PHOTOPRODUCTION OF NEUTRAL PIONS IN COMPLEX NUCLEI

Thesis by

Christian Albertus Engelbrecht

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ABSTRACT

The photoproduction of neutral pions from complex nuclei is expressed in terms of the photoproduction from single nucleons by means of the direct interaction model and the impulse approximation. A summation over all final states of the nucleus is performed by means of a closure approximation.

Within the framework of the independent particle model of the nucleus, the nuclear matrix elements are evaluated exactly for closed shell nuclei, the final results being expressed in terms of the nucleon density and the two-particle correlation functions in the nuclear ground state. For quantitative purposes the correlation functions are calculated both in the case of a Fermi gas and for harmonic oscillator wave functions.

The differential cross section is decomposed into an elastic part, which leaves the nucleus in its ground state, and an inelastic part. The elastic cross section has a coherent peak close to the forward direction. The inelastic part is suppressed at small angles because of the contribution from the correlation functions, which is a manifestation of the exclusion principle.

The interactions of the produced pion with the nucleus are included by means of the optical model. Numerical integrations are performed using an approximate wave function for the pion.

Neutral pions can also be produced by the interaction with the electrostatic field of the nucleus. This process is related to the lifetime of the pion for decay into two photons.

When compared with the available experimental results, the theoretically predicted cross sections seem to be too low. Possible reasons for this discrepancy are discussed.

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1. Introduction

At low energies the photoproduction of pions from single nucleons is well known (2) and has been explained field theoretically (3) as due to the interaction of the electromagnetic field with the pion-nucleon current. Recently the cross section for the production of neutral pions from protons has been measured (4) at energies up to about 1 Bev. Although the details of the high energy process are still very imperfectly known (5), it is presumably still due to the direct coupling with the pion-nucleon current.

According to the direct interaction model (6) the photoproduction of pions from a complex nucleus can be considered as due to the interaction of the photon with the individual nucleons as long as:

$$k r_0 \gg 1 \tag{1.1}$$

where k is the photon momentum while r_0 is given in terms of the mass number A and the nuclear radius R by:

$$R = r_0 A^{1/3} \tag{1.2}$$

We shall always be using units such that $\hbar = c = 1$. If the impulse approximation (7) is valid, the interaction with the bound nucleons can be taken to be equal to that with free nucleons. The process which would occur if for the latter we use the direct coupling with the pion-nucleon current only, will be called the nuclear production.

The nuclear production of positive pions was treated by Lax and Feshbach (8) for low energies. These authors were mainly interested in

the pion energy spectrum at 90°. At such a large angle the coherent production is very small so that they confined their attention to the incoherent cross section.

Primakoff (13) some time ago proposed that the lifetime of the neutral pion be determined by measuring the photoproduction of neutral pions from heavy nuclei. Since the π^0 decays into two gamma-rays, it should be possible for a neutral pion to be produced when the incident photon interacts with a virtual photon corresponding to the electrostatic field of the nucleus. This process will be called the coulomb production of neutral pions.

A successful direct measurement of the lifetime of the neutral pion was reported for the first time (14) when this thesis was already being written. Primakoff's suggestion provides an indirect way of measuring the lifetime. The coulomb production would have a pole when the virtual photon in question becomes real. This condition is approached in the forward direction where the momentum transfer to the nucleus reaches a minimum value. The photoproduction cross section must thus be measured at very small angles and then corrected for the nuclear production in order to extract a value for the lifetime. It is therefore of considerable interest to obtain a theoretical estimate of the nuclear production at small angles. In this region the coherent cross section can no longer be neglected and actually becomes dominant.

The cross section for photoproduction of neutral pions from nuclei at 250 Mev has been measured by Davidson (15). He treated the coherent nuclear production where the nucleus remains in its ground state, by including a source term in the Klein-Gordon equation for the pion and fitting various parameters to his experimental results. The deviations

from his fit at small angles were ascribed to coulomb production and used to obtain a value for the lifetime of the neutral pion. Inelastic processes were not included. The lifetime determination was also made difficult by the fact that the coulomb production happens to be much smaller than the nuclear production at 250 Mev. A similar experiment is at present in progress at the Caltech synchrotron (16) at an energy of 900 Mev at which the coulomb production is much larger, particularly from heavy nuclei.

In the present work the differential cross section for the photoproduction of neutral pions from complex nuclei in fairly forward directions is expressed in terms of the interactions of pions with single nucleons and of certain properties of the nuclear ground state. Some of these quantities are already known and others could be determined in the near future. This is done within the framework of the direct interaction model since condition 1.1 is satisfied at the energies under consideration. The further basic assumption is the validity of the impulse approximation.

In chapter I a summation is performed over the cross sections which leave the nucleus in specific final states. This is achieved by means of a closure approximation based on the work of Placzek and Wick (9) on the scattering of neutrons by molecules and the work of Fowler and Watson (10) on the scattering of particles by nuclei. The differential cross section is expressed in terms of expectation values of two-nucleon operators with respect to the ground state of the nucleus. The part which is diagonal with respect to the two nucleons, is completely incoherent. The remaining part can be decomposed into a correlated and a non-correlated contribution. The latter is proportional to the square

of the mass number A and produces a diffraction peak near the forward direction.

In chapter II the nuclear ground state is decomposed into states of definite symmetry. The expectation values of the two-nucleon operators are expressed in terms of form factors and spin-isospin matrix elements. The two-particle correlation functions are calculated using an independent particle model and it is demonstrated that the correlated cross section partly cancels the diagonal contribution. This suppression is due to the fact that certain final states of the struck nucleon are forbidden by the exclusion principle.

The final state interactions of the produced pion are introduced in chapter III. The pion plane wave is replaced by the wave function for a pion inside a complex potential well. The effects of the pion absorption and scattering inside the nucleus are computed numerically. The cross sections are attenuated and the diffraction minima are smoothed out.

In chapter IV the decay of the neutral pion is considered. The coulomb production from heavy nuclei is expressed in terms of the pion lifetime. Absorption of the pion is included and the determination of the lifetime of the neutral pion is discussed.

CHAPTER I

PHOTOPRODUCTION IN A COMPLEX NUCLEUS

2. Cross section for photoproduction from a nucleus

The differential cross section in the laboratory system for the production of a neutral pion from a nucleus of mass number A by a photon of momentum $\vec{k}$, is given by:

$$\frac{d\sigma}{d\Omega} = \int \frac{q^2\, dq}{(2\pi)^2} \langle 0 | T_A^\dagger \, \delta(E_i - H_0)\, T_A | 0 \rangle \qquad (2.1)$$

This expression follows from A-23* if we assume that the nucleus is initially at rest and in its ground state $|0\rangle$. All possible final states of the nucleus have been included and we integrate over the values of the final momentum q of the pion. The argument of the delta function is:

$$E_i - H_0 = k + W_0 - \omega - H_A \qquad (2.2)$$

where the terms on the right are the photon energy, the ground state energy of the nucleus, the pion energy $\omega = \sqrt{m^2 + q^2}$, and the nuclear hamiltonian, respectively. T_A is the transition operator (see appendix A) corresponding to the photoproduction of a neutral pion. It operates on the nuclear variables only. At this point expression 2.1 is still exact.

The purpose of this work is to investigate how well the photoproduction cross section (2.1) can be explained in terms of the

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*An expression such as A.23 refers to an equation in appendix A.

interactions of a single nucleon with the pion and electromagnetic fields and of certain properties of the nuclear ground state such as the spatial nucleon density, the momentum distribution, and the correlation functions. The direct interaction model was introduced by Serber (6) to explain high energy nuclear reactions. In this model the interaction of an incident particle with the nucleus is expressed in terms of its interactions with the individual nucleons. The conditions under which this model is valid, are that the wave length of the incident particle does not exceed the distance between nucleons (condition 1.1) and that the absorption is small enough so that the incident wave is not appreciably attenuated when it passes through the nucleus.

These conditions are both satisfied in our problem so that we may express T_A as a sum of single nucleon transition operators t_n. In order to satisfy the requirement of translational invariance, we have to multiply each t_n by a phase factor which can also be considered as the product of the photon and meson wave functions at the position of the nucleon (appendix B):

$$T_A = \sum_{n=1}^{A} e^{i\vec{p}\cdot\vec{x}_n} t_n \qquad (2.3).$$

Here $\vec{x}_n$ is the spatial coordinate of nucleon n and $\vec{p}$ is the momentum transfer:

$$\vec{p} = \vec{k} - \vec{q} \qquad (2.4).$$

The single nucleon operator t_n no longer depends on the position of the nucleon but only on its spin, isospin, and momentum. It does not operate on the photon or the pion fields but does depend on their quantum numbers such as the polarization of the photon and the angle θ between

the pion and photon moments.

The usefulness of 2.3 becomes apparent when it is combined with the impulse approximation (7) according to which the single particle transition operator for a nucleon bound inside the nucleus is taken to be equal to the corresponding transition operator for a free nucleon. The relative error involved in this approximation has been estimated (7) to be of order:

$$\frac{f}{\lambda}\ \frac{\langle u_B \rangle}{\varepsilon} \qquad (2.5)$$

where f is the transition amplitude, λ the wave length of the incident particle, $\langle u_B \rangle$ the average potential energy of a bound nucleon (in the neighborhood of 40 Mev), and ε the kinetic energy in the initial state (in our case the total photon energy k). Even at the 33-resonance 2.5 is only about 1 % so that it is very reasonable to make the impulse approximation. The validity of the form 2.3 actually depends on another approximation which is usually treated together with the impulse approximation namely the neglect of multiple scattering. In our problem multiple scattering involves repetitive absorption and re-emission of the incident photon. This can be completely ignored because of the smallness of the fine structure constant.

The transition operator for the nucleus can thus be expressed as 2.3 where the t_n are the free nucleon transition operators corresponding to the photoproduction of a neutral pion. The most general form that t_n can assume, is given by (B.30):

$$t_n = K + \vec{L}\cdot\vec{\sigma}_n + M\tau_{3n} + \vec{N}\cdot\vec{\sigma}_n\tau_{3n} \qquad (2.6)$$

where $\vec{\sigma}_n$ and τ_{3n} are the spin and isospin operators for nucleon n

(for notation see appendix B) and where K, $\vec{L}$, M, $\vec{N}$ may depend on the momentum operator for the nucleon and on the photon and pion energies. At low energies K, $\vec{L}$, M, $\vec{N}$ are to a good approximation given by the static theory (see appendix B). At high energies no satisfactory theory exists as yet and K, $\vec{L}$, M, $\vec{N}$ have to be determined by a phenomenological analysis (see appendix C). In principle, however, we have expressed T_A in terms of known quantities.

When 2.3 is substituted into the expression 2.1 for the differential cross section, the following expression is obtained:

$$\frac{d\sigma}{d\Omega} = \int \frac{q^2 dq}{(2\pi)^2} \sum_m \sum_n \langle 0 | t_m^\dagger e^{-i\vec{p}\cdot\vec{x}_m} \delta(E_i - H_0) e^{i\vec{p}\cdot\vec{x}_n} t_n | 0 \rangle \tag{2.7}$$

It is convenient to decompose this into two parts which are respectively diagonal and non-diagonal in the nucleon indices:

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_D + \left(\frac{d\sigma}{d\Omega}\right)_{ND} \tag{2.8}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_D = \int \frac{q^2 dq}{(2\pi)^2} \sum_{n=1}^{A} \langle 0 | t_n^\dagger e^{-i\vec{p}\cdot\vec{x}_n} \delta(E_i - H_0) e^{i\vec{p}\cdot\vec{x}_n} t_n | 0 \rangle \tag{2.9}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{ND} = \int \frac{q^2 dq}{(2\pi)^2} \sum_{m \neq n} \sum_n \langle 0 | t_m^\dagger e^{-i\vec{p}\cdot\vec{x}_m} \delta(E_i - H_0) e^{i\vec{p}\cdot\vec{x}_n} t_n | 0 \rangle \tag{2.10}$$

If the delta function did not contain any operators, the exponential terms in the diagonal contribution would have cancelled directly so that 2.9 would have been independent of the spatial distribution of the nucleons. An approximate treatment of the delta function will be presented in the next section.

3. Closure approximation

In order to reduce the expressions 2.9 and 2.10 for the nuclear photoproduction cross sections to a more tractable form, the delta functions have to be removed from inside the matrix elements. The sum over final nuclear states contained in 2.1 was performed by exact closure. This will now essentially be replaced by a closure approximation. The method which we shall use was developed by Placzek and by Wick (9) in their treatments of the scattering of neutrons by molecular systems and was later applied by Fowler and Watson (10) to the scattering of elementary particles by nuclei.

First the diagonal cross section (2.9) will be considered. Our aim is to commute the delta function through the factor $e^{i\vec{p}\cdot\vec{x}_n}\, t_n$ to the right so that it can operate directly on the nuclear ground state. From 2.2 we can see that the only operator contained in the delta function is the nuclear hamiltonian H_A. After commutation, the argument of the delta function will therefore contain additional terms due to the fact that H_A does not commute with $e^{i\vec{p}\cdot\vec{x}_n}\, t_n$. The terms which are to be expected, are listed below.

A) Since H_A contains the kinetic energy operator $-\frac{1}{2M}\sum_m \nabla_m^2$, there will definitely be a term arising from the commutation of this operator with the exponential $e^{i\vec{p}\cdot\vec{x}_n}$.

B) If H_A contains space-exchange and momentum-dependent potentials, they will not commute with the exponential and will therefore introduce additional terms.

C) When H_A is commuted through t_n, a contribution will arise from the potential energy terms because of the momentum dependence of t_n.

D) Finally there will be a contribution from the commutation of the spin and isospin operators in t_n with the corresponding potential terms in H_A.

The latter three terms (B-D) exist as a result of the nuclear binding potential U_B and would be absent in a system of non-interacting nucleons. Their contribution to the total cross section was estimated by Placzek, Wick (9), and Fowler and Watson (10). Relative to the remaining cross section, their magnitude was found to be of order:

$$\frac{\langle U_B \rangle}{\mathcal{E}} \qquad (3.1),$$

the average potential energy of a nucleon divided by the initial kinetic energy which in our case is just the photon energy k. These terms will therefore be unimportant at the energies we are considering and will henceforth be neglected.

The contribution (A) from the kinetic energy can easily be evaluated by expanding the delta function in a Taylor series and using:

$$\frac{1}{2M}\nabla_n^2\, e^{i\vec{p}\cdot\vec{x}_n} = e^{i\vec{p}\cdot\vec{x}_n}\left(\frac{1}{2M}\nabla_n^2 - \frac{p^2}{2M} - \frac{\vec{p}\cdot\nabla_n}{iM}\right) \qquad (3.2).$$

If the series is again summed and terms of order 3.1 are neglected, the diagonal cross section reduces to:

$$\left(\frac{d\sigma}{d\Omega}\right)_D = \int \frac{q^2 dq}{(2\pi)^2} \sum_n \langle 0 | t_n^+ t_n\, \delta\left(k - \omega - \frac{p^2}{2M} - \frac{\vec{p}\cdot\nabla_n}{iM}\right) | 0 \rangle \qquad (3.3).$$

The exponential factors have here been combined to give unity whereas $(W_o - H_A)$ when operating on the ground state was set equal to zero.

The next step is to Fourier analyze the nuclear ground state in terms of nucleon n only:

$$\langle x_1 .. x_n .. x_A | 0 \rangle = \int \frac{d^3 p_{in}}{(2\pi)^3} \langle x_n | p_{in} \rangle \langle x_1 .. p_{in} .. x_A | 0 \rangle \tag{3.4}$$

In this way the gradient operator ∇_n in 3.3 is replaced by the momentum $i\vec{p}_{in}$. Since t_n does not depend on x_n, we thus obtain:

$$\left(\frac{d\sigma}{d\Omega}\right)_D = \int \frac{q^2 dq}{(2\pi)^2} \sum_n \int \frac{d^3 p_{in}}{(2\pi)^3} P(p_{in}) \overline{\langle i | t_n^+ t_n | i \rangle} \, \delta\left(k - \omega - \frac{p^2}{2M} - \frac{\vec{p}\cdot\vec{p}_{in}}{M}\right) \tag{3.5}$$

where the momentum distribution in the ground state is defined by:

$$P(p_{iA}) = \sum_{spins} \int d^3 x_1 \ldots d^3 x_{A-1} \left| \langle x_1, \ldots x_{A-1}, p_{iA} | 0 \rangle \right|^2 \tag{3.6}.$$

If a "final momentum" $\vec{p}_{fn}$ is defined by:

$$\vec{p}_{fn} = \vec{p}_{in} + \vec{p} \tag{3.7},$$

the argument of the delta function becomes:

$$k + \frac{p_{in}^2}{2M} - \omega - \frac{p_{fn}^2}{2M} \tag{3.8}.$$

Within this approximation, the kinematics for the diagonal cross section is thus the same as that for production from a free nucleon. The matrix element of $t_n^+ t_n$ is averaged over the single nucleon spin-isospin states $|i\rangle$ occurring in the ground state of the nucleus. Because of the antisymmetry of the nuclear wave function, we can replace the sum over n by a factor A and write the matrix element as the expectation value of $t_1^+ t_1$ in the nuclear ground state:

$$\left(\frac{d\sigma}{d\Omega}\right)_D = A \int \frac{d^3 p_i}{(2\pi)^3} P(p_i) \int \frac{q^2 dq}{(2\pi)^2} \langle 0 | t_1^+ t_1 | 0 \rangle \, \delta\left(k + \frac{p_i^2}{2M} - \omega - \frac{p_f^2}{2M}\right) \tag{3.9}.$$

In order to interpret this expression, we compare it with the differential cross section in the laboratory system for the production of a neutral pion by the interaction of a photon with momentum $\vec{k}$ and a single free nucleon with momentum $\vec{p}_i$ in the spin-isospin state $|i\rangle$

$$\left(\frac{d\sigma}{d\Omega}\right)_{free} = \frac{1}{\left(1-\frac{\vec{k}\cdot\vec{p}_i}{kM}\right)}\int\frac{q^2dq}{(2\pi)^2}\langle i|t^+t|i\rangle\,\delta\left(k+\frac{p_i^2}{2M}-\omega-\frac{p_f^2}{2M}\right) \qquad (3.10)$$

By comparison of these last two expressions, it becomes clear that the diagonal cross section is, apart from the flux factor in 3.10, just A times the free nucleon cross section, averaged over the spin and momentum states which are encountered in the nuclear ground state. As far as the one particle contribution (2.9) is concerned, each nucleon therefor behaves like a free independent particle, the only effect of the other nucleons appearing in the specification of its initial state. At this point the question may arise whether the exclusion principle has not erroneously been neglected somewhere. This is, in fact, not the case. As we shall see in the next chapter, a part of the non-diagonal cross section (2.10) will partly cancel 3.9 whenever the exclusion principle becomes effective.

We turn next to the non-diagonal or two particle contribution (2.10). The structure of the matrix element:

$$\langle 0|\, t_m^+ e^{-i\vec{p}\cdot\vec{x}_m}\,\delta(k+W_0-\omega-H_A)\,e^{i\vec{p}\cdot\vec{x}_n}\,t_n\,|0\rangle$$

is such that the nucleus is first excited to some state by an interaction with nucleon n and then de-excited back to the ground state by an interaction with a different nucleon m. This will in general be

possible only for low excitation energies. To a good approximation we may therefore for high energy processes replace H_A in the delta function by W_o . The non-diagonal cross section can then be written:

$$\left(\frac{d\sigma}{d\Omega}\right)_{ND} = \int \frac{q^2 dq}{(2\pi)^2} \sum_{m \neq n} \sum \langle o | e^{i\vec{p}\cdot(\vec{x}_n - \vec{x}_m)} t_m^+ t_n | o \rangle \, \delta(k-\omega) \qquad (3.11).$$

We assume that the transition operators in 3.11 contain spin and isospin operators only. Their dependence on the momentum operators $-i\nabla_n$ will therefore give rise to correction terms. Further correction terms come from values other than W_o for H_A inside the delta function. These terms were estimated by the same authors as before (9), (10) to be of the order of the average kinetic energy of a nucleon divided by the "kinetic" energy ε in the initial state. They will from now on be neglected.

If the single nucleon transition operator did not contain any spin or isospin operators, in other words if the only non-vanishing term on the right hand side of 2.6 were the first one, the matrix element in 3.11 could be written:

$$\sum_{m \neq n} \sum \langle o | e^{i\vec{p}\cdot(\vec{x}_n - \vec{x}_m)} t_m^+ t_n | o \rangle$$
$$= A(A-1) \langle o | t_2^+ t_1 | o \rangle \langle o | e^{i\vec{p}\cdot(\vec{x}_1 - \vec{x}_2)} | o \rangle \qquad (3.12)$$

where $\langle o | t_2^+ t_1 | o \rangle$ is in this case of course just the number $|K|^2$. Equation 3.12 would also hold for the more general transition operator (2.6) if the nuclear wave function could be written as a single product of a spin-isospin wave function and a wave function in coordinate space. This latter condition can be realized only in nuclei with less than five nucleons or in a completely unphysical nucleus consisting, say, of A neutrons with parallel spins. The alternative

condition for the validity of 3.12 is also apparently violated in the energy range under consideration since it seems that the photoproduction from single nucleons does not vanish in the forward direction (2). Nevertheless we shall for the present suppose that 3.12 holds since the non-spin-isospin dependent part of t is actually treated correctly and since the results which follow from 3.12 can be corrected by a slight modification. This will be discussed in the next chapter where the matrix element in 3.11 will be evaluated exactly for closed shell nuclei.

If we define a two-particle probability density:

$$\rho(\vec{x}_1, \vec{x}_2) = \sum_{Spins} \int |\Psi(1\ldots A)|^2 d^3x_3 .. d^3x_A \qquad (3.13)$$

where Ψ is the ground state nuclear wave function:

$$\Psi(1\ldots A) = \langle x_1 \ldots x_A | 0 \rangle \qquad (3.14)$$

while $\sum_{Spins}$ indicates that the inner product with respect to all the spin-isospin variables must be taken, then the matrix element on the extreme right of 3.12 can be written:

$$\langle 0 | e^{i\vec{p}\cdot(\vec{x}_1 - \vec{x}_2)} | 0 \rangle = \int \rho(\vec{x}_1, \vec{x}_2)\, e^{i\vec{p}\cdot(\vec{x}_1 - \vec{x}_2)} d^3x_1 d^3x_2 \qquad (3.15).$$

It is convenient to express 3.13 in terms of the single nucleon density $\rho(\vec{x})$ and the two-particle correlation function $g(\vec{x}_1, \vec{x}_2)$:

$$\rho(\vec{x}_1, \vec{x}_2) = \frac{A}{A-1} \rho(\vec{x}_1)\rho(\vec{x}_2)\left[1 + g(\vec{x}_1, \vec{x}_2)\right] \qquad (3.16)$$

$$\rho(\vec{x}_1) = \int \rho(\vec{x}_1, \vec{x}_2)\, d^3x_2 \qquad (3.17).$$

The non-diagonal cross section (3.11) can then again be decomposed into two parts:

$$\left(\frac{d\sigma}{d\Omega}\right)_{ND} = \left(\frac{d\sigma}{d\Omega}\right)_{NC} + \left(\frac{d\sigma}{d\Omega}\right)_{C} \tag{3.18}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{NC} = A^2 \int \frac{q^2 dq}{(2\pi)^2} \delta(k-\omega) \langle o | t_2^+ t_1 | o \rangle |F(p)|^2 \tag{3.19}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{C} = A \int \frac{q^2 dq}{(2\pi)^2} \delta(k-\omega) \langle o | t_2^+ t_1 | o \rangle G(p) \tag{3.20}$$

where the non-correlated cross section (3.19) is the part which does not depend on the correlation function.

The two form factors appearing in 3.19 and 3.20 are defined as follows:

$$F(p) = \int \rho(\vec{x})\, e^{i\vec{p}\cdot\vec{x}}\, d^3x \tag{3.21}$$

$$G(p) = A \int \rho(\vec{x}_1)\rho(\vec{x}_2)\, g(\vec{x}_1, \vec{x}_2)\, e^{i\vec{p}\cdot(\vec{x}_1 - \vec{x}_2)}\, d^3x_1\, d^3x_2 \tag{3.22}.$$

They are normalized in such a manner that:

$$\left.\begin{aligned} F(0) &= -G(0) = 1 \\ \lim_{p\to\infty} F(p) &= \lim_{p\to\infty} G(p) = 0 \end{aligned}\right\} \tag{3.23}.$$

The first one (3.21) is the same form factor which is determined from the scattering of nucleons by nuclei and is probably very nearly equal to that determined from the scattering of electrons by nuclei (11). The correlation function $g(\vec{x}_1, \vec{x}_2)$ is a manifestation of the way in which the position of nucleon 1 is influenced by the position of nucleon 2 (or vice versa) due to the exclusion principle and the nuclear forces. If the nucleons had been completely independent, the correlation function would have had the constant value $-1/A$ in order that both

sides of 3.16 have the same normalisation.

Using expression 2.6 for the single nucleon transition operator, one can easily show that the matrix element which occurs in 3.9, is given by:

$$t_D^2 \equiv \langle o | t_1^\dagger t_1 | o \rangle = |K|^2 + |\vec{L}|^2 + |M|^2 + |\vec{N}|^2 \qquad (3.24)$$

for a closed shell nucleus with equal numbers of neutrons and protons. In the next chapter the matrix element which occurs in 3.19 and 3.20 will be shown to be given by:

$$\langle o | t_2^\dagger t_1 | o \rangle = |K|^2 - \frac{|\vec{L}|^2 + |M|^2 + |\vec{N}|^2}{A-1} \qquad (3.25),$$

again for a closed shell nucleus with $A=2Z$.

Equations 3.19 and 3.20 were derived from 3.12 which, as has already been stated, will in general be incorrect. In the next chapter we shall see, however, that equations 3.19 and 3.20 may still be used as long as the matrix element 3.25 is replaced by modified quantities t_{NC}^2 and t_C^2 . For closed shell nuclei with $A=2Z$ one finds:

$$t_{NC}^2 = |K|^2 \qquad (3.26)$$

$$t_C^2 = |K|^2 + |\vec{L}|^2 + |M|^2 + |\vec{N}|^2 = t_D^2 \qquad (3.27).$$

Only the contribution from $|K|^2$ is thus correctly given by 3.25.

4. Preliminary estimate of differential cross section

The three parts into which the differential cross section for the nuclear production of neutral pions was decomposed in the previous section, each contains an integral over an energy delta function:

$$\Gamma = \int \frac{q^2 dq}{(2\pi)^2} \delta(E_i - E_f) \qquad (4\cdot1)$$

which is just 2π times the appropriate density of final states. According to the Placzek-Wick-Fowler-Watson approximation (9)(10), this density of states was found to be equal to that for photoproduction from a free nucleon in the case of the diagonal cross section. In the case of the non-diagonal cross section the approximation was made that the nuclear excitation energy Δ vanishes so that the density of states became equal to that for production from a tightly bound nucleon. In a closed shell nucleus, the relation $\Delta=0$ holds rigorously for the non-correlated contribution. In the correlated cross section all excitation energies up to the Fermi energy are in principle possible so that the approximation $\Delta=0$ may no longer be very good for this part. On the other hand, although the procedure leading to equation 3.3 for the diagonal cross section could also be applied to the correlated cross section, the resulting expression cannot be interpreted in physical terms analogous to 3.5. It is therefore somewhat uncertain what treatment of the correlated cross section would be the most satisfactory.

The momentum transfer p to the nucleus is given in terms of the photon and pion momenta k and q and the angle θ between them, by:

$$p^2 = (k-q)^2 + 2kq(1-\cos\theta) \qquad (4\cdot2a).$$

At high energies and small angles, this is approximately equal to:

$$p^2 \approx \left(\Delta + \frac{m^2}{2k}\right)^2 + k^2\theta^2 \qquad (4\cdot 2b)$$

where m is the pion mass while Δ is the nuclear excitation energy. In the forward direction the momentum transfer is thus insensitive to excitation energies small compared to $m^2/2k$. At 250 Mev this quantity has the value 40 Mev whereas at 900 Mev its value is about 11 Mev. If an independent particle model of the nucleus is adopted, the maximum excitation energy which could contribute to the correlated cross section is the Fermi energy which has the value 33 Mev if the nuclear radius parameter $r_o = 1.2$ fermis. The mean excitation energy would probably be much lower. If the mean excitation energy due to recoil alone is $p^2/2M$, it does not become equal to $m^2/2k$ until θ is about 9° at $k = 900$ Mev or more than 60° at $k = 250$ Mev. For small values of θ, it may therefore be neglected. The excitation energy associated with spin flip is probably of the order of the spin-orbit splitting of nuclear levels which may be several Mev. Although this is not negligible compared to $m^2/2k$, it should not affect the value of p too much.

