A DIGITAL-COMPUTER-PROGRAMMED TOPOLOGICAL METHOD OF COORDINATE SELECTION FOR NUMERICAL COMPUTATIONS IN AN ELECTRICAL NETWORK

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I thank my wife, Cecilia, for her understanding attitude and for typing the first draft. If the work were to be dedicated, no persons would be more deserving than my loving parents.

ABSTRACT

In this thesis an algorithm is developed for setting up the differential equations and initial conditions of an electrical network of arbitrarily connected capacitors, resistors, inductors, multi-winding ideal transformers, and ideal voltage and current sources that topologically represents a large class of systems. The algorithm formulates the equation in a set of coordinates such that all matrices to be inverted are nonsingular. The topological description of the circuit is used to select a nonsingular set of coordinates which enables the computation of the transient responses and the short circuit admittances to a set of arbitrarily chosen ports of a network. Transformers are accounted for by appropriately selecting a set of dependent variables from the set of transformer linear equations. The algorithm for selecting a nonsingular set of coordinates, being mainly symbol manipulations, is coded in LISP. It is also shown that the same method may be applied to systems with nonlinear parameter matrices.

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CHAPTER 1

INTRODUCTION

1.1 Electrical Network as an Analogy of a Large Class of Systems

The analogy between two systems has often been used to study one system by means of the other. In the extreme, one can consider that mathematics is a system of language, consisting of a set of postulates, a set of rules and, consequently, a set of theorems, by means of which one may transform one equation into another. When mathematics is used to analyze a physical system, a set of symbols in the mathematical language is taken to represent a set of quantities in the physical system. From the observed basic relations among the physical variables, namely the physical laws, the set of mathematical symbols is correspondingly correlated. What is known as mathematical analysis becomes nothing more than setting up an analogy between the system of mathematical language and the physical system under analysis. A rich mathematical concept is concise and yet comprehensive. However, for complex systems, there is no assurance that simple mathematical models can be constructed so that the subsequent analysis can be successively carried out in analytical form; even if one succeeds in obtaining the result, the mathematical form may be so complex that information cannot be extracted without going through a long evaluation procedure, probably with the assistance of computers.

Systems, describable by partial differential equations with irregular boundary conditions, fall in this category. One may argue that this drawback is due to the inadequacy of the present mathematical languages which cannot describe complex systems in simple terms, and that when some super-mathematical language is established in the future, all these difficulties may be resolved. However, until that time, other methods are employed to obtain the solution.

Other methods of analysis, also, use the analogy between the system under study and some other system whose properties can be more readily explored. Currently there are two models being used most commonly in system analysis. They employ analog and digital computer principles. The former uses electrical quantities, namely voltages and currents, to represent variables and the latter uses the discrete states in a switching circuit. Analogies have been established between electrical networks and other systems which may be the actual physical systems or the mathematical models of the systems in the form of a set of differential equations. The former often employs the direct topological analogy (1) that gives a model consisting of electrical elements - representing the intrinsic properties of the system - interconnected in topologically the same form as the physical system variables are related. Examples are the finite difference analogies of beam (2) and plate (3), heat diffusion (4), electromagnetic wave (5), composite structures (6), and any other systems which can be approximated by a finite difference model describable by ordinary differential equations (7). The latter often uses a differential analyzer which is an interconnection of integrators, summers and constant coefficient multipliers (8).

One of the important criteria in judging the effectiveness of the model is the ease of making observations and varying parameters.

With the present art of electronic instrumentation, observation of any quantity in an electrical network can be made quickly and accurately. If the parameter to be varied is simply the coefficient in a differential equation, the differential analyzer offers a simple scheme of making parameter changes. However, if the parameter is the value of a certain element in the direct analog model, the use of the topological model will be preferred. In both cases once the model is constructed as a network of electrical elements, measurements can be made to analyze the system.

1.2 Digital Computers as Simulators

As distinct from analog computers, digital computers employ a set of coded multistate elements (mainly binary elements) to represent different states. Each state may be assigned to represent a symbol which specifically may be a number. A digital computer has a set of built-in mechanisms to operate on the symbols. As far as the programmer is concerned these mechanisms are the machine programming commands.

The task of specifying the steps is known as "programming" and the set of sequenced steps as the "algorithm"**. In all cases, one has to know the algorithm before implementing the process on a digital computer. The digital computer together with the programmed

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** The Webster New International Dictionary, 2nd Edition, defines algorism (algorithm) as follows:

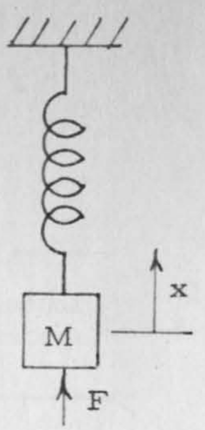
"1. The art of calculating by means of nine figures and zero; arithmetic.
2. The art of calculating with any species of notation; as the algorithm of fractions, proportion, surds, etc. Cf. Euclid's algorithm."

algorithm form a digital model that simulates the physical system. Figure 1-1 therefore shows four different ways to represent the same system. We may say that any one of the four is a model of the others. They are equivalent within the limit of interest, in the sense that if (d) is a model of (b) and (b) is a model of (a), then (d) is a model of (a). This equivalence property is often used to set up the digital model as shown by the path (1, 2) in fig. 1-2. The use of a differential analyzer takes the path (1, 3); the direct analog topological model, the path (4). Most of the systems analyzed by using the path (1, 2) in fig. 1-2 require a human being to derive the mathematical equations into the form that is acceptable to the programmed digital computer. The human being's task is mainly symbol manipulation according to a set of rules (as specified by the mathematics).

A digital computer can be programmed to do more general symbol manipulation than that defined as numerical computation. It is conceivable that we may program the symbol manipulation part of the link (1, 2) and do away with the human being who derives the equations from the system. This gives a direct path, (5), from the physical system to the digital model. In achieving this goal, there are two requisites:

** Footnote (continued)

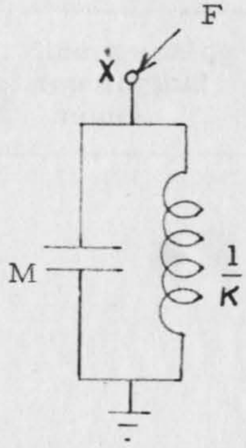
"Algorithm" is used here to denote the sequence of <u>operations</u> which when performed on the initial data will provide the <u>end solution</u>. The initial data and end solution are represented by some symbols and their association, and the operations are expressed as the transformation on the symbols and their association. The algorithm consists of the description of the initial data, the final solution and the complete sequence of steps that transform the input symbol into the solution symbol.



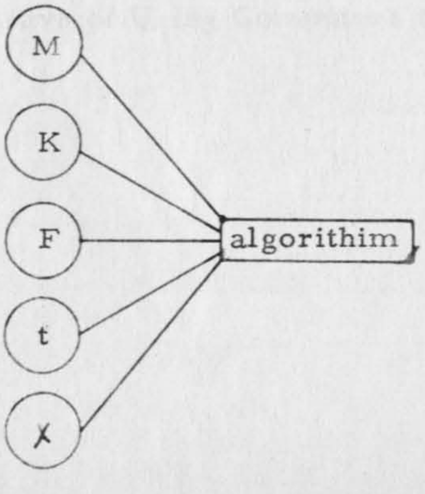
(a) the mechanical system

$$M\frac{d^2x}{dt^2} + Kx = F$$

(b) the mathematical equation

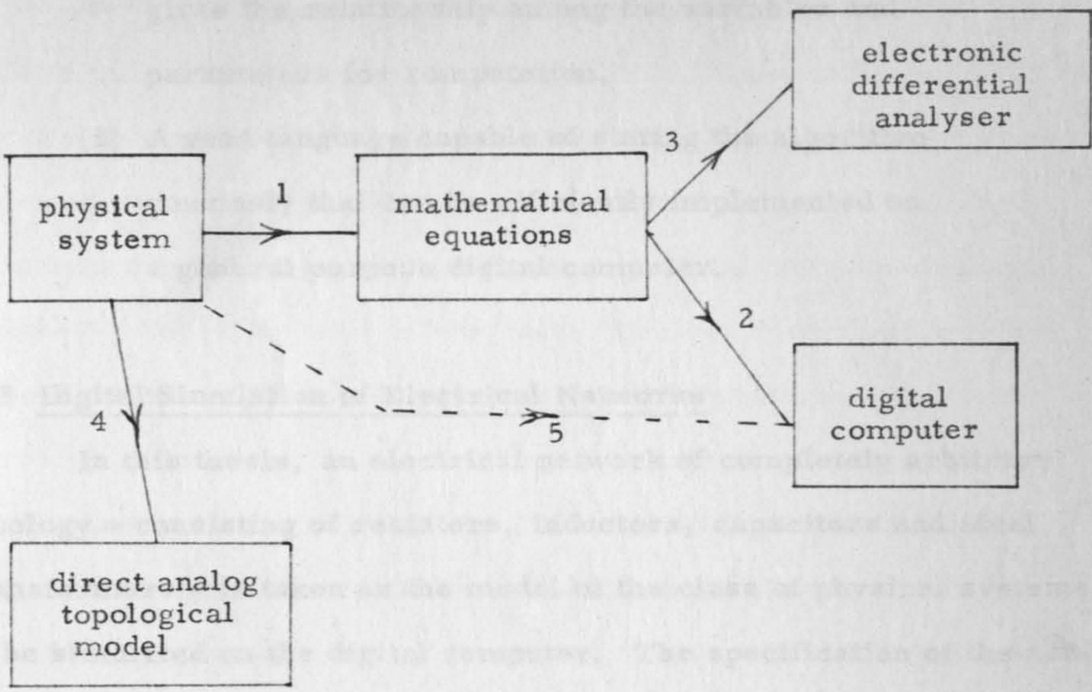


(c) the direct analog nodal analogy



(d) the digital computer programmed in terms of M, K, F, t, x.

FIGURE 1-1



Various Ways of Using Computers to Analyse a Physical System

FIGURE 1-2

(1) The algorithm that accepts the physical system description in its natural form as input data and gives the relationship among the variables and parameters for computation,

(2) A good language capable of stating the algorithm concisely that can be efficiently implemented on a general purpose digital computer.

1.3 Digital Simulation of Electrical Networks

In this thesis, an electrical network of completely arbitrary topology - consisting of resistors, inductors, capacitors and ideal transformers - is taken as the model of the class of physical systems to be simulated on the digital computer. The specification of the network consists of three parts:

(1) The passive structure

This takes the form of a list of elements giving the values of their defining parameters and connections in the network.

(2) The active components

This consists of ideal voltage and current sources across any node pair in the network.

(3) The initial conditions

These are the complete specification of the energy distribution in the network at the time from which the transient response is to be computed. In an electrical network they are simply the charges in capacitors (electrostatic energy) and the currents in inductors (magnetic energy).

With (1), (2) and (3) completely specified this is a well-defined initial value problem. An algorithm is presented to select a set of nonsingular coordinates* in terms of which equations may be systematically derived to describe the network completely. The equations can be subsequently solved on a digital computer. For transient studies numerical integration methods of various order of approximation (10) can be employed; for the determination of network functions in the complex plane, matrix manipulations are used.

In the succeeding sections of this thesis, chapter 2 gives a review of the general coordinate transformation theory and derives, specifically, the equations of linear coordinate transformation which are used to develop the materials in the following chapters. Chapter 3 describes the governing factors that dictate the choice of coordinates and the algorithm for selecting a set of nonsingular coordinates in a network of arbitrary topology consisting of R, L, C elements only. Chapter 4 discusses the inclusion of the two most general types of forcing functions, namely, the ideal voltage and current sources, and the systematic way of setting up the initial conditions (charges in capacitors and currents in inductors). Chapter 5 extends the scope of the network to include multiple winding ideal transformers. Chapter 6 considers a slightly different problem.

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* A nonsingular set of coordinates is any set of coordinates in terms of which the method of numerical computation does not encounter the situation of inverting singular matrices.

In this case, only the passive structure of the network is given, and the problem is to determine the pole-zero distributions in the complex plane of the short circuit input admittance and the short circuit transfer admittance between any two node pairs in the network. Chapter 7 describes the computer program coded in one of the currently available symbol manipulating languages, LISP. This program selects the nonsingular coordinates according to the algorithm described in the earlier chapters. Chapter 8 concludes this thesis by indicating the scope of this thesis and suggesting several related areas of research that are worth further investigations. Appendix A gives a method to evaluate the determinant of a matrix polynomial. The actual LISP program listing is given in appendix B. Appendix C gives several worked out examples.

CHAPTER 2

COORDINATE TRANSFORMATION

The linear coordinate transformation equations are reviewed in this Chapter to provide an immediate reference for the subsequent chapters. Most of the material for this is drawn from references (11), (12), (13) and (14).

2.1 Hamilton's principle

Hamilton's variation principle is equivalent to the Newtonian equations of motion and can be derived from them. Instead of describing the motion of a particle directly in terms of its acceleration, this principle describes the path in terms of a quantity whose integral along the path has a stationary value compared with other possible paths. The variation principle is of little or no assistance in solving the equations, but it does provide a convenient means of writing the equations in any desired coordinates.

Hamilton's principle states that for the motion of a mechanical system

$$\delta \int_{t'}^{t''} L(q_1, q_2 \dots q_n, \dot{q}_1, \dot{q}_2 \dots \dot{q}_n, t)dt = 0 \qquad (2\text{-}1)$$

The q's in equation 2-1 are the coordinates necessary to specify the configuration of the system completely; the $\dot{q}$'s are their first time derivatives; t is the time variable; and L is the Lagrangian function of the system as defined by

$$L = T - V \tag{2-2}$$

where T is the kinetic energy and V is the potential energy in the system.

When all the coordinates are independent the path is described by the set of differential equations, with Q_i as the generalized force,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i \tag{2-3}$$

$$i = 1, 2 \ldots n.$$

When the set of coordinates are not completely independent, there exists a set of equations of constraint. In general, the time dependent relation may be written as

$$\emptyset_j(q_1, q_2 \ldots q_n, t) = 0 \tag{2-4}$$

$$j = 1, 2 \ldots m$$

The corresponding Lagrangian equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \sum_{j=1}^{m} \lambda_j(t)\frac{\partial \emptyset_j}{\partial q_i} = Q_i \tag{2-5}$$

$$i = 1, 2 \ldots n$$

The Lagrangian multipliers, $\lambda_j(t)$, are unknown functions of time. In simple cases they are constants. From equation 2-4 and equation 2-5 the λ's may be eliminated and the equations describing the trajectory in the space of the set of independent coordinates can be derived.

2.2 Generalized coordinate elimination

Let the set of coordinates $(q_1, q_2 \dots q_n)$ be divided into two subsets

$$q^1 = q^1_1, q^1_2 \dots q^1_{n-m}$$
$$q^2 = q^2_1, q^2_2 \dots q^2_m \qquad (2\text{-}6)$$

where m is the number of constraints among the coordinates. These two subsets are such that q^2 may be expressed as a function of q^1, and equation 2-4 may be written as

$$q^2_j + F_j(q^1, t) = 0. \qquad (2\text{-}7)$$

$$j = 1, 2 \dots m$$

Substituting equation 2-7 into equation 2-5 and separating q into q^1 and q^2,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^1_k}\right) - \frac{\partial L}{\partial q^1_k} + \sum_{j=1}^{m} \lambda_j \frac{\partial F_j(q^1, t)}{\partial q^1_k} = Q^1_k \qquad (2\text{-}8)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^2_\ell}\right) - \frac{\partial L}{\partial q^2_\ell} + \lambda_\ell = Q^2_\ell \qquad (2\text{-}9)$$

where $k = 1, 2 \dots (n-m)$ and $\ell = 1, 2 \dots m$.

The unknown multiplier, λ_ℓ, from equation 2-9 may then be substituted into equation 2-8 giving

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^1_k}\right) - \frac{\partial L}{\partial q^1_k} + \sum_{j=1}^{m} \left(Q^2_j - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^2_j}\right) + \frac{\partial L}{\partial q^2_j} \; \frac{\partial F_j(q^1, t)}{\partial q^1_k}\right)$$

$$= Q^1_k \qquad (2\text{-}10)$$

A set of (n - m) differential equations in the independent coordinates $(q_1^1, q_2^1 \ldots q_{n-m}^1)$ can be derived from equation 2-10 by substituting $F_j(q^1, t)$ for q_j^2 and $\frac{d}{dt}F_j(q_t^1)$ for $\dot{q}_j^2$.

At this point the type of constraint that relates q^2 to q^1 and the function dependence of the Lagrangian on q and $\dot{q}$ can take any form. The general result in equation 2-10 will apply to a large class of systems. The next section treats specifically the transformation under time independent linear constraints.

2.3 Coordinate elimination under linear constraint

When equation 2-7 is linear and time independent, we may write it as

$$q^2 - [\,F\,]\,q^1 = 0 \tag{2-11}$$

where q^2 is a column vector of m components $(q_1^2, q_2^2 \ldots q_m^2)$, and F is a matrix of m rows and (n - m) columns that represents the linear dependence between q^2 and q^1. Substituting equation 2-11 into equation 2-10, we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k^1}\right) - \frac{\partial L}{\partial q_k^1} \sum_{j=1}^{m} \left(Q_j^2 - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j^2}\right) + \frac{\partial L}{\partial q_j^2} \quad f_{jk}\right)$$

$$= Q_k^1 \tag{2-12}$$

where f_{jk} is the j^{th} row and k^{th} column element in the matrix F.

When the elements in the system are linear, we may write

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k^1}\right) = [\, C_{11} \,]\ddot{q}^1 + [\, C_{12} \,]\ddot{q}^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j^2}\right) = [\, C_{21} \,]\ddot{q}^1 + [\, C_{22} \,]\ddot{q}^2 \tag{2-13}$$

$$- \frac{\partial L}{\partial q_k^1} = [\, L_{11} \,]q^1 + [\, L_{12} \,]q^2$$

$$- \frac{\partial L}{\partial q_j^2} = [\, L_{21} \,]q^1 + [\, L_{22} \,]q^2$$

where $[\, C_{11} \,]$, $[\, L_{11} \,]$ are $(n-m) \times (n-m)$ matrices

$[\, C_{22} \,]$, $[\, L_{22} \,]$ are $m \times m$ matrices

$[\, C_{12} \,]$, $[\, L_{12} \,]$, $[\, C_{21} \,]^T [\, L_{21} \,]^T$ are $(n-m) \times m$ matrices,

equation 2-12 may be written in matrix form as

$$[\, C_{11} \,]\ddot{q}^1 + [\, C_{12} \,]\ddot{q}^2 + [\, F \,]^T [\, C_{21} \,]\ddot{q}^1 + [\, F \,]^T [\, C_{22} \,]\ddot{q}^2$$

$$+ [\, L_{11} \,]q^1 + [\, L_{12} \,]q^2 + [\, F \,]^T [\, L_{21} \,]q^1 + [\, F^T \,][\, L_{22} \,]q^2$$

$$= Q_1 + [\, F \,]^T Q^2 \tag{2-14}$$

The following equation is obtained by eliminating q^2 in equation 2-14 by equation 2-11,

$$\left([C_{11}] + [C_{12}][F] + [F]^T[C_{21}] + [F]^T[C_{22}][F]\right)\ddot{q}^1$$
$$+\left([L_{11}] + [L_{12}][F] + [F]^T[L_{21}] + [F]^T[L_{22}][F]\right)q^1$$
$$= \quad Q^1 + [F]^T Q^2 \qquad (2\text{-}15)$$

Equation 2-15 is the system of equations when a set of constraints is imposed on the coordinates. If there were no constraint, the system of equations should be derived from

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^1}\right) - \frac{\partial L}{\partial q^1} = Q^1$$
$$(2\text{-}16)$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^2}\right) - \frac{\partial L}{\partial q^2} = Q^2$$

to give

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}\begin{bmatrix} \ddot{q}^1 \\ \ddot{q}^2 \end{bmatrix} + \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}\begin{bmatrix} q^1 \\ q^2 \end{bmatrix} = \begin{bmatrix} Q^1 \\ Q^2 \end{bmatrix}. \qquad (2\text{-}17)$$

When the constraint, equation 2-11, is written in the following manner

$$\begin{bmatrix} q^1 \\ q^2 \end{bmatrix} = \begin{bmatrix} E \\ F \end{bmatrix} q^1 = [A]\, q^1 \qquad (2\text{-}18)$$

and E is the identity matrix, equation 2-15, may be written as

$$[A]^T \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} [A]\ddot{q}^1 + [A]^T \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} [A]q^1$$

$$= [A]^T \begin{bmatrix} Q^1 \\ Q^2 \end{bmatrix} \tag{2-19}$$

The result of a linear transformation may be stated as follows:

Let the system be originally described by a set of generalized independent coordinates, q_1, generalized forces, Q_1, and the system equation

$$[Y_1]q_1 = Q_1, \tag{2-20}$$

where $[Y_1]$ is a linear differential operator. When constraints are applied to q_1, such that the resulting system is specified by another set of independent coordinates, q_2, then from the equations of constraint,

$$q_1 = [A]q_2, \tag{2-21}$$

the equation of the constrainted system in q_2 coordinates is

$$[Y_2]q_2 = Q_2 \tag{2-22}$$

where

$$[Y_2] = [A]^T[Y_1][A] \tag{2-23}$$

and

$$Q_2 = [A]^T Q_1 \tag{2-24}$$

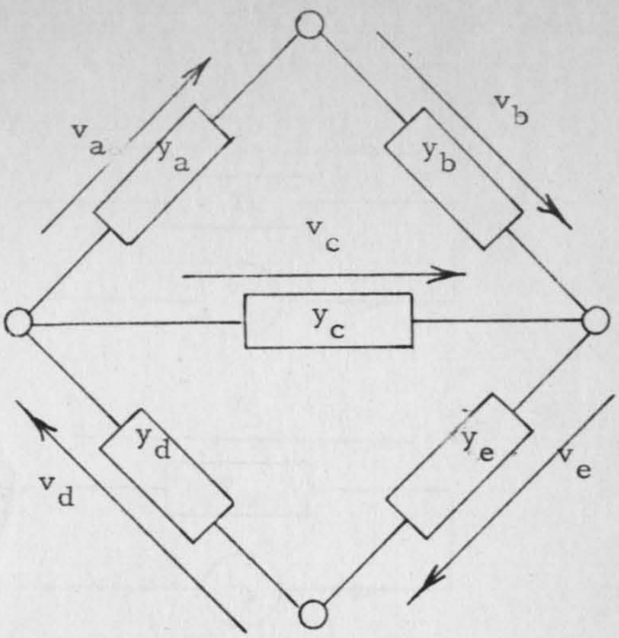
$[A]$ may be a nonsingular matrix, in which case the equations of constraint merely specify a set of coordinate transformations. An independent set of coordinates has the minimum number of coordinates which can completely describe the state of the system.

2.4 Coordinate transformation in electrical networks

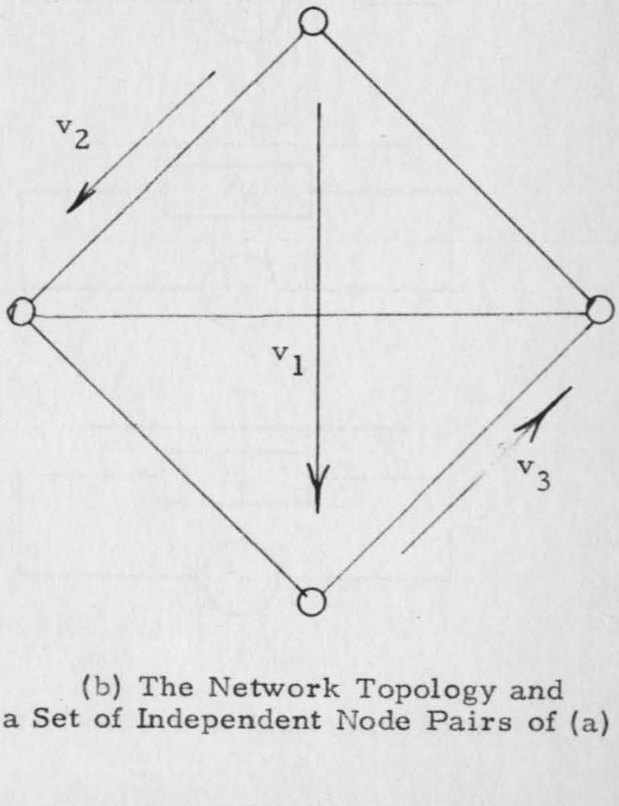
Using the "definitions of terms in network topology" as published in the IRE proceeding, January, 1951, (15), a "network" is a combination of "elements". An "element" is any electrical device (such as inductor, resistor, capacitor, generator, line, electron tube) with terminals at which it may be directly connected to other electrical devices. Topologically, a network consists of a cluster of O-dimension members, namely, the nodes and a collection of one-dimensional members, namely, the branches. (Fig. 2-1-a) is an example of a network whose topology is shown in (fig. 2-1-b). The branches of a network form the original set of coordinates, in terms of which the Lagrangian may be formulated and the system equation in the absence of other constraints may be written as

$$[Y_B]v_B = i_B \tag{2-25}$$

where the subscript B denotes branch quantities. This system of disconnected branches forms the primitive network (11), (12), from which all other networks using the same branches may be constructed. The primitive network for (fig. 2-1-a) is shown in (fig. 2-2) whose system equation is

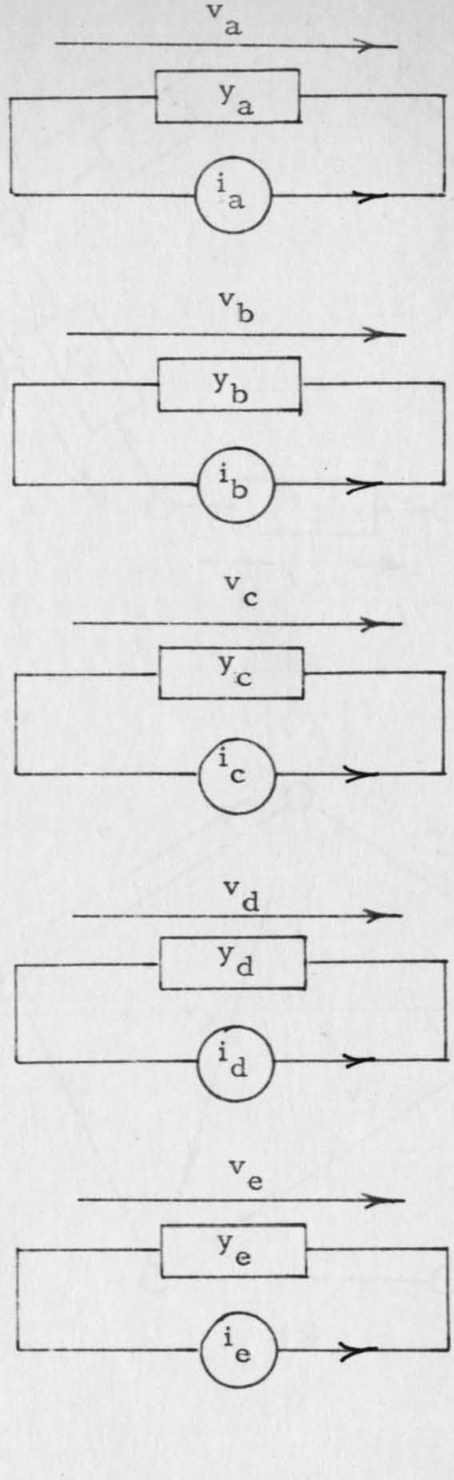


(a) A Network with Five Branches



(b) The Network Topology and a Set of Independent Node Pairs of (a)

FIGURE 2-1



The Primitive Network of Individual Branches

FIGURE 2-2

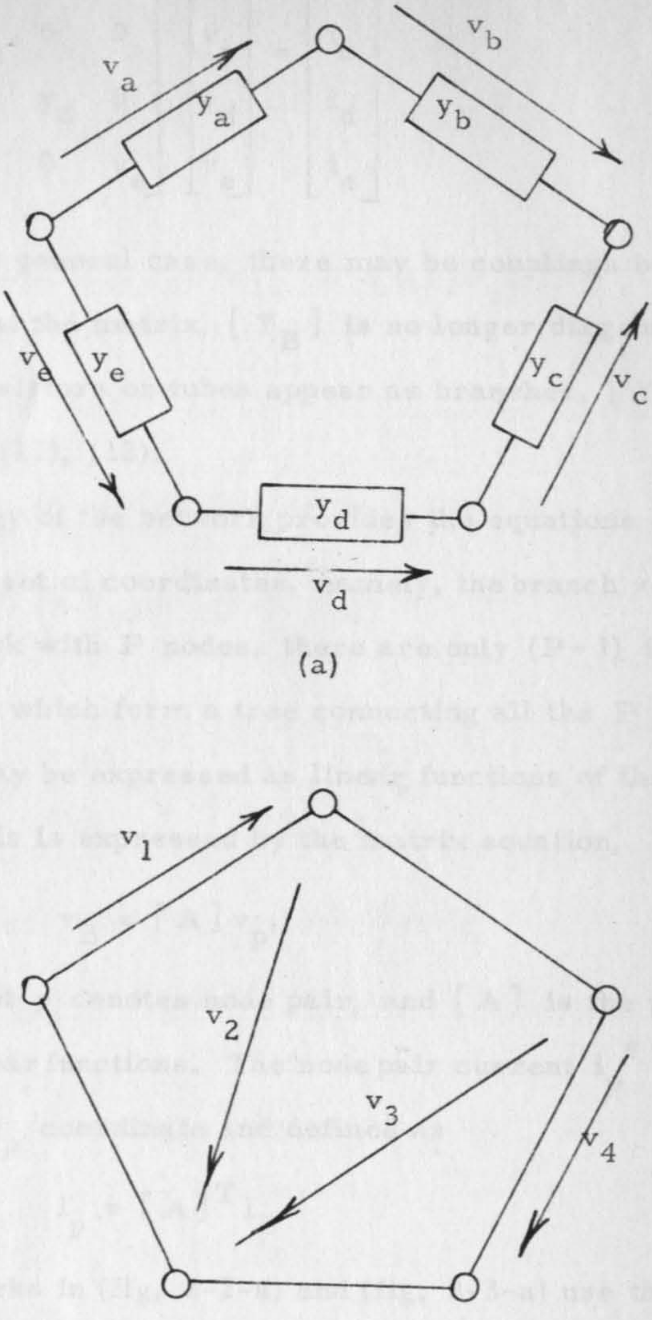


FIGURE 2-3

$$\begin{bmatrix} y_a & 0 & 0 & 0 & 0 \\ 0 & y_b & 0 & 0 & 0 \\ 0 & 0 & y_c & 0 & 0 \\ 0 & 0 & 0 & y_d & 0 \\ 0 & 0 & 0 & 0 & y_e \end{bmatrix} \begin{bmatrix} v_a \\ v_b \\ v_c \\ v_d \\ v_e \end{bmatrix} = \begin{bmatrix} i_a \\ i_b \\ i_c \\ i_d \\ i_e \end{bmatrix} \tag{2-26}$$

In the most general case, there may be couplings between the branches, such that the matrix, $[\, Y_B \,]$ is no longer diagonal. When elements like transistors or tubes appear as branches, $[\, Y_B \,]$ is not even symmetrical (11), (12).

The topology of the network provides the equations of constraint among the original set of coordinates, namely, the branch voltages. For a connected network with P nodes, there are only (P - 1) independent node-pair voltages which form a tree connecting all the P nodes. All branch voltages may be expressed as linear functions of the (P - 1) node pair voltages. This is expressed by the matrix equation,

$$v_B = [\, A \,]\, v_p, \tag{2-27}$$

where the subscript p denotes node pair, and $[\, A \,]$ is the matrix that represents the linear functions. The node pair current i_p* is the generalized current in v_p coordinate and defined as

$$i_p = [\, A \,]^T i_B \tag{2-28}$$

Both networks in (fig. 2-1-a) and (fig. 2-3-a) use the same branches. They are only different in the topology as shown in

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*This corresponds to the generalized force in the original Hamilton formulation.

(fig. 2-3-b). For the network in (fig. 2-1) we may choose any three node pairs that form a tree spanning the four nodes. Using the selection in (fig. 2-1-b), we have

$$\begin{bmatrix} v_a \\ v_b \\ v_c \\ v_d \\ v_e \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = [A_1]\, v_{p1} \tag{2-29}$$

The equation in the node pair coordinates is given by

$$[Y_{p1}]\, v_{p1} = i_{p1} \tag{2-30}$$

where

$$\begin{aligned} [Y_{p1}] &= [A_1]^T [Y_B][A_1] \\ i_{p1} &= [A_1]^T i_B \end{aligned} \tag{2-31}$$

Similarly for the network in (fig. 2-3) we have

$$\begin{bmatrix} v_a \\ v_b \\ v_c \\ v_d \\ v_e \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = [A_2]\, v_{p2} \tag{2-32}$$

then

$$[Y_{p2}]\, v_{p2} = i_{p2} \tag{2-33}$$

and

$$[Y_{p2}] = [A_2]^T [Y_B][A_2]$$

$$i_{p2} = [A_2]^T i_B$$

2.5 Tensorial concept of electrical network

It can be seen from equation 2-21, 2-22, 2-23 and 2-24, that q, the generalized coordinates, obey the transformation rule of a covariant vector (or tensor of the first rank), and Q, the generalized forces, obey the transformation rule of a contravariant vector (16). The quantity $[Y]$ which we have called a matrix, transforms like a contravariant second rank tensor. Having established the tensorial concept of a network any transformation other than linear ones can be handled automatically by using the rule of tensor transformation (11). The same result may be obtained from equation 2-10 in which the term $\frac{\partial F_j}{\partial q_k^1}$ will eventually lead to the tensor transformation. The concept of representing a stationary network by tensors does not help to solve the network equations; however, the concept offers a unified approach to a much larger class of system not specified by stationary linear transformations. Kron (17) initiated the idea and applied it to the analysis of electrical machinery. It is conceivable that the same approach may be used in the study of magneto-hydrodynamics.

CHAPTER III

COORDINATE SELECTION IN RLC NETWORKS

Definitions for some of the terms used form the first section of this chapter. They are followed by some topological theorems pertinent to the remaining discussions. Then the necessity of selecting a nonsingular set of node pair coordinates is pointed out, followed by a discussion of the algorithm that selects the nonsingular set of coordinates in a completely passive network with no transformers.

3.1 Definitions

Node: A terminal of any branch of a network or a terminal common to two or more branches of a network.

Branch: A portion of a network consisting of one or more two-terminal elements in parallel that have the same terminal nodes.

Element: Any electrical device. An active element can be either an ideal voltage source or an ideal current source. A passive element can be a resistor, capacitor, inductor or a winding belonging to an ideal transformer.

Network: A combination of elements.

Loop (mesh): A set of elements forming a closed path in a network, provided that if any one element is omitted from the set, the remaining elements of the set do not form a closed path.

Node pair:	A pair of nodes whose voltage difference is used to describe the state of the network.
Terminal pair:	An associated pair of accessible terminals, such as input pair, output pair and the like.
Tree:	A set of connected branches including no meshes.
Connected:	A network is connected if there exists at least one path, composed of branches of the network, between every pair of nodes of the network.
Separated:	Two networks are separated if they are not connected.
Resistive elements:	The passive elements whose currents are proportional to the voltages across them. They have the dimension "ohms" as impedances and "mhos" as admittances.
Capacitive elements:	The passive elements whose currents are proportional to the first time derivative of the voltages across them. They have the dimension "Farad" as admittances and $(\text{Farad})^{-1}$ as impedances.
Inductive elements:	The passive elements whose terminal voltages are proportional to the first time derivative of the currents in them. They have the dimension "Henry" as impedances and $(\text{Henry})^{-1}$ as admittances.
Voltage sources:	The voltages across the voltage sources are independent of the currents in them.

Current sources: The currents from current sources are independent of the voltages across their terminals.

Ideal transformer: An electrical device with several two terminal windings. Each two terminal winding W_i is characterized by a relative number of turns n_i. The current i_i in the i^{th} winding must satisfy the condition,

$$\sum n_i i_i = 0 .$$

Between any two windings, the voltages across the winding terminals, v_i and v_j, must satisfy the condition,

$$n_i v_j = n_j v_i .$$

3.2 Some Topological Theorems in Networks

Theorem 1

(i) At least $(P - 1)$ branches are required to connect P nodes, and (ii) any more than $P - 1$ branches connected among P nodes form at least one loop.

Corollary 1

When there are P nodes forming D separated networks, then the minimum number of branches among the P nodes is $(P - D)$.

Corollary 2

In a network of D separated parts, there are D sub-trees that connect the nodes within each connected group of nodes. If a tree is constructed to connect the D parts together, then the resulting connected network is still a tree.

Corollary 3

In a connected network of P nodes and B branches, there are $(B - P + 1)$ loops (meshes).

Theorem 2

Two trees, each of $(P - 1)$ branches, connecting the same P nodes are different if at least one of their branches is different. Then the number of different trees one can form among the P nodes, $S(P)$, is given by the expression,

$$S(P) = T_P(1) = \sum_{i=1}^{P-1} \binom{P-1}{i} T_{P-1}(i)$$

where $\binom{a}{b}$ is the coefficient of x^b in the binomial expression of $(1+x)^a$, and $T_{P-1}(i)$ is recursively defined as

$$T_m(n) = \sum_{j=1}^{m-n} \binom{m-n}{j} \frac{(n+j-1)!}{j!(n-1)!} T_{m-n}(j) \qquad \text{for } m > n$$

and $T_m(n) = 1$ for $m = n$.

Proof:

An arbitrary node is taken as the reference, then $\binom{P-1}{i} T_{P-1}(i)$ is the number of different trees that have i of the remaining $P - 1$ nodes connected to the reference node. The total number of different trees is then given by

$$S(P) = \sum_{i=1}^{P-1} \binom{P-1}{i} T_{P-1}(i). \qquad (3\text{-}1)$$

$T_m(n)$ in equation 3-1 is defined as the number of different trees that can be constructed among $m - n$ distinct nodes and a reference datum of indistinguishable n nodes. The j branches that connect to the datum can be distributed indistinguishably among the n nodes in $\frac{(n+j-1)!}{j!(n-1)!}$ ways. Therefore, $T_m(n)$ can be recursively defined as

$$T_m(n) = \sum_{j=1}^{m-n} \binom{m-n}{j} \frac{(n+j-1)!}{j!(n-1)!} T_{m-n}(j) \qquad (3\text{-}2)$$

for $m > n$

and

$$T_m(n) = 1. \qquad \text{for } m = n \qquad (3\text{-}3)$$

The recursive function defined in equation 3-2 and 3-3 always converges for $m \geq n$. From the very definition of $T_m(n)$, $S(P)$ can be defined as

$$S(P) = T_p(1). \tag{3-4}$$

Here are a few evaluated values:

$$S(2) = T_2(1) = T_1(1) = 1$$

$$S(3) = T_3(1) = \binom{2}{1}T_2(1) + \binom{2}{2}T_2(2) = 3$$

$$S(4) = T_4(1) = \binom{3}{1}T_3(1) + \binom{3}{2}T_3(2) + \binom{3}{3}T_3(3)$$

$$= 3 \cdot 3 + 3(2 \cdot T_1(1)) + 1 = 16$$

(Fig. 3-1) gives the sixteen different trees that one can construct to connect 4 nodes. They are divided into the subsets $T_3(3)$, $T_3(2)$ and $T_3(1)$, taking node 4 as the reference. The next two values are

$$S(5) = 119$$

$$S(6) = 1136.$$

The growth factor is approximately given by

$$S(P) \simeq (P-1)!\ 2^P \tag{3-5}$$

which is a very fast growing number.

Theorem 3

The transformation matrix, [A], that transforms a vector, V_i, whose (P - 1) elements are the voltages across the tree

The Sixteen Different Trees that Connect
Four Nodes
Constructed from subsets $T_3(3)$, $T_3(2)$, $T_3(1)$

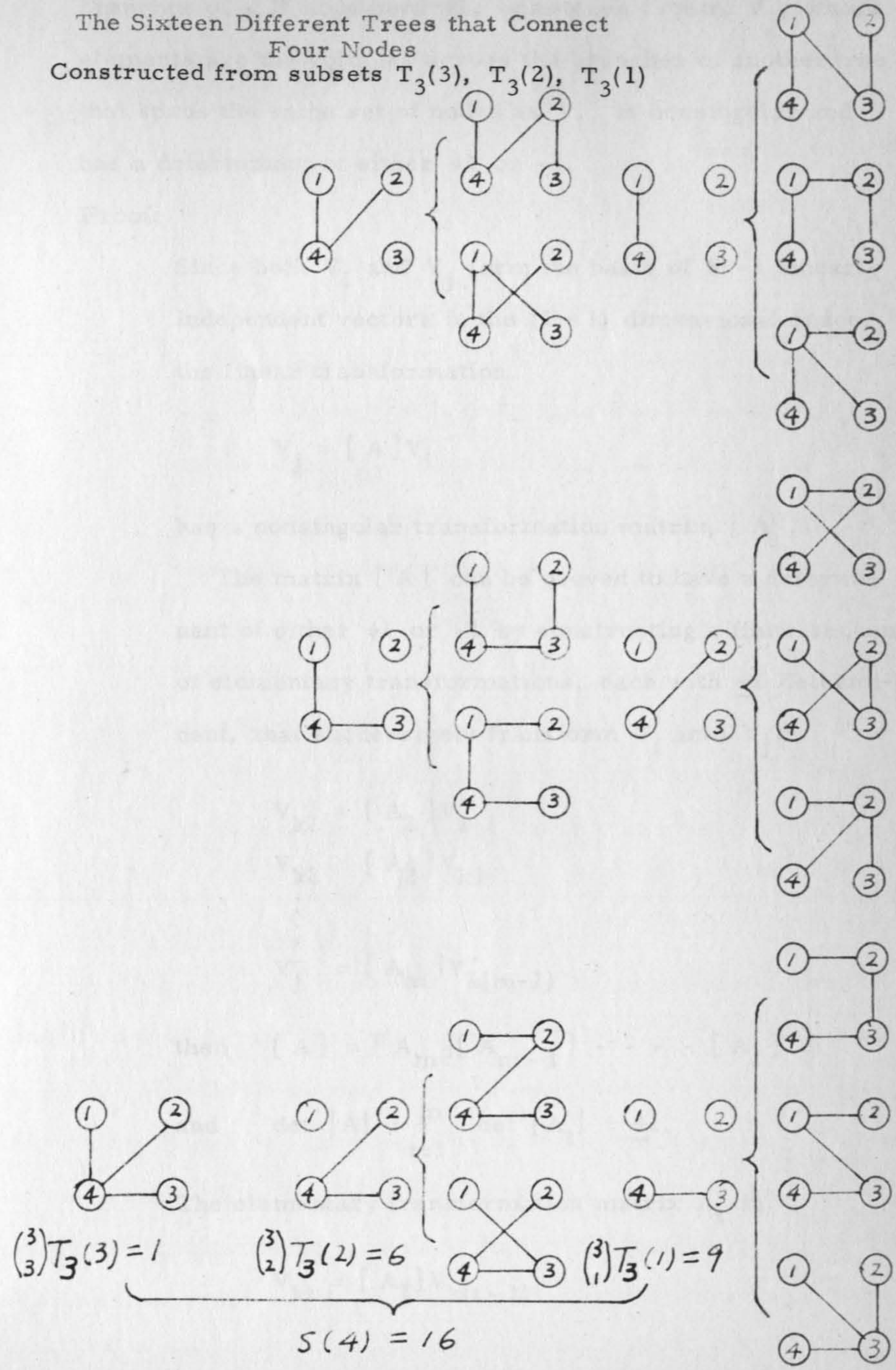


FIGURE 3-1

branches of a P node network, to another vector, V_j, whose elements are the voltages across the branches of another tree that spans the same set of nodes as V_i, is nonsingular and has a determinant of either +1 or -1.

Proof:

Since both V_i and V_j form the basis of P -1 linearly independent vectors in the (P - 1) dimensional space, the linear transformation

$$V_j = [A] V_i$$

has a nonsingular transformation matrix, [A].

The matrix [A] can be proved to have a determinant of either +1 or -1 by constructing a finite sequence of elementary transformations, each with ± 1 determinant, that successively transform V_i into V_j.

$$\begin{aligned} V_{k1} &= [A_1] V_i \\ V_{k2} &= [A_2] V_{k1} \\ &\vdots \\ V_j &= [A_m] V_{k(m-1)} \end{aligned}$$

then $[A] = [A_m][A_{m-1}] - - - - [A_1]$

and $\det |A| = \prod_{t=1}^{m} \det |A_t| = \pm 1.$

The elementary transformation matrix A_t in

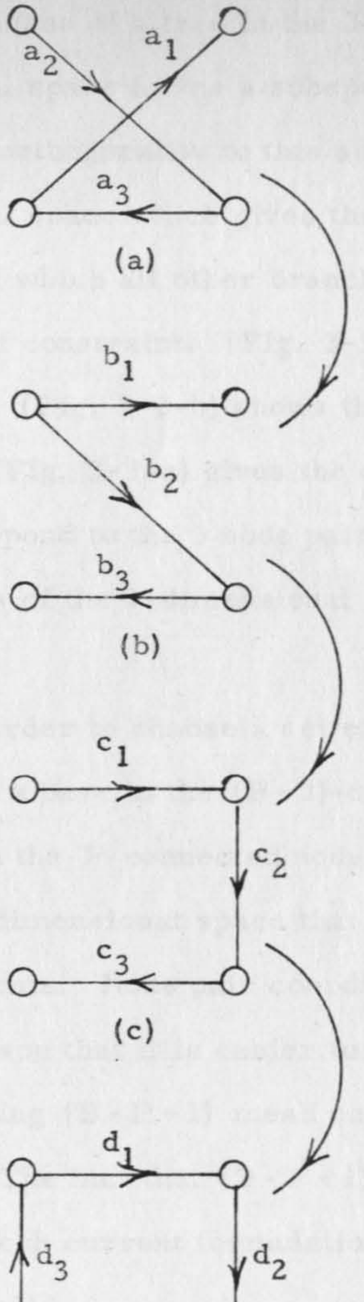
$$V_{kt} = [A_t] V_{k(t-1)}$$

is such that V_{kt} and $V_{k(t-1)}$ have only one different branch in their corresponding trees. The differing branch in the V_{kt} tree is a branch in the tree of V_j and the differing branch in the $V_{k(t-1)}$ tree is not. Since there are at most $P - 1$ different branches between any two trees that span the same set of P nodes, the sequence of transformations, $[A_t]$, has a finite length of at most $P - 1$. Each elementary transformation matrix, $[A_t]$, will have $P - 2$ rows with ± 1 on the diagonal and zero off diagonal terms, and a single row with some ± 1 off diagonal terms in addition to the ± 1 diagonal term. Such a matrix has a determinant of ± 1, hence the matrix $[A]$, which is the product of these elementary matrices, has a determinant of ± 1.

(Fig. 3-2) shows the successive transformation from the tree in (a) into the tree in (d).

3.3 Network Solution in Node Pair Coordinates

In Chapter 2, two networks, with the same branches, but connected into different topologies, are considered as the same object subject to different constraints on their independent coordinates, namely, the branch voltages or currents. This object (network) with B branches may be considered to span a B-dimensional space. Upon constraint, the object is restricted in such a way that fewer than B vectors in the B-dimensional space can define the object uniquely. In a connected network of P nodes, (P - 1) node pair voltages form a (P - 1)-dimensional



$$\begin{array}{c|ccc} & a_1 & a_2 & a_3 \\ \hline b_1 & 1 & 1 & 1 \\ b_2 & 0 & 1 & 0 \\ b_3 & 0 & 0 & 1 \end{array}$$

$$\begin{array}{c|ccc} & b_1 & b_2 & b_3 \\ \hline c_1 & 1 & 0 & 0 \\ c_2 & -1 & 1 & 0 \\ c_3 & 0 & 0 & 1 \end{array}$$

$$\begin{array}{c|ccc} & c_1 & c_2 & c_3 \\ \hline d_1 & 1 & 0 & 0 \\ d_2 & 0 & 1 & 0 \\ d_3 & -1 & -1 & -1 \end{array}$$

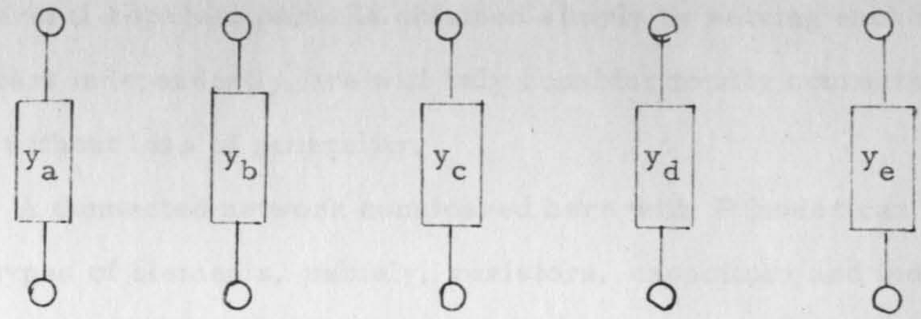
Successive Elementary Transformations from One Tree to Another

FIGURE 3-2

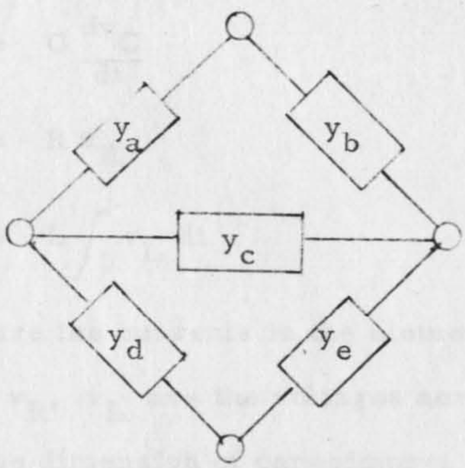
space that defines the network. If these (P - 1) node pairs are taken as the branches of a tree in the P connected nodes, this (P - 1)-dimensional space forms a subspace of the original B-dimensional space. In orthogonality to this subspace, there is a (B - P + 1)-dimensional space which gives the set of independent branch currents, in terms of which all other branch currents can be computed from the condition of constraint. (Fig. 3-3-a) shows the five branches that form the object. (Fig. 3-3-b) shows the object (network) under a set of constraints. (Fig. 3-3-c) gives the components of the 3-dimensional space that correspond to the 3 node pair voltages, and (fig. 3-3-d) gives the components of the 2-dimensional space that correspond to the 2 mesh currents.

In order to choose a set of independent coordinates, one may either pick a base in the (P - 1)-dimensional space that corresponds to a tree in the P connected nodes, or one may define a base in the (B - P + 1)-dimensional space that corresponds to a set of independent mesh currents. Node pair coordinates are used in the present work for the reason that it is easier to detect (P - 1) independent node pairs than selecting (B - P + 1) mesh currents in a network with arbitrary topology. The fact that (B - P + 1) may be less than (P - 1), in which case the mesh current formulation has fewer variables, is not considered at all.

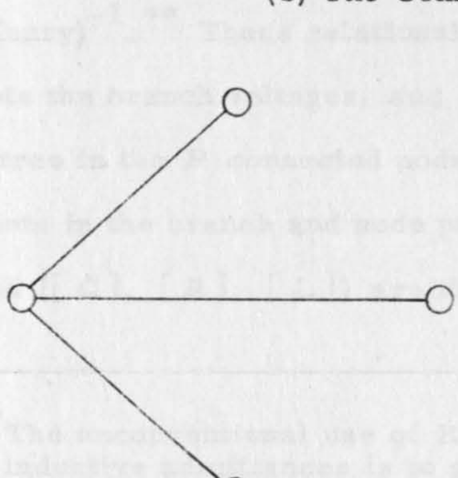
Networks consisting of purely passive elements without ideal transformer are considered in this chapter. In the subsequent chapters, active elements and transformers are included by extending the results from the present simplified model. Since the solution of the network

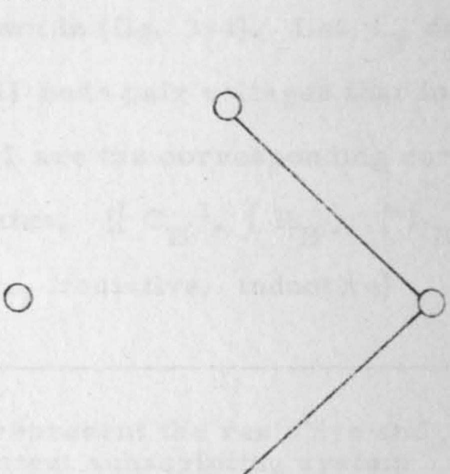


(a) The Individual Branches



(b) The Connected Network





(c) A Set of Independent Node-Pairs Voltages

(d) A Set of Independent Branch Currents

FIGURE 3-3

with several separate parts is obtained simply by solving each separated part independently, we will only consider totally connected networks without loss of generality.

A connected network considered here with P nodes can have three types of elements, namely, resistors, capacitors and inductors. Each branch may consist of any parallel combination of the three types of elements. The basic physical laws describing the three types of element are

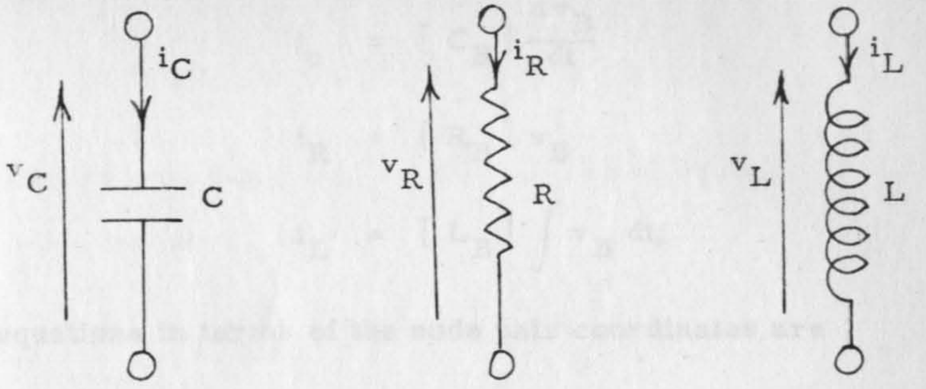
$$i_c = C \frac{dv_C}{dt} \qquad (3\text{-}6)$$

$$i_R = R\, v_R \qquad (3\text{-}7)$$

$$i_L = L \int v_L \, dt \qquad (3\text{-}8)$$

where i_c, i_R, i_L are the currents in the elements: capacitor, resistor, inductor; v_C, v_R, v_L are the voltages across the corresponding elements. C has the dimension of capacitance, namely, Farad; R is the resistive admittance in Mho; and L is the inductive admittance in $(\text{Henry})^{-1}$.** These relationships are shown in (fig. 3-4). Let v_B denote the branch voltages, and V the (P - 1) node pair voltages that form a tree in the P connected nodes; i_B and I are the corresponding currents in the branch and node pair coordinates. ($[C_B]$, $[R_B]$, $[L_B]$) and ($[C]$, $[R]$, $[L]$) are the (capacitive, resistive, inductive)

** The unconventional use of R and L to represent the resistive and inductive admittances is to give a consistent subscripting system such that the C, R, L subscripts denote the quantities in capacitive, resistive and inductive elements. Secondly, although G is often used to denote conductance, no universally accepted symbol denotes the inductive admittance.



