' (the lower one) $\overset{G}{\longmapsto} x \overset{F}{\longmapsto}$ between y and $b \overset{G^{-1}}{\longmapsto}$ between y' and $e \longmapsto$ between x' and $z' \overset{H}{\longmapsto} \cdots$. So the curve through the lower x' is an infinite spiral, limiting either to γ or possibly some other closed curve γ'. We choose F so that there is no γ', i.e. all curves limit to γ. Similarly for the other side of γ.

Now we modify F again so that γ thickens up to an annulus A with two closed boundary curves γ_1 and γ_2, and with all interior curves of A spirals limiting on γ_1 and γ_2 (figure 2.28). On A, F is of course still increasing, but "relative to γ_1 and γ_2", F looks decreasing inside A, which is what gives us the *middle spiral with the opposite sense.*

This proves the claim that the branched surface of figure 2.25 carries laminations of all boundary slopes.

Remark 2.2.3 Actually, this branched surface fully carries only laminations of slope -7 (see figure 2.24). In diagram 2 of figure 2.25, before doing the saddle, we can get any integer slope by twisting one of the inner spirals the desired number of times. And then we can get all (rational) slopes between any two integer slopes by branching the inner spiral and twisting a copy of it, as shown in figure 2.29. □(Remark)

By modifying this branched surface appropriately, we get laminations for all knots of type (II):

Step 2. (II) $\cap$ (b).

First we start with those in class (b), namely $[2m+1, -2n]$, with $m \geq 1$ and $n \geq 2$ because we are excluding type (I). The $2m+1$ twists are realized in going from diagram 2 to 3 (figure 2.25), where instead of the $+3\pi$ rotation we have $(2m+1)\pi$. This obviously makes no difference in whether or not the branched surface carries laminations. The $-2n$ twists, on the other hand, are realized in diagram 6 before doing the saddle by rotating its bottom portion counterclockwise the required number of times, i.e. $-(n-2)2\pi$ (for $n = 2$ we don't need any more twists than the $-2\pi - 2\pi$ already present in diagrams 4-5 and 8-9) (figure 2.30).

That the branched surface still carries laminations is no longer obvious, but easy to check, as follows. In figure 2.27, the only thing that changes is that the upper x' is renamed x'', and a point just above it becomes the new upper x', as in figure 2.31. However the argument given to show that the old branched surface carries laminations still goes through word for word.

Step 3. (II) $\cap$ (c).

For (c), the knot $[2m+1, -2n+1, 2p+1]$ can also be represented as $[2m+1, -2n, -2, -2, \ldots, -2]$ with $2p$ -2's (see footnote 1) with $m, p \geq 1$ and $n \geq 2$ since $n = 1$ gives type (I). The $2m+1$ and the $-2n$ are taken care of the same way as for (b) above. The $2p$ -2's are realized by adding $2p$ pairs of counterclockwise half-twists and saddles after diagram 10 of figure 2.25 before reaching diagram 11, as in figure 2.32. Obviously this does not affect whether or not the branched surface carries laminations. $\square$ (Lemma 2.2.2)

Chapter 3

Proofs of Essentiality

In this chapter we finish the proof of Theorems 1 and 2 by checking that the branched surfaces (and therefore laminations) constructed in the previous chapter are essential.

3.1 Some Generalities

Suppose M is obtained by $p/q \in \mathbf{Q}$ surgery on a knot $K \subset S^3$. What we did in the previous chapter was to construct a branched surface $B \subset S^3 - \mathring{N}(K)$ which carries a lamination of boundary slope p/q. Then after p/q-Dehn filling, to get a lamination in M we capped off the boundary curves of the lamination with meridional discs in the Dehn filling solid torus. Now to show this lamination is essential, we complete B to a branched surface $B' \subset M$ carrying the lamination, and show B' is essential.

Construction 3.1.1 With notation as above, let $T = \partial(S^3 - \mathring{N}(K))$, and let V be the solid torus being glued in the Dehn filling. Let $T \times [0,1]$ be a one sided collar of T in $V \subset M$, with $T \times 0 = T \simeq \partial V$. Now $\tau = \partial B$ is a train track in T which carries a lamination of slope p/q, so it can be split into a finite number of simple closed curves of slope p/q in T, or equivalently slope ∞ (i.e. = meridian) in ∂V. So now the natural way to get B' is to first extend B to the collar $T \times [0,1]$ according to the splitting of τ, and then cap it off with discs in $V - T \times [0,1]$. Thus, $\{B' \cap (T \times t) \mid t \in [0,1]\}$ gives us a "movie" for the splitting of τ.

Actually though, we construct B' a little differently: assume $\tau \subset T$ is "nontrivial" enough that $T - \tau$ consists of digons, i.e. a disc with two branch points in the boundary, as in figure 3.33. Then, before splitting τ across the collar $T \times [0,1]$, first we take a copy of one side of each digon and move it parallel to itself across the digon to the opposite side as t goes from 0 to 1/2 in the collar (figure 3.34). Thus $B' \cap (T \times 1/2) = \tau$ again, as in level $t = 0$. Then for $1/2 \leq t \leq 1$ we extend B' as before, i.e. by splitting τ and then capping off with discs. □ (construction 3.1.1)

The reason for 'moving copies of sides of digons across from one side to the other' is to make the analysis of the complementary components of $B' \subset M$ (and therefore checking essentiality of $B' \subset M$) simpler, as explained in the following three lemmas (which are proven, although not in so many words, in proposition 1 of [H]).

Lemma 3.1.2 *With notation as above, every complementary component of B' which lies entirely in V is an open 3-ball.*

Proof: Let X be such a complementary component. Then $X \cap (T \times [0,1/2])$ is a disjoint union of balls. As t goes from 1/2 to 1, only a splitting of the branched surface as shown in figure 3.35 can affect the topology of $X \cap (T \times [0,t])$: it either joins two disjoint balls into a new ball, which does not change the topology, or it joins two "parts" of the same ball to form a solid torus, intersecting $T \times 1$ in an annulus. $B' \cap (T \times 1)$ consists of a finite union of parallel simple closed curves (of slope p/q), which are capped off with meridional discs in V. Thus to each solid torus $X \cap (T \times [0,1])$ a 2-handle $D^2 \times I$ is being attached along an annulus in $T \times 1$, resulting in a 3-ball. □ (Lemma 3.1.2)

Lemma 3.1.3 *With notation as above, if $B \subset (S^3 - \mathring{N}(K))$ has essential horizontal boundary, then so does $B' \subset M$. (B is assumed to satisfy the hypothesis necessary for construction 3.1.1, namely that ∂B is a train track in $\partial(S^3 - \mathring{N}(K))$ whose complementary regions are all digons.)*

Proof: Let X be a component of S^3- interior$(N(K) \cup N(B))$. Then $\partial X \subset \partial_h N(B) \cup \partial_V(B) \cup \partial N(K)$, and $\partial X - \partial_h N(B)$ is a disjoint union of annuli, which we call the *suture.* Each annulus consists of a finite union of discs $D_i \simeq I \times I$ with each D_i contained alternately in $\partial_v N(B)$ and $\partial N(K)$. Figure 3.36 shows an example where X is a solid torus, with the suture consisting of one annulus. Each of D_2 and D_4 is a complementary digon of the train track $\tau \subset \partial N(K)$.

Now, the result of 'moving copies of sides of digons across from one side to the other' (in the $T \times [0,1/2]$ part of construction 3.1.1) is that to each complementary component X we are gluing one $D^2 \times I$ for each $D_i \subset \partial N(K)$ in the suture of X. The suture on the $D^2 \times I$ is an annulus half of which is in $\partial_v N(B')$, the other half glued to D_i. Therefore, X remains topologically the same, and further, its suture also remains the same except that it is now entirely $\subset \partial_v N(B')$.

This shows $\partial_h N(B')$ has no compressing discs or monogons in the enlarged X, since any such compressing disc or monogon could be isotoped to lie in the old X, violating the hypothesis that $\partial_h N(B)$ is essential. And $\partial_h N(B')$ has no properly embedded spheres or discs since neither does $\partial_h N(B)$.

So to finish the proof, we need to show $\partial_h N(B')$ has no compressing discs or monogons or properly embedded spheres or discs in the remaining complementary components, namely the ones which lie entirely in the Dehn filling solid torus V glued to $\partial(S^3 - \mathring{N}(K))$. But it is easily seen that this is satisfied because by Lemma 3.1.2 all these complementary components are just $D^2 \times I$ products, each with the annulus $\partial D^2 \times I$ as suture. □ (Lemma 3.1.3)

Lemma 3.1.4 *With notation as above, suppose (the closure) of every $D^2 \times I$ complementary component of $B \subset (S^3 - \mathring{N}(K))$ intersects T, and ∂B is a train track transverse to the meridians of T. Then B' contains no Reeb branched surfaces.*

Proof: Suppose towards contradiction that there is a Reeb branched surface $C \subset B'$. Then by definition C is contained in a solid torus R with $\partial R \subset C$. ∂R intersects T (or $T \times \{1/2\}$, if $R \subset V$) in one or more simple closed curves α_i transverse to meridians of T. Let β be a simple closed curve on T intersecting α_1 transversely once. Then β must intersect ∂R in at least one more point (upon entering and exiting R), which shows there are at least two simple closed curves $\alpha_1 \cup \alpha_2 \subset T \cap \partial R$ transverse to meridians of T. By renumbering if necessary, we can assume α_1 and α_2 are adjacent, i.e. bound an annulus A on T which is properly embedded in R. By hypothesis, A is a union of digons on T, transverse to meridians. So as a meridian m crosses A from α_1 to α_2, in each digon it goes from one side of a $D^2 \times I$ complementary component of $R - C$ to the other.

Each meridional disc of C has two distinguished sides: the first side is an entire side, say $D^2 \times 0$, of a $D^2 \times I$, while the second side is a proper subset of a $D^2 \times 1$ side of a $D^2 \times I$, the rest lying on ∂R. So m starts from α_1 at a point on ∂R, goes to the first side of a meridional disc, emerges from its second side, goes to the first side of perhaps another meridional disc, emerges from its second side, and so on. So just before reaching α_2 on ∂R, it must have emerged from the second side of a meridional disc, which lies on the *same* side of a $D^2 \times I$ as $m \cap \alpha_2$, so we have a contradiction (with the last sentence of the last paragraph). □ (Lemma 3.1.4)

Now we are ready to finish the proofs of Theorems 1 and 2. That the branched surface of Theorem 1 is essential is basically proven in Proposition 1 of [H]. There are two adjustments however, in case 3 of step 4, and in step 5, below.

3.2 Proof of Theorem 1

Let $K \subset S^3$ be the $T_{2,q}$ torus knot, with exterior $X = S^3 - \mathring{N}(K)$.

Step 1. First we enlarge the branched surface constructed in the previous chapter by adding, for each saddle, a "complementary saddle" in the same level 2-sphere; figure 3.37 shows the case $K =$ trefoil.

This enlarged branched surface B contains the original branched surface B_0 as a subset. Therefore any lamination carried by B_0 is also carried by B; so it is enough to show B is essential. The reason for enlarging B_0 to B is that B has "smaller" complementary components, which are easier to work with. The disadvantage, however, is that it may be harder to show that B *fully* carries a lamination.

Step 2. B has essential horizontal boundary in X.

The two saddles of diagram 1 (figure 3.37) "add up" to the whole level 2-sphere at that level; therefore B has a complementary component "above" diagram 1 which is just a ball $D^2 \times I$ with $\partial D^2 \times I$ as the suture; so its horizontal boundary contains no compressing discs or monogons or properly embedded spheres or discs.

The two saddles of diagram 3 also add up to a level 2-sphere. The complementary component of B trapped between the level 2-spheres of diagrams 1 and 3 is also seen (with a little visualization) to be topologically a ball $D^2 \times I$ with $\partial D^2 \times I$ as the suture, as shown in figure 3.38 (where the suture is drawn as just a curve rather than an annulus).

The complementary component between diagrams 3 and 7 is exactly the same as the one between 1 and 3, except that it is upside down. (This can be seen without any visualization by simply noting that to go "upward" from diagram 7 to 3 one makes the same "saddle moves" as going "downward" from 1 to 3.)

The complementary component between diagrams 7 and 10 is a solid torus $A \times I$ with the suture consisting of two parallel annuli $\partial A \times I$ of slope 1 on the boundary of the solid torus, as in figure 3.39 which is equal to figure 3.36 (again the suture is drawn as curves). Therefore the horizontal boundary $A \times \partial I$ is essential in $A \times I$.

Remark 3.2.1 In going from diagram 9 to10, if instead of the $+2\pi$ rotation we had $2n\pi$ rotation, n a nonzero integer, we would still get an $A \times I$ solid torus complementary component, except that the suture would now consist of two parallel annuli $\partial A \times I$ of slope $1/n$. For a $(2n+1)\pi$ rotation, $n \geq 1$ or ≤ -2, we get an $A \times I$ solid torus complementary component, with suture consisting of one annulus $A \times 0$ of slope $2/(2n+1)$. Both these cases arise for the non-torus 2-bridge knots, which we are concerned with in Lemma 2.2.1. In either case, the horizontal boundary ($A \times \partial I$ for $2n\pi$, $\partial(A \times I) - (A \times 0)$ for $(2n+1)\pi$) is essential in the solid torus. □ (remark)

Finally, the complementary component below diagram 10 is the same as the one above diagram 1, i.e. a ball.

For the other torus knots (and in fact all 2-bridge knots with the branched surfaces of [H]) we get the same complementary components since we are using the same "saddle moves", which yield branched surfaces made up of the same "building blocks". (There are nice pictures of these building blocks in figure 3.1 of [FH]—for 2-bridge knots, only Σ_A, Σ_C, and Σ_D are used.)

Step 3. B' has essential horizontal boundary, and no Reeb branched surfaces.

∂B is a train track on ∂X transverse to meridians, with digons as complementary components. So if M is obtained by Dehn surgery on K, we can apply construction 3.1.1 to B to get $B' \subset M$. By step 2, the hypotheses of Lemmas 3.1.2,3.1.3, and 3.1.4 are satisfied, so B' has essential horizontal boundary in M, and has no Reeb branched surfaces.

Step 4. B' has no discs of contact.

Suppose towards contradiction that D is a disc of contact for B.

Case 1. ∂D lies in the suture $\partial D^2 \times I$ of a $D^2 \times I$ complementary component of B'.

Then D together with $D^2 \times 0$ give a (immersed) sphere carried by B'. By cutting and pasting at the branch locus, we can assume this sphere is embedded in $N(B')$. This sphere cannot lie entirely in the Dehn filling solid torus since if it did, we could split B' along it to get an $S^2 \times I$ complementary component which contradicts Lemma 3.1.2. Therefore B carries a genus zero surface in X, but this can happen only if in figure 2.13, $\epsilon = 1$, i.e. B is one of the incompressible branched surfaces of [HT]. But all the incompressible surfaces in $S^3 - \mathring{N}$(2-bridge knot) are of genus ≥ 1 (except for Mobius bands in the case of torus knots), which gives us a contradiction.

Case 2. ∂D lies in the suture $A \times 0$ of an $A \times I$ complementary component.

Then D, together with the horizontal boundary annulus $\partial A \times I \cup A \times 1$, plus one more copy of D give a sphere carried by B, which again leads to a contradiction.