Since the value of the momentum transfer is not very sensitive to the nuclear excitation energy, particularly in the forward direction, the uncertainty about the kinematics of the correlated cross section which was discussed above, is probably not very important. Near the forward direction the value of the argument of the form factor will from now on be assumed to be equal to that for production from a tightly bound nucleon and no distinction will be made between the value of Γ for the diagonal, correlated, and non-correlated cross sections.

The expressions for the three parts of the photoproduction cross section thus become:

$$\left(\frac{d\sigma}{d\Omega}\right)_{NC} = A^2 \left|F(p)\right|^2 t_{NC}^2 \Gamma \qquad (4\cdot3)$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{C} = A \; G(p) \; t_C^2 \Gamma \qquad (4\cdot4)$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{D} = A \; t_D^2 \Gamma \qquad (4\cdot5)$$

where the values of t_{NC}^2, t_C^2, and t_D^2 were given by equations 3.27, 3.26 and 3.24 and will be derived in the next chapter.

In the non-correlated cross section, t_{NC}^2 must be determined from the single nucleon cross section in the laboratory system at the same energy as that at which the production from the nucleus is studied. In the diagonal cross section, on the other hand, the single nucleon cross section must be averaged over the momentum distribution in the nucleus. The correlated cross section again suffers from the uncertainty discussed above. It is, however, extremely likely that t_C^2 should be calculated in the same way as t_D^2 as we shall presently see. This will from now on be assumed to be the case.

When $G(p)$ is not negligible, it is always negative. Since the expressions for t_C^2 and t_D^2 are identical, at least for closed shell nuclei with $A=2Z$, the quantity:

$$1 + G(p) \qquad (4\cdot6)$$

acts as a suppression factor which vanishes when p vanishes and tends to unity when p becomes very large. For small momentum transfer, the diagonal cross section is nearly completely cancelled by the correlated cross section. Such a suppression would be expected to result from the

exclusion principle since most of the available final states are occupied. In the next chapter we shall see that for a Fermi gas 4.6 has exactly the dependence on p which would be expected from the exclusion principle. When p vanishes (which cannot quite be realized physically), the cancellation should be exact. This would only happen if t_C^2 and t_D^2 are identical. As shown in the next chapter, the expressions for t_C^2 and t_D^2 do agree. It is thus very likely that they should actually be calculated in the same manner (namely averaged over the nucleon momentum distribution) as conjectured in the previous paragraph.

The role of the diagonal and correlated cross sections can now be interpreted in a very direct way. The diagonal cross section arises when the production processes from the individual nucleons are added incoherently, without regard to their identity or the residual nucleon-nucleon interactions. The effects of the exclusion principle and the two-body forces are all contained in the correlation function so that the correlated cross section then takes them into account. In spite of the partial cancellation which is sometimes nearly complete, the separation is thus actually very natural.

The non-correlated cross section also allows a very simple interpretation. The differential cross section for elastic photoproduction, where the nucleus remains in its ground state, is given by:

$$\left(\frac{d\sigma}{d\Omega}\right)_{el} = \left|\langle o|\sum_n t_n e^{i\vec{p}\cdot\vec{x}_n}|o\rangle\right|^2 \Gamma$$

$$= A^2 \left|F(p)\right|^2 t_{el}^2 \Gamma \qquad (4.7)$$

where t_{el}^2 is defined by:

$$t_{el}^2 = \left|\langle o|\, t_1 |o\rangle\right|^2 \qquad (4.8).$$

Equation (4.7) follows from the antisymmetry of the nuclear wave function and from the relation:

$$\langle o|\, t_i\, e^{i\vec{p}\cdot\vec{x}_i}\,|o\rangle = \langle o|\, t_i\,|o\rangle\, \langle o|\, e^{i\vec{p}\cdot\vec{x}_i}\,|o\rangle \qquad (4.9)$$

which obviously holds for a closed shell nucleus. Using expression 2.6 for t_i, one finds:

$$t_{el}^2 = |K|^2 = t_{NC}^2 \qquad (4.10).$$

By comparing 4.3 and 4.7, one can thus conclude that the non-correlated cross section is precisely the elastic cross section where the nucleus remains in its ground state and recoils as a whole when the pion is produced.

It is shown in appendix C that the K-term in 2.6 must always contain $\sin\theta$ as a factor. This follows simply from the conservation of the longitudinal component of the total angular momentum in the elementary production process. Thus t_{NC}^2 vanishes in the forward direction and then increases with θ. On the other hand $F(p)$ decreases with increasing p and therefore with increasing θ. The combination of these two effects results in the non-correlated cross section exhibiting a peak at some angle. Using (4.2) and (4.12), a very rough estimate of the position of this angle is found to be:

$$\theta_{peak} \approx 2/kR$$

where R is the nuclear radius.

If the spatial matter distribution in the nucleus is assumed to be uniform inside a sphere of radius R, and zero outside:

$$\left.\begin{aligned} \rho(r) &= \frac{3}{4\pi R^3} && r < R \\ &= 0 && r > R \end{aligned}\right\} \qquad (4\cdot11),$$

the form factor (3.2) is given by:

$$F(p) = \mathcal{L}(pR) \qquad (4\cdot12)$$

$$\mathcal{L}(x) = \frac{2}{x^3}\left[\sin x - x\cos x\right] \equiv \frac{3}{x} j_1(x) \qquad (4\cdot13).$$

The behavior of the function $\mathcal{L}(x)$ for small values of x is given by:

$$\mathcal{L}(x) \approx 1 - \frac{1}{10}x^2 + \frac{1}{280}x^4 \ldots\ldots \qquad (4\cdot14).$$

The first zero occurs at $x = 4.49$ after which it oscillates with rapidly decreasing amplitude.

Experimentally (11) it is known that 4.11 is not a very realistic distribution since nuclei apparently have diffuse edges. To take this into account, 4.11 can be replaced by a trapezoidal model:

$$\left.\begin{aligned} \rho(r) &= N && r < R_h - a \\ &= \frac{N}{2a}(R_h + a - r) && R_h - a < r < R_h + a \\ &= 0 && R_h + a < r \end{aligned}\right\} \qquad (4\cdot15).$$

N is a normalisation constant, R_h is the half-density radius, and a is related to Hofstadter's "skin thickness" by $t = 1\cdot6\,a$. The form factor becomes:

$$F(p) = \frac{R_h^2}{R_h^2 + a^2}\left[\left(\frac{\sin pa}{pa}\right)\mathcal{L}(pR_h) + \left(\frac{a}{R_h}\right)^2\left(\frac{\sin pR_h}{pR_h}\right)\mathcal{L}(pa)\right] \qquad (4.16).$$

In order to compare 4.12 and 4.16, the parameters were adjusted to yield the root mean square radii as measured by Hofstadter (11):

$$\left.\begin{array}{lllll} \text{for } C^{12}: & R = 3.05 & R_h = 2.42 & a = 1.3 \\ \text{for } Pb^{208}: & R = 7.10 & R_h = 6.73 & a = 1.5 \end{array}\right\} \qquad (4.17).$$

The radii are here given in units of one fermi (10^{-13} cm.). Using these parameters, the form factors were calculated for different values of p. In the case of carbon, a significant difference did not appear until the value of p was about 220 Mev/c at which stage $F(p)$ had decreased to about 0.3. In the case of lead, the form factors remained equal until beyond p=110 Mev/c when $F(p)$ was already smaller than 0.1.

Since the coherent peak usually occurs where $F(p)$ is still about 0.6, the uniform model (4.11) is perfectly satisfactory up to at least some distance beyond the coherent peak. The positions of the zeros of 4.12 and 4.16 do not coincide. On the other hand, a more smoothed out distribution than 4.15 may not have any zeros at all. Since the precise details of the diffraction minima are thus uncertain anyway, there does not seem to be enough reason to discard the uniform model as long as we keep in mind that the minima in the angular distribution may well be smoothed out somewhat. Such a smearing of minima will actually emerge automatically when the absorption of the produced pion is included. In this connection it must be noted that the abrupt surface of 4.11 may produce too much internal reflection when the final

state interaction of the pion is included. We shall, however, introduce this interaction by a method which avoids this source of trouble.

Apart from the changes wrought by the inclusion of the pion interactions, and from modifications of 4.7 when nuclei with $A \neq 2Z$ are considered, the non-correlated cross section is thus given by:

$$\left(\frac{d\sigma}{d\Omega}\right)_{NC} = A^2 \, \mathcal{L}^2(pR) \, |K|^2 \, \Gamma \qquad (4\cdot18)$$

where Γ is given by 4.1. The correlated and diagonal cross sections will be discussed in the next chapter, after which the pion interactions will be introduced.

CHAPTER II

NUCLEAR MATRIX ELEMENTS

5. Nuclear matrix elements of two-nucleon operators

In this chapter the nuclear matrix element which occurred in 3.11 will be calculated. The previously used result (3.12) only held under very special circumstances and will now be generalized. Although it will still be necessary to make certain restrictive assumptions, the results of this chapter will have a much wider range of validity and may also be expected to remain approximately true even for certain nuclei which do not satisfy the conditions used in the derivation.

In order to evaluate the expectation value of a general operator of the form $R_{12}O_{12}$ in the state Ψ, where R_{12} contains the position operators of nucleons 1 and 2 only whereas O_{12} contains the spin and isospin operators of nucleons 1 and 2 only, it is convenient to define exchange operators for the spin and for the isospin variables of nucleons 1 and 2:

$$\left.\begin{aligned} P_s &= \tfrac{1}{2}\left(1 + \vec{\sigma}_1 \cdot \vec{\sigma}_2\right) \\ P_t &= \tfrac{1}{2}\left(1 + \vec{\tau}_1 \cdot \vec{\tau}_2\right) \end{aligned}\right\} \qquad (5.1).$$

Here $\vec{\sigma}_n$ and $\vec{\tau}_n$ are the usual spin and isospin operators for nucleon n. These permutation operators have the following properties:

$$\left.\begin{aligned} P_s P_s &= P_t P_t = 1 \\ P_s P_t &= P_t P_s \\ P_s \vec{\sigma}_1 &= \vec{\sigma}_2 P_s \quad \text{etc.} \end{aligned}\right\} \qquad (5.2).$$

One can also define an operator P_r which exchanges the spatial coordinates of nucleons 1 and 2. The complete interchange of nucleons 1 and 2 is then accomplished by the product $P_r P_s P_t$. Since nucleons obey Fermi statistics, all allowed nuclear states must be antisymmetric with respect to the interchange of any pair of nucleons. This implies:

$$P_r P_s P_t = -1 \qquad (5.3)$$

which can be combined with (5.2) to yield:

$$P_r = -P_s P_t \qquad (5.4).$$

Operators can be defined which project any state onto a state with definite symmetry with respect to nucleons 1 and 2. Because of 5.3, only the symmetry with respect to space and spin have to be specified, the symmetry with respect to isospin being then uniquely determined. The four projection operators are:

$$\left.\begin{aligned} D_1 = D_{ss} &= \tfrac{1}{4}(1+P_s)(1-P_t) \\ D_2 = D_{sa} &= \tfrac{1}{4}(1-P_s)(1+P_t) \\ D_3 = D_{as} &= \tfrac{1}{4}(1+P_s)(1+P_t) \\ D_4 = D_{aa} &= \tfrac{1}{4}(1-P_s)(1-P_t) \end{aligned}\right\} \qquad (5.5).$$

The first index indicates the space symmetry, the second the spin symmetry. For example: D_{sa} projects any state onto a state which is symmetric with respect to interchange of the spatial coordinates of nucleons 1 and 2 but antisymmetric with respect to spin-exchange. It is easy to show, using 5.2, that these projection operators have the properties:

$$\left.\begin{aligned} \sum_{\lambda} D_{\lambda} &= 1 \\ D_{\lambda_1} D_{\lambda_2} &= \delta_{\lambda_1 \lambda_2} D_{\lambda_1} \end{aligned}\right\} \quad (5.6).$$

An independent particle model will be used for the nucleus. Thus the wave function Ψ can be expressed as a superposition of single nucleon wave functions which will for briefness be called orbitals. We can choose the complete set of orbitals in such a way that each one is given by a product of a spatial wave function, an isospin wave function, and a spin wave function:

$$\varphi_i(1) = \varphi_j^r(1)\, \varphi_k^t(1)\, \varphi_l^s(1) \qquad (5.7).$$

The indices k and l can each have two values only, corresponding to the two eigenvalues of τ_3 and of σ_3. The index i denotes the triplet jkl.

We now make the restrictive assumption that our nucleus can be described by the single Slater determinant:

$$\Psi = \frac{1}{\sqrt{A!}} \begin{vmatrix} \varphi_1(1) & \cdots & \varphi_A(1) \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \varphi_1(A) & \cdots & \varphi_A(A) \end{vmatrix} \qquad (5.8)$$

where the set of orbitals is saturated in the sense that each φ_j^r which is occupied, is occupied by 4 nucleons with the 4 possible combinations of $\varphi_k^t \varphi_l^s$. Within the framework of the independent particle model, this assumption will be rigorously satisfied by nuclei such as O^{16} and Ca^{40}.

It follows from 5.6 that the expectation value of our operator $R_{12}O_{12}$ in the state Ψ is given by:

$$\langle\Psi|R_{12}O_{12}|\Psi\rangle = \sum_{\lambda}\langle\Psi|R_{12}D_{\lambda}D_{\lambda}O_{12}|\Psi\rangle \qquad (5.9).$$

The next step is to insert a complete set of states between the two projection operators D_{λ}:

$$\langle\Psi|R_{12}O_{12}|\Psi\rangle = \sum_{\lambda}\sum_{n}\langle\Psi|R_{12}D_{\lambda}|\Psi_n\rangle\langle\Psi_n|D_{\lambda}O_{12}|\Psi\rangle \qquad (5.10).$$

At this point it is necessary to proceed with great caution.

At first sight it may seem as if the proper set of orthonormal states Φ_n to use is the set of all possible nuclear wave functions which are completely antisymmetric with respect to all A nucleons. If this is done, one argues as follows. The second factor on the right of equation 5.10 does not contain any position operators. It will therefore be nonvanishing only if Φ_n contains the same set of space wave functions $\varphi_j^{\dagger}$ as Ψ. With the restrictive assumption which we made, however, this determines Φ_n completely so that the only state which can contribute to the sum, is Ψ itself. Hence one would obtain:

$$\langle\Psi|R_{12}O_{12}|\Psi\rangle = \sum_{\lambda}\langle\Psi|R_{12}D_{\lambda}|\Psi\rangle\langle\Psi|D_{\lambda}O_{12}|\Psi\rangle \qquad (5.11).$$

This result, slightly modified in some respects, was apparently used by Fowler and Watson (10). It is incorrect, however, as can be seen by setting $R_{12} = O_{12} = 1$:

$$\langle\Psi|\Psi\rangle = \sum_{\lambda}\langle\Psi|D_{\lambda}|\Psi\rangle\langle\Psi|D_{\lambda}|\Psi\rangle \qquad (5.12).$$

The left hand side is unity by normalization so that the result is inconsistent with the relation:

$$\langle \Psi | \Psi \rangle = \sum_{\lambda} \langle \Psi | D_{\lambda} | \Psi \rangle \qquad (5.13)$$

unless Ψ is a simultaneous eigenstate of the D_{λ}, which cannot be satisfied beyond He^4. The same argument could as a matter of fact be used to justify the erroneous result (3.12) used in the previous chapter.

The source of these paradoxical results, is the fact that the set of Φ_n chosen is in fact not a complete set (17). Instead of states which are antisymmetrical with respect to all A nucleons, one should use states which are antisymmetric with respect to nucleons 1 and 2 and completely antisymmetric with respect to the other $A-2$ nucleons, but which have no definite symmetry between nucleon 1 say, and any of the $A-2$. This can be achieved by writing:

$$\Phi_n = \sum_{\alpha} \chi_{\alpha} \psi_{\beta} \qquad (5.14)$$

where the χ_{α} are a complete set of orthonormal states of the $A-2$ nucleons (antisymmetrised) whereas the ψ_{β} are a complete set of orthonormal states for nucleons 1 and 2 (antisymmetrised).

For the χ_{α} we shall use Slater determinants of any $A-2$ orbitals (5.7) so that the index α denotes a specific collection of orbitals. The states ψ_{β}, on the other hand, will be chosen to be simultaneous eigenstates of the four D_{λ}. It follows from 5.6 that:

$$\left.\begin{aligned} D_{\lambda} \psi_{\beta} &= \psi_{\beta} && \text{for one value of } \lambda \\ &= 0 && \text{for the other three values of } \lambda \end{aligned}\right\} (5.15).$$

The index β therefore specifies a set of two φ_j^r, two φ_k^t, and two φ_l^s, as well as the index λ for which $D_\lambda \psi_\beta$ does not vanish. Corresponding to each choice of φ_j^r, φ_k^t, and φ_l^s, there will therefore be one, two, or four ψ_β. If the set of orbitals occupied in X_α is a subset of the set of orbitals occupied in Ψ, then β will be called complementary to α if the remaining two orbitals in Ψ are composed of the φ_j^r, φ_k^t, and φ_l^s specified by β. In this case β is determined by α apart from the index λ so that we may write the state vector as $\Phi_{\alpha\lambda}$.

The state vector Ψ can be expanded in terms of the $\Phi_{\alpha\beta}$:

$$\Psi = \sum_{\alpha'} \sum_{\beta'} N_{\alpha'\beta'} \Phi_{\alpha'\beta'} \qquad (5.16).$$

The only non-vanishing $N_{\alpha'\beta'}$ will be those for which the orbitals α' are occupied in Ψ and for which β' is complementary to α'. Equation 5.10 can now be written:

$$\langle \Psi | R_{12} C_{12} | \Psi \rangle = \sum_\lambda \sum_\alpha \sum_\beta \sum_{\alpha'} \sum_{\beta'} \sum_{\alpha''} \sum_{\beta''} \langle \Phi_{\alpha'\beta'} | N^*_{\alpha'\beta'} R_{12} D_\lambda | \Phi_{\alpha\beta} \rangle \cdot \langle \Phi_{\alpha\beta} | D_\lambda C_{12} N_{\alpha''\beta''} | \Phi_{\alpha''\beta''} \rangle \qquad (5.17)$$

Since the operators all refer to nucleons 1 and 2 only, it is clear that the only non-vanishing terms on the right hand side will be those for which α is equal to α' and to α''. Furthermore, the only spin-isospin operators in the first factor on the right hand side occur in D_λ and the $\Phi_{\alpha\beta}$ have been defined in such a way that $D_\lambda \Phi_{\alpha\beta}$ must either vanish or just equal $\Phi_{\alpha\beta}$. Hence we can also conclude that β must equal β' and must therefore be complementary to α.

The matrix element under consideration can thus be expressed as:

$$\langle \Psi | R_{12} C_{12} | \Psi \rangle$$

$$= \sum_{\lambda} \sum_{\alpha} \langle \Psi | R_{12} D_{\lambda} | \Psi_{\alpha\lambda} \rangle \langle \Psi_{\alpha\lambda} | D_{\lambda} C_{12} | \Psi \rangle \qquad (5.18)$$

where the orbitals specified by α must be occupied in the state Ψ. We can therefore write:

$$\langle \Psi | R_{12} C_{12} | \Psi \rangle = \sum_{\lambda} N_{\lambda} \langle \Psi | R_{12} D_{\lambda} | \Psi \rangle \langle \Psi | D_{\lambda} C_{12} | \Psi \rangle \qquad (5.19).$$

To determine the constant N_{λ}, we set $R_{12} = 1$, $C_{12} = D_{\lambda}$, obtaining:

$$N_{\lambda} = \langle \Psi | D_{\lambda} | \Psi \rangle^{-1} \qquad (5.20).$$

For nuclear wave functions which satisfy the criteria stated above, we thus obtain the general result:

$$\langle \Psi | R_{12} C_{12} | \Psi \rangle = \sum_{\lambda} \frac{\langle \Psi | R_{12} D_{\lambda} | \Psi \rangle \langle \Psi | D_{\lambda} C_{12} | \Psi \rangle}{\langle \Psi | D_{\lambda} | \Psi \rangle} \qquad (5.21)$$

which will in general differ from 5.11 and from 5.14.

This equation can now be applied to the matrix element which occurs in 3.11. The result is that the right hand side of (3.12) must be replaced by:

$$A(A-1) \sum_{\lambda} \frac{\langle 0 | D_{\lambda} t_2^{\dagger} t_1 | 0 \rangle}{\langle 0 | D_{\lambda} | 0 \rangle} \langle 0 | e^{i \vec{p} \cdot (\vec{x}_1 - \vec{x}_2)} D_{\lambda} | 0 \rangle \qquad (5.22).$$

The factor containing the exponential will be evaluated in the next section whereas the spin-isospin matrix element will be discussed in section 7.

6. Correlation functions for independent particle model

In this section we shall consider the matrix element:

$$\langle 0 | e^{i\vec{p}\cdot(\vec{x}_1-\vec{x}_2)} D_\lambda | 0 \rangle \qquad (6.1).$$

By analogy with 3.15, this matrix element can be expressed as:

$$\int \rho_\lambda(\vec{x}_1,\vec{x}_2)\, e^{i\vec{p}\cdot(\vec{x}_1-\vec{x}_2)} d^3x_1 d^3x_2 \qquad (6.2)$$

where the two-particle densities, by analogy with 3.13, are given by:

$$\rho_\lambda(\vec{x}_1,\vec{x}_2) = \sum_{\text{Spins}} \int \Psi^*(1..A) D_\lambda \Psi(1..A) d^3x_3 .. d^3x_A \qquad (6.3).$$

They obviously satisfy the condition:

$$\sum_\lambda \rho_\lambda(\vec{x}_1,\vec{x}_2) = \rho(\vec{x}_1,\vec{x}_2) \qquad (6.4).$$

If in 6.3 we use for Ψ the single saturated Slater determinant (5.8) which was discussed in the previous section, the two-particle densities can be expressed in terms of the "mixed density":

$$d(\vec{x}_1,\vec{x}_2) = \sum_{j=1}^{A} \varphi_j(\vec{x}_1)\varphi_j^*(\vec{x}_2) \qquad (6.5).$$

Here the $\varphi_j(\vec{x}_1)$ are the single nucleon spatial wave functions $\varphi_j^\gamma(1)$ which appear on the right hand side of 5.7. Each distinct wave function is counted four times so that the sum contains A terms. The result of this substitution is:

$$\left.\begin{aligned} \rho_{ss}(\vec{x}_1,\vec{x}_2) &= \frac{3}{16A(A-1)}\left[d(\vec{x}_1,\vec{x}_1)d(\vec{x}_2,\vec{x}_2) + d(\vec{x}_1,\vec{x}_2)d(\vec{x}_2,\vec{x}_1)\right] \\ \rho_{sa}(\vec{x}_1,\vec{x}_2) &= \frac{3}{16A(A-1)}\left[d(\vec{x}_1,\vec{x}_1)d(\vec{x}_2,\vec{x}_2) + d(\vec{x}_1,\vec{x}_2)d(\vec{x}_2,\vec{x}_1)\right] \end{aligned}\right\} \qquad (6.6a)$$

$$\left.\begin{aligned}\rho_{as}(\vec{x}_1,\vec{x}_2) &= \frac{9}{16A(A-1)}\left[d(\vec{x}_1,\vec{x}_1)\,d(\vec{x}_2,\vec{x}_2) - d(\vec{x}_1,\vec{x}_2)\,d(\vec{x}_2,\vec{x}_1)\right]\\ \rho_{aa}(\vec{x}_1,\vec{x}_2) &= \frac{1}{16A(A-1)}\left[d(\vec{x}_1,\vec{x}_1)\,d(\vec{x}_2,\vec{x}_2) - d(\vec{x}_1,\vec{x}_2)\,d(\vec{x}_2,\vec{x}_1)\right]\end{aligned}\right\} \quad (6\text{-}6b)$$

Alternatively the two-particle densities may be expressed in terms of the single nucleon density:

$$\rho(\vec{x}) = \frac{1}{A}\,d(\vec{x},\vec{x}) \qquad (6\text{-}7)$$

and the quantity:

$$\left.\begin{aligned}h(\vec{x}_1,\vec{x}_2) &= \frac{d(\vec{x}_1,\vec{x}_2)\,d(\vec{x}_2,\vec{x}_1)}{d(\vec{x}_1,\vec{x}_1)\,d(\vec{x}_2,\vec{x}_2)}\\ &= \frac{\sum_i\sum_j \varphi_i^*(x_1)\varphi_j^*(x_2)\,P_r\,\varphi_i(x_1)\varphi_j(x_2)}{\sum_i\sum_j \varphi_i^*(x_1)\varphi_j^*(x_2)\,\varphi_i(x_1)\varphi_j(x_2)}\end{aligned}\right\} \quad (6\text{-}8)$$

by the relations:

$$\left.\begin{aligned}\rho_{ss}(\vec{x}_1,\vec{x}_2) &= \frac{3}{16}\,\frac{A}{A-1}\,\rho(\vec{x}_1)\rho(\vec{x}_2)\left[1 + h_1(\vec{x}_1,\vec{x}_2)\right]\\ \rho_{sa}(\vec{x}_1,\vec{x}_2) &= \frac{3}{16}\,\frac{A}{A-1}\,\rho(\vec{x}_1)\rho(\vec{x}_2)\left[1 + h_1(\vec{x}_1,\vec{x}_2)\right]\\ \rho_{as}(\vec{x}_1,\vec{x}_2) &= \frac{9}{16}\,\frac{A}{A-1}\,\rho(\vec{x}_1)\rho(\vec{x}_2)\left[1 - h_1(\vec{x}_1,\vec{x}_2)\right]\\ \rho_{aa}(\vec{x}_1,\vec{x}_2) &= \frac{1}{16}\,\frac{A}{A-1}\,\rho(\vec{x}_1)\rho(\vec{x}_2)\left[1 - h_1(\vec{x}_1,\vec{x}_2)\right]\end{aligned}\right\} \quad (6\text{-}9)$$

Using 6.4 and 6.9, one can easily see that the two-particle correlation function defined by 3.16, is given by:

$$g(\vec{x}_1,\vec{x}_2) = -\frac{1}{4}\,h(\vec{x}_1,\vec{x}_2) \qquad (6\text{-}10)$$

for the nuclear model which is being used. It also follows from the second of equations 6.8 that an integral over the correlation function is essentially a measure of the ratio between exchange and ordinary

integrals.

By substituting 6.9 into 6.2, the following expressions are found

$$A(A-1)\langle o|e^{i\vec{p}\cdot(\vec{x}_1-\vec{x}_2)}D_\lambda|o\rangle = \mathcal{L}_\lambda\left\{A^2|F(p)|^2 \pm A\,H(p)\right\} \qquad (6.11)$$

where the plus sign must be taken for $\lambda = 1$ or 2 and the minus sign for $\lambda = 3$ or 4 and where the four values of $\mathcal{L}_\lambda$ are $\frac{3}{16}$, $\frac{3}{16}$, $\frac{9}{16}$, $\frac{1}{16}$. The form factor $F(p)$ has already been defined (3.21) whereas:

$$H(p) = A\int \varphi(\vec{x}_1)\varphi(\vec{x}_2)h(\vec{x}_1,\vec{x}_2)\,e^{i\vec{p}\cdot(\vec{x}_1-\vec{x}_2)}\,d^3x_1\,d^3x_2 \qquad (6.12)$$

differs from $G(p)$ by a factor -4. The two terms on the right hand side of 6.11 thus give rise to the non-correlated and the correlated cross sections, respectively.

