A PERTURBATION THEORY FOR UNSTEADY CAVITY FLOWS

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ABSTRACT

This investigation deals with a perturbation theory for unsteady cavity flows in which the time-dependent part of the flow may be considered as a small perturbation superimposed on an established steady cavity flow of an ideal fluid, the gravity effect being neglected in this study. In order to make a comparison between the various existing steady-cavity-flow models when applied to unsteady motions, some of these models have been employed to evaluate the small time behavior of, and the initial reaction to an unsteady disturbance. Furthermore, the mechanism by which the cavity volume may be changed with time is studied and the initial hydrodynamic force resulting from such change is calculated.

The second kind of unsteady cavity flow problems treated here is characterized by the fact that the disturbances are limited to be small for all time instants. Based on a systematic linearization with respect to the steady basic flow, a general perturbation theory for unsteady cavity flows is formulated. From this perturbation theory the generation of surface waves along the cavity boundary is revealed, much in the same way as the classical gravity waves in water, except with the centrifugal acceleration due to the curvature of the free-streamlines in the basic flow playing the role of an equivalent gravity effect.

TABLE OF CONTENTS

	Page No.
INTRODUCTION	1
PART I. SMALL-TIME BEHAVIOR OF UNSTEADY CAVITY FLOWS	6
1. General Formulation	6
2. Solution of the Perturbed Flow	12
3. Inclined Lamina in Kirchhoff Flow	15
4. Re-entrant Jet Model	22
5. Riabouchinsky's Model	27
6. A Wake Model for an Oblique Plate with a Finite Cavity	35
PART II. A PERTURBATION THEORY FOR UNSTEADY CAVITY FLOWS	41
7. General Theory	41
8. General Methods of Solution	51
9. Surface Waves on a Hollow Vortex	52
REFERENCES	56
APPENDIX	58
FIGURES	65

INTRODUCTION

The first solution of a problem of steady, irrotational, two-dimensional cavitating, or wake-forming flows of an ideal fluid was given by Helmholtz (1) and Kirchhoff (2) in the eighteen-sixties. These pioneering works have since stimulated much interest in the subject, leading to the development of general methods and several mathematical models for dealing with such problems. In spite of such a long history of steady cavitating flows and its applications to engineering problems, the subject of unsteady cavitating flows has received attention only in the past fourteen years. Some of the difficulties involved in the latter problem can be envisaged as follows. The theoretical treatment of irrotational, two-dimensional cavitating flows of an ideal fluid is usually based on certain proposed physical models, for example, the Kirchhoff-Helmholtz model. If the flow is steady, the exact solution of such a problem, within the assumption of the proposed model, is usually obtained by using the hodograph method, since in this case a surface of constant pressure is also one of constant speed. This property, however, no longer holds valid in the case when the flow is unsteady. Consequently, in order to investigate some of the characteristics of unsteady cavitating flows, different approaches and approximations have been adopted by various authors.

In 1949 von Kármán (3) treated an accelerated flow normal to a flat plate which is held fixed in an inertial frame such that with given acceleration of the flow, the flow will separate from the plate to form a closed cavity of constant shape behind the plate, and he obtained a solution which is valid only at a particular Froude number characterizing the acceleration. Later, Gilbarg (4) approximated the free surface of an unsteady cavity flow by a streamline, and obtained two types of solutions for a flow with finite cavity behind a symmetrical polygon (of which the flat plate is a special case). One type of the solutions contains a doubly covered cavity subregion in the flow plane, which, as pointed out by him, is physically unrealistic; the other type solution has a cusped cavity which, as pointed out by Woods (5), is also physically unrealistic. In order to remove the cusped cavity Woods (5) introduced a second fictitious body at the rear of the cavity, as in the Riabouchinsky model for steady cavitating flows (6); the problem was then solved with the same approximation as used by Gilbarg (4), that is, the free surface of the unsteady cavity flow is approximated by a streamline. Recently, Yih (7) treated several special types of unsteady cavity flows by generalizing von Kármán's approach; and he also gave the solution of a different problem which is concerned with an accelerating body, to which a finite cavity is attached, passing through a fluid which is otherwise at rest in an inertial frame.

Another method of dealing with the unsteady cavity flows is by regarding the unsteady motion as a perturbation of an established steady cavity flow. With this approach Ablow and Hayes (8) have developed a perturbation theory which was later employed by Fox and Morgan (9) to investigate stability problems of some free surface flows. In this category it may be mentioned that a somewhat different perturbation theory with further simplifying assumptions has been applied to treat several different problems by Woods (10), Parkin (11), Wu (12), Timman (13) and Geurst (14), (15).

It is of importance to note an essential difference between the unsteady flows with and without a free surface. In the determination of the velocity field of an unsteady flow without a free surface the time appears only as a parameter. Therefore, the kinematics of the unsteady motion will not be basically different from its corresponding steady flow. On the other hand, for an unsteady flow with a free surface, the flow will depend on its previous time history because the explicit role played by the time enters the problem through the boundary conditions on the free surface. Consequently, the unsteady motion may completely change the character of the flow, for example, by the creation of surface waves in the flow. However, when an established steady cavitating flow (basic flow) is given a sudden acceleration, there will be no past history of any time-varying disturbance at the moment of the application of this sudden change. Therefore, the problem of finding the flow characteristics at the initial instant of the unsteady cavity flow is expected to be not any more complicated than that of the general unsteady flow without a free surface. With this formulation Gurevich (16) treated the impact problem of the Kirchhoff-Helmholtz flow. Wu (17), adopting the wake model as the basic flow, evaluated the initial time behavior of a finite cavity flow.

The purpose of this investigation is to study the general features of unsteady, irrotational, two-dimensional cavity flows of an ideal fluid without including the gravity effect. As was mentioned in the beginning, the theoretical investigations of the cavity flows are based on proposed artificial models. It is known that in a positive range of the cavitation number up to moderate values of order unity the agreement between the existing models for steady flows may be considered very close (18). No comparison, however, has ever been made between these models when applied to unsteady flows. Part I of this investigation is mainly devoted to make a comparison of the small-time behavior of unsteady cavity flows when different existing models are used to describe the basic flow. Furthermore, Part I includes an evaluation of the initial effect of removing fluid at infinity on an existing steady cavity flow. For cavity flows of an incompressible liquid surrounding a vapor cavity, it is obvious that, when the cavity volume changes, the conservation of mass and conservation of volume of the entire flow become incompatible because of the difference in the density of these two phases. Consequently, any variation of the cavity volume must come from a source distribution in the flow with its net strength depending on the time rate-of change of the cavity volume. For simplicity we assume that the change in cavity volume is affected by a source or sink located at the point of infinity. A direct consequence of this source with a time-varying strength in a two-dimensional flow of infinite extent is that it generates a pressure field with is logarithmically singular at infinite distances. Such a flow requires an infinite amount of energy to be created. In reality, however, the flow is usually finite in extent and never two-dimensional in the large. What formulation, or which model, then, gives a good approximation to account for the change in the cavity volume? A part of this work in Part I is devoted to clarify this question.

In the previous works on the unsteady cavity flows the assumption has usually been introduced that the free surface of the cavity may be approximated by a streamline in order to avoid some mathematical difficulties, and it is hoped that such an approximation will give a satisfactory result, at least for slowly varying flows. Based on such an approximation the resulting flow has been interpreted (10) to contain the effect that an unsteady disturbance applied on the solid body will produce two vortex sheets leaving the separation points and propagating downstream on the free surface of the cavity with a velocity equal to that of the free stream of the basic flow. In Part II of the present work the formulation of a rigorous perturbation theory is presented, which is based on a systematic linearization and without assuming that the displaced free surface of the cavity be approximated by a streamline. From the general formulation it is seen that the unsteady motion of the solid body produces in general a free surface wave propagating along the cavity boundary, much the same as the gravity waves generated by a floating body in oscillation. The centrifugal force due to the curvature of the free surface in the basic flow now plays the role of an equivalent gravity in the classical water wave problem. In this sense, the unsteady cavity flows are similar in nature to the radiation of gravity waves over a flat water surface, only now in a much more complex form since the centrifugal acceleration varies along the cavity surface. Such a wave phenomenon cannot be found in the formulation obtained by adopting the streamline-approximation mentioned previously. Also, in Part II, a simple illustration of this theory will be carried out for the surface waves over a hollow vortex.

PART I. SMALL-TIME BEHAVIOR OF UNSTEADY CAVITY FLOWS

1. General Formulation

To fix ideas, we suppose that for the time $t < 0$, a steady two-dimensional cavity flow past a solid body has been established, the solution of which is assumed to be given, or can be determined with the aid of some cavity flow models. Suppose now the solid body to which the cavity is attached is given for $t > 0$ a sudden accelerated motion; the problem is to evaluate the small-time behavior of the resultant unsteady cavity flow.

In general, the motion of the rigid boundary may consist of a translation and a rotation. Let (x_o, y_o) be a point on the rigid surface $S_o(x_o, y_o) = 0$ in the basic steady flow, and let it be displaced in time t to the position (x, y) with translational velocity $(V_1(t), V_2(t))$ and angular velocity $\omega(t)$. In terms of the complex variable $z = x + iy$ and $V(t) = V_1 + i V_2$, the motion of z may be written

$$\frac{dz}{dt} = V(t) + i\,\omega(t) z . \tag{1}$$

We shall assumed that the acceleration $d^2 z/dt^2$ is continuous at $t = 0$ so that for small positive t, V and ω may be expanded in power series of t, starting with the linear term as

$$\frac{dz}{dt} = c_1 t + c_2 t^2 + i(\omega_1 t + \omega_2 t^2) z + O(t^3) \tag{2}$$

where $c_n = a_n + i b_n$, a_n, b_n and ω_n being real constants. It then follows that for small t,

$$z = z_0 + \frac{1}{2}(c_1 + i\omega_1 z_0)t^2 + \frac{1}{3}(c_2 + i\omega_2 z_0)t^3 + o(t^4). \tag{3}$$

The displaced surface will be denoted by $S(x, y, t) = 0$. In fact, we have

$$S(x_0(x,y,t),\ y_0(x,y,t)) = S(x,y,t), \tag{4}$$

regarding equation 1 as to provide the canonical transformation $x_o = x_o(x, y, t)$, $y_o = y_o(x, y, t)$.

From the nature of the body motion it also follows that the complex velocity potential of the flow

$$f(z,t) = \varphi(x,y,t) + i\psi(x,y,t), \tag{5}$$

will assume for small t the expansion

$$f(z,t) = f_0(z) + t f_1(z) + \frac{1}{2}t^2 f_2(z) + \text{-------}, \tag{6}$$

where $f_n(z) = \varphi_n(x, y) + i\psi_n(x, y)$, $n = 0, 1, 2,$, and $f_o(z)$ is the complex velocity potential of the basic flow. The function $\varphi_1(x, y)$ may be called the initial acceleration potential. While ψ_o gives the streamlines of the basic flow, the function ψ_n for $n > 1$, being complex conjugate of φ_n, are introduced so that $f_n(z)$ are analytic functions of z. The velocity components are, as usual,

$$u(x,y,t) = \frac{\partial\varphi}{\partial x}, \qquad v(x,y,t) = \frac{\partial\varphi}{\partial y}. \tag{7a}$$

We shall introduce the complex velocity $w = u - iv$, and $w_n = \partial\varphi_n/\partial x - i\,\partial\varphi_n/\partial y$. Then from equation 6

$$w(z,t) = w_0(z) + t w_1(z) + \frac{1}{2}t^2 w_2(z) + \text{-------}. \tag{7b}$$

Similarly the pressure $p(x, y, t)$ may be assumed to possess the expansion

$$p(x,y,t) = p_o(x,y) + p_1(x,y) + t\, p_2(x,y) + t^2 p_3(x,y) + \cdots\cdots \tag{8}$$

where p_o denotes the pressure field of the basic flow, p_1 the impulsive pressure due to the sudden acceleration. Then from the Bernoulli equation,

$$\frac{p}{\rho} + \frac{\partial \varphi}{\partial t} + \frac{1}{2}(\nabla\varphi)^2 = \frac{p_\infty}{\rho} + \frac{1}{2}U^2 \tag{9}$$

where p_∞ and U are the pressure and speed of the basic flow at infinity, we obtain, by equating the coefficients of same powers of t, the following relations:

$$\begin{aligned} &\frac{p_o}{\rho} + \frac{1}{2}(\nabla\varphi_o)^2 = \frac{p_\infty}{\rho} + \frac{1}{2}U^2, \\ &\frac{p_1}{\rho} = -\varphi_1, \\ &\frac{p_2}{\rho} = -\varphi_2 - (\nabla\varphi_o)\cdot(\nabla\varphi_1) \qquad \text{and so on.} \end{aligned} \tag{10}$$

The boundary conditions of the problem are as follows:

(i) At the solid surface, the normal component of the flow velocity relative to the moving boundary must vanish. An alternative way of stating this condition is that the fluid particles originally on $S(x, y, t)=0$, at small time interval apart, will remain on it. That is,

$$\frac{DS}{Dt} = \frac{\partial S}{\partial t} + \frac{\partial\varphi}{\partial x}\frac{\partial S}{\partial x} + \frac{\partial\varphi}{\partial y}\frac{\partial S}{\partial y} = 0 \qquad \text{on} \quad S(x,y,t) = 0,$$

which becomes, upon using equation 4

$$\frac{\partial S_o}{\partial x_o}\left[\frac{\partial\varphi}{\partial x}\frac{\partial x_o}{\partial x} + \frac{\partial\varphi}{\partial y}\frac{\partial x_o}{\partial y} + \frac{\partial x_o}{\partial t}\right] + \frac{\partial S_o}{\partial y_o}\left[\frac{\partial\varphi}{\partial x}\frac{\partial y_o}{\partial x} + \frac{\partial\varphi}{\partial y}\frac{\partial y_o}{\partial y} + \frac{\partial y_o}{\partial t}\right] = 0 \tag{11}$$

on $S(x, y, t) = 0$. Here the functions $x_o(x, y, t)$ and $y_o(x, y, t)$ can be written down immediately from equation 3 by interchanging z and z_o, and by changing the signs of a_n, b_n, and ω_n, as can readily be seen from equation 3. Equation 11 is in a form convenient for manipulation since $\partial S_o/\partial x_o$ and $\partial S_o/\partial y_o$ already correspond to the components of the normal to the initial surface $S_o(x_o, y_o) = 0$. Substituting equations 3 and 6 in equation 11, expanding the various quantities about (x_o, y_o) and $t = 0$, and equating the coefficients of different powers of t, we obtain the conditions that on $S_o(x_o, y_o) = 0$,

$$\frac{\partial \mathcal{Q}_o}{\partial n_o} = 0, \tag{12a}$$

$$\frac{\partial \mathcal{Q}_1}{\partial n_o} = n_{o_1}(a_1 - \omega_1 y_o) + n_{o_2}(b_1 + \omega_1 x_o) \tag{12b}$$

$$\begin{aligned}\frac{\partial \mathcal{Q}_2}{\partial n_o} = {} & 2n_{o_1}(a_2 - \omega_2 y_o) + 2n_{o_2}(b_2 + \omega_2 x_o) - \left(n_{o_1}\frac{\partial \mathcal{Q}_o}{\partial y_o} - n_{o_2}\frac{\partial \mathcal{Q}_o}{\partial x_o}\right) \\ & - (a_1 - \omega_1 y_o)\left(n_{o_1}\frac{\partial^2 \mathcal{Q}_o}{\partial x_o^2} + n_{o_2}\frac{\partial^2 \mathcal{Q}_o}{\partial x_o \partial y_o}\right) - (b_1 + \omega_1 x_o)\left(n_{o_1}\frac{\partial^2 \mathcal{Q}_o}{\partial x_o \partial y_o} + n_{o_2}\frac{\partial^2 \mathcal{Q}_o}{\partial y_o^2}\right),\end{aligned} \tag{12c}$$

where $\vec{n}_o = (n_{o_1}, n_{o_2})$ is the unit outward normal to the surface $S_o(x_o, y_o) = 0$.

(ii) There are two boundary conditions at the free surface of the cavity. Suppose the free surface may be expressed as

$$F(x, y, t) = y - h(x, t) = 0,$$

then the kinematic condition that the fluid particles on the free surface will remain on it requires

$$v = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} \qquad \text{on} \qquad y = h(x, t) \tag{13}$$

We assume that for small t, $h(x, t)$ may be expanded as

$$h(x,t) = h_0(x) + t h_1(x) + \frac{1}{2} t^2 h_2(x) + O(t^3), \tag{14}$$

where $y = h_o(x)$ denotes the cavity boundary of the basic flow on which

$$\frac{dh_0}{dx} = \frac{v_0(x, h_0)}{u_0(x, h_0)} \, . \tag{15}$$

Substituting equations 7 and 14 in equation 13, and expanding u_n and v_n on $y = h(x, t)$ about $y = h_o(x)$ and $t = 0$, we obtain

$$\begin{aligned} h_1 &= v_0(x, h_0) - u_0(x, h_0) \frac{dh_0}{dx} = 0, \\ h_2 &= v_1(x, h_0) - u_1(x, h_0) \frac{dh_0}{dx} \end{aligned} \tag{16}$$

Since $h_1 \equiv 0$, the free surface will not be displaced in the time of order t, as should be expected.

In the basic flow, the cavity boundary is a surface of constant pressure, and hence also one of constant velocity, say

$$p_0 = p_c \, , \qquad |\nabla Q_0| = q_c \, , \qquad \text{on} \qquad y = h_0(x), \tag{17}$$

so that by the Bernoulli equation,

$$q_c = U(1 + \sigma)^{1/2} \tag{18a}$$

where σ is the cavitation number, defined as

$$\sigma = \frac{(p_\infty - p_c)}{\frac{1}{2}\rho U^2} \, . \tag{18b}$$

We shall assume that the pressure in the cavity of the unsteady flow will be maintained at the same constant value p_c, that is

$$p(x, y, t) = p_c \qquad \text{on} \qquad y = h(x,t). \tag{19}$$

By expanding the left side of this dynamic condition about $y = h_o(x)$ and

$t = 0$, using equations 8, 14 and 16, the following conditions result

$$p_1 = p_2 = 0 \qquad \text{on} \qquad y = h_0(x). \tag{20}$$

Hence from equation 10,

$$\begin{aligned} \varphi_1 &= 0 \qquad \text{on} \qquad y = h_0(x) \\ \varphi_2 &= -(\nabla\varphi_0)\cdot(\nabla\varphi_1) \qquad \text{on} \qquad y = h_0(x). \end{aligned} \tag{21a}$$

Upon differentiating equation 21a along $y = h_o(x)$ and using equation 15, it is readily seen that $(\nabla\varphi_o)\cdot(\nabla\varphi_1) = 0$, and hence

$$\varphi_2 = 0 \qquad \text{on} \qquad y = h_0(x). \tag{21b}$$

From equations 21a and 21b it therefore follows that

$$(\nabla\varphi_0)\cdot(\nabla\varphi_1) = (\nabla\varphi_0)\cdot(\nabla\varphi_2) = 0 \qquad \text{on} \qquad y = h_0(x) \tag{22}$$

which asserts that the perturbation velocity is normal to the original cavity boundary up to the time of order t^2.

(iii) At the point of infinity we require the perturbation velocity to vanish,

$$|\nabla\varphi_n| \longrightarrow 0 \qquad \text{as} \qquad |z| \longrightarrow \infty, \qquad \text{for} \quad n = 1, 2, \cdots\cdots . \tag{23}$$

If, in addition to the sudden acceleration of the solid body and the assumption of the constancy of the cavity pressure, a certain amount of fluid is removed at infinity with a source strength $Q_1 t + \frac{1}{2} Q_2 t^2 +$ so that the cavity volume may be changed arbitrarily, then, aside from condition 23, we must impose additional conditions at infinity as

$$\operatorname{Im} \int_\Gamma w_n(z)\,dz = Q_n \quad , \qquad n = 1, 2, \cdots\cdots\cdots , \tag{24}$$

where Γ is a contour around the point of infinity. However, we impose no boundedness condition on p at infinity since, if such Q_n can be arbitrarily chosen, $p_n (n > 1)$ will be logarithmically singular at infinity. It will be seen later that, when the cavity is taken to be infinitely long (Helmholtz flow, corresponding to the cavitation number $\sigma = 0$), the solution exists only when $Q_n = 0$. Consequently the effect of change in cavity volume can be sought only in the case of finite cavities. However, the limit of the hydrodynamic forces in such case as the cavity becomes infinitely long, with Q_n held fixed, is seen to exist.