$i_C = C\frac{dv_C}{dt}$ $\quad$ $i_R = R\,v_R$ $\quad$ $i_L = L\int v_L dt$

Capacitive, Resistive and Inductive Elements with C, R, L as their Respective Admittances

FIGURE 3-4

matrices - or tensors, if Kron's terminology is used, in the branch and node pair coordinates. The equations in terms of branch coordinates are

$$i_c = [C_B] \frac{dv_B}{dt} \tag{3-9}$$

$$i_R = [R_B] v_B \tag{3-10}$$

$$i_L = [L_B] \int v_B \, dt; \tag{3-11}$$

the equations in terms of the node pair coordinates are

$$I_c = [C] \frac{dV}{dt} \tag{3-12}$$

$$I_R = [R] V \tag{3-13}$$

$$I_L = [L] \int V \, dt. \tag{3-14}$$

In the absence of active elements, the resulting current in the generalized node pair coordinate must be zero,

$$I_c + I_R + I_L = 0. \tag{3-15}$$

If v_B is related to V by a transformation matrix, [A]

$$v_B = [A] V \tag{3-16}$$

then from equation 2-23 and equation 2-24 we have

$$I_c = [A]^T i_c \tag{3-17}$$

$$I_R = [A]^T i_R \tag{3-18}$$

$$I_L = [A]^T i_L \tag{3-19}$$

and

$$[\,C\,] = [\,A\,]^T[\,C_B\,][\,A\,] \tag{3-20}$$

$$[\,R\,] = [\,A\,]^T[\,R_B\,][\,A\,] \tag{3-21}$$

$$[\,L\,] = [\,A\,]^T[\,L_B\,][\,A\,] \tag{3-22}$$

Substituting equation 3-12, equation 3-13 and equation 3-14 into equation 3-15, we have

$$[\,C\,]\frac{dV}{dt} + [\,R\,]\,V + [\,L\,]\int V\,dt = 0. \tag{3-23}$$

Equation 3-23 is the second order matrix differential equation one has to solve. When solving equation 3-23 on a digital computer, the method of numerical integration (10) converts it into the canonical form as shown in equation 3-25 where y is defined as

$$y = \int V\,dt \tag{3-24}$$

$$\left.\begin{aligned} \frac{dV}{dt} &= [\,C\,]^{-1}(-[\,R\,]\,V - [\,L\,]\,y) \\ \frac{dy}{dt} &= V. \end{aligned}\right. \tag{3-25}$$

The method of numerical integration works provided that the right hand side of equations 3-25 are evaluable. Since the evaluation involves the inversion of $[\,C\,]$, which may be singular, equations 3-25 cannot be applied directly. The rank of $[\,C\,]$ is an invariant property of the network, depending only on the topology of the capacitor connections. However, if one can select the coordinates in such a way that $[\,C\,]$ is in the form

$$[\,C\,] = \begin{bmatrix} C_{11} & 0 \\ 0 & 0 \end{bmatrix} \tag{3-26}$$

where $[\,C_{11}\,]$ is a submatrix of $[\,C\,]$ and has a dimension equal to the rank of $[\,C\,]$, then we may write equation 3-23 in the partitioned coordinates (page 48 of reference (18)),

$$V = \begin{bmatrix} V^1 \\ V^x \end{bmatrix} \qquad y = \begin{bmatrix} y^1 \\ y^x \end{bmatrix} \tag{3-27}$$

$$\begin{bmatrix} C_{11} & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} V^1 \\ V^x \end{bmatrix} + \begin{bmatrix} R_{11} & R_{1x} \\ R_{x1} & R_{xx} \end{bmatrix} \begin{bmatrix} V^1 \\ V^x \end{bmatrix} + \begin{bmatrix} L_{11} & L_{1x} \\ L_{x1} & L_{xx} \end{bmatrix} \begin{bmatrix} y^1 \\ y^x \end{bmatrix} = 0 \tag{3-28}$$

The canonical form for numerical integration becomes

$$\frac{dV^1}{dt} = [\,C_{11}\,]^{-1}\Big(- [\,R_{11}\,]\,V^1 - [\,R_{1x}\,]\,V^x - [\,L_{11}\,]\,y^1 - [\,L_{1x}\,]\,y^x \Big)$$

$$\frac{dy^1}{dt} = V^1 \tag{3-29}$$

$$\frac{dy^x}{dt} = V^2 = [\,R_{xx}\,]^{-1}\Big(- [\,R_{x1}\,]\,V^1 - [\,L_{x1}\,]\,y^1 - [\,L_{xx}\,]\,y^x \Big)$$

By the choice of $[\,C\,]$ in equation 3-26, $[\,C_{11}\,]^{-1}$ always exists. However, if $[\,R_{xx}\,]^{-1}$ does not exist, equations 3-29 are still not completely evaluable. Once the coordinate V^1 is chosen to give a nonsingular $[\,C_{11}\,]$, the rank of $[\,R_{xx}\,]$ is invariant to the choice of V^x. Therefore, it is necessary to choose V^x in such a way that V^x may be partitioned into the form

$$V^x = \begin{bmatrix} V^2 \\ V^3 \end{bmatrix}, \tag{3-30}$$

and [R] partitioned into

$$[R] = \begin{bmatrix} R_{11} & R_{12} & 0 \\ R_{21} & R_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3-31}$$

where $[R_{22}]^{-1}$ always exists. Equation 3-28 is then developed into the form

$$\begin{bmatrix} C_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} V^1 \\ V^2 \\ V^3 \end{bmatrix} + \begin{bmatrix} R_{11} & R_{12} & 0 \\ R_{21} & R_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V^1 \\ V^2 \\ V^3 \end{bmatrix} + \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} y^1 \\ y^2 \\ y^3 \end{bmatrix} = 0 \tag{3-32}$$

and equations 3-29 become

$$\begin{aligned} \frac{dV^1}{dt} &= [C_{11}]^{-1}\left(-[R_{11}]V^1 - [R_{12}]V^2 - [L_{11}]y^1 - [L_{12}]y^2 - [L_{13}]y^3\right) \\ \frac{dy^1}{dt} &= V^1 \\ \frac{dy^2}{dt} &= V^2 = [R_{22}]^{-1}\left(-[R_{21}]V^1 - [L_{21}]y^1 - [L_{22}]y^2 - [L_{23}]y^3\right) \\ y^3 &= [L_{33}]^{-1}\left(-[L_{31}]y^1 - [L_{32}]y^2\right) \end{aligned} \tag{3-33}$$

Since the network is connected, $[L_{33}]^{-1}$ must exist and equations 3-33 are completely evaluable. Therefore, from the initial conditions which will be discussed in the next chapter, the state of the network at all subsequent times may be computed.

In setting up a digital model of any arbitrary network by using numerical integration, the important factor is the choice of coordinates in such a way that $[C_{11}]^{-1}$, $[R_{22}]^{-1}$ and $[L_{33}]^{-1}$ exist. Any set of coordinates that satisfies the above condition is called a "nonsingular set of coordinates". The next section will discuss two different methods of selecting a nonsingular set of coordinates; one uses matrix operations and the other uses topological properties of the network. It will be shown that the former method is not always applicable when there are excessive round off errors during matrix operations.

3.4 Two Methods of Deriving a Nonsingular Set of Coordinates

There are two general methods of deriving a nonsingular set of coordinates. One assumes a base set of (P - 1) node-pair coordinates, and by matrix algebra such as the congruent transformation (page 89 of reference (19) on the $[C]$, $[R]$, $[L]$ matrices in sequence, the base set of coordinates which may be singular, is transformed into a set of nonsingular coordinates. The other method which is developed in this thesis takes the circuit topology as a starting point, and selects V^1, V^2, V^3 in sequence. When one class of coordinates is selected the network is reduced so that the selection of the next class of coordinates is from a simpler network.

(a) The transformation method:

Let V_o be the initial base set of coordinates and $[C_o]$, $[R_o]$, $[L_o]$ be the capacitive, resistive and inductive matrices in the base coordinates. Transformation is first applied to $[C_o]$ which is a symmetrical matrix of unknown rank. The transformation reduced $[C_o]$ to a matrix of the form (page 89 of reference (19)),

$$[\, C_1 \,] = [\, P^1 \,]^T [\, C_o \,][\, P^1 \,] = \begin{bmatrix} C_{11} & 0 \\ 0 & 0 \end{bmatrix} \tag{3-34}$$

where $[\, C_{11} \,]$ is a diagonal matrix.

After $[\, P^1 \,]$ is obtained to give equation 3-34, the same transformation is applied to $[\, R_o \,]$ and $[\, L_o \,]$ giving

$$[\, R_1 \,] = [\, P^1 \,]^T [\, R_o \,][\, P^1 \,] = \begin{bmatrix} R_{11} & R_{1x} \\ R_{x1} & R_{xx} \end{bmatrix} \tag{3-35}$$

$$[\, L_1 \,] = [\, P^1 \,]^T [\, L_o \,][\, P^1 \,] = \begin{bmatrix} L_{11} & L_{1x} \\ L_{x1} & L_{xx} \end{bmatrix} . \tag{3-36}$$

Then $[\, R_{xx} \,]$ of unknown rank is subject to the same treatment as $[\, C_o \,]$, giving the transformation matrix $[\, P^{2'} \,]$ such that

$$[\, P^{2'} \,]^T [\, R_{xx} \,][\, P^{2'} \,] = \begin{bmatrix} R_{22} & 0 \\ 0 & 0 \end{bmatrix} . \tag{3-37}$$

Now we define

$$[\, P^2 \,] = \begin{bmatrix} E_1 & 0 \\ 0 & P^{2'} \end{bmatrix} \tag{3-38}$$

where $[\, E_1 \,]$ is the identity matrix of the same dimension as $[\, C_{11} \,]$, and compute

$$[\, C_2 \,] = [\, P^2 \,]^T [\, P^1 \,]^T [\, C_o \,][\, P^1 \,][\, P^2 \,]$$

$$= \begin{bmatrix} C_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3-39}$$

$$[R_2] = [P^2]^T [P^1]^T [R_o][P^1][P^2]$$

$$= \begin{bmatrix} R_{11} & R_{12} & 0 \\ R_{21} & R_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3-40}$$

$$[L_2] = [P^2]^T [P^1]^T [L_o][P^1][P^2]$$

$$= \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \tag{3-41}$$

where $[C_{11}]$ and $[R_{22}]$ are diagonal, therefore nonsingular, and $[L_{33}]$ is nonsingular since the original base V_o is a set of independent coordinates. The new set of nonsingular coordinates is given by

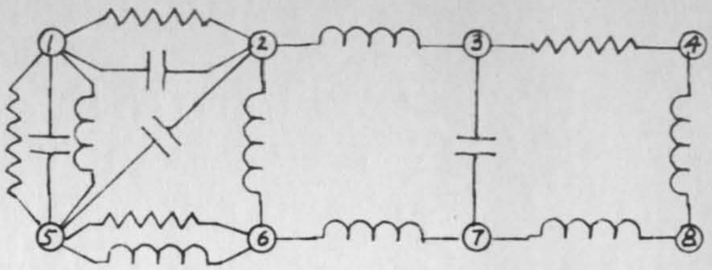
$$\begin{bmatrix} V^1 \\ V^2 \\ V^3 \end{bmatrix} = [P^1][P^2]^{-1}(V_o) \tag{3-42}$$

This process of deriving a set of nonsingular coordinates by using congruent transformation is practicable if no appreciable round-off error is developed in the arithmetic operations which may change the rank of the matrices. The altering of the rank may change a coordinate originally in class V^2 or V^3 into a component of V^1 or V^2, in which cases, although the final matrices $[C_{11}]$, $[R_{22}]$ and $[L_{33}]$ are nonsingular, they will introduce large round off and truncation errors in subsequent computations, since they are ill-conditioned. Such difficulties may be resolved by pre-determining the ranks and introducing special control in the arithmetic computations which will complicate the algorithm considerably.

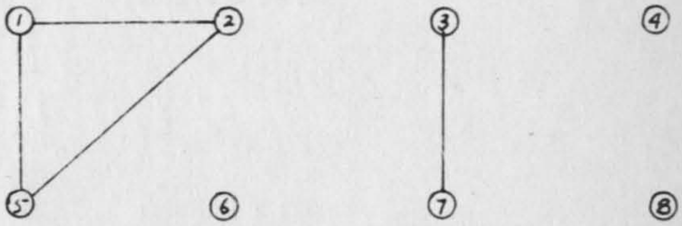
(b) The topological method

The starting point of the topological method developed in this thesis is the network topology itself. Since the network considered here has three types of elements, namely, resistive, capacitive and inductive elements, there are three topologies corresponding to the three types of elements. (Fig. 3-5-a) shows a network whose capacitive, resistive and inductive topologies are given in (fig. 3-5-b, -c and -d) respectively. The topological method of selecting V^1, V^2 and V^3 is stated first, followed by the proof of its validity.

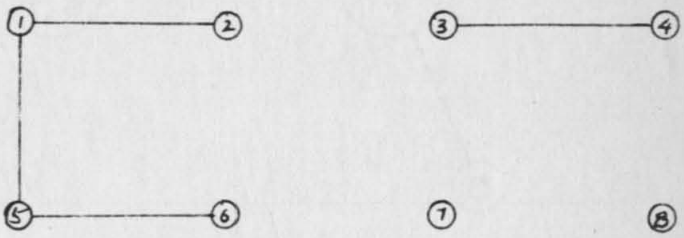
When the network consists only of R-, L- and C-elements, the first step of selecting V^1 is to draw the capacitive topology diagram such as (fig. 3-5-b). In this diagram trees are selected to connect all the connected nodes. The branches of the trees with arbitrary orientation form the components in V^1. For the example in (fig. 3-5-b) the node pairs $v_{1,2}$, $v_{5,2}$ and $v_{3,7}$ may serve as the components of V^1. When several nodes are connected together, there are a large number of ways to form a tree connecting these nodes as shown in Theorem 2. Any one of these trees may be used to provide a coordinate V^1 such that $[C_{11}]^{-1}$ exists; however, these trees may differ in other respects. One important consideration in performing matrix operations on a digital computer is the control of round off errors. The next section will discuss the criterion of selecting the tree among a large set that will give the minimum r.m.s. round off errors. After selecting V^1 the topological method proceeds to select V^2 from the reduced



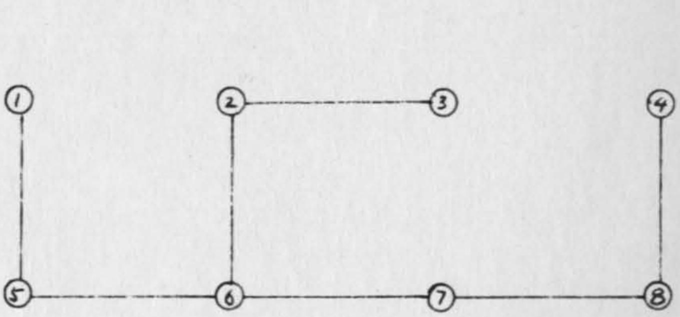
(a) The Complete Network



(b) The Capacitive Topology

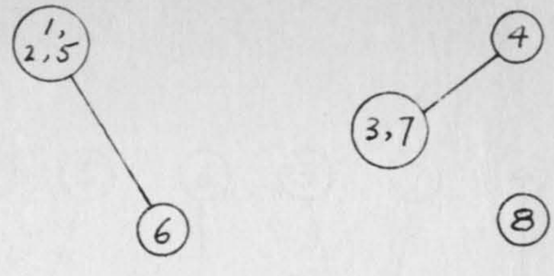


(c) The Resistive Topology

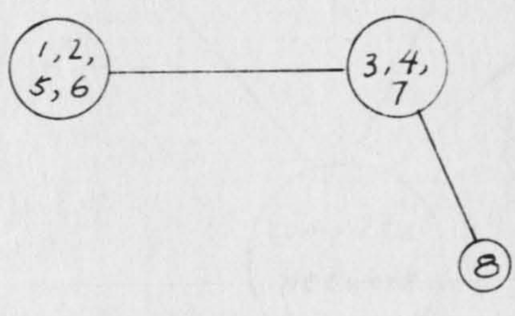


(d) The Inductive Topology

FIGURE 3-5

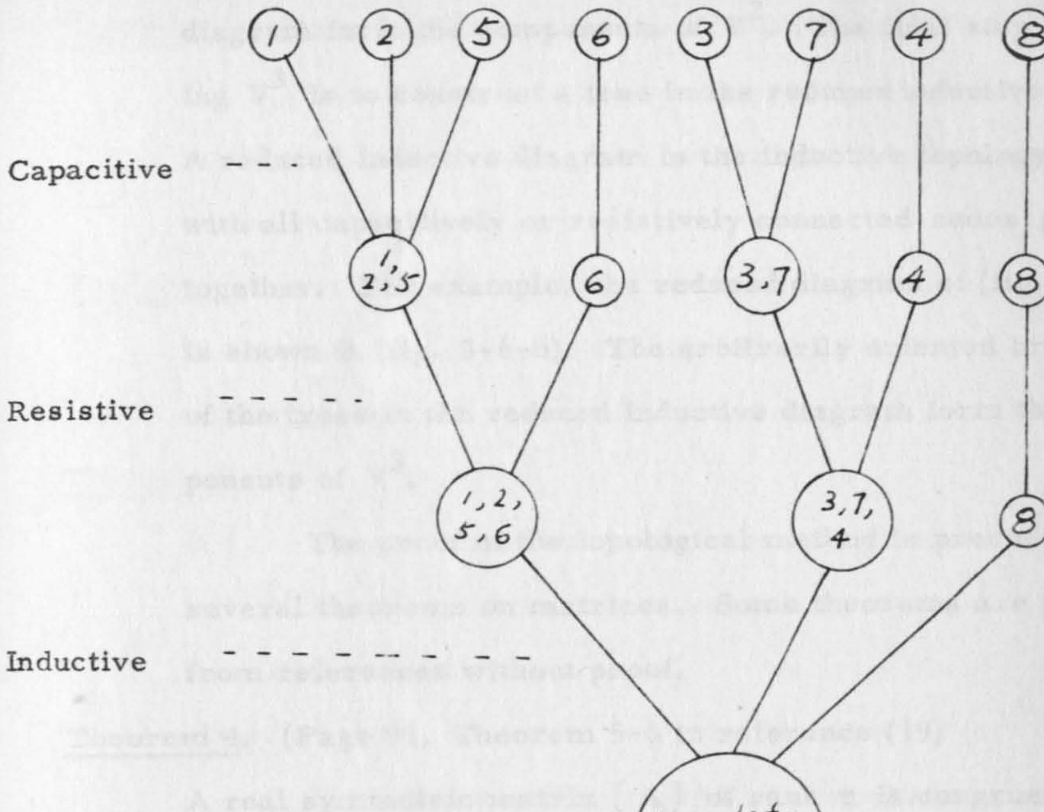


(a) The Reduced Resistive Diagram of the Network in Figure 3-5



(b) The Reduced Inductive Diagram of the Network in Figure 3-5

FIGURE 3-6



(c) The Grouping of Nodes during the Node-Pair Selection in the Network in Figure 3-5

FIGURE 3-6 (continued)

resistive topology diagram with all the nodes that are connected in the capacitive topology diagram short circuited. For example. (fig. 3-6-a) gives the reduced diagram of (fig. 3-5-c). The arbitrarily oriented branches of the trees in the reduced resistive diagram form the components of V^2. The final step of selecting V^3 is to construct a tree in the reduced inductive diagram. A reduced inductive diagram is the inductive topology diagram with all capacitively or resistively connected nodes grouped together. For example, the reduced diagram of (fig. 3-5-d) is shown in (fig. 3-6-b). The arbitrarily oriented branches of the trees in the reduced inductive diagram form the components of V^3.

The proof of the topological method is preceeded by several theorems on matrices. Some theorems are quoted from references without proof.

Theorem 4. (Page 91, Theorem 5-6 in reference (19)

A real symmetric matrix $[A]$ of rank r is congruent to a matrix

$$[B] = \begin{bmatrix} E_p & 0 & 0 \\ 0 & -E_{r-p} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3-43}$$

The integer p is uniquely determined by A.

Definition

The integer p in equation 3-43 is called the index of the symmetric, real matrix $[A]$.

Theorem 5. (page 94, Theorem 5-8 in reference (19))

An nxn real, symmetric matrix of rank r and index p is positive semidefinite if and only if $p = r$, and positive definite if and only if $p = r = n$.

Theorem 6. (page 94, Theorem 5-10 in reference (19))

If $[A]$ is positive definite, every principal submatrix is positive definite. Also, $|A|$ and all principal subdeterminants are positive.

Theorem 7. (page 94, Theorem 5-9 in reference (19))

A real matrix $[A]$ is positive definite if and only if there is a nonsingular real matrix $[P]$ such that $[A] = [P]^T[P]$.

Theorem 8.

If $[A]$ is positive definite, then any congruence of $[A]$, $[Q]^T[A][Q]$, is also positive definite where $[Q]$ is nonsingular.

Proof:

From theorem 7, the positive definite matrix, $[A]$, may be written as

$$[A] = [P]^T[P]$$

and

$$[Q]^T[A][Q] = [Q]^T[P]^T[P][Q]. \tag{3-44}$$

Since $[P]$ and $[Q]$ are nonsingular, their product, $[S]$ is also nonsingular,

$$[Q]^T[A][Q] = [S]^T[S], \tag{3-45}$$

and by Theorem 7, we proved that $[Q]^T[A][Q]$ is positive definite.

Theorem 9.

If $[A]$ is an nxn positive definite matrix, and $[B]$ is an nxs $(s<n)$ rectangular matrix of rank s, then the sxs matrix, $[B]^T[A][B]$, is positive definite.

Proof:

Since $[B]$ has rank $[s]$, it has s independent vectors. It is always possible to find $(n-s)$ additional independent vectors orthogonal to the column vectors of $[B]$, and call them $[B_1]$. The nxn matrix $[B, B_1]$ has n independent column vectors, therefore nonsingular. When $[A]$ is congruent transformed by $[B, B_1]$, we have

$$[B, B_1]^T[A][B, B_1] = \begin{bmatrix} B^T A B & B^T A B_1 \\ B_1^T A B & B_1^T A B_1 \end{bmatrix}, \tag{3-46}$$

Since $[B, B_1]$ is nonsingular, by Theorem 8, the right-hand side of equation 3-46 is positive definite. Furthermore by Theorem 6, the submatrix $[B]^T[A][B]$ is positive definite irrespective of $[B_1]$. This proves the theorem.

The following notations are defined:

v_C, v_R, v_L	=	the set of branch voltages of all the capacitors, resistors and inductors.
$v_R^{\ 1}$	=	the set of branch voltages of all the resistors whose terminal nodes are both connected by the tree that gives V^1.
$v_R^{\ 2}$	=	the set of branch voltages of all the resistors whose terminal nodes are either both connected by the tree that gives V^2, or one in the tree that gives V^2 and the other in the tree that gives V^1.
$v_L^{\ 1}$, $v_L^{\ 2}$	=	the inductor branch voltages similarly defined as $v_R^{\ 1}$ and $v_R^{\ 2}$.
$v_L^{\ 3}$	=	the set of inductor branch voltages which has at least one terminal node connected by the tree that gives V^3.
d_1	=	the number of components in V^1.
d_2	=	the number of components in V^2.
d_3	=	the number of components in V^3.
B_c	=	the number of capacitors in the network.
B_R	=	the number of resistors in the network.
B_L	=	the number of inductors in the network.

$[C_B]$, $[R_B]$, $[L_B]$ = the capacitive, resistive and inductive admittance matrices in coordinate v_C, v_R and v_L.

$[R_B^1]$, $[R_B^2]$ = the resistive admittance matrices in coordinates ${v_R}^1$, ${v_R}^2$.

$[L_B^1]$, $[L_B^2]$, $[L_B^3]$ = the inductive admittance matrices in coordinates ${v_L}^1$, ${v_L}^2$, ${v_L}^3$.

With the above introduced theorems and notations, the validity of the topological method is proved as follows:

(1) To prove that $[C_{11}]^{-1}$ exists:

Since V^1 is selected to connect all the capacitors, we may write

$$[v_C] = [U]V^1 \tag{3-47}$$

where $[U]$ is a $B_c \times d_1$ matrix of rank d_1. If all capacitances are positive, the capacitive matrix $[C_B]$ in v_C coordinate is positive definite. Therefore, from Theorem 9, the matrix

$$[C_{11}] = [U]^T[C_B][U] \tag{3-48}$$

is positive definite and nonsingular.

(2) To prove that $[R_{22}]^{-1}$ exists:

Since all resistors are connected by V^1 and V^2, we may write

$$\begin{aligned} {v_R}^1 &= [W_{11}]V^1 \\ {v_R}^2 &= [W_{21}]V^1 + [W_{22}]V^2 \end{aligned} \tag{3-49}$$

where $[W_{22}]$ is a rectangular matrix of d_2 independent columns, that is, $[W_{22}]$ has rank $= d_2$.

The resistor matrix in (V^1, V^2) coordinates is given by

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \begin{bmatrix} W_{11} & 0 \\ W_{21} & W_{22} \end{bmatrix}^T [R_B] \begin{bmatrix} W_{11} & 0 \\ W_{21} & W_{22} \end{bmatrix}. \qquad (3\text{-}50)$$

with $[R_B]$ written as

$$[R_B] = \begin{bmatrix} R_B^1 & 0 \\ 0 & R_B^2 \end{bmatrix} \qquad (3\text{-}51)$$

where both $[R_B^1]$ and $[R_B^2]$ are positive definite, we have

$$[R_{11}] = [W_{11}]^T[R_B^1][W_{11}] + [W_{21}]^T[R_B^2][W_{21}] \qquad (3\text{-}52)$$

$$[R_{12}] = [R_{21}]^T = [W_{21}]^T[R_2][W_{22}] \qquad (3\text{-}53)$$

$$[R_{22}] = [W_{22}]^T[R_B^2][W_{22}]. \qquad (3\text{-}54)$$

In equation 3-49, the rank of $[W_{11}]$ and $[W_{21}]$ are not known, therefore, we cannot conclude whether $[R_{11}]$ is positive definite or positive semidefinite. However, the rank of $[W_{22}]$ is d_2, therefore the $d_2 \times d_2$ matrix, $[R_{22}]$, is positive definite.

(3) To prove that $[L_{33}]^{-1}$ exists:

$$\begin{aligned} v_L^{\,1} &= [S_{11}]V^1 \\ v_L^{\,2} &= [S_{21}]V^1 + [S_{22}]V^2 \\ v_L^{\,3} &= [S_{31}]V^1 + [S_{32}]V^2 + [S_{33}]V^3 \end{aligned} \qquad (3\text{-}55)$$

From the way we select V^3, the transformation matrix $[S_{33}]$ has a rank of d_3. The inductor matrix in (V^1, V^2, V^3) coordinates is given by

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} = \begin{bmatrix} S_{11} & 0 & 0 \\ S_{21} & S_{22} & 0 \\ S_{31} & S_{32} & S_{33} \end{bmatrix}^T \begin{bmatrix} L_B^1 & 0 & 0 \\ 0 & L_B^2 & 0 \\ 0 & 0 & L_B^3 \end{bmatrix} \begin{bmatrix} S_{11} & 0 & 0 \\ S_{21} & S_{22} & 0 \\ S_{31} & S_{32} & S_{33} \end{bmatrix} . \quad (3\text{-}56)$$

From equation 3-56 we have

$$[L_{33}] = [S_{33}]^T [L_B^3] [S_{33}] \quad (3\text{-}57)$$

since the matrix $[S_{33}]$ has rank $= d_3$, the $d_3 \times d_3$ matrix, $[L_{33}]$, is positive definite.

3.5 Coordinate Selection to Minimize Round Off Errors in Matrix Computations

The transformation method described in Section 3.4 reduces the matrices to be inverted into diagonal forms. The round off error in inverting a diagonal matrix is in the last significant digit the computer can represent. However, the transformation procedure that leads to the diagonal matrices involves many arithmetic operations which can introduce appreciable amount of error in the diagonal terms. The transformation method has even a greater disadvantage of altering the rank of $[C_{11}]$, $[R_{22}]$ and $[L_{33}]$; its use in coordinate selection will not be considered further. In the topology method, the selection of V^1, V^2, and V^3 are made by constructing trees among a set of nodes. There are $S(P)$ different trees one can construct to connect P nodes.

S(P), as evaluated in Theorem 2 is a very fast growing function of P. This section will discuss the selection of one tree among this large set of permissible ones such that the r.m.s. round off errors in subsequent matrix operations may be minimized.

Turing (page 298, reference 20) gave the following statements:

$$\frac{\text{r.m.s. error of coefficients of solution}}{\text{r.m.s. error of solution}}$$
$$= \frac{1}{n} N(A)\,N(A^{-1})\,\frac{\text{r.m.s. error of coefficient of } [A]}{\text{r.m.s. coefficient of } [A]} \qquad (3\text{-}58)$$

where the matrix under consideration is $[A]$ and $N(A)$ is the norm of $[A]$ as defined by

$$N(A) = (\text{trace } A^T A)^{1/2} = \Big(\sum_{i,j} a_{ij}^2\Big)^{1/2} \qquad (3\text{-}59)$$

He called $\frac{1}{n}N(A)\,N(A^{-1})$ the N-number of the matrix $[A]$. Similarly, he defined an M-number as $nM(A)\,M(A^{-1})$ where $M(A)$ is the maximum coefficient of the matrix $[A]$.

$$M(A) = \max_{i,j} |a_{ij}|. \qquad (3\text{-}60)$$

From equation 3-58 we can see that the N-number is a measure of the ill conditioning in a matrix $[A]$ with randomly distributed coefficients. If we want to compare this property of two matrices with the same dimension n, the value $N(A)\,N(A^{-1})$ will suffice. Given the matrix $[A]$, $N(A^{-1})$ varies inversely as $\det|A|$. Therefore, for the comparison of two matrices, instead of deriving the r.m.s. error relationship as in equation 3-58, we may use $\frac{N(A)}{\det|A|}$ as a measure of relative round off error in matrix computation. If we further define $\bar{M}(A)$

$$\bar{M}(A) = \sum_{ij} |a_{ij}| , \tag{3-61}$$

as the sum of all the absolute values of elements in matrix [A], then we can also use

$$\frac{\bar{M}(A)}{\det|A|} \tag{3-62}$$

as the round off error measure. If

$$\frac{\bar{M}(A)}{\det|A|} > \frac{\bar{M}(B)}{\det|B|} , \tag{3-63}$$

we say that the matrix operations in [A] will introduce more round off errors than in matrix [B]. In the selection of the tree such that the matrix operations introduce the least round off errors, equation 3-62 is used instead of equation 3-58, because the latter, involving quadratic forms, in hard to implement into a selection algorithm.

The following theorem is given before describing the algorithm that selects the optimum coordinates.

<u>Theorem 10.</u>

> The determinants of the admittance matrices formulated in the node pair voltages are invariant to the choice of the trees from which the node pair voltages are selected.

Proof:

Theorem 3 states that the transformation matrix, $[A]$, in

$$V_j = [A]\, V_i \tag{3-64}$$

has a determinant of ± 1 where V_j and V_i are any two vectors whose elements are the node pair voltages across the branches of any two trees. From equations 2-21, 2-22 and 2-23 we have

$$[K_i] = [A]^T [K_j][A] \tag{3-65}$$

where $[K_i]$ and $[K_j]$ are the admittance matrices in the coordinates V_i and V_j. Hence we have

$$\det|K_i| = (\pm 1)^2 \det|K_j| \tag{3-66}$$

Returning to the problem of selecting the tree that minimizes the value given in equation 3-62, it can be seen that since the determinant is invariant to the tree selection, the optimum set of coordinates will minimize the value $\overline{M}(A)$.

Let there be B admittance branches whose branch voltages are $(v_1, v_2 \ldots v_B)$ and the admittances of the branches be $(k_1, k_2 \ldots k_B)$, then the admittance matrix in branch coordinates is the diagonal matrix

$$[K_B] = \begin{bmatrix} k_1 & 0 & --- & 0 \\ 0 & k_2 & --- & 0 \\ \vdots & & & \\ 0 & & & kB \end{bmatrix} \tag{3-67}$$

Assume that these B branches are connected into P nodes, and a set of (P - 1) node pairs are selected $(V_1, V_2 \ldots V_{p-1})$, then we may write

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_B \end{bmatrix} = [A] \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_{p-1} \end{bmatrix} \tag{3-68}$$

and $[K]$, the admittance matrix in $(V_1, V_2 \ldots V_{p-1})$ coordinates is given by

$$[K] = [A]^T [K_B][A], \tag{3-69}$$

or

$$k_{ij} = \sum_{m=1}^{B} a_{mi} a_{mj} k_m \tag{3-70}$$

where k_{ij}, a_{ij} are the elements in i^{th} row and j^{th} column of matrices $[K]$ and $[A]$, and k_m is defined in equation 3-67.

Equation 3-70 may be interpreted differently by freezing the dummy index m. Then we can say that each branch v_m which has the branch admittance k_m contributed the amount $a_{mi}a_{mj}k_m$ to the element k_{ij}, and the resultant k_{ij} is the sum of the contributions from all branches $(v_1, v_2 \ldots v_B)$. Since we want to minimize $\overline{M}(K)$, the sum of the absolute values of all the elements in $[K]$, we wish to keep the contributions from the largest k_m to a minimum number of terms of k_{ij}. The elements a_{ij} can either be +1, - 1 or 0, therefore, the contribution due to k_M, the maximum branch admittance, may be limited to a single term k_{ii} if we set

$$a_{M\ell} = 0 \qquad \text{for } \ell \neq i$$
$$a_{M\ell} = \pm 1 \qquad \text{for } \ell = i \quad . \qquad (3\text{-}71)$$

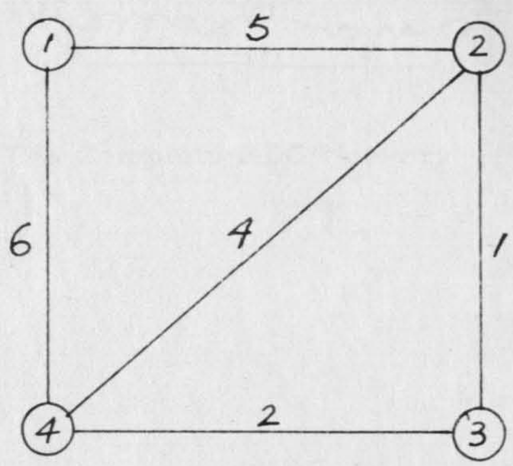
When the result in equation 3-71 is substituted into equation 3-68 v_M is selected as a branch of the tree. After the branch with the largest admittance is selected as a branch of the tree, the branch with the next largest admittance is selected, provided it does not violate the tree topology. For example, after selecting the branches (4, 1) (1, 2) that correspond to the first two largest branches in (fig. 3-7), the third branch cannot be (4, 2) which, although its admittance is the largest among the remaining branches, violates the tree topology. With the tree branches selected according to this algorithm, the value $\bar{M}(K)$ will be minimized, hence reducing the round off error in subsequent matrix computations as formulated in equation 3-33.

3.6 The Algorithm to Select the Optimum Coordinates in a Passive RLC Network.

The algorithm to select an optimum set of node pair coordinates is summarized below with the supplementing example in (fig. 3-8) for illustration.

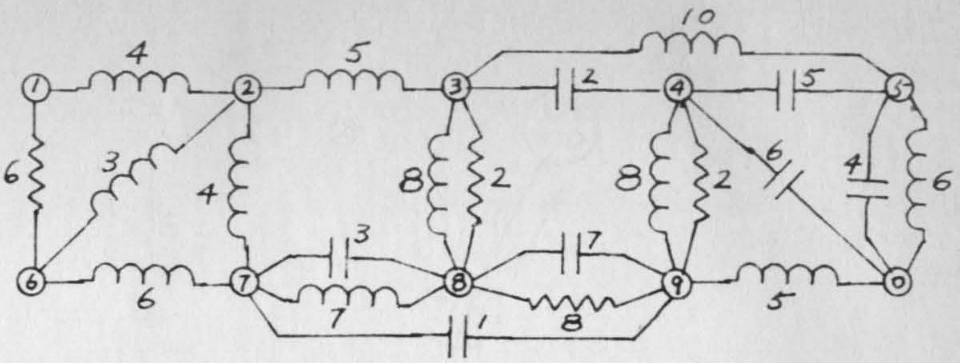
(Fig. 3-8-a) shows an arbitrary network of resistors, inductors and capacitors. The values of all the elements are given to guide the selection of the optimum set of coordinates.

(1) From the given network, the capacitive topology in (fig. 3-8-b) is constructed as several connected branches weighted according to the capacitances. The most weighted branch, v_{89} is selected as the first component in V^1. Then with the terminal nodes of

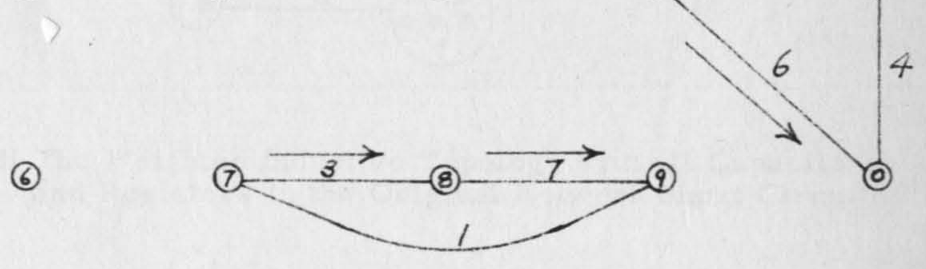


The Selection of Node-pair Coordinates in a Weighted Topology to Minimize the Round-off Errors in Matrix Operations

FIGURE 3-7



(a) The Complete RLC Network

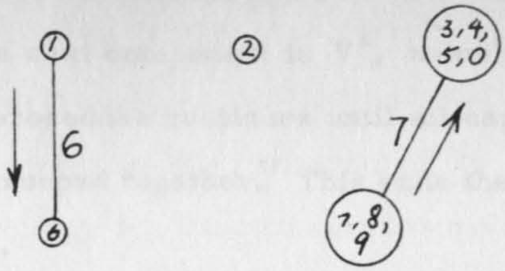


(b) The Weighted Capacitive Topology

$$V^1 = \begin{bmatrix} v_{89} \\ v_{40} \\ v_{45} \\ v_{78} \\ v_{43} \end{bmatrix}$$

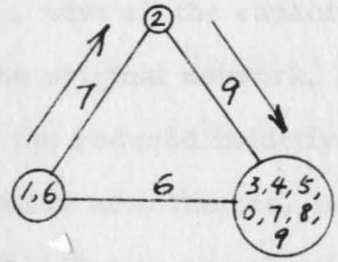
An Example

FIGURE 3-8



$$V^2 = \begin{bmatrix} v_{16} \\ v_{73} \end{bmatrix}$$

(c) The Weighted Resistive Topology with all Capacitors in the Network Short Circuited



$$V^3 = \begin{bmatrix} v_{23} \\ v_{12} \end{bmatrix}$$

(d) The Weighted Inductive Topology with all Capacitors and Resistors in the Original Network Short Circuited

An Example

FIGURE 3-8 (continued)

the selected branch grouped together, and all the parallel branches combined, the next most weighted branch is selected as the next component in V^1, namely, $^{v}40$ in (fig. 3-8-b). The procedure continues until all capacitively connected nodes are grouped together. This ends the selection of node pairs in V^1.

(2) From the given network, with all the capacitors replaced by short circuiting wires, and all the parallel resistors combined together, the reduced resistive topology, weighted according to the resistive admittance, is constructed as shown in (fig. 3-8-c). The same selection criterion used to select V^1 from the capacitive topology is used on the reduced resistive topology to give all the node pairs in V^2.

(3) Finally, with all the capacitors and resistors short circuited in the original network, and all the parallel inductors combined, the reduced inductive topology, weighted according to the inductive admittances, is constructed as shown in (fig. 3-8-d). The same selection criterion used to select V^1 and V^2 from the capacitive topology and the reduced resistive topology is used to select all the node pairs in V^3 from the reduced inductive topology.

CHAPTER 4

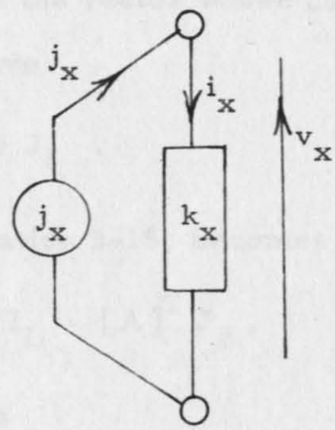
FORCING FUNCTIONS AND INITIAL CONDITIONS

Chapter 3 dealt with the selection of an optimum set of coordinates for a passive RLC network. This chapter extends the method to include voltage and current sources and gives a systematic procedure of setting up the initial conditions for the differential equations.

4.1 Voltage and Current Sources

Voltage and current sources are also considered as two terminal elements. Voltage sources introduce additional constraints to the set of independent node pairs; however, current sources merely add additional terms to the current summation equation, equation 3-15. In order to have a unified approach, voltage and current sources are represented as J_v and J_i in the current summation equation, J_v representing the current vector in voltage-source elements, and J_i, the current vector in current-source elements. By definitions, J_i is known and J_v is unknown. In (fig. 3-3-a), each passive branch is represented by an admittance which may be any parallel combination of R, L, C elements. With the addition of voltage and current sources, each branch is represented as shown in (fig. 4-1). The branch x has branch voltage v_x and two current components: i_x, the current in the passive elements; and j_x, the current in the active elements, namely the voltage and current sources.

$$j_x = j_v + j_i \quad . \tag{4-1}$$



A Generalized Branch Representation with a Passive Admittance k_x, and an External Forcing Current j_x.

FIGURE 4-1

When several of these branches are connected together, a set of independent node pairs V is selected, and the branch voltages v_B can be related to the set of node-pair voltages V by a transformation matrix, [A] ,

$$v_B = [A]\, V \quad . \tag{4-2}$$

If we use J_B to represent the vector whose components are the branch current j_x , we have

$$J_B = J_v + J_i \quad . \tag{4-3}$$

The current equation, equation 3-15, becomes

$$I_C + I_R + I_L = [A]^T J_B , \tag{4-4}$$

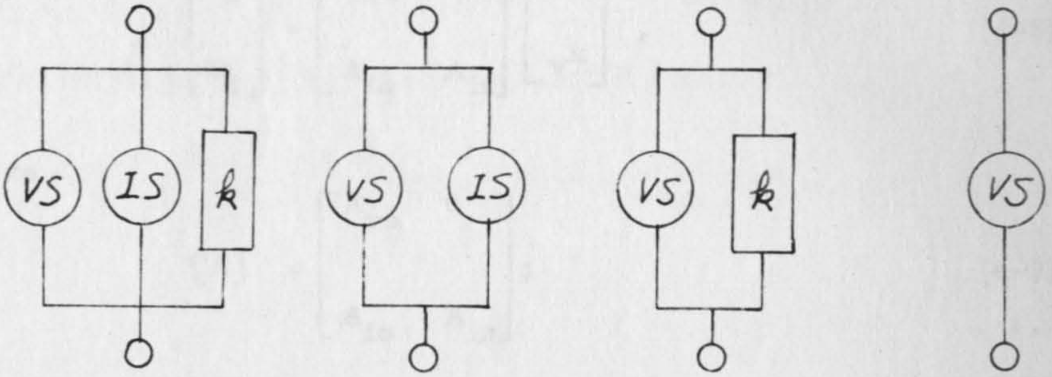
and equation 3-23 becomes

$$[C]\frac{dV}{dt} + [R]V + [L]\int V dt = [A]^T J_B \quad . \tag{4-5}$$

In order to factor out the unknown current, J_v , in equation 4-5, all branches that contain voltage sources are denoted by v_v and all other branches by v_i . A branch in v_v contains a voltage source in parallel with or without any combination of other elements in the network. The four types of v_v are shown in (fig. 4-2) where VS represents voltage source; IS, current source; and k , any parallel combination of resistors, capacitors or inductors.

If we can pick a set of node pairs V, such that

$$V = \begin{bmatrix} V^o \\ V^x \end{bmatrix} , \tag{4-6}$$



VS = Voltage Source

IS = Current Source

k = Passive Admittance

Four Types of Voltage-source Branches

FIGURE 4-2

and the branch voltages

$$v_B = \begin{bmatrix} v_v \\ v_i \end{bmatrix} \tag{4-7}$$

are related to V by the transformation matrix $[A]$ such that

$$\begin{bmatrix} v_v \\ v_i \end{bmatrix} = \begin{bmatrix} A_{vo} & 0 \\ A_{io} & A_{ix} \end{bmatrix} \begin{bmatrix} V^o \\ V^x \end{bmatrix}, \tag{4-8}$$

where

$$[A] = \begin{bmatrix} A_{vo} & 0 \\ A_{io} & A_{ix} \end{bmatrix}; \tag{4-9}$$

we can write equation 4-5 in the partitioned form as

$$\begin{bmatrix} C_{oo} & C_{ox} \\ C_{xo} & C_{xx} \end{bmatrix} \frac{d}{dt} \begin{bmatrix} V^o \\ V^x \end{bmatrix} + \begin{bmatrix} R_{oo} & R_{ox} \\ R_{xo} & R_{xx} \end{bmatrix} \begin{bmatrix} V^o \\ V^x \end{bmatrix} + \begin{bmatrix} L_{oo} & L_{ox} \\ L_{xo} & L_{xx} \end{bmatrix} \int \begin{bmatrix} V^o \\ V^x \end{bmatrix} dt$$

$$= \begin{bmatrix} I^o \\ I^x \end{bmatrix}, \tag{4-10}$$

where

$$I^o = [A_{vo}]^T J_v + [A_{io}]^T J_i \tag{4-11}$$

$$I^x = [A_{ix}]^T J_i . \tag{4-12}$$

Since by definition J_v, the current from voltage sources, is unknown, I^o is not defined. Therefore, the first equation in equation 4-10,

$$[C_{oo}]\frac{dV^o}{dt} + [C_{ox}]\frac{dV^x}{dt} + [R_{oo}]V^o + [R_{ox}]V^x + [L_{oo}]\int V^o dt + [L_{ox}]\int V^x dt = I^o, \quad (4\text{-}13)$$

cannot be integrated; however, the second equation,

$$[C_{xo}]\frac{dV^o}{dt} + [C_{xx}]\frac{dV^x}{dt} + [R_{xo}]V^o + [R_{xx}]V^x + [L_{xo}]\int V^o dt + [L_{xx}]\int V^x dt = I^x, \quad (4\text{-}14)$$

is integrable to give the value of $V^x(t)$ as a function of $I^x(t)$ and $V^o(t)$, where $I^x(t)$ is defined solely by the current sources, J_i, as shown in equation 4-12, and $V^o(t)$ is related to v_v in equation 4-8,

$$V^o = [A_{vo}]^{-1} v_v , \quad (4\text{-}15)$$

where $[A_{vo}]^{-1}$ always exists as a consequence of the Kirchhoff voltage law that the set of branches, v_v, which contain voltage sources must not form loops. For numerical computations, equation 4-14 can be written as

$$\frac{dV^x}{dt} = [C_{xx}]^{-1}\Big(I^x - [C_{xo}]\frac{dV^o}{dt} - [R_{xo}]V^o - [L_{xo}]y^o - [R_{xx}]V^x - [L_{xx}]y^x\Big),$$

$$\frac{dy^x}{dt} = V^x \quad . \quad (4\text{-}16)$$

Equation 4-16 is in the same form as equation 3-25; therefore, all the discussions on the evaluation of equation 3-25 apply to equation 4-16. Since $[\ C_{xx}\ ]^{-1}$ does not always exist, it is necessary to select the coordinates within V^x into three classes: V^1, V^2 and V^3

$$V^x = \begin{bmatrix} V^1 \\ V^2 \\ V^3 \end{bmatrix} \tag{4-17}$$

such that the correspondingly partitioned submatrices $[\ C_{11}\ ]$, $[\ R_{22}\ ]$ and $[\ L_{33}\ ]$ are nonsingular.

Equation 4-13 is not integrable since I^o is undefined; however, it may be used to compute the currents in voltage sources.

$$J_v = [\ A_{vo}^{\ T}\ ]^{-1} \left(I^o - [\ A_{io}\ ]^T J_i \right). \tag{4-18}$$

In concluding this part of the discussion, the above analysis is summarized as follows:

(1) Restrictions on voltage- and current-source topologies

(a) The voltage-source topology derived from the complete network with all elements removed except voltage sources must contain no loops. This restriction follows directly from the Kirchhoff voltage law.

(b) The current-source topology derived from the complete network with all elements except current sources replaced by short circuiting wires must contain no branches. This restriction follows directly from the Kirchhoff current law that the total algebraic sum of currents entering any set of internally connected nodes must be zero.

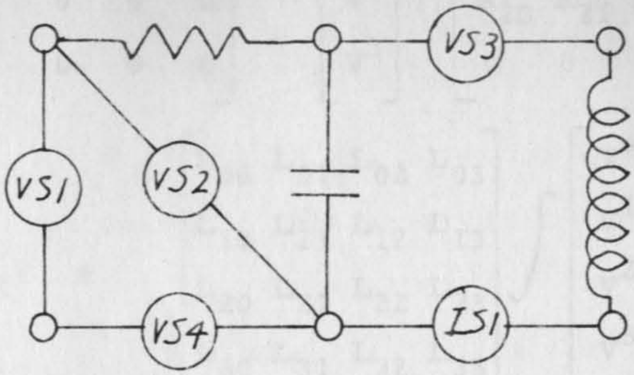
(Fig. 4-3-a) shows an example of a forbidden voltage-source topology, and (fig. 4-3-b) shows an example of a forbidden current-source topology.

(2) In a network consisting of voltage sources, current sources and RLC elements connected in any arbitrary topology to form P nodes and D separate parts, within the restriction imposed on the voltage and current sources stated in (1) above, a set of (P - D) node pairs V, subdividing into four classes, may be selected,

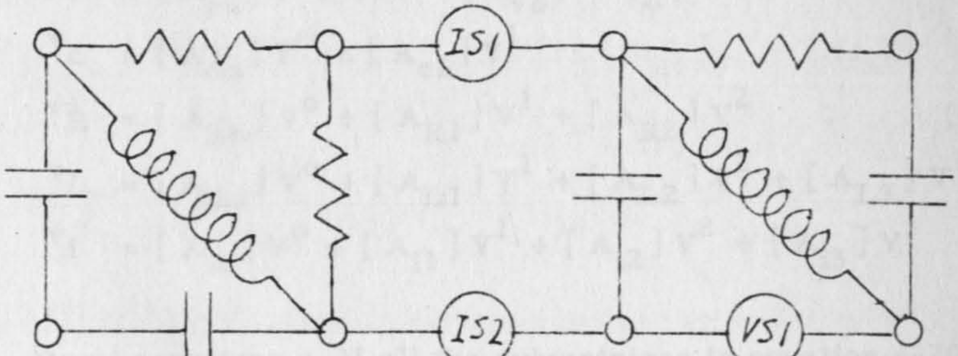
$$V = \begin{bmatrix} V^o \\ V^1 \\ V^2 \\ V^3 \end{bmatrix} . \qquad (4\text{-}19)$$

V^o is selected from the voltage-source topology; V^1 is selected from the reduced capacitive topology with all nodes connected by V^o being grouped together; V^2 is selected from the reduced resistive topology with all nodes connected by V^o and V^1 being grouped together; V^3 is selected from the reduced inductive topology with all nodes connected by V^o, V^1 and V^2 being grouped together.

(3) In terms of the coordinates (V^o, V^1, V^2, V^3) selected in (2), the current equations may be formulated as



(a) An Example of Forbidden Voltage-source Connections



(b) An Example of Forbidden Current-source Connections

The Forbidden Voltage- and Current-source Connections that Violate the Physical Laws

FIGURE 4-3

$$\begin{bmatrix} C_{00} & C_{01} & 0 & 0 \\ C_{10} & C_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} V^o \\ V^1 \\ V^2 \\ V^3 \end{bmatrix} + \begin{bmatrix} R_{00} & R_{01} & R_{02} & 0 \\ R_{10} & R_{11} & R_{12} & 0 \\ R_{20} & R_{21} & R_{22} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V^o \\ V^1 \\ V^2 \\ V^3 \end{bmatrix}$$

$$+ \begin{bmatrix} L_{00} & L_{01} & L_{02} & L_{03} \\ L_{10} & L_{11} & L_{12} & L_{13} \\ L_{20} & L_{21} & L_{22} & L_{23} \\ L_{30} & L_{31} & L_{32} & L_{33} \end{bmatrix} \int \begin{bmatrix} V^o \\ V^1 \\ V^2 \\ V^3 \end{bmatrix} dt = \begin{bmatrix} I^o \\ I^1 \\ I^2 \\ I^3 \end{bmatrix} . \tag{4-20}$$

Let (v_v, v_i, v_c, v_R, v_L) represent the branch voltages of (voltage-source, current-source, capacitor, resistor, inductor) elements, then their linear dependence on the selected coordinates (V^o, V^1, V^2, V^3) are as follows:

$$\begin{aligned} v_v &= [\,A_{vo}\,]\, V^o; \quad (V^o = [\,A_{vo}\,]^{-1}\, v_v) \\ v_c &= [\,A_{co}\,]\, V^o + [\,A_{c1}\,]\, V^1 \\ v_R &= [\,A_{Ro}\,]\, V^o + [\,A_{R1}\,]\, V^1 + [\,A_{R2}\,]\, V^2 \quad . \\ v_L &= [\,A_{Lo}\,]\, V^o + [\,A_{L1}\,]\, V^1 + [\,A_{L2}\,]\, V^2 + [\,A_{L3}\,]\, V^3 \\ v_i &= [\,A_{io}\,]\, V^o + [\,A_{i1}\,]\, V^1 + [\,A_{i2}\,]\, V^2 + [\,A_{i3}\,]\, V^3 \end{aligned} \tag{4-21}$$

From equations 4-21 all the submatrices in equation 4-20 are defined.

$$[\,C_{kj}\,] = [\,A_{ck}\,]^T [\,C_B\,][\,A_{cj}\,] \qquad \begin{matrix} k = 0,\ 1 \\ j = 0,\ 1 \end{matrix} \tag{4-22}$$

$$[\,R_{kj}\,] = [\,A_{Rk}\,]^T [\,R_B\,][\,A_{Rj}\,] \qquad \begin{matrix} k = 0,\ 1,\ 2 \\ j = 0,\ 1,\ 2 \end{matrix} \tag{4-23}$$

$$[L_{kj}] = [A_{Li}]^T [L_B][A_{Lj}] \qquad \begin{matrix} k = 0, 1, 2, 3 \\ i = 0, 1, 2, 3 \end{matrix} \qquad (4\text{-}24)$$

$$I^k = [A_{ik}]^T J_i \qquad k = 1, 2, 3 \qquad (4\text{-}25)$$

$$I^o = [A_{io}]^T J_i + [A_{vo}] J_v \qquad (4\text{-}26)$$

where $[C_B]$, $[R_B]$, $[L_B]$ are the branch admittance matrices; J_i the current sources and J_v the current vector from voltage sources.