The form factor $F(p)$ has already been discussed in section 4 so that we may confine our attention to $H(p)$. Before this is done, however, it will be convenient at this point to briefly sketch a proof of a statement which was made in the first paragraph of section 4, namely that the excitation energy $\Delta = 0$ for the non-correlated cross section. We were at that time considering a matrix element of the form:

$$\langle o|\, f_2\, \delta(k-\omega+W_0-H_A)\, f_1\,|o\rangle \qquad (6.13)$$

where f_1 operated on nucleon 1 only while f_2 operated on nucleon 2 only; In the independent particle model the nuclear hamiltonian H_A is given by a sum of single nucleon hamiltonians $H_1 + H_2 + \ldots\ldots$

By expanding the Slater determinant (5.8) which is used for the ground state $|o\rangle$, we can write 6.13 as a sum of ordinary terms of the form:

$$\langle \varphi_i(1)\varphi_j(2)| f_2\, \delta(k-\omega+W_0-E_{ij}-H_1-H_2) f_1 |\varphi_i(1)\varphi_j(2)\rangle \qquad (6.14)$$

and a sum of exchange terms of the form:

$$\langle \varphi_i(1)\varphi_j(2)| f_2\, \delta(k-\omega+W_0-E_{ij}-H_1-H_2) f_1 |\varphi_j(1)\varphi_i(2)\rangle \qquad (6.15).$$

Here the $\varphi_i(1)$ are the orbitals 5.7 and:

$$E_{ij} = W_0 - \varepsilon_i - \varepsilon_j \qquad (6.16).$$

The single nucleon energies ε_i are defined by:

$$H_1 \varphi_i(1) = \varepsilon_i \varphi_i(1) \qquad (6.17).$$

In 6.14 H_1 can be commuted through f_2 to the left to operate on $\varphi_i(1)$ while H_2 can be commuted to the right to operate on $\varphi_j(2)$. Using 6.17, one finds that the argument of the delta function becomes:

$$k-\omega \qquad (6.18)$$

so that the excitation energy Δ vanishes in this case. If the same procedure is carried out in 6.15, one finds instead that the argument becomes:

$$k-\omega+\varepsilon_i-\varepsilon_j \qquad (6.19).$$

All that can be said in this case is therefore that the excitation energy cannot be greater than the maximum possible difference in single particle energies namely the Fermi energy. As we have shortly ago seen, however, the ordinary terms (6.14) all contribute to the non-correlated cross section while the exchange terms (6.15) all contribute to the

correlated cross section. It follows therefore that, within the independent particle model:

$$\Delta_{NC} = 0 \qquad \Delta_C \leq E_F \tag{6.20}$$

The form factor $H(p)$ will first be calculated for infinite nuclear matter. Infinite nuclear matter can be described by the Fermi gas model. The wave functions are plane waves:

$$\varphi_k(\vec{x}) = \frac{1}{\sqrt{V}} e^{i\vec{k}\cdot\vec{x}} \tag{6.21}$$

where V is the normalization volume. The summation in 6.5 becomes an integral over all values of the momentum less than or equal to the Fermi momentum p_F which is related to the normalization volume V and the number of nucleons A inside this volume by:

$$p_F^3 = \frac{3\pi^2 A}{2V} \tag{6.22}$$

The mixed density becomes:

$$d(\vec{x}_1, \vec{x}_2) = \frac{A}{V} \mathcal{L}(p_F r) \tag{6.23}$$

where $r = |\vec{x}_1 - \vec{x}_2|$ and where the function $\mathcal{L}(x)$ has been defined previously (4.13). Using 6.8, one finds that:

$$h_1(\vec{x}_1, \vec{x}_2) = \mathcal{L}^2(p_F r) \tag{6.24}$$

is a function of the distance between $\vec{x}_1$ and $\vec{x}_2$ only.

The form factor 6.12 is given by:

$$H(p) = A\int d^3r\; \mathcal{L}^2(p_F r)\, e^{i\vec{p}\cdot\vec{r}} \int d^3x_2\, \rho(\vec{r}+\vec{x}_2)\rho(\vec{x}_2)$$

$$= \frac{4\pi A}{V}\int_0^\infty \mathcal{L}^2(p_F r)\,\frac{\sin pr}{pr}\, r^2\, dr$$

$$= \left.\begin{aligned} &4\left[1 - \tfrac{3}{4}\left(\tfrac{p}{p_F}\right) + \tfrac{1}{16}\left(\tfrac{p}{p_F}\right)^3\right], && p<2p_F \\ &0, && p>2p_F \end{aligned}\right\} \quad (6\cdot 25).$$

Using 6.10 and 3.22, one finds that the suppression factor (4.6) for infinite nuclear matter is given by:

$$1 + G(p) = \left.\begin{aligned} &\tfrac{3}{4}\left(\tfrac{p}{p_F}\right) - \tfrac{1}{16}\left(\tfrac{p}{p_F}\right)^3 && p<2p_F \\ &1 && p>2p_F \end{aligned}\right\} \quad (6\cdot 26).$$

This is exactly the suppression factor which one obtains by allowing only the part of the Fermi sphere (in momentum space) which lies outside another Fermi sphere whose center is displaced from the center of the first sphere by a distance p. Thus we see that in the infinite nuclear matter case the partial cancellation of $\left(\frac{d\sigma}{d\Omega}\right)_D$ by $\left(\frac{d\sigma}{d\Omega}\right)_C$ corresponds exactly to the suppression expected from the exclusion principle. This latter suppression cannot be determined so simply in the case of a finite nucleus but is presumably then also obtained by just adding the negative contribution of the correlated cross section.

The suppression factor for a finite nucleus can be expected to differ from that for nuclear matter. Presumably this difference would be most important for light nuclei where the ratio of surface to volume is relatively large. The mixed density (6.5) for a finite nucleus can be calculated if the single particle wave functions are known. Harmonic oscillator wave functions are usually employed for light nuclei because of their simple analytical properties. We shall also do so. Apart from He^4 the lightest nucleus which satisfies the restrictions imposed in

this chapter, is O^{16}. Since deviations from the results for infinite nuclear matter are most likely to occur in light nuclei, we shall choose this nucleus for the detailed calculation.

The single particle wave functions in rectangular coordinates are:

$$\left.\begin{aligned} \varphi_{n_1 n_2 n_3}(\vec{r}) &= \psi_{n_1}(x)\,\psi_{n_2}(y)\,\psi_{n_3}(z) \\ \psi_n(x) &= \left(\frac{\alpha}{\sqrt{\pi}}\right)^{1/2} e^{-\frac{1}{2}\alpha^2 x^2} h_n(x) \end{aligned}\right\} \quad (6.27)$$

where the $h_n(x)$ are proportional to hermite polynomials. For example:

$$h_0 = 1 \qquad h_2 = \sqrt{2}\,x \qquad h_3 = \frac{1}{\sqrt{2}}(2\alpha^2 x^2 - 1) \qquad (6.28).$$

The α is related to the distance ω between energy levels and to the radius R of the equivalent uniform nucleus by:

$$\alpha = \sqrt{M\omega} \qquad R^2 = \frac{15}{4\alpha^2} \qquad (6.29).$$

Using these wave functions, the following expression is obtained for the mixed density (6.4):

$$\rho(\vec{r}_1, \vec{r}_2) = 4\left(\frac{\alpha}{\sqrt{\pi}}\right)^3 e^{-\frac{1}{2}\alpha^2(r_1^2 + r_2^2)}\left[1 + 2\alpha^2\,\vec{r}_1 \cdot \vec{r}_2\right] \qquad (6.30).$$

The single nucleon density and the two-particle correlation function can be determined from 6.7, 6.8, and 6.10. The integral in 6.12 is performed by transforming according to:

$$\left.\begin{aligned} \vec{r}_1 + \vec{r}_2 &= \vec{R} \\ \vec{r}_1 - \vec{r}_2 &= \vec{\rho} \end{aligned}\right\} \qquad (6.31).$$

The Jacobian for this transformation is:

$$\frac{\partial(\vec{r}_1, \vec{r}_2)}{\partial(\vec{\rho}, \vec{R})} = \frac{1}{8} \qquad (6.32).$$

The integral can easily be performed with the result:

$$H(p) = \frac{1}{8}\left(\frac{\alpha}{\sqrt{\pi}}\right)^6 \int e^{-\frac{1}{2}\alpha^2(R^2+\rho^2)}\left[1+\tfrac{1}{2}\alpha^2(R^2-\rho^2)\right] e^{i\vec{p}\cdot\vec{\rho}}\, d^3R\, d^3\rho$$

$$= 4\left[1+\left(\frac{p}{2\alpha}\right)^4\right] e^{-p^2/2\alpha^2} \qquad (6.33).$$

Hence the suppression factor (4.6) is found to be:

$$\left.\begin{aligned} 1+G(p) &= 1-\left[1+\left(\tfrac{p}{2\alpha}\right)^4\right] e^{-p^2/2\alpha^2} \\ &\approx \tfrac{2}{15}(pR)^2 \approx 1.9\left(p/p_F\right)^2 \end{aligned}\right\} \quad (6.34)$$

As one would expect from the structure of the integral in 6.33, the form factor is an even function of the momentum transfer p. The odd dependence (6.25) in the case of nuclear matter comes about because of the integration to infinity. When 6.26 and 6.34 are compared, we observe that for small values of the momentum transfer the cross section is suppressed more in a finite nucleus than in nuclear matter. The ratio:

$$\frac{\left[1+G(p)\right]_{\text{finite nucleus}}}{\left[1+G(p)\right]_{\text{nuclear matter}}} \qquad (6.35)$$

vanishes when $p=0$ and then starts to rise linearly. Using the following parameters for O^{16} :

$$R = 3.35 \text{ fermi} \qquad p_F = 221 \text{ Mev/c} \qquad (6.36),$$

one finds by a numerical calculation that the ratio 6.25 rises steadily until:

$$p \geq 120 \text{ Mev/c} \qquad (\epsilon\text{-}37)$$

and then remains within 5% from unity.

7. Calculation of spin-isospin matrix elements

In order to evaluate the spin-isospin matrix element:

$$\frac{\langle 0| D_\lambda t_2^\dagger t_1 |0\rangle}{\langle 0| D_\lambda |0\rangle} \qquad (7.1)$$

we shall make use of the fact that the total spin S and the total isospin T both vanish in the nuclear ground state $|0\rangle$ if this state satisfies the conditions imposed in section 5. According to the Wigner-Eckart theorem (18):

$$\langle j'm'|P_{LM}|jm\rangle = C(j,L,j';m,M,m')\, P_L(j',j) \qquad (7.2)$$

where P_{LM} is a component of an irreducible tensor which transforms like the spherical harmonic Y_{LM} under rotations, j and m are the angular momentum and angular momentum projection quantum numbers of the state vector, and $C(jLj';mMm')$ is a Clebsch-Gordan coefficient. This theorem can now separately be applied to the spin and the isospin in 7.1. In each case j, m, j', and m' will all be zero so that the matrix element $\langle 0|P_{LM}|0\rangle$ will be proportional to $C(0L0;0M0)$. This Clebsch-Gordan coefficient vanishes unless $L=M=0$. The only operators whose expectation values in the state $|0\rangle$ would not vanish, must therefore under rotations transform like a scalar, both in ordinary space and in charge space.

The spin and isospin operators occurring in D_λ all occur in the combinations $\vec{\sigma}_1\cdot\vec{\sigma}_2$ and $\vec{\tau}_1\cdot\vec{\tau}_2$ so that D_λ already transforms like a scalar in both spaces. All that remains, therefore, is to select the scalar parts of the operator $t_2^\dagger t_1$. The spin and isospin parts may be treated independently. For simplicity we shall just consider:

$$t_2^\dagger t_1 = (A + \vec{B}\cdot\vec{\sigma}_2)(C + \vec{D}\cdot\vec{\sigma}_1) \qquad (7.3)$$

where A and $\vec{B}$ are the complex conjugates of C and $\vec{D}$. The right hand side contains four terms:

$$t_2^\dagger t_1 = AC + A(\vec{D}\cdot\vec{\sigma}_1) + C(\vec{B}\cdot\vec{\sigma}_2) + (\vec{B}\cdot\vec{\sigma}_2)(\vec{D}\cdot\vec{\sigma}_1) \qquad (7.4)$$

The first is already a scalar. The second and third terms transform like vectors and will therefore not contribute to 7.1. The last term can be written $B_i D_j \sigma_i^2 \sigma_j^1$ where i and j run from 1 to 3 and the summation convention is assumed. We can now decompose:

$$\sigma_i^2\sigma_j^1 = \left[\tfrac{1}{3}\sigma_k^2\sigma_k^1\delta_{ij}\right] + \left[\tfrac{1}{2}(\sigma_i^2\sigma_j^1 - \sigma_j^2\sigma_i^1)\right]$$
$$+ \left[\tfrac{1}{2}(\sigma_i^2\sigma_j^1 + \sigma_j^2\sigma_i^1) - \tfrac{1}{3}\sigma_k^2\sigma_k^1\delta_{ij}\right] \qquad (7.5)$$

where the three terms within the square brackets transform like irreducible tensors of rank 0, 1, and 2, respectively. Again, only the first one can contribute. The only part of 7.3 which will have a non-vanishing expectation value in the state $|0\rangle$, is therefore:

$$AC + \tfrac{1}{3}(\vec{B}\cdot\vec{D})(\vec{\sigma}_1\cdot\vec{\sigma}_2) \qquad (7.6).$$

If the results of the preceding paragraph are applied to the transition operator:

$$t = K + \vec{L}\cdot\vec{\sigma} + M\tau_3 + \vec{N}\cdot\vec{\sigma}\tau_3 \qquad (7.7),$$

it can easily be shown that we may replace $t_2^\dagger t_1$ in 7.1 by:

$$t^2 = |K|^2 + \tfrac{1}{3}|L|^2(\vec{\sigma}_1\cdot\vec{\sigma}_2) + \tfrac{1}{3}|M|^2(\vec{\tau}_1\cdot\vec{\tau}_2) + \tfrac{1}{9}|N|^2(\vec{\sigma}_1\cdot\vec{\sigma}_2)(\vec{\tau}_1\cdot\vec{\tau}_2)$$
$$(7.8).$$

It follows from:

$$\left.\begin{aligned}(\vec{\sigma}_1\cdot\vec{\sigma}_2)\left(\frac{1+P_s}{2}\right)|0\rangle &= \left(\frac{1+P_s}{2}\right)|0\rangle \\ (\vec{\sigma}_1\cdot\vec{\sigma}_2)\left(\frac{1-P_s}{2}\right)|0\rangle &= -3\left(\frac{1-P_s}{2}\right)|0\rangle\end{aligned}\right\} \quad (7.9),$$

that we can now directly write down the values of:

$$t_\lambda^2 = \frac{\langle 0|\, D_\lambda\, t_2^\dagger t_1\,|0\rangle}{\langle 0|\, D_\lambda\,|0\rangle} \qquad (7.10)$$

for the four values of λ, namely:

$$\left.\begin{aligned}t_{ss}^2 &= |K|^2 + \tfrac{1}{3}|L|^2 - |M|^2 - \tfrac{1}{3}|N|^2 \\ t_{sa}^2 &= |K|^2 - |L|^2 + \tfrac{1}{3}|M|^2 - \tfrac{1}{3}|N|^2 \\ t_{as}^2 &= |K|^2 + \tfrac{1}{3}|L|^2 + \tfrac{1}{3}|M|^2 + \tfrac{1}{9}|N|^2 \\ t_{aa}^2 &= |K|^2 - |L|^2 - |M|^2 + |N|^2\end{aligned}\right\} \quad (7.11).$$

Equations 5.22 and 6.11 can now be combined to yield the result:

$$\sum_{m\neq n}\sum_{n}\langle 0|\, e^{i\vec{p}\cdot(\vec{x}_n-\vec{x}_m)} t_m^\dagger t_n\,|0\rangle$$

$$= \sum_\lambda t_\lambda^2 \mathcal{D}_\lambda \left\{A^2|F(p)|^2 \pm A H(p)\right\}$$

$$= A^2|F(p)|^2\left\{|K|^2\right\} - \tfrac{1}{4}A H(p)\left\{|K|^2+|L|^2+|M|^2+|N|^2\right\} \qquad (7.12).$$

Since $H(p) = -4\,\sigma_t(p)$, we have thus now proved the relations 3.26 and 3.27 which were used in the expressions 4.3, 4.4, and 4.5 for the differential cross section. By replacing $F(0)=1$ and $H(0)=4$ in 7.12 and dividing by $A(A-1)$, expression 3.25 for $\langle 0|\, t_2^\dagger t_1\,|0\rangle$ also follows directly.

CHAPTER III

FINAL STATE INTERACTIONS OF THE PRODUCED PION

8. Introduction of the final state interactions

Until now the photoproduction of a neutral pion from a single nucleon has been described by the matrix element (A.10):

$$\langle \Phi_f | T | \Phi_i \rangle \qquad (8.1)$$

where T is the transition operator for the whole system. The initial state vector $|\Phi_i\rangle$ describes a nucleon in state i and a plane wave photon with momentum $\vec{k}$ whereas $|\Phi_f\rangle$ describes a nucleon in state f and a free pion with momentum $\vec{q}$. The transition operator T contains the photon and pion field operators. It is shown in appendix B that if these are both expanded in terms of plane wave states, 8.1 can be replaced by:

$$\langle f | e^{-i\vec{q}\cdot\vec{x}}\, t\, e^{i\vec{k}\cdot\vec{x}} | i \rangle \qquad (8.2)$$

where t now operates on the variables of the nucleon only and no longer contains the nucleon coordinate $\vec{x}$. Thus 8.1 is replaced by the matrix element between nucleon states $|i\rangle$ and $|f\rangle$ of the operator:

$$e^{-i\vec{q}\cdot\vec{x}}\, t\, e^{i\vec{k}\cdot\vec{x}} \qquad (8.3).$$

The factor $e^{-i\vec{q}\cdot\vec{x}}$ is just the value of the pion wave function at the position of the nucleon. This form of the single nucleon transition

operator was used (2.3) in chapter I.

When photoproduction occurs in a nucleus, the produced pion is not free since it interacts with the other nucleons. The matrix element 8.1 must therefore be replaced by (A.18):

$$\langle \chi_f^{(-)} | \; T \; | \Phi_i \rangle \qquad (8.4)$$

where in $|\chi_f^{(-)}\rangle$ the pion is not in a plane wave state but in a state which includes the interactions with the nucleus and which has ingoing spherical waves only (see appendix A). In order to perform the inner product with respect to the pion variables, the pion field operator contained in T should thus not be expanded in terms of plane waves but in terms of the wave functions $\psi(\vec{x})$ of a pion interacting with the nucleus in question. Instead of (8.3), the following expression should thus be used for the nucleon transition operator:

$$\psi_q^{(-)}(\vec{x}) \; t \; e^{i\vec{k}\cdot\vec{x}} \qquad (8.5).$$

The interactions of a pion with a system of nucleons may be divided into elastic scattering, inelastic scattering, and true absorption. The pion wave function including elastic scattering may be determined from the Schrödinger equation:

$$\sqrt{m^2+q^2}\,\psi_q(\vec{x}) - \sqrt{m^2-\nabla^2}\,\psi_q(\vec{x}) = \int \langle \vec{x}|V|\vec{x}'\rangle \psi_q(\vec{x}')\, d^3x' \qquad (8.6)$$

where $\langle \vec{x}|V|\vec{x}'\rangle$ is a non-local optical model potential (12). The non-locality of V can be taken into account by using a local potential with an appropriate energy dependence. If terms of second order in the potential are neglected, (8.6) can then be replaced by the integral equation:

$$\psi_q^{(-)}(\vec{x}) = e^{-i\vec{q}\cdot\vec{x}} + \frac{2q}{v}\int G_q^{(-)}(\vec{x},\vec{x}')\, V(\vec{x}')\, \psi_q^{(-)}(\vec{x}')\, d^3x' \qquad (8\cdot7).$$

Here $G_q^{(-)}(\vec{x},\vec{x}')$ is the Green's function for the scalar Helmholtz equation while q and v are the values of the pion's momentum and velocity outside the range of the interaction.

The optical model potential $V(\vec{x})$ at a position $\vec{x}$ inside the nucleus can be expressed (12) in terms of the nucleon density at that point and the diagonal element of the transition matrix for the scattering in the laboratory system of a pion by a single nucleon:

$$V(\vec{x}) = \rho(\vec{x}) \langle t_s \rangle \qquad (8\cdot8).$$

The transition operator in the laboratory system can be expressed in terms of that in the center of mass system by means of relation A.22.

In infinite nuclear matter $\rho(\vec{x})$ is constant so that $V(\vec{x})$ is just a complex number:

$$V(\vec{x}) = V_R + i V_I \qquad (8\cdot9).$$

The solution of 8.7 can then be written as:

$$\psi_q^{(-)}(\vec{x}) = e^{-in\vec{q}\cdot\vec{x}}\, e^{\hat{q}\cdot\vec{x}/2\lambda} \qquad (8\cdot10)$$

where n and λ are the real refractive index and mean free path for pions in nuclear matter and are given by:

$$n = 1 - \frac{V_R}{qv} \quad , \quad \frac{1}{\lambda} = -\frac{2}{v} V_I \qquad (8\cdot11).$$

These relations are only useful when the values of V_R and V_I are small compared to the total pion energy. When this is not the case, more

complicated expressions involving higher powers of the potential must be used.

By means of 8.8 and 8.11 the optical model parameters may be expressed in terms of the forward scattering amplitude and the total cross section for pion-nucleon scattering:

$$\left.\begin{aligned} n &= 1 + \frac{2\pi}{q^2}\,\rho\ \mathrm{Re}\ \overline{f(0)} \\ \frac{1}{\lambda_s} &= \frac{4\pi}{q}\,\rho\ \mathrm{Im}\ \overline{f(0)} = \rho\,\overline{\sigma} \end{aligned}\right\} \qquad (8\cdot12).$$

The cross section σ and forward scattering amplitude $f(0)$ are averaged over all the nucleons. In general these averages will differ for positive and negative pions while the averages for neutral pions will just be the mean of the other two. In a nucleus with the same number of neutrons and protons the optical model potential is independent of the charge state of the pion.

Elastic scattering can in principle be treated exactly by the optical model. Inelastic scattering, on the other hand, is treated as absorption. Furthermore equation 8.8 does not include the effects of the true absorption of a pion by a pair of nucleons which will become very important when the mean free path λ_s for inelastic scattering becomes long. The value of λ to be used in 8.10 should thus be determined from:

$$\frac{1}{\lambda} = \frac{1}{\lambda_s} + \frac{1}{\lambda_a} \qquad (8\cdot13)$$

where λ_s is given by 8.12 while λ_a is the mean free path for true absorption.

The optical model potential for pions in nuclear matter has been calculated by Frank, Gammel, and Watson (12). The forward scattering

amplitude was determined using dispersion relations while λ_a was estimated by means of a method due to Brueckner, Serber, and Watson. As can be seen from 8.12, the potential is proportional to the nucleon density so that it is very sensitive to the value of the nuclear radius parameter r_0 (nuclear radius $R = r_0 A^{1/3}$). Using 8.11 in conjunction with the potentials calculated by Watson et al., the following values were found:

<u>With $r_0 = 1.4$ fermis:</u>

At 250 Mev: $n = 1.20$ $\lambda = 1.75$ fermis
At 900 Mev: $n = 1.00$ $\lambda = 3.05$ fermis } (8.14)

<u>With $r_0 = 1.2$ fermis:</u>

At 250 Mev: $n = 1.24$ $\lambda = 0.90$ fermis
At 900 Mev: $n = 1.00$ $\lambda = 1.90$ fermis } (8.15)

The energies are the mean energies of the photoproduction experiments at M.I.T. (15) and Caltech (16) and are in both cases 10 Mev more than the total pion energy outside the nucleus. Although the matter density in nuclei seems to correspond more to the lower value of r_0 , the optical model parameters determined experimentally (19) from the scattering of pions by nuclei agree much better with the parameters of (8.14) at low energies. At higher energies only the theoretical parameters above are as yet available.

The solution of 8.7 for a finite nucleus would presumably contain diffracted waves in addition to the quasi-classical approximation 8.10. Nevertheless 8.10 can still be expected to be a good approximation if qR is large compared to unity (R is the nuclear radius) which pertains in all the cases to be considered. A closer approximation to the exact solution could be obtained by substituting 8.10 for $\psi_q^{(-)}$ under the

integral in 8.7. The principal effect is to introduce internally reflected waves. The amplitude of these waves are very sensitive to the thickness $2a$ of the nuclear surface (see equation 4.17). Since qa is also large compared to unity, the reflected waves should be unimportant in the present problem. Further attempts to approximate the exact $\psi_q^{(-)}$ did not produce any significant results.

In order to take the final state interactions into account, we shall therefore use the transition operator 8.5 with $\psi_q^{(-)}$ given by 8.10. The plane wave is thus modified by changing the momentum and by multiplying by a growing exponential. This is just the time reversed state of the usual scattering wave function with a decaying exponential.

9. Effect on non-correlated cross section

Equation 4.3 for the non-correlated cross section can be written in the following form:

$$\left(\frac{d\sigma}{d\Omega}\right)_{NC} = A^2 \left| \sin\theta \; F(p) \right|^2 \left\{ \frac{\left(\frac{d\sigma}{d\Omega}\right)_{NSF}}{\sin^2\theta} \right\} \qquad (9.1)$$

where $\left(\frac{d\sigma}{d\Omega}\right)_{NSF}$ is related to the non-spin flip photoproduction cross section from a single nucleon in the laboratory system. In appendix C it is shown that the non-spin flip cross section must always be proportional to $\sin^2\theta$ so that it is convenient to factor this term out and include it with the form factor.

Using expression 8.3 for the single nucleon transition operator, the form factor was previously (3.21) shown to be given by:

$$F_o(p) = \int e^{-i\vec{q}\cdot\vec{x}} e^{i\vec{k}\cdot\vec{x}} \rho(\vec{x}) \, d^3x \qquad (9.2).$$

We now wish to determine $F(p)$ when the transition operator appropriate to the production of an interacting pion (8.5) is used instead of 8.3. As discussed in section 4, we shall use a uniform model of the nuclear density:

$$\left.\begin{aligned} \rho(\vec{x}) &= \frac{3}{4\pi R^3} \qquad x < R \\ &= 0 \qquad\qquad x > R \end{aligned}\right\} \qquad (9.3).$$

In order to be consistent with the usual definition of cross sections, the wave function (8.10) of the outgoing pion must be normalized to unit volume in the asymptotic region. At a point $\vec{x}_o$ on the nuclear surface, the spin wave function is thus just:

$$\psi_q^{(-)}(\vec{x}_0) = e^{-i\vec{q}\cdot\vec{x}_0} \qquad (9.4)$$

since the refractive index $n=1$ just outside the nucleus. At a point $\vec{x}$ inside the nucleus, the wave function (8.10) is:

$$\psi_q^{(-)}(\vec{x}) = N\, e^{\frac{1}{2\lambda}\hat{q}\cdot\vec{x}}\, e^{-in\vec{q}\cdot\vec{x}} \qquad (9.5).$$

The normalization constant N is determined by the requirement that 9.4 and 9.5 agree on the nuclear surface if $(\vec{x}_0-\vec{x})$ is parallel to $\vec{q}$. If N is found by setting $\vec{x}=\vec{x}_0$ in 9.5 and equating the result to 9.4, and if this value of N is substituted back into 9.5, one finds:

$$\psi_q^{(-)}(\vec{x}) = e^{i(n-1)q\ell}\, e^{-\ell/2\lambda}\, e^{-i\vec{q}\cdot\vec{x}} \qquad (9.6)$$

where ℓ is defined by:

$$\left.\begin{aligned} \vec{x}_0 &= \vec{x} + \ell\hat{q} \\ x_c &= R \end{aligned}\right\} \qquad (9.7).$$

We have here taken the center of the nucleus as origin.

When the final state interactions of the produced pion are included, the form factor is thus given by:

$$F_1(p) = \frac{3}{4\pi R^3}\int e^{i(\vec{k}-\vec{q})\cdot\vec{x}}\, e^{i(n-1)q\ell}\, e^{-\ell/2\lambda}\, d^3x \qquad (9.8)$$

where the volume of integration is the interior of the nucleus. To perform this integral, we choose dimensionless cylindrical coordinates with the direction of the pion momentum as axis. If we define:

$$\left.\begin{aligned} \alpha &= kR\cos\theta - qR \\ \beta &= kR\sin\theta \\ \gamma &= R/2\lambda \\ \delta &= (n-1)\,qR \end{aligned}\right\} \qquad (9.9),$$

it is easy to verify that:

$$\left.\begin{aligned} (\vec{k}-\vec{q})\cdot\vec{x} &= \alpha z + \beta r\cos\varphi \\ -\ell/2\lambda &= \gamma z - \gamma\sqrt{1-r^2} \\ (n-1)\,q\,\ell &= \delta\sqrt{1-r^2} - \delta z \end{aligned}\right\} \qquad (9.10).$$

The coordinates of the point $\vec{x}$ are (zR, rR, φ) and θ is again the angle between $\vec{q}$ and $\vec{k}$.

The integral over φ is standard:

$$\int_0^{2\pi} e^{i\beta r\cos\varphi}\,d\varphi = 2\pi\,J_0(\beta r) \qquad (9.11).$$

The remaining integral is:

$$F_1(p) = \frac{3}{2}\int_0^1 dr \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} dz\; e^{(\gamma+i\alpha-i\delta)z}\, e^{(i\delta-\gamma)\sqrt{1-r^2}}\, r\, J_0(\beta r)$$

$$= \frac{3}{2}\int_0^1 dr \left\{ r\, J_0(\beta r)\, \frac{K(r)+iL(r)}{\gamma+i\alpha-i\gamma} \right\} \qquad (9.12)$$

$$\left.\begin{aligned} K(r) &= \cos\left[\alpha\sqrt{1-r^2}\right] - e^{-2\gamma\sqrt{1-r^2}}\cos\left[(\alpha-2\delta)\sqrt{1-r^2}\right] \\ L(r) &= \sin\left[\alpha\sqrt{1-r^2}\right] + e^{-2\gamma\sqrt{1-r^2}}\sin\left[(\alpha-2\delta)\sqrt{1-r^2}\right] \end{aligned}\right\} \qquad (9.13).$$

All attempts to integrate 9.12 analytically, were unsuccessful. Instead, 9.12 was integrated numerically on an electronic computer for different energies. The results of this numerical integration are presented in

appendix D and will be discussed later.