On the other hand, from Kelvin's theorem on the conservation of circulation, the circulation around the point of infinity cannot be changed in the unsteady motion for $t < \infty$. By combining this condition with equation 24, we may write

$$\int_{\Gamma} w_n(z)\,dz = i\,Q_n \quad , \qquad n = 1, 2, 3, \cdots\cdots . \tag{25}$$

This completes our formulation of the small-time perturbation theory.

2. Solution of the Perturbed Flow

For the moment we assume that the solution of the basic steady flow is given. We note that in the basic flow the entire boundary of the body-cavity system belongs to a streamline, the form of which is known. Therefore, by the general theory of conformal mapping, it is always possible to find an analytic function

$$\zeta(z) = \xi(x,y) + i\,\eta(x,y) \tag{26}$$

such that it maps the entire flow region in the z-plane into the upper half of the ζ-plane with the entire boundary of the body-cavity system mapped onto the entire ξ-axis. For simplicity, we shall make the part $|\xi| < 1$ of the real ζ-axis correspond to the wetted solid surface and the part $|\xi| > 1$ correspond to the cavity boundary.

After the transformation to the ζ-plane the boundary value problem formulated in the last section can be stated as a Hilbert problem, as will be shown below, the solution of which is readily attained. It is noted from the last section that the problems of different orders in the perturbation are expressed in a similar form. That is, the normal velocities $\partial\varphi_n/\partial n_o$ are given at the initial solid surface and the potentials φ_n are prescribed at the unperturbed cavity boundary. It is therefore sufficient to treat one as the typical case. To save writing we shall denote $f_n = \varphi_n + i\psi_n$ by $F = \Phi + i\Psi$.

It is convenient to introduce the analytic function

$$G(\zeta) = \frac{dF}{d\zeta} = \frac{dF}{dz}\cdot\frac{dz}{d\zeta} \tag{27}$$

which is defined for $\operatorname{Im}\zeta > 0$. Then on the solid surface, $\eta = 0+$, $|\xi| < 1$,

$$\operatorname{Im} G = -\frac{\partial\Phi}{\partial\eta} = -\frac{\partial\Phi}{\partial n_o}\left|\frac{dz}{d\zeta}\right| = g_1(\xi) \qquad \text{say,} \tag{28}$$

which is known, by equations 12 and 26. Furthermore, on the cavity boundary, $\eta = 0+$, $|\xi| > 1$,

$$\operatorname{Re} G = \frac{\partial\Phi}{\partial\xi} = g_2(\xi) \qquad \text{say,} \tag{29}$$

which is also known, by equation 21. In particular, $g_2 = 0$ for φ_1 and φ_2. Finally, equation 25 becomes

$$\int_\Gamma \frac{dF}{dz}\,dz = \int_{\Gamma'} G(\zeta)\,d\zeta = iQ \tag{30}$$

where Γ' is the image of Γ in the ζ-plane, the subscript n of Q is dropped to show a typical case.

It is possible to transform the above boundary value problems of a mixed type into a Hilbert problem by extending the unknown function $G(\zeta)$ to a sectionally analytic function, defined on the whole ζ-plane (excluding the real ζ-axis). Since φ_3 and higher terms will not be treated explicitly here, we shall demonstrate the method by taking g_2 of equation 29 to be zero; the general case of $g_2 \neq 0$ can be carried out similarly. First, the function $G(\zeta)$ may be continued into the lower half ζ-plane by

$$G(\bar{\zeta}) = -\overline{G(\zeta)} \tag{31}$$

For the case $g_2 = 0$, $G(\bar{\zeta})$ is the analytical continuation of $G(\zeta)$ through the interval $|\xi| > 1$. In the following, by $G_{\pm}(\xi)$ will be denoted the limiting values of $G(\zeta)$ as $\eta \longrightarrow \pm 0$. From equations 28, 29 and 31 it then follows that

$$\begin{aligned} G_+ + G_- &= 2i \,\mathrm{Im}\, G_+ = 2i g_1(\xi) && \text{for } |\xi| < 1, \\ G_+ - G_- &= 2 \,\mathrm{Re}\, G_+ = 0 && \text{for } |\xi| > 1. \end{aligned} \tag{32}$$

The above Hilbert problem is well-known (19), its general solution can be written

$$G(\zeta) = \frac{i}{\pi}\frac{1}{\sqrt{\zeta^2-1}}\int_{-1}^{1}\frac{\sqrt{1-\xi^2}\,g_1(\xi)}{\xi-\zeta}\,d\xi + \frac{i\,P(\zeta)}{\sqrt{\zeta^2-1}} \qquad \text{for } \eta > 0 \tag{33}$$

where the function $(\zeta^2 - 1)^{\frac{1}{2}}$ is defined on the entire ζ-plane with the branch so chosen that $(\zeta^2 - 1)^{\frac{1}{2}} \rightarrow \zeta$ as $|\zeta| \rightarrow \infty$, and $P(\zeta)$ is an arbitrary Laurent's series with real coefficients. The last term in equation 33 is the general solution to the homogeneous problem (with $g_1 = 0$ also). The real coefficients of $P(\zeta)$, and hence also $G(\zeta)$, can be determined uniquely, when equation 30 is satisfied and the conditions that the pressure is nowhere less than the cavity pressure and further is integrable over the rigid boundary are observed.

In the following the above perturbation theory will be carried out for several basic steady cavity flows.

3. Inclined Lamina in Kirchhoff Flow

As a simple example we consider the basic flow to be that past a flat plate held at angle α, with a cavity formation of infinite length. The solution of this problem is known (20), which we simply cited below for the subsequent use. The coordinate system in the z-plane and its conformal mapping planes are shown in figure 1. For simplicity the free stream velocity U and the plate length ℓ, are normalized to unity. The solution $w_o = w_o(z_o)$ can be written parametrically as

$$w_o = \frac{\sqrt{1-\zeta^2} - 1}{\zeta}, \tag{34a}$$

$$z_o = -K \int_1^{\zeta} \frac{(\sqrt{1-\zeta^2} + 1)}{(\zeta + \sec\alpha)^3} d\zeta \tag{34b}$$

$$= \frac{K}{2} \left\{ \frac{1 + \sqrt{1-\zeta^2}}{(\zeta + \sec\alpha)^2} - \frac{\cot\alpha}{\sin\alpha} \cdot \frac{\sqrt{1-\zeta^2}}{(\zeta + \sec\alpha)} + 2\cot^3\alpha \tan\left(\sqrt{\frac{1-\zeta}{1+\zeta}} \tan\frac{\alpha}{2}\right) - \frac{1}{(1+\sec\alpha)^2} \right\}$$

where

$$K = 2\tan^3\alpha \frac{\sin\alpha}{(4 + \pi\sin\alpha)} . \tag{34c}$$

The entire flow region is mapped into the upper half ζ-plane with the corresponding boundary as prescribed in the last section (i.e. on $\eta = 0$, $|\xi| < 1$ corresponds to the plate and $|\xi| > 1$ to the cavity surface). The coefficient of the normal force N_o of the basic flow is

$$C_{N_o} = \frac{N_o}{\frac{1}{2}\rho U^2 \ell} = \frac{2\pi \sin\alpha}{4 + \pi \sin\alpha} . \tag{35}$$

(a) The first order solution

The unit normal to the plate is now $\vec{n}_o = (0, -1)$. We suppose the rotation is referred to the leading edge of the plate. Then the boundary condition equations 12b, 21a and 23 become

$$\frac{\partial \mathcal{Q}_1}{\partial y_o} = (b_1 + \omega_1 x_o) \qquad \text{on} \qquad \eta = 0 , \quad |\xi| < 1 , \tag{36a}$$

$$\mathcal{Q}_1 = 0 \qquad \text{and hence} \qquad \frac{\partial \mathcal{Q}_1}{\partial \xi} = 0 \qquad \text{on } \eta = 0, \ |\xi| > 1 . \tag{36b}$$

$$|\nabla \mathcal{Q}_1| \longrightarrow 0 \qquad \text{as} \qquad \zeta \longrightarrow -\sec\alpha . \tag{36c}$$

It is noted that a_1, the first term of the x-component of acceleration, drops out from condition equation 36a, implying that the acceleration of the plate parallel to itself has no effect on the flow up to time of order t. Furthermore, equation 25 cannot be satisfied unless $Q_n = 0$, $n = 1, 2, \ldots$. This can be seen as follows. If $Q_n \neq 0$, then equation 25 implies that $w_n = Q_n / 2\pi z + o(|z|^{-1})$, and hence $\varphi_n \sim (Q_n / 2\pi) \log |z|$, as $|z| \to \infty$. It follows that p_n will be logarithmically singular at $z = \infty$, which contradicts the conditions of p_n, such as equation 20, on the undisturbed free boundary $y = h_o(x)$ which extends to infinity. This indicates that the Kirchhoff-Helmholtz model with an infinite cavity is not a realistic model for the consideration of change of cavity volume. The problem when $Q_n \neq 0$ will be considered later when other finite-cavity models are adopted.

By making use of equations 36a, 36b and 34b, equations 28 and 29 become

$$\mathrm{Im}\, G = \frac{\partial Q_1}{\partial y_0}\left|\frac{dz}{d\xi}\right| = K\,(b_1 + \omega_1 x_0)(\sqrt{1-\xi^2}+1)(\xi+\sec\alpha)^{-3}, \quad |\xi|<1,$$

$$\mathrm{Re}\, G = 0, \qquad |\xi|>1,$$

where $x_0(\xi)$ is given by equation 34b, and K is given by equation 34c. Finally, by equation 33, we obtain

$$\frac{df_1}{d\zeta} = \frac{K}{\pi\sqrt{1-\zeta^2}}\left\{\int_{-1}^{1}\frac{(b_1+\omega_1 x_0)(1-\xi^2+\sqrt{1-\xi^2})}{(\xi-\zeta)(\xi+\sec\alpha)^3}\,d\xi + \sum_{n=-\infty}^{\infty} C_n(\zeta+\sec\alpha)^n\right\} \tag{37}$$

where C_n are real coefficients. The Laurentz series in the last term is expanded about $\zeta = -\sec\alpha$ (or $z = \infty$) for the convenience of application of the boundary conditions at $z = \infty$. The first term in equation 37 behaves like ζ^{-2} as $|\zeta| \longrightarrow \infty$ and is regular everywhere except at $\zeta = \pm 1$. Now by using equation 34b,

$$w_1 = \frac{df_1}{d\zeta}\cdot\frac{d\zeta}{dz} = -\frac{(\zeta+\sec\alpha)^3}{K(\sqrt{1-\zeta^2}+1)}\cdot\frac{df_1}{d\zeta}. \tag{38}$$

From equations 37 and 38 we readily see that $C_n = 0$ except for $n = -1, -2$ in order that $w_1 \rightarrow 0$ as $z \rightarrow \infty$ (or $\zeta \rightarrow -\sec\alpha$) and w_1 is regular as $\zeta \longrightarrow \infty$. Furthermore, with $Q_n = 0$, as explained above, equation 30 requires that

$$z w_1 \rightarrow 0 \qquad \text{as} \qquad |z| \rightarrow \infty. \tag{39}$$

From equation 34b, we find that as $|z| \rightarrow \infty$,

$$z \sim \frac{K}{2}\cdot\frac{1+i\tan\alpha}{(\zeta+\sec\alpha)^2}.$$

Making use of this result, we readily deduce from equations 37 and 38 that equation 39 is satisfied if, and only if, $C_{-1} = C_{-2} = 0$ also. Therefore all the coefficients C_n vanish, thereby the first order solution given by equations 37 and 38 is uniquely determined.

It may be noted that w_1 has a singularity at the edges of the plate, physically corresponding to a narrow spray sheet. For, with $\zeta = \pm(1+\epsilon)$, $|\epsilon| << 1$, we deduce from equations 37 and 38 that as $\zeta \to \pm 1$,

$$w_1 \sim \frac{i}{\pi} \cdot \frac{(\pm 1 + \sec\alpha)^3}{(2\epsilon)^{1/2}} \int_{-1}^{1} \left(\frac{1+\xi}{1-\xi}\right)^{\pm\frac{1}{2}} \frac{(b_1 + \omega_1 x_0)(1+\sqrt{1-\xi^2})}{(\xi + \sec\alpha)^3} d\xi \left[1 + O(|\epsilon|^{1/2})\right].$$

From the behavior of w_1 on the free surface (ϵ positive small) it may be seen that the free surface starts to move into the cavity when the plate accelerates into the fluid (e.g., with $b_1 < 0$, $\omega_1 = 0$), and vice versa. Furthermore, it is noted that w_1 is of order $|z|^{-3/2}$ for large values of $|z|$. Consequently it follows that there will be no net change in the cavity volume.

Since the spray sheets do not produce any singular force (like the leading edge suction on a thin airfoil), the normal force acting on the plate can be obtained by integrating the pressure along the plate so that for small t, in view of equation 3,

$$N = \int_0^1 p(x_0, 0^-, t)\, dx_0 = \int_0^1 [p_0(x_0, 0) + p_1(x_0, 0) + t\, p_2(x_0, 0)]\, dx_0 + O(t^2)$$

$$= N_0 + N_1 + t\, N_2 + O(t^2). \tag{40a}$$

The first term N_0 is given by equation 35. Now, from equation 10

$$N_1 = \int_0^1 p_1(x_0,0)\,dx_0 = -\rho\int_0^1 Q_1(x_0,0)\,dx_0 = -\rho\int_{-1}^1 x_0(\xi)\,\frac{\partial x_1(\xi)}{\partial \xi}\,d\xi\,. \tag{40b}$$

By substituting the real part of equation 37 for $\partial\varphi_1/\partial\xi$ in the above integral, the normal force coefficient may be expressed as

$$C_{N_1} = \frac{N_1}{\frac{1}{2}\rho U^2 \ell} = -\left(\frac{\ell b_1}{U^2}\right)\Gamma_{b_1} - \left(\frac{\ell^2\omega_1}{U^2}\right)\Gamma_{\omega_1}\,, \tag{41a}$$

where

$$\Gamma_{b_1}(\alpha) = -\frac{2}{\pi}\int_{-1}^1 \frac{x_0(\xi)\,d\xi}{\sqrt{1-\xi^2}}\int_{-1}^1 \frac{\sqrt{1-s^2}}{s-\xi}\cdot\frac{\partial x_0(s)}{\partial s}\,ds\,, \tag{41b}$$

$$\Gamma_{\omega_1}(\alpha) = -\frac{1}{\pi}\int_{-1}^1 \frac{x_0(\xi)\,d\xi}{\sqrt{1-\xi^2}}\int_{-1}^1 \frac{\sqrt{1-s^2}}{s-\xi}\cdot\frac{\partial x_0^2(s)}{\partial s}\,ds\,, \tag{41c}$$

the above integrals being interpreted by their Cauchy principal values. In equation 41a, b_1 and ω_1 are expressed in the physical units, and the plate length ℓ and free stream velocity U are restored for completeness. The integrals in equations 41b and 41c cannot be expressed in terms of elementary functions of α; they are integrated numerically and the results are plotted versus α in figure 2. In particular, we have

$$\Gamma_{b_1}(\pi/2) = 0.8448\,, \tag{42}$$

which is the special case treated by Gurevich (16). The quantity C_{N_1} represents the jump in the normal force coefficient due to the acceleration.

(b) <u>The second order solution for $\alpha = \pi/2$, $\omega_1 = b_2 = \omega_2 = 0$.</u>

To facilitate investigation of the behavior of the second order solution, let us choose the special case: $\alpha = \pi/2$ (the flat plate being held normal to the stream) and ω_1, b_2, ω_2 all vanish. Then the boundary conditions (equations 12 c, 21a and 23) become

$$\frac{\partial \varphi_2}{\partial y_0} = -b_1 \frac{\partial^2 \varphi_0}{\partial y_0^2}, \qquad \text{on} \quad \eta = 0, \qquad |\xi| < 1, \tag{43a}$$

$$\varphi_2 = 0 \qquad \text{on} \quad \eta = 0, \qquad |\xi| > 1, \tag{43b}$$

$$|\nabla \varphi_2| \longrightarrow 0 \qquad \text{as} \quad |z| \longrightarrow \infty. \tag{43c}$$

Since the component a_1 of the acceleration parallel to the plate does not appear in the above conditions, the flow will not be affected by it up to the second order terms. Now, in the limit as $\alpha \longrightarrow \pi/2$, the zeroth order solution becomes

$$z_0 = \frac{1}{(4+\pi)}\left[2(1-\zeta) - \zeta\sqrt{1-\zeta^2} + \cos^{-1}\zeta\right] \tag{44}$$

and w_0 is still given by equation 34a. Consequently, from equations 28 and 29

$$Im\, G = b_1 \frac{\partial u_0}{\partial x}\left|\frac{\partial x}{\partial \xi}\right| = \frac{b_1}{\xi^2}\cdot\left[\frac{1}{\sqrt{1-\xi^2}} - 1\right], \qquad \eta = 0, \quad |\xi| < 1,$$

$$Re\, G = 0, \qquad \eta = 0, \quad |\xi| > 1.$$

Hence, by equation 33,

$$\frac{df_2}{d\zeta} = \frac{b_1}{\pi\sqrt{1-\zeta^2}} \int_{-1}^{1} \frac{1-\sqrt{1-\xi^2}}{(\xi-\zeta)\,\xi^2}\, d\xi \tag{45}$$

in which the complementary solution is zero, as can be shown by the same argument given previously for φ_1. Carrying out the integration, we then obtain

$$\frac{df_2}{d\zeta} = \frac{b_1}{\pi\zeta^2}\cdot\frac{\left[2\zeta + \log\frac{\zeta-1}{\zeta+1} - i\pi\sqrt{1-\zeta^2}\right]}{\sqrt{1-\zeta^2}}, \tag{46a}$$

$$w_2 = \frac{df_2}{dz} = -\frac{i(4+\pi)}{4\zeta^2}\left[1-\sqrt{1-\zeta^2}\right]\frac{df_2}{d\zeta}. \tag{46b}$$

It is readily verified that w_2 also behaves like $z^{-3/2}$ for large $|z|$.

The second order normal force, by equations 40a and 10, can be obtained from

$$N_2 = \int_0^1 p_2(x_o, o)\,dx_o = \rho\int_{-1}^1 [\phi_2 + u_o u_1]_{\eta=0}\frac{\partial x_o}{\partial \xi}d\xi$$

$$= \rho\int_{-1}^1 \left[u_o(\xi)u_1(\xi)\frac{\partial x_o}{\partial \xi} - x_o(\xi)\frac{\partial \phi_2}{\partial \xi}\right]d\xi .$$

After substituting the various terms and evaluating the resulting integral, we find the simple result:

$$N_2 = 0. \tag{47}$$

For this special case we therefore have the normal force coefficient

$$C_N(t) = C_{N_o} + C_{N_1} + C_{N_2}t + O[(Ut/\ell)^2]$$

$$= 0.8798 + 0.8448\,(\ell b_1/U^2) + O[(Ut/\ell)^2]. \tag{48}$$

Thus for constant acceleration, C_N has, aside from the stepwise change, the following behavior

$$\left(\frac{dC_N}{dt}\right)_{t=0^+} = 0 .$$

However, N_2 may not vanish when b_2, ω_1, and ω_2 are different from zero.