Let

$$y^k = \int V^k dt \qquad k = 0, 1, 2, 3 \qquad (4\text{-}47)$$

then the following equations are derived from equation 4-20

$$\begin{aligned}
\frac{dV^1}{dt} &= [C_{11}]^{-1}\Big(I^{1*} - [R_{11}]V^1 - [R_{12}]V^2 - [L_{11}]y^1 \\
&\qquad - [L_{12}]y^2 - [L_{13}]y^3\Big) \\
\frac{dy^1}{dt} &= V^1 \\
\frac{dy^2}{dt} &= V^2 = [R_{22}]^{-1}\Big(I^{2*} - [R_{21}]V^1 - [L_{21}]y^1 - [L_{22}]y^2 \\
&\qquad - [L_{23}]y^3\Big) \\
y^3 &= [L_{33}]^{-1}\Big(I^{3*} - [L_{31}]y^1 - [L_{32}]y^2\Big)
\end{aligned} \qquad , (4\text{-}28)$$

where I^{1*}, I^{2*}, I^{3*} are the equivalent source currents in V^1, V^2, and V^3 coordinates and they are defined as:

$$\begin{aligned}
I^{1*} &= I^1 - [C_{10}]\frac{dV^o}{dt} - [R_{10}]V^o - [L_{10}]y^o \\
I^{2*} &= I^2 - [R_{20}]V^o - [L_{20}]y^o \\
I^{3*} &= I^3 - [L_{30}]y^o
\end{aligned} \qquad . (4\text{-}29)$$

Since equations 4-28 are in the canonical form for applying various numerical integration formulae and the coordinates are so selected that $[C_{11}]^{-1}$, $[R_{22}]^{-1}$ and $[L_{33}]^{-1}$ always exist; given the initial values of V^1, y^1 and y^2, the state of the network at all subsequent times can be computed. This leads to the next unsolved task of deriving the initial values of V^1, y^1 and y^2 from the energy distribution in the system, namely, the charges in capacitors and the currents in inductors.

4.2 Initial Conditions.

Bryant (21) and Bers (22) have discussed the problem of evaluating the number of natural frequencies which the first author called "the order of complexity of the network", and the number of nonzero natural frequencies which the second author called "the degrees of freedom in RLC networks". They disagreed on the terminologies used as can be observed in their correspondences (23). Here the same subject is touched upon again, but from a different point of view. Based on the coordinates selected, namely V^o, V^1, V^2 and V^3, the results obtained are the same as those of Bryant and Bers. The emphasis here, however, is not merely a number that represents the complexity of the network but on the systematic way of incorporating the energy distribution into the differential equations of equation 4-28.

In a network of RLC elements under the excitation of arbitrary voltage and current sources, a set of coordinates (V^o, V^1, V^2, V^3) is selected as described in section 4.1. The matrix equation in these coordinates is given in equation 4-20. All branches that are connected within V^o coordinates are not allowed to take arbitrary initial conditions on the voltage across capacitors, since the voltages across all these branches are completely specified by the voltage sources that connect the same set of nodes. All inductors connected within V^o can have any initial currents without changing the subsequent transient response since the effect of the inductor current in V^o is absorbed by I^o which itself is an undefined quantity. Therefore, all subsequent discussions concern coordinates (V^1, V^2, V^3) only, without losing generality to networks that have V^o coordinates as well.

Since all capacitors are connected within V^1, independent of the number of capacitors in the network, there are only d_1 independent parameters to specify all the voltages across capacitors, where d_1, d_2, d_3 are the number of components in the vectors V^1, V^2, V^3 respectively. These d_1 independent parameters are the voltages across the d_1 branches of any one of the $S(d_1 + 1)$ trees that can be constructed from the $(d_1 + 1)$ nodes connected by the V^1 coordinates. With these d_1 branch voltages, the electrostatic energy distribution in the system is uniquely defined.

In order to determine the number of independent parameters that uniquely specify the magnetostatic energy distribution in the system, the following equations derived from equation 4-20 are considered:

$$[C_{11}]\frac{dV^1}{dt} + [R_{11}]V^1 + [R_{12}]V^2 = I^{1*} - I_L^1 \tag{4-30}$$

$$[R_{21}]V^1 + [R_{22}]V^2 = I^{2*} - I_L^2 \tag{4-31}$$

$$0 = I^{3*} - I_L^3 \tag{4-32}$$

where $[C_{11}]$ is defined in equation 4-22,

$[R_{21}]$, $[R_{22}]$ are defined in equation 4-23,

I^{1*}, I^{2*}, I^{3*} are defined in equation 4-29,

and I_L^1, I_L^2, I_L^3 are the current components in V^1, V^2, V^3 coordinates due to the currents in the inductive elements. If i_L is the current vector of all inductive elements in the network, then using the transformation matrices defined in equation 4-21 and the law of transformation in equation 2-24, we have

$$\begin{aligned} I_L^1 &= [A_{L1}]^T i_L \\ I_L^2 &= [A_{L2}]^T i_L \quad . \\ I_L^3 &= [A_{L3}]^T i_L \end{aligned} \tag{4-33}$$

After substituting for I_L^3, equation 4-32 may be written as

$$[A_{L3}]^T i_L = I^{3*} , \tag{4-34}$$

which is a set of d_3 independent linear equations in the variables, i_L. In equation 4-31, $[R_{21}]V^1$ is specified by the voltages across capacitors; and I^{2*}, by the voltage and current sources; however, $[R_{22}]V^2$ is not constrainted. Therefore, I_L^2 can take on any arbitrary value. In equation 4-30, $[R_{11}]V^1$ and I^{1*} are prescribed in the same way as $[R_{21}]V^1$ and I^{2*} in equation 4-31; $[R_{12}]V^2$ is determined by the arbitrary choice of I_L^2 in equation 4-31; however, $[C_{11}]\frac{dV^1}{dt}$ is not constrainted such that I_L^1 may take on any arbitrary value. Therefore, there are only d_3 linear equations relating the B_L components of i_L where B_L is the number of inductors in the network.

In other words, there are $(B_L - d_3)$ independent parameters to specify the initial conditions on the currents in inductive elements. The total number of independent parameters to specify completely the initial energy distribution in the network is

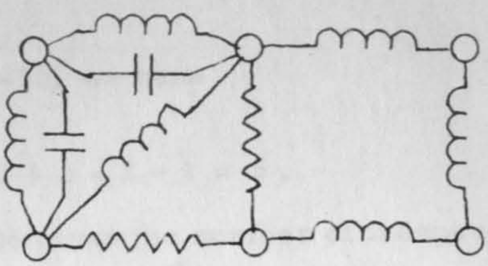
$$\sigma = d_1 + B_L - d_3, \tag{4-35}$$

which agrees with Bryant's results (21).

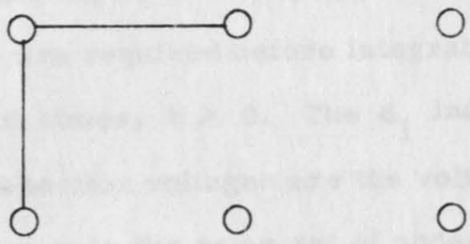
Whenever several inductors form a loop, an arbitrary d.c. current may flow in the loop without changing the dynamic response of the network. When $(B_L - d_3)$ independent parameters are used to specify the complete magnetostatic energy distribution, as many of them are used to specify the d.c. loop currents as there are loops in the inductive topology. Each d.c. loop current constitutes a node of zero frequency. If one is only interested in the number of nonzero frequency as Bers (22) was, the number of inductor loops has to be subtracted from equation 4-35 which gives the number of all the modes, including multiple zero frequency mode. The number of nonzero frequence modes is, therefore, given by

$$p = d_1 + B_L - d_3 - \ell_L \tag{4-36}$$

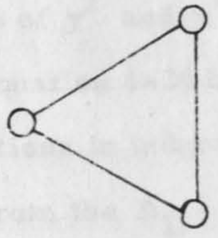
where ℓ_L is the number of loops in the inductive topology of the network. (Fig. 4-4-a) shows an arbitrary network; (fig. 4-4-b) shows its capacitive topology which gives $d_1 = 2$; (fig. 4-4-c) shows the reduced inductive topology which gives $d_3 = 2$; (fig. 4-4-d) shows the inductive topology which gives $\ell_L = 1$. With $B_L = 6$ from equation 4-35 we have



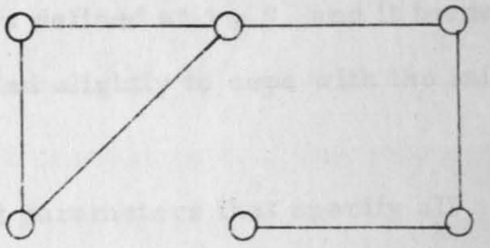
(a) The Complete Network, $B_L = 6$



(b) The Capacitive Topology, $d_1 = 2$



(c) The Reduced Inductive Topology, $d_3 = 2$



(d) The Inductive Topology, $\ell_L = 1$

The Determination of σ and p of an Arbitrary RLC Network

FIGURE 4-4

$$\sigma = 2 + 6 - 2 = 6;$$

and from equation 4-36, we have

$$p = 2 + 6 - 2 - 1 = 5.$$

Equation 4-35 gives the number of independent parameters one may use to specify the initial condition of the network. The next step is to incorporate these σ independent parameters into equation 4-28, the set of equations we wish to integrate.

In the way equation 4-28 is formulated, the values of V^1, y^1 and y^2 at time $t = 0$ are required before integrating to determine the state of the network at times, $t > 0$. The d_1 independent parameters that specify all the capacitor voltages are the voltages across the branches of a tree that span the same set of nodes as V^1, then from Theorem 3 in Chapter 3, we know that these d_1 branch voltages and the components of V^1 are related by a nonsingular transformation which gives a unique value of V^1 from the d_1 parameters. However, the values of y^1 and y^2 are not always defined at $t = 0$, and it turns out that equation 4-28 has to be modified slightly to cope with the initial conditions in inductor currents.

From the $B_L - d_3$ independent parameters that specify all B_L inductor currents, we may compute

$$\begin{aligned} I_L^1 &= [A_{L1}]^T i_L \\ I_L^2 &= [A_{L2}]^T i_L \quad . \qquad\qquad (4\text{-}33) \\ I_L^3 &= [A_{L3}]^T i_L \end{aligned}$$

Although I_L is related to y by

$$\begin{bmatrix} I_L^1 \\ I_L^2 \\ I_L^3 \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} y^1 \\ y^2 \\ y^3 \end{bmatrix}, \tag{4-37}$$

we cannot in general compute y's from I_L's because the rank of the inductor matrix in equation 4-37 is invariant to the choice of coordinates, and the inductor matrix is singular whenever the inductive topology of the network does not connect all the nodes of a connected network. This difficulty is resolved by writing equation 4-28 as follows:

$$\begin{aligned} \frac{dV^1}{dt} &= [C_{11}]^{-1}\left(I^{1*} - I_L^1 - [R_{11}]V^1 - [R_{12}]V^2\right) \\ \frac{dy^1}{dt} &= V^1 \\ \frac{dy^2}{dt} &= V^2 = [R_{22}]^{-1}\left(I^{2*} - I_L^2 - [R_{21}]V^1\right) \end{aligned} \tag{4-38}$$

$$y^3 = [L_{33}]^{-1}\left(I^{3*} - [L_{31}]y^1 - [L_{32}]y^2\right) \tag{4-39}$$

I_L^1 and I_L^2 in equation 4-38 are defined in equation 4-37 and they are readily evaluable from i_L at $t = 0$ by using equation 4-33, therefore, equations 4-38 are completely evaluable at $t = 0$ to give $\frac{dV^1}{dt}$, $\frac{dy^1}{dt}$, $\frac{dy^2}{dt}$ at $t = 0$. Depending on the numerical integration method used (10), ΔV^1, Δy^1 Δy^2 may be computed where the Δ-operator is defined as

$$V^1(t + \Delta t) = V^1(t) + \Delta V^1(t). \tag{4-40}$$

However, in order to repeat the procedure to advance the computation from $t = \Delta t$ to $t = 2\Delta t$, the values of I_L^1 and I_L^2 at $t = \Delta t$ are required. They are computed in the following way.

Assuming that the inductor matrix is time independent, from equation 4-37 we have

$$\begin{bmatrix} \Delta I_L^1 \\ \Delta I_L^2 \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \end{bmatrix} \begin{bmatrix} \Delta y^1 \\ \Delta y^2 \\ \Delta y^3 \end{bmatrix} \tag{4-41}$$

where Δy^1, Δy^2 are computed from $\frac{dy^1}{dt}$, $\frac{dy^2}{dt}$ and Δy^3 is computed by taking the differential of equation 4-39,

$$\Delta y^3 = [L_{33}]^{-1}\left(\Delta I^{3*} - [L_{31}]\Delta y^1 - [L_{32}]\Delta y^2\right). \tag{4-42}$$

Finally the values of I_L^1 and I_L^2 at $t = \Delta t$ are computed by

$$\begin{aligned} I_L^1(\Delta t) &= I_L^1(0) + \Delta I_L^1(0) \\ I_L^2(\Delta t) &= I_L^2(0) + \Delta I_L^2(0) \end{aligned} \tag{4-43}$$

At this point, all quantities at the right hand side of equations 4-38 are defined at $t = \Delta t$, from which the same procedure is repeated to compute the variables at $t = 2\Delta t$ and so on.

There are two points that are worth mentioning:

(1) Although y's are introduced as $\int V dt$ in the original formulation in order to give a unified approach to the problem, their values are never defined during integration. This follows from the fact that no knowledge is assumed on the value of V for $t < 0$, and from

$$y(t) = \int Vdt = \int_{o}^{t} Vdt + y(0), \tag{4-44}$$

there is no way to determine $y(0)$ which depends on the values of V for $t<0$.

(2) I_L^3 is not used in computing equation 4-36; however, it serves as a check on the computation procedure since it must, at all time including $t = 0$, satisfy equation 4-32,

$$I^{3*} - I_L^3 = 0.$$

In concluding this chapter, the sequence of computation procedure is stated. It accepts the data of a network consisting of RLC elements interconnected in any arbitrary topology, voltage and current sources across any node pairs provided that they do not violate Kirchhoff's voltage and current laws, a set of d_1 initial conditions on capacitor voltages, and a set of $B_L - d_3$ initial conditions on inductor currents.

(1) Select V^o, V^1, V^2 and V^3 coordinates.

(2) Compute $[A_{v0}]$; $[A_{C0}]$; $[A_{C1}]$; $[A_{R0}]$; $[A_{R1}]$; $[A_{R2}]$; $[A_{L0}]$; $[A_{L1}]$; $[A_{L2}]$; $[A_{L3}]$; $[A_{i0}]$; $[A_{i1}]$; $[A_{i2}]$; $[A_{i3}]$ as defined in equation 4-21.

(3) Compute $[C_{ij}]$; $[R_{ij}]$; $[L_{ij}]$ as defined in equation 4-22, equation 4-23 and equation 4-24.

(4) Compute $V^1(t)$ and $I_L^1(t)$, $I_L^2(t)$ from the initial conditions; $I_L^3(t)$ may be computed to check with $I^{3*}(t)$.

(5) Compute $I^k(t)$ and $I^{k*}(t)$ in equation 4-25 and equation 4-29.

(6) Compute ΔV^1, Δy^1, Δy^2 from equation 4-36.

(7) Compute $V^1(t + \Delta t)$ from equation 4-38 and $I_L^1(t + \Delta t)$, $I_L^2(t + \Delta t)$ from equation 4-41, equation 4-40 and equation 4-39.

(8) Compute the particular quantities to be observed by using equation 4-21 for all branch voltages and equation 4-18 for the currents from voltage sources.

(9) Increment t by Δt and advance the computation by returning to step (5).

CHAPTER 5

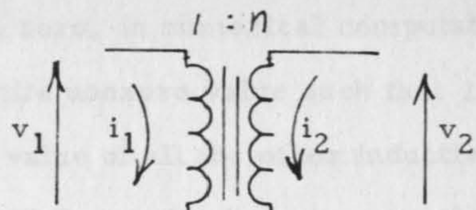
IDEAL TRANSFORMERS

In this chapter, the network studied in Chapter 4 is further generalized to contain multiple-winding ideal transformers, interconnected in any permissible topology. A non-permissible connection violates either the Kirchhoff's voltage or the Kirchhoff's current laws. A nonsingular set of node-pair coordinates is selected by appropriately removing the dependent node pairs due to the transformer constraints. Section 5.1 points out the inadequacy of using the equivalent circuit of an ideal transformer (24). 5.2 lists the forbidden transformer-winding connections. 5.3 discusses the algorithm of selecting a set of nonsingular coordinates in the presence of ideal transformers. 5.4 gives the evaluation of σ and p in a network containing ideal transformers, where σ, as used by Bryant (21), is the degree of complexity of the network and p, as used by Bers (22), is the number of nonzero frequency modes.

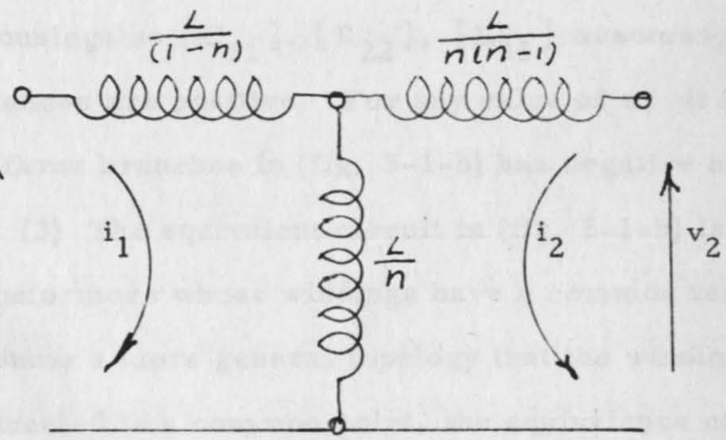
5.1 Equivalent Circuit of an Ideal Transformer.

Crosby (24) offered an equivalent circuit for a two-winding common ground transformer as shown in (fig. 5-1). In the limit that the inductive admittance L in (fig. 5-1-b) approaches to zero, the following relations are satisfied:

$$\begin{aligned} v_2 &= nv_1 \\ i_1 &= ni_2 \end{aligned} \tag{5-1}$$



(a) A Two-Winding Common-Ground Transformer



(b) The Equivalent Circuit of (a) in the Limit that $L \to 0$. All Inductors are Valued as Inductive Admittances.

Equivalent Circuit of a Two-Winding Common-Ground Transformer

FIGURE 5-1

where n is the turns ratio and v_1, v_2, i_1, i_2 are defined in (fig. 5-1-a). There are three major objections to the use of such an equivalent circuit in numerical computations:

(1) When the inductive admittance L in (fig. 5-1-b) approaches to zero, in numerical computations, it is approximated by a finite nonzero value such that L is much less than the minimum value of all the other inductive admittances in the network. This results in a very poorly conditioned inductive admittance matrix, $[L_{33}]$, such that the resulting numerical computations will introduce excessive round-off errors. This objection does not arise in purely analytical manipulations which may retain the expression L during computation and apply the limit $L \rightarrow 0$ to the end result.

(2) The algorithm that selects V^o, V^1, V^2, V^3 to give nonsingular $[C_{11}]$, $[R_{22}]$, $[L_{33}]$ assumes that all admittances are positive. For any value of n, at least one of the three branches in (fig. 5-1-b) has negative admittance.

(3) The equivalent circuit in (fig. 5-1-b) is restricted to transformers whose windings have a common terminal. If we assume a more general topology that the windings need not be connected to a common point, the equivalence cannot be applied.

In Section 5.3, a different approach is presented to select the set of nonsingular coordinates for numerical computations in a network containing multi-winding transformers connected in any permissible topology. The three types of forbidden transformer connections are listed in the following section.

5.2 Forbidden Transformer Connections.

There are three types of forbidden transformer connections:

(1) The violation of voltage law

The voltages across the transformer windings must satisfy the equation

$$\frac{e_i}{n_i} = \text{constant} \qquad (5.2)$$

where e_i is the voltage across the i^{th} winding and n_i, the relative number of turns. Equation 5-2 forbids the connection of more than one winding in the same transformer to any arbitrary voltage sources. (Fig. 5-2-a) shows a forbidden connection of this kind.

(2) The violation of current law

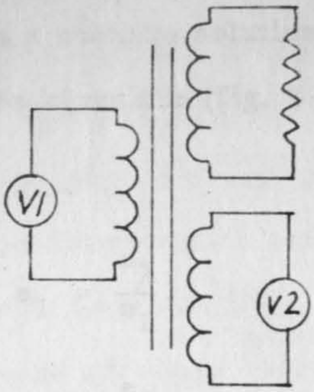
The current relationship among the windings of a transformer is

$$\sum n_i u_i = 0 \qquad (5\text{-}3)$$

where u_i is the current in the i^{th} winding. Therefore, in any transformer, at least one winding must not be connected to a current source. (Fig. 5-2-b) shows such a forbidden current relationship.

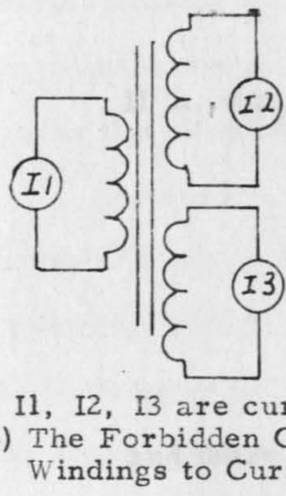
(3) Over specified dependence among winding voltages

A transformer with m windings specifies (m - 1) independent linear relationships among the m winding voltages. If the (m - 1) equations relate (m - 1) variables, then the (m - 1) x (m - 1) matrix, consisting of the coefficients in the



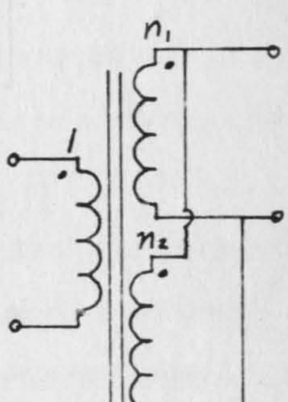
V1, V2 are voltage sources

(a) The Forbidden Connection of More Than One Winding to Voltage Sources



I1, I2, I3 are current sources

(b) The Forbidden Connection of all Windings to Current Sources



(c) Over Specified Constraint Between e_1 and e_2 where $n_1 \neq n_2$

Forbidden Transformer Connections

FIGURE 5-2-c

linear equations, must have a rank less than (m - 1) such that there exists a nonzero solution for the (m - 1) variables. For example, the circuit in (fig. 5-2-c) is over specified as such that

$$e_1 = \frac{e_2}{n_1}$$
$$e_1 = \frac{e_2}{n_2} \tag{5-4}$$

or

$$\begin{bmatrix} 1 & -\frac{1}{n_1} \\ 1 & -\frac{1}{n_2} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 0 \tag{5-5}$$

If $n_1 \neq n_2$, then

$$\det \begin{vmatrix} 1 & -\frac{1}{n_1} \\ 1 & -\frac{1}{n_2} \end{vmatrix} \neq 0$$

and there exists no solution for (e_1, e_2).

5.3 Coordinates Selection in the Presence of Ideal Transformers

The network under analysis consists of RLC elements, voltage sources, current sources and ideal transformers. The RLC elements may be interconnected in any arbitrary topology; voltage and current sources must not encounter the forbidden connections in (fig. 4-3); transformer connections must satisfy the conditions discussed in Section 5.2. Since the restrictions on the connections of voltage sources, current sources and transformer windings will not be encountered by any physical network, the method of analysis presented here will apply to any physical system that has a topological analogy to a physical network of RLC elements, voltage sources, current sources and ideal transformers.

In developing the concept of coordinate transformation in a network, Section 4 in Chapter 2 assumes a primitve network consisting of all the individual branches such as (fig. 2-2), and a set of equations are set up in terms of these primitive coordinates, namely, the branch voltages. After the branches are interconnected, a transformation matrix is obtained to relate the original branch voltages to a new set of independent coordinates. Using the results developed in Section 2.3, namely, equations 2-21, 2-23 and 2-24, the network equation in the independent coordinates is derived systematically. The same concept of coordinate transformation will be used to set up the equations of a network containing transformers, which merely introduce additional linear constraints among the coordinates.

Analogous to the procedures in Section 2.4, a primitive network is here defined as the connected network with all transformer constraints removed. Such a network consists of voltage sources, current sources, RLC elements and uncoupled transformer windings. In supplementing the description of coordinate selection, an example which represents a finite difference model of a cantilevered beam under bending is used (25). The complete circuit, including the voltage source, VS, is shown in (fig. 5-3-a). Two three-winding transformers are used to relate the deflections of the beam to its slopes. This example brings out all the features to be discussed, and it also serves to indicate the application of the generalized network study to the analysis of systems which are topologically analogous to electrical networks (2), (3), (4), (5), (6), (7).

From the primitve network with transformer constraints removed, we proceed to select the coordinates V^o, V^1, V^2, V^3, V^4 in the following sequence:

(o) From the voltage source topology, a set of independent node pairs is selected to specify all the voltage source branches. They form the components of V^o. The example in (fig. 5-3-b) has v_{07} as the only component of V^o.

(1) From the reduced capacitive topology with all voltage sources replaced by short circuiting wires, a set of independent node pairs, specifying all the capactive branch voltages in the reduced topology, is selected to form the components of V^1. The example in (fig. 5-3-c) has v_{03}, v_{05}, v_{06} as the components of V^1.

(2) From the reduced resistive topology with all voltage sources and capacitors short circuited, a set of independent node pairs, specifying all the resistive branch voltages in the reduced topology, is selected to form the components of V^2. The example in (Fig. 5-3-d) has v_{27} as the component of V^2.

(3) From the reduced inductive topology with all voltage sources, capacitors and resistors short circuited, a set of independent node pairs, specifying all the inductor branch voltages in the reduced topology, is selected to form the components of V^3. The example in (fig. 5-3-e) has v_{01} as the component of V^3.

(4) From the reduced winding topology with all elements except transformer windings short circuited, a set of independent node pairs, specifying all the winding voltages in the reduced topology, is selected to form the components of V^4. The example in (fig. 5-3-f) has v_{34} as the component of V^4.

In terms of these five vectors (V^o, V^1, V^2, V^3, V^4), the matrix equation equating the currents in the network is

$$\left(\begin{bmatrix} C_{00} & C_{01} & 0 & 0 & 0 \\ C_{10} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \frac{d}{dt} + \begin{bmatrix} R_{00} & R_{01} & R_{02} & 0 & 0 \\ R_{10} & R_{11} & R_{12} & 0 & 0 \\ R_{20} & R_{21} & R_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right.$$

$$\left. + \begin{bmatrix} L_{00} & L_{01} & L_{02} & L_{03} & 0 \\ L_{10} & L_{11} & L_{12} & L_{13} & 0 \\ L_{20} & L_{21} & L_{22} & L_{23} & 0 \\ L_{30} & L_{31} & L_{32} & L_{33} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \int dt \right) \begin{bmatrix} V^o \\ V^1 \\ V^2 \\ V^3 \\ V^4 \end{bmatrix} \tag{5-6}$$

$$+ \begin{bmatrix} I_w^o \\ I_w^1 \\ I_w^2 \\ I_w^3 \\ I_w^4 \end{bmatrix} = \begin{bmatrix} I^o \\ I^1 \\ I^2 \\ I^3 \\ I^4 \end{bmatrix}.$$

In equation 5-6 the four terms on the left hand side correspond to the currents in capacitors, resistors, inductors and transformer windings, transformed into the $(V^o,\ V^1,\ V^2,\ V^3,\ V^4)$ coordinates; the current vector on the right hand side is due to voltage and current sources. The transformation between the various branch voltages and the selected coordinates may be written as

$$\begin{bmatrix} v_V \\ v_C \\ v_R \\ v_L \\ v_W \\ v_i \end{bmatrix} = \begin{bmatrix} A_{Vo} & 0 & 0 & 0 & 0 \\ A_{Co} & A_{C1} & 0 & 0 & 0 \\ A_{Ro} & A_{R1} & A_{R2} & 0 & 0 \\ A_{Lo} & A_{L1} & A_{L2} & A_{L3} & 0 \\ A_{Wo} & A_{W1} & A_{W2} & A_{W3} & A_{W4} \\ A_{io} & A_{i1} & A_{i2} & A_{i3} & A_{i4} \end{bmatrix} \begin{bmatrix} V^o \\ V^1 \\ V^2 \\ V^3 \\ V^4 \end{bmatrix} \tag{5-7}$$

where v_W is the vector whose components are the branch voltages across transformer windings and v_V, v_C, v_R, v_L and v_i are defined in equation 4-21. Let v_B represent $(v_V,\ v_C,\ v_R,\ v_L,\ v_W,\ v_i)$ and V, $(V^o,\ V^1,\ V^2,\ V^3,\ V^4)$, equation 5-7 may be concisely written as

$$v_B = [\ A_B\ ]\ V \tag{5-8}$$

where $[\ A_B\ ]$ is the matrix in equation 5-7.

Equation 5-8 represents the constraints on the branch voltages v_B due to the interconnection of various elements into P nodes. The (P - 1) node pairs that constitute the components of V are so chosen such that in equation 5-6, $[\ C_{11}\ ]^{-1}$, $[\ R_{22}\ ]^{-1}$ and $[\ L_{33}\ ]^{-1}$ always exist. The constraints on node-pair voltages due to transformers can now be introduced on the coordinates V. Each transformer T_k

with m_k windings introduces $(m_k - 1)$ constraints on the node-pair voltages. The total number of constraints introduced by the transformers is

$$M = \sum_k (m_k - 1). \tag{5-9}$$

From these M linear constraints, a new independent set of (P - M) coordinates, V^*, is selected from the (P - 1) dimensional space which has V as a base,

$$V = [A_T] V^*. \tag{5-10}$$

$[A_T]$ in equation 5-10 is the transformation matrix derived from the transformer constraints. The important thing about $[A_T]$ is that the resulting current equation in V^* coordinate must be separable into V^{o*}, V^{1*}, V^{2*}, V^{3*} such that $[C_{11}{}^*]^{-1}$, $[R_{22}{}^*]^{-1}$, $[L_{33}{}^*]^{-1}$ always exist. This condition is satisfied if $[A_T]$ has the form in equation 5-11

$$\begin{bmatrix} V^o \\ V^1 \\ V^2 \\ V^3 \\ V^4 \end{bmatrix} = \begin{bmatrix} E & 0 & 0 & 0 \\ A_{10} & A_{11} & 0 & 0 \\ A_{20} & A_{21} & A_{22} & 0 \\ A_{30} & A_{31} & A_{32} & A_{33} \\ A_{40} & A_{41} & A_{42} & A_{43} \end{bmatrix} \begin{bmatrix} V^{o*} \\ V^{1*} \\ V^{2*} \\ V^{3*} \end{bmatrix} \tag{5-11}$$

where E is the identity matrix.

Proof:

Let $[Z]$ represent $[C]$, $[R]$ or $[L]$

then $$[Z^*] = [A_T]^T [Z] [A_T] \tag{5-12}$$

and $$[Z^*_{ij}] = \sum_{k=0}^{4} \sum_{\ell=0}^{4} [A_{ik}]^T [Z_{k\ell}] [A_{\ell j}] \tag{5-13}$$

$$i = 0,\ 1,\ 2,\ 3$$
$$j = 0,\ 1,\ 2,\ 3$$

the transformed capacitor matrices in V^* coordinates are

$$[C^*_{ij}] = 0 \qquad \text{for i or j} = 2, 3 \tag{5-14}$$

$$\begin{aligned}[C^*_{11}] = [A_{10}]^T [C_{00}] [A_{10}] &+ [A_{10}]^T [C_{01}] [A_{11}] \\ &+ [A_{11}]^T [C_{10}] [A_{10}] \\ &+ [A_{11}]^T [C_{11}] [A_{11}]\end{aligned} \tag{5-15}$$

From Theorem 9 in Chapter 3, the last term in equation 5-15, $[A_{11}]^T [C_{11}] [A_{11}]$, is positive definite and the first three terms are positive semidefinite, therefore, $[C^*_{11}]$ is positive definite and $[C^*_{11}]^{-1}$ always exists. Substituting $[R^*]$ for $[Z^*]$ in equation 5-13 we have

$$[R^*_{ij}] = 0 \qquad \text{for i or j} = 3$$

$$[R^*_{22}] = \sum_{k=0}^{2} \sum_{\ell=0}^{2} [A_{2k}]^T [R_{k\ell}] [A_{\ell 2}] \tag{5-16}$$

From Theorem 9, the term $[A_{22}]^T[R_{22}][A_{22}]$ in equation 5-16 is positive definite with all other terms positive semidefinite, therefore, $[R_{22}^*]^{-1}$ always exists.

Substituting $[L^*]$ for $[Z^*]$ in equation 5-13, we have

$$[L_{33}^*] = \sum_{k=0}^{3} \sum_{\ell=0}^{3} [A_{3k}]^T [L_{k\ell}][A_{\ell 3}] \tag{5-17}$$

The term $[A_{33}]^T[L_{33}][A_{33}]$ in equation 5-17 is positive definite with all the other terms positive semidefinite, therefore, $[L_{33}^*]^{-1}$ always exists.

The results in equations 5-14, 5-15, 5-16 and 5-17 prove that $[C_{11}^*]^{-1}$, $[R_{22}^*]^{-1}$ and $[L_{33}^*]^{-1}$ always exist if V^* is related to V by the transformation matrix in equation 5-11.

The task that remains is to determine the matrix $[A_T]$ from the M constraints introduced by the transformers.

For a transformer, T_k, with m_k windings, whose terminal voltages are denoted by $(e_{k1}, e_{k2} \ldots e_{km_k})$, there exist $(m_k - 1)$ independent relations among the m_k winding voltages. If e_{k1} is taken as the reference, then we have, for B_T transformers,

$$e_{k\ell} = \frac{n_{k\ell}}{n_{k1}} e_{k1}. \qquad \begin{array}{l} \ell = 2, 3 \ldots m_k \\ k = 1, 2 \ldots B_T \end{array} \tag{5-18}$$

There are a total of

$$M = \sum_{k=1}^{B_T} (m_k - 1)$$

independent linear constraints among the winding voltages v_w. Since each e_{kj}, (for $k = 1, 2 \ldots B_T$ and $j = 1, 2 \ldots m_k$), is a member of the set v_w, the set of M constraints among v_w may be translated into a set of M constraints among $(V^o, V^1, V^2, V^3, V^4)$ by using the transformation matrices in equation 5-7.

$$v_w = [\, A_{wo} \,] V^o + [\, A_{w1} \,] V^1 + [\, A_{w2} \,] V^2 + [\, A_{w3} \,] V^3 + [\, A_{w4} \,] V^4 \quad . \tag{5-19}$$

After substituting equation 5-19 into equation 5-18, we have the following M linear equations in $(V^o, V^1, V^2, V^3, V^4)$.

$$f_{ko}(V^o) + f_{k1}(V^1) + f_{k2}(V^2) + f_{k3}(V^3) + f_{k4}(V^4) = 0 \tag{5-20}$$
$$k = 1, 2 \ldots M$$

where f's are linear functions of their arguments.

At this point it is convenient to introduce a hierarchy among the coordinates $(V^o, V^1, V^2, V^3, V^4)$. We say that V^k is of a higher hierarchy than V^j, if $j > k$, and it is denoted as $V^k > V^j$, therefore, we have

$$V^o > V^1 > V^2 > V^3 > V^4. \tag{5-21}$$

With the hierarchy defined in equation 5-21, the algorithm that gives $[\, A_T \,]$ from the M linear equations in equation 5-20 is described below.

The algorithm aims to divide the original coordinates into two parts,

$$
\begin{aligned}
V^{o} &= V^{o*} \\
V^{1} &= \begin{bmatrix} V^{1d} \\ V^{1*} \end{bmatrix} \\
V^{2} &= \begin{bmatrix} V^{2d} \\ V^{2*} \end{bmatrix} \\
V^{3} &= \begin{bmatrix} V^{3d} \\ V^{3*} \end{bmatrix} \\
V^{4} &= V^{4d}
\end{aligned}
\tag{5-22}
$$

where the d superscript denotes the dependent components to be eliminated by using the M equations in equation 5-20 and the * superscript denotes the components to be retained. In order to obtain $[\,A_T\,]$ in the form specified by equation 5-11, we must have

$$
V^{kd} = \sum_{\ell=0}^{k} A'_k \; V^{\ell *} \qquad \text{for } k = 1,\ 2,\ 3,\ 4 \tag{5-23}
$$

such that

$$
V^{k} = \begin{bmatrix} V^{kd} \\ V^{k*} \end{bmatrix} = \begin{bmatrix} A'_{ko} & A'_{k1} & - - - & A'_{kk} \\ 0 & 0 & & E \end{bmatrix} \begin{bmatrix} V^{o*} \\ V^{1*} \\ \vdots \\ V^{k*} \end{bmatrix} \tag{5-24}
$$

In equation 5-22 we must have $V^{o} = V^{o*}$ such that none of the node pairs that specify the voltage sources may be eliminated. This follows directly from the first forbidden transformer connections stated in Section 5.2. In equation 5-22, we also have $V^{4} = V^{4d}$, that is all the node pairs selected from the reduced winding topology with all elements, except windings, short circuited, can be eliminated from the M equations in equation 5-20. This condition is always satisfied if there is no redundant transformer winding in the circuit. A redundant transformer winding is defined to be the winding which can be removed from the network without changing the network characteristic. (Fig. 5-4) shows a network with two redundant transformers.

The algorithm that gives the equation of the form in equation 5-23 is best described as a recursive function on two objects, L1 and L2. L1 is the object that consists of a set of (M - K) linear equations in the form of equation 5-20, and L2 is the object that consists of a set of K linear equations in the form of equation 5-23. Then the recursive function F(L1, L2) is defined as follows:

If L1 contains no equation (i.e., M - K = 0), then

F(L1, L2) = L2;

otherwise,

F(L1, L2) = F(L1*, L2*)

where L1* and L2* are derived from L1 and L2 in the following way:

(1) One equation is taken from the (M - K) equations in L1, and define the remaining (M - K - 1) equations to be L1'.

(2) The equation taken from L1 has the form in equation 5-20. Using the hierarchy defined in equation 5-21, express one component V^{kd} of the lowest hierarchy coordinate in terms of all other components with the same or higher hierarchy.

(3) The expression obtained in (2) is substituted for all appearances of V^{kd} in L1' and L2. L1* is then defined as the new L1'; and L2* is defined to be the union of the new L2 and the equation obtained in (2).

The object L1* contains M - K - 1 equations, and the object L2* contains K + 1 equations.

With the recursive function F(L1, L2) defined as above, the set of equations in equation 5-23 is derived from the M equations in equation 5-20 by setting

$$\text{equation 5-23} = F(\text{equation 5-20, NIL}) \tag{5-25}$$

where NIL represents an empty object L2, i.e., an L2 that contains no equation at all.

The algorithm that eliminates the appropriate set of node pairs is deliberately described in the recursive language, since it is concise and easy to implement in a symbol manipulating language for a digital computer such as LISP (26), or IPL (27).

With equation 5-8 and equation 5-10, we may transform v_B directly into V*,

$$v_B = [\, A_B \,][\, A_T \,]\, V^* . \tag{5-26}$$

Let $\qquad [\, A \,] = [\, A_B \,][\, A_T \,],$

then we may compute $[\, C^* \,]$, $[\, R^* \,]$, $[\, L^* \,]$ and the corresponding currents in (V^{1*}, V^{2*}, V^{3*}) coordinates directly from the branch matrices: $[\, C_B \,]$, $[\, R_B \,]$, $[\, L_B \,]$ and the current sources, J_i.

In concluding this section, the example in (fig. 5-3) will be used to illustrate the working principle of coordinate transformation introduced by transformer windings.

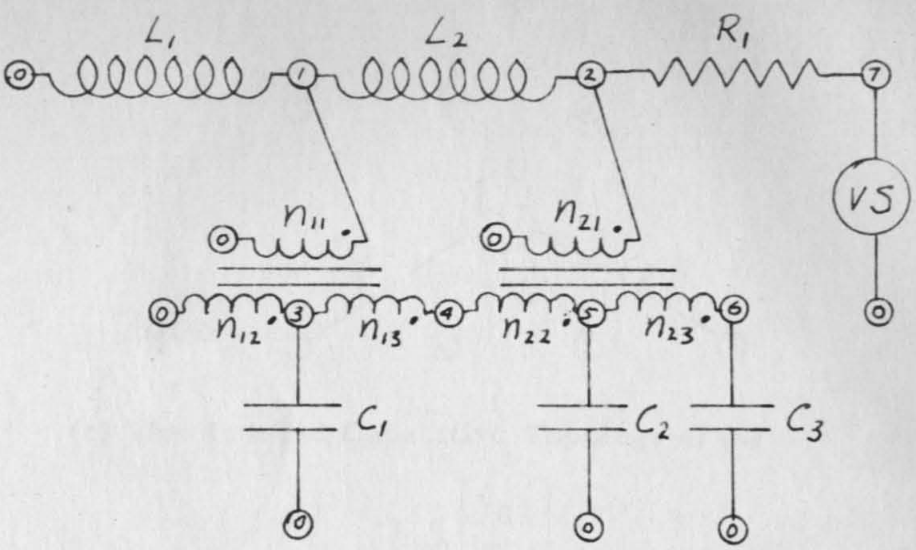
From (fig. 5-3), we have

$$
\begin{aligned}
V^{o} &= (v_{07}) \\
V^{1} &= (v_{03}, v_{05}, v_{06}) \\
V^{2} &= (v_{27}) \\
V^{3} &= (v_{01}) \\
V^{4} &= (v_{34})
\end{aligned}
\qquad (5\text{-}28)
$$

From (fig. 5-3-a), we have

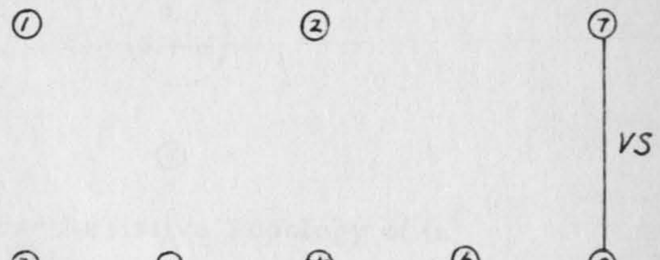
$$
\begin{aligned}
v_{v} &= (v_{07}) \\
v_{C} &= (v_{03}, v_{05}, v_{06}) \\
v_{R} &= (v_{27}) \\
v_{L} &= (v_{01}, v_{12}) \\
v_{W} &= (v_{01}, v_{03}, v_{34}, v_{02}, v_{45}, v_{56}) \\
v_{i} &= 0
\end{aligned}
\qquad (5\text{-}29)
$$

From equation 5-29 and equation 5-28, the transformation matrices are computed as



VS - Voltage Source

(a) The Circuit Analogy of a Cantilevered Beam Under Bending

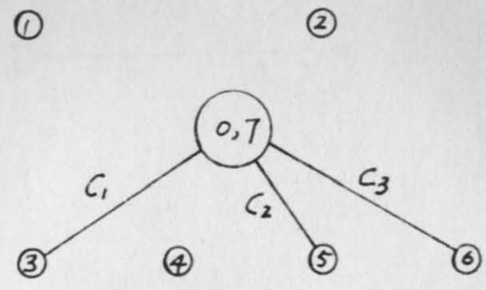


(b) The Voltage Source Topology of the Circuit in (a)

$$V^{o} = (v_{07})$$

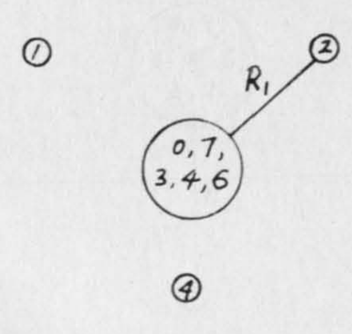
AN EXAMPLE

FIGURE 5-3



(c) The Reduced Capacitive Topology of (a)

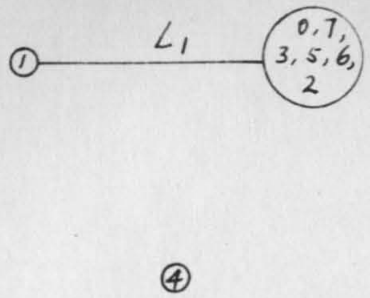
$$V^1 = \begin{bmatrix} v_{03} \\ v_{05} \\ v_{06} \end{bmatrix}$$



(d) The Reduced Resistive Topology of (a)

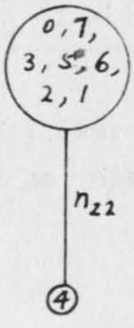
$$V^2 = [\, v_{27} \,]$$

FIGURE 5-3 (continued)



(e) The Reduced Inductive Topology of (a)

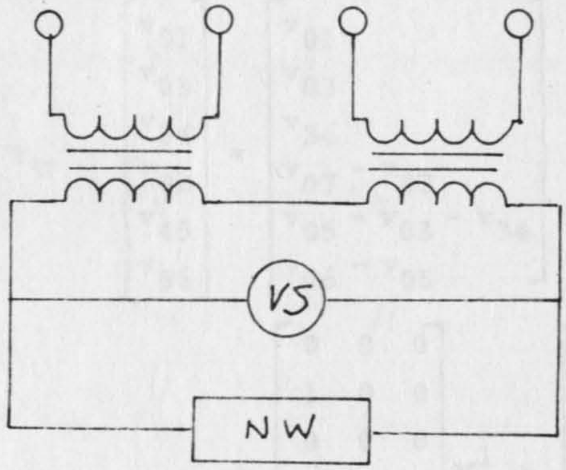
$$V^3 = (v_{01})$$



(f) The Reduced Transformer Winding Topology of (a)

$$V^4 = (v_{34})$$

FIGURE 5-3 (continued)



VS - Voltage Source

NW - Any other network

A Circuit with Two Redundant Transformers

FIGURE 5-4

$$v_V = [v_{07}] = [E]\,V^o$$

$$v_C = \begin{bmatrix} v_{03} \\ v_{05} \\ v_{06} \end{bmatrix} = [E]\,V^1$$

$$v_R = [v_{27}] = [E]\,V^2$$

$$v_L = \begin{bmatrix} v_{01} \\ v_{12} \end{bmatrix} = \begin{bmatrix} v_{01} \\ v_{07} - v_{27} - v_{01} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} V^o + \begin{bmatrix} 0 \\ -1 \end{bmatrix} V^2 + \begin{bmatrix} 1 \\ -1 \end{bmatrix} V^3$$

$$v_W = \begin{bmatrix} v_{01} \\ v_{03} \\ v_{34} \\ v_{02} \\ v_{45} \\ v_{56} \end{bmatrix} = \begin{bmatrix} v_{01} \\ v_{03} \\ v_{34} \\ v_{07} - v_{27} \\ v_{05} - v_{03} - v_{34} \\ v_{06} - v_{05} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} V^o \tag{5-30}$$

$$+ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} V^1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} V^2$$

$$+ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} V^3 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} V^4$$

The two transformers give the following linear equations in terms of v_W:

$$
\begin{aligned}
v_{03} &= \frac{n_{12}}{n_{11}} v_{01} \\
v_{34} &= \frac{n_{13}}{n_{11}} v_{01} \\
v_{45} &= \frac{n_{22}}{n_{21}} v_{02} \\
v_{56} &= \frac{n_{23}}{n_{21}} v_{02}
\end{aligned}
\tag{5-31}
$$

After substituting V^o, V^1, V^2, V^3, V^4 into equation 5-31 by using the transformation matrices in equation 5-30, we have

$$
\begin{aligned}
v_{03} &= \frac{n_{12}}{n_{11}} v_{01} \\
v_{34} &= \frac{n_{13}}{n_{11}} v_{01} \\
v_{05} - v_{03} - v_{34} &= \frac{n_{22}}{n_{21}} (v_{07} - v_{27}) \\
v_{06} - v_{05} &= \frac{n_{23}}{n_{21}} (v_{07} - v_{27})
\end{aligned}
\quad . \tag{5-32}
$$

The algorithm, defined as a recursive function in equation 5-25, is applied to equation 5-32.

L1	L2
equation 5-32	NIL
$v_{34} = \frac{n_{13}}{n_{12}} v_{03}$ $v_{05} - v_{03} - v_{34} = \frac{n_{22}}{n_{21}} (v_{07} - v_{27})$ $v_{06} - v_{05} = \frac{n_{23}}{n_{21}} (v_{07} - v_{27})$	$v_{01} = \frac{n_{11}}{n_{12}} v_{03}$
$v_{05} - v_{03} - \frac{n_{13}}{n_{12}} v_{03} = \frac{n_{22}}{n_{21}} (v_{07} - v_{27})$ $v_{06} - v_{05} = \frac{n_{23}}{n_{21}} (v_{07} - v_{27})$	$v_{01} = \frac{n_{11}}{n_{12}} v_{03}$ $v_{34} = \frac{n_{13}}{n_{12}} v_{03}$
$v_{06} - v_{05} = \frac{n_{23}}{n_{22}} \quad v_{05} - \frac{n_{12} + n_{13}}{n_{12}} v_{03}$	$v_{01} = \frac{n_{11}}{n_{12}} v_{03}$ $v_{34} = \frac{n_{13}}{n_{12}} v_{03}$ $v_{27} = \frac{n_{21}}{n_{22}} \left(-v_{05} + \frac{n_{12} + n_{13}}{n_{12}} v_{03}\right) + v_{07}$
NIL	$v_{01} = \frac{n_{11}}{n_{12}} v_{03}$ $v_{34} = \frac{n_{13}}{n_{12}} v_{03}$ $v_{27} = \frac{n_{21}}{n_{22}} \left(\frac{n_{22}(n_{12} + n_{13})}{n_{12}(n_{22} + n_{23})} v_{03} - \frac{n_{22}}{n_{22} + n_{23}} v_{06} \right) + v_{07}$ $v_{05} = \frac{n_{22}}{n_{22} + n_{23}} \left(v_{06} + \frac{n_{23}(n_{12} + n_{13})}{n_{22} n_{12}} v_{03} \right)$

TABLE 5-1

The successive changes in L1 and L2 as the recursive function in equation 5-25 is applied, are listed in Table 5-1. The final coordinates are

$$
\begin{aligned}
V^{o*} &= v_{07} \\
V^{1*} &= \begin{bmatrix} v_{03} \\ v_{06} \end{bmatrix} \\
V^{2*} &= 0 \\
V^{3*} &= 0
\end{aligned}
\tag{5-33}
$$

With the last entry of L2 in Table 5-1 substituted into equation 5-30, and defined

$$
\begin{aligned}
n_1 &= \frac{n_{11}}{n_{12}} \\
n_2 &= \frac{n_{13}}{n_{12}} \\
n_3 &= \frac{n_{21}(n_{12}+n_{13})}{n_{12}(n_{22}+n_{23})} \\
n_4 &= -\frac{n_{21}}{n_{22}+n_{23}} \\
n_5 &= \frac{n_{22}}{n_{22}+n_{23}} \\
n_6 &= \frac{n_{23}(n_{12}+n_{13})}{n_{12}(n_{22}+n_{23})}
\end{aligned}
\tag{5-34}
$$

we have

$$
\begin{aligned}
v_v &= \{v_{07}\} = [\,E\,]\,V^{o*} \\
v_L &= \begin{bmatrix} v_{03} \\ v_{05} \\ v_{06} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ n_6 & n_5 \\ 0 & 1 \end{bmatrix} V^{1*} \\
v_R &= \{v_{27}\} = [\,E\,]\,V^{o*} \qquad (5\text{-}35) \\
v_L &= \begin{bmatrix} v_{01} \\ v_{12} \end{bmatrix} = \begin{bmatrix} n_1 v_{03} \\ \cancel{v_{07}} - n_3 v_{03} - n_4 v_{06} - \cancel{v_{07}} - n_1 v_{03} \end{bmatrix} \\
&= \begin{bmatrix} n_1 & 0 \\ -(n_1 + n_3) & -n_4 \end{bmatrix} V^{1*}
\end{aligned}
$$

The matrices $[\,C^*\,]$, $[\,R^*\,]$, $[\,L^*\,]$ in the coordinates $(V^{o*},\ V^{1*})$ are computed as follows:

$$
[\,C^*\,] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & C_1 + n_6^{\,2} C_2 & n_6 n_5 C_2 \\ 0 & n_6 n_5 C_2 & n_5^{\,2} C_2 + C_3 \end{bmatrix}
$$

$$
[\,R^*\,] = \begin{bmatrix} R_1 & n_3 R_1 & n_4 R_1 \\ n_3 R_1 & n_3^{\,2} R_1 & n_3 n_4 R_1 \\ n_4 R_1 & n_3 n_4 R_1 & n_4^{\,2} R_1 \end{bmatrix} \qquad (5\text{-}36)
$$

$$
[\,L^*\,] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \begin{matrix} n_1^{\,2} L_1 \\ +(n_1 + n_3)^2 L_2 \end{matrix} & (n_1 + n_3) n_4 L_2 \\ 0 & (n_1 + n_3) n_4 L_2 & n_4 L_2 \end{bmatrix}
$$

With the admittance matrices in equation 5-36, the current equation in the form of equation 4-28 can be formulated. When the initial conditions are given, and the time dependence of the voltage source specified, the independent set of variables, V^{1*} may be integrated.

5.4 <u>σ and p of a Network with Ideal Transformers</u>

σ is the number of independent parameters to specify the complete energy distribution in the network. The expression σ given by equation 4-32 also holds for a network with ideal transformers.

$$\sigma = d_1^{\,*} + B_L - d_3^{\,*} \tag{5-37}$$

where $d_1^{\,*}$ and $d_3^{\,*}$ are the number of components of V^{1*} and V^{3*}.

p is the number of nonzero frequency modes of the network. The expression of p given in equation 4-33 also applies to networks with ideal transformers, if the value $(B_L - \ell_L)$ is replaced by $RK([L^*])$, which is defined to be the rank of the matrix $[L^*]$.

$$p = d_1^{\,*} - d_3^{\,*} + RK([L^*]) \tag{5-38}$$

The results in equation 5-37 and equation 5-38 are stronger than the ones given by Bryant (21) and Bers (22) since equations 5-37 and 5-38 apply to a larger class of networks that contain transformers, with Bryant's and Bers' model as a special case.

The example in (fig. 5-3) has

$$\sigma = 4$$

$$p = 4.$$

CHAPTER 6

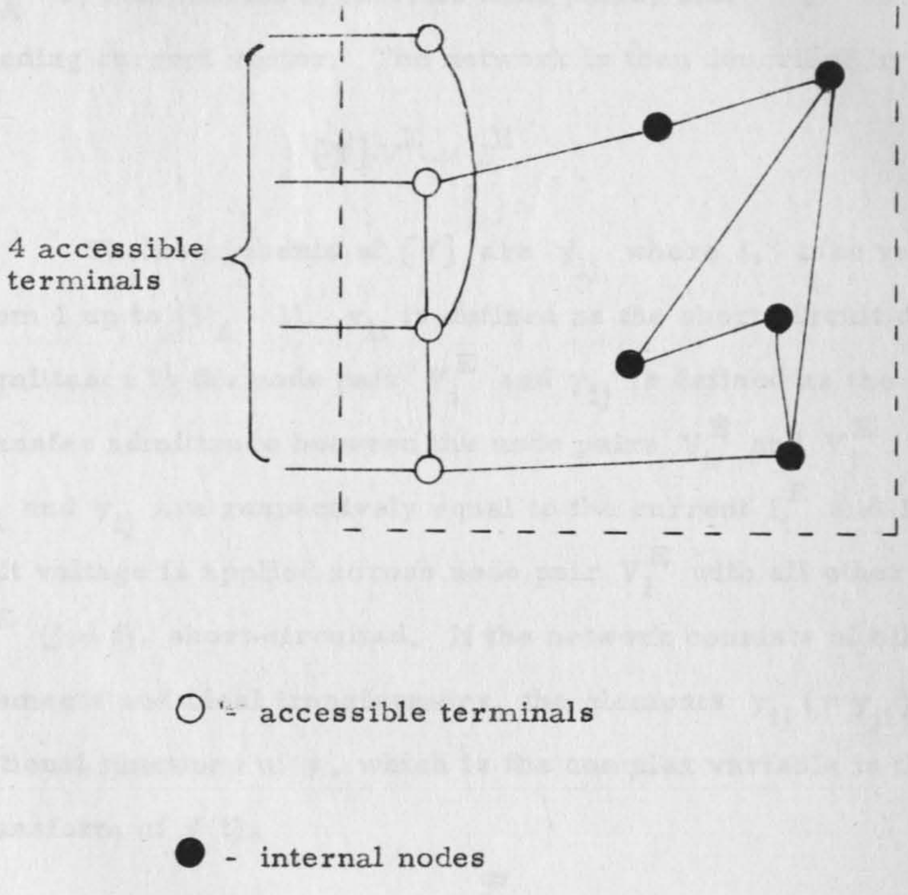
COMPUTATIONS OF DRIVING-POINT AND TRANSFER ADMITTANCES

This chapter develops the method that computes the poles and zeros of the short circuit driving-point and transfer admittances (page 153, reference 28), associated with an arbitrarily selected independent set of accessible node pairs in a network which consists of an arbitrary interconnection of resistors, inductors, capacitors and ideal transformers. The problem is first defined in Section 6.1, followed by a discussion on the inadequacy of applying conventional recursive formulae to networks with arbitrary topology. Here also are formulated the methods of admittances determination in terms of polynomial matrix operations. Section 6.2 solves the matrix polynomial equation developed in Section 6.1. The method of solution requires a nonsingular set of coordinates selected in the same way as in Chapters 3 and 5. Section 6.3 works out an example of a two-port network.