Case 3. ∂D lies in the suture $\partial A \times I$ of an $A \times I$ complementary component.

But as noted in remark 3.2.1, the annuli $\partial A \times I$ have slope $1/n$, $|n| \geq 1$, on the solid torus, and by figure 3.39 this solid torus is unknotted in X, so ∂D is homotopic in $X - (A \times I)$ to a meridian, which cannot bound a disc in X. Nor can it bound a disc in M (after the Dehn filling), because of the following. $D \cap \partial X$ is a union of simple closed curves $c_1, \ldots, c_n$ on ∂X. By figure 3.39, any curve on ∂X, and in particular any c_i, has linking number zero with the core of the solid torus (K is a 2-bridge knot, so it goes down through the "hole" of the solid torus, and then up, exactly two times). By renumbering if necessary, assume that $c_1, \ldots, c_m$, $1 \leq m \leq n$, are in the same component of $D \cap X$ as ∂D. Then ∂D is homotopic in $X - (A \times I)$ to a connected sum of $c_1, \ldots, c_m$, so ∂D also has linking number zero with the core of the solid torus, which contradicts ∂D being homotopic to a meridian. (This part of the proof is different from that given in [H]; there, it is argued that because of a symmetry the branched surface has, the other suture on the solid torus must also bound a disc of contact, thus again yielding a sphere carried by the branched surface, which gives a contradiction. Here, however, the branched surfaces constructed for the torus knots as in figure 2.20 no longer possess this symmetry—the symmetry exists as long as at most one of the saddles of diagrams 2, 5, or 8 are present—and so we use a different argument.)

Step 5. B' fully carries a lamination.

This requires showing that B fully carries a lamination with *closed* boundary curves. We showed in the previous chapter that B_0 (the branched surface before the enlarging of step 1) fully carries a lamination with closed boundary curves, and the whole argument was basically qualitative: the main point was to get in figure 2.18 fixed-point free self-homeomorphisms g and h of the interval.

But this is achieved for B also, simply by assigning small enough weights to all portions of $B - B_0$, since under small enough perturbations g and h will still satisfy the (open) property of being fixed-point free. Also, in figure 2.18 the conjugating map f_1 will now be distributed over the saddles of diagram 7 of figure 3.37. □ (Theorem 1)

3.3 Proof of Theorem 2

The laminations of Lemma 2.2.1, as stated in the note following it, were proven to be essential in the proof of Theorem 1. So here we prove the branched surfaces of Lemma 2.2.2 are essential. Let $K \subset S^3$ be a knot of type (II).

Step1. If $K = [2m + 1, -2]$ ((II) $\cap$ (b)), we modify the branched surface of figure 2.25 according to remark 2.2.3 and step 2 of proof of Lemma 2.2.2;

the resulting branched surface B is shown in figure 3.40.

With some visualization one sees that $N(K) \cup N(B)$ is just a solid torus with suture as shown in figure 3.41 (this "pictorial computation" is done in figures 3.44-3.50).

If $K = [2m+1, -2n+1, 2p+1]$ ((II) $\cap$ (c)), we further modify B according to step 3 of proof of Lemma 2.2.2, as in figure 2.32. Then $N(K) \cup N(B)$ is obtained by adding $2p$ handles to figure 3.41. Figure 3.42 shows the case $p = 1$.

Step 2. B has essential horizontal boundary in X.

For the case of figure 3.41, B has exactly one complementary component. It is an unknotted solid torus $A \times I$ with one annulus as suture (the curve in figure 3.41 is connected). A meridional disc of $A \times I$ must bound a longitude on the solid torus of figure 3.41, which intersects the suture in $2m+1$ points, $m \geq 1$; so there are no compressing discs or monogons.

For the case of figure 3.42 also, B has exactly one complementary component. It is a handlebody of genus $2p + 1$. We use the disc decomposition technology of [G2] (explained below in section 3.4) to simplify figure 3.42 to figure 3.43 by filling in its $2p$ extra holes; by Lemma 3.4.1 one has essential horizontal boundary if and only if so does the other one. But figure 3.43 is the same as figure 3.41, so it has no compressing discs or monogons.

Step 3. B' has essential horizontal boundary, and no Reeb branched surfaces.

∂B is a train track on ∂X transverse to meridians, with digons as complementary components. So if M is obtained by Dehn surgery on K, we can apply construction 3.1.1 to B to get $B' \subset M$. By step 2, the hypotheses of Lemmas 3.1.2,3.1.3, and 3.1.4 are satisfied, so B' has essential horizontal boundary in M, and has no Reeb branched surfaces.

Step 4. B' has no discs of contact.

To show essentiality for the particular laminations (carried by B') we have in mind, it is in fact enough to show B' has no disc of contact *which is isotopic to a leaf of the lamination* [G3]. This is clear since the laminations constructed in Lemma 2.2.2 have infinite spirals. However, we will also show that B' has no discs of contact (isotopic to a leaf, or not).

Assume D is a disc of contact for B'. As in case 1 of step 4 in the proof of Theorem 1, $D \cap X$ is not empty. Therefore the spiral regions of B must be finite spirals, i.e. they can be opened up by unspiraling, and B will still carry $D \cap X$. So again as in case 1 of step 4, B degenerates into one of the branched surfaces of [HT], say B_1. Then D is a disc of contact for B_1', which we know is impossible by proof of Theorem 1 (or Proposition 1 of [H]).

Step 5. B' fully carries a lamination.

This was already shown when B was constructed. □ (Theorem 2)

3.4 Essentiality is Preserved by Disc Decomposition

For a general definition of disc decomposition see [G2]. Here we consider only the special case in figure 3.51: We have a handlebody $H \subset S^3$ with a suture on it, and we attach to it a 2-handle $D^2 \times I$ along the annulus $A = \partial D^2 \times I$ such that the suture (thought of as a curve rather than an annulus) intersects A in two arcs $x \times I$ and $y \times I$, $x, y \in \partial D^2$. The suture in the new handlebody is obtained by replacing $x \times I$ and $y \times I$ by two (properly embedded) arcs in $D^2 \times 0$ and $D^2 \times 1$, connecting $x \times 0$ to $y \times 0$ and $x \times 1$ to $y \times 1$.

Suppose we want to show that $G = S^3 - \mathring{H}$ has no compressing discs or monogons. Attaching a 2-handle to H as above is equivalent to removing a 1-handle from G (and changing the suture accordingly), which simplifies G. By the following lemma, it is enough to check that this simplified G has no compressing discs or monogons.

Lemma 3.4.1 *Let H_1 be obtained by disc decomposition from a sutured handlebody $H \subset S^3$ (as above). Then $G_1 = S^3 - \mathring{H}_1$ has compressing discs or monogons if and only if so does $G = S^3 - \mathring{H}$.*

Proof: The 'only if' direction is clear. For the converse, let D be a compressing disc or monogon in G. Let $D^2 \times I$ be the 1-handle being removed from G, with $A = \partial D^2 \times I$. If D is disjoint from $D^2 \times I$, we are done. Otherwise, isotope D near its boundary so that ∂D is transverse to ∂A, has minimal intersection with it, and is disjoint from the suture inside A. Using a standard innermost disc argument and the irreducibility of G, we further isotope D (rel ∂) so that $D \cap (D^2 \times \partial I)$ has no circle components, so it is a union of disjoint arcs in D. Then $D \cap G_1$ is a finite union of properly embedded discs in G_1, at least two of which, say D_1 and D_2, each intersect $D^2 \times I$ in only one arc, which intersects the (new) suture in G_1 in at most one point. So at least one of D_1 or D_2 is a compressing disc or monogon for G_1 (we needed to make $\partial D \cap \partial A$ minimal to ensure that now we get a "true" compressing disc). □ (Lemma 3.4.1)

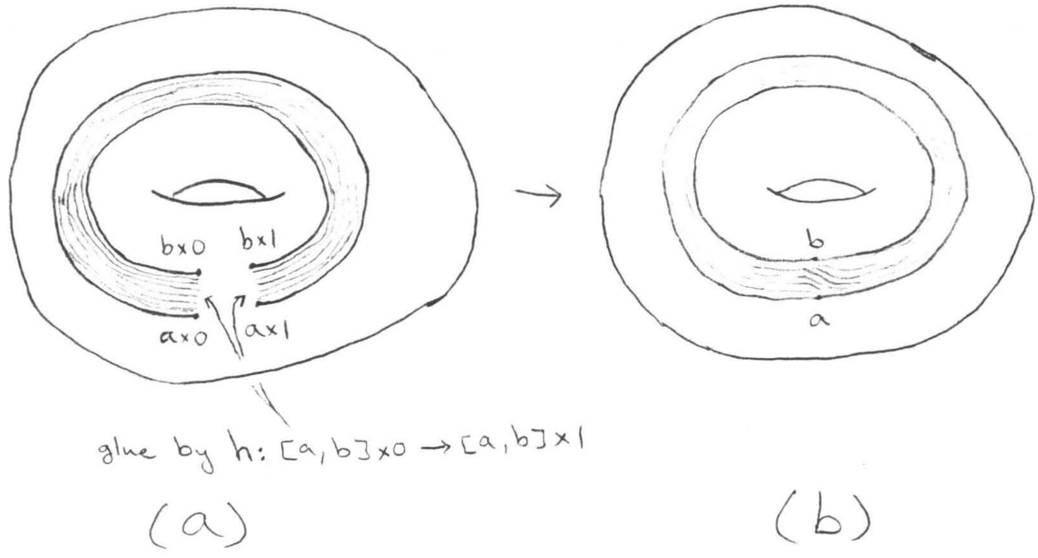

Figure 1.1: A lamination with two closed curves of slope 0, with nontrivial holonomy.

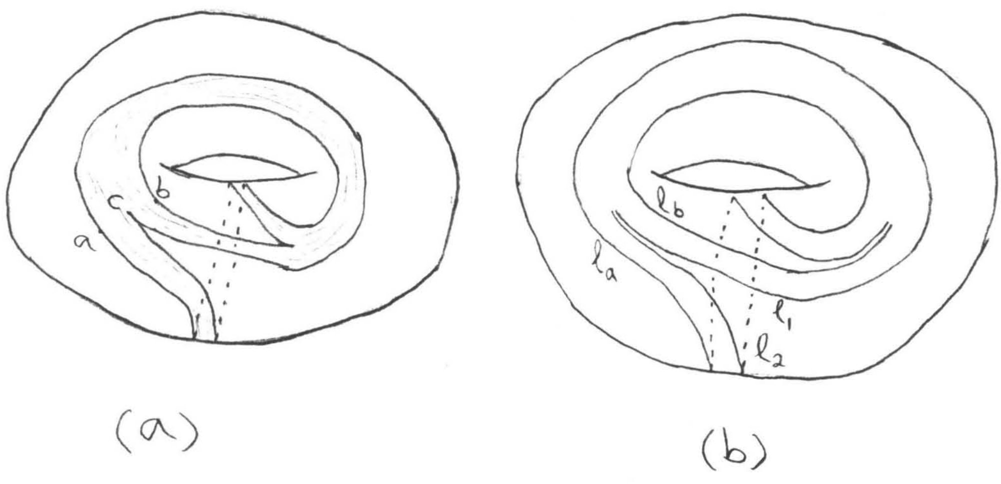

Figure 1.2: A lamination with leaves of slope $(c-a)/(b-a)$.

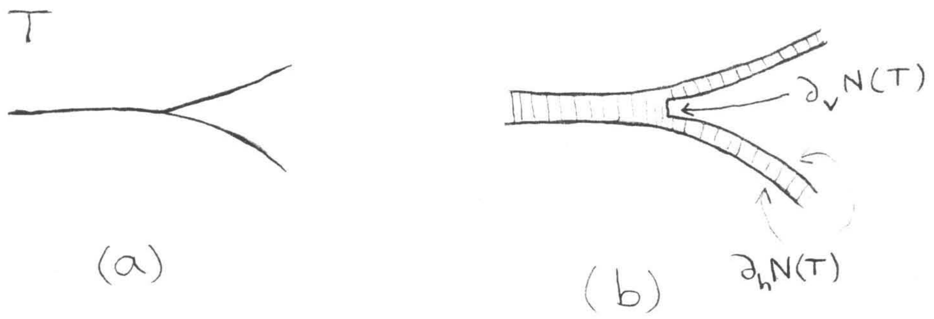

Figure 1.3: Train Track

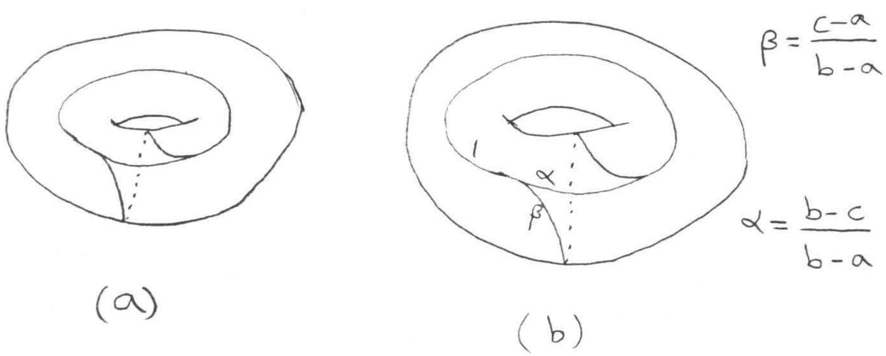

Figure 1.4: A train track carrying the lamination of figure 1.2.

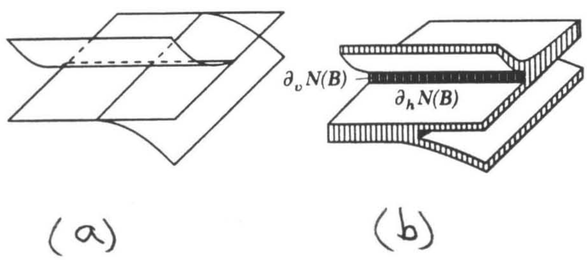

Figure 1.5: Branched Surface

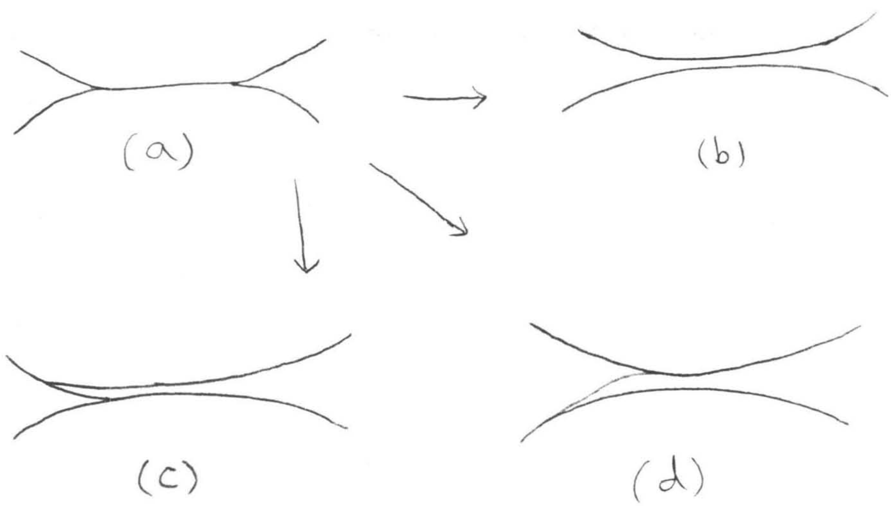

Figure 1.6: Splitting a train track (or branched surface).