The quantity $m_1 = N_1/b_1$ may be called the initial induced mass of this cavity flow, then by equation 48

$$m_1 = \frac{\rho U^2 \ell C_{N_1}}{2b_1} = 0.4224\,\rho\ell^2. \tag{49}$$

If the flat plate had undergone an acceleration b_1 normal to the flow without wake formation (a postulated Dirichlet flow), then the induced mass would be

$$m_0 = \pi\rho(\ell/2)^2. \tag{50a}$$

Thus

$$\frac{m_1}{m_0} = 0.5377\,. \tag{50b}$$

This ratio is less than unity, as should be expected, since the cavitated side of the plate, being exposed to constant pressure, has no capacity of imparting kinetic energy to the exterior flow.

4. Re-entrant Jet Model

We proceed to consider the effect of the cavitation number σ (defined by equation 18) on the unsteady flow when the cavity of the basic flow is finite in size. An additional degree of freedom achieved in this type of flow problem is that the cavity volume can now be changed arbitrarily by prescribing a flow source at infinity. In order that basic steady flow is tractable to analysis, various cavity-flow models have been introduced, such as the Riabouchinsky model (6), the re-entrant jet model (21), the wake model proposed independently by Joukowsky (22), Roshko (23) and Eppler (24), and the modified wake model introduced by Wu (25). In each of these flow models an artifice of some sort is introduced to admit σ as a free parameter, and to replace the actual wake flow of a real fluid by a simplified model within the framework of an equivalent potential flow. It has been found that in a positive range of σ

up to moderate values of order unity, the agreement between these models may be considered very close (see Wu (18)). Furthermore, the validity of these models in predicting the hydrodynamic forces acting on the body has been supported by experimental observations. The purpose of the following several sections is to make a comparison between the resultant unsteady flows when different models are used for the basic cavity flow, for such a task should be of fundamental value in the study of unsteady cavity flows.

For simplicity the basic flow is taken to be that past a flat plate held normal to the stream of unperturbed velocity U and pressure p_∞, forming a finite cavity with a prescribed cavity pressure $p_c (< p_\infty)$. According to the re-entrant jet model, the free streamlines eventually reverse their direction at the rear part of the cavity, forming a re-entrant jet which disappears on another "Riemann Sheet" (see figure 3). Due to the assumed symmetry of the flow, it suffices to consider only the flow in the left half physical z-plane. To save writing, both the half plate length, $\ell/2$, and the constant speed q_c (see equation 18) on the cavity surface will be normalized to unity.

It is convenient to introduce the variable

$$\Omega = \log \frac{dz}{df_0} = \log \frac{1}{w_0} = \log \frac{1}{q_0} + i\theta \tag{51}$$

where f_o is the complex potential of the basic flow, q_o is the flow speed, and θ the flow direction to the x-axis. The part of the flow under consideration in the z, f_o, and Ω-planes is shown in figure 3. By applying the Schwarz-Cristoffel transformation, the flow region can be mapped conformally into the upper half ζ-plane, with the point of infinity I, front stagnation E, plate edge A, jet infinity B and the rear stagnation C corresponding to $\zeta = \infty$, 0, 1, b, and c respectively. The required transformation is given by

$$\frac{df_o}{d\zeta} = A\,\frac{\zeta - c}{\zeta - b}\,. \tag{52}$$

$$\frac{d\Omega}{d\zeta} = \frac{B}{\zeta(\zeta - c)\left[(\zeta-1)(\zeta-b)\right]^{1/2}} \tag{53}$$

where A, B are two coefficients. From the local behavior of Ω at $\zeta = 0$ and c, we find the relations

$$B = -\frac{1}{2}cb^{1/2}\,, \qquad b = c(c-1)(c-3/4)^{-1}. \tag{54}$$

Integrating equation 53, and by making use of equation 54, we obtain

$$\frac{dz}{df_o} = e^{\Omega} = \frac{\sqrt{\dfrac{(c-b)(\zeta-1)}{(c-1)(\zeta-b)}} + 1}{\sqrt{\dfrac{(c-b)(\zeta-1)}{(c-1)(\zeta-b)}} - 1} \cdot \left[\frac{\sqrt{\dfrac{\zeta-b}{b(\zeta-1)}} + 1}{\sqrt{\dfrac{\zeta-b}{b(\zeta-1)}} - 1}\right]^{1/2}. \tag{55}$$

As $|\zeta| \longrightarrow \infty$, $df_o/dz \rightarrow U = (1+\sigma)^{\frac{1}{2}}$. From this condition and equation 55 it results

$$\frac{2+\sigma}{\sigma} = \frac{1}{2}\cdot\frac{c^{3/2}(c-1)^{1/2}}{(c-3/4)^{3/2}}\,. \tag{56}$$

Finally, by integrating equations 55 and 52 to obtain $z = z(\zeta)$, the coefficient A is determined by the unit plate length, giving

$$A = \frac{(b-1)^{3/2}}{[K(1) - K(0)]} \tag{57a}$$

where

$$K(\xi) = (2c-b)\sqrt{b(1-\xi)\xi} + (2c-1)\sqrt{(b-\xi)\xi} + (c - \tfrac{1}{2}b)\sin^{-1}\frac{2\xi - b}{b}$$
$$+ \sqrt{b}\left[b(b-\tfrac{3}{2}) + c(2-b)\right]\sin^{-1}(2\xi - 1) + (b-1)^{3/2}(c-b)\sin^{-1}\frac{(2b-1)\xi - b}{b-\xi}. \tag{57b}$$

This completes the zeroth order solution for prescribed σ. In particular, on the half plate, $y = 0$, $-1 < x < 0$ (or $\eta = 0+$, $0 < \xi < 1$), we have

$$X(\xi) = \frac{-[K(\xi) - K(0)]}{[K(1) - K(0)]} , \tag{58a}$$

$$\frac{dx}{d\xi} = A\left(\frac{c-\xi}{b-\xi}\right)\cdot\frac{\left[\frac{(c-b)(1-\xi)}{(c-1)(b-\xi)}\right]^{1/2}+1}{\left[\frac{(c-b)(1-\xi)}{(c-1)(b-\xi)}\right]^{1/2}-1}\cdot\left\{\frac{\left[\frac{b-\xi}{b(1-\xi)}\right]^{1/2}+1}{\left[\frac{b-\xi}{b(1-\xi)}\right]^{1/2}-1}\right\}^{1/2}. \tag{58b}$$

This expression is needed for the first order solution.

When a sudden acceleration of magnitude b_1 is applied normal to the plate, the boundary conditions corresponding to equation 32 can be written

$$G_+ + G_- = 2ib_1\frac{dx}{d\xi} \qquad \text{for} \qquad 0 < \xi < 1, \tag{59a}$$

$$G_+ - G_- = 0 \qquad \text{for} \qquad 1 < \xi < b, \tag{59b}$$

$$G_+ + G_- = 0 \qquad \text{for} \qquad -\infty < \xi < 0 \text{ and } b < \xi < \infty. \tag{59c}$$

The last condition (equation 59c) expresses the assumption that the perturbed flow preserves the basic flow symmetry about the y-axis, that is, $v_1 = 0$ on $\eta = 0$, for $\xi < 0$ and $\xi > b$. The above conditions are expressed in forms different from equation 32. The solution, however, can be written down directly by applying the same principle. It is obvious that $H(\zeta) = [(\zeta-1)(\zeta-b)]^{\frac{1}{2}}$, defined with branch cuts from $\zeta = -\infty$ to 1 and from b to ∞ so that $H \to \zeta$ as $|\zeta| \to \infty$, $0 < \arg\zeta < \pi$, is a homogeneous solution of the present problem. Therefore the general solution can be written

$$G(\zeta) = \frac{-b_1}{\pi H(\zeta)}\int_0^1 \frac{[(1-\xi)(b-\xi)]^{1/2}}{\xi-\zeta}\cdot\frac{dx}{d\xi}d\xi + \frac{1}{H(\zeta)}\left[\sum_{-\infty}^{\infty} c_n\zeta^n\right] \tag{60}$$

where $dx/d\xi$ is given by equation 58b and c_n are arbitrary real constants. To determine c_n, we note first that the pressure must be finite at $\zeta = 0$, hence $c_n = 0$ for $n < 0$. Furthermore, we note that the first term in equation 60 is of order of ζ^{-2} as $|\zeta| \to \infty$, this implies that the behavior of df_1/dz at large distances is determined by the last term in equation 60. By applying equation 30 and using the symmetry property of the flow, we obtain $c_0 = Q_1/2\pi$, $c_1 = c_2 = \ldots = 0$, where Q_1 is the source strength defined in equation 24.

The integral in equation 60 can be integrated explicitly. For the determination of hydrodynamic forces, however, only the real part of G on the plate (where $\eta = 0$, $0 < \xi < 1$) is needed. The final result is

$$\begin{aligned} \operatorname{Re} G &= -\frac{b_1}{\pi} \cdot \frac{[(1-\xi)(b-\xi)]^{-1/2}}{[K(1) - K(0)]} \left\{ A_1 + B_1(\xi - 1) + \sqrt{b}\, \frac{(\xi-1)[(2c-b-2)(\xi-1) - c(b-1)]}{[\xi(b-\xi)]^{1/2}} \cdot \right. \\ &\qquad \left. \log \frac{(b-1)\xi + (b-\xi) + 2\sqrt{(b-1)\xi(b-\xi)}}{b(1-\xi)} \right\} - \frac{Q_1 [(1-\xi)(b-\xi)]^{-1/2}}{2\pi}, \\ A_1 &= \pi\left[b(c-1) + \tfrac{1}{2}\right] + \frac{\sqrt{b}}{2}(b^2 - 2b + 2c)\left(\frac{\pi}{2} + \sin^{-1}\frac{2-b}{b}\right) - \sqrt{b(b-1)}\,(b-2c), \\ B_1 &= \sqrt{b}\,(b-2c)\left(\frac{\pi}{2} + \sin^{-1}\frac{2-b}{b}\right) - \pi(2c-1). \end{aligned} \tag{61}$$

By using equation 40b and the symmetry property of the flow, the first order normal force N_1 is given by

$$N_1 = 2\rho \int_0^1 x(\xi)\{\operatorname{Re} G\}\, d\xi, \tag{62a}$$

where $x(\xi)$ and $R\ell\ G$ are given by equations 58a and 61. The result can be expressed in terms of nondimensional parameters as

$$C_{N_1} = \frac{N_1}{\frac{1}{2}\rho U^2 \ell} = -\left(\frac{\ell b_1}{U^2}\right)\Gamma_{b_1} - \left(\frac{Q_1}{U^2}\right)\Gamma_{Q_1}, \tag{62b}$$

in which the coefficients Γ_{b_1} and Γ_{Q_1} are functions of the cavitation number σ. Analytic evaluation of the above integral is too tedious to be practical. The numerical computation of these coefficients has been carried out with an IBM 7090, the final result is plotted versus σ in figures 4 and 5 to show the effect of (1) the finite cavity size and (2) the change in the cavity volume. From this result several salient features of significance may be pointed out.

First, it is noted that as $\sigma \to 0$, the value of Γ_{b_1} tends to 0.8448, which is the limit of Kirchhoff-Helmholtz case (see equation 42). For small to moderately large values of σ, Γ_{b_1} increases very slowly with increasing σ compared with the increase of the zeroth order drag which increases approximately with the factor $(1 + \sigma)$ (see, e.g., Wu (18)).

Another point of interest is that $\Gamma_{Q_1} \to 0$ rather rapidly as $\sigma \to 0$, this limit being independent of Q_1 so long as Q_1 is finite. This result shows that the effect on the drag force by removing fluid at infinity is insignificant when the cavity is sufficiently long. Furthermore, it shows that the limit of the solution as $\sigma \to 0$ is non-uniform with respect to Q_1 since the solution in the Kirchhoff case does not exist unless $Q_1 = 0$.

5. Riabouchinsky's Model

The essential feature of the Riabouchinsky model is the introduction of an appropriate image body downstream of the real body so that the free boundaries of the cavity are connected by this pair of solid boundaries. Let us apply this model to consider the cavity flow past a flat plate set normal to the stream, the flow in the physical z-plane is shown in figure 6. Again, as in the problem stated in the previous section, the unperturbed velocity and pressure are U and p_∞ respectively, the cavity pressure is p_c, corresponding to the cavitation number σ. Also, the half plate length, $\ell/2$, and the constant speed q_c on the cavity will be normalized to unity. Furthermore, due to the assumed symmetry, only the left half z-plane of the flow needs to be considered.

For the present case it is convenient to denote by ζ the complex velocity potential. We further introduce an auxiliary complex variable τ defined by

$$\tau = \frac{1}{2}(w_o + w_o^{-1}) \tag{63}$$

where $w_o = d\zeta/dz$ is the hodograph plane of the basic flow. The flow field under consideration in the z-plane and conformal mapping planes w_o, ζ, τ, are shown in figure 6. By the assumed symmetry of the flow, we may choose the potential at the front and rear stagnation points to be $\zeta = -n$ and n respectively, and $\zeta_B = -m$, $\zeta_D = m$. We further define k' and k by

$$k' = \frac{1}{1+\sigma}, \qquad k = (1-k'^2)^{1/2} = (2\sigma+\sigma^2)^{1/2}/(1+\sigma). \tag{64}$$

Then, at $z = \infty$, we have

$$w_I = -i\sqrt{k'}, \qquad \tau_I = \frac{i}{2}(k'^{-1/2} - k'^{1/2}). \tag{65}$$

The upper half τ-plane is mapped into the upper half ζ-plane by the Schwarz-Christoffel transformation

$$\zeta = 2n\sqrt{k'}\,\frac{\tau}{\sqrt{4k'\tau^2 + (1-k')^2}}\,. \tag{66}$$

From the local conformal behavior of $\zeta(\tau)$ at the point D, i.e. $\zeta_D = m$, $\tau_D = 1$, we find the relationship

$$m = \frac{2\sqrt{k'}\, n}{(1+k')} . \tag{67}$$

Let us introduce another auxiliary variable $\omega = \mu + i\nu$ defined by

$$w_o = \frac{d\zeta}{dz} = -i\,\frac{\sqrt{k'}}{\mathrm{dn}\,\omega} , \tag{68}$$

where $\mathrm{dn}\,\omega$ is the Jacobian elliptic function, delta amplitude of ω, with modulus k. In the following analysis the conventional notations for the elliptic functions and integrals will be used without specification and the modulus k will always be omitted to save writing. By substituting equation 68 into equation 66 it gives

$$\zeta = \frac{ni}{k} \cdot \frac{\mathrm{dn}^2\omega - k'}{\mathrm{sn}\,\omega\,\mathrm{cn}\,\omega} . \tag{69}$$

From equations 68 and 69 we deduce that on both the front and the image half plate $-1 < x < 0$,

$$x = \frac{n}{\sqrt{k'}}\left[k'\mu - E(\mu) + k'(1-k')\frac{\mathrm{sn}\,\mu\,\mathrm{cn}\,\mu}{\mathrm{dn}\,\mu}\right], \qquad 0 < \mu < \frac{K}{2} . \tag{70}$$

Evaluating the above result at $\mu = K/2$, which corresponds to $x = -1$, we obtain

$$n = \frac{2\sqrt{k'}}{E - k'K + \frac{(1-k')^2}{1+k'}} , \tag{71}$$

which completes the necessary calculation for the basic flow.

Due to the presence of the image plate, an additional assumption is needed for this model in the study of its small time behavior. It is given that at $t = 0$ a sudden acceleration b_1 is applied on the front plate directed in the positive y-direction. For small $t > 0$ the image plate may be assumed to move in the y-direction with speed

$$v = -\beta b_1 t + O(t^2), \tag{72}$$

where β is an unknown constant. To determine this unknown constant β, we shall assume that on the image plate the jump in the drag due to the suddenly applied acceleration is zero. The physical significance of this assumption may be explained as follows. It has been pointed out by Wu (18) that the image plate in the basic flow may be regarded as a means to represent the energy dissipation in the wake flow of a real fluid. In fact, in a frame of reference at rest with respect to the fluid at infinite, the work done by the moving image plate is negative and numerically equal to the work done by the real plate since the total force on the pair of plates vanishes. This negative work done by the image plate therefore corresponds to the mechanical energy removed from the system in unit time as there is no other means of dissipating energy in potential flow. Now in the unsteady motion, it is conceivable that the rate of dissipation in the wake (of the actual flow) cannot be affected at small time. This implies that the initial change of momentum at the image plate must vanish. It is this physical meaning that underlies the above assumption. For small $t > 0$, the boundary condition on the part of the ξ-axis $(\eta = 0^+)$ corresponding to the image plate can be written by using equation 72 as

$$\operatorname{Im} G = \beta b_1 \operatorname{Re}\left(\frac{dz}{d\zeta}\right),$$

where $G = df_1/d\zeta$. Or, by applying the continuation (equation 31), we have on the part of the ξ-axis corresponding to the image plate the condition

$$G_+ + G_- = 2i\beta b_1 \operatorname{Re}\left(\frac{dz}{d\zeta}\right). \tag{73}$$

Referring to equations 32 and 73 and using equation 68, the boundary conditions of this problem are

$$G_+ + G_- = -2ib_1\left(\frac{dz}{d\xi}\right) = 2i\frac{b_1}{\sqrt{k'}}\frac{cn\mu(\xi)}{sn\mu(\xi)} \qquad \text{for} \qquad -n<\xi<-m,$$

$$G_+ - G_- = 0 \qquad \text{for} \qquad -m<\xi<m,$$

$$G_+ + G_- = 2i\beta b_1 Re\left(\frac{dz}{d\xi}\right) = 2i\frac{\beta b_1}{\sqrt{k'}}\frac{cn\mu(\xi)}{sn\mu(\xi)} \qquad \text{for} \qquad m<\xi<n,$$

$$G_+ + G_- = 0 \qquad \text{for} \quad -\infty<\xi<-n, \quad n<\xi<\infty$$

where m and n are given by equations 67 and 71.

