APPLICATIONS OF
AN EDGE- AND CORNER-LAYER TECHNIQUE
TO ELASTIC PLATES AND SHELLS

Thesis by

Roger Edwin Messick

In Partial Fulfillment of the Requirements
For the Degree of
Doctor of Philosophy

California Institute of Technology
Pasadena, California
1962

ACKNOWLEDGEMENTS

The author very gratefully acknowledges his indebtedness to Professor James K. Knowles, not only for many hours of consultation but also for a subtler kind of assistance which took less tangible and less easily acknowledged forms -- assistance requiring human understanding and personal honesty. Both his professional competence and friendship have been great sources of inspiration.

Thanks are also due Mrs. Barbara E. Rickert for her competence and perseverance in producing the manuscript.

Part of this work was conducted with the financial assistance of a Ford Foundation Fellowship.

ABSTRACT

This paper contains several problems that can be formulated mathematically as two-dimensional boundary value problems for partial differential equations containing a parameter. A method is given which leads directly to asymptotic solutions for large values of the parameter without resorting to the exact solutions. The examples discussed involve linear differential equations and are drawn primarily from various problems in the theory of elasticity.

The method involves consideration of what are termed corner-layers in addition to the well known boundary-layers. The need for considering these corner-layers arises from the fact that the problems treated lead to boundary-layer differential equations which contain derivatives, not only with respect to the boundary-layer variable, but also with respect to the remaining independent variable. Thus, the solution of such boundary-layer equations requires knowledge of boundary conditions in addition to those needed in standard boundary-layer problems.

The applications include: a heat conduction problem, two problems with transverse bending of stretched plates, and two problems from elastic shell theory.

The shell problems concern the bending of both the shallow and the non-shallow helicoidal shell. It is found that these shells have boundary-layers whose characteristic length is proportional to the one-third power of the thickness parameter. This may be contrasted with shells of revolution, where this characteristic length is proportional to the one-half power of the thickness parameter.

TABLE OF CONTENTS

LIST OF FIGURES

I. PRESENTATION OF THE METHOD

A. Introduction

Theoretically, the solution of linear partial differential equations - with appropriate subsidiary conditions - can be systematically worked out. However, in general, the complete mathematical representation of the exact solution is very cumbersome. It is fortunate that the practical application and physical significance of many problems are restricted to cases where some parameter approximates a critical or extreme value. Such cases have a special appeal to applied mathematicians because of the possibility of obtaining adequate approximate representations for the solutions.

As an example, there is the mathematical theory of thin elastic shells. In this theory, the thickness dimension h of the elastic body is assumed to be small in comparison with some other characteristic dimension L. That is, a shell is described as a body that has one of its dimensions small in comparison with its two others. L might be a radius of curvature. In such cases, the parameter L/h is assumed to be large relative to unity (1).

Another example occurs in fluid dynamics in which the parameter VL/ν, known as the Reynolds number, is of importance. Here V and L represent a characteristic velocity and length, respectively, while ν is the kinematic viscosity of the fluid. When the Reynolds number is large inertial effects are dominant while a Reynolds number near zero indicates that viscous forces are dominant (2).

In such problems, it is frequently found that there are portions of the exact solution which are insignificant when the relevant parameter is, say, large. Furthermore, it may be that these insignificant portions add considerably to the difficulty of finding an exact solution. It is, therefore, very desirable to have a technique which will provide the significant portion of such a solution by direct means, i.e., without having to find the exact solution first and then making approximations. An important set of techniques which perform this service are classified as belonging to the study of asymptotics or asymptotic methods. We will consider one such technique in this thesis. Because of the nature of the approximations obtained, this technique is known as a boundary layer or edge layer method.

The current and world wide interest in such asymptotic problems and procedures is exemplified by most of the papers in the references.

Friedrichs and Dressler (3) are concerned with a systematic and logical derivation of an approximate two-dimensional theory of thin elastic plates from the exact three-dimensional theory for elastic bodies. In particular, they have considered the boundary layers in linear plate theory due to an arbitrary system of edge loads. Among other things their boundary layer treatment derives the classical Kirchhoff boundary conditions for thin plates.

In (4), Goldenveizer presents an over-all view and summary of some asymptotic methods used in the linear theory of thin elastic shells. Broadly speaking, the contents are indicated by a quotation from the paper itself:

"All the questions enumerated ... are considered in this article and their discussion takes a form partly of an account of methods leading to solutions and partly of a description of the results obtained.

Sometimes only a statement of the problem is formulated. The author made it his aim to draw the attention of mathematicians to problems, inadequately dealt with in the literature ..."

Some problems related to those in this thesis are briefly mentioned in (4). Free oscillation and stability problems are among those discussed.

Lagerstrom and Cole (5) illustrate some asymptotic expansion procedures for the non-linear equations of fluid dynamics: specifically, for certain solutions of the Navier-Stokes equations when the Reynolds number is either large or small. As part of their method they employ the technique of first transforming their equations so that they depend upon a new set of independent variables, which are themselves functions of both the Reynolds number and the original coordinates, and then formally taking limits for large or small Reynolds numbers while holding the new variables fixed. This technique is employed in this thesis. Kaplan and Lagerstrom in (6), (7) and (8) use and discuss the basic ideas in (5) for problems involving small Reynolds numbers.

Fife (9), Mahony (10) and several of their references investigate certain asymptotic expansions of the non-linear von Kármán equations for thin elastic plates.

This thesis considers several problems which can be formulated as linear two-dimensional boundary value problems containing a large parameter. With one exception, they are taken from the linear theory of thin elastic plates and shells. A method is given which leads directly to formal asymptotic approximations of the exact solutions of these problems. In two of the problems considered, a comparison is made between certain exact solutions and the approximations obtained by this method. The most important applications are, of course, to problems where the exact solutions are unknown.

There is a class of edge effect problems, particularly in thin elastic shell theory, which possess features not present in the problems which have been treated by boundary layer techniques in this area. The method to be presented here represents a modification and extension of methods previously used, and makes it possible to treat these problems. These "previous" methods and the problems to which they have been applied will be referred to as classical methods and problems if only to distinguish them from those discussed here. The distinction between the classical problems and those considered here will be explained in the following sections.

B. Heuristic Description

A heuristic presentation of the concepts and arguments of the asymptotic method can be given at this stage which will clarify both the subsequent procedures and terminology. The complex structure of the asymptotic solutions has necessitated the introduction of extensive notation. However, as in the classical method, there is considerable dependence upon intuitive reasoning by means of a conceptual model or picture. Because the problems treated here are formulated as two-dimensional boundary value problems, a model is easily visualized. Consider a typical boundary value problem as having been formulated for a <u>rectangular domain</u> in the (x, y) plane and that the solution consists of a single dependent variable $u = u(x, y)$ which can be plotted upon the z-axis. Thus, the solution is a surface plotted above the rectangular domain. For ease of explanation, assume that the boundary conditions are merely requirements as to the height of the surface above the circumference of the rectangle (fig. 1). The edges are numbered and will be designated as 1-edge, 2-edge, etc.

Characteristically, the differential equation is assumed to involve a large parameter λ in its coefficients. Let the coefficients be constant and assume that there is an asymptotic representation of the exact solution as λ tends to infinity. It is also typical of such problems that the coefficients of some of the higher order derivatives in the differential equation become negligible in comparison with the other coefficients as λ becomes large. Nevertheless, the terms involving these relatively negligible coefficients are not, in general, themselves negligible. Intuitively, such terms are important wherever the solution "varies rapidly", i.e., wherever the higher derivatives in question are large enough to offset the smallness of their coefficients. In boundary layer or edge effect problems such higher derivatives become sufficiently large only in regions near the boundary of the domain. Thus, the domain of the problem can be considered in terms of vaguely defined sub-domains. There is a central or inner domain in which certain

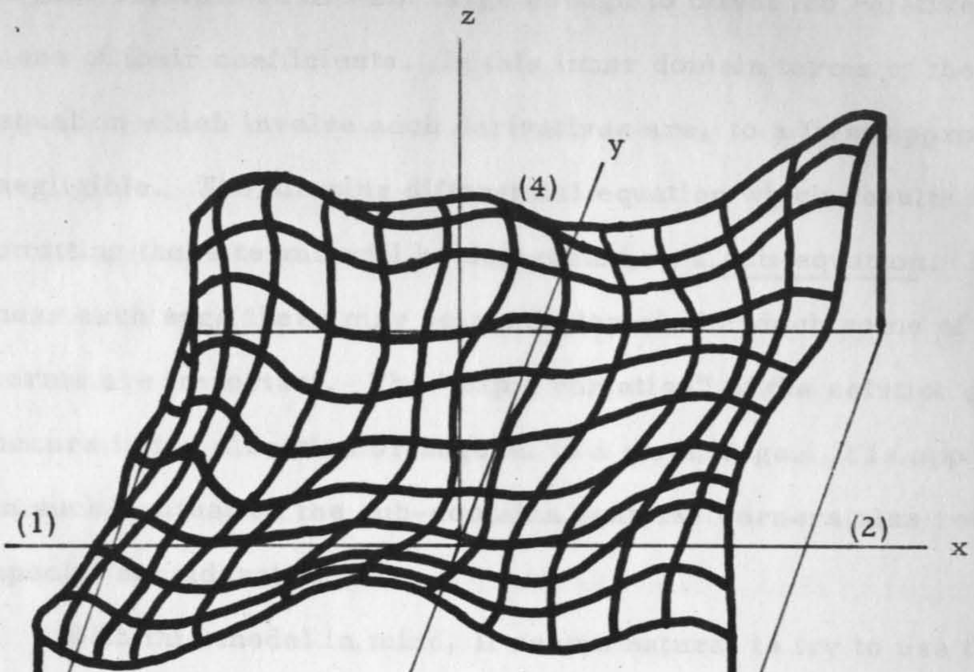


Figure 1. Model for a Boundary Value Problem

higher derivatives are not large enough to offset the relative smallness of their coefficients. In this inner domain terms of the differential equation which involve such derivatives are, to a first approximation, negligible. The simpler differential equation which results from omitting these terms will be designated as a sub-equation. However, near each edge there may be a sub-domain in which some of these terms are important. The "rapid variation" of the solution generally occurs in the direction orthogonal to a given edge. It is apparent that in such a situation the sub-domains near the corners also require special consideration.

With this model in mind, it seems natural to try to use the sub-equation which, to a first approximation, governs the solution in the inner domain. Assuming that appropriate boundary conditions could be found, we could solve a boundary value problem using this simpler differential equation. The approximate solution would be defined everywhere in the (entire) domain but would only have validity in the inner domain. Near the edges or corners it would be expected that corrections would have to be added in order to get a valid first approximation for these sub-domains. These considerations lead one to view the exact solution as a kind of geological formation which consists of layers superimposed or stratified one upon the other.* There is the inner solution which is a valid approximation in the inner or 0-domain.

* This analogy, however useful, has the obvious flaw that functions representing layers may have negative values and so subtract from each other.

In addition, along the l-edge there is a boundary layer* or l-layer that must be added to the inner solution in order to obtain a valid approximation in the l-domain. Such a function diminishes rapidly in magnitude as the distance from the l-edge increases. Similar statements apply concerning the other three edges. Also there are corner layers which must be superimposed to give valid representations in corner domains. These layers, of course, become negligible as the distance from the corner increases. This conceptual model, although stated somewhat differently, is essentially the classical one. It has been utilized in various fields, such as in the boundary layer theory for viscous fluids, and in edge effect problems in elastic shell theory. It applies to the problems of this thesis.

However, the distinction between the classical method and that of this thesis is essentially in the analysis and utility of the layer structure in the corner domains. The <u>classical problems</u> are those problems in which boundary conditions for the inner solution and the boundary layers can be determined <u>without</u> consideration of the corner layers. The classical method recognizes that approximate solutions involving only the boundary layers and the inner solution are not valid in the corners, but for many purposes this is unimportant. For the problems in this thesis (with one exception), the inner solution and boundary layers can

---

* It is convenient to use the term <u>layer</u> to mean a specific function as well as the graph of that function. Discussion of the origin and interpretation of the term boundary layer in fluid dynamics is omitted here. It is related to L. Prandtl's observation of a layer-like region which develops along the surface of a body when it is submerged in a stream of viscous fluid. There is a large velocity gradient in this region. The boundary layer theory in fluid dynamics is due to Prandtl (2).

apparently only be determined by consideration of the corner layers. As might be expected, in general, the boundary layers, corner layers and inner solution are all more or less interdependent. Thus, even though the behavior of the solution in the corner sub-domains may not be of primary physical importance in such problems, it is necessary to consider it in order to determine the appropriate approximations in the remaining portions of the domain. This situation will be illustrated by an example in the following section.

A description of the layers in the corner domains can also be given in a heuristic manner. To do this, let us return to a consideration of the inner solution. It represents the gross behavior of the exact solution. Now to first approximation, it satisfies a differential equation (a sub-equation) which is of lower order in certain derivatives than the exact differential equation. This means that the inner solution is, in general, unable to satisfy all of the boundary conditions placed upon the exact solution. In this sense then, the boundary layers are used to patch or match the inner solution to the boundary conditions at the edges. However, this matching may not be complete. To see this, visualize the sub-domains for the boundary layers as if they were thin rectangular strips along their respective edges.

For example in figure 2, the sub-domain for the 1-layer is depicted as a strip adjacent to the 1-edge. Also, for simplicity assume that there are no boundary layers along the other edges, i.e., assume that the inner solution satisfies the boundary conditions at all edges except the 1-edge. A 1-layer is required to match the inner solution to the 1-edge. However, now the 1-layer may violate the conditions where its sub-domain intersects the 3- and 4-edges. This situation can be remedied by using corner layers to match the 1-layer. These corner sub-domains are depicted in figure 3. The corner layers which match the 1-layer to the 3- and 4-edges are denoted as the 13-layer and the 14-layer, respectively. It now may occur that these corner layers violate the conditions where their sub-domains intersect the 1-edge. Analogously, the corner layer which matches the 13-layer to the 1-edge is denoted as the 131-layer. The 131- and 141-domains have been included in figure 4. It is obvious that this process might continue indefinitely.

In more general cases there are boundary layers along all edges and in a given corner there can be an infinite set of corner layers for each of the boundary layers that meet there. We will refer to the n-layer (n = 1, 2, 3, 4) itself together with its associated corner layers as the n-layers. Note that the 31-layer is the corner layer that matches the 3-layer to the 1-edge and is distinct from the 13-layer. The superposition of three layers is represented in the three-dimensional drawing of figure 5. In the figure, the inner solution and the boundary conditions are at different but constant levels.

C. An Example

a. Formulation of the problem

An adequate quantitative description of the method in general terms is difficult at this point. For clarity, the details will be presented by using a specific problem as an example. Although this problem is much simpler than those to follow, it does display some

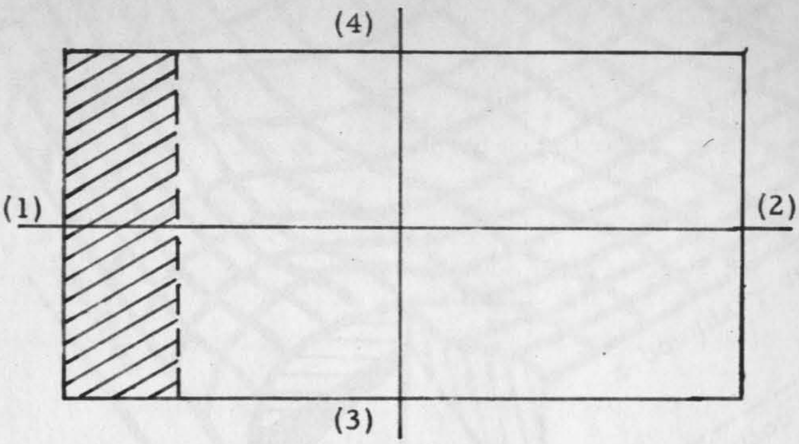


Figure 2. Model Depicting the 1-Domain

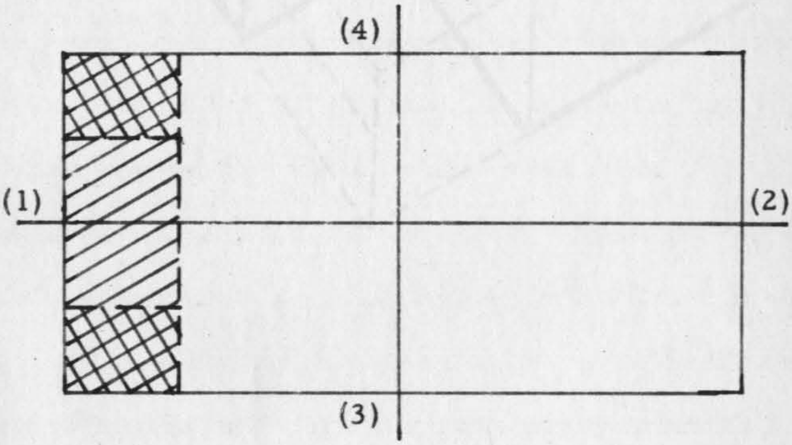


Figure 3. Model Depicting 13- and 14-Domains

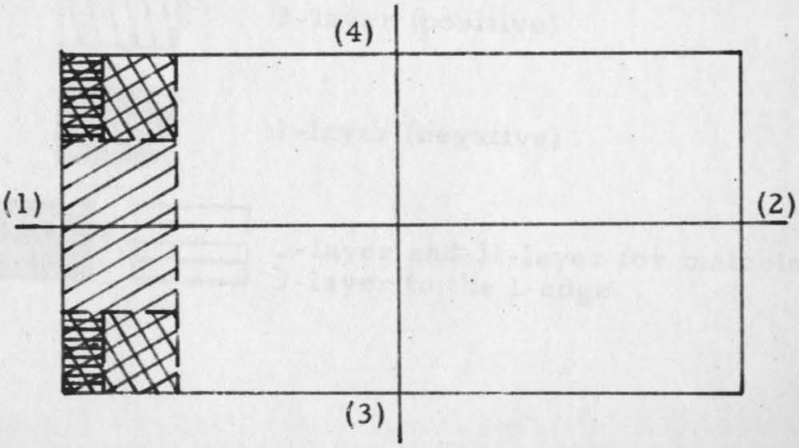


Figure 4. Model Depicting 131- and 141-Domains

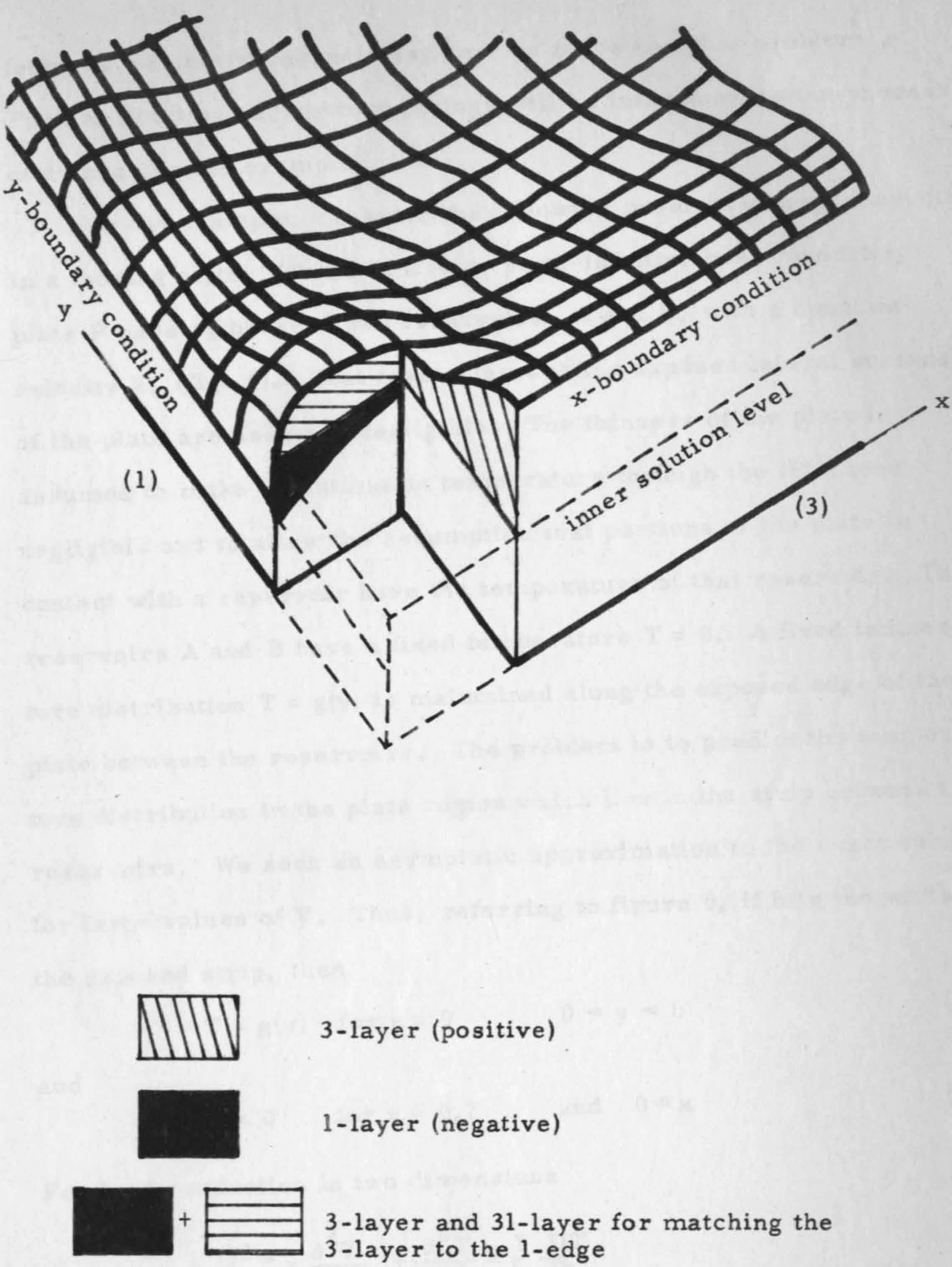


Figure 5. Superposition of Three Layers

features which are characteristic of the more complex problems. Further notation and/or terminology will be introduced in the process of discussing the example.

For the example, consider the following problem of heat conduction in a moving plate. There is a thin, semi-infinite, heat conducting plate P passing between two reservoirs, A and B, with a constant velocity V, (fig. 6). Heat losses through the exposed lateral surfaces of the plate are assumed negligible. The thinness of the plate is assumed to make variations in temperature through the thickness negligible and to allow the assumption that portions of the plate in contact with a reservoir have the temperature of that reservoir. The reservoirs A and B have a fixed temperature T = 0. A fixed temperature distribution T = g(y) is maintained along the exposed edge of the plate between the reservoirs. The problem is to predict the temperature distribution in the plate region which lies in the strip between the reservoirs. We seek an asymptotic approximation to the exact solution for large values of V. Thus, referring to figure 6, if b is the width of the exposed strip, then

$$T = g(y) \quad \text{for } x = 0 \qquad 0 < y < b \tag{1.1}$$

and

$$T = 0 \qquad \text{for } y = 0, 1 \qquad \text{and} \quad 0 < x \tag{1.2}$$

For heat conduction in two dimensions

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{1}{k}\frac{DT}{Dt} \tag{1.3}$$

where k is the thermal diffusivity and DT/Dt is the material or

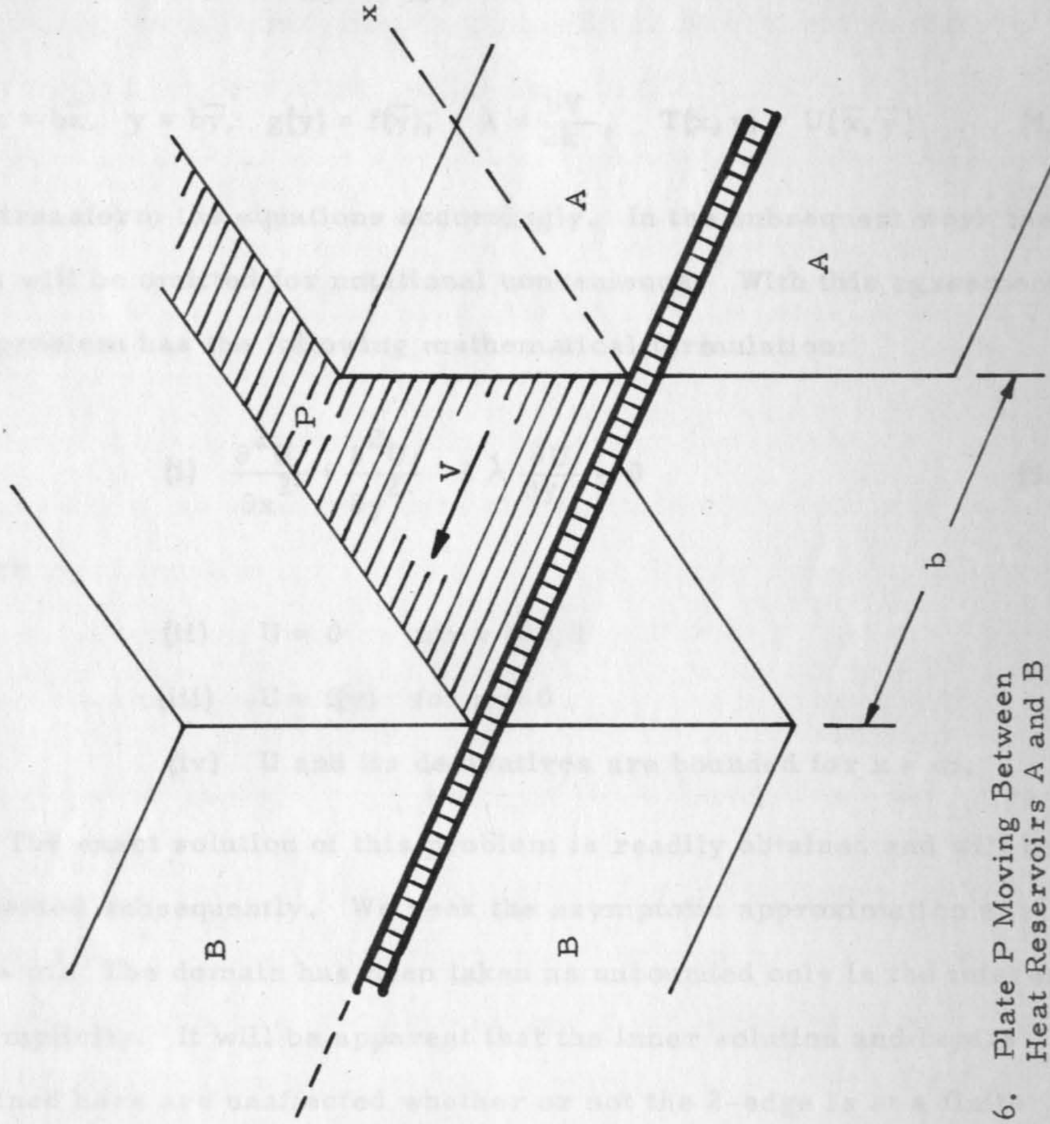


Figure 6. Plate P Moving Between Heat Reservoirs A and B

Eulerian derivative, (11) p. 13. If we assume that the temperature is steady, i.e., $\partial T/\partial t = 0$ but that the medium is moving in the y direction with constant velocity V then

$$\frac{DT}{Dt} = V\frac{\partial T}{\partial y} \tag{1.4}$$

Let

$$x = b\bar{x}, \quad y = b\bar{y}, \quad g(y) = f(\bar{y}), \quad \lambda = \frac{bV}{2k}, \quad T(x, y) = U(\bar{x}, \bar{y}) \tag{1.5}$$

and transform the equations accordingly. In the subsequent work the bars will be omitted for notational convenience. With this agreement the problem has the following mathematical formulation:

$$\text{(i)} \quad \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - 2\lambda\frac{\partial U}{\partial y} = 0 \tag{1.6}$$

where

(ii) $U = 0$ for $y = 0, 1$

(iii) $U = f(y)$ for $x = 0$

(iv) U and its derivatives are bounded for $x = \infty$.

The exact solution of this problem is readily obtained and will be presented subsequently. We seek the asymptotic approximation as $\lambda \to \infty$. The domain has been taken as unbounded only in the interests of simplicity. It will be apparent that the inner solution and layers obtained here are unaffected whether or not the 2-edge is at a finite distance.

Wasow (12) has considered the behavior of the solution of the Dirichlet problem in a bounded domain for the differential equation 1.6 (i) as $\lambda \to \infty$. A more general related problem has been treated by Levinson (13) and will be discussed in greater detail in section f.

Concerning heat conduction in media moving relative to their boundaries, Wilson (14) gives an example where the conducting medium is moving in a direction perpendicular to the boundary which has a given temperature distribution. Other references on moving media with heat conduction can be found in (11).

b. Procedure and criteria

As described in section 2, the asymptotic approximation is conceived as the sum of an inner solution and numerous layers. The layers are associated with the importance of certain terms of the differential equation which involve the higher derivatives. It will be assumed that the relative orders of magnitude of the terms in the differential equation can be displayed explicitly by proper transformations from x and y to new independent variables ξ and η. We will use transformations of the form,

$$\xi = \lambda^{\alpha} (a \pm x) \qquad \text{and} \qquad \eta = \lambda^{\beta} (b \pm y) \tag{1.7}$$

where α, β, a and b are real constants. For our purposes α and β will be positive.* Such a transformation will be designated as a layer transformation and the variables ξ and η as layer variables. When the differential equation is written in terms of layer variables it will be designated as a transformed differential equation.

If either α or β are zero then no transformation is made on the relevant variable, i.e., if $\alpha = 0$ then $\xi = x$ or if $\beta = 0$ then $\eta = y$ and

* Negative values will be discussed subsequently.

such variables are not layer variables. The constants a and b and the signs of x and y are chosen so that ξ and η are positive for points in the domain and zero on the relevant edges which are adjacent to the layer's sub-domain. Geometrically, the transformations are essentially magnifications of the relevant sub-domains in directions orthogonal to the edges.

For any differential equation with constant coefficients, the estimation of the relative orders of magnitude of the terms in the transformed differential equation is independent of the constants a and b. Thus, in the present problem let $\xi = \lambda^{\alpha}x$ and $\eta = \lambda^{\beta}y$. The transformed differential equation can then be written as

$$\lambda^{2\alpha} \frac{\partial^2 U}{\partial \xi^2} + \lambda^{2\beta} \frac{\partial^2 U}{\partial \eta^2} - 2\lambda^{1+\beta} \frac{\partial U}{\partial \eta} = 0 \tag{1.8}$$

If for a fixed α and β all derivatives of U are considered to be of the same order of magnitude in λ, then, to a first approximation for large λ, terms in the transformed differential equation can be deleted or retained - depending upon the values of α and β.* There are seven possible combinations of the three terms in this transformed differential equation. Any differential equation which is formed by one of these combinations will be designated as a <u>sub-equation</u>. The seven sub-equations derivable from equation 1.3 by letting $\lambda \to \infty$ are as follows:

---

* The author has seen a procedure somewhat similar to this one employed by Professor P. A. Lagerstrom in lectures at the California Institute of Technology in connection with an ordinary differential equation.

$$\text{(a)}\quad \frac{\partial U}{\partial \eta} = 0 \qquad \text{if } 1+\beta > 2\alpha,\ 2\beta,\ \text{i.e., } \alpha,\ \beta < 1 \text{ and } 2\alpha < 1+\beta$$

$$\text{(b)}\quad \frac{\partial^2 U}{\partial \eta^2} = 0 \qquad \text{if } 2\beta > 2\alpha,\ 1+\beta,\ \text{i.e., } \alpha,\ 1 < \beta$$

$$\text{(c)}\quad \frac{\partial^2 U}{\partial \xi^2} = 0 \qquad \text{if } 2\alpha > 2\beta,\ 1+\beta,\ \text{i.e., } 1+\beta < 2\alpha \text{ and } \beta < \alpha$$

$$\text{(d)}\quad \frac{\partial^2 U}{\partial \xi^2} - 2\frac{\partial U}{\partial \eta} = 0 \qquad \text{if } 2\alpha = 1+\beta > 2\beta,\ \text{i.e., } 2\alpha = \beta+1 \text{ and } \beta < \alpha < 1 \tag{1.9}$$

$$\text{(e)}\quad \frac{\partial^2 U}{\partial \eta^2} - 2\frac{\partial U}{\partial \eta} = 0 \qquad \text{if } 2\beta = 1+\beta > 2\alpha,\ \text{i.e., } \alpha < \beta = 1$$

$$\text{(f)}\quad \frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} = 0 \qquad \text{if } 2\alpha = 2\beta > 1+\beta,\ \text{i.e., } 1 < \alpha = \beta$$

$$\text{(g)}\quad \frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} - 2\frac{\partial U}{\partial \eta} = 0 \qquad \text{if } \alpha = \beta = 1$$

Each layer in the asymptotic approximation is associated with one or two layer variables and will be considered as depending explicitly upon its layer variable(s). Thus, there is an ordered pair (α, β) associated with each layer. By means of equation 1.9, (α, β) determines a sub-equation to be associated with the layer. The pair (α, β) will be designated as the exponent pair. The inner solution, as a degenerate layer, can be considered as having the exponent pair $(0, 0)$. A boundary layer has an exponent pair of the form $(\alpha, 0)$ or $(0, \beta)$ and is designated, respectively, as a x-layer or y-layer where α and β are greater than zero. All other exponent pairs are associated with corner layers.