6.1 The Problem of Driving-Point- and Transfer-Admittances Computation

(1) Definitions of short circuit driving-point- and transfer-admittances

A network consisting of P connected nodes has (P - 1) independent node pairs. When a subset of the P nodes, say P_A nodes, are accessible, there are (P_A - 1) independent accessible terminal-pairs (or node pairs, ports). (Fig. 6-1) shows an arbitrary network with four accessible terminals. From theorem 2 in Chapter 3, we



An Arbitrary Network with 4 Accessible Terminals

FIGURE 6-1

know that there are $S(P_A)$ different ways to pick a set of $(P_A - 1)$ independent node pairs. These different sets are related by a group of nonsingular transformations, and any one of them may be used to describe the network property at the accessible ports. Let V^E be the vector whose components $(V_1^E, V_2^E, \ldots, V_{P_A-1}^E)$ are the $(P_A - 1)$ independent accessible node pairs, and I^E be the corresponding current vector. The network is then described by the equation

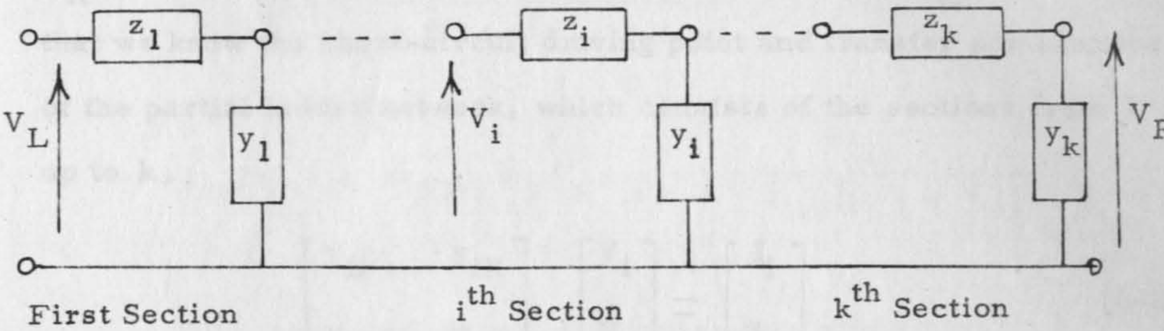
$$[Y]\, V^E = I^E \tag{6-1}$$

The components of $[Y]$ are y_{ij} where i, j take values ranging from 1 up to $(P_A - 1)$. y_{ii} is defined as the short circuit driving point admittance to the node pair V_i^E and y_{ij} is defined as the short-circuit transfer admittance between the node pairs V_i^E and V_j^E. Literally, y_{ii} and y_{ij} are respectively equal to the current I_i^E and I_j^E when a unit voltage is applied across node pair V_i^E with all other node pairs, V_j^E $(j \neq i)$, short-circuited. If the network consists of bilateral RLC elements and ideal transformers, the elements y_{ij} $(= y_{ji})$ are rational functions of s, which is the complex variable in the Laplace transform of $f(t)$,

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt \quad .$$

(2) Evaluation of y_{ij} for a ladder network

A ladder network has a highly regular topology. It is an iterative connection of many sections with identical topology, namely, T-sections or π sections. (Fig. 6-2) shows a ladder network consisting of K sections, where the i^{th} section is characterized by z_i,



A Ladder Network

FIGURE 6-2

the series impedance, and y_i, the parallel admittance. A ladder network is often characterized by two accessible terminal pairs, one at each end. Due to the regularity of the ladder topology, y_{ij} may be evaluated by adding one section at a time, and each time the same recursive formula is used. For example, if we let the terminal-pair voltage at the right side end of the ladder network in (fig. 6-2) be V_R, and the terminal-pair voltage at the left side be V_L, and assume that we know the short-circuit driving point and transfer admittances of the partial ladder network, which consists of the sections from i up to k,

$$\begin{bmatrix} y_{ii} & y_{iR} \\ y_{Ri} & y_{RR} \end{bmatrix}_i \begin{bmatrix} V_i \\ V_R \end{bmatrix} = \begin{bmatrix} I_i \\ I_R \end{bmatrix} \tag{6-2}$$

The recursive formula will give the short circuit driving point and transfer admittances of the augmented network which consists of the sections from (i-1) up to K.

$$\begin{bmatrix} y_{i-1,\,i-1} & y_{i-1,\,R} \\ y_{R,\,i-1} & y_{R,\,R} \end{bmatrix}_{i-1} \begin{bmatrix} V_{i-1} \\ V_R \end{bmatrix} = \begin{bmatrix} I_{i-1} \\ I_R \end{bmatrix} \tag{6-3}$$

In equation 6-3, each y_{kj} (k, j = i-1, R) is a function of z_{i-1}, and y_{i-1} of the $(i-1)^{th}$ section, and the y_{kj} (k, j = i, R) in equation 6-2. This may be stated in a functional form as

$$\begin{bmatrix} y_{i-1,\,i-1} & y_{i-1,\,R} \\ y_{R,\,i-1} & y_{R,\,R} \end{bmatrix}_{i-1} = F_i \left(\begin{bmatrix} y_{ii} & y_{iR} \\ y_{Ri} & y_{RR} \end{bmatrix}_i ,\ z_{i-1},\ y_{i-1} \right). \tag{6-4}$$

The function F_i in equation 6-4 only depends on the topology of connection of the $(i-1)^{th}$ section. For a ladder network of iterative sections, F_i is independent of i, and defined as

$$y_{i-1,\,i-1} = \frac{y_{i-1} + y_{ii}}{1 + z_{i-1}(y_{i-1} + y_{ii})}$$

$$y_{i-1,\,R} = \frac{y_{iR}}{1 + z_{i-1}(y_{i-1} + y_{ii})} \tag{6-5}$$

$$(y_{RR})_{i-1} = (y_{RR})_i + y_{iR}\, z_{i-1} \frac{y_{iR}}{1 + z_{i-1}(y_{i-1} + y_{ii})} \quad .$$

With the trivial case of the last section alone,

$$\begin{bmatrix} y_{KK} & y_{KR} \\ y_{RK} & y_{RR} \end{bmatrix}_K \begin{bmatrix} V_K \\ V_R \end{bmatrix} = \begin{bmatrix} I_K \\ I_R \end{bmatrix} \tag{6-6}$$

where

$$\begin{aligned} y_{KK} &= \frac{1}{z_K} \\ y_{KR} &= \frac{1}{z_K} \\ y_{RR} &= y_K + \frac{1}{z_K} \quad , \end{aligned} \tag{6-7}$$

equation 6-4 may be applied repeatedly until all sections are included.

The example of the ladder network illustrates one way of evaluating y_{ij}. However, this method is highly restrictive. It requires a regular iterative network topology, and the iterative topology must be simple enough such that the iterative function F_i in equation 6-4 is derivable in reasonably simple form. It is obvious that such a method cannot be applied to networks with general irregular topology. The next paragraph presents a unified approach which formulates y_{ij} as matric polynomials in s.

(3) Evaluation of y_{ij} for a network with arbitrary topology

When the current equation

$$[C] \frac{d V(t)}{dt} + [R] V(t) + [L] \int V(t)\, dt = I(t) \qquad (6\text{-}8)$$

in the time domain is transformed to the complex frequency domain by the Laplace transform (30),

$$[C] s V(s) + [R] V(s) + [L] \frac{1}{s} V(s) = I(s) , \qquad (6\text{-}9)$$

the differential equation is transformed into an algebraic equation. For the remainder of this chapter, we concern ourselves with the algebraic equation in s.

A P-node network of arbitrarily interconnected resistors, inductors, capacitors and ideal transformers is taken as the model. It is assumed that $(P_A - 1)$ independent node pairs, forming the vector V^E, can be selected from the P_A accessible terminals. (If transformers are so connected that some of the accessible node pairs are constrainted, the resulting set of unconstrainted node pairs are taken.) From the remaining nodes, another $(P - P_A - M)$ independent node pairs, forming the vector V^*, may be selected where M is the number of constraints introduced by ideal transformers. In terms of the coordinates (V^E, V^*) the matrix equation is formulated as

$$\left(\begin{bmatrix} C_{EE} & C_{E*} \\ C_{*E} & C_{**} \end{bmatrix} s + \begin{bmatrix} R_{EE} & R_{E*} \\ R_{*E} & R_{**} \end{bmatrix} + \begin{bmatrix} L_{EE} & L_{E*} \\ L_{*E} & L_{**} \end{bmatrix} \frac{1}{s} \right) \begin{bmatrix} V^E \\ V^* \end{bmatrix} = \begin{bmatrix} I^E \\ I^* \end{bmatrix} \quad (6\text{-}10)$$

Equation 6-10 can be transformed to the form of equation 6-1 by eliminating V^* from the first equation in equations 6-10. Since V^* is selected completely outside of the accessible node pairs, we have

$$I^* = 0 \quad . \quad (6\text{-}11)$$

The second equation in the partitioned matrix equation, equation 6-10, can be written as

$$V^* = - \left([C_{**}] s + [R_{**}] + [L_{**}] \frac{1}{s} \right)^{-1} \left([C_{*E}] s + [R_{*E}] + [L_{*E}] \frac{1}{s} \right) V^E \quad . \quad (6\text{-}12)$$

In order to keep the presentation simple to read, matrices with polynomial coefficients are defined as

$$[H] = [C] s^2 + [R] s + [L] \quad . \quad (6\text{-}13)$$

Then equation 6-12 becomes

$$V^* = -[H_{**}]^{-1} [H_{*E}] V^E \quad . \quad (6\text{-}14)$$

Substituting equation 6-14 into the first equation in equations 6-10, we have

$$\frac{1}{s}\left([H_{EE}] - [H_{E*}][H_{**}]^{-1}[H_{*E}]\right)V^E = I^E \quad . \qquad (6\text{-}15)$$

Comparing equation 6-15 with equation 6-1, the following relation is established.

$$s[Y] = [H_{EE}] - [H_{E*}][H_{**}]^{-1}[H_{*E}] \quad . \qquad (6\text{-}16)$$

Equation 6-16 will give the short circuit driving point and transfer admittances of the selected set of accessible node pairs, V^E. Nothing has been mentioned about the feasibility and the algorithm of computing the inverse of $[H_{**}]$, whose elements are polynomials in s . This is treated in the next section.

6.2 Matrix Operations Over the Field of Rational Functions

(1) The concept of a field (Chapter 3, reference 31)

We assume as given a non-empty set F of elements a, b, c, etc. F is a field if we can define two binary operations on its elements such that the following laws hold:

I. Laws of Addition

(a) The commutative law -

$$a + b = b + a$$

(b) The associative law -

$$a + (b + c) = (a + b) + c$$

(c) The reversibility of addition, i. e., the equation

$$a + x = b$$

is always solvable in F for x .

II. Laws of Multiplication

(d) The commutative law -

$$a \cdot b = b \cdot a$$

(e) The associative law -

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

(f) The reversibility of multiplication, i.e., the equation

$$a \cdot x = b$$

is always solvable in F for x, if $a \neq 0$.

(g) The existence of an element different from 0.

III. Distributive Law

(h) If a, b, c are any three elements in F, then

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

We can see easily that all rational numbers form the elements of a field, as do all the complex numbers.

(2) Calculation with matric polynomials (page 298, reference 31)

Since the theorems in matrices and the determinant theory are derived solely on the assumption that their entries were elements of a field, we may apply all the theorems to the calculations of matric polynomials if we can set up a field whose elements contain all polynomials. The domain of all polynomials is itself not a field because the axiom of reversibility of multiplication (i.e., the possibility of division) is not always satisfied. However, the domain of all rational functions constitutes a field, and the domain of polynomials is imbedded in this field with the denominator polynomial being equal to one.

A rational function is defined as the ratio of two polynomials; therefore,

$$\frac{f_1}{g_1}\ ,\ \frac{f_2}{g_2}$$

are rational functions where f_1, f_2, g_1, g_2 are polynomials.

The binary operation of addition is defined as

$$\frac{f_1}{g_1} + \frac{f_2}{g_2} = \frac{f_1 g_2 + f_2 g_1}{g_1 g_2} . \tag{6-17}$$

The polynomial f is defined as

$$f = \frac{f}{1} . \tag{6-18}$$

The binary operation of multiplication is defined as

$$\frac{f_1}{g_1} \cdot \frac{f_2}{g_2} = \frac{f_1 \cdot f_2}{g_1 \cdot g_2} . \tag{6-19}$$

The inverse is defined as

$$\left(\frac{f_1}{g_1}\right)^{-1} = \frac{g_1}{f_1} , \tag{6-20}$$

Two rational functions are equal if

$$f_1 g_2 = f_2 g_1 . \tag{6-21}$$

All matrix theorems apply to matrices whose entries are rational functions, which include polynomials as special cases. The calculations involving polynomial matrices may lead outside the domain of polynomial matrices, however always within the domain of rational functions. It is easy to see now that the inverse of a polynomial matrix may very well have entries which are rational functions. In fact, the inverse of a polynomial matrix is also a polynomial matrix only when the determinant of the matrix is equal to a scalar.

(3) Computation of $[Y]$ in equation 6-16

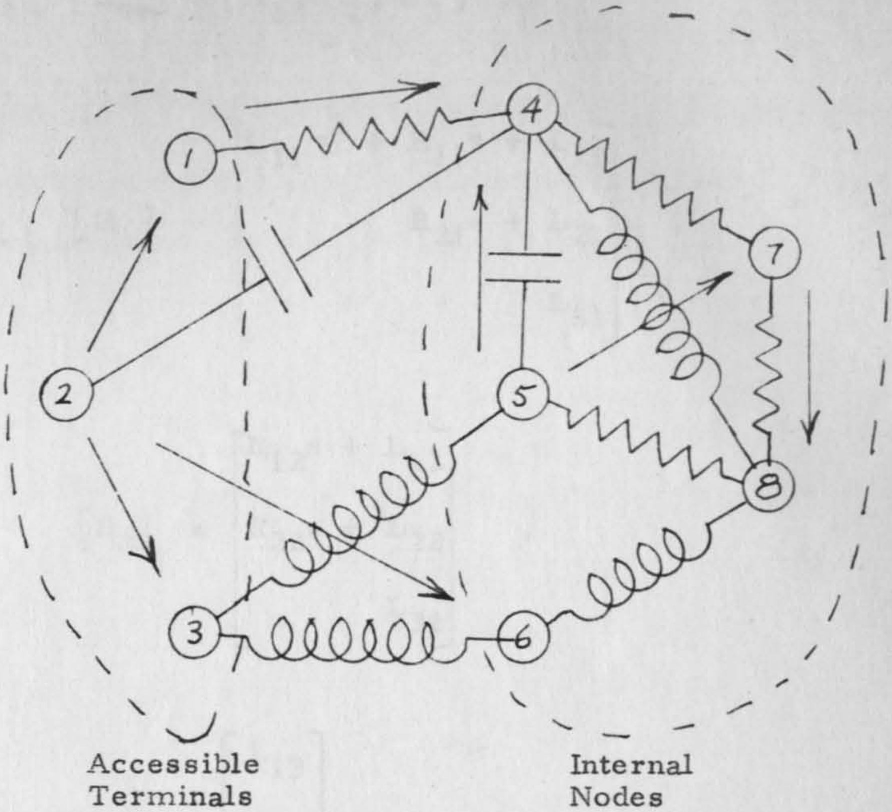
From the definition of $[H]$ in equation 6-13, the equation we want to solve, equation 6-16, is a matric polynomial in s , the solution of which is in general a matric rational function. The necessary and sufficient condition that $[H_{**}]^{-1}$ exists is that det $|H_{**}| \neq 0$, or that $[H_{**}]$ must have a rank equal to its dimension. It will first be proved that $[H_{**}]$ in equation 6-16 has nonzero determinant, and then a method is described to compute $[H_{**}]^{-1} \cdot [H_{*E}]$.

(a) Existence of H_{**}^{-1}

$[H_{**}]$ is the admittance matrix in the set of independent coordinates V^*. The $(P - P_A - M)$ independent node pairs selected from the $(P - P_A)$ internal nodes plus the grouped node, consisting of all the accessible nodes, can always be divided into three classes, (V^1, V^2, V^3), such that the partitioned $[H_{**}]$ has the form

$$[H_{**}] = \begin{bmatrix} C_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} s^2 + \begin{bmatrix} R_{11} & R_{12} & 0 \\ R_{21} & R_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} s + \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \tag{6-22}$$

and $[C_{11}]^{-1}$, $[R_{22}]^{-1}$, and $[L_{33}]^{-1}$ always exist (Chapters 3 and 5). For example, (fig. 6-3) shows a set of V^1, V^2, V^3 that will partition $[H_{**}]$ into the form of equation 6-22. $[H_{**}]$ can now be written as



$$V^E = \begin{bmatrix} v_{21} \\ v_{23} \end{bmatrix}$$

$$V^* = \begin{bmatrix} V^1 \\ V^2 \\ V^3 \end{bmatrix}$$

where

$$V^1 = \begin{bmatrix} v_{14} \\ v_{54} \end{bmatrix}$$

$$V^2 = \begin{bmatrix} v_{57} \\ v_{78} \end{bmatrix}$$

$$V^3 = [v_{26}]$$

The Node-pair Selection in Terms of Accessible Terminals and Internal Nodes.

FIGURE 6-3

$$[H_{**}] = [H_1, H_2, H_3] \quad , \tag{6-23}$$

where

$$[H_1] = \begin{bmatrix} C_{11}s^2 + R_{11}s + L_{11} \\ R_{21}s + L_{21} \\ L_{31} \end{bmatrix} \quad , \tag{6-24}$$

$$[H_2] = \begin{bmatrix} R_{12}s + L_{12} \\ R_{22}s + L_{22} \\ L_{32} \end{bmatrix} \quad , \tag{6-25}$$

$$[H_3] = \begin{bmatrix} L_{13} \\ L_{23} \\ L_{33} \end{bmatrix} \quad . \tag{6-26}$$

Since $[C_{11}]$ is positive definite, $[H_1]$ has rank d_1 where d_1 is the number of components in V^1, which is also the rank and dimension of $[C_{11}]$. Likewise, $[H_2]$ and $[H_3]$ have ranks of d_2 and d_3, respectively. We want to prove that the matrix $[H_{**}]$ has rank of $(d_1 + d_2 + d_3)$, or in other words, we want to prove that every column vector in $[H_i]$ is independent from all the column vectors in $[H_j]$ where $i, j = (1, 2, 3)$ and $i \neq j$.

Let us first take a column in $[H_2]$ and prove that it cannot be a linear combination of the columns in $[H_1]$. Let $h_{j1} \dots h_{jd_j}$ be the column vectors of the matrix $[H_j]$, $j = (1, 2, 3)$, and assume that

$$h_{2k} = a_{11}h_{11} + a_{12}h_{12} + \ldots + a_{1d_1}h_{1d_1} \tag{6-27}$$

represents the k^{th} column vector in $[H_2]$ being a linear combination of the column vectors in $[H_1]$. From the fact that $[H_2]$ has no s^2 terms, we must have

$$\begin{bmatrix} C_{11} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ \cdot \\ \cdot \\ a_{1d_1} \end{bmatrix} = 0 \tag{6-28}$$

or

$$\begin{bmatrix} C_{11} \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ \cdot \\ \cdot \\ \cdot \\ a_{1d_1} \end{bmatrix} = 0 \tag{6-29}$$

Since $[C_{11}]$ is positive definite, equation 6-29 is false and the assumption in equation 6-27 is not valid. This proves that every vector h_{2k}, $k = (1, 2, \ldots, d_2)$, in $[H_2]$ is independent from every vector in $[H_1]$. In the same way we may prove that every vector h_{3k}, $k = (1, 2, \ldots, d_3)$ is independent from the vectors in $[H_1]$ and $[H_2]$. This proves that the matrix $[H_{**}]$ has $(d_1 + d_2 + d_3)$ independent vectors and hence a rank of $(d_1 + d_2 + d_3)$.

(b) Solution of $[H_{**}]^{-1} \cdot [H_{*E}]$

$[H_{**}]$ is a matrix over the field of rational function; therefore, the following theorem also applies:

Theorem 11 (page 79, theorem 4-8, reference 19)

A square matrix $[A]$ is nonsingular if and only if $\det |A| \neq 0$. In this case

$$[A]^{-1} = \frac{1}{|A|} \cdot [\mathrm{adj}\ A] \quad .$$

$[\mathrm{adj}\ A]$ is defined to be the matrix, such that the i^{th} row and j^{th} column element of its transpose is the cofactor c_{ij} of A, where c_{ij} is defined as

$$c_{ij} = (-1)^{i+j} \quad M_{ij}$$

with M_{ij} being the determinant of the matrix $[A]$ with i^{th} row and j^{th} column deleted.

Applying theorem 11 to $[H_{**}]$, we have

$$[H_{**}]^{-1} = \frac{1}{\det |H_{**}|} \cdot [\mathrm{adj}\ H_{**}] = \frac{[F(s)]}{G(s)} \quad . \qquad (6\text{-}30)$$

If $[H_{**}]$ is a matric polynomial, $[\mathrm{adj}\ H_{**}]$ will also be a matric polynomial and $\det |H_{**}|$ will be a single polynomial, $G(s)$. From this, we can see that the entries in $[H_{**}]^{-1}$ are rational functions with the denominator polynomials equal to $\det |H_{**}|$.

Equation 6-30 offers a scheme to compute $[H_{**}]^{-1}$ which involves many determinant evaluations. Since determinant evaluation is a long process, especially when the entries of the matrix are polynomials, this workable scheme is not practical, and instead the following method will be used.

Assume that the polynomial $\det|H_{**}|$ is known, (Appendix A gives a method to evaluate $\det|H_{**}|$),

$$\det|H_{**}| = G(s) = g_n s^n + g_{n-1} s^{n-1} \dots g_1 s + g_o \; , \qquad (6\text{-}31)$$

then $[H_{**}]^{-1} \cdot [H_{*E}]$ may be written as

$$[H_{**}]^{-1} [H_{*E}] = \frac{[F(s)]}{G(s)} [H_{*E}] \; . \qquad (6\text{-}32)$$

Multiplying both sides of equation 6-32 by $[H_{**}]$, we have

$$[H_{*E}] = [H_{**}] \frac{[F(s)]}{G(s)} [H_{*E}] \; . \qquad (6\text{-}33)$$

If we write

$$[Q] = [F] [H_{*E}] \; , \qquad (6\text{-}34)$$

and substitute into equation 6-32 and equation 6-33, we have

$$[H_{**}^{-1}][H_{*E}] = \frac{1}{G(s)} [Q] \qquad (6\text{-}35)$$

and

$$G(s) [H_{*E}] = [H_{**}] [Q] \; . \qquad (6\text{-}36)$$

From equation 6-36 we will solve for $[Q]$, a matric polynomial, and then substitute it into equation 6-35 to obtain the solution of $[H_{**}]^{-1} \cdot [H_{*E}]$.

Both the left hand side and the right hand side of equation 6-36 are polynomials in s with matrix coefficients; hence we may solve for $[Q]$ by equating coefficients for the same power of s.

$[Q]$ is a matric polynomial and can be written as

$$[Q] = [Q_n]\, s^n + [Q_{n-1}]\, s^{n-1} + \ldots + [Q_1]\, s + [Q_o] \quad (6\text{-}37)$$

Writing $[H_{**}]$ and $[H_{*E}]$ in polynomial form as they are defined in equation 6-13, we can expand equation 6-36 into the form

$$(g_n s^n + g_{n-1} s^{n-1} \ldots g_1 s + g_o)\,(\,[C_{*E}]\, s^2 + [R_{*E}]\, s + [L_{*E}]\,)$$

$$= (\,[C_{**}]\, s^2 + [R_{**}]\, s + [L_{**}]\,)(\,[Q_n]\, s^n + [Q_{n-1}]\, s^{n-1}$$

$$\ldots [Q_1]\, s + [Q_o]\,) \quad (6\text{-}38)$$

Then by equating the coefficients for the same power in s, we obtain the following set of equations:

$\underline{s^{n+2} \text{ terms}}$

$$[C_{**}]\ [Q_n] = g_n\, [C_{*E}] \quad (6\text{-}39)$$

$\underline{s^{n+1} \text{ terms}}$

$$[C_{**}]\ [Q_{n-1}] + [R_{**}]\ [Q_n] = g_n\, [R_{*E}] + g_{n-1}\, [C_{*E}] \quad (6\text{-}40)$$

$\underline{s^j \text{ terms},}$ where $n \geqslant j \geqslant 2$

$$[C_{**}]\ [Q_{j-2}] + [R_{**}]\ [Q_{j-1}] + [L_{**}]\ [Q_j]$$

$$= g_j\, [L_{*E}] + g_{j-1}\, [R_{*E}] + g_{j-2}\, [C_{*E}] \quad (6\text{-}41)$$

$\underline{s^1 \text{ terms}}$

$$[R_{**}]\ [Q_o] + [L_{**}]\ [Q_1] = g_1\, [L_{*E}] + g_o\, [R_{*E}] \quad (6\text{-}42)$$

s^o terms

$$[L_{**}]\,[Q_o] = g_o\,[L_{*E}] \tag{6-43}$$

When det $|H_{**}|$ is a polynomial of n^{th} order, ~~then~~ we have, from equation 6-39 up to equation 6-43, $(n + 3)$ matric equations to solve for $(n + 1)$ unknown matrices $([Q_n], [Q_{n-1}] \ldots [Q_o])$. If $[C_{**}]^{-1}$ exists, they may be solved starting from the s^{n+2} terms in equation 6-39 for $[Q_n]$, which is subsequently substituted into equation 6-40 to solve for $[Q_{n-1}]$. The recursive relation to solve for $[Q_{j-2}]$ from $[Q_{j-1}]$ and $[Q_j]$ is given in equation 6-41. If $[L_{**}]^{-1}$ exists, the process is reversed by solving first for $[Q_o]$ in equation 6-43, then $[Q_1]$ from equation 6-42. The recursive relation that solves for $[Q_j]$ from $[Q_{j-1}]$ and $[Q_{j-2}]$ is also given by equation 6-41. Equation 6-44 up to equation 6-46 give the equations for $[Q]$ when $[C_{**}]^{-1}$ exists.

$$[Q_n] = [C_{**}]^{-1}\, g_n\, [C_{*E}] \tag{6-44}$$

$$[Q_{n-1}] = [C_{**}]^{-1}\left(g_n\,[R_{*E}] + g_{n-1}\,[C_{*E}] - [R_{**}]\,[Q_n]\right) \tag{6-45}$$

$$[Q_{j-2}] = [C_{**}]^{-1}\left(g_j\,[L_{*E}] + g_{j-1}\,[R_{*E}] + g_{j-2}\,[C_{*E}] - [R_{**}]\,[Q_{j-1}] - [L_{**}]\,[Q_j]\right) \tag{6-46}$$

$$n \geqslant j \geqslant 2\ .$$

However, very often neither $[C_{**}]^{-1}$ nor $[L_{**}]^{-1}$ exists such as the network in (fig. 6-3). Under such circumstances, we cannot use equations 6-44, 6-45, and 6-46 derived from equations 6-39 up to 6-43. Instead, the V^* coordinate must be picked such that

$$V^* = \begin{bmatrix} V^1 \\ V^2 \\ V^3 \end{bmatrix} \tag{6-47}$$

and the corresponding $[C_{**}]$, $[R_{**}]$, and $[L_{**}]$ matrices become

$$[C_{**}] = \begin{bmatrix} C_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[R_{**}] = \begin{bmatrix} R_{11} & R_{12} & 0 \\ R_{21} & R_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6-48}$$

$$[L_{**}] = \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix}$$

where $[C_{11}]^{-1}$, $[R_{22}]^{-1}$ and $[L_{33}]^{-1}$ always exist. The existence of such a set of coordinates and the topological algorithm that selects them are given in Chapters 3 and 5.

According to the partition in V^*, $[Q]$ and $[H_{*E}]$ are similarly partitioned:

$$[Q_j] = \begin{bmatrix} Q_j^1 \\ Q_j^2 \\ Q_j^3 \end{bmatrix}, \quad j = 0, 1, \ldots, n \qquad (6\text{-}49)$$

$$[Z_{*E}] = \begin{bmatrix} Z_{1E} \\ Z_{2E} \\ Z_{3E} \end{bmatrix}, \quad Z = C, R, L \quad . \qquad (6\text{-}50)$$

With the partitioning scheme, each equation in equation 6-39 up to equation 6-43 contains three equations. Each of the partitioned equations is denoted by two indices: the first one gives the power of s whose coefficients are equated; the second index gives the order of sequence due to partitioning. For example in the following equations, equation (n + 1, 2) is the second equation partitioned from equation 6-40 which equates the coefficients of s^{n+1} terms.

$$[C_{11}][Q_n^1] = g_n [C_{1E}] \qquad (n+2, 1)$$

$$[C_{11}][Q_{n-1}^1] + [R_{11}][Q_n^1] + [R_{12}][Q_n^2] = g_n [R_{1E}] + g_{n-1}[C_{1E}] \qquad (n+1, 1)$$

$$[R_{21}][Q_n^1] + [R_{22}]\, Q_n^2 = g_n [R_{2E}] + g_{n-1}[C_{2E}] \qquad (n+1, 2)$$

$$\left[C_{11}\right]\left[Q^1_{j-2}\right] + \left[R_{11}\right]\left[Q^1_{j-1}\right] + \left[R_{12}\right]\left[Q^2_{j-1}\right] + \left[L_{11}\right]\left[Q^1_{j}\right] + \left[L_{12}\right]\left[Q^2_{j}\right] + \left[L_{13}\right]\left[Q^3_{j}\right] = g_j\left[L_{1E}\right] + g_{j-1}\left[R_{1E}\right] + g_{j-2}\left[C_{1E}\right] \qquad (j,\ 1)$$

$$\left[R_{21}\right]\left[Q^1_{j-1}\right] + \left[R_{22}\right]\left[Q^2_{j-1}\right] + \left[L_{21}\right]\left[Q^1_{j}\right] + \left[L_{22}\right]\left[Q^2_{j}\right] + \left[L_{23}\right]\left[Q^3_{j}\right] = g_j\left[L_{2E}\right] + g_{j-1}\left[R_{2E}\right] + g_{j-2}\left[C_{2E}\right] \qquad (j,\ 2)$$

$$\left[L_{31}\right]\left[Q^1_{j}\right] + \left[L_{32}\right]\left[Q^2_{j}\right] + \left[L_{33}\right]\left[Q^3_{j}\right] = g_j\left[L_{3E}\right] + g_{j-1}\left[R_{3E}\right] + g_{j-2}\left[C_{3E}\right] \qquad (j,\ 3)$$

Since $\left[C_{11}\right]^{-1}$ and $\left[R_{22}\right]^{-1}$ exist, $\left[Q^1_n\right]$ can be evaluated from equation (n + 2, 1), and $\left[Q^1_{n-1}\right]$, $\left[Q^2_n\right]$ can be evaluated from equations (n + 1, 1) and (n + 1, 2), respectively. By substituting $\left[Q^1_n\right]$, $\left[Q^2_n\right]$ and $\left[Q^1_{n-1}\right]$ into equations (n, 1), (n, 2) and (n, 3), we can evaluate $\left[Q^3_n\right]$, $\left[Q^2_{n-1}\right]$ and $\left[Q^1_{n-2}\right]$. Now we will prove the induction process, that knowing

$$\left[Q^1_{j-1}\right], \left[Q^1_j\right], \left[Q^1_{j+1}\right] \cdots\cdots \left[Q^1_n\right]$$

$$\left[Q^2_j\right], \left[Q^2_{j+1}\right] \cdots\cdots \left[Q^2_n\right] \tag{6-51}$$

$$\left[Q^3_{j+1}\right] \cdots\cdots \left[Q^3_n\right],$$

then by substituting into equations (j, 1), (j, 2) and (j, 3) we can compute $\left[Q^1_{j-2}\right]$, $\left[Q^2_{j-1}\right]$, and $\left[Q^3_j\right]$.

From equation (j, 1) we may compute $\left[Q^1_{j-2}\right]$ as

$$\begin{aligned}\left[Q^1_{j-2}\right] = \left[C_{11}\right]^{-1} \big(g_j \left[L_{1E}\right] &+ g_{j-1}\left[R_{1E}\right] + g_{j-2}\left[C_{1E}\right] \\ &- \left[R_{11}\right]\left[Q^1_{j-1}\right] - \left[R_{12}\right]\left[Q^2_{j-1}\right] - \left[L_{11}\right]\left[Q^1_j\right] \\ &- \left[L_{12}\right]\left[Q^2_j\right] - \left[L_{13}\right]\left[Q^3_j\right]\big)\end{aligned} \tag{6-52}$$

where everything on the right hand side is known and $\left[C_{11}\right]^{-1}$ exists. From equation (j, 2) we may compute $\left[Q^2_{j-1}\right]$ as

$$\begin{aligned}\left[Q^2_{j-1}\right] = \left[R_{22}\right]^{-1} \big(g_j \left[L_{2E}\right] &+ g_{j-1}\left[R_{2E}\right] + g_{j-2}\left[C_{2E}\right] \\ &- \left[R_{21}\right]\left[Q^1_{j-1}\right] - \left[L_{21}\right]\left[Q^1_j\right] - \left[L_{22}\right]\left[Q^2_j\right] \\ &- \left[L_{13}\right]\left[Q^3_j\right]\big)\end{aligned} \tag{6-53}$$

where $\left[R_{22}\right]^{-1}$ exists and every term on the right hand side is known.

Finally from equation (j, 3), we compute $[Q_j^3]$ as

$$[Q_j^3] = [L_{33}]^{-1}(g_j [L_{3E}] + g_{j-1} [R_{3E}] + g_{j-2} [C_{3E}] - [L_{31}] [Q_j^1] - [L_{32}] [Q_j^2]), \tag{6-54}$$

where $[L_{33}]^{-1}$ exists.

This proves that with the selection of V^1, V^2, V^3 coordinates, all $[Q_j]$ $(j = 0, 1, \ldots, n)$ may be computed. With the substitution of $[Q]$ into equation 6-35 which is further substituted into equation 6-16, the short circuit driving point and transfer admittances, $[Y]$ can be evaluated.

The steps to compute $[Y]$ are now summarized:

The problem is to compute the short circuit driving point and transfer admittances between a specified set of node pairs in a network. The network may consist of RLC elements and ideal transformers interconnected into any arbitrary topology. The systematic steps of computation are as follows:

(1) Check to see that the specified set of accessible node pairs are independent. If otherwise, remove the dependent ones. This may occur when the problem is badly specified or some of the node pairs are constrainted by transformers.

(2) Form V^E whose components are the specified set of independent accessible node pairs.

(3) Select V^1, V^2, V^3 from the network with all node pairs used in V^E short-circuited. In the presence of ideal transformers, the algorithm in Chapter 5 is used to reduce them to an independent set.

(4) Compute $[Z_{ij}]$ where $Z = (C, R, L)$

and $i, j = (E, 1, 2, 3)$

(5) Compute $\det|H_{**}|$ as a polynomial $G(s)$, where $[H_{**}]$ is defined as

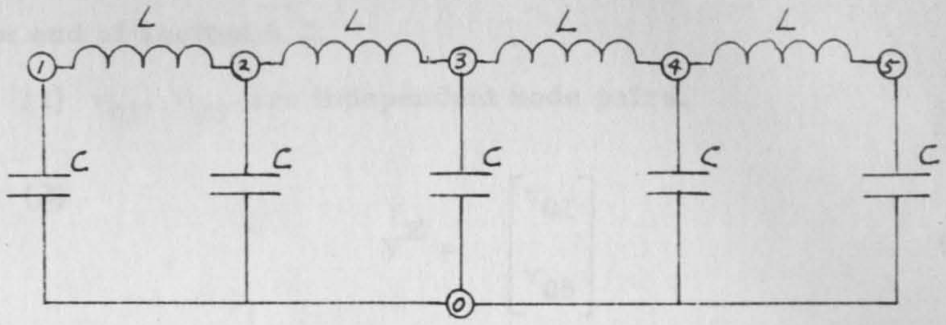
$$[H_{**}] = \begin{bmatrix} C_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} s^2 + \begin{bmatrix} R_{11} & R_{12} & 0 \\ R_{21} & R_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} s + \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix}$$

(6) Solve for $[Q_j^1]$ $[Q_j^2]$ $[Q_j^3]$ $(j = 0, 1, 2, \ldots, n)$ by using the recursive formulae in equations 6-52, 6-53, and 6-54.

(7) Compute $[Y] = \left([H_{EE}] - [H_{E*}][Q](g(s))^{-1}\right) s^{-1}$.

6.3 An Example

A ladder network is used because of its regular topology so that for comparison, an independent solution can be obtained with the conventional method described in section 6.1. The network, shown in (fig. 6-4), consists of four series branch inductors of equal admittance value, L in $(\text{henry})^{-1}$, and five parallel branch capacitors of equal capacitances C. The choice of equal parameter values is purely for easier manipulation by hand. When the algorithm is programmed on a



$$V^E = \begin{bmatrix} v_{01} \\ v_{05} \end{bmatrix}$$

$$V^1 = \begin{bmatrix} v_{02} \\ v_{03} \\ v_{04} \end{bmatrix}$$

$$V^2 = 0$$

$$V^3 = 0$$

$$[C_{EE}] = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \qquad [R_{EE}] = 0 \qquad [L_{EE}] = \begin{bmatrix} L & 0 \\ 0 & L \end{bmatrix}$$

$$[C_{11}] = \begin{bmatrix} C & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & C \end{bmatrix} \qquad [R_{11}] = 0 \qquad [L_{11}] = \begin{bmatrix} 2L & -L & 0 \\ -L & 2L & -L \\ 0 & -L & 2L \end{bmatrix}$$

$$[C_{E1}] = 0 \qquad [R_{E1}] = 0 \qquad [L_{E1}] = \begin{bmatrix} -L & 0 & 0 \\ 0 & 0 & -L \end{bmatrix}$$

An Example in Ladder Network

FIGURE 6-4

computer, any parameter values may be used; even mutual couplings between branches are allowed. v_{01} and v_{05} are the two accessible terminal-pairs. The computation now follows the steps summarized at the end of section 6.2.

(1) v_{01}, v_{05} are independent node pairs.

(2)
$$V^E = \begin{bmatrix} v_{01} \\ v_{05} \end{bmatrix} \tag{6-55}$$

(3)
$$V^1 = \begin{bmatrix} v_{02} \\ v_{03} \\ v_{04} \end{bmatrix} \tag{6-56}$$

$$V^2 = 0$$

$$V^3 = 0 \quad .$$

(4)
$$[C_{EE}] = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix}$$

$$[R_{EE}] = 0$$

$$[L_{EE}] = \begin{bmatrix} L & 0 \\ 0 & L \end{bmatrix}$$

$$[C_{11}] = \begin{bmatrix} C & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & C \end{bmatrix}$$

(6-57)
to be continued

$$[R_{11}] = 0$$

$$[L_{11}] = \begin{bmatrix} 2L & -L & 0 \\ -L & 2L & -L \\ 0 & -L & 2L \end{bmatrix}$$

$$[C_{E1}] = 0 \qquad \text{(6-57) cont.}$$

$$[R_{E1}] = 0$$

$$[L_{E1}] = [L_{1E}]^T = \begin{bmatrix} -L & 0 & 0 \\ 0 & 0 & -L \end{bmatrix} .$$

(5) $$\det \left| [C_{11}]\, s^2 + [L_{11}] \right| = c^3 s^6 + 6LC^2 s^4 + 10L^2 C s^2 + 4L^3 . \qquad \text{(6-58)}$$

(6) Solve for $[Q_6]$, $[Q_5]$, $[Q_4]$ ... $[Q_o]$

$$[Q_6] = [C_{11}]^{-1} C^3 \underset{0}{[\cancel{C_{1E}}]} = 0$$

$$[Q_5] = [C_{11}]^{-1} \left(C^3 \underset{0}{[\cancel{R_{1E}}]} + 0 \underset{0}{[\cancel{C_{1E}}]} - \underset{0}{[\cancel{R_{11}}]} \underset{0}{[\cancel{Q_6}]} \right) = 0$$

$$[Q_4] = [C_{11}]^{-1} \left(C^3 \begin{bmatrix} -L & 0 \\ 0 & 0 \\ 0 & -L \end{bmatrix} + 0 \right) = \begin{bmatrix} -C^2L & 0 \\ 0 & 0 \\ 0 & -C^2L \end{bmatrix}$$

$$[Q_3] = 0 \qquad \text{(6-59) to be continued}$$

$$[Q_2] = [C_{11}]^{-1}\left(6LC^2 \begin{bmatrix} -L & 0 \\ 0 & 0 \\ 0 & -L \end{bmatrix} - \begin{bmatrix} 2L & -L & 0 \\ -L & 2L & -L \\ 0 & -L & 2L \end{bmatrix} \begin{bmatrix} -C^2L & 0 \\ 0 & 0 \\ 0 & -C^2L \end{bmatrix} \right)$$

$$= \begin{bmatrix} -4L^2C & 0 \\ -L^2C & -L^2C \\ 0 & -4L^2C \end{bmatrix}$$

$$[Q_1] = 0$$

$$[Q_o] = [C_{11}]^{-1}\left(10L^2C \begin{bmatrix} -L & 0 \\ 0 & 0 \\ 0 & -L \end{bmatrix} - \begin{bmatrix} 2L & -L & 0 \\ -L & 2L & -L \\ 0 & -L & 2L \end{bmatrix} \begin{bmatrix} -4L^2C & 0 \\ -L^2C & -L^2C \\ 0 & -4L^2C \end{bmatrix} \right)$$

$$= \begin{bmatrix} -3L^3 & -L^3 \\ -2L^3 & -2L^3 \\ -L^3 & -3L^3 \end{bmatrix}. \tag{6-59 cont.}$$

(7) Compute $[Y]$

$$[Y] = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} s + \begin{bmatrix} L & 0 \\ 0 & L \end{bmatrix} s^{-1} - \begin{bmatrix} -L & 0 & 0 \\ 0 & 0 & -L \end{bmatrix} \left([Q_4]\, s^4 + [Q_2]\, s^2 + [Q_o] \right) (C^3 s^7 + 6LC^2 s^5 + 10L^2Cs^3 + 4L^3 s)^{-1} \tag{6-60}$$

With the values of $[Q_4]$, $[Q_2]$, $[Q_o]$ from equation 6-59 substituted into equation 6-60, and putting everything under the same denominator, we have

$$[Y] = (C^3s^7 + 6LC^2s^5 + 10L^2Cs^3 + 4L^3s)^{-1}\left(\begin{bmatrix} C^4 & 0 \\ 0 & C^4 \end{bmatrix} s^8 + \begin{bmatrix} 7LC^3 & 0 \\ 0 & 7LC^3 \end{bmatrix} s^6 + \begin{bmatrix} 15L^2C^2 & 0 \\ 0 & 15L^2C^2 \end{bmatrix} s^4 + \begin{bmatrix} 10L^3C & 0 \\ 0 & 10L^3C \end{bmatrix} s^2 + \begin{bmatrix} L^4 & -L^4 \\ -L^4 & L^4 \end{bmatrix}\right) \tag{6-61}$$

Then,

$$y_{11} = \frac{C^4s^8 + 7LC^3s^6 + 15L^2C^2s^4 + 10L^3Cs^2 + L^4}{C^3s^7 + 6LC^2s^5 + 10L^2Cs^3 + 4L^3s}$$

$$y_{22} = y_{11} \tag{6-62}$$

$$y_{12} = y_{21} = \frac{-L^4}{C^3s^7 + 6LC^2s^5 + 10L^2Cs^3 + 4L^3s}$$

To check the solution obtained in equation 6-62, the same network is evaluated using the recursive function F in equation 6-4.

$$y_{11} = \epsilon s + \cfrac{1}{\frac{1}{L}s + \cfrac{1}{\epsilon s + \cfrac{1}{\frac{1}{L}s + \cfrac{1}{\epsilon s + \cfrac{1}{\frac{1}{L}s + \cfrac{1}{\epsilon s + \frac{L}{s}}}}}}}$$

$$= \frac{\epsilon^4 s^8 + 7LC^3 s^6 + 15L^2 Cs^4 + 10L^3 Cs^2 + L^4}{\epsilon^3 s^7 + 6LC^2 s^5 + 10L^2 Cs^3 + 4L^3 s} \qquad (6\text{-}63)$$

The expression computed in equation 6-63 agrees with the expression for y_{11} in equation 6-62.

CHAPTER 7

THE COMPUTER PROGRAM

The first part of this chapter is to introduce the concept of symbol manipulation as the most universal data processor, followed by a brief description of a currently available symbol manipulating language, namely, the LISP. The final section of the chapter presents the program organization of the coordinate selection algorithm in terms of symbol manipulation on list structures. The actual LISP coding appears in Appendix B. Examples of the LISP program output are included in Appendix C.

7.1 Symbol Manipulation as the Universal Processor

The Oxford dictionary (The Concise Oxford Dictionary, fourth edition) gives the following definition for "Symbol":

1. Things regarded by general consent as naturally typifying or representing or recalling something by possession of analogous qualities or by association in fact or thought.

2. Mark or character taken as the conventional sign of some object or idea or process, e.g., the astronomical signs for the planets, the letter standing for chemical elements, letter of the alphabet, the mathematical signs for addition and infinity, the asterisk; hence or cong.

Symbols are used to represent ideas, concepts and objects. They may stand for themselves or they may be the names of some objects. The word Bridge in the sentence

"There is a B in bridge."

stands for itself, and the same word Bridge in the sentence

"Washington Bridge is in New York."

denotes the physical structure known as bridge. We will classify symbols that stand for themselves as atomic symbols; and the others, name symbols. All symbols are different and their associated meanings are defined by the person who created them. When a set of symbols is used as the communication between two parties, the symbol meaning must be understood by both parties.

In formal mathematics symbols are used to represent concepts, objects and operations. They are given rigorous definition so that their subsequent appearances with other symbols can be appropriately interpreted. A postulate or a theorem in mathematics is a string of defined symbols, for example, the equation,

$$2 + 3 = 5 \tag{7-1}$$

is a string of five symbols whose associated meanings must be understood before the whole string of them can be interpreted. Spoken language is also composed of a set of defined symbols.

When the computer is used to solve a numerical problem, the latter is transformed into the symbol domain that consists of numerical numbers and arithmetic operations. The process of computing the end result from the input data can be interpreted as the transformation of input data symbols into solution symbols. The transformation is specified by a sequence of arithmetic operations which are themselves represented by symbols. The string of symbols that represent the computing process, namely the programs, forms an object that is subject to transformation just like the string of symbols that represents the input data to the numerical problem. This is the basic idea of automatic programming, which has internally stored programs that can be modified as well as the numerical data the program works on.

When the problem is not numeric, such as the analytical evaluation of an integral, the simulation of human thought process, the study of biological system behavior, etc., we cannot use the symbol manipulation of arithmetic operation since in these problems the symbols used to describe the objects are not in the class of numerical numbers. For example,

$$\int \cos x \, dx \tag{7-2}$$

and

$$\text{HE SAW THE CAT.} \tag{7-3}$$

are merely strings of symbols associated to each other in some special way. The processes that operate on these symbols transform them into different strings of symbols that represent the results, such as

$$\sin x \tag{7-4}$$

and

$$\text{HE WAS SCARED.} \tag{7-5}$$

We can say in general that any system whether it be mathematical, physical, behavioristic or philosophical, which can be described by a set of defined symbols and their associates - numerical numbers, arithmetic operations, topological propoerties or plain English description - can be studied or simulated as symbol manipulation. How the input symbols should be manipulated to give the correct output symbols constitutes the algorithm pertinent to that particular system under study or simulation. The present day compiler is a symbol manipulating process that transforms the compiler statements which are strings of symbols into the machine program which is also a string of symbols. It is not hard to see that symbol manipulation is indeed the most universal processor.

Before describing a symbol manipulating processor in the next section, some of their important characteristics are discussed here.

(1) The processor must have the ability to represent and differentiate a large number of symbols.

(2) The processor must be able to associate any arbitrary number of symbols together in any arbitrary manner. We have the concept of a string of symbols that itself forms an entity and can be represented by a name symbol which can again be one of the elements in some other string of symbols. This can best be described by the recursive definition of symbol:

> A symbol can be either an atomic symbol or a name symbol.
>
> A name symbol is a string (or list) of symbols.
>
> The arbitrary symbol association, called the list structure, is defined as a list of elements which can be atomic symbols or list structures.

(3) The processor must be independent of the data and, for convenience, it allows recursive definition of functions such as

$$n! = n.(n-1)! \qquad (7\text{-}6)$$

7.2 LISP (26)

The LISP is one of the currently available computer languages for symbol manipulation. It has been coded for the IBM 704, 709 and 7090 series of machines. This section will only outline some of its characteristics. A detailed method of coding and implementation can be found in its manual.*

(1) Atoms or atomic symbols:

An infinite set of distinguishable atomic symbols are represented by strings of capital English letters and digits. For example,

A
AA (7-8)
CZ5

are all atomic symbols.

(2) S - expressions (S stands for symbolic):

An S - expression is either an atom or an ordered pair, the terms of which may be atomic symbols or S - expressions. If we use "." to form pairs, examples of S - expressions are

(A. B) (7-9)

(AB · (A · B)) (7-10)

---

*The latest is LISP 1.5 Programmer's Manual, July 14, 1961, distributed by the Computation Center and Research Laboratory of Electronics, Massachusetts Institute of Technology.

Both terms in equation 7-8 are atomic symbols. The first term in equation 7-10 is atomic, while the second term is an S - expression.

With the definition of S - expressions given above, a list of symbols M1, M2 . . . Mn as denoted by

$$(M1,\ M2\ .\ .\ .\ Mn) \tag{7-11}$$

is represented by the S - expression

$$(M1 \cdot (M2 \cdot (.\ .\ .\ (Mn \cdot NIL)\ .\ .\ .\ ))) \tag{7-12}$$

where NIL is an atomic symbol used to terminate lists.

(3) S - functions:

All transformations on S - expressions are represented as functions applied on the S - expressions to be transformed as their arguments. These S - functions are written in a conventional functional notation. In order to distinguish the expression representing functions from S - expressions, a sequence of lower case letters and digits is used for function names and variables. Brackets are used to enclose the arguments and arguments are separated by semicolons. Examples are

$$\mathrm{car}\ [\ x\ ] \tag{7-13}$$

$$\mathrm{cdr}\ [\ \mathrm{cons}\ [\ x;\ (A \cdot B)\ ]\ ] \tag{7-14}$$

In these expressions, any S - expressions that occur stand for themselves such as the $(A \cdot B)$ in equation 7-14.

(4) Propositional expressions and predicates:

A propositional expression is an expression whose possible values are T (for truth) and F (for falsity). Typical propositional expressions are

$$5 > 8 \tag{7-15}$$

$$167 \text{ is prime} \tag{7-16}$$

A predicate is a function whose range consists of the truth values T and F.

(5) Conditional expressions:

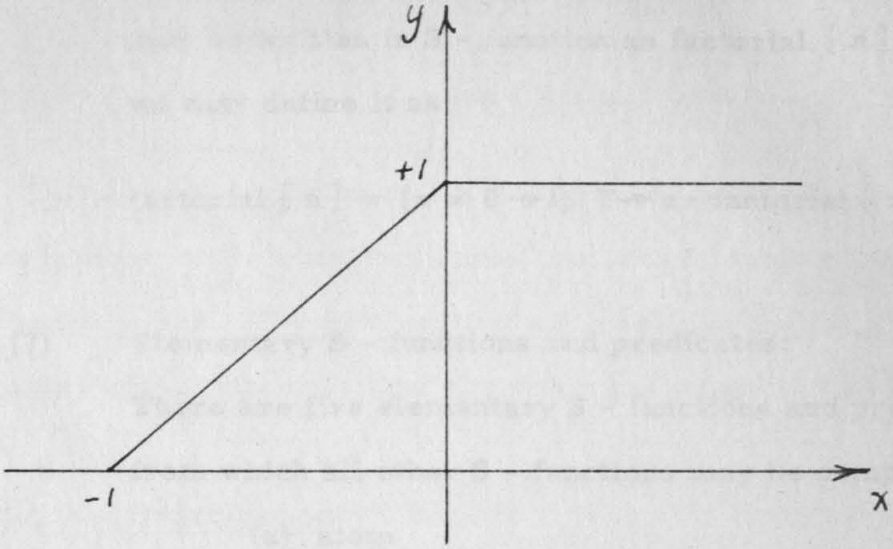
A conditional expression is used to express the dependence of an object on some propositional expressions. A conditional expression has the form

$$(p_1 \rightarrow e_1;\ p_2 \rightarrow e_2;\ \ldots;\ p_n \rightarrow e_n) \tag{7-17}$$

where p's are propositional expressions and e's are any kind of S - expression. Equation 7-17 may be read as "If p_1 then e_1, otherwise if p_2 then e_2 . . . , otherwise if p_n then e_n."

Equation 7-18 is an example of the use of conditional expression in defining the functional dependence of y on x in (fig. 7-1).

$$y[x] = (x < -1 \rightarrow 0;\ x \geqslant -1 \rightarrow 1 + x;\ x \geqslant 0 \rightarrow 1) \tag{7-18}$$



A Function Describable in Conditional Expression

$y(x) = (x < -1 \rightarrow 0;\ -1 \leqslant x < 0 \rightarrow -1 + x;\ T \rightarrow 1)$

FIGURE 7-1

(6) Recursive function definitions:

By using conditional expressions, functions may be defined by formulae in which the defined functions occur. For example, the factorial of an integer, n, may be written in S - function as factorial [n], then we may define it as

$$\text{factorial}[\,n\,] = (n = 0 \rightarrow 1;\ T \rightarrow n \cdot \text{factorial}[\,n-1\,]) \tag{7-19}$$

(7) Elementary S - functions and predicates:

There are five elementary S - functions and predicates from which all other S - functions may be composed.

(a) atom

atom [x] has the value of T or F, accordingly as x is an atomic symbol or not.

(b) eq

eq [x; y] is defined if and only if either x or y is atomic. eq [x; y] = T if x and y are the same symbol, and eq [x; y] = F, otherwise.