Consider the function $h(\zeta) = \sqrt{\zeta^2 - m^2}$ with the branch cuts from $-\infty$ to $-m$ and from m to ∞, and $h(\zeta) \rightarrow \zeta$ as $|\zeta| \rightarrow \infty$, $0 < \arg\zeta < K$. It is obviously a homogeneous solution of the above Hilbert problem and satisfies equation 31. There, by using equation 33 the solution may be written

$$G(\zeta) = -\frac{b_1 k'^{-1/2}}{\pi\sqrt{\zeta^2-m^2}}\left\{\int_{-n}^{-m}\frac{\sqrt{\xi^2-m^2}\,\frac{cn\mu(\xi)}{sn\mu(\xi)}}{\xi-\zeta}d\xi - \beta\int_{m}^{n}\frac{\sqrt{\xi^2-m^2}\,\frac{cn\mu(\xi)}{sn\mu(\xi)}}{\xi-\zeta}d\xi\right\} + \frac{P(\zeta)}{\sqrt{\zeta^2-m^2}} \tag{74}$$

where $P(\zeta) = \sum_{-\infty}^{\infty} c_n \zeta^n$, and c_n are arbitrary real constants. Again, since the pressure at $\zeta = 0$ should be finite and the first term in equation 74 is of order of ζ^{-2} as $|\zeta| \rightarrow \infty$, we have, by applying condition (30)and the symmetric property of the flow, $P(\zeta) = Q_1/2\pi$. By using equation 69, the variable ξ in equation 74 can be transformed into the variable μ, giving

$$G(\zeta) = \frac{b_1(1-k')n}{\pi\sqrt{k'}\sqrt{\zeta^2-m^2}}\left\{\int_0^{K/2}\frac{cn^2\mu\, dn^2\mu\left(1-\frac{k'}{dn^2\mu}\right)^2}{dn^2\mu+\frac{1+k'}{n}\zeta\, dn\mu+k'}\,d\mu\right.$$

$$\left.+\beta\int_0^{K/2}\frac{cn^2\mu\, dn^2\mu\left(1-\frac{k'}{dn^2\mu}\right)^2}{dn^2\mu-\frac{1+k'}{n}\zeta\, dn\mu+k'}\,d\mu\right\}+\frac{Q_1}{2\pi\sqrt{\zeta^2-m^2}}. \tag{75}$$

After integrating part of the above expression and some simplification, we obtain

$$G(\zeta) = \frac{b_1}{\pi\sqrt{\zeta^2-m^2}}\left\{(1+\beta)A_2-(1-\beta)B_2\zeta+\frac{(1-k')}{\sqrt{k'}}\left[\frac{(1+k')^2}{n}\zeta^2-4nk'\right]I(\zeta)\right\}$$

$$+\frac{Q_1}{2\pi\sqrt{\zeta^2-m^2}}, \tag{76a}$$

where

$$A_2 = \frac{1-k'}{1+k'}\cdot\frac{E+k'K+(1+k')}{E-k'K+\frac{(1-k')^2}{(1+k')}} \tag{76b}$$

$$B_2 = \frac{1+k'}{2\sqrt{k'}}\left[\frac{\pi}{2}\cdot\frac{1-k'}{1+k'}+\sin^{-1}\frac{1-k'}{1+k'}\right] \tag{76c}$$

and

$$I(\zeta) = \int_0^{K/2} cn^2\mu\left[\frac{1}{dn^2\mu+\frac{1+k'}{n}\zeta\, dn\mu+k'}+\frac{\beta}{dn^2\mu-\frac{1+k'}{n}\zeta\, dn\mu+k'}\right]d\mu \tag{76d}$$

From the above results and equation 69, we deduce that

$$\operatorname{Re} G = \mp \frac{b_1}{\pi}\Big\{ (1+\beta)\Big[A_2 \frac{1+k'}{n}\Big(\frac{dn\mu}{dn^2\mu - k'}\Big) + \frac{K}{2\sqrt{k'}}\Big(\frac{dn^2\mu - k'}{dn\mu}\Big)$$

$$- \frac{1}{\sqrt{k'}} \cdot \frac{dn\mu\, cn^2\mu}{sn^2\mu} \text{ P.V. } \Pi\Big(\sin^{-1}\frac{1}{\sqrt{1+k'}}, \frac{1}{sn^2\mu}\Big) - k'^{5/2}\frac{sn^2\mu}{dn\mu\, cn^2\mu} \cdot$$

$$\Pi\Big(\sin^{-1}\frac{1}{\sqrt{1+k'}}, \frac{dn^2\mu}{cn^2\mu}\Big)\Big] \pm (1-\beta)\Big[B_2 \frac{dn^2\mu + k'}{dn^2\mu - k'} + \frac{1}{2} \cdot \frac{dn^2\mu - k'}{(1-k')sn\mu\, cn\mu} \cdot$$

$$\log\frac{cn\mu + \sqrt{k'}\, sn\mu}{cn\mu - \sqrt{k'}\, sn\mu}\Big]\Big\} \mp \frac{Q_1}{2\pi} \cdot \frac{1+k'}{n}\Big(\frac{dn\mu}{dn^2\mu - k'}\Big) \qquad 0 < \mu < \frac{K}{2}, \tag{77}$$

where the upper sign is used for the front plate, the lower sign for the image plate. In particular, when $\sigma = 0$, the above expression reduces to

$$\operatorname{Re} G = \frac{-b_1}{\pi \cos 2\theta}\Big[(\pi+2) + 2\frac{\cos^2 2\theta}{\sin 2\theta}\log\frac{1+\tan\theta}{1-\tan\theta}\Big] \quad \text{for the front plate}$$

$$= \frac{\beta b_1}{\pi \cos 2\theta}\Big[(\pi+2) + 2\frac{\cos^2 2\theta}{\sin 2\theta}\log\frac{1+\tan\theta}{1-\tan\theta}\Big] \quad \text{for the image plate,}$$

where $0 < \theta = \sin^{-1} sn\mu < \frac{\pi}{4}$. By applying equations 40b, 70, 77 and 66, we obtain the normal force N_1 on the front plate as

$$N_1 = 2\rho\big\{ b_1[(1+\beta)I_1 + (1-\beta)I_2] + Q_1 I_3 \big\} \tag{78}$$

and the normal force N_1 (in the y-direction) on the image plate as

$$D_y = 2\rho\big\{ b_1[(1+\beta)I_1 - (1-\beta)I_2] + Q_1 I_3 \big\} \tag{79}$$

where

$$I_1 = \frac{(1-k')n}{\pi\sqrt{k'}}\int_0^{K/2} x(\mu)\frac{sn\mu\, cn\mu}{dn\mu}\Big\{ \frac{(1+k')\sqrt{k'}}{n}A_2 + \frac{K}{2}\cdot\frac{(dn^2\mu - k')^2}{dn^2\mu}$$

$$+ (dn^2\mu - k')\Big[\frac{cn^2\mu}{sn^2\mu}\text{ P.V. }\Pi\Big(\sin^{-1}\frac{1}{\sqrt{1+k'}}, \frac{1}{sn^2\mu}\Big)$$

$$- k'^3\frac{sn^2\mu}{dn^2\mu\, cn^2\mu}\Pi\Big(\sin^{-1}\frac{1}{\sqrt{1+k'}}, \frac{dn^2\mu}{cn^2\mu}\Big)\Big]\Big\} d\mu, \tag{80a}$$

$$I_2 = \frac{(1-k')n}{\pi\sqrt{k'}}\int_0^{K/2} x(\mu)\Big[\sqrt{k'}\, B_2\Big(\frac{dn^2\mu + k'}{dn^2\mu}\Big) sn\mu\, cn\mu + \frac{1}{2}\cdot\frac{(dn^2\mu - k')^2}{(1-k')dn^2\mu}\cdot \tag{80b}$$

$$\log\frac{cn\mu + \sqrt{k'}\, sn\mu}{cn\mu - \sqrt{k'}\, sn\mu}\Big] d\mu,$$

$$I_3 = \frac{1-k'^2}{2\pi}\int_0^{K/2} x(\mu)\,\frac{sn\mu\, cn\mu}{dn\mu}\, d\mu \tag{80c}$$

and $x(\mu)$ is given by equation 70.

Now we apply the condition that on the image plate the change in the drag due to the applied acceleration is zero, it results

$$2\rho b_1\left[(1+\beta)I_1 - (1-\beta)I_2\right] = 0,$$

or

$$\beta = \frac{I_2 - I_1}{I_1 + I_2}. \tag{81}$$

Substituting equation 81 into equation 78, we obtain the drag on the front plate as

$$N_1 = 4\rho\left[b_1\frac{2I_1I_2}{I_1+I_2} - \frac{Q_1}{2}I_3\right]. \tag{82}$$

This result can also be expressed in the following non-dimensional form:

$$C_{N_1} = \frac{N_1}{\frac{1}{2}\rho U^2 \ell} = -\left(\frac{\ell b_1}{U^2}\right)\Gamma_{b_1} - \left(\frac{Q_1}{U^2}\right)\Gamma_{Q_1} \tag{83a}$$

where

$$\Gamma_{b_1} = -4\frac{I_1 I_2}{I_1+I_2}, \qquad \Gamma_{Q_1} = I_3. \tag{83b}$$

The integrals I_1, I_2 and I_3 are computed numerically with an IBM 7090, and the final results of Γ_{b_1} and Γ_{Q_1} are shown in figures 4 and 5 as a comparison with the results of other flow models. It is further noted that as $\sigma \longrightarrow 0$, Γ_{b_1} reduces to equation 42 of the Kirchhoff case and $\Gamma_{Q_1} \longrightarrow 0$.

6. A Wake Model for an Oblique Plate with a Finite Cavity

Thus far we have considered the accelerating motion of an inclined plate in Kirchhoff flow, and of the finite-cavity flow past a plate broadwise to the stream. Application of either the re-entrant jet model or the Riabouchinsky model to an oblique plate with a finite cavity formation leads to very complicated analyses. The task is considerably simplified, however, if we adopt a modified wake model recently proposed by Wu (25) to describe the basic steady flow. The purpose of this section is to determine the effect of acceleration on the hydrodynamic forces with the cavitation number σ and the angle of attack α as two free parameters.

The basic flow is taken to be a uniform stream of infinite extent impinging on a flat plate at an incidence angle α, to which a finite cavity is attached. According to this modified wake model, the incoming stagnation streamline branches off the plate at the leading edge A and the trailing edge B, forming two free streamlines ACI and BC'I which are assumed to become asymptotically parallel to the main flow at downstream infinity (see figure 7). The pressure on the parts AC and BC' of the free streamlines is assumed to take the constant cavity pressure p_c, and the space within the closed curve ACC'BA is regarded to represent approximately the cavity. The space in between the free streamlines CI and C'I represents a crude model of the dissipating wake, along its boundary the uniform stream conditions are eventually restored at downstream infinity. The flow outside this infinite wake strip is assumed to be irrotational. The locations of the points C and C' are determined with two assumptions; the first is that both the velocity potential and the flow inclination at C and C' are equal, and the second assumption is the so-called "hodograph-slit condition" that the free streamlines CI and C'I form a slit of undetermined shape in the hodograph plane. With these two additional assumptions the whole flow field outside the wake is then completely determined. For the convenience of subsequent application, the solution of the basic flow is reproduced briefly in the following. The plate length ℓ and the constant speed q_c on AC and BC' are again normalized to unity.

The flow in the physical z-plane, the complex potential f_o-plane, and the hodograph w_o-plane are shown in figure 7. The subscript of w_o will be omitted for simplicity. We further introduce the parametric ζ-plane defined by

$$\zeta = \frac{1}{2}(w + w^{-1}) \tag{83a}$$

or

$$w = \zeta - (\zeta^2 - 1)^{1/2}, \tag{83b}$$

in which the function $(\zeta^2 - 1)^{\frac{1}{2}}$ is defined with a branch cut made between the points $\zeta = -1$ and 1 so that $(\zeta^2 - 1)^{\frac{1}{2}} \longrightarrow \zeta$ as $|\zeta| \longrightarrow \infty$. At the point of infinity, the complex velocity takes the value

$$w = W = U e^{-i\alpha}, \qquad U = (1 + \sigma)^{-1/2}. \tag{84}$$

The corresponding value of ζ is

$$\zeta_\infty = \frac{1}{2}(U^{-1} e^{i\alpha} + U e^{-i\alpha}). \tag{85}$$

Since $\mathrm{Im}\, f_o = 0$ on the entire real ζ-axis, the complex potential $f(\zeta)$ can be continued analytically into the lower-half ζ-plane by

$$f_0(\bar{\zeta}) = \overline{f_0(\zeta)}. \tag{86}$$

Now from the asymptotic behavior of the streamlines ψ_o = const. near the point $\zeta = \zeta_\infty$, it is evident that f_o must have there a simple pole. Furthermore, from the local conformal behavior of f_o at $f_o = 0$, it is obvious that $f_o = O(\zeta^{-2})$ as $|\zeta| \longrightarrow \infty$. Therefore the solution must be of the form

$$f_0 = \frac{A}{4} \cdot \frac{1}{(\zeta - \zeta_\infty)(\zeta - \bar{\zeta}_\infty)} \tag{87a}$$

where A is a real constant. Or, expressing in terms of w by equation 83,

$$f_0 = \frac{A w^2}{(w - W)(w - \overline{W})(w - W^{-1})(w - \overline{W}^{-1})}. \tag{87b}$$

The z-plane is determined by integration of $dz/df_o = 1/w$, giving

$$z + a = \frac{f_0(w)}{w} + iB\Big\{ \Big(\frac{1}{\overline{W}} - \overline{W}\Big)\Big[\frac{1}{W}\log(w - W) - W\log\Big(w - \frac{1}{W}\Big)\Big]$$

$$- \Big(\frac{1}{W} - W\Big)\Big[\frac{1}{\overline{W}}\log(w - \overline{W}) - \overline{W}\log\Big(w - \frac{1}{\overline{W}}\Big)\Big]\Big\}, \tag{88a}$$

where the constant B is related to A by

$$\frac{A}{B} = 2(U^{-1} - U)\sin\alpha\big[(U^{-1} + U)^2 - (2\cos\alpha)^2\big]. \tag{88b}$$

Finally, the constant A is determined by the plate length as

$$A = \big[(U^{-1} + U)^2 - (2\cos\alpha)^2\big]\big/K, \tag{89a}$$

$$K = 2\,\frac{(U^{-1}+U)^2 + (2\cos\alpha)^2}{(U^{-1}+U)^2 - (2\cos\alpha)^2} + \frac{\pi(U^{-1}+U)}{2\sin\alpha} + \frac{(U^{-1}+U)^2 - (2\cos\alpha)^2}{(U^{-1}-U)\sin\alpha}\tan^{-1}\Big(\frac{U^{-1}-U}{2\sin\alpha}\Big). \tag{89b}$$

For the unsteady motion we shall confine ourselves to the simple case of constant acceleration so that for any point z of the plate

$$\frac{dz}{dt} = (a_1 + i b_1) t . \tag{90}$$

Then the boundary conditions of this problem (see equation 32) become

$$\begin{aligned} G_+ + G_- &= -2 i b_1 \operatorname{Re} \frac{dz}{d\zeta} && \text{for} \quad |\xi| > 1, \\ G_+ - G_- &= 0 && \text{for} \quad |\xi| < 1, \end{aligned} \tag{91}$$

where $G = df_1/d\zeta$. In the first condition, $dz/d\zeta = w^{-1}\, df_0/d\zeta$ can be deduced from equations 83 and 87.

Similar to the general solution of the problem with the boundary conditions given in equation 32, the general solution of the above boundary value problem, stated in equation 91, can be written

$$G(\zeta) = -\frac{b_1}{\pi(\zeta^2-1)^{1/2}} \left\{ \int_{-\infty}^{-1} + \int_{1}^{\infty} \right\} \frac{(\xi^2-1)^{1/2}\left(\frac{dx}{d\xi}\right)}{\xi - \zeta} d\xi + \frac{P(\zeta)}{(\zeta^2-1)^{1/2}} \tag{92}$$

where the function $(\zeta^2 - 1)^{\frac{1}{2}}$ is defined in the entire ζ-plane with branch cuts from $-\infty$ to -1 and from 1 to ∞ along the ξ-axis so that $(\zeta^2 - 1)^{\frac{1}{2}} \to \zeta$ as $|\zeta| \to \infty$, for $0 < \arg \zeta < \pi$. The arbitrary function $P(\zeta)$ is real on the real ζ-axis, and hence can be expanded into the Laurent's series

$$P(\zeta) = \sum_{n=-\infty}^{\infty} \left[c_n (\zeta - \zeta_\infty)^n + \bar{c}_n (\zeta - \bar{\zeta}_\infty)^n \right].$$

Since $z \sim (\zeta - \zeta_\infty)^{-1}$ as $|z| \to \infty$, it is necessary to have $c_n = 0$ for $n > -2$ in order that the perturbation velocity w_1 vanishes at infinity. Furthermore, we must impose $c_n = 0$ for $n \geqslant 0$ if we require the pressure at $|\zeta| = \infty$ to be finite. Finally, application of equation 30 yields

$$P(\zeta) = C_{-1}(\zeta - \zeta_\infty)^{-1} + \overline{C}_{-1}(\zeta - \overline{\zeta}_\infty)^{-1}, \tag{93a}$$

$$C_{-1} = -\frac{Q_1}{2\pi}(\zeta_\infty^2 - 1)^{1/2} = -\frac{Q_1}{2\pi}(W^{-1} - W), \tag{93b}$$

where use has been made of equations 84 and 85.

Transforming the variable ζ in the above solution into w by equation 83, we obtain

$$G = \frac{2b_1 w^2}{\pi(1-w^2)}\int_{-1}^{1} g(w,u)\,du - \frac{Q_1 w^2}{\pi(1-w^2)}\left[\frac{1-W^2}{(w-W)(wW-1)} + \frac{1-\overline{W}^2}{(w-\overline{W})(w\overline{W}-1)}\right] \tag{94a}$$

where

$$g(w,u) = \frac{1}{u}\frac{df_0}{du}\cdot\frac{1-u^2}{(u-w)(uw-1)}\,. \tag{94b}$$

In particular, on the rigid plate, $-1 < w < 1$, $R\ell\, G = \partial\varphi_1/\partial\xi$, and

$$\frac{\partial\varphi_1}{\partial\xi} = \frac{2w^2}{\pi(1-w^2)}\left\{b_1 \ \text{P.V.}\int_{-1}^{1} g(w,u)\,du - \frac{Q_1}{2}\left[\frac{1-W^2}{(w-W)(wW-1)} + \frac{1-\overline{W}^2}{(w-\overline{W})(wW-1)}\right]\right\} \tag{95}$$

where (P. V.) denotes the Cauchy principal value of the integral.

The normal force N_1 acting on the plate due to the acceleration is given by

$$N_1 = \rho\int_{-1}^{1} x(w)\,\frac{\partial\varphi_1}{\partial\xi}\,\frac{\partial\xi}{\partial w}\,dw \tag{96}$$

where $\partial\varphi_1/\partial\xi$ is given by equation 95, and $\partial\xi/\partial w$ can be obtained from equation 83. As in the previous cases, we write

$$C_{N_1} = -\left(\frac{\ell b_1}{U^2}\right)\Gamma_{b_1} - \left(\frac{Q_1}{U^2}\right)\Gamma_{Q_1}\,, \tag{97}$$

then, from equation 96,

$$\Gamma_{b_1} = \frac{2}{\pi} (P.V.) \int_{-1}^{1} \chi(w) dw \int_{-1}^{1} g(w, u) du, \tag{98}$$

$$\Gamma_{Q_1} = -\frac{1}{\pi} \int_{-1}^{1} \chi(w) \left[\frac{1 - W^2}{(w - W)(wW - 1)} + \frac{1 - \overline{W}^2}{(w - \overline{W})(w\overline{W} - 1)} \right] dw. \tag{99}$$

These integrals have been computed numerically with an IBM 7090, the final result of Γ_{b_1} and Γ_{Q_1} is shown in figures 8 and 9 for $\alpha = 75^\circ$. The result of the special case $\alpha = \pi/2$ is also compared with the other flow models in figure 4. For the special case $\alpha = \pi/2$, we deduce from equation 98 the expression which can be readily integrated after the expansion for small σ is made; the final result is

$$\Gamma_{b_1} = 0.8448 \left[1 + 0.067\, \sigma^2 (1 - \sigma) + O(\sigma^4) \right] \tag{100}$$

which is the special case already noted by Wu (17).

PART II. A PERTURBATION THEORY FOR UNSTEADY CAVITY FLOWS

7. General Theory

It is assumed that for the time $t < 0$ a steady, irrotational, two-dimensional cavity flow past a solid body has been established, the solution of which is assumed to be given. Suppose now for $t > 0$ the solid body to which the cavity is attached is given a time dependent small disturbance in such a way that the resulting flow remains irrotational, and the small disturbance is characterized by a small parameter ϵ. We shall establish a perturbation theory, of the first order in ϵ, for the resulting unsteady cavity flow; the perturbation is made with respect to the basic steady cavity flow.

Generally speaking, the physical space in a cavitating flow is occupied by a solid body, an air-filled cavity and the flow field under consideration. Consequently, the velocity potential of the basic flow, denoted by $\varphi_o(x, y)$ (where x, y are the Cartesian coordinates in the physical plane), will be a function defined only at points within the region of the basic steady flow. For the purpose of later application, we shall assume that the function $\varphi_o(x, y)$ can be extended analytically into the region of the solid body and the air-filled cavity in the basic flow. That such analytic continuation is possible has been demonstrated, for example, by Shiffman (26).

Under the assumptions mentioned above it follows that for $t > 0$ there exists in the flow field a time-dependent velocity potential, $\varphi(x, y, t)$, which may be expanded as

$$\varphi(x,y,t) = \varphi_o(x,y) + \epsilon\, \varphi_1(x,y,t) + O(\epsilon^2), \tag{101a}$$

where $\varphi_o(x, y)$ is the "extended velocity potential" of the steady basic flow, $\varphi_1(x, y, t)$ is a function of x, y and t but independent of ϵ. The necessity of constructing conceptually the analytically extended velocity potential is quite clearly seen from equation 101a in which the function $\varphi(x, y, t)$ is defined for all points, possibly with the exception of a finite number of isolated singular points, of the unsteady flow region whose boundary is a function of t and therefore, in the course of time it may cross into the region of the solid body and the air-filled cavity of the original steady basic flow. If $\varphi_o(x, y)$ were not extended, equation 1 could only be held valid over the part of the unsteady flow field which is in common with the original steady basic flow region and the problem of defining $\varphi(x, y, t)$ would become considerably complicated.