It is now seen that there must be some kind of rule by means of which the relevant exponent pairs can be determined. Such a rule is incorporated in a set of criteria which form the operational rules of our technique. These criteria are simply the result of an attempt to codify procedures and considerations that have been successful in a number of problems. The first three criteria are as follows:

(A): Layer

A layer and its derivatives must become, at least, exponentially small as any of its layer variables tend to infinity. This must also be true of each layer component.

(B): Matching

(i) When two layers are matched to a boundary, they must both have the same layer variable as measured tangentially to that boundary.

(ii) Matching progresses from inner solution, to boundary layer, to corner layer, etc. When matching layers, the matching is done progressively for layers having increasing values of $\alpha^2 + \beta^2$.

(iii) Matching continues indefinitely or until a layer is reached which has the entire transformed differential equation as its sub-equation. (In certain cases matching might stop when a sub-equation is reached which includes all terms of the transformed differential equation involving the higher derivatives.)

(C): Uniqueness

The pertinent sub-equations are among those which are associated with unique values of $\alpha \geq 0$ and $\beta \geq 0$: provided that these values are obtained systematically in accordance with criterion B.

There are also sub-equations associated with unique values of $\alpha \leq 0$ and $\beta \leq 0$ which can be obtained in a similar manner. Remarks concerning these will be made subsequently.

Before these criteria can be applied the term component must be defined. Now in asymptotic methods such as this each layer, as well as the inner solution, is a function which is determined by an iterative procedure. For example, the inner solution U^0 is assumed to be of the form;

$$U^0 = U^0(x, y, \lambda) = \sum_{n=0}^{\infty} U_n^0(x, y)\, \lambda^{-\mu_0 n} \tag{1.10}$$

where μ_0 is some constant greater than zero. The superscript 0 is not an exponent but identifies the function as the inner solution. Thus U^1 will be the 1-layer, U^2 the 2-layer, U^{13} the 13-layer, etc. In practice each of the terms of which the series in equation 1.10 is composed is found successively. Each function U_n^0 which appears in these terms will be designated as a component of U^0 or as an 0-component. Each component is found as a solution of a boundary value problem. In other words, the inner solution and each layer are determined by an iterative procedure which entails a sequence of boundary value problems.

Let us use the criteria to determine the pertinent exponent pairs and sub-equations.

First, for boundary layers along the 1- and 2-edges, we take $\beta = 0$. The following implications result from equations 1.9;

(a) $\Rightarrow \quad \alpha < \frac{1}{2}$

(b), (e), (f) and (g) are not applicable

(c) $\Rightarrow \quad \alpha > \frac{1}{2}$

(d) $\Rightarrow \quad \alpha = \frac{1}{2}$

By criterion C, the pertinent sub-equation is given by (d) with $\alpha = \frac{1}{2}$. Now, by criterion B(ii) the corner layers for matching these boundary layers must have $\alpha = \frac{1}{2}$ and $\beta > 0$. Again using criterion C, it follows that the pertinent sub-equation is given by (e) with the exponent pair $(\frac{1}{2}, 1)$. Continuing this procedure, we seek a corner layer with an exponent pair $(\alpha, 1)$ where $\alpha > \frac{1}{2}$. The same criteria determine the sub-equation (g) with $\alpha = \beta = 1$. Since (g) is the entire transformed differential equation, as stated in criterion B(iii), no further matching is necessary.

Second, for boundary layers along the 3- and 4-edges, we take $\alpha = 0$. In this case both (a) and (c) satisfy the uniqueness criterion. However, (c) with $\beta = 1$ gives a positive β.* Thus, sub-equation (c) with $(\alpha, \beta) = (0, 1)$ pertains to these boundary layers. The sub-equation for the matching corner layer is found to be (g) with $(\alpha, \beta) = (1, 1)$ and

---

* As stated before, the use of negative values of α and β will be discussed subsequently.

again the matching process terminates. We see that boundary layers along the 3- and 4-edges have only one set of corner layers. The pertinent variable transformations are now known, as is the general structure of the layers in the corners.

As described in the heuristic discussion, we assume that the exact solution U can be asymptotically approximated for large λ by a sum of functions, i.e.,

$$U \sim U^0 + U^1 + U^2 + U^3 + U^4 + U^{13} + U^{14} + U^{23} + U^{24} + U^{31} + U^{32} + U^{41} + U^{42} + U^{131} + U^{141} + U^{232} + U^{242} \tag{1.11}$$

Layers such as U^{313}, U^{323}, U^{414} and U^{424} are not included since the discussion above implies that U^{31}, U^{32}, U^{41} and U^{42} need no layers to match them to the boundary conditions. In solving this problem the basic technique is used repeatedly. For example, the procedure that shows the 3- and 4-layers to be zero is quite analogous to the procedure for determining the 1-layers. This statement also applies to the 2-layers since we could take a (finite) rectangle and deduce that the 2-layers were also zero — no matter how great the rectangle's length was. Therefore, for simplicity, it is deemed best to assume that only the inner solution and 1-layers represent the asymptotic approximation for large λ. Let

$$U \sim U^0 + U^1 + U^{13} + U^{14} + U^{131} + U^{141} \tag{1.12}$$

where $U^0 = U^0(x, y, \lambda)$. Since the exponent pair for the 1-layer is $(\frac{1}{2}, 0)$ the layer variable is $\xi = \sqrt{\lambda}\, x$. Similarly, the exponent pair $(\frac{1}{2}, 1)$ for the 13- and 14-layers leads to the layer variables $\xi = \sqrt{\lambda}\, x$,

$\eta = \lambda y$ and $\xi = \sqrt{\lambda}\, x$, $\eta = \lambda(1-y)$, respectively. Let

$$\xi_1 = \sqrt{\lambda}\, x, \quad \xi_2 = \lambda x, \quad \eta_1 = \lambda y, \quad \eta_2 = \lambda(1-y) \tag{1.13}$$

then $U \sim V$ where

$$V = U^0(x, y; \lambda) + U^1(\xi_1, y; \lambda) + U^{13}(\xi_1, \eta_1; \lambda) + U^{14}(\xi_1, \eta_2; \lambda)$$

$$+ U^{131}(\xi_2, \eta_1; \lambda) + U^{141}(\xi_2, \eta_2; \lambda).$$

Now criterion A is used in determining the relations which each of the terms of V must satisfy. For example, at a given point interior to the domain $V \sim U^0(s, y; \lambda)$ as $\lambda \to \infty$ since the ξ_n and η_n must tend to ∞ and make the layers exponentially small. This is true for all points of any bounded closed sub-domain strictly interior to the domain. Since this is assumed to also apply to all derivatives of V it follows that U^0 must satisfy equation 1.6. Similarly, if we consider all points such that y is bounded away from 0 and 1, and ξ_1 is bounded away from zero and infinity, then $V \sim U^0(0+, y; \lambda) + U^1(\xi_1 y; \lambda)$ since ξ_2, η_1 and $\eta_2 \to \infty$. Under this assumption, because U^0 already satisfies equation 1.6 then U^1 will satisfy the transformed form of 1.1 in terms of ξ_1 and y. Similar statements apply to the other layers. At a boundary point, e.g., at x = 0, with y bounded away from 0 or 1, we have $V \sim U^0(0, y; \lambda)$ $+ U^1(0, y; \lambda)$, since the layer variables other than ξ_1 tend to infinity. This result simply states our intuitive notion that at points on 1-edge, which are away from the corners, only the inner solution and the 1-layer should be significant. Reasoning such as this will be implicit throughout this thesis and results will generally be derived formally without detailed discussion. Note also that in the subsequent work subscripts will not be used on the layer variables in order to avoid excessive notation. There need be no confusion since the superscripts on a function will indicate how its independent variables are to be interpreted.

c. Differential equations

The inner solution

Since λ appears in equation 1.6 to the first power, it is sufficient to take $\mu_o = 1$ in equation 1.10. Thus, we assume

$$U^0 = \sum_{n=0}^{\infty} U_n{}^0(x, y)\,\lambda^{-n} \tag{1.14}$$

and formally substitute U^0 for U in equation 1.6. Upon equating coefficients of equal powers of λ, we find that the 0-components satisfy the sequence of differential equations given by

$$2\,\frac{\partial U_n{}^0}{\partial y} = \frac{\partial^2 U_{n-1}{}^0}{\partial x^2} + \frac{\partial^2 U_{n-1}{}^0}{\partial y^2} \qquad n = 0, 1, 2, \ldots \tag{1.15}$$

Throughout the sequel functions with negative subscripts will be defined to be zero and n will be a non-negative integer.

The 1-layer

As stated before, this layer's exponent pair is $(\frac{1}{2}, 0)$. Analogous to the form of equation 1.14, we assume that the 1-layer, U^1, can be written in the series form

$$U^1 = U^1(\xi, y, \lambda) = \sum_{n=0}^{\infty} U_n{}^1(\xi, y)\,\lambda^{-n} \tag{1.16}$$

where $\xi = \sqrt{2\lambda}\,x$. The $\sqrt{2}$ has been introduced here merely for convenience. With this transformation, the transformed form of equation 1.3 is:

$$\lambda \frac{\partial^2 U}{\partial \xi^2} + \frac{1}{2} \frac{\partial^2 U}{\partial y^2} - \lambda \frac{\partial U}{\partial y} = 0 \tag{1.17}$$

We now formally substitute U^1 for U in equation 1.17 and by equating coefficients of equal powers of λ, we obtain the sequence of differential equations satisfied by the 1-components. They are

$$\frac{\partial^2 U_n^1}{\partial \xi^2} - \frac{\partial U_n^1}{\partial y} = -\frac{1}{2} \frac{\partial^2 U_{n-1}^1}{\partial y^2} \tag{1.18}$$

The 13-layer

In accordance with criterion B(i), the layer variables for the 13-layer are taken to be

$$\xi = \sqrt{2\lambda}\,x \quad \text{and} \quad \eta = \lambda y \tag{1.19}$$

The transformed differential equation is therefore

$$\frac{\partial^2 U}{\partial \eta^2} - 2 \frac{\partial U}{\partial \eta} + \frac{2}{\lambda} \frac{\partial^2 U}{\partial \xi^2} = 0 \tag{1.20}$$

Analogous to the preceding work, assume that

$$U^{13} = \sum_{n=0}^{\infty} U_n^{13}(\xi, \eta)\lambda^{-n} \tag{1.21}$$

and formally substitute U^{13} for U in equation 1.20. It follows that

$$\frac{\partial^2 U_n^{13}}{\partial \eta^2} - 2 \frac{\partial U_n^{13}}{\partial \eta} = -2 \frac{\partial^2 U_{n-1}^{12}}{\partial \xi^2} \tag{1.22}$$

The 14-layer

For this layer the layer variables will be

$$\xi = \sqrt{2\lambda}\, x \quad \text{and} \quad \eta = \lambda(1-y) \tag{1.23}$$

and the assumed form for U^{14} is

$$U^{14} = \sum_{n=0}^{\infty} U_n^{14}(\xi, \eta)\lambda^{-n} \tag{1.24}$$

It is obvious that the equations analogous to equation 1.20 and equation 1.22 are found by replacing η by $-\eta$. Thus,

$$\frac{\partial^2 U_n^{14}}{\partial \eta^2} + 2\frac{\partial U_n^{14}}{\partial \eta} = -2\frac{\partial^2 U_{n-1}^{14}}{\partial \xi^2} \tag{1.25}$$

As stated before, there need be no confusion due to the use of η as the second independent variable for both U^{13} and U^{14}. It will be assumed throughout that the superscript on a function indicates how its independent variables are to be interpreted.

We now seek the appropriate boundary conditions for the differential equations derived here.

d. Boundary conditions

At points with y bounded away from 0 and 1, the boundary condition for x = 0 becomes

$$U^0(0, y; \lambda) + U^1(0, y; \lambda) = f(y)$$

or

$$U_0^0(0, y) + U_0^1(0, y) = f(y) \qquad U_n^0(0, y) + U_n^1(0, y) = 0$$

Now U_0^0 can quickly be disposed of because equation 1.15 shows that it is a function of x only. Thus, the boundary conditions at the 3- and 4-edges require it to be zero. It then follows that all $U_n^0 = 0$ by induction. With the inner solution identically zero, the boundary condition for $x = \xi = 0$ becomes

$$U_0^1 = f(y) \qquad U_n^1 = 0 \tag{1.26}$$

In particular then, using equation 1.18, the first component for the 1-layer must satisfy the following equations:

$$\boxed{\frac{\partial^2 U_0^1}{\partial \xi^2} - \frac{\partial U_0^1}{\partial y} = 0,} \qquad U_0^1(0, y) = f(y) \tag{1.27}$$

A short digression is in order here to point out how the present example requires extensions of the procedures used in the classical method. It is clear that equations 1.26 and 1.27 have been obtained without using information concerning the structure or the detailed behavior of the corner layers. Thus, these equations could have been obtained by what has been termed the classical method. However, the boundary condition for U_0^1 given by equation 1.26 together with the requirement of exponential decay for $\xi \rightarrow \infty$ (as required in criterion A) are <u>not sufficient</u> conditions to determine U_0^1. This is due to the presence of the y derivative in equation 1.27, which indicates that another boundary condition is required.

If the original differential equation 1.6 were replaced by

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - 2\lambda U = 0$$

while the boundary conditions remained the same, it is clear that the first component, U_0^1, in the boundary layer at x = 0 would satisfy the differential equation

$$\frac{\partial^2 U_0^1}{\partial \xi^2} - U_0^1 = 0$$

and the boundary condition

$$U_0^1(0, y) = f(y)$$

This boundary condition, together with the requirement of exponential decay as $\xi \to \infty$, uniquely determines this U_0^1 as

$$U_0^1 = f(y)e^{-\xi}$$

In such a case U_0^1 is therefore determined without reference to the corner layers. This is typical of what we will term a classical problem.

Wittrick (15) provides an excellent example of the use of the classical method. He obtains a solution for the boundary layer stress system near a circular edge of a thin elastic shell whose middle surface is a surface of revolution. In doing so, (15) page 250, he arrives at the differential equation

$$\frac{\partial^4 W_o}{\partial \xi^4} + 4W_o = 0 \tag{1.28}$$

for the first component W_o in a boundary layer expansion of the displacement normal to the shell. The stresses in the boundary layer are given in terms of W_o. ξ is the layer variable measured normal to the circular edge along a meridian. $W_o = W_o(\xi, \theta)$ where θ measures the angle of the meridian plane. Equation 1.28 is essentially an ordinary differential equation. As Wittrick indicates, a solution which has exponential decay for increasing ξ is $W_0 = e^{-\xi}\left[F(\theta)\cos\xi + G(\theta)\sin\xi\right]$ where $F(\theta)$ and $G(\theta)$ are arbitrary functions. They are readily determined by conditions at $\xi = 0$. Again then, the classical method is successful where boundary conditions at $\xi = 0$, together with the requirement of exponential decay, are sufficient to determine the boundary layer.

In the present example, the required additional boundary condition is obtained by considering the matching corner layers. At y = 1 the boundary layer and corner layer are superimposed so that formal substitution of $U^1 + U^{14}$ into the condition U = 0 gives

$$U_n^1(\xi, 1) + U_n^{14}(\xi, 0) = 0 \tag{1.29}$$

Similarly, at y = 0 we obtain

$$U_n^1(\xi, 0) + U_n^{13}(\xi, 0) = 0 \tag{1.30}$$

Equation 1.22 gives

$$\frac{\partial^2 U_0^{13}}{\partial \eta^2} - 2\frac{\partial U_0^{13}}{\partial \eta} = 0$$

which is readily integrated. Its solution is

$$U_0^{13} = f_0(\xi)\, e^{2\eta} + f_1(\xi)$$

Equation 1.25 gives

$$\frac{\partial^2 U_0^{14}}{\partial \eta^2} + 2\frac{\partial U_0^{14}}{\partial \eta} = 0$$

or

$$U_0^{14} = g_0(\xi)e^{-2\eta} + g_1(\xi)$$

However, in accordance with criterion A, for these to be corner layers they must decay exponentially in η. Thus, we must have $f_0=f_1=g_1=0$. This means, by obvious induction on n, that there is no 13-layer to match the 1-layer to the 3-edge, but that

$$U_0^{14} = -U_0^1(\xi, 1)e^{-2\eta} \tag{1.31}$$

matches U_0^1 to the 4-edge whatever the value of $U_0^1(\xi, 1)$. It follows that we must have

$$U_n^1(\xi, 0) = 0 \tag{1.32}$$

This is the necessary additional boundary condition for U_0^1.

e. First components

In summary then the boundary value problem for U_0^1 is

$$\frac{\partial^2 U_0^1}{\partial \xi^2} - \frac{\partial U_0^1}{\partial y} = 0$$

$$U_0^1(0, y) = f(y) \tag{1.33}$$

$$U_0^1(\xi, y) \rightarrow 0 \quad \text{as} \quad \xi \rightarrow \infty$$

The boundary value problem is described in the ξ, y plane in figure 7.

If the differential equation 1.33 were required to hold in the entire quadrant $\xi > 0$, $y > 0$, instead of in the strip $0 < y < 1$, $\xi > 0$, and if the boundary datum f(y) were given along the entire half-line $y > 0$, then the resulting boundary value problem for U_0^1 would be a standard problem of transient heat conduction. With the variable y representing

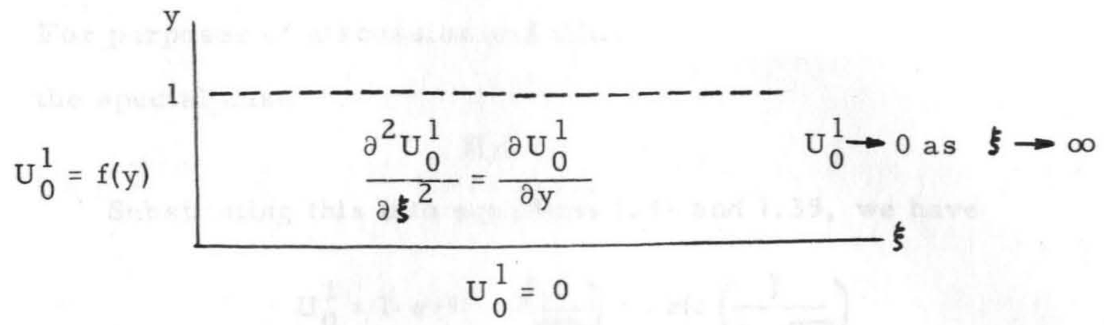


Figure 7. Schematic of Boundary Value Problem

the time, the problem would be the determination of the temperature U_0^1 in a semi-infinite rod subject to a given time-dependent end temperature $U_0^1 = f(y)$ and a zero initial temperature. The solution to this heat conduction problem is known to be*

$$U_0^1 = \frac{\xi}{\sqrt{4\pi}} \int_0^y \frac{f(p) e^{-\frac{\xi^2}{4(y-p)}}}{(y-p)^{3/2}} \, dp = \sqrt{\frac{2}{\pi}} \int_{\frac{\xi}{\sqrt{2y}}}^{\infty} f\left(y - \frac{\xi^2}{2p^2}\right) e^{-p^2/2} \, dp \tag{1.34}$$

Since it is clear that the values of this solution, for $0 < y < 1$, depend only on the values of f(y) for $0 < y < 1$, we expect that our present problem (1.33) is well posed and that its solution is given by 1.34. Thus, from equation 1.31, the first corner layer component is found to be

$$U_0^{14} = \frac{-\xi e^{-2\eta}}{\sqrt{4\pi}} \int_0^1 \frac{f(p)}{(1-p)^{3/2}} e^{-\frac{\xi^2}{4(1-p)}} \, dp = -\sqrt{\frac{2}{\pi}} e^{-2\eta} \int_{\frac{\xi}{\sqrt{2}}}^{\infty} f\left(1 - \frac{\xi^2}{2p}\right) e^{-p^2/2} dp \tag{1.35}$$

---

* Further assumptions about U_0^1 are necessary to insure uniqueness. One such condition is that U_0^1 be bounded uniformly in ξ as y tends to zero. This would exclude functions such as the doublet $(\xi/y^{3/2}) \exp(-\xi^2/4y)$, (11) p. 35.

For purposes of discussion and illustration, we restrict attention to the special case

$$f(y) = 1 \tag{1.36}$$

Substituting this into equations 1.34 and 1.35, we have

$$U_0^1 = 1-\mathrm{erf}\left(\frac{\xi}{2\sqrt{y}}\right) = \mathrm{erfc}\left(\frac{\xi}{2\sqrt{y}}\right) \tag{1.37}$$

and

$$U_0^{14} = -e^{-2\eta}\ \mathrm{erfc}\left(\frac{\xi}{2}\right) \tag{1.38}$$

For large values of $\xi = \sqrt{2\lambda}\ x$, there is the asymptotic expansion

$$U_0^1 \sim \frac{2\sqrt{y}}{\xi\sqrt{\pi}}\, e^{-\frac{\xi^2}{4y}} \left\{1 - \frac{1}{2\left[\frac{\xi}{2\sqrt{y}}\right]^2}\right\} + \ldots \tag{1.39}$$

This shows a decay rate for increasing ξ which is greater than the exponential decay that we have required.

f. Results and discussion

In terms of the original variables, the temperature distribution in the plate region between the reservoirs has the asymptotic approximation

$$T \sim 1 - \mathrm{erf}\left[\left(\frac{\xi}{2}\right)\sqrt{\frac{b}{y}}\right] + O\left(\frac{1}{\lambda}\right) \tag{1.40}$$

provided that corners of the region are excluded and that $\lambda = bV/2k$ is large. Here $\xi = \sqrt{V/kb}\ x$ and $T = 1$ for $x = 0$. This is an approximation in which terms sufficiently small in comparison with unity have been neglected.

Let us consider the possibility of plotting isothermal contours. We note that for small temperatures the approximation becomes of the same order as the neglected terms. Thus, the shape of the contours can only be accurately estimated where the temperature differs significantly from zero. This means that such contours are inside the boundary layer sub-domain, i.e., the 1-domain. However, it is physically clear that all contours meet in the corners. To see this consider the exact solution of the heat conduction problem for V = 0; it can be shown that

$$T = 1 - \frac{2}{\pi} \arctan \frac{\sinh \frac{\pi x}{b}}{\sin \frac{\pi y}{b}} \tag{1.41}$$

Equation 1.41 gives isothermal contours which are symmetric about the line $y = b/2$ and concave toward the 1-edge. These contours are sketched in figure 8. Intuitively, it is expected that for $V > 0$ these curves would shift in the positive y direction. In accordance with the above discussion, we see that 1.40 implies that the contours have the shape of parabolas for the larger values of temperature when λ is large. These contours are sketched as solid lines in figure 9. However, since all contours meet in the origin, there will be contours there that represent low temperatures for which 1.40 is inadequate. In this sense then, we may consider the boundary layer domain to be bounded by a parabola, i.e., to have a semi-parabolic shape rather than that of a rectangle as was employed in the heuristic discussion. Of course, the shape of the boundary layer domains can be expected to vary with the type of problem considered.

The differential equation 1.6(i) is a special case of the equation treated by Wasow (12), i.e., $\Delta U + \lambda \frac{\partial U}{\partial x} = \lambda f(x, y)$. Wasow considers

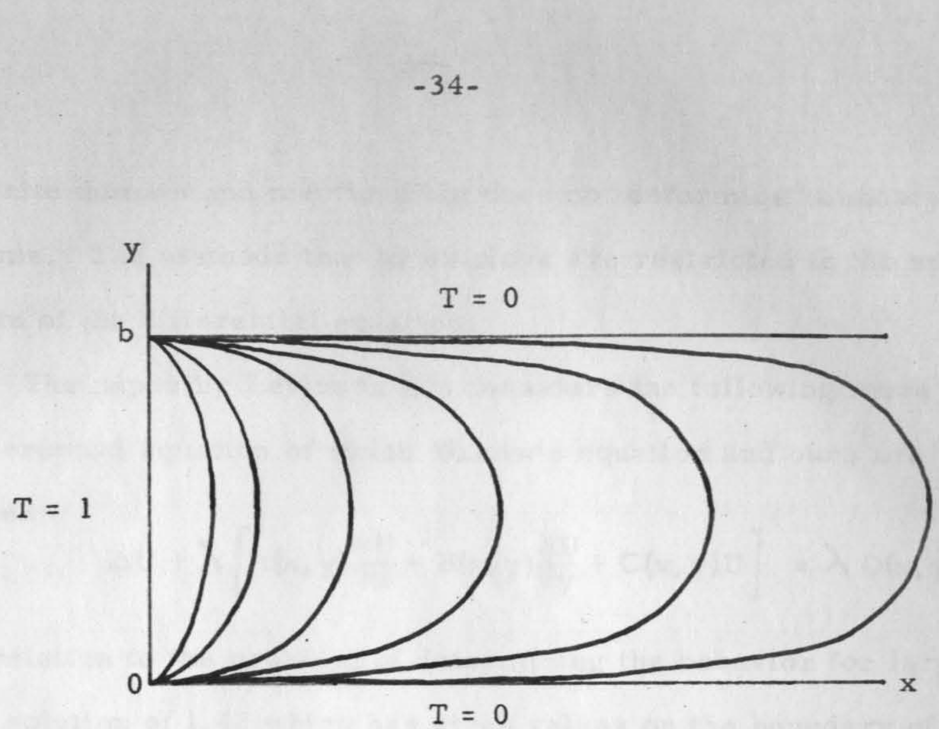


Figure 8. Isothermal Contours for $\lambda = 0$ (The Static Case)

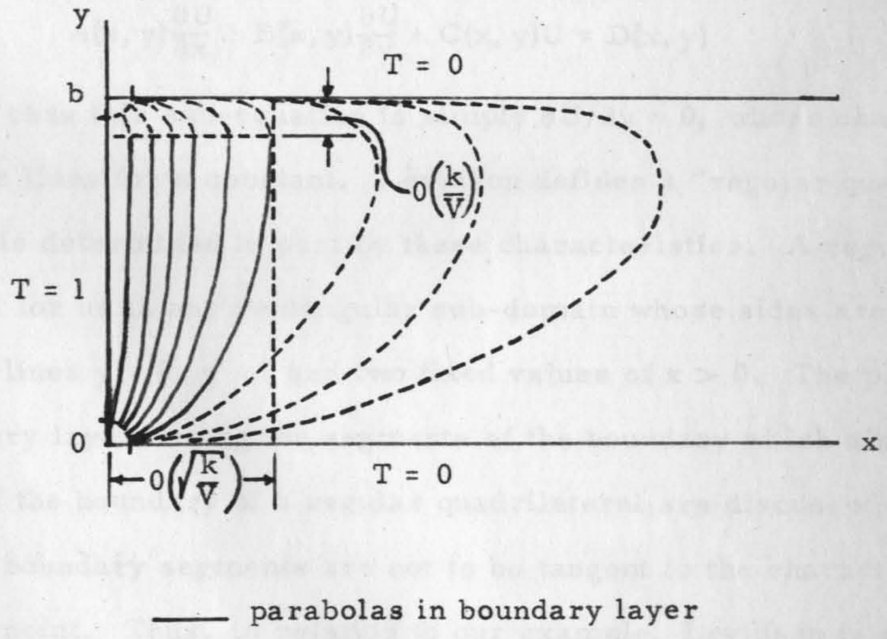


Figure 9. Isothermal Contours for Large λ

a finite domain and mentions but does not determine boundary layer terms. The methods that he employs are restricted to the special form of the differential equation.

The paper by Levinson (13) considers the following more general differential equation of which Wasow's equation and ours are special cases;

$$\Delta U + \lambda\left[A(x,y)\frac{\partial U}{\partial x} + B(x,y)\frac{\partial U}{\partial y} + C(x,y)U\right] = \lambda D(x,y) \qquad (1.42)$$

In relation to the problem of determining the behavior for large λ of the solution of 1.42 which has given values on the boundary of a domain, Levinson shows that an important role is played by the characteristics of the sub-equation

$$A(x,y)\frac{\partial U}{\partial x} + B(x,y)\frac{\partial U}{\partial y} + C(x,y)U = D(x,y)$$

In our case this sub-equation is simply $\partial U/\partial y = 0$, whose characteristics are the lines for x constant. Levinson defines a "regular quadrilateral" which is determined in part by these characteristics. A regular quadrilateral for us is any rectangular sub-domain whose sides are determined by the lines $y = 0$, $y = 1$ and two fixed values of $x > 0$. The presence of boundary layers along the segments of the boundary which also form part of the boundary of a regular quadrilateral are discussed in (13). These boundary segments are not to be tangent to the characteristics at any point. Thus, in relation to our example, Levinson is concerned with boundary layers on the 3- and 4-edges. His very general results, when applied to our case, predict the possible presence of a boundary layer on the 4-edge and none on the 3-edge (or vice versa, depending upon the sign of λ). Furthermore, the functional form of what we call the first boundary layer component is given for the 4-edge. Because of our special boundary conditions the 4-layer is zero. A point of significance for this thesis is that both Wasow and Levinson show that what we call the inner solution can in general only be made to satisfy boundary conditions over part of the boundary and that this part depends upon the sign of λ. By reasoning similar to that used here for the 1-layer, we can arrive at this conclusion. Since the exponent pair for the 3- and 4-layers is (0, 1), the pertinent sub-equation is 1.9(e) which can be written

$$\frac{\partial^2 U^q}{\partial \eta^2} + (-1)^q \frac{\partial U^q}{\partial \eta} = 0 \tag{1.43}$$

where $\eta = \lambda y$ or $\lambda(1-y)$ if $q = 3$ or 4, respectively. Integration of this equation, as was done in connection with equation 1.31, shows that there can be no 3-layer since 1.43 with $q = 3$ does not yield any exponentially decreasing functions for increasing η. It is obvious that a change of sign for λ in 1.6(i) would yield the reverse situation: there would be no 4-layer. We thus conclude that the inner solution must satisfy the boundary conditions where there is no boundary layer and, of course, in general it would then not satisfy the boundary conditions where a boundary layer can exist since the inner solution satisfies a differential equation of too low an order.

It should be noted that both papers have sought to study asymptotic approximations in a rigorous manner, e.g., Levinson's results can be used to prove that the first component of the inner solution is zero in our example. However, although both authors consider domains of very general shape, the 1-edge of our problem represents a portion of the boundary which is not discussed in their papers in relation to a boundary layer. The 1-edge is a boundary segment which coincides with one of the characteristics mentioned above.

g. The exact solution

The problem as formulated in equations 1.6 is readily solved by use of the sine transform. Define

$$\overline{U} = \int_0^\infty U(x,y) \sin \omega x \, dx \tag{1.44}$$

so that equation 1.6(i) can be transformed into the form

$$\frac{d^2\overline{U}}{dy^2} - 2\lambda \frac{d\overline{U}}{dy} - \omega^2 \overline{U} = -\omega f(y) \tag{1.45}$$

wherein equations 1.6(iii) and 1.6(iv) have been utilized. Using equation 1.6(ii), it follows that

$$\overline{U} = \frac{\omega e^{\lambda y}}{\rho \sinh \rho} \left\{ \sinh \rho(1-y) \int_0^y e^{-\lambda\tau} f(\tau) \sinh \rho\tau \, d\tau + \sinh \rho y \int_y^1 e^{-\lambda\tau} f(\tau) \sinh \rho(1-\tau) d\tau \right\} \tag{1.46}$$

where $\rho = (\lambda^2 + \omega^2)^{1/2}$. If we now assume that equation 1.36 holds, i.e., $f(y) = 1$, then it can be shown that (16),

$$U = 1 - \frac{2}{\pi} e^{\lambda y} \int_0^\infty \frac{\sinh \rho(1-y)}{\sinh \rho} \frac{\sin \omega x}{\omega} dw - \frac{2}{\pi} e^{-\lambda(1-y)} \int_0^\infty \frac{\sinh \rho y}{\sinh \rho} \frac{\sin \omega x}{\omega} dw \tag{1.47}$$

In order to make a comparison with the results in section (e), we must obtain the asymptotic approximation to equation 1.47. The integrals involved are not directly amendable to the usual asymptotic methods for approximation, such as the methods of steepest decent, stationary phase or Laplace. To obtain the boundary layer components, we set $\xi = \sqrt{2\lambda}\, x$ and formally obtain the asymptotic approximation by taking λ to be large even in comparison with ω. Thus, for example, we take

$$\rho = \lambda + \frac{1}{2}\frac{\omega^2}{\lambda} - \frac{1}{8}\frac{\omega^4}{\lambda^3} + \cdots$$

By making such approximations, we may write

$$U \sim 1 - \frac{2}{\pi}\int_0^\infty \frac{\sin\left(\frac{\omega\xi}{\sqrt{2\lambda}}\right)}{\omega}\, d\omega \Bigg\{ e^{-\frac{y\omega^2}{2\lambda}}\left[1 + \frac{y\omega^4}{8\lambda^3} - \frac{y}{16}\frac{\omega^6}{\lambda^5} + O\left(\frac{\omega^8}{\lambda^6}\right)\right]$$

$$-e^{-\lambda(1-y)}\, e^{-\frac{(2-y)\omega^2}{2\lambda}}\left[1 + O\left(\frac{\omega^4}{\lambda^3}\right)\right] + e^{-2\lambda(1-y)}\, e^{-\frac{(1-y)\omega^2}{2\lambda}}\left[1 + O\left(\frac{\omega^4}{\lambda^3}\right)\right]$$

$$-e^{-2\lambda}\, e^{-\frac{(1+y)\omega^2}{2\lambda}}\left[1 + O\left(\frac{\omega^4}{\lambda^3}\right)\right] + O\left(e^{-(1+y)\lambda}\right) + O\left(e^{-2\lambda(2-y)}\right)\Bigg\} \qquad (1.48)$$

It follows that all terms in equation 1.48 below the first line are negligible for y bounded away from one. Further evaluation can be made if we use the following relations (16) p. 74,

$$\frac{2}{\pi}\int_0^\infty e^{-\frac{y\omega^2}{2\lambda}}\, \frac{\sin\omega x}{\omega}\, dw = \text{erf}\left(\sqrt{\frac{\lambda}{2y}}\, x\right) \qquad (1.49)$$

$$\frac{2}{\pi}\int_0^\infty e^{-\frac{y\omega^2}{2\lambda}}\, \omega^3 \sin\omega x\, dw = -\sqrt{\frac{2}{\pi}}\, e^{-\frac{\lambda x^2}{2y}}\, \lambda^2\left[\frac{(\sqrt{\lambda}\, x)^3}{y^{7/2}} - \frac{3\sqrt{\lambda}\, x}{y^{5/2}}\right] \qquad (1.50)$$

Appropriate substitutions then give the result,

$$U \sim 1-\mathrm{erf}\left(\frac{\xi}{2\sqrt{y}}\right)+\frac{1}{\lambda}\left\{\frac{1}{8\sqrt{\pi}}e^{-\frac{\xi^2}{4y}}\left[\frac{\xi^3}{2y^{5/2}}-\frac{3\xi}{y^{3/2}}\right]\right\}+O\left(\frac{1}{\lambda^2}e^{-\frac{\xi^2}{4y}}\right) \tag{1.51}$$

provided that y is bounded away from one and $\xi^2 + y^2$ is bounded away from zero. The first term is U_0^1 as given in equation 1.37. The second term is also readily obtained by our asymptotic method, but again as stated in connection with equation 1.34 further assumptions are necessary to insure uniqueness.