(c) car

car [x] is defined if and only if x is not atomic, and car [x] equals to the first term in the S - expression pair x. Thus

$$\text{car}[\,(e_1 \cdot e_2)\,] = e_1 \tag{7-20}$$

(d) cdr

cdr [x] is defined if and only if x is not atomic, and cdr [x] equals to the second term in the S - expression pair, x. Thus

$$\mathrm{cdr}\ [(e_1 \cdot e_2)] = e_2 \tag{7-21}$$

(e) cons

cons [x; y] is defined for any x and y, and the result is the S - expression $(x \cdot y)$. Thus

$$\mathrm{cons}\ [e_1; e_2] = (e_1 \cdot e_2) \tag{7-22}$$

The above description of LISP is by no means complete. For a full insight into its working principle, its programmer's manual (26) should be consulted. The next section will describe the program organization of the coordinate selection algorithm presented in Chapters 3 - 6.

7. 3 Program Organization

A program is written to select the set of nonsingular coordinates for a network of arbitrary topology consisting of RLC elements, ideal transformers, voltage sources and current sources. The complete program is given the name of an S - function, "corsel", and its argument is the S - expression that describes the network, say "NETWORK". After applying "corsel" to "NETWORK", the S - expression "NETWORK" is transformed into a different S - expression that lists the selected Node Pair Coordinates, say "NPCORD". Then we have

$$\mathrm{corsel}\ [\mathrm{NETWORK}] = \mathrm{NPCORD}. \tag{7-23}$$

The program organization that performs the transformation in equation 7-23 is divided into three aspects, namely, the S - expression format of NETWORK, the S - expression format of NPCORD and the S - function corsel (for coordinate Selection).

(1) The S - expression of NETWORK:

NETWORK is represented as an S - expression in the form of

$$\text{NETWORK} = (\text{CLIST, RLIST, LLIST, TLIST, VLIST, ILIST}) \qquad (7\text{-}23)$$

where the equivalence of "," in representing a list of elements and "." in representing a pair is given by equations 7-11 and 7-12. The individual elements in equation 7-23 are defined as follows:

CLIST:

Capacitor list. It is the name of the S - expression whose elements represent the capacitors in the network.

$$\text{CLIST} = (\text{C1, C2} \ldots \text{CB}_C) \qquad (7\text{-}24)$$

The elements of CLIST are also S - expressions and the i^{th} capacitor, CI, has the form

$$\text{CI} = (\text{n1, n2, VCI, QCI}). \qquad (7\text{-}25)$$

The elements in CI are atomic symbols. n1, n2 are the symbols used to represent the two terminal nodes the capacitor, CI, is connected to; VCI is the atomic symbol that represents the capacitance of CI; QCI is the atomic symbol that represents the initial condition of CI.

RLIST

Resistor list. It is the name of the S - expression whose elements represent the resistors in the network.

$$\text{RLIST} = (\text{R1, R2} \ldots \text{RB}_C) \qquad (7\text{-}26)$$

where the general term RI has the form,

$$RI = (n1, \ n2, \ VRI). \qquad (7\text{-}27)$$

In equation 7-27, n1, n2 are the terminal nodes of RI, and VRI is the admittance value of the resistor RI.

LLIST:

Inductor list. It is the name of the S - expression whose elements represent the inductors in the network.

$$LLIST = (L1, \ L2 \ . \ . \ . \ LB_L) \qquad (7\text{-}28)$$

where the general term LI has the form,

$$LI = (n1, \ n2, \ VLI, \ ILI). \qquad (7\text{-}29)$$

In equation 7-29, n1, n2 are the terminal nodes of LI; VLI, its inductive admittance; ILI, its initial condition.

TLIST:

Transformer list. It is the name of the S - expression whose elements are the transformers in the network.

$$TLIST = (T1, \ T2 \ . \ . \ . \ TB_T) \qquad (7\text{-}30)$$

The i^{th} transformer TI is characterized by its windings,

$$TI = (WI1, \ WI2 \ . \ . \ . \ WI_{mI}) \qquad (7\text{-}31)$$

The j^{th} winding of the i^{th} transformer is characterized by

$$WIJ = (n1, \ n2, \ VWIJ). \qquad (7\text{-}32)$$

In equation 7-32, n1, n2 are the terminal nodes of the winding WIJ and VWIJ is its relative turns ratio.

VLIST:

Voltage source list. It is the list of voltage sources in the network.

$$\text{VLIST} = (\text{V1, V2} \ldots \text{VB}_V) \qquad (7\text{-}33)$$

$$\text{VI} = (\text{n1, n2, VVI}) \qquad (7\text{-}34)$$

In equation 7-34, n1, n2 are the terminal nodes the i^{th} voltage source, VI, is connected to, and VVI is the name of the S - expression such that when applied on by "evalsf" (for Evaluate Source Function) will give the value of VI at time, t,

$$\text{evalsf}\,[\,\text{VVI; t}\,] = \text{value of VI at time t.} \qquad (7\text{-}35)$$

ILIST:

Current source list. It is the list of current sources in the network.

$$\text{ILIST} = (\text{I1, I2} \ldots \text{IB}_I) \qquad (7\text{-}36)$$

$$\text{II} = (\text{n1, n2, VII}). \qquad (7\text{-}37)$$

In equation 7-37, n1, n2 are the terminal nodes of II, and VII is the name of the S - expression such that

$$\text{evalsf}\,[\,\text{VII; t}\,] = \text{value of II at time t.} \qquad (7\text{-}38)$$

We can see that any arbitrary network consisting of linear time independent RLC elements, ideal transformers, time dependent voltage sources and current sources, can be described completely by the S - expression in equation 7-23. It will only be a simple modification to include nonlinearities in RLC elements. All we have to do is to replace the atomic symbols in equations 7-24, 7-27 and 7-29 that give the element values by S - expressions specifying the nonlinearities. For example, a nonlinear capacitor, CI, will be represented as

$$CI = (n1,\ n2,\ NCI,\ QCI), \tag{7-39}$$

where NCI is the S - expression such that

$$\begin{aligned} &\text{evalnl}\ [\ NCI;\ P1;\ P2;\ \ldots\ ] \qquad (7\text{-}40) \\ &\quad = \text{capacitance of CI evaluated at the} \\ &\quad\quad \text{parameters } P1,\ P2,\ \ldots\ . \end{aligned}$$

"evalnl" is the S - function that evaluates the value of nonlinear elements at the specified parameters.

(2) The S - expression of NPCORD:
NPCORD is the S - expression that represents the Node Pair Coordinates.

$$NPCORD = (INDNP,\ DEPNP) \tag{7-41}$$

In equation 7-41, INDNP is the S - expression of the independent set of node pairs; DEPNP is the S - expression of the dependent node pairs introduced by ideal transformers. They are defined as follows:

$$\text{INDNP} = (\text{VZ, V1, V2, V3}) \tag{7-42}$$

$$\text{DEPNP} = (\text{EQ1, EQ2, . . . EQM}) \tag{7-43}$$

In equation 7-42, VZ is the S - expression containing the components of V^o; V1, the components of V^1; V2, the components of V^2; V3, the components of V^3. The components of V^o, V^1, V^2, V^3 have the same form - they are pairs of two atomic symbols representing the terminal nodes of the node pairs. For example, the coordinates

$$V^o = \begin{bmatrix} v_{13} \\ v_{32} \end{bmatrix} \tag{7-44}$$

is represented in S - expressions as

$$\text{VZ} = ((\text{N1} \cdot \text{N3}), (\text{N3} \cdot \text{N2})). \tag{7-45}$$

In equation 7-43, the general term EQI is the S - expression that represents the linear equation which eliminates the i^{th} dependent node pair coordinate.

$$\text{EQI} = (\text{DNPI, EXPI}) \tag{7-46}$$

DNPI in equation 7-46 is the name of the i^{th} Dependent Node Pair, and EXPI is the linear expression in the independent node pairs to which DNPI is equal.

$$\text{EXPI} = (\text{EXPVZ, EXPV1, EXPV2, EXPV3}) \tag{7-47}$$

The S - expression, EXPI, is divided into four components according to the classification of the independent node pairs in its expression. Each sub-expression is a list of pairs, the first term of which is the coefficient and the second term is the S - expression of the independent node pair. For example, a network has only one dependent node pair, (N4 · N5), and the set of independent node pairs,

$$\begin{aligned} VZ &= ((N1 \cdot N3),\ (N3 \cdot N2)) \\ V1 &= NIL \\ V2 &= NIL \\ V3 &= ((N2 \cdot N4),\ (N6 \cdot N4)). \end{aligned} \qquad (7\text{-}48)$$

Let the linear equation expressing the dependence be

$$(N4 \cdot N5) = -4(N1 \cdot N3) + 3 \cdot 3(N6 \cdot N4) \qquad (7\text{-}49)$$

then DEPNP defined in equation 7-43 becomes

$$DEPNP = (EQ1) \qquad (7\text{-}50)$$

$$EQ1 = ((N4 \cdot N5),\ EXP1) \qquad (7\text{-}51)$$

$$\begin{aligned} EXP1 = (((-4,\ (N1 \cdot N3))),\ NIL,\ NIL, \\ ((3 \cdot 3,\ (N6 \cdot N4)))) \end{aligned} \qquad (7\text{-}52)$$

The complete S - expression for NPCORD of the network is given by

```
NPCORD = ((((N1 · N3), (N3 · N2))
             NIL,
             NIL,
             ((N2 · N4), (N6 · N4)))
             (((N4 · N5), (((-4, (N1 · N3)))
                            NIL
                            NIL
                            ((3 · 3, (N6 · N4)))))))
```

(7-53)

(3) The S - function corsel

The S - function that performs the corrdinate selection from a completely specified network, NETWORK, is defined as "corsel".

The S - function "corsel" is defined in terms of several sub - S - functions. They will now be defined.

(a) vnpgen [NETWORK] = NPLIST (7-54)

vnpgen (Voltage Node Pair Generator) is the S - function whose argument is the S - expression that specifies the network and whose value is NPLIST (Node Pair LIST). NPLIST is the S - expression whose elements are lists of components in the V^o, V^1, V^2, V^3, V^4 coordinates. The algorithm for selection is described in steps (o) - (4) just prior to equation 5-6 in Section 5.3. In the selection of V^1, V^2, V^3, the criterion that minimizes the round-off errors in subsequent matrix computation as described in Sections 3.5 and 3.6 is also incorporated.

NPLIST = (VZLT, V1LT, V2LT, V3LT, V4LT)

(7-55)

(b) lqgen [TLIST; NPLIST] = LQLIST (7-56)

lqgen (Linear eQuations GENeration) is the S - function with two arguments. The first argument is the S - expression, TLIST, which as defined in equations 7-30, 7-31, 7-32, specifies all the transformers and their connections in the network. The second argument is NPLIST, which is the S - expression computed from equation 7-54. The value of lqgen is LQLIST (Linear eQuation LIST). LQLIST is the S - expression in the form of a list of sub - S - expressions each of which represents a linear equation with the node pairs in NPLIST as variables.

LQLIST = (LQ1, LQ2, . . . LQM) (7-57)

The S - expression LQI that describes the i^{th} linear equation is defined as follows:

LQI = (LHSLQI, RHSLQI) (7-58)

In equation 7-58, both LHSLQI and RHSLQI have the same form. They are the S - expressions that represent the Left Hand Side and Right Hand Side of the equation, LQI.

LHSLQI = (EXPVZ, EXPV1, EXPV2,
EXPV3, EXPV4) (7-59)

where each component of LHSLQI, say EXPV2, is an S - expression in the form of a list of pairs. The first term in the pair is the coefficient of the variable in the linear form and the second term in the pair is the name of the variable which is a node pair in, say the V2LT in NPLIST. For example, the network in (fig. 5-3-a) has V^o, V^1, V^2, V^3, V^4 selected as shown in (fig. 5-3-b, c, d, e, f) then its NPLIST, as defined in equation 7-55 has the following S - expressions as its elements:

$$
\begin{aligned}
\mathrm{VZLT} &= ((\mathrm{NZ}\cdot \mathrm{N7}))\\
\mathrm{V1LT} &= ((\mathrm{NZ}\cdot \mathrm{N3}),\ (\mathrm{NZ}\cdot \mathrm{N5}),\ (\mathrm{NZ}\cdot \mathrm{N6})\\
\mathrm{V2LT} &= ((\mathrm{N2}\cdot \mathrm{N7}))\\
\mathrm{V3LT} &= ((\mathrm{NZ}\cdot \mathrm{N1}))\\
\mathrm{V4LT} &= ((\mathrm{N3}\cdot \mathrm{N4}))
\end{aligned} \tag{7-60}
$$

NZ is the atomic symbol for node zero and Ni is the atomic symbol for node i (i being numeric).

The network has two transformers, each with three windings, therefore introducing four linear equations as given in equations 5-32. The LQLIST of this network as defined in equation 7-57 has the following S - expressions as its elements:

$$
\begin{aligned}
\mathrm{LQ1} &= (\overbrace{\underline{(\mathrm{NIL},\ ((1.0,\ (\mathrm{NZ}\cdot \mathrm{N3}))),\ \mathrm{NIL},\ \mathrm{NIL},\ \mathrm{NIL})}}^{\mathrm{LHSLQ1}},\\
&\quad \underbrace{(\mathrm{NIL},\ \mathrm{NIL},\ \mathrm{NIL},\ ((\frac{n_{12}}{n_{11}},\ (\mathrm{NZ}\cdot \mathrm{N1}),\ \mathrm{NIL}))}_{\mathrm{RHSLQ1}}\\
\mathrm{LQ2} &= (\underline{(\mathrm{NIL},\ \mathrm{NIL},\ \mathrm{NIL},\ \mathrm{NIL},\ ((1.0,\ (\mathrm{N3}\cdot \mathrm{N4}))))},\\
&\quad (\mathrm{NIL},\ \mathrm{NIL},\ \mathrm{NIL},\ ((\frac{n_{13}}{n_{11}},\ (\mathrm{NZ}\cdot \mathrm{N1})),\ \mathrm{NIL}))\\
\mathrm{LQ3} &= (\underline{(\mathrm{NIL},\ ((1.0,\ (\mathrm{NZ}\cdot \mathrm{N5})),\ (-1.0,\ (\mathrm{NZ}\cdot \mathrm{N3}))),}\\
&\quad \underline{\mathrm{NIL},\ \mathrm{NIL},\ ((-1.0,\ (\mathrm{N3}\cdot \mathrm{N4}))))},\\
&\quad ((\frac{n_{22}}{n_{21}},\ (\mathrm{NZ}\cdot \mathrm{N7}))),\ \mathrm{NIL},\ ((-\frac{n_{22}}{n_{21}},\ (\mathrm{NZ}\cdot \mathrm{N7}))),\\
&\quad \mathrm{NIL},\ \mathrm{NIL}))\\
\mathrm{LQ4} &= (\underline{(\mathrm{NIL},\ ((1.0,\ (\mathrm{NZ}\cdot \mathrm{N6})),\ (-1.0,\ (\mathrm{NZ}\cdot \mathrm{N5}))),}\\
&\quad \underline{\mathrm{NIL},\ \mathrm{NIL},\ \mathrm{NIL})},\\
&\quad ((\frac{n_{23}}{n_{21}},\ (\mathrm{NZ}\cdot \mathrm{N7}))),\ \mathrm{NIL}\ ((-\frac{n_{23}}{n_{21}},\ (\mathrm{NZ}\cdot \mathrm{N7}))),\\
&\quad \mathrm{NIL},\ \mathrm{NIL}))
\end{aligned} \tag{7-61}
$$

For ease of reading, the expressions in equations 7-61 that represent the left hand side of the equation are underlined.

(c) sedv [LQLIST] = DEPNP (7-62)

sedv (SElect Dependent Variables) is the S - function that computes the list of dependent node pairs, DEPNP from the list of linear equation, LQLIST. LQLIST is the S - expression computed from equation 7-56 and DEPNP is the S - expression defined in equation 7-43. sedv is defined according to the algorithm in equation 5-25.

(d) rednp [NPLIST; DEPNP] = INDNP (7-63)

rednp (REmove Dependent Node Pairs) is the S - function that removes the dependent node pairs from NPLIST which is the S - expression computed in equation 7-54. The dependent node pairs are given as DNPI in equation 7-46 which is the S - expression of the i^{th} term in DEPNP as defined in equation 7-43. The value of rednp is the S- expression, INDNP that specifies all the final selected independent node pairs. The definition of INDNP is given in equation 7-42.

With the functions defined in equations 7-54, 7-56, 7-62, the S - function corsel is now defined in terms of the dummy variable k:

```
corsel [ k ] =
cons [
rednp [ vnpgen [ k ];                                        (7-64)
        sedv [ lqgen [ caddddr [ k ]; vnpgen [ k ]]]];
sedv [ lqgen [ caddddr [ k ]; vnpgen [ k ]]]]
```

In equation 7-64 cons is the elementary S - function defined in equation 7-22, and the S - function caddddr is defined as

```
caddddr [ x ] = car [ cdr [ cdr [ cdr [ cdr [ x ]]]]]
```

(7-65)

With the S - function corsel in equation 7-64 defined in terms of the S - functions vnpgen, lqgen, sedv and rednp, there still remains the task of defining them in terms of the five elementary S - functions described in Section 7.2(7). Their definitions are given in Appendix B. It is assumed that the LISP working principles are the prerequisite before tracing the definitions in Appendix B.

Since this thesis is primarily concerned with the algorithm of selecting a set of nonsingular coordinates suitable for various digital computations on the network, the detailed method of implementing the algorithm by using symbol manipulating language is not included. The purpose of this chapter is to illustrate the use of symbol manipulation as a universal data processor.

CHAPTER 8

CONCLUSION

In this thesis, an algorithm expressed in terms of network topology has been derived to select an independent set of coordinates. The ordinary differential equation in the chosen coordinates describes the electrical network of RLC elements, ideal transformers, ideal voltage- and current-sources, which is topologically analogous to a large class of systems with linear constant coefficient parameters. The algorithm insures that parameter matrices requiring inversion will always be nonsingular in the application of conventional methods of numerical analysis to integration methods for transient response calculations and matric polynomial manipulations for driving point and transfer admittance determinations. A modified Turing's criterion (20) is incorporated in the algorithm to minimize the round-off errors in matrix operations.

Because of the non-numeric nature of the algorithm, a symbol manipulating language such as the LISP (20) (coded on IBM 7090 computers) is chosen to implement it. The LISP is found efficient in describing the algorithm in which the search of a path in a network of arbitrary topology and the manipulations of linear equations introduced by ideal transformers are programmed as operations on list structures. The program has been successfully applied to the networks in Appendix C of various complexity. For the example on the plate analogy of a delta wing, the network of thirty nodes and fifteen two-winding transformers took about ten minutes to give the set of independent coordinates and almost exceeded the core memory capacity of 32K on the 7090. This indicates the need of more efficient digital computers oriented towards non-numeric computations.

Section 8.1 extends the algorithm to networks with nonlinear elements and coupled branches. Alternative methods of evaluating the matrix expressions are discussed in Section 8.2. Some related research topics are outlined in Section 8.3.

8.1. Nonlinear Elements, Coupled Branches and Nonbilateral Elements

(1) Nonlinear Elements

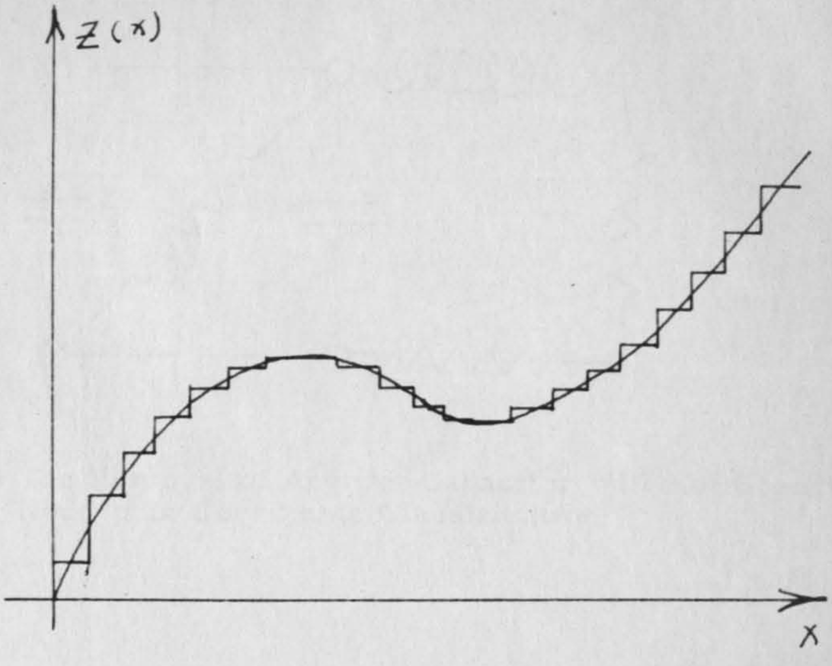
In the preceding chapters, the selection of coordinates and the subsequent formulation into the canonical form for numerical integration (equations 4-28) assumes that all RLC elements are linear, time independent and positive. These methods can, however, be extended to nonlinear systems.

The nonlinearities in element values introduce nonlinear parameter matrices in equations 4-28. Although the analytical treatment of nonlinear mechanics is difficult and rather restricted, it is a simple matter to numerically integrate a nonlinear differential equation (10). From the initial state of the system, at $t = 0$, enough parameters are available to compute all the nonlinear element values. The method of numerical integration assumes that the system remains linear during the time interval of Δt and evaluates the state of the system at time $t = \Delta t$. Due to the change of state, all nonlinear elements are re-evaluated to correspond to the new set of parameters, and hence compute the state of the system at $t = 2\Delta t$. The approximation is to replace the continuous nonlinear dependence by the staircase-like function as shown in (fig. 8-1). It is evident that the closer the intervals, the better is the approximation; however, it is difficult to estimate the absolute error introduced due to such an approximation. By using the same method, equations 4-28 with nonlinear elements can be integrated step by step; and at each step all nonlinear matrices are adjusted to correspond to the change of state, provided that $[C_{11}]^{-1}$, $[R_{22}]^{-1}$, and $[L_{33}]^{-1}$ remain nonsingular at all times. These conditions are satisfied if

(a) nonlinear elements always have values greater than zero; or

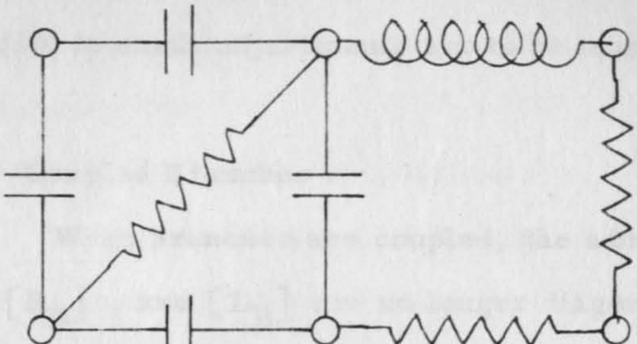
(b) if the nonlinear element does become zero, then the removal of which must not effect the V^o, V^1, V^2, V^3, V^4 coordinates classification. For example, the disappearance of any one capacitor in the circuit in (fig. 8-2-a) will not effect the coordinates classification, and the removal of any one capacitor in (fig. 8-2-b) decreases d_1 by 1.

If some elements are negative in value and others positive, it is not possible to conclude on the existence of $[C_{11}]^{-1}$, $[R_{22}]^{-1}$, and $[L_{33}]^{-1}$. However, if all elements of one type have negative values, then $[C_{11}]$, $[R_{22}]$, or $[L_{33}]$ will be negative definite and also

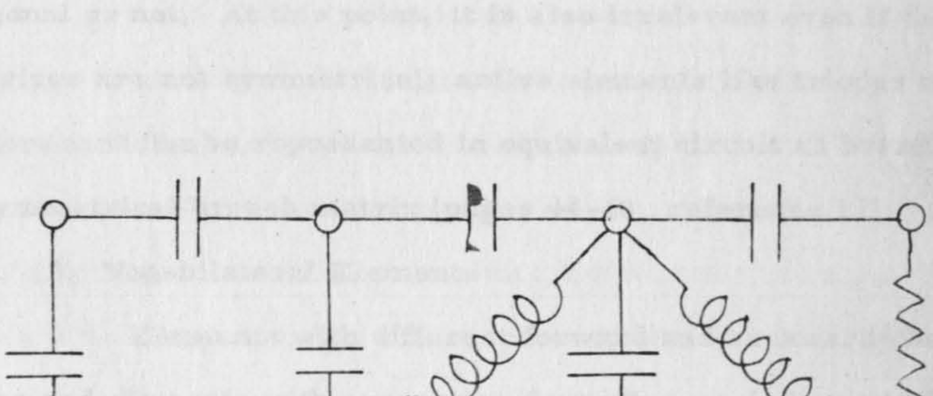


The Staircase Approximation of a Nonlinear Element, Z (x)

FIGURE 8-1



(a) The Removal of Any One Capacitor Will Not Change the Node-pair Coordinate Classification



(b) The Removal of Any One Capacitor Will Alter the Node-pair Coordinate Classification

FIGURE 8-2

possess inverses. Therefore, we can say that if the nonlinear elements always satisfy conditions (a) or (b), the method of coordinate selection developed for linear systems is also applicable with the additional work of adjusting the nonlinear matrices in equations 4-28 at every interval. (If higher order numerical integration formula (10) is used, adjustments are to be made even at mid-interval points.)

(2) Coupled Branches

When branches are coupled, the admittance matrices $[C_B]$, $[R_B]$, and $[L_B]$ are no longer diagonal. This condition does not effect the computation of matrices used in equations 4-28. Equations 4-22, 4-23, and 4-24 give the admittance matrices transformation. It is irrelevant whether $[C_B]$, $[R_B]$, and $[L_B]$ are diagonal or not. At this point, it is also irrelevant even if the branch matrices are not symmetrical; active elements like triodes or transistors can often be represented in equivalent circuit as branches with unsymmetrical branch matrix (pages 44-48, reference 12).

(3) Non-bilateral Elements

Elements with different forward and backward characteristics and elements with properties depending on their past history such as the hysteresis loop are all special cases of nonlinear elements. The discussions on nonlinear elements apply directly.

8.2 Alternative Methods of Evaluating Matrix Expressions

After the coordinates are selected as V^o, V^1, V^2, V^3, V^4, then with all V^4 and some of V^1, V^2, V^3 eliminated due to transformer constraints, we set up the equations to be integrated in the form of equations 4-28. The evaluation of the matrix expressions in equation 4-28 involves matrix multiplication and matrix inversion. Although these matrix operations are commonly coded as subroutines so that one can call for their service readily, alternative ways of evaluating these matrix operations are worth the consideration under special circumstances.

(1) Sparsely Distributed Matrices

When an m x n matrix is stored as n consecutive columns each with m elements, (m x n) memory cells are used irrespective of the element distribution within the matrix. If most of its elements are nonzero, this is almost the best way to store matrices in computers. However, if the matrix were only sparsely distributed such that a larger portion of its elements is equal to zero, the columnwise storage of a matrix would be wasteful in memory utilization and computing time. In this case, matrices may be stored by specifying only their nonzero elements, each of which is specified by three quantities: the row index, the column index, and the value. In networks with a large number of nodes, each node is usually only connected to a few other nodes through RLC elements. The matrices in equation 4-28 for such networks are sparsely distributed, and the scheme of storing only nonzero elements in the computer deserves consideration.

(2) Use of Relaxation Methods

If the network is such that all the matrices in equation 4-28 are sparsely distributed, we may store only the nonzero elements.

However, even if $[C_{11}]$, $[R_{22}]$, and $[L_{33}]$ are only sparsely distributed, their inverses are in general full matrices. In order to retain the virtue of efficient memory utilization, these matrices are not to be inverted, and instead equations 4-28 are evaluated by relaxation methods (33) to which the following features are related.

(a) Convergence

When the Gauss-Siedel (34) relaxation method is used, convergence is assured whenever the matrix is positive definite. The choice of coordinates in Chapter 3 (that minimizes the round-off error) will, in general, also give the fastest convergence rate.

(b) Trial Solution

When the relaxation method is used, a trial solution is usually assumed from which the method will iterate towards the actual solution. If the trial solution is close to the actual solution, only a few steps of relaxation would converge on the final solution. When equations 4-28 are integrated, variables are continuously varying provided that there are no discontinuities such as step changes in forcing function; then the values at time t can be used as the trial solutions for relaxing the values at time $t + \Delta t$.

(c) Nonlinearities

When the network has nonlinear $[C_{11}]$, $[R_{22}]$, and $[L_{33}]$ matrices which are to be adjusted at every integration time interval, the use of relaxation methods does not require additional computations in evaluating equations 4-28. If $[C_{11}]^{-1}$, $[R_{22}]^{-1}$, and $[L_{33}]^{-1}$ are used, they have to be inverted at every time interval, whereas if the network is linear they would only be inverted once. This feature suggests that the relaxation method is more suitable than the matrix inversion method for non-linear systems.

8.3 Related Research Topics

(1) Network Synthesis in Terms of More General Topological Configurations

This thesis has presented a systematic way of analyzing networks with RLC elements, ideal transformers, ideal voltage- and current-sources interconnected in any arbitrary topology. The algorithm is rigorous and can be programmed on digital computers. The most closely related subject is to extend the approach to network synthesis in more general topology other than the usual ladder or lattice configurations. Topological properties such as the number of nodes, the number of branches and the physical layout of elements may be of practical interest. It is desirable to have control over these parameters by finding the most suitable topology besides satisfying the usual input-output transfer functions. The importance of synthesis leading to more general topological configurations has already been initiated in the literature (35) (36) (37).

(2) Unified Approach to System Analysis

This thesis reports a unified approach to the analysis of any electrical network which topologically represents a large class of systems described by a set of ordinary differential equations. The systematic procedure from accepting basic information about the system to setting up the appropriate equations for computation is algorithmically programmable. It will be encouraging to take some other classes of systems and, from the basic physical laws, derive all the steps that accept the physical description of the system and provide the computed quantities that characteristically represent the system properties. With the algorithm programmed on the computers, the computers extend their capabilities a step further toward supplementing human beings' mental effort in system analysis. The significance of searching for a unified approach to system analysis is analogous to the physicist's effort to search for a unified field theory.

(3) Machine Organization Oriented Toward Symbol Manipulation

Although a symbol manipulating language such as LISP is found efficient to express the algorithm in this thesis, its implementation on computers leaves much to be desired. As computers are used more and more to solve non-numeric problems such as the one in this thesis, some thought should be given to the organization of a digital computer oriented toward symbol manipulations rather than high speed arithmetic operations.

APPENDIX A

DETERMINANT EVALUATION FOR CERTAIN CLASSES OF MATRIC POLYNOMIALS

Consider the matric polynomial

$$[H] = [H_n] s^n + [H_{n-1}] s^{n-1} + \dots [H_1] s + [H_0] \quad \text{(A-A-1)}$$

and we want to evaluate det $|H|$ which is a polynomial in s. The straightforward method is to expand along one row or column to give

$$\det |H| = \sum_i (-1)^{i+j} h_{ij} M_{ij} \quad \text{for any } j \quad \text{(A-A-2)}$$

where h_{ij} is the i^{th} row, j^{th} column element of $[H]$, and M_{ij} is the determinant of the matrix $[H]$ with i^{th} row and j^{th} column deleted. However, the process in equation A-A-2 is a long one and especially when the elements are, in general, polynomials, the arithmetic involved is complicated. For certain classes of matric polynomials, alternative methods can be used.

The approach of the method described below is to convert the determinant evaluation of a matric polynomial into the problem of determining the eigenvalues of a matrix constructed from the coefficient matrices in the matric polynomial.

Let G(s) be the polynomial evaluated as the determinant of the matric polynomial $[H]$, then G(s) can be factored into the form

$$G(s) = \prod_{j=1}^{n} (s - \lambda j) \quad \text{(A-A-3)}$$

where n is the order of the polynomial $G(s)$, and λj are the roots of $G(s)$. If $G(s)$ has real coefficients, then λj must be all real or in complex conjugate pairs. λj are the values of s at which $G(s)$, the determinant of $[H]$, vanishes. The evaluation of $G(s)$ from $[H]$ is reduced to the problem of determining the values of s at which $\det|H| = 0$.

When the matrix $[H]$ is of the special form

$$[H] = [H_1]s + [H_0] \qquad \text{(A-A-3a)}$$

the determination of the values of s at which $\det|H| = 0$ can be treated as the determination of the eigenvalues of the matrix $[H_1]^{-1}[H_0]$ if $[H_1]^{-1}$ exists; or as the inverses of the eigenvalues of $[H_0]^{-1}[H_1]$ if $[H_0]^{-1}$ exists. The values of s and the elements of the matrices $[H_1]$, $[H_0]$ are scalars, hence the eignevalues can be efficiently computed by using various kinds of iterative procedures (46). However, when the matrix $[H]$ is of order higher than linear, such as equation A-A-1, additional transformation is required.

Let the matric polynomial be normalized to have identity matrix as its leading coefficient by multiplying the whole polynomial by $[H_n]^{-1}$ if it exists, then equation A-A-1 becomes

$$s^n + [H^*_{n-1}]s^{n-1} + \ldots [H^*_1]s + [H^*_0] \qquad \text{(A-A-4)}$$

where

$$[H^*_j] = [H^{-1}_n][H_j] \qquad \text{(A-A-5)}$$

for $j = 0, 1, 2 \ldots n-1$.

The determinant of the polynomial in equation A-A-1 is only different from the determinant of the polynomial in equation A-A-4 by a scalar, det $|H_n|$. Let E be the identity matrix, then by expanding the determinant of the matrix in equation A-A-6, we obtain equation A-A-4.

$$\begin{bmatrix} H^*_{n-1} + Es & H^*_{n-2} & & H^*_1 & H^*_0 \\ -E & +Es & 0 & 0 & 0 \\ 0 & -E & +Es & 0 & \\ & & & & \\ & & -E & +Es & 0 \\ & & & -E & +Es \end{bmatrix} \qquad \text{(A-A-6)}$$

Hence, we have the following classes of matric polynomials whose determinants may be evaluated by the eigenvalue method:

(1) Proper matric polynomial:

A proper matric polynomial has a nonsingular leading coefficient matrix, therefore, we may normalize the leading coefficient to unity as in equation A-A-4. With the equivalence of equation A-A-6, its determinant can be evaluated by expanding equation A-A-3 where λj are the eigenvalues of the matrix

$$\begin{bmatrix} -H^*_{n-1} & -H^*_{n-2} & -H^*_{n-3} & -H^*_1 & -H^*_0 \\ E & 0 & 0 & 0 & 0 \\ 0 & E & 0 & & \\ & & & & \\ 0 & 0 & 0 & E & 0 \end{bmatrix} \qquad \text{(A-A-7)}$$

The eigenvalues of a matrix $[A]$ are defined as the values of λ that satisfy the equation

$$\det \left| [A] - \lambda [E] \right| = 0 . \tag{A-A-8}$$

(2) Nonsingular H_0 matric polynomial:

If H_0 of the matric polynomial in equation A-A-1 is nonsingular, we may introduce a change of variable

$$s' = \frac{1}{s} \tag{A-A-9}$$

such that the new normalized matric polynomial is

$$s'^{n} + [H_1'] s'^{n-1} + \ldots [H_n'] \tag{A-A-10}$$

where

$$[H_j'] = [H_0]^{-1} [H_j] \tag{A-A-11}$$

for $j = 1, 2 \ldots n$.

Then the determinant of the original matric polynomial is given by equation A-A-3 where λj are the inverses of the eigenvalues of the matrix

$$\begin{bmatrix} -H_1' & -H_2' & & -H_{n-1}' & -H_n' \\ E & 0 & & 0 & 0 \\ 0 & E & & 0 & 0 \\ & & & & \\ & & 0 & E & 0 \end{bmatrix} \tag{A-A-12}$$

(3) Neither H_n nor H_0 exists:

When the matric polynomial is such that neither the leading coefficient nor the coefficient of the lowest order term is nonsingular, then neither (1) nor (2) can be used and special procedure is required to get around the singularities.

For example, the matric quadratic in equation 6-13 is used,

$$[\,H\,] = [\,C\,]\,s^2 + [\,R\,]\,s + [\,L\,] \qquad \text{(A-A-13)}$$

where $[\,C\,]^{-1}$ and $[\,L\,]^{-1}$ do not exist.

If det $|H| = 0$, then there is a nonzero vector y such that

$$[\,H\,]\,y = 0. \qquad \text{(A-A-14)}$$

Let us introduce coordinate transformation on y, or congruent transformation on $[\,C\,]$, $[\,R\,]$ and $[\,L\,]$, such that after partitioning y into three subvectors

$$y = \begin{bmatrix} y^1 \\ y^2 \\ y^3 \end{bmatrix} \qquad \text{(A-A-15)}$$

equation A-A-14 is transformed to

$$\left(\begin{bmatrix} C_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} s^2 + \begin{bmatrix} R_{11} & R_{12} & 0 \\ R_{21} & R_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} s + \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \right) \begin{bmatrix} y^1 \\ y^2 \\ y^3 \end{bmatrix} = 0$$

(A-A-16)

where $[C_{11}]^{-1}$, $[R_{22}]^{-1}$ and $[L_{33}]^{-1}$ always exist. To solve for the values of s in equation A-A-16, we may first eliminate the variable y^3 which is uniquely related to y^1 and y^2 by

$$y^3 = -[L_{33}]^{-1}\Big([L_{31}]y^1 + [L_{32}]y^2\Big) \qquad \text{(A-A-17)}$$

The independent equations in equations A-A-16 become

$$\left(\begin{bmatrix} C_{11} & 0 \\ 0 & 0 \end{bmatrix} s^2 + \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} s + \begin{bmatrix} L_{11}-L_{13}L_{33}^{-1}L_{31} & L_{12}-L_{13}L_{33}^{-1}L_{32} \\ L_{21}-L_{23}L_{33}^{-1}L_{31} & L_{22}-L_{23}L_{33}^{-1}L_{32} \end{bmatrix}\right)\begin{bmatrix} y^1 \\ y^2 \end{bmatrix} = 0$$

(A-A-18)

To solve for s in equation A-A-18, we may solve for the following simultaneous equations:

$$[C_{11}]s^2y^1 + [R_{11}]sy^1 + [R_{12}]sy^2 + [L_{11}']y^1 + [L_{12}']y^2 = 0$$

(A-A-19)

$$[R_{21}]sy^1 + [R_{22}]sy^2 + [L_{21}']y^1 + [L_{22}']y^2 = 0$$

(A-A-20)

where $[L_{11}']$, $[L_{12}']$, $[L_{21}']$, $[L_{22}']$ are the corresponding elements in equation A-A-18.

In order to write the equations into first order form so that the eigenvalue method can be used, we introduce the variables

$$\begin{aligned} V^1 &= sy^1 \\ V^2 &= sy^2 \end{aligned} , \qquad \text{(A-A-21)}$$

then equations A-A-19 and A-A-20 may be written as

$$
\begin{aligned}
sV^1 = & -[C_{11}]^{-1}\Big([R_{11}-R_{12}R_{22}{}^{-1}R_{21}]\,V^1 \\
& +[L'_{11}-R_{12}R_{22}{}^{-1}L'_{21}]\,y^1 \\
& +[L'_{12}-R_{12}R_{22}{}^{-1}L'_{22}]\,y^2\Big) \\
sy^1 = & \; V^1 \\
sy^2 = & -[R_{22}]^{-1}\Big([R_{21}]\,V^1+[L'_{21}]\,y^1+[L'_{22}]\,y^2\Big)
\end{aligned}
\qquad \text{(A-A-22)}
$$

The left hand sides of equations A-A-22 have sV^1, sy^1, sy^2, and the right hand sides only have variables in V^1, y^1, y^2, therefore, we may write it as

$$
s\begin{bmatrix} V^1 \\ y^1 \\ y^2 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & A_3 \\ E & 0 & 0 \\ A_4 & A_5 & A_6 \end{bmatrix}\begin{bmatrix} V^1 \\ y^1 \\ y^2 \end{bmatrix} = [A]\begin{bmatrix} V^1 \\ y^1 \\ y^2 \end{bmatrix} \qquad \text{(A-A-23)}
$$

and solve for s as the eigenvalues of the matrix $[A]$ in equation A-A-23, where

$$
\begin{aligned}
[A_1] &= -[C_{11}]^{-1}[R_{11}-R_{12}R_{22}{}^{-1}R_{21}] \\
[A_2] &= -[C_{11}]^{-1}[L'_{11}-R_{12}R_{22}{}^{-1}L'_{21}] \\
[A_3] &= -[C_{11}]^{-1}[L'_{12}-R_{12}R_{22}{}^{-1}L'_{22}] \\
[A_4] &= -[R_{22}]^{-1}[R_{21}] \\
[A_5] &= -[R_{22}]^{-1}[L'_{21}] \\
[A_6] &= -[R_{22}]^{-1}[L'_{22}]
\end{aligned}
\qquad \text{(A-A-24)}
$$

and

$$
\begin{aligned}
[L'_{11}] &= [L_{11}] - [L_{13}][L_{33}]^{-1}[L_{31}] \\
[L'_{12}] &= [L_{12}] - [L_{13}][L_{33}]^{-1}[L_{32}] \\
[L'_{21}] &= [L_{21}] - [L_{23}][L_{33}]^{-1}[L_{31}] \\
[L'_{22}] &= [L_{22}] - [L_{23}][L_{33}]^{-1}[L_{32}]
\end{aligned} \quad . \qquad \text{(A-A-25)}
$$

The evaluation of the determinant of the matrix $[H]$ in equation A-A-13 by the eigenvalue method is hinged on the coordinate transformation in equation A-A-15 to give equation A-A-16. The algorithm of selecting V^1, V^2, V^3 discussed in this thesis gives the coordinates in equation A-A-15 directly.

Equation A-A-23 will give $2d_1 + d_2$ eigenvalues where (d_1, d_2) are the numbers of components in the vector (y^1, y^2). The actual number of nonzero roots in equation A-A-13, is given by equation 5-38 as

$$p = d_1 - d_3 + RK([L]). \qquad \text{(A-A-26)}$$

When the rank of the matrix $[L]$ is equal to its dimension, $(d_1 + d_2 + d_3)$, the actual number of roots is exactly $2d_1 + d_2$. If otherwise, $p < (2d_1 + d_2)$, and the extraneous zero roots computed from equation A-A-23 should not be included into equation A-A-31 in evaluating the determinant $G(s)$.

The same procedure, described starting at equation A-A-13 to equation A-A-23, can be used to compute the roots (natural frequencies) of any arbitrary passive linear network.