Since both $\varphi(x, y, t)$ and $\varphi_o(x, y)$ are harmonic functions of x, y, and since ϵ is an arbitrary small parameter, it follows that $\varphi_1(x, y, t)$ must also be a harmonic function of x, y. In a similar way, we shall assume, for the purpose of later application, that $\varphi_1(x, y, t)$ has also been analytically extended, at fixed time, into the instantaneous region of the solid body and the cavity.

For the convenience of derivation, instead of using the Cartesian coordinates (x, y) the intrinsic coordinates (s, n) of the extended steady basic flow will be used in the following formulation, where s is the distance measured along a streamline in the basic flow and n is the distance measured normal to a streamline such that on the cavity boundary it is directed into the flow as shown in figure 10. By using this coordinate system equation 101a becomes

$$\phi(s,n,t) = \phi_0(s,n) + \epsilon\,\phi_1(s,n,t) + O(\epsilon^2). \tag{101b}$$

The pressure field of the resulting flow, $p(s, n, t)$, is related to the velocity potential through the Bernoulli equation which may be written in the form

$$\frac{p}{\rho} + \frac{1}{2}(\nabla\phi)^2 + \frac{\partial\phi}{\partial t} = \frac{p_c}{\rho} + \frac{1}{2}q_c^2, \tag{102}$$

where ρ is the density of the fluid under consideration, p_c and q_c are the cavity pressure and the constant speed of the fluid particles on the free surface of the cavity in the basic flow respectively. By use of equation 101b and the property that $\frac{\partial\varphi_o}{\partial n} = 0$, we have

$$(\nabla\phi)^2 = \left(\frac{\partial\phi_0}{\partial s}\right)^2 + 2\epsilon\left(\frac{\partial\phi_0}{\partial s}\right)\left(\frac{\partial\phi_1}{\partial s}\right) + O(\epsilon^2), \tag{103}$$

and

$$\frac{\partial\phi}{\partial t} = \epsilon\,\frac{\partial\phi_1}{\partial t} + O(\epsilon^2). \tag{104}$$

Using the results of equations 103 and 104, we can write equation 102 as

$$\frac{p}{\rho} = \frac{p_c}{\rho} + \frac{1}{2}\left[q_c^2 - \left(\frac{\partial\phi_0}{\partial s}\right)^2\right] - \epsilon\left[\left(\frac{\partial\phi_0}{\partial s}\right)\left(\frac{\partial\phi_1}{\partial s}\right) + \frac{\partial\phi_1}{\partial t}\right] + O(\epsilon^2), \tag{105}$$

which gives the pressure field of the perturbed flow.

The boundary conditions of the problem are as follows:

(i) There are two boundary conditions at the free surface of the cavity, the kinematic boundary condition and the pressure condition. To express the kinematic boundary condition, let us denote the deviation of the perturbed free surface of the cavity, S_f', from that of the basic flow, S_f, by

$$n = \epsilon h(s,t), \tag{106}$$

where s is measured along S_f and S_f is given by $n = 0$ (see figure 10). Then, the kinematic condition that the fluid particles on the free surface will remain on it requires

$$\frac{\partial \varphi}{\partial n} = \epsilon \left[\frac{\partial \varphi}{\partial s} \cdot \frac{\partial h}{\partial s} + \frac{\partial h}{\partial t} \right] \qquad \text{on } S_f' \ . \tag{107}$$

By using the expansion of φ in equation 101b equation 107 becomes

$$\epsilon \frac{\partial \varphi_1}{\partial n} = \epsilon \left[\frac{\partial \varphi_0}{\partial s} \cdot \frac{\partial h}{\partial s} + \frac{\partial h}{\partial t} \right] + O(\epsilon^2) \qquad \text{on } S_f' \tag{108}$$

where $\dfrac{\partial \varphi_o}{\partial n} = 0$ has been used. Since $\varphi_o(s, n)$ is the "extended velocity potential" of the basic flow, we may write the value of $\dfrac{\partial \varphi_o}{\partial s}$ on S_f' as

$$\left(\frac{\partial \varphi_0}{\partial s}\right)_{S_f'} = \left(\frac{\partial \varphi_0}{\partial s}\right)_{S_f} + \epsilon h(s,t) \left[\frac{\partial}{\partial n}\left(\frac{\partial \varphi_0}{\partial s}\right)\right]_{S_f} + O(\epsilon^2), \tag{109a}$$

or, using the notation $q_c = \left(\dfrac{\partial \varphi_o}{\partial s}\right)_{S_f}$,

$$\left(\frac{\partial \varphi_0}{\partial s}\right)_{S_f'} = q_c + \epsilon h(s,t) \left[\frac{\partial}{\partial n}\left(\frac{\partial \varphi_0}{\partial s}\right)\right]_{S_f} + O(\epsilon^2) . \tag{109b}$$

Similarly,

$$\left(\frac{\partial \varphi_1}{\partial n}\right)_{S_f'} = \left(\frac{\partial \varphi_1}{\partial n}\right)_{S_f} + \epsilon h(s,t) \left[\frac{\partial}{\partial n}\left(\frac{\partial \varphi_1}{\partial n}\right)\right]_{S_f} + O(\epsilon^2) . \tag{110}$$

By using equations 109b and 110, equation 108 can be written as

$$\epsilon \left(\frac{\partial \varphi_1}{\partial n}\right) = \epsilon \left[q_c \frac{\partial h}{\partial s} + \frac{\partial h}{\partial t} \right] + O(\epsilon^2) \qquad \text{on } S_f \ . \tag{111}$$

Retaining first order in ϵ, we have

$$\frac{\partial \phi_1}{\partial n} = q_c \frac{\partial h}{\partial s} + \frac{\partial h}{\partial t} \qquad \text{on } S_f \ . \tag{112}$$

The pressure condition on S_f' is stated by the assumption that the cavity pressure will be maintained at the same constant p_c, that is

$$p(s,n,t) = p_c \qquad \text{on } S_f' \ . \tag{113}$$

From equations 105 and 113, it follows that

$$\frac{1}{2}\left[q_c^2 - \left(\frac{\partial \phi_0}{\partial s}\right)^2\right] - \epsilon\left[\left(\frac{\partial \phi_0}{\partial s}\right)\left(\frac{\partial \phi_1}{\partial s}\right) + \left(\frac{\partial \phi_1}{\partial t}\right)\right] + O(\epsilon^2) = 0 \qquad \text{on } S_f' \ . \tag{114}$$

By expanding $\left(\frac{\partial \varphi_1}{\partial s}\right)_{S_f'}$ and $\left(\frac{\partial \varphi_1}{\partial t}\right)_{S_f'}$ about their values at S_f and using equation 109b, equation 114 gives

$$\epsilon\left\{ q_c h(s,t)\left[\frac{\partial}{\partial n}\left(\frac{\partial \phi_0}{\partial s}\right)\right] + \left(\frac{\partial \phi_0}{\partial s}\right)\left(\frac{\partial \phi_1}{\partial s}\right) + \left(\frac{\partial \phi_1}{\partial t}\right)\right\}_{S_f} + O(\epsilon^2) = 0 \ . \tag{115}$$

Retaining the first order terms in ϵ, we have

$$q_c h(s,t)\left[\frac{\partial}{\partial n}\left(\frac{\partial \phi_0}{\partial s}\right)\right] + \left(\frac{\partial \phi_0}{\partial s}\right)\left(\frac{\partial \phi_1}{\partial s}\right) + \frac{\partial \phi_1}{\partial t} = 0 \qquad \text{on } S_f \ . \tag{116}$$

The vorticity, ζ, of a steady two-dimensional flow can be expressed in terms of the intrinsic coordinates as [cf. Ref. (27)]

$$\zeta = \frac{\partial q}{\partial n} + \frac{q}{R} \ , \tag{117}$$

where R is the radius of curvature of a streamline at a point considered, and n is in the direction of increasing R. By the assumption of irrotational flow and making use of equation 117, we can write (cf. figure 10)

$$\left[\frac{\partial}{\partial n}\left(\frac{\partial \phi_0}{\partial s}\right)\right]_{S_f} = -\frac{q_c}{R_f} \ , \tag{118}$$

where R_f represents the radius of curvature of S_f. Substituting equation 118 into equation 116, we obtain

$$\frac{q_c^2}{R_f} h(s,t) = q_c \frac{\partial \varphi_1}{\partial s} + \frac{\partial \varphi_1}{\partial t} \qquad \text{on } S_f \ , \qquad (119)$$

which represents the first order pressure condition on the perturbed free surface.

Equations 112 and 119 are the two conditions on the cavity free boundary. It is of interest to note that, if q_c^2/R_f is regarded as an equivalent gravitational acceleration and if the s-coordinate is rectilinear, then these two conditions are in the same form as those in the classical water wave problems. Thus, the centrifugal acceleration q_c^2/R_f here plays the role of gravity in producing surface waves on the curved cavity boundary.

Furthermore, it is noted that h can be eliminated from equations 112 and 119, thereby we obtain a single boundary condition containing φ_1 only. Since both $\left(\frac{\partial q_c}{\partial s}\right)_{S_f}$ and $\left(\frac{\partial q_c}{\partial t}\right)_{S_f}$ are zero, we may first normalize q_c to unity and then perform the operation of substitution, giving

$$\frac{\partial \varphi_1}{\partial n} = R_f \cdot \left(\frac{\partial}{\partial s} + \frac{\partial}{\partial t}\right)^2 \varphi_1 + \left(\frac{dR_f}{ds}\right)\left(\frac{\partial}{\partial s} + \frac{\partial}{\partial t}\right)\varphi_1 \qquad \text{on } S_f . \ (120)$$

Dividing equation 120 by R_f, we obtain

$$\left(\frac{\partial}{\partial s} + \frac{\partial}{\partial t}\right)^2 \varphi_1 + \left[\frac{d}{ds}\log R_f\right]\left(\frac{\partial}{\partial s} + \frac{\partial}{\partial t}\right)\varphi_1 - \frac{1}{R_f}\frac{\partial \varphi_1}{\partial n} = 0 \quad \text{on } S_f, \ (121)$$

which is the free surface boundary condition of φ_1.

We may also express equation 121 in a complex variable form. To do so, we shall choose the complex velocity potential of the steady basic flow, $f_o(z) = \varphi_o(x, y) + i\,\psi_o(x, y)$, as the spatial independent variable, in terms of which the first order complex perturbation velocity potential, $f_1(z, t) = \varphi_1 + i\psi_1$, will be expressed. The introduction of $f_1(z, t)$ is justified by the reason that φ_1 is a harmonic function of (x, y) as mentioned previously. Since both φ_o and ψ_o are functions of x and y, and therefore, functions of s and n, so we can express the operators

$$\frac{\partial}{\partial s} = \left(\frac{\partial \varphi_o}{\partial s}\right)\frac{\partial}{\partial \varphi_o} + \left(\frac{\partial \psi_o}{\partial s}\right)\frac{\partial}{\partial \psi_o}, \tag{122}$$

$$\frac{\partial}{\partial n} = \left(\frac{\partial \varphi_o}{\partial n}\right)\frac{\partial}{\partial \varphi_o} + \left(\frac{\partial \psi_o}{\partial n}\right)\frac{\partial}{\partial \psi_o}. \tag{123}$$

But,

$$\frac{\partial \varphi_o}{\partial n} = \frac{\partial \psi_o}{\partial s} = 0, \tag{124}$$

and

$$\frac{\partial \psi_o}{\partial n} = \pm\frac{\partial \varphi_o}{\partial s}, \tag{125}$$

where the upper sign will be used when the local positive n-direction is at 90^o counter-clockwise from the local positive s-direction, and the lower sign will be used when the positive n-direction is at 90^o clockwise from the positive s-direction. Substitution of equations 124 and 125 into equations 122 and 123 gives

$$\frac{\partial}{\partial s} = \left(\frac{\partial \varphi_o}{\partial s}\right)\frac{\partial}{\partial \varphi_o} \tag{126}$$

$$\frac{\partial}{\partial n} = \pm\left(\frac{\partial \varphi_o}{\partial s}\right)\frac{\partial}{\partial \psi_o} \tag{127}$$

To express R_f, the radius of curvature of the free surface of the basic flow, in complex variable form we note that

$$\frac{1}{R_f} = \mp \frac{\partial \theta}{\partial s}, \tag{128}$$

where θ denotes the angle of the tangent to the free surface measured counter-clockwise from a fixed direction, say, from positive x-axis, and the $\mp$ sign will be chosen so that the right-hand side of the last equation is always positive. It is worthy to note that on S_f both the signs in equations 128 and 125 are interrelated; that is, when the upper (or lower) sign in equation 128 is required at a place on S_f, the corresponding positive n-direction is at 90° counter-clockwise (or clockwise) from the positive s-direction at that place. This is the result due to the assumption that the cavity pressure is a minimum pressure in the flow field* and due to the convention we adopted for the positive directions of s and n (see figure 10). Also, we note that, since the speed of the fluid particles on the free surface of the basic flow has been normalized to unity, the complex velocity on the free surface of the basic flow can be written as

$$w_0 = \frac{df_0}{dz} = e^{-i\theta} \qquad \text{on } S_f . \tag{129}$$

Hence

$$\theta = i \log w_0 \qquad \text{on } S_f . \tag{130}$$

Substitution of equation 130 into equation 128 gives

$$\frac{1}{R_f} = \mp i \left(\frac{1}{w_0}\right) \frac{dw_0}{ds} \qquad \text{on } S_f , \tag{131}$$

* Which implies that the streamlines of the basic flow are convex seen from the fluid.

which also implies that $\frac{1}{w_o}\frac{dw_o}{ds}$ is purely imaginary on S_f. By the aid of the results obtained in equations 126, 127 and 131, and the relation $\frac{\partial\phi_1}{\partial\psi_o} = -\frac{\partial\psi_1}{\partial\varphi_o}$ we may write equation 121 in the following form:

$$\mathrm{Re}\left\{\left(\frac{\partial}{\partial f_o}+\frac{\partial}{\partial t}\right)^2 f_1(f_o,t) - \left(\frac{d}{df_o}\log R_f\right)\left(\frac{\partial}{\partial f_o}+\frac{\partial}{\partial t}\right)f_1(f_o,t) \mp \frac{i}{R_f}\frac{\partial f_1(f_o,t)}{\partial f_o}\right\} = 0, \text{on } S_f \;, \tag{132}$$

or,

$$\mathrm{Re}\left\{\left[\left(\frac{\partial}{\partial f_o}+\frac{\partial}{\partial t}\right)^2 - \left\{\frac{d}{df_o}\log\left(\frac{1}{w_o}\frac{dw_o}{df_o}\right)\right\}\left(\frac{\partial}{\partial f_o}+\frac{\partial}{\partial t}\right) - \frac{1}{w_o}\frac{dw_o}{df_o}\cdot\frac{\partial}{\partial f_o}\right]f_1(f_o,t)\right\} = 0 \text{ on } S_f \;, \tag{133}$$

which is the free surface boundary condition expressed in complex variable form.

The result expressed by equation 133 can also be obtained in a different way as shown in the Appendix in which the perturbation is applied not only to the velocity potential and pressure, but also to the positions of fluid particles.

(ii) At the solid surface the normal component of the flow velocity relative to the moving boundary must vanish. This condition can be derived analogous to the kinematic boundary condition on the free surface. If we denote the displacement of the wetted side of the solid body from its steady position S_o by

$$n = \epsilon \mathcal{S}(s,t) \tag{134}$$

where (s, n) are the intrinsic coordinates of the basic flow and s is measured along S_o from one end to the other (see figure 10), then the boundary condition on the solid body can be described by

$$\frac{\partial \phi}{\partial n} = \epsilon\left[\left(\frac{\partial \phi}{\partial s}\right)\frac{\partial S}{\partial s} + \frac{\partial S}{\partial t}\right] \qquad \text{on} \qquad n = \epsilon\, S(s,t). \tag{135}$$

After making use of equation 101b and expanding the involved quantities about S_o, we have, up to the first order in ϵ,

$$\frac{\partial \phi_1}{\partial n} = \left(\frac{\partial \phi_0}{\partial s}\right)\frac{\partial S}{\partial s} + \frac{\partial S}{\partial t} \qquad \text{on } S_o\ . \tag{136}$$

But,

$$\frac{\partial \phi_1}{\partial n} = -\frac{\partial \psi_1}{\partial s}, \tag{137}$$

hence the boundary condition on the solid body can be written as

$$\psi_1 = -\int^{s}\left[q_s\frac{\partial S}{\partial s} + \frac{\partial S}{\partial t}\right]ds \qquad \text{on } S_o\ , \tag{138}$$

where
$$q_s = \left(\frac{\partial \phi_0}{\partial s}\right)_{S_0}$$

(iii) At the point of infinity we require the perturbation velocity to vanish, or

$$\left|\frac{\partial f_1}{\partial z}\right| \longrightarrow 0 \qquad \text{as} \qquad |z| \rightarrow \infty \tag{139}$$

Also, the applied disturbance cannot induce a source or sink at infinity; that is

$$Im \oint_{\Gamma} \frac{\partial f_1}{\partial z}dz = 0, \tag{140}$$

where Γ is a countour around the point of infinity. Furthermore, from Kelvin's theorem on the conservation of circulation, the perturbed unsteady motion cannot, for any finite time, result in a net vortex at infinity; that is

$$\operatorname{Re}\oint_{\Gamma} \frac{\partial f_1}{\partial z} dz = 0 \qquad \text{for} \qquad t < \infty. \tag{141}$$

Equations 140 and 141 can be combined to give

$$\oint_{\Gamma} \frac{\partial f_1}{\partial z} dz = 0 \tag{142}$$

This completes the formulation of the perturbation theory.

8. General Methods of Solution

In view of the complexity of the boundary condition on the free surface, the details involved in solving the problem formulated above will depend on the basic flow and hence will differ for each individual case. An outline of the general method, however, can still be given. Since the wetted side of the solid body together with the free surface of the basic flow form a streamline, say, $\psi_o = 0$, then, by the aid of the f_o-plane it is always possible to transform the whole flow field into the upper-half of a certain ζ-plane such that the streamline $\psi_o = 0$ coincides with the real axis of the ζ-plane. With the boundary conditions expressed in the form of equations 133, 138 and 142, the solution of f_1 can be obtain in the ζ-plane by applying either of the methods described below:

(i) If the solid body is a flat plate, the quantity $\frac{1}{w_o} \frac{dw_o}{df_o}$ will be purely real on it. Then, by applying the operator

$$L = \left(\frac{\partial}{\partial f_o} + \frac{\partial}{\partial t}\right)^2 - \left[\frac{d}{df_o} \log\left(\frac{1}{w_o}\frac{dw_o}{df_o}\right)\right]\left(\frac{\partial}{\partial f_o} + \frac{\partial}{\partial t}\right) - \frac{1}{w_o}\frac{dw_o}{df_o}\frac{\partial}{\partial f_o} \tag{143}$$

on both sides of equation 138, we get

$$\operatorname{Im}\{L[f_1]\} = g(f_o, t) \qquad \text{on } S_o , \tag{144}$$

where $g(f_o, t)$ is a known function. With the boundary conditions on the free surface and the solid body expressed in the form of equations 133 and 144 a Hilbert problem can be constructed in the ζ-plane by the method described in section 2, Part I. The solution of the so constructed Hilbert problem will lead us to a second order linear partial differential equation with variable coefficients, the solution of which gives f_1.