This process can also be repeated using the two transformations $\xi = \sqrt{2\lambda}\, x$ and $\eta = \lambda(1-y)$ so as to confirm the corner layer component in equation 1.35.

D. Commentary

a. An additional criterion

It seems advisable to add one more criterion to the three already given in part C, section b:

(D): Consistency

All boundary conditions and differential equations involving a layer component must be mutually consistent.

This is to mean, for example, that if certain derivatives of a component satisfy given conditions at a boundary, then these conditions must not violate the relations between these same derivatives that are implied by the differential equations that govern the component. Another example occurs where relations can be obtained by requiring a set of boundary conditions to be mutually consistent.

This criterion seems obvious as stated especially if we assume all functions to be sufficiently differentiable and the pertinent series to have proper rates of convergence. However, it has been found to be of such practical importance that it deserves explicit statement.

b. Negative exponents

In relation to the criteria and the sub-equations, the possibility of exponent pairs, (α, β), where α and/or β may be negative has been pointed out but not discussed. The criteria indicate how these exponent pairs are to be deduced. From the nature of the layer variables corresponding to the exponent pairs, it is apparent that any functions of them must in some sense be considered as wide or super layers. Johnson and Reissner (17) obtained one such wide boundary layer. The first problem in chapter II is a more general treatment of the problem that they considered. At this stage of investigation it must be stated as a conjecture that the exponent pairs with negative α and β (which are analogous to those for corner layers) are important in certain problems and can be treated in a systematic manner. Since the method presented here, using positive α and β, is apparently adequate for certain problems with bounded domains, it is expected that the use of negative α and β is necessary for treating corresponding problems with infinite domains. Thus, we may predict the existence of wide of super corner layers for such problems wherever our method indicates exponent pairs with both α and β negative. An example of a possible domain for this would be a quadrant.

II. TWO APPLICATIONS IN THE THEORY OF ELASTIC PLATES

A. A Stretched Plate with a Transverse Load

a. Introduction

Consider the problem of the deflection of a thin plate whose middle surface has the shape of a semi-infinite strip. It is subjected to a uniform tension N_x parallel to its infinite edges and a transverse load that varies across its width but which is independent of the lengthwise position.* The two infinite edges are simply supported and the finite edge is clamped (fig. 10). The plate is of uniform thickness h and width 2b. It consists of elastic, isotropic material. This problem, for a special sinusoidal transverse load, has been treated by Johnson and Reissner (17). In their paper they demonstrate the existence of a wide- or super-layer in the asymptotic approximation. This layer is so named because it has a sub-domain with a width which is large in comparison with the finite edge of the plate. The usual boundary layer is one whose sub-domain has a width that is small compared with the characteristic dimensions of the plate. The importance of this problem is that it affords an example of the use of an exponent pair (α, β) where, say, α is negative. Thus, the layer transformation for this super layer is a condensation or contraction of the domain in the direction orthogonal to the finite edge, rather than a magnification as is the case for a boundary layer. Because of the large scale of this super layer and the believed applicability of the method as presented in

---

* N_x is a stress resultant in as much as it is the integral of the corresponding stress taken over the plate thickness.

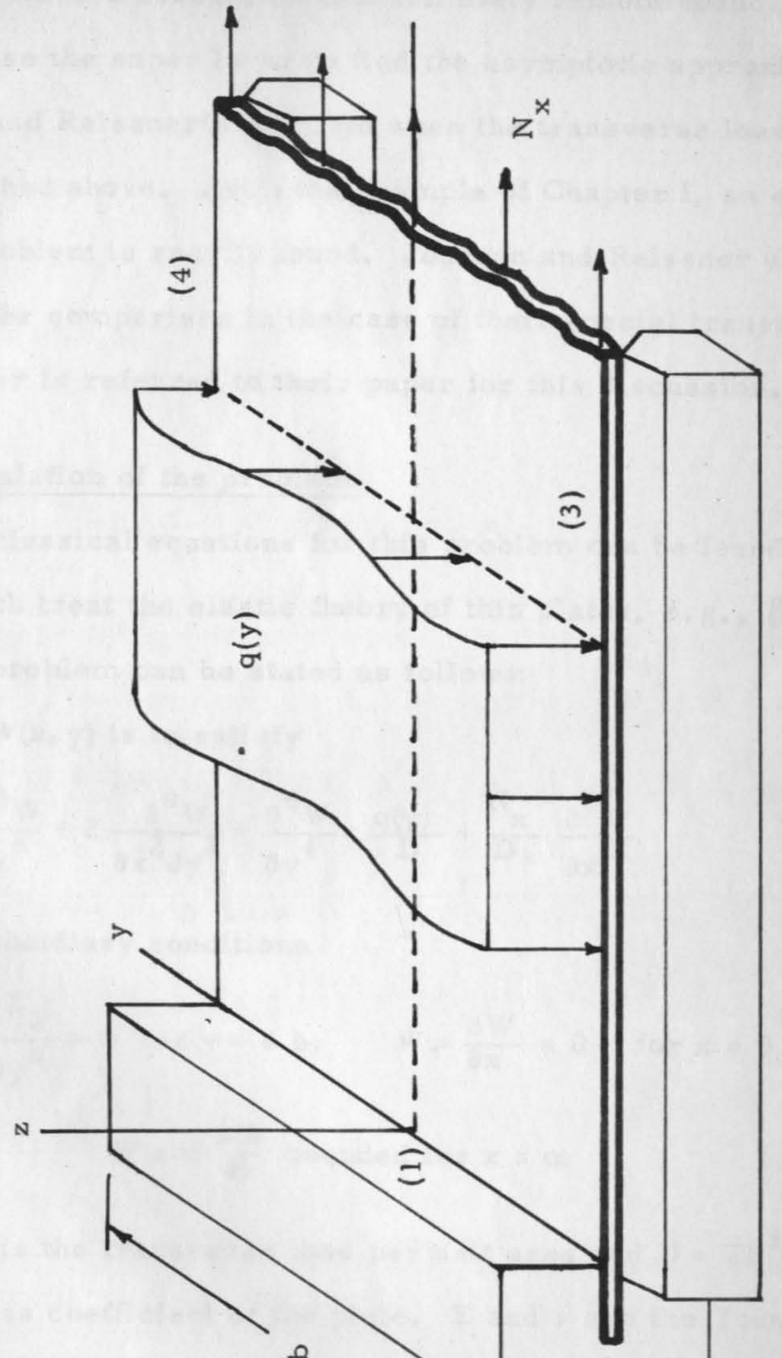


Figure 10. A Stretched Plate with a Transverse Load

Chapter I for finite rectangles, it is expected that such super layers are only useful in treating problems with unbounded domains. Thus, super layers are associated with infinitely remote boundaries or edges. We will use the super layer to find the asymptotic approximation for Johnson and Reissner's problem when the transverse load is arbitrary as described above. As in the example of Chapter I, an exact solution of this problem is readily found. Johnson and Reissner use an exact solution for comparison in the case of their special transverse load. The reader is referred to their paper for this discussion.

b. Formulation of the problem

The classical equations for this problem can be found in standard texts which treat the elastic theory of thin plates, e.g., (18) p. 299 et al.

The problem can be stated as follows:

$W = W(x, y)$ is to satisfy

$$\frac{\partial^4 W}{\partial x^4} + 2\frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} = \frac{q(y)}{D} + \frac{N_x}{D}\frac{\partial^2 W}{\partial x^2} \tag{2.1}$$

and the subsidiary conditions

$$W = \frac{\partial^2 W}{\partial y^2} = 0 \quad \text{for } y = \pm b; \qquad W = \frac{\partial W}{\partial x} = 0 \quad \text{for } x = 0 \tag{2.2}$$

$$W \text{ and } \frac{\partial W}{\partial x} \text{ bounded for } x = \infty$$

Here $q(y)$ is the transverse load per unit area and $D = Eh^3/12(1-\nu^2)$ is the stiffness coefficient of the plate. E and ν are the Young's modulus and Poisson's ratio, respectively, for the material of the plate. The differential equation can be put into a homogeneous, non-dimensional form by the following substitutions;

Let

$$x = b\bar{x}, \quad y = b\bar{y}, \quad \lambda^2 = \frac{b^2 N_x}{D}, \quad W = U(\bar{x}, \bar{y}) - f(\bar{y}) \tag{2.3}$$

where

$$f(\bar{y}) = Q(\bar{y}) - \frac{Q''(1)}{12}(\bar{y}^2-1)(\bar{y}+3) - \frac{Q(1)}{2}(\bar{y}+1)$$

and

$$Q(\bar{y}) = -\frac{b^4}{6D}\int_{-1}^{\bar{y}} (\bar{y}-t)^3 \, q(bt)dt; \quad \frac{d^2Q}{d\bar{y}^2} = Q''$$

so that

$$f(\bar{y}) = \frac{d^2 f(\bar{y})}{d\bar{y}^2} = 0 \text{ for } y = \pm 1 \tag{2.4}$$

We assume in the sequel, that these substitutions have been made. For convenience, the bar notation will be omitted. We now have;

$$\frac{\partial^4 U}{\partial x^4} + 2\frac{\partial^4 U}{\partial x^2 \partial y^2} + \frac{\partial^4 U}{\partial y^4} - \lambda^2 \frac{\partial^2 U}{\partial x^2} = 0 \tag{2.5}$$

with the subsidiary conditions

$$U = \frac{\partial^2 U}{\partial y^2} = 0 \text{ for } y = \pm 1; \quad U = f(y) \text{ and } \frac{\partial U}{\partial x} = 0 \text{ for } x = 0 \tag{2.6}$$

c. Structure of the layers

We seek an asymptotic approximation for large λ. As discussed in Chapter I, the transformed differential equation is obtained by letting $\xi = \lambda^a x$ and $\eta = \lambda^\beta y$. It is

$$\lambda^{4a}\frac{\partial^4 U}{\partial \xi^4} + 2\lambda^{2(a+\beta)}\frac{\partial^4 U}{\partial \xi^2 \partial \eta^2} + \lambda^{4\beta}\frac{\partial^4 U}{\partial \eta^4} - \lambda^{2(a+1)}\frac{\partial^2 U}{\partial \xi^2} = 0 \tag{2.7}$$

Again, proceeding as in Chapter I, a sub-equation analysis gives,

$$\text{(a)}\quad \frac{\partial^2 U}{\partial \xi^2} = 0 \qquad \text{if } \alpha,\ \beta,\ 2\beta-\alpha < 1$$

$$\text{(b)}\quad \frac{\partial^4 U}{\partial \xi^4} = 0 \qquad \text{if } 1,\ \beta < \alpha$$

$$\text{(c)}\quad \frac{\partial^4 U}{\partial \eta^4} = 0 \qquad \text{if } \alpha,\ \frac{1+\alpha}{2} < \beta$$

$$\text{(d)}\quad \frac{\partial^4 U}{\partial \xi^4} - \frac{\partial^2 U}{\partial \xi^2} = 0 \qquad \text{if } \beta < \alpha = 1 \tag{2.8}$$

$$\text{(e)}\quad \frac{\partial^4 U}{\partial \eta^4} - \frac{\partial^2 U}{\partial \xi^2} = 0 \qquad \text{if } \alpha < \beta = \frac{1+\alpha}{2}$$

$$\text{(f)}\quad \frac{\partial^4 U}{\partial \xi^4} + 2\frac{\partial^4 U}{\partial \xi^2 \partial \eta^2} + \frac{\partial^4 U}{\partial \eta^4} = 0 \qquad \text{if } 1 < \alpha = \beta$$

$$\text{(g)}\quad \frac{\partial^4 U}{\partial \xi^4} + 2\frac{\partial^4 U}{\partial \xi^2 \partial \eta^2} + \frac{\partial^4 U}{\partial \eta^4} - \frac{\partial^2 U}{\partial \xi^2} = 0 \qquad \text{if } 1 = \alpha = \beta$$

All other sub-equations are impossible.

For the x-layers, assume $\beta = 0$. Criterion C then indicates that the appropriate sub-equation is 2.8(d) with $\alpha=1$. However, there is also 2.8(e) with $\alpha=-1$. This latter sub-equation must describe the super layer. It is seen that there are no other $\beta \neq 0$ which are uniquely determined for $\alpha = -1$. This indicates that there are no layers to match the super layer to the 3- and 4-edges. Such layers would obviously be unnecessary since 2.8(e) is of fourth order in the η derivative. Continuing the discussion for the x-layers, we seek a sub-equation having an exponent pair $(1, \beta)$ with $\beta > 0$. Again by criterion C, this exponent pair is (1, 1) which corresponds to equation 2.8(g). Since this sub-equation represents the entire equation, no further matching is necessary.

For the y-layers, assume $\alpha = 0$. Criterion C indicates that $\beta = \frac{1}{2}$, which corresponds to equation 2.8(e). Accordingly, the next exponent pair is $(1, \frac{1}{2})$ corresponding to equation 2.8(d). This procedure continues to one more sub-equation; the entire transformed differential equation 2.8(g) with $(\alpha, \beta) = (1, 1)$.

We see that had the plate had its infinite dimension parallel to the y-axis that there would have been no super layer. The super layer obtained by Johnson and Reissner is the only one connected with this class of problems. This super layer can be considered to replace the inner solution since it intuitively "lies over" the inner domain. The order of the differential equation 2.8(e) in relation to the ξ derivative indicates that the super layer will require matching to the l-edge just as the inner solution (equation 2.8(a)) would for a finite plate. Thus, the asymptotic approximation is considered to consist of the super layer and the l-layers. We will use the notation U^{-1} to denote the super layer. It is important to note here that the sub-equation 2.8(d) is essentially an ordinary differential equation. Thus, the l-layer can be determined without consideration of the relevant corner layers, i.e., the treatment of this layer is exactly that used in the classical method. However, we also see that in general these corner layers do exist and that our method indicates how they can be found.

d. The l-layer

We assume

$$U^{l} = \sum_{n=0}^{\infty} U_n^{l}(\xi, y)\,\lambda^{-2n} \tag{2.9}$$

where $\xi = \lambda x$ since the exponent pair is (1, 0). With these variables the transformed differential equation for U^{l} is

$$\frac{\partial^4 U^l}{\partial \xi^4} - \frac{\partial^2 U^l}{\partial \xi^2} = -\frac{2}{\lambda^2}\frac{\partial^4 U^l}{\partial \xi^2 \partial y^2} - \frac{1}{\lambda^4}\frac{\partial^4 U^l}{\partial y^4} \tag{2.10}$$

Formal substitution of equation 2.9 into 2.10 gives

$$\frac{\partial^4 U_n^l}{\partial \xi^4} - \frac{\partial^2 U_n^l}{\partial \xi^2} = -2\frac{\partial^4 U_{n-1}^l}{\partial \xi^2 \partial y^2} - \frac{\partial^4 U_{n-2}^l}{\partial y^4} \tag{2.11}$$

as the sequence of equations to be satisfied by the l-layer components. In particular,

(i) $$\frac{\partial^4 U_0^l}{\partial \xi^4} - \frac{\partial^2 U_0^l}{\partial \xi^2} = 0$$

and (2.12)

(ii) $$\frac{\partial^4 U_1^l}{\partial \xi^4} - \frac{\partial^2 U_1^l}{\partial \xi^2} = -2\frac{\partial^4 U_0^l}{\partial \xi^2 \partial y^2}$$

These two are readily integrated to give;

$$U_0^l = g_0(y)e^{-\xi} \quad \text{and} \quad U_1^l = \xi\, g_0^2(y)e^{-\xi} + g_1(y)e^{-\xi} \quad \text{where} \quad g_0^2(y) = \frac{d^2 g_0(y)}{dy^2} \tag{2.13}$$

as the only solutions in compliance with criterion A, i.e., the solutions which decay at least exponentially as $\xi \to \infty$.

e. The super layer

As for other layers, we assume

$$U^{-1} = \sum_{n=0}^{\infty} U_n^{-1}(\xi, y)\lambda^{-2n} \tag{2.14}$$

where $\xi = x/\lambda$ is the transformation indicated by the exponential pair (-1, 0). With these variables the transformed differential equation for U^{-1} is

$$\frac{\partial^4 U^{-1}}{\partial y^4} - \frac{\partial^2 U^{-1}}{\partial \xi^2} = -\frac{2}{\lambda^2}\frac{\partial^4 U^{-1}}{\partial \xi^2 \partial y^2} - \frac{1}{\lambda^4}\frac{\partial^4 U^{-1}}{\partial \xi^4} \tag{2.15}$$

Formal substitution of equation 2.14 into 2.15 gives

$$\frac{\partial^4 U_n^{-1}}{\partial y^4} - \frac{\partial^2 U_n^{-1}}{\partial \xi^2} = -2\frac{\partial^4 U_{n-1}^{-1}}{\partial \xi^2 \partial y^2} - \frac{\partial^4 U_{n-2}^{-1}}{\partial \xi^4} \tag{2.16}$$

f. Boundary conditions

The absence of boundary layers, along the 3- and 4-edges, to match the U^{-1} layer means that each component of U^{-1} must satisfy the boundary conditions on these edges. Thus,

$$U_n^{-1} = \frac{\partial^2 U_n^{-1}}{\partial y^2} = 0 \qquad \text{for } y = \pm 1 \tag{2.17}$$

Furthermore, U^{-1} must be bounded for ξ large. The boundary conditions on these layers at the 1-edge are obtained by formally substituting $U^1 + U^{-1}$ into the conditions at x = 0 and equating coefficients of equal powers of λ. Doing this, we derive the conditions:

$$U_0^1(0,y) + U_0^{-1}(0,y) = f(y) \text{ and } U_n^1(0,y) + U_n^{-1}(0,y) = 0 \quad \text{for } n \neq 0 \tag{2.18}$$

also

$$\frac{\partial U_0^1(0,y)}{\partial \xi} = 0 \text{ and } \frac{\partial U_n^1(0,y)}{\partial \xi} + \frac{\partial U_{n-1}^{-1}(0,y)}{\partial \xi} = 0 \quad \text{for } n \neq 0 \tag{2.19}$$

Let the reader be cautioned again that the two ξ 's appearing in the second equation of 2.19 are <u>not</u> the same. There should be no confusion since the independent variable ξ for U^1 is given in section (c) while that for U^{-1} is given in section (d).

g. Determination of the components

The form of U_0^1 given in 2.13 and the first condition in 2.19 imply that

$$U_0^1 \equiv 0, \qquad U_1^1 = g_1(y)e^{-\xi} \quad , \quad g_1(y) = \frac{\partial U_0^{-1}(0,y)}{\partial \xi} \tag{2.20}$$

The boundary value problem for U_0^{-1} can be formulated as follows: let $V = U_0^{-1}$ then

$$\frac{\partial^2 V}{\partial y^4} - \frac{\partial^2 V}{\partial \xi^2} = 0 \tag{2.21}$$

where

$$V = \frac{\partial^2 V}{\partial y^2} = 0 \quad \text{for } y = \pm 1$$

$$V = f(y) \quad \text{for } \xi = 0$$

and V is bounded for ξ large. Assume

$$V = \sum_{n=1}^{\infty} V_n(\xi) \sin \frac{n\pi(1+y)}{2} \tag{2.22}$$

then

$$\frac{d^2V_n}{d\xi^2} - \left(\frac{n\pi}{2}\right)^4 V_n = 0$$

or

$$V_n = a_n e^{-\frac{n^2\pi^2}{4}\xi} \tag{2.23}$$

so that the solution is bounded. Using the orthogonality and completeness properties of the functions $\sin \frac{n\pi}{2}(1+y)$, it follows from the condition at $\xi = 0$ that

$$U_0^{-1} = V = \sum_{n=1}^{\infty} a_n e^{-\frac{n^2\pi^2}{4}\xi} \sin \frac{n\pi}{2}(1+y) \tag{2.24}$$

where

$$a_n = \int_{-1}^{1} f(t) \sin \frac{n\pi}{2}(1+t)\,dt$$

Now V satisfies the differential equation 2.21

$$\left(\frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial \xi}\right)\left(\frac{\partial^2}{\partial y^2} - \frac{\partial}{\partial \xi}\right) V = 0 \tag{2.25}$$

by making the first operator zero, i.e., $\partial V^2/\partial y^2 = \partial V/\partial \xi$ as can be verified by the series 2.24 since it converges uniformly for $0 < \xi$.*
Thus, as far as this solution is concerned, $V = 0$ at $y = \pm 1$ implies $\partial^2 V/\partial y^2 = 0$ at $y = \pm 1$. If ξ represents the time, V is the solution for the temperature distribution in a rod of length 2. The ends of the rod are maintained at zero temperature and the initial temperature distri-

---

* ξ need not be bounded away from the origin since d^3f/dy^3 is continuous by construction.

bution inside the rod is given by f(y). Various special cases are discussed in Carslaw and Jaeger (11) p. 58. The second derivative of f(y) exists due to the special nature of its construction. This means that the boundary layer component U_1^1 can be deduced from equation 2.20 as simply

$$U_1^1 = \frac{d^2 f}{dy^2} e^{-\xi} \tag{2.26}$$

The corner layers U_n^{13} and U_n^{14} for n = 0,1 can readily be shown to be zero because $U_0^1 \equiv 0$ and $d^2f/dy^2 = 0$ for $y = \pm 1$. It also follows from equations 2.16 and 2.18 that

$$\frac{\partial^4 U_1^{-1}}{\partial y^4} - \frac{\partial^2 U_1^{-1}}{\partial \xi^2} = -2\,\frac{\partial^3 U_0^{-1}}{\partial \xi^3} = +2\sum_{n=1}^{\infty}\left(\frac{n\pi}{2}\right)^6 a_n e^{-\frac{n^2\pi^2}{4}\xi} \sin\frac{n\pi}{2}(1+y) \tag{2.27}$$

and

$$U_1^{-1}(0,y) = -\frac{d^2 f}{dy^2}$$

The solution, that satisfies the boundary conditions at $y = \pm 1$ and is bounded for large ξ, is readily shown to be

$$U_1^{-1} = \sum_{n=1}^{\infty}\left[b_n + \left(\frac{n\pi}{2}\right)^4 a_n \xi\right] e^{-\frac{n^2\pi^2}{4}\xi} \sin\frac{n\pi}{2}(1+y) \tag{2.28}$$

where

$$b_n = -\int_{-1}^{1} \frac{df^2(t)}{dt^2} \sin\frac{n\pi}{2}(1+t)dt$$

Collecting results,

$$U \sim \sum_{n=1}^{\infty} a_n e^{-\frac{n^2\pi^2 x}{4\lambda}} \sin\frac{n\pi}{2}(1+y) \ +$$

$$+\frac{1}{\lambda^2}\left\{\frac{d^2f}{dy^2}e^{-\lambda x}+\sum_{n=1}^{\infty}\left[b_n+\left(\frac{n\pi}{2}\right)^4 a_n\frac{x}{\lambda}\right]e^{-\frac{n^2\pi^2 x}{4\lambda}}\sin\frac{n\pi}{2}(1+y)\right\} \quad (2.29)$$

$$+\,0\left(\frac{1}{\lambda^4}\right)$$

where x and y are understood to be barred.

h. Results

In the original notation, we have:

$$W\sim -f\left(\frac{y}{b}\right)+\sum_{n=1}^{\infty}a_n e^{-\frac{n^2\pi^2 x}{4b\lambda}}\sin\frac{n\pi}{2b}(b+y)$$

$$+\frac{1}{\lambda^2}\left\{b^2\frac{d^2f\left(\frac{y}{b}\right)}{dy^2}e^{-\frac{\lambda x}{b}}+\sum_{n=1}^{\infty}\left[b_n+\left(\frac{n\pi}{2}\right)^4 a_n\frac{x}{b\lambda}\right]e^{-\frac{n^2\pi^2 x}{4b\lambda}}\sin\frac{n\pi}{2b}(b+y)\right\}$$

$$+\,O\left(\frac{1}{\lambda^4}\right) \quad (2.30)$$

where

$$a_n=\int_{-1}^{1}f(t)\sin\frac{n\pi}{2}(1+t)dt, \qquad b_n=-\int_{-1}^{1}\frac{d^2f(t)}{dt^2}\sin\frac{n\pi}{2}(1+t)dt$$

$$\lambda=\sqrt{\frac{b^2N_x}{D}}$$

$$f(y)=Q(y)-\frac{Q''(1)}{12}(y^2-1)(y+3)-\frac{Q(1)}{2}(y+1)$$

$$Q(y)=-\frac{b^4}{6D}\int_{-1}^{y}(y-t)^3\,q(bt)dt$$

This is the asymptotic approximation for large λ of the transverse deflection of the stretched plate. The deflection is produced by the arbitrary transverse load $q = q(y)$. The terms for the corner layers have been omitted so that this approximation is not accurate in regions near the corners.

If we assume as a special case that

$$q(y) = k \cos \frac{\pi y}{2b} \tag{2.31}$$

then from equations 2.3

$$Q(y) = -\frac{kb^4}{3\pi D}\left[(y+1)^3 - \frac{24}{\pi^2}(y+1) - \frac{48}{\pi^3}\cos\frac{\pi y}{2}\right]$$

which makes

$$f(y) = -\frac{16kb^4}{\pi^4 D}\cos\frac{\pi y}{2}, \qquad a_1 = \frac{4}{\pi^2} b_1 = -\frac{16kb^4}{\pi^4 D}$$

and

$$a_n = b_n = 0 \qquad \text{for } n \neq 1.$$

Thus, the substitution of $q(y)$ from 2.31 into 2.30 gives

$$W \sim \frac{16kb^4}{\pi^4 D}\left\{1 - \left[1 + \frac{1}{\lambda^2}\frac{\pi^2}{4} + \frac{1}{\lambda^3}\frac{\pi^4}{16}\frac{x}{b}\right]e^{-\frac{\pi^2 x}{4b\lambda}} - \frac{1}{\lambda^2}\frac{\pi^2}{4}e^{-\frac{\lambda x}{b}}\right\}\cos\frac{\pi y}{2b}$$

$$+ O\left(\frac{1}{\lambda^4}\right) \tag{2.32}$$

This formula for the transferse deflection of the stretched plate was found by Johnson and Reissner (17). We note that the super-layer decays exponentially with increasing x.

B. Bending of a Stretched Rectangular Plate

a. Introduction

This problem treats the bending of a thin, rectangular plate which is subjected, on two opposite edges, to a uniform tension N_y parallel to the mid-plane of the plate. There are arbitrarily distributed bending moments M_1 and M_2 upon the other two edges (fig. 11).* The plate is of uniform thickness h, length 2a and width 2b. As given in standard texts, e.g., (18), the differential equation which governs this problem for small displacements is,

$$\frac{\partial^4 W}{\partial x^4} + 2\frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} = \frac{N_y}{D}\frac{\partial^2 W}{\partial y^2} \qquad (2.33)$$

where W is the transverse deflection and $D = Eh^3/12(1-\nu^2)$ is the stiffness coefficient of the plate. E and ν are the Young's modulus and Poisson's ratio, respectively, for the elastic isotropic material of the plate. This differential equation is readily obtained from equation 2.1 if q(y) is set equal to zero and the variables x and y are interchanged. Thus, the structure of the layers associated with this equation has already been discussed in part A of this chapter: provided allowance is made for the variable interchange.

The boundary conditions assumed in this problem require special comment. They are not given explicitly in the standard texts on plates and shells. Specifically, the usual Kirchhoff boundary condition for an edge which is free of vertical shear is modified due to the presence of

---

* N_y is a stress resultant in as much as it is the integral of the corresponding stress taken over the plate thickness. M_1 and M_2 are moments per unit length of perimeter of the middle surface.

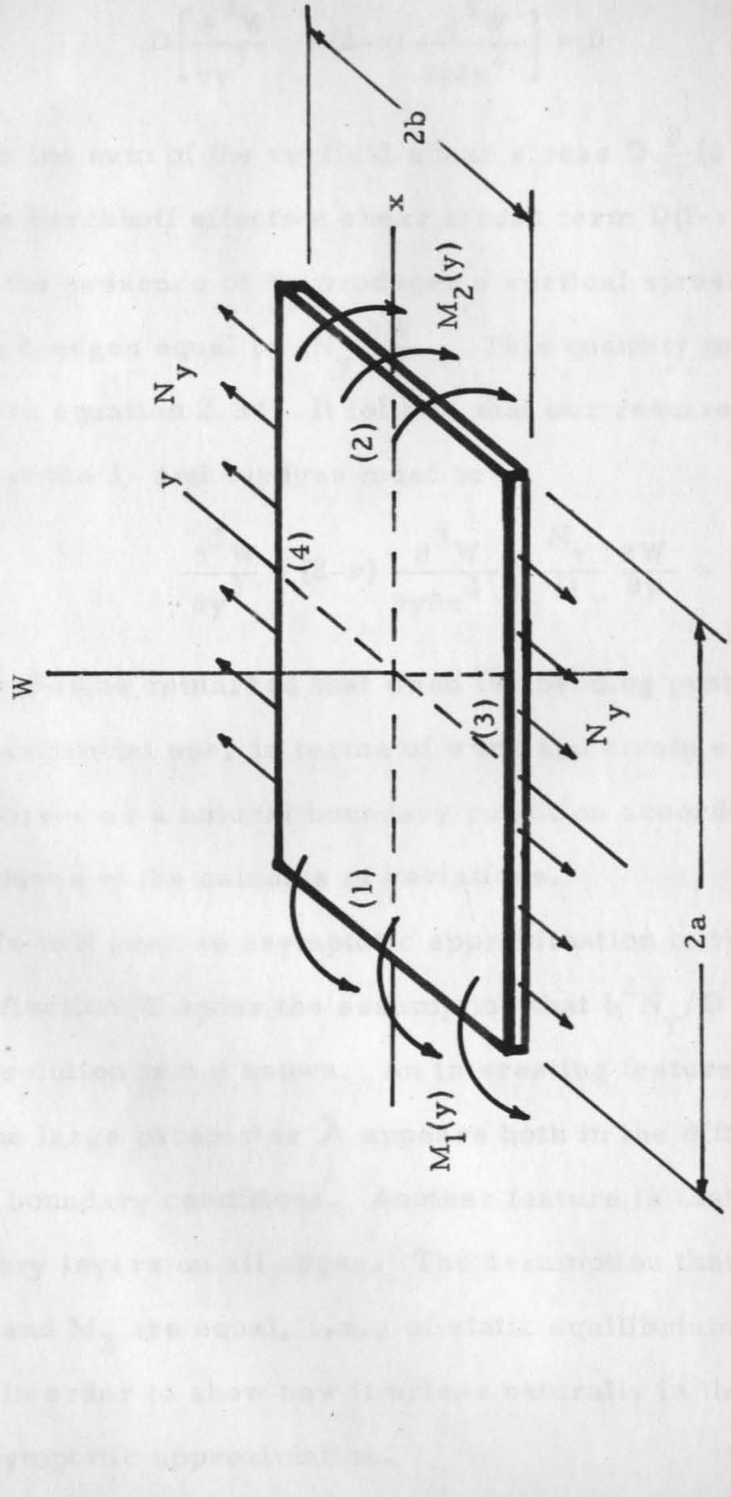


Figure 11. Bending of a Stretched Plate

the in-surface stress N_y. As discussed in the texts, the Kirchhoff condition for no vertical shear on a y-edge is

$$D\left[\frac{\partial^3 W}{\partial y^3} + (2-\nu)\frac{\partial^3 W}{\partial y \partial x^2}\right] = 0 \qquad (2.34)$$

This is the sum of the vertical shear stress $D\frac{\partial}{\partial y}(\partial^2 W/\partial y^2 + \partial^2 W/\partial x^2)$ and the Kirchhoff effective shear stress term $D(1-\nu)\,\partial^3 W/\partial y \partial x^2$. However, the presence of N_y produces a vertical stress component at the 3- and 4-edges equal to $-N_y \frac{\partial W}{\partial y}$. This quantity must be added to the terms in equation 2.34. It follows that our requirement for no vertical shear at the 3- and 4-edges must be

$$\frac{\partial^3 W}{\partial y^3} + (2-\nu)\frac{\partial^3 W}{\partial y \partial x^2} - \frac{N_y}{D}\frac{\partial W}{\partial y} = 0 \qquad (2.35)$$

It may also be remarked that when the bending problem is formulated as a variational one, in terms of work and strain energy, the condition 2.35 arises as a natural boundary condition according to the standard procedures of the calculus of variations.