In the preceding chapters the pion was treated as if it did not interact with the nucleus at all after it had been produced. By means of 9.12 we have taken the absorption of the pion as well as the changed value of its momentum inside the nucleus into account. The direction of $\vec{q}$ has so far been considered to remain unchanged, however, whereas the pion could certainly undergo non-forward elastic scattering from the other nucleons. In the formalism introduced in the previous section, the results of such deflections are concentrated in the regions where the refractive index changes. In our model there should thus be a deflection at the nuclear surface so that the pion may be produced at an angle θ_1 (with respect to $\vec{k}$) which is different from the asymptotic direction θ. Another consequence of this deflection is that the pion is produced with an azimuthal angle χ_1 which differs from the asymptotic value χ (in order to prevent confusion with the nuclear coordinate φ, which is measured with respect to the final pion direction, the azimuthal angle, which is measured with respect to the photon direction, is denoted χ). At 900 Mev these effects will not occur since the refractive index is about unity (8.14). We shall nevertheless discuss them.

Instead of $[e^{i\chi} \sin\theta \; F(p)]$, the following expression should thus be used:

$$\int \psi_{\vec{q}}^{(-)}(\vec{x})\, e^{i\chi_1} \sin\theta_1\, e^{i\vec{k}\cdot\vec{x}}\, \rho(\vec{x})\, d^3x \qquad (9.14)$$

where θ_1 and χ_1 depend on $\vec{x}$ and must be kept in the integrand. The deflection depends on the parameter:

$$\xi = n^2 - 1 \qquad (9.15)$$

where n is the index of refraction. After a substantial amount of geometry, one finds that to lowest order in ξ :

$$\left.\begin{aligned} \sin\theta_1 &= \sin\theta - g(r)\cos\theta\cos\varphi \qquad \sin\theta \geq |g(r)| \\ &= g(r) - \sqrt{1-g^2(r)}\sin\theta\cos\varphi \qquad \sin\theta < |g(r)| \end{aligned}\right\} \qquad (9.16)$$

$$\left.\begin{aligned} e^{i(\lambda_1-\lambda)} &= 1 \qquad \sin\theta \geq |g(r)\cos\varphi| \\ &= e^{i\varphi} \qquad \sin\theta < |g(r)\cos\varphi| \end{aligned}\right\} \qquad (9.17)$$

where:

$$\left.\begin{aligned} g(r) &= \frac{\xi r}{2\sqrt{1-r^2}} \qquad r\sqrt{|\xi|} < 2\sqrt{1-r^2} \\ &= \frac{\xi}{\sqrt{|\xi|}} \qquad r\sqrt{|\xi|} > 2\sqrt{1-r^2} \end{aligned}\right\} \qquad (9.18).$$

If the division in 9.17 did not depend on $\cos\varphi$, the integrals over φ and z could again be done analytically, leaving just the integral over r to be done numerically. We may, however, assume first that this is the case and then correct for the error by integrating the difference $(1 - e^{i\varphi})$ over the region where $\sin\theta$ lies between $|g(r)|$ and $|g(r)\cos\varphi|$. The integrals over φ can be done using 10.14 while those over z are trivial. The result is that 9.14 can be written as:

$$e^{i\lambda}\left\{(F_2 + F_3)\sin\theta + F_4\cos\theta + F_5\right\} \qquad (9.19)$$

where:

$$\left.\begin{aligned} F_2 &= \int_{\mathrm{I}} J_0(\beta r)\, M(r)\, dr \\ F_3 &= \int_{\mathrm{II}} \left\{ \frac{2}{\pi |g(r)|} - \sqrt{1-g^2(r)} \left[J_0(\beta r) - \frac{J_1(\beta r)}{\beta r} \right] \right\} M(r)\, dr \\ F_4 &= -i \int_{\mathrm{I}} g(r)\, J_1(\beta r)\, M(r)\, dr \\ F_5 &= i \int_{\mathrm{II}} g(r)\, J_1(\beta r)\, M(r)\, dr \end{aligned}\right\} \quad (9.20).$$

Here we have defined:

$$M(r) = \frac{3r}{2} \frac{k(r) + iL(r)}{\gamma + i\alpha - iS} \qquad (9.21)$$

and the regions of integration:

$$\left.\begin{aligned} \mathrm{I}: &\quad |g(r)| < \sin\vartheta \;, \quad 0 < r < 1 \\ \mathrm{II}: &\quad |g(r)| > \sin\vartheta \;, \quad 0 < r < 1 \end{aligned}\right\} \qquad (9.22)$$

The integrals 9.20 were computed numerically. From 9.19 one might expect that the production will no longer vanish in the forward direction since a pion could be produced at some angle and then deflected towards $\theta = 0$. The terms which would make this possible, F_4 and F_5, both contain a $J_1(kRr\sin\vartheta)$, however, which also vanishes in the forward direction. This is caused by the variation of the azimuthal angle (9.17) when θ becomes small. The non-correlated cross section thus still vanishes in the forward direction as is required by angular momentum conservation.

In appendix D the results of the numerical computation of the following quantities are presented:

$$\left.\begin{aligned} X_0 &= A^2 \left| F_0(p) \sin\theta \right|^2 \\ X_1 &= A^2 \left| F_1(p) \sin\theta \right|^2 \\ X_2 &= A^2 \left| (F_2+F_3)\sin\theta + F_4 \cos\theta + F_5 \right|^2 \end{aligned}\right\} \quad (9.23).$$

The first one is proportional to the non-correlated cross section when the pion interactions are ignored, the second includes the absorption of the pion and the third also includes deflections. At 900 Mev the index of refraction is practically one so that the last two are equal. As expected, the principal effect of the interactions is to attenuate the cross section and to smear out the diffraction minima.

10. Effect on correlated and diagonal cross sections

In section 6 we saw that the correlated cross section is proportional to the quantity (6.12):

$$H(p) = A\int \rho(\vec{x}_1)\rho(\vec{x}_2)\, h(\vec{x}_1,\vec{x}_2)\, e^{i\vec{p}\cdot(\vec{x}_1-\vec{x}_2)}\, d^3x_1\, d^3x_2 \qquad (10.1)$$

if an independent particle model of the nucleus is adopted. The quantity $h(\vec{x}_1,\vec{x}_2)$ was defined in 6.8.

As in the non-correlated case, the exponential is really the product of photon and pion wave functions:

$$e^{i\vec{p}\cdot(\vec{x}_1-\vec{x}_2)} = e^{i\vec{k}\cdot\vec{x}_1} e^{-i\vec{q}\cdot\vec{x}_1} e^{-i\vec{k}\cdot\vec{x}_2} e^{i\vec{q}\cdot\vec{x}_2} \qquad (10.2).$$

To introduce the final state interactions of the pion, 10.2 must therefore be replaced by:

$$e^{i\vec{k}\cdot\vec{x}_1}\, \psi_q^{(-)}(\vec{x}_1)\, e^{-i\vec{k}\cdot\vec{x}_2}\, \psi_q^{(-)*}(\vec{x}_2) \qquad (10\cdot3)$$

where for the pion wave functions the approximation 9.6 will be used.

In the Fermi gas model of the nucleus (6.23) the first part of the integrand of 10.1 is given by:

$$A\,\rho(\vec{x}_1)\rho(\vec{x}_2)\,h(\vec{x}_1,\vec{x}_2) = \frac{A}{V^2}\,\mathcal{L}^2\left(p_F|\vec{x}_1-\vec{x}_2|\right) \qquad (10\cdot4).$$

In section 6 the integral 10.1 was performed by changing the variables of integration to $\vec{x}_2$ and $\vec{r} = \vec{x}_1 - \vec{x}_2$. When the pion plane waves have been replaced by the wave functions $\psi_q^{(-)}(\vec{x})$ of 9.6, however, such a change of variables no longer enables one to do the integral and attempts to integrate 10.1 directly were also unsuccessful when 10.4 was used.

In section 6 we also introduced harmonic oscillator wave functions for O^{16} (6.30). In that case:

$$A\,\varphi(\vec{x}_1)\varphi(\vec{x}_2)h(\vec{x}_1,\vec{x}_2) = \left(\frac{\mu}{\sqrt{\pi}}\right)^6 e^{-\mu^2(x_1^2+x_2^2)}\left[1+2\mu^2\vec{x}_1\cdot\vec{x}_2\right]^2 \qquad (10\cdot5).$$

The parameter α of section 6 is here denoted μ. In terms of the radius R of the equivalent uniform model:

$$\mu R = s = \sqrt{\frac{15}{4}} \qquad (10\cdot6).$$

Both for $\vec{x}_1$ and $\vec{x}_2$ we shall use dimensionless cylindrical coordinates with the pion momentum as axis:

$$\left.\begin{aligned} \vec{x}_1 &: \quad u = \mu z_1\,,\ x = \mu r_1\,,\ \varphi_1 \\ \vec{x}_2 &: \quad v = \mu z_2\,,\ y = \mu r_2\,,\ \varphi_2 \end{aligned}\right\} \qquad (10\cdot7)$$

which means that:

$$\left.\begin{aligned} \mu^2(x_1^2+x_2^2) &= x^2+u^2+y^2+v^2 \\ 2\mu^2\vec{x}_1\cdot\vec{x}_2 &= 2uv+2xy\cos(\varphi_1-\varphi_2) \\ \mu^6\int d^3x_1\,d^3x_2 &= \int_0^\infty dx\int_0^\infty dy\int_{-\infty}^{\infty}du\int_{-\infty}^{\infty}dv\int_0^{2\pi}x\,d\varphi_1\int_0^{2\pi}y\,d\varphi_2 \end{aligned}\right\} \qquad (10\cdot8).$$

In this model the density of the nucleons, from which the pion is produced, is proportional to:

$$e^{-\frac{1}{2}(x^2+u^2)}\left[1+2(x^2+u^2)\right] \qquad (10\cdot9)$$

so that the nucleus has a diffuse surface. In order to make the calculation feasible, we shall, however, assume that the optical model potential, which causes the absorption of the pion, has a uniform

distribution:

$$\left.\begin{aligned} V(x,u) &= V_R + i V_I \qquad x^2+u^2 < S^2 \\ &= 0 \qquad\qquad\quad x^2+u^2 > S^2 \end{aligned}\right\} \qquad (10.10)$$

where S **is the constant defined in 10.6. By using the pion wave functions corresponding to 9.6, the products in 10.3 are found to be:**

$$\left.\begin{aligned} e^{i\vec{k}\cdot\vec{x}_1}\psi_q^{(-)}(\vec{x}_1) &= e^{i\alpha u + i\beta x\cos\varphi_1} e^{\gamma u - \gamma\sqrt{S^2-x^2}}, \quad u^2+x^2<S^2 \\ &= e^{i\alpha u + i\beta x\cos\varphi_1} e^{-2\gamma\sqrt{S^2-x^2}} \quad \begin{cases} x < S \\ u < -\sqrt{S^2-x^2} \end{cases} \\ &= e^{i\alpha u + i\beta x\cos\varphi_1} \qquad \text{elsewhere} \end{aligned}\right\} (10.11)$$

and similarly for $e^{-i\vec{k}\cdot\vec{x}_2}\psi_q^{(-)*}(\vec{x}_2)$ **. The parameters** α **,** β **, and** γ **differ from those of equation 9.9 by a factor** S **:**

$$\left.\begin{aligned} \alpha &= (k\cos\theta - q)/\mu \\ \beta &= (k\sin\theta)/\mu \\ \gamma &= 1/2\lambda\mu \end{aligned}\right\} \qquad (10.12).$$

For simplicity only the absorption of the pion will be considered.

It can be seen from 10.8 and 10.11 that the integrals over φ_1 **and** φ_2 **are of the form:**

$$\int_0^{2\pi}\int_0^{2\pi} \left[1 + 2uv + 2xy\cos(\varphi_1-\varphi_2)\right]^2 e^{i\beta(x\cos\varphi_1 - y\cos\varphi_2)} \cdot xy\, d\varphi_1\, d\varphi_2 \qquad (10.13).$$

This double integral can be reduced to sums and products of the following integrals:

$$\left.\begin{aligned}
\int_0^{2\pi} e^{iz\cos u}\,du &= 2\pi J_0(z)\\
\int_0^{2\pi} e^{iz\cos u}\cos u\,du &= 2\pi i J_1(z)\\
\int_0^{2\pi} e^{iz\cos u}\sin u\,du &= 0\\
\int_0^{2\pi} e^{iz\cos u}\cos u\sin u\,du &= 0\\
\int_0^{2\pi} e^{iz\cos u}\sin^2 u\,du &= \frac{2\pi}{z} J_1(z)\\
\int_0^{2\pi} e^{iz\cos u}\cos^2 u\,du &= 2\pi J_0(z) - \frac{2\pi}{z} J_1(z)
\end{aligned}\right\} \quad (10\cdot14).$$

These results are obtainable from any of the usual books on Bessel functions.

The remaining integrations over x , y , u , **and** v **could now be performed numerically. In order to reduce the results to products of simple integrals instead of two-fold integrals, we shall, however, replace 10.11 by the following approximation:**

$$e^{i\vec{k}\cdot\vec{x}_1}\,\psi_q^{(-)}(\vec{x}_1) = e^{i\alpha u + i\beta x\cos\varphi_1}\,h(x)\,f(u) \qquad (10\cdot15)$$

$$\left.\begin{aligned}
h(x) &= e^{-\gamma\sqrt{s^2-x^2}} && x < s\\
&= 1 && x > s
\end{aligned}\right\} \qquad (10\cdot16)$$

$$\left.\begin{aligned}
f(u) &= e^{-\gamma s} && u < -s\\
&= e^{\gamma u} && u^2 < s^2\\
&= e^{\gamma s} && u > s
\end{aligned}\right\} \qquad (10\cdot17)$$

This is identical with 10.11 in the region $x^2+u^2<s^2$. **Outside this region, however, the value of** $e^{i\vec{k}\cdot\vec{x}_1}\,\psi_q^{(-)}(\vec{x}_1)$ **is enhanced in the forward hemisphere (** $u>0$ **) and suppressed in the backward**

hemisphere ($u<0$). Since the density 10.9 decreases very rapidly outside the uniform radius $\sqrt{x^2+u^2} = S$, the error arising from the use of 10.15 can, however, be expected to be quite small.

With this approximation, 10.1 can now be expressed in terms of the following integrals:

$$\left.\begin{aligned} F_0 &= \int_{-\infty}^{\infty} e^{i\alpha u} e^{-u^2} f(u)\, du \\ F_1 &= \int_{-\infty}^{\infty} e^{i\alpha u} e^{-u^2} f(u)\, u\, du \\ F_2 &= \int_{-\infty}^{\infty} e^{i\alpha u} e^{-u^2} f(u)\, u^2\, du \end{aligned}\right\} \qquad (10\cdot18)$$

$$\left.\begin{aligned} H_0 &= \int_0^{\infty} e^{-x^2} h(x)\, J_0(\beta x)\, x\, dx \\ H_1 &= \int_0^{\infty} e^{-x^2} h(x)\, J_1(\beta x)\, x^2\, dx \\ H_2 &= \int_0^{\infty} e^{-x^2} h(x) \left[\frac{J_1(\beta x)}{\beta x}\right] x^3\, dx \\ H_3 &= \int_0^{\infty} e^{-x^2} h(x) \left[J_0(\beta x) - \frac{J_1(\beta x)}{\beta x}\right] x^3\, dx \end{aligned}\right\} \qquad (10\cdot19)$$

The integrals in 10.18 can further be reduced to:

$$\left.\begin{aligned} F_0 &= \sqrt{\pi}\, e^{-\alpha^2/4} \cosh \gamma s + \int_{-s}^{s} e^{i\alpha u - u^2} g(u)\, du \\ F_1 &= e^{-\alpha^2/4}\left(\sinh \gamma s + i\alpha \tfrac{\sqrt{\pi}}{2} \cosh \gamma s\right) + \int_{-s}^{s} e^{i\alpha u - u^2} g(u)\, u\, du \\ F_2 &= e^{-\alpha^2/4}\left(\tfrac{\sqrt{\pi}}{2}\left[1 - \tfrac{\alpha^2}{2}\right]\cosh \gamma s + i\alpha \sinh \gamma s\right) \\ &\quad + \int_{-s}^{s} e^{i\alpha u - u^2} g(u)\, u^2\, du \end{aligned}\right\} \qquad (10\cdot20)$$

where:

$$\left.\begin{aligned} g(u) &= e^{\gamma u} - e^{-\gamma s} \qquad u<0 \\ &= e^{\gamma u} - e^{\gamma s} \qquad u>0 \end{aligned}\right\} \qquad (10\cdot21).$$

The original integral 10.1 is then given by:

$$\frac{4}{\pi}\left\{|F_0|^2\left(H_0^2+4H_1^2+4H_2^2+4H_3^2\right)+|F_1|^2\left(4H_2^2+8H_1^2\right)+4|F_2|^2H_2^2\right\} \qquad (10.22).$$

The integrals in 10.19 and 10.20 were done numerically on a digital computer. Since the values of the integrands in 10.19 decrease rapidly with increasing x, the point $x=3$ was chosen as the upper limit in the actual calculations. This represents a radial distance of more than 1½ times the uniform radius.

To calculate the effect of absorption on the diagonal cross section, we can return to the uniform density used for the non-correlated cross section in section 9. Using the pion wave function 9.6, it is easy to see that the diagonal cross section must just be multiplied by a factor:

$$\zeta = \int e^{-\ell(\vec{x})/\lambda}\,\rho(\vec{x})\,d^3x \qquad (10.23)$$

where $\ell(\vec{x})$ was defined in 9.7. If the same coordinates as in section 9 are used, one finds:

$$\zeta = \frac{3}{2}\int_0^1\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} e^{\left[z-\sqrt{1-r^2}\right]R/\lambda}\,r\,dz\,dr$$

$$= \frac{3}{4\gamma}\left\{\frac{1}{2}-\frac{1}{16\gamma^2}\left[1-(1+4\gamma)e^{-4\gamma}\right]\right\} \qquad (10.24)$$

where $\gamma = R/2\lambda$ is the γ of equation 9.9 and differs from the γ of equation 10.12 by a factor S. Thus ζ can be calculated directly for any value of the nuclear radius R and the mean free path λ. Using the parameters of 8.14, one finds the following values for ζ :

A	12	40	64	208
At 250 Mev	0.37	0.28	0.25	0.18
At 900 Mev	0.53	0.42	0.38	0.30

To a good approximation these values can be represented by the formulae:

$$\left.\begin{aligned} \text{At } 250 \text{ Mev}: \quad & \zeta = 0.69\, A^{-1/4} \\ \text{At } 900 \text{ Mev}: \quad & \zeta = 0.87\, A^{-1/5} \end{aligned}\right\} \quad (10.25).$$

These formulae are purely empirical and no significance should be attached either to the exponents or to the coefficients.

The effect of the final state interactions on the non-correlated, the correlated, and the diagonal cross sections have now been estimated. In the case of the correlated cross section this has been done for the special case of O^{16} only but in the other two cases the results are completely general, within the approximations mentioned from time to time of course.

The results of the numerical integrations are presented in appendix D. At this time, however, it may be worthwhile to compare the attenuation of the correlated cross section as compared to that of the diagonal cross section.

The correlated cross section is proportional to the function $G(p)$ of 3.22 which has unit magnitude when p vanishes and vanishes when p tends to infinity. The numerical integrations using the parameters of 8.14 produce the result that in the forward direction,

$G(p)$ is attenuated by factors:

$$\left.\begin{array}{lll} 0.32 & \text{at} & 250 \text{ Mev} \\ 0.48 & \text{at} & 900 \text{ Mev} \end{array}\right\} \qquad (10.26).$$

On the other hand equation 10.24 yields for O^{16} the values:

$$\left.\begin{array}{lll} \zeta = 0.34 & \text{at} & 250 \text{ Mev} \\ \zeta = 0.50 & \text{at} & 900 \text{ Mev} \end{array}\right\} \qquad (10.27).$$

The slight difference between 10.26 and 10.27 probably reflects more on deficiencies in the treatment of $G(p)$ than on a fundamental difference and is in fact perhaps completely ascribable to the fact that p does not quite vanish in the forward direction. One can therefore deduce that the cancellation between the diagonal and correlated cross sections for vanishing momentum transfers is not destroyed by the pion interactions.

At the angles at which the magnitude of $G(p)$ without interaction falls to the value 0.5, the attenuation factors of (10.26) are replaced by:

$$\left.\begin{array}{lll} 0.27 & \text{at} & 250 \text{ Mev} \\ 0.44 & \text{at} & 900 \text{ Mev} \end{array}\right\} \qquad (10.28)$$

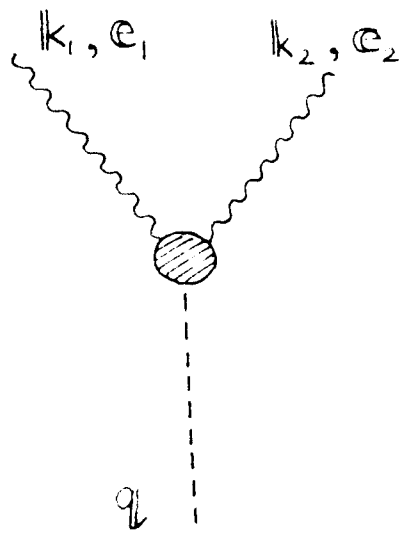
so that the absorption causes $G(p)$ to decrease slightly more rapidly with increasing p.

CHAPTER IV

INVERSE DECAY OF THE NEUTRAL PION

11. Photoproduction by the coulomb field

The decay of the neutral pion into two photons is presumably due to the strong (B.3) and the electromagnetic (B.4) couplings. Apart from the method of dispersion relations, no way has yet been found of extracting quantitative results from the relativistic field theory of the strong couplings so that it is as yet impossible to represent the lifetime of the π^0 in terms of known constants.



Although the Feynman-Dyson perturbation technique is not applicable to the problem in question, the structure of the sum of all Feynman diagrams containing one external pion line and two external photon lines only, is determined uniquely by the conditions that it should depend linearly on each of the two photon polarization vectors and that the result should be a pseudoscalar. This sum is represented by the diagram above. All relevant renormalizations are supposed to have been performed. This vertex may thus be supposed to represent a fundamental coupling which may, however, be used in irreducible diagrams only.

Four-vectors are denoted q_μ or q. The pion's four-momentum is q whereas those of the two photons are k_1 and k_2. The photon polarisations are e_1 and e_2. Only four of these five four-vectors are independent since:

$$q = k_1 + k_2 \qquad (11\cdot1).$$

Only one pseudoscalar can be constructed out of four four-vectors so that the diagram above must be represented by the following vertex operator in momentum space:

$$\varepsilon_{\kappa\lambda\mu\nu}\, e_{1\kappa}\, e_{2\lambda}\, k_{1\mu}\, k_{2\nu}\, Y(k_1\cdot k_2,\ k_1^2,\ k_2^2) \qquad (11\cdot2).$$

Here $\varepsilon_{\kappa\lambda\mu\nu}$ is the completely antisymmetric unit tensor in four dimensions. The form factor Y may still depend on the three scalars which do not involve the polarization vectors. We shall use the relativistic metric and the γ-matrices defined in appendix B.

By means of the Feynman rules the S-matrix element for the diagram in question can be written down directly:

$$\frac{(2\pi)^4\,\delta^4(q-k_1-k_2)}{\sqrt{2k_1}\,\sqrt{2k_2}\,\sqrt{2\omega}}\ \varepsilon_{\kappa\lambda\mu\nu}\, e_{1\kappa} e_{2\lambda} k_{1\mu} k_{2\nu} Y(k_1\cdot k_2, 0, 0) \qquad (11\cdot3)$$

since the photons are real so that $k_1^2 = k_2^2 = 0$. The pion energy is $\omega = q_0$. The transition rate is thus given by:

$$\Gamma = (2\pi)^4\ \frac{1}{2} \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3}\ \frac{\delta^4(q-k_1-k_2)}{8k_1k_2\omega}$$

$$\times \sum_{e_1}\sum_{e_2} \left| \varepsilon_{\kappa\lambda\mu\nu}\, e_{1\kappa} e_{2\lambda} k_{1\mu} k_{2\nu}\, Y \right|^2 \qquad (11\cdot4).$$

The factor $\frac{1}{2}$ in the density of states is necessary because of the identity of the photons. When the sums over polarizations are performed,

one finds:

$$\sum_{e_1}\sum_{e_2}\left|\varepsilon_{\kappa\lambda\mu\nu}e_{1\kappa}e_{2\lambda}k_{1\mu}k_{2\nu}Y\right|^2 = 2(k_1\cdot k_2)^2\,|Y|^2 \qquad (11\cdot5).$$

Four of the six integrations in 11.4 just remove the delta function and yield the conservation law 11.1. If θ is defined by:

$$\vec{k}_1\cdot\vec{q} = k_1 q\cos\theta \qquad (11\cdot6),$$

the following expression is obtained for the decay rate:

$$\Gamma = \int \frac{d\Omega_1}{(2\pi)^2}\,\frac{k_1^2 k_2}{\omega - q\cos\theta}\,\frac{(k_1\cdot k_2)^2}{8k_1k_2\omega}\left|Y(k_1\cdot k_2, 0, 0)\right|^2 \qquad (11\cdot7).$$

The rate Γ is not an invariant. The lifetime of the pion in its own system is obtained by choosing:

$$\left.\begin{aligned} &\omega = m \qquad q = 0 \qquad k_1 = k_2 = \tfrac{1}{2}m \\ &k_1\cdot k_2 = k_1\cdot q = -k_1\omega = -\tfrac{1}{2}m^2 \end{aligned}\right\} \qquad (11\cdot8).$$

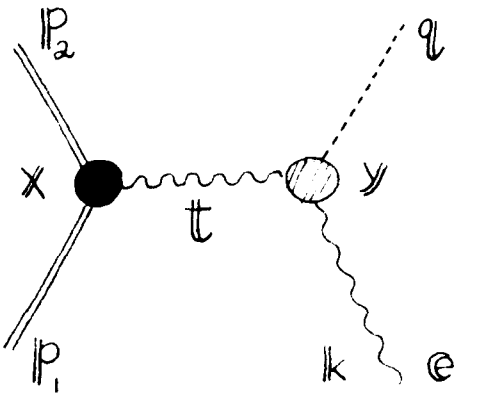
The integral over solid angle $\int d\Omega_1$ just introduces a factor 4π so that the final expression for the lifetime is:

$$\frac{1}{\tau} = \Gamma = \frac{m^3}{64\pi}\left|Y\left(-\tfrac{m^2}{2}, 0, 0\right)\right|^2 \qquad (11\cdot9).$$

The value of the form factor in 11.2 at one specific point can thus be expressed in terms of the mass m and lifetime τ of the neutral pion.

To lowest order in the electromagnetic coupling (B.4) photoproduction by the inverse decay of the neutral pion can

now occur through the diagram on the right. The initial and final four-momenta of the nucleus are denoted $\mathbb{P}_1$ and $\mathbb{P}_2$ while those of the incident photon and the produced pion are $\mathbb{k}$ and $\mathbb{q}$, respectively. The four-momentum of the virtual photon is denoted $\mathbb{t}$ while $\mathbb{e}$ is the polarisation of the incident photon. The vertex on the right is the one we discussed before (11.2). The one on the left represents the interaction of the virtual photon with the whole nucleus and will be denoted Γ_μ.