(ii) For a solid body arbitrary in shape, we express the free surface boundary condition in the form

$$L[f_1] = i G(f_o, t), \tag{145}$$

where $G(f_o, t)$ is an analytic function of f_o, continuous in t, which is real for f_o on the basic flow free boundary but is otherwise unknown a priori. The solution of equation 145 gives both the real and imaginary parts of f_1 on the free surface with G as an unknown function, which together with the other boundary conditions may be expected to determine $f_1(f_o, t)$ uniquely.

9. Surface Waves on a Hollow Vortex

This relatively simple problem was chosen to demonstrate the application of this general theory to a very special case; this application is partly meant for the verification of the complicated expression of the free surface boundary condition, since the problem has already been solved by Lord Kelvin (28) in a completely different way (the solution is obtained by making use of the axial symmetry of the basic flow field).

The basic flow is an irrotational, circulating motion about a point, say $z = 0$, as center. The free surface will be denoted by $|z| = a$ on which the speed of flow is normalized to unity. The velocity potential can easily be seen as

$$f_0(z) = -ia \log z \tag{146}$$

and therefore,

$$w_0(z) = -\frac{ia}{z}. \tag{147}$$

The transformation

$$\zeta = i \frac{1-w_0}{1+w_0} \tag{148}$$

maps the entire basic flow in the hodograph w_0-plane, $|w_0| < 1$, into the upper half- of the ζ-plane (see figure 11).

The solution of the perturbation flow is obtained by using the first method of solution stated in the last section. The boundary conditions of this problem are

$$\mathrm{Re}\{L[f_1]\} = 0 \qquad \text{on} \quad \eta = 0^+ \quad \text{all } \xi \tag{149}$$

$$\left|\frac{\partial f_1}{\partial z}\right| = 0 \qquad \text{as} \quad |z| \to \infty, \tag{150}$$

and

$$\oint_\Gamma \frac{\partial f_1}{\partial z} dz = \oint_\Gamma \frac{\partial f_1}{\partial \zeta} d\zeta = 0, \tag{151}$$

where the operator L is defined by equation 143 and Γ is a contour around the point of infinity, or $\zeta = i$. A particular solution of the above boundary value problem is

$$L[f_1] = 0. \tag{152}$$

The complementary solution can be written as

$$L[f_1] = \sum_{n=-\infty}^{\infty} \left[c_n(t)(\zeta - i)^n - \overline{c_n(t)}(\zeta + i)^n \right], \tag{153}$$

where $c_n(t)$ are unknown functions of t. In order to satisfy equations 150 and 151, that is, to satisfy the condition

$$f_1 \sim O[(\zeta - i)^\alpha], \qquad \alpha > 0 \qquad \text{as} \qquad \zeta \to i, \tag{154}$$

it is necessary that

$$c_n = 0 \qquad \text{for} \qquad n \leq 0. \tag{155}$$

But, $|L[f_1]| < \infty$ at $\omega_0 = -1$, or, at $\zeta = \infty$, so we must have

$$c_n = 0 \qquad \text{for} \qquad n > 0. \tag{156}$$

Therefore, the solution of the boundary value problem is

$$L[f_1] = 0. \tag{157}$$

From equations 146 and 147, we have

$$\begin{aligned} &\frac{1}{w_0}\frac{dw_0}{df_0} = -\frac{i}{a}, &\qquad &\frac{d}{df_0}\left(\frac{1}{w_0}\frac{dw_0}{df_0}\right) = 0, \\ &\frac{\partial}{\partial f_0} = \frac{i}{a} z \frac{\partial}{\partial z}, &\qquad &\text{—} = -\frac{1}{a^2}\left(z^2 \frac{\partial^2}{\partial z^2} + z \frac{\partial}{\partial z}\right). \end{aligned} \tag{158}$$

Substitution of these expressions into equation 157 gives

$$\left[\frac{z^2}{a^2}\frac{\partial^2}{\partial z^2} + \frac{2z}{a^2}\frac{\partial}{\partial z} - \frac{2i}{a} z \frac{\partial^2}{\partial z \partial t} - \frac{\partial^2}{\partial t^2} \right] f_1(z,t) = 0. \tag{159}$$

Since $f_1(z, t)$ should be regular everywhere outside $|z| = a$, we may write

$$f_1(z,t) = \sum_{n=1,2,\cdots}^{\infty} A_n(t) \frac{1}{z^n} \tag{160}$$

Substituting equation 160 into equation 159 gives

$$\sum_{n=1,2,\cdots}^{\infty} \left[\frac{n(n+1)}{a^2} A_n(t) - \frac{2n}{a^2} A_n(t) + \frac{2in}{a} \dot{A}(t) - \ddot{A}(t) \right] \frac{1}{z^n} = 0 , \tag{161}$$

or,

$$\ddot{A}_n(t) - \frac{2in}{a} \dot{A}_n(t) - \frac{n}{a^2}(n-1) A_n(t) = 0 , \tag{162}$$

where dot represents the differentiation with respect to t. The solution of the last equation is

$$A_n = a_n e^{\frac{i}{a}(n \pm \sqrt{n})t} ,$$

where a_n are constants. Finally, we have

$$f_1(z,t) = \sum_{n=1,2,\cdots}^{\infty} \left[a_n e^{\frac{i}{a}(n+\sqrt{n})t} + b_n e^{\frac{i}{a}(n-\sqrt{n})t} \right] \frac{1}{z^n} , \tag{163}$$

where the constants a_n and b_n can be determined by appropriate initial conditions. The result given in equation 163 agrees with that obtained by Lord Kelvin (28).

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APPENDIX

The free surface boundary condition as expressed by equation 133, Part II, may also be obtained by considering perturbations of both the velocity potential and the positions of fluid particles. Let $z_o = x_o + iy_o$ denote the position of a fluid particle in a two-dimensional, irrotational, steady flow (the basic flow), and let $f_o(z_o) = \varphi_o + i\psi_o$ denote the complex velocity potential of the basic flow. We assume that at the time $t = 0$, a small disturbance is applied on the basic flow so that the resulting unsteady irrotational motion may be considered as a perturbation of the basic flow. Then the position of a fluid particle and the complex velocity potential of the resulting flow may be written as

$$z = z_o + \epsilon z_1(x_o, y_o, t) + O(\epsilon^2) \tag{163}$$

and

$$f(z,t) = f_o(z) + \epsilon f_1(z,t) + O(\epsilon^2), \tag{164}$$

where ϵ is a small parameter, and both z_1 and f_1 are independent of ϵ. Since both $f(z, t)$ and $f_o(z)$ are analytic functions of z, and ϵ is an arbitrary small parameter, it follows that $f_1(z, t)$ is also an analytic function of z.

We denote

$p(x, y, t)$ = the pressure field of the resulting perturbed flow,

$w(z, t)$ = $\partial f(z, t)/\partial z$ = complex velocity of the resulting flow,

p_c = cavity pressure of the basic flow,

q_c = constant speed of the fluid particles on the free surface of the basic flow, which will be normalized to unity in the following derivation.

The Bernoulli equation may be written as

$$\frac{p}{\rho} + \frac{1}{2} w(z,t)\overline{w(z,t)} + Re\left[\frac{\partial f(z,t)}{\partial t}\right] = \frac{p_c}{\rho} + \frac{1}{2} q_c^2 = \frac{p_c}{\rho} + \frac{1}{2} \tag{165}$$

where $\overline{w(z, t)}$ = the complex conjugate of $w(z, t)$, $R\ell\,[\partial f(z, t)/\partial t]$ = real part of $\partial f(z, t)/\partial t$.

There are two boundary conditions on the free surface of the cavity; they are (i) that the pressure is prescribed to be the constant value p_c, and (ii) that during a small time interval a fluid particle originally on the free surface will remain on it. To express the pressure condition we put $p(x, y, t) = p_c$ in equation 165, giving

$$\frac{1}{2} w(z,t)\overline{w(z,t)} + Re\left[\frac{\partial f_1(z,t)}{\partial t}\right] = \frac{1}{2} \tag{166}$$

on the perturbed free surface S_f'. In the following, we shall proceed to express the quantities in equation 166 in terms of the basic flow and the perturbation variables. From equations 164 and 163, we can write

$$\begin{aligned} w(z,t) &= \frac{df_0(z)}{dz} + \epsilon\,\frac{\partial f_1(z,t)}{\partial z} + O(\epsilon^2) \\ &= \frac{df_0(z_0)}{dz_0} + \epsilon\left[\frac{\partial f_1(z_0,t)}{\partial z_0} + \frac{d^2 f_0(z_0)}{dz_0^2}\, z_1(x_0,y_0,t)\right] + O(\epsilon^2) \\ &= w_0(z_0) + \epsilon\left[\frac{\partial f_1(f_0,t)}{\partial f_0}\cdot\frac{df_0(z_0)}{dz_0} + z_1\,\frac{dw_0(f_0)}{df_0}\cdot\frac{df_0(z_0)}{dz_0}\right] + O(\epsilon^2) \\ &= w_0(z_0)\left\{1 + \epsilon\left[\frac{\partial f_1(f_0,t)}{\partial f_0} + z_1\,\frac{dw_0(f_0)}{df_0}\right]\right\} + O(\epsilon^2), \end{aligned} \tag{167}$$

and,

$$\frac{\partial f(z,t)}{\partial t} = \epsilon \frac{\partial f_1(z,t)}{\partial t} + O(\epsilon^2) = \epsilon \frac{\partial f_1(z_0,t)}{\partial t} + O(\epsilon^2). \tag{168}$$

Substitution of equations 167 and 168 into equation 166 leads to

$$\frac{1}{2} w_0(z_0)\overline{w_0(z_0)}\left[1+\epsilon\left(\frac{\partial f_1}{\partial f_0} + z_1\frac{dw_0}{df_0}\right)\right]\left[1+\epsilon\overline{\left(\frac{\partial f_1}{\partial f_0} + z_1\frac{dw_0}{df_0}\right)}\right]$$

$$+\epsilon \mathrm{Re}\left[\frac{\partial f_1}{\partial t}\right] = \frac{1}{2}, \tag{169}$$

where z_o is evaluated on the free surface of the basic flow, S_f. Retaining the terms of the first order in ϵ and using the relation $[w_o(z_o)\overline{w_o(z_o)}]_{S_f} = q_c^2 = 1$, we obtain the pressure condition on S_f' as

$$\frac{1}{2}\left[\left(\frac{\partial f_1(f_0,t)}{\partial f_0} + z_1\frac{dw_0(f_0)}{df_0}\right) + \overline{\left(\frac{\partial f_1(f_0,t)}{\partial f_0} + z_1\frac{dw_0(f_0)}{df_0}\right)}\right]$$

$$+\mathrm{Re}\left[\frac{\partial f_1(f_0,t)}{\partial t}\right] = 0 \qquad \text{for } f_o \text{ on } S_f, \tag{170}$$

or,

$$\mathrm{Re}\left[\frac{\partial f_1(f_0,t)}{\partial f_0} + z_1(x_0,y_0,t)\frac{dw_0(f_0)}{df_0}\right.$$

$$\left. + \frac{\partial f_1(f_0,t)}{\partial t}\right] = 0 \qquad \text{for } f_o \text{ on } S_f. \tag{171}$$

To express the kinematic boundary condition (ii) stated previously, we may use the relation $dz/dt = \overline{w(z,t)}$ when evaluated on S_f'. In the following derivation, instead of using the form of equation 163, it will be more convenient to write the position of the perturbed free surface in the form

$$z = z_0 + \epsilon z_1(s_0,t) + O(\epsilon^2)$$

where z_o is the position of a particle on S_f, and s_o is the arc length measured along S_f in the flow direction. Then,

$$\frac{dz}{dt} = \frac{dz_0}{dt} + \epsilon \frac{dz_1(s_0,t)}{dt} + O(\epsilon^2) = \overline{w_0(z_0)} + \epsilon \left[\frac{\partial z_1}{\partial s_0}\frac{ds_0}{dt} + \frac{\partial z_1}{\partial t} \right] + O(\epsilon^2). \quad (172)$$

Since along S_f, $\frac{\partial}{\partial s_o} = \frac{\partial \phi_o}{\partial s_o}\frac{\partial}{\partial \varphi_o} = q_c \frac{\partial}{\partial f_o} = \frac{\partial}{\partial f_o}$, and

$\frac{ds_o}{dt} = q_c = 1$, hence by using equation 167, we obtain the kinematic boundary condition on S_f' as

$$\frac{dz}{dt} = \overline{w_0(z_0)} + \epsilon \left(\frac{\partial z_1}{\partial f_0} + \frac{\partial z_1}{\partial t} \right) + O(\epsilon^2)$$

$$= \overline{w(z,t)} = \overline{w_0(z_0)} \left\{ 1 + \epsilon \left[\frac{\partial f_1}{\partial f_0} + z_1 \frac{dw_0}{df_0} \right] \right\} + O(\epsilon^2) \quad \text{for } f_o \text{ on } S_f, \quad (173)$$

or,

$$\overline{\left(\frac{\partial f_1}{\partial f_0} + z_1 \frac{dw_0}{df_0} \right)} = w_0(z_0)\left(\frac{\partial z_1}{\partial f_0} + \frac{\partial z_1}{\partial t} \right) \quad \text{for } f_o \text{ on } S_f, \quad (174)$$

where use has been made of the relation $[w_o(z_o)\,\overline{w_o(z_o)}]_{S_f} = 1$.

Substitution of equation 174 into equation 170 gives

$$\frac{1}{2}\left[\frac{\partial}{\partial f_0}(f_1 + w_0 z_1) + \frac{\partial}{\partial t}(f_1 + w_0 z_1) - \frac{\partial f_1}{\partial t} \right] + Re\left[\frac{\partial f_1}{\partial t} \right] = 0$$

$$\text{for } f_o \text{ on } S_f, \quad (175)$$

which can be rewritten as

$$\left(\frac{\partial}{\partial f_0} + \frac{\partial}{\partial t} \right)(f_1 + w_0 z_1) + \frac{\partial \overline{f_1}}{\partial t} = 0 \quad \text{for } f_o \text{ on } S_f. \quad (176)$$

Integrating the last equation from $t = 0^+$ to t with f_o held fixed, we obtain for f_o on S_f

$$\overline{f_1(f_o,t)} - \overline{f_1(f_o,0^+)} = -\left\{ \frac{\partial}{\partial f_o}\int_{0^+}^{t} (f_1 + w_o z_1)\,dt + [f_1(f_o,t) + w_o z_1(f_o,t)] \right.$$

$$\left. - [f_1(f_o,0^+) + w_o z_1(f_o,0^+)] \right\} \qquad (177)$$

where $\overline{f}_1$ denotes the complex conjugate of f_1. By denoting

$$H = \int_{0^+}^{t} (f_1 + w_o z_1)\,dt\,, \qquad (178a)$$

or,

$$\frac{\partial H}{\partial t} = f_1 + w_o z_1 \quad , \qquad (178b)$$

we obtain for f_o on S_f the result

$$\overline{f_1(f_o,t)} = -\left\{ \frac{\partial H}{\partial f_o} + \frac{\partial H}{\partial t} - 2\phi_1(f_o, t=0^+) - w_o(f_o) z_1(f_o, t=0^+) \right\}. \qquad (179)$$

Since it is not possible to have a pressure jump across a free surface at any time we must have

$$Q_1(f_o, t=0^+) = 0 \qquad \text{for } f_o \text{ on } S_f\,. \qquad (180)$$

Furthermore it is obvious that

$$z_1(f_o, t=0^+) = 0 \qquad \text{for } f_o \text{ on } S_f\,, \qquad (181)$$

whence

$$\overline{f_1(f_o,t)} = -\left(\frac{\partial H}{\partial f_o} + \frac{\partial H}{\partial t}\right) \qquad \text{for } f_o \text{ on } S_f\,. \qquad (182)$$

From equations 178b and 182 z_1 can be expressed in terms of H as

$$z_1 = \frac{1}{w_0}\left[\frac{\partial \bar{H}}{\partial f_0} + \frac{\partial}{\partial t}(H + \bar{H})\right] \qquad \text{for } f_0 \text{ on } S_f \, . \tag{183}$$

Substituting equation 183 into equation 171 we have

$$\mathrm{Re}\left\{\frac{\frac{dw_0}{df_0}}{w_0}\left[\frac{\partial \bar{H}}{\partial f_0} + \frac{\partial}{\partial t}(H + \bar{H})\right] + \frac{\partial f_1}{\partial f_0} + \frac{\partial f_1}{\partial t}\right\} = 0 \qquad \text{for } f_0 \text{ on } S_f \, . \tag{184}$$

As pointed out in equation 131 that $\frac{1}{w_0}\frac{dw_0}{df_0}$ is purely imaginary on S_f, therefore equation 184 reduces to

$$\mathrm{Re}\left[\frac{1}{w_0}\cdot\frac{dw_0}{df_0}\cdot\frac{\partial \bar{H}}{\partial f_0} + \frac{\partial f_1}{\partial f_0} + \frac{\partial f_1}{\partial t}\right] = 0 \qquad \text{for } f_0 \text{ on } S_f \, . \tag{185}$$

We now eliminate H by differentiating equation 185 with respect to f_0 on S_f and t respectively, thereby we obtain

$$\mathrm{Re}\left[\frac{\partial^2 f_1}{\partial f_0^2} + \frac{\partial^2 f_1}{\partial f_0 \partial t} + \frac{1}{w_0}\cdot\frac{dw_0}{df_0}\cdot\frac{\partial^2 \bar{H}}{\partial f_0^2} + \frac{d}{df_0}\left(\frac{1}{w_0}\cdot\frac{dw_0}{df_0}\right)\frac{\partial \bar{H}}{\partial f_0}\right] = 0$$
$$\text{for } f_0 \text{ on } S_f \, , \tag{186}$$

and

$$\mathrm{Re}\left[\frac{\partial^2 f_1}{\partial t \partial f_0} + \frac{\partial^2 f_1}{\partial t^2} + \frac{1}{w_0}\cdot\frac{dw_0}{df_0}\cdot\frac{\partial^2 \bar{H}}{\partial t \partial f_0}\right] = 0 \qquad \text{for } f_0 \text{ on } S_f \, . \tag{187}$$

Adding equation 187 to equation 186 and using equation 182, we obtain

$$\mathrm{Re}\left[\frac{\partial^2 f_1}{\partial f_0^2} + 2\frac{\partial^2 f_1}{\partial f_0 \partial t} + \frac{\partial^2 f_1}{\partial t^2} - \frac{1}{w_0}\cdot\frac{dw_0}{df_0}\cdot\frac{\partial f_1}{\partial f_0} + \frac{d}{df_0}\left(\frac{1}{w_0}\cdot\frac{dw_0}{df_0}\right)\frac{\partial \bar{H}}{\partial f_0}\right] = 0$$
$$\text{for } f_0 \text{ on } S_f \, . \tag{188}$$

From equation 185 we can deduce for f_0 on S_f the relation

$$\mathrm{Re}\left[\frac{d}{df_0}\left(\frac{1}{w_0}\cdot\frac{dw_0}{df_0}\right)\frac{\partial \bar{H}}{\partial f_0}\right] = -\,\mathrm{Re}\left[\frac{\frac{d}{df_0}\left(\frac{1}{w_0}\cdot\frac{dw_0}{df_0}\right)}{\frac{1}{w_0}\cdot\frac{dw_0}{df_0}}\left(\frac{\partial f_1}{\partial f_0} + \frac{\partial f_1}{\partial t}\right)\right]. \tag{189}$$

Substitution of equation 189 into equation 188 gives

$$\mathrm{Re}\left[\frac{\partial^2 f_1}{\partial f_0^2} + 2\frac{\partial^2 f_1}{\partial f_0 \partial t} + \frac{\partial^2 f_1}{\partial t^2} - \frac{\frac{d}{df_0}\left(\frac{1}{w_0}\frac{dw_0}{df_0}\right)}{\frac{1}{w_0}\cdot\frac{dw_0}{df_0}}\left(\frac{\partial f_1}{\partial f_0} + \frac{\partial f_1}{\partial t}\right) - \frac{1}{w_0}\cdot\frac{dw_0}{df_0}\cdot\frac{\partial f_1}{\partial f_0}\right] = 0$$

for f_o on S_f , (190)

which is the free surface boundary condition obtained in equation 133, Part II.