APPENDIX B

The complete listing of the LISP program, "corsel", and all the subfunctions used in its definition are included in alphabetical order in this appendix. Starting with the LISP 1.5 tape (26), this listed deck of cards will produce a new LISP tape with the defined function "corsel", and many of the unused functions in the LISP system removed to give more working memory. As many functions are compiled as possible to provide speedier computations. The LISP manual (26) should be consulted for the notations and function definitions given in this appendix.

```
KXL000    LISP PROG  ARB NETWORK COORD SELECTION
 SETSET WILL SET EVEN WHEN THERE IS AN ERROR - LISP
DEFINE ((
   (APELTST (LAMBDA (A  L) (COND
      ((NULL L) F) ((ATOM L) (COND ((EQ A L) T) (T F)))
      (T (OR (APELTST A (CAR L)) (APELTST A (CDR L)))))))
      ))
DEFINE ((
   (APPEAR (LAMBDA (L1 L2) (COND
      ((NULL L2) (LIST L1))
      ((OR (AND (EQUAL (CAAR L2) (CAR L1))
                (EQUAL (CADAR L2) (CADR L1)))
           (AND (EQUAL (CAAR L2) (CADR L1))
                (EQUAL (CADAR L2) (CAR L1)))) L2)
      (T (CONS (CAR L2) (APPEAR L1 (CDR L2)))))))
      ))
DEFINE ((
   (APPEAR3 (LAMBDA (L1 L2) (COND
      ((NULL L2) F)
      ((EQUAL L1 (CAR L2)) T)
      (T (APPEAR3 L1 (CDR L2))))))
      ))
DEFINE ((
   (ARRANGE (LAMBDA (N LPR) (PROG (A B)
            (SETQ A NIL)
            (SETQ B LPR)
      H1    (COND ((NULL B) (RETURN (CONS F A)))
               ((EQUAL N (CDAR B)) (RETURN (CONS T (APPEND B A)))))
            (SETQ A (CONS (CAR B) A))
            (SETQ B (CDR B))
            (GO H1))))
      ))
DEFINE ((
   (BVEV (LAMBDA (L1 L2 L3)
      (COND ((EQ (CAR L1) (CADR L1)) NIL)
            (T (INCF (CAR L1) (CADR L1)
                     (PATHFD1 (CONS (CAR L1) (CADR L1)) L3)
                     L2)))))
      ))
DEFINE ((
   (CADDDDR (LAMBDA (L) (CAR (CDDDDR L))))
   (CDAADDDDR (LAMBDA (L) (CDAAR (CDDDDR L))))
   (CAAADDDDR (LAMBDA (L) (CAAAR (CDDDDR L))))
   (CAADDDDR (LAMBDA (L) (CAAR (CDDDDR L))))
   (CDADDDDR (LAMBDA (L) (CDAR (CDDDDR L))))
   (CDAADDDR (LAMBDA (L) (CDAAR (CDDDR L))))
   (CAAADDDR (LAMBDA (L) (CAAAR (CDDDR L))))
   (CAADDDR (LAMBDA (L) (CAAR (CDDDR L))))
   (CDADDDR (LAMBDA (L) (CDAR (CDDDR L))))
   (CDAADDR (LAMBDA (L) (CDAAR (CDDR L))))
   (CAAADDR (LAMBDA (L) (CAAAR (CDDR L))))
      ))
DEFINE ((
   (CONNECT1 (LAMBDA (G N1 N2) (COND
      ((INCLUDE N1 G) (COND ((INCLUDE N2 G) (QUOTE ERROR2))
                            (T (CONS T N2))))
      ((INCLUDE N2 G) (CONS T N1))
      (T (LIST F)))))
      ))
DEFINE ((
   (CONNECT2 (LAMBDA (N1 N2 L) (PROG (A B)
            (SETQ A (FACTOR N1 L))
```

```
            (SETQ B (STRING1 N1 (CAR A)))
            (COND ((NULL B) (RETURN F))
                  ((INCLUDE N2 B) (RETURN T)))
            (RETURN (CONLT2 B N2 (CDR A))))))
    (CONLT2 (LAMBDA (L1 N L2) (COND
      ((NULL L1) F) ((CONNECT2 (CAR L1) N L2) T)
      (T (CONLT2 (CDR L1) N L2)))))
    (STRING1 (LAMBDA (N L) (COND
      ((NULL L) NIL) (T (CONS (COND ((EQ N (CAAR L)) (CDAR L))
                                    (T (CAAR L)))
                              (STRING1 N (CDR L)))))))
   ))
DEFINE ((
   (CORSEL (LAMBDA (NETWORK) (PROG
            (NPLIST LQLIST DEPNP NETWORK1)
            (SETQ NETWORK1 (CAR (GNLTST NETWORK)))
            (SETQ NPLIST (VNPGEN NETWORK1))
            (SETQ LQLIST (LQGEN (CADDDR NETWORK1) NPLIST))
            (SETQ DEPNP (SEDV LQLIST))
            (RETURN (CONS (REDNP NPLIST DEPNP) DEPNP)))))
   ))
DEFINE ((
   (ELIM (LAMBDA (L1 L3) (PROG (A B C)
            (SETQ B NIL)
            (SETQ A L1)
      H1    (COND ((NULL A) (RETURN B))
                  ((NULL (CAR A)) (GO H2)))
            (SETQ C (TAKAY (CAR A) L3))
            (COND ((CAR C) (GO H3)))
      H2    (SETQ B (APPEND B (LIST (CAR A))))
            (SETQ A (CDR A))
            (GO H1)
      H3    (SETQ B (APPEND B (LIST (CDDR C))))
            (SETQ A (CDR A))
            (GO H1))))
   ))
DEFINE ((
   (FACTOR (LAMBDA (N L) (PROG (A TLIST1 TLIST2)
            (SETQ TLIST1 NIL)
            (SETQ TLIST2 NIL)
            (SETQ A L)
      H1    (COND ((NULL A) (RETURN (CONS TLIST1 TLIST2)))
                  ((OR (EQ N (CAAR A)) (EQ N (CDAR A)))
                   (GO H2)))
            (SETQ TLIST2 (CONS (CAR A) TLIST2))
      H3    (SETQ A (CDR A))
            (GO H1)
      H2    (SETQ TLIST1 (CONS (CAR A) TLIST1))
            (GO H3))))
   ))
DEFINE ((
   (GNLTST (LAMBDA (L) (SUBSUBLIS (NORMAL L))))
   ))
DEFINE ((
   (INCF (LAMBDA (N1 N2 L1 L2) (PROG (TZ T1 T2 T3 T4 B C D NN)
            (SETQ TZ NIL)
            (SETQ T1 NIL)
            (SETQ T2 NIL)
            (SETQ T3 NIL)
            (SETQ T4 NIL)
            (SETQ NN N1)
            (SETQ D L1)
```

```
        HH    (SETQ B (SEARCH3 NN D))
              (COND ((EQ (CAAR B) NN) (GO HH1)))
              (SETQ C -1.0)
              (SETQ NN (CAAR B))
              (GO HH2)
        HH1   (SETQ C 1.0)
              (SETQ NN (CDAR B))
        HH2   (COND ((APPEAR3 (CAR B) (CAR L2)) (GO HZ))
                    ((APPEAR3 (CAR B) (CADR L2)) (GO H1))
                    ((APPEAR3 (CAR B) (CADDR L2)) (GO H2))
                    ((APPEAR3 (CAR B) (CADDDR L2)) (GO H3))
                    ((APPEAR3 (CAR B) (CADDDDR L2)) (GO H4))
                    (T (RETURN (QUOTE ERROR5))))
        HZ    (SETQ TZ (CONS (CONS C (CAR B)) TZ))
              (GO HH3)
        H1    (SETQ T1 (CONS (CONS C (CAR B)) T1))
              (GO HH3)
        H2    (SETQ T2 (CONS (CONS C (CAR B)) T2))
              (GO HH3)
        H3    (SETQ T3 (CONS (CONS C (CAR B)) T3))
              (GO HH3)
        H4    (SETQ T4 (CONS (CONS C (CAR B)) T4))
        HH3   (COND ((NULL (CDR B)) (RETURN (LIST TZ T1 T2 T3 T4))))
              (SETQ D (CDR B))
              (GO HH))))
      ))
DEFINE ((
   (INCLUDE (LAMBDA (B L) (COND
      ((NULL L) F)
      ((EQ B (CAR L)) T)
      (T (INCLUDE B (CDR L))))))
      ))
DEFINE ((
   (LOGEN (LAMBDA (L1 L2) (PROG (TLIST A B C)
              (SETQ TLIST NIL)
              (SETQ C (STRING L2))
              (SETQ A L1)
        H     (COND ((NULL A) (RETURN TLIST)))
              (SETQ B (TFRED1 (CAAR A) (CDAR A) L2 C))
              (SETQ TLIST (NCONC B TLIST))
              (SETQ A (CDR A))
              (GO H))))
      ))
DEFINE ((
   (MAXCF (LAMBDA (L) (PROG (TLIST A B)
              (SETQ TLIST NIL)
              (SETQ B (CAR L))
              (SETQ A (CDR L))
        HH    (COND ((NULL A) (RETURN (CONS B TLIST)))
                    ((LESSP (CAR B) (CAAR A))
                     (COND ((MINUSP (PLUS (CAR B) (CAAR A)))
                            (GO BGRA))
                           (T (GO AGRB))))
                           ((MINUSP (PLUS (CAR B) (CAAR A)))
                            (GO AGRB))
                           (T (GO BGRA)))
        BGRA (SETQ TLIST (CONS (CAR A) TLIST))
              (SETQ A (CDR A))
              (GO HH)
        AGRB (SETQ TLIST (CONS B TLIST))
              (SETQ B (CAR A))
              (SETQ A (CDR A))
```

```
            (GO HH))))
     ))
DEFINE ((
   (MINUS1 (LAMBDA (L1 L2) (PLUS1 L1 (TIMES1 -1.0 L2))))
     ))
DEFINE ((
   (NLEQ (LAMBDA (L) (COND
      ((NULL L) NIL)
      (T (CONS (MINUS1 (CAAR L) (CADAR L)) (NLEQ (CDR L)))))))
     ))
DEFINE ((
   (NORMAL (LAMBDA (L) (COND ((NULL (CDR L)) L)
      ((APELTST (CAAR L) (CDR L)) (NORMAL (APPEND (CDR L) (LIST (CAR L))
      )))
      (T (APPEND (CDR L) (LIST (CAR L)))))))
     ))
DEFINE ((
   (PATHFD1 (LAMBDA (L1 L2) (PROG (A)
            (SETQ A (FACTOR (CAR L1) L2))
      H1    (COND ((NULL (CAR A)) (RETURN (QUOTE ERROR3)))
                  ((NOT (CONNECT2 (CAR L1) (CDR L1)
            (COND ((NULL (CDAR A)) (CDR A))
                  (T (APPEND (CDAR A) ( CDR A))))))
                  (COND ((OR (EQ (CAAAR A) (CDR L1))
                             (EQ (CDAAR A) (CDR L1))))
                        (RETURN (LIST (CAAR A))))
                        (T (RETURN (CONS (CAAR A)
            (PATHFD1 (COND ((EQ (CAAAR A) (CAR L1))
                             (CONS (CDR L1) (CDAAR A)))
                            ((EQ (CAAAR A) (CDR L1))
                             (CONS (CAR L1) (CDAAR A)))
                            ((EQ (CDAAR A) (CAR L1))
                             (CONS (CDR L1) (CAAAR A)))
                            (T (CONS (CAR L1) (CAAAR A))))
                  (CDR A)))))))))
            (SETQ A (CONS (CDAR A) (CDR A)))
            (GO H1))))
     ))
DEFINE ((
   (PICK (LAMBDA (L) (PROG (LL A B)
            (SETQ LL (COND ((NOT (NULL (CADDDDR L))) (CDDDDR L))
                           ((NOT (NULL (CADDDR L))) (CDDDR L))
                           ((NOT (NULL (CADDR L))) (CDDR L))
                           ((NOT (NULL (CADR L))) (CDR L))
                           (T (RETURN (QUOTE ERROR6)))))
            (SETQ B (MAXCF (CAR LL)))
            (SETQ A (CAR B))
            (RPLACA LL (CDR B))
            (RETURN (CONS (CDR A) (LIST
                  (TIMES1 (RECIP (MINUS (CAR A))) L)))))))
     ))
DEFINE ((
   (PLUS1 (LAMBDA (L1 L2) (COND
      ((NULL L2) L1)
      ((NULL L1) L2)
      (T (CONS (PLUS2 (CAR L1) (CAR L2))
               (PLUS1 (CDR L1) (CDR L2)))))))
     ))
DEFINE ((
   (PLUS2 (LAMBDA (L3 L4) (PROG (A B)
            (COND ((NULL L3) (RETURN L4))
                  ((NULL L4) (RETURN L3)))
```

```
            (SETQ A (ARRANGE  (CDAR L3) L4))
            (COND ((CAR A) (GO HH)))
            (RETURN (CONS (CAR L3) (PLUS2 (CDR L3) L4)))
     HH     (SETQ B (PLUS (CAAR L3) (CAADR A)))
            (COND ((ZEROP B) (RETURN (PLUS2 (CDR L3) (CDDR A)))))
            (RETURN (CONS (CONS B (CDAR L3))
                          (PLUS2 (CDR L3) (CDDR A)))))))
    ))
DEFINE ((
  (PSEND (LAMBDA (A L) (PROG (B C)
            (SETQ B L)
     H1     (COND ((NULL B) (RETURN A)))
            (SETQ C (CAR B))
            (COND ((INCLUDE A C) (RETURN C)))
            (SETQ B (CDR B))
            (GO H1)))))
    ))
DEFINE ((
  (REDEMLT (LAMBDA (L) (PROG (A E G K J K1 TLIST)
            (SETQ TLIST NIL)
            (SETQ A L)
     H1     (COND ((NULL A) (RETURN TLIST)))
            (SETQ E (CAR A))
            (SETQ G TLIST)
            (SETQ K NIL)
     H5     (COND ((NOT (NULL G)) (GO H2)))
            (SETQ TLIST (CONS E K))
     H4     (SETQ A (CDR A))
            (GO H1)
     H2     (SETQ J (CAR G))
            (COND ((OR (AND (EQ (CAAR E) (CAAR J))
                            (EQ (CAADR E) (CAADR J)))
                       (AND (EQ (CAAR E) (CAADR J))
                            (EQ (CAADR E) (CAAR J))))
                   (COND ((LESSP  (CADDR E) (CADDR J)) (GO H4))
                         (T (GO H6))))))
            (SETQ K1 (LIST J))
            (RPLACD K1 K)
            (SETQ K K1)
            (SETQ G (CDR G))
            (GO H5)
     H6     (SETQ TLIST (APPEND (CDR G) (CONS E K)))
            (GO H4))))
    ))
DEFINE ((
 (REDEMLT1 (LAMBDA (L1 L2) (PROG (A B C D TLIST)
            (SETQ TLIST NIL)
            (SETQ A L1)
     H1     (COND ((NULL A) (RETURN TLIST)))
            (SETQ B (CAR A))
            (SETQ C (PSEND (CAR B) L2))
            (SETQ D (PSEND (CADR B) L2))
            (COND ((NOT (EQ C D)) (GO H2)))
     H4     (SETQ A (CDR A))
            (GO H1)
     H2     (SETQ TLIST (CONS (LIST (CONS C (CAR B))
                                    (CONS D (CADR B))
                                    (CADDR B)) TLIST))
            (GO H4))))
    ))
DEFINE ((
  (REDEMLT2 (LAMBDA (L1 L2) (PROG (A B C TLIST)
```

```
          (SETQ TLIST NIL)
          (SETQ A L1)
     H1   (COND ((NULL A) (RETURN TLIST)))
          (SETQ B (CAR A))
          (COND ((OR (EQ (CAAR B) (CAR L2))
                     (EQ (CAAR B) (CDR L2)))
                 (GO H2)))
          (SETQ C (CAR B))
     H4   (COND ((OR (EQ (CAADR B) (CAR L2))
                     (EQ (CAADR B) (CDR L2)))
                 (GO H3)))
          (SETQ TLIST (CONS (CONS C (CDR B)) TLIST))
     H5   (SETQ A (CDR A))
          (GO H1)
     H2   (SETQ C (CONS L2 (CDAR B)))
          (GO H4)
     H3   (COND ((EQUAL (CAR C) L2) (GO H5)))
          (SETQ TLIST (CONS (CONS C (CONS (CONS L2 (CDADR B))
                        (CDDR B))) TLIST))
          (GO H5))))
  ))
DEFINE ((
  (REDLTLT (LAMBDA (L) (COND
    ((NULL L) NIL)
    (T (CONS (REDEMLT (CAR L)) (REDLTLT (CDR L)))))))
  ))
DEFINE ((
  (REDLTLT1 (LAMBDA (L1 L2) (COND
    ((NULL L1) NIL)
    (T (CONS (REDEMLT1 (CAR L1) L2) (REDLTLT1 (CDR L1) L2))))))
  ))
DEFINE ((
  (REDLTLT2 (LAMBDA (L1 L2) (COND
    ((NULL L1)) NIL)
    (T (CONS (REDEMLT2 (CAR L1) L2)
             (REDLTLT2 (CDR L1) L2))))))
  ))
DEFINE ((
  (REDNP (LAMBDA (L1 L2) (COND
    ((NULL L2) L1)
    (T (REDNP (ELIM L1 (CAAR L2)) (CDR L2))))))
  ))
DEFINE ((
  (REDUCE (LAMBDA (L) (PROG (A B)
          (SETQ B L)
          (SETQ A NIL)
     C    (COND ((NULL B) (RETURN A))
                ((EQUAL (CAAR B) (CADAR B)) (GO D)))
          (SETQ A (APPEAR (CAR B) A))
     D    (SETQ B (CDR B))
          (GO C))))
  ))
DEFINE ((
  (REMPROPLT (LAMBDA (A L) (PROG (B)
          (SETQ B L)
     H    (COND ((NULL B) (RETURN NIL)))
          (REMPROP (CAR B) A)
          (SETQ B (CDR B))
          (GO H))))
  ))
DEFINE ((
  (SEARCH3 (LAMBDA (N L) (COND
```

```
        ((OR (EQ N (CAAR L)) (EQ N (CDAR L))) L)
        (T (APPEND (SEARCH3 N (CDR L)) (LIST (CAR L)))))))
        ))
DEFINE ((
   (SEDV (LAMBDA (L) (PROG (TLIST A B)
            (SETQ TLIST NIL)
            (SETQ A (MLEQ L))
      H1    (COND ((NULL A) (RETURN TLIST)))
            (SETQ B (PICK (CAR A)))
            (SETQ TLIST (CONS B (SUBSTT B TLIST)))
            (SETQ A (SUBSTT1 B (CDR A)))
            (GO H1))))
      ))
DEFINE ((
   (SELECT (LAMBDA (L) (PROG (A B C D)
            (COND ((NULL L) (RETURN NIL)))
            (SETQ A L)
            (SETQ B NIL)
            (SETQ C (CAR A))
            (SETQ D (CDR A))
      H1    (COND ((NULL D) (RETURN (CONS C B)))
                  ((NOT (LESSP   (CADDAR D) (CADDR C))) (GO H2)))
            (SETQ B (CONS (CAR D) B))
      H3    (SETQ D (CDR D))
            (GO H1)
      H2    (SETQ B (CONS C B))
            (SETQ C (CAR D))
            (GO H3))))
      ))
DEFINE ((
   (STRING (LAMBDA (L) (COND
      ((NULL L) NIL)
      (T (APPEND (CAR L) (STRING (CDR L)))))))
      ))
DEFINE ((
   (SUBSTT (LAMBDA (L1 L2) (COND
      ((NULL L2) NIL)
      (T (CONS (CONS (CAAR L2) (LIST (SUBSTT2 L1 (CADAR L2))))
               (SUBSTT L1 (CDR L2))))))))
      ))
DEFINE ((
   (SUBSTT1 (LAMBDA (L1 L2) (COND
      ((NULL L2) NIL)
      (T (CONS (SUBSTT2 L1 (CAR L2)) (SUBSTT1 L1 (CDR L2)))))))
      ))
DEFINE ((
   (SUBSTT2 (LAMBDA (L1 L2) (PROG (TLIST A D)
            (SETQ TLIST NIL)
            (SETQ A L2)
      H1    (COND ((NULL A) (RETURN TLIST)))
            (SETQ D (ARRANGE (CAR L1) (CAR A)))
            (COND ((CAR D) (GO H4)))
            (SETQ TLIST (NCONC TLIST (LIST (CAR A))))
            (SETQ A (CDR A))
            (GO H1)
      H4    (RETURN (PLUS1 (TIMES1 (CAADR D) (CADR L1))
                           (CONC TLIST (LIST (CDDR D)) (CDR A)))))))
      ))
DEFINE ((
   (SUBSUBLIS (LAMBDA (L) (COND
      ((NULL (CDR L)) (CDAR L))
      (T (SUBSUBLIS (SUBST (CADAR L) (CAAR L) (CDR L)))))))
```

```
    ))
DEFINE ((
   (TAKAY (LAMBDA (L1 L2) (PROG (A B)
            (SETQ A NIL)
            (SETQ B L1)
      H1    (COND ((NULL B) (RETURN (CONS F A)))
                  ((EQUAL L2 (CAR B)) (RETURN (CONS T (APPEND B A)))))
            (SETQ A (CONS (CAR B) A))
            (SETQ B (CDR B))
            (GO H1))))
    ))
DEFINE ((
   (TFRED1 (LAMBDA (L1 L2 L3 D) (PROG (TLIST A B C)
            (SETQ TLIST NIL)
            (SETQ A (TIMES1 (RECIP (CADDR L1)) (BVEV L1 L3 D)))
            (SETQ B L2)
      H1    (COND ((NULL B) (RETURN TLIST)))
            (SETQ C (TIMES1 (RECIP (CADDAR B)) (BVEV (CAR B) L3 D)))
            (SETQ TLIST (CONS (CONS A (LIST C)) TLIST))
            (SETQ B (CDR B))
            (GO H1))))
    ))
DEFINE ((
   (TIMES1 (LAMBDA (K L) (COND
      ((NULL L) NIL)
      (T (CONS (TIMES2 K (CAR L)) (TIMES1 K (CDR L)))))))
    ))
DEFINE ((
   (TIMES2 (LAMBDA (K L) (COND
      ((NULL L) NIL)
      (T (CONS (CONS (TIMES K (CAAR L)) (CDAR L))
               (TIMES2 K (CDR L)))))))
    ))
DEFINE ((
   (VGEN (LAMBDA (L1 L2) (PROG (A B C D E G)
            (SETQ C NIL)
            (SETQ A L1)
            (SETQ D L2)
      H1    (COND ((NULL A) (RETURN (CONS C D))))
            (SETQ B (SELECT A))
            (SETQ E (CONS (CDAAR  B) (CDADAR B)))
            (SETQ G (CONS (CAAAR B) (CAADAR B)))
            (SETQ A (REDEMLT (REDEMLT2 (CDR B) G)))
            (SETQ D (REDLTLT (REDLTLT2 D G)))
            (SETQ C (CONS E C))
            (GO H1))))
    ))
DEFINE ((
   (VNPGEN (LAMBDA (L) (PROG (A B C D E)
            (SETQ A (VZGEN (CADDDDR L)))
            (SETQ B (CAR A))
            (SETQ C (CDR A))
            (SETQ D (CONS (CAR L)
                          (CONS (CADR L)
                                (CONS (CADDR L)
                                      (LIST (REDUCE (STRING (CADDDR L))))))))
            (SETQ D (REDLTLT (REDLTLT1 D C)))
            (SETQ E (V1234GEN D))
            (RETURN (CONS B E)))))
    ))
DEFINE ((
   (VZGEN (LAMBDA (L) (PROG (A B C D E G J)
```

```
          (SETQ A L)
          (SETQ B NIL)
          (SETQ C NIL)
          (SETQ E NIL)
          (SETQ G NIL)
     H7   (COND ((NULL A) (RETURN (CONS B C))))
          (SETQ D (CONS (CAAR A) (LIST (CADAR A))))
          (SETQ G D)
          (GO H8)
     H3   (COND ((NULL G) (GO H4)))
     H1   (COND ((NULL A) (GO H2)))
          (SETQ J (CONNECT1 G (CAAR A) (CADAR A)))
          (COND ((ATOM J) (RETURN (QUOTE ERROR2)))
                ((CAR J) (GO H5)))
          (SETQ E (CONS (CAR A) E))
     H6   (SETQ A (CDR A))
          (GO H1)
     H2   (SETQ G (CDR G))
          (SETQ A E)
          (SETQ E NIL)
          (GO H3)
     H4   (SETQ C (CONS D C))
          (GO H7)
     H5   (NCONC D (LIST (CDR J)))
     H8   (SETQ B (CONS (CONS (CAAR A) (CADAR A)) B))
          (GO H6))))
     ))
DEFINE ((
   (V1234GEN (LAMBDA (L) (PROG (A B C D)
          (COND ((NULL L) (RETURN NIL)))
          (SETQ A (CAR L))
          (SETQ B (CDR L))
          (SETQ D NIL)
     H1   (SETQ C (VGEN A B))
          (SETQ D (CONS (CAR C) D))
          (COND ((NULL (CDR C)) (RETURN (REVERSE D))))
          (SETQ A (CADR C))
          (SETQ B (CDDR C))
          (GO H1))))
     ))
   TRACLIS ((
      CORSEL
      VNPGEN
     LOGEN
      SEDV
      ))
   COMDEF ((
     CADDAR
     STRING
     CADDDDR
     ))
   COMDEF ((
     APPEAR
     APPEAR3
     ARRANGE
     INCLUDE
     STRING1
     REDEMLT
     REDEMLT1
     REDEMLT2
     SELECT
     ))
```

```
COMDEF ((
  CONNECT1
  PSEND
  REDLTLT
  REDLTLT1
  REDLTLT2
  REDUCE
  ))
COMDEF ((
  FACTOR
  SEARCH3
  VGEN
  ))
COMDEF ((
  CONNECT2
  CONLT2
  PATHFD1
  ))
REMPROPLT (
   SUBR
   (ATTRIB
    PROP
    COPY
    PAIR
    SASSOC
    SEARCH
    EXPT
    FIXP
    FLOATP
    LEFTSHIFT
    ARRAY
    COMPILE
    SAP
    COMPSAP
    OPDEFINE
    READ
    PUNCH
    PROG2
    CPL
    GENSYM
    TEMPUS-FUGIT))
REMPROPLT (
   FSUBR
   (LOGOR
    LOGAND
    LOGXOR))
REMPROPLT (
   EXPR
   (CONSTVAL
    COMDEF
    PRINTPROP
    PUNCHDEF
    MAKCBLR
    FORMAT))
      STOP)))))))))STOP
```

APPENDIX C

EXAMPLES

The example in A. C. 1 is the dynamic circuit analogy of a 4-cell finite difference cantilevered beam (13). The coordinate selection is worked out manually in detail and followed by the actual LISP program (corsel) output. The example in A. C. 2 is the dynamic circuit analogy of an airplane wing, represented as a 6-cell finite difference model (reference (13), Chapter 5) of a mass coupled bending and torsion beam, with Russell analogy in the bending mode. The example in A. C. 3 is the dynamic circuit analogy of a delta wing, represented as a 6-cell finite difference model of a plate with Poisson's lateral coupling (3). A. C. 4 gives an example of a network with many irregular transformer interconnections. Only the LISP program output are provided for the last three examples.

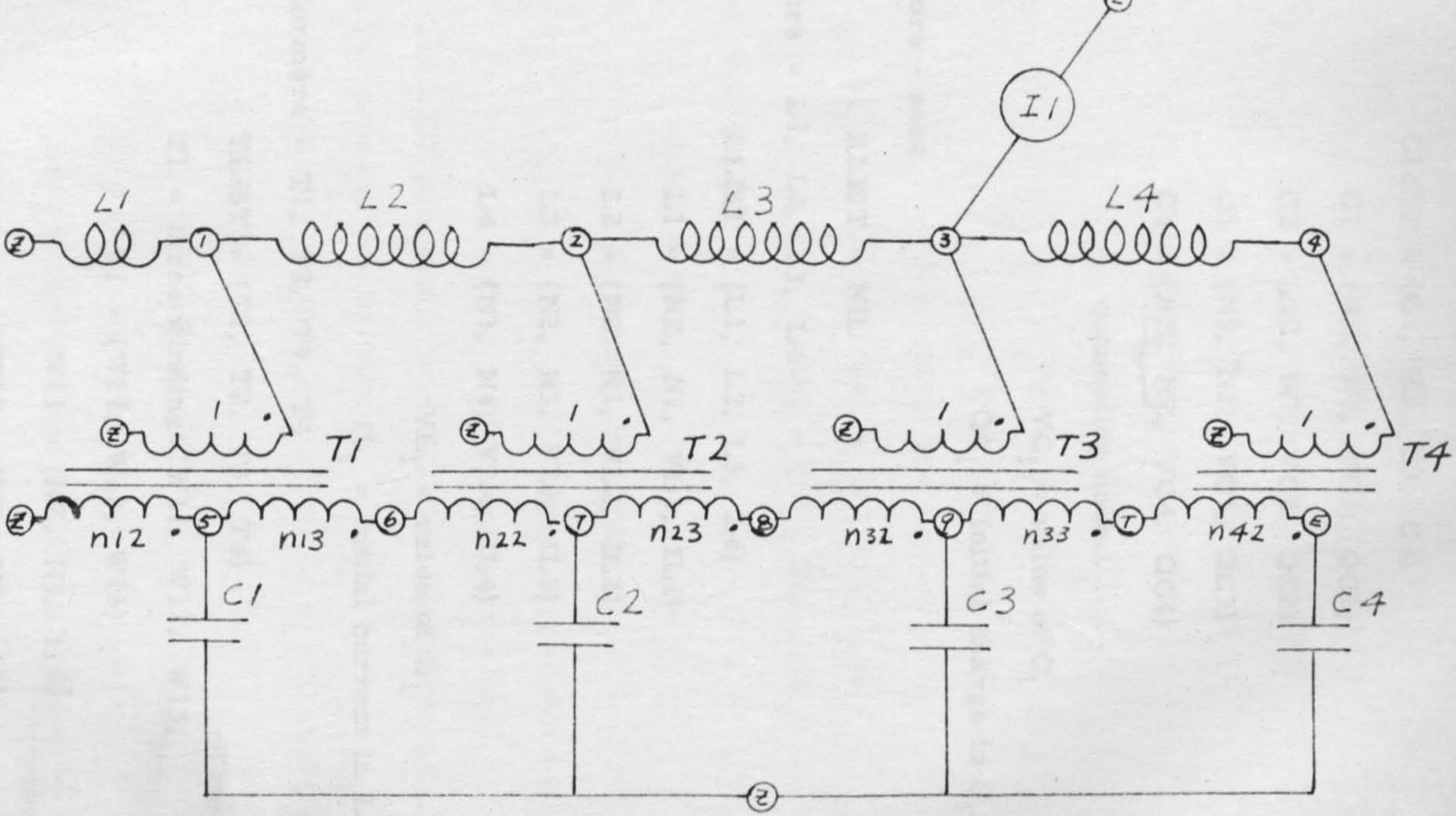
A. C. 1

The circuit in (fig. A-C-1) is taken as an example. It is the dynamic analog circuit of a 4-cell finite difference cantilevered beam (reference (13), Chapter 5) with torque I_1 applied at the point that corresponds to node 3.

The complete description of the network is as follows:

Voltage sources - none

VLIST = NIL



The Dynamic Analog Circuit of a Cantilevered Beam with a Moment Applied at Point 3

FIGURE A - C - 1

Capacitors C1, C2, C3, C4

CLIST = (C1, C2, C3, C4)

C1 = (NZ, N5, VC1, QC1)

C2 = (NZ, N7, VC2, QC2)

C3 = (N9, NZ, VC3, QC3)

C4 = $(\underbrace{\text{NE, NZ}}$, VC4, QC4)

connecting nodes.

VC_i = value of C_i

QC_i = initial charge in C_i.

Resistors - none

RLIST = NIL

Inductors = L1, L2, L3, L4

LLIST = (L1, L2, L3, L4)

L1 = (NZ, N1, VL1, IL1)

L2 = (N2, N1, VL2, IL2)

L3 = (N2, N3, VL3, IL3)

L4 = (N3, N4, VL4, IL4)

VL_i = value of L_i

IL_i = initial current in L_i.

Transformers = T1, T2, T3, T4

TLIST = (T1, T2, T3, T4)

T1 = three windings W11, W12, W13,

T1 = (W11, W12, W13)

W11 = (NZ, N1, 1.0)

W12 = (NZ, N5, n12)

W13 = (N5, N6, n13)

Similarly for T2, T3, T4

T2 = (W21, W22, W23)

W21 = (NZ, N2, 1.0)

W22 = (N6, N7, n22)

W23 = (N7, N8, n23)

T3 = (W31, W32, W33)

W31 = (NZ, N3, 1.0)

W32 = (N8, N9, n32)

W33 = (N9, NT, n33)

T4 = (W41, W42)

W41 = (NZ, N4, 1.0)

W42 = (NT, NE, n42)

Current sources: I1

ILIST = (I1)

I1 = (N3, NZ, VI1)

VI1 contains the information on the time dependence of I1.

The LISP program, "corsel", that algorithmically selects the coordinates proceeds as follows:

(o) VLIST is taken and V^o is set to NIL since VLIST is empty.

VZLT = NIL

(1) All nodes connected by the node pairs in V^o are grouped together and CLIST is reduced by removing all the elements whose two terminals are connected to the same node and only the one with the largest value among the several that are connected to the

common pair of nodes is retained. From the reduced CLIST, V^1 is selected according to the criterion of minimizing round-off errors.

$$V1LT = (v_{z5}, v_{z1}, v_{9z}, v_{EZ})$$

(2) RLIST is empty, $V^2 = 0$

$$V2LT = NIL$$

(3) LLIST is reduced by grouping all the nodes connected by V^o, V^1 and V^2, then V^3 is selected according to the minimum round-off error criterion.

$$V3LT = (v_{Z1}, v_{21}, v_{23}, v_{34})$$

(4) TLIST is stringed together to form a winding list, and the winding list is reduced by grouping all the nodes connected by V^o, V^1, V^2, V^3. From the non-empty winding list, V^4 is selected.

$$V4LT = (v_{67}, v_{78}, v_{TE})$$

(5) Form the node pair list NPLIST

$$NPLIST = (VZLT, V1LT, V2LT, V3LT, V4LT)$$

(6) From TLIST one transformer is taken at a time and from its m windings, (m - 1) linear equations are constructed. The variables in the equations are the components of V^o, V^1, V^2, V^3, V^4. The list of equations from all the transformers form the LQLIST. T1 gives

$$LQ1 = v_{Z1} = \frac{1}{n12} v_{Z5}$$

$$LQ2 = v_{Z1} = \frac{1}{n13} (v_{Z7} - v_{Z5} - v_{67})$$

T2 gives

$$LQ3 = v_{Z1} - v_{21} = \frac{1}{n22}(v_{67})$$

$$LQ4 = v_{Z1} - v_{21} = \frac{1}{n23} v_{78}$$

T3 gives

$$LQ5 = v_{Z1} - v_{21} + v_{23} = \frac{1}{n32}(-v_{9Z} - v_{Z7} - v_{78})$$

$$LQ6 = v_{Z1} - v_{21} + v_{23} = \frac{1}{n33}(-v_{EZ} + v_{9Z} - v_{TE})$$

T4 gives

$$LQ7 = v_{Z1} - v_{21} + v_{23} + v_{34} = \frac{1}{n42}(v_{TE})$$

LQLIST = (LQ1, LQ2, LQ3, LQ4, LQ5, LQ6, LQ7)

(7) From LQLIST and the hierarch of variables defined in NPLIST,

$$V^o > V^1 > V^2 > V^3 > V^4$$

express the variables of the lowest hierarchy in terms of variables of higher hierarchy. Using the seven equations in LQLIST, seven variables are listed, together with their dependences on the remaining four variables.

LQ1 in LQLIST is taken first and the lowest hierarchy variable, v_{Z1}, is expressed in terms of the other higher hierarchy variables in the equation. The expression of v_{Z1} is put into DEPNP, and the new LQLIST has the old LQ1 removed and every v_{Z1} substituted by its equivalent expression in the higher hierarchy variables. Then from the new LQLIST, one equation is taken and the expression of one of its lowest hierarchy variables is substituted for the appearance of the variable in both DEPNP and LQLIST. Its expression is also added to DEPNP. This process continues until the final LQLIST is empty. For our example, we have the DEPNP as follows:

DEPNP = (EQ1, EQ2, EQ3, EQ4, EQ5, EQ6, EQ7)

Let all n's = 0.5, then we have

$$EQ1 = v_{Z1} = 2v_{Z5}$$
$$EQ2 = v_{67} = -v_{Z5} + v_{Z7}$$
$$EQ3 = v_{21} = 6v_{Z5} - 2v_{Z7}$$
$$EQ4 = v_{78} = -2v_{Z5} + v_{Z7}$$
$$EQ5 = v_{23} = 8v_{Z5} - 6v_{Z7} - 2v_{9Z}$$
$$EQ6 = v_{TE} = -2v_{Z5} + 2v_{Z7} - 2v_{9Z} - v_{EZ}$$
$$EQ7 = v_{34} = -8v_{Z5} + 8v_{Z7} + 6v_{9Z} - 2v_{EZ}$$

(8) The final set of independent node pairs are obtained by removing all the dependent node pairs from NPLIST.

INDNP = (VZ, V1, V2, V3)

VZ = NIL

V1 = $(v_{Z5}, v_{Z7}, v_{9Z}, v_{EZ})$

V2 = NIL

V3 = NIL

(9) The [C], [R], [L] matrices in the coordinate $(v_{Z5}, v_{Z7}, v_{9Z}, v_{EZ})$ are computed.

$$[C_{11}] = \begin{bmatrix} C1 & 0 & 0 & 0 \\ 0 & C2 & 0 & 0 \\ 0 & 0 & C3 & 0 \\ 0 & 0 & 0 & C4 \end{bmatrix}$$

$[R_{11}] = 0$

$$[L_{11}] = \begin{bmatrix} 2 & 6 & 8 & -8 \\ 0 & -2 & -6 & 8 \\ 0 & 0 & -2 & 6 \\ 0 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} L1 & 0 & 0 & 0 \\ 0 & L2 & 0 & 0 \\ 0 & 0 & L3 & 0 \\ 0 & 0 & 0 & L4 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 6 & -2 & 0 & 0 \\ 8 & -6 & -2 & 0 \\ -8 & 8 & 6 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} & \ell_{14} \\ \ell_{21} & \ell_{22} & \ell_{23} & \ell_{24} \\ \ell_{31} & \ell_{32} & \ell_{33} & \ell_{34} \\ \ell_{41} & \ell_{42} & \ell_{43} & \ell_{44} \end{bmatrix}$$

$[Z_{ij}] = 0$ for $[Z] = [C]$, $[R]$, $[L]$ and

$i, j = 0, 2, 3$

where

$$\ell_{11} = 4(L1) + 36(L2) + 64(L3) + 64(L4)$$
$$\ell_{12} = -12(L2) - 48(L3) - 64(L4)$$
$$\ell_{13} = -16(L3) - 48(L4)$$
$$\ell_{14} = 16(L4)$$
$$\ell_{22} = 4(L2) + 36(L3) + 64(L4)$$
$$\ell_{23} = 12(L3) + 48(L4)$$
$$\ell_{24} = -16(L4)$$
$$\ell_{33} = 4(L3) + 36(L4)$$
$$\ell_{34} = -12(L4)$$
$$\ell_{44} = 4(L4)$$
$$\ell_{ij} = \ell_{ji}$$

(10) The forcing function (I^o, I^1, I^2, I^3) is computed. The current source is connected between nodes N3 and NZ,

$$v_{3Z} = v_{Z1} - v_{21} + v_{23}$$

Then from (EQ1, EQ3, EQ5) v_{3Z} is expressed as a linear function of the chosen independent node pairs:

$$v_{3Z} = 4v_{Z5} - 4v_{Z7} - 2v_{9Z}.$$

The forcing function I^1 becomes

$$I^1 = \begin{bmatrix} 4 \\ -4 \\ -2 \\ 0 \end{bmatrix} I1$$

where I1 is the time dependent current source as specified in ILIST.

(11) After choosing the coordinates and setting up the appropriate matrices, various numerical integration methods may be used to compute the transient response of the network.

The example does not give the actual result from numerical computation since the purpose of the example is to illustrate the algorithm in coordinate selection. The program in Appendix B coded in LISP restricts itself to the selection of coordinates. Once the coordinates are selected comparatively straightforward programs can be written to do the actual numerical computations.

The LISP program input card listing is given in (fig. A-C-2-a) and the output is included in (fig. A-C-2-b)

The selected coordinates are:

$$V^1 = \begin{matrix} NE \cdot NZ \\ N9 \cdot NZ \\ NZ \cdot N5 \\ NZ \cdot N7 \end{matrix}$$

```
  KXL000   LISP  FOUR CELL FINITE DIFFERENCE  BEAM ANALOGY
   TEST  THIS IS AN OVERLORD CARD - LISP
CORSEL  ((
   (NETWORK (CLIST RLIST LLIST TLIST VLIST ILIST))
   (CLIST (C1 C2 C3 C4))
   (C1 (NZ N5 1.0))
   (C2 (NZ N7 1.0))
   (C3 (N9 NZ 1.0))
   (C4 (NE NZ 0.5))
   (RLIST NIL)
   (LLIST (L1 L2 L3 L4))
   (L1 (NZ N1 0.5))
   (L2 (N2 N1 1.0))
   (L3 (N2 N3 1.0))
   (L4 (N3 N4 1.0))
   (TLIST (T1 T2 T3 T4))
   (T1 (W11 W12 W13))
   (W11 (NZ N1 1.0))
   (W12 (NZ N5 0.5))
   (W13 (N5 N6 0.5))
   (T2 (W21 W22 W23))
   (W21 (NZ N2 1.0))
   (W22 (N6 N7 0.5))
   (W23 (N7 N8 0.5))
   (T3 (W31 W32 W33))
   (W31 (NZ N3 1.0))
   (W32 (N8 N9 0.5))
   (W33 (N9 NT 0.5))
   (T4 (W41 W42))
   (W41 (NZ N4 1.0))
   (W42 (NT NE 0.5))
   (ILIST (I1))
   (I1 (VI1))
   (VLIST NIL)
  ))
     STOP))))))))))STOP
```

FIGURE A - C - 2 - a

```
        KXL000   LISP  FOUR CELL FINITE DIFFERENCE  BEAM ANALOGY              000000000000

         TEST  THIS IS AN OVERLORD CARD - LISP                                 000000000000

 THE TIME ( 0/ 0  000.0) HAS COME, THE WALRUS SAID, TO TALK OF MANY THINGS .....   -LEWIS CARROLL-

EVALQUOTE OPERATOR AS OF 1 MARCH 1961.    INPUT LISTS NOW BEING READ.

 THE TIME ( 0/ 0  000.0) HAS COME, THE WALRUS SAID, TO TALK OF MANY THINGS .....   -LEWIS CARROLL-



 FUNCTION   EVALQUOTE   HAS BEEN ENTERED, ARGUMENTS..
CORSEL
(((NETWORK (CLIST RLIST LLIST TLIST VLIST ILIST)) (CLIST (C1 C2 C3 C4)) (C1 (NZ N5  1.0000000+00)) (C2 (NZ N7  1.000000
0+00)) (C3 (N9 NZ  1.0000000+00)) (C4 (NE NZ  5.0000000-01)) (RLIST NIL) (LLIST (L1 L2 L3 L4)) (L1 (NZ N1  5.0000000-01
)) (L2 (N2 N1  1.0000000+00)) (L3 (N2 N3  1.0000000+00)) (L4 (N3 N4  1.0000000+00)) (TLIST (T1 T2 T3 T4)) (T1 (W11 W12
W13)) (W11 (NZ N1  1.0000000+00)) (W12 (NZ N5  5.0000000-01)) (W13 (N5 N6  5.0000000-01)) (T2 (W21 W22 W23)) (W21 (NZ N
2  1.0000000+00)) (W22 (N6 N7  5.0000000-01)) (W23 (N7 N8  5.0000000-01)) (T3 (W31 W32 W33)) (W31 (NZ N3  1.0000000+00)
) (W32 (N8 N9  5.0000000-01)) (W33 (N9 NT  5.0000000-01)) (T4 (W41 W42)) (W41 (NZ N4  1.0000000+00)) (W42 (NT NE  5.000
0000-01)) (ILIST (I1)) (I1 (VI1)) (VLIST NIL)))


(ARGUMENTS OF CORSEL)
((NETWORK (CLIST RLIST LLIST TLIST VLIST ILIST)) (CLIST (C1 C2 C3 C4)) (C1 (NZ N5  1.0000000+00)) (C2 (NZ N7  1.0000000
+00)) (C3 (N9 NZ  1.0000000+00)) (C4 (NE NZ  5.0000000-01)) (RLIST NIL) (LLIST (L1 L2 L3 L4)) (L1 (NZ N1  5.0000000-01)
) (L2 (N2 N1  1.0000000+00)) (L3 (N2 N3  1.0000000+00)) (L4 (N3 N4  1.0000000+00)) (TLIST (T1 T2 T3 T4)) (T1 (W11 W12 W
13)) (W11 (NZ N1  1.0000000+00)) (W12 (NZ N5  5.0000000-01)) (W13 (N5 N6  5.0000000-01)) (T2 (W21 W22 W23)) (W21 (NZ N2
  1.0000000+00)) (W22 (N6 N7  5.0000000-01)) (W23 (N7 N8  5.0000000-01)) (T3 (W31 W32 W33)) (W31 (NZ N3  1.0000000+00))
 (W32 (N8 N9  5.0000000-01)) (W33 (N9 NT  5.0000000-01)) (T4 (W41 W42)) (W41 (NZ N4  1.0000000+00)) (W42 (NT NE  5.0000
000-01)) (ILIST (I1)) (I1 (VI1)) (VLIST NIL))

RECLAIMER ENTRY   0/ 0  000.0 MARK  000.0      SWEEP   000.0  FULL WORDS   404 FREE   4034  PUSH DOWN DEPTH    121

(ARGUMENTS OF VNPGEN)
(((NZ N5  1.0000000+00) (NZ N7  1.0000000+00) (N9 NZ  1.0000000+00) (NE NZ  5.0000000-01)) NIL ((NZ N1  5.0000000-01) (
N2 N1  1.0000000+00) (N2 N3  1.0000000+00) (N3 N4  1.0000000+00)) (((NZ N1  1.0000000+00) (NZ N5  5.0000000-01) (N5 N6
 5.0000000-01)) ((NZ N2  1.0000000+00) (N6 N7  5.0000000-01) (N7 N8  5.0000000-01)) ((NZ N3  1.0000000+00) (N8 N9  5.00
00000-01) (N9 NT  5.0000000-01)) ((NZ N4  1.0000000+00) (NT NE  5.0000000-01))) NIL ((VI1)))

RECLAIMER ENTRY   0/ 0  000.0 MARK  000.0      SWEEP   000.0  FULL WORDS   404 FREE   4460  PUSH DOWN DEPTH     64

(VALUE OF VNPGEN)
(NIL ((NE . NZ) (N9 . NZ) (NZ . N5) (NZ . N7)) NIL ((NZ . N1) (N2 . N1) (N3 . N4) (N2 . N3)) ((NT . NE) (N7 . N8) (N6 .
 N7)))

(ARGUMENTS OF LOGEN)
(((NZ N1  1.0000000+00) (NZ N5  5.0000000-01) (N5 N6  5.0000000-01)) ((NZ N2  1.0000000+00) (N6 N7  5.0000000-01) (N7 N
8  5.0000000-01)) ((NZ N3  1.0000000+00) (N8 N9  5.0000000-01) (N9 NT  5.0000000-01)) ((NZ N4  1.0000000+00) (NT NE  5.
0000000-01)))
(NIL ((NE . NZ) (N9 . NZ) (NZ . N5) (NZ . N7)) NIL ((NZ . N1) (N2 . N1) (N3 . N4) (N2 . N3)) ((NT . NE) (N7 . N8) (N6 .
 N7)))

(VALUE OF LOGEN)
(((NIL NIL NIL (( 1.0000000+00 N3 . N4) ( 1.0000000+00 N2 . N3) (-1.0000000+00 N2 . N1) ( 1.0000000+00 NZ . N1)) NIL) (
NIL NIL NIL NIL (( 2.0000000+00 NT . NE)))) ((NIL NIL NIL (( 1.0000000+00 N2 . N3) (-1.0000000+00 N2 . N1) ( 1.0000000+
00 NZ . N1)) NIL) (NIL ((-2.0000000+00 NE . NZ) ( 2.0000000+00 N9 . NZ)) NIL NIL ((-2.0000000+00 NT . NE)))) ((NIL NIL
NIL (( 1.0000000+00 N2 . N3) (-1.0000000+00 N2 . N1) ( 1.0000000+00 NZ . N1)) NIL) (NIL ((-2.0000000+00 N9 . NZ) (-2.00
```

FIGURE A - C - 2 - b

```
00000+00 NZ . N7)) NIL NIL ((-2.0000000+00 N7 . N8)))) ((NIL NIL NIL ((-1.0000000+00 N2 . N1) ( 1.0000000+00 NZ . N1))
NIL) (NIL NIL NIL NIL (( 2.0000000+00 N7 . N8)))) ((NIL NIL NIL ((-1.0000000+00 N2 . N1) ( 1.0000000+00 NZ . N1)) NIL)
(NIL NIL NIL NIL (( 2.0000000+00 N6 . N7)))) ((NIL NIL NIL (( 1.0000000+00 NZ . N1)) NIL) (NIL (( 2.0000000+00 NZ . N7)
(-2.0000000+00 NZ . N5)) NIL NIL ((-2.0000000+00 N6 . N7)))) ((NIL NIL NIL (( 1.0000000+00 NZ . N1)) NIL) (NIL (( 2.00
00000+00 NZ . N5)) NIL NIL NIL)))

(ARGUMENTS OF SEDV)
((((NIL NIL NIL (( 1.0000000+00 N3 . N4) ( 1.0000000+00 N2 . N3) (-1.0000000+00 N2 . N1) ( 1.0000000+00 NZ . N1)) NIL) (
NIL NIL NIL NIL (( 2.0000000+00 NT . NE)))) ((NIL NIL NIL (( 1.0000000+00 N2 . N3) (-1.0000000+00 N2 . N1) ( 1.0000000+
00 NZ . N1)) NIL) (NIL ((-2.0000000+00 NE . NZ) ( 2.0000000+00 N9 . NZ)) NIL NIL ((-2.0000000+00 NT . NE)))) ((NIL NIL
NIL (( 1.0000000+00 N2 . N3) (-1.0000000+00 N2 . N1) ( 1.0000000+00 NZ . N1)) NIL) (NIL ((-2.0000000+00 N9 . NZ) (-2.00
00000+00 NZ . N7)) NIL NIL ((-2.0000000+00 N7 . N8)))) ((NIL NIL NIL ((-1.0000000+00 N2 . N1) ( 1.0000000+00 NZ . N1))
NIL) (NIL NIL NIL NIL (( 2.0000000+00 N7 . N8)))) ((NIL NIL NIL ((-1.0000000+00 N2 . N1) ( 1.0000000+00 NZ . N1)) NIL)
(NIL NIL NIL NIL (( 2.0000000+00 N6 . N7)))) ((NIL NIL NIL (( 1.0000000+00 NZ . N1)) NIL) (NIL (( 2.0000000+00 NZ . N7)
(-2.0000000+00 NZ . N5)) NIL NIL ((-2.0000000+00 N6 . N7)))) ((NIL NIL NIL (( 1.0000000+00 NZ . N1)) NIL) (NIL (( 2.00
00000+00 NZ . N5)) NIL NIL NIL)))

RECLAIMER ENTRY   0/ 0  000.0 MARK  000.0     SWEEP  000.0  FULL WORDS  375 FREE  4110  PUSH DOWN DEPTH   130

RECLAIMER ENTRY   0/ 0  000.0 MARK  000.0     SWEEP  000.0  FULL WORDS  346 FREE  3796  PUSH DOWN DEPTH   163

RECLAIMER ENTRY   0/ 0  000.0 MARK  000.0     SWEEP  000.0  FULL WORDS  341 FREE  3768  PUSH DOWN DEPTH   162

RECLAIMER ENTRY   0/ 0  000.0 MARK  000.0     SWEEP  000.0  FULL WORDS  350 FREE  3864  PUSH DOWN DEPTH   164

(VALUE OF SEDV)
(((N2 . N1) (NIL ((-2.0000000+00 NZ . N7) ( 6.0000000+00 NZ . N5)) NIL NIL NIL)) ((N3 . N4) (NIL (( 8.0000000+00 NZ . N
7) (-8.0000000+00 NZ . N5) ( 6.0000000+00 N9 . NZ) (-2.0000000+00 NE . NZ)) NIL NIL NIL)) ((N6 . N7) (NIL (( 1.0000000+
00 NZ . N7) (-2.0000000+00 NZ . N5)) NIL NIL NIL)) ((N2 . N3) (NIL ((-6.0000000+00 NZ . N7) ( 8.0000000+00 NZ . N5) (-2
.0000000+00 N9 . NZ)) NIL NIL NIL)) ((N7 . N8) (NIL (( 1.0000000+00 NZ . N7) (-2.0000000+00 NZ . N5)) NIL NIL NIL)) ((N
Z . N1) (NIL (( 2.0000000+00 NZ . N5)) NIL NIL NIL)) ((NT . NE) (NIL (( 2.0000000+00 NZ . N7) (-2.0000000+00 NZ . N5) (
 2.0000000+00 N9 . NZ) (-1.0000000+00 NE . NZ)) NIL NIL NIL)))

(VALUE OF CORSEL)
((NIL ((NE . NZ) (N9 . NZ) (NZ . N5) (NZ . N7)) NIL NIL NIL) ((N2 . N1) (NIL ((-2.0000000+00 NZ . N7) ( 6.0000000+00 NZ
 . N5)) NIL NIL NIL)) ((N3 . N4) (NIL (( 8.0000000+00 NZ . N7) (-8.0000000+00 NZ . N5) ( 6.0000000+00 N9 . NZ) (-2.0000
000+00 NE . NZ)) NIL NIL NIL)) ((N6 . N7) (NIL (( 1.0000000+00 NZ . N7) (-2.0000000+00 NZ . N5)) NIL NIL NIL)) ((N2 . N
3) (NIL ((-6.0000000+00 NZ . N7) ( 8.0000000+00 NZ . N5) (-2.0000000+00 N9 . NZ)) NIL NIL NIL)) ((N7 . N8) (NIL (( 1.00
00000+00 NZ . N7) (-2.0000000+00 NZ . N5)) NIL NIL NIL)) ((NZ . N1) (NIL (( 2.0000000+00 NZ . N5)) NIL NIL NIL)) ((NT .
 NE) (NIL (( 2.0000000+00 NZ . N7) (-2.0000000+00 NZ . N5) ( 2.0000000+00 N9 . NZ) (-1.0000000+00 NE . NZ)) NIL NIL NIL
)))

END OF EVALQUOTE, VALUE IS ..
((NIL ((NE . NZ) (N9 . NZ) (NZ . N5) (NZ . N7)) NIL NIL NIL) ((N2 . N1) (NIL ((-2.0000000+00 NZ . N7) ( 6.0000000+00 NZ
 . N5)) NIL NIL NIL)) ((N3 . N4) (NIL (( 8.0000000+00 NZ . N7) (-8.0000000+00 NZ . N5) ( 6.0000000+00 N9 . NZ) (-2.0000
000+00 NE . NZ)) NIL NIL NIL)) ((N6 . N7) (NIL (( 1.0000000+00 NZ . N7) (-2.0000000+00 NZ . N5)) NIL NIL NIL)) ((N2 . N
3) (NIL ((-6.0000000+00 NZ . N7) ( 8.0000000+00 NZ . N5) (-2.0000000+00 N9 . NZ)) NIL NIL NIL)) ((N7 . N8) (NIL (( 1.00
00000+00 NZ . N7) (-2.0000000+00 NZ . N5)) NIL NIL NIL)) ((NZ . N1) (NIL (( 2.0000000+00 NZ . N5)) NIL NIL NIL)) ((NT .
 NE) (NIL (( 2.0000000+00 NZ . N7) (-2.0000000+00 NZ . N5) ( 2.0000000+00 N9 . NZ) (-1.0000000+00 NE . NZ)) NIL NIL NIL
)))

 THE TIME ( 0/ 0  000.0) HAS COME, THE WALRUS SAID, TO TALK OF MANY THINGS .....   -LEWIS CARROLL-
```

FIGURE A - C - 2 - b (continued)

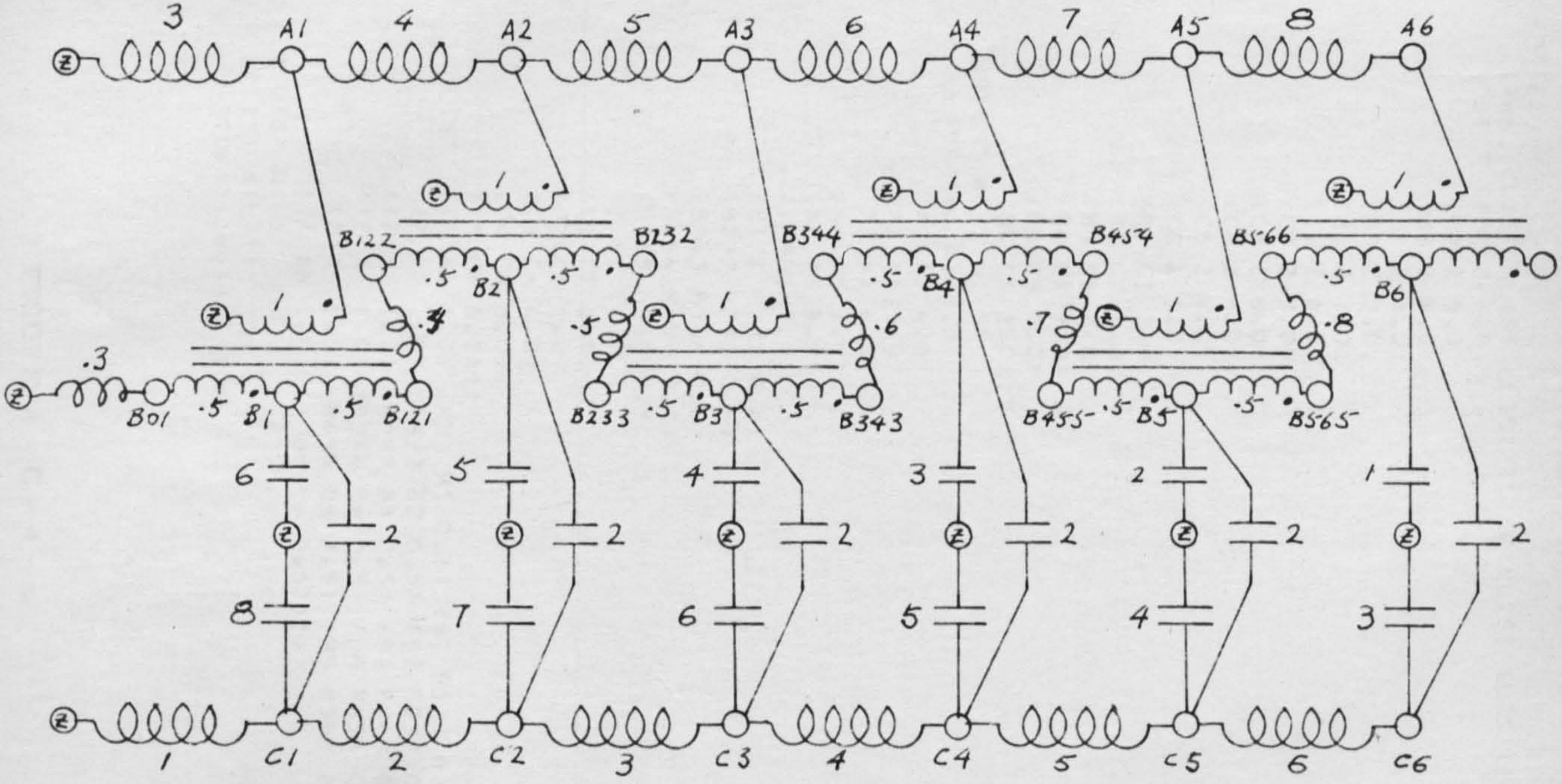
The dependent node pairs are:

$$
\begin{aligned}
(N2 \cdot N1) &= -2(NZ \cdot N7) \\
&\quad +6(NZ \cdot N5) \\
(N3 \cdot N4) &= +8(NZ \cdot N7) \\
&\quad -8(NZ \cdot N5) \\
&\quad +6(N9 \cdot NZ) \\
&\quad -2(NE \cdot NZ) \\
(N6 \cdot N7) &= (NZ \cdot N7) \\
&\quad -2(NZ \cdot N5) \\
(N2 \cdot N3) &= -6(NZ \cdot N7) \\
&\quad +8(NZ \cdot N5) \\
&\quad -2(N9 \cdot NZ) \\
(N7 \cdot N8) &= (NZ \cdot N7) \\
&\quad -2(NZ \cdot N5) \\
(NZ \cdot N1) &= 2(NZ \cdot N5) \\
(NT \cdot NE) &= +2(NZ \cdot N7) \\
&\quad -2(NZ \cdot N5) \\
&\quad +2(N9 \cdot NZ) \\
&\quad -2(NE \cdot NZ)
\end{aligned}
$$

The atomic symbol NX corresponds to the node X in (fig. A-C-1). (NX • NY) represents the node-pair voltage v_{xy}.

A.C.2

The circuit of a six-cell finite difference mass coupled bending and torsion beam model of an airplane wing (reference (13), Chapter 5) is shown in (fig. A-C-3). Russell analogy (2) is used in the bending mode.