The S-matrix element can again be written down directly:

$$\int \frac{d^4t}{(2\pi)^4} \frac{1}{i\mathbb{t}^2} \frac{(2\pi)^4 \delta^4(\mathbb{k}+\mathbb{t}-\mathbb{q})}{\sqrt{2k'}\sqrt{2\omega'}} \varepsilon_{\mu\nu\lambda\kappa} \Gamma_\mu e_\nu t_\lambda k_\kappa \times Y(\mathbb{k}\cdot\mathbb{t}, \mathbb{t}^2, \mathbb{k}^2) \qquad (11.10).$$

The result of the integration is that the components of $\mathbb{t}$ are just the momentum and energy transfer to the nucleus:

$$\left.\begin{aligned} \vec{t} &= \vec{q}-\vec{k} = -\vec{p} \\ t_0 &= \omega - k = -\Delta \end{aligned}\right\} \qquad (11.11).$$

Since the other photon is real, $e_0 = 0$ so that:

$$\varepsilon_{\mu\nu\lambda\kappa}\Gamma_\mu e_\nu t_\lambda k_\kappa = \Gamma_0 \vec{e}\cdot(\vec{q}\times\vec{k}) + (\vec{p}k - \vec{k}\Delta)\cdot(\vec{\Gamma}\times\vec{e}) \qquad (11.12).$$

If the nucleus had been a scalar particle with a point charge Ze, the vertex function Γ_μ would have been:

$$\Gamma_\mu = \int \frac{1}{\sqrt{2E_2}} e^{-i\mathbf{p}_2\cdot x}\left[Ze(p_{1\mu}+p_{2\mu})e^{-i\mathbf{q}\cdot x}\right]\frac{1}{\sqrt{2E_1}} e^{i\mathbf{p}_1\cdot x} d^4x$$

$$= Ze\,\frac{p_{1\mu}+p_{2\mu}}{2\sqrt{E_1E_2}}(2\pi)^4\,\delta^4(\mathbf{p}_1-\mathbf{p}_2-\mathbf{q}) \qquad (11\cdot13)$$

where $E_1 = p_{10}$ and $E_2 = p_{20}$. Similarly for a single Dirac point proton (B.4):

$$\Gamma_\mu = \int e^{-i\mathbf{p}_2\cdot x}\left[\bar{u}_f\, e\gamma_\mu u_i\, e^{-i\mathbf{q}\cdot x}\right] e^{i\mathbf{p}_1\cdot x} d^4x$$

$$= e(\bar{u}_f\gamma_\mu u_i)(2\pi)^4\,\delta^4(\mathbf{p}_1-\mathbf{p}_2-\mathbf{q}) \qquad (11\cdot14)$$

where u_i and u_f are the initial and final spinors. If instead of in plane wave states the proton were initially and finally in two stationary states of a potential, then

$$\Gamma_\mu = \int \psi_f^*(\vec{x})\, e^{iE_2t}\left[\bar{u}_f e\gamma_\mu u_i\, e^{i\vec{p}\cdot\vec{x}} e^{-i\Delta t}\right]\psi_i(\vec{x})\, e^{-iE_1t}\, dt\, d^3x$$

$$= e(\bar{u}_f\gamma_\mu u_i)(2\pi)\,\delta(E_1+\Delta-E_2)$$

$$\times \int \psi_f^*(\vec{x})\, e^{i\vec{p}\cdot\vec{x}}\,\psi_i(\vec{x})\, d^3x \qquad (11\cdot15).$$

For simplicity we supposed that the spinor components have the same spatial dependence. If the initial and final wave functions are the same, the last integral is just the charge form factor:

$$F(p) = \int |\psi(x)|^2 e^{i\vec{p}\cdot\vec{x}}\, d^3x \qquad (11\cdot16).$$

In general the three-momentum delta functions in 11.13 and 11.14 are replaced by 11.16 or something similar when the charge is not concentrated at a point.

In a real nucleus the current Γ_μ is very complex. Its spatial components, however, are usually of the same order as the magnetic moments of single nucleons whereas the charge is always additive. We shall therefore for the moment keep Γ_0 only and assume that the nucleus remains in its ground state so that Δ is negligible and:

$$\Gamma_0 = Ze\,(2\pi)\,\delta(E_1-E_2)\,F(p) \qquad (11.17)$$

where $F(p)$ is the form factor defined in 3.20. The discussion of possible corrections will be left to section 13.

The T-matrix element can then be written (A.9) as:

$$\frac{Ze\,F(p)}{2\sqrt{k\omega'}}\;\frac{\vec{e}\cdot(\vec{q}\times\vec{k})}{p^2}\;Y(\vec{k}\cdot\vec{p},\,p^2,\,0) \qquad (11.18).$$

Comparison with 11.9 shows that the value of the form factor Y which enters the photoproduction matrix element will in general not be identical to that encountered in the decay process. Especially in forward directions, however, these two can be expected to be approximately equal since it is extremely unlikely that the non-locality of Y has a range larger than the Compton wave length of the proton. We shall thus substitute for Y in 11.18 the value:

$$Y = \frac{8\sqrt{\pi}}{\sqrt{m^3\tau}}\;e^{i\delta} \qquad (11.19)$$

where δ is an unknown phase factor.

The differential cross section for photoproduction by the inverse decay of the neutral pion, now follows simply from A.20:

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha}{m^3\tau}\int \frac{q^2\,dq}{(2\pi)^2}\,\delta(k-\omega)\,\frac{64\pi^2}{k\omega}\,|F(p)|^2\,\frac{1}{2}\sum_{\epsilon}\left|\frac{\vec{e}\cdot(\vec{q}\times\vec{k})}{p^2}\right|^2$$

$$= \left(\frac{8Z^2\alpha\hbar}{mc^2\tau}\right)\left(\frac{\hbar}{mc}\right)^2 |F(p)|^2\,\frac{q^3 k\sin^2\theta}{p^4} \qquad (11.20)$$

where $\alpha = e^2/4\pi\hbar c$ is the fine structure constant and where we have temporarily restored $\hbar$ and c to indicate the dimensions.

The factor $\frac{\sin^2\theta}{p^4}$ has a maximum at an angle:

$$\theta_p = \frac{k-q}{\sqrt{kq}} \qquad (11.21)$$

which becomes very small at high energies. At 900 Mev, for example, its value is less than two thirds of a degree. From then onwards, this factor decreases monotonically but becomes modulated by the $|F(p)|^2$. The coulomb production should thus exhibit a strong peak at a very small angle. The magnitude of this peak is inversely proportional to the pion lifetime so that τ could be determined by detecting this peak and correcting for the nuclear production.

The production of neutral pions by its inverse decay was first suggested by Primakoff (13) who proposed that its lifetime be determined by looking for a Z^2 - term in the total cross section.

12. Effect of final state interactions on coulomb production

The introduction of the final state interactions of the produced pion follows in the most direct way if the S-matrix element (11.10) is expressed in configuration space instead of momentum space. Following the usual Feynman rules, we now need an integral over the coordinates x and y in the diagram on page 68, instead of an integral over the momentum q of the virtual photon. The integration over the time-components produces a delta function of energy which can be factored out (A.9) to yield the transition matrix element:

$$T_{fi} = \int d^3x\, d^3y\; G(\vec{x})\, \frac{1}{4\pi |\vec{x}-\vec{y}|} \left[\frac{e^{-i\vec{q}\cdot\vec{y}}}{\sqrt{2\omega}} \sqrt{\frac{64\pi}{m^3\tau}}\, e^{i\delta}\, \vec{e}\cdot(\vec{q}\times\vec{k})\, \frac{e^{i\vec{k}\cdot\vec{y}}}{\sqrt{2k}} \right] \qquad (12.1).$$

We have here made the same approximations as in the previous section, such as using 11.19 for the "pion form factor" and keeping the contribution from Γ_0 only (11.12).

Apart from some factors, $G(\vec{x})$ is the Fourier transform of Γ_0. For a single nucleon, for example:

$$G(\vec{x}) = \bar{\psi}(\vec{x})\, e\gamma_0\, \psi(\vec{x}) = e\, \psi^*(\vec{x})\, \psi(\vec{x}) \qquad (12.2).$$

Under the same conditions as before, $G(\vec{x})$ for a whole nucleus will be given by:

$$G(\vec{x}) = Ze\, \rho(\vec{x}) \qquad (12.3).$$

Hence the transition matrix element becomes:

$$T_{fi} = \frac{Ze \cdot e^{i\delta}}{\sqrt{m^3 \tau \pi}} \frac{\hat{e}\cdot(\vec{q}\times\vec{k})}{\sqrt{k\omega}} \int \frac{\rho(\vec{x})\, e^{i\vec{k}\cdot\vec{y}}\, e^{-i\vec{q}\cdot\vec{y}}}{|\vec{x}-\vec{y}|} d^3x\, d^3y \qquad (12\cdot4)$$

which is of course identical to 11.18 since nothing new has yet been introduced.

Already in this form it is possible to see roughly what the effect of the absorption of the pion will be. Let us in 12.4 once again use a uniform distribution (4.11) and perform the integral:

$$\left.\begin{aligned} \int \frac{\rho(\vec{x})}{|\vec{x}-\vec{y}|} d^3x &= \frac{3}{2R} - \frac{y}{2R^3} \qquad y<R \\ &= \frac{1}{y} \qquad y>R \end{aligned}\right\} \qquad (12\cdot5).$$

The remaining integral then yields:

$$\int d^3y\, e^{i\vec{p}\cdot\vec{y}} \int d^3x\, \frac{\rho(\vec{x})}{|\vec{x}-\vec{y}|} = \frac{4\pi}{p^2}\left\{\left[\mathcal{L}(pR) - \cos pR\right] + \cos pR\right\} \qquad (12\cdot6)$$

where the contribution $\left[\mathcal{L}(pR) - \cos pR\right]$ came from the interior region ($y<R$). The function $\mathcal{L}(x)$ has already been defined (4.13). A rough idea of the consequences of absorption can now be obtained by multiplying the interior contribution by a factor μ which is less than unity. Effectively the form factor:

$$F(p) = \mathcal{L}(pR) \qquad (12\cdot7)$$

thus seems to be replaced by:

$$\mu\left[\mathcal{L}(pR) - \cos pR\right] + \cos pR \qquad (12\cdot8).$$

Since the first zero of $\mathcal{L}(x)$ occurs at $x = 4\cdot49$ whereas

$\cos x$ has a zero already at $x = 1.57$, the first minimum in the differential cross section for coulomb production can thus be expected to be shifted appreciably towards the forward direction. In the case of coulomb production it is probably even more so than for nuclear production that the results cannot be trusted beyond the first minimum since the marked oscillatory behavior contained in 12.6 will be modified if a smoother nuclear density is used. After this preliminary investigation, we can now proceed to include the final state interactions in a more systematic manner.

As discussed in section 8, the pion field operator at the y-vertex should be expanded in terms of the true wave functions $\psi_q^{(-)}(\vec{y})$ rather than the plane wave $e^{-i\vec{q}\cdot\vec{y}}$. To account for the final state interactions, we shall thus replace the exponential in 12.4 by $\psi_q^{(-)}(\vec{y})$. The transition amplitude thus becomes:

$$T_{fi} = \frac{Ze^{i\delta}}{m}\sqrt{\frac{\alpha}{m\tau}}\,\frac{\hat{\epsilon}\cdot(\vec{q}\times\vec{k})}{\sqrt{k\omega}}\,\frac{4\pi}{p^2}\,H(p) \qquad (12.9)$$

which agrees with 11.18 except for the replacement of $F(p)$ by:

$$H(p) = \frac{p^2}{4\pi}\int d^3y\; \psi_q^{(-)}(\vec{y})\, e^{i\vec{k}\cdot\vec{y}} \int \frac{\rho(\vec{x})}{|\vec{x}-\vec{y}|}\, d^3x \qquad (12.10).$$

It is convenient to decompose $H(p)$ into $H_{in}(p)$ and $H_{ex}(p)$ which are the contributions from $y<R$ and $y>R$, respectively.

The interior contribution, $H_{in}(p)$, can be obtained by using for $\psi_q^{(-)}(\vec{x})$ the same nuclear matter approximation 9.6 that was used for the nuclear production, and using the first of the two equations 12.5. Proceeding just as in section 9, one obtains:

$$H_{in}(p) = \frac{(pR)^2}{4} \int_0^1 dr \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} dz \; e^{(\gamma + i\alpha - i\delta)z} \, e^{(i\delta - \gamma)\sqrt{1-r^2}}$$

$$\times J_0(\beta r)\left[3r - r^3 - r z^2\right] \qquad (12.11).$$

The parameters α , β , γ , and δ have been defined in 9.9 and should not be confused with the α (fine structure constant) and δ (unknown phase) occurring in 12.9. The integral over z is quite trivial but the resulting expression is not instructive and rather tedious and will be omitted. The remaining integral over r was again done numerically on a digital computer.

If the nuclear matter (8.10) wave function is used for the pion, its value inside the nucleus has been shown to be given by 9.9. In a similar manner one finds that outside the nucleus:

$$\left.\begin{aligned} \psi_q^{(-)}(\hat{x}) &= e^{2(i\delta - \gamma)\sqrt{1-r^2}} \, e^{-iqRz}, \quad \begin{matrix} 0 < r < 1 \\ -\infty < z < -\sqrt{1-r^2} \end{matrix} \\ &= e^{-iqRz}, \quad \text{everywhere else where } r^2 + z^2 > 1 \end{aligned}\right\} (12.12).$$

Since the value of $H_{ex}(p)$ is just $\cos pR$ when there is no absorption or refraction, it is convenient to write:

$$H_{ex}(p) = \cos pR + H_{corr}(p) \qquad (12.13).$$

Proceeding just as with $H_{in}(p)$, one would then write:

$$H_{corr}(p) = \frac{(pR)^2}{4} \int_0^1 2r f(r) J_0(\beta r)\left[e^{2(i\delta - \gamma)\sqrt{1-r^2}} - 1\right] dr \qquad (12.14)$$

$$f(r) = \int_{-\infty}^{-\sqrt{1-r^2}} \frac{e^{i\alpha z}}{\sqrt{r^2 + z^2}} \, dz = \int_1^{\infty} \frac{e^{-i\alpha\sqrt{x^2 - r^2}}}{\sqrt{x^2 - r^2}} \, dx \qquad (12.15).$$

This latter integral, however, diverges logarithmically when α tends to zero, i.e. when there is no momentum transfer parallel to the direction of the produced pion.

At last we are forced to pay for the use of the approximate wave function 12.12. The separation between 12.12a and 12.12b is probably fairly good up to some distance away from the nucleus, but to maintain it to infinity is completely unrealistic since the diffracted waves which should be included in $\psi_q^{(-)}(\vec{x})$, will gradually obliterate the distinction between the values of the wave function inside and outside the cylinder $\tau = 1$. The only completely satisfactory remedy would be to use the exact $\psi_q^{(-)}(\vec{x})$ and to integrate the difference $\left[\psi_q^{(-)}(\vec{x}) - e^{-iqRz}\right]$ over all space outside the nucleus. It is then also necessary, however, to use the correct density $\rho(\vec{x})$ since $\psi_q^{(-)}(\vec{x})$ will otherwise contain too much reflected wave. Instead of this, we shall replace the integration over the complete infinite cylinder "behind" the nucleus in 12.15 by an integration over the shadow region only. This is not a precisely defined region but since the region "behind the nucleus" does not even under the most favorable conditions comprise as much as 15% of the total production volume, we shall choose a convenient boundary extending to $z = qR$ along the axis, this being the maximum extension of the shadow region obtained from considering the uncertainty principle or the Fraunhofer diffraction behind a sphere. The second integral in 12.15 is thus replaced by:

$$f(\tau) = \int_1^{g(\tau)} \frac{e^{-i\alpha\sqrt{x^2-\tau^2}}}{\sqrt{x^2-\tau^2}}\, dx \qquad (12.16)$$

where $g(\tau)$ must vary smoothly from $g(1) = 1$ to $g(0) = qR$. For convenience the expression:

$$g(r) = qR(1-r^2) + r^2 \qquad (12.17)$$

was used in the numerical computations.

A second approximation that was made for the purpose of the numerical integration, is to replace 12.16 by:

$$f(r) = \int_1^{g(r)} \frac{e^{-ix}}{x}\, dx \qquad (12.18)$$

which is not too unreasonable since we are interested in the region $r<1$ whereas the region of integration contains only $x>1$. For qR less than unity, which never occurred in the cases considered, $f(r)$ could be put equal to zero. Since 12.18 can be expressed in terms of sine and cosine integrals, it is now possible to integrate 12.14 numerically. This was also done on the computer.

Just as in the case of nuclear production, one can expect corrections due to the deflection of the produced pion to have some effect. These corrections vanish at 900 Mev though, since the refractive index of nuclear matter for pions is then practically one. Since coulomb production will be important mainly at these higher energies, the corrections due to deflections were omitted.

In appendix D the results of the numerical computation of the following quantities are presented:

$$\left.\begin{aligned} \bar{Y}_0 &= \frac{Z^2 q^3 k \sin^2\theta}{p^4}\, |F(p)|^2 \\ \bar{Y}_1 &= \frac{Z^2 q^3 k \sin^2\theta}{p^4}\, |H_{in}(p) + \cos pR|^2 \\ \bar{Y}_2 &= \frac{Z^2 q^3 k \sin^2\theta}{p^4}\, |H_{in}(p) + H_{ex}(p)|^2 \end{aligned}\right\} \qquad (12.19)$$

The first is proportional to the differential cross section (11.20) without final state interactions. The second expression includes the absorption of pions produced inside the nucleus only. As conjectured previously (12.8), this shifts the diffraction minima appreciably towards smaller angles. The third expression contains in addition the absorption of pions produced in the shadow region.

13. Further corrections to coulomb production

The transition operator for the nuclear production of neutral pions was expressed (2.3) in terms of single nucleon transition operators:

$$T^{(N)} = \sum_{n=1}^{A} e^{i\vec{p}\cdot\vec{x}_n} t_n^{(N)} \qquad (13.1)$$

so that the matrix element in the ground state is given by:

$$\langle 0| T^{(N)} |c\rangle = \sum_{n=1}^{A} \langle c| e^{i\vec{p}\cdot\vec{x}_n} t_n^{(N)} |c\rangle$$

$$= A\, F(p) \langle c| t^{(N)} |c\rangle \qquad (13.2).$$

The single nucleon operator is expressed as:

$$t^{(N)} = K^{(N)} + M^{(N)} \tau_3 + (\vec{L}^{(N)} + \vec{N}^{(N)} \tau_3)\cdot\vec{\sigma} \qquad (13.3).$$

A slightly different approach has so far been used for the coulomb production. By inspection of equations 11.18, 12.1, and 12.4, one can see, however, that a similar formalism could be used for the coulomb production, namely by writing:

$$T^{(c)} = \sum_{n=1}^{A} e^{i\vec{p}\cdot\vec{x}_n} \sqrt{\frac{4x'}{m^3 c}} \left(\frac{4\pi}{p^2}\right) \hat{\epsilon}\cdot(\vec{q}\times\vec{k}) \, e^{(3)} \qquad (13.4).$$

By introducing the projection operator $\frac{1}{2}(1 - \tau_3^n)$, the sum can be extended over all the nucleons:

$$T^{(c)} = \sum_{n=1}^{A} e^{i\vec{p}\cdot\vec{x}_n} t_n^{(c)} \qquad (13.5)$$

$$t^{(c)} = K^{(c)} + M^{(c)} \tau_3 \qquad (13.6)$$

$$K^{(c)} = -M^{(c)} = \frac{1}{2} N m^3 \tau \left(\frac{4\pi}{p^2}\right) \frac{\hat{\epsilon}\cdot(\vec{q}\times\vec{k})}{\sqrt{k\omega}} e^{i\delta} \qquad (13\cdot 1).$$

Exactly the same analysis as in chapters I and II can now be carried out for the coulomb production. The previous results can thus be taken over directly. We expect namely that the interaction with the electrostatic field of the nucleus will also lead to a non-correlated cross section:

$$\left(\frac{d\sigma}{d\Omega}\right)^{(c)}_{NC} = A^2 \, |F(p)|^2 \, |K^{(c)}|^2 \, \Gamma \qquad (13\cdot 8)$$

and an incoherent cross section:

$$\left(\frac{d\sigma}{d\Omega}\right)^{(c)}_{D} + \left(\frac{d\sigma}{d\Omega}\right)^{(c)}_{C} = A\left[1+G(p)\right]\left\{|K^{(c)}|^2 + |M^{(c)}|^2\right\}\Gamma \qquad (13\cdot 9).$$

By taking the factor ½ in 13.7 into account, it is easy to see that 13.8 is just the cross section (11.20) considered in section 11 with Z^2 replaced by $\frac{1}{4}A^2$, which follows from our assumption that $A = 2Z$. In addition to the nuclear cross sections of chapter I and the coulomb cross sections of equations 13.8 and 13.9, there will in general be interference terms since the correct procedure would be to consider the total transition operator:

$$t = t^{(N)} + t^{(c)} \qquad (13\cdot 10).$$

The non-correlated production would thus for example be given by:

$$\left(\frac{d\sigma}{d\Omega}\right)_{NC} = A^2 \, |F(p)|^2 \, |K^{(N)} + K^{(c)}|^2 \qquad (13\cdot 11).$$

There will thus also be interference terms:

$$\left(\frac{d\sigma}{d\Omega}\right)^{(Int)}_{NC} = A^2 |F(p)|^2 \, 2 \, |K^{(N)}| \cdot |K^{(C)}| \cos(\delta^N - \delta^C) \qquad (13.12).$$

Since the coupling is weak, the coulomb amplitude is probably real (i.e. δ^C equals 0° or 180°). Since pions are mostly produced outside the nucleus when the momentum transfer is small, the interaction with the nucleus is also unable to produce a different phase. The pions which do in fact come close to the nucleus, are mostly absorbed. At low energies δ^N would be given by the scattering phase shift in the 33-state (C.33). At high energies the only theorems of this nature which are available, are much weaker and at present the nuclear phase is unknown.

When the absorption of the pion is included, an additional phase difference arises because the form factors for coulomb and nuclear production are then no longer identical. These phases due to the absorption were computed along with the values of the form factors and their values are included in appendix D.

In section 11 we assumed that the excitation energy Δ is negligible. In the new formalism 13.8 is the elastic production where $\Delta = 0$ while the contributions from nuclear excitations are included in 13.9. Another approximation which was made in section 11, was to neglect the contribution $\vec{\Gamma}$ from the spatial current. This contribution will now be discussed.

The value of $\vec{\Gamma}$ for a Dirac proton is given by 11.14. In the non-relativistic approximation the interaction $e\phi$ with the electrostatic field ϕ is replaced by the interaction of the space and spin currents with the vector potential $\vec{A}$:

$$\frac{e}{c}\frac{d\vec{x}}{dt}\cdot\vec{A} + \frac{e\hbar}{2Mc}\vec{\sigma}\cdot\nabla\times\vec{A} \qquad (13.13).$$

The magnetic moment of a real free proton is modified by a factor $\mu_p = 2.79$ due to the strong interactions whereas a real free neutron acquires a moment given by $\mu_n = -1.91$. Although the virtual pion clouds may be damped inside the nucleus, we shall use the free nucleon moments in our discussion.

The transition amplitude can then be expressed in terms of the multiple amplitudes given, for example, on page 599 of Blatt and Weisskopf (20):

$$T_{fi} = \frac{4\pi}{p^2} e^{i\delta} \frac{\hat{e}\times(\vec{p}k - \vec{k}\Delta)}{\sqrt{k\,\omega m^3 c\pi}} \cdot \sum_{\sigma,i} \vec{B}^{\sigma}_{i}(f,i) \qquad (13.4).$$

The constants follow from 11.12 and 13.4 and the definition of the multipole amplitudes, which differs slightly from Blatt and Weisskopf. As an illustration, the spin part of the M1 amplitude and the z-component of the space part of the E1 amplitude will be exhibited:

$$\vec{B}^{m}_{1}(f,i) = \left(\frac{p}{\hbar}\right)\frac{e\hbar}{2Mc}\sum_{s=1}^{A}\mu_s\langle f|\vec{\sigma}_s|i\rangle \qquad (13.15)$$

$$B^{e}_{1,z}(f,i) = \left(\frac{\Delta}{\hbar}\right)\frac{e}{c}\sum_{s=1}^{A} q_s\langle f|z_s|i\rangle \qquad (13.16)$$

where μ_s and q_s are the magnetic moment and electric charge of nucleon s in units of the nuclear magneton and the protonic charge, respectively. The factors $(p/\hbar)$ and $(\Delta/\hbar)$ come from the derivatives in 13.13.

The initial state i will again be taken to be the ground state of the closed shell nucleus which we considered. The elastic transition

(i → i) will therefore be forbidden except for internal pair conversion. Since this process is not related to the lifetime of the pion for decay into two photons but to the much longer lifetime for decay into a photon and an electron-position pair, we shall neglect it. This means that $\vec{\Gamma}$ can only contribute by exciting the nucleus. Its contribution can therefore only be of order A and not A^2 like the elastic nuclear and coulomb production.

The magnetic dipole contribution (13.15) is only the first term in an expansion of the exponential $e^{i\vec{p}\cdot\vec{x}}$. Actually the terms arising from the nucleon spins can again be treated using our previous formalism. In this way the whole exponential can be taken into account. The magnetic moment of nucleon S can be written:

$$\mu_s = -\frac{1}{2}(\mu_p - \mu_n)\,\tau_{3s} + \frac{1}{2}(\mu_p + \mu_n) \qquad (13.17).$$

The square of the first term is thirty times as large as that of the second so that the latter may be discarded. Once again equations analogous to 13.5 and 13.7 can be written down:

$$T^{(s)} = \sum_{n=1}^{A} e^{i\vec{p}\cdot\vec{x}_n}\,\vec{N}^{(s)}\cdot\vec{\sigma}_n\,\tau_{3n} \qquad (13.18)$$

$$\vec{N}^{(s)} = \left(\frac{\mu_n-\mu_p}{2}\right)\left(\frac{p}{2M}\right)\sqrt{\frac{4\pi}{m^3\omega}}\left(\frac{4\pi}{\mu^2}\right)\frac{\hat{\epsilon}\times(\vec{p}\vec{k}-\vec{k}\Delta)}{\sqrt{k\omega}}\,e^{i\delta} \qquad (13.19),$$

leading to a cross section:

$$\left(\frac{d\sigma}{d\Omega}\right)^{(s)}_{D} + \left(\frac{d\sigma}{d\Omega}\right)^{(s)}_{C} = A\left\{1 + G(p)\right\}\,|\vec{N}^{(s)}|^2\,\Gamma \qquad (13.20).$$

In addition to the contributions from the spin operators, the multipole moments in 13.14 contain contributions from the orbital motion of the nucleons. These cannot be calculated by means of the formalism which has been used until now and the various terms in 13.14 have to be

considered separately. At 900 Mev pR may already at 3° become equal to unity and even with no excitation in carbon, pR has a value of one sixth in the forward direction. Nevertheless, we shall discuss only the electric dipole contribution (13.16) because of its dominant and simple behavior (21). At larger angles one should also include higher multipoles such as $E2$.

Inside the nucleus the protons and neutrons have effective charges:

$$q_p = \left(1 - \frac{Z}{A}\right) \quad , \quad q_n = -\frac{Z}{A} \qquad (13.21)$$

due to the recoil effects. If this is taken into account, the Thomas-Reiche-Kuhn sum rule becomes (20):

$$\sum_f (E_f - E_i) \left| \langle f | \sum_{n=1}^{A} e q_n z_n | i \rangle \right|^2 = \frac{\hbar^2 e^2}{2M} \frac{Z}{A}(A-Z) \qquad (13.22).$$

The usual derivation is no longer valid when exchange forces are operative and Levinger and Bethe (21) have shown that one can correct for this by multiplying 13.22 by a factor (1 + 0.8 x) where x is the fraction of the nuclear force which consists of exchange forces.

It has been found experimentally (21) that the cross section for absorption of electric dipole radiation by nuclei contains a so-called "giant resonance" with a width of about 5 Mev centered around 20 Mev. Furthermore, the area under the cross section-energy curve practically exhausts the sum rule. To a fairly good approximation we may thus set $\Delta = E_f - E_i = 20$ Mev for E1-transitions so that we may write:

$$\sum_f \left| B^e_{1z}(f,i) \right|^2 = \frac{\hbar^2 e^2 \Delta}{2M} \left(\frac{A}{4}\right)(1.4) \qquad (13.23).$$

Here we have set (A-Z)Z/A = A/4 and assumed a Serber force mixture x = 0.5. The cross section becomes:

$$\left(\frac{d\sigma}{d\Omega}\right)^{E1} = \frac{1{\cdot}4\,\Delta A}{8M}\left(\frac{4x}{m^3\tau}\right)\frac{16\pi^2}{k_0 p^4}\Gamma\frac{1}{2}\sum_{\hat{e}}\left|\hat{e}\times(\vec{p}k-\vec{k}\Delta)\right|^2 \qquad (13{\cdot}24).$$

The value of the momentum transfer at small angles is given by:

$$p^2 \approx (\Delta+\varepsilon)^2 + k^2\theta^2 \qquad (13{\cdot}25)$$

where Δ is the excitation energy and $\varepsilon = m^2/2k$ has a value of 11 Mev when $k = 900$ Mev . Equations 13.8 and 13.9 contain the quantity:

$$\frac{1}{2}\sum_{\hat{e}}\left|\hat{e}\cdot(\vec{q}\times\vec{k})\right|^2 = \frac{1}{2}q^2k^2\sin^2\theta \qquad (13{\cdot}26)$$

while 13.20 and 13.24 contain the quantity:

$$\frac{1}{2}\sum_{\hat{e}}\left|\hat{e}\times(\vec{p}k-\vec{k}\Delta)\right|^2 \approx k^2\left(\varepsilon^2+\tfrac{1}{2}q^2\sin^2\theta\right) \qquad (13{\cdot}27).$$

We shall now compare the magnitudes of the various corrections discussed in this section. As a convenient unit we shall choose the quantity:

$$\sigma_0 = \frac{8x}{m^3\tau} = \left[\frac{3{\cdot}04\times10^{-16}}{\tau \text{ in sec.}}\right]\times 2\times10^{-5}\ \mu b \qquad (13{\cdot}28).$$

The recent measurement (14) of the pion lifetime has yielded a value $\tau = (3{\cdot}2 \pm 1{\cdot}0)\times10^{-16}$ sec. so that the quantity in brackets is about unity.