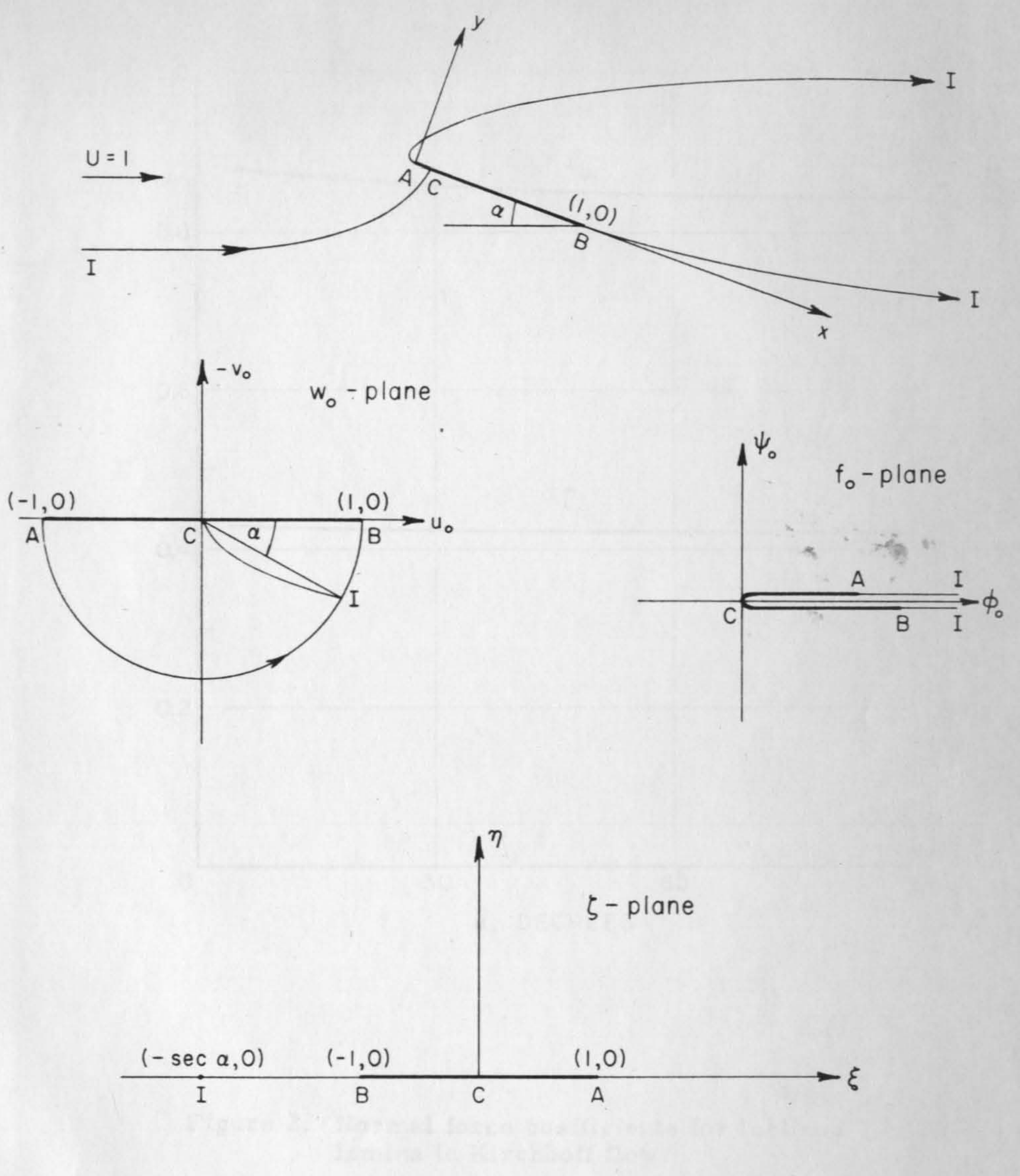

Figure 1. Inclined lamina in Kirchhoff flow and its conformal mapping planes.

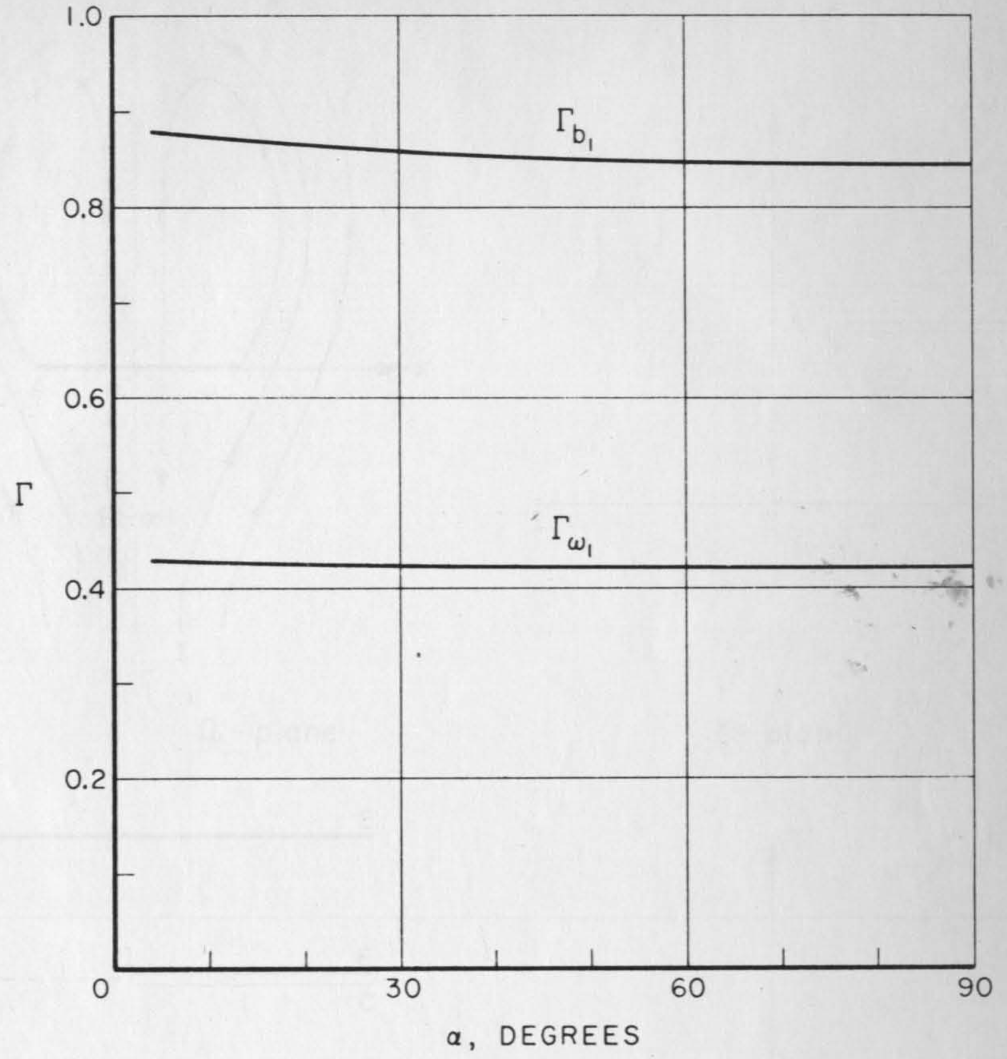

Figure 2. **Normal force coefficients for inclined lamina in Kirchhoff flow.**

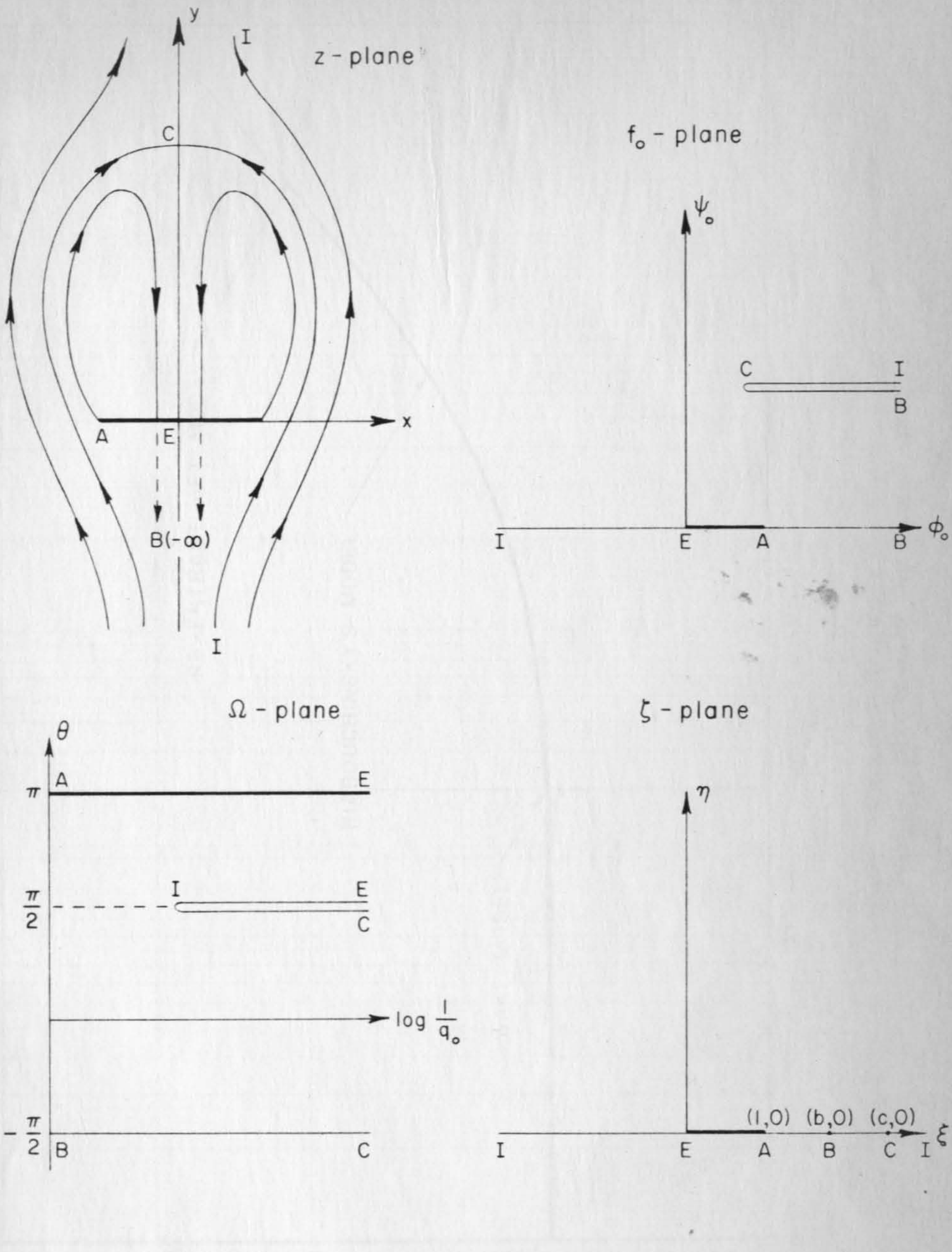

Figure 3. Re-entrant jet flow past a flat plate and its conformal mapping planes.

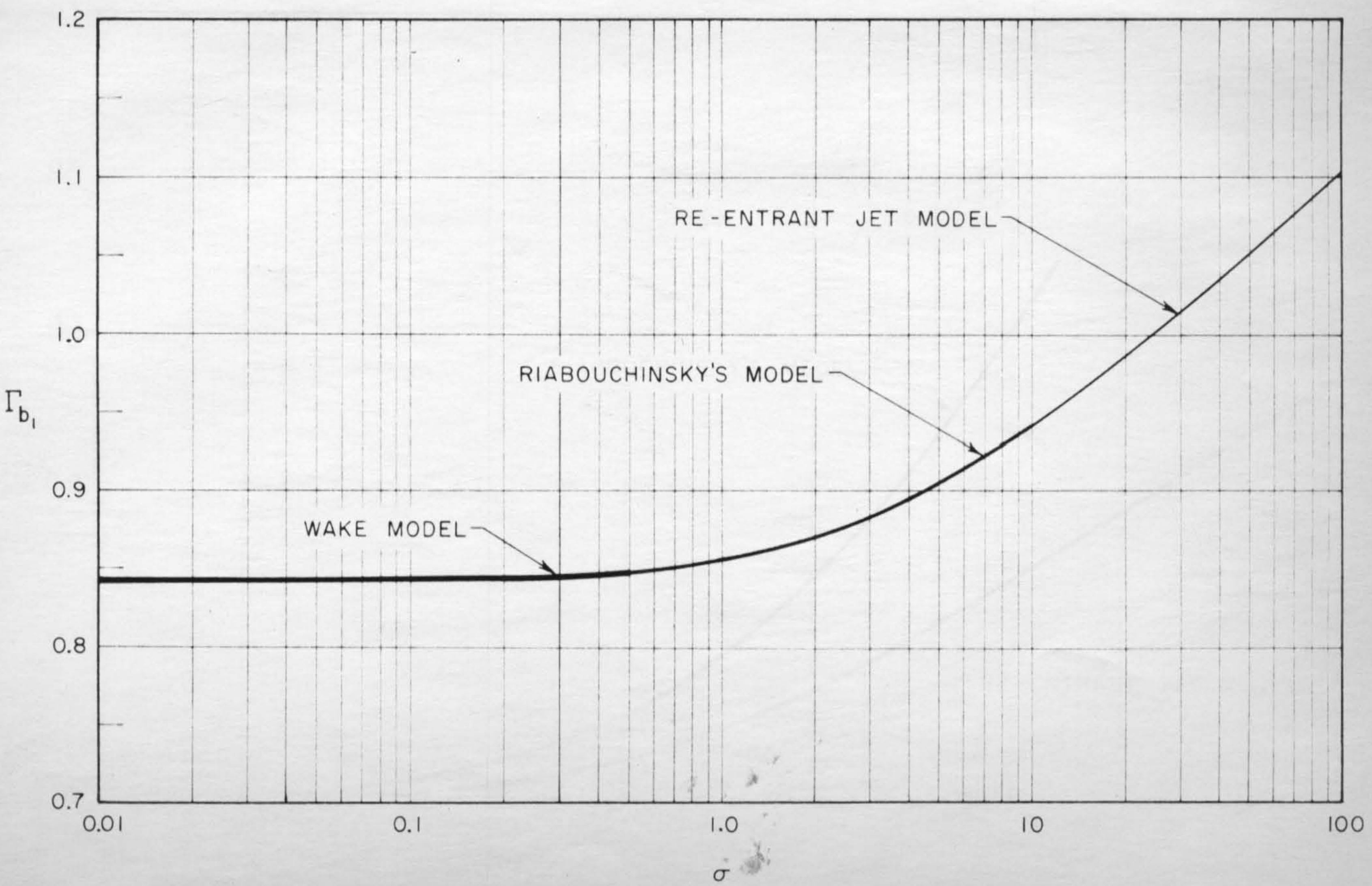

Figure 4. Normal force coefficients for the sudden acceleration of a flat plate placed normally to the incoming stream.

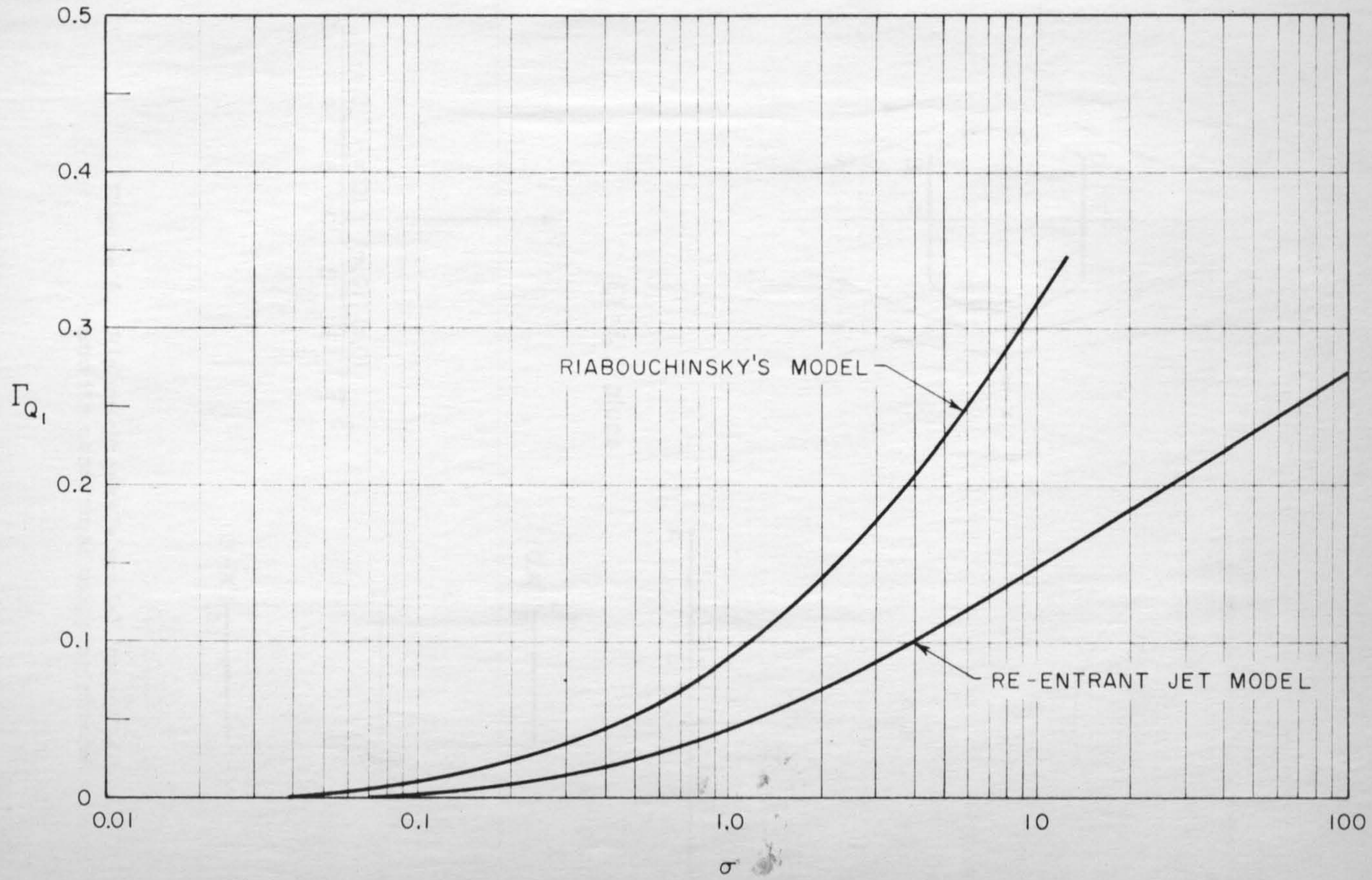

Figure 5. Normal force coefficients due to a source at the point of infinity for a flat plate placed normally to the incoming stream.