The Dynamic Analog Circuit of a Six-Cell Finite Difference Mass Coupled Bending and Torsion Beam Model of an Airplane Wing

FIGURE A - C - 3

```
   KXL000  LISP  6 CELL RUSSEL ANALOGY FINITE DIF WING MODEL
    TEST  THIS IS AN OVERLORD CARD - LISP
CORSEL ((
   (RW6 (CLIST RLIST LLIST TLIST VLIST ILIST))
   (CLIST ((B1 Z 6.0)
           (B2 Z 5.0)
           (B3 Z 4.0)
           (B4 Z 3.0)
           (B5 Z 2.0)
           (B6 Z 1.0)
           (C1 Z 8.0)
           (C2 Z 7.0)
           (C3 Z 6.0)
           (C4 Z 5.0)
           (C5 Z 4.0)
           (C6 Z 3.0)
           (B1 C1 2.0)
           (B2 C2 2.0)
           (B3 C3 2.0)
           (B4 C4 2.0)
           (B5 C5 2.0)
           (B6 C6 2.0)))
   (RLIST NIL)
   (LLIST ((Z A1 3.0)
           (A1 A2 4.0)
           (A2 A3 5.0)
           (A3 A4 6.0)
           (A4 A5 7.0)
           (A5 A6 8.0)
           (Z B01 0.3)
           (B121 B122 0.4)
           (B232 B233 0.5)
           (B343 B344 0.6)
           (B454 B455 0.7)
           (B565 B566 0.8)
           (Z C1 1.0)
           (C1 C2 2.0)
           (C2 C3 3.0)
           (C3 C4 4.0)
           (C4 C5 5.0)
           (C5 C6 6.0)))
   (TLIST (((Z A1 1.0) (B01 B1 0.5) (B1 B121 0.5))
           ((Z A2 1.0) (B122 B2 0.5) (B2 B232 0.5))
           ((Z A3 1.0) (B233 B3 0.5) (B3 B343 0.5))
           ((Z A4 1.0) (B344 B4 0.5) (B4 B454 0.5))
           ((Z A5 1.0) (B455 B5 0.5) (B5 B565 0.5))
           ((Z A6 1.0) (B566 B6 0.5))))
   (VLIST NIL)
   (ILIST NIL)))
      STOP))))))))))STOP
```

FIGURE A - C - 4 - a

```
KAL000  LISP  6 CELL RUSSEL ANALOGY FINITE DIF WING MODEL          000000000000

TEST  THIS IS AN OVERLORD CARD - LISP                              000000000000

THE TIME ( 0/ 0  000.0) HAS COME, THE WALRUS SAID, TO TALK OF MANY THINGS .....   -LEWIS CARROLL-

EVALQUOTE OPERATOR AS OF 1 MARCH 1961.    INPUT LISTS NOW BEING READ.

THE TIME ( 0/ 0  000.0) HAS COME, THE WALRUS SAID, TO TALK OF MANY THINGS .....   -LEWIS CARROLL-


FUNCTION   EVALQUOTE   HAS BEEN ENTERED, ARGUMENTS..
CORSEL
(((RW6 (CLIST RLIST LLIST TLIST VLIST ILIST)) (CLIST ((B1 Z  6.0000000+00) (B2 Z  5.0000000+00) (B3 Z  4.0000000+00) (B
4 Z  3.0000000+00) (B5 Z  2.0000000+00) (B6 Z  1.0000000+00) (C1 Z  8.0000000+00) (C2 Z  7.0000000+00) (C3 Z  6.0000000
+00) (C4 Z  5.0000000+00) (C5 Z  4.0000000+00) (C6 Z  3.0000000+00) (B1 C1  2.0000000+00) (B2 C2  2.0000000+00) (B3 C3
 2.0000000+00) (B4 C4  2.0000000+00) (B5 C5  2.0000000+00) (B6 C6  2.0000000+00))) (RLIST NIL) (LLIST ((Z A1  3.0000000
+00) (A1 A2  4.0000000+00) (A2 A3  5.0000000+00) (A3 A4  6.0000000+00) (A4 A5  7.0000000+00) (A5 A6  8.0000000+00) (Z B
01  3.0000000-01) (B121 B122  4.0000000-01) (B232 B233  5.0000000-01) (B343 B344  5.9999999-01) (B454 B455  7.0000000-0
1) (B565 B566  8.0000000-01) (Z C1  1.0000000+00) (C1 C2  2.0000000+00) (C2 C3  3.0000000+00) (C3 C4  4.0000000+00) (C4
 C5  5.0000000+00) (C5 C6  6.0000000+00))) (TLIST (((Z A1  1.0000000+00) (B01 B1  5.0000000-01) (B1 B121  5.0000000-01)
) ((Z A2  1.0000000+00) (B122 B2  5.0000000-01) (B2 B232  5.0000000-01)) ((Z A3  1.0000000+00) (B233 B3  5.0000000-01)
(B3 B343  5.0000000-01)) ((Z A4  1.0000000+00) (B344 B4  5.0000000-01) (B4 B454  5.0000000-01)) ((Z A5  1.0000000+00) (
B455 B5  5.0000000-01) (B5 B565  5.0000000-01)) ((Z A6  1.0000000+00) (B566 B6  5.0000000-01)))) (VLIST NIL) (ILIST NIL
)))

(ARGUMENTS OF CORSEL)
((RW6 (CLIST RLIST LLIST TLIST VLIST ILIST)) (CLIST ((B1 Z  6.0000000+00) (B2 Z  5.0000000+00) (B3 Z  4.0000000+00) (B4
 Z  3.0000000+00) (B5 Z  2.0000000+00) (B6 Z  1.0000000+00) (C1 Z  8.0000000+00) (C2 Z  7.0000000+00) (C3 Z  6.0000000+
00) (C4 Z  5.0000000+00) (C5 Z  4.0000000+00) (C6 Z  3.0000000+00) (B1 C1  2.0000000+00) (B2 C2  2.0000000+00) (B3 C3
2.0000000+00) (B4 C4  2.0000000+00) (B5 C5  2.0000000+00) (B6 C6  2.0000000+00))) (RLIST NIL) (LLIST ((Z A1  3.0000000+
00) (A1 A2  4.0000000+00) (A2 A3  5.0000000+00) (A3 A4  6.0000000+00) (A4 A5  7.0000000+00) (A5 A6  8.0000000+00) (Z B0
1  3.0000000-01) (B121 B122  4.0000000-01) (B232 B233  5.0000000-01) (B343 B344  5.9999999-01) (B454 B455  7.0000000-01
) (B565 B566  8.0000000-01) (Z C1  1.0000000+00) (C1 C2  2.0000000+00) (C2 C3  3.0000000+00) (C3 C4  4.0000000+00) (C4
C5  5.0000000+00) (C5 C6  6.0000000+00))) (TLIST (((Z A1  1.0000000+00) (B01 B1  5.0000000-01) (B1 B121  5.0000000-01))
 ((Z A2  1.0000000+00) (B122 B2  5.0000000-01) (B2 B232  5.0000000-01)) ((Z A3  1.0000000+00) (B233 B3  5.0000000-01) (
B3 B343  5.0000000-01)) ((Z A4  1.0000000+00) (B344 B4  5.0000000-01) (B4 B454  5.0000000-01)) ((Z A5  1.0000000+00) (B
455 B5  5.0000000-01) (B5 B565  5.0000000-01)) ((Z A6  1.0000000+00) (B566 B6  5.0000000-01)))) (VLIST NIL) (ILIST NIL)
)

RECLAIMER ENTRY   0/ 0  000.0 MARK  000.0     SWEEP   000.0  FULL WORDS   369 FREE   4494  PUSH DOWN DEPTH    121

(ARGUMENTS OF VNPGEN)
(((B1 Z  6.0000000+00) (B2 Z  5.0000000+00) (B3 Z  4.0000000+00) (B4 Z  3.0000000+00) (B5 Z  2.0000000+00) (B6 Z  1.000
0000+00) (C1 Z  8.0000000+00) (C2 Z  7.0000000+00) (C3 Z  6.0000000+00) (C4 Z  5.0000000+00) (C5 Z  4.0000000+00) (C6 Z
  3.0000000+00) (B1 C1  2.0000000+00) (B2 C2  2.0000000+00) (B3 C3  2.0000000+00) (B4 C4  2.0000000+00) (B5 C5  2.00000
00+00) (B6 C6  2.0000000+00)) NIL ((Z A1  3.0000000+00) (A1 A2  4.0000000+00) (A2 A3  5.0000000+00) (A3 A4  6.0000000+0
0) (A4 A5  7.0000000+00) (A5 A6  8.0000000+00) (Z B01  3.0000000-01) (B121 B122  4.0000000-01) (B232 B233  5.0000000-01
) (B343 B344  5.9999999-01) (B454 B455  7.0000000-01) (B565 B566  8.0000000-01) (Z C1  1.0000000+00) (C1 C2  2.0000000+
00) (C2 C3  3.0000000+00) (C3 C4  4.0000000+00) (C4 C5  5.0000000+00) (C5 C6  6.0000000+00)) (((Z A1  1.0000000+00) (B0
1 B1  5.0000000-01) (B1 B121  5.0000000-01)) ((Z A2  1.0000000+00) (B122 B2  5.0000000-01) (B2 B232  5.0000000-01)) ((Z
 A3  1.0000000+00) (B233 B3  5.0000000-01) (B3 B343  5.0000000-01)) ((Z A4  1.0000000+00) (B344 B4  5.0000000-01) (B4 B
454  5.0000000-01)) ((Z A5  1.0000000+00) (B455 B5  5.0000000-01) (B5 B565  5.0000000-01)) ((Z A6  1.0000000+00) (B566
B6  5.0000000-01))) NIL NIL)

RECLAIMER ENTRY   0/ 0  000.0 MARK  000.0     SWEEP   000.0  FULL WORDS   369 FREE   3636  PUSH DOWN DEPTH    61
```

FIGURE A - C - 4 - b

```
RECLAIMER ENTRY   0/ 0  000.0 MARK  000.0      SWEEP   000.0  FULL WORDS   369 FREE   3637  PUSH DOWN DEPTH      64

(VALUE OF VNPGEN)
(NIL ((B6 . C6) (B5 . Z) (B4 . Z) (C6 . Z) (C5 . Z) (B3 . Z) (B2 . Z) (C4 . Z) (C3 . Z) (B1 . Z) (C2 . Z) (C1 . Z)) NIL
 ((Z . B01) (B121 . B122) (B232 . B233) (B343 . B344) (B454 . B455) (B565 . B566) (Z . A1) (A1 . A2) (A2 . A3) (A3 . A4
) (A4 . A5) (A5 . A6)) ((B1 . B121) (B5 . B565) (B2 . B232) (B3 . B343) (B455 . B5)))

(ARGUMENTS OF LOGEN)
(((Z A1  1.0000000+00) (B01 B1  5.0000000-01) (B1 B121  5.0000000-01)) ((Z A2  1.0000000+00) (B122 B2  5.0000000-01) (B
2 B232  5.0000000-01)) ((Z A3  1.0000000+00) (B233 B3  5.0000000-01) (B3 B343  5.0000000-01)) ((Z A4  1.0000000+00) (B3
44 B4  5.0000000-01) (B4 B454  5.0000000-01)) ((Z A5  1.0000000+00) (B455 B5  5.0000000-01) (B5 B565  5.0000000-01)) ((
Z A6  1.0000000+00) (B566 B6  5.0000000-01)))
(NIL ((B6 . C6) (B5 . Z) (B4 . Z) (C6 . Z) (C5 . Z) (B3 . Z) (B2 . Z) (C4 . Z) (C3 . Z) (B1 . Z) (C2 . Z) (C1 . Z)) NIL
 ((Z . B01) (B121 . B122) (B232 . B233) (B343 . B344) (B454 . B455) (B565 . B566) (Z . A1) (A1 . A2) (A2 . A3) (A3 . A4
) (A4 . A5) (A5 . A6)) ((B1 . B121) (B5 . B565) (B2 . B232) (B3 . B343) (B455 . B5)))

RECLAIMER ENTRY   0/ 0  000.0 MARK  000.0      SWEEP   000.0  FULL WORDS   365 FREE   3624  PUSH DOWN DEPTH      45

RECLAIMER ENTRY   0/ 0  000.0 MARK  000.0      SWEEP   000.0  FULL WORDS   354 FREE   3402  PUSH DOWN DEPTH      91

RECLAIMER ENTRY   0/ 0  000.0 MARK  000.0      SWEEP   000.0  FULL WORDS   341 FREE   3304  PUSH DOWN DEPTH      88

RECLAIMER ENTRY   0/ 0  000.0 MARK  000.0      SWEEP   000.0  FULL WORDS   330 FREE   3180  PUSH DOWN DEPTH      85

(VALUE OF LOGEN)
(((NIL NIL NIL (( 1.0000000+00 A5 . A6) ( 1.0000000+00 A4 . A5) ( 1.0000000+00 A3 . A4) ( 1.0000000+00 A2 . A3) ( 1.000
0000+00 A1 . A2) ( 1.0000000+00 Z . A1)) NIL) (NIL ((-2.0000000+00 B6 . C6) (-2.0000000+00 C6 . Z) ( 2.0000000+00 B5 .
Z)) NIL ((-2.0000000+00 B565 . B566)) ((-2.0000000+00 B5 . B565)))) ((NIL NIL NIL (( 1.0000000+00 A4 . A5) ( 1.0000000+
00 A3 . A4) ( 1.0000000+00 A2 . A3) ( 1.0000000+00 A1 . A2) ( 1.0000000+00 Z . A1)) NIL) (NIL NIL NIL NIL (( 2.0000000+
00 B5 . B565)))) ((NIL NIL NIL (( 1.0000000+00 A4 . A5) ( 1.0000000+00 A3 . A4) ( 1.0000000+00 A2 . A3) ( 1.0000000+00
A1 . A2) ( 1.0000000+00 Z . A1)) NIL) (NIL NIL NIL NIL (( 2.0000000+00 B455 . B5)))) ((NIL NIL NIL (( 1.0000000+00 A3 .
 A4) ( 1.0000000+00 A2 . A3) ( 1.0000000+00 A1 . A2) ( 1.0000000+00 Z . A1)) NIL) (NIL ((-2.0000000+00 B5 . Z) ( 2.0000
000+00 B4 . Z)) NIL ((-2.0000000+00 B454 . B455)) ((-2.0000000+00 B455 . B5)))) ((NIL NIL NIL (( 1.0000000+00 A3 . A4)
( 1.0000000+00 A2 . A3) ( 1.0000000+00 A1 . A2) ( 1.0000000+00 Z . A1)) NIL) (NIL ((-2.0000000+00 B4 . Z) ( 2.0000000+0
0 B3 . Z)) NIL ((-2.0000000+00 B343 . B344)) ((-2.0000000+00 B3 . B343)))) ((NIL NIL NIL (( 1.0000000+00 A2 . A3) ( 1.0
000000+00 A1 . A2) ( 1.0000000+00 Z . A1)) NIL) (NIL NIL NIL NIL (( 2.0000000+00 B3 . B343)))) ((NIL NIL NIL (( 1.00000
00+00 A2 . A3) ( 1.0000000+00 A1 . A2) ( 1.0000000+00 Z . A1)) NIL) (NIL ((-2.0000000+00 B3 . Z) ( 2.0000000+00 B2 . Z)
) NIL ((-2.0000000+00 B232 . B233)) ((-2.0000000+00 B2 . B232)))) ((NIL NIL NIL (( 1.0000000+00 A1 . A2) ( 1.0000000+00
 Z . A1)) NIL) (NIL NIL NIL NIL (( 2.0000000+00 B2 . B232)))) ((NIL NIL NIL (( 1.0000000+00 A1 . A2) ( 1.0000000+00 Z .
 A1)) NIL) (NIL ((-2.0000000+00 B2 . Z) ( 2.0000000+00 B1 . Z)) NIL ((-2.0000000+00 B121 . B122)) ((-2.0000000+00 B1 .
B121)))) ((NIL NIL NIL (( 1.0000000+00 Z . A1)) NIL) (NIL NIL NIL NIL (( 2.0000000+00 B1 . B121)))) ((NIL NIL NIL (( 1.
0000000+00 Z . A1)) NIL) (NIL ((-2.0000000+00 B1 . Z)) NIL ((-2.0000000+00 Z . B01)) NIL)))

(ARGUMENTS OF SEDV)
(((NIL NIL NIL (( 1.0000000+00 A5 . A6) ( 1.0000000+00 A4 . A5) ( 1.0000000+00 A3 . A4) ( 1.0000000+00 A2 . A3) ( 1.000
0000+00 A1 . A2) ( 1.0000000+00 Z . A1)) NIL) (NIL ((-2.0000000+00 B6 . C6) (-2.0000000+00 C6 . Z) ( 2.0000000+00 B5 .
Z)) NIL ((-2.0000000+00 B565 . B566)) ((-2.0000000+00 B5 . B565)))) ((NIL NIL NIL (( 1.0000000+00 A4 . A5) ( 1.0000000+
00 A3 . A4) ( 1.0000000+00 A2 . A3) ( 1.0000000+00 A1 . A2) ( 1.0000000+00 Z . A1)) NIL) (NIL NIL NIL NIL (( 2.0000000+
00 B5 . B565)))) ((NIL NIL NIL (( 1.0000000+00 A4 . A5) ( 1.0000000+00 A3 . A4) ( 1.0000000+00 A2 . A3) ( 1.0000000+00
A1 . A2) ( 1.0000000+00 Z . A1)) NIL) (NIL NIL NIL NIL (( 2.0000000+00 B455 . B5)))) ((NIL NIL NIL (( 1.0000000+00 A3 .
 A4) ( 1.0000000+00 A2 . A3) ( 1.0000000+00 A1 . A2) ( 1.0000000+00 Z . A1)) NIL) (NIL ((-2.0000000+00 B5 . Z) ( 2.0000
000+00 B4 . Z)) NIL ((-2.0000000+00 B454 . B455)) ((-2.0000000+00 B455 . B5)))) ((NIL NIL NIL (( 1.0000000+00 A3 . A4)
( 1.0000000+00 A2 . A3) ( 1.0000000+00 A1 . A2) ( 1.0000000+00 Z . A1)) NIL) (NIL ((-2.0000000+00 B4 . Z) ( 2.0000000+0
0 B3 . Z)) NIL ((-2.0000000+00 B343 . B344)) ((-2.0000000+00 B3 . B343)))) ((NIL NIL NIL (( 1.0000000+00 A2 . A3) ( 1.0
000000+00 A1 . A2) ( 1.0000000+00 Z . A1)) NIL) (NIL NIL NIL NIL (( 2.0000000+00 B3 . B343)))) ((NIL NIL NIL (( 1.00000
00+00 A2 . A3) ( 1.0000000+00 A1 . A2) ( 1.0000000+00 Z . A1)) NIL) (NIL ((-2.0000000+00 B3 . Z) ( 2.0000000+00 B2 . Z)
) NIL ((-2.0000000+00 B232 . B233)) ((-2.0000000+00 B2 . B232)))) ((NIL NIL NIL (( 1.0000000+00 A1 . A2) ( 1.0000000+00
 Z . A1)) NIL) (NIL NIL NIL NIL (( 2.0000000+00 B2 . B232)))) ((NIL NIL NIL (( 1.0000000+00 A1 . A2) ( 1.0000000+00 Z .
 A1)) NIL) (NIL ((-2.0000000+00 B2 . Z) ( 2.0000000+00 B1 . Z)) NIL ((-2.0000000+00 B121 . B122)) ((-2.0000000+00 B1 .
B121)))) ((NIL NIL NIL (( 1.0000000+00 Z . A1)) NIL) (NIL NIL NIL NIL (( 2.0000000+00 B1 . B121)))) ((NIL NIL NIL (( 1.
0000000+00 Z . A1)) NIL) (NIL ((-2.0000000+00 B1 . Z)) NIL ((-2.0000000+00 Z . B01)) NIL)))

RECLAIMER ENTRY   0/ 0  000.0 MARK  000.0      SWEEP   000.0  FULL WORDS   299 FREE   2925  PUSH DOWN DEPTH      184

RECLAIMER ENTRY   0/ 0  000.0 MARK  000.0      SWEEP   000.0  FULL WORDS   261 FREE   2581  PUSH DOWN DEPTH      130
```

FIGURE A - C - 4 - b (continued)

```
RECLAIMER ENTRY   0/ 0  000.0 MARK  000.0     SWEEP   000.0  FULL WORDS   272 FREE   2682  PUSH DOWN DEPTH    144

RECLAIMER ENTRY   0/ 0  000.0 MARK  000.0     SWEEP   000.0  FULL WORDS   253 FREE   2499  PUSH DOWN DEPTH    229

RECLAIMER ENTRY   0/ 0  000.0 MARK  000.0     SWEEP   000.0  FULL WORDS   255 FREE   2525  PUSH DOWN DEPTH    105

RECLAIMER ENTRY   0/ 0  000.0 MARK  000.0     SWEEP   000.0  FULL WORDS   256 FREE   2542  PUSH DOWN DEPTH     99

RECLAIMER ENTRY   0/ 0  000.0 MARK  000.0     SWEEP   000.0  FULL WORDS   249 FREE   2453  PUSH DOWN DEPTH    174

(VALUE OF SEDV)
(((Z . B01) (NIL (( 5.0000000-01 B2 . Z) (-1.5000000+00 B1 . Z)) NIL (( 5.0000000-01 B121 . B122) ( 2.5000000-01 A1 . A
2)) NIL)) ((Z . A1) (NIL ((-1.0000000+00 B2 . Z) ( 1.0000000+00 B1 . Z)) NIL ((-1.0000000+00 B121 . B122) (-5.0000000-0
1 A1 . A2)) NIL)) ((B1 . B121) (NIL ((-5.0000000-01 B2 . Z) ( 5.0000000-01 B1 . Z)) NIL ((-5.0000000-01 B121 . B122) (-
2.5000000-01 A1 . A2)) NIL)) ((B232 . B233) (NIL (( 1.5000000+00 B2 . Z) (-5.0000000-01 B1 . Z) (-1.5000000+00 B3 . Z)
( 5.0000000-01 B4 . Z)) NIL (( 5.0000000-01 B121 . B122) (-2.5000000-01 A1 . A2) ( 5.0000000-01 B343 . B344) ( 2.500000
0-01 A3 . A4)) NIL)) ((B2 . B232) (NIL ((-5.0000000-01 B2 . Z) ( 5.0000000-01 B1 . Z)) NIL ((-5.0000000-01 B121 . B122)
( 2.5000000-01 A1 . A2)) NIL)) ((A2 . A3) (NIL (( 1.0000000+00 B2 . Z) (-1.0000000+00 B1 . Z) (-1.0000000+00 B4 . Z) (
1.0000000+00 B3 . Z)) NIL (( 1.0000000+00 B121 . B122) (-5.0000000-01 A1 . A2) (-5.0000000-01 A3 . A4) (-1.0000000+00
B343 . B344)) NIL)) ((B3 . B343) (NIL ((-5.0000000-01 B4 . Z) ( 5.0000000-01 B3 . Z)) NIL ((-5.0000000-01 B343 . B344)
(-2.5000000-01 A3 . A4)) NIL)) ((B454 . B455) (NIL (( 1.5000000+00 B4 . Z) (-5.0000000-01 B3 . Z) (-1.5000000+00 B5 . Z
) ( 5.0000000-01 C6 . Z) ( 5.0000000-01 B6 . C6)) NIL (( 5.0000000-01 B343 . B344) (-2.5000000-01 A3 . A4) ( 5.0000000-
01 B565 . B566) ( 2.5000000-01 A5 . A6)) NIL)) ((B455 . B5) (NIL ((-5.0000000-01 B6 . C6) (-5.0000000-01 C6 . Z) ( 5.00
00000-01 B5 . Z)) NIL ((-5.0000000-01 B565 . B566) (-2.5000000-01 A5 . A6)) NIL)) ((A4 . A5) (NIL (( 1.0000000+00 B4 .
Z) (-1.0000000+00 B3 . Z) (-1.0000000+00 B6 . C6) (-1.0000000+00 C6 . Z) ( 1.0000000+00 B5 . Z)) NIL (( 1.0000000+00 B3
43 . B344) (-5.0000000-01 A3 . A4) (-1.0000000+00 B565 . B566) (-5.0000000-01 A5 . A6)) NIL)) ((B5 . B565) (NIL ((-5.00
00000-01 B6 . C6) (-5.0000000-01 C6 . Z) ( 5.0000000-01 B5 . Z)) NIL ((-5.0000000-01 B565 . B566) (-2.5000000-01 A5 . A
6)) NIL)))

(VALUE OF CORSEL)
((NIL ((B6 . C6) (B5 . Z) (B4 . Z) (C6 . Z) (C5 . Z) (B3 . Z) (B2 . Z) (C4 . Z) (C3 . Z) (B1 . Z) (C2 . Z) (C1 . Z)) NI
L ((A3 . A4) (A1 . A2) (A5 . A6) (B565 . B566) (B121 . B122) (B343 . B344)) NIL) ((Z . B01) (NIL (( 5.0000000-01 B2 . Z
) (-1.5000000+00 B1 . Z)) NIL (( 5.0000000-01 B121 . B122) ( 2.5000000-01 A1 . A2)) NIL)) ((Z . A1) (NIL ((-1.0000000+0
0 B2 . Z) ( 1.0000000+00 B1 . Z)) NIL ((-1.0000000+00 B121 . B122) (-5.0000000-01 A1 . A2)) NIL)) ((B1 . B121) (NIL ((-
5.0000000-01 B2 . Z) ( 5.0000000-01 B1 . Z)) NIL ((-5.0000000-01 B121 . B122) (-2.5000000-01 A1 . A2)) NIL)) ((B232 . B
233) (NIL (( 1.5000000+00 B2 . Z) (-5.0000000-01 B1 . Z) (-1.5000000+00 B3 . Z) ( 5.0000000-01 B4 . Z)) NIL (( 5.000000
0-01 B121 . B122) (-2.5000000-01 A1 . A2) ( 5.0000000-01 B343 . B344) ( 2.5000000-01 A3 . A4)) NIL)) ((B2 . B232) (NIL
((-5.0000000-01 B2 . Z) ( 5.0000000-01 B1 . Z)) NIL ((-5.0000000-01 B121 . B122) ( 2.5000000-01 A1 . A2)) NIL)) ((A2 .
A3) (NIL (( 1.0000000+00 B2 . Z) (-1.0000000+00 B1 . Z) (-1.0000000+00 B4 . Z) ( 1.0000000+00 B3 . Z)) NIL (( 1.0000000
+00 B121 . B122) (-5.0000000-01 A1 . A2) (-5.0000000-01 A3 . A4) (-1.0000000+00 B343 . B344)) NIL)) ((B3 . B343) (NIL (
(-5.0000000-01 B4 . Z) ( 5.0000000-01 B3 . Z)) NIL ((-5.0000000-01 B343 . B344) (-2.5000000-01 A3 . A4)) NIL)) ((B454 .
 B455) (NIL (( 1.5000000+00 B4 . Z) (-5.0000000-01 B3 . Z) (-1.5000000+00 B5 . Z) ( 5.0000000-01 C6 . Z) ( 5.0000000-01
 B6 . C6)) NIL (( 5.0000000-01 B343 . B344) (-2.5000000-01 A3 . A4) ( 5.0000000-01 B565 . B566) ( 2.5000000-01 A5 . A6)
) NIL)) ((B455 . B5) (NIL ((-5.0000000-01 B6 . C6) (-5.0000000-01 C6 . Z) ( 5.0000000-01 B5 . Z)) NIL ((-5.0000000-01 B
565 . B566) (-2.5000000-01 A5 . A6)) NIL)) ((A4 . A5) (NIL (( 1.0000000+00 B4 . Z) (-1.0000000+00 B3 . Z) (-1.0000000+0
0 B6 . C6) (-1.0000000+00 C6 . Z) ( 1.0000000+00 B5 . Z)) NIL (( 1.0000000+00 B343 . B344) (-5.0000000-01 A3 . A4) (-1.
0000000+00 B565 . B566) (-5.0000000-01 A5 . A6)) NIL)) ((B5 . B565) (NIL ((-5.0000000-01 B6 . C6) (-5.0000000-01 C6 . Z
) ( 5.0000000-01 B5 . Z)) NIL ((-5.0000000-01 B565 . B566) (-2.5000000-01 A5 . A6)) NIL)))

END OF EVALQUOTE, VALUE IS ..
((NIL ((B6 . C6) (B5 . Z) (B4 . Z) (C6 . Z) (C5 . Z) (B3 . Z) (B2 . Z) (C4 . Z) (C3 . Z) (B1 . Z) (C2 . Z) (C1 . Z)) NI
L ((A3 . A4) (A1 . A2) (A5 . A6) (B565 . B566) (B121 . B122) (B343 . B344)) NIL) ((Z . B01) (NIL (( 5.0000000-01 B2 . Z
) (-1.5000000+00 B1 . Z)) NIL (( 5.0000000-01 B121 . B122) ( 2.5000000-01 A1 . A2)) NIL)) ((Z . A1) (NIL ((-1.0000000+0
0 B2 . Z) ( 1.0000000+00 B1 . Z)) NIL ((-1.0000000+00 B121 . B122) (-5.0000000-01 A1 . A2)) NIL)) ((B1 . B121) (NIL ((-
5.0000000-01 B2 . Z) ( 5.0000000-01 B1 . Z)) NIL ((-5.0000000-01 B121 . B122) (-2.5000000-01 A1 . A2)) NIL)) ((B232 . B
233) (NIL (( 1.5000000+00 B2 . Z) (-5.0000000-01 B1 . Z) (-1.5000000+00 B3 . Z) ( 5.0000000-01 B4 . Z)) NIL (( 5.000000
0-01 B121 . B122) (-2.5000000-01 A1 . A2) ( 5.0000000-01 B343 . B344) ( 2.5000000-01 A3 . A4)) NIL)) ((B2 . B232) (NIL
((-5.0000000-01 B2 . Z) ( 5.0000000-01 B1 . Z)) NIL ((-5.0000000-01 B121 . B122) ( 2.5000000-01 A1 . A2)) NIL)) ((A2 .
A3) (NIL (( 1.0000000+00 B2 . Z) (-1.0000000+00 B1 . Z) (-1.0000000+00 B4 . Z) ( 1.0000000+00 B3 . Z)) NIL (( 1.0000000
+00 B121 . B122) (-5.0000000-01 A1 . A2) (-5.0000000-01 A3 . A4) (-1.0000000+00 B343 . B344)) NIL)) ((B3 . B343) (NIL (
(-5.0000000-01 B4 . Z) ( 5.0000000-01 B3 . Z)) NIL ((-5.0000000-01 B343 . B344) (-2.5000000-01 A3 . A4)) NIL)) ((B454 .
 B455) (NIL (( 1.5000000+00 B4 . Z) (-5.0000000-01 B3 . Z) (-1.5000000+00 B5 . Z) ( 5.0000000-01 C6 . Z) ( 5.0000000-01
 B6 . C6)) NIL (( 5.0000000-01 B343 . B344) (-2.5000000-01 A3 . A4) ( 5.0000000-01 B565 . B566) ( 2.5000000-01 A5 . A6)
) NIL)) ((B455 . B5) (NIL ((-5.0000000-01 B6 . C6) (-5.0000000-01 C6 . Z) ( 5.0000000-01 B5 . Z)) NIL ((-5.0000000-01 B
565 . B566) (-2.5000000-01 A5 . A6)) NIL)) ((A4 . A5) (NIL (( 1.0000000+00 B4 . Z) (-1.0000000+00 B3 . Z) (-1.0000000+0
0 B6 . C6) (-1.0000000+00 C6 . Z) ( 1.0000000+00 B5 . Z)) NIL (( 1.0000000+00 B343 . B344) (-5.0000000-01 A3 . A4) (-1.
0000000+00 B565 . B566) (-5.0000000-01 A5 . A6)) NIL)) ((B5 . B565) (NIL ((-5.0000000-01 B6 . C6) (-5.0000000-01 C6 . Z
) ( 5.0000000-01 B5 . Z)) NIL ((-5.0000000-01 B565 . B566) (-2.5000000-01 A5 . A6)) NIL)))

THE TIME ( 0/ 0  000.0) HAS COME, THE WALRUS SAID, TO TALK OF MANY THINGS .....   -LEWIS CARROLL-
```

FIGURE A - C - 4 - b (continued)

The LISP program output is in (fig. A-C-4-b) while the input card listing is included in (fig. A-C-4-a).

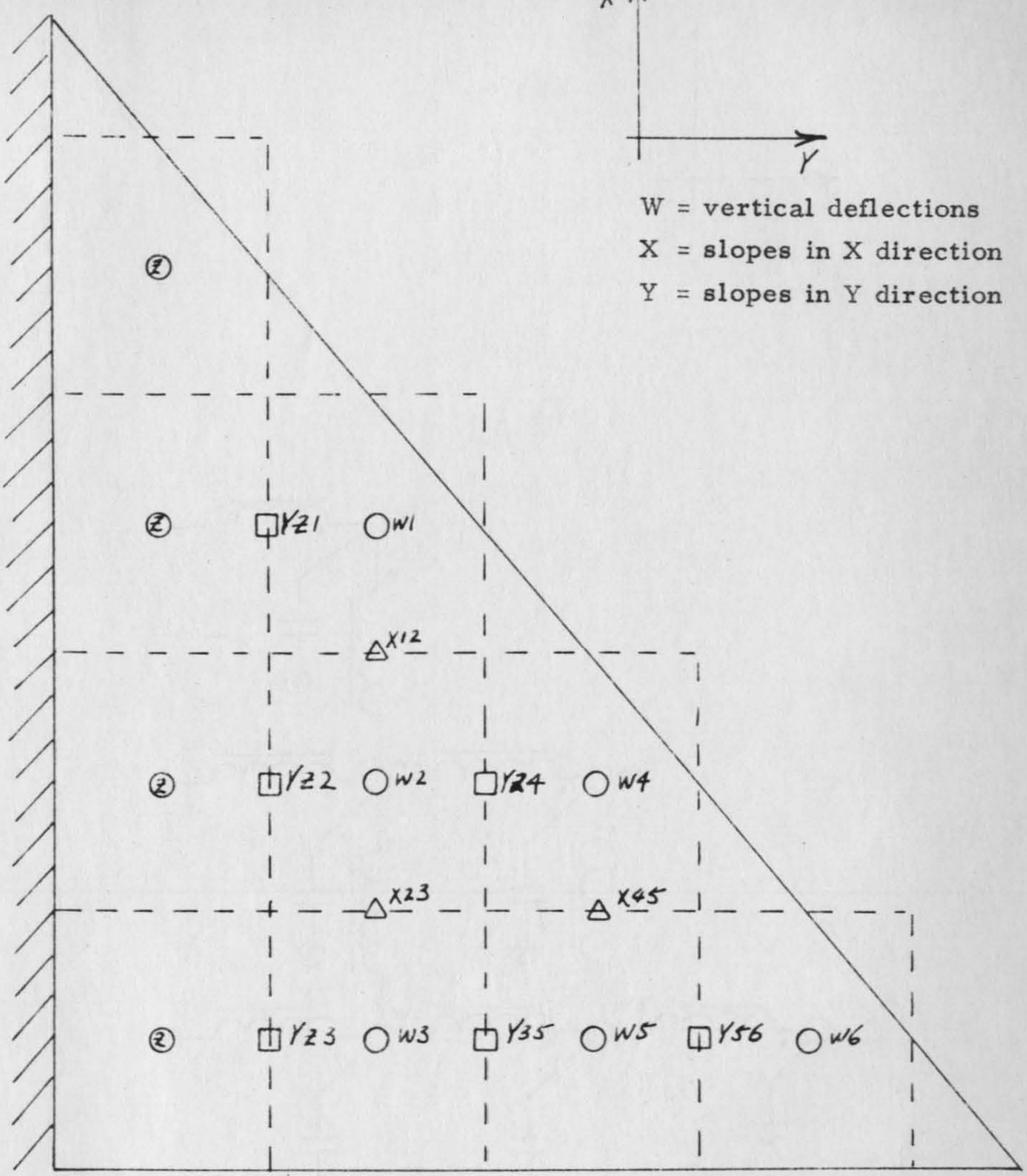
The selected coordinates are:

$$
\begin{aligned}
V^1 = \ & (B6 \cdot C6) \\
& (B5 \cdot Z) \\
& (B4 \cdot Z) \\
& (C6 \cdot Z) \\
& (C5 \cdot Z) \\
& (B3 \cdot Z) \\
& (B2 \cdot Z) \\
& (C4 \cdot Z) \\
& (C3 \cdot Z) \\
& (B1 \cdot Z) \\
& (C2 \cdot Z) \\
& (C1 \cdot Z)
\end{aligned}
$$

$$
\begin{aligned}
V^3 = \ & (A3 \cdot A4) \\
& (A1 \cdot A2) \\
& (A5 \cdot A6) \\
& (B565 \cdot B566) \\
& (B121 \cdot B122) \\
& (B343 \cdot B344)
\end{aligned}
$$

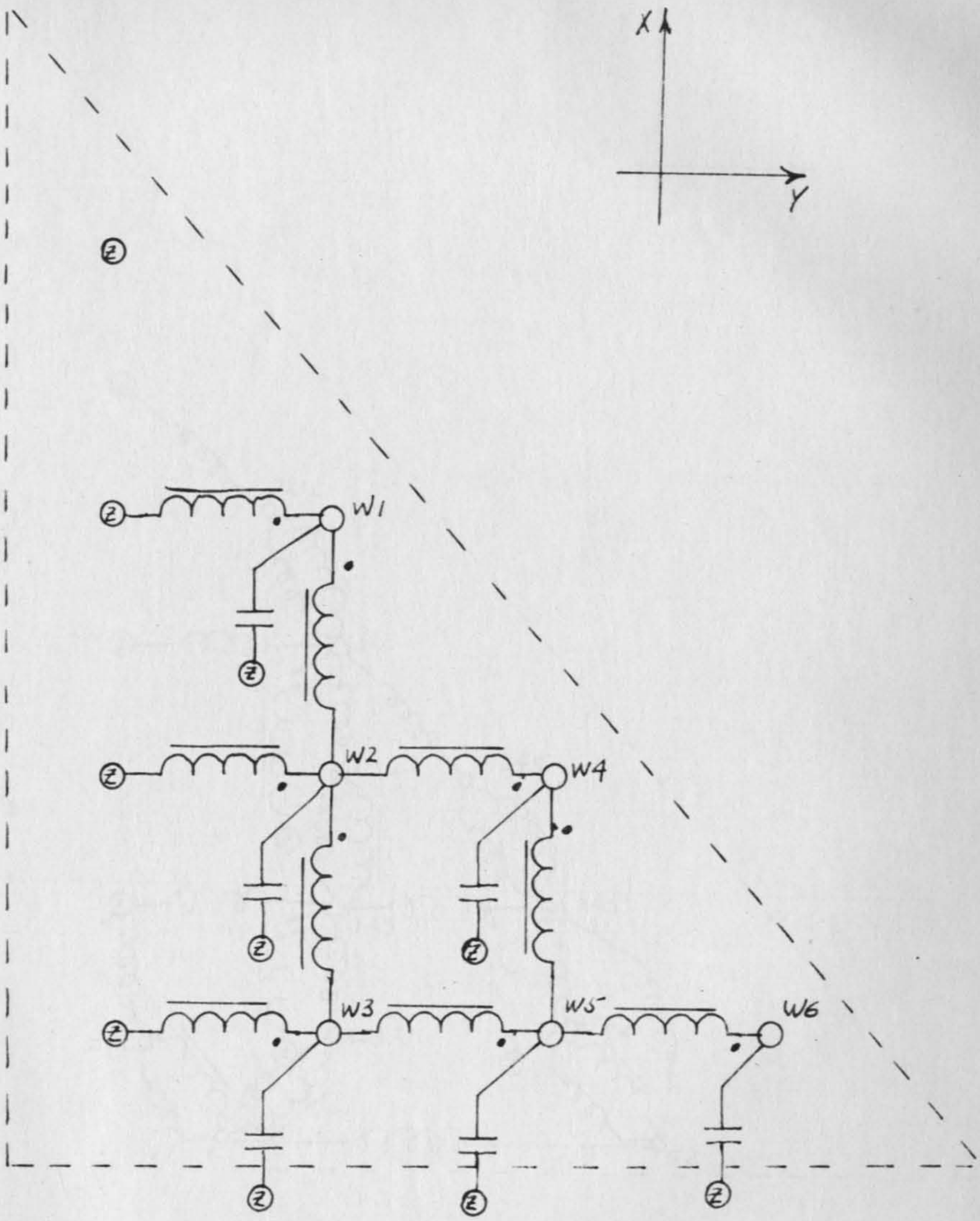
A. C. 3

The analog circuit of a six-cell finite difference plate analogy of a delta wing with Poisson's lateral coupling (reference 13, Chapter 5), (3) is shown in (fig. A-C-5-a, b, c, d). The LISP program input cards



The Cell-Division and Coordinate Representation of a Delta Wing

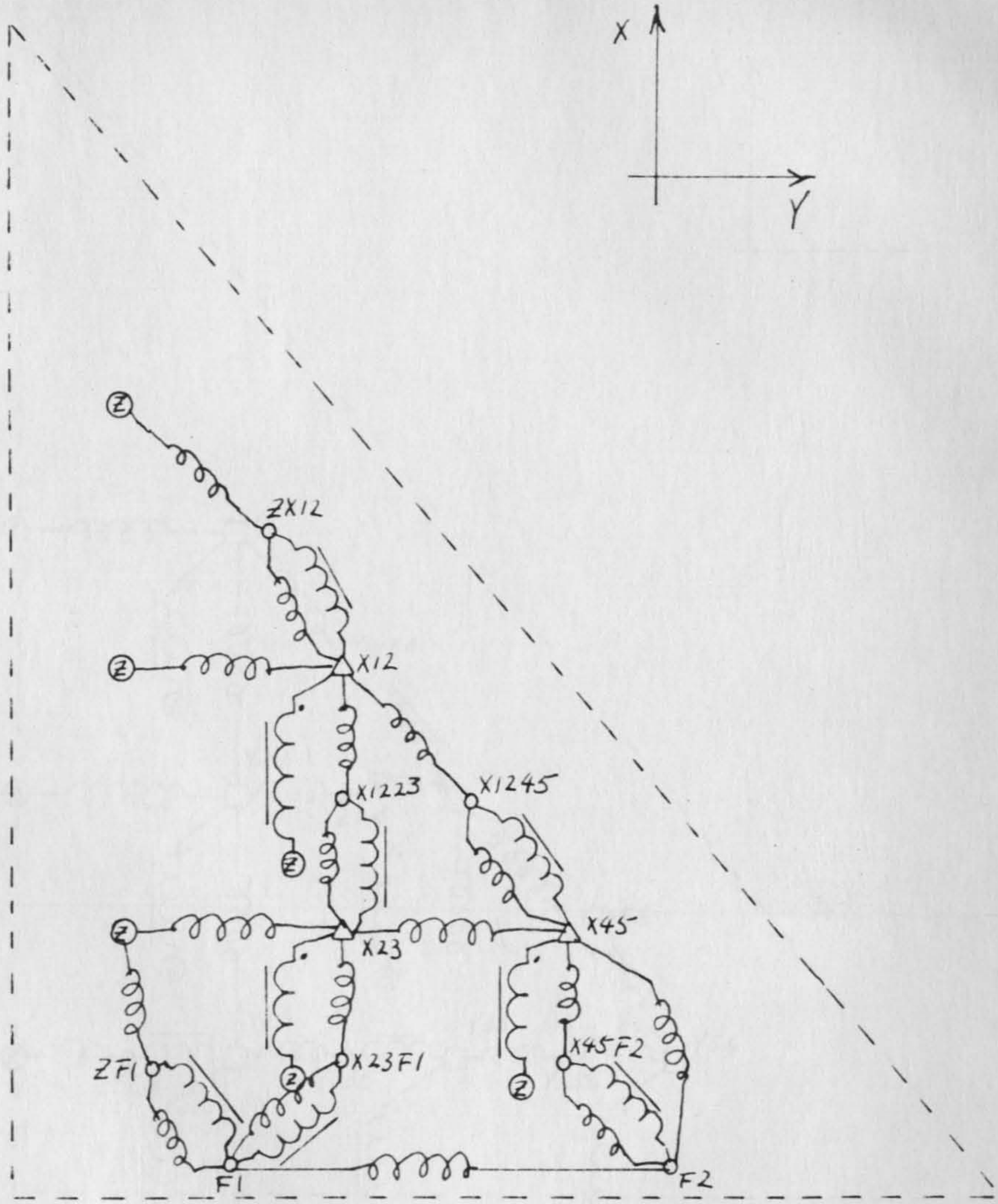
FIGURE A - C - 5 - a



The W-Circuit of the Delta Wing in (a)

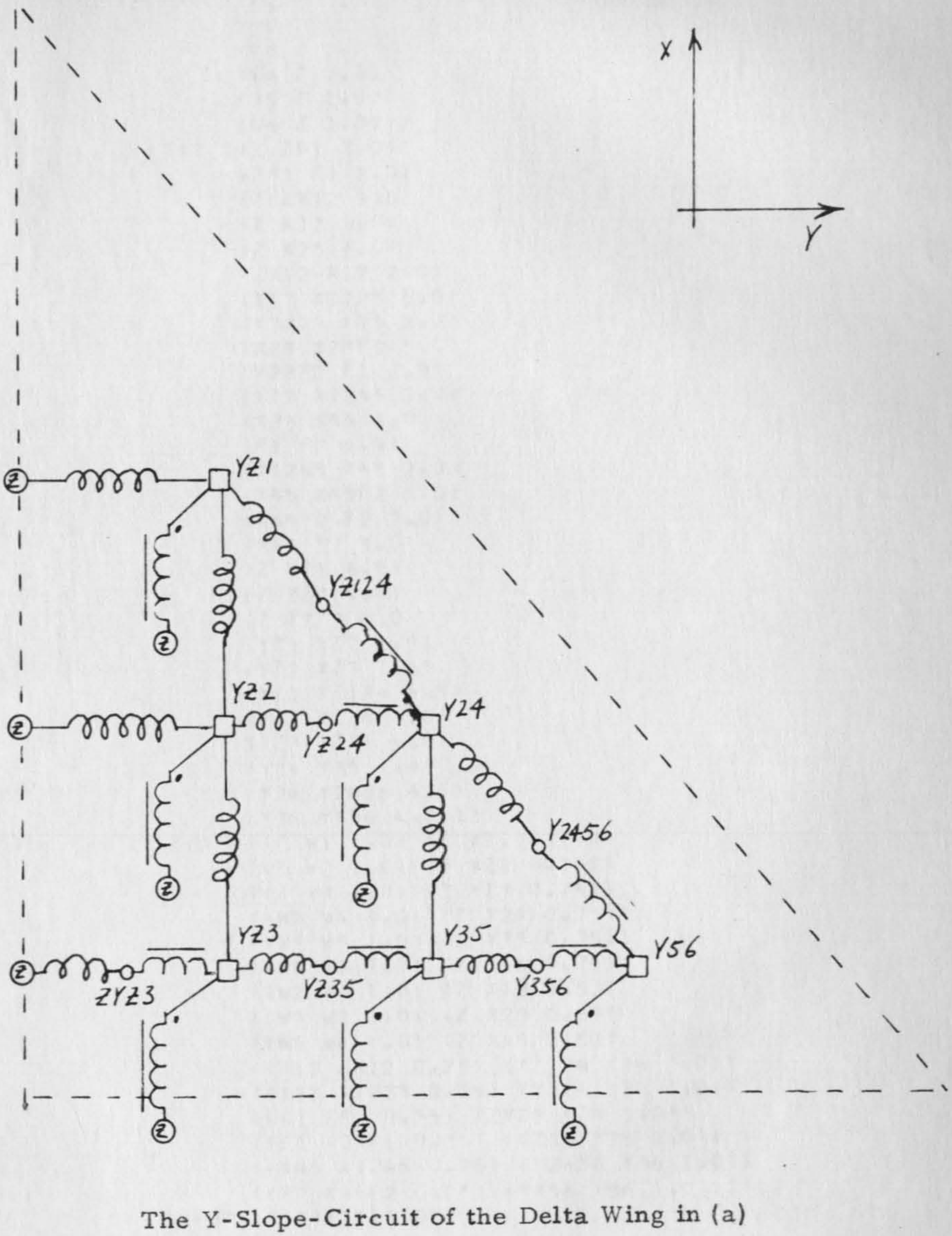
The Capacitors Represent the Translational Masses of Individual Cells; the Transformer Windings Specify the Coordinate Transformation Between the Vertical Deflections and Slopes.

FIGURE A - C - 5 - b



The X-Slope-Circuit of the Delta Wing in (a)

FIGURE A - C - 5 - c



The Y-Slope-Circuit of the Delta Wing in (a)

FIGURE A - C - 5 - d

```
KXLOOO   LISP   SIMPLIFIED DELTA WING PLATE ANALOGY
 TEST  THIS IS AN OVERLORD CARD - LISP
CORSEL ((
   (DTWING (CLIST NIL LLIST TLIST NIL NIL))
   (CLIST ((W1 Z 1.0)
           (W2 Z 1.0)
           (W3 Z 1.0)
           (W4 Z 1.0)
           (W5 Z 1.0)
           (W6 Z 1.0)))
   (LLIST ((Z ZF1 7.0)
           (ZF1 F1 2.0)
           (Z ZX12 3.0)
           (Z X12 1.0)
           (Z X23 1.0)
           (ZX12 X12 2.0)
           (X12 X1223 5.0)
           (X1223 X23 2.0)
           (X23 X23F1 5.0)
           (X23F1 F1 2.0)
           (X12 X1245 3.0)
           (X23 X45 1.0)
           (F1 F2 4.0)
           (X1245 X45 2.0)
           (X45 X45F2 5.0)
           (X45F2 F2 2.0)
           (X45 F2 3.0)
           (Z YZ1 4.5)
           (Z YZ2 4.5)
           (Z ZYZ3 4.0)
           (YZ1 YZ2 1.5)
           (YZ2 YZ3 1.5)
           (YZ1 YZ124 4.5)
           (YZ2 YZ24 4.0)
           (YZ3 YZ35 4.0)
           (Y24 Y35 1.5)
           (Y24 Y2456 4.5)
           (Y35 Y356 4.0)))
   (TLIST (((Z W1 1.0) (Z YZ1 0.75))
           ((Z W2 1.0) (Z YZ2 0.75))
           ((Z W3 1.0) (Z YZ3 0.75))
           ((W2 W4 1.0) (Z Y24 0.75))
           ((W3 W5 1.0) (Z Y35 0.75))
           ((W5 W6 1.0) (Z Y56 0.75))
           ((W2 W1 1.0) (Z X12 0.5))
           ((W3 W2 1.0) (Z X23 0.5))
           ((W5 W4 1.0) (Z X45 0.5))
           ((X12 ZX12 0.25) (YZ124 Y24 1.0))
           ((X23 X1223 0.25) (YZ24 Y24 1.0))
           ((F1 ZF1 0.25) (ZYZ3 YZ3 1.0))
           ((F1 X23F1 0.25) (YZ35 Y35 1.0))
           ((X45 X1245 0.25) (Y2456 Y56 1.0))
           ((F2 X45F2 0.25) (Y356 Y56 1.0))))))
     STOP)))))))))))STOP
```

FIGURE A - C - 6 - a

```
           XXLO00   LISP   SIMPLIFIED DELTA WING PLATE ANALOGY                     000000000000

            TEST  THIS IS AN OVERLORD CARD - LISP                                  000000000000

 THE TIME ( 0/ 0  000.0) HAS COME, THE WALRUS SAID, TO TALK OF MANY THINGS .....   -LEWIS CARROLL-

EVALQUOTE OPERATOR AS OF 1 MARCH 1961.    INPUT LISTS NOW BEING READ.

 THE TIME ( 0/ 0  000.0) HAS COME, THE WALRUS SAID, TO TALK OF MANY THINGS .....   -LEWIS CARROLL-

CORSEL
(((DTWING (CLIST NIL FLIST TLIST NIL NIL)) (CLIST ((W1 Z  1.0000000+00) (W2 Z  1.0000000+00) (W3 Z  1.0000000+00) (W4 Z
  1.0000000+00) (W5 Z  1.0000000+00) (W6 Z  1.0000000+00))) (LLIST ((Z ZF1  7.0000000+00) (ZF1 F1  2.0000000+00) (Z ZX1
2  3.0000000+00) (Z X12  1.0000000+00) (Z X23  1.0000000+00) (ZX12 X12  2.0000000+00) (X12 X1223  5.0000000+00) (X1223
X23  2.0000000+00) (X23 X23F1  5.0000000+00) (X23F1 F1  2.0000000+00) (X12 X1245  3.0000000+00) (X23 X45  1.0000000+00)
 (F1 F2  4.0000000+00) (X1245 X45  2.0000000+00) (X45 X45F2  5.0000000+00) (X45F2 F2  2.0000000+00) (X45 F2  3.0000000+
00) (Z YZ1  4.5000000+00) (Z YZ2  4.5000000+00) (Z ZYZ3  4.0000000+00) (YZ1 YZ2  1.5000000+00) (YZ2 YZ3  1.5000000+00)
(YZ1 YZ124  4.5000000+00) (YZ2 YZ24  4.0000000+00) (YZ3 YZ35  4.0000000+00) (Y24 Y35  1.5000000+00) (Y24 Y2456  4.50000
00+00) (Y35 Y356  4.0000000+00))) (TLIST (((Z W1  1.0000000+00) (Z YZ1  7.5000000-01)) ((Z W2  1.0000000+00) (Z YZ2  7.
5000000-01)) ((Z W3  1.0000000+00) (Z YZ3  7.5000000-01)) ((W2 W4  1.0000000+00) (Z Y24  7.5000000-01)) ((W3 W5  1.0000
000+00) (Z Y35  7.5000000-01)) ((W5 W6  1.0000000+00) (Z Y56  7.5000000-01)) ((W2 W1  1.0000000+00) (Z X12  5.0000000-0
1)) ((W3 W2  1.0000000+00) (Z X23  5.0000000-01)) ((W5 W4  1.0000000+00) (Z X45  5.0000000-01)) ((X12 ZX12  2.5000000-0
1) (YZ124 Y24  1.0000000+00)) ((X23 X1223  2.5000000-01) (YZ24 Y24  1.0000000+00)) ((F1 ZF1  2.5000000-01) (ZYZ3 YZ3  1
.0000000+00)) ((F1 X23F1  2.5000000-01) (YZ35 Y35  1.0000000+00)) ((X45 X1245  2.5000000-01) (Y2456 Y56  1.0000000+00))
 ((F2 X45F2  2.5000000-01) (Y356 Y56  1.0000000+00))))))


(ARGUMENTS OF CORSEL)
((DTWING (CLIST NIL LLIST TLIST NIL NIL)) (CLIST ((W1 Z  1.0000000+00) (W2 Z  1.0000000+00) (W3 Z  1.0000000+00) (W4 Z
 1.0000000+00) (W5 Z  1.0000000+00) (W6 Z  1.0000000+00))) (LLIST ((Z ZF1  7.0000000+00) (ZF1 F1  2.0000000+00) (Z ZX12
  3.0000000+00) (Z X12  1.0000000+00) (Z X23  1.0000000+00) (ZX12 X12  2.0000000+00) (X12 X1223  5.0000000+00) (X1223 X
23  2.0000000+00) (X23 X23F1  5.0000000+00) (X23F1 F1  2.0000000+00) (X12 X1245  3.0000000+00) (X23 X45  1.0000000+00)
(F1 F2  4.0000000+00) (X1245 X45  2.0000000+00) (X45 X45F2  5.0000000+00) (X45F2 F2  2.0000000+00) (X45 F2  3.0000000+0
0) (Z YZ1  4.5000000+00) (Z YZ2  4.5000000+00) (Z ZYZ3  4.0000000+00) (YZ1 YZ2  1.5000000+00) (YZ2 YZ3  1.5000000+00) (
YZ1 YZ124  4.5000000+00) (YZ2 YZ24  4.0000000+00) (YZ3 YZ35  4.0000000+00) (Y24 Y35  1.5000000+00) (Y24 Y2456  4.500000
0+00) (Y35 Y356  4.0000000+00))) (TLIST (((Z W1  1.0000000+00) (Z YZ1  7.5000000-01)) ((Z W2 .1.0000000+00) (Z YZ2  7.5
000000-01)) ((Z W3  1.0000000+00) (Z YZ3  7.5000000-01)) ((W2 W4  1.0000000+00) (Z Y24  7.5000000-01)) ((W3 W5  1.00000
00+00) (Z Y35  7.5000000-01)) ((W5 W6  1.0000000+00) (Z Y56  7.5000000-01)) ((W2 W1  1.0000000+00) (Z X12  5.0000000-01
)) ((W3 W2  1.0000000+00) (Z X23  5.0000000-01)) ((W5 W4  1.0000000+00) (Z X45  5.0000000-01)) ((X12 ZX12  2.5000000-01
) (YZ124 Y24  1.0000000+00)) ((X23 X1223  2.5000000-01) (YZ24 Y24  1.0000000+00)) ((F1 ZF1  2.5000000-01) (ZYZ3 YZ3  1.
0000000+00)) ((F1 X23F1  2.5000000-01) (YZ35 Y35  1.0000000+00)) ((X45 X1245  2.5000000-01) (Y2456 Y56  1.0000000+00))
((F2 X45F2  2.5000000-01) (Y356 Y56  1.0000000+00))))))

RECLAIMER ENTRY   0/ 0  000.0 MARK  000.0      SWEEP   000.0  FULL WORDS   376 FREE   4554  PUSH DOWN DEPTH    178

(ARGUMENTS OF VNPGEN)
(((W1 Z  1.0000000+00) (W2 Z  1.0000000+00) (W3 Z  1.0000000+00) (W4 Z  1.0000000+00) (W5 Z  1.0000000+00) (W6 Z  1.000
0000+00)) NIL ((Z ZF1  7.0000000+00) (ZF1 F1  2.0000000+00) (Z ZX12  3.0000000+00) (Z X12  1.0000000+00) (Z X23  1.0000
000+00) (ZX12 X12  2.0000000+00) (X12 X1223  5.0000000+00) (X1223 X23  2.0000000+00) (X23 X23F1  5.0000000+00) (X23F1 F
1  2.0000000+00) (X12 X1245  3.0000000+00) (X23 X45  1.0000000+00) (F1 F2  4.0000000+00) (X1245 X45  2.0000000+00) (X45
 X45F2  5.0000000+00) (X45F2 F2  2.0000000+00) (X45 F2  3.0000000+00) (Z YZ1  4.5000000+00) (Z YZ2  4.5000000+00) (Z ZY
Z3  4.0000000+00) (YZ1 YZ2  1.5000000+00) (YZ2 YZ3  1.5000000+00) (YZ1 YZ124  4.5000000+00) (YZ2 YZ24  4.0000000+00) (Y
Z3 YZ35  4.0000000+00) (Y24 Y35  1.5000000+00) (Y24 Y2456  4.5000000+00) (Y35 Y356  4.0000000+00)) (((Z W1  1.0000000+0
0) (Z YZ1  7.5000000-01)) ((Z W2  1.0000000+00) (Z YZ2  7.5000000-01)) ((Z W3  1.0000000+00) (Z YZ3  7.5000000-01)) ((W
2 W4  1.0000000+00) (Z Y24  7.5000000-01)) ((W3 W5  1.0000000+00) (Z Y35  7.5000000-01)) ((W5 W6  1.0000000+00) (Z Y56
 7.5000000-01)) ((W2 W1  1.0000000+00) (Z X12  5.0000000-01)) ((W3 W2  1.0000000+00) (Z X23  5.0000000-01)) ((W5 W4  1.
0000000+00) (Z X45  5.0000000-01)) ((X12 ZX12  2.5000000-01) (YZ124 Y24  1.0000000+00)) ((X23 X1223  2.5000000-01) (YZ2
4 Y24  1.0000000+00)) ((F1 ZF1  2.5000000-01) (ZYZ3 YZ3  1.0000000+00)) ((F1 X23F1  2.5000000-01) (YZ35 Y35  1.0000000+
00)) ((X45 X1245  2.5000000-01) (Y2456 Y56  1.0000000+00)) ((F2 X45F2  2.5000000-01) (Y356 Y56  1.0000000+00))) NIL NIL
)

RECLAIMER ENTRY   0/ 0  000.0 MARK  000.0      SWEEP   000.0  FULL WORDS   376 FREE   3898  PUSH DOWN DEPTH     45
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FIGURE A - C - 6 - b

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RECLAIMER ENTRY   07 0  000.0 MARK  000.0     SWEEP   000.0  FULL WORDS   376 FREE   3533  PUSH DOWN DEPTH     61

RECLAIMER ENTRY   07 0  000.0 MARK  000.0     SWEEP   000.0  FULL WORDS   376 FREE   3437  PUSH DOWN DEPTH     55

RECLAIMER ENTRY   07 0  000.0 MARK  000.0     SWEEP   000.0  FULL WORDS   376 FREE   3439  PUSH DOWN DEPTH     55

(VALUE OF VSPOIN)
(NIL ((W3 . Z) (W4 . Z) (W5 . Z) (W1 . Z) (W6 . Z) (W2 . Z)) NIL ((YZ2 . YZ3) (Y24 . Y35) (ZF1 . F1) (ZX12 . X12) (X122
3 . X23) (X45 . F2) (X12 . X1245) (Z . ZX12) (F1 . F2) (YZ3 . YZ35) (YZ2 . YZ24) (Z . ZYZ3) (Y35 . Y356) (Z . YZ1) (YZ4
 . Y2456) (YZ1 . YZ124) (Z . YZ2) (X45 . X45F2) (X12 . X1223) (X23 . X23F1) (Z . ZF1)) ((Y2456 . Y56) (YZ35 . Y35)))

(ARGUMENTS OF LOGEN)
((((Z W1  1.0000000+00) (Z YZ1  7.5000000-01)) ((Z W2  1.0000000+00) (Z YZ2  7.5000000-01)) ((Z W3  1.0000000+00) (Z YZ3
 7.5000000-01)) ((W2 W4  1.0000000+00) (Z Y24  7.5000000-01)) ((W3 W5  1.0000000+00) (Z Y35  7.5000000-01)) ((W5 W6  1
.0000000+00) (Z Y56  7.5000000-01)) ((W2 W1  1.0000000+00) (Z X12  5.0000000-01)) ((W3 W2  1.0000000+00) (Z X23  5.0000
000-01)) ((W5 W4  1.0000000+00) (Z X45  5.0000000-01)) ((X12 ZX12  2.5000000-01) (YZ124 Y24  1.0000000+00)) ((X23 X1223
 2.5000000-01) (YZ24 Y24  1.0000000+00)) ((F1 ZF1  2.5000000-01) (ZYZ3 YZ3  1.0000000+00)) ((F1 X23F1  2.5000000-01) (
YZ35 Y35  1.0000000+00)) ((X45 X1245  2.5000000-01) (Y2456 Y56  1.0000000+00)) ((F2 X45F2  2.5000000-01) (Y356 Y56  1.0
000000+00)))
(NIL ((W3 . Z) (W4 . Z) (W5 . Z) (W1 . Z) (W6 . Z) (W2 . Z)) NIL ((YZ2 . YZ3) (Y24 . Y35) (ZF1 . F1) (ZX12 . X12) (X122
3 . X23) (X45 . F2) (X12 . X1245) (Z . ZX12) (F1 . F2) (YZ3 . YZ35) (YZ2 . YZ24) (Z . ZYZ3) (Y35 . Y356) (Z . YZ1) (YZ4
 . Y2456) (YZ1 . YZ124) (Z . YZ2) (X45 . X45F2) (X12 . X1223) (X23 . X23F1) (Z . ZF1)) ((Y2456 . Y56) (YZ35 . Y35)))

RECLAIMER ENTRY   07 0  000.0 MARK  000.0     SWEEP   000.0  FULL WORDS   375 FREE   3667  PUSH DOWN DEPTH     78

RECLAIMER ENTRY   07 0  000.0 MARK  000.0     SWEEP   000.0  FULL WORDS   369 FREE   3593  PUSH DOWN DEPTH     48

RECLAIMER ENTRY   07 0  000.0 MARK  000.0     SWEEP   000.0  FULL WORDS   353 FREE   3372  PUSH DOWN DEPTH     76

RECLAIMER ENTRY   07 0  000.0 MARK  000.0     SWEEP   000.0  FULL WORDS   340 FREE   3197  PUSH DOWN DEPTH     82

RECLAIMER ENTRY   07 0  000.0 MARK  000.0     SWEEP   000.0  FULL WORDS   329 FREE   3077  PUSH DOWN DEPTH     86

RECLAIMER ENTRY   07 0  000.0 MARK  000.0     SWEEP   000.0  FULL WORDS   315 FREE   2955  PUSH DOWN DEPTH    137

RECLAIMER ENTRY   07 0  000.0 MARK  000.0     SWEEP   000.0  FULL WORDS   304 FREE   2839  PUSH DOWN DEPTH     91

(VALUE OF LOGEN)
(((NIL NIL NIL (( 4.0000000+00 X45 . X45F2) (-4.0000000+00 X45 . F2)) NIL) (NIL NIL NIL (( 1.0000000+00 Y24 . Y2456) (-
1.0000000+00 Y24 . Y35) (-1.0000000+00 Y35 . Y356)) (( 1.0000000+00 Y2456 . Y56)))) ((NIL NIL NIL (( 4.0000000+00 X12 .
 X1245) ( 4.0000000+00 ZX12 . X12) ( 4.0000000+00 Z . ZX12) (-4.0000000+00 Z . ZF1) (-4.0000000+00 ZF1 . F1) (-4.000000
0+00 F1 . F2) ( 4.0000000+00 X45 . F2)) NIL) (NIL NIL NIL NIL (( 1.0000000+00 Y2456 . Y56)))) ((NIL NIL NIL (( 4.000000
0+00 X23 . X23F1) ( 4.0000000+00 X1223 . X23) ( 4.0000000+00 X12 . X1223) ( 4.0000000+00 ZX12 . X12) ( 4.0000000+00 Z .
 ZX12) (-4.0000000+00 Z . ZF1) (-4.0000000+00 ZF1 . F1)) NIL) (NIL NIL NIL NIL (( 1.0000000+00 YZ35 . Y35)))) ((NIL NIL
 NIL ((-4.0000000+00 ZF1 . F1)) NIL) (NIL NIL NIL (( 1.0000000+00 YZ2 . YZ3) ( 1.0000000+00 Z . YZ2) (-1.0000000+00 Z .
 ZYZ3)) NIL)) ((NIL NIL NIL ((-4.0000000+00 X1223 . X23)) NIL) (NIL NIL NIL ((-1.0000000+00 Y24 . Y35) ( 1.0000000+00 Y
Z3 . YZ35) ( 1.0000000+00 YZ2 . YZ3) (-1.0000000+00 YZ2 . YZ24)) (( 1.0000000+00 YZ35 . Y35)))) ((NIL NIL NIL ((-4.0000
000+00 ZX12 . X12)) NIL) (NIL NIL NIL ((-1.0000000+00 Y24 . Y35) ( 1.0000000+00 YZ3 . YZ35) ( 1.0000000+00 YZ2 . YZ3) (
 1.0000000+00 Z . YZ2) (-1.0000000+00 Z . YZ1) (-1.0000000+00 YZ1 . YZ124)) (( 1.0000000+00 YZ35 . Y35)))) ((NIL ((-1.0
000000+00 W4 . Z) ( 1.0000000+00 W5 . Z)) NIL NIL NIL) (NIL NIL NIL ((-2.0000000+00 X45 . F2) ( 2.0000000+00 F1 . F2) (
 2.0000000+00 ZF1 . F1) ( 2.0000000+00 Z . ZF1)) NIL)) ((NIL ((-1.0000000+00 W2 . Z) ( 1.0000000+00 W3 . Z)) NIL NIL NI
L) (NIL NIL NIL (( 2.0000000+00 X1223 . X23) ( 2.0000000+00 X12 . X1223) ( 2.0000000+00 ZX12 . X12) ( 2.0000000+00 Z .
ZX12)) NIL)) ((NIL ((-1.0000000+00 W1 . Z) ( 1.0000000+00 W2 . Z)) NIL NIL NIL) (NIL NIL NIL (( 2.0000000+00 ZX12 . X12
) ( 2.0000000+00 Z . ZX12)) NIL)) ((NIL ((-1.0000000+00 W6 . Z) ( 1.0000000+00 W5 . Z)) NIL NIL NIL) (NIL NIL NIL (( 1.
3333333+00 Y24 . Y2456) (-1.3333333+00 Y24 . Y35) ( 1.3333333+00 YZ3 . YZ35) ( 1.3333333+00 YZ2 . YZ3) ( 1.3333333+00 Z
 . YZ2)) (( 1.3333333+00 Y2456 . Y56) ( 1.3333333+00 YZ35 . Y35)))) ((NIL ((-1.0000000+00 W5 . Z) ( 1.0000000+00 W3 . Z
)) NIL NIL NIL) (NIL NIL NIL (( 1.3333333+00 YZ3 . YZ35) ( 1.3333333+00 YZ2 . YZ3) ( 1.3333333+00 Z . YZ2)) (( 1.333333
3+00 YZ35 . Y35)))) ((NIL ((-1.0000000+00 W4 . Z) ( 1.0000000+00 W2 . Z)) NIL NIL NIL) (NIL NIL NIL ((-1.3333333+00 Y24
 . Y35) ( 1.3333333+00 YZ3 . YZ35) ( 1.3333333+00 YZ2 . YZ3) ( 1.3333333+00 Z . YZ2)) (( 1.3333333+00 YZ35 . Y35)))) ((
NIL ((-1.0000000+00 W3 . Z)) NIL NIL NIL) (NIL NIL NIL (( 1.3333333+00 YZ2 . YZ3) ( 1.3333333+00 Z . YZ2)) NIL)) ((NIL
((-1.0000000+00 W2 . Z)) NIL NIL NIL) (NIL NIL NIL (( 1.3333333+00 Z . YZ2)) NIL)) ((NIL ((-1.0000000+00 W1 . Z)) NIL N
IL NIL) (NIL NIL NIL (( 1.3333333+00 Z . YZ1)) NIL)))

(ARGUMENTS OF SEGV)
(((NIL NIL NIL (( 4.0000000+00 X45 . X45F2) (-4.0000000+00 X45 . F2)) NIL) (NIL NIL NIL (( 1.0000000+00 Y24 . Y2456) (-
1.0000000+00 Y24 . Y35) (-1.0000000+00 Y35 . Y356)) (( 1.0000000+00 Y2456 . Y56)))) ((NIL NIL NIL (( 4.0000000+00 X12 .
```