The elastic cross section (13.8) contains a slowly varying form factor $|F(p)|^2$ and a rapidly varying factor:

$$S_0(\theta) \approx \frac{k^4\theta^2}{[\varepsilon^2+k^2\theta^2]^2} \qquad (13{\cdot}29)$$

which has a maximum value of $(k/2\varepsilon)^2$ when $\theta = \varepsilon/k$. At 900

Mev this means that $S_o(\theta)$ reaches a peak value of about 1680 at $\theta = 0.012$ which is about 0.7 degrees. At this angle $|F(p)|^2$ has only decreased to about 0.99. Inserting the density of states:

$$\frac{1}{2\pi}\Gamma = \frac{q\omega}{(2\pi)^3} \qquad (13.30),$$

one thus finds for the elastic cross section a value:

$$\left(\frac{d\sigma}{d\Omega}\right)_{elastic} = 1.610\ Z^2\ \sigma_o \qquad (13.31)$$

at the peak.

The incoherent cross section (13.9) also contains a factor similar to 13.29 except that ε is replaced by $\varepsilon+\Delta$. Since the mean excitation energy Δ is unknown, we shall estimate an upper bound to the cross section by setting $\Delta=0$. In addition, the form factor $|F(p)|^2$ is replaced by the suppression factor $1+G(p)$. This factor will vary from nucleus to nucleus but we shall use the value calculated (6.34) for O^{16}:

$$1+G(p) \approx \frac{2}{15} R^2 \left[\varepsilon^2 + k^2 t^2\right] \qquad (13.32)$$

where Δ has again been set equal to zero. Using the calcium radius of 4.44 fermis, one finds that the incoherent cross section has a peak value of:

$$\left(\frac{d\sigma}{d\Omega}\right)_{incoherent} = 24.5\ Z\ \sigma_o \qquad (13.33).$$

The "spin" cross section (13.20) has the second instead of the fourth power of p in the denominator. This changes the sharply peaked profile of 13.29 to the very flat function:

$$S_1(\theta) \approx \frac{\varepsilon^2 + \frac{1}{2}k^2\theta^2}{(\varepsilon+\Delta)^2 + k^2\theta^2} \qquad (13.24)$$

which has the value $\left(\varepsilon/\varepsilon+\Delta\right)^2$ in the forward direction and then decreases or increases monotonically to a value of 0.5. The factor $(\mu_p-\mu_n)^2 kq \,/4M^2$ has a value of about 5 while $1+G_E(p)$ is given by 13.32. At the position of the peak in the elastic cross section, the "spin" cross section has a magnitude of:

$$\left(\frac{d\sigma}{d\Omega}\right)_{spin} = 0.04 \; Z \sigma_0 \qquad (13.35).$$

The electric dipole cross section (13.24) again contains the factor 13.34, with Δ now known to be about 20 Mev. In addition to the small factor $0.7\,\Delta/M \approx 0.015$, this cross section contains a factor p^2 in the denominator, which causes a rapid increase towards small angles. Even in the forward direction, however, the El cross section has a value of only:

$$\left(\frac{d\sigma}{d\Omega}\right)_{El} = 7 \; Z \sigma_0 \qquad (13.36).$$

By comparison of 13.31, 13.33, 13.35, and 13.36, it becomes obvious that at small angles all the correction terms discussed in this section may be neglected in comparison with the elastic coulomb production discussed in sections 11 and 12. By the same token, the only interference term which has to be considered, is the term 13.12, which may be written as:

$$\left(\frac{d\sigma}{d\Omega}\right)^{(Int)}_{NC} = 2\sqrt{\left(\frac{d\sigma}{d\Omega}\right)^{(N)}_{NC} \cdot \left(\frac{d\sigma}{d\Omega}\right)^{(C)}_{NC}} \cos\left(\delta^N - \delta^C\right) \qquad (13.37).$$

The complete cross section for the photoproduction of neutral pions

from complex nuclei is then:

$$\left(\frac{d\sigma}{d\Omega}\right)^{(C)}_{NC} + \left(\frac{d\sigma}{d\Omega}\right)^{(Int)}_{NC} + \left(\frac{d\sigma}{d\Omega}\right)^{(N)}_{NC} + \left(\frac{d\sigma}{d\Omega}\right)^{(N)}_{Inc} \qquad (13\cdot38),$$

where the fourth term indicates the sum of the nuclear diagonal and correlated cross sections.

14. CONCLUSIONS

The only experimental results with which the theoretical predictions can at present be compared, are those of measurements at 250 Mev done at M.I.T. (15) and the preliminary results of the 900 Mev experiment at Caltech (16).

At 250 Mev the photoproduction of neutral pions from single nucleons is well understood. According to the static theory, the single nucleon transition operator t must be proportional to (B.27):

$$(2+\sigma_z)\, e^{if} \sin\theta \;-\; 2\sigma_+ \cos\theta \qquad (14\cdot1)$$

where (θ, f) are the spherical polar coordinates of the pion in the center of mass system. Equation 14.1 holds for production by right-handedly polarized photons.

The free nucleon differential cross section in the center of mass system is then:

$$\left(\frac{d\sigma}{d\Omega}\right)_f = \xi\,(5\sin^2\theta + 2\cos^2\theta)$$

$$= \xi\,(5 - 3\cos^2\theta) \qquad (14\cdot2)$$

where ξ is an energy-dependent constant which can be determined empirically. We shall adopt the value:

$$\xi = 2\cdot35 \;\mu b \qquad (14\cdot3)$$

which leads to the differential cross section:

$$\left(\frac{d\sigma}{d\Omega}\right)_f = (11.75 - 7.05\ \cos^2\theta)\ \mu b/ster \qquad (14.4).$$

The so-called non-spin flip cross section is proportional to $|K|^2$ where K is the part of 14.1 which does not contain any spin or isospin operators. Hence:

$$\left(\frac{d\sigma}{d\Omega}\right)_{NSF} = 4\xi\ \sin^2\theta \qquad (14.5).$$

It is not simply the part of 14.2 which does not flip the spin, namely $5\xi\ \sin^2\theta$.

The square of the transition amplitude is approximately given by an invariant divided by $k\omega$, where k and ω are the photon and pion energies in a given system (see B.26). In addition to this matrix element squared, the cross section contains the density of states which is approximately proportional to $q\omega$, where q is the pion momentum. Hence the following quantity:

$$\frac{k}{q}\ \frac{d\sigma}{d\Omega} \qquad (14.6)$$

can to a good approximation be treated as an invariant. For photoproduction from a single nucleon, the quantity (q/k) has the value 0.96 in the center of mass system. In the case of photoproduction from a heavy nucleon which recoils as a whole, (q/k) has the value 0.84 in the laboratory system. Hence at 250 Mev the single nucleon cross sections should be multiplied by a factor $\frac{0.84}{0.96} = 0.875$ before their values are substituted into the expressions for photoproduction from complex nuclei. At 900 Mev this factor is practically unity so that this point may be disregarded.

In order to avoid possible uncertainties due to non-closed shell effects, the comparison with Davidson's M.I.T. data will be made for calcium. The non-correlated or elastic cross section is given by (D.1):

$$\left(\frac{d\sigma}{d\Omega}\right)_{NC} = 8.23\ \bar{X} \quad \mu b/ster \qquad (14.7)$$

where the 8.23 is the product of the 4ξ and the factor 0.875 while the computed values of $\bar{X}$ appear in appendix D. We shall use both $\bar{X}_0$, the value without final state interactions, and $\bar{X}_2$, the value which includes absorption and deflections of the pion.

The incoherent cross section can be expressed in terms of the free nucleon cross section (14.2) as:

$$\left(\frac{d\sigma}{d\Omega}\right)^0_{Inc} = A\left[1 + G_0(p)\right]\left(\frac{d\sigma}{d\Omega}\right)_f \qquad (14.8)$$

in the case of no absorption and:

$$\left(\frac{d\sigma}{d\Omega}\right)^2_{Inc} = A\left[\xi + G_2(p)\right]\left(\frac{d\sigma}{d\Omega}\right)_f \qquad (14.9)$$

when absorption is included. We used for $G_0(p)$ the Fermi gas expression (6.26) with a Fermi momentum $p_F = 250$ Mev. At small angles this will be larger than the finite nucleus values as one can see from the tables in appendix D. In addition, since $G_2(p)$ was calculated for O^{16} only, the expression in the square brackets in 14.9 was replaced by:

$$\xi\left[1 + G_0(p)\right] \qquad (14.10)$$

which can be seen from appendix D to give a fairly good estimate. The value used for ξ was 0.28 (see 10.25).

If the experimental value (14) of the pion lifetime is at all

correct, the coulomb production will be negligibly small at 250 Mev. The elastic cross section is:

$$\left(\frac{d\sigma}{d\Omega}\right)_{coul} = Y\,\sigma_0 \qquad (14\cdot11)$$

where Y appears in the tables of appendix D while σ_0 was defined in 13.28 and appears to have the value:

$$\sigma_0 = 2 \times 10^{-5} \quad \mu b \qquad (14\cdot12).$$

From the table of calcium at 250 Mev it follows that at most 0.04 μb/sterad can be expected to arise from the coulomb production. The interference with 14.7 will also obviously be negligible.

In the following table we shall tabulate the values of 14.7, 14.8, and:

$$\left(\frac{d\sigma}{d\Omega}\right)_{tot} = \left(\frac{d\sigma}{d\Omega}\right)_{Nc} + \left(\frac{d\sigma}{d\Omega}\right)_{Inc} \qquad (14\cdot13),$$

both in the case of no absorption and in the case of absorption according to the parameters D.3. In addition, interpolated values from the M.I.T. data will be presented. The units of all cross sections will be microbarns per steradian while the angle θ is measured in radians.

Calcium at 250 Mev

θ	0.1	0.2	0.3	0.4	0.5	0.6	0.8
$\left(\frac{d\sigma}{d\Omega}\right)^0_{Inc}$	23	32	44	60	78	100	152
$\left(\frac{d\sigma}{d\Omega}\right)^0_{NC}$	105	353	588	673	573	362	28
$\left(\frac{d\sigma}{d\Omega}\right)^0_{tot}$	128	385	632	733	651	462	180
$\left(\frac{d\sigma}{d\Omega}\right)^2_{Inc}$	6	9	12	17	22	28	43
$\left(\frac{d\sigma}{d\Omega}\right)^2_{NC}$	12	68	144	175	150	79	4
$\left(\frac{d\sigma}{d\Omega}\right)^2_{tot}$	18	77	156	192	172	107	47
$\left(\frac{d\sigma}{d\Omega}\right)_{Exp.}$	350±70	510±80	800±70	850±60	740±80	560±70	380±40

Inspection of the table reveals the fact that the theoretical predictions are far too low to account for the experimental results. As a matter of fact, if one allows for the fact that certain production modes are not taken into account by the theory, the predictions when absorption is not taken into account, namely $\left(\frac{d\sigma}{d\Omega}\right)^0_{tot}$, agree amazingly well with the experiment. Both $\left(\frac{d\sigma}{d\Omega}\right)^0_{tot}$ and $\left(\frac{d\sigma}{d\Omega}\right)^2_{tot}$ agree quite well with $\left(\frac{d\sigma}{d\Omega}\right)_{Exp}$ as far as shape is concerned, $\left(\frac{d\sigma}{d\Omega}\right)^2_{tot}$ just being too small by a factor 4.

At 900 Mev the results are not yet available in a form which permits a similar analysis. Partly this is due to the fact that the cross section is not measured directly as a function of the angle θ but of another variable. Some results are already available, however, and will now be discussed.

Unfortunately the single nucleon cross sections at high energies are only very incompletely understood at present (5). Even the available experimental information is inconsistent. The separation into spin flip and non-spin flip or the resolution into multipole amplitudes is quite sensitive to measurements at small angles. The closest values to the forward direction which are available, are due to Vette and Berkelman and Waggoner (4). They disagree, however:

$$\left.\begin{array}{lll} \text{Vette:} & 912 \text{ Mev} \quad (31.5°) & 2.42 \pm 0.42 \ \mu\text{b/ster} \\ \text{B \& W:} & 940 \pm 100 \ (27.6 \pm 2°) & 1.0 \pm 0.3 \ \mu\text{b/ster} \end{array}\right\} \quad (14\cdot14).$$

It seems (16) that Berkelman and Waggoner's result is probably closer to the truth.

Since the details of the angular distribution are still quite uncertain, it would be safest to use the values 14.14 only to place upper or lower limits on the cross sections. One could for example hope to place an upper limit on the non-spin flip cross section by assuming that the cross section at 30° is 1.0 μb/ster and that it is all due to a D-wave non-spin flip amplitude:

$$\left(\frac{d\sigma}{d\Omega}\right)_+ = \eta \cos^2\theta \sin^2\theta \qquad (14\cdot15).$$

This would imply that η has the value 5.3 μb/ster. Using the tables of appendix D, one would thus expect the nuclear production from lead to have the following maximum values:

$$\left.\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)^{Pb}_{NC} &< 360 \quad \mu b/ster \quad \text{without absorption} \\ \left(\frac{d\sigma}{d\Omega}\right)^{Pb}_{NC} &< 76 \quad \mu b/ster \quad \text{with absorption}\end{aligned}\right\} \quad (14\cdot16).$$

The photoproduction from lead has actually been analyzed (16). The cross section was represented as the sum of:

$$\left.\begin{aligned}\sigma_N(\theta) &= A^2\,\sigma_H\,\sin^2\theta\; e^{-\frac{1}{10}(pR)^2} \\ \sigma_C(\theta) &= Z^2\,\sigma_0\left(k/p\right)^4 \sin^2\theta\; e^{-\frac{1}{10}(pR)^2} \\ \sigma_I(\theta) &= 2\,A'\,\overline{\sigma_N(\theta)}\cdot\sigma_C(\theta)\cos\delta\end{aligned}\right\} \quad (14\cdot17)$$

and the values of σ_H , σ_0 , and δ were sought which gave a best fit to the results. The values found, were:

$$\left.\begin{aligned}\sigma_H &= (8.50 \pm 1.13)\ \mu b/ster \\ \sigma_0 \rightarrow\ \tau_\pi &= \left(1.1\ {}^{+14}_{-0.6}\right)\times 10^{-16}\ sec \\ \delta &= 107^\circ\end{aligned}\right\} \quad (14\cdot18).$$

The results were, however, also consistent with keeping σ_N only. The value of σ_H is slightly larger than the upper limit $\eta = 5.3$ derived from Berkelman and Waggoner's data. One might hope to be able to stretch the values a little, or to use Vette's values, in order to obtain agreement. The expressions 14.17, however, correspond to the theory without absorption. When absorption of the pion is included, an additional factor of 4.7 has to be explained.

In both the low and the high energy regions it thus appears as if the Born approximation (no absorption) yields results which come surprisingly close to the experimental values. The calculations which take

the pion interactions into account, on the other hand seem to produce values which are far too low. Since this tends to cast doubt on the validity of the theory, it will be worth while to review all the assumptions and approximations which were made.

The basic assumption was that of the validity of the direct interaction model (6) and the impulse approximation (7). This general mode of attacking high energy problems seems to be fairly well established. It is, in fact, so far the only type of approach which has met with a reasonable amount of success in explaining the properties of many-body systems in terms of more fundamental constants. It pions can be produced by some process in which it is essential that many nucleons must take part, such pions will, however, not be included in our formalism.

The next approximation that was made, was the closure approximation (9) (10), in order to sum over all the final states. This is, admittedly, an approximation, but it so happens that the dominant elastic part of the cross section is treated exactly which makes it unlikely that an improvement of this approximation would remove the disagreement with the experiments.

In the evaluation of the nuclear matrix elements, a two-fold approximation was made. The first was the adoption of an independent particle model in which the nucleons occupy single particle orbitals. The second was the restriction to closed shell nuclei. These two assumptions made an exact calculation of the nuclear matrix elements possible.

Although the expressions for the spin-isospin matrix elements were rigorously derived for closed shell nuclei only, they can probably

to a reasonably good approximation be used for non-closed shell nuclei also since such nuclei can be considered as a closed shell core plus additional nucleons and the number of nucleon pairs of which both members belong to the core will usually outweigh the remaining pairs. This could in any case not be the source of trouble since Ca^{40} is a closed shell nucleus.

The other approximation, that of independent particle motion, is more serious. It neglects strong correlations due to the residual forces between nucleons. Especially processes where the nucleus is disrupted, may be appreciably enhanced due to the presence of high momentum components arising from the hard core repulsion between nucleons. Once again, however, the elastic process where the nucleus recoils as a whole, is the least likely to be affected much.

The final state interactions of the produced pion were treated by means of the optical model (12). In this model a pion which is scattered inelastically, is treated as if it has been absorbed, whereas it may in fact still be detected and included in the measured cross section. It is thus almost certain that such a treatment would underestimate the observed cross section. Another process which is also neglected in the treatment, is the initial production of a charged pion which is then later converted into a neutral pion by charge exchange scattering with the nucleons in the nucleus. On the whole the experimental cross sections can thus be expected to be higher than the theoretical estimates. It is extremely improbable that this could explain the factors of 4 by which theory and experiment disagreed, however.

Even within the optical model a further approximation was used

for the pion wave function (section 8). In the damped plane wave that was used, scattered waves were disregarded. This could possibly lead to appreciable errors in the calculations relating to the coulomb production and perhaps also in the calculations involving deflections at the nuclear surface. It is hard to see how an improved calculation of the latter could completely remove the discrepancy at 250 Mev, however, since the final result would have to be that the effect of absorption is nearly negligible. The place where it is least likely that this method may be misleading, is in the calculation of the effect of absorption on the nuclear production at 900 Mev. This is the place where the largest amount of confidence in the treatment is felt. A calculation is in progress (17) to replace the approximate wave function by an expansion into partial waves which satisfy the Schrödinger equation (8.6). It would be interesting to see how the results compare with the results obtained using the approximate wave function 8.10.

The interaction parameters used were those of calculated by Frank, Gammel, and Watson (12). The set (8.14) which was used, agrees with empirical determinations at low energies (19). The other set (8.15) would make the discrepancy in the present work even greater. The values of the interaction parameters at 900 Mev have not yet been checked experimentally and they may be incorrect, particularly since it seems as if certain low energy results, such as the Brueckner-Serber-Watson model, may just have been extrapolated. The results are quite sensitive to the mean free path so that a change in these values may change the predictions considerably.

The possibility must finally not be overlooked that there is in fact no discrepancy after all. In this connection one might mention

that the latest M.I.T. results (15) differ by a factor 6 from their previously reported measurements (see Osborne: CERN, 1956). It appears (16), however, that their earlier results were probably due to faulty calibration and that it is very unlikely that their latest results contain any serious error.

The apparent discrepancy at 900 Mev could possibly be explained by a fortuitous cancellation. The quantity that enters measurements such as those of Vette and of Berkelman and Waggoner, is:

$$|K - M|^2 \qquad (14.19)$$

(see B.31) whereas the quantity which enters the elastic nuclear cross section, is $|K|^2$ alone. It is thus still possible, even if it may be improbable, that $|K|^2$ may be large although the quantity 14.19 happens to be small. This uncertainty could be removed by also measuring photoproduction from neutrons and adding it to 14.19 so that the cross terms cancel.

If the discrepancy between theory and experiment persists, it may be necessary to investigate explanations in terms of processes which have not yet been considered, such as the virtual three pion bound state suggested by Chew, which could introduce a retardation term into the photoproduction of neutral pions from single nucleons (16).

APPENDIX A

Some results from the formal theory of scattering (1)

The time-independent Schrödinger equation:

$$H \psi_i = E_i \psi_i \qquad (A\cdot 1)$$

is usually solved by introducing as basis the eigenvectors of some appropriately chosen "free" hamiltonian H_o:

$$H_o = H - V \qquad (A\cdot 2)$$

$$H_o \phi_i = E_i \phi_i \qquad (A\cdot 3).$$

In scattering problems H_o is usually chosen in such a manner that it has the same continuous spectrum as H. Equation A.1 can then be replaced by the Lippman-Schwinger integral equation which incorporates the boundary condition on ψ:

$$\psi_i^{(\pm)} = \phi_i + \frac{1}{E_i - H_o \pm i\eta} V \psi_i^{(\pm)} \qquad (A\cdot 4).$$

Here η is an infinitesimal positive number which prescribes the treatment of the pole. The plus and minus signs correspond to solutions which contain only outgoing and only incoming spherical waves, respectively, in the asymptotic region.

The wave matrix operator of Møller transforms each ϕ_i into the corresponding $\psi_i^{(+)}$:

$$\psi_i^{(+)} = \Omega \phi_i \qquad (A\cdot 5).$$

Equation A.4 is satisfied if:

$$\Omega = 1 + \frac{1}{E_i - H_0 + i\eta} V\Omega \qquad (A\cdot 6).$$

This equation can be demonstrated to have the following solution, due to Chew and Goldberger (7):

$$\Omega = 1 + \frac{1}{E_i - H_0 - V + i\eta} V \qquad (A\cdot 7).$$

In this formalism any reaction can be considered to be a transition from an "uncoupled" state Φ_i to another state Φ_f as a result of the "interaction hamiltonian" V. Heisenberg's S-matrix is then defined by:

$$S_{fi} = \langle \Phi_f | S | \Phi_i \rangle \equiv \langle \psi_f^{(-)} | \psi_i^{(+)} \rangle \qquad (A\cdot 8).$$

By using A.4 and A.7, one can easily demonstrate that:

$$S_{fi} = \delta_{fi} - 2\pi i\, \delta(E_i - E_f)\, T_{fi} \qquad (A\cdot 9)$$

where δ_{fi} and $\delta(E_i - E_f)$ are the Kronecker delta and Dirac's delta function while the transition matrix T_{fi} is defined by:

$$T_{fi} = \langle \Phi_f | T | \Phi_i \rangle \equiv \langle \Phi_f | V | \psi_i^{(+)} \rangle \qquad (A\cdot 10).$$

The transition operator T can therefore be written:

$$T = V\Omega \qquad (A\cdot 11).$$

It satisfies the integral equation:

$$T = V + V \frac{1}{E - H_0 + i\eta} T \qquad (A\cdot 12)$$

which has the solution:

$$T = V + V \frac{1}{E - H_0 - V + i\eta} V \qquad (A\cdot13).$$

It is often convenient to split the interaction into two parts. Equation A.2 is then replaced by:

$$H = H_0 + U + V \qquad (A\cdot14).$$

By analogy with A.4, we define:

$$\psi_i^{(\pm)} = \phi_i + \frac{1}{E_i - H_0 \pm i\eta} (U+V) \psi_i^{(\pm)} \qquad (A\cdot15)$$

$$\chi_i^{(\pm)} = \phi_i + \frac{1}{E_i - H_0 \pm i\eta} U \chi_i^{(\pm)} \qquad (A\cdot16)$$

where ϕ_i are the eigenfunctions of H_0, as before. The transition matrix can then be written as:

$$T_{fi} \equiv \langle \phi_f | (U+V) | \psi_i^{(+)} \rangle$$

$$= \langle \phi_f | U | \chi_i^{(+)} \rangle + \langle \chi_f^{(-)} | V | \psi_i^{(+)} \rangle \qquad (A\cdot17).$$

If V is the photoproduction interaction whereas U is completely non-electromagnetic, and if the state ϕ_f contains one photon less than the state ϕ_i, only the second term can contribute:

$$T_{fi} = \langle \chi_f^{(-)} | V | \psi_i^{(+)} \rangle \qquad (A\cdot18).$$

This formula is very useful for the study of final state interactions.

If the basis states ϕ_i are eigenstates of operators corresponding to constants of the motion, then the transition matrix becomes diagonal. In particular, for reactions which conserve the total linear momentum:

$$T_{fi} = \delta^3(\vec{P}_i - \vec{P}_f)\, M_{fi} \qquad (A\cdot 19).$$

The only exception to this rule occurs in the case of energy conservation. Matrix elements of T between states of different energy do, in general, not vanish. This exception does not occur for the S-matrix, as can be seen from A.9.

If all plane wave states are normalized to unit volume, the cross section for a reaction whose initial and final states Φ_i and Φ_f are both two-particle states, is given by:

$$\sigma_{fi} = \frac{2\pi}{v_i} \int \frac{d^3q}{(2\pi)^3} |T_{fi}|^2\, \delta(E_i - E_f) \qquad (A\cdot 20).$$

Here $E_i = E_1 + E_2$ and $E_f = E_3 + E_4$ are the values of the total energy in the initial and final states and $\vec{q}$ is the momentum of one of the final particles. If the incident particles have velocities $\vec{u}_1$ and $\vec{u}_2$, the incident flux v_i is given by:

$$v_i^2 = (1 - \vec{u}_1 \cdot \vec{u}_2)^2 - (1 - u_1^2)(1 - u_2^2) \qquad (A\cdot 21).$$

The following two quantities are Lorentz invariant:

$$\left.\begin{aligned} E_1 E_2 v_i &= \text{invariant} \\ E_1 E_2 E_3 E_4 |T_{fi}|^2 &= \text{invariant} \end{aligned}\right\} \qquad (A\cdot 22).$$

The total cross section from a state Φ_i to all final states Φ_f can be obtained from A.20 by closure. At the same time the total cross section can be converted into a differential cross section:

$$\frac{d\sigma}{d\Omega} = \frac{2\pi}{v_i} \int \frac{q^2 dq}{(2\pi)^3} \sum_f T_{if}^\dagger \, \delta(E_i - E_f) \, T_{fi}$$

$$= \frac{2\pi}{v_i} \int \frac{q^2 dq}{(2\pi)^3} \langle \phi_i | T^\dagger \, \delta(E_i - H_0) \, T | \phi_i \rangle \qquad (A.23)$$

APPENDIX B

Field theoretical description of the pion-nucleon interaction

The theoretical analysis of processes in which particles are created or destroyed, is usually formulated in terms of a quantized field theory or some equivalent formalism. In a completely relativistic theory the interactions between pions and nucleons, for example, is described by introducing field operators ϕ_i and ψ for the pion and nucleon fields. Of the three pion field operators

ϕ_0 destroys the π^0 or creates the π^0,
ϕ destroys the π^- or creates the π^+,
$\phi^\dagger$ destroys the π^+ or creates the π^-.

The following summation conventions in charge space will be used:

$$\left.\begin{aligned} \phi_i \phi_i &\equiv 2\phi^\dagger\phi + \phi_0\phi_0 \\ \tau_i \phi_i &\equiv \sqrt{2}\left(\tau_+\phi + \tau_-\phi^\dagger\right) + \tau_3\phi_0 \end{aligned}\right\} \quad (B\cdot 1)$$

where for the isospin matrices we use the convention:

$$\tau_+|p\rangle = |n\rangle \qquad \tau_3|p\rangle = -|p\rangle \qquad \tau_-|p\rangle = 0 \qquad (B\cdot 2)$$

which is more convenient in nuclear physics than the high energy physics convention in which the roles of proton and neutron are interchanged.

If the pseudoscalar coupling between the pion and the nucleon fields is assumed, all the properties of the interaction can in principle be determined from the commutation rules and the coupled field equations:

$$\left.\begin{aligned} (\partial_\mu \partial_\mu - m^2)\,\phi_i &= i g_0 \bar{\psi}\,\gamma_5 \tau_i \psi \\ (\gamma_\mu \partial_\mu + M)\,\psi &= -i g_0 \gamma_5 \tau_i \phi_i \psi \end{aligned}\right\} \qquad (B\cdot 3)$$

where m and M are the bare masses of the pion and the nucleon. We define $\partial_\mu \equiv \frac{\partial}{\partial x_\mu}$ ($\mu = 0,1,2,3,$) and are using the metric in which $x_\mu x_\mu \equiv \vec{x}\cdot\vec{x} - t^2$. The γ-matrices are defined in terms of the Dirac matrices by $\vec{\gamma} = -i\beta\vec{\alpha}$, $\gamma_0 = -i\beta$, and $\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3$. For pseudovector coupling one gets similar but slightly different field equations. These field equations look very similar to the equations for the interaction between the electromagnetic field and any charged fermion field:

$$\left.\begin{aligned} \partial_\mu \partial_\mu A_\nu &= -i e_0 \bar{\psi}\,\gamma_\nu \psi \\ (\gamma_\mu \partial_\mu + M)\,\psi &= i e_0 \gamma_\mu A_\mu \psi \end{aligned}\right\} \qquad (B\cdot 4)$$

which has been very successfully treated by the Feynman-Dyson perturbation technique. The relatively large coupling constant ($g^2 \approx 15$ as compared to $e^2 \approx 1/137$) renders the usual perturbation theory useless for relativistic meson theory, however.