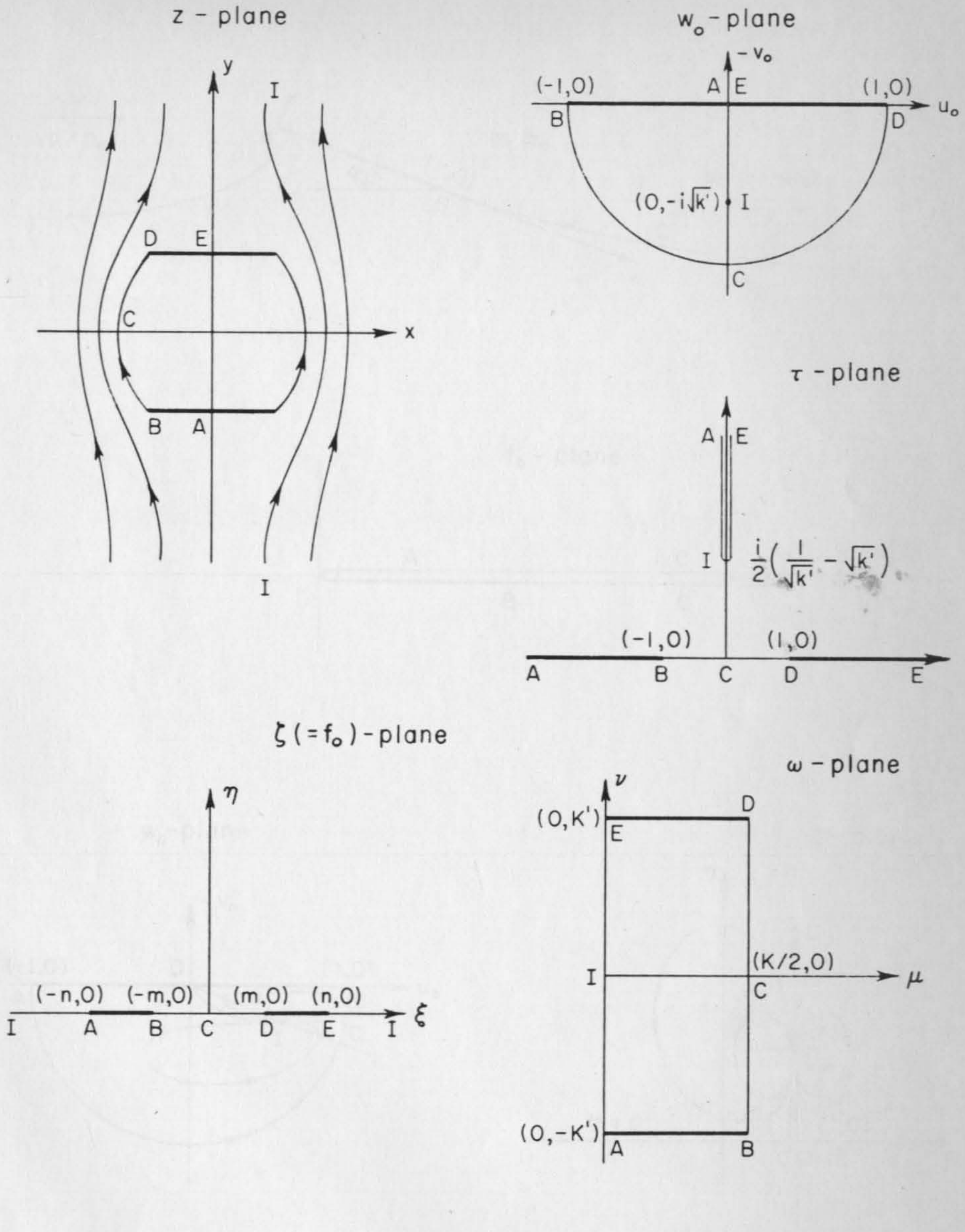

Figure 6. Riabouchinsky's model for a flat plate and its conformal mapping planes.

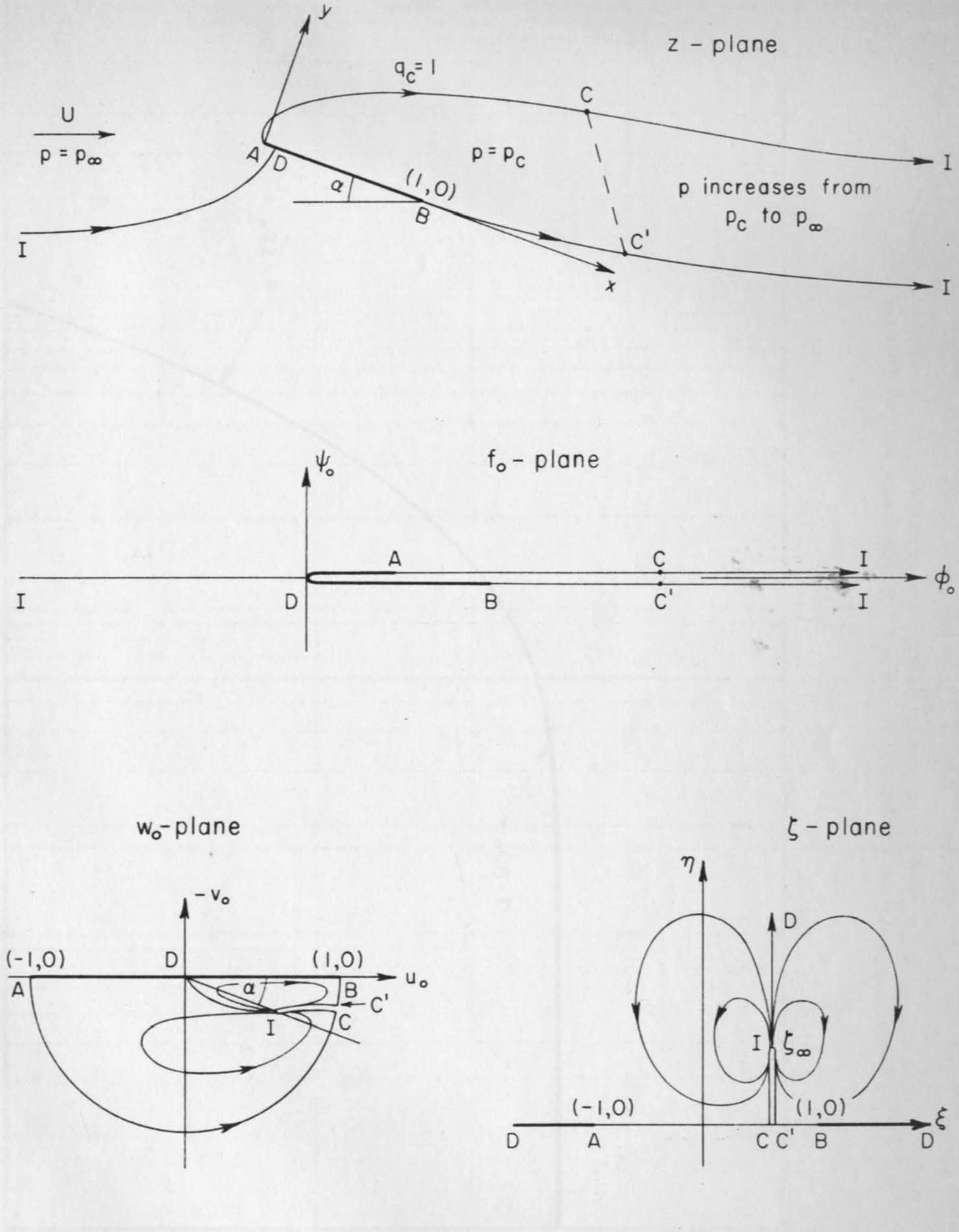

Figure 7. A wake model for an oblique plate with a finite cavity and its conformal mapping planes.

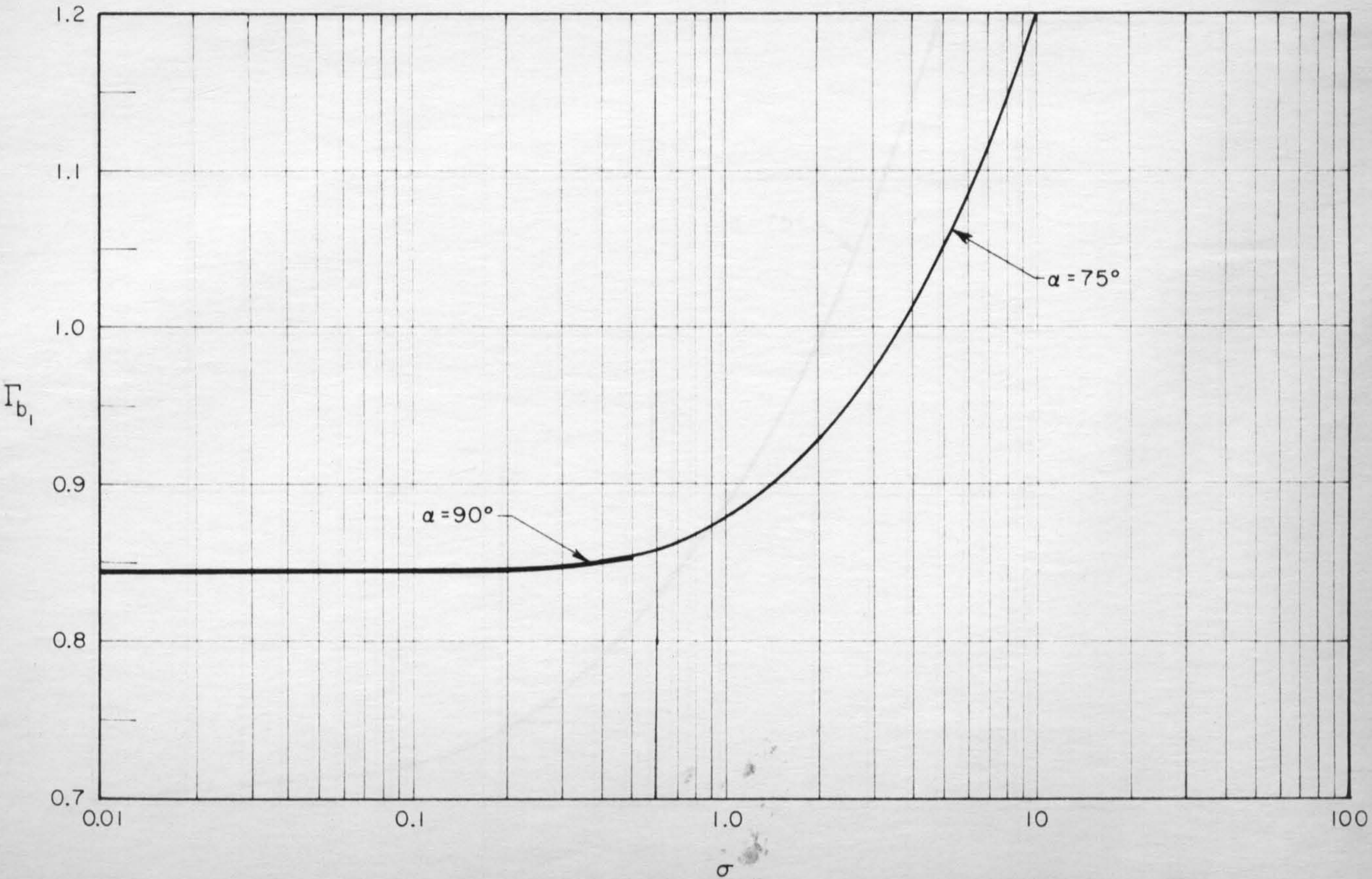

Figure 8. Normal force coefficients for the sudden acceleration of a flat plate, at an angle of attack α, in the wake model.

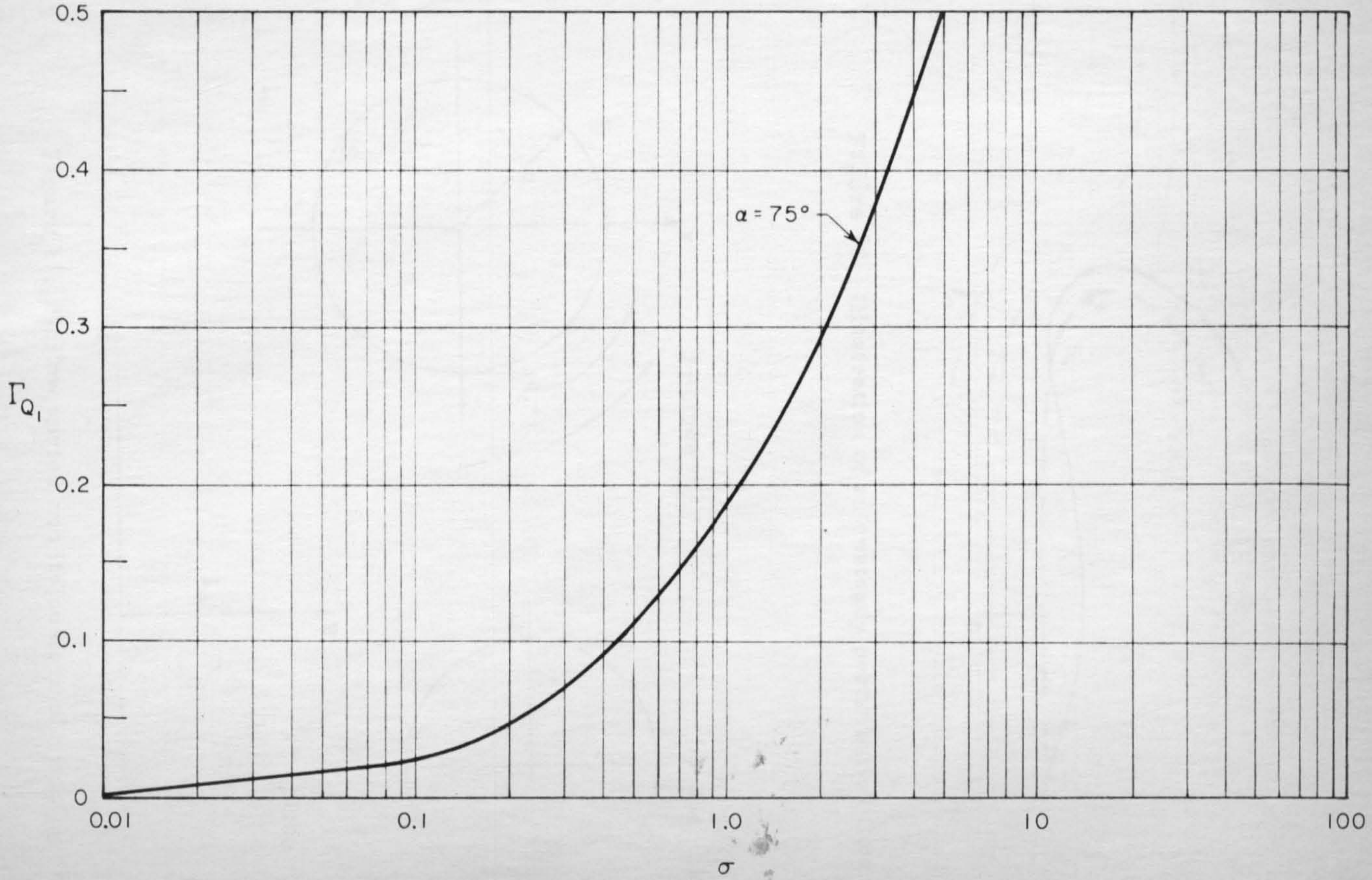

Figure 9. Normal force coefficient due to a source at the point of infinity for a flat plate, at an angle of attack α, in the wake model.

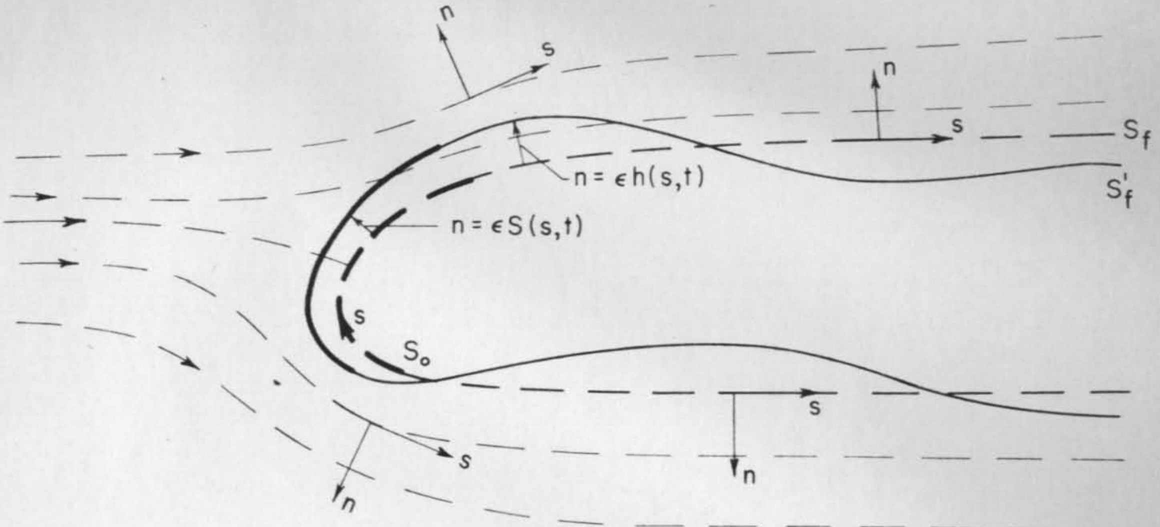

Figure 10. Illustration of an unsteady perturbation flow.

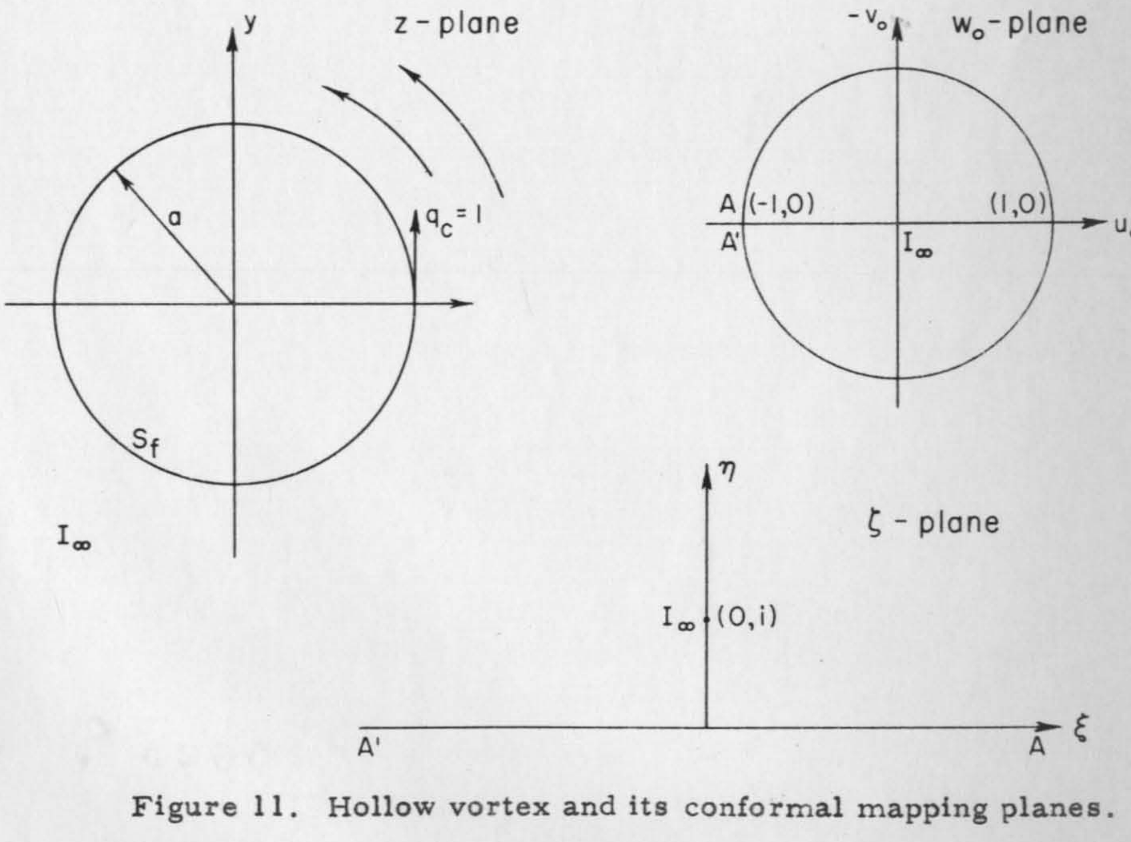

Figure 11. Hollow vortex and its conformal mapping planes.