We will seek an asymptotic approximation to the exact solution for the deflection W under the assumption that $b^2 N_y / D = \lambda$ is large. The exact solution is not known. An interesting feature of this problem is that the large parameter λ appears both in the differential equation and in the boundary conditions. Another feature is that the problem exhibits boundary layers on all edges. The assumption that the net moments due to M_1 and M_2 are equal, i.e., of static equilibrium, is not explicitly made in order to show how it arises naturally in the process of finding the asymptotic approximation.

b. Formulation of the problem

In view of the preceding discussion, the problem may be formulated as follows:

W = W(x, y) is to satisfy

$$\frac{\partial^4 W}{\partial x^4} + 2\frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} - \frac{N_y}{D}\frac{\partial^2 W}{\partial y^2} = 0 \tag{2.36}$$

and the subsidiary conditions

$$-D\left(\frac{\partial^2 W}{\partial x^2} + \nu\frac{\partial^2 W}{\partial y^2}\right) = M_1(y); \quad \frac{\partial^3 W}{\partial x^3} + (2-\nu)\frac{\partial^3 W}{\partial x \partial y^2} = 0 \quad \text{at } x = -a$$

$$-D\left(\frac{\partial^2 W}{\partial x^2} + \nu\frac{\partial^2 W}{\partial y^2}\right) = M_2(y); \quad \frac{\partial^3 W}{\partial x^3} + (2-\nu)\frac{\partial^3 W}{\partial x \partial y^2} = 0 \quad \text{at } x = a$$

$$\frac{\partial^2 W}{\partial y^2} + \nu\frac{\partial^2 W}{\partial x^2} = \frac{\partial^3 W}{\partial y^3} + (2-\nu)\frac{\partial^3 W}{\partial y \partial x^2} - \frac{N_y}{D}\frac{\partial W}{\partial y} = 0 \quad \text{at } y = \pm b$$

To put the problem in a non-dimensional form, let

$$x = b\bar{x}, \quad y = b\bar{y}, \quad \lambda^2 = \frac{b^2 N_y}{D}, \quad \gamma = \frac{a}{b}, \quad W(x, y) = \overline{W}(\bar{x}, \bar{y}) \tag{2.37}$$

$$M_1(y) = \frac{-D}{b^2} m_1(\bar{y}), \quad M_2(y) = \frac{-D}{b^2} m_2(\bar{y})$$

If x, y and W are understood in the sequel to be the barred variables, the problem then becomes:

$$\text{(i)} \quad \frac{\partial^4 W}{\partial x^4} + 2\frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} - \lambda^2 \frac{\partial^2 W}{\partial y^2} = 0 \tag{2.38}$$

where

(ii) $\frac{\partial^2 W}{\partial x^2} + \nu \frac{\partial^2 W}{\partial y^2} = m_1(y);\ \frac{\partial^3 W}{\partial y^3} + (2-\nu)\frac{\partial^3 W}{\partial x \partial y^2} = 0$ at $x = -\gamma$

(iii) $\frac{\partial^2 W}{\partial x^2} + \nu \frac{\partial^2 W}{\partial y^2} = m_2(y);\ \frac{\partial^3 W}{\partial x^3} + (2-\nu)\frac{\partial^3 W}{\partial x \partial y^2} = 0$ at $x = \gamma$

(iv) $\frac{\partial^2 W}{\partial y^2} + \nu \frac{\partial^2 W}{\partial x^2} = \frac{\partial^3 W}{\partial y^3} + (2-\nu)\frac{\partial^3 W}{\partial y \partial x^2} - \lambda^2 \frac{\partial W}{\partial y} = 0$ at $y = \pm 1$.

Figure 12 depicts this formulation. We seek an asymptotic approximation to the solution for large λ.

c. Structure of the layers

Since equation 2.38 is merely equation 2.5 with x and y interchanged, the layer structure given in section b of part A in this chapter suffices for this section also. The same sub-equations apply if ξ and η as well as α and β are interchanged. The situation may be summarized as follows;

For the x-layers: each boundary layer has an exponent pair $(\frac{1}{2}, 0)$ while the exponent pair for the matching corner layers is $(\frac{1}{2}, 1)$. These corner layers are in turn matched by a final set of corner layers with the exponent pair $(1, 1)$.

For the y-layers; each boundary layer has an exponent pair $(0, 1)$ while the exponent pair for the matching corner layers is $(1, 1)$. No further corner layers are needed. Figure 13 gives schematic diagrams for the sub-domains of these layers. These diagrams are analogous to those in figures 2, 3 and 4.

d. The inner solution

Using the form of the boundary conditions in 2.38 as a guide, we assume that the inner solution may be written in the form

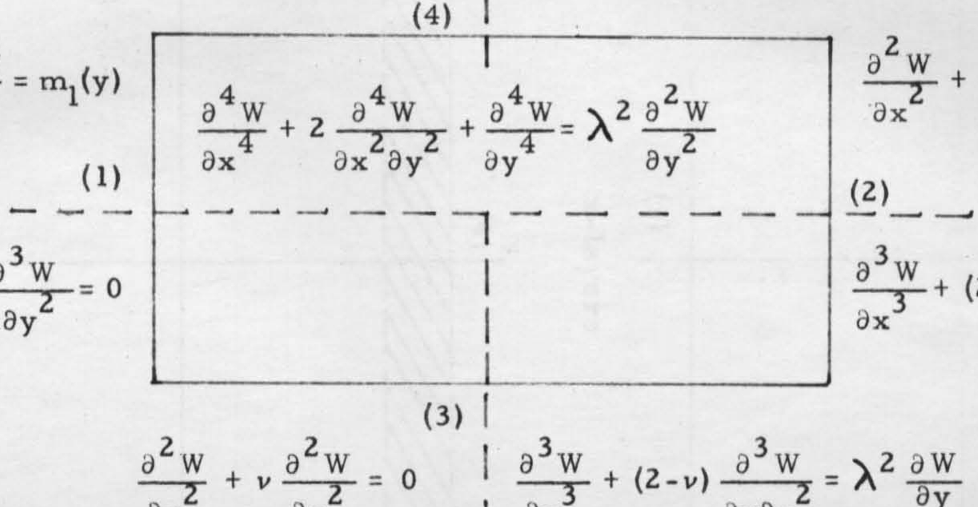
Edge (4):

$$\frac{\partial^2 W}{\partial y^2} + \nu \frac{\partial^2 W}{\partial x^2} = 0 \qquad \frac{\partial^3 W}{\partial y^3} + (2-\nu)\frac{\partial^3 W}{\partial y \partial x^2} = \lambda^2 \frac{\partial W}{\partial y}$$

Edge (1):

$$\frac{\partial^2 W}{\partial x^2} + \nu \frac{\partial^2 W}{\partial y^2} = m_1(y)$$

$$\frac{\partial^3 W}{\partial x^3} + (2-\nu)\frac{\partial^3 W}{\partial x \partial y^2} = 0$$



$$\frac{\partial^4 W}{\partial x^4} + 2\frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} = \lambda^2 \frac{\partial^2 W}{\partial y^2}$$

Edge (2):

$$\frac{\partial^2 W}{\partial x^2} + \nu \frac{\partial^2 W}{\partial y^2} = m_2(y)$$

$$\frac{\partial^3 W}{\partial x^3} + (2-\nu)\frac{\partial^3 W}{\partial x \partial y^2} = 0$$

Edge (3):

$$\frac{\partial^2 W}{\partial y^2} + \nu \frac{\partial^2 W}{\partial x^2} = 0 \qquad \frac{\partial^3 W}{\partial y^3} + (2-\nu)\frac{\partial^3 W}{\partial y \partial x^2} = \lambda^2 \frac{\partial W}{\partial y}$$

Figure 12. Mathematical Formulation for the Rectangular Plate

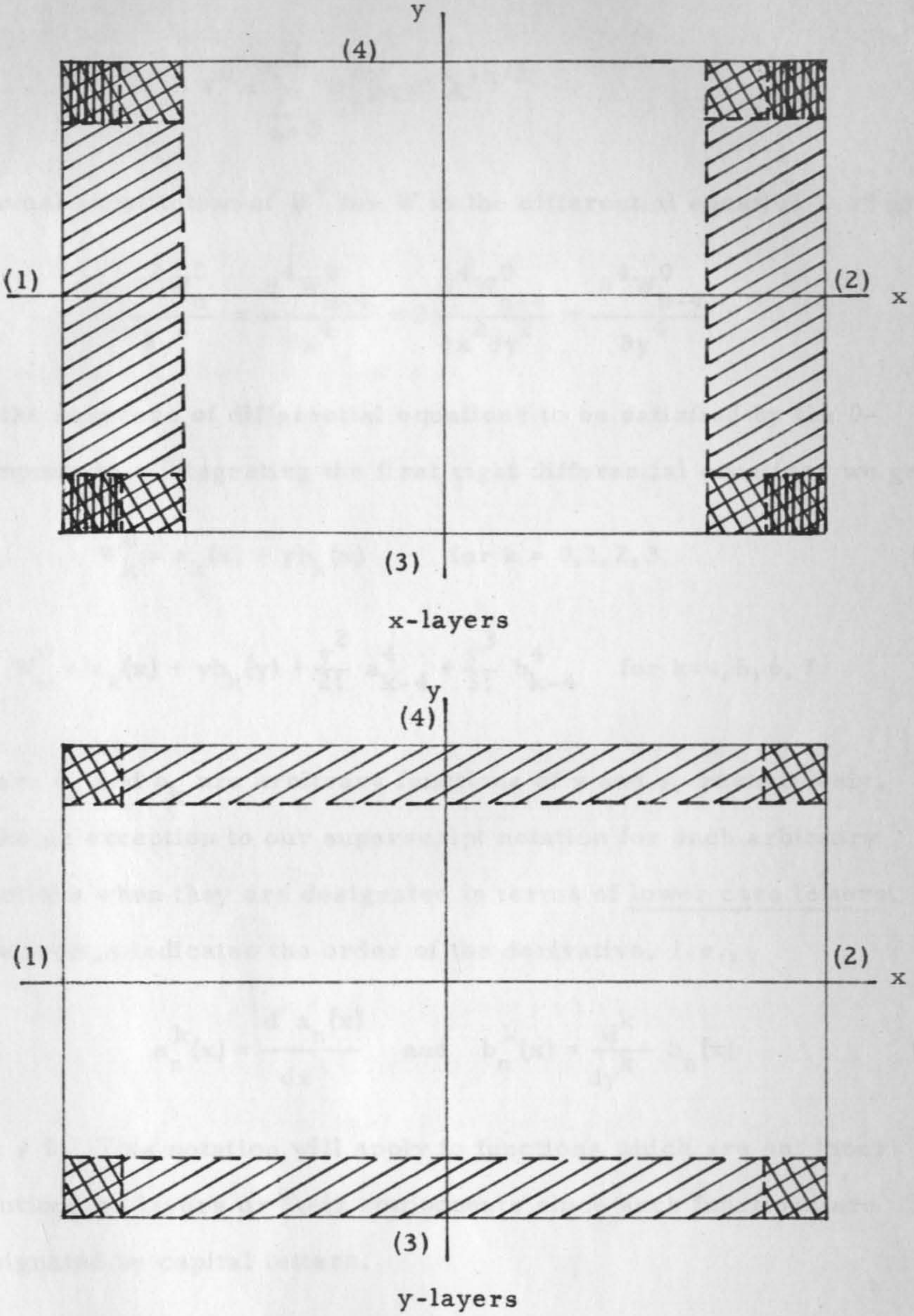


Figure 13. Diagrams of the Layer Sub-Domains

$$W^0 = \sum_{n=0}^{\infty} W_n^0(x, y) \lambda^{-n/2} \tag{2.39}$$

Formal substitution of W^0 for W in the differential equation 2.38 gives

$$\frac{\partial^2 W_n^0}{\partial y^2} = \frac{\partial^4 W_{n-4}^0}{\partial x^4} + 2 \frac{\partial^4 W_{n-4}^0}{\partial x^2 \partial y^2} + \frac{\partial^4 W_{n-4}^0}{\partial y^4} \tag{2.40}$$

as the sequence of differential equations to be satisfied by the 0-components. Integrating the first eight differential equations we get

$$W_k^0 = a_k(x) + yb_k(x) \qquad \text{for } k = 0, 1, 2, 3 \tag{2.41}$$

$$W_k^0 = a_k(x) + yb_k(y) + \frac{y^2}{2!} a_{k-4}^4 + \frac{y^3}{3!} b_{k-4}^4 \qquad \text{for } k=4, 5, 6, 7$$

where a_k and b_k are arbitrary functions of x and y, respectively. We make an exception to our superscript notation for such arbitrary functions when they are designated in terms of <u>lower case letters</u>. The superscript indicates the order of the derivative, i.e.,

$$a_n^k(x) = \frac{d^k a_n(x)}{dx^k} \quad \text{and} \quad b_n^k(x) = \frac{d^k}{dy^k} b_n(x) \tag{2.42}$$

if $k \neq 0$. This notation will apply to functions which are not inner solutions or layers or their components since such functions are designated by capital letters.

e. The x-boundary layers

Let p equal either 1 or 2 and

$$W^p = W^p(\xi, y) \tag{2.43}$$

where if p = 1 then $\xi = \lambda^{1/2}(\gamma+x)$ or if p = 2 then $\xi = \lambda^{1/2}(\gamma-x)$, i.e.,

$$\xi = \lambda^{1/2}\left(\gamma-(-1)^p\, x\right) \tag{2.44}$$

Thus, ξ is the layer variable for the x-boundary layers in accordance with the exponent pair $(\frac{1}{2}, 0)$. If we make either of these transformations on the x variable in the differential equation 2.38, we obtain the transformed differential equation

$$\frac{\partial^4 W}{\partial \xi^4} - \frac{\partial^2 W}{\partial y^2} = -\frac{2}{\lambda}\frac{\partial^4 W}{\partial \xi^2 \partial y^2} - \frac{1}{\lambda^2}\frac{\partial^4 W}{\partial y^4} \tag{2.45}$$

Let us assume that

$$W^p = \sum_{n=0}^{\infty} W_n^p(\xi, y)\,\lambda^{-\left(1+\frac{n}{2}\right)} \tag{2.46}$$

Formal substitution of W^p for W in equation 2.45 gives

$$\frac{\partial^4 W_n^p}{\partial \xi^4} - \frac{\partial^2 W_n^p}{\partial y^2} = -2\,\frac{\partial^4 W_{n-2}^p}{\partial \xi^2 \partial y^2} - \frac{\partial^4 W_{n-4}^p}{\partial y^4} \tag{2.47}$$

as the sequence of differential equations satisfied by the p-components.

The boundary conditions at a p-edge which involve the inner solution and a p-layer are obtained by formally substituting the sum $W^0 + W^p$ for W in the relevant equations in 2.38. Thus,

$$\lambda\frac{\partial^2 W^p}{\partial \xi^2} + \nu\,\frac{\partial^2 W^p}{\partial y^2} + \frac{\partial^2 W^0}{\partial x^2} + \nu\,\frac{\partial^2 W^0}{\partial y^2} = m_p(y) \qquad \text{at } \xi = 0$$

and

$$\lambda^{3/2}\,\frac{\partial^3 W^p}{\partial \xi^3} + \lambda^{1/2}(2-\nu)\frac{\partial^3 W^p}{\partial \xi\,\partial y} - (-1)^p\,\frac{\partial^3 W^0}{\partial x^3} - (2-\nu)(-1)^p\,\frac{\partial^3 W^0}{\partial x \partial y^2} = 0 \qquad \text{at } \xi = 0$$

Now substituting the series in equations 2.39 and 2.46 for W^0 and W^p, we obtain boundary conditions upon the components,

$$\frac{\partial^2 W_0^p}{\partial \xi^2} + \frac{\partial^2 W_0^0}{\partial x^2} + \nu \frac{\partial^2 W_0^0}{\partial y^2} = m_p(y) \text{ and } \frac{\partial^2 W_n^p}{\partial \xi^2} + \frac{\partial^2 W_n^0}{\partial y^2} + \nu \frac{\partial^2 W_{n-2}^p}{\partial y^2} = 0; \tag{2.48}$$

$$\frac{\partial^3 W_n^p}{\partial \xi^3} - (-1)^p \frac{\partial^3 W_{n-1}^p}{\partial x^3} - (2-\nu)(-1)^p \frac{\partial^3 W_{n-1}^0}{\partial x \partial y^2} + (2-\nu) \frac{\partial^3 W_{n-2}^2}{\partial \xi \, \partial y^2} = 0 \tag{2.49}$$

which are valid when the relevant ξ is zero.

f. <u>The y-boundary layers</u>

Let q equal either 3 or 4 and

$$W^q = W^q(x, \eta) \tag{2.50}$$

where if q = 3 then $\eta = \lambda(1+y)$ or if q = 4 then $\eta = \lambda(1-y)$, i.e.,

$$\eta = \lambda\left(1-(-1)^q \, y\right) \tag{2.51}$$

Thus, η is the layer variable for the y-boundary layers in accordance with the exponent pair (0, 1). If we make either of these transformations on the y variable in the differential equation 2.38, we obtain the transformed differential equation

$$\frac{\partial^4 W}{\partial \eta^4} - \frac{\partial^2 W}{\partial \eta^2} = -\frac{2}{\lambda^2} \frac{\partial^4 W}{\partial x^2 \partial \eta^2} - \frac{1}{\lambda^4} \frac{\partial^2 W}{\partial x^4} \tag{2.52}$$

Let us assume that

$$W^q = \sum_{n=0}^{\infty} W_n^q(x, \eta) \, \lambda^{-\left(1+\frac{n}{2}\right)} \tag{2.53}$$

Formal substitution of W^q for W in equation 2.52 gives

$$\frac{\partial^4 W^q_n}{\partial \eta^4} - \frac{\partial^2 W^q_n}{\partial \eta^2} = -2 \frac{\partial^4 W^q_{n-4}}{\partial x^2 \partial \eta^2} - \frac{\partial^4 W^q_{n-8}}{\partial x^4} \tag{2.54}$$

as the sequence of differential equations satisfied by the q-components.

The boundary conditions at a q-edge which involve the inner solution and a q-layer are obtained by formally substituting the sum $W^0 + W^q$ for W in the relevant equations in 2.38. Thus,

$$\lambda^2 \frac{\partial^2 W^q}{\partial \eta^2} + \nu \frac{\partial^2 W^q}{\partial x^2} + \frac{\partial^2 W^0}{\partial y^2} + \nu \frac{\partial^2 W^0}{\partial x^2} = 0 \quad \text{at } \eta = 0$$

and

$$\lambda^3 \frac{\partial^3 W^q}{\partial \eta^3} + (2-\nu)\lambda \frac{\partial^3 W^q}{\partial \eta \partial x^2} - \lambda^3 \frac{\partial^3 W^q}{\partial \eta^3} = (-1)^q \left[\frac{\partial^3 W^0}{\partial y^3} + (2-\nu) \frac{\partial^3 W^0}{\partial y \partial x^2} - \lambda^2 \frac{\partial W^0}{\partial y} \right] \quad \text{at } \eta = 0$$

Now substituting the series in equations 2.39 and 2.53 for W^0 and W^q, we obtain boundary conditions upon the components,

$$\frac{\partial^2 W^q_n}{\partial \eta^2} + \frac{\partial^2 W^0_{n-2}}{\partial y^2} + \nu \frac{\partial^2 W^0_{n-2}}{\partial x^2} + \nu \frac{\partial^2 W^q_{n-4}}{\partial x^2} = 0 \tag{2.55}$$

$$\frac{\partial^3 W^q_n}{\partial \eta^3} - \frac{\partial W^q_n}{\partial \eta} + (-1)^q \frac{\partial W^0_n}{\partial y} + (2-\nu) \frac{\partial^3 W^q_{n-4}}{\partial \eta \partial x^2} = (-1)^q \left[\frac{\partial^3 W^0_{n-4}}{\partial y^3} + (2-\nu) \frac{\partial^3 W^0_{n-4}}{\partial y \partial x^2} \right] \tag{2.56}$$

which are valid when the relevant η is zero.

g. The x-corner layers

Let

$$W^{pq} = W^{pq}(\xi, \eta) \tag{2.57}$$

where p, q, ξ and η are defined by equations 2.43, 2.44, 2.50 and 2.51. If we make any of these transformations on x and y in the differential equation 2.38, we obtain the transformed differential equation

$$\frac{\partial^4 W}{\partial \eta^4} - \frac{\partial^2 W}{\partial \eta^2} = -\frac{2}{\lambda}\frac{\partial^4 W}{\partial \xi \partial \eta^2} - \frac{1}{\lambda^2}\frac{\partial^4 W}{\partial \xi^4} \tag{2.58}$$

We will assume that

$$W^{pq} = \sum_{n=0}^{\infty} W_n^{pq}(\xi, \eta)\lambda^{-\left(2+\frac{n}{2}\right)} \tag{2.59}$$

Formal substitution of W^{pq} into equation 2.58 gives

$$\frac{\partial^4 W_n^{pq}}{\partial \eta^4} - \frac{\partial^2 W_n^{pq}}{\partial \eta^2} = -2\frac{\partial^4 W_{n-2}^{pq}}{\partial \xi^2 \partial \eta^2} - \frac{\partial^4 W_{n-4}^{pq}}{\partial \xi^4} \tag{2.60}$$

as the sequence of differential equations satisfied by the pq-components.

The boundary conditions at a q-edge which involve a p-layer and a pq-layer are obtained by formally substituting the sum $W^p + W^{pq}$ for W in the relevant equations in 2.38. Thus,

$$\lambda^2\frac{\partial^2 W^{pq}}{\partial \eta^2} + \nu\lambda\frac{\partial^2 W^{pq}}{\partial \xi^2} + \frac{\partial^2 W^p}{\partial y^2} + \nu\lambda\frac{\partial^2 W^p}{\partial \xi^2} = 0 \quad \text{for } \eta = 0$$

and

$$\lambda^3\frac{\partial^3 W^{pq}}{\partial \eta^3} + (2-\nu)\lambda^2\frac{\partial^3 W^{pq}}{\partial \eta \partial \xi^2} - \lambda^3\frac{\partial W^{pq}}{\partial \eta} = (-1)^q\left[\frac{\partial^3 W^p}{\partial y^3} + (2-\nu)\lambda\frac{\partial^3 W^p}{\partial y \partial \xi^2} - \lambda^2\frac{\partial W^p}{\partial y}\right] \quad \text{for } \eta = 0$$

Now substituting the series in equation 2.46 and 2.69 for W^p and W^{pq}, we obtain boundary conditions upon the components,

$$\frac{\partial^2 W_n^{pq}}{\partial \eta^2} + \nu \frac{\partial^2 W_n^p}{\partial \xi^2} + \nu \frac{\partial^2 W_{n-2}^{pq}}{\partial \xi^2} + \frac{\partial^2 W_{n-2}^p}{\partial y^2} = 0 \tag{2.61}$$

$$\frac{\partial^3 W_n^{pq}}{\partial \eta^3} - \frac{\partial W_n^{pq}}{\partial \eta} + (-1)^q \frac{\partial W_n^p}{\partial y} + (2-\nu) \frac{\partial^3 W_{n-2}^{pq}}{\partial \eta \partial \xi^2} = (-1)^q \left[(2-\nu) \frac{\partial^3 W_{n-2}^p}{\partial y \partial \xi^2} + \frac{\partial^3 W_{n-4}^p}{\partial y^3} \right] \tag{2.62}$$

which are valid when the relevant η is zero.

The x-corner layers necessary to match W^{pq}, i.e., W^{pqp} will not be treated here. These layers have the exponent pair (1,1). No equations involving their components are necessary in order to find the boundary layers and inner solution.

h. The y-corner layers

This set of corner layers, W^{qp}, need not be treated in detail here because the sub-equation for W^q is essentially an ordinary differential equation as is found in the classical problems. However, these equations are necessary if one is interested in the asymptotic approximation in the corner. Note that W^{qp} has essentially the same sub-domain as W^{pqp} and it is expected that only one layer of this order is needed in each corner. We note that the sub-equation is the entire transformed differential equation. The components of this layer are related only through their boundary conditions and not by means of the sequence of differential equations for the components.

i. Summary of several relationships

For n = 0 From equation 2.41, we have $W_0^0 = a_0(x) + yb_0(x)$. Substituting this into equations 2.48, 2.49, 2.55 and 2.56 for n = 0, we have

for $\xi = 0$;

$$\text{(i)}\ \frac{\partial^2 W_0^1}{\partial \xi^2} + a_0^2(-\gamma) + yb_0^2(-\gamma) = m_1(y) \qquad \text{(iii)}\ \frac{\partial^3 W_0^1}{\partial \xi^3} = 0$$

$$\text{(ii)}\ \frac{\partial^2 W_0^2}{\partial \xi^2} + a_0^2(\gamma) + yb_0^2(\gamma) = m_2(y) \qquad \text{(iv)}\ \frac{\partial^3 W_0^2}{\partial \xi^3} = 0 \tag{2.63}$$

for $\eta = 0$:

$$\text{(i)}\ \frac{\partial^2 W_0^3}{\partial \eta^2} = 0 \qquad \text{(iii)}\ \frac{\partial^3 W_0^3}{\partial \eta^3} - \frac{\partial W_0^3}{\partial \eta} - b_0(x) = 0$$

$$\text{(ii)}\ \frac{\partial^2 W_0^4}{\partial \eta^2} = 0 \qquad \text{(iv)}\ \frac{\partial^3 W_0^4}{\partial \eta^3} - \frac{\partial W_0^4}{\partial \eta} + b_0(x) = 0 \tag{2.64}$$

For n = 1 Except for the equations analogous to 2.63(iii) and 2.63 (iv), the equations in this case can be obtained from those for n = 0 by increasing the subscripts by one and omitting $m_p(y)$. The two exceptions give; for $\xi = 0$

$$\text{(iii)}\ \frac{\partial^3 W_1^1}{\partial \xi^3} + a_0^3(-\gamma) + yb_0^3(-\gamma) = 0$$

$$\text{(iv)}\ \frac{\partial^3 W_1^1}{\partial \xi^3} - a_0^3(\gamma) - yb_0^3(\gamma) = 0 \tag{2.65}$$

For n = 2 The corresponding equations are $W_2^0 = a_2(x) + yb_2(x)$ and for $\xi = 0$:

$$\text{(i)}\quad \frac{\partial^2 W_2^1}{\partial\xi^2} + a_2^2(-\gamma) + yb_2^2(-\gamma) + \nu\frac{\partial^2 W_0^1}{\partial y^2} = 0$$

$$\text{(ii)}\quad \frac{\partial^2 W_2^2}{\partial\xi^2} + a_2^2(\gamma) + yb_2^2(\gamma) + \nu\frac{\partial^2 W_0^2}{\partial y^2} = 0 \qquad (2.66)$$

$$\text{(iii)}\quad \frac{\partial^3 W_2^1}{\partial\xi^3} + a_1^3(-\gamma) + yb_1^3(-\gamma) + (2-\nu)\frac{\partial^3 W_0^1}{\partial\xi\partial y^2} = 0$$

$$\text{(iv)}\quad \frac{\partial^3 W_2^2}{\partial\xi^3} - a_1^3(\gamma) - yb_1^3(\gamma) + (2-\nu)\frac{\partial^3 W_0^2}{\partial\xi\,\partial y^2} = 0$$

for $\eta = 0$:

$$\text{(i)}\quad \frac{\partial^2 W_2^3}{\partial\eta^2} + \nu a_0^2(x) - \nu b_0^2(x) = 0 \qquad \text{(iii)}\quad \frac{\partial^3 W_2^3}{\partial\eta^3} - \frac{\partial W_2^3}{\partial\eta} - b_2(x) = 0$$

$$\text{(ii)}\quad \frac{\partial^2 W_2^4}{\partial\eta^2} + \nu a_0^2(x) + \nu b_0^2(x) = 0 \qquad \text{(iv)}\quad \frac{\partial^2 W_2^4}{\partial\eta^3} - \frac{\partial W_2^4}{\partial\eta} + b_2(x) = 0 \qquad (2.67)$$

For n = 3 The equations are analogous to those in 2.66 and 2.67 except that all subscripts are increased by one.

For n = 4 Only two of the rather complex equations that are obtained in this case will be needed in the subsequent work. The pertinent equations corresponding to those given above are

$$W_4^0 = a_4(x) + yb_4(x) + \frac{y^2}{2!}\, a_0^4(x) + \frac{y^3}{3!}\, b_0^4(x)$$

and for $\eta = 0$:

$$\text{(iii)}\ \frac{\partial^3 W_4^3}{\partial \eta^3} + (2-\nu)\frac{\partial^3 W_0^3}{\partial \eta\, \partial x^2} + (2-\nu)\, b_0^2(x) - \frac{\partial W_4^3}{\partial \eta} - b_4(x) + a_0^4(x) - \frac{1}{2!}\, b_0^4(x) = 0$$

$$(2.68)$$

$$\text{(iv)}\ \frac{\partial^3 W_4^4}{\partial \eta^3} + (2-\nu)\frac{\partial^3 W_0^4}{\partial \eta\, \partial x^2} - (2-\nu)\, b_0^2(x) - \frac{\partial W_4^4}{\partial \eta} + b_4(x) + a_0^4(x) + \frac{1}{2!}\, b_0^4(x) = 0$$

For $n = 5$ The equations are analogous to those in 2.68 except that the subscripts are increased by one.

j. First components

Consider first a q-layer. For $n = 0$ equation 2.54 gives

$$\frac{\partial^4 W_0^q}{\partial \eta^4} - \frac{\partial^2 W_0^q}{\partial \eta^2} = 0 \qquad (2.69)$$

By the layer criterion A, these equations are readily integrated to give

$$W_0^q = f_q(x) e^{-\eta} \qquad (2.70)$$

However, equations 2.64(i) and 2.64(ii) imply that

$$W_0^q \equiv 0 \qquad (2.71)$$

Using this result, and referring to equation 2.54 for $n = 4$, we see that W_4^q satisfies equation 2.69. For our purposes we need only perform one integration, i.e.,

$$\frac{\partial^3 W_4^q}{\partial \eta^3} - \frac{\partial W_4^q}{\partial \eta} = 0 \tag{2.72}$$

We have used criterion A again. We note further that if equations 2.71 are substituted into equations 2.64(iii) and 2.64(iv) then it is found that

$$b_0(x) = 0 \tag{2.73}$$

Now substituting from equations 2.71, 2.72, and 2.73 into equations 2.68 we find that

$$a_0^4(x) - b_4(x) = a_0^4(x) + b_4(x) = 0$$

or that

$$a_0^4(x) = b_4(x) = 0 \tag{2.74}$$

Thus, we have shown that W_0^0 is a cubic in x and independent of y. Linear or constant terms in W_0^0 will be ignored since they represent rigid body displacements. We now turn our attention to the p-layers. Equation 2.47 for n = 0 gives

$$\boxed{\frac{\partial^4 W_0^p}{\partial \xi^4} - \frac{\partial^2 W_0^p}{\partial y^2} = 0} \tag{2.75}$$

Because of the presence of the y-derivative in this equation, it cannot be treated as essentially an ordinary differential equation as is done in what we have designated as classical problems. Remarks similar to those in connection with equation 1.27 can also be made here. It will now be demonstrated how our method leads to a well defined boundary value problem for equation 2.74. We note that equation 2.74 appears in Gol'denveizer's paper (4) p. 17. Equations 2.48 and 2.49

give boundary conditions at a p-edge,

$$\frac{\partial^2 W_0^p}{\partial \xi^2} + a_0^2 = m_p(y), \qquad \frac{\partial^3 W_0^p}{\partial \xi^3} = 0 \qquad \text{at } \xi = 0 \tag{2.76}$$

These boundary conditions are <u>not</u> sufficient to determine W_0^p and we must use our method to determine additional conditions from the x-corner layers. Equations 2.61 and 2.62 give the boundary conditions for W_0^p at the q-edges, i.e., for n = 0,

$$\frac{\partial^2 W_0^{pq}}{\partial \eta^2} + \nu \frac{\partial^2 W_0^p}{\partial \xi^2} = 0 \text{ and } \frac{\partial^3 W_0^{pq}}{\partial \eta^3} - \frac{\partial W_0^{pq}}{\partial \eta} + (-1)^q \frac{\partial W_0^p}{\partial y} = 0 \quad \text{at } \eta = 0 \tag{2.77}$$

Also for n = 0, equation 2.60 provides the differential equation for W_0^{pq},

$$\frac{\partial^4 W_0^{pq}}{\partial \eta^4} - \frac{\partial^2 W_0^{pq}}{\partial \eta^2} = 0 \tag{2.78}$$

This is the same differential equation that we encountered before and a first integration gives

$$\frac{\partial^3 W_0^{pq}}{\partial \eta^3} - \frac{\partial W_0^{pq}}{\partial \eta} = 0 \tag{2.79}$$

We now assume, as described in the consistency criterion D, that the second boundary condition in equation 2.77 must be consistent with equation 2.79. We thus conclude that

$$\frac{\partial W_0^p}{\partial y} = 0 \quad \text{at } y = \pm 1 \tag{2.80}$$

These form the additional boundary conditions, which in conjunction with equations 2.75 and 2.76 determine the boundary value problem for W_0^p.