In order to produce a real antinucleon from a free nucleon which is at rest, pions or photons of more than 3.7 Bev are required. We shall, on the other hand, only be considering energies up to a Bev or so. Although virtual antinucleon-nucleon pairs are probably very important even at these energies, their effect can presumably to a good approximation be included by modifying the basic interactions. If this is done, it is no longer necessary to quantize the nucleon field. Instead

the nucleon is described by an ordinary wave function just as in non-relativistic quantum mechanics.

In such a formalism, the free hamiltonian H_0 is given by the sum of terms like:

$$\left.\begin{aligned} H_N &= M + \frac{p^2}{2M} + \cdots\cdots \\ H_\pi &= \int d^3x \left[\tfrac{1}{2}m^2\phi_i\phi_i + \tfrac{1}{2}(\nabla\phi_i)\cdot(\nabla\phi_i) + \tfrac{1}{2}\dot{\phi}_i\dot{\phi}_i\right] \end{aligned}\right\} \quad (B\cdot 5).$$

The interaction hamiltonian is the sum of terms with the following structure:

$$\left.\begin{aligned} &H_1 \quad \text{contains the pion field operator linearly} \\ &H_2 \quad \text{contains the pion field operator quadratically} \\ &\quad \text{etc.} \end{aligned}\right\} \quad (B\cdot 6).$$

If the pion field is expanded in plane waves:

$$\phi_i(\vec{x}) = \sum_k N_k \left(a_{ik} e^{i\vec{k}\cdot\vec{x}} + a_{ik}^\dagger e^{-i\vec{k}\cdot\vec{x}}\right) \qquad (B\cdot 7)$$

where a_{ik} and $a_{ik}^\dagger$ are the destruction and creation operators for a pion of momentum $\vec{k}$ in a charge state i while N_k is a normalization constant, then the terms containing $\phi(\vec{x})$ will reduce to the following form:

$$\left.\begin{aligned} H_\pi &= \sum_i \sum_k a_{ik}^\dagger a_{ik}\,\omega_k \\ H_1 &= \sum_i \sum_k \left(a_{ik}V_{ik} + a_{ik}^\dagger V_{ik}^\dagger\right) \\ H_2 &= \sum_i\sum_j\sum_k\sum_q \left(a_{ik}a_{jq}V_{ikjq}^{(1)} + a_{ik}a_{jq}^\dagger V_{ikjq}^{(2)}\right) + \text{herm. conj.} \end{aligned}\right\} \quad (B\cdot 8).$$

Here $\omega_k = \sqrt{m^2+k^2}$ is the pion energy while V_{ik} etc. no longer contain any creation or destruction operators but operate on the nucleon

variables only.

At energies appreciably smaller than the nucleon rest mass, the motion of the nucleon may be neglected. The V_{ik} in B.8 can then contain the spin and isospin operators of the nucleon only. The structure of H_1 is then completely determined by the conservation of angular momentum, parity, and isospin. For a nucleon whose center is at $\vec{x}_n$, one obtains:

$$V_{ik} = \frac{f_0}{m} N_k v(k) \left[i \vec{\sigma} \cdot \vec{k} \right] \tau_i e^{i \vec{k} \cdot \vec{x}_n} \qquad (B \cdot 9)$$

where $v(k)$ is the Fourier transform of the spatial distribution of the nucleon while f_0 is a dimensionless coupling constant which can be related to the relativistic coupling constant g_0 of B.3 by:

$$f_0 = \frac{m}{2M} g_0 \qquad (B \cdot 10).$$

The non-relativistic coupling constant is therefore much smaller than the relativistic one so that perturbation theory is not completely useless.

The transition amplitude (A.10) for the scattering of a pion by a single nucleon can be expressed as the matrix element between initial and final states of the complete non-interacting system, of a transition operator T_S for the whole system:

$$\langle f, \vec{k}_f \mid T_S \mid i, \vec{k}_i \rangle \qquad (B \cdot 11).$$

Here i and f denote the initial and final states of the nucleon. The pion is initially in a plane wave state with momentum $\vec{k}_i$ and finally in one with momentum $\vec{k}_f$. The charge variable of the pion is suppressed. If the only interaction term which exists, is H_1, T_S

is according to A.12 and A.13 given by the perturbation expansion:

$$T_s = H_1 + H_1 \frac{1}{E - H_0 + i\eta} H_1 + \ldots \qquad (B.12).$$

The interaction operators H_1 contain creation and annihilation operators for pions in plane wave states (B.8) so that the inner product in B.11 can easily be performed with respect to the pion variables. It merely involves the replacement of one of the H_1's by a $V(k_i)$ and another by a $V^+(k_f)$ where these quantities are given by B.9 in the static theory. The transition amplitude B.11 can therefore also be considered as the matrix element between the initial and final states of the nucleon only of a different transition operator which only operates on the nucleon variables:

$$\langle f | t_s(\vec{p}, \vec{x}) | i \rangle \qquad (B.13).$$

Using B.9, we see that $t_s(\vec{p}, \vec{x})$ can be written as:

$$t_s(\vec{p}, \vec{x}) = e^{-i\vec{k}_f \cdot \vec{x}} t_s e^{i\vec{k}_i \cdot \vec{x}}$$

$$= e^{i\vec{p} \cdot \vec{x}} t_s \qquad (B.14)$$

where $\vec{p}$ is the momentum transfer and where t_s no longer depends on the nucleon coordinate $\vec{x}$.

When the transition operator T_s for the whole system is replaced by a transition operator t_s for the nucleon only, the wave functions of the incoming and outgoing field particles thus appear as an explicit factor. Such a factor is also to be expected from translational invariance. If the initial and final spin-isospin states of the nucleon are denoted $|\chi_i\rangle$ and $|\chi_f\rangle$, and the initial and final spatial states are plane waves with momenta $\vec{p}_i$ and $\vec{p}_f$, the transition

matrix becomes:

$$\int d^3x \; e^{-i\vec{p}_f\cdot\vec{x}} \langle \lambda_f | t_s(\vec{p},\vec{x}) | \lambda_i \rangle e^{i\vec{p}_i\cdot\vec{x}} \qquad (B\cdot15)$$

where $\vec{x}$ is the nucleon coordinate. Since, according to (A.19), this matrix element must be proportional to $\delta^3(\vec{p}_i+\vec{p}-\vec{p}_f)$, we must have:

$$t_s(\vec{p},\vec{x}) = e^{i\vec{p}\cdot\vec{x}} t_s \qquad (B\cdot16).$$

Chew, Low, and Wick (3) have investigated the implications of the static theory in which B.9 is used in H_1 while all higher terms H_2, H_3, ... are neglected. By adjusting the renormalized coupling constant and the cutoff momentum ($v(k)$ being 1 when k is less than the cutoff momentum and 0 otherwise), very good agreement is obtained with the observed p-wave pion scattering below 450 Mev. The small amount of s-wave scattering which is in fact observed, can be ascribed to nucleon recoil. The main feature of the p-wave scattering is the resonance in the state with total isotopic spin I and total angular momentum J both equal to $3/2$.

In the energy region near the resonance, one can to a good approximation assume that this is the only state which contributes to the scattering. For the non-charge exchange scattering of neutral pions, the transition operator will not contain any isospin operators so that all the operators are contained in the projection operator on the state which is given by (3):

$$P_s = \tfrac{3}{2}(\vec{\sigma}\cdot\hat{k}_i)(\vec{\sigma}\cdot\hat{k}_f) + \tfrac{1}{2}(\vec{\sigma}\cdot\hat{k}_f)(\vec{\sigma}\cdot\hat{k}_i)$$
$$= 2\hat{k}_i\cdot\hat{k}_f + i\vec{\sigma}\cdot(\hat{k}_i\times\hat{k}_f) \qquad (B\cdot17).$$

Here $\vec{k}_i$ and $\vec{k}_f$ are again the initial and final pion momenta. If $\hat{k}_i$ is chosen as axis of spin quantization and as polar axis for a system of spherical polar coordinates in which $\hat{k}_f$ has the directional coordinates (θ , φ), B.17 reduces to:

$$P_s = \alpha \cos\theta + \left(\sigma_+ e^{-i\varphi} - \sigma_- e^{i\varphi}\right) \sin\theta \qquad (B\cdot 18)$$

Hence the spin flip amplitude vanishes in the forward direction while the non-spin flip amplitude vanishes at 90° in the center of mass system. The convention $\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$ has been used for the nucleon spin operators.

In order to characterize the reaction by an invariant number which depends on the total energy in the center of mass system only, one should write (A.22):

$$t_s = \frac{\text{Invariant}}{\sqrt{\omega_i E_i E_f \omega_f}} P_s \qquad (B\cdot 19)$$

where ω_i , ω_f are the pion energies while E_i , E_f are the nucleon energies. Since the value of $\sqrt{E_i E_f}$ varies only slightly from laboratory to center of mass sytem in the energy range under consideration, we can to a good approximation write:

$$t_s = \frac{\mathcal{J}_s}{\sqrt{\omega_i \omega_f}} P_s \qquad (B\cdot 20)$$

where $\mathcal{J}_s$ is now an (approximately) invariant complex number which contains all the intrinsic energy dependence of the process, such as the resonance behavior. The differential cross section for non-charge exchange scattering of neutral pions from nucleons is thus given by:

$$\left(\frac{d\sigma}{d\omega}\right)_S = \left(\frac{\gamma_S}{2\pi}\right)^2 \frac{1 + 3\cos^2\theta}{\left(1 + \frac{\omega}{\sqrt{M^2 + p^2}}\right)^2}$$

in the center of mass system.

In order to describe the photoproduction of pions field theoretically, the coupling of the electromagnetic field to the total pion-nucleon current $\vec{J}$ must be added to B.6. This is usually described by the interaction hamiltonian:

$$H_\gamma = -\int \vec{J} \cdot \vec{A}\, d^3x \qquad (B\cdot 22)$$

where $\vec{A}$ is the vector potential associated with the photon field.

Chew and Low (3) decomposed the current operator $\vec{J}$ into two parts, $\vec{J}_N$ and $\vec{J}_\pi$, in such a way that $\vec{J}_N$ commutes with the pion creation operators while $\vec{J}_\pi$ has a vanishing expectation value in the state containing a single real nucleon and no pions. The coupling with $\vec{J}_\pi$ corresponds to making the usual substitution:

$$\vec{k} \longrightarrow \vec{k} - e\vec{A} \quad \text{and} \quad \nabla \longrightarrow \nabla - ie\vec{A}$$

in B.9 and B.5, respectively. In the case of charged pion production these two replacements give rise to the direct s-wave production and to the retardation term, respectively. For the production of neutral pions, however, the contribution from $\vec{J}_\pi$ vanishes so that we need only consider $\vec{J}_N$.

The "real nucleon current" $\vec{J}_N$ can again be decomposed into an isoscalar part $\vec{J}_S$ and an isovector part $\vec{J}_V$. In the static theory the contribution of the latter dominates because only it can lead to the $I = \frac{3}{2}$ state. This term is equivalent to the usual

$-\vec{\mu}\cdot\vec{H}$ -coupling of the magnetic field $\vec{H}$ with the true (i.e. including anomalous parts) magnetic moment $\vec{\mu}$ of the nucleon.

By analogy with B.11, the transition amplitude for the photoproduction of a pion of momentum $\vec{q}$ by a photon of momentum $\vec{k}$ can be written as:

$$\langle f, \vec{q} \mid T_p \mid i, \vec{k} \rangle \qquad (B\cdot 23).$$

The transition operator T_p will now contain a destruction operator for a photon in a plane wave state $|\vec{k}\rangle$ in addition to the creation operator for a pion in state $|\vec{q}\rangle$. These operators can again be removed by replacing B.23 by:

$$\langle f \mid t_p(\vec{p}, \vec{x}) \mid i \rangle \qquad (B\cdot 24)$$

where $t_p(\vec{p}, \vec{x})$, by analogy with B. 14, can be written as:

$$t_p(\vec{p}, \vec{x}) = e^{-i\vec{q}\cdot\vec{x}}\, t_p\, e^{+i\vec{k}\cdot\vec{x}}$$
$$= e^{i\vec{p}\cdot\vec{x}}\, t_p \qquad (B\cdot 25).$$

The transition operator t_p again operates on the nucleon only.

If only the contribution from $\vec{J}_V$ is kept, one finds (3) that the operators in t_p may be separated out as in B.20:

$$t_p = \frac{g_p}{\sqrt{k\omega}}\, \mathcal{P}_p \qquad (B\cdot 26)$$

where the projection operator $\mathcal{P}_p$ is obtained from $\mathcal{P}_s$ by replacing the momentum of the incident pion by the quantity $i\,\hat{k}\times\hat{\varepsilon}$ ($\vec{k}$ and $\hat{\varepsilon}$ are the momentum and the polarization vector of the photon). Using the same coordinates as in the scattering case and assuming that the photon is right-handedly polarised (this does not involve any loss of

generality since the strong and electromagnetic interactions are invariant under reflections), one finds:

$$P_p = \frac{1}{\sqrt{2}}\left[(2+\sigma_z)e^{i\varphi}\sin\theta - 2\sigma_+\cos\theta\right] \qquad (B\cdot 27).$$

Unlike the scattering case, it is the non-spin flip amplitude which now vanishes in the forward direction. The differential cross section for photoproduction from unpolarized nucleons becomes:

$$\left(\frac{d\sigma}{d\Omega}\right)_p = \left(\frac{\gamma_p}{2\pi}\right)^2 \frac{\frac{1}{2}(5-3\cos^2\theta)}{\left(1+\frac{k}{\sqrt{M^2+k^2}}\right)\left(1+\frac{q}{\sqrt{M^2+q^2}}\right)} \qquad (B\cdot 28)$$

in the center of mass system.

The principal result of the Chew-Low theory is that the $l=1$, $J=\frac{3}{2}$, $I=\frac{3}{2}$ state is enhanced in all pion-nucleon processes at low energies. If only this state is kept, the theory gives a very simple relationship between the scattering and photoproduction of neutral pions namely:

$$\frac{\gamma_p}{\gamma_s} = \frac{e}{f}\,\frac{m}{4M}\,(\mu_p - \mu_n) \qquad (B\cdot 29)$$

where e and f are the renormalized coupling constants $\left(\frac{e^2}{4\pi} = \frac{1}{137},\ \frac{f^2}{4\pi} = 0.08\right)$, m and M are the pion and nucleon masses and $\mu_p = 2.79$ and $\mu_n = -1.91$ are proton and neutron magnetic moments. The scattering of neutral pions cannot be studied directly but the component that goes through the $I=\frac{3}{2}$ state can be inferred from the scattering of positive pions by protons (appendix C). If this is done, one finds that B.29 gives a good approximation to the low energy experimental results. The low energy photoproduction of neutral pions can thus be described quite well by the static theory.

The most general form that the transition operator t_p for the photoproduction of neutral pions can have, is:

$$t_p = K + \vec{L}\cdot\vec{\sigma} + M\tau_3 + \vec{N}\cdot\vec{\sigma}\tau_3 \qquad (B\cdot 30)$$

where K, $\vec{L}$, M, and $\vec{N}$ are energy-dependent complex numbers. By measuring the differential cross section B. 28 for the photoproduction of neutral pions from unpolarized protons, one determines the quantity:

$$|K-M|^2 + |\vec{L}-\vec{N}|^2 \qquad (B\cdot 31).$$

(For the production from neutrons, the minus signs would be replaced by plus signs.) At low energies, the simple static theory leads to the result that M and $\vec{N}$ vanish (this corresponds to the neglect of the isoscalar current $\vec{J}_s$) and that K and $\vec{L}$ are related by B.27. Thus the single determination of B.31 specifies the transition operator B.30 completely, the energy dependence of both K and $\vec{L}$ being contained in J_p. As a matter of fact, by means of relations B.29 and B.21 the transition operator for the production of neutral pions can at low energies be determined by measuring the <u>total</u> cross section for the scattering of positive pions from hydrogen! Such a determination is of course correct only to the extent that all states except the $J=\frac{3}{2}$, $I=\frac{3}{2}$ state may be neglected.

At high energies there is unfortunately not yet a theory which provides simplifications corresponding to those at low energies. The photoproduction from hydrogen determines the quantity B.31 only whereas for the production from complex nuclei we need the quantities:

$$\left.\begin{array}{l} |K|^2 \\ |K|^2 + |\vec{L}|^2 + |M|^2 + |\vec{N}|^2 \end{array}\right\} \qquad (B\cdot 32).$$

Even if the isoscalar contributions are negligible at high energies too (i.e. M and $\vec{N}$ both vanish), it is still necessary to separate the spin flip from the non-spin flip contributions which is not possible from a measurement of the angular distribution alone. Experiments with polarized nucleons may be used to obtain more information.

Although no field theoretical prediction at high energies is as yet in existence, it is possible that a phenomenological analysis of the experimental results may indicate that a small number of multipole amplitudes dominate the photoproduction process. If this happens, it may become possible to separate the spin flip from the non-spin flip terms. The relation B.27 for example follows just from the dominance of the M_{13} -amplitude and historically the 33-resonance was discovered by phenomenological analysis before the advent of the successful field theoretical description. The relation B.29, however, is a consequence of the Chew-Low theory. The description in terms of multipole amplitudes is considered in appendix C.

APPENDIX C

Phenomenological analysis of pion scattering and photoproduction (2)

In any reaction which conserves the total angular momentum J and the parity ω, the transition matrix must be diagonal in a representation where eigenstates of the total angular momentum and parity are chosen as basis. In addition, the matrix elements must be independent of the value of the projection M_J of J.

In practice measurements of cross sections are performed in terms of plane wave states rather than angular momentum eigenstates. The transition matrix for the scattering of a pion by a nucleon is therefore written as:

$$\langle S_f, \vec{q} | T_S | S_i, \vec{p} \rangle \qquad (C.1)$$

where S_i and S_f are the initial and final values of the nucleon spin projection while $\vec{p}$ and $\vec{q}$ are the initial and final momenta of the pion, respectively. The coordinates will usually be chosen in such a manner that the momentum of the incident particle $\vec{p}=p\hat{z}$ where $\hat{z}$ denotes a unit vector in the z-direction. This direction will also be chosen as axis of quantization for the angular momentum. The transition operator (A.12) for pion scattering is denoted T_S.

In order to fully exploit the invariance properties of the transition operator, C.1 can be written as the following sum:

$$\sum \langle \hat{q} | l, m \rangle \langle S_f, l, m | J, M_J \rangle \langle q, J, M_J, l | T_S | p, J', M'_J, l' \rangle \times \langle J', M'_J | S_i, l', m' \rangle \langle l', m' | \hat{p} \rangle \qquad (C.2)$$

where the summation runs over all possible values of ℓ, m, J, M_J, J', M_J', ℓ', and m'; ℓ and m denote the pion's orbital angular momentum and its projection, respectively.

The first and last factors of each term in C.2 are the transformation coefficients between plane and spherical waves. The well known expansion:

$$e^{i\vec{k}\cdot\vec{x}} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} 4\pi i^{\ell} j_{\ell}(kx) Y_{\ell m}(\hat{x}) \cdot Y^{*}_{\ell m}(\hat{k}) \qquad (C\cdot 3)$$

can also be written as:

$$\langle \vec{x} | \vec{k} \rangle = \sum_{\ell} \sum_{m} \langle \vec{x} | k, \ell, m \rangle \langle \ell, m | \hat{k} \rangle \qquad (C\cdot 4)$$

in which form it is obvious that the transformation coefficients must be identified with the values of the corresponding spherical harmonics in the direction $\hat{k}$ of the momentum. With our choice of coordinates, we thus have:

$$\langle \ell', m' | \hat{p} \rangle = \delta_{m'0} \sqrt{\frac{2\ell'+1}{4\pi}} \qquad (C\cdot 5)$$

$$\langle \hat{q} | \ell, m \rangle = Y_{\ell,m}(\theta, \varphi) \qquad (C\cdot 6)$$

where (θ, φ) are the angular coordinates of the direction $\hat{q}$ in which the final pion is emitted.

The second and fourth factors of each term in C.2 are the transformation coefficients between a representation in which J and M_J are diagonal and one in which m and S are diagonal, namely the Clebsch-Gordan coefficients:

$$\langle s_f, \ell, m | J, M_J \rangle = C\left(\ell, \tfrac{1}{2}, J ; m, s_f, M_J\right) \qquad (C\cdot 7).$$

The phase of these coefficients will always be chosen in accordance with

the convention of Condon and Shortley.

The third factor in C.2 is the transition matrix in the desired representation. Since the pion is pseudoscalar, the parity is given by:

$$\omega = -(-1)^{\ell}$$

Conservation of total angular momentum and parity therefore implies:

$$J' = J$$

$$\ell' = \ell, \ell \pm 2, \ell \pm 4, \ldots$$

Because $J = \ell \pm \frac{1}{2}$, conservation of parity is in this case equivalent to conservation of orbital angular momentum so that:

$$\langle q, J, M_J, \ell | T_s | p, J', M_J', \ell' \rangle = 4\pi \delta_{JJ'} \delta_{M_J M_J'} \delta_{\ell\ell'} t_s(J, \ell, p) \qquad (C.8).$$

The eightfold sum in C.2 has thus been reduced to:

$$\sum_J \sum_\ell \sqrt{4\pi(2\ell+1)}\, Y_{\ell,(s_i - s_f)}(\theta, \varphi)\, C_f C_i\, t_s(J, \ell, p) \qquad (C.9)$$

where C_f and C_i are the final and initial Clebsch-Gordan coefficients.

If no inelastic processes can occur, i.e., below the threshold for the production of two pions, the scattering in each state can be represented by a real phase shift $\delta(J, \ell, p)$. If ω is the total pion energy, the relation between t_s and the phase shift is:

$$t_s(J, \ell, p) = -\frac{2\pi e^{i\delta} \sin\delta}{\omega p} \qquad (C.10).$$

Using C.10 and C.9, one can now obtain expressions for the differential cross-sections for pion scattering with and without flipping of the nucleon spin, by substituting C.1 into A.20. If one defines:

$$\alpha_\ell^{\pm} = \frac{e^{i\delta}\sin\delta}{p} \qquad (C.11)$$

where the ($\pm$) signs correspond to $J=\ell\pm\frac{1}{2}$, respectively, then:

$$\left(\frac{d\sigma}{d\Omega}\right)(\text{spin flip}) = \left|\sum_\ell (\alpha_\ell^+ - \alpha_\ell^-)\sin\theta \ \frac{d P_\ell(\cos\theta)}{d(\cos\theta)}\right|^2 \qquad (C.12)$$

$$\left(\frac{d\sigma}{d\Omega}\right)(\text{no flip}) = \left|\sum_\ell [(\ell+1)\alpha_\ell^+ + \ell\,\alpha_\ell^-]\ P_\ell(\cos\theta)\right|^2 \qquad (C.13).$$

An interesting case occurs if the only nonvanishing phase shift is $\delta(\frac{3}{2},1)$, namely the p-wave phase shift for the $J=\frac{3}{2}$ state. In that case C.12 and C.13 become:

$$\frac{d\sigma}{d\Omega}(\text{spin flip}) = \frac{\sin^2\delta}{p^2}\sin^2\theta$$

$$\frac{d\sigma}{d\Omega}(\text{no flip}) = 4\,\frac{\sin^2\delta}{p^2}\cos^2\theta$$

Adding these, one finds for the differential and total cross sections:

$$\frac{d\sigma}{d\Omega} = \frac{\sin^2\delta}{p^2}\left(1 + 3\cos^2\theta\right) \qquad (C.14)$$

$$\sigma = \frac{8\pi}{p^2}\sin^2\delta \qquad (C.15).$$

Here p is the momentum in the center of mass system.

At total pion energies up to at least 450 Mev in the laboratory, the angular distribution in the center of mass system of positive pions scattered from hydrogen agrees very closely with C.14. In addition, the total cross section reaches a value of $8\pi/p^2$ · at a total pion energy of about 330 Mev. From these facts it can be deduced that the low energy interactions between pions and nucleons are completely dominated by the interaction in the $J=\frac{3}{2}$, $\ell=1$ state and that the scattering phase shift for this state goes through 90° at a laboratory

energy of around 330 Mev. The existence of such a resonance was contemplated by Brueckner in 1952.

By means of the isospin formalism, reactions involving different charge states of the pion and the nucleon can be handled together. The usefulness of this formalism is due to the fact that the total isospin I and its 3-component I_3 are both apparently conserved by all pion-nucleon interactions when electromagnetic effects are neglected. The physical pion-nucleon states $|\pi, N\rangle$ can be expanded in terms of eigenstates $|I, I_3\rangle$ by means of Clebsch-Gordan coefficients and the summation in C.2 can be extended to include I and I_3. The transition matrix C.9 then becomes:

$$\sum_I \sum_J \sum_l \sqrt{4\pi(2l+1)}\, Y_{l,(s_i-s_f)}(\theta,\varphi)\, C_f C_i C_f^I C_i^I\, t_S(I,J,l,p) = \frac{2\pi}{\omega}\,(f_1 + f_3) \qquad (C.16)$$

where f_1 and f_3 are scattering amplitudes corresponding to $I=\frac{1}{2}$ and $I=\frac{3}{2}$, respectively.

In terms of f_1 and f_3, the differential cross sections for the ten possible scattering processes become:

$$\left.\begin{aligned}
\frac{d\sigma}{d\Omega}(\pi^+p\to\pi^+p) &= \frac{d\sigma}{d\Omega}(\pi^-n\to\pi^-n) = |f_3|^2 \\
\frac{d\sigma}{d\Omega}(\pi^-p\to\pi^-p) &= \frac{d\sigma}{d\Omega}(\pi^+n\to\pi^+n) = \tfrac{1}{9}|f_3|^2 + \tfrac{4}{9}|f_1|^2 + \tfrac{4}{9}\mathrm{Re}\, f_3^* f_1 \\
\frac{d\sigma}{d\Omega}(\pi^-p\rightleftarrows\pi^0n) &= \frac{d\sigma}{d\Omega}(\pi^+n\rightleftarrows\pi^0p) = \tfrac{2}{9}|f_3|^2 + \tfrac{2}{9}|f_1|^2 - \tfrac{4}{9}\mathrm{Re}\, f_3^* f_1 \\
\frac{d\sigma}{d\Omega}(\pi^0p\to\pi^0p) &= \frac{d\sigma}{d\Omega}(\pi^0n\to\pi^0n) = \tfrac{4}{9}|f_3|^2 + \tfrac{1}{9}|f_1|^2 + \tfrac{4}{9}\mathrm{Re}\, f_3^* f_1
\end{aligned}\right\} \qquad (C.17).$$

At low energies the total cross section (including charge exchange) for

scattering of negative pions by hydrogen is very close to one third of that for positive pions. This implies that $|f_1|$ is much smaller than $|f_3|$ so that Brueckner's enhanced state has the additional quantum number $I = \frac{3}{2}$.

At low energies the scattering of pions can therefore, apart from small s-wave effects, be described quite well by the single parameter δ_{33} (we are using the notation $\delta_{2I,2J}$ for p-wave phase shifts). The resonance in the 33-state completely dominates the other three p-states and the two s-states.

At higher energies the situation becomes much more complicated. The $(\pi^- + p)$ cross section passes through two peaks at total laboratory energies of about 800 Mev and 1100 Mev. The $(\pi^+ + p)$ cross section does not exhibit these peaks. This has led some investigators to assume that these peaks are due to resonances in two single states with $I = \frac{1}{2}$. As yet, however, it has not yet been established whether these are true resonances, whether these peaks are due to a single state each, or even how purely $I = \frac{1}{2}$ they are.