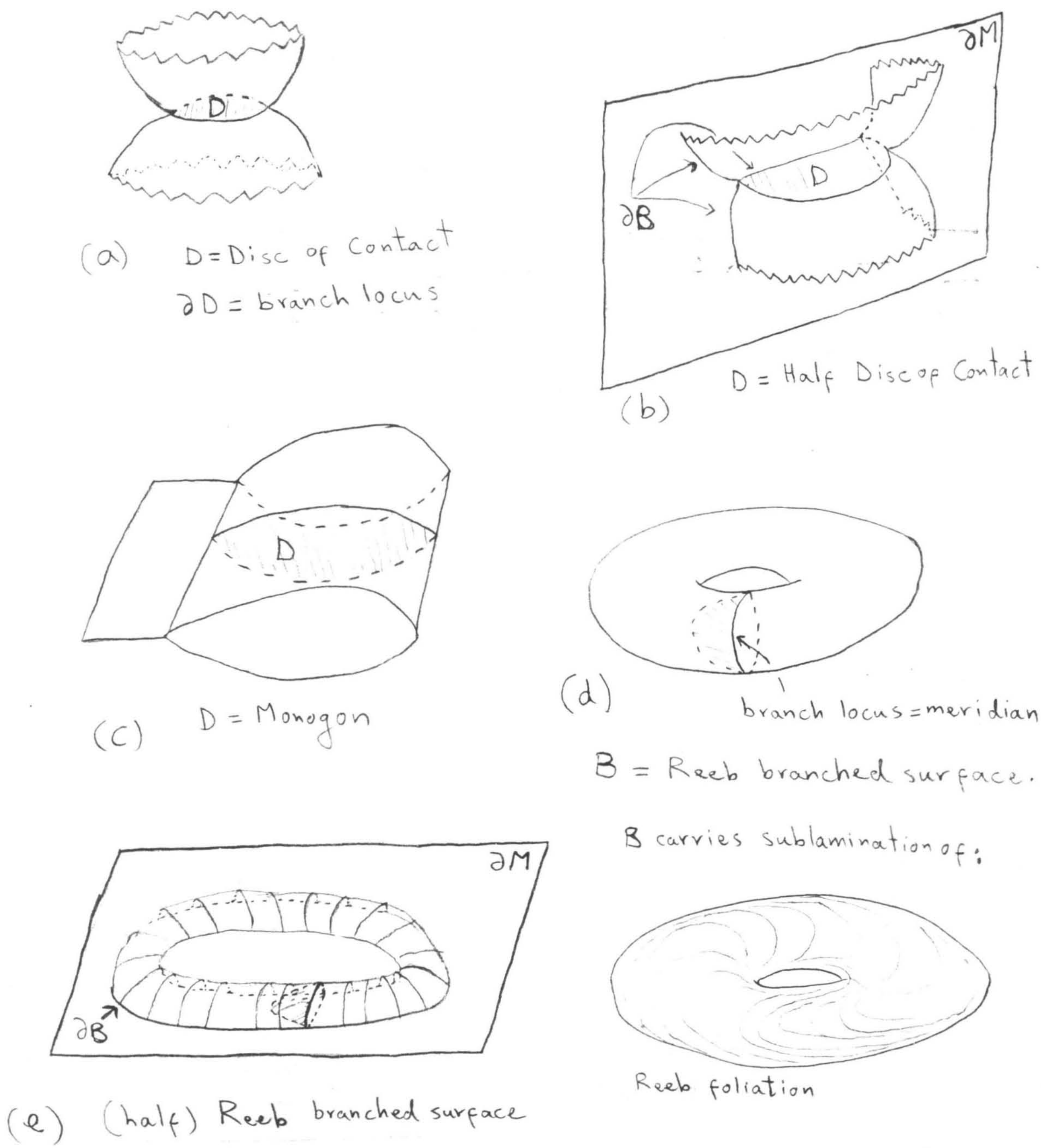

Figure 1.7: Discs of constact, monogons, and Reeb branched surfaces.

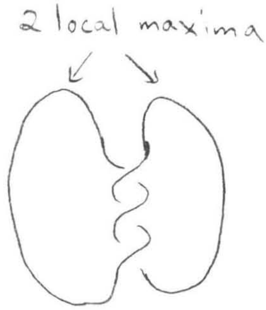

Figure 1.8: The trefoil is a 2-bridge knot.

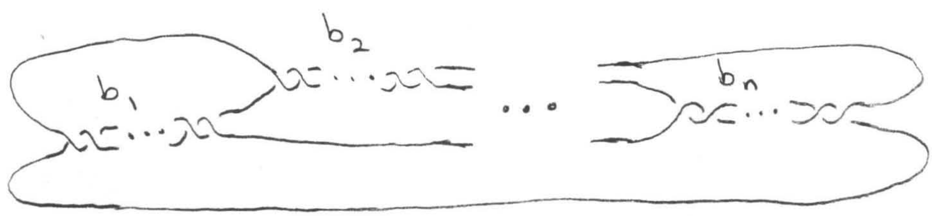

Figure 1.9: Every 2-bridge knot or link can be put into this "nice" form, denoted $[b_1, b_2, \ldots, b_n]$.

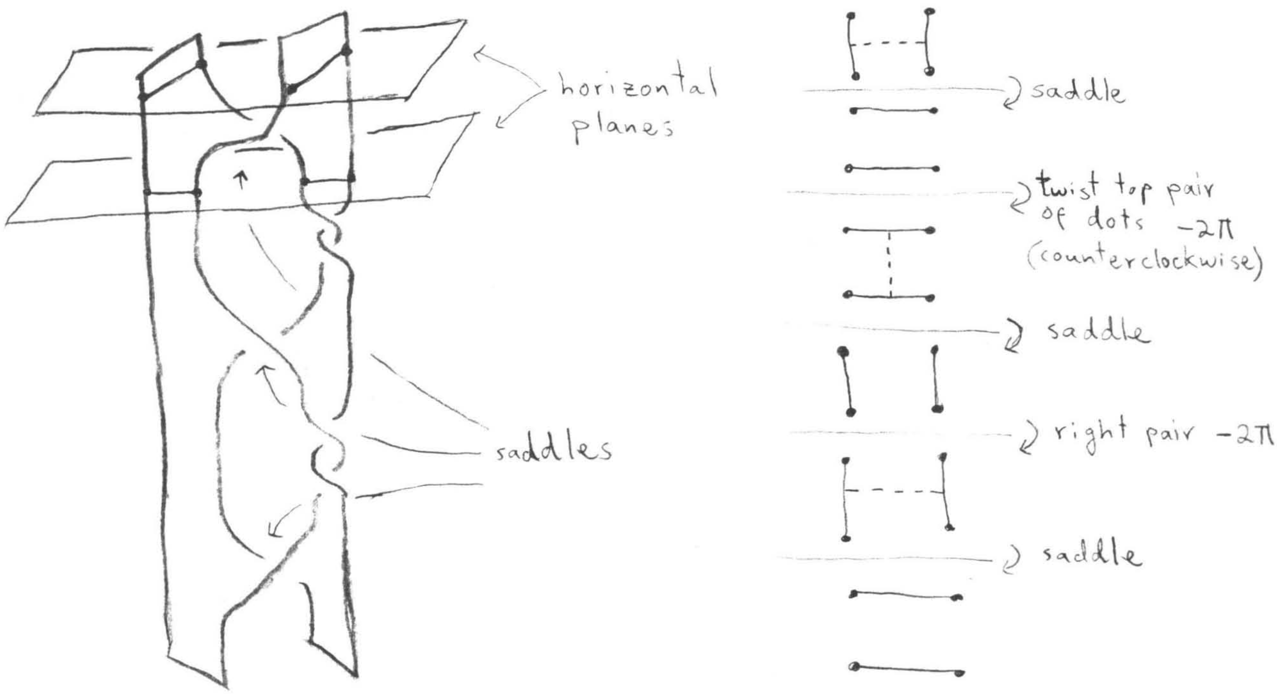

Figure 1.10: Seifert surface for trefoil.

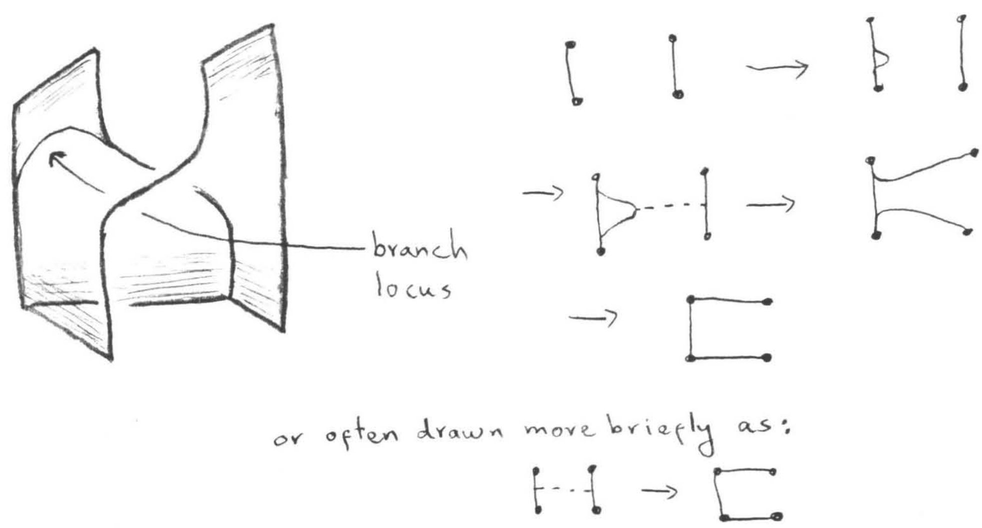

Figure 1.11: Intersection of a branched surface with horizontal planes.

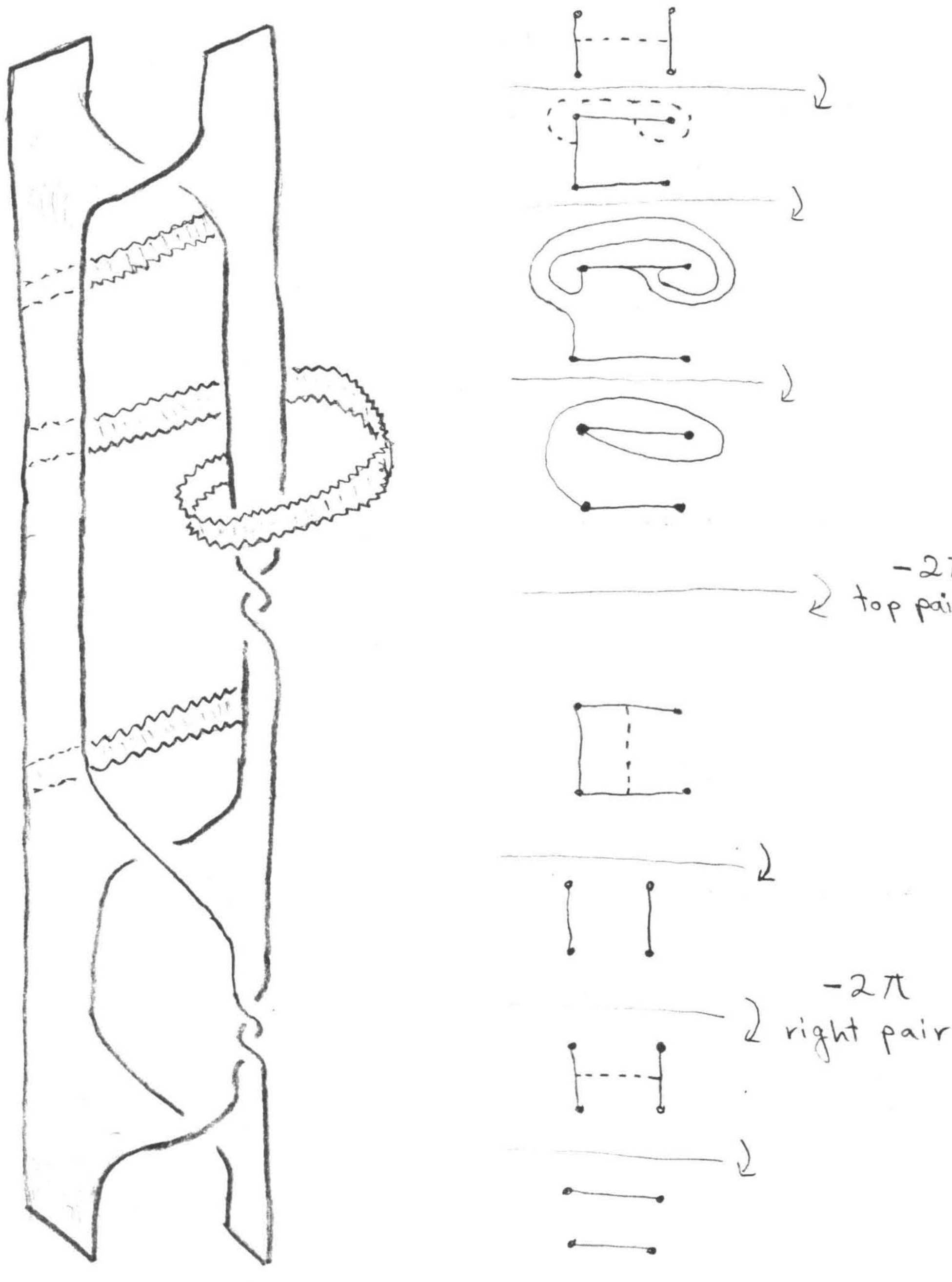

Figure 2.12: [H]'s branched surface for the trefoil.

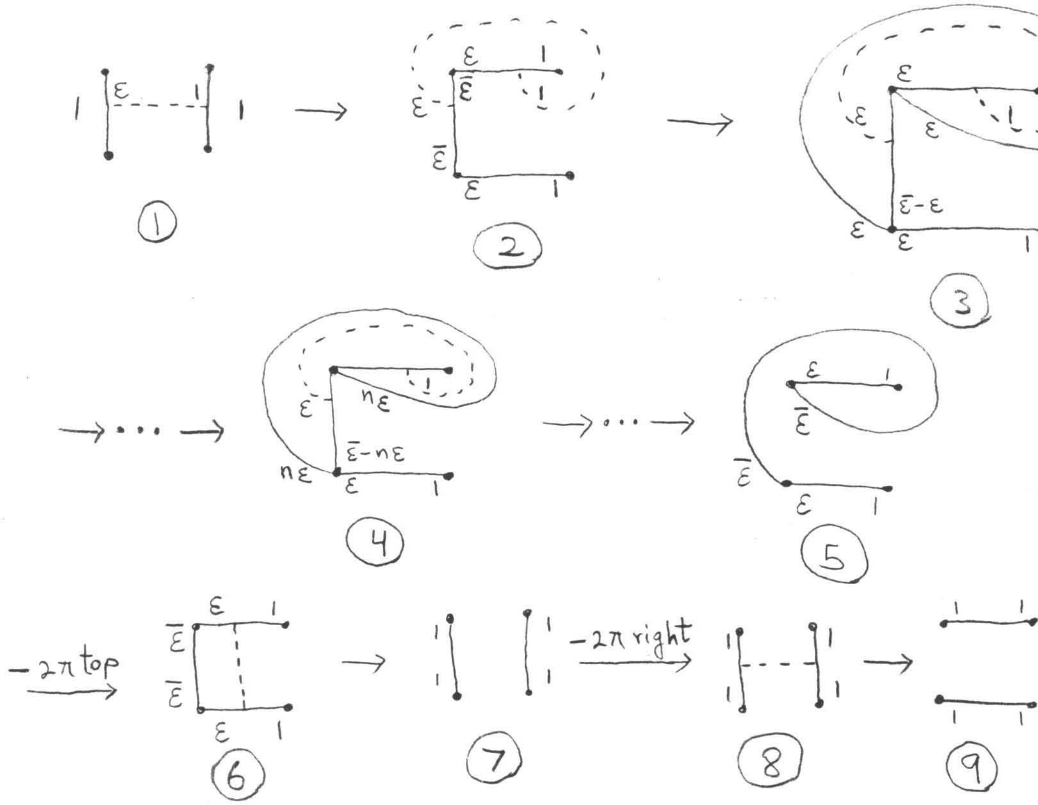

Figure 2.13: [H]'s construction of laminations of slopes $(-\infty, 0]$.