Before we attempt to solve for W_0^p, we note that the first equations in 2.76 together with 2.75 and 2.80 allow us to determine $W_0^0 = a_0(x)$. Let a tilda on a function indicate the average of that function taken over y, e.g.,

$$\widetilde{W}_0^{\,0} = \frac{1}{2}\int_{-1}^{1} W_0^{\,0}\, dy$$

Thus, if we integrate equation 2.75 with respect to y from -1 to 1 and use the conditions 2.80, we have

$$\frac{\partial^4 \widetilde{W}_0^p}{\partial \xi^4} = 0 \qquad \text{or by criterion A,} \qquad \widetilde{W}_0^p = 0 \tag{2.81}$$

Applying this result to the first boundary conditions in 2.76, we obtain

$$a_0^2(-\gamma) = \widetilde{m}_1 = \frac{1}{2}\int_{-1}^{1} m_1(y)dy \quad \text{and} \quad a_0^2(\gamma) = \widetilde{m}_2 = \frac{1}{2}\int_{-1}^{1} m_2(y)dy \tag{2.82}$$

It follows that

$$W_0^{\,0} = a_0(x) = \frac{\widetilde{m}_2+\widetilde{m}_1}{4}\, x^2 + \frac{\widetilde{m}_2-\widetilde{m}_1}{12\gamma}\, x^3 \tag{2.83}$$

to within a linear term in x. Restriction upon the m_p will arise out of equations for the higher order components.

The boundary value problems described by equations 2.75, 2.76 and 2.80 for W_0^p are easily solved. For example, assume that

$$W_0^1 = \sum_{n=0}^{\infty} C_n(\xi) \cos\frac{n\pi}{2}(1+y) \qquad (2.84)$$

since these eigen functions, $\cos\frac{n\pi}{2}(1+y)$, form a complete set and satisfy equation 2.80. The condition that equation 2.75 be satisfied is

$$\frac{d^4 C_n}{d\xi^4} + \left(\frac{n\pi}{2}\right)^2 C_n = 0 \qquad (2.85)$$

which has four solutions of the form $e^{\pm\alpha_n(1\pm i)\xi}$ where $\alpha_n = \sqrt{n\pi/2}$.

The condition that W_0^1 be a layer, criterion A, requires that $C_0 = 0$ and

$$C_n = A_n e^{-\alpha_n r\xi} + B_n e^{-\alpha_n s\xi} \qquad \text{for } n \neq 0$$

where $r = 1+i$ and $s = 1-i$. The condition that $\partial^3 W_0^1/\partial\xi^3 = 0$ at $\xi = 0$ is satisfied if $B_n = iA_n$, thus

$$W_0^1 = (1+i)\sum_{n=1}^{\infty} A_n e^{-\alpha_n\xi}\left(\cos\alpha_n\xi - \sin\alpha_n\xi\right)\cos\frac{n\pi}{2}(1+y) \qquad (2.86)$$

where

$$A_n = \frac{2}{n\pi(1+i)} \int_{-1}^{1} m_1(t)\cos\frac{n\pi}{2}(1+t)dt$$

since

$$\frac{\partial^2 W_0^1}{\partial\xi^2} = m_1(y) - \widetilde{m}_1 \quad \text{at } \xi = 0.$$

The problem for W_0^2 is analogous; both solutions can be written

$$W_0^p = 2\sum_{n=1}^{\infty} \frac{e^{-a_n\xi}}{n\pi}\left[\cos a_n\xi - \sin a_n\xi\right]\left[\int_{-1}^{1} m_p(t)\cos\frac{n\pi}{2}(1+t)dt\right]\cos\frac{n\pi}{2}(1+y) \tag{2.87}$$

where $a_n = \sqrt{n\pi/2}$ and $p = 1, 2$.

k. Higher order terms

By analogy with the discussion in section j, it may be seen that $b_1(x) = W_1{}^3 = W_1{}^4 = 0$. Since the conditions upon $a_1(x)$ are the same as those on $a_0(x)$ provided that $m_1(y)$ is taken to be zero, it follows that $a_1(x) = 0$. Now substituting $b_0(x) \equiv 0$ and $a_0(x)$ from equation 2.83 into the equations of 2.65, we obtain for $\xi = 0$

$$\frac{\partial^3 W_1^1}{\partial \xi^3} + \frac{\widetilde{m}_2 - \widetilde{m}_1}{\gamma} = \frac{\partial^3 W_1^2}{\partial \xi^3} - \frac{\widetilde{m}_2 - \widetilde{m}_1}{\gamma} = 0 \tag{2.88}$$

However, again analogous to the result in equation 2.81, we see that

$$\frac{\partial^4 W_1^p}{\partial \xi^4} = 0. \tag{2.89}$$

Thus, the boundary layers have no average value over y. Equations 2.88 and 2.89 then imply that

$$\widetilde{m}_1 = \widetilde{m}_2 = \widetilde{m} \tag{2.90}$$

This is the physical condition for static equilibrium of the plate under the applied bending moments. From the fact that equations 2.88 are now homogeneous conditions upon $W_1{}^p$ as are all other conditions on $W_1{}^p$, it follows that $W_1{}^p \equiv 0$. Also, analogous to equation 2.71 we have that $W_1^q \equiv 0$.

We will seek W_2^q in order to exhibit some non-zero components for the boundary layers on the 3- and 4-edges. Corresponding to equation 2.70, we have

$$W_2^{\,q} = f_q(x)\, e^{-\eta} \tag{2.91}$$

so that the conditions given by equations 2.67 for $\eta = 0$ become

$$\text{(i)} \quad \frac{\partial^2 W_2^{\,q}}{\partial \eta^2} = f_q(x) = -\nu\, \widetilde{m} \tag{2.92}$$

$$\text{(ii)} \quad b_2(x) = 0$$

Therefore,

$$W_2^{\,q} = -\nu\, \widetilde{m}\, e^{-\eta} \tag{2.93}$$

Concerning higher order components; there will be equations analogous to equations 2.88 and 2.89 which are not necessarily homogeneous. Such equations will lead to restrictions on the $m_p(y)$ at $y = \pm 1$. For example, it can be shown that

$$m_1(1) + m_1(-1) = m_2(1) + m_2(-1) \tag{2.94}$$

is the next restriction that is obtained. However, if this restriction is violated only higher order terms in the asymptotic approximation would be affected. It seems that there are an infinite number of such restrictions and that there is no way of eliminating them.

1. Results

In the original notation: if corner domains are excluded, we have the following asymptotic approximation for the transverse deflection of the plate as λ becomes large

$$W \sim -\frac{M}{4bD}x^2 - \frac{1}{\lambda}\frac{\nu^2}{4bD}\left(\bar{M}-M\right)x^2 + \ldots$$

$$+\frac{1}{\lambda}W_0^1 + \frac{1}{\lambda^2}W_2^1 + \ldots + \frac{1}{\lambda}W_0^2 + \frac{1}{\lambda^2}W_2^2 + \ldots$$

$$+\frac{1}{\lambda^2}\frac{b\nu M}{2D}e^{-\lambda(1+\frac{y}{b})} + \ldots + \frac{1}{\lambda^2}\frac{b\nu M}{2D}e^{-\lambda(1-\frac{y}{b})} + \ldots \tag{2.95}$$

where

$$M = \int_{-b}^{b} M_1(y)dy = \int_{-b}^{b} M_2(y)dy, \quad \lambda = \frac{b^2 N_y}{D} \tag{2.96}$$

$$\bar{M} = b\left[M_1(b) + M_1(-b)\right] = b\left[M_2(b) + M_2(-b)\right]$$

and

$$W_0^p = -\frac{2b}{D}\sum_{n=1}^{\infty}\frac{e^{-a_n\xi}}{n\pi}\left[\cos a_n\xi - \sin a_n\xi\right]\left[\int_{-b}^{b} M_p(t)\cos\frac{n\pi}{2}(1+\frac{t}{b})dt\right]\cdot$$

$$\cos\frac{n\pi}{2}(1+\frac{y}{b})$$

with

$$a_n = \sqrt{\frac{n\pi}{2}} \quad \text{and} \quad \xi = \frac{\lambda^{1/2}}{b}\left[a-(-1)^p x\right] \text{ for } p=1 \text{ or } 2 \tag{2.97}$$

The second component in the inner solution was obtained in a manner similar to that presented here for the first component. Although the boundary layers are of higher order in λ than the first term of the inner solution, they are essential for calculating the moments in the vicinity of the edges. It is interesting to note that the transverse deflection W for a flat plate under these same moments but without stretching is

$$W = -\frac{M}{4b(1-\nu^2)D}(x^2 - \nu y^2) \tag{2.98}$$

provided that certain sub-domains along the 1- and 2-edges are excluded. This gives the moments $M_{xx} = M/2b$ and $M_{yy} = 0$. If the 1- and 2-domains are excluded in equation 2.95, we obtain

$$\text{(i)} \quad M_{xx} = \frac{M}{2b}\left[1-\nu^2 e^{-\lambda(1-\frac{y}{b})} - \nu^2 e^{-\lambda(1+\frac{y}{b})}\right] \tag{2.99}$$

and

$$\text{(ii)} \quad M_{yy} = \frac{\nu M}{2b}\left[1-e^{-\lambda(1-\frac{y}{b})} - e^{-\lambda(1+\frac{y}{b})}\right]$$

as first approximations for large λ. Thus, in the interior there is a moment $M_{yy} = \nu M/2b$ which tends to keep the plate flat in the y-direction when there is stretching. Stretching also reduces the displacement of the line $y = 0$ by a factor of $(1-\nu^2)$.

III. TWO APPLICATIONS IN THE THEORY OF ELASTIC SHELLS

A. Bending of a Shallow Hyperbolic Paraboloidal Shell

a. Introduction

The shell studied here can be considered to be a thin, pretwisted rectangular plate of width 2b, length 2a and uniform thickness h, (fig. 14). Its undeflected middle surface is given by

$$z = kxy; \qquad k \text{ a constant} \tag{3.1}$$

When undeflected, the shell is assumed to be stress free. Two opposite edges of the shell are subjected to distributed bending moments $M_1(y)$ and $M_2(y)$.* The other edges are free from stress. The material of the shell is elastic and isotropic with a Young's modulus E and a Poisson's ratio ν. This problem has been treated by Maunder and Reissner (19). However, they did not consider the behavior of the solution in the vicinity of the edges of the shell. They found what we designate as the inner solution. For the moments considered here, their result for the transverse deflection W is

$$W = -\frac{M}{4b(1-\nu^2)D}(x^2-\nu y^2) \tag{3.2}$$

where

$$M = \int_{-b}^{b} M_1(y)dy = \int_{-b}^{b} M_2(y)dy$$

and $D = Eh^3/12(1-\nu^2)$. This result is the same as that obtained for the deflection of a flat plate under these moments (see equation 2.98). Such a result, of course, is valid only within the approximations used in

* M_1 and M_2 are moments per unit length of perimeter of the middle surface.

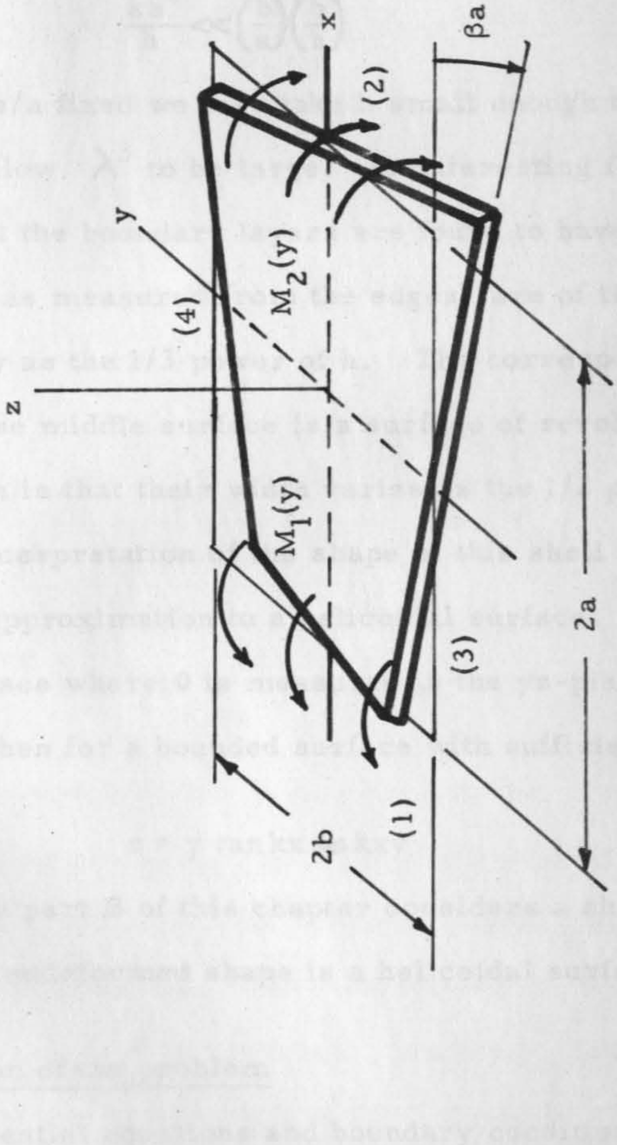


Figure 14. Bending of a Pretwisted Plate

formulating the problem. The formulation, as given in (19), is valid for ka $\ll$ 1. We will seek an asymptotic approximation to the problem with this formulation under the assumption that the parameter

$$\lambda^3 = \frac{2\sqrt{12(1-\nu^2)}}{h}\left(kb^2\right)$$

is large. Thus, we must have

$$\frac{kb^2}{h} \ll \left(\frac{b}{a}\right)\left(\frac{b}{h}\right) \tag{3.3}$$

However, for b/a fixed we may take h small enough to make this bound large and so allow λ^3 to be large. An interesting feature of this problem is that the boundary layers are found to have sub-domains whose widths, as measured from the edges, are of the order of λ^{-1}, i.e., they vary as the 1/3 power of h. The corresponding sub-domains for shells whose middle surface is a surface of revolution are of narrower width in that their width varies as the 1/2 power of h (15).

Another interpretation of the shape of this shell is that its middle surface is an approximation to a helicoidal surface. If $\theta = kx$ is the helicoidal surface where θ is measured in the yz-plane with the y axis as base line, then for a bounded surface with sufficiently small k, we have;

$$z = y \tan kx \approx kxy \tag{3.4}$$

The problem in part B of this chapter considers a shell with a middle surface whose undeformed shape is a helicoidal surface.

b. Formulation of the problem

The differential equations and boundary conditions for this problem are obtained by specializing the more general equations of the linear theory of shallow shells as given by Marguerre (20). This specialization was performed by Maunder and Reissner (19). The differential equations are

$$\Delta^2 F = 2Ehk \frac{\partial^2 W}{\partial x \partial y} \qquad D\Delta^2 W = -2k \frac{\partial^2 F}{\partial x \partial y} \tag{3.5}$$

where F is an Airy stress function and W the transverse deflection of the middle surface. The symbol Δ^2 denotes the bi-harmonic operator

$$\frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

F and W are related to the stress resultants as follows:

$$N_{xx} = \frac{\partial^2 F}{\partial y^2}, \quad N_{yy} = \frac{\partial^2 F}{\partial x^2}, \quad N_{xy} = -\frac{\partial^2 F}{\partial x \partial y}, \quad M_{xx} = -D\left(\frac{\partial^2 W}{\partial x^2} + \nu \frac{\partial^2 W}{\partial y^2}\right)$$

$$M_{yy} = -D\left(\frac{\partial^2 W}{\partial y^2} + \nu \frac{\partial^2 W}{\partial x^2}\right), \quad M_{xy} = -(1-\nu) D \frac{\partial^2 W}{\partial x \partial y} \tag{3.6}$$

$$V_x = -D\left(\frac{\partial^3 W}{\partial x^3} + (2-\nu)\frac{\partial^3 W}{\partial x \partial y^2}\right) + ky\,N_{xx} + kx\,N_{yy}$$

$$V_y = -D\left(\frac{\partial^3 W}{\partial y^3} + (2-\nu)\frac{\partial^3 W}{\partial y \partial x^2}\right) + ky\,N_{xy} + kx\,N_{yy}$$

The resultants denoted by N are tangential to the middle surface of the shell. The resultants denoted by V are the Kirchhoff effective shear resultants and are perpendicular to the xy-plane. Within the approximations of this formulation, W and the M resultants can be considered to be in the directions of the relevant coordinate axes. The strains and displacements of the middle surface are related to F and W as follows:

$$Eh\epsilon_x = Eh\left(\frac{\partial U}{\partial x} + ky\frac{\partial W}{\partial x}\right) = N_{xx} - \nu N_{yy}$$

$$Eh\epsilon_y = Eh\left(\frac{\partial V}{\partial y} + kx\frac{\partial W}{\partial y}\right) = N_{yy} - \nu N_{xx} \tag{3.7}$$

$$\frac{Eh}{2(1+\nu)}\gamma_{xy} = \frac{Eh}{2(1+\nu)}\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} + ky\frac{\partial W}{\partial y} + kx\frac{\partial W}{\partial x}\right) = N_{xy}$$

where U and V are parallel to the x and y axis, respectively. We prescribe the boundary conditions to be

$$\text{(i)}\ N_{yy} = N_{xy} = M_{yy} = V_y = 0 \qquad \text{for } y = \pm b \tag{3.8}$$

and

$$\text{(ii)}\ N_{xx} = N_{xy} = V_x = 0, \quad M_{xx} + kxy\,N_{xx} = M_p(y) \quad \text{at } x = a(-1)^p$$

where p = 1 or 2.

It will be convenient to put the equations in terms of non-dimensional quantities except for the dependent variables F and W which will be transformed into variables with the dimensions of a moment. Let

$$x = b\bar{x}, \quad y = b\bar{y}, \quad W = \frac{b}{D}G, \quad F = \frac{\sqrt{12(1-\nu^2)}\,b}{h}\bar{F}$$

$$\gamma = \frac{a}{b}, \quad \lambda^3 = \frac{2\sqrt{12(1-\nu^2)}\,kb^2}{h}, \quad M_p(y) = -\frac{1}{b}m_p(\bar{y}) \tag{3.9}$$

$$\bar{\Delta}^2 = \frac{1}{b^4}\left(\frac{\partial^4}{\partial\bar{x}^4} + 2\frac{\partial^4}{\partial\bar{x}^2\partial\bar{y}^2} + \frac{\partial^4}{\partial\bar{y}^4}\right)$$

and substitute these quantities into equations 3.3, 3.4 and 3.6. The bar notation will be understood in the subsequent work, so that the problem now has the formulation

$$\Delta^2 F - \lambda^3 \frac{\partial^2 G}{\partial x \partial y} = 0, \qquad \Delta^2 G + \lambda^3 \frac{\partial^2 F}{\partial x \partial y} = 0 \tag{3.10}$$

and

$$\text{(i)} \quad \frac{\partial^2 F}{\partial x^2} = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 G}{\partial y^2} + \nu \frac{\partial^2 G}{\partial x^2} = \frac{\partial^3 G}{\partial y^3} + (2-\nu) \frac{\partial^3 G}{\partial y \partial x^2} = 0 \quad \text{for } y = \pm 1$$

$$\text{(ii)} \quad \frac{\partial^2 F}{\partial y^2} = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^3 G}{\partial x^3} + (2-\nu) \frac{\partial^3 G}{\partial x \partial y^2} = 0; \quad \frac{\partial^2 G}{\partial x^2} + \nu \frac{\partial^2 G}{\partial y^2} = m_p(y) \tag{3.11}$$

$$\text{for } x = (-1)^p$$

and p = 1 or 2.

The two equations 3.10 can be conveniently written as one by forming the complex function H = F + iG so that

$$\Delta^2 H + i \lambda^3 \frac{\partial^3 H}{\partial x \partial y} = 0 \tag{3.12}$$

c. Structure of the layers

Although we are dealing with two differential equations of fourth order or equivalently with a single differential equation of eighth order, the complex function H makes it necessary only to consider the fourth order differential equation 3.12 when making a sub-equation analysis. Thus, the transformed differential equation for this problem is obtained by letting $\xi = \lambda^\alpha x$ and $\eta = \lambda^\beta y$ in equation 3.12. This gives

$$\lambda^{4\alpha} \frac{\partial^4 H}{\partial \xi^4} + 2\lambda^{2(\alpha+\beta)} \frac{\partial^4 H}{\partial \xi^2 \partial \eta^2} + \lambda^{4\beta} \frac{\partial^4 H}{\partial \eta^4} + i \lambda^{3+\alpha+\beta} \frac{\partial^2 H}{\partial \xi \partial \eta} = 0 \tag{3.13}$$

A sub-equation analysis provides the following equations.

$$\text{(a)}\quad \frac{\partial^2 H}{\partial\xi\,\partial\eta} = 0 \qquad \text{if } 2\alpha,\ 2\beta,\ \alpha+\beta < 3$$

$$\text{(b)}\quad \frac{\partial^4 H}{\partial\xi^4} = 0 \qquad \text{if } 3\beta,\ 3+\beta < 3\alpha$$

$$\text{(c)}\quad \frac{\partial^4 H}{\partial\eta^4} = 0 \qquad \text{if } 3\alpha,\ 3+\alpha < 3\beta$$

$$\text{(d)}\quad \frac{\partial^4 H}{\partial\xi^4} + i\,\frac{\partial^2 H}{\partial\xi\,\partial\eta} = 0 \qquad \text{if } 3\alpha = 3+\beta \text{ and } \beta < \alpha < \frac{3}{2} \tag{3.14}$$

$$\text{(e)}\quad \frac{\partial^4 H}{\partial\eta^4} + i\,\frac{\partial^2 H}{\partial\xi\,\partial\eta} = 0 \qquad \text{if } 3\beta = 3+\alpha \text{ and } \alpha < \beta < \frac{3}{2}$$

$$\text{(f)}\quad \frac{\partial^4 H}{\partial\xi^4} + 2\,\frac{\partial^4 H}{\partial\xi^2\,\partial\eta^2} + \frac{\partial^4 H}{\partial\eta^4} = 0 \qquad \text{if } \frac{3}{2} < \alpha = \beta$$

$$\text{(g)}\quad \frac{\partial^4 H}{\partial\xi^4} + 2\,\frac{\partial^4 H}{\partial\xi^2\,\partial\eta^2} + \frac{\partial^4 H}{\partial\eta^4} + i\,\frac{\partial^2 H}{\partial\xi\,\partial\eta} = 0 \qquad \text{if } \frac{3}{2} = \alpha = \beta$$

All other sub-equations are impossible.

Consider the x-layers with $\beta = 0$. Using the criteria, equation 3.14(d) $\alpha = 1$ for the positive value, while (e) gives $\alpha = -3$ as the negative value. Since we are dealing with a finite domain, only $\alpha = 1$ is of interest. This sub-equation for the matching corner layers must have an exponent pair $(1, \beta)$ where $\beta > 0$. This leads to equation 3.14(e) and the exponent pair $(1, \frac{4}{3})$. Continuing according to criteria B and C the next sub-equation is 3.14(d) with the exponent pair $(\frac{13}{9}, \frac{4}{3})$. It is apparent that this process may be continued indefinitely. If we denote the successive values of α and β by α_n and β_n, respectively, then we observe that they satisfy two iteration equations:

$$\alpha_n = 1 + \frac{\beta_n}{3} \qquad \text{and} \qquad \beta_{n+1} = 1 + \frac{\alpha_n}{3} \qquad \text{where } \beta_0 = 0 \tag{3.15}$$

or eliminating α_n,

$$\beta_{n+1} = \frac{4}{3} + \frac{\beta_n}{3} \qquad \text{where } \beta_0 = 0 \tag{3.16}$$

This first order difference equation gives the solution

$$\alpha_n = \frac{3}{2}\left(1 - \frac{1}{3^{2n+1}}\right), \quad \beta_n = \frac{3}{2}\left(1 - \frac{1}{3^{2n}}\right) \tag{3.17}$$

Sub-equation 3.14(d) corresponds to the exponent pairs (α_n, β_n), while 3.14(e) corresponds to the exponent pairs (α_n, β_{n+1}). Obviously, $\alpha_\infty = \beta_\infty = \frac{3}{2}$ which is the condition for 3.14(g). We see, therefore, that there is an infinite sequence of corner sub-equations and that matching continues indefinitely without ever achieving the entire differential equation as a sub-equation.

For the y-layers, beginning with $\alpha = 0$, the discussion is completely analogous since the differential equation 3.13 is symmetrical in ξ and η. We will assume that the y-layers are zero for our particular boundary conditions.

For completeness, we note here that if the domain were unbounded and the exponent pair (-3, 0) were used, we would obtain an infinite set of exponent pairs corresponding to

$$\alpha_n = -\frac{3}{2}\left(3^{2n+1} - 1\right) \quad \text{and} \quad \beta_n = -\frac{3}{2}\left(3^{2n} - 1\right) \tag{3.18}$$

These correspond to sub-equations for infinite domains. A similar statement applies to starting with the exponent pair (0, -3), again, because of the symmetry of the differential equation.

d. The inner solution

Consideration of the differential equation and boundary conditions leads us to assume that

$$H^0 = \sum_{n=0}^{\infty} H_n^0(x, y)\, \lambda^{-3n} = F^0 + iG^0 \tag{3.19}$$

where $H_n^0 = F_n^0 + iG_n^0$. Now H^0 must satisfy equation 3.11, i.e.,

$$\frac{\partial^2 H^0}{\partial x \partial y} = \frac{i}{\lambda^3}\left(\frac{\partial^4 H^0}{\partial x^4} + 2\frac{\partial^4 H^0}{\partial x^2 \partial y^2} + \frac{\partial^4 H^0}{\partial y^2}\right) \tag{3.20}$$

Formal substitution of equation 3.19 into 3.20 gives

$$\frac{\partial^2 H_n^0}{\partial x \partial y} = i\left(\frac{\partial^4 H_{n-1}^0}{\partial x^4} + 2\frac{\partial^4 H_{n-1}^0}{\partial x^2 \partial y^2} + \frac{\partial^4 H_{n-1}^0}{\partial y^4}\right) \tag{3.21}$$

as the sequence of differential equations satisfied by the 0-components. Separating the real and imaginary parts of equation 3.21, we get

$$-\frac{\partial^2 F_n^0}{\partial x \partial y} = \frac{\partial^4 G_{n-1}^0}{\partial x^4} + 2\frac{\partial^4 G_{n-1}^0}{\partial x^2 \partial y^2} + \frac{\partial^4 G_{n-1}^0}{\partial y^4} \tag{3.22}$$

$$\frac{\partial^2 G_n^0}{\partial x \partial y} = \frac{\partial^4 F_{n-1}^0}{\partial x^4} + 2\frac{\partial^4 F_{n-1}^0}{\partial x^2 \partial y^2} + \frac{\partial^4 F_{n-1}^0}{\partial y^4} \tag{3.23}$$

Integrating the first two pairs of equations and applying the boundary conditions 3.11(i) to each F_n^0 and G_n^0, we obtain

(i) $F_0^0 = \nu f_0(y)$ $\quad G_0^0 = -g_0^2(1)\frac{x^2}{2} + \nu g_0(y)$ (3.24)

(ii) $F_1^0 = \nu\left[f_1(y) - x g_0^3(y)\right]$ $\quad G_1^0 = -g_1^2(1)\frac{x^2}{2} + \nu g_1(y) - f_0^5(1)\frac{x^3}{3!} + \nu f_0^3(y)x$

provided that terms linear in x or y are ignored and

$$\text{(i) } f_0^{\,5}(1) = f_0^{\,5}(-1), \quad f_0^{\,6}(1) = f_0^{\,6}(-1) = 0 \tag{3.25}$$

$$\text{(ii) } g_k^{\,2}(1) = g_k^{\,2}(-1), \quad g_k^{\,3}(1) = g_k^{\,3}(-1) = g_0^{\,4}(1) = g_0^{\,5}(-1) = 0 \quad \text{for } k=1 \text{ or } 2$$

Here the superscripts on the "lower case" functions such as f_n and g_n are to indicate the order of their derivatives, e.g., $d^k f_n / dy^k = f_n^{\,k}$ if $k \neq 0$. This notation will be used in the sequel. It is obvious that equations 3.25 do not exhaust the conditions for $y = \pm 1$ which would be obtained for these functions if we continued the process to components with higher subscripts. However, all further conditions will be upon higher derivatives of the functions.

e. The x-boundary layers

Let

$$H^p = H^p(\xi, y) \quad \text{where } \xi = \lambda\left[\gamma - (-1)^p x\right] \quad \text{for } p = 1 \text{ or } 2 \tag{3.26}$$

so that ξ is the layer variable corresponding to the exponent pair (1, 0). Using this transformation, equation 3.11 gives the transformed differential equation

$$\frac{\partial^4 H}{\partial \xi^4} - i(-1)^p \frac{\partial^2 H}{\partial \xi \partial y} = -\frac{2}{\lambda^2}\frac{\partial^4 H}{\partial \xi^2 \partial y^2} - \frac{1}{\lambda^4}\frac{4H}{\partial y^4} \tag{3.27}$$

Let us assume that

$$H^p = \sum_{n=0}^{\infty} H_n^{\,p}(\xi, y)\,\lambda^{-(2+n)} = F^p + iG^p \quad \text{where } H_n^{\,p} = F_n^{\,p} + iG_n^{\,p} \tag{3.28}$$

so that formal substitution of H^p for H in equation 3.27 gives

$$\frac{\partial^4 H_n^p}{\partial \xi^4} - i(-1)^p \frac{\partial^2 H_n^p}{\partial \xi \partial y} = -2 \frac{\partial^4 H_{n-2}^p}{\partial \xi^2 \partial y^2} - \frac{\partial^4 H_{n-4}^p}{\partial y^4} \tag{3.29}$$

as the sequence of differential equations satisfied by the p-components. Separating the real and imaginary parts of equation 3.29, we get

$$\frac{\partial^4 F_n^p}{\partial \xi^4} + (-1)^p \frac{\partial^2 G_n^p}{\partial \xi \partial y} = -2 \frac{\partial^4 F_{n-2}^p}{\partial \xi^2 \partial y^2} - \frac{\partial^4 F_{n-4}^p}{\partial y^4} \tag{3.30}$$

and

$$\frac{\partial^4 G_n^p}{\partial \xi^4} - (-1)^p \frac{\partial^2 F_n^p}{\partial \xi \partial y} = -2 \frac{\partial^4 G_{n-2}^p}{\partial \xi^2 \partial y^2} - \frac{\partial^4 G_{n-4}^p}{\partial y^4} \tag{3.31}$$

We note here that for n = 0 equations 3.30 and 3.31 form a set of differential equations which contain y-derivatives and so cannot be treated as essentially ordinary differential equations in ξ as is done in what we have designated as classical problems. Our method will provide the additional boundary conditions for these equations which are necessary to determine a well defined boundary value problem. The statements made in connection with equation 1.27 illustrate this distinction between our problems and the classical ones. The boundary conditions at a p-edge which involve the inner solution and a p-layer are obtained by formally substituting the sums $F^0 + F^p$ and $G^0 + G^p$ for F and G, respectively, in equations 3.11(ii). Thus,

$$\text{(i)}\ \frac{\partial^2 F^0}{\partial y^2} + \frac{\partial^2 F^p}{\partial y^2} = 0 \qquad \text{(ii)}\ \frac{\partial^2 F^0}{\partial x \partial y} - \lambda(-1)^p \frac{\partial^2 F^p}{\partial \xi \partial y} = 0$$

$$\text{(iii)}\ \frac{\partial^3 G^0}{\partial x^3} + (2-\nu)\frac{\partial^3 G^0}{\partial x \partial y^2} - (-1)^p \left(\lambda^3 \frac{\partial^3 G^p}{\partial \xi^3} + (2-\nu)\lambda \frac{\partial^3 G^p}{\partial \xi \partial y^2} \right) = 0 \tag{3.32}$$

$$\text{(iv)}\ \frac{\partial^2 G^0}{\partial x^2} + \nu \frac{\partial^2 G^0}{\partial y^2} + \lambda^2 \frac{\partial^2 G^p}{\partial \xi^2} + \nu \frac{\partial^2 G^p}{\partial y^2} = m_p(y)$$

all at $x = (-1)^p \gamma$. Now substituting the series in equations 3.19 and 3.28 into 3.32 we obtain the boundary conditions upon the components.