FIGURE A - C - 6 - b (continued)

```
X1245) ( 4.0000000+00 ZX12 . X12) ( 4.0000000+00 Z . ZX12) (-4.0000000+00 Z . ZF1) (-4.0000000+00 ZF1 . F1) (-4.000000
0+00 F1 . F2) ( 4.0000000+00 X45 . F2)) NIL) (NIL NIL NIL NIL (( 1.0000000+00 Y2456 . Y56)))) ((NIL NIL NIL (( 4.000000
0+00 X23 . X23F1) ( 4.0000000+00 X1223 . X23) ( 4.0000000+00 X12 . X1223) ( 4.0000000+00 ZX12 . X12) ( 4.0000000+00 Z .
ZX12) (-4.0000000+00 Z . ZF1) (-4.0000000+00 ZF1 . F1)) NIL) (NIL NIL NIL NIL (( 1.0000000+00 YZ35 . Y35)))) ((NIL NIL
NIL ((-4.0000000+00 ZF1 . F1)) NIL) (NIL NIL NIL (( 1.0000000+00 YZ2 . YZ3) ( 1.0000000+00 Z . YZ2) (-1.0000000+00 Z .
ZYZ3)) NIL)) ((NIL NIL NIL ((-4.0000000+00 X1223 . X23)) NIL) (NIL NIL NIL ((-1.0000000+00 Y24 . Y35) ( 1.0000000+00 Y
Z3 . YZ35) ( 1.0000000+00 YZ2 . YZ3) (-1.0000000+00 YZ2 . YZ24)) (( 1.0000000+00 YZ35 . Y35)))) ((NIL NIL NIL ((-4.0000
000+00 ZX12 . X12)) NIL) (NIL NIL NIL ((-1.0000000+00 Y24 . Y35) ( 1.0000000+00 YZ3 . YZ35) ( 1.0000000+00 YZ2 . YZ3) (
1.0000000+00 Z . YZ2) (-1.0000000+00 Z . YZ1) (-1.0000000+00 YZ1 . YZ124)) (( 1.0000000+00 YZ35 . Y35)))) ((NIL ((-1.0
000000+00 W4 . Z) ( 1.0000000+00 W5 . Z)) NIL NIL NIL) (NIL NIL NIL ((-2.0000000+00 X45 . F2) ( 2.0000000+00 F1 . F2) (
2.0000000+00 ZF1 . F1) ( 2.0000000+00 Z . ZF1)) NIL)) ((NIL ((-1.0000000+00 W2 . Z) ( 1.0000000+00 W3 . Z)) NIL NIL NI
L) (NIL NIL NIL (( 2.0000000+00 X1223 . X23) ( 2.0000000+00 X12 . X1223) ( 2.0000000+00 ZX12 . X12) ( 2.0000000+00 Z .
ZX12)) NIL)) ((NIL ((-1.0000000+00 W1 . Z) ( 1.0000000+00 W2 . Z)) NIL NIL NIL) (NIL NIL NIL (( 2.0000000+00 ZX12 . X12
) ( 2.0000000+00 Z . ZX12)) NIL)) ((NIL ((-1.0000000+00 W6 . Z) ( 1.0000000+00 W5 . Z)) NIL NIL NIL) (NIL NIL NIL (( 1.
3333333+00 Y24 . YZ456) (-1.3333333+00 Y24 . Y35) ( 1.3333333+00 YZ3 . YZ35) ( 1.3333333+00 YZ2 . YZ3) ( 1.3333333+00 Z
. YZ2)) (( 1.3333333+00 YZ456 . Y56) ( 1.3333333+00 YZ35 . Y35)))) ((NIL ((-1.0000000+00 W5 . Z) ( 1.0000000+00 W3 . Z
)) NIL NIL NIL) (NIL NIL NIL (( 1.3333333+00 YZ3 . YZ35) ( 1.3333333+00 YZ2 . YZ3) ( 1.3333333+00 Z . YZ2)) (( 1.333333
3+00 YZ35 . Y35)))) ((NIL ((-1.0000000+00 W4 . Z) ( 1.0000000+00 W2 . Z)) NIL NIL NIL) (NIL NIL NIL ((-1.3333333+00 Y24
. Y35) ( 1.3333333+00 YZ3 . YZ35) ( 1.3333333+00 YZ2 . YZ3) ( 1.3333333+00 Z . YZ2)) (( 1.3333333+00 YZ35 . Y35)))) ((
NIL ((-1.0000000+00 W3 . Z)) NIL NIL NIL) (NIL NIL NIL (( 1.3333333+00 YZ2 . YZ3) ( 1.3333333+00 Z . YZ2)) NIL)) ((NIL
((-1.0000000+00 W2 . Z)) NIL NIL NIL) (NIL NIL NIL (( 1.3333333+00 Z . YZ2)) NIL)) ((NIL ((-1.0000000+00 W1 . Z)) NIL N
IL NIL) (NIL NIL NIL (( 1.3333333+00 Z . YZ1)) NIL))))
```

```
RECLAIMER ENTRY   07 0  000.0 MARK  000.0   SWEEP  000.0  FULL WORDS  266 FREE  2713  PUSH DOWN DEPTH  127
RECLAIMER ENTRY   07 0  000.0 MARK  000.0   SWEEP  000.0  FULL WORDS  249 FREE  2353  PUSH DOWN DEPTH  250
RECLAIMER ENTRY   07 0  000.0 MARK  000.0   SWEEP  000.0  FULL WORDS  225 FREE  2036  PUSH DOWN DEPTH  243
RECLAIMER ENTRY   07 0  000.0 MARK  000.0   SWEEP  000.0  FULL WORDS  211 FREE  1959  PUSH DOWN DEPTH  190
RECLAIMER ENTRY   07 0  000.0 MARK  000.0   SWEEP  000.0  FULL WORDS  198 FREE  1854  PUSH DOWN DEPTH  204
RECLAIMER ENTRY   07 0  000.0 MARK  000.0   SWEEP  000.0  FULL WORDS  177 FREE  1639  PUSH DOWN DEPTH  249
RECLAIMER ENTRY   07 0  000.0 MARK  000.0   SWEEP  000.0  FULL WORDS  171 FREE  1673  PUSH DOWN DEPTH  145
RECLAIMER ENTRY   07 0  000.0 MARK  000.0   SWEEP  000.0  FULL WORDS  162 FREE  1565  PUSH DOWN DEPTH  193
RECLAIMER ENTRY   07 0  000.0 MARK  000.0   SWEEP  000.0  FULL WORDS  151 FREE  1349  PUSH DOWN DEPTH  318
RECLAIMER ENTRY   07 0  000.0 MARK  000.0   SWEEP  000.0  FULL WORDS  143 FREE  1288  PUSH DOWN DEPTH  534
RECLAIMER ENTRY   07 0  000.0 MARK  000.0   SWEEP  000.0  FULL WORDS  141 FREE  1397  PUSH DOWN DEPTH  337
RECLAIMER ENTRY   07 0  000.0 MARK  000.0   SWEEP  000.0  FULL WORDS  126 FREE  1260  PUSH DOWN DEPTH  256
RECLAIMER ENTRY   07 0  000.0 MARK  000.0   SWEEP  000.0  FULL WORDS  101 FREE   946  PUSH DOWN DEPTH  423
RECLAIMER ENTRY   07 0  000.0 MARK  000.0   SWEEP  000.0  FULL WORDS  148 FREE  1465  PUSH DOWN DEPTH  274
RECLAIMER ENTRY   07 0  000.0 MARK  000.0   SWEEP  000.0  FULL WORDS  121 FREE  1061  PUSH DOWN DEPTH  630
RECLAIMER ENTRY   07 0  000.0 MARK  000.0   SWEEP  000.0  FULL WORDS  115 FREE  1171  PUSH DOWN DEPTH  334
RECLAIMER ENTRY   07 0  000.0 MARK  000.0   SWEEP  000.0  FULL WORDS  132 FREE  1316  PUSH DOWN DEPTH  177
RECLAIMER ENTRY   07 0  000.0 MARK  000.0   SWEEP  000.0  FULL WORDS  118 FREE  1227  PUSH DOWN DEPTH  247
RECLAIMER ENTRY   07 0  000.0 MARK  000.0   SWEEP  000.0  FULL WORDS   90 FREE   804  PUSH DOWN DEPTH  477
RECLAIMER ENTRY   07 0  000.0 MARK  000.0   SWEEP  000.0  FULL WORDS   80 FREE   767  PUSH DOWN DEPTH  417
RECLAIMER ENTRY   07 0  000.0 MARK  000.0   SWEEP  000.0  FULL WORDS   70 FREE   725  PUSH DOWN DEPTH  364
RECLAIMER ENTRY   07 0  000.0 MARK  000.0   SWEEP  000.0  FULL WORDS  130 FREE  1376  PUSH DOWN DEPTH  201
```

FIGURE A - C - 6 - b (continued)

RECLAIMER ENTRY	0/0	000.0	MARK	000.0	SWEEP	000.0	FULL WORDS	[illegible]	FREE	1141	PUSH DOWN DEPTH	114
RECLAIMER ENTRY	0/0	000.0	MARK	000.0	SWEEP	000.0	FULL WORDS	84	FREE	953	PUSH DOWN DEPTH	229
RECLAIMER ENTRY	0/0	000.0	MARK	000.0	SWEEP	000.0	FULL WORDS	58	FREE	716	PUSH DOWN DEPTH	237
RECLAIMER ENTRY	0/0	000.0	MARK	000.0	SWEEP	000.0	FULL WORDS	47	FREE	532	PUSH DOWN DEPTH	360
RECLAIMER ENTRY	0/0	000.0	MARK	000.0	SWEEP	000.0	FULL WORDS	49	FREE	643	PUSH DOWN DEPTH	257
RECLAIMER ENTRY	0/0	000.0	MARK	000.0	SWEEP	000.0	FULL WORDS	28	FREE	334	PUSH DOWN DEPTH	438
RECLAIMER ENTRY	0/0	000.0	MARK	000.0	SWEEP	000.0	FULL WORDS	19	FREE	381	PUSH DOWN DEPTH	267
RECLAIMER ENTRY	0/0	000.0	MARK	000.0	SWEEP	000.0	FULL WORDS	85	FREE	950	PUSH DOWN DEPTH	355
RECLAIMER ENTRY	0/0	000.0	MARK	000.0	SWEEP	000.0	FULL WORDS	61	FREE	929	PUSH DOWN DEPTH	226
RECLAIMER ENTRY	0/0	000.0	MARK	000.0	SWEEP	000.0	FULL WORDS	70	FREE	870	PUSH DOWN DEPTH	90
RECLAIMER ENTRY	0/0	000.0	MARK	000.0	SWEEP	000.0	FULL WORDS	43	FREE	434	PUSH DOWN DEPTH	375
RECLAIMER ENTRY	0/0	000.0	MARK	000.0	SWEEP	000.0	FULL WORDS	58	FREE	735	PUSH DOWN DEPTH	250
RECLAIMER ENTRY	0/0	000.0	MARK	000.0	SWEEP	000.0	FULL WORDS	38	FREE	359	PUSH DOWN DEPTH	586
RECLAIMER ENTRY	0/0	000.0	MARK	000.0	SWEEP	000.0	FULL WORDS	42	FREE	600	PUSH DOWN DEPTH	250
RECLAIMER ENTRY	0/0	000.0	MARK	000.0	SWEEP	000.0	FULL WORDS	26	FREE	336	PUSH DOWN DEPTH	414
RECLAIMER ENTRY	0/0	000.0	MARK	000.0	SWEEP	000.0	FULL WORDS	17	FREE	378	PUSH DOWN DEPTH	207
RECLAIMER ENTRY	0/0	000.0	MARK	000.0	SWEEP	000.0	FULL WORDS	[illegible]	FREE	1175	PUSH DOWN DEPTH	132
RECLAIMER ENTRY	0/0	000.0	MARK	000.0	SWEEP	000.0	FULL WORDS	84	FREE	996	PUSH DOWN DEPTH	66
RECLAIMER ENTRY	0/0	000.0	MARK	000.0	SWEEP	000.0	FULL WORDS	79	FREE	901	PUSH DOWN DEPTH	205
RECLAIMER ENTRY	0/0	000.0	MARK	000.0	SWEEP	000.0	FULL WORDS	75	FREE	859	PUSH DOWN DEPTH	159
RECLAIMER ENTRY	0/0	000.0	MARK	000.0	SWEEP	000.0	FULL WORDS	56	FREE	665	PUSH DOWN DEPTH	249
RECLAIMER ENTRY	0/0	000.0	MARK	000.0	SWEEP	000.0	FULL WORDS	50	FREE	549	PUSH DOWN DEPTH	345
RECLAIMER ENTRY	0/0	000.0	MARK	000.0	SWEEP	000.0	FULL WORDS	48	FREE	575	PUSH DOWN DEPTH	370
RECLAIMER ENTRY	0/0	000.0	MARK	000.0	SWEEP	000.0	FULL WORDS	32	FREE	285	PUSH DOWN DEPTH	489
RECLAIMER ENTRY	0/0	000.0	MARK	000.0	SWEEP	000.0	FULL WORDS	123	FREE	1276	PUSH DOWN DEPTH	175
RECLAIMER ENTRY	0/0	000.0	MARK	000.0	SWEEP	000.0	FULL WORDS	110	FREE	1133	PUSH DOWN DEPTH	159
RECLAIMER ENTRY	0/0	000.0	MARK	000.0	SWEEP	000.0	FULL WORDS	[illegible]	FREE	994	PUSH DOWN DEPTH	243
RECLAIMER ENTRY	0/0	000.0	MARK	000.0	SWEEP	000.0	FULL WORDS	151	FREE	1483	PUSH DOWN DEPTH	272
RECLAIMER ENTRY	0/0	000.0	MARK	000.0	SWEEP	000.0	FULL WORDS	137	FREE	1349	PUSH DOWN DEPTH	207
RECLAIMER ENTRY	0/0	000.0	MARK	000.0	SWEEP	000.0	FULL WORDS	128	FREE	1198	PUSH DOWN DEPTH	291
RECLAIMER ENTRY	0/0	000.0	MARK	000.0	SWEEP	000.0	FULL WORDS	174	FREE	1687	PUSH DOWN DEPTH	108
RECLAIMER ENTRY	0/0	000.0	MARK	000.0	SWEEP	000.0	FULL WORDS	187	FREE	1834	PUSH DOWN DEPTH	156
RECLAIMER ENTRY	0/0	000.0	MARK	000.0	SWEEP	000.0	FULL WORDS	[illegible]	FREE	1563	PUSH DOWN DEPTH	156

FIGURE A - C - 6 - b (continued)

```
RECLAIMER ENTRY   0/ 0  000.0 MARK  000.0     SWEEP   000.0  FULL WORDS   180 FREE   1776  PUSH DOWN DEPTH    124
RECLAIMER ENTRY   0/ 0  000.0 MARK  000.0     SWEEP   000.0  FULL WORDS   154 FREE   1450  PUSH DOWN DEPTH    306
RECLAIMER ENTRY   0/ 0  000.0 MARK  000.0     SWEEP   000.0  FULL WORDS   152 FREE   1455  PUSH DOWN DEPTH    289
RECLAIMER ENTRY   0/ 0  000.0 MARK  000.0     SWEEP   000.0  FULL WORDS   137 FREE   1269  PUSH DOWN DEPTH    300
RECLAIMER ENTRY   0/ 0  000.0 MARK  000.0     SWEEP   000.0  FULL WORDS   192 FREE   1823  PUSH DOWN DEPTH     64
RECLAIMER ENTRY   0/ 0  000.0 MARK  000.0     SWEEP   000.0  FULL WORDS   199 FREE   1848  PUSH DOWN DEPTH    202
RECLAIMER ENTRY   0/ 0  000.0 MARK  000.0     SWEEP   000.0  FULL WORDS   216 FREE   2018  PUSH DOWN DEPTH    202
RECLAIMER ENTRY   0/ 0  000.0 MARK  000.0     SWEEP   000.0  FULL WORDS   209 FREE   1847  PUSH DOWN DEPTH    285
RECLAIMER ENTRY   0/ 0  000.0 MARK  000.0     SWEEP   000.0  FULL WORDS   219 FREE   1965  PUSH DOWN DEPTH    295
(VALUE OF SEQV)
(((Z . YZ1) (NIL ((-7.5000000-01 W1 . Z)) NIL NIL NIL)) ((YZ2 . YZ3) (NIL ((-7.5000000-01 W3 . Z) ( 7.5000001-01 W2 . Z
)) NIL NIL NIL)) ((Z . YZ2) (NIL ((-7.5000000-01 W2 . Z)) NIL NIL NIL)) ((Y24 . Y35) (NIL ((-7.4999996-01 W2 . Z) ( 7.4
999994-01 W3 . Z) (-7.4999999-01 W5 . Z) ( 7.5000001-01 W4 . Z)) NIL NIL NIL)) ((X12 . X1223) (NIL (( 5.0000001-01 W3 .
 Z) (-6.2500000-01 W2 . Z) ( 5.0000000-01 W1 . Z) (-1.8750000-01 W4 . Z)) NIL ((-2.5000000-01 YZ2 . YZ24)) NIL)) ((X12
 . X1245) (NIL ((-6.8750000-01 W2 . Z) ( 6.8749999-01 W5 . Z) (-3.1250000-01 W4 . Z) ( 5.0000000-01 W1 . Z) (-1.8750000-
01 W6 . Z)) NIL ((-2.5000000-01 Y24 . Y2456)) NIL)) ((Z . ZX12) (NIL ((-3.1250000-01 W1 . Z) (-1.8750000-01 W4 . Z) ( 6
.8750000-01 W2 . Z)) NIL ((-2.5000000-01 YZ1 . YZ124)) NIL)) ((X23 . X23F1) (NIL (( 6.2499950-02 W3 . Z) (-6.2499978-02
 W5 . Z) ( 1.8750000-01 W6 . Z) (-5.0000000-01 W4 . Z) ( 5.0000003-01 W2 . Z)) NIL ((-9.9999999-01 F1 . F2) (-2.5000000
-01 YZ3 . YZ35) ( 9.9999999-01 X45 . X45F2) ( 2.5000000-01 Y35 . Y356)) NIL)) ((Z . ZF1) (NIL (( 1.2500001-01 W5 . Z) (
 1.8750000-01 W6 . Z) (-5.0000000-01 W4 . Z)) NIL ((-2.5000000-01 Z . ZYZ3) (-9.9999999-01 F1 . F2) ( 9.9999999-01 X45
 . X45F2) ( 2.5000000-01 Y35 . Y356)) NIL)) ((ZX12 . X12) (NIL ((-1.8750000-01 W1 . Z) (-1.8749998-01 W2 . Z) ( 1.875000
0-01 W4 . Z)) NIL (( 2.5000000-01 YZ1 . YZ124)) NIL)) ((X1223 . X23) (NIL ((-3.7499998-01 W2 . Z) ( 1.8750000-01 W4 . Z
)) NIL (( 2.4999999-01 YZ2 . YZ24)) NIL)) ((ZF1 . F1) (NIL (( 1.8750000-01 W3 . Z)) NIL (( 2.5000000-01 Z . ZYZ3)) NIL)
) ((YZ35 . Y35) (NIL (( 1.4999999+00 W3 . Z) (-7.4999998-01 W5 . Z)) NIL ((-9.9999999-01 YZ3 . YZ35)) NIL)) ((X45 . F2)
 (NIL (( 1.8749997-01 W3 . Z) (-3.7499997-01 W5 . Z) ( 1.8750000-01 W6 . Z)) NIL (( 9.9999999-01 X45 . X45F2) ( 2.50000
00-01 Y35 . Y356)) NIL)) ((Y2456 . Y56) (NIL ((-7.4999993-01 W2 . Z) ( 7.4999995-01 W5 . Z) ( 7.4999998-01 W4 . Z) (-7.
4999999-01 W6 . Z)) NIL ((-9.9999998-01 Y24 . Y2456)) NIL)))

RECLAIMER ENTRY-- 0/ 0  000.0 MARK  000.0     SWEEP   000.0  FULL WORDS   230 FREE   2115  PUSH DOWN DEPTH     76

(VALUE OF CONSEL)
((NIL ((W3 . Z) (W4 . Z) (W5 . Z) (W1 . Z) (W6 . Z) (W2 . Z)) NIL ((F1 . F2) (YZ3 . YZ35) (YZ2 . YZ24) (Z . ZYZ3) (Y35
 . Y356) (YZ1 . YZ124) (Y24 . Y2456) (X45 . X45F2)) NIL) ((Z . YZ1) (NIL ((-7.5000000-01 W1 . Z)) NIL NIL NIL)) ((YZ2 .
YZ3) (NIL ((-7.5000000-01 W3 . Z) ( 7.5000001-01 W2 . Z)) NIL NIL NIL)) ((Z . YZ2) (NIL ((-7.5000000-01 W2 . Z)) NIL NI
L NIL)) ((Y24 . Y35) (NIL ((-7.4999996-01 W2 . Z) ( 7.4999994-01 W3 . Z) (-7.4999999-01 W5 . Z) ( 7.5000001-01 W4 . Z))
 NIL NIL NIL)) ((X12 . X1223) (NIL (( 5.0000001-01 W3 . Z) (-6.2500000-01 W2 . Z) ( 5.0000000-01 W1 . Z) (-1.8750000-01
 W4 . Z)) NIL ((-2.5000000-01 YZ2 . YZ24)) NIL)) ((X12 . X1245) (NIL ((-6.8750000-01 W2 . Z) ( 6.8749999-01 W5 . Z) (-3
.1250000-01 W4 . Z) ( 5.0000000-01 W1 . Z) (-1.8750000-01 W6 . Z)) NIL ((-2.5000000-01 Y24 . Y2456)) NIL)) ((Z . ZX12)
(NIL ((-3.1250000-01 W1 . Z) (-1.8750000-01 W4 . Z) ( 6.8750000-01 W2 . Z)) NIL ((-2.5000000-01 YZ1 . YZ124)) NIL)) ((X
23 . X23F1) (NIL (( 6.2499950-02 W3 . Z) (-6.2499978-02 W5 . Z) ( 1.8750000-01 W6 . Z) (-5.0000000-01 W4 . Z) ( 5.00000
03-01 W2 . Z)) NIL ((-9.9999999-01 F1 . F2) (-2.5000000-01 YZ3 . YZ35) ( 9.9999999-01 X45 . X45F2) ( 2.5000000-01 Y35 .
 Y356)) NIL)) ((Z . ZF1) (NIL (( 1.2500001-01 W5 . Z) ( 1.8750000-01 W6 . Z) (-5.0000000-01 W4 . Z)) NIL ((-2.5000000-0
1 Z . ZYZ3) (-9.9999999-01 F1 . F2) ( 9.9999999-01 X45 . X45F2) ( 2.5000000-01 Y35 . Y356)) NIL)) ((ZX12 . X12) (NIL ((
-1.8750000-01 W1 . Z) (-1.8749998-01 W2 . Z) ( 1.8750000-01 W4 . Z)) NIL (( 2.5000000-01 YZ1 . YZ124)) NIL)) ((X1223 .
X23) (NIL ((-3.7499998-01 W2 . Z) ( 1.8750000-01 W4 . Z)) NIL (( 2.4999999-01 YZ2 . YZ24)) NIL)) ((ZF1 . F1) (NIL (( 1.
8750000-01 W3 . Z)) NIL (( 2.5000000-01 Z . ZYZ3)) NIL)) ((YZ35 . Y35) (NIL (( 1.4999999+00 W3 . Z) (-7.4999998-01 W5 .
 Z)) NIL ((-9.9999999-01 YZ3 . YZ35)) NIL)) ((X45 . F2) (NIL (( 1.8749997-01 W3 . Z) (-3.7499997-01 W5 . Z) ( 1.8750000
-01 W6 . Z)) NIL (( 9.9999999-01 X45 . X45F2) ( 2.5000000-01 Y35 . Y356)) NIL)) ((Y2456 . Y56) (NIL ((-7.4999993-01 W2
 . Z) ( 7.4999995-01 W5 . Z) ( 7.4999998-01 W4 . Z) (-7.4999999-01 W6 . Z)) NIL ((-9.9999998-01 Y24 . Y2456)) NIL)))

END OF EVALQUOTE, VALUE IS ..
((NIL ((W3 . Z) (W4 . Z) (W5 . Z) (W1 . Z) (W6 . Z) (W2 . Z)) NIL ((F1 . F2) (YZ3 . YZ35) (YZ2 . YZ24) (Z . ZYZ3) (Y35
 . Y356) (YZ1 . YZ124) (Y24 . Y2456) (X45 . X45F2)) NIL) ((Z . YZ1) (NIL ((-7.5000000-01 W1 . Z)) NIL NIL NIL)) ((YZ2 .
YZ3) (NIL ((-7.5000000-01 W3 . Z) ( 7.5000001-01 W2 . Z)) NIL NIL NIL)) ((Z . YZ2) (NIL ((-7.5000000-01 W2 . Z)) NIL NI
L NIL)) ((Y24 . Y35) (NIL ((-7.4999996-01 W2 . Z) ( 7.4999994-01 W3 . Z) (-7.4999999-01 W5 . Z) ( 7.5000001-01 W4 . Z))
 NIL NIL NIL)) ((X12 . X1223) (NIL (( 5.0000001-01 W3 . Z) (-6.2500000-01 W2 . Z) ( 5.0000000-01 W1 . Z) (-1.8750000-01
 W4 . Z)) NIL ((-2.5000000-01 YZ2 . YZ24)) NIL)) ((X12 . X1245) (NIL ((-6.8750000-01 W2 . Z) ( 6.8749999-01 W5 . Z) (-3
.1250000-01 W4 . Z) ( 5.0000000-01 W1 . Z) (-1.8750000-01 W6 . Z)) NIL ((-2.5000000-01 Y24 . Y2456)) NIL)) ((Z . ZX12)
(NIL ((-3.1250000-01 W1 . Z) (-1.8750000-01 W4 . Z) ( 6.8750000-01 W2 . Z)) NIL ((-2.5000000-01 YZ1 . YZ124)) NIL)) ((X
23 . X23F1) (NIL (( 6.2499950-02 W3 . Z) (-6.2499978-02 W5 . Z) ( 1.8750000-01 W6 . Z) (-5.0000000-01 W4 . Z) ( 5.00000
03-01 W2 . Z)) NIL ((-9.9999999-01 F1 . F2) (-2.5000000-01 YZ3 . YZ35) ( 9.9999999-01 X45 . X45F2) ( 2.5000000-01 Y35 .
 Y356)) NIL)) ((Z . ZF1) (NIL (( 1.2500001-01 W5 . Z) ( 1.8750000-01 W6 . Z) (-5.0000000-01 W4 . Z)) NIL ((-2.5000000-0
1 Z . ZYZ3) (-9.9999999-01 F1 . F2) ( 9.9999999-01 X45 . X45F2) ( 2.5000000-01 Y35 . Y356)) NIL)) ((ZX12 . X12) (NIL ((
-1.8750000-01 W1 . Z) (-1.8749998-01 W2 . Z) ( 1.8750000-01 W4 . Z)) NIL (( 2.5000000-01 YZ1 . YZ124)) NIL)) ((X1223 .
X23) (NIL ((-3.7499998-01 W2 . Z) ( 1.8750000-01 W4 . Z)) NIL (( 2.4999999-01 YZ2 . YZ24)) NIL)) ((ZF1 . F1) (NIL (( 1.
8750000-01 W3 . Z)) NIL (( 2.5000000-01 Z . ZYZ3)) NIL)) ((YZ35 . Y35) (NIL (( 1.4999999+00 W3 . Z) (-7.4999998-01 W5 .
 Z)) NIL ((-9.9999999-01 YZ3 . YZ35)) NIL)) ((X45 . F2) (NIL (( 1.8749997-01 W3 . Z) (-3.7499997-01 W5 . Z) ( 1.8750000
-01 W6 . Z)) NIL (( 9.9999999-01 X45 . X45F2) ( 2.5000000-01 Y35 . Y356)) NIL)) ((Y2456 . Y56) (NIL ((-7.4999993-01 W2
 . Z) ( 7.4999995-01 W5 . Z) ( 7.4999998-01 W4 . Z) (-7.4999999-01 W6 . Z)) NIL ((-9.9999998-01 Y24 . Y2456)) NIL)))

THE TIME ( 0/ 0  000.0) HAS COME, THE WALRUS SAID, TO TALK OF MANY THINGS .....   -LEWIS CARROLL-
```

FIGURE A - C - 6 - b (continued)

are listed in (fig. A-C-6-a) and the <u>corsel</u> output, in (fig. A-C-6-b). The selected coordinates are:

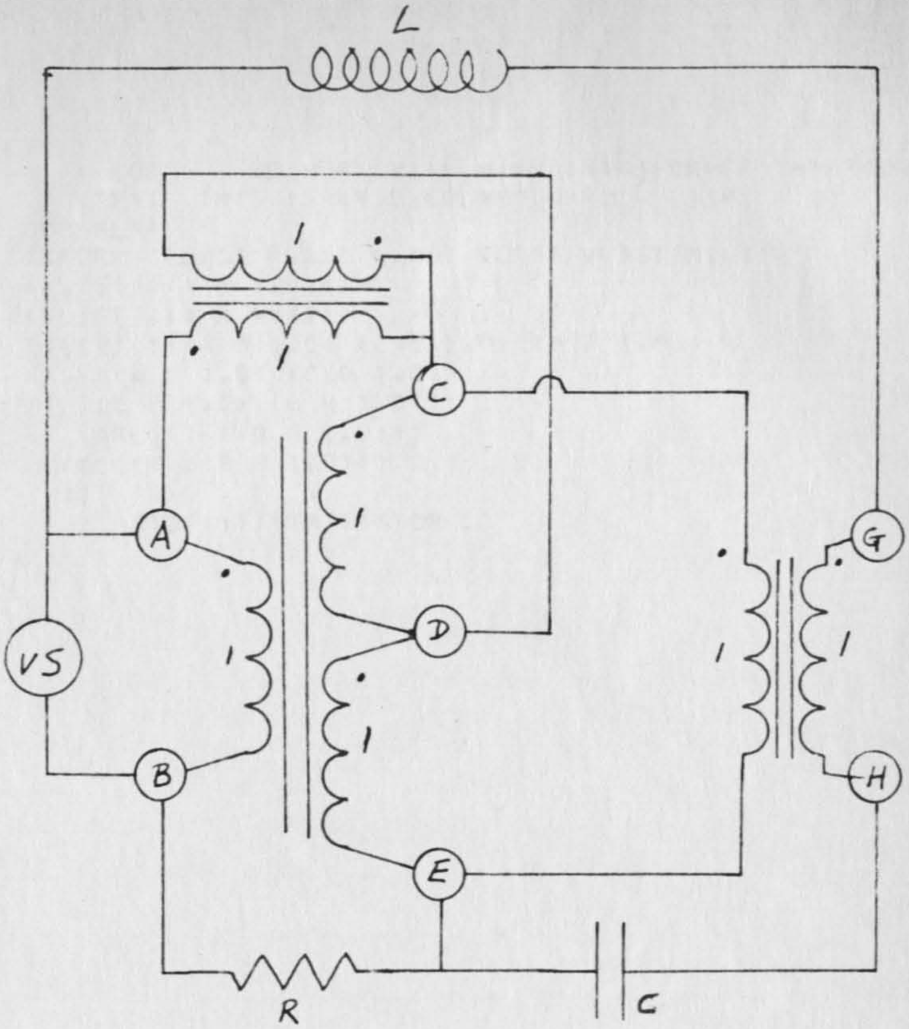
$$
\begin{aligned}
V^1 = \; & (W3 \cdot Z) \\
& (W4 \cdot Z) \\
& (W5 \cdot Z) \\
& (W1 \cdot Z) \\
& (W6 \cdot Z) \\
& (W2 \cdot Z)
\end{aligned}
$$

$$
\begin{aligned}
V^3 = \; & (F1 \cdot F2) \\
& (YZ3 \cdot YZ35) \\
& (YZ2 \cdot YZ24) \\
& (Z \cdot ZYZ3) \\
& (Y35 \cdot Y356) \\
& (YZ1 \cdot YZ124) \\
& (Y24 \cdot Y2456) \\
& (X45 \cdot X45F2)
\end{aligned}
$$

A. C. 4

The circuit in (fig. A-C-7) shows an arbitrary irregular transformer interconnection. The resulting selected coordinates by the LISP program appear in (fig. A-C-8-b) while the input cards are listed in (fig. A-C-8-a). The independent coordinates are:

$$
\begin{aligned}
V^o &= (A \cdot B) \\
V^1 &= (E \cdot H).
\end{aligned}
$$



An Arbitrary Network with Irregular Transformer Constraints

FIGURE A - C - 7

```
KXL000     NETWORK WITH MANY TRANSFORMER INTERCONNECTIONS
 TEST  THIS IS AN OVERLORD CARD - LISP
CORSEL((
(XFORM (CLIST RLIST LLIST TLIST VLIST NIL))
(LLIST ((A G 1.0)))
(VLIST ((A B VVA)))
(TLIST (((A B 1.0) (C D 1.0) (D E 1.0))
   ((A C 1.0) (C D 1.0))
   ((C E 1.0) (G H 1.0))))
   (RLIST ((B E 1.0)))
  (CLIST ((E H 1.0)))
 ))
     STOP)))))))))))STOP
```

FIGURE A - C - 8 - a

```
END OF EVALQUOTE OPERATOR

        KALOGG    NETWORK WITH MANY TRANSFORMER INTERCONNECTIONS            000000000000

         TEST  THIS IS AN OVERLORD CARD - LISP                              000000000000

 THE TIME ( 0/ 0  000.0) HAS COME, THE WALRUS SAID, TO TALK OF MANY THINGS .....   -LEWIS CARROLL-

EVALQUOTE OPERATOR AS OF 1 MARCH 1961.    INPUT LISTS NOW BEING READ.

 THE TIME ( 0/ 0  000.0) HAS COME, THE WALRUS SAID, TO TALK OF MANY THINGS .....   -LEWIS CARROLL-



 FUNCTION   EVALQUOTE   HAS BEEN ENTERED, ARGUMENTS..
CURSEL
(((XFORM (CLIST RLIST LLIST TLIST VLIST NIL)) (LLIST ((A G  1.0000000+00))) (VLIST ((A B VVA))) (TLIST (((A B  1.000000
0+00) (C D  1.0000000+00) (D E  1.0000000+00)) ((A C  1.0000000+00) (C D  1.0000000+00)) ((C E  1.0000000+00) (G H  1.0
000000+00)))) (RLIST ((B E  1.0000000+00))) (CLIST ((E H  1.0000000+00)))))


(ARGUMENTS OF CURSEL)
((XFORM (CLIST RLIST LLIST TLIST VLIST NIL)) (LLIST ((A G  1.0000000+00))) (VLIST ((A B VVA))) (TLIST (((A B  1.0000000
+00) (C D  1.0000000+00) (D E  1.0000000+00)) ((A C  1.0000000+00) (C D  1.0000000+00)) ((C E  1.0000000+00) (G H  1.00
00000+00)))) (RLIST ((B E  1.0000000+00))) (CLIST ((E H  1.0000000+00))))

(ARGUMENTS OF VNPGEN)
(((E H  1.0000000+00)) ((B E  1.0000000+00)) ((A G  1.0000000+00)) (((A B  1.0000000+00) (C D  1.0000000+00) (D E  1.00
00000+00)) ((A C  1.0000000+00) (C D  1.0000000+00)) ((C E  1.0000000+00) (G H  1.0000000+00))) ((A B VVA)) NIL)

(VALUE OF VNPGEN)
(((A . B)) ((E . H)) ((B . E)) ((A . G)) ((C . E) (D . E)))

(ARGUMENTS OF LOGEN)
(((A B  1.0000000+00) (C D  1.0000000+00) (D E  1.0000000+00)) ((A C  1.0000000+00) (C D  1.0000000+00)) ((C E  1.00000
00+00) (G H  1.0000000+00)))
(((A . B)) ((E . H)) ((B . E)) ((A . G)) ((C . E) (D . E)))

(VALUE OF LOGEN)
(((NIL NIL NIL NIL (( 1.0000000+00 C . E))) ((( 1.0000000+00 A . B)) (( 1.0000000+00 E . H)) (( 1.0000000+00 B . E)) ((
-1.0000000+00 A . G)) NIL)) (((( 1.0000000+00 A . B)) NIL (( 1.0000000+00 B . E)) NIL ((-1.0000000+00 C . E))) (NIL NIL
 NIL NIL ((-1.0000000+00 D . E) ( 1.0000000+00 C . E)))) (((( 1.0000000+00 A . B)) NIL NIL NIL NIL) (NIL NIL NIL NIL ((
 1.0000000+00 D . E)))) (((( 1.0000000+00 A . B)) NIL NIL NIL NIL) (NIL NIL NIL NIL ((-1.0000000+00 D . E) ( 1.0000000+
00 C . E)))))

(ARGUMENTS OF SEDV)
(((NIL NIL NIL NIL (( 1.0000000+00 C . E))) ((( 1.0000000+00 A . B)) (( 1.0000000+00 E . H)) (( 1.0000000+00 B . E)) ((
-1.0000000+00 A . G)) NIL)) (((( 1.0000000+00 A . B)) NIL (( 1.0000000+00 B . E)) NIL ((-1.0000000+00 C . E))) (NIL NIL
 NIL NIL ((-1.0000000+00 D . E) ( 1.0000000+00 C . E)))) (((( 1.0000000+00 A . B)) NIL NIL NIL NIL) (NIL NIL NIL NIL ((
 1.0000000+00 D . E)))) (((( 1.0000000+00 A . B)) NIL NIL NIL NIL) (NIL NIL NIL NIL ((-1.0000000+00 D . E) ( 1.0000000+
00 C . E)))))

RECLAIMER ENTRY   0/ 0  000.0 MARK  000.0      SWEEP   000.0  FULL WORDS   421 FREE   4691  PUSH DOWN DEPTH    109

RECLAIMER ENTRY   0/ 0  000.0 MARK  000.0      SWEEP   000.0  FULL WORDS   416 FREE   4575  PUSH DOWN DEPTH    235

(VALUE OF SEDV)
(((B . E) ((( 2.0000000+00 A . B)) NIL NIL NIL NIL)) ((A . G) ((( 1.0000000+00 A . B)) (( 1.0000000+00 E . H)) NIL NIL
NIL)) ((D . E) ((( 1.0000000+00 A . B)) NIL NIL NIL NIL)) ((C . E) ((( 2.0000000+00 A . B)) NIL NIL NIL NIL)))

(VALUE OF CURSEL)
((((A . B)) ((E . H)) NIL NIL NIL) ((B . E) ((( 2.0000000+00 A . B)) NIL NIL NIL NIL)) ((A . G) ((( 1.0000000+00 A . B)
) (( 1.0000000+00 E . H)) NIL NIL NIL)) ((D . E) ((( 1.0000000+00 A . B)) NIL NIL NIL NIL)) ((C . E) ((( 2.0000000+00 A
 . B)) NIL NIL NIL NIL)))

END OF EVALQUOTE, VALUE IS ..
((((A . B)) ((E . H)) NIL NIL NIL) ((B . E) ((( 2.0000000+00 A . B)) NIL NIL NIL NIL)) ((A . G) ((( 1.0000000+00 A . B)
) (( 1.0000000+00 E . H)) NIL NIL NIL)) ((D . E) ((( 1.0000000+00 A . B)) NIL NIL NIL NIL)) ((C . E) ((( 2.0000000+00 A
 . B)) NIL NIL NIL NIL)))

 THE TIME ( 0/ 0  000.0) HAS COME, THE WALRUS SAID, TO TALK OF MANY THINGS .....   -LEWIS CARROLL-
```

FIGURE A - C - 8 - b

The dependent coordinates are:

$$(B \cdot E) = 2(A \cdot B)$$

$$(A - G) = (A \cdot B) + (E \cdot H)$$

$$(D \cdot E) = (A \cdot B)$$

$$(C - E) = 2(A \cdot B)$$

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LIST OF SYMBOLS

t	the time variable
s	the Laplace transform complex variable
q_i	the i^{th} generalized coordinate
q^1	the set of independent generalized coordinates
q^2	the set of dependent generalized coordinates
L	the Lagragian, $L = T - V$ (chapter 2)
T	the kinetic energy in the system (chapter 2)
V	the potential energy in the system (chapter 2)
	the Lagragian multiplier (chapter 2)
$[A]$	the coordinate transformation matrix
$[Y_B]$	the admittance matrix of a network in its branch voltage coordinates
v_B	the set of branch voltages
i_B	the set of currents in v_B
$[Y_P]$	the node pair admittance matrix (chapter 2)
v_P	the set of node pair voltages (chapter 2)
i_P	the currents in v_P (chapter 2)
P	the number of nodes (or terminals) in the network
D	the number disjointed parts in a network
B	the number of elements in the network
$S(P)$	the number of different trees that connect the same set of P nodes
i_C	the set of currents in capacitive elements
i_R	the set of currents in resistive elements
i_L	the set of currents in inductive elements
v_C	the set of capacitor branch voltages

v_R	the set of resistor branch voltages
v_L	the set of inductor branch voltages
C	capacitance in FARAD
R	resistive admittance in mho
L	inductive admittance in $(\text{henry})^{-1}$
$[C_B]$, $[R_B]$, $[L_B]$	the capacitive, resistive, inductive admittance matrices in the branch voltage coordinates, v_C, v_R, v_L
$[C]$, $[R]$, $[L]$	the capacitive, resistive, inductive admittance matrices in the node pair voltage coordinates, V
V	the set of node pair voltages
I	the set of currents in V
y	the time integral of V, $y = \int V dt$
V^o, I^o, y^o	the partitioned components of V, I, y, that correspond to the node pairs connected by voltage sources
V^1, I^1, y^1	the partitioned components of V, I, y, that correspond to the node pairs connected by capacitors, with voltage sources short-circuited
V^2, I^2, y^2	the partitioned components of V, I, y, that correspond to the node pairs connected by resistors, with voltage sources and capacitors short-circuited
V^3, I^3, y^3	the partitioned components of V, I, y, that correspond to the node pairs connected by inductors, with voltage sources, capacitors, and resistors short-circuited
V^4, I^4, y^4	the partitioned components of V, I, y, that correspond to the node pairs connected by transformer windings, with voltage sources, capacitors, resistors, and inductors short-circuited

$[C_{ij}]$, $[R_{ij}]$, $[L_{ij}]$	for i, j = (0, 1, 2, 3), the submatrices in $[C]$, $[R]$, $[L]$, partitioned according to the partitioning of V into V^o, V^1, V^2, V^3
V_o	the arbitrary set of independent node pairs to form the base for transformation
$[P^1]$	the congruent transformation that changes V_o into V_1 such that the nonsingular submatrix, C_{11}, is partitioned out of C_1, the capacitor matrix in V_1
V_1	the base coordinate after P^1 being applied on V_o
$[P^2]$	the congruent transformation that changes V_1 into V_2 such that the nonsingular R_{22} is partitioned out of R_2, the resistor matrix in V_2
V_2	the base coordinate after P^2 being applied on V_1
$[C_o]$, $[R_o]$, $[L_o]$	the capacitive, resistive, and inductive admittance matrices in V_o coordinates
$[C_1]$, $[R_1]$, $[L_1]$	the capacitive, resistive, and inductive admittance matrices in V_1 coordinates
$[C_2]$, $[R_2]$, $[L_2]$	the capacitive, resistive, and inductive admittance matrices in V_2 coordinates
v_R^1	the resistor branch voltages whose terminals are connected within V^1
v_R^2	the resistor branch voltages which have at most one terminal connected within V^1
v_L^1, v_L^2	the inductor branch voltages similarly defined as v_R^1 and v_R^2
v_L^3	the inductor branch voltages which have at least one terminal connected within V^3
d_1	the number of components in V^1
d_2	the number of components in V^2

d_3	the number of components in V^3
B_C	the number of capacitors in the network
B_R	the number of resistors in the network
B_L	the number of inductors in the network
B_T	the number of transformers in the network
$[R_B^1]$, $[R_B^2]$	the resistor matrices in the coordinates v_R^1, v_R^2
$[L_B^1]$, $[L_B^2]$, $[L_B^3]$	the inductor matrices in the coordinates v_L^1, v_L^2, v_L^3
j_x	the current component in branch x due to external sources, voltage and current sources
v_v	the set of voltage source branch voltages
v_i	the set of branch voltages the current sources are connected to
J_v	the current vector in v_v (unknown)
J_i	the current vector in v_i (known)
J_B	the current vector in branch coordinates, V_B, due to external sources, voltage and current
I_L^1, I_L^2, I_L^3	the components of current in coordinates V^1, V^2, V^3, due to the inductive elements in the network
I^{1*}, I^{2*}, I^{3*}	the equivalent source currents, the combined result of current and voltage sources
σ	the number of independent parameters that specifies completely the energy distribution in the network
p	the number of nonzero roots of the network
l_L	the number of loops formed by the inductors in the network alone

$e_{k\ell}$	the voltage across the ℓ^{th} winding of the k^{th} transformer in the network
m_k	the number of windings of the k^{th} transformer
$n_{k\ell}$	the relative turns ratio of the ℓ^{th} winding of the k^{th} transformer
M	the total number of linear constraints introduced by ideal transformers
I_W	the current component in I, due to the transformer winding connections
$[A_B]$	the coordinate transformation matrix due to branch connections
$[A_T]$	the coordinate transformation matrix due to ideal transformers
V^d	the subset of V, chosen to be dependent variables due to transformer constraints
V^*	the subset of V, chosen to remain independent in the presence of transformers (chapter 5)
P_A	the number of accessible nodes
V^E	the set of externally accessible node pairs
V^*	the remaining inaccessible node pairs that, in complement to V^E, form the complete set of independent node pairs in the network
I^E	the current vector in V^E
$[Y]$	the short circuit driving point and transfer admittances in V^E coordinates
$[H]$	the matrix polynomial defined as $H = C s^2 + R s + L$
$G(s)$	the polynomial in s, evaluated as the determinant of H^{**}
$g_n, g_{n-1}, \ldots g_o$	the coefficients of the polynomial $G(s) = g_n s^n + g_{n-1} s^{n-1} + \ldots + g_1 s + g_o$

$[F(s)]$	the numerator matrix polymonial of the inverse of $[H^{**}]$ $[H^{**}]^{-1} = \frac{1}{G(s)}[F(s)]$
$[Q]$	the matrix polynomial, defined as $[Q] = [F][H^*_E]$
$[Q_n] \ldots [Q_o]$	the matrix coefficients of Q $[Q] = [Q_n]\, s^n + \ldots + [Q_1]\, s + [Q_o]$
S	the number of separate networks to be connected together
$V_1^E, V_2^E \ldots V_3^E$	the sets of externally accessible node pairs of the S separate networks
$I_1^E, I_2^E, \ldots, I_3^E$	the current vectors in $V_1^E, V_2^E, \ldots, V_3^E$
$[Y_1], [Y_2], \ldots, [Y_S]$	the short circuit driving point and transfer admittance matrices of the S networks
V^A	the set of accessible node pairs after interconnecting the S separate networks together
V^C	the set of node pairs that interconnect the S separate networks
$[A_C]$	the connection transformation matrix that connects the S separate networks together
PN	a partition operator on matrices