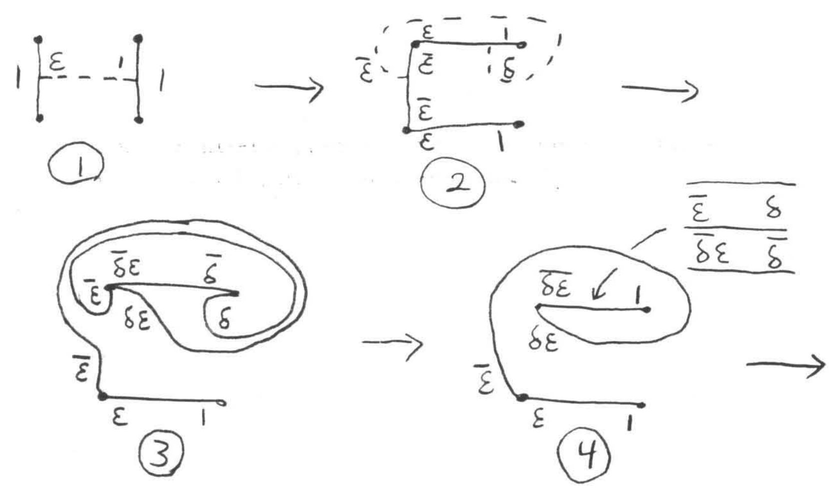

Figure 2.14: Modification of figure 2.13 giving slopes (0,1) on $T_{2,3}$.

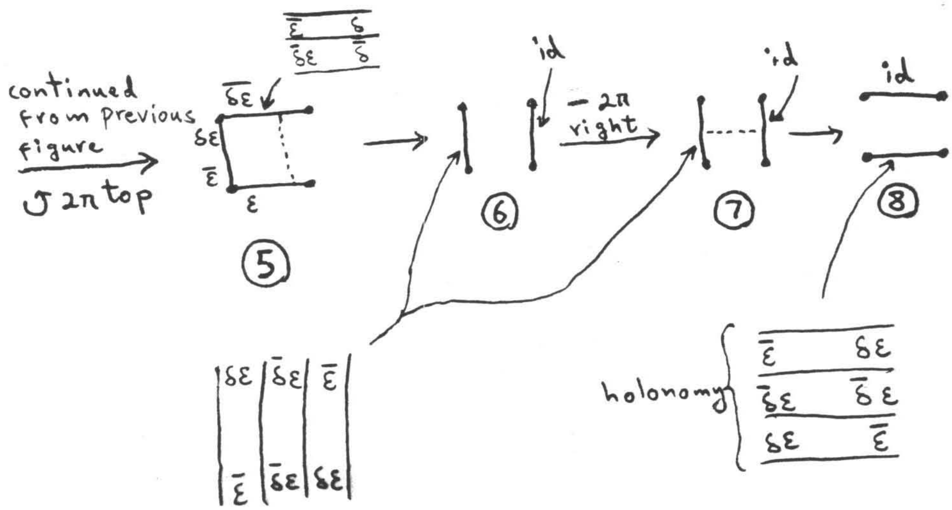

Figure 2.15: One gets holonomy if all saddles are just linear.

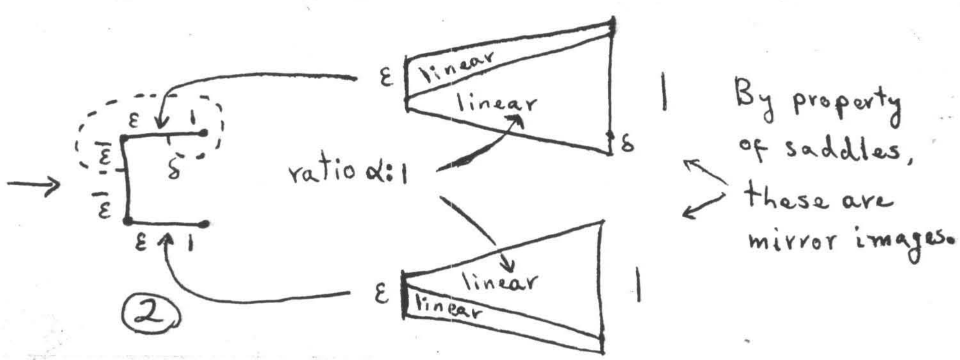

Figure 2.16: Avoiding holonomy.

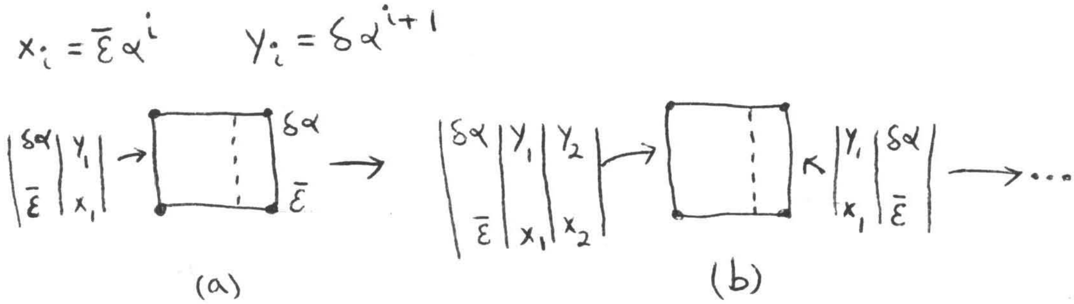

Figure 2.17: This process converges to figure 2.18.

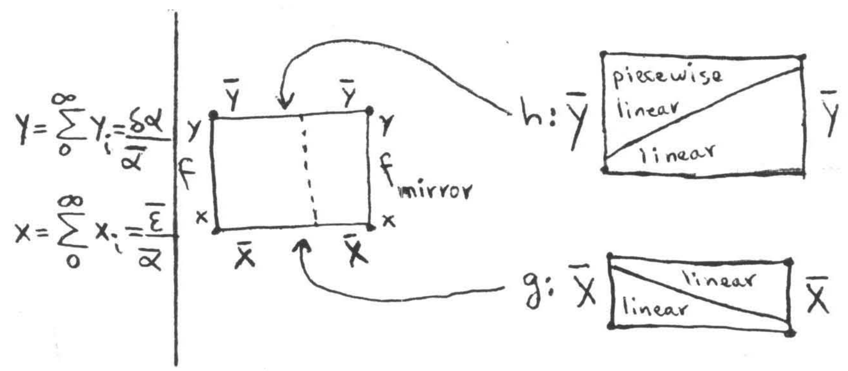

Figure 2.18: g and h have no interior fixed points.

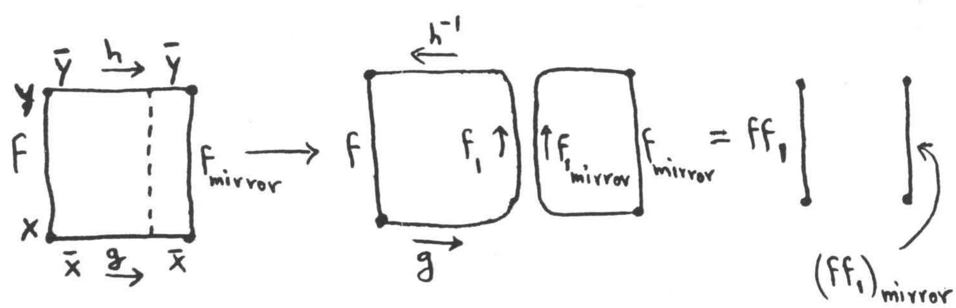

Figure 2.19: f_1 conjugates g to h.

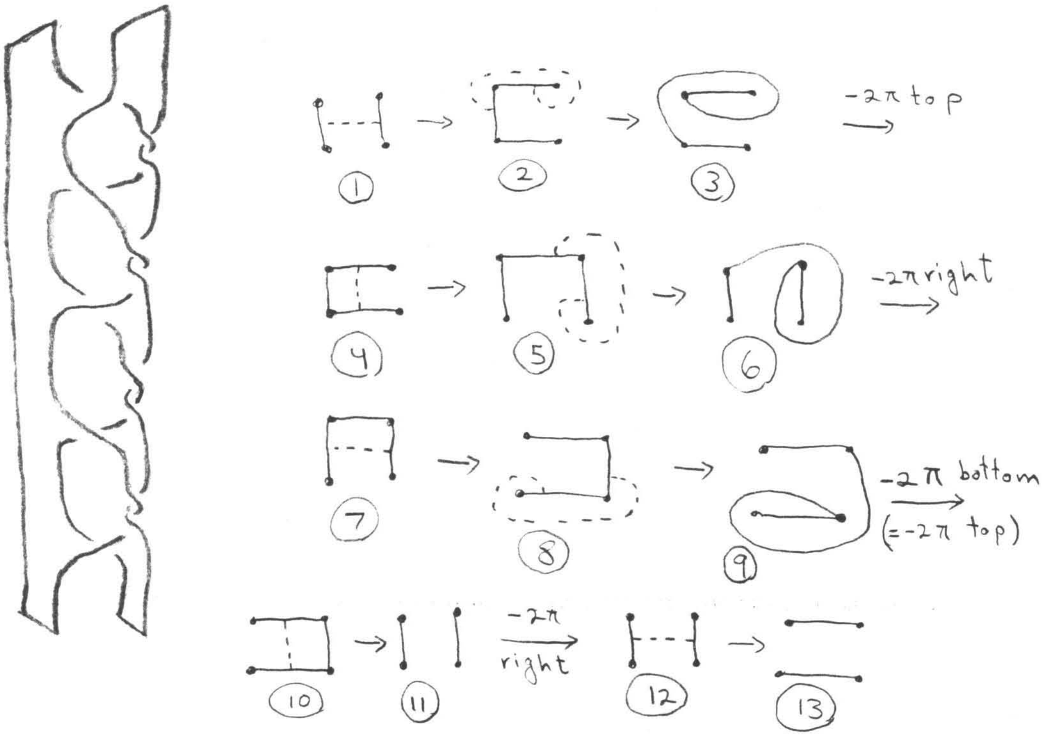

Figure 2.20: Branched Surface for $T_{2,5}$.

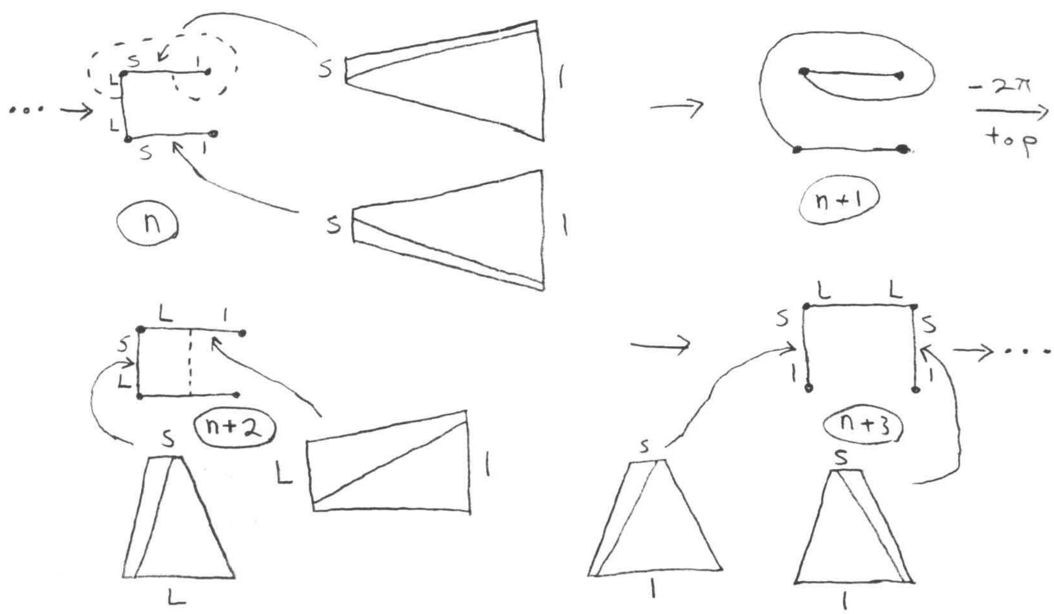

Figure 2.21: Induction step for avoiding holonomy for $T_{2,5}$.

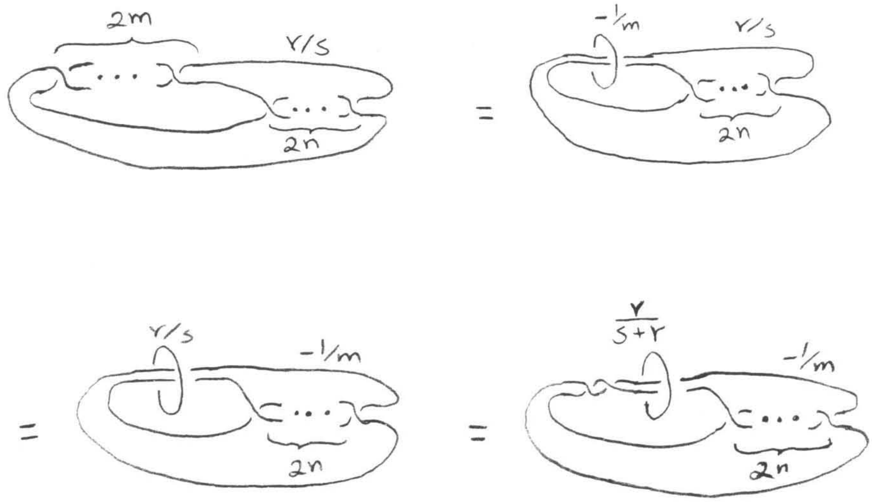

Figure 2.22: For knots of type (I).

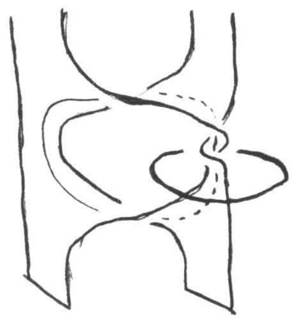

Figure 2.23: The unknot is isotopic to the curve on the surface.