$$\text{(i)} \quad \frac{\partial^2 F_n^0}{\partial y^2} + \frac{\partial^2 F_{3n-2}^p}{\partial y^2} = 0 \qquad \text{(ii)} \quad \frac{\partial^2 F_n^0}{\partial x \partial y} - (-1)^p \frac{\partial^2 F_{3n-1}^p}{\partial \xi \partial y} = 0$$

$$\text{(iii)} \quad \frac{\partial^3 G_{3n-2}^p}{\partial \xi^3} - (-1)^p \left[\frac{\partial^3 G_{n-1}^0}{\partial x^3} + (2-\nu) \frac{\partial^3 G_{n-1}^0}{\partial x \partial y^2} \right] + (2-\nu) \frac{\partial^3 G_{3n-4}^p}{\partial \xi \partial y^2} = 0 \quad (3.33)$$

$$\text{(iv)} \quad \frac{\partial^2 G_n^0}{\partial x^2} + \nu \frac{\partial^2 G_n^0}{\partial y^2} + \frac{\partial^2 G_{3n}^p}{\partial \xi^2} + \nu \frac{\partial^2 G_{3n-2}^p}{\partial y^2} = m_p(y)$$

and

$$\frac{\partial^2 F_{k-2}^p}{\partial y^2} = \frac{\partial^2 F_{k-1}^p}{\partial \xi \partial y} = \frac{\partial^3 G_{k-2}^p}{\partial \xi^3} + (2-\nu) \frac{\partial^3 G_{k-4}^p}{\partial \xi \partial y^2} = \frac{\partial^2 G_k^p}{\partial \xi^2} + \nu \frac{\partial^2 G_{k-2}^p}{\partial y^2} = 0 \quad (3.34)$$

for the relevant $\xi = 0$ and k a positive integer not equal to 3n.

f. <u>The x-corner layers</u>

Let

$$H^{pq} = H^{pq}(\xi, \eta) \quad (3.35)$$

where ξ and η are the layer variables corresponding to the exponent pair $(1, \frac{4}{3})$. Thus, in accordance with criterion B(i), we take ξ as defined by equation 3.26. The η variable for these layers must now have the form,

$$\eta = \lambda^{4/3} \left[1 - (-1)^q y \right] \quad (3.36)$$

Using these layer variables, equation 3.11 gives the transformed differential equation

$$\frac{\partial^4 H}{\partial \eta^4} + i(-1)^{p+q} \frac{\partial^2 H}{\partial \xi \partial \eta} = -\frac{2}{\lambda^{2/3}} \frac{\partial^2 H}{\partial \xi^2 \partial \eta^2} - \frac{1}{\lambda^{1+\frac{1}{3}}} \frac{\partial^4 H}{\partial \xi^4} \tag{3.37}$$

Let us assume that

$$H^{pq} = \sum_{n=0}^{\infty} H_n^{pq}(\xi, \eta) \lambda^{-(2+\frac{n}{3})} = F^{pq} + iG^{pq} \tag{3.38}$$

where $H_n^{pq} = F_n^{pq} + iG_n^{pq}$. Thus, formal substitution of H^{pq} for H in equation 3.43 gives

$$\frac{\partial^4 H_n^{pq}}{\partial \eta^4} + i(-1)^{p+q} \frac{\partial^2 H_n^{pq}}{\partial \xi \partial \eta} = -2 \frac{\partial^4 H_{n-2}^{pq}}{\partial \xi^2 \partial \eta^2} - \frac{\partial^4 H_{n-4}^{pq}}{\partial \xi^4} \tag{3.39}$$

as the sequence of differential equations satisfied by the pq-components. In terms of the real quantities these equations become,

$$\frac{\partial^4 F_n^{pq}}{\partial \eta^4} - (-1)^{p+q} \frac{\partial^2 G_n^{pq}}{\partial \xi \partial \eta} = -2 \frac{\partial^4 F_{n-2}^{pq}}{\partial \xi^2 \partial \eta^2} - \frac{\partial^4 F_{n-4}^{pq}}{\partial \xi^4} \tag{3.40}$$

and

$$\frac{\partial^4 G_n^{pq}}{\partial \eta^4} + (-1)^{p+q} \frac{\partial^2 F_n^{pq}}{\partial \xi \partial \eta} = -2 \frac{\partial^4 G_{n-2}^{pq}}{\partial \xi^2 \partial \eta^2} - \frac{\partial^4 G_{n-4}^{pq}}{\partial \xi^4} \tag{3.41}$$

The boundary conditions at a q-edge which involve a p-layer and a pq-layer are obtained by formally substituting the sums $F^p + F^{pq}$ and $G^p + G^{pq}$ for F and G, respectively, in equations 3.11(i). Thus,

(i) $\frac{\partial^2 F^p}{\partial \xi^2} + \frac{\partial^2 F^{pq}}{\partial \xi^2} = 0$ (ii) $\frac{\partial^2 F^p}{\partial \xi \partial y} - (-1)^q \lambda^{4/3} \frac{\partial^2 F^{pq}}{\partial \xi \partial \eta} = 0$

$$\text{(iii)} \ \frac{\partial^2 G^p}{\partial y^2} + \nu\lambda^2 \frac{\partial^2 G^p}{\partial \xi^2} + \lambda^{8/3} \frac{\partial^2 G^{pq}}{\partial \eta^2} + \nu \lambda^2 \frac{\partial^2 G^{pq}}{\partial \xi^2} = 0 \tag{3.42}$$

$$(iv)\ \frac{\partial^3 G^p}{\partial y^3} + (2-\nu)\,\lambda^2 \frac{\partial^2 G^p}{\partial y \partial \xi^2} - (-1)^q \left[\lambda^4 \frac{\partial^3 G^{pq}}{\partial \eta^3} + (2+\nu)\,\lambda^{10/3} \frac{\partial^3 G^{pq}}{\partial \eta \partial \xi^2}\right] = 0$$

for $y = (-1)^q$. Now substituting the series in equations 3.28 and 3.44 into 3.48, we obtain the boundary conditions upon the components. At $\eta = 0$,

$$(i)\ \frac{\partial^2 F_n^p}{\partial \xi^2} + \frac{\partial^2 F_{3n}^{pq}}{\partial \xi^2} = 0 \qquad (ii)\ \frac{\partial^2 F_{3n+1}^{pq}}{\partial \xi \partial \eta} - (-1)^q \frac{\partial^2 F_{n-1}^p}{\partial \xi \partial y} = 0$$

$$(iii)\ \frac{\partial^2 G_{3n+2}^{pq}}{\partial \eta^2} + \nu \frac{\partial^2 G_{3n}^{pq}}{\partial \xi^2} + \nu \frac{\partial^2 G_n^p}{\partial \xi^2} + \frac{\partial^2 G_{n-2}^p}{\partial y^2} = 0 \tag{3.43}$$

$$(iv)\ \frac{\partial^3 G_{3n}^{pq}}{\partial \eta^3} + (2-\nu) \frac{\partial^3 G_{3n-2}^{pq}}{\partial \eta \partial \xi^2} - (-1)^q (2-\nu) \frac{\partial^2 G_{n-4}^p}{\partial y \partial \xi^2} - (-1)^q \frac{\partial^3 G_{n-6}^p}{\partial y^3} = 0$$

and

$$\frac{\partial^2 F_k^{pq}}{\partial \xi^2} = \frac{\partial^2 F_{k-2}^{pq}}{\partial \xi \partial \eta} = \frac{\partial^2 G_{k-1}^{pq}}{\partial \eta^2} + \nu \frac{\partial^2 G_{k-3}^{pq}}{\partial \xi^2} = \frac{\partial^3 G_k^{pq}}{\partial \eta^3} + (2-\nu) \frac{\partial^3 G_{k-2}^{pq}}{\partial \eta \partial \xi^2} = 0 \tag{3.44}$$

for k a positive integer and $k \neq 3n$.

g. First components

It will be convenient to list some of the results for the first few components. In the following equations the expressions for F_n^0 and G_n^0 which are given in 3.24 will be substituted into equations 3.33 and 3.34. Thus, if $\xi = 0$ we have;

(i) $f_0^2(y) = 0$ (ii) and (iii) are automatically satisfied

$$(iv)\ \frac{\partial^2 G_0^p}{\partial \xi^2} - g_0^2(1) + \nu^2 g_0^2(y) = m_p(y) \tag{3.45}$$

for n = 1 in 3.33

$$\text{(i)}\quad \nu\left[f_1^{\,2}(y) - \gamma(-1)^p g_0^{\,5}(y)\right] + \frac{\partial^2 F_1^p}{\partial y^2} = 0 \qquad \text{(ii)}\quad \nu g_0^{\,4}(y) + (-1)^p \frac{\partial^2 F_2^p}{\partial \xi\,\partial y} = 0$$

$$\text{(iii)}\quad \frac{\partial^3 G_1^p}{\partial \xi^3} = 0 \qquad \text{(iv) not needed} \tag{3.46}$$

from 3.34

$$\frac{\partial^2 F_0^p}{\partial y^2} = \frac{\partial^2 F_0^p}{\partial \xi\,\partial y} = \frac{\partial^2 F_1^p}{\partial \xi\,\partial y} = \frac{\partial^3 G_0^p}{\partial \xi^3} = \frac{\partial^3 G_0^p}{\partial \xi^3} = \frac{\partial^2 G_1^p}{\partial \xi^2} = 0 \tag{3.47}$$

The other equations pertain to higher order components. From 3.45(i) it follows that $F_0^0 \equiv 0$. We now seek to determine $\partial^2 F_1^p/\partial y^2$ in 3.46(i). From equation 3.31 for n=1, we have $\partial^4 G_1^p/\partial \xi^4 - (-1)^p\ \partial^2 F_1^p/\partial \xi\,\partial y = 0$. Since the dependent functions are layers as described in criterion A, we may integrate once to get

$$\frac{\partial^3 G_1^p}{\partial \xi^3} - (-1)^p \frac{\partial F_1^p}{\partial y} = 0 \tag{3.48}$$

The arbitrary function of y that would appear here must be zero since the solution and its derivatives must decay exponentially as ξ increases.

Now, as described in criterion D, this equation is required to be consistent with the boundary conditions. Therefore, if $\partial^3 G_1^p/\partial \xi^3 = 0$ as given in 3.46(iii) then $\partial F_1^p/\partial y = 0$ also at $\xi = 0$. Differentiating this with respect to y, we have

$$\frac{\partial^2 F_1^p}{\partial y^2} = 0 \quad \text{at} \quad \xi = 0 \tag{3.49}$$

It follows that $f_1^{\,2}(y) + \gamma g_0^{\,5}(y) = f_1^{\,2}(y) - \gamma g_0^{\,5}(y) = 0$ or that

$$f_1^2(y) = g_0^5(y) = 0 \tag{3.50}$$

Thus, $g_0^3(y)$ is linear in y but conditions 3.25(ii) require that this linear function be identically zero. This means that $F_1^0 \equiv 0$, see equation 3.24(ii), and that $g_0^2(y)$ is a constant. The equations in 3.25(ii) are all satisfied. We may now write 3.45(iv) in the form

$$\frac{\partial^2 G_0^p}{\partial \xi^2} - (1-\nu^2)\, g_0^2 = m_p(y) \tag{3.51}$$

where g_0^2 is a constant. Before determining this constant, we consider the boundary value problem for F_0^p and G_0^p. We seek boundary conditions for $y = \pm 1$. To find such conditions we must investigate the boundary conditions involving the corner layers. From equations 3.43 and 3.44 we obtain the four pertinent equations;

$$\text{(i)}\ \frac{\partial^2 F_0^p}{\partial \xi^2} + \frac{\partial^2 F_0^{pq}}{\partial \xi^2} = 0 \quad \text{(ii)}\ \frac{\partial^2 F_0^{pq}}{\partial \xi\, \partial \eta} = 0 \quad \text{(iii)}\ \frac{\partial^2 G_0^{pq}}{\partial \eta^2} = 0 \quad \text{(iv)}\ \frac{\partial^3 G_0^{pq}}{\partial \eta^3} = 0 \tag{3.52}$$

all for $\eta = 0$. Equation 3.41 for n = 0 may be integrated once to give

$$\frac{\partial^3 G_0^{pq}}{\partial \eta^3} + (-1)^{p+q} \frac{\partial F_0^{pq}}{\partial \xi} = 0 \tag{3.53}$$

As before, the arbitrary function of ξ that would appear here must be zero since it must decay exponentially with η. As described in criterion D, we require the differential equation 3.53 to be consistent with the boundary condition 3.52(iv). This gives $\partial F_0^{pq}/\partial \xi = 0$ at $\eta = 0$, which may be integrated to the form

$$F_0^{pq} = 0$$

However, such integration can also be performed on equation 3.52(i) to give $F_0^p + F_0^{pq} = 0$ at $y = \pm 1$. It follows that

$$F_0^p = 0 \quad \text{at } y = \pm 1. \tag{3.54}$$

Now the constant g_0^2 in equation 3.51 can be evaluated by finding the mean value of $\partial^2 G_0^p/\partial \xi^2$. The equation analogous to equation 3.48 which involves G_0^p and F_0^p is

$$\frac{\partial^3 G_0^p}{\partial \xi^3} - (-1)^p \frac{\partial F_0^p}{\partial y} = 0 \tag{3.55}$$

Define

$$\tilde{f} = \int_{-1}^{1} f(y)dy \tag{3.56}$$

so that we may integrate equation 3.55 with respect of y to get

$$\frac{\partial^3 \tilde{G}_0^p}{\partial \xi^3} = (-1)^p \left[F_0^p(1) - F_0^p(-1) \right] = 0$$

or integrating this with respect to ξ, we get

$$\tilde{G}_0^p = 0 \tag{3.57}$$

It follows from equation 3.51 that

$$g_0^2 = -\frac{1}{2(1-\nu^2)} \int_{-1}^{1} m_p(y)dy = -\frac{m}{2(1-\nu^2)} \tag{3.58}$$

where

$$m = \int_{-1}^{1} m_1(y)dy = \int_{-1}^{1} m_2(y)dy$$

The boundary value problem for the first components of the p-boundary layers may be formulated as follows:

(i) $\dfrac{\partial^3 F_0^p}{\partial \xi^3} + (-1)^p \dfrac{\partial G_0^p}{\partial y} = 0$	$\dfrac{\partial^3 G_0^p}{\partial \xi^3} - (-1)^p \dfrac{\partial F_0^p}{\partial y} = 0$

and for $\xi = 0$ (3.59)

(ii) $\dfrac{\partial^2 G_0^p}{\partial \xi^2} = m_p(y) - \dfrac{m}{2}$ (iii) $\dfrac{\partial^2 F_0^p}{\partial \xi \partial y} = \dfrac{\partial^3 G_0^p}{\partial \xi^3} = 0$

while at $y = \pm 1$

(iv) $F_0^p = 0$

Equations 3.59(i) follow from 3.30 and 3.31. Equation 3.59(ii) follows from equations 3.51 and 3.58. Equation 3.59(iii) comes from 3.47. Note that $\partial^2 F_0^p / \partial y^2 = 0$ is a redundant condition which follows from equation 3.59(iii) and the requirement that the differential equations be consistent with this. This is analogous to equation 3.49.

To solve this problem we assume that

$$F_0^p = \sum_{n=1}^{\infty} a_n(\xi) \sin \frac{n\pi}{2}(1+y); \quad G_0^p = \sum_{n=1}^{\infty} b_n(\xi) \cos \frac{n\pi}{2}(1+y) \tag{3.60}$$

and substitute into equations 3.59(i). We get

$$a_n^{3}(\xi) - \left(\frac{n\pi}{2}\right)(-1)^p b_n(\xi) = 0 \quad \text{and} \quad b_n^{3}(\xi) - \left(\frac{n\pi}{2}\right)(-1)^p a_n(\xi) = 0$$

If $a_n(\xi)$ is eliminated the single equation

$$b_n^{6}(\xi) - \left(\frac{n\pi}{2}\right)^2 b_n(\xi) = 0 \tag{3.61}$$

results. Now b_n must be a sum of functions of the form $e^{\gamma_n r \xi}$ where $\gamma_n = (n\pi/2)^{1/3}$ and $r^6 = 1$. Thus, $r = 1, -1, \frac{1}{2}(1 \pm i\sqrt{3}), -\frac{1}{2}(1 \pm i\sqrt{3})$ but we can use only those r which have negative real parts in order to insure exponential decay. We conclude, therefore, that

$$b_n(\xi) = A_n e^{-\gamma_n \xi} + B_n e^{-\gamma_n \omega_+ \xi} + C_n e^{-\gamma_n \omega_- \xi} \tag{3.62}$$

and

$$a_n(\xi) = \left[A_n e^{-\gamma_n \xi} - B_n e^{-\gamma_n \omega_+ \xi} - C_n e^{-\gamma_n \omega_- \xi}\right](-1)^{p+1}$$

where

$$\omega_\pm = \frac{1}{2}(1 \pm i\sqrt{3})$$

The condition that $\partial^2 F_0^p / \partial \xi \, \partial y = 0$ at $\xi = 0$ can be satisfied if we take

$$A_n - \omega_+ B_n - \omega_- C_n = 0 \tag{3.63}$$

The condition $\partial^3 G_0^p / \partial \xi^3 = 0$ at $\xi = 0$ is satisfied if

$$A_n - B_n - C_n = 0 \tag{3.64}$$

Solving equations 3.63 and 3.64 for B_n and C_n, we have

$$B_n = \frac{1+\omega_-}{3} A_n \quad \text{and} \quad C_n = \frac{1+\omega_+}{3} A_n \tag{3.65}$$

Thus,

$$F_0^p = (-1)^{p+1} \sum_{n=1}^{\infty} A_n \left(e^{-\gamma_n \xi} - \frac{1+\omega_-}{3} e^{-\gamma_n \omega_+ \xi} - \frac{1+\omega_+}{3} e^{-\gamma_n \omega_- \xi} \right) \sin \frac{n\pi}{2}(1+y) \tag{3.66}$$

$$G_0^p = \sum_{n=1}^{\infty} A_n \left(e^{-\gamma_n \xi} + \frac{1+\omega_-}{3} e^{-\gamma_n \omega_+ \xi} + \frac{1+\omega_+}{3} e^{-\gamma_n \omega_- \xi} \right) \cos \frac{n\pi}{2} (1+y)$$

where

$$A_n = \frac{1}{\gamma_n^2} \int_{-1}^{1} m_p(y) \cos \frac{n\pi}{2} (1+y) dy \quad \text{and} \quad \gamma_n = \left(\frac{n\pi}{2}\right)^{1/3}$$

in order that condition 3.59(ii) be satisfied. In summary then, if the corners are excluded,

$$F \sim \frac{1}{2} (F_0^1 + F_0^2) + 0(\frac{1}{\lambda^3}) \text{ and } G \sim - \frac{m}{4(1-\nu^2)} (x^2 - \nu y^2) + \frac{1}{\lambda^2} (G_0^1 + G_0^2) + 0(\frac{1}{\lambda^3}) \tag{3.67}$$

h. Results

If equation 3.67 is put in terms of the original variables then

$$F \sim \frac{\sqrt{12(1-\nu^2)}b}{h\lambda^2} (F_0^1 + F_0^2) + 0\left(\frac{b}{h\lambda^3}\right) \tag{3.68}$$

and

$$W \sim - \frac{M}{4b(1-\nu^2)D} (x^2 - \nu y^2) + \frac{b}{D\lambda^2} (G_0^1 + G_0^2) + 0(\frac{b}{D\lambda^2}) \tag{3.69}$$

where

$$F_0^1 = \sum_{n=1}^{\infty} A_n^1 \left\{ e^{-2\delta_n \lambda (a+x)} - \frac{2\sqrt{3}}{3} e^{-\delta_n \lambda (a+x)} \cos\left(\delta_n \lambda \sqrt{3} (a+x) + \frac{\pi}{6}\right) \right\} \sin \frac{n\pi}{2b} (b+y)$$

$$F_0^2 = -\sum_{n=1}^{\infty} A_n^2 \left\{ e^{-2\delta_n \lambda (a-x)} - \frac{2\sqrt{3}}{3} e^{-\delta_n \lambda (a-x)} \cos\left(\delta_n \lambda \sqrt{3}\,(a-x) + \frac{\pi}{6}\right) \right\} \sin \frac{n\pi}{2b}(b+y)$$

$$G_0^1 = \sum_{n=1}^{\infty} A_n^1 \left\{ e^{-2\delta_n \lambda (a+x)} + \frac{2\sqrt{3}}{3} e^{-\delta_n \lambda (a+x)} \cos\left(\delta_n \lambda \sqrt{3}\,(a+x) + \frac{\pi}{6}\right) \right\} \cos \frac{n\pi}{2b}(b+y)$$

$$G_0^2 = \sum_{n=1}^{\infty} A_n^2 \left\{ e^{-2\delta_n \lambda (a-x)} + \frac{2\sqrt{3}}{3} e^{-\delta_n \lambda (a-x)} \cos\left(\delta_n \lambda \sqrt{3}\,(a-x) + \frac{\pi}{6}\right) \right\} \cos \frac{n\pi}{2b}(b+y)$$

$$A_n^p = -\frac{1}{4b^2 \delta_n^2} \int_{-b}^{b} M_p(y) \cos \frac{n\pi}{2b}(b+y)\,dy$$

$$\delta_n = \frac{1}{2b}\left(\frac{n\pi}{2}\right)^{1/3} ; \qquad \lambda = \left[12(1-\nu^2)\right]^{1/6} \left(\frac{2kb^2}{h}\right)^{1/3}$$

$$M = \int_{-b}^{b} M_1(y)\,dy = \int_{-b}^{b} M_2(y)\,dy$$

B. Bending of a Helicoidal Shell

a. Introduction

Let us consider the bending of a shell whose undeformed middle surface is a right helicoid described by the equations

$$z = a\theta, \qquad x = r\cos\theta, \qquad y = r\sin\theta \tag{3.70}$$

where

$$-b \leq r \leq b \quad \text{and} \quad -\gamma \leq \theta \leq \gamma$$

The shell represents a pretwisted strip having a total twist 2γ, a width $2b$, a pitch $2\pi a$ and an axial length $2a\gamma$. We assume that it consists of elastic isotropic material and has a uniform thickness h. For simplicity, we will consider only the cases where $\lambda = n\pi$ with n a positive integer. Let the shell be subjected only to the distributed bending moments $M_1(r)$ and $M_2(r)$ along the edges $\theta = -\gamma$ and $\theta = \gamma$, respectively.* The conditions of static equilibrium require that

$$M = \int_{-b}^{b} M_1(r)dr = \int_{-b}^{b} M_2(r)dr \tag{3.71}$$

Figure 15 depicts the shell schematically.

The equations governing this problem have been given by Knowles and Reissner (22). They have used these equations to solve the rotationally symmetric problems for the helicoidal shell under axial torsion and tension (23). Their solutions are given in terms of series expansions in powers of the parameter $k = \frac{b}{a}$, i.e., their

---

* M_1 and M_2 are moments per unit length of the edge of the middle surface and will be designated as stress couples.

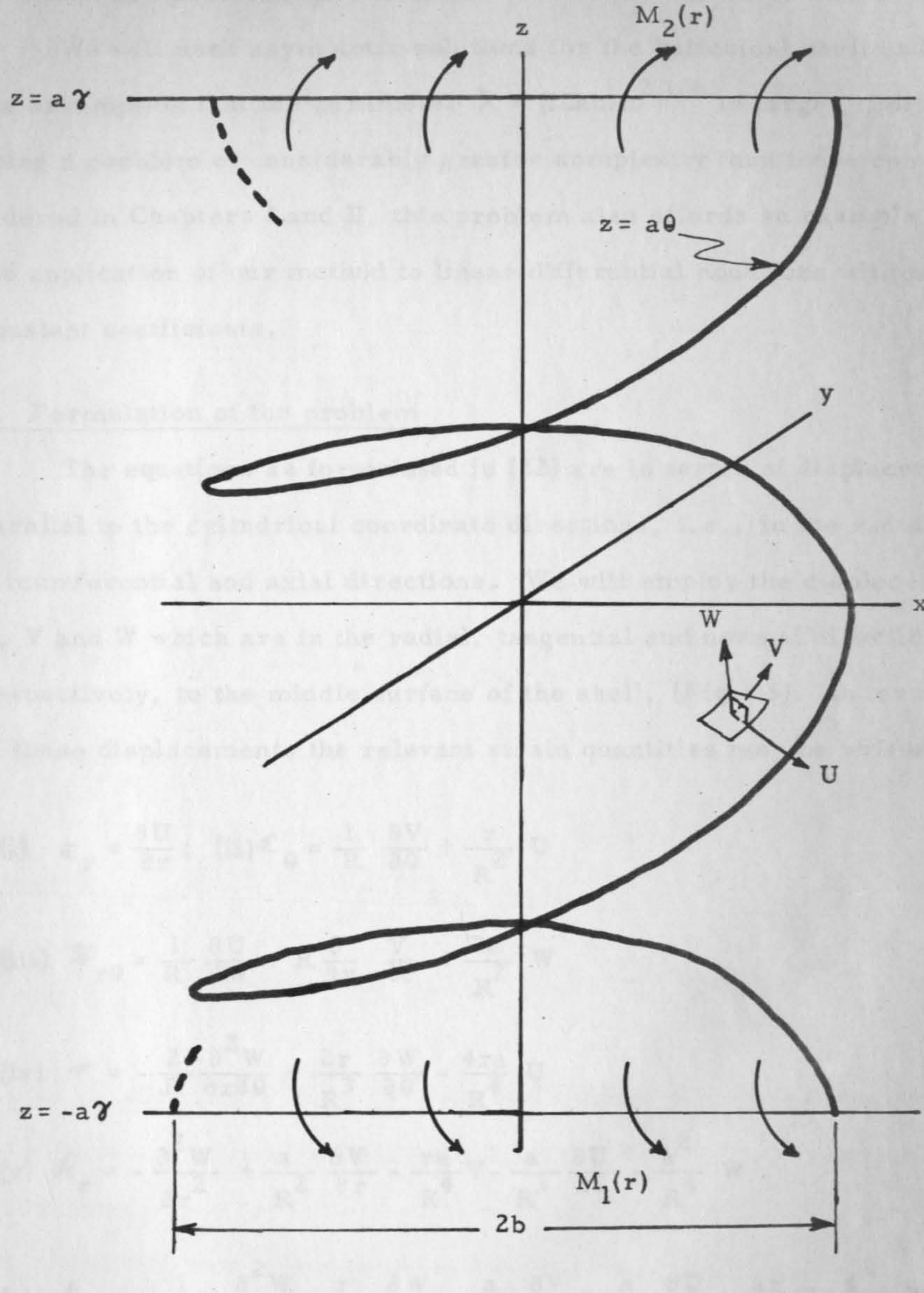


Figure 15. Bending of a Helicoidal Shell

solutions are perturbations from the case of the untwisted strip.

We will seek asymptotic solutions for the helicoidal shell under the assumption that the parameter $\lambda = (12ab/h^2)^{1/6}$ is large. Besides being a problem of considerably greater complexity than those considered in Chapters I and II, this problem also affords an example of the application of our method to linear differential equations with non-constant coefficients.

b. Formulation of the problem

The equations as formulated in (22) are in terms of displacements parallel to the cylindrical coordinate directions, i.e., in the radial, circumferential and axial directions. We will employ the displacements U, V and W which are in the radial, tangential and normal directions, respectively, to the middle surface of the shell, (Fig. 15). In terms of these displacements the relevant strain quantities may be written:

$$\text{(i)}\quad \epsilon_r = \frac{\partial U}{\partial r}\,; \quad \text{(ii)}\ \epsilon_\theta = \frac{1}{R}\frac{\partial V}{\partial \theta} + \frac{r}{R^2}U$$

$$\text{(iii)}\quad \gamma_{r\theta} = \frac{1}{R}\frac{\partial U}{\partial \theta} + R\frac{\partial}{\partial r}\frac{V}{R} + \frac{2a}{R^2}W$$

$$\text{(iv)}\quad \tau = -\frac{2}{R}\frac{\partial^2 W}{\partial r \partial \theta} + \frac{2r}{R^3}\frac{\partial W}{\partial \theta} - \frac{4ra}{R^4}U \qquad (3.72)$$

$$\text{(v)}\quad k_r = -\frac{\partial^2 W}{\partial r^2} + \frac{a}{R^2}\frac{\partial V}{\partial r} - \frac{ra}{R^4}V - \frac{a}{R^3}\frac{\partial U}{\partial \theta} - \frac{a^2}{R^4}W$$

$$\text{(vi)}\quad k_\theta = -\frac{1}{R^2}\frac{\partial^2 W}{\partial \theta^2} - \frac{r}{R^2}\frac{\partial W}{\partial r} - \frac{a}{R^2}\frac{\partial V}{\partial r} + \frac{a}{R^3}\frac{\partial U}{\partial \theta} - \frac{ar}{R^4}V - \frac{a^2}{R^4}W$$

where $R = (a^2 + r^2)^{1/2}$

The stress resultants (N) and the stress couples (M) are related to the strain quantities as follows:

$$\text{(i)}\quad N_r = C(\epsilon_r + \nu\epsilon_\theta) - \frac{a}{R^2} D\left[\tau - (1-\nu)\frac{a}{R^2}(\epsilon_r - \epsilon_\theta)\right]$$

$$\text{(ii)}\quad N_\theta = C(\epsilon_\theta + \nu\epsilon_r) - \frac{a}{R^2} D\left[\tau - (1-\nu)\frac{a}{R^2}(\epsilon_\theta - \epsilon_r)\right]$$

$$\text{(iii)}\quad N_{r\theta} = C\frac{(1-\nu)}{2}\gamma_{r\theta} - \frac{a}{2R^2} D\left[(1+\nu)\left[k_r + (3-\nu)k_\theta\right]\right]$$

$$\text{(iv)}\quad N_{\theta r} = C\frac{(1-\nu)}{2}\gamma_{r\theta} - \frac{a}{2R^2} D\left[(1+\nu)k_\theta + (3-\nu)k_r\right] \qquad (3.73)$$

$$\text{(v)}\quad M_r = D\left[k_r + \nu k_\theta - \frac{1-\nu}{2}\frac{a}{R^2}\gamma_{r\theta}\right]$$

$$\text{(vi)}\quad M_\theta = D\left[k_\theta + \nu k_r - \frac{1-\nu}{2}\frac{a}{R^2}\gamma_{r\theta}\right]$$

$$\text{(vii)}\quad M_{r\theta} = D\left[\frac{1-\nu}{2}\tau - \frac{a}{R^2}(\epsilon_\theta + \nu\epsilon_r)\right]$$

$$\text{(viii)}\quad M_{\theta r} = D\left[\frac{1-\nu}{2}\tau - \frac{a}{R^2}(\epsilon_r + \nu\epsilon_\theta)\right]$$

where $D = Eh^3/12(1-\nu^2)$ and $C = Eh/1-\nu^2$. Here E and ν are the Young's modulus and Poisson's ratio, respectively, for the material of the plate.