The photoproduction of pions from nucleons can be analysed in a similar manner. The incident photon can in this case be in different states of polarization. If the direction $\hat{k}$ of the photon momentum is again chosen as z-axis and as axis of quantization, the photon's spin projection can be either parallel or antiparallel to its momentum. Because of invariance under space inversion, we can without loss of generality restrict the discussion to right handedly polarized photons, i.e. photons having spin projection $+1$. The wave function of the incident photon will thus be taken to be:

$$\langle \vec{x} \mid \vec{k}, \hat{\varepsilon} \rangle = \vec{A} = C \hat{\varepsilon} e^{ikz} \qquad (C.18)$$

where C is the normalising factor to unit volume and:

$$\hat{\varepsilon} = \frac{1}{\sqrt{2}} (\hat{x} + i\hat{y}) \qquad (C.19).$$

Just as before, the plane wave state can be expanded in terms of eigenstates of the total photon angular momentum j. Since the orbital angular momentum of the photon can have no component along its momentum, the projection of j will also be $+1$. Corresponding to C.4 one can write:

$$\langle \vec{x} | \vec{k}, \hat{\varepsilon} \rangle = \sum_{j=1}^{\infty} \sum_{\sigma} \langle \vec{x} \mid k, j, \sigma \rangle \langle j, \sigma \mid \hat{k}, \hat{\varepsilon} \rangle \qquad (C.20).$$

The eigenfunctions $\langle \vec{x} \mid k, j, \sigma \rangle$ are the so-called multipole fields which we shall define as:

$$\langle \vec{x} \mid k, j, m \rangle = \frac{C}{\sqrt{2}} \cdot 4\pi i^j j_j(kx) \vec{Y}_{j(j)+1} \qquad (C.21)$$

$$\langle \vec{x} \mid k, j, e \rangle = -\frac{C}{\sqrt{2}} 4\pi \left[\sqrt{\frac{j+1}{2j+1}} \, i^{j-1} j_{j-1}(kx) \vec{Y}_{j(j-1)+1} + \sqrt{\frac{j}{2j+1}} \, i^{j+1} j_{j+1}(kx) \vec{Y}_{j(j+1)+1} \right] \qquad (C.22)$$

Here $j_\ell(kx)$ are spherical Bessel functions and $\vec{Y}_{jlm}(\hat{x})$ are the vector spherical harmonics discussed, for example, in Blatt and Weisskopf (20). With this definition, the expansion coefficients are:

$$\langle j, \sigma \mid \hat{k} = \hat{z}, \hat{\varepsilon} \rangle = \sqrt{\frac{2j+1}{4\pi}} \qquad (C.23).$$

The summation over σ contains two terms, $\sigma = e$ and $\sigma = m$.

The photoproduction transition matrix $\langle s_t, \vec{q} \mid T_p \mid s_i, \vec{k}, \hat{\varepsilon} \rangle$

can now be expanded in a sum similar to C.2:

$$\sum \langle \hat{q} | \ell, m \rangle \langle s_f, \ell, m | J, M_J \rangle \langle q, J, M_J, \ell | T_p | k, J', M'_J, j, \sigma \rangle$$
$$\times \langle J', M'_J | s_i, j, +1 \rangle \langle j, \sigma | \hat{k}, \hat{\varepsilon} \rangle \qquad (C.24).$$

The summation runs over ℓ, m, J, M_J, J', M'_J, j, and σ. The last factor is given by C.23 and the fourth is again a Clebsch-Gordan coefficient:

$$\langle J', M'_J | s_i, j, +1 \rangle = C\left(j, \tfrac{1}{2}, J'; +1, s_i, M'_J\right)$$

The third factor can be written:

$$\langle q, J, M_J, \ell | T_p | k, J', M'_J, j, \sigma \rangle$$
$$= 4\pi \delta_{JJ'} \delta_{M_J M'_J} \delta_{\omega\omega'} t_p(J, j, \sigma, k) \qquad (C.25).$$

As before, the parity on the left hand is given by:

$$\omega = -(-1)^{\ell}$$

The parity on the right hand is different for electric and magnetic multipoles:

$$\omega' = +(-1)^j \qquad (\sigma = e)$$
$$\omega' = -(-1)^j \qquad (\sigma = m).$$

Parity conservation therefore implies that s-wave pions can be produced by electric dipole ($\sigma = e$, $j = 1$) radiation only whereas p-wave pions can be produced by magnetic dipole ($\sigma = m$, $j = 1$) or electric quadrupole ($\sigma = e$, $j = 2$) radiation.

Keeping in mind the restrictions imposed by parity considerations, one can write C.24 as:

$$\sum_J \sum_j \sum_\sigma \sum_\ell \sqrt{4\pi(2j+1)}\, Y_{\ell,(1+S_i-S_f)}(\theta,\varphi)\, C_f C_i\, \delta_{\omega\omega'} \times t_p(J,j,\sigma,k) \qquad (C.26)$$

where (θ,φ) again define the direction $\hat{q}$ of the momentum of the emitted pion while C_f and C_i are the appropriate Clebsch-Gordan coefficients. It is customary to denote the matrix elements of T_p as follows:

$$\left.\begin{aligned} t_p(J,j,\sigma,k) &= M_{j,2J} \quad \text{if} \quad \sigma = m \\ t_p(J,j,\sigma,k) &= E_{j,2J} \quad \text{if} \quad \sigma = e \end{aligned}\right\} \qquad (C.27).$$

In terms of these quantities one can write down the matrix elements of t_p between definite initial and final spin states of the nucleon. Denoting the matrix element between $S_i = -\frac{1}{2}$ and $S_f = +\frac{1}{2}$ by $\langle + | t_p | - \rangle$ and so forth, one gets:

$$\left.\begin{aligned}
\langle +|t_p|+\rangle &= \sqrt{4\pi}\Big[Y_{11}\left(\sqrt{3}M_{13}-E_{23}\right) + Y_{21}\left(-\sqrt{\tfrac{3}{5}}E_{13}+\sqrt{\tfrac{1}{5}}M_{23}+\sqrt{\tfrac{16}{5}}M_{25}\right. \\
&\qquad \left. -\sqrt{\tfrac{8}{5}}E_{35}\right) + Y_{31}\left(-\sqrt{\tfrac{8}{7}}E_{25}+\sqrt{\tfrac{4}{7}}M_{35}+\ldots\right)+\ldots.\Big] \\
\langle -|t_p|-\rangle &= \sqrt{4\pi}\Big[Y_{11}\left(\sqrt{\tfrac{4}{3}}M_{11}+\sqrt{\tfrac{1}{3}}M_{13}+E_{23}\right) + Y_{21}\left(\sqrt{\tfrac{3}{5}}E_{13}+\sqrt{\tfrac{9}{5}}M_{23}\right. \\
&\qquad \left. +\sqrt{\tfrac{4}{5}}M_{25}+\sqrt{\tfrac{8}{5}}E_{35}\right) + Y_{31}\left(\sqrt{\tfrac{8}{7}}E_{25}+\sqrt{\tfrac{16}{7}}M_{35}+\ldots\right)+\ldots\Big] \\
\langle +|t_p|-\rangle &= \sqrt{4\pi}\Big[Y_{00}\left(\sqrt{2}E_{11}\right) + Y_{10}\left(-\sqrt{\tfrac{2}{3}}M_{11}+\sqrt{\tfrac{2}{3}}M_{13}+\sqrt{2}E_{23}\right) \\
&\qquad + Y_{20}\left(-\sqrt{\tfrac{2}{5}}E_{13}-\sqrt{\tfrac{6}{5}}M_{23}+\sqrt{\tfrac{6}{5}}M_{25}+\sqrt{\tfrac{12}{5}}E_{35}\right) \\
&\qquad + Y_{30}\left(-\sqrt{\tfrac{6}{7}}E_{25}-\sqrt{\tfrac{12}{7}}M_{35}+\ldots\right)+\ldots.\Big] \\
\langle -|t_p|+\rangle &= \sqrt{4\pi}\Big[Y_{22}\left(\sqrt{\tfrac{12}{5}}E_{13}-\sqrt{\tfrac{4}{5}}M_{23}+\sqrt{\tfrac{4}{5}}M_{25}-\sqrt{\tfrac{2}{5}}E_{35}\right) \\
&\qquad + Y_{32}\left(\sqrt{\tfrac{20}{7}}E_{25}-\sqrt{\tfrac{10}{7}}M_{35}+\ldots..\right)+\ldots..\Big]
\end{aligned}\right\} \qquad (C.28).$$

The Y_{lm}'s are spherical harmonics with the angular direction of the emitted pion as argument. All terms up to and including $J=\frac{5}{2}$ were kept. These expressions are valid when the incident photon has $m_j = +1$. The corresponding expressions for left handedly polarized photons can be obtained by interchanging $|+>$ and $|->$ and changing the signs of the m-subscripts of the spherical harmonics.

Since the spherical harmonic $Y_{lm}(\theta,f)$ contains $\sin^m\theta$ as a factor, it is obvious from C.28 that the non-spin-flip amplitude must always be proportional to $\sin\theta$ and will therefore vanish in the forward direction. This is of course a direct consequence of the conservation of M_J.

The differential cross section for a specific transition in the center of mass system is given by:

$$\frac{d\sigma}{d\Omega}(f,i) = \frac{qk}{\left(1+\frac{k}{M}\right)^2} \frac{|<f|t_p|i>|^2}{(2\pi)^2} \qquad (C.29)$$

if the energy transfer to the nucleon is neglected. Here M is the nucleon mass while k and q are the center of mass momenta of the photon and the pion, respectively. If the only nonvanishing multipole amplitude (C.27) is M_{13}, one obtains from C.28 that:

$$\left.\begin{aligned} \frac{d\sigma}{d\Omega}(++) &\sim \frac{9}{2}\sin^2\theta \\ \frac{d\sigma}{d\Omega}(--) &\sim \frac{1}{2}\sin^2\theta \\ \frac{d\sigma}{d\Omega}(+-) &\sim 2\cos^2\theta \end{aligned}\right\} \qquad (C.30).$$

For unpolarized nucleons one would observe:

$$\frac{d\sigma}{d\Omega} = \frac{1}{2}\sum_i\sum_f \frac{d\sigma}{d\Omega}(f,i) \sim \frac{1}{2}(5-3\cos^2\theta) \qquad (C.31).$$

This differential cross section is actually observed in the production of neutral pions from protons by photons of energy up to at least 450 Mev. The total cross section has a peak at 320 Mev and its energy dependence in general is very similar to that of the cross sections for scattering of positive and negative pions by protons. This fact, combined with the fact that the M_{13} -amplitude produces p-wave pions with $J=\frac{3}{2}$, leads one to conclude that the photoproduction of the neutral pions is also dominated by the resonance in the 33-state.

The production of charged mesons is more complicated, a strong E_{11} -amplitude and the so-called retardation term being added to the M_{13} -amplitude. If the interaction hamiltonian can be written as the sum of an isoscalar and the 3-component of an isovector, then the four photoproduction amplitudes can be expressed in terms of three parameters:

$$\left.\begin{aligned} t_p(\gamma p \to \pi^+ n) &= \sqrt{\tfrac{2}{3}}\, V_3 + \sqrt{\tfrac{1}{6}}\, V_1 - \sqrt{2}\, S \\ t_p(\gamma p \to \pi^0 p) &= \sqrt{\tfrac{4}{3}}\, V_3 - \sqrt{\tfrac{1}{12}}\, V_1 + S \\ t_p(\gamma n \to \pi^0 n) &= \sqrt{\tfrac{4}{3}}\, V_3 - \sqrt{\tfrac{1}{12}}\, V_1 - S \\ t_p(\gamma n \to \pi^- p) &= \sqrt{\tfrac{2}{3}}\, V_3 + \sqrt{\tfrac{1}{6}}\, V_1 + \sqrt{2}\, S \end{aligned}\right\} \quad (c.32).$$

Here S is the matrix element of the isoscalar term in the $I=\frac{1}{2}$ state while V_1 and V_3 are the matrix elements of the isovector term for final states with $I=\frac{1}{2}$ and $I=\frac{3}{2}$, respectively. If the isoscalar term is negligible, the production of neutral pions from protons and neutrons should therefore be equal. If the final state with $I=\frac{3}{2}$ dominates, the productions of neutral pions from protons should be twice that of positive pions. This agrees with experiment which lends further

support to the 33-hypothesis. The production of charge pions will not be considered further.

At higher energies the $(\gamma p \to \pi^0 p)$ cross section also becomes more complicated. That the close relation to the scattering is maintained, is shown by the appearance of two peaks at photon energies of 800 Mev and 1100 Mev. As in the case of scattering, many investigators consider these peaks to be due to resonances. The angular distribution at 800 Mev could possibly be explained by a single state with $J=\frac{3}{2}$. Wilson and Stoppini have assumed that the 800 Mev peak is due to an M_{13} -amplitude (with $I=\frac{1}{2}$, however, in contradistinction to the $I=\frac{3}{2}$ peak at 320 Mev). Peierls and Sakurai, on the other hand, favor an E_{13} -amplitude (5).

Below the threshold for the production of two pions, the matrix elements C.27 have been shown to have the following form:

$$M^{2I}_{j,2J} = m(j,J,I)\, e^{i\delta(l,J,I)} \qquad (C.33)$$

where $m(j,J,I)$ is a real number while $\delta(l,J,I)$ is the pion scattering phase shift in the corresponding state. This theorem is a consequence of invariance under time reversal and of the unitarity of the S-matrix. At high energies, where inelastic pion scattering is possible, equation C.33 is no longer valid.

APPENDIX D

Results of numerical computations

When the final state interactions of the produced pion with the nucleus were introduced, various radial integrals were encountered which could not be done analytically. Their values were computed using the IBM 709 digital computer at the Western Data Processing Center, Graduate School of Business Administration, University of California at Los Angeles. The results of the computations are presented in this chapter.

The non-correlated cross section for nuclear production can be expressed in terms of the non-spin flip cross section from single nucleons (9.1):

$$\left(\frac{d\sigma}{d\Omega}\right)_{NC} = \frac{\left(\frac{d\sigma}{d\Omega}\right)_{NSF}}{\sin^2\theta}\left\{A^2\left|e^{i\varphi}\sin\theta\ F(p)\right|^2\right\} \qquad (D.1).$$

The quantity inside the curly brackets was computed numerically. Its value when there are no interactions, will be denoted X_0; its value when the final state interactions are included without allowing for deflections of the produced pion, will be denoted X_1; its value when deflections are also included, will be denoted X_2. In order to discuss the interference with the coulomb production, one also has to know the phase caused by the absorption. In the second case, for example, we define:

$$\tan\delta_1 = \frac{\text{Im } F_1(p)}{\text{Re } F_1(p)} \qquad (D.2)$$

and similarly in the other cases. When there is no absorption, the

phase δ_0 vanishes. The values of the phases will always be given in radians.

The coulomb production cross section can be expressed in terms of the lifetime τ of the neutral pion, the pion mass m and the fine structure constant α by (11.20):

$$\left(\frac{d\sigma}{d\Omega}\right)_{coul} = \frac{8\alpha\hbar}{mc^2}\left(\frac{\hbar}{mc}\right)^2 \left\{ \frac{q^3 k}{p^4} Z^2 \sin^2\theta \left|F(p)\right|^2 \right\} \qquad (D\cdot 3).$$

The quantity inside the curly brackets was computed numerically. Its value when there are no interactions, will be denoted Y_0 ; its value when interactions are included for pions produced inside the nucleus, will be denoted Y_1 ; its value when pions produced behind the nucleus are also allowed to interact, will be denoted Y_2 . The phase of Y_i is defined just as in the nuclear case but is distinguished by a superscript c , e.g. δ_2^c corresponding to Y_2 .

Calculations were performed for carbon, calcium, copper, and lead and the following values were used for the mass number A and for the uniform radius R :

$$\left.\begin{array}{lll} \text{C :} & A = 12, & R = 3.05 \text{ fermis} \\ \text{Ca:} & A = 40, & R = 4.44 \text{ fermis} \\ \text{Cu:} & A = 64, & R = 5.00 \text{ fermis} \\ \text{Pb:} & A = 208, & R = 7.10 \text{ fermis} \end{array}\right\} \qquad (D\cdot 4).$$

The interaction parameters (section 8) chosen at 250 Mev, were:

$$\eta = 1.20 \quad , \quad \lambda = 1.75 \text{ fermis} \qquad (D\cdot 5)$$

corresponding to the optical model potential (8.9):

$$V = -(42.5 + 49\, i) \text{ Mev} \qquad (D\cdot 6).$$

The results of the computations at 250 Mev will now be presented for various values of the angle θ (in radians) between the pion and the photon directions in the laboratory.

Carbon at 250 Mev								
θ	X_0	X_1	X_2	δ_2	Y_0	Y_1	Y_2	δ_2^C
0.05	0.33	0.11	0.02	1.40	62.3	52.6	51.7	0.08
0.10	1.29	0.44	0.12	1.18	162.6	131.9	128.8	0.10
0.15	2.81	0.94	0.40	0.97	207.2	157.0	151.1	0.14
0.20	4.75	1.57	0.93	0.79	199.1	136.7	128.0	0.19
0.25	6.95	2.26	1.65	0.66	168.4	101.2	90.5	0.27
0.30	9.24	2.95	2.53	0.55	134.2	67.9	56.2	0.39
0.40	13.37	4.05	4.14	0.38	78.7	24.5	15.2	0.85
0.50	15.96	4.51	5.06	0.30	44.1	7.2	6.5	-1.14
0.60	16.41	4.24	4.94	0.19	23.8	3.6	14.4	-0.27
0.70	14.79	3.40	4.25	0.13	12.4	5.0	24.3	0.15
0.80	11.75	2.31	2.50	-0.04	6.1	6.9	27.0	0.46
0.90	8.16	1.30	1.46	-0.13	2.8	7.7	21.0	0.76
1.00	4.83	0.57	0.70	-0.31	1.1	7.0	28.4	-1.56
1.10	2.30	0.17	0.28	-0.65	0.4	5.5	59.8	1.41
1.20	0.76	0.06	0.14	-1.34	0.1	3.8	49.7	1.45
Calcium at 250 Mev								
0.05	3.3	0.8	0.3	-1.44	629	385	496	0.24
0.10	12.8	3.0	1.5	1.51	1606	869	1200	0.30
0.15	26.8	6.2	4.1	1.29	1973	857	1351	0.42
0.20	42.9	9.6	8.3	1.07	1799	563	1114	0.63
0.25	58.6	12.7	13.1	0.91	1420	290	841	0.93
0.30	71.5	14.7	17.6	0.74	1037	154	712	1.34
0.35	79.5	15.4	20.6	0.61	721	140	754	-1.37
0.40	81.8	14.7	21.3	0.49	482	198	936	-0.99
0.45	78.2	12.8	20.1	0.40	310	276	1184	-0.69
0.50	69.7	10.1	18.2	0.34	192	339	1401	-0.43
0.60	44.0	4.4	9.6	0.11	64	360	1399	0.06
0.70	18.5	0.7	3.9	-0.11	15	260	747	0.58
0.80	3.4	0.1	0.5	1.50	2	128	950	-1.35
0.90	0.1	1.2	1.4	0.66	0	37	439	-1.53

Copper at 250 Mev								
θ	X_0	X_1	X_2	δ_2	Y_0	Y_1	Y_2	δ_2^c
0.05	8.1	1.7	0.8	-1.34	1259	630	1007	0.27
0.10	30.8	6.4	3.9	-1.53	3180	1326	2432	0.35
0.15	63.2	12.8	10.3	1.39	3831	1157	2761	0.50
0.20	98.6	19.3	19.7	1.17	3396	656	2375	0.73
0.25	129.9	24.2	29.7	0.99	2584	376	1998	1.07
0.30	150.9	26.3	37.5	0.81	1799	422	1973	1.47
0.35	158.1	25.4	41.0	0.66	1177	666	2345	-1.29
0.40	150.9	21.7	38.8	0.52	729	939	3007	-0.97
0.45	131.2	16.4	33.3	0.41	427	1121	3632	-0.68
0.50	103.5	10.6	27.3	0.32	235	1159	3983	-0.41
0.60	44.5	2.0	9.6	0.01	53	859	2990	0.15
0.70	7.3	0.1	2.3	-0.49	5	387	758	0.89
0.80	0.8	2.6	2.9	0.58	0	81	2589	1.43
0.90	11.6	4.9	5.1	0.29	3	17	405	0.78
Lead at 250 Mev								
0.04	44	6.6	6.1	-1.02	5420	830	3710	0.35
0.08	163	24.0	26.4	-1.12	15330	2010	9930	0.43
0.12	325	46.3	62.7	-1.28	20720	3210	12420	0.58
0.16	487	66.1	110.4	-1.47	19940	6450	11240	0.81
0.20	605	77.0	156.8	1.45	15760	12390	9350	1.14
0.24	651	76.1	188.3	1.25	10910	19230	8520	1.53
0.28	616	64.3	196.2	1.06	6790	24350	8680	-1.25
0.32	513	45.8	181.7	0.89	3810	25980	8650	-0.98
0.36	370	26.5	148.5	0.71	1910	23790	7570	-0.79
0.40	222	11.5	94.1	0.50	810	18760	5620	-0.70
0.44	101	3.7	65.1	0.34	270	12530	3670	-0.70
0.48	25	3.3	30.0	0.06	50	6760	2120	-0.71
0.52	0	7.9	11.9	-0.35	0	2620	670	-0.40
0.56	16	14.1	5.8	-0.94	18	540	420	-1.05
0.60	55	19.0	5.2	1.27	50	300	5700	-0.32

At 900 Mev the interaction parameters which were used, are:

$$\eta = 1.00 \quad , \quad \lambda = 3.05 \text{ fermis} \qquad (D.7)$$

corresponding to a complex potential:

$$V = (3.5 - 32i) \text{ Mev} \qquad (D.8).$$

Since the refractive index is about 1.0, the quantities X_1 and X_2 are equal. The results of the computations are presented in the next four tables. The quantities Y_i given in the tables must be multiplied by a factor 10^3.

θ	X_0	X_2	S_2	Y_0	Y_1	S_1^c	Y_2	S_2^c
Carbon at 900 Mev								
.004	0.002	0.001	0.02	0.253	0.252	0.00	0.249	0.00
.008	0.009	0.005	0.02	0.580	0.575	0.00	0.567	0.00
.012	0.021	0.010	0.02	0.659	0.650	0.00	0.638	0.01
.016	0.036	0.018	0.02	0.592	0.581	0.00	0.565	0.01
.020	0.056	0.028	0.02	0.490	0.478	0.00	0.460	0.01
.030	0.125	0.062	0.01	0.288	0.275	0.00	0.254	0.03
.040	0.215	0.107	0.01	0.177	0.163	0.00	0.143	0.04
.050	0.325	0.162	0.01	0.116	0.102	0.00	0.082	0.07
.075	0.646	0.319	0.01	0.048	0.036	0.00	0.020	0.17
.100	0.963	0.470	0.01	0.023	0.013	0.01	0.003	0.44
.150	1.274	0.599	0.00	0.006	0.001	0.00	0.003	-0.02
.200	0.987	0.432	-0.02	0.002	0.000	0.29	0.006	0.48
.250	0.416	0.153	-0.05	0.000	0.000	0.13	0.002	-0.69
.300	0.045	0.006	-0.30	0.000	0.000	0.16	0.003	-1.38
.350	0.028	0.037	0.08	0.000	0.000	0.24	0.000	-0.27
Calcium at 900 Mev								
.004	0.03	0.01	0.03	2.80	2.75	0.00	2.69	0.01
.008	0.10	0.04	0.03	6.39	6.23	0.00	6.07	0.02
.012	0.23	0.09	0.03	7.23	6.98	0.00	6.74	0.02
.016	0.40	0.15	0.03	6.47	6.15	0.00	5.85	0.03
.020	0.61	0.23	0.03	5.32	4.96	0.00	4.64	0.05
.030	1.32	0.50	0.03	3.06	2.67	0.00	2.33	0.10
.040	2.22	0.83	0.03	1.83	1.44	0.01	1.13	0.19
.050	3.22	1.20	0.03	1.15	0.79	0.01	0.53	0.33
.060	4.22	1.56	0.03	0.75	0.43	0.01	0.25	0.58
.090	6.40	2.28	0.02	0.23	0.05	0.04	0.13	-1.11
.120	6.26	2.08	0.01	0.07	0.01	-0.38	0.32	-0.44
.150	4.08	1.19	0.00	0.02	0.00	0.00	0.38	-0.02
.180	1.52	0.32	-0.03	0.00	0.01	0.03	0.18	0.45
.210	0.13	0.00	-1.33	0.00	0.01	0.07	0.13	-1.44

Copper at 900 Mev								
θ	X_0	X_2	δ_2	$\overline{Y_0}$	$\overline{Y_1}$	δ_1^c	$\overline{Y_2}$	δ_2^c
.004	0.06	0.02	0.04	5.86	5.71	0.00	5.60	0.02
.008	0.26	0.09	0.04	13.37	12.91	0.00	12.57	0.02
.012	0.57	0.19	0.04	15.11	14.39	0.00	13.85	0.03
.016	1.01	0.34	0.04	13.48	12.57	0.00	11.91	0.05
.020	1.55	0.53	0.04	11.06	10.05	0.00	9.32	0.07
.030	3.31	1.12	0.04	6.29	5.20	0.00	4.48	0.14
.040	5.47	1.83	0.04	3.69	2.64	0.01	2.05	0.27
.050	7.76	2.58	0.04	2.27	1.33	0.01	0.92	0.49
.060	9.92	3.25	0.04	1.45	0.63	0.02	0.46	0.88
.090	13.39	4.15	0.03	0.40	0.02	0.12	0.56	-0.97
.120	10.71	2.98	0.02	0.10	0.03	-0.06	1.09	-0.48
.150	4.80	1.02	0.00	0.02	0.06	0.00	0.93	-0.03
.180	0.64	0.02	-0.14	0.00	0.04	0.03	0.17	0.65
.210	0.21	0.32	0.04	0.00	0.01	0.08	0.35	1.36
Lead at 900 Mev								
.004	0.7	0.2	0.08	46.2	43.2	0.00	42.0	0.04
.008	2.7	0.6	0.08	104.7	93.9	0.00	92.5	0.05
.012	5.9	1.4	0.08	117.3	103.5	0.00	98.4	0.08
.016	10.2	2.4	0.08	103.4	86.5	0.01	80.7	0.11
.020	15.5	3.6	0.08	83.6	65.1	0.01	59.3	0.16
.030	31.3	7.2	0.08	45.0	26.8	0.02	23.2	0.37
.040	47.8	10.8	0.07	24.5	9.1	0.04	9.2	0.79
.050	61.2	13.4	0.07	13.6	2.2	0.09	6.1	1.41
.060	68.3	14.4	0.07	7.5	0.1	0.37	7.3	-1.24
.080	60.4	11.3	0.07	2.2	0.9	-0.13	10.3	-0.80
.100	32.8	4.7	0.08	0.5	1.7	-0.06	7.2	-0.62
.120	7.9	0.4	0.13	0.1	1.2	-0.04	2.5	-0.74
.140	0.0	0.7	-0.02	0.0	0.4	-0.02	0.4	-1.11

The effect of the absorption on the diagonal cross section is expressed by the constant factor ξ by which the cross section without absorption must be multiplied. Using the interaction parameters D.5 and D.7, and for O^{16} the radius D.10, one finds:

$$\left.\begin{aligned} \xi &= 0.344 \text{ at } 250 \text{ Mev} \\ \xi &= 0.500 \text{ at } 900 \text{ Mev} \end{aligned}\right\} \qquad (D\cdot 9).$$

The form factor $G(p)$ for the correlated cross section without absorption was discussed in section 6. Its value using the Fermi gas model will

be denoted G_0 and its value using harmonic oscillator wave functions will be denoted G_1. The form factor G_2 obtained by including absorption in the description by means of harmonic oscillator wave functions, was discussed in section 10. All numerical computations were done for O^{16} for which the following uniform radius and Fermi momentum were used:

$$O^{16}: \quad R = 3.35 \text{ fermis}, \quad p_F = 225 \text{ Mev/c} \qquad (D.1C).$$

In the following two tables the values of the recoil momentum p (in Mev/c), the form factors $-G_0$, $-G_1$, $-G_2$, and the suppression factors $Z_0 = 1 + G_0$, $Z_1 = 1 + G_1$, and $Z_2 = \zeta + G_2$ will be presented for various values of the angle θ (in radians).

Oxygen at 250 Mev							
θ	p	$-G_0$	$-G_1$	$-G_2$	Z_0	Z_1	Z_2
0.00	40	0.867	0.938	0.299	0.133	0.062	0.045
0.05	42	0.860	0.935	0.297	0.140	0.065	0.047
0.10	46	0.848	0.923	0.291	0.152	0.077	0.053
0.15	53	0.824	0.900	0.282	0.176	0.100	0.062
0.20	61	0.797	0.868	0.270	0.203	0.132	0.074
0.25	70	0.768	0.833	0.255	0.232	0.167	0.089
0.30	79	0.739	0.793	0.239	0.261	0.207	0.105
0.40	100	0.672	0.705	0.206	0.328	0.295	0.138
0.50	120	0.609	0.615	0.174	0.391	0.385	0.170
0.60	141	0.545	0.530	0.146	0.455	0.470	0.198
0.80	183	0.424	0.388	0.106	0.576	0.612	0.238
1.00	223	0.319	0.280	0.076	0.681	0.720	0.268
Oxygen at 900 Mev							
0.00	11	0.963	0.995	0.477	0.037	0.005	0.023
0.05	46	0.848	0.918	0.437	0.152	0.082	0.063
0.10	90	0.704	0.750	0.344	0.296	0.250	0.156
0.15	135	0.563	0.558	0.249	0.437	0.442	0.251
0.20	179	0.433	0.402	0.175	0.567	0.598	0.325
0.25	223	0.319	0.280	0.120	0.681	0.720	0.380
0.30	267	0.214	0.188	0.075	0.786	0.812	0.425
0.40	356	0.060	0.052	0.017	0.940	0.948	0.483
0.50	444	0.001	0.008	0.002	0.999	0.992	0.498

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