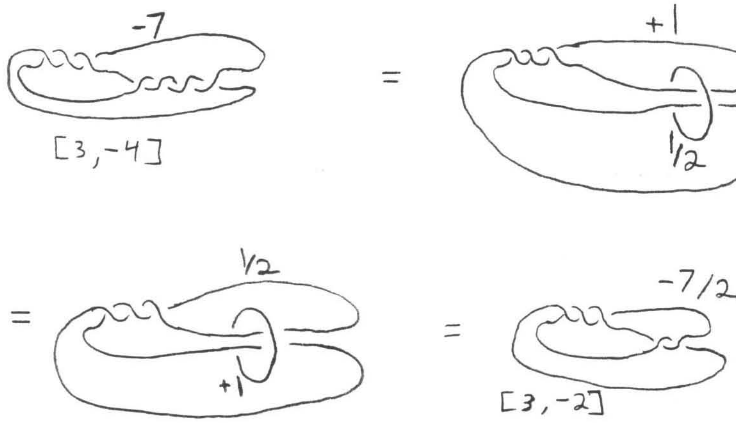

Figure 2.24: -7 on $7_3 = -7/2$ on 5_2.

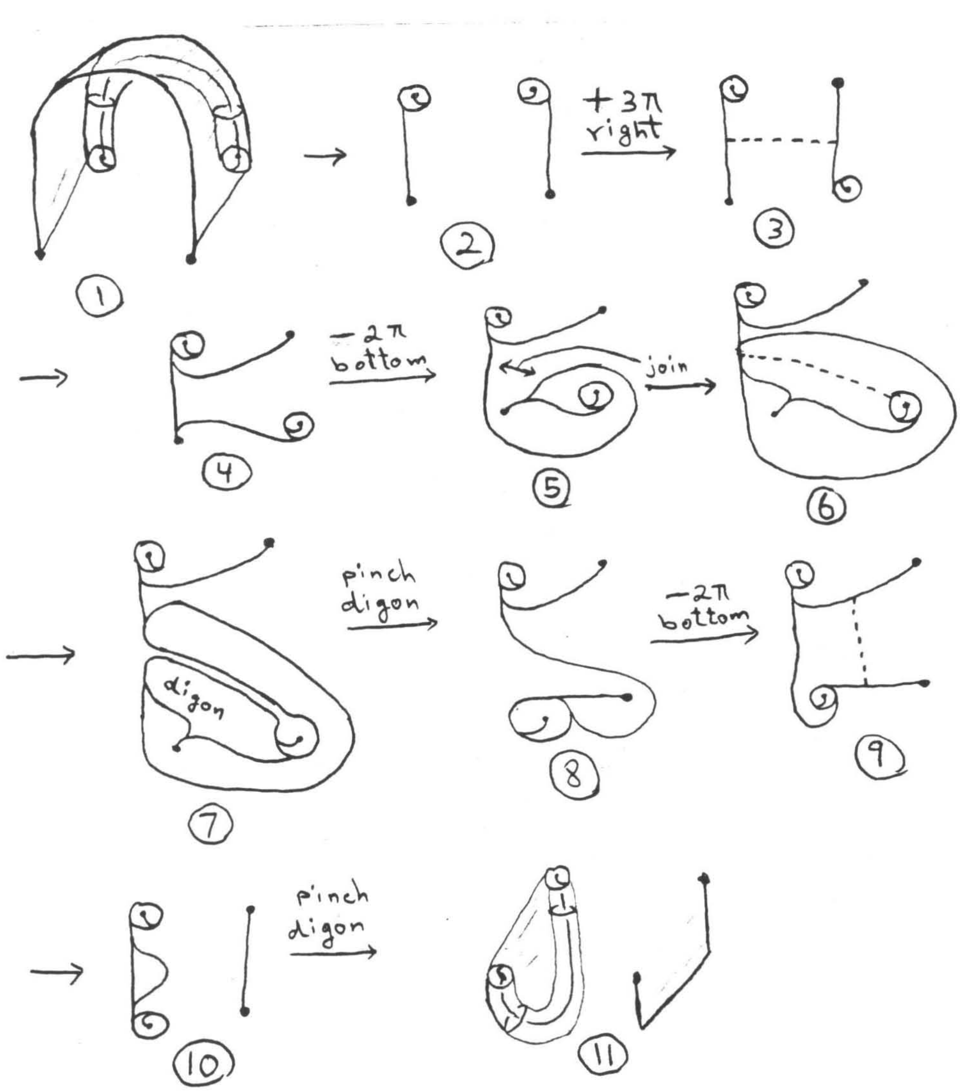

Figure 2.25: Essential branched surface for the [3,-4] knot (7_3 in [R]) carrying laminations of all (finite) boundary slopes.

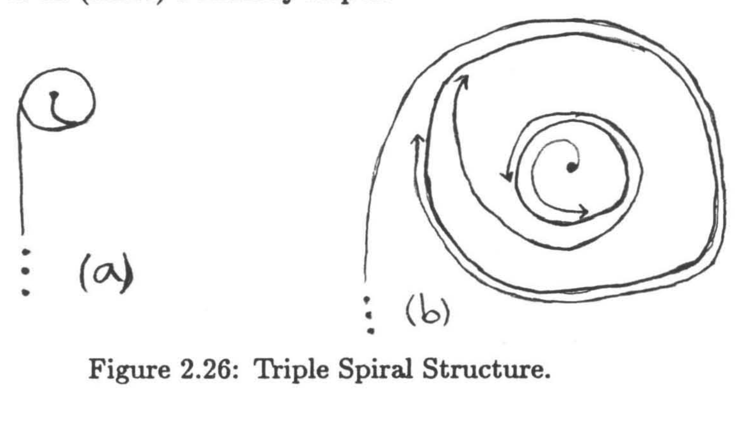

Figure 2.26: Triple Spiral Structure.

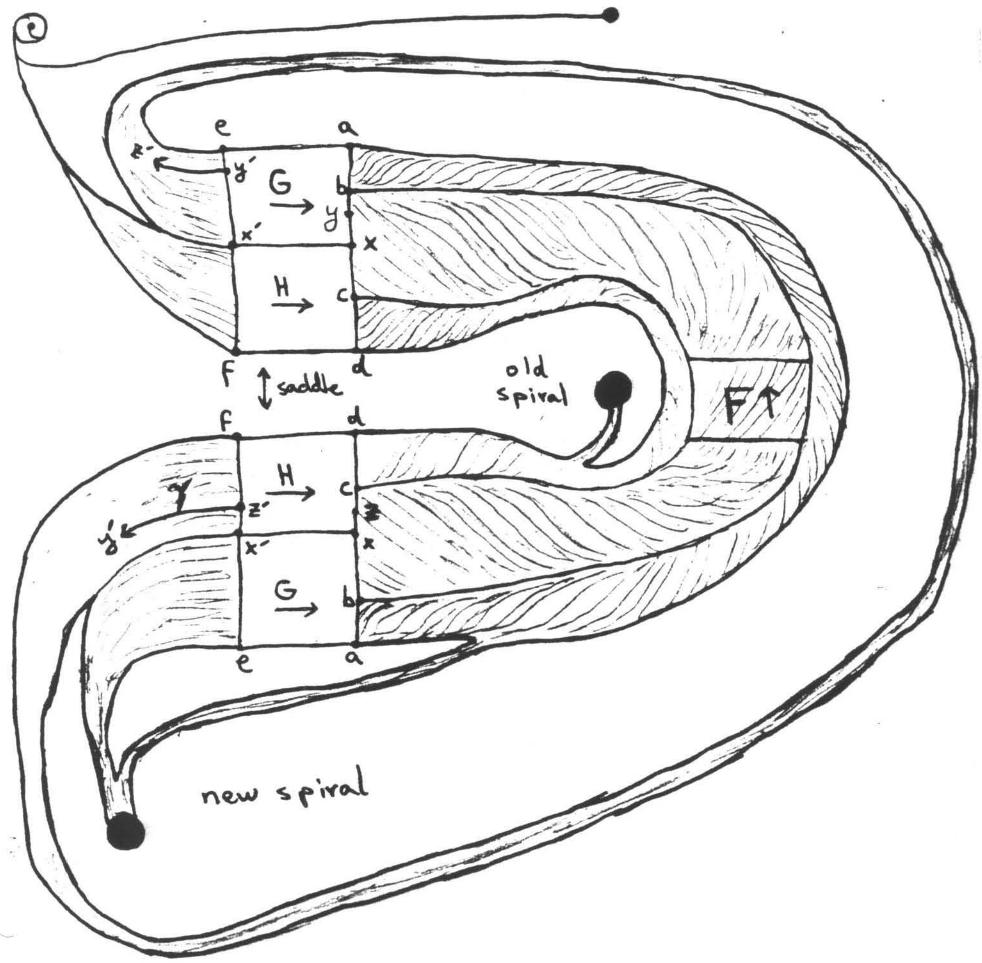

Figure 2.27: Blow up of diagram 7 of figure 2.25.

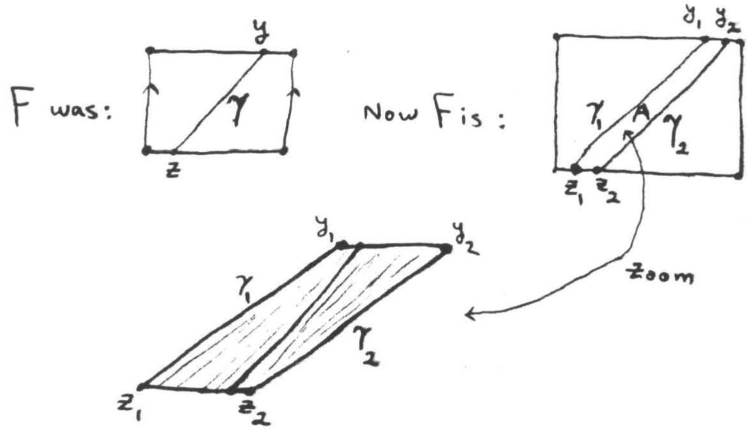

Figure 2.28: To get the middle spiral with opposite sense.

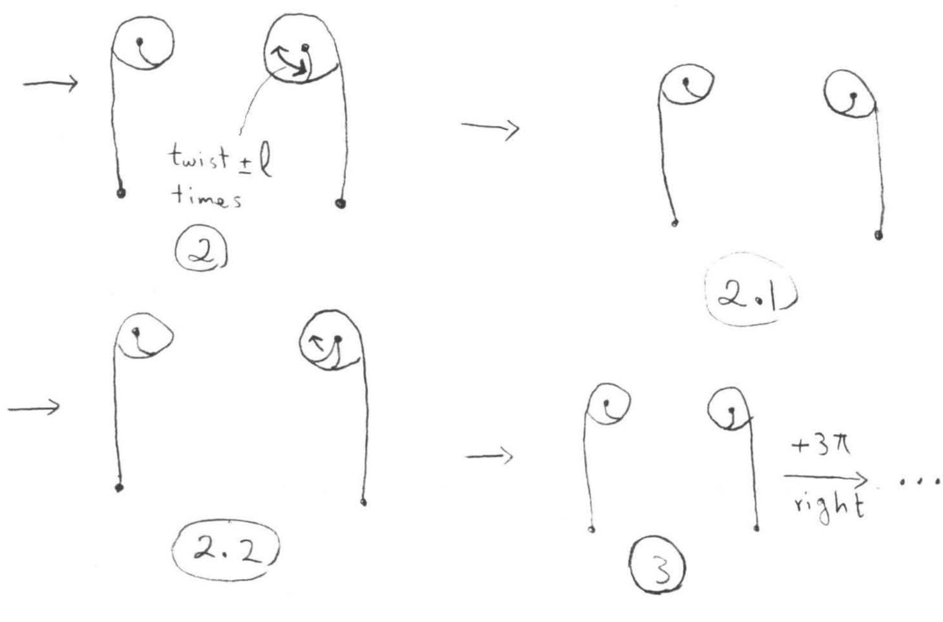

Figure 2.29: To get all slopes in figure 2.25.

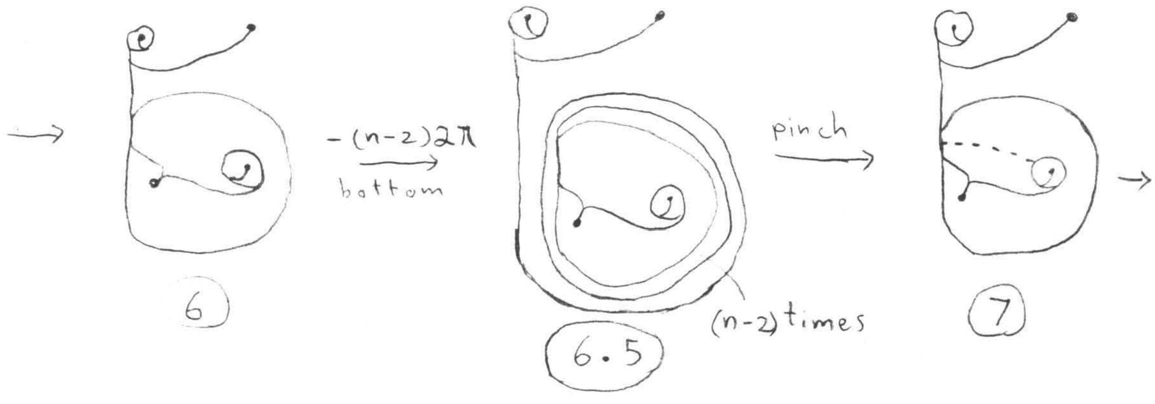

Figure 2.30: Modifying figure 2.25 for knots in class (b).

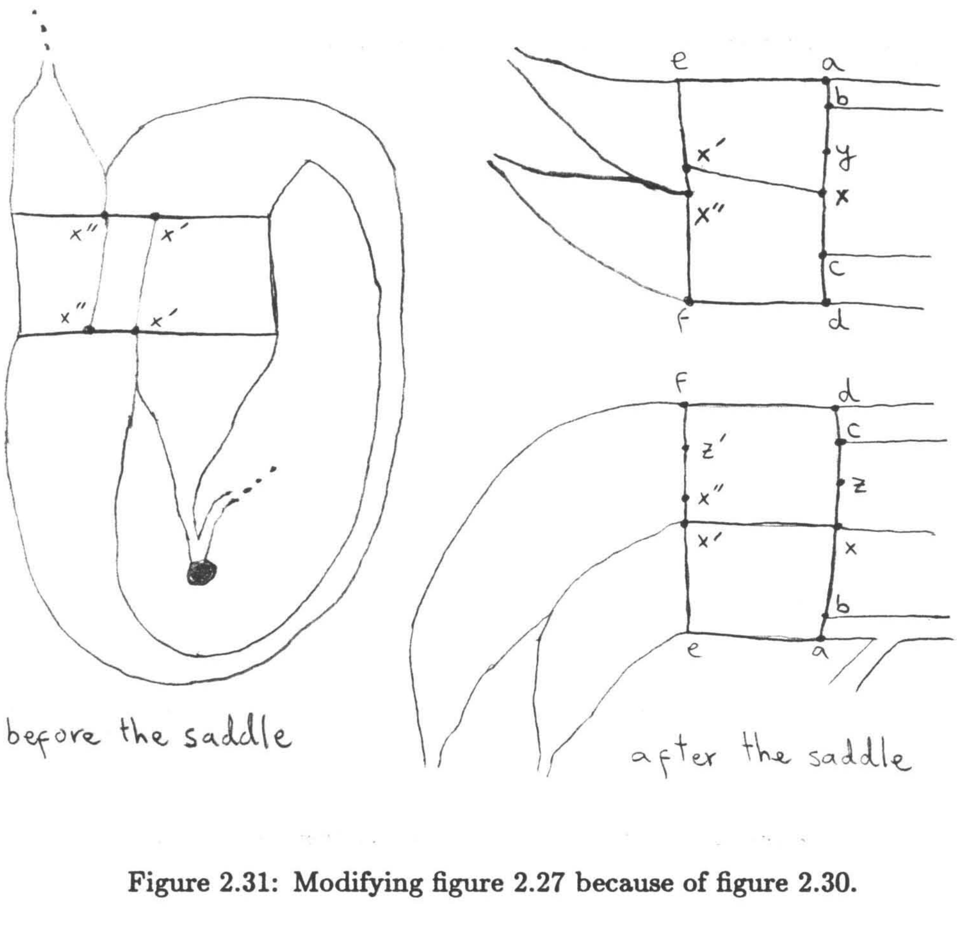

Figure 2.31: Modifying figure 2.27 because of figure 2.30.

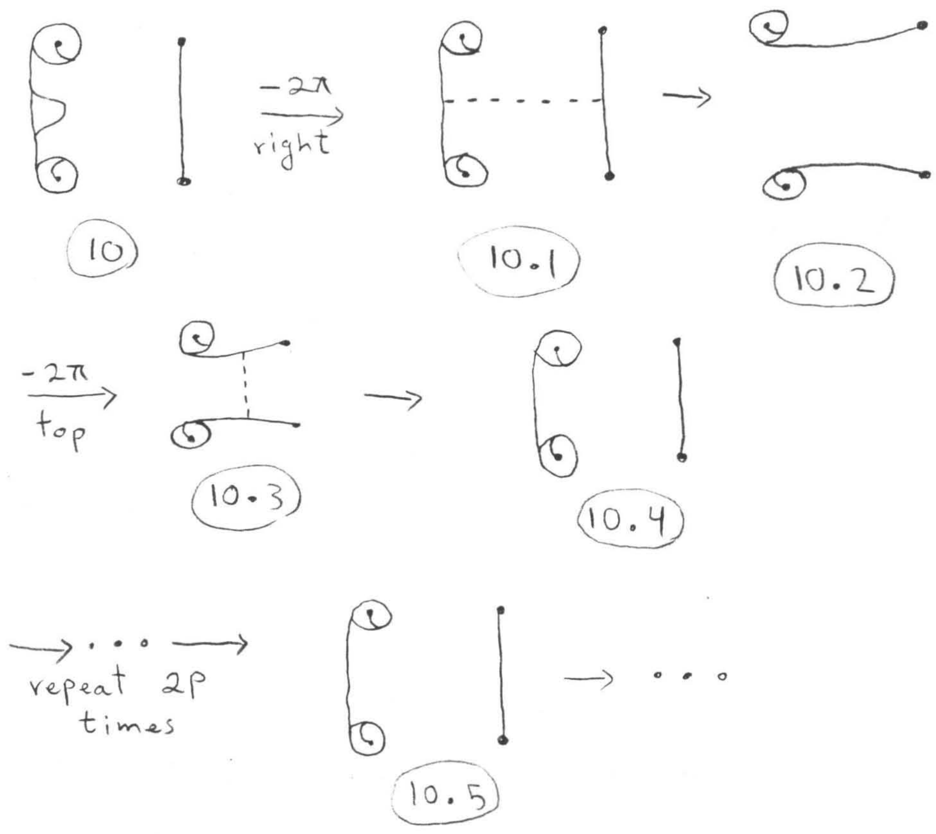

Figure 2.32: For knots in class (c).

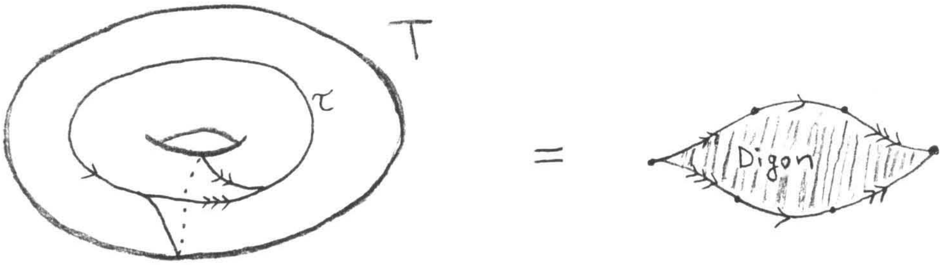

Figure 3.33: Digon.

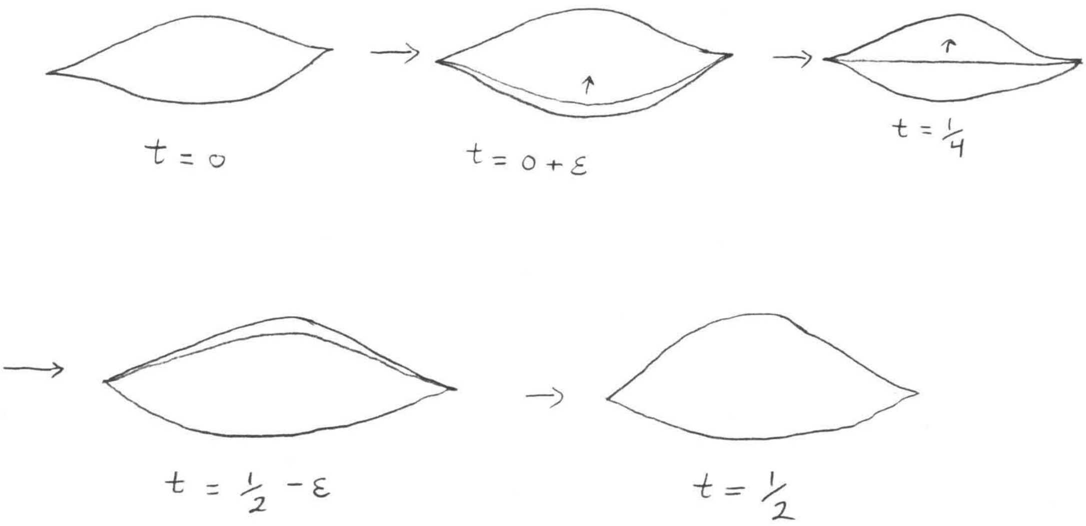

Figure 3.34: 'Moving copies of sides of digons across from one side to the other'.

Figure 3.35: Only this type of splitting in construction 3.1.1 can affect the topology of the complementary components.

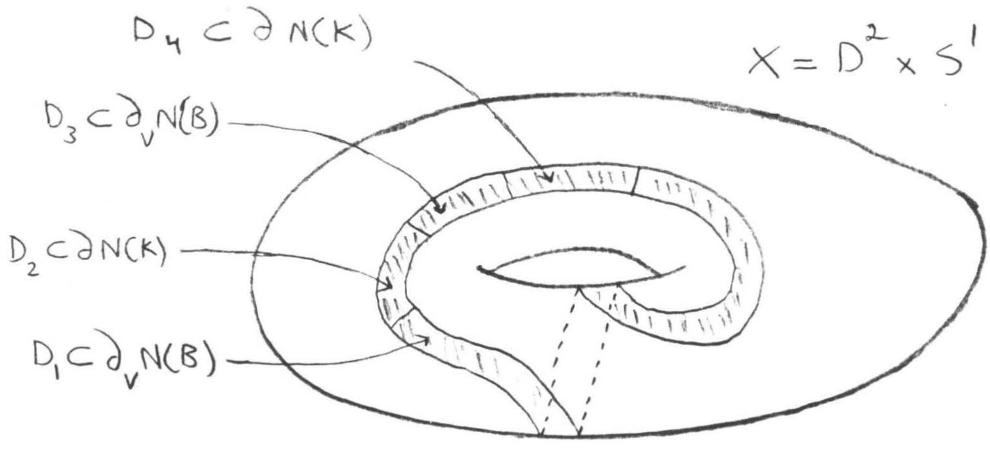

Figure 3.36: A solid torus complementary component with one annulus as suture.

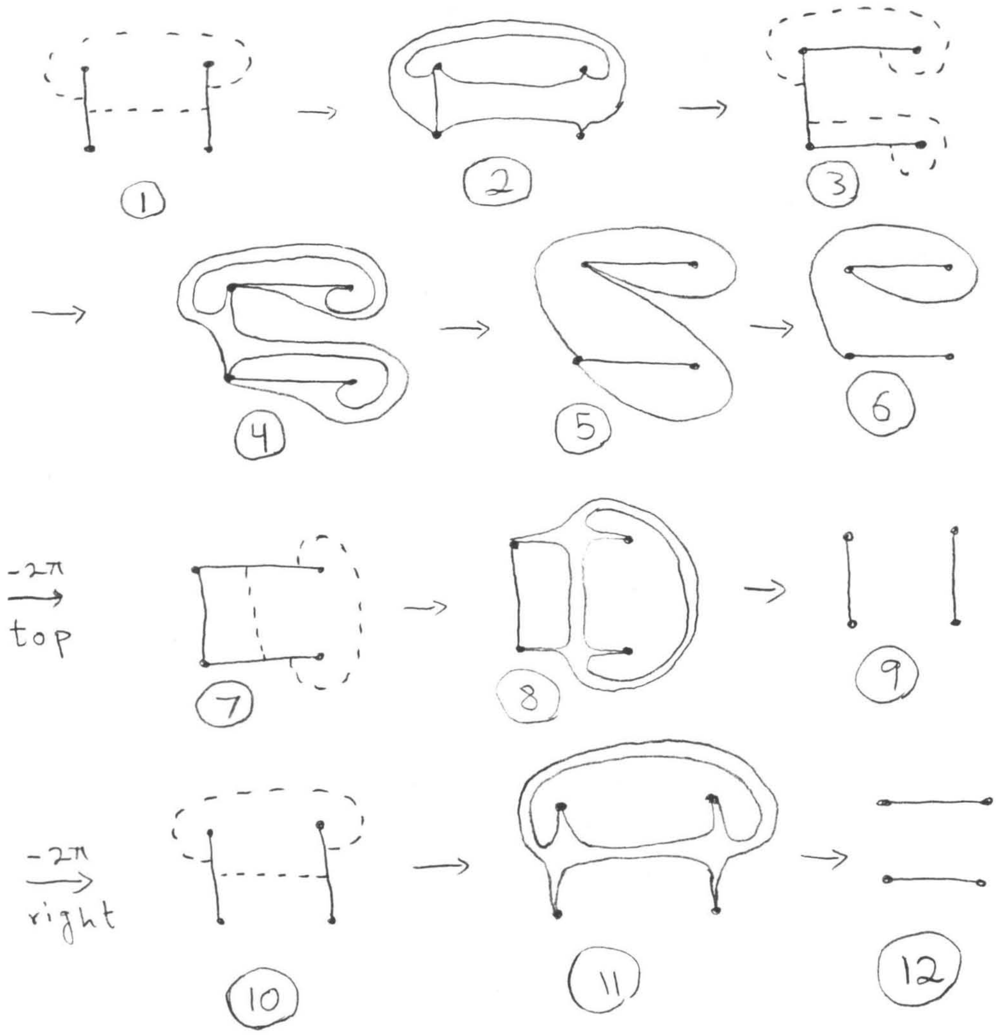

Figure 3.37: Branched surface for trefoil, with complementary saddles added in.

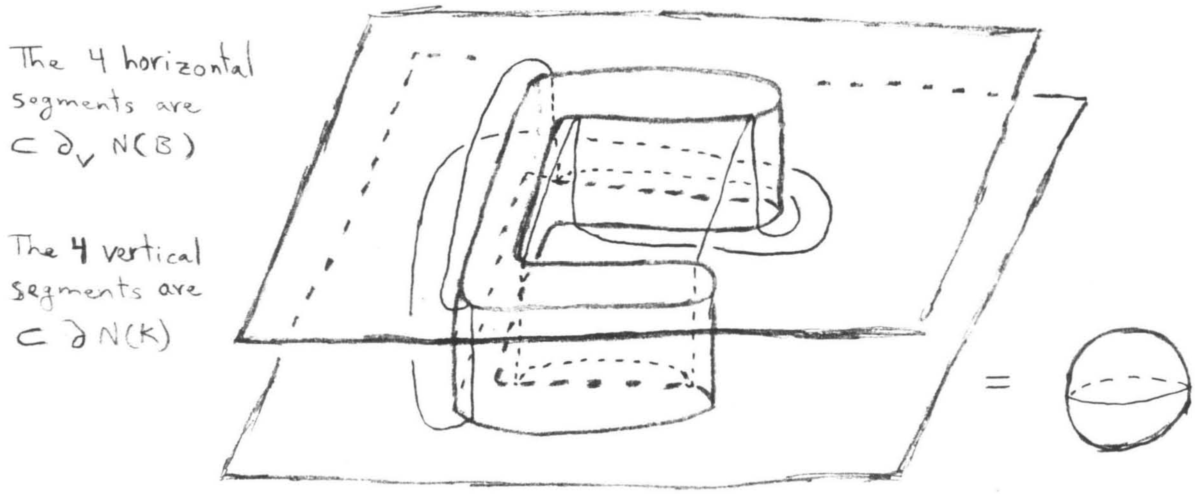

Figure 3.38: Complementary component of B between diagrams 1 and 3 of figure 3.37.

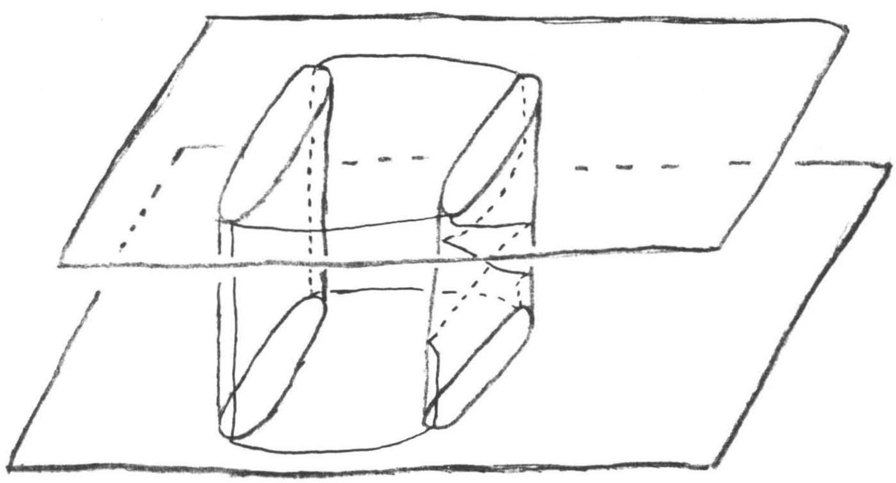

Figure 3.39: Complementary component of B between diagrams 7 and 10 of figure 3.37.

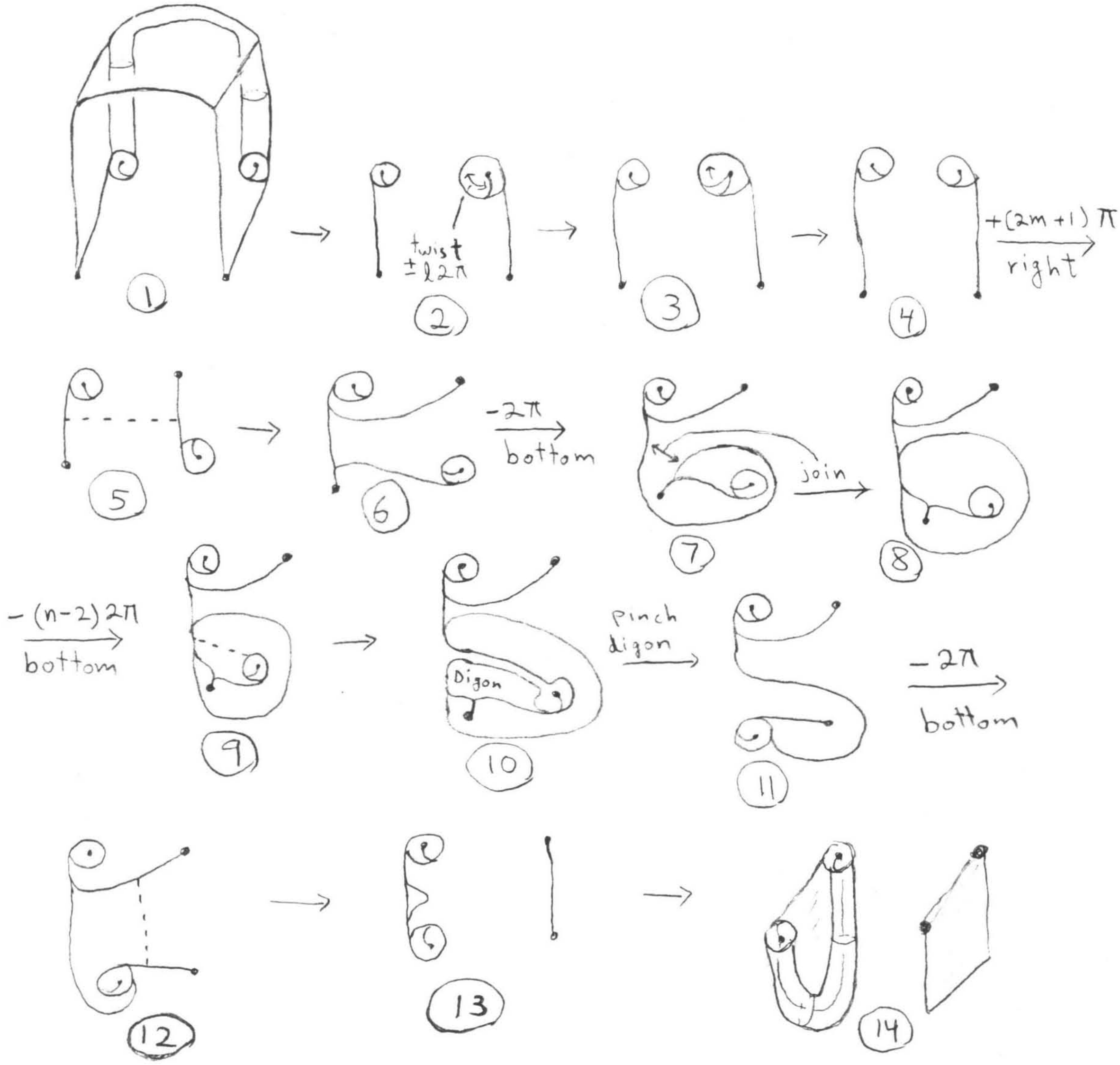

Figure 3.40: Branched surface for knots of (II) ∩ (b).

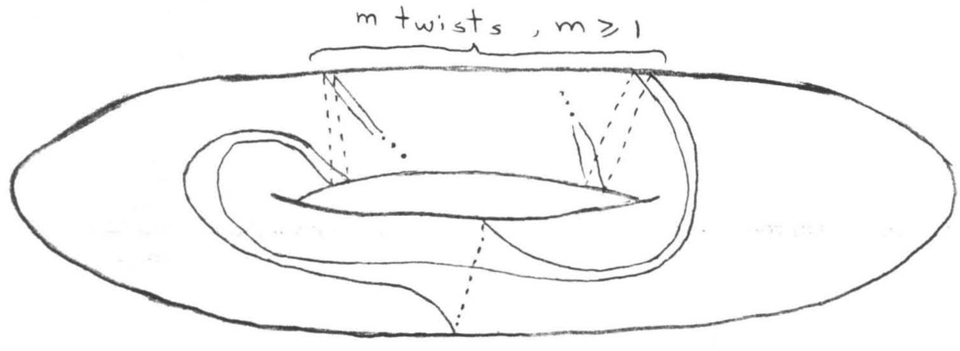

Figure 3.41: Neighborhood of branched surface of figure 3.40.

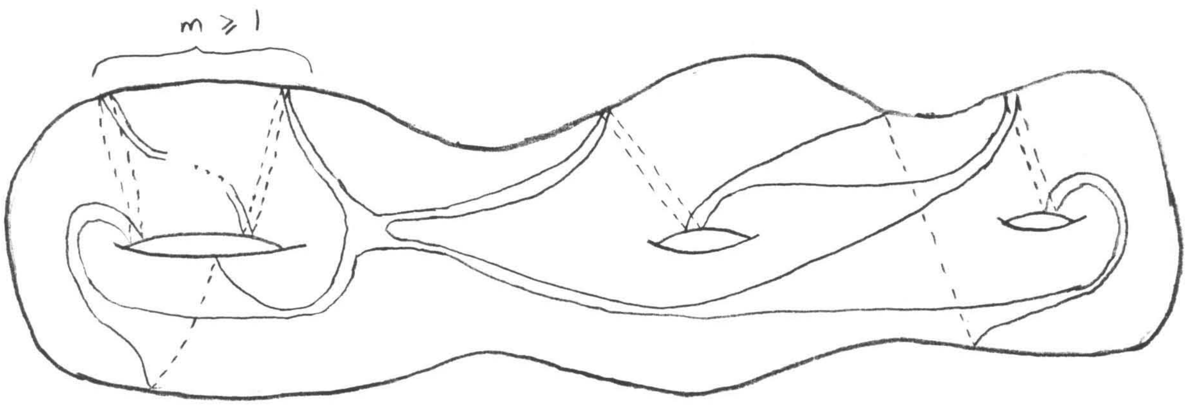

Figure 3.42: Neighborhood of branched surface for a knot in (II) ∩ (c).

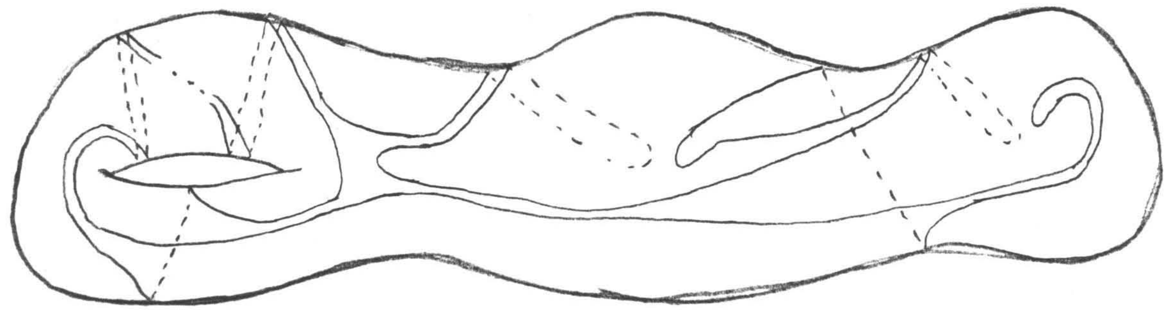

Figure 3.43: Result of disk decomposition on figure 3.42.

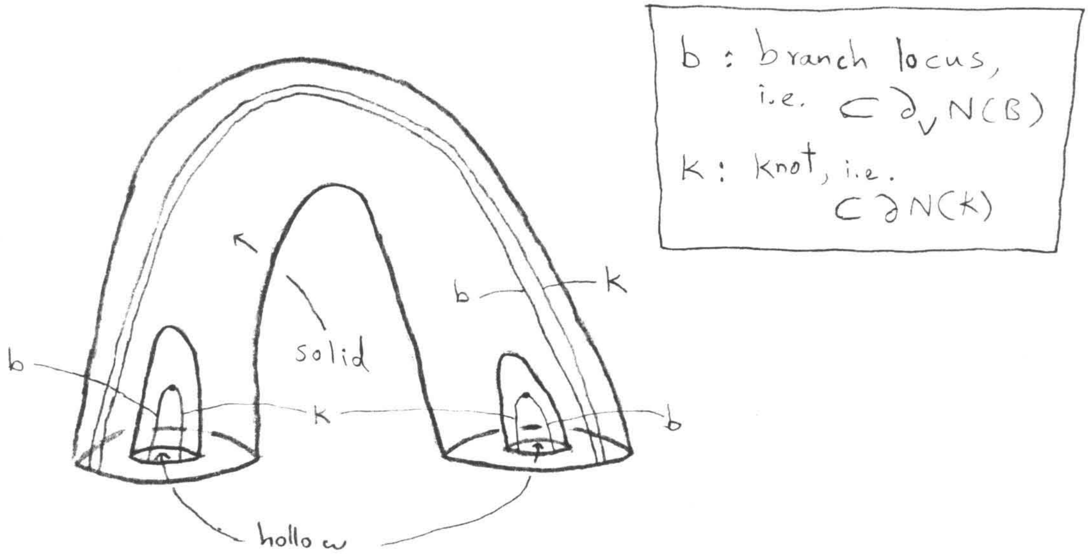

Figure 3.44: Neighborhood of branched surface between diagrams 1 and 4 of figure 3.40.

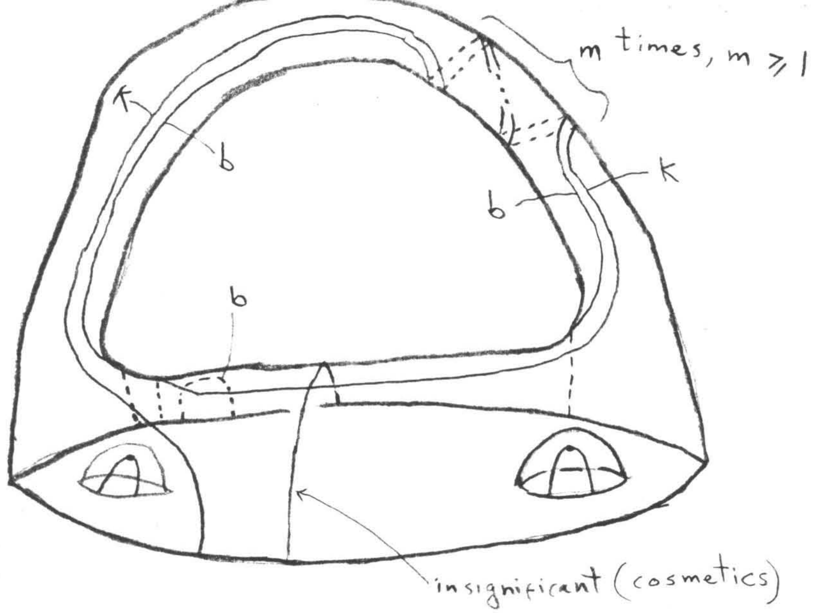

Figure 3.45: Neighborhood of branched surface between diagrams 1 and 6 of figure 3.40.

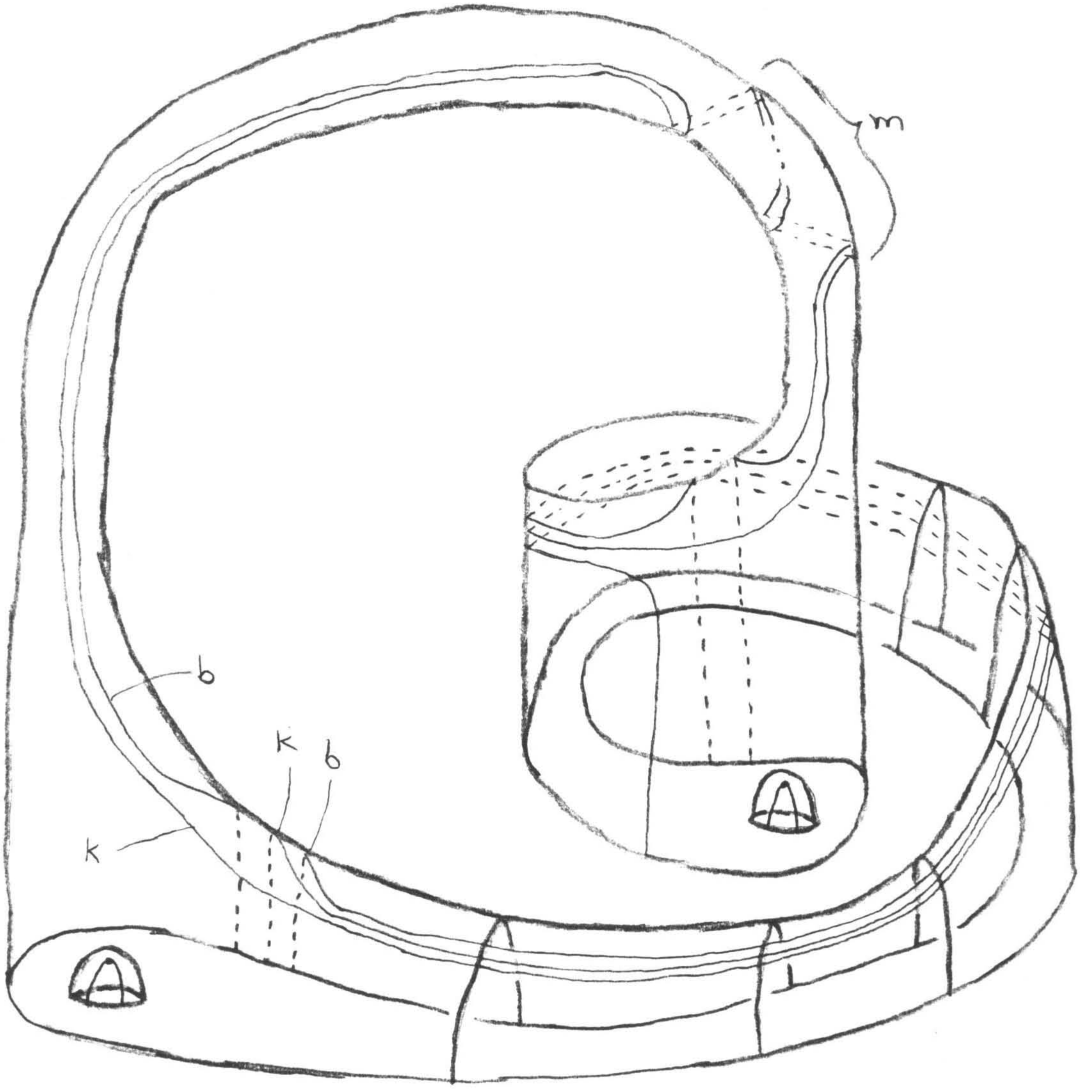

Figure 3.46: Neighborhood of branched surface between diagrams 1 and 7 of figure 3.40.

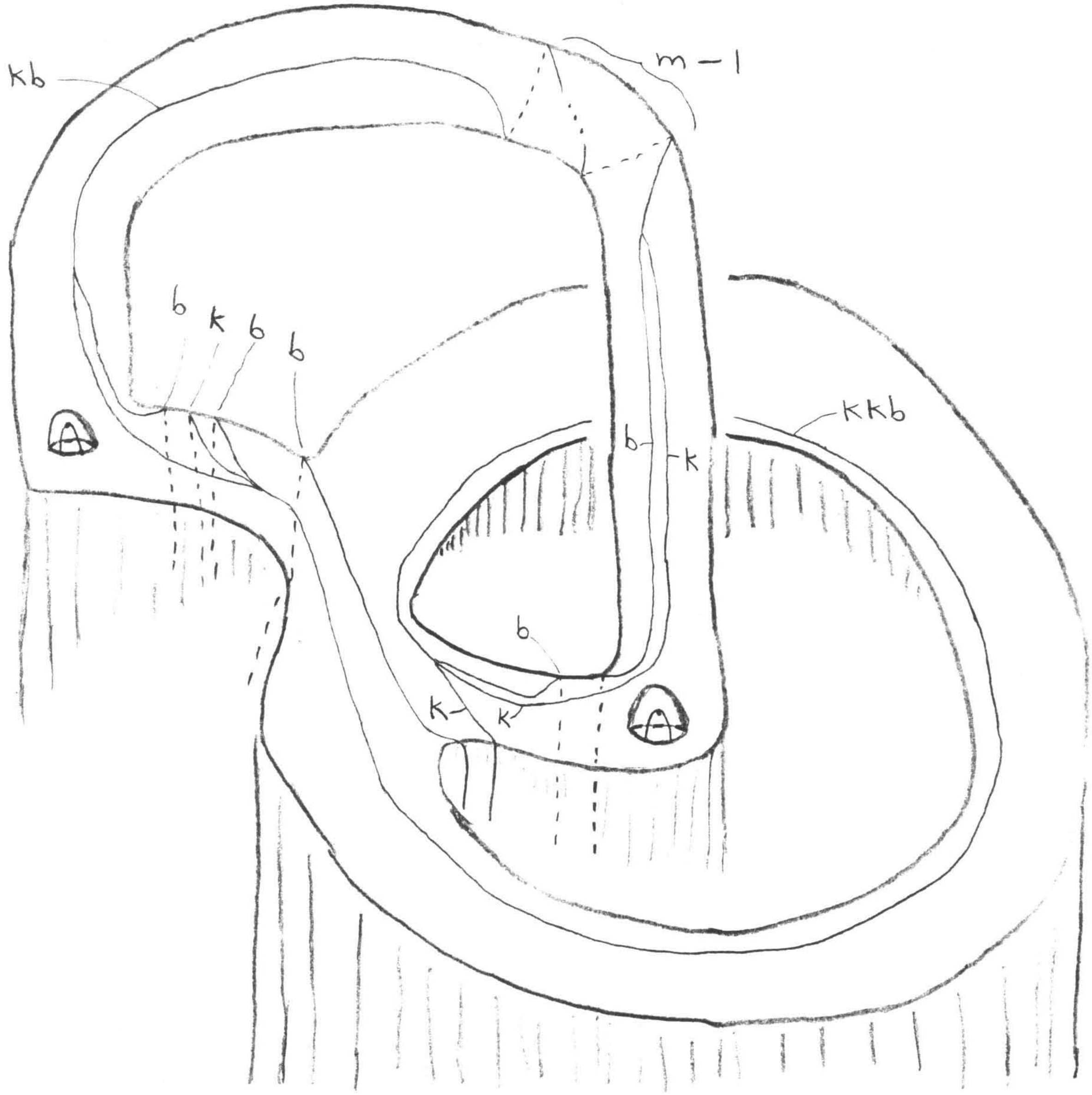

Figure 3.47: Neighborhood of branched surface between diagrams 1 and 8 of figure 3.40.

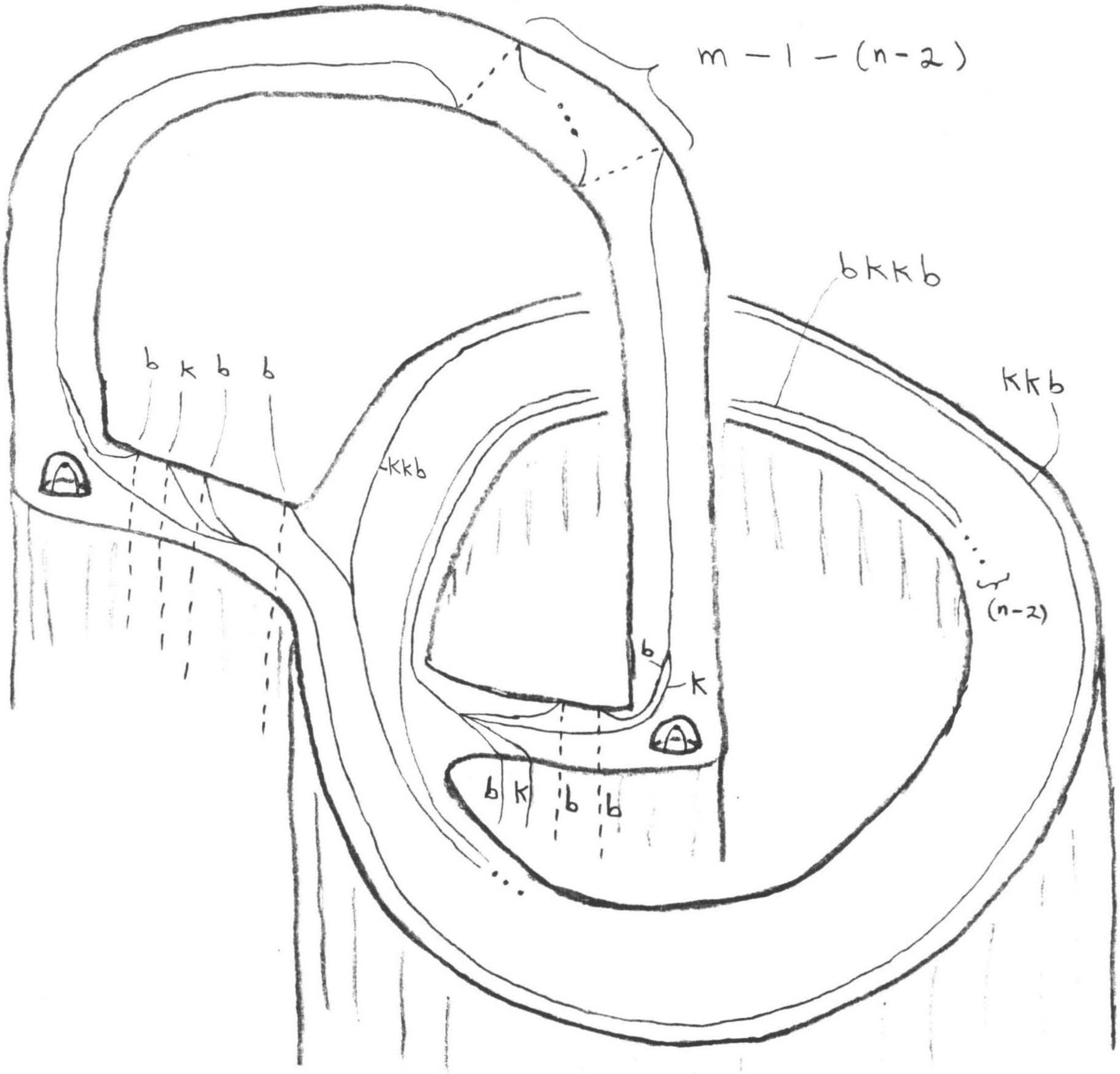

Figure 3.48: Neighborhood of branched surface between diagrams 1 and 9 of figure 3.40.

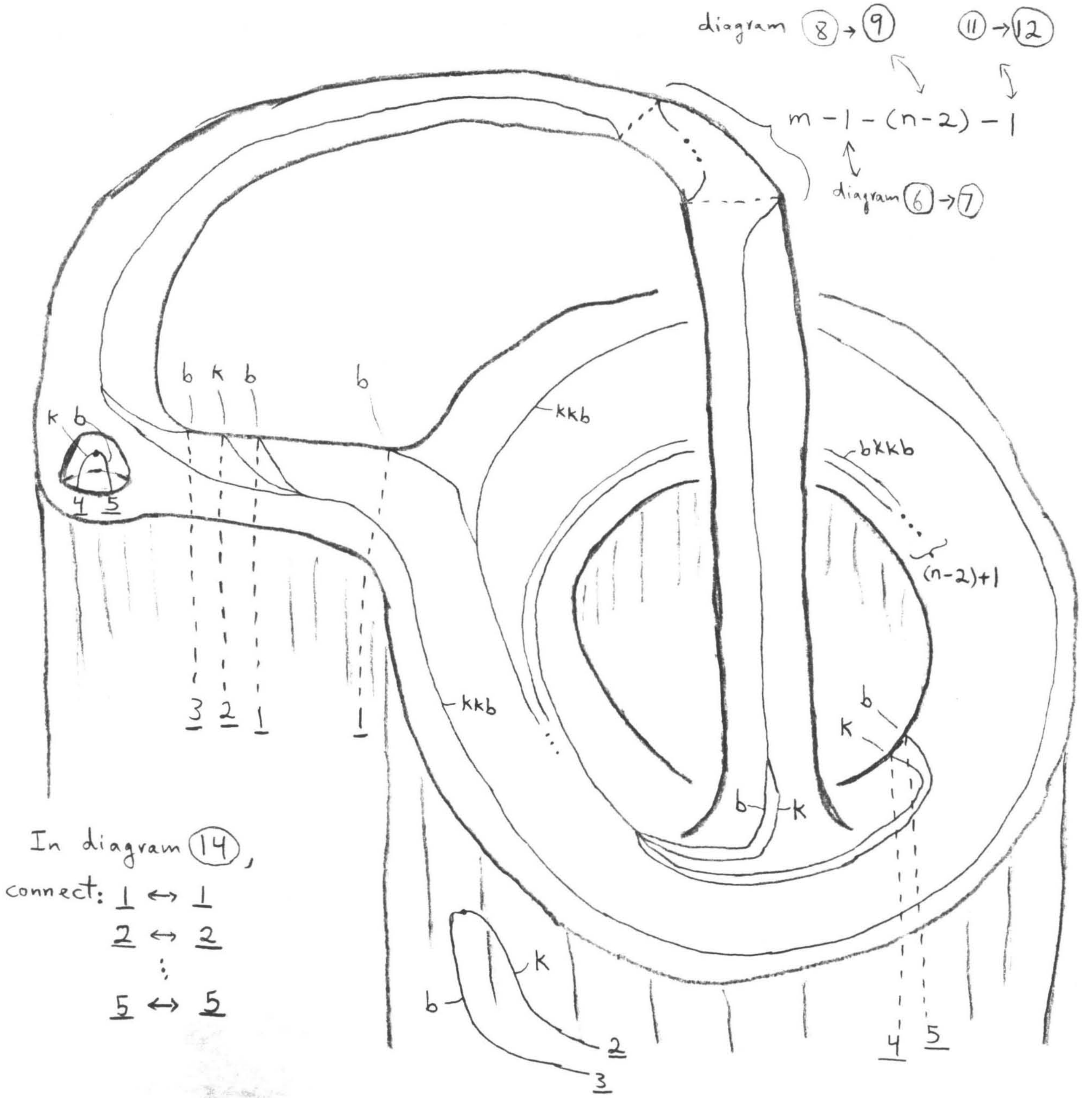

Figure 3.49: Neighborhood of branched surface between diagrams 1 and 12 of figure 3.40.

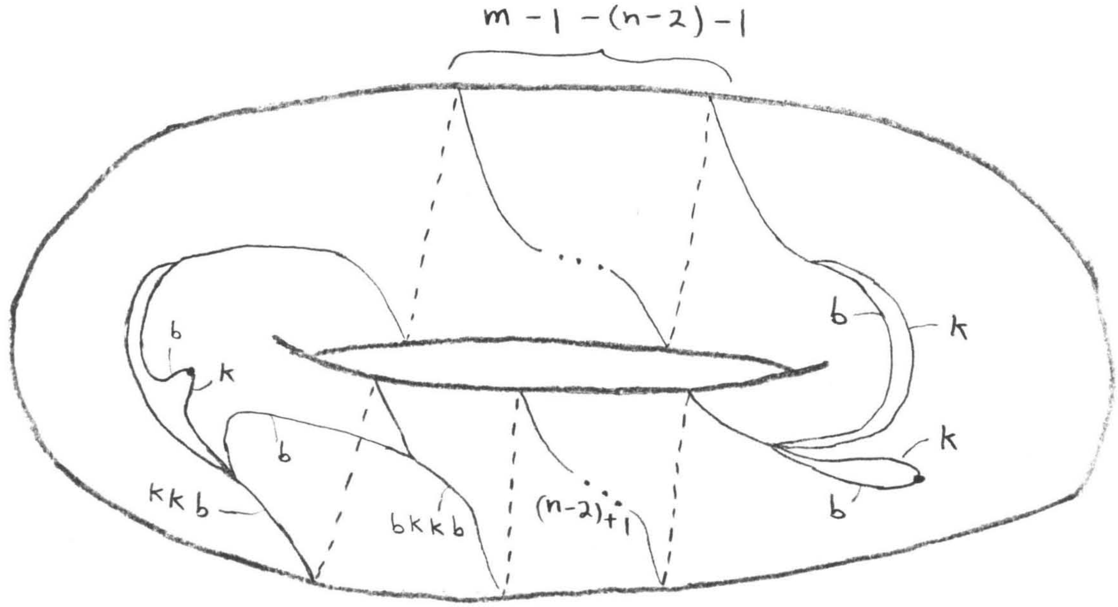

Figure 3.50: Neighborhood of branched surface of figure 3.40 which simplifies to figure 3.41.

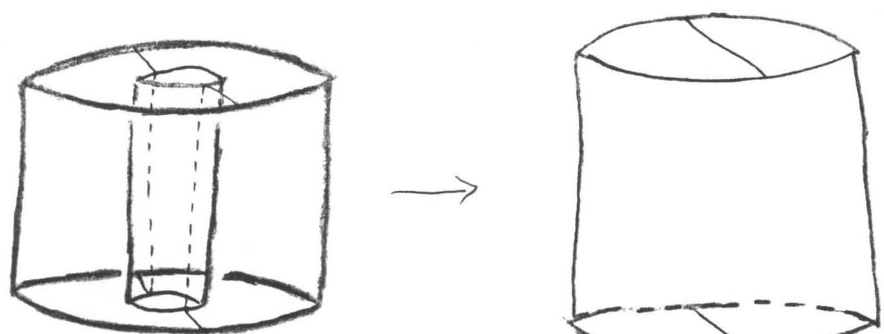

Figure 3.51: Disk Decomposition.

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