The differential equations of the problem are given by the equilibrium equations:

$$\text{(i)}\ \frac{\partial}{\partial r}(RN_r) + \frac{\partial N_{\theta r}}{\partial \theta} - \frac{r}{R} N_\theta + \frac{a}{R} Q_\theta = 0$$

$$\text{(ii)}\ \frac{\partial}{\partial r}(RN_{r\theta}) + \frac{\partial N_\theta}{\partial \theta} + \frac{r}{R} N_{\theta r} + \frac{a}{R} Q_r = 0$$

$$\text{(iii)}\ \frac{\partial}{\partial r}(RQ_r) + \frac{\partial Q_\theta}{\partial \theta} - \frac{a}{R}(N_{r\theta} + N_{\theta r}) = 0 \qquad (3.74)$$

$$\text{(iv)}\ \frac{\partial}{\partial r}(RM_r) + \frac{\partial M_{\theta r}}{\partial \theta} - \frac{r}{R} M_\theta - RQ_r = 0$$

$$\text{(v)}\ \frac{\partial}{\partial r}(RM_{r\theta}) + \frac{\partial M_\theta}{\partial \theta} + \frac{r}{R} M_{\theta r} - RQ_\theta = 0$$

$$\text{(vi)}\ N_{r\theta} - N_{\theta r} + \frac{a}{R^2}(M_\theta - M_r) = 0$$

Equations 3.74(iv) and (v) can be considered as defining the Q stress resultants. Equation 3.74(vi) is an identity which is useful in calculations. It is physically a consequence of the equilibrium of moments about the normal to the shell. The boundary conditions are:

at $r = \pm b$:

$$\text{(i)}\ N_r + \frac{a}{R^2} M_{r\theta} = 0 \qquad \text{(zero radial stress)}$$

$$\text{(ii)}\ N_{r\theta} = 0 \qquad \text{(zero tangential stress)} \qquad (3.75)$$

$$\text{(iii)}\ Q_r + \frac{1}{R}\frac{\partial M_{r\theta}}{\partial \theta} = 0 \qquad \text{(zero transverse shear stress)}$$

$$\text{(iv)}\ M_r = 0 \qquad \text{(zero bending moment)}$$

at $\theta = (-1)^p \gamma$:

(i) $N_\theta + \frac{a}{R^2} M_{\theta r} = 0$ (zero tangential stress)

(ii) $N_{\theta r} = 0$ (zero radial stress)

(iii) $Q_\theta + \frac{\partial M_{\theta r}}{\partial r} = 0$ (zero transverse shear stress) (3.76)

(iv) $M_\theta = M_p(r)$ for p = 1 or 2. (prescribed bending couple)

We now put the equations in a more convenient form by the transformations:

$$d = b\overline{d}, \quad N = C\overline{N}, \quad M = \frac{D}{b}\overline{M}, \quad Q = D(b^2\overline{R})^{-1}\overline{Q}$$

$$R = a\overline{R}, \quad k = \frac{b}{a}, \quad \lambda = \left(\frac{12ab}{h^2}\right)^{1/6} \tag{3.77}$$

where d stands for the quantities x, y, z, r, U, V, W, τ, κ_r and κ_θ. Also N, Q and M stand for any of the corresponding stress resultants and stress couples. Assuming that these transformations have been made, we will drop the bar notation in the sequel and take the equations to be formulated as follows;

For the strain quantities,

(i) $\epsilon_r = \frac{\partial U}{\partial r}$ (ii) $\epsilon_\theta = k \frac{\partial}{\partial \theta}\left(\frac{V}{R}\right)$

(iii) $\gamma_{r\theta} = k \frac{\partial}{\partial \theta}\left(\frac{U}{R}\right) + R \frac{\partial}{\partial r}\left(\frac{V}{R}\right) + \frac{2kW}{R^2}$

$$\text{(iv)} \quad \tau = -2k \frac{\partial^2}{\partial r \partial \theta}\left(\frac{W}{R}\right) - \frac{4rk^3}{R^4} U$$

$$\text{(v)} \quad \kappa_r = -\frac{\partial^2 W}{\partial r^2} - \frac{k^2 W}{R^4} - \frac{k}{R}\frac{\partial}{\partial r}\left(\frac{V}{R}\right) - \frac{k^2}{R^3}\frac{\partial U}{\partial \theta} \tag{3.78}$$

$$\text{(vi)} \quad \kappa_\theta = -\frac{k^2}{R^2}\frac{\partial^2 W}{\partial \theta^2} - \frac{rk^2}{R^2}\frac{\partial W}{\partial r} - \frac{k^2}{R^4} W - \frac{k}{R}\frac{\partial}{\partial r}\left(\frac{V}{R}\right) + \frac{k^2}{R^3}\frac{\partial U}{\partial \theta}$$

where $R = (1 + k^2 r^2)^{1/2}$

For the stress resultants and couples:

$$H = N_r = \epsilon_r + \nu \epsilon_\theta - \frac{1}{R^2 \lambda^6}\left[\tau - (1-\nu)\frac{k}{R^2}(\epsilon_r - \epsilon_\theta)\right]$$

$$I = N_\theta = \epsilon_\theta + \nu \epsilon_r - \frac{1}{R^2 \lambda^6}\left[\tau - (1-\nu)\frac{k}{R^2}(\epsilon_\theta - \epsilon_r)\right]$$

$$J = N_{r\theta} = \frac{1-\nu}{2}\gamma_{r\theta} - \frac{1}{2R^2\lambda^6}\left[(1+\nu)\kappa_r + (3-\nu)\kappa_\theta\right]$$

$$K = N_{\theta r} = \frac{1-\nu}{2}\gamma_{r\theta} - \frac{1}{2R^2\lambda^6}\left[(1+\nu)\kappa_\theta + (3-\nu)\kappa_r\right] \tag{3.79}$$

$$L = M_r = \kappa_r + \nu \kappa_\theta - \frac{1-\nu}{2}\frac{k}{R^2}\gamma_{r\theta}$$

$$M = M_\theta = \kappa_\theta + \nu \kappa_r - \frac{1-\nu}{2}\frac{k}{R^2}\gamma_{r\theta}$$

$$N = M_{r\theta} = \frac{1-\nu}{2}\tau - \frac{k}{R^2}(\epsilon_\theta + \nu \epsilon_r)$$

$$O = M_{\theta r} = \frac{1-\nu}{2}\tau - \frac{k}{R^2}(\epsilon_r + \nu \epsilon_\theta)$$

$$P = Q_r = \frac{\partial}{\partial r}(RL) + k\frac{\partial O}{\partial \theta} - \frac{rk^2}{R} M$$

$$Q = Q_\theta = \frac{\partial}{\partial r}(RN) + k\frac{\partial M}{\partial \theta} + \frac{rk^2}{R} O$$

where we have introduced new notation which will be convenient for discussing the layers. The equations for P and Q come from the two equilibrium equations 3.74(iv) and (v). The remaining equilibrium equations can now be written as

$$\text{(i)}\quad \frac{\partial}{\partial r}(RH) + k\frac{\partial K}{\partial \theta} - \frac{rk^2}{R} I + \frac{1}{R^2\lambda^6} Q = 0$$

$$\text{(ii)}\quad \frac{\partial}{\partial r}(RJ) + k\frac{\partial I}{\partial \theta} + \frac{rk^2}{R} K + \frac{1}{R^2\lambda^6} P = 0$$

$$\text{(iii)}\quad J + K - \frac{R}{k^2\lambda^6}\frac{\partial P}{\partial r} - \frac{1}{k\lambda^6}\frac{\partial Q}{\partial \theta} = 0 \qquad (3.80)$$

$$\text{(iv)}\quad J - K + \frac{1}{R^2\lambda^6}(M-L) = 0 \qquad \text{(identity)}$$

An alternative form of equations 3.78 can be obtained by solving for J and K between 3.78(iii) and (iv) and eliminating them in equations 3.78(i) and (iii). The alternative forms are

$$\text{(i)}\quad \frac{\partial}{\partial r}(RH) + \frac{R}{2k\lambda^6}\frac{\partial^2 P}{\partial r\partial \theta} + \frac{1}{2\lambda^6}\frac{\partial^2 Q}{\partial \theta^2} - \frac{rk^2}{R} I + \frac{k}{2R^2\lambda^6}\frac{\partial}{\partial \theta}(M+L) + \frac{Q}{R^2\lambda^6} = 0$$

(ii) = 3.80(ii)

$$\text{(iii)}\quad \frac{\partial}{\partial r}\left[\frac{R^2}{k^2\lambda^6}\frac{\partial P}{\partial r}+\frac{R}{k\lambda^6}\frac{\partial Q}{\partial \theta}-\frac{1}{R\lambda^6}(M-L)\right]+2k\frac{\partial I}{\partial \theta} \tag{3.81}$$

$$+\frac{r}{\lambda^6}\frac{\partial P}{\partial r}+\frac{rk}{R\lambda^6}\frac{\partial Q}{\partial \theta}+\frac{rk^2}{R^3\lambda^6}(M-L)+\frac{2P}{R^2\lambda^6}=0$$

Finally, the transformed boundary conditions become;

at $r = \pm 1$:

(i) $H + \frac{1}{R^2\lambda^6} N = 0$ (zero radial stress)

(ii) $J = 0$ (zero tangential stress)

(iii) $P + k\frac{\partial W}{\partial \theta} = 0$ (zero transverse shear stress) (3.82)

(iv) $L = 0$ (zero bending moment)

at $\theta = (-1)^p \gamma$:

(i) $I + \frac{1}{R^2\lambda^6} O = 0$ (zero tangential stress)

(ii) $K = 0$ (zero radial stress)

(3.83)

(iii) $Q + R\frac{\partial O}{\partial r} = 0$ (zero transverse shear stress)

(iv) $M = m_p(r)$ for $p = 1$ or 2 (prescribed bending couple) where we have set $M_p(r) = (Cb/k\lambda^6)m_p(r)$ -- this represents the applied bending moments at the ends of the strip.

c. Structure of the layers

The determination of the exponent pairs and sub-equations cannot be made as simply as in the previous problems. The fact that the differential equations have variable coefficients makes the structure of the layers depend upon the location of the boundaries. In particular, a boundary at r = 0 requires special consideration. However, for the shell considered here this is not of importance. The principles employed here are the same as before although the calculations are somewhat more laborious and are carried out in a somewhat different manner. The method for finding the exponent pairs and sub-equations for the corner layers is entirely analogous to that for finding exponent pairs and sub-equations for the boundary layers. The calculations for the boundary layers will be omitted here. It can be shown that the exponent pair for the θ-boundary layers is (1, 0) and that for the r-boundary layers is (0,1). We will now seek the exponent pairs for the corner layers. Assume α and β greater than zero and set

$$\xi = \lambda^{\alpha}(\gamma + \theta) \qquad \eta = \lambda^{\beta}(1+r) \qquad s = (1+k^2)^{1/2} \tag{3.84}$$

The procedure is to transform all equations into new ones involving the layer variables ξ and η. It is only necessary to retain terms involving each variable U, V and W that have the highest powers of in their coefficients. Thus, if we transform the equations 3.78 and substitute from them into 3. 79 and yet again substitute from these equations into 3.81, we obtain to first approximation

$$s\lambda^{2\beta}\frac{\partial^2 U}{\partial\eta^2} - \frac{5-\nu}{4}\,\frac{k}{s}\lambda^{2\alpha+2\beta-6}\,\frac{\partial^4 U}{\partial\xi^2\partial\eta^2} + \frac{1-\nu}{4}\,\frac{k^3}{s^3}\lambda^{4\alpha-6}\,\frac{\partial^4 U}{\partial\xi^4}$$

$$+ \nu k\lambda^{\alpha+\beta}\frac{\partial^2 V}{\partial\xi\,\partial\eta} + \frac{1-\nu}{4}\lambda^{\alpha+3\beta}\,\frac{\partial^4 V}{\partial\xi\,\partial\eta^3}$$

$$- \frac{5-\nu}{4}\,\frac{k^2}{s^2}\lambda^{3\alpha+\beta-6}\,\frac{\partial^4 V}{\partial\xi^3\partial\eta} - \frac{s^2}{2k}\lambda^{\alpha+4\beta-6}\,\frac{\partial^5 W}{\partial\xi\,\partial\eta^4} - k\lambda^{3\alpha+2\beta-6}\,\frac{\partial^5 W}{\partial\xi^3\partial\eta^2}$$

$$- \frac{k^3}{2s^2}\lambda^{5\alpha-6}\,\frac{\partial^5 W}{\partial\xi^5} = 0 \qquad (3.85)$$

$$\frac{1+\nu}{2}\,k\,\lambda^{\alpha+\beta}\,\frac{\partial^2 U}{\partial\xi\,\partial\eta} + \frac{1-\nu}{2}\,s\,\lambda^{2\beta}\,\frac{\partial^2 V}{\partial\eta^2} + \frac{k^2}{s}\lambda^{2\alpha}\,\frac{\partial^2 V}{\partial\xi^2} - \frac{1-\nu}{2s}\lambda^{3\beta-6}\,\frac{\partial^3 W}{\partial\eta^3}$$

$$+ \frac{1-\nu}{s}\,k\lambda^{\beta}\frac{\partial W}{\partial\eta} + \frac{1-\nu}{2}\,\frac{k^2}{s^3}\lambda^{2\alpha+\beta-6}\,\frac{\partial^3 W}{\partial\xi^2\partial\eta} = 0 \qquad (3.86)$$

$$2\nu k\,\lambda^{\alpha+\beta}\,\frac{\partial^2 U}{\partial\xi\,\partial\eta} - \frac{5-\nu}{2}\lambda^{\alpha+3\beta-6}\,\frac{\partial^4 U}{\partial\xi\,\partial\eta^3} + \frac{1-\nu}{2}\,\frac{k^2}{s^2}\lambda^{3\alpha+\beta-6}\,\frac{\partial^4 U}{\partial\xi^3\partial\eta}$$

$$+ \frac{2k^2}{s}\lambda^{2\alpha}\,\frac{\partial^2 V}{\partial\xi^2} + \frac{1-\nu}{2}\,\frac{s}{k}\lambda^{4\beta-6}\,\frac{\partial^4 V}{\partial\eta^4}$$

$$- \frac{5-\nu}{2}\,\frac{k}{s}\lambda^{2\alpha+2\beta-6}\,\frac{\partial^4 V}{\partial\xi^2\partial\eta^2} - \frac{s^3}{k^2}\lambda^{5\beta-6}\,\frac{\partial^5 W}{\partial\eta^5}$$

$$-2s\lambda^{2\alpha+3\beta-6}\,\frac{\partial^5 W}{\partial\xi^2\partial\eta^3} - \frac{k^2}{s}\lambda^{4\alpha+\beta-6}\,\frac{\partial^5 W}{\partial\xi^4\partial\eta} = 0 \qquad (3.87)$$

We may now make various assumptions about the magnitudes of α and β. For example, if we assume that α and β are each greater than or equal to one and less than two we may omit several terms from the preceding equations. To further restrict this example assume that $\alpha > \beta$, then we have from equations 3.85, 3.86 and 3.87;

$$\text{(i)}\quad s\lambda^{\alpha+\beta}\frac{\partial^2 U}{\partial\eta^2} + \nu k\lambda^{2\alpha}\frac{\partial^2 V}{\partial\xi\,\partial\eta} - \frac{k^3}{2s^2}\lambda^{6\alpha-\beta-6}\frac{\partial^5 W}{\partial\xi^5} = 0$$

$$\text{(ii)}\quad (1+\nu)s\lambda^{\alpha+\beta}\frac{\partial^2 U}{\partial\xi\,\partial\eta} + 2k\lambda^{2\alpha}\frac{\partial^2 V}{\partial\xi^2} + 2(1-\nu)\lambda^{\beta}\frac{\partial W}{\partial\eta} = 0 \qquad (3.88)$$

$$\text{(iii)}\quad 2\nu s\lambda^{\alpha+\beta}\frac{\partial^2 U}{\partial\xi\,\partial\eta} + 2k\lambda^{2\alpha}\frac{\partial^2 V}{\partial\xi^2} - k\lambda^{4\alpha+\beta-6}\frac{\partial^5 W}{\partial\xi^4\partial\eta} = 0$$

Let us introduce the "O" or order notation. To illustrate, we will write for the first term in equation 3.88(i), $O\left(\lambda^{\alpha+\beta}U\right)$ and mean by this "the order of $\lambda^{\alpha+\beta}U$ with respect to λ as λ becomes large." The argument goes as follows: Assume $O\left(\lambda^{\alpha+\beta}U\right) > O\left(\lambda^{2\alpha}V\right)$ then in equations 3.88(i) and (iii) the first terms and third terms must be of equal order; otherwise, either U or W would be zero under the assumption that the dependent variables and their derivatives must tend to zero exponentially as the layer variables tend to infinity (Criterion A). However, if this is so then $O\left(\lambda^{6\alpha-\beta-6}W\right) = O\left(\lambda^{4\alpha+\beta-6}W\right)$ or $\alpha = \beta$ contrary to our assumption. Similarly, the assumption that $O\left(\lambda^{\alpha+\beta}U\right) < O\left(\lambda^{2\alpha}V\right)$ also leads to this contradiction. The only remaining possibility is that $O\left(\lambda^{\alpha+\beta}U\right) = O\left(\lambda^{2\alpha}V\right)$. By arguing in the manner indicated, it can be shown that we must have $\alpha = \frac{\beta}{3} + 1$ with $\alpha < \frac{3}{2}$ and $O\left(\lambda^{\alpha+\beta} U\right) = O\left(\lambda^{2\alpha} V\right) =$ $O\left(\lambda^{6\alpha-\beta-6} W\right)$. Indeed, if we set $\lambda^{\alpha} U = u$, $\lambda^{2\alpha-\beta} V = v$ and $W = w$ then to first approximation equations 3.88 become

$$\text{(i)} \quad s \frac{\partial^2 u}{\partial \eta^2} + \nu k \frac{\partial^2 v}{\partial \xi \partial \eta} - \frac{k^3}{2 s^2} \frac{\partial^5 w}{\partial \xi^5} = 0$$

$$\text{(ii)} \quad (1+\nu) s \frac{\partial^2 u}{\partial \xi \partial \eta} + 2k \frac{\partial^2 v}{\partial \xi^2} + 2(1-\nu) \frac{\partial w}{\partial \eta} = 0 \qquad (3.89)$$

$$\text{(iii)} \quad \nu s \frac{\partial^2 u}{\partial \xi \partial \eta} + k \frac{\partial^2 v}{\partial \xi^2} = 0$$

The third of these equations is readily integrated, in accordance with criterion A, to give $\nu s \frac{\partial u}{\partial \eta} + k \frac{\partial v}{\partial \xi} = 0$. Similarly, when 3.89(iii) is used to eliminate v in 3.89(ii), we can deduce that $s \frac{\partial u}{\partial \xi} + 2w = 0$. In summary, equations 3.89 lead to the sub-equations:

$$\text{(i)} \quad \nu s \frac{\partial u}{\partial \eta} + \frac{\partial v}{\partial \xi} = 0$$

$$\text{(ii)} \quad s \frac{\partial u}{\partial \xi} + 2w = 0 \qquad (3.90)$$

$$\text{(iii)} \quad \frac{\partial^6 w}{\partial \xi^6} + \frac{4(1-\nu^2)s^2}{k^3} \frac{\partial^2 w}{\partial \eta^2} = 0$$

with $\alpha = \frac{\beta}{3} + 1$.

In the previous discussion, if we had assumed $\beta > \alpha$ then we would have obtained $\beta = \frac{\alpha}{3} + 1 < \frac{3}{2}$ and

(i) $s\frac{\partial u}{\partial \eta} + \nu k \frac{\partial v}{\partial \xi} = 0$

(ii) $s^2 \frac{\partial v}{\partial \eta} + 2kw = 0$ (3.91)

(iii) $\frac{\partial^6 w}{\partial \eta^6} + \frac{4(1-\nu^2)k^5}{s^6} \frac{\partial^2 w}{\partial \xi^2} = 0$

provided that here $\lambda^{2\beta-\alpha}U = u$, $\lambda^{\beta}V = v$ and $W = w$.

We will omit the discussion for other magnitudes of α and β, e.g., for α and β not bounded between one and two. Note that the relations between α and β that have been found here are those appearing in equations 3.14(d) and 3.14(e) in the first problem of this chapter. Thus, the exponential pairs are the same as those found there and so the layers will have the thickness h appearing (by means of λ) to the same powers as in the previous problem. Thus, there are infinite sets of corner layers.

d. The inner solution

Since λ appears both in the differential equations and in the boundary conditions to the sixth power, we assume

$$U^0 = \sum_{n=0}^{\infty} U_n^0(\theta, r)\,\lambda^{-6n}$$

$$V^0 = \sum_{n=0}^{\infty} V_n^0(\theta, r)\,\lambda^{-6n} \qquad (3.92)$$

$$W^0 = \sum_{n=0}^{\infty} W_n^0(\theta, r)\,\lambda^{-6n}$$

Let S denote a representative function for the functions H through Q and also assume that

$$S^0 = \sum_{n=0}^{\infty} S_n^0(\theta, r)\, \lambda^{-n} \tag{3.93}$$

Now formally substituting U^0, V^0 and W^0 for U, V and W in equations 3.89 and substituting these in turn into equations 3.90, we have for the first 0-components of the stress resultants and couples:

$$H_0^0 = \frac{\partial U}{\partial r} + \nu \frac{rk^2}{R^2} U + \frac{\nu k}{R}\frac{\partial V}{\partial \theta}$$

$$I_0^0 = \nu \frac{\partial U}{\partial r} + \frac{rk^2}{R^2} U + \frac{k}{R}\frac{\partial V}{\partial \theta}$$

$$J_0^0 = K_0^0 = \frac{1-\nu}{2}\left[\frac{k}{R}\frac{\partial U}{\partial \theta} + R\frac{\partial}{\partial r}\left(\frac{V}{R}\right) + \frac{2kW}{R^2}\right]$$

$$L_0^0 = -\frac{1-\nu}{2}\frac{3k^2}{R^3}\frac{\partial U}{\partial \theta} + \frac{1-\nu}{2}\frac{k}{R}\frac{\partial}{\partial r}\left(\frac{V}{R}\right) - \frac{2k^2}{R^4} W - \frac{\partial^2 W}{\partial r^2} - \frac{\nu k^2}{R^2}\frac{\partial^2 W}{\partial \theta^2}$$

$$- \frac{\nu rk^2}{R^2}\frac{\partial W}{\partial r} \tag{3.94}$$

$$M_0^0 = \frac{1-\nu}{2}\frac{k^2}{R^3}\frac{\partial U}{\partial \theta} - \frac{1-\nu}{2}\frac{3k}{R}\frac{\partial}{\partial r}\left(\frac{V}{R}\right) - \frac{2k^2}{R^4} W - \nu\frac{\partial^2 W}{\partial r^2} - \frac{k^2}{R^2}\frac{\partial^2 W}{\partial \theta^2}$$

$$- \frac{rk^2}{R^2}\frac{\partial W}{\partial r}$$

$$N_0^0 = -\frac{3-2\nu}{R^4} rk^3 U - \frac{\nu k}{R^2}\frac{\partial U}{\partial r} - \frac{k^2}{R^3}\frac{\partial V}{\partial \theta} - (1-\nu)k\frac{\partial^2}{\partial r \partial \theta}\left(\frac{W}{R}\right)$$

$$O_0^0 = -\frac{2-\nu}{R^4}\ rk^3 U - \frac{k}{R^2}\frac{\partial U}{\partial r} - \frac{\nu k^2}{R^3}\frac{\partial V}{\partial \theta} - (1-\nu)k\frac{\partial^2}{\partial r \partial \theta}\left(\frac{W}{R}\right)$$

$$P_0^0 = \frac{\partial}{\partial r}(RL_0^0) + k\frac{\partial O_0^0}{\partial \theta} - \frac{rk^2}{R}\ M_0^0$$

$$Q_0^0 = \frac{\partial}{\partial r}(RN_0^0) + k\frac{\partial M_0^0}{\partial \theta} + \frac{rk^2}{R}\ O_0^0$$

where the U, V and W occurring in equations 3.94 are understood to be U_0^0, V_0^0 and W_0^0, respectively.

If we substitute S^0 for S in equations 3.80, where S is a representative function for the functions H through Q, then the inner solution components satisfy

$$\text{(i)}\quad \frac{\partial}{\partial r}(RH_n^0) + k\frac{\partial K_n^0}{\partial \theta} - \frac{rk^2}{R}\ I_n^0 + \frac{1}{R^2}\ Q_{n-1}^0 = 0$$

$$\text{(ii)}\quad \frac{\partial}{\partial r}(RJ_n^0) + k\frac{\partial I_n^0}{\partial \theta} + \frac{rk^2}{R}\ K_n^0 + \frac{1}{R^2}\ P_{n-1}^0 = 0 \qquad (3.95)$$

$$\text{(iii)}\quad J_n^0 + K_n^0 - \frac{R}{k^2}\ \frac{\partial P_{n-1}^0}{\partial r} - \frac{1}{k}\ \frac{\partial Q_{n-1}^0}{\partial \theta} = 0$$

$$\text{(iv)}\quad J_n^0 - K_n^0 + \frac{1}{R^2}(M_{n-1}^0 - L_{n-1}^0) = 0$$

Also since we assume no boundary layers at $r = \pm 1$, we have by a similar substitution into equations 3.82 that for $r = \pm 1$,

(i) $R^2 H_n^0 + N_{n-1}^0 = 0$ (zero radial stress)

(ii) $J_n^0 = 0$ (zero tangential stress)

(3.96)

(iii) $P_n^0 + k \frac{\partial N_n^0}{\partial \theta} = 0$ (zero transverse shear stress)

(iv) $L_n^0 = 0$ (zero bending moment)

e. The θ-boundary layers

As stated in section c the exponent pair for these layers is (1, 0). In the terms of the O notation, it can also be shown that

$$O(\lambda U) = O(\lambda^2 V) = O(W).$$

Accordingly, we assume that for p = 1 or 2,

$$U^p = \sum_{n=0}^{\infty} U_n^p(\xi, r)\, \lambda^{-(3+n)}$$

$$V^p = \sum_{n=0}^{\infty} V_n^p(\xi, r)\, \lambda^{-(4+n)} \quad \text{where } \xi = \lambda\left[\gamma - (-1)^p \theta\right] \tag{3.97}$$

$$W^p = \sum_{n=0}^{\infty} W_n^p(\xi, r)\, \lambda^{-(2+n)}$$

Let S denote a representative function for the functions H through Q and write

$$S^p = \sum_{n=0}^{\infty} S_n^p(\xi, r)\, \lambda^{-(\mu+n)} \tag{3.98}$$

then we also assume the following correspondences

S	H	I	J	K	L	M	N	O	P	Q
μ	3	3	2	2	0	0	1	1	0	-1

Formal substitution of U^p, V^p and W^p for U, V and W in equations 3.78 and use of the series forms of equations 3.97 provides the following relations for the first components:

$$\text{(i)}\quad H_0^p = \frac{\partial U}{\partial r} + \frac{\nu r k^2}{R^2} U - \frac{\nu k}{R}(-1)^p \frac{\partial V}{\partial \xi}$$

$$\text{(ii)}\quad I_0^p = \nu \frac{\partial U}{\partial r} + \frac{r k^2}{R^2} U - \frac{k}{R}(-1)^p \frac{\partial V}{\partial \xi}$$

$$\text{(iii)}\quad J_0^p = \frac{1-\nu}{2}\left[-\frac{k}{R}(-1)^p \frac{\partial U}{\partial \xi} + \frac{2k}{R^2} W\right] = K_0^p$$

$$\text{(iv)}\quad L_0^p = -\frac{\nu k^2}{R^2}\,\frac{\partial^2 W}{\partial \xi^2}$$

$$\text{(v)}\quad M_0^p = -\frac{k^2}{R^2}\,\frac{\partial^2 W}{\partial \xi^2} \qquad (3.99)$$

$$\text{(vi)}\quad N_0^p = \frac{1-\nu}{2}\, k(-1)^p \frac{\partial^2}{\partial \xi\, \partial r}\left(\frac{W}{R}\right) = O_0^p$$

$$\text{(vii)}\quad P_0^p = -\frac{1+\nu}{2}\, k^2 \frac{\partial^3}{\partial \xi^2 \partial r}\left(\frac{W}{R}\right) + \frac{r k^4}{R^3}\,\frac{\partial^2 W}{\partial \xi^2}$$

$$\text{(viii)}\quad Q_0^p = \frac{k^3}{R^2}(-1)^p \frac{\partial^3 W}{\partial \xi^3}$$

where U, V and W are understood to stand for U_0^p, V_0^p and W_0^p, respectively.

Thus, we have the first p-components of the stresses and couples in terms of the first p-components of the displacements. We may now find the differential equations satisfied by these displacement components by first substituting from equations 3.98 into equations 3.80. This gives

$$\text{(i)}\quad \frac{\partial}{\partial r}(RH_0^p) + \frac{1}{2}\frac{\partial^2 Q_0^p}{\partial \xi^2} - \frac{rk^2}{R} I_0^p = 0$$

$$\text{(ii)}\quad \frac{\partial}{\partial r}(RJ_0^p) - k(-1)^p \frac{\partial I_0^p}{\partial \xi} + \frac{rk^2}{R} K_0^p = 0 \qquad (3.100)$$

$$\text{(iii)}\quad J_0^p + K_0^p = 0 \qquad \text{(iv)}\quad J_0^p - K_0^p = 0$$

for the first p-components.

Now 3.100(iii) and (iv) imply $J_0^p = K_0^p = 0$ and from 3.100(ii) we get $\partial I_0^p/\partial \xi = 0$ which can be integrated, in accordance with criterion A, to give $I_0^p = 0$. Thus, noting that $H_0^p - \nu I_0^p = (1-\nu^2)\,\partial U_0^p/\partial r$, we have

$$\text{(i)}\quad 2(1-\nu^2)\frac{\partial}{\partial r}\left(R\frac{\partial U_0^p}{\partial r}\right) + \frac{\partial^2 Q_0^p}{\partial \xi^2} = 0$$

$$\text{(ii)}\quad J_0^p = 0 \qquad (3.101)$$

$$\text{(iii)}\quad I_0^p = 0$$

or rewriting these in terms of displacements by means of equation 3.99 we get

$$\text{(i)}\quad \nu\frac{\partial U_0^p}{\partial r} + \frac{rk^2}{R^2} U_0^p - \frac{k}{R}(-1)^p \frac{\partial V_0^p}{\partial \xi} = 0$$

$$\text{(ii)}\quad 2W_0^p - R(-1)^p \frac{\partial U_0^p}{\partial \xi} = 0 \qquad (3.102)$$

$$\boxed{\text{(iii)}\quad \frac{\partial^6 U_0^p}{\partial \xi^6} + \frac{4(1-\nu^2)}{k^3} R\frac{\partial}{\partial r}\left(R\frac{\partial U_0^p}{\partial r}\right) = 0}$$

These are the basic equations for the boundary layers at an edge determined by constant θ. The differential equations for the other p-components will be at most inhomogeneous forms of these. Note that equation 3.102(iii) includes derivatives with respect to both ξ and r. This means that it is not one of the classical type as described in relation to equation 1.27.

The equations for the second p-components can be obtained from equation 3.102 by increasing the subscripts by one. The equations for the third p-components which are obtained analogously to equations 3.101 are:

$$\text{(i)}\quad \frac{\partial}{\partial r}(RH_2^p) - \frac{R}{2k}(-1)^p \frac{\partial^2 P_0^p}{\partial \xi \, \partial r} + \frac{1}{2}\frac{\partial^2 Q_2^p}{\partial \xi^2} - \frac{rk^2}{R} I_2^p - \frac{k}{2R^2}(-1)^p \left(M_0^p - L_0^p\right) = 0$$

$$\text{(ii)}\quad \frac{\partial}{\partial r}(RJ_2^p) - k(-1)^p \frac{\partial I_2^p}{\partial \xi} + \frac{rk^2}{R} K_2^p = 0 \tag{3.103}$$

$$\text{(iii)}\quad J_2^p + K_2^p + \frac{1}{k}(-1)^p \frac{\partial Q_0^p}{\partial \xi} = 0$$

$$\text{(iv)}\quad J_2^p - K_2^p = 0$$

it follows that

$$J_2^p = K_2^p = -\frac{1}{2k}(-1)^p \frac{\partial Q_0^p}{\partial \xi} \tag{3.104}$$

and substitution of 3.104 into 3.103 (ii) gives, after an integration with respect to ξ ,

$$-2k^2 I_2^p = \frac{\partial}{\partial r}(RQ_0^p) + Q_0^p \tag{3.105}$$

The equations for the fourth p-components can be obtained from these by increasing the subscripts by one.

The boundary conditions at a p-edge which involve the inner solution and a p-layer are obtained by formally substituting sums of the form $S^0 + S^p$ for S, where S is a representative function, into equations 3.83 . Thus, for $\theta = \gamma(-1)^p$

$$\begin{aligned} &(i) \quad R(I^0 + I^p) + \frac{1}{R\lambda^6}\left(O^0 + O^p\right) = 0 \\ &(ii) \quad K^0 + K^p = 0 \\ &(iii) \quad Q^0 + Q^p + R\frac{\partial O^0}{\partial r} + R\frac{\partial O^p}{\partial r} = 0 \\ &(iv) \quad M^0 + M^p = m_p(r) \end{aligned} \qquad (3.106)$$

Substitution from equations 3.93 and 3.97 gives for the relevant $\xi = 0$

$$\begin{aligned} &(i) \quad R^2(I_n^0 + I_{6n-3}^p) + O_{n-1}^0 + O_{6n-9}^p = 0 \\ &(ii) \quad K_n^0 + K_{6n-2}^p = 0 \\ &(iii) \quad Q_n^0 + Q_{6n+1}^p + R\frac{\partial O_n^0}{\partial r} + R\frac{\partial O_{6n-1}^p}{\partial r} = 0 \\ &(iv) \quad M_n^0 + M_{6n}^p = m_p(r) \end{aligned} \qquad (3.107)$$

and

$$\begin{aligned} &(i) \quad I_{k-3}^p + O_{k-9}^p = 0 \quad (ii) \quad K_{k-2}^p = 0 \quad (iii) \quad Q_{k-5}^p + R\frac{\partial O_{6n-7}^p}{\partial r} = 0 \\ &(iv) \quad M_k^p = 0 \end{aligned} \qquad (3.108)$$

for k a positive integer not equal to 6n.

f. The r-layers

Although these layers will not be used here, we present their basic formulation for completeness. As stated in section c, the boundary layers have the exponent pair (0,1). In terms of the O notation, it can also be shown that

$$O\left(\lambda^2 U\right) = O\left(\lambda V\right) = O\left(W\right)$$

Accordingly, we might assume that for q = 3 or 4,

$$U^q = \sum_{n=0}^{\infty} U_n^q(\theta, \eta)\,\lambda^{-(2+n)}$$

$$V^q = \sum_{n=0}^{\infty} V_n^q(\theta, \eta)\,\lambda^{-(1+n)} \qquad (3.109)$$

$$W^q = \sum_{n=0}^{\infty} W_n^q(\theta, \eta)\,\lambda^{-n}$$

and then perform the calculations which are analogous to those in section e. If this is done, the equations corresponding to equations 3.102 are found to be

$$\text{(i)}\quad s\,\frac{\partial U_0^q}{\partial \eta} + \nu k\,\frac{\partial V_0^q}{\partial \theta} = 0$$

$$\text{(ii)}\quad s^2\,\frac{\partial V_0^q}{\partial \eta} + 2k\,W_0^q = 0 \qquad (3.110)$$

$$\boxed{\text{(iii)}\quad \frac{\partial^6 W_0^q}{\partial \eta^6} + \frac{4(1-\nu^2)k^5}{s^6}\,\frac{\partial^2 W_0^q}{\partial \theta^2} = 0}$$

where $s = (1+k^2)^{1/2}$.

These are the basic equations for the boundary layers at an edge determined by constant $r \neq 0$. The differential equations for the other q-components will be at most inhomogeneous forms of these. As with equation 3.102(iii), derivatives in both variables appear in 3.110(iii) so that it is not of the classical type. Our method would be needed to get sufficient boundary conditions for it.

The basic equations for the r-corner layers are given by equations 3.90 and 3.91 and we note that equations 3.110 are analogous to equations 3.91.

g. The θ-corner layers

As deduced in section c, the first set of matching corner layers have the exponent pair $(1, \frac{4}{3})$ and in the O notation

$$O\left(\lambda^{5/3} U\right) = O\left(\lambda^{4/3} V\right) = O\left(W\right)$$

Accordingly, we assume that for p = 1 or 2 and q = 3 or 4,

$$U^{pq} = \sum_{n=0}^{\infty} U_n^{pq}(\xi, \eta)\, \lambda^{-\frac{13+n}{3}}$$

$$V^{pq} = \sum_{n=0}^{\infty} V_n^{pq}(\xi, \eta)\, \lambda^{-\frac{12+n}{3}} \tag{3.111}$$

$$W^{pq} = \sum_{n=0}^{\infty} W_n^{pq}(\xi, \eta)\, \lambda^{-\frac{8+n}{3}}$$

where $\xi = \lambda\left[\gamma - (-1)^p \theta\right]$ and $\eta = \lambda^{4/3}\left[1 - (-1)^q r\right]$.

Let S denote a representative function for the functions H through Q and write

$$S^{pq} = \sum_{n=0}^{\infty} S_n^{pq}(\xi, \eta)\, \lambda^{-\frac{\mu+n}{3}} \tag{3.112}$$

Then we also assume the following correspondences:

S	H	I	J	K	L	M	N	O	P	Q
μ	9	9	8	8	0	0	1	1	-4	-3

Formal substitution of U^{pq}, V^{pq} and W^{pq} for U, V and W in equations 3.78, and substitution in turn of these equations into equations 3.79 gives for the first components:

$$\text{(i)}\quad H_0^{pq} = -(-1)^q \frac{\partial U}{\partial \eta} - \frac{\nu k}{s}(-1)^p \frac{\partial V}{\partial \xi}$$

$$\text{(ii)}\quad I_0^{pq} = -\nu(-1)^q \frac{\partial U}{\partial \eta} - \frac{k}{s}(-1)^p \frac{\partial V}{\partial \xi}$$

$$\text{(iii)}\quad J_0^{pq} = \frac{1-\nu}{2}\left[-(-1)^q \frac{\partial V}{\partial \eta} + \frac{2kW}{s^2}\right] = K_0^{pq}$$

$$\text{(iv)}\quad L_0^{pq} = -\frac{\partial^2 W}{\partial \eta^2}$$

$$\text{(v)}\quad M_0^{pq} = -\nu \frac{\partial^2 W}{\partial \eta^2} \tag{3.113}$$

$$\text{(vi)}\quad N_0^{pq} = -(1-\nu)\frac{k}{s}(-1)^{p+q} \frac{\partial^2 W}{\partial \xi \partial \eta} = O_0^{pq}$$

$$\text{(vii)}\quad P_0^{pq} = s(-1)^q \frac{\partial^3 W}{\partial \eta^3}$$

$$\text{(viii)}\quad Q_0^{pq} = k(-1)^p \frac{\partial^3 W}{\partial \xi \partial \eta^2}$$

where $s = (1+k^2)^{1/2}$ and U, V and W are understood to stand for U_0^{pq}, V_0^{pq} and W_0^{pq}, respectively.

The differential equations which are obtained by substituting S^{pq} for S, where S is a representative function for H through Q, from equations 3.113 into equations 3.81 can be deduced from equations 3.91,

$$\text{(i)}\quad s\frac{\partial U}{\partial \eta} + \nu k(-1)^{p+q}\frac{\partial V}{\partial \xi} = 0$$

$$\text{(ii)}\quad s^2\frac{\partial V}{\partial \eta} - 2k(-1)^q W = 0 \qquad (3.114)$$

$$\text{(iii)}\quad \frac{\partial^6 W}{\partial \eta^6} + \frac{4(1-\nu^2)}{s^6}k^5\frac{\partial^2 W}{\partial \xi^2} = 0$$

where again U, V and W stand for U_0^{pq}, V_0^{pq} and W_0^{pq}, respectively. The boundary conditions at a q-edge which involve a p-layer and a pq-layer are obtained by formally substituting sums of the form $S^p + S^{pq}$ for S, where S is a representative function, into equations 3.82 . Thus, for the relevant $\eta = 0$, i.e., for $r = (-1)^q$, we obtain

$$\text{(i)}\quad s^2(H^p+H^{pq}) + \lambda^{-6}(N^p+N^{pq}) = 0 \qquad \text{(zero radial stress)}$$

$$\text{(ii)}\quad J^p + J^{pq} = 0 \qquad \text{(zero tangential stress)}$$

$$\text{(iii)}\quad P^p+P^{pq} + k\lambda\left(\frac{\partial N^p}{\partial \xi} + \frac{\partial N^{pq}}{\partial \xi}\right) = 0 \qquad \text{(zero transverse shear stress)} \qquad (3.115)$$

$$\text{(iv)}\quad L^p + L^{pq} = 0 \qquad \text{(zero bending moment)}$$

Substitution from equations 3.98 and 3.112 into equations 3.115 gives

$$
\begin{aligned}
&\text{(i)}\quad s^2(H_n^p + H_{3n}^{pq}) + N_{n-4}^p + N_{3n-10}^{pq} = 0 \\
&\text{(ii)}\quad J_n^p + J_{3n-2}^{pq} = 0 \\
&\text{(iii)}\quad P_n^p + P_{3n+4}^{pq} + k\frac{\partial N_n^p}{\partial \xi} + k\frac{\partial N_{3n-2}^{pq}}{\partial \xi} = 0 \\
&\text{(iv)}\quad L_n^p + L_{3n}^{pq} = 0
\end{aligned}
\tag{3.116}
$$

and

$$
\text{(i)}\quad s^2 H_m^{pq} + N_{m-10}^{pq} = 0 \qquad \text{(ii)}\quad J_{m-2}^{pq} = 0 \qquad \text{(iii)}\quad P_{m-2}^{pq} + k\frac{\partial N_{m-8}^{pq}}{\partial \xi} = 0
$$

$$
\text{(iv)}\quad L_m^{pq} = 0 \tag{3.117}
$$

form a positive integer not equal to 3n.

h. Equations for first components

We first attempt to formulate a boundary layer problem for the first components of the p-layers by deducing boundary conditions at the q-edges. We note that equation 3.114(i) together with 3.113(i) imply that $H_0^{pq} \equiv 0$. Similarly, 3.102(ii) together with 3.99(iii) imply that $J_0^p = K_0^p \equiv 0$. Thus, for n = 0 equation 3.116(ii) is satisfied automatically and the other conditions from 3.116 and 3.117 are, at $\eta = 0$,

$$
\begin{aligned}
&\text{(i)}\quad H_0^p = 0 \\
&\text{(ii)}\quad J_0^{pq} = 0 \quad \text{satisfied by 3.114(ii)} \\
&\text{(iii)}\quad P_0^{pq} = 0 \\
&\text{(iv)}\quad L_0^p + L_0^{pq} = 0
\end{aligned}
\tag{3.118}
$$

The condition we seek is 3.118(i), which gives

$$H_0^p = (1-\nu^2)\frac{\partial U_0^p}{\partial r} = 0 \qquad \text{at } r = \pm 1 \tag{3.119}$$

This is another example of how our method of considering the layers in the corners leads to boundary conditions which are necessary for determining the boundary layer. Now we may integrate 3.102(iii) over r from -1 to 1 and get

$$\frac{\partial^6 \tilde{U}_0^p}{\partial \xi^6} = -\frac{4(1-\nu^2)}{k^3}\, s\left[\frac{\partial U_0^p(1)}{\partial r} - \frac{\partial U_0^p(-1)}{\partial r}\right] = 0$$

where $\tilde{U}_0^p = \int_{-1}^{1} \frac{U_0^p}{R}\, dr$. Thus, since U_0^p and all its derivatives must tend to zero exponentially with ξ, we conclude that

$$\tilde{U}_0^p = 0 \tag{3.120}$$

The relevant boundary conditions for $\theta = \gamma(-1)^p$ from equations 3.107 and 3.108 are

$$\begin{array}{llll} \text{(i) } I_0^0 = 0 & \text{(ii) } K_0^0 = 0 & \text{(iii) } M_0^0 + M_0^p = m_p(r) & \text{(iv) } I_0^p = 0 \\ \text{(v) } K_0^p = 0 & \text{(vi) } Q_0^p = 0 & \text{(vii) } K_2^p = 0 & \end{array} \tag{3.121}$$

of which (ii), (iv) and (v) are satisfied automatically. Note that $K_2^p = -\frac{1}{2k}(-1)^p\,(\partial Q_0^p/\partial \xi)$ from equation 3.104. Thus, equations 3.121(iii), (vi) and (vii) together with 3.99 provide the following boundary conditions on W_0^p,

$$-\frac{k^2}{R^2}\frac{\partial^2 W_0^p}{\partial \xi^2} + M_0^0 = m_p(r); \quad \frac{\partial^3 W_0^p}{\partial \xi^3} = \frac{\partial^4 W_0^p}{\partial \xi^4} = 0 \quad \text{at } \xi = 0.$$

Using equation 3.102(ii), these may be written in terms of U_0^p as

$$k^2 \frac{\partial^3 U_0^p}{\partial \xi^3} = 2R(-1)^p \left[M_0^0 - m_p(r)\right]; \quad \frac{\partial^4 U_0^p}{\partial \xi^4} = \frac{\partial^5 U_0^p}{\partial \xi^5} = 0 \qquad (3.122)$$

when the relevant ξ is zero.

Equations 3.102(iii), 3.119 and 3.122 determine the boundary value problem for U_0^p provided M_0^0 is known to within its mean value.

We now turn our attention to the inner solution. Equations 3.95 for n = 0 give the differential equations

$$\text{(i)} \quad \frac{\partial}{\partial r}(RH_0^0) + k\frac{\partial K_0^0}{\partial \theta} - \frac{rk^2}{R} I_0^0 = 0$$

$$\text{(ii)} \quad \frac{\partial}{\partial r}(RJ_0^0) + k\frac{\partial I_0^0}{\partial \theta} + \frac{rk^2}{R} K_0^0 = 0 \qquad (3.123)$$

$$\text{(iii)} \quad J_0^0 + K_0^0 = 0 \qquad \text{(iv)} \quad J_0^0 - K_0^0 = 0$$

and for $r = \pm 1$, from equation 3.96

$$\text{(i)} \quad H_0^0 = 0 \quad \text{(ii)} \quad J_0^0 = 0 \quad \text{(iii)} \quad P_0^0 + k\frac{\partial N_0^0}{\partial \theta} = 0 \quad \text{(iv)} \quad L_0^0 = 0 \qquad (3.124)$$

Also, from 3.121(i), we have

$$I_0^0 = 0 \quad \text{at } \theta = \gamma(-1)^p \qquad (3.125)$$

Now equations 3.123(ii),(iii) and (iv) together imply that I_0^0 is a function of r only, so that 3.125 requires $I_0^0 \equiv 0$. Similarly, from 3.123(i) we deduce that RH_0^0 is a function of θ only but 3.124(i) requires this function to be zero. Thus, we have

$$H_0^0 = I_0^0 = J_0^0 = K_0^0 = 0 \qquad (3.126)$$

and the two boundary conditions at $r = \pm 1$,

$$L_0^0 = P_0^0 + k \frac{\partial N_0^0}{\partial \theta} = 0 \tag{3.127}$$

Since $H_0^0 - \nu I_0^0 = (1-\nu^2)(\partial U_0^0/\partial r)$, we deduce that

$$U_0^0 = -2 \frac{\partial F}{\partial \theta} \tag{3.128}$$

where $F = F(\theta)$ is an arbitrary function. Putting $H_0^0 = 0$ in 3.94, we thus deduce that

$$V_0^0 = 2 \frac{rk}{R} F(\theta) - 2kR \int^r \frac{G(r)}{R^3} dr \tag{3.129}$$

where G is an arbitrary function of r. Similarly, $J_0^0 = 0$ gives

$$W_0^0 = R \frac{d^2 F}{d\theta^2} + \left(R - \frac{2}{R}\right) F + G \tag{3.130}$$

The inner solution now depends upon the determination of F and G. Substitution of equations 3.128, 3.129 and 3.130 into 3.94 gives

$$L_0^0 = \frac{-\nu k^2}{R} \mathcal{L} F - \frac{d^2 G}{dr^2} - \frac{\nu r k^2}{R^2} \frac{dG}{dr} - \frac{3-\nu}{R^4} k^2 G$$

$$M_0^0 = -\frac{k^2}{R} \mathcal{L} F - \nu \frac{d^2 G}{dr^2} - \frac{rk^2}{R^2} \frac{dG}{dr} + \frac{1-3\nu}{R^4} k^2 G \tag{3.131}$$

$$N_0^0 = 0$$

$$O_0^0 = 0$$

where $\mathcal{L} F = (d^4 F/d\theta^4) + 2(d^2 F/d\theta^2) + F$. Now equation 3.124(iv) implies that

$$\mathcal{L} F = C_0 \quad \text{a constant} \tag{3.132}$$

Thus it follows that

$$Q_0^0 = 0 \tag{3.133}$$

Only the components L_0^0, M_0^0 and P_0^0 may differ from zero. These correspond respectively to M_r, M_θ and Q_r. We note that they are independent of θ and that equation 3.127 requires that

$$L_0^0 = P_0^0 = 0 \quad \text{at } r = \pm 1. \tag{3.134}$$

Since we have used all of the equations for n = 0 in equations 3.95, we now consider the equations for n = 1. Using our previous results, we have

$$\begin{aligned}
&\text{(i)} \quad \frac{\partial}{\partial r}(RH_1^0) + k\frac{\partial K_1^0}{\partial \theta} - \frac{rk^2}{R} I_1^0 = 0 \\
&\text{(ii)} \quad \frac{\partial}{\partial r}(RJ_1^0) + k\frac{\partial I_1^0}{\partial \theta} + \frac{rk^2}{R} K_1^0 + \frac{1}{R^2} P_0^0 = 0 \\
&\text{(iii)} \quad J_1^0 + K_1^0 = \frac{R}{k^2}\frac{\partial P_0^0}{\partial r} \\
&\text{(iv)} \quad J_1^0 - K_1^0 = \frac{1}{R^2}(L_0^0 - M_0^0)
\end{aligned} \tag{3.135}$$

It follows that since the first components here are independent of θ then so are J_1^0, K_1^0 and $\partial I_1^0/\partial\theta$. We now seek conditions on I_1^0. Two relevant conditions from equations 3.107 are (i) with n = 1 and (iii) with n = 0. These give

$$\text{(i)} \quad I_1^0 + I_3^p = 0 \qquad \text{(ii)} \quad Q_1^p = 0 \tag{3.136}$$

for $\theta = \gamma(-1)^p$. As stated in relation to equation 3.105,

$$-2k^2 I_3^p = \frac{\partial}{\partial r}(RQ_1^p) + Q_1^p$$

Thus, it follows from equation 3.136 that

$$I_1^0 = 0 \quad \text{for } \theta = \gamma(-1)^p \tag{3.137}$$

However, I_1^0 is linear in θ and so must be zero. Equations 3.135(i) and 3.96(i) imply $H_1^0 = 0$. Equations 3.135(ii), (iii) and (iv) now provide us with a differential equation involving L_0^0, M_0^0 and P_0^0. Also, P_0^0 is given in terms of L_0^0 and M_0^0 in equation 3.94. Eliminating J_1^0 and K_1^0 between equations 3.135(ii), (iii) and (iv), we get

$$\frac{d}{dr}\left(R^3 \frac{dP_0^0}{dr}\right) + k^2 \frac{d}{dr}(L_0^0 - M_0^0) - \frac{2rk^4}{R^2}(L_0^0 - M_0^0) + \frac{2k^2}{R} P_0^0 = 0 \tag{3.138}$$

but

$$P_0^0 = R\frac{dL_0^0}{dr} + \frac{rk^2}{R}(L_0^0 - M_0^0) \tag{3.139}$$

so that we may integrate and obtain

$$R^3 \frac{dP_0^0}{dr} + k^2(L_0^0 - M_0^0) + 2k^2 L_0^0 = -2k^2 C_2 \tag{3.140}$$

where C_2 is a constant of integration. Again, eliminating $L_0^0 - M_0^0$ between equations 3.139 and 3.140, we obtain

$$\frac{d}{dr}\left(\frac{r}{R} P_0^0\right) - \frac{d}{dr}\left(\frac{L_0^0}{R^2}\right) = -\frac{2k^2 r}{R^4} C_2$$

which integrates into

$$rRP_0^0 - L_0^0 = C_1(s^2 - R^2) = C_1 k^2 (1 - r^2) \tag{3.141}$$

where C_1 is an arbitrary constant of integration and C_2 has been eliminated by use of the conditions 3.134. Let

$$T = \frac{dG}{dr} - \frac{1}{rR^2} G = \frac{r}{R}\frac{d}{dr}\left(\frac{RG}{r}\right) \qquad (3.142)$$

then from equations 3.131, we may write

$$\text{(i)} \quad -L_0^0 = \frac{dT}{dr} + \frac{1+\nu k^2 r^2}{rR^2} T + \frac{\nu k^2}{R} C_0$$

$$\text{(ii)} \quad -M_0^0 = \nu \frac{dT}{dr} + \frac{\nu + k^2 r^2}{rR^2} T + \frac{k^2}{R} C_0 \qquad (3.143)$$

Using equations 3.139, 3.141 and 3.143, we obtain

$$\frac{d^2T}{dr^2} + \frac{k^2 r}{R^2}\frac{dT}{dr} - \left[\frac{2}{r^2R^2} + \frac{\nu k^2 + k^4 r^2}{R^4}\right] T = -\frac{k^2(1-r^2)}{rR^2} C_1 + \frac{k^2(\nu + r^2k^2)}{rR^3} C_0 \qquad (3.144)$$

and

$$-P_0^0 = \frac{R}{r^2}\left[r^2\frac{d^2T}{dr^2} + r\frac{dT}{dr} - T\right] - \frac{C_0 k^4 r}{R^2} \qquad (3.145)$$

Now equation 3.141 assures us that $P_0^0 = 0$ at $r = \pm 1$ if L_0^0 does also. Thus, we may consider equation 3.144 and the condition that $L_0^0 = 0$ for $r = \pm 1$ as determining T in terms of C_0 and C_1. To see this, let

$$T = C_0 T_0(r) - C_1 T_1(r) \qquad (3.146)$$

then if

$$\overline{L} = \frac{d^2}{dr^2} + \frac{k^2 r}{R^2}\frac{d}{dr} - \left[\frac{2}{r^2R^2} + \frac{\nu k^2 + k^4 r^2}{R^4}\right] \qquad (3.147)$$

we have

$$\text{(i)} \quad \overline{L}\, T_0 = \frac{k^2(\nu + r^2k^2)}{rR^3}$$

$$\text{(ii)} \quad \frac{dT_0}{dr} + \frac{1+\nu k^2 r^2}{rR^2} T_0 = \frac{-\nu k^2}{R} \quad \text{for } r = \pm 1 \qquad (3.148)$$

and

$$\text{(i)} \quad \bar{L}\,T_1 = + \frac{k^2(1-r^2)}{rR^2}$$

$$\text{(ii)} \quad \frac{dT_1}{dr} + \frac{1+\nu k^2 r^2}{rR^2} T_1 = 0 \quad \text{for } r = \pm 1 \tag{3.149}$$

Thus, T_0 and T_1 are determinate. It is interesting to compare these equations with those found by Knowles and Reissner in (23) p. 416. If $\bar{K}$ is their operator with S_0 and S_1 as the dependent functions, their equations may be written

$$\bar{K} = \frac{d^2}{dr^2} + \frac{k^2 r}{R^2}\frac{d}{dr} + \frac{\nu k^2 - k^4 r^2}{R^4}$$

so that

$$\text{(i)} \quad \bar{K}S_0 = \frac{(1-\nu)kr}{R^2} - \frac{(1+\nu)kr}{R^4}$$

$$\text{(ii)} \quad \frac{dS_0}{dr} + \frac{\nu k^2 r}{R^2} S_0 = \frac{-\nu k}{R^2} \qquad \text{for } r = \pm 1$$

and

$$\text{(i)} \quad \bar{K}S_1 = \frac{(1+\nu)k^2 r}{R^4}$$

$$\text{(ii)} \quad \frac{dS_1}{dr} + \frac{\nu k^2 r}{R^2} S_1 = \frac{-\nu}{R^2} \qquad \text{for } r = \pm 1$$

The operator $\bar{L}$ has a regular singular point at the origin, while both $\bar{K}$ and $\bar{L}$ have singularities at $kr = \pm i$. However, because of the way that G depends upon T, there is no such singularity in the stresses or displacements.

Inspection of equations 3.148 and 3.149 shows that both T_0 and T_1 are odd functions of r. Now it is readily verified that

$$T_0 = -\frac{R}{r} \tag{3.150}$$

is the solution of equations 3.148. Thus, substitution of T from equation 3.146 into equations 3.143 and 3.145 gives

$$\text{(i)}\quad L_0^0 = C_1\left[\frac{dT_1}{dr} + \frac{1+\nu k^2 r^2}{rR^2} T_1\right]$$

$$\text{(ii)}\quad M_0^0 = C_1\left[\nu\frac{dT_1}{dr} + \frac{\nu+k^2 r^2}{rR^2} T_1\right] \tag{3.151}$$

$$\text{(iii)}\quad P_0^0 = \frac{C_1 R}{r^2}\left[r^2\frac{d^2T_1}{dr^2} + r\frac{dT_1}{dr} - T_1\right]$$

This means that M_0^0 is completely determined to within the multiplicative constant C_1 which is found from the condition that

$$\int_{-1}^{1} M_0^0 dr = \int_{-1}^{1} m_p(r)\, dr \tag{3.152}$$

It is readily shown that the constant C_0 enters through F and G in such a way that all functions are unaffected. Indeed, it may be defined to be zero.

We may now solve the boundary value problem for the p-boundary layer. If we let $u = U_0^p$, the formulation of the problem is:

$$\text{(i)}\quad \frac{\partial^6 u}{\partial \xi^6} + \frac{4(1-\nu^2)}{k^3} R\frac{\partial}{\partial r}\left(R\frac{\partial u}{\partial r}\right) = 0$$

where for $\xi = 0$,

$$\text{(ii)} \quad \frac{\partial^3 u}{\partial \xi^3} = \frac{2R}{k^2}(-1)^p \left[M_0^0(r) - m_p(r) \right] \tag{3.153}$$

$$\text{(iii)} \quad \frac{\partial^4 u}{\partial \xi^4} = 0 \qquad \text{(iv)} \quad \frac{\partial^5 u}{\partial \xi^5} = 0$$

and

$$\text{(v)} \quad \frac{\partial u}{\partial r} = 0 \quad \text{for } r = \pm 1$$

Let

$$w = \frac{\sinh^{-1} kr}{\sinh^{-1} k} \quad \text{and} \quad \frac{2R}{k^2}(-1)^p \left[M_0^0(r) - m_p(r) \right] = f(w) \tag{3.154}$$

then equations 3.153(i), (ii) and (v) become

$$\text{(i)} \quad \frac{\partial^6 u}{\partial \xi^6} + A^2 \frac{\partial^2 u}{\partial w^2} = 0$$

where

$$\text{(ii)} \quad \frac{\partial^3 u}{\partial \xi^3} = f(w) \text{ at } \xi = 0; \quad \frac{\partial u}{\partial w} = 0 \text{ at } w = \pm 1 \tag{3.155}$$

and

$$A = \left[\frac{4(1-\nu^2)}{k} \right]^{1/2} \frac{1}{\sinh^{-1} k}$$

We also have from equation 3.120 that

$$\int_{-1}^{1} \frac{u\,dr}{R} = \frac{\sinh^{-1} k}{k} \int_{-1}^{1} u\,dw = 0 \tag{3.156}$$

Thus, we may assume that

$$u = \sum_{n=1}^{\infty} a_n(\xi) \cos \frac{n\pi}{2}(1+w) \tag{3.157}$$

and so

$$\frac{d^6 \alpha_n}{d\xi^6} - \left(\frac{n\pi}{2} A\right)^2 \alpha_n = 0$$

similar to the result in equation 3.62, we conclude that

$$\alpha_n = A_n e^{-\gamma_n \xi} + B_n e^{-\gamma_n w_+ \xi} + C_n e^{-\gamma_n w_- \xi}$$

where $\gamma_n = (n\pi A/2)^{1/3}$ and $w_\pm = \frac{1}{2}(1 \pm i\sqrt{3})$. Conditions 3.153(iii) and (iv) require that

$$A_n - w_+ B_n - w_- C_n = 0$$

and

$$A_n + w_- B_n + w_+ C_n = 0$$

respectively. Solving for B_n and C_n, we get

$$-B_n = C_n = \frac{iAn}{\sqrt{3}} \tag{3.158}$$

Thus,

$$u = \sum_{n=1}^{\infty} A_n \left[e^{-\gamma_n \xi} - \frac{i}{\sqrt{3}} e^{-\gamma_n w_+ \xi} + \frac{i}{\sqrt{3}} e^{-\gamma_n w_- \xi} \right] \cos \frac{n\pi}{2}(1+w) \tag{3.159}$$

where

$$A_n = -\frac{1}{\gamma_n^3} \int_{-1}^{1} f(w) \cos \frac{n\pi}{2}(1+w) dw$$

in order that equation 3.155(ii) be satisfied. This equation, together with equations 3.102(i) and (ii), determine the displacements due to the boundary layers. Knowing the boundary layer displacements, we can calculate the boundary layer stresses using equations 3.99.

i. Conclusions

By substituting 3.159 into equation 3.102(ii), we obtain

$$W_0^p = \sum_{n=1}^{\infty} RA_n \left[e^{-\gamma_n \xi} + \frac{1+w_-}{3} e^{-\gamma_n w_+ \xi} + \frac{1+w_+}{3} e^{-\gamma_n w_- \xi} \right] \cos \frac{n\pi}{2} (1+w) \tag{3.160}$$

where

$$A_n = \frac{1}{\gamma_n^2 k \sinh^{-1} k} \int_{-1}^{1} \left[M_0^0(r) - m_p(r) \right] \left[\cos \frac{n\pi}{2} \left(1 + \frac{\sinh^{-1} kr}{\sinh^{-1} k} \right) \right] dr$$

This may be compared for small k with G_0^p which is given in connection with equation 3.66.

In the original notation, from equations 3.151 we may write for the inner solution, to first approximation:

$$M_r = C_1 \left[\frac{dT}{dr} + \frac{a^2 + \nu r^2}{rR^2} T \right]$$

$$M_\theta = C_1 \left[\nu \frac{dT}{dr} + \frac{a^2 \nu + r^2}{rR^2} T \right] \tag{3.161}$$

$$Q_r = \frac{C_1}{r^2} \left[r^2 \frac{d^2 T}{dr^2} + r \frac{dT}{dr} - T \right]$$

where

$$C_1 = \frac{\int_{-b}^{b} M_p(r) dr}{\int_{-b}^{b} \left(\nu \frac{dT}{dr} + \frac{a^2 \nu + r^2}{rR^2} T \right) dr} , \qquad R = (a^2 + r^2)^{1/2}$$

and T is a solution of the equation

$$\frac{d^2 T}{dr^2} + \frac{r}{R^2} \frac{dT}{dr} - \left(\frac{2a^2}{r^2 R^2} + \frac{\nu a^2 + r^2}{R^4} \right) T = \frac{b^2 - r^2}{brR^2}$$

with the conditions

$$\frac{dT}{dr} + \frac{a^2+\nu r^2}{rR^2} T = 0 \qquad \text{at } r = \pm b$$

Further calculation is necessary to determine the displacements uniquely.

REFERENCES

1. Love, A.E.H., A Treatise on the Mathematical Theory of Elasticity, New York: Dover Publications, (1944).

2. Goldstein, S. (ed.), Modern Developments in Fluid Dynamics, Oxford: Oxford University Press, (1938).

3. Friedrichs, K.O. and Dressler, R.F., "A Boundary-Layer Theory for Elastic Plates", Communications on Pure and Applied Mathematics, Vol. 14 No. 1, (Feb. 1961), pp. 1-33.

4. Gol'denveizer, A.L., "Some Mathematical Problems in the Linear Theory of Thin Elastic Shells," Russian Mathematical Surveys, (trans. of Uspekhi Mat. Nauk), Vol. 15 No. 5 (Sept.-Oct. 1960).

5. Lagerstrom, P.A. and Cole, J.D., "Examples Illustrating Expansion Procedures for the Navier-Stokes Equations," Journal of Rational Mechanics and Analyses, Vol. 4 No. 6 (Nov. 1955) pp. 817-882.

6. Kaplan, S. and Lagerstrom, P.A., "Asymptotic Expansions of Navier-Stokes Solutions for Small Reynolds Numbers," Journal of Mathematics and Mechanics, Vol. 6 No. 5 (1957) pp. 585-593.

7. Kaplan, S., "Low Reynolds Number Flow Past a Circular Cylinder," Journal of Mathematics and Mechanics, Vol. 6 No. 5 (1957) pp. 595-603.

8. Lagerstrom, P.A., "Note on the Preceding Two Papers," Journal of Mathematics and Mechanics, Vol. 6 No. 5 (1957) pp. 605-606.

9. Fife, P., "Non-Linear Deflection of Thin Elastic Plates Under Tension," Communications on Pure and Applied Mathematics, Vol. 14 No. 2 (May 1961) pp. 81-112.

10. Mahony, J.J., "The Large Deflection of Thin Cantilevered Plates," Parts I and II, The Quarterly Journal of Mechanics and Applied Mathematics, Vol. 14 pt. 3 (Aug. 1961) pp. 257-282.

11. Carslaw, H.S. and Jaeger, J.C., Conduction of Heat in Solids, Oxford; Oxford University Press, (1959) Second Edition.

12. Wasow, W., "Asymptotic Solution of Boundary Value Problems for the Differential Equation $\Delta U + \lambda \frac{\partial U}{\partial x} = \lambda f(x, y)$," Duke Mathematical Journal, Vol. 11 (1944) pp. 405-415.

13. Levinson, N., "The First Boundary Value Problem for $\epsilon \Delta U + A(x, y)U_x + B(x, y)U_y + C(x, y)U = D(x, y)$ for Small ϵ," Annals of Mathematics, Vol. 51 No. 2 (Mar. 1950) pp. 428-445.

14. Wilson, H.A., "On Convection of Heat," Proceedings of the Cambridge Philosophical Society, Vol. 12 (1902-04) pp. 406-423.

15. Wittrick, W.H., "Edge Stresses in Thin Shells of Revolution," Australian Journal of Applied Science, Vol. 8 No. 1, (Mar. 1957) pp. 235-260.

16. Erdelyi, A. (ed.), Tables of Integral Transforms, Vol. I, New York: McGraw-Hill Book Company, Inc. (1954).

17. Johnson, M.W. and Reissner, E., "Parametric Expansions for a Class of Boundary Value Problems of Partial Differential Equations," Journal of the Society for Industrial and Applied Mathematics, Vol. 8 No. 2 (Jun. 1960) pp. 389-396.

18. Timoshenko, S., Theory of Plates and Shells, New York; McGraw-Hill Book Company, Inc., (1940).

19. Maunder, L. and Reissner, E., "Pure Bending of Pretwisted Rectangular Plates," Journal of the Mechanics and Physics of Solids, Vol. 5 (1957) pp. 261-266.

20. Marguerre, K., "Zur Theorie der gekrummten Platte grosser Formanderung," Proceedings of the Fifth International Congress of Applied Mechanics, (1938) pp. 93-101.

21. Reissner, E., "On Some Aspects of the Theory of Thin Elastic Shells," Journal of the Boston Society of Civil Engineers, Vol. 42 No. 2 (Apr. 1955) pp. 100-133.

22. Knowles, J.K. and Reissner, E., "Note on Stress-Strain Relations for Thin Elastic Shells," Journal of Mathematics and Physics, Vol. 37 No. 3 (Oct. 1958) pp. 269-282.

23. Knowles, J.K. and Reissner, E., "Torsion and Extension of Helicoidal Shells," Quarterly of Applied Mathematics, Vol. 17 No. 4, (Jan. 1960) pp. 